Inaugural-Dissertation

zur
Erlangung der Doktorwürde
der
Naturwissenschaftlich-Mathematischen Gesamtfakultät
der
Ruprecht-Karls-Universität
Heidelberg

vorgelegt von

Diplom-Mathematiker Heiko Kröner
aus: Aachen
Tag der mündlichen Prüfung:

Thema

The inverse F-curvature flow in ARW spaces

Gutachter: Prof. Dr. Claus Gerhardt

Abstract: In this thesis we consider the so-called inverse F-curvature flow (IFCF)

$$\dot{x} = -F^{-1}\nu \tag{0.1}$$

in ARW spaces, i.e. in Lorentzian manifolds with a special future singularity. Here, F denotes a curvature function of class (K^*), which is homogenous of degree one, e.g. the n-th root of the Gaussian curvature, and ν the past directed normal. We prove existence of the IFCF for all times and convergence of the rescaled scalar solution in $C^\infty(S_0)$ to a smooth function. Using the rescaled IFCF we maintain a transition from big crunch to big bang into a mirrored spacetime.

Zusammenfassung: In dieser Arbeit betrachten wir den sogenannten inversen F-Krümmungsfluß (IFCF)

$$\dot{x} = -F^{-1}\nu \tag{0.1}$$

in ARW-Räumen, d.h. in Lorentzmannigfaltigkeiten mit einer speziellen Zukunfts-singularität. Hierbei bezeichnet F eine Krümmungsfunktion der Klasse (K^*), die homogen vom Grade eins ist, z.B. die n-te Wurzel aus der Gaußkrümmung, und ν die vergangenheitsgerichtete Normale. Wir beweisen Existenz des IFCF für alle Zeiten und Konvergenz der reskalierten skalaren Lösung in $C^\infty(S_0)$ gegen eine glatte Funktion. Mittels des IFCF erhalten wir einen Übergang von big crunch nach big bang in eine gespiegelte Raumzeit.

Danksagung

Ich möchte mich bei Herrn Prof. Dr. Claus Gerhardt, der jederzeit für Fragen zur Verfügung stand, für die exzellente Betreuung während meiner Promotion sowie für das Heranführen an ein interessantes und auch sehr gute Zukunftsperspektiven eröffnendes Forschungsgebiet bedanken. Weiterhin bedanke ich mich bei meiner Familie und Svenja Reith für ihre Unterstützung.

THE INVERSE F-CURVATURE FLOW IN ARW SPACES

CONTENTS

1. INTRODUCTION

Let $N = N^{n+1}$ be a ARW space with respect to the future, i.e. N is a globally hyperbolic spacetime and a future end N_+ of N can be written as a product $[a, b) \times S_0$, where S_0 is a Riemannian space and there exists a future directed time function $\tau = x^0$ such that the metric in N_+ can be written as

$$
(1.1) \qquad d\breve{s}^2 = e^{2\tilde{\psi}}\{-(dx^0)^2 + \sigma_{ij}(x^0, x)dx^i dx^j\},
$$

where S_0 corresponds to

$$
(1.2) \qquad x^0 = a,
$$

$\tilde{\psi}$ is of the form

$$
(1.3) \qquad \tilde{\psi}(x^0, x) = f(x^0) + \psi(x^0, x),
$$

and we assume that there exists a positive constant c_0 and a smooth Riemannian metric $\bar{\sigma}_{ij}$ on S_0 such that

$$
(1.4) \qquad \lim_{\tau \to b} e^\psi = c_0 \quad \wedge \quad \lim_{\tau \to b} \sigma_{ij}(\tau, x) = \bar{\sigma}_{ij}(x) \quad \wedge \quad \lim_{\tau \to b} f(\tau) = -\infty.
$$

W.l.o.g. we may assume $c_0 = 1$. Then N is ARW with respect to the future, if the derivatives of arbitrary order with respect to space and time of $e^{-2f}\breve{g}_{\alpha\beta}$ converge uniformly to the corresponding derivatives of the following metric

$$
(1.5) \qquad -(dx^0)^2 + \bar{\sigma}_{ij}(x)dx^i dx^j
$$

when x^0 tends to b.

We assume furthermore, that f satisfies the following five conditions

$$
(1.6) \qquad 0 < -f',
$$

there exists $\omega \in \mathbb{R}$ such that

$$
(1.7) \qquad n + \omega - 2 > 0 \quad \wedge \quad \lim_{\tau \to b} |f'|^2 e^{(n+w-2)f} = m > 0.
$$

Set $\tilde{\gamma} = \frac{1}{2}(n + \omega - 2)$, then there exists the limit

$$
(1.8) \qquad \lim_{\tau \to b}(f'' + \tilde{\gamma}|f'|^2)
$$

and

$$
(1.9) \qquad |D_\tau^m(f'' + \tilde{\gamma}|f'|^2)| \leq c_m|f'|^m \quad \forall m \geq 1,
$$

as well as

$$
(1.10) \qquad |D_\tau^m f| \leq c_m|f'|^m \quad \forall m \geq 1.
$$

If S_0 is compact, then we call N a normalized ARW spacetime, if

$$
(1.11) \qquad \int_{S_0} \sqrt{\det \bar{\sigma}_{ij}} = |S^n|.
$$

In the following S_0 is assumed to be compact.

Remark 1.1. (i) If these assumptions are satisfied, then we shall show that the range of τ is finite, hence we may–and shall–assume w.l.o.g. that $b = 0$, i.e.

$$
(1.12) \qquad a < \tau < 0.
$$

(ii) Any ARW space with compact S_0 can be normalized as one easily checks.

To guarantee the C^3-regularity for the transition flow, see Section 11, especially (11.37), we have to impose another technical assumption, namely that the following limit exists

$$(1.13) \qquad \lim_{\tau \to 0} (f'' + \tilde{\gamma}|f'|^2)' \tau.$$

We furthermore assume that in the case $\tilde{\gamma} < 1$ the limit metric $\bar{\sigma}_{ij}$ has non-negative sectional curvature.

We can now state our main theorem, cf. also Section 2 for notations.

Theorem 1.2. *Let N be as above and let $F \in C^\infty(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$ be a curvature function of class (K^*), cf. Definition (2.3), in the positive cone $\Gamma_+ \subset \mathbb{R}^n$, which is in addition positiv homogenous of degree one and normalized such that*

$$(1.14) \qquad F(1, ..., 1) = n.$$

Let M_0 be a smooth, closed, spacelike hypersurface in N which can be written as a graph over S_0 for which we furthermore assume that it is convex and that it satisfies

$$(1.15) \qquad -\epsilon < \inf_{M_0} x^0 < 0,$$

where

$$(1.16) \qquad \epsilon = \epsilon(N, \breve{g}_{\alpha\beta}) > 0.$$

(i) Then the so-called inverse F-curvature flow (IFCF) given by the equation

$$(1.17) \qquad \dot{x} = -\frac{1}{F}\nu$$

with initial surface $x(0) = M_0$ exists for all times. Here, ν denotes the past directed normal.

(ii) If we express the flow hypersurfaces $M(t)$ as graphs over S_0

$$(1.18) \qquad M(t) = \operatorname{graph} u(t, \cdot),$$

and set

$$(1.19) \qquad \tilde{u} = u e^{\gamma t},$$

where $\gamma = \frac{1}{n}\tilde{\gamma}$, then there are positive constants c_1, c_2 such that

$$(1.20) \qquad -c_2 \le \tilde{u} \le -c_1 < 0,$$

and $\tilde{u}$ converges in $C^\infty(S_0)$ to a smooth function, if t goes to infinity.

(iii) Let (g_{ij}) be the induced metric of the leaves $M(t)$ of the inverse F-curvature flow, then the rescaled metric

$$(1.21) \qquad e^{\frac{2}{n}t} g_{ij}$$

converges in $C^\infty(S_0)$ to

$$(1.22) \qquad (\tilde{\gamma}^2 m)^{\frac{1}{\tilde{\gamma}}} (-\tilde{u})^{\frac{2}{\tilde{\gamma}}} \bar{\sigma}_{ij},$$

where we are slightly ambiguous by using the same symbol to denote $\tilde{u}(t, \cdot)$ and $\lim \tilde{u}(t, \cdot)$.

(iv) The leaves $M(t)$ of the IFCF get more umbilical, if t tends to infinity, namely

$$(1.23) \qquad F^{-1}|h_i^j - \frac{1}{n}H\delta_i^j| \le c e^{-2\gamma t}.$$

In case $n + \omega - 4 > 0$, we even get a better estimate, namely

$$(1.24) \qquad |h_i^j - \frac{1}{n} H \delta_i^j| \le c e^{-\frac{1}{2n}(n+\omega-4)t}.$$

In [4] together with [5] this theorem is proved when the curvature F is replaced by the mean curvature of the flow hypersurfaces.

In our proof we go along the lines of [4] and [5] as far as possible, for Section 5 we use [2].

The thesis is organized as follows. In the remainder of the present section we list some well-known properties of f, cf. [8, section 7.3], which will be used later. In Section 2 we introduce some notations and definitions. In Section 3, 4 and 5 we prove Theorem 1.2 (i), in Section 6, 7, 8, 9 and 10 we prove Theorem 1.2 (ii)-(iv) and in Section 11 we will define a so-called transition from big crunch to big bang via the rescaled IFCF into a mirrored universe.

Let us briefly compare our case with the mean curvature case.

Concerning the proof of the existence of the flow the C^0-estimates are similar to the mean curvature case and the C^1-estimates are even easier in our case, since they follow immediately from the convexity of the flow hypersurfaces. For the C^2-estimates we prove the important Lemma 4.11 and obtain with it in Lemma 5.2 the optimal lower bound for the F-curvature of the flow hypersurfaces, at which optimality is not seen until Section 8. The remaining part of the C^2-estimates is different from the mean curvature case but can be found in [2].

Concerning the asymptotic behaviour of the flow the C^0-estimates are similar to the mean curvature case. But the C^1-estimates in Section 7 and particularly the crucial C^2-estimates in Section 8 differ essentially from the mean curvature case. Using the homogeneity of F the C^2-estimates lead to very good decay properties of the derivatives of F, so that from this time on the difference between our and the mean curvature case is only formal.

I would like to thank Claus Gerhardt for many helpful hints.

Lemma 1.3. *Let $f \in C^2([a,b))$ satisfy the conditions*

$$(1.25) \qquad \lim_{\tau \to b} f(\tau) = -\infty$$

and

$$(1.26) \qquad \lim_{\tau \to b} |f'|^2 e^{2\tilde{\gamma} f} = m,$$

where $\tilde{\gamma}, m$ are positive, then b is finite.

Corollary 1.4. *We may–and shall–therefore assume that $b = 0$, i.e., the time interval I is given by $I = [a, 0)$.*

Lemma 1.5. *(i)*

$$(1.27) \qquad \lim_{\tau \to 0} \frac{e^{\tilde{\gamma} f}}{\tau} = -\tilde{\gamma} \sqrt{m}.$$

(ii) There holds

$$(1.28) \qquad f' e^{\tilde{\gamma} f} + \sqrt{m} \sim c\tau^2,$$

where c is a constant, and where the relation

$$(1.29) \qquad \varphi \sim c\tau^2$$

means

$$\text{(1.30)} \qquad \lim_{\tau \to 0} \frac{\varphi(\tau)}{\tau^2} = c.$$

Lemma 1.6. *The asymptotic relation*

$$\text{(1.31)} \qquad \tilde{\gamma} f' \tau - 1 \sim c\tau^2$$

is valid.

2. Notations and definitions

In this section, where we want to introduce some general notations, we assume for N all properties listed from the beginning of Section 1 as far as equation (1.2) except for being ARW and we write ψ instead of $\tilde{\psi}$. Let $M \subset N$ be a connected and spacelike hypersurface with differentiable normal ν (which is then timelike). Geometric quantities in N are denoted by $(\bar{g}_{\alpha\beta})$, $(\bar{R}_{\alpha\beta\gamma\delta})$ etc. and those in M by (g_{ij}), (R_{ijkl}) etc.. Greek indices range from 0 to n, Latin indices from 1 to n; summation convention is used. Coordinates in N and M are denoted by (x^α) and (ξ^i) respectively. Covariant derivatives are written as indices, only in case of possibly confusion we precede them by a semicolon, i.e. for a function u the gradient is (u_α) and $(u_{\alpha\beta})$ the hessian, but for the covariant derivative of the Riemannian curvature tensor we write $\bar{R}_{\alpha\beta\gamma\delta;\epsilon}$.

In local coordinates, (x^α) in N and (ξ^i) in M, the following four important equations hold; the Gauss formular

$$\text{(2.1)} \qquad x_{ij}^\alpha = h_{ij}\nu^\alpha.$$

In this implicit definition (h_{ij}) is the second fundamental form of M with respect to ν. Here and in the following a covariant derivative is always a full tensor, i.e.

$$\text{(2.2)} \qquad x_{ij}^\alpha = x_{,ij}^\alpha - \Gamma_{ij}^k x_k^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha x_i^\beta x_j^\gamma$$

and the comma denotes ordinary partial derivatives.

The second equation is the Weingarten equation

$$\text{(2.3)} \qquad \nu_i^\alpha = h_i^k x_k^\alpha,$$

where ν_i^α is a full tensor. The third equation is the Codazzi equation

$$\text{(2.4)} \qquad h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_j^\gamma x_k^\delta$$

and the fourth is the Gauß equation

$$\text{(2.5)} \qquad R_{ijkl} = -\{h_{ik}h_{jl} - h_{il}h_{jk}\} + \bar{R}_{\alpha\beta\gamma\delta}x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta.$$

As an example for the covariant derivative of a full tensor we give

$$\text{(2.6)} \qquad \bar{R}_{\alpha\beta\gamma\delta;i} = \bar{R}_{\alpha\beta\gamma\delta;\epsilon}x_i^\epsilon,$$

where this identity follows by applying the chain rule from the definition of the covariant derivative of a full tensor; it can be generalized obviously to other quantities.

Let (x^α) be a future directed coordinate system in N, then the contravariant vector $(\xi^\alpha) = (1, 0, ..., 0)$ is future directed; as well its covariant version $(\xi_\alpha) = e^{2\psi}(-1, 0, ..., 0)$.

Now we want to express normal, metric and second fundamemtal form for space-like hypersurfaces, which can be written as graphs over the Cauchyhypersurface. Let $M = \operatorname{graph} u|_{S_0}$ be a spacelike hypersurface in N, i.e.

$$(2.7) \qquad M = \left\{ \left(x^0, x\right) : x^0 = u(x), \ x \in S_0 \right\},$$

then the induced metric is given by

$$(2.8) \qquad g_{ij} = e^{2\psi} \left\{ -u_i u_j + \sigma_{ij} \right\},$$

where σ_{ij} is evaluated at (u, x) and the inverse $\left(g^{ij}\right) = \left(g_{ij}\right)^{-1}$ is given by

$$(2.9) \qquad g^{ij} = e^{-2\psi} \left\{ \sigma^{ij} + \frac{u^i u^j}{v^2} \right\},$$

where $\left(\sigma^{ij}\right) = \left(\sigma_{ij}\right)^{-1}$ and

$$(2.10) \qquad \begin{aligned} u^i &= \sigma^{ij} u_j \\ v^2 &= 1 - \sigma^{ij} u_i u_j \equiv 1 - |Du|^2, \quad v > 0. \end{aligned}$$

We define $\tilde{v} = v^{-1}$.

From (2.8) we conclude that $\operatorname{graph} u$ is spacelike if and only if $|Du| < 1$.

The covariant version of the normal of a graph is

$$(2.11) \qquad (\nu_\alpha) = \pm v^{-1} e^\psi \left(1, -u_i\right)$$

and the contravariant version

$$(2.12) \qquad (\nu^\alpha) = \mp v^{-1} e^{-\psi} \left(1, u^i\right).$$

We have

Remark 2.1. Let M be a spacelike graph in a future directed coordinate system, then

$$(2.13) \qquad (\nu^\alpha) = v^{-1} e^{-\psi} \left(1, u^i\right)$$

is the contravariant future directed normal and

$$(2.14) \qquad (\nu^\alpha) = -v^{-1} e^{-\psi} \left(1, u^i\right)$$

the past directed.

In the following we choose ν always as the past directed normal.

Let us consider the component $\alpha = 0$ in (2.1), so we have due to (2.14) that

$$(2.15) \qquad e^{-\psi} v^{-1} h_{ij} = -u_{ij} - \bar{\Gamma}^0_{00} u_i u_j - \bar{\Gamma}^0_{0j} u_i - \bar{\Gamma}^0_{0i} u_j - \bar{\Gamma}^0_{ij},$$

where u_{ij} are covariant derivatives with respect to M. Choosing $u \equiv \mathrm{const}$, we deduce

$$(2.16) \qquad e^{-\psi} \bar{h}_{ij} = -\bar{\Gamma}^0_{ij},$$

where $\bar{h}_{ij}$ is the second fundamental form of the hypersurface $\left\{x^0 = \mathrm{const}\right\}$. An easy calculation shows

$$(2.17) \qquad e^{-\psi} \bar{h}_{ij} = -\frac{1}{2} \dot{\sigma}_{ij} - \dot{\psi} \sigma_{ij},$$

where the dot indicates differentiation with respect to x^0.

Now we define the classes (K) and (K^*), which are special classes of curvature functions; for a more detailed treatment of these classes we refer to [8, Section 2.2].

For a curvature function F (i.e. symmetric in its variables) in the positive cone $\Gamma_+ \subset \mathbb{R}^n$ we define

(2.18)
$$F(h_{ij}) = F(\kappa_i),$$

where the κ_i are the eigenvalues of an arbitrary symmetric tensor (h_{ij}), whose eigenvalues are in Γ_+.

Definition 2.2. A symmetric curvature function $F \in C^{2,\alpha}(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$, positively homogeneous of degree $d_0 > 0$, is said to be of class (K), if

(2.19)
$$F_i = \frac{\partial F}{\partial \kappa^i} > 0 \quad \text{in } \Gamma_+,$$

(2.20)
$$F_{|\partial\Gamma_+} = 0,$$

and

(2.21)
$$F^{ij,kl}\eta_{ij}\eta_{kl} \le F^{-1}(F^{ij}\eta_{ij})^2 - F^{ik}\tilde{h}^{jl}\eta_{ij}\eta_{kl} \quad \forall\, \eta \in S,$$

where F is evaluated at an arbitrary symmetric tensor (h_{ij}), whose eigenvalues are in Γ_+ and S denotes the set of symmetric tensors. Here, F_i is a partial derivative of first order with respect to κ_i and $F^{ij,kl}$ are second partial derivatives with respect to (h_{ij}). Furthermore $(\tilde{h}^{ij})$ is the inverse of (h_{ij}).

In Theorem 1.2 the κ_i in (2.18) are the eigenvalues of the second fundamental form (h_{ij}) with respect to the metric (g_{ij}), i.e. the principal curvatures of the flow hypersurfaces.

Definition 2.3. A curvature function $F \in (K)$ is said to be of class (K^*), if there exists $0 < \epsilon_0 = \epsilon_0(F)$ such that

(2.22)
$$\epsilon_0 F H \le F^{ij} h_{ik} h_j^k,$$

for any symmetric (h_{ij}) with all eigenvalues in Γ_+, where F is evaluated at (h_{ij}). H represents the mean curvature, i.e. the trace of (h_{ij}).

In the following a '+' sign attached to the symbol of a metric of the ambient space refers to the corresponding Riemannian background metric, if attached to an induced metric, it refers to the induced metric relative to the corresponding Riemannian background metric. Let us consider as an example the metrics $\breve{g}_{\alpha\beta}$ and g_{ij} introduced as above, then

(2.23)
$$\overset{+}{g}_{\alpha\beta} = e^{2\tilde{\psi}}\{(dx^0)^2 + \sigma_{ij}(x^0,x)dx^i dx^j\}, \quad \overset{+}{g}_{ij} = \overset{+}{\breve{g}}_{\alpha\beta}\, x_i^\alpha x_j^\beta.$$

3. C^0-ESTIMATES–EXISTENCE FOR ALL TIMES

Let $M_\tau = \{x^0 = \tau\}$ denote the coordinate slices. Then

(3.1)
$$|M_\tau| = \int_{S_0} e^{n\tilde{\psi}(\tau,x)}\sqrt{|\det \sigma_{ij}(\tau,x)|}dx \longrightarrow 0, \quad \tau \to 0.$$

And for the second fundamental form $\bar{h}_{ij}$ of the M_τ we have

(3.2)
$$\bar{h}_j^i = -e^{-\tilde{\psi}}(\frac{1}{2}\sigma^{ik}\dot{\sigma}_{kj} + \dot{\tilde{\psi}}\delta_j^i),$$

hence there exists τ_0 such that M_τ is convex for all $\tau \ge \tau_0$.

Choosing τ_0 if necessary larger we have

$$(3.3) \qquad e^{\tilde{\psi}} F|_{M_\tau} = e^{\tilde{\psi}} F(\bar{h}_j^i) = F(-\frac{1}{2}\sigma^{ik}\dot{\sigma}_{kj} - \dot{\tilde{\psi}}\delta_j^i) \geq -\delta_0 f' =: \varphi(\tau) \quad \forall \tau \geq \tau_0,$$

where $\delta_0 > 0$ is a constant.

We will show that the flow does not run into the future singularity within finite time.

Lemma 3.1. *There exists a time function $\tilde{x}^0 = \tilde{x}^0(x^0)$, so that the F-curvature $\bar{F}$ of the slices $\{\tilde{x}^0 = const\}$ satisfies*

$$(3.4) \qquad\qquad\qquad\qquad e^{\tilde{\tilde{\psi}}}\bar{F} \geq 1.$$

$e^{\tilde{\tilde{\psi}}}$ *is the conformal factor in the representation of the metric with respect to the coordinates $(\tilde{x}^0, x^i)$, i.e.*

$$(3.5) \qquad\qquad\qquad d\breve{s} = e^{2\tilde{\tilde{\psi}}}\{-(d\tilde{x}^0)^2 + \tilde{\sigma}_{ij}(\tilde{x}^0, x)dx^i dx^j\}.$$

Furthermore there holds

$$(3.6) \qquad\qquad\qquad\qquad \tilde{x}^0(\{\tau_0 \leq x^0 < 0\}) = [0, \infty)$$

and the future singularity corresponds to $\tilde{x}^0 = \infty$.

Proof. Define $\tilde{x}^0$ by

$$(3.7) \qquad \tilde{x}^0 = \int_{\tau_0}^{\tau} \varphi(s)ds = -\int_{\tau_0}^{\tau} \epsilon_0 f' = \epsilon_0 f(\tau_0) - \epsilon_0 f(\tau) \to \infty, \quad \tau \to 0,$$

where φ is chosen as in (3.3). For the conformal factor in (3.5) we have

$$(3.8) \qquad\qquad\qquad e^{2\tilde{\tilde{\psi}}} = e^{2\tilde{\psi}}\frac{\partial x^0}{\partial \tilde{x}^0}\frac{\partial x^0}{\partial \tilde{x}^0} = e^{2\tilde{\psi}}\varphi^{-2}$$

and therefore

$$(3.9) \qquad\qquad\qquad e^{\tilde{\tilde{\psi}}}\bar{F} = e^{\tilde{\psi}}\bar{F}\varphi^{-1} \geq 1.$$

$\square$

The evolution problem (1.17) is a parabolic problem, hence a solution exists on a maximal time interval $[0, T^*)$, $0 < T^* \leq \infty$.

Lemma 3.2. *For any finite $0 < T \leq T^*$ the flow stays in a precompact set Ω_T for $0 \leq t < T$.*

Proof. For the proof we choose with Lemma 3.1 a time function x^0 such that

$$(3.10) \qquad\qquad\qquad\qquad e^{\tilde{\psi}}\bar{F} \geq 1$$

for the coordinate slices $\{x^0 = const\}$. Let

$$(3.11) \qquad\qquad\qquad\qquad M(t) = \text{graph } u(t, \cdot)$$

be the flow hypersurfaces in this coordinate system and

$$(3.12) \qquad\qquad\qquad \varphi(t) = \sup_{S_0} u(t, \cdot) = u(t, x_t)$$

with suitable $x_t \in S_0$. It is well-known that φ is Lipschitz continuous and that for a.e. $0 \leq t < T$

$$(3.13) \qquad\qquad\qquad\qquad \dot{\varphi}(t) = \frac{\partial}{\partial t}u(t, x_t).$$

From (2.15) we deduce in x_t the relation

$$h_{ij} \geq \bar{h}_{ij}, \tag{3.14}$$

hence

$$F \geq \bar{F}. \tag{3.15}$$

We look at the component $\alpha = 0$ in (1.17) and get

$$\dot{u} = \frac{\tilde{v}}{Fe^{\tilde{\psi}}}, \tag{3.16}$$

where

$$\dot{u} = \frac{\partial u}{\partial t} + u_i \dot{x}^i \tag{3.17}$$

is a total derivative. This yields

$$\frac{\partial u}{\partial t} = e^{-\tilde{\psi}} v \frac{1}{F}, \tag{3.18}$$

so that we have in x_t

$$\frac{\partial u}{\partial t} = \frac{1}{e^{\tilde{\psi}} F} \leq \frac{1}{e^{\tilde{\psi}} \bar{F}} \leq 1. \tag{3.19}$$

With (3.13) we conclude

$$\varphi \leq \varphi(0) + t \quad \forall 0 \leq t < T^*, \tag{3.20}$$

which proves the lemma, since the future singularity corresponds to $x^0 = \infty$. □

Remark 3.3. If we choose

$$\varphi(t) = \inf_{S_0} u(t, \cdot) \tag{3.21}$$

in the proof of Lemma 3.2, we can easily derive that the flow runs into the future singularity, which means–in the coordinate system chosen there–

$$\lim_{t \to \infty} \inf_{S_0} u(t, \cdot) = \infty, \tag{3.22}$$

provided the flow exists for all times.

4. C^1-ESTIMATES–EXISTENCE FOR ALL TIMES

As a direct consequence of [8, Theorem 2.7.11] and the convexity of the flow hypersurfaces we have the following

Lemma 4.1. *As long as the flow stays in a precompact set Ω the quantity $\tilde{v}$ is uniformly bounded by a constant, which only depends on Ω.*

Due to later demand our aim in the remainder of this section will be to prove an estimate for $\tilde{v}$ for the leaves of the IFCF on the maximal existence interval $[0, T^*)$, cf. Lemma 4.5 and to prove Lemma 4.11.

To prove this we consider the flow to be embedded in N with the conformal metric

$$\bar{g}_{\alpha\beta} = e^{-2\tilde{\psi}} \breve{g}_{\alpha\beta} = -(dx^0)^2 + \sigma_{ij}(x^0, x) dx^i dx^j. \tag{4.1}$$

This point of view will be later on also a key ingredient in the proof of the convergence results for the flow. Though, formally we have a different ambient space we still denote it by the same symbol N and distinguish only the metrics $\breve{g}_{\alpha\beta}$ resp.

$\bar{g}_{\alpha\beta}$ and the corresponding quantities of the hypersurfaces $\breve{h}_{ij}$, $\breve{g}_{ij}$, $\breve{\nu}$ resp. h_{ij}, g_{ij}, ν, etc., i.e., the standard notations now apply to the case when N is equipped with the metric (4.1).

The second fundamental forms $\breve{h}_i^j$ and h_i^j are related by

$$(4.2) \qquad e^{\tilde{\psi}}\breve{h}_i^j = h_i^j + \tilde{\psi}_\alpha\nu^\alpha\delta_i^j = h_i^j - \tilde{v}f'\delta_i^j + \psi_\alpha\nu^\alpha =_{def} \breve{h}_i^j,$$

cf. [8, Proposition 1.1.11]. When we insert $\breve{h}_i^j$ into F we will denote the result in accordance with our convention as $\breve{F}$. Due to a lack of convexity it would not make any sense to insert h_i^j into the curvature function F, so that we stipulate that the symbol F will stand for

$$(4.3) \qquad F = e^{\tilde{\psi}}\breve{F} = F(h_i^j - \tilde{v}f'\delta_i^j + \psi_\alpha\nu^\alpha),$$

which will be useful, cf. (4.5).

Quantities like $\tilde{v}$, that are not different if calculated with respect to $\breve{g}_{\alpha\beta}$ or $\bar{g}_{\alpha\beta}$ are denoted in the usual way.

These notations introduced above will be used in the present section as well as from the beginning of Section 6 to the end of this thesis.

Due to

$$(4.4) \qquad \breve{\nu} = e^{-\tilde{\psi}}\nu$$

the evolution equation $\dot{x} = -\frac{1}{\breve{F}}\breve{\nu}$ can be written as

$$(4.5) \qquad \dot{x} = -\frac{1}{F}\nu.$$

Lemma 4.2. *(Evolution of $\tilde{v}$) Consider the flow (4.5). Then $\tilde{v}$ satisfies the evolution equation*

$$
\begin{aligned}
(4.6) \quad &\dot{\tilde{v}} - F^{-2}F^{ij}\tilde{v}_{ij} = -F^{-2}F^{ij}h_{kj}h_i^k\tilde{v} + F^{-2}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l \\
&\quad - F^{-2}F^{ij}h_{ij}\eta_{\alpha\beta}\nu^\alpha\nu^\beta - F^{-1}\eta_{\alpha\beta}\nu^\alpha\nu^\beta \\
&\quad - F^{-2}(F^{ij}\eta_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma + 2F^{ij}\eta_{\alpha\beta}x_k^\alpha x_i^\beta h_j^k) \\
&\quad - F^{-2}(-\tilde{v}f''\|Du\|^2F^{ij}g_{ij} - \tilde{v}_k u^k f'F^{ij}g_{ij} \\
&\quad + \psi_{\alpha\beta}\nu^\alpha x_k^\beta u^k F^{ij}g_{ij} + \psi_\alpha x_l^\alpha h_k^l u^k F^{ij}g_{ij}),
\end{aligned}
$$

where $\eta = (\eta_\alpha) = (-1, 0, ..., 0)$ is a covariant unit vectorfield.

Proof. We have

$$(4.7) \qquad \tilde{v} = \eta_\alpha\nu^\alpha.$$

Let (ξ^i) be local coordinates for $M(t)$; differentiating $\tilde{v}$ covariantly yields

$$(4.8) \qquad \tilde{v}_i = \eta_{\alpha\beta}x_i^\beta\nu^\alpha + \eta_\alpha\nu_i^\alpha$$

and

$$
\begin{aligned}
(4.9) \quad \tilde{v}_{ij} = &\eta_{\alpha\beta\gamma}x_i^\beta x_j^\gamma\nu^\alpha + \eta_{\alpha\beta}\nu_j^\alpha x_i^\beta + \eta_{\alpha\beta}\nu^\alpha\nu^\beta h_{ij} + \eta_\alpha x_{rj}^\alpha h_i^r \\
&+ \eta_\alpha x_r^\alpha h_{i;j}^r + \eta_{\alpha\beta}x_j^\beta x_r^\alpha h_i^r.
\end{aligned}
$$

As usual, cf. [8, Lemma 2.3.2], the evolution equation for the normal is

$$(4.10) \qquad \dot{\nu}^\alpha = g^{ij}(-\frac{1}{F})_i x_j^\alpha = \frac{1}{F^2}g^{ij}F_i x_j^\alpha$$

and for the time derivative of $\tilde{v}$ we get

$$
\begin{aligned}
\dot{\tilde{v}} &= \eta_{\alpha\beta}\nu^\alpha \dot{x}^\beta + \eta_\alpha \dot{\nu}^\alpha \\
&= -\frac{1}{F}\eta_{\alpha\beta}\nu^\alpha\nu^\beta - \frac{1}{F^2}g^{ij}F_i u_j.
\end{aligned}
$$
(4.11)

Writing

$$
\begin{aligned}
F_k &= F^{ij}h_{ij;k} - \tilde{v}_k f' F^{ij}g_{ij} - \tilde{v}f'' u_k F^{ij}g_{ij} \\
&\quad + \psi_{\alpha\beta}\nu^\alpha x_k^\beta F^{ij}g_{ij} + \psi_\alpha x_r^\alpha h_k^r F^{ij}g_{ij}
\end{aligned}
$$
(4.12)

and using the Codazzi equation

$$
h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_j^\gamma x_k^\delta
$$
(4.13)

we deduce the desired evolution equation for $\tilde{v}$ by putting together the above equations. $\qquad\square$

We now present some auxiliary estimates which will be needed in the following.

Lemma 4.3. *Let $|||\cdot|||$ denote the norm of a tensor with respect to the Riemannian metric $\overset{+}{g}_{\alpha\beta}$, cf Section 2, then*
(i)

$$
\begin{aligned}
|\eta_{\alpha\beta}\nu^\alpha\nu^\beta| &\leq c\tilde{v}^2 |||\eta_{\alpha\beta}|||, \\
|F^{ij}\eta_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma| &\leq c\tilde{v}^3 |||\eta_{\alpha\beta\gamma}||| F^{ij}g_{ij}, \\
|\psi_{\alpha\beta}\nu^\alpha x_k^\beta u^k| &\leq c|||\eta_{\alpha\beta}|||\tilde{v}^3.
\end{aligned}
$$
(4.14)

(ii) For any $\epsilon > 0$ we have

$$
|F^{ij}\eta_{\alpha\beta}x_k^\alpha x_i^\beta h_j^k| \leq c\epsilon\tilde{v}F^{ij}h_{kj}h_i^k |||\eta_{\alpha\beta}||| + c_\epsilon\tilde{v}^3 F^{ij}g_{ij}|||\eta_{\alpha\beta}|||.
$$
(4.15)

(iii)

$$
|F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l| \leq c\tilde{v}^3 F^{ij}g_{ij}.
$$
(4.16)

(iv) Furthermore

$$
|\psi_\alpha x_k^\alpha h_i^k u^i| \leq c|||D\psi|||\tilde{v}^3
$$
(4.17)

in points where $\tilde{v}_i = 0$.

Remark 4.4. These are tensor estimates, i.e. not depending on the special local coordinates of the hypersurface and S_0. But to prove these estimates we sometimes choose special coordinates such that in a fixed point $g_{ij} = \delta_{ij}$, $\overset{+}{g}_{ij}= diagonal$.

Proof of Lemma 4.3. We have $|||\nu^\alpha||| \leq 2\tilde{v}$,

$$
\overset{+}{g}_{ij}\leq 2\sigma_{ij} \leq 2\tilde{v}^2 g_{ij} \quad \wedge \quad g^{ij} \leq c\tilde{v}^2\sigma^{ij} \quad \wedge \quad u^i = \tilde{v}^2\check{u}^i
$$
(4.18)

and $\|Du\|^2 = \tilde{v}^2|Du|^2$.

Proof of (i): Using these properties together with Schwarz inequality proves (i).

Proof of (ii):

$$|||F^{ij}x_k^\alpha x_i^\beta h_j^k|||^2 = F^{ij}F^{\bar{i}\bar{j}}h_j^k h_{\bar{j}}^{\bar{k}} \overset{+}{g}_{k\bar{k}} \overset{+}{g}_{i\bar{i}}, \quad g_{ij} = \delta_{ij}, \overset{+}{g}_{ij} = diagonal$$

$$\leq c\tilde{v}^4 F^{ij}F^{\bar{i}\bar{j}}h_j^k h_{\bar{j}}^{\bar{k}} g_{k\bar{k}} g_{i\bar{i}}, \quad g_{ij} = \delta_{ij}, h_{ij} = \kappa_i\delta_{ij}, F^{ij} = diagonal$$

(4.19)
$$\leq c\tilde{v}^4 \sum_i (F^{ii})^2 (h_{ii})^2$$

$$\leq c\tilde{v}^4 (\sum_i F^{ii}|h_{ii}|)^2$$

$$\leq c\tilde{v}^4 (\sum_i F^{ii}(\frac{\epsilon}{\tilde{v}}h_{ii}^2 + c_\epsilon\tilde{v}g_{ii}))^2,$$

taking the square root yields the result.

Proof of (iii): The following proof can be found in [8, Lemma 5.4.5]. Let $p \in M(t)$ be arbitrary. Let (x^α) be the special Gaussian coordinate system of N and (ξ^i) local coordinates around p such that

$$x_i^\alpha = \begin{cases} u_i & , \quad \alpha = 0 \\ \delta_i^k & , \quad \alpha = k. \end{cases}$$

All indices are raised with respect to g^{ij} with exception of

(4.20)
$$\check{u}^i = \sigma^{ij}u_j.$$

We point out that

(4.21)
$$\|Du\|^2 = g^{ij}u_iu_j = \tilde{v}^2\sigma^{ij}u_iu_j = \tilde{v}^2|Du|^2$$
$$(\nu^\alpha) = -\tilde{v}(1,\check{u}^i)$$

and

(4.22)
$$\eta_\epsilon x_l^\epsilon g^{kl} = -u^k.$$

We have

(4.23)
$$-F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_k^\gamma x_j^\delta u^k = F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_k^\gamma x_j^\delta \eta_\epsilon x_l^\epsilon g^{kl}.$$

Let

(4.24)
$$a_{ij} = \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_k^\gamma x_j^\delta \eta_\epsilon x_l^\epsilon g^{kl}.$$

We shall show that the symmetrization $\tilde{a}_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ of a_{ij} satisfies

(4.25)
$$-c\tilde{v}^3 g_{ij} \leq \tilde{a}_{ij} \leq c\tilde{v}^3 g_{ij}$$

with a uniform constant c. We have $F^{ij}\tilde{a}_{ij} = F^{ij}a_{ij}$, and assuming (4.25) as true the claim then follows by chosing a coordinate system such that $g_{ij} = \delta_{ij}$ and $\tilde{a}_{ij} = diagonal$.

Now we prove (4.25). For this let e_r, $1 \leq r \leq n$, be an orthonormal basis of $T_p(M(t))$ and let $\lambda^r e_r$ be an arbitrary vector in $T_p(M(t))$ then we have with $e_r = (e_r^i)$ that

(4.26)
$$|\tilde{a}_{ij}\lambda^r e_r^i \lambda^s e_s^j| \leq n \max_{r,s} |\tilde{a}_{ij}e_r^i e_s^j| \sum_r |\lambda^r|^2$$

and

(4.27)
$$g_{ij}\lambda^r e_r^i \lambda^s e_s^j = \sum_r |\lambda^r|^2$$

so that it will suffice to show that

$$(4.28) \qquad \max_{r,s} |\tilde{a}_{ij} e_r^i e_s^j| \leq c\tilde{v}^3$$

for some special choice of orthonormal basis e_r.

To prove (4.28) we may assume $Du \neq 0$ so that we can specialize our orthonormal basis by requiring that

$$(4.29) \qquad e_1 = \frac{Du}{\|Du\|},$$

here more precisely we had to write down the contravariant version of Du.

For $2 \leq k \leq n$, the e_k are also orthonormal with respect to the metric σ_{ij} and it is also valid that

$$(4.30) \qquad \sigma_{ij} \breve{u}^i e_k^j = 0 \quad \forall 2 \leq k \leq n.$$

In view of (4.22) and the symmetry properties of the Riemann curvature tensor we have

$$(4.31) \qquad a_{ij} u^j = 0.$$

Next we shall expand the right side of (4.24) explicitly yielding

$$(4.32) \qquad \begin{aligned} a_{ij} =& \bar{R}_{0i0j} \tilde{v} \|Du\|^2 + \bar{R}_{0ik0} \tilde{v} u_j u^k + \bar{R}_{0ikj} \tilde{v} u^k \\ &+ \bar{R}_{l0k0} \tilde{v} u^k \breve{u}^l u_i u_j + \bar{R}_{l00j} \tilde{v} \breve{u}^l u_i \|Du\|^2 \\ &+ \bar{R}_{lokj} \tilde{v} u^k \breve{u}^l u_i + \bar{R}_{li0j} \tilde{v} \breve{u}^l \|Du\|^2 \\ &+ \bar{R}_{lik0} \tilde{v} u^k \breve{u}^l u_j + \bar{R}_{likj} \tilde{v} u^k \breve{u}^l. \end{aligned}$$

For $2 \leq r, s \leq n$, we deduce from (4.32)

$$(4.33) \qquad \begin{aligned} a_{ij} e_r^i e_s^j =& \bar{R}_{0i0j} \tilde{v} \|Du\|^2 e_r^i e_s^j + \bar{R}_{0ikj} \tilde{v} u^k e_r^i e_s^j \\ &+ \bar{R}_{li0j} \tilde{v} \breve{u}^l \|Du\|^2 e_r^i e_s^j + \bar{R}_{likj} \tilde{v} u^k \breve{u}^l e_r^i e_s^j \end{aligned}$$

and hence

$$(4.34) \qquad |a_{ij} e_r^i e_s^j| \leq c\tilde{v}^3 \quad \forall\, 2 \leq r, s \leq n.$$

It remains to estimate $a_{ij} e_1^i e_r^j$ for $2 \leq r \leq n$ because of (4.31).

We deduce from (4.32)

$$(4.35) \qquad a_{ij} e_1^i e_r^j = \bar{R}_{0i0j} \tilde{v} \|Du\|^2 \tilde{v}^{-2} e_1^i e_r^j + \bar{R}_{0ikj} \tilde{v}^{-1} u^k e_1^i e_r^j,$$

where we used the symmetry properties of the Riemann curvature tensor.

Hence, we conclude

$$(4.36) \qquad |a_{ij} e_1^i e_r^j| \leq c\tilde{v}^2 \quad \forall 2 \leq r \leq n,$$

and the relation (4.28) is proved.

Proof of (iv): Differentiating the equation

$$(4.37) \qquad \tilde{v}^2 = 1 + \|Du\|^2$$

with respect to i yields

$$(4.38) \qquad 0 = 2\tilde{v}\tilde{v}_i = 2u_{ij} u^j$$

which implies in view of

$$(4.39) \qquad \tilde{v} h_{ij} = -u_{ij} + \bar{h}_{ij},$$

cf. Section 2, that

$$(4.40) \qquad h_{ij}u^j = \tilde{v}\bar{h}_{ij}\check{u}^j$$

hence

$$(4.41) \qquad \begin{aligned} \psi_\alpha x_k^\alpha h_i^k u^i &= \psi_\alpha x_k^\alpha g^{kl} h_{li} u^i = \tilde{v}\psi_\alpha x_k^\alpha g^{kl} \bar{h}_{li}\check{u}^i \\ &= \tilde{v}\psi_\alpha x_k^\alpha (\sigma^{kl} + \tilde{v}^2 \check{u}^k \check{u}^l)\bar{h}_{li}\check{u}^i. \end{aligned}$$

Applying Schwarz inequality finishes the proof. $\qquad\square$

Lemma 4.5. *$\tilde{v}$ is uniformly bounded on $[0, T^*)$ namely*

$$(4.42) \qquad \sup_{[0,T^*)} \tilde{v} \le c = c(\sup_{M_0} \tilde{v}, (N, \check{g}_{\alpha\beta})).$$

Proof. We have (1.15) in mind. For $0 < T < T^*$ assume that there are $0 < t_0 \le T$ and $x_0 \in S_0$ such that

$$(4.43) \qquad \sup_{[0,T]} \sup_{M(t)} \tilde{v} = \tilde{v}(t_0, x_0) \ge 2.$$

In (t_0, x_0) we have $\|Du\|^2 \ge \frac{1}{4}\tilde{v}^2$,

$$(4.44) \qquad 0 \le \dot{\tilde{v}} - F^{-2}F^{ij}\tilde{v}_{ij},$$

and after multiplying this inequality by F^2 we get if $\epsilon > 0$ sufficiently small that

$$(4.45) \qquad \begin{aligned} 0 \le & - F^{ij}h_{kj}h_i^k\tilde{v} + F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l - F^{ij}h_{ij}\eta_{\alpha\beta}\nu^\alpha\nu^\beta \\ & - F\eta_{\alpha\beta}\nu^\alpha\nu^\beta - F^{ij}\eta_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma + 2F^{ij}\eta_{\alpha\beta}x_k^\alpha x_i^\beta h_j^k \\ & + \tilde{v}f''\|Du\|^2 F^{ij}g_{ij} + \tilde{v}_k u^k f' F^{ij}g_{ij} - \psi_{\alpha\beta}\nu^\alpha x_k^\beta u^k F^{ij}g_{ij} \\ & - \psi_\alpha x_l^\alpha h_k^l u^k F^{ij}g_{ij} \\ \le & - \frac{1}{2}F^{ij}h_{kj}h_i^k\tilde{v} + c\tilde{v}^3|f'|F^{ij}g_{ij} + \tilde{v}f''\|Du\|^2 F^{ij}g_{ij}, \end{aligned}$$

which is a contradiction if $\epsilon > 0$ very small.

Hence

$$(4.46) \qquad \tilde{v}(t_0, x_0) \le \max(\sup_{M_0}\tilde{v}, 2).$$

$\qquad\square$

We prove a decay property of certain tensors.

Lemma 4.6. *(i) Let $\varphi \in C^\infty([a,0))$, $a < 0$, and assume*

$$(4.47) \qquad \lim_{\tau \to 0} \varphi^{(k)}(\tau) = 0 \quad \forall k \in \mathbb{N},$$

then for every $k \in \mathbb{N}$ there exists a $c_k > 0$ such that

$$(4.48) \qquad |\varphi(\tau)| \le c_k|\tau|^k.$$

(ii) Let T be a tensor such that for all $k \in \mathbb{N}$

$$(4.49) \qquad |||D^k T(x^0, x)||| \longrightarrow 0 \quad as \quad x^0 \longrightarrow 0 \quad uniformly\ in\ x$$

then

$$(4.50) \qquad \forall_{k\in\mathbb{N}} \quad \exists_{c_k>0} \quad \forall_{x\in S_0} \quad |||T(x^0, x)||| \le c_k|x^0|^k$$

(iii) For $T = (\eta_{\alpha\beta})$ the relation (4.50) is true, analogously for $|||\eta_{\alpha\beta\gamma}|||$, $|||D\psi|||$, $|||\bar{R}_{\alpha\beta\gamma\delta}\eta^{\alpha}|||$, or more generally for any tensor that would vanish identically, if it would have been formed with respect to the product metric

$$(4.51) \qquad\qquad -(dx^0)^2 + \bar{\sigma}_{ij}dx^i dx^j.$$

Proof. (i) From the assumptions it follows that

$$(4.52) \qquad\qquad \sup_{[a,0)} |\varphi^{(k)}| \leq c_k.$$

From the mean value theorem we get

$$(4.53) \qquad\qquad \sup_{[\tau,\tau_0]} |\varphi^{(k)}| \leq |\varphi^{(k)}(\tau_0)| + |\tau| \sup_{[\tau,\tau_0]} |\varphi^{(k+1)}|$$

and therefore

$$(4.54) \qquad\qquad \sup_{[\tau,\tau_0]} |\varphi| \leq \sum_{l=0}^{k-1} |\tau|^l |\varphi^{(l)}(\tau_0)| + |\tau|^k \sup_{[\tau,\tau_0]} |\varphi^{(k)}|,$$

hence taking the limit $\tau_0 \to 0$ yields

$$(4.55) \qquad\qquad |\varphi(\tau)| \leq c_k |\tau|^k.$$

(ii) For simplicity we only consider $T = (T^{\alpha})$. Choose $x \in S_0$ arbitrary and define

$$(4.56) \qquad\qquad \varphi(\tau) = |||T(\tau,x)|||^2 = T^{\alpha}T^{\beta} \overset{+}{g}_{\alpha\beta}$$

then we have

$$(4.57) \qquad\qquad \varphi^{(1)}(\tau) = 2T^{\alpha}_{;\gamma}\eta^{\gamma}T^{\beta} \overset{+}{g}_{\alpha\beta} + T^{\alpha}T^{\beta} \overset{+}{g}_{\alpha\beta;\delta} \eta^{\delta}$$

so that one easily checks that φ satisfies (4.47) and (4.52) with c_k not depending on x. The claim now follows by (i).

(iii) The tensor $T = \eta_{\alpha\beta}$ is a covariant derivative of η_{α} with respect to the metric $\bar{g}_{\alpha\beta}$. If we would have calculated this covariant derivative with respect to the limit metric

$$(4.58) \qquad\qquad -(dx^0)^2 + \bar{\sigma}_{ij}(x)dx^i dx^j$$

then it would vanish identically, as well as all its derivatives of arbitrary order. From this together with the convergence properties of $\bar{g}_{\alpha\beta}$ we deduce that T satisfies the assumptions in (ii), so that the claim follows. The remaining estimates are similarly proved via (ii). □

Now we prove a result for general convex, spacelike graphs.

Lemma 4.7. *Let $\epsilon > 0$ be arbitrary, then there exists $\delta = \delta((N, \breve{g}_{\alpha\beta}), \epsilon) > 0$ such that for every closed, spacelike, convex hypersurface M in the end $N_{\delta}^+ = \{x^0 > -\delta\}$ holds*

$$(4.59) \qquad\qquad \tilde{v} \leq \epsilon |f'|^{\frac{1}{\gamma}}.$$

Proof. Let $p > \tilde{\gamma}^{-1}$ and define

$$(4.60) \qquad\qquad w = \tilde{v}\{e^f + |u|^p\}$$

and look at a point, where w attains its maximum, and infer

$$
\begin{aligned}
0 = w_i &= \tilde{v}_i \{e^f + |u|^p\} + \tilde{v}\{e^f f^{'} - p|u|^{p-1}\}u_i \\
&= \{-h_{ik}u^k + \tilde{v}^{-1}\bar{h}_{ik}u^k\}\{e^f + |u|^p\} + \tilde{v}\{e^f f^{'} - p|u|^{p-1}\}u_i \\
&= \{-\breve{h}_{ik}u^k e^{-\breve{\psi}} - \tilde{v}f^{'}u_i + \tilde{\epsilon}\tilde{v}u_i\}\{e^f + |u|^p\} + \tilde{v}\{e^f f^{'} - p|u|^{p-1}\}u_i,
\end{aligned}
$$
(4.61)

where

$$|\tilde{\epsilon}| \le c_m |u|^m \quad \forall\, m \in \mathbb{N}.$$
(4.62)

Multiplying by u^i and assuming $Du \ne 0$ we get the inequality

$$
\begin{aligned}
0 &\le (-f^{'} + \tilde{\epsilon})\{e^f + |u|^p\} + e^f f^{'} - p|u|^{p-1} \\
&= -f^{'}|u|^p + \tilde{\epsilon}\{e^f + |u|^p\} - p|u|^{p-1} < 0,
\end{aligned}
$$
(4.63)

if $\delta > 0$ small, since

$$f^{'}u \le \tilde{\gamma}^{-1} + cu^2.$$
(4.64)

This is a contradiction, hence $Du = 0$.

Since

$$\varphi(\tau) = e^{f(\tau)} + |\tau|^p, \quad a \le \tau < 0,$$
(4.65)

is monotone decreasing we conclude

$$\tilde{v} \le \frac{e^{f(u_{\min})} + |u_{\min}|^p}{e^{f(u)} + |u|^p} \le (e^{f(u_{\min})} + |u_{\min}|^p)e^{-f(u)},$$
(4.66)

where $u_{\min} = \inf u$. Choosing δ appropiately small finishes the proof, where we used Lemma 1.5 (ii). $\qquad\square$

Remark 4.8. We also could have chosen

$$w = \tilde{v}\{|u|^{\frac{1}{\tilde{\gamma}}} + |u|^p\}$$
(4.67)

in (4.60).

Corollary 4.9. *Let $\delta > 0$ be small and N_δ^+ and M be as in Lemma 4.7, then*

$$F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta \ge -c\delta F^{ij}g_{ij},$$
(4.68)

if the limit metric $\bar{\sigma}_{ij}$ has non-negative sectional curvature.

Proof. We define

$$\bar{R}_{\alpha\beta\gamma\delta}(0,\cdot) = \lim_{\tau\uparrow 0}\bar{R}_{\alpha\beta\gamma\delta}(\tau,\cdot)$$
(4.69)

and have

$$
\begin{aligned}
F^{ij}&\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta \\
&= F^{ij}(\bar{R}_{\alpha\beta\gamma\delta}(0,\cdot) + \bar{R}_{\alpha\beta\gamma\delta}(u,\cdot) - \bar{R}_{\alpha\beta\gamma\delta}(0,\cdot))\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta \\
&\ge F^{ij}(\bar{R}_{\alpha\beta\gamma\delta}(u,\cdot) - \bar{R}_{\alpha\beta\gamma\delta}(0,\cdot))\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta \\
&\ge -|||F^{ij}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta||| \cdot |||\bar{R}_{\alpha\beta\gamma\delta}(u,\cdot) - \bar{R}_{\alpha\beta\gamma\delta}(0,\cdot)||| \\
&\ge -c_m|u|^m F^{ij}g_{ij},
\end{aligned}
$$
(4.70)

for arbitrary $m \in \mathbb{N}$ and suitable c_m. Note that we used for the last inequality that

$$\bar{R}_{\alpha\beta\gamma\delta}(x^0,\cdot) - \bar{R}_{\alpha\beta\gamma\delta}(0,\cdot)$$
(4.71)

satisfies (4.49). □

We want to formulate the relation of the curvature tensors for conformal metrics.

Lemma 4.10. *The curvature tensors of the metrics $\breve{g}_{\alpha\beta}, \bar{g}_{\alpha\beta}$ are related by*

$$
\begin{aligned}
e^{-2\tilde{\psi}}\breve{R}_{\alpha\beta\gamma\delta} =&\bar{R}_{\alpha\beta\gamma\delta} - \bar{g}_{\alpha\gamma}\tilde{\psi}_{\beta\delta} - \bar{g}_{\beta\delta}\tilde{\psi}_{\alpha\gamma} + \bar{g}_{\alpha\delta}\tilde{\psi}_{\beta\gamma} + \bar{g}_{\beta\gamma}\tilde{\psi}_{\alpha\delta} \\
&+ \bar{g}_{\alpha\gamma}\tilde{\psi}_{\beta}\tilde{\psi}_{\delta} + \bar{g}_{\beta\delta}\tilde{\psi}_{\alpha}\tilde{\psi}_{\gamma} - \bar{g}_{\alpha\delta}\tilde{\psi}_{\beta}\tilde{\psi}_{\gamma} - \bar{g}_{\beta\gamma}\tilde{\psi}_{\alpha}\tilde{\psi}_{\delta} \\
&+ \{\bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma} - \bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta}\}\|D\tilde{\psi}\|^2.
\end{aligned}
\tag{4.72}
$$

Now we are able to prove the following lemma which is necessary for the C^2-estimates in the next section.

Lemma 4.11. *There exists a constant $\tilde{c} > 0$ such that we have for the leaves of the IFCF*

$$
\breve{F}^{ij}\breve{R}_{\alpha\beta\gamma\delta}\breve{\nu}^{\alpha}x_i^{\beta}\breve{\nu}^{\gamma}x_j^{\delta} \geq \tilde{c}|f'|^2 e^{-2\tilde{\psi}}
\tag{4.73}
$$

provided

$$
-\epsilon < \inf_{M_0} x^0 < 0,
\tag{4.74}
$$

where $\epsilon = \epsilon(N, \breve{g}_{\alpha\beta})$. Here $\breve{F}^{ij}$ is evaluated at $\breve{h}_i^j$.

Proof. In view of the homogeneity of F we have

$$
F_j^i = \breve{F}_j^i,
\tag{4.75}
$$

hence

$$
F^{ij} = e^{2\tilde{\psi}}\breve{F}^{ij}.
\tag{4.76}
$$

We have due to Lemma 4.10

$$
\begin{aligned}
e^{2\tilde{\psi}}&\breve{F}^{ij}\breve{R}_{\alpha\beta\gamma\delta}\breve{\nu}^{\alpha}x_i^{\beta}\breve{\nu}^{\gamma}x_j^{\delta} \\
&= F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^{\alpha}x_i^{\beta}\nu^{\gamma}x_j^{\delta} + F^{ij}x_i^{\beta}x_j^{\delta}\tilde{\psi}_{\beta\delta} - F^{ij}g_{ij}\tilde{\psi}_{\alpha\gamma}\nu^{\alpha}\nu^{\gamma} \\
&\quad - F^{ij}x_i^{\beta}x_j^{\delta}\tilde{\psi}_{\beta}\tilde{\psi}_{\delta} + F^{ij}g_{ij}\tilde{\psi}_{\alpha}\tilde{\psi}_{\gamma}\nu^{\alpha}\nu^{\gamma} + F^{ij}g_{ij}\|D\tilde{\psi}\|^2.
\end{aligned}
\tag{4.77}
$$

We have

$$
\overset{+}{g}_{ij} \leq 2\sigma_{ij} \leq 2\tilde{v}^2 g_{ij}.
\tag{4.78}
$$

Now we estimate each summand in (4.77) separately with the help of the Riemannian background metric $\overset{+}{g}_{\alpha\beta}$, namely

$$
|F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^{\alpha}x_i^{\beta}\nu^{\gamma}x_j^{\delta}| \leq c\tilde{v}^2(F^{ij}F^{\bar{i}\bar{j}}\overset{+}{g}_{i\bar{i}}\overset{+}{g}_{j\bar{j}})^{\frac{1}{2}} \leq c\tilde{v}^2 F^{ij}\sigma_{ij} \leq c\tilde{v}^4 F^{ij}g_{ij},
\tag{4.79}
$$

$$
F^{ij}x_i^{\beta}x_j^{\delta}\tilde{\psi}_{\beta\delta} = F^{ij}u_iu_jf'' + F^{ij}x_i^{\beta}x_j^{\delta}\psi_{\beta\delta} \geq F^{ij}u_iu_jf'' - c\tilde{v}^2 F^{ij}g_{ij},
\tag{4.80}
$$

$$
\begin{aligned}
-F^{ij}g_{ij}\tilde{\psi}_{\alpha\gamma}\nu^{\alpha}\nu^{\gamma} &= -\tilde{v}^2 F^{ij}g_{ij}f'' - F^{ij}g_{ij}\psi_{\alpha\gamma}\nu^{\alpha}\nu^{\gamma} \\
&\geq -\tilde{v}^2 F^{ij}g_{ij}f'' - c\tilde{v}^2 F^{ij}g_{ij},
\end{aligned}
\tag{4.81}
$$

$$-F^{ij}x_i^\beta x_j^\delta \tilde{\psi}_\beta \tilde{\psi}_\delta = -F^{ij}u_i u_j (\psi_0 + f^{'})^2 - F^{ij}\psi_i \psi_j - 2F^{ij}u_j \psi_i (\psi_0 + f^{'})$$

$$\geq -F^{ij}u_i u_j (\psi_0 + f^{'})^2 - c(1 + |f^{'}||Du|)F^{ij}\sigma_{ij}|D\psi|$$

(4.82)

$$\geq -F^{ij}u_i u_j (\psi_0 + f^{'})^2 - c\tilde{v}^2(1 + |f^{'}||Du|)F^{ij}g_{ij}|D\psi|$$

$$\geq -F^{ij}u_i u_j (\psi_0 + f^{'})^2 - c|f^{'}|\tilde{v}^2 F^{ij}g_{ij},$$

where $|D\psi|^2 = \sigma^{ij}\psi_i \psi_j$,

(4.83) $$F^{ij}g_{ij}\tilde{\psi}_\alpha \tilde{\psi}_\gamma \nu^\alpha \nu^\gamma \geq \tilde{v}^2(\psi_0 + f^{'})^2 F^{ij}g_{ij} - c\tilde{v}^2 |f^{'}|F^{ij}g_{ij},$$

$$F^{ij}g_{ij}\|D\tilde{\psi}\|^2 = -(f^{'} + \psi_0)^2 F^{ij}g_{ij} + \sigma^{ij}\psi_i \psi_j F^{ij}g_{ij}$$

(4.84)

$$\geq -(f^{'} + \psi_0)^2 F^{ij}g_{ij} - cF^{ij}g_{ij}.$$

Thus we conclude (using $u_i u_j \leq (\tilde{v}^2 - 1)g_{ij}$)

$$e^{2\tilde{\psi}}\breve{F}^{ij}\breve{R}_{\alpha\beta\gamma\delta}\breve{\nu}^\alpha x_i^\beta \breve{\nu}^\gamma x_j^\delta \geq -c\tilde{v}^4 F^{ij}g_{ij} + F^{ij}u_i u_j f^{''} - \tilde{v}^2 f^{''}F^{ij}g_{ij}$$

$$- c\tilde{v}^2 |f^{'}|F^{ij}g_{ij}$$

(4.85)

$$+ (\psi_0 + f^{'})^2 F^{ij}(\tilde{v}^2 g_{ij} - u_i u_j - g_{ij})$$

$$\geq -c\tilde{v}^4 F^{ij}g_{ij} - \tilde{v}^2 f^{''}F^{ij}g_{ij} - c|f^{'}|\tilde{v}^2 F^{ij}g_{ij}.$$

Now, the claim follows with Lemma 4.7 if $\tilde{\gamma} \geq 1$, cf. (1.8).

Let us now consider the case $\tilde{\gamma} < 1$. Due to assumption the limit metric $\bar{\sigma}_{ij}$ has non-negative sectional curvature. Now we use Corollary 4.9 to bound the first summand of the right side of (4.77) from below by the term $-cF^{ij}g_{ij}$, one easily checks that this term replaces the summand with $\tilde{v}^4$ in (4.85) completing the proof.   □

Remark 4.12. Lemma 4.11 is also true for general convex, spacelike graphs over S_0 in a future end of N, we did not use in the proof that the hypersurfaces are flow hypersurfaces of the IFCF.

Before we consider the C^2-estimates in the next section we show that N satisfies the timelike convergence condition with respect to the future.

Corollary 4.13. *Lemma 4.11 remains valid, if we replace inequality (4.73) by*

(4.86) $$\breve{R}_{\alpha\beta}\breve{\nu}^\alpha \breve{\nu}^\beta \geq \tilde{c}|f^{'}|^2 e^{-2\tilde{\psi}}$$

Proof. We substitute $\breve{F}^{ij}$ by $\breve{g}^{ij}$ and F^{ij} by g^{ij} in the proof of Lemma 4.11. The proof even simplifies, since we have the estimate

(4.87) $$|g^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta| = |\bar{R}_{\alpha\beta}\nu^\alpha \nu^\beta| \leq c\tilde{v}^2,$$

especially the assumption, that the limit metric $\bar{\sigma}_{ij}$ has non-negative sectional curvature in case $\tilde{\gamma} < 1$, is not needed.   □

5. C^2-ESTIMATES–EXISTENCE FOR ALL TIMES

In this section we consider N to be equipped only with the metric $\breve{g}_{\alpha\beta}$ and will––for simplicity–apply standard notation to this case, i.e. no $\breve{\ }$ is written down. In the next section we will go back to the notation of the previous section until the end oft this thesis.

Lemma 5.1. *The following evolution equation holds*

$$(5.1) \qquad \frac{d}{dt}\left(\frac{1}{F}\right) - \frac{1}{F^2}F^{ij}\left(\frac{1}{F}\right)_{ij} = -\frac{1}{F^3}F^{ij}h_{ik}h_j^k - \frac{1}{F^3}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta.$$

Proof. cf. [8, Lemma 2.3.4]. $\qquad\square$

Lemma 5.2. *Assume (4.74), then*

$$(5.2) \qquad\qquad F \geq \inf_{M_0} F$$

as long as the flow exists. If in addition the IFCF exists for all times, there even holds

$$(5.3) \qquad\qquad F \geq c_0 e^{(\gamma+\frac{1}{n})t}$$

with $c_0 = c_0(M_0) > 0$.

Proof. We define

$$(5.4) \qquad\qquad \varphi(t) = \inf_{M(t)} F$$

and infer from Lemma 5.1

$$(5.5) \qquad \begin{aligned} \frac{d}{dt}F - F^{-2}F^{ij}F_{ij} =& \frac{1}{F}F^{ij}h_{ik}h_j^k + \frac{1}{F}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta \\ & - \frac{2}{F^3}F^{ij}F_i F_j, \end{aligned}$$

hence using Lemma 4.11 we deduce

$$(5.6) \qquad\qquad \dot{\varphi}(t) \geq \tilde{c}\frac{|f'|^2}{F}e^{-2f},$$

especially $\dot{\varphi}(t) \geq 0$ for a.e. $0 < t < T^*$.

If the flow exists for all times, we know from Remark 3.3 that the flow runs into the future singularity

$$(5.7) \qquad\qquad \liminf_{t \to \infty} u(t, \cdot) = 0.$$

A careful view of the proofs of Lemma 6.1 and Theorem 6.2 shows that everything needed there is available at this point, so that we infer from (5.6)

$$(5.8) \qquad\qquad \dot{\varphi}(t) \geq \tilde{c}\frac{|f'|^2}{\varphi}e^{-2f}$$

and

$$(5.9) \qquad\qquad \frac{d}{dt}(\varphi^2) \geq ce^{2(\gamma+\frac{1}{n})t}$$

for a.e. $t > 0$ and a positive constant $c > 0$. This implies

$$(5.10) \qquad \varphi(t)^2 \geq \varphi(0)^2 + \frac{c}{2(\gamma+\frac{1}{n})}(e^{2(\gamma+\frac{1}{n})t} - 1)$$

for all $t > 0$. $\qquad\square$

Remark 5.3. Due to [8, Lemma 1.8.3], and the remark at the beginning of Section 3, especially inequality (3.2), for every relative compact subset Ω of N lying sufficiently far in the future of N, i.e. $|\inf_\Omega x^0|$ close to 0, there exists a strictly convex function $\chi \in C^2(\bar{\Omega})$, this means

$$(5.11) \qquad\qquad \chi_{\alpha\beta} \geq c_0 \bar{g}_{\alpha\beta}$$

with a constant $c_0 > 0$.

Lemma 5.4. *The following evolution equation holds*

$$(5.12) \qquad \dot{\chi} - \frac{1}{F^2} F^{ij} \chi_{ij} = -\frac{2}{F} \chi_\alpha \nu^\alpha - \frac{1}{F^2} F^{ij} \chi_{\alpha\beta} x_i^\alpha x_j^\beta$$

Proof. Direct calculation. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 5.5. *The following evolution equation holds*

$$(5.13) \qquad (\log F)' - \frac{1}{F^2} F^{ij} (\log F)_{ij} = \frac{1}{F^2} F^{ij} h_{ik} h_j^k + \frac{1}{F^2} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta$$
$$- \frac{1}{F^4} F^{ij} F_i F_j$$

Proof. Use Lemma 5.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 5.6. *The following evolution equation holds*

$$(5.14) \qquad \dot{\tilde{v}} - \frac{1}{F^2} F^{ij} \tilde{v}_{ij} = -\frac{1}{F^2} F^{ij} h_{ik} h_j^k \tilde{v} - \frac{2}{F} \eta_{\alpha\beta} \nu^\alpha \nu^\beta - \frac{2}{F^2} F^{ij} h_j^k x_i^\alpha x_k^\beta \eta_{\alpha\beta}$$
$$- \frac{1}{F^2} F^{ij} \eta_{\alpha\beta\gamma} x_i^\beta x_j^\gamma \nu^\alpha - \frac{1}{F^2} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_k^\gamma x_j^\delta \eta_\epsilon x_l^\epsilon g^{kl},$$

where $(\eta_\alpha) = e^{\tilde{\psi}}(-1, 0, ..., 0)$.

Proof. cf. [8, Lemma 2.4.4]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 5.7. *Let $\Omega \subset N$ be precompact and assume that the flow stays in Ω for $0 \leq t \leq T < T^*$, then the F-curvature of the flow hypersurfaces is bounded from above,*

$$(5.15) \qquad\qquad 0 < F < c(\Omega).$$

Proof. Consider the function

$$(5.16) \qquad\qquad w = \log F + \lambda \tilde{v} + \mu \chi,$$

where $\lambda, \mu > 0$ will be chosen later appropiately. Assume

$$(5.17) \qquad\qquad w(t_0, x_0) = \sup_{[0,T]} \sup_{M(t)} w$$

with $0 < t_0 \leq T$, then we have in (t_0, x_0)

$$
\begin{aligned}
0 \leq & \dot{w} - \frac{1}{F^2} F^{ij} w_{ij} \\
= & \frac{1}{F^2} F^{ij} h_{ik} h_j^k + \frac{1}{F^2} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta - \frac{1}{F^4} F^{ij} F_i F_j \\
& - \frac{\lambda}{F^2} F^{ij} h_{ik} h_j^k \tilde{v} - \frac{2\lambda}{F} \eta_{\alpha\beta} \nu^\alpha \nu^\beta - \frac{2\lambda}{F^2} F^{ij} h_j^k x_i^\alpha x_k^\beta \eta_{\alpha\beta} \\
& - \frac{\lambda}{F^2} F^{ij} \eta_{\alpha\beta\gamma} x_i^\beta x_j^\gamma \nu^\alpha - \frac{\lambda}{F^2} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_k^\gamma x_j^\delta \eta_\epsilon x_l^\epsilon g^{kl} \\
& - \frac{2\mu}{F} \chi_\alpha \nu^\alpha - \frac{\mu}{F^2} F^{ij} \chi_{\alpha\beta} x_i^\alpha x_j^\beta \\
\leq & - \epsilon_0 (\frac{\lambda}{2} - 1) \tilde{v} + \frac{c\lambda}{F^2} F^{ij} g_{ij} + c(\mu + \lambda) \frac{1}{F} - c_0 \frac{\mu}{F^2} F^{ij} g_{ij}.
\end{aligned}
$$

(5.18)

Now we choose $\lambda > 2$ arbitrary and $\mu >> 1$ large and we deduce that F is a priori bounded from above in (t_0, x_0) from which we conclude the Lemma. $\qquad \square$

Let $\Omega \subset N$ be precompact and assume that the flow stays in Ω for $0 \leq t \leq T < T^*$, then there exist–as we have just proved–constants $0 < c_1(\Omega) < c_2(\Omega)$ such that

$$
(5.19) \qquad c_1(\Omega) < F < c_2(\Omega)
$$

(concerning the lower bound we proved even more, cf. Lemma 5.2). It remains to prove that there also holds an estimate for the principal curvatures from above

$$
(5.20) \qquad \kappa_i \leq c_3(\Omega),
$$

yielding

$$
(5.21) \qquad 0 < c_4(\Omega) \leq \kappa_i \leq c_3(\Omega)
$$

due to the convexity of the flow hypersurfaces and (5.19).

Lemma 5.8. *The mixed tensor h_i^j satisfies the parabolic equation*

$$
\begin{aligned}
\dot{h}_i^j - \frac{1}{F^2} F^{kl} h_{i;kl}^j = & -F^{-2} F^{kl} h_{rk} h_l^r h_i^j + \frac{1}{F} h_{ri} h^{rj} + \frac{1}{F} h_i^k h_k^j \\
& + \frac{1}{F^2} F^{kl,rs} h_{kl;i} h_{rs}^{;j} - \frac{2}{F^3} F_i F^j + \frac{2}{F^2} F^{kl} \bar{R}_{\alpha\beta\gamma\delta} x_m^\alpha x_i^\beta x_k^\gamma x_r^\delta h_l^m g^{rj} \\
& - \frac{1}{F^2} F^{kl} \bar{R}_{\alpha\beta\gamma\delta} x_m^\alpha x_k^\beta x_r^\gamma x_l^\delta h_i^m g^{rj} - \frac{1}{F^2} F^{kl} \bar{R}_{\alpha\beta\gamma\delta} x_m^\alpha x_k^\beta x_i^\gamma x_l^\delta h^{mj} \\
& - \frac{1}{F^2} F^{kl} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_k^\beta \nu^\gamma x_l^\delta h_i^j + \frac{2}{F} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_m^\delta g^{mj} \\
& + \frac{1}{F^2} F^{kl} \bar{R}_{\alpha\beta\gamma\delta;\epsilon} \left\{ \nu^\alpha x_k^\beta x_l^\gamma x_i^\delta x_m^\epsilon g^{mj} + \nu^\alpha x_i^\beta x_k^\gamma x_m^\delta x_l^\epsilon g^{mj} \right\}.
\end{aligned}
$$

(5.22)

Proof. cf. [8, Lemma 2.4.1]. $\qquad \square$

Lemma 5.9. *Let $\Omega \subset N$ be precompact and assume that the flow stays in Ω for $0 \leq t < T^*$, then there exists $c_3(\Omega)$ such that*

$$
(5.23) \qquad \kappa_i \leq c_3(\Omega).
$$

Proof. Let φ and w be defined respectively by

$$
(5.24) \qquad \begin{aligned}
\varphi &= \sup\{h_{ij} \eta^i \eta^j : \|\eta\| = 1\}, \\
w &= \log \varphi + \lambda \tilde{v} + \mu \chi,
\end{aligned}
$$

where λ, μ are large positive parameters to be specified later. We claim that w is bounded for a suitable choice of λ, μ.

Let $0 < T < T^*$, and $x_0 = x_0(t_0)$, with $0 < t_0 \leq T$, be a point in $M(t_0)$ such that

$$(5.25) \qquad \sup_{M_0} w < \sup\{\sup_{M(t)} w : 0 < t \leq T\} = w(x_0).$$

We then introduce a Riemannian normal coordinate system (ξ^i) at $x_0 \in M(t_0)$ such that at $x_0 = x(t_0, \xi_0)$ we have

$$(5.26) \qquad g_{ij} = \delta_{ij} \quad and \quad \varphi = h_n^n.$$

Let $\tilde\eta = (\tilde\eta^i)$ be the contravariant vector field defined by

$$(5.27) \qquad \tilde\eta = (0, ..., 0, 1),$$

and set

$$(5.28) \qquad \tilde\varphi = \frac{h_{ij}\tilde\eta^i\tilde\eta^j}{g_{ij}\tilde\eta^i\tilde\eta^j}.$$

$\tilde\varphi$ is well defined in a neighbourhood of (t_0, ξ_0).

Now, define $\tilde w$ by replacing φ by $\tilde\varphi$ in (5.24); then $\tilde w$ assumes its maximum at (t_0, ξ_0). Moreover, at (t_0, ξ_0) we have

$$(5.29) \qquad \dot{\tilde\varphi} = \dot h_n^n,$$

and the spatial derivatives do also coincide; in short, at (t_0, ξ_0) $\tilde\varphi$ satisfies the same differential equation (5.22) as h_n^n. For the sake of greater clarity, let us therefore treat h_n^n like a scalar and pretend that w is defined by

$$(5.30) \qquad w = \log h_n^n + \lambda\tilde v + \mu\chi.$$

W.l.o.g. we assume that μ, λ and h_n^n are larger than 1.

At (t_0, ξ_0) we have $\dot w \geq 0$ and in view of the maximum principle, we deduce from $(5.22), (5.14), (5.12)$ and (5.19)

$$(5.31) \qquad \begin{aligned} 0 \leq &ch_n^n + c\lambda F^{ij}g_{ij} - \frac{\lambda}{2}\epsilon_0\tilde v\frac{H}{F} + \mu c - c_0\frac{\mu}{F^2}F^{ij}g_{ij} \\ &+ \frac{1}{F^2}F^{ij}(\log h_n^n)_i(\log h_n^n)_j - \frac{2}{h_n^n F^3}F^n F_n + \frac{1}{h_n^n F^2}F^{kl,rs}h_{kl;n}h_{rs;i}g^{ni}. \end{aligned}$$

Because of [8, Lemma 2.2.6] we have

$$(5.32) \qquad F^{kl,rs}h_{kl;n}h_{rs;n} \leq F^{-1}(F^{ij}h_{ij;n})^2 - \frac{1}{h_n^n}F^{ij}h_{in;n}h_{jn;n}$$

so that we can estimate the last two summands of (5.31) from above by

$$(5.33) \qquad -\frac{1}{(h_n^n)^2}\frac{1}{F^2}F^{ij}(h_{n;i}^n + \bar R_i)(h_{n;j}^n + \bar R_j);$$

here

$$(5.34) \qquad \bar R_i = \bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_n^\beta x_i^\gamma x_n^\delta = h_{in;n} - h_{nn;i}$$

denotes the correction term which comes from the Codazzi equation when changing the indices from $h_{in;n}$ to $h_{nn;i}$.

Thus the terms in (5.31) containing derivatives of h_n^n are estimated from above by

$$(5.35) \qquad -2\frac{1}{(h_n^n)^2 F^2} F^{ij} h_{n;i}^n \bar{R}_j = -2\frac{1}{h_n^n F^2} F^{ij} (\log h_n^n)_i \bar{R}_j.$$

Moreover Dw vanishes at ξ_0, i.e.,

$$(5.36) \qquad \begin{aligned} (\log h_n^n)_i &= -\lambda \tilde{v}_i - \mu \chi_i \\ &= -\lambda \eta_{\alpha\beta} x_i^\beta \nu^\alpha - \lambda \eta_\alpha x_k^\alpha h_i^k - \mu \chi_\alpha x_i^\alpha. \end{aligned}$$

Hence we conclude from (5.31) that

$$(5.37) \qquad \begin{aligned} 0 \leq{}& c h_n^n + c\lambda F^{ij} g_{ij} - \frac{\lambda}{2}\epsilon_0 \tilde{v}\frac{H}{F} + \mu c + \mu\frac{c}{h_n^n} F^{ij} g_{ij} - c_0 \frac{\mu}{F^2} F^{ij} g_{ij} \\ \leq{}& c_1 h_n^n + c_2 \lambda F^{ij} g_{ij} - \lambda c_3 h_n^n + \mu c_4 + \mu \frac{c_5}{h_n^n} F^{ij} g_{ij} - c_0 \mu F^{ij} g_{ij}, \end{aligned}$$

where c_i, $i = 0, ..., 5$, are positive constants and the value of c_0 changed. We note that we used the estimate

$$(5.38) \qquad F^{ij} \bar{R}_j \eta_\alpha x_k^\alpha h_i^k \leq cF,$$

which can be immediately proved.

Now suppose h_n^n to be so large that

$$(5.39) \qquad \frac{c_5}{h_n^n} < \frac{1}{2}c_0,$$

and choose λ, μ such that

$$(5.40) \qquad \frac{\lambda}{2}c_3 > c_1 \quad and \quad \frac{1}{4}c_0\mu > c_2\lambda$$

yielding that estimating the right side of (5.37) yields

$$(5.41) \qquad 0 \leq -\frac{\lambda}{2}c_3 h_n^n - \frac{c_0}{4}\mu F^{ij} g_{ij} + \mu c_4,$$

hence h_n^n is apriori bounded at (t_0, ξ_0). $\qquad \square$

Remark 5.10. Now all neccessary apriori estimates are proved so that we can deduce existence of the flow for all times in the usual way. In view of Remark 3.3 the flow runs into the future singularity.

The latter property can also be proved as follows. Using Lemma 5.2 and $F \leq H$ we infer

$$(5.42) \qquad \infty \longleftarrow \inf_{M(t)} F \leq \inf_{M(t)} H \quad as \quad t \longrightarrow \infty.$$

The timelike convergence condition with respect to the future, cf. Corollary 4.13, together with

$$(5.43) \qquad \lim_{t\to\infty} \inf_{M(t)} H = \infty$$

implies that the flow runs into the future singularity. To see this we argue as in the proof of [9, Lemma 4.2].

6. C^0-estimates–Asymptotic behaviour of the flow

From now on until the end of this thesis we go back to the notation introduced in Section 4 and consider the flow as embedded in $(N, \bar{g}_{\alpha\beta})$, i.e. standard notations apply to this case.

We prove that the flow runs exponentially fast into the future singularity, which means more precisely that there are constants $c_1, c_2 > 0$ such that

$$(6.1) \qquad\qquad -c_1 e^{-\gamma t} < u < -c_2 e^{-\gamma t}.$$

The first step for this will be the following Lemma.

Lemma 6.1. *Let u be the scalar solution of the inverse F-curvature flow, then for every $0 < \lambda < \gamma$ there is $c(\lambda) > 0$ such that*

$$(6.2) \qquad\qquad |u e^{\lambda t}| \le c(\lambda).$$

Proof. Define

$$(6.3) \qquad\qquad \varphi(t) = \inf_{x \in S_0} u(t, x)$$

and

$$(6.4) \qquad\qquad w = \log(-\varphi) + \lambda t.$$

In x_t we have, we remind that $h_{ij} = -u_{ij} - \frac{1}{2}\dot{\sigma}_{ij}$,

$$(6.5) \qquad \begin{aligned} F &= F(h_{ij} - \tilde{v}f' g_{ij} + \psi_\alpha \nu^\alpha g_{ij}) \\ &\le F(c g_{ij} - f' g_{ij}) \quad \text{(where } c > 0\text{)} \\ &= (c - f')F(g_{ij}) \\ &= n(c - f') \end{aligned}$$

and

$$(6.6) \qquad \begin{aligned} \dot{w} &= \frac{\dot{\varphi}}{\varphi} + \lambda = \frac{\frac{\partial u}{\partial t}}{u} + \lambda = \frac{1}{Fu} + \lambda \\ &\le \frac{1}{nu(c - f')} + \lambda \quad a.e., \end{aligned}$$

cf. (3.13). Now we observe that the argument of f' is u and

$$(6.7) \qquad\qquad \lim_{t \to \infty} \inf_{x \in S_0} u(t, x) = 0$$

because of Remark 5.10. On the other hand

$$(6.8) \qquad\qquad \lim_{t \to \infty} f' u = \tilde{\gamma}^{-1} = \frac{1}{n\gamma},$$

in view of (1.31), and we infer

$$(6.9) \qquad\qquad \frac{1}{nu(c - f')} \to -\gamma,$$

hence $\dot{w}(t) \le 0$ for a.e. $t \ge t_\lambda$, $t_\lambda > 0$ suitable.

Therefore, we deduce

$$(6.10) \qquad\qquad w \le w(t_\lambda) \quad \forall t \ge t_\lambda,$$

i.e.

$$(6.11) \qquad\qquad -u e^{\lambda t} \le c(\lambda) \quad \forall t \in \mathbb{R}_+.$$

$\square$

We are now able to prove the exact exponential velocity.

Theorem 6.2. *There are constants $c_1, c_2 > 0$ such that*

$$\text{(6.12)} \qquad -c_1 \leq \tilde{u} = ue^{\gamma t} \leq -c_2 < 0.$$

Proof. (i) We prove the estimate from above. Define

$$\text{(6.13)} \qquad \varphi(t) = \sup_{x \in S_0} u(t, x)$$

and

$$\text{(6.14)} \qquad w = \log(-\varphi) + \gamma t.$$

Reasoning similar as in the proof of the previous lemma, we obtain for a.e. $t \geq t_0$, t_0 sufficiently large,

$$\text{(6.15)} \qquad \begin{aligned} \dot{w} &\geq \frac{1}{nu(-c - f')} + \gamma \quad \text{(where } c > 0\text{)} \\ &= u\frac{\frac{1-\tilde{\gamma}uf'}{u} - cn\gamma}{nu(-c - f')} \\ &\geq \tilde{c}u, \end{aligned}$$

where $\tilde{c}$ is a positive upper bound for the fraction; note that this fraction converges due to the assumptions, cf. (1.31).

The previous lemma now yields

$$\text{(6.16)} \qquad \dot{w} \geq \tilde{c}u \geq -\tilde{c}c_\lambda e^{-\lambda t} \quad a.e. \ t \geq t_\lambda$$

for any $0 < \lambda < \gamma$. Hence w is bounded from below, or equivalently,

$$\text{(6.17)} \qquad \tilde{u} \leq -c_2 < 0.$$

(ii) Now, we prove the estimate from below. Define

$$\text{(6.18)} \qquad \varphi(t) = \inf_{x \in S_0)} u(t, x)$$

and w as in (6.14), then we obtain analogously that

$$\text{(6.19)} \qquad -c_1 \leq \tilde{u}.$$

$\square$

Lemma 6.3. *For any $k \in \mathbb{N}^*$ there exists $c_k > 0$ such that*

$$\text{(6.20)} \qquad |f^{(k)}| \leq c_k e^{k\gamma t},$$

where $f^{(k)}$ is evaluated at u.

Proof. In view of the assumption (1.10) there holds

$$\text{(6.21)} \qquad |f^{(k)}| \leq c_k|f'|^k = c_k|f'|^k u^k \tilde{u}^{-k} e^{k\gamma t}.$$

Then use (1.31) and the preceding theorem. $\square$

7. C^1-ESTIMATES–ASYMPTOTIC BEHAVIOUR OF THE FLOW

In Section 4 we proved that $\tilde{v}$ is uniformly bounded for all times, cf. Lemma 4.5. We recall that

$$(7.1) \qquad \tilde{u} = ue^{\gamma t}.$$

Our final goal is to show that $\|D\tilde{u}\|^2$ is uniformly bounded, but this estimate has to be deferred to Section 8. At the moment we only prove an exponential decay for any $0 < \lambda < \gamma$, i.e., we shall estimate $\|Du\|e^{\lambda t}$.

We remember that we have

$$(7.2) \qquad F = F(\check{h}_i^j) = F(e^{\check{\psi}}\check{h}_i^j) = F(h_i^j - \tilde{v}f'\delta_i^j + \psi_\alpha\nu^\alpha\delta_i^j).$$

We need in the following a slightly different estimate from the one in (4.16).

Lemma 7.1.
$$(7.3) \qquad \begin{aligned} F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l = & -\tilde{v}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\eta^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l \\ & -\tilde{v}F^{ij}\bar{R}_{r\beta\gamma\delta}\tilde{u}^r x_i^\beta x_l^\gamma x_j^\delta u^l. \end{aligned}$$

With the help of the boundedness of $\tilde{v}$, cf. Lemma 4.5, we prove the following estimate.

Lemma 7.2. *There exists $\epsilon > 0$ and a constant c_ϵ such that*

$$(7.4) \qquad \|Du\|e^{\epsilon t} \le c_\epsilon.$$

Proof. We have

$$(7.5) \qquad \tilde{v}^2 = 1 + \|Du\|^2.$$

Taking the log yields since $\tilde{v}$ is bounded

$$(7.6) \quad \|Du\|^2(1 - c_1\|Du\|^2) \le 2\log\tilde{v} = \log(1 + \|Du\|^2) \le \|Du\|^2(1 + c_1\|Du\|^2),$$

where c_1 is a positive constant, i.e., it is sufficient to prove that $\log\tilde{v}e^{2\epsilon t}$ is uniformly bounded.

Let $\epsilon > 0$ be small and set

$$(7.7) \qquad \varphi = \log\tilde{v}e^{2\epsilon t},$$

then φ satisfies

$$(7.8) \qquad \dot{\varphi} - F^{-2}F^{ij}\varphi_{ij} = \frac{1}{\tilde{v}}(\dot{\tilde{v}} - F^{-2}F^{ij}\tilde{v}_{ij})e^{2\epsilon t} + F^{-2}\frac{1}{\tilde{v}^2}F^{ij}\tilde{v}_i\tilde{v}_j e^{2\epsilon t} + 2\epsilon\varphi$$

hence (cf. Lemma 4.2)

$$(7.9) \qquad \begin{aligned} F^2 e^{-2\epsilon t}(\dot{\varphi} - F^{-2}F^{ij}\varphi_{ij}) = & -F^{ij}h_{kj}h_i^k + \frac{1}{\tilde{v}}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l \\ & -\frac{1}{\tilde{v}}F^{ij}h_{ij}\eta_{\alpha\beta}\nu^\alpha\nu^\beta - \frac{1}{\tilde{v}}F\eta_{\alpha\beta}\nu^\alpha\nu^\beta \\ & -\frac{1}{\tilde{v}}F^{ij}\eta_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma - \frac{2}{\tilde{v}}F^{ij}\eta_{\alpha\beta}x_k^\alpha x_i^\beta h_j^k \\ & + f''\|Du\|^2 F^{ij}g_{ij} + \frac{1}{\tilde{v}}\tilde{v}_k u^k f' F^{ij}g_{ij} \\ & -\psi_{\alpha\beta}\nu^\alpha x_k^\beta u^k\frac{1}{\tilde{v}}F^{ij}g_{ij} - \frac{1}{\tilde{v}}\psi_\alpha x_l^\alpha h_k^l u^k F^{ij}g_{ij} \\ & + F^{ij}\tilde{v}_i\tilde{v}_j\frac{1}{\tilde{v}^2} + 2\epsilon F^2\log\tilde{v}. \end{aligned}$$

For T, $0 < T < \infty$, assume that

$$\sup_{[0,T]} \sup_{M(t)} \varphi = \varphi(t_0, x_0), \tag{7.10}$$

where $0 < t_0 \le T$ large, $x_0 \in S_0$.

Applying the maximum principle we deduce in (t_0, x_0) using Lemma 4.3, Lemma 4.6 and Lemma 7.1 that (note that $\tilde{u} = u e^{\gamma t}$ is bounded) for t_0 large and $\epsilon > 0$ small.

$$
\begin{aligned}
0 \le{}& -\frac{1}{2} F^{ij} h_{kj} h_i^k + c u^2 F^{ij} g_{ij} + c|u| \|Du\| F^{ij} g_{ij} \\
& + c\|Du\|^2 F^{ij} g_{ij} + f'' \|Du\|^2 F^{ij} g_{ij} + c\epsilon |f'|^2 \log \tilde{v} F^{ij} g_{ij},
\end{aligned} \tag{7.11}
$$

here we used that we have

$$F^2 \le c(F^{ij} h_{ik} h_j^k + |f'|^2 F^{ij} g_{ij}) \tag{7.12}$$

due to $\epsilon_0 F^2 \le F^{ij} \check{h}_{kj} \check{h}_i^k$, cf. Definition 2.3.

The $\log \tilde{v}$ in (7.11) can be estimated by $c\|Du\|^2$ yielding

$$0 \le -\frac{1}{2} F^{ij} h_{kj} h_i^k + c u^2 F^{ij} g_{ij} + \frac{1}{2} f'' \|Du\|^2 F^{ij} g_{ij}, \tag{7.13}$$

where we have chosen $\epsilon > 0$ small and assumed that $t_0 > 0$ large.

Hence in (t_0, x_0)

$$\varphi = \log \tilde{v} e^{2\epsilon t} \le c\|Du\|^2 e^{2\epsilon t} \le \frac{c u^2}{|f''|} e^{2\epsilon t} \le c. \tag{7.14}$$

$\square$

Lemma 7.3. *(Evolution of u)*

$$\dot{u} - F^{-2} F^{ij} u_{ij} = 2 F^{-1} \tilde{v} + F^{-2} \tilde{v}^2 f' F^{ij} g_{ij} - F^{-2} \tilde{v} \psi_\alpha \nu^\alpha F^{ij} g_{ij} - F^{-2} F^{ij} \bar{h}_{ij}$$

Proof. The claim follows from the three identities

$$
\begin{aligned}
\dot{u} &= \frac{\tilde{v}}{F} \\
u_{ij} &= -\tilde{v} h_{ij} + \bar{h}_{ij} \\
-F^{ij} h_{ij} &= -F - \tilde{v} f' F^{ij} g_{ij} + \psi_\alpha \nu^\alpha F^{ij} g_{ij}.
\end{aligned} \tag{7.15}
$$

$\square$

Lemma 7.4. *For any $0 < \lambda < \gamma$, there exists c_λ such that*

$$\|Du\| e^{\lambda t} \le c_\lambda. \tag{7.16}$$

Proof. Define

$$\varphi = \log \tilde{v} - \frac{\mu}{2} |u|^{2-\epsilon}, \tag{7.17}$$

with $0 < \epsilon < 1$ arbitrary and $\mu \gg 1$ chosen appropriately later. The interesting case is, when ϵ is close to 0.

Then φ satisfies the following evolution equation, cf. Lemma 4.2 and Lemma 7.3,

$$
\begin{aligned}
\dot{\varphi} - F^{-2}F^{ij}\varphi_{ij} =& \frac{1}{\tilde{v}}(\dot{\tilde{v}} - F^{-2}F^{ij}\tilde{v}_{ij}) + \frac{2-\epsilon}{2}\mu|u|^{1-\epsilon}(\dot{u} - F^{-2}F^{ij}u_{ij}) \\
& + F^{-2}\frac{1}{\tilde{v}^2}F^{ij}\tilde{v}_i\tilde{v}_j + \frac{2-\epsilon}{2}\mu(1-\epsilon)|u|^{-\epsilon}F^{-2}F^{ij}u_iu_j \\
=& -F^{-2}F^{ij}h_{kj}h_i^k + \frac{1}{\tilde{v}}F^{-2}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l \\
& - \frac{1}{\tilde{v}}F^{-2}F^{ij}h_{ij}\eta_{\alpha\beta}\nu^\alpha\nu^\beta - \frac{1}{\tilde{v}}F^{-1}\eta_{\alpha\beta}\nu^\alpha\nu^\beta \\
& - \frac{1}{\tilde{v}}F^{-2}F^{ij}\eta_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma + \frac{2}{\tilde{v}}F^{-2}F^{ij}\eta_{\alpha\beta}x_k^\alpha x_i^\beta h_j^k \\
& + F^{-2}f''\|Du\|^2F^{ij}g_{ij} + \frac{1}{\tilde{v}}F^{-2}\tilde{v}_k u^k f' F^{ij}g_{ij} \\
& - F^{-2}\psi_{\alpha\beta}\nu^\alpha x_k^\beta u^k\frac{1}{\tilde{v}}F^{ij}g_{ij} - \frac{1}{\tilde{v}}F^{-2}\psi_\alpha x_l^\alpha h_k^l u^k F^{ij}g_{ij} \\
& + (2-\epsilon)\mu|u|^{(1-\epsilon)}\frac{\tilde{v}}{F} + (1-\frac{\epsilon}{2})\mu|u|^{1-\epsilon}F^{-2}\tilde{v}^2 f' F^{ij}g_{ij} \\
& - (1-\frac{\epsilon}{2})\mu|u|^{1-\epsilon}F^{-2}\tilde{v}\psi_\alpha\nu^\alpha F^{ij}g_{ij} \\
& - (1-\frac{\epsilon}{2})\mu|u|^{1-\epsilon}F^{-2}F^{ij}\bar{h}_{ij} \\
& + F^{-2}F^{ij}\tilde{v}_i\tilde{v}_j\frac{1}{\tilde{v}^2} + (1-\frac{\epsilon}{2})(1-\epsilon)\mu|u|^{-\epsilon}F^{-2}F^{ij}u_iu_j \\
=& \quad RHS.
\end{aligned}
$$

$\square$

We will show

$$\varphi < 0 \quad \forall t \geq 0. \tag{7.19}$$

Assume that this is not the case. Let $t_0 > 0$ be minimal such that

$$\sup_{S_0}\varphi(t_0,\cdot) = 0 \tag{7.20}$$

and $x_0 \in S_0$ such that

$$\varphi(t_0, x_0) = 0, \tag{7.21}$$

which implies that in (t_0, x_0) the RHS in (7.18) is ≥ 0,

$$\frac{1}{2}\|Du\|^2 \geq \log\tilde{v} = \frac{\mu}{2}|u|^{2-\epsilon} \tag{7.22}$$

for $\mu > 0$ large (which implies t_0 large) and

$$\tilde{v}_i = -(1-\frac{\epsilon}{2})\mu\tilde{v}|u|^{1-\epsilon}u_i. \tag{7.23}$$

We now show that RHS in (7.18) is negative, if t_0 is sufficiently large, which can be guaranteed by increasing μ accordingly.

We use

$$F \leq |u|^{1-\beta}\delta F^{ij}h_{ik}h_j^k + \frac{c(\delta)}{|u|^{1-\beta}}F^{ij}g_{ij} + \tilde{v}|f'|F^{ij}g_{ij} \tag{7.24}$$

where $\beta > 0$ is chosen according to Lemma 7.2 such that

$$(7.25) \qquad \log \tilde{v} \leq c|u|^{\beta}$$

and $\delta > 0$ is small, $c(\delta)$ also depends on the upper bound of $|\psi_{\alpha}\nu^{\alpha}|$.

We find

$$0 \leq F^2(RHS)$$

$$\leq -\frac{1}{2}F^{ij}h_{kj}h_i^k + c\|Du\|^2 F^{ij}g_{ij} + c\mu|u|F^{ij}g_{ij} + f^{''}\mu|u|^{2-\epsilon}F^{ij}g_{ij}$$

$$+ (1 - \frac{\epsilon}{2})\mu|u|^{-\epsilon}\|Du\|^2(\frac{1}{\tilde{\gamma}} + cu^2)F^{ij}g_{ij}$$

$$(7.26) \qquad + (2 - \epsilon)\mu|u|^{1-\epsilon}\tilde{v}(|u|^{1-\beta}\delta F^{ij}h_{ik}h_j^k + \frac{c(\delta)}{|u|^{1-\beta}}F^{ij}g_{ij} + \tilde{v}|f^{'}|F^{ij}g_{ij})$$

$$+ (1 - \frac{\epsilon}{2})\mu|u|^{1-\epsilon}\tilde{v}^2 f^{'}F^{ij}g_{ij} + 2(1 - \frac{\epsilon}{2})^2\mu\log\tilde{v}|u|^{-\epsilon}|Du|^2 F^{ij}g_{ij}$$

$$+ (1 - \frac{\epsilon}{2})\mu(1 - \epsilon)|u|^{-\epsilon}|Du|^2 F^{ij}g_{ij}$$

$$< \frac{\mu}{\tilde{\gamma}}|u|^{-\epsilon}(-1 + c\|Du\| + (1 - \frac{\epsilon}{2})\tilde{v}^2 + c\tilde{\gamma}|u|^{\epsilon} + c|u|^{\beta})F^{ij}g_{ij}$$

$$< 0,$$

where we have chosen μ large ($\Rightarrow t_0$ large). Here we used

$$(7.27) \qquad |f^{'}u - \frac{1}{\tilde{\gamma}}| \leq cu^2, \quad ||f^{''}|u^2 - \frac{1}{\tilde{\gamma}}| \leq cu^2$$

and

$$(7.28) \qquad \mu = \frac{2\log\tilde{v}}{|u|^{2-\epsilon}} \leq 2|u|^{\beta+\epsilon-2}.$$

8. C^2-ESTIMATES–ASYMPTOTIC BEHAVIOUR OF THE FLOW

F grows exponentially fast in time, more precisely we have the following

Theorem 8.1. *The estimate*

$$(8.1) \qquad F \geq ce^{\gamma t}$$

is valid, where $c > 0$ depends on M_0.

Proof. Use Lemma 5.2 (note that we used a different notation there) and (4.3). $\square$

For later purposes we obtain an evolution equation for F.

As usual we have (we remark that in our case the evolution equations are the same as in [8, Lemma 2.3.2, Lemma 2.3.3], see also (7.15))

$$\dot{h}_i^j = (-\frac{1}{F})_i^j + \frac{1}{F}h_i^k h_k^j + \frac{1}{F}\bar{R}_{\alpha\beta\gamma\delta}\nu^{\alpha}x_i^{\beta}\nu^{\gamma}x_k^{\delta}g^{kj}$$

$$\dot{\nu}^{\alpha} = g^{ij}\frac{F_i}{F^2}x_j^{\alpha}$$

$$(8.2) \qquad \dot{\tilde{v}} = -\frac{1}{F}\eta_{\alpha\beta}\nu^{\alpha}\nu^{\beta} - g^{ij}\frac{F_i}{F^2}u_j$$

$$\dot{u} = \frac{\tilde{v}}{F}$$

$$\dot{g}_{ij} = -\frac{2}{F}h_{ij}$$

and, furthermore, since

(8.3)
$$F = F(\check{h}_i^j) = F(h_i^j - \tilde{v}f'\delta_i^j + \psi_\alpha \nu^\alpha \delta_i^j)$$

we infer

(8.4)
$$\dot{F} = F_j^i \dot{\check{h}}_i^j,$$

and finally

Lemma 8.2.

(8.5)
$$
\begin{aligned}
\dot{F} - \frac{1}{F^2}F^{ij}F_{ij} = &-\frac{2}{F^3}F^{ij}F_iF_j + \frac{1}{F}F^{ij}h_i^k h_{kj} + \frac{1}{F}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta \\
&+\frac{1}{F}\eta_{\alpha\beta}\nu^\alpha \nu^\beta f' F^{ij}g_{ij} - \frac{1}{F}\tilde{v}^2 f'' F^{ij}g_{ij} + \frac{1}{F^2}f' F_k u^k F^{ij}g_{ij} \\
&-\frac{1}{F}\psi_{\alpha\beta}\nu^\alpha \nu^\beta F^{ij}g_{ij} + \frac{1}{F^2}\psi_\alpha x_k^\alpha F^k F^{ij}g_{ij}.
\end{aligned}
$$

In the following lemma we prove the important evolution equation for the second fundamental form (h_l^k).

Lemma 8.3.

(8.6)
$$
\begin{aligned}
\dot{h}_l^k - F^{-2}F^{ij}h_{l;ij}^k = &-2F^{-3}F^k F_l + F^{-1}h^{kr}h_{rl} + F^{-1}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_l^\beta \nu^\gamma x_r^\delta g^{rk} \\
&- F^{-2}F^{ij}h_{aj}h_i^a h_l^k + F^{-2}F^{ij}h_{ij}h_{al}h^{ak} + 2F^{-2}g^{pk}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}x_r^\alpha x_p^\beta x_i^\gamma x_l^\delta h_j^r \\
&- F^{-2}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}x_a^\alpha x_i^\beta x_l^\gamma x_j^\delta h^{ak} - F^{-2}g^{pk}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}x_r^\alpha x_i^\beta x_p^\gamma x_j^\delta h_l^r \\
&- F^{-2}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta h_l^k + F^{-2}g^{pk}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_p^\beta \nu^\gamma x_l^\delta h_{ij} \\
&+ F^{-2}g^{pk}F^{ij}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x_p^\beta x_i^\gamma x_l^\delta x_j^\epsilon + F^{-2}g^{pk}F^{ij}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x_i^\beta x_j^\gamma x_p^\delta x_l^\epsilon \\
&+ F^{-2}g^{pk}F^{ij,rs}\check{h}_{ij;p}\check{h}_{rs;l} \\
&+ F^{-2}F^{ij}g_{ij}(-u_l u^k \tilde{v}f''' + g^{pk}\psi_{\alpha\beta\gamma}\nu^\alpha x_p^\beta x_l^\gamma + \psi_{\alpha\beta}\nu^\alpha \nu^\beta h_l^k \\
&\quad + g^{pk}\psi_{\alpha\beta}x_r^\alpha x_p^\beta h_l^r + \psi_{\alpha\beta}x_r^\alpha x_l^\beta h^{rk} + \psi_\alpha \nu^\alpha h_{lr}h^{rk} + \psi_\alpha x_r^\alpha h^{rk}_{\;;l}) \\
&+ F^{-2}F^{ij}g_{ij}(-g^{pk}f'\eta_{\alpha\beta\gamma}\nu^\alpha x_p^\beta x_l^\gamma - g^{pk}f'\eta_{\alpha\beta}x_r^\alpha x_p^\beta h_l^r - f'\eta_{\alpha\beta}\nu^\alpha \nu^\beta h_l^k \\
&\quad - f'\eta_{\alpha\beta}x_r^\alpha x_l^\beta h^{rk} - f'h_{rl}h^{rk}\tilde{v} + f'u^r h_{l;r}^k + f'u^r g^{kp}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_p^\beta x_r^\gamma x_l^\delta \\
&\quad - f''(\tilde{v}^k u_l + \tilde{v}_l u^k) + f''\tilde{v}^2 h_l^k + f''\tilde{v}\eta_{\alpha\beta}x_l^\alpha x_r^\beta g^{rk}).
\end{aligned}
$$

Proof. The starting point of the proof is the equation for $\dot{h}_i^j$ given in (8.2), which contains the summand

(8.7)
$$\left(-\frac{1}{F}\right)_i^j = \frac{1}{F^2}F_i^j - \frac{2}{F^3}F_i F^j.$$

To finish the proof, we only have to calculate the covariant derivative F_i^j in detail. Deriving the purely covariant version of this tensor we first get

(8.8)
$$F_{kl} = F^{ij}\check{h}_{ij;kl} + F^{ij,rs}\check{h}_{ij;k}\check{h}_{rs;l},$$

then $\check{h}_{ij;kl}$ will be expressed as

(8.9)
$$\check{h}_{ij;kl} = h_{ij;kl} + additional\ terms$$

and interchanging indices in the usual way (which is technical using the Codazzi equations and the Ricci identities, cf. the proof of [8, Lemma 2.4.1]) leads to the

representation

$$(8.10) \qquad \check{h}_{ij;kl} = h_{kl;ij} + additional\ terms,$$

with different *additional terms*. □

We already know the estimate

$$(8.11) \qquad -c|f'| \leq \kappa_i,$$

$c > 0$, because of the fact that the $\check{\kappa}_i$ are positive, remember $\check{\kappa}_i = \kappa_i - \tilde{v}f' + \psi_\alpha \nu^\alpha$.
Now we prove an estimate from above.

Theorem 8.4. *We have*

$$(8.12) \qquad \kappa_i \leq c.$$

Proof. Let φ be defined by

$$(8.13) \qquad \varphi = \sup\{h_{ij}\eta^i\eta^j : \|\eta\| = 1\}.$$

We shall prove that

$$(8.14) \qquad w = \log \varphi + \lambda \tilde{v}$$

is uniformly bounded from above, if λ is large enough.

The proof is devided into two steps:

(i) There is a $\mu > 0$ such that if a maximum of $w|_{[0,T]}$ (where $0 < T < \infty$ arbitrary but fixed) is attained in (t_0, x_0), $0 < t_0 \leq T$, $x_0 \in S_0$, then there holds in (t_0, x_0)

$$(8.15) \qquad h_n^n \leq \mu|f'|$$

(h_n^n denotes as usual the largest principal curvature).

(ii) Secondly we prove that

$$(8.16) \qquad h_n^n \leq c$$

in (t_0, x_0), where, without loss of generality, we may assume that t_0 is large.

Now we prove (i) by contradiction. Introducing Riemannian normal coordinates around (t_0, x_0) and arguing as usual, i.e. second derivatives of φ with respect to space and the first derivative with respect to time coincide with the corresponding ones of h_n^n, furthermore $g_{ij} = \delta_{ij}$ and h_i^j is diagonal, we may assume that w is defined by

$$(8.17) \qquad w = \log h_n^n + \lambda \tilde{v}.$$

Moreover, we assume $h_n^n > \mu|f'|$ in (t_0, x_0), where μ is large and will be chosen later. Applying the maximum principle we obtain

$$(8.18) \qquad 0 \leq \dot{w} - \frac{1}{F^2} F^{ij} w_{ij}.$$

in (t_0, x_0).

Using $F = F^{ij}\check{h}_{ij}$ and $F \in (K^*)$ we have, cf. Definition 2.3,

$$\epsilon_0 F \check{H} \leq F^{ij}\check{h}_i^k\check{h}_{kj} = F^{ii}(\check{h}_{ii})^2 \leq F^{ii}(h_{ii} + \tilde{v}|f'|g_{ii} + \psi_\alpha\nu^\alpha g_{ii})^2$$

$$(8.19) \qquad \leq (1+\epsilon)F^{ii}h_{ii}^2 + 2\tilde{v}|f'|F^{ij}h_{ij} + \tilde{v}^2|f'|^2 F^{ij}g_{ij} + c_\epsilon|u|F^{ij}g_{ij}$$

$$\leq (1+\epsilon)F^{ii}h_{ii}^2 + 2\tilde{v}|f'|F,$$

where $\epsilon > 0$. In view of

(8.20) $$\check{h}_n^n = h_n^n - \tilde{v} f' + \psi_\alpha \nu^\alpha.$$

we infer

$$-(1+\epsilon) F^{ii} h_{ii}^2 \leq -\epsilon_0 F \check{H} + 2\tilde{v} |f'| F$$

(8.21)
$$\leq -\frac{\epsilon_0}{2} F h_n^n - \frac{\epsilon_0}{2} F h_n^n + 2\tilde{v} |f'| F$$

$$\leq -\frac{\epsilon_0}{2} F h_n^n - \frac{\epsilon_0}{2} \mu F |f'| + 2\tilde{v} |f'| F$$

$$\leq -\frac{\epsilon_0}{2} F h_n^n,$$

where we assume that μ is large; hence there is $\delta_0 > 0$ such that

(8.22) $$-F^{ij} h_{ik} h_j^k \leq -\delta_0 F h_n^n$$

in (t_0, x_0).

In (t_0, x_0) we have

(8.23) $$h_{n;i}^n = -\lambda \tilde{v}_i h_n^n$$

and in view of (8.18)

(8.24) $$0 \leq \frac{1}{h_n^n}(\dot{h}_n^n - F^{-2} F^{ij} h_{n;ij}^n) + \lambda(\dot{\tilde{v}} - F^{-2} F^{ij} \tilde{v}_{ij}) + \frac{\lambda^2}{F^2} F^{ij} \tilde{v}_i \tilde{v}_j.$$

Multiplying this inequality by F^2, inserting the evolution equations for h_n^n and $\tilde{v}$, cf. Lemma 8.3 and Lemma 4.2, as well as some trivial estimates yield (no summation with respect to n)

$$0 \leq -2\frac{1}{h_n^n} F^{-1} F^n F_n + 2F h_n^n + \frac{c}{h_n^n} F + \frac{1}{h_n^n} F^{ij,rs} \check{h}_{ij;n} \check{h}_{rs;n}$$

(8.25)
$$+ c|f'|^{\frac{3}{2}} F^{ij} g_{ij} + \tilde{v}^2 f'' F^{ij} g_{ij} + \lambda |u| F^{ij} g_{ij}$$

$$- \frac{\lambda}{2} \tilde{v} F^{ii} h_{ii}^2 + \lambda^2 F^{ij} \tilde{v}_i \tilde{v}_j.$$

We remark that we have estimated the term arising from the second term in the second line of equation (8.6) together with two other terms arising from (8.6) by employing the homogeneity of F, namely, $F = F^{ij} h_{ij} - \tilde{v} f' F^{ij} g_{ij} + \psi_\alpha \nu^\alpha F^{ij} g_{ij}$.

Terms arising from the two terms in (8.6) depending linearly on the derivatives of the second fundamental form are first rewritten with the help of the Codazzi equation (the correction terms can be estimated very easily) such that we obtain the derivative of h_n^n. The resulting terms can be estimated as follows:

(8.26) $$\frac{1}{h_n^n} \psi_\alpha x_r^\alpha h_n^{n;r} F^{ij} g_{ij} \leq \lambda c|u| F^{ij} g_{ij} + \lambda \psi_\alpha x_r^\alpha u_s h^{sr} F^{ij} g_{ij}$$

for the first term, where we used

(8.27) $$\tilde{v}_i = \eta_{\alpha\beta} \nu^\alpha x_i^\beta - u_r h_i^r,$$

with (η_α) as in Lemma 4.2, cf. also Lemma 4.6, and

(8.28) $$\frac{1}{h_n^n} f' u^r h_{n;r}^n F^{ij} g_{ij} = -\lambda f' u^r \tilde{v}_r F^{ij} g_{ij}$$

for the second one. Both last summands in the previous inequalities appear among the terms coming from the evolution equation of $\tilde{v}$ with opposite sign.

Since $F \in (K)$ and homogenous of degree 1 we deduce from [8, Lemma 2.2.14] that F is concave, hence [8, Proposition 2.1.23] implies

$$(8.29) \qquad F^{ij,rs}(\check{h}_{ij})\check{h}_{ij;n}\check{h}_{rs;n} \le 0.$$

Together with

$$(8.30) \qquad F^{ij}\tilde{v}_i\tilde{v}_j \le c|u|F^{ij}g_{ij} + c\|Du\|^2 F^{ij}h_{ik}h_j^k$$

(which follows by using $\|x-y\|^2 \le 2\|x\|^2 + 2\|y\|^2$, here $\|\cdot\|$ is the norm induced by the quadratic form F^{ij}) we conclude

$$(8.31) \qquad \begin{aligned} 0 \le & 2Fh_n^n + \frac{c}{h_n^n}F + c|f'|^{\frac{3}{2}}F^{ij}g_{ij} + \tilde{v}^2 f'' F^{ij}g_{ij} + c\lambda^2|u|F^{ij}g_{ij} \\ & - \frac{\lambda}{4}\tilde{v}\delta_0 Fh_n^n. \end{aligned}$$

For $\lambda > 0$ large we get a contradiction, which finishes the proof of (i).

We now prove (ii). From (i) we deduce that the largest principal curvature of $M(t)$ is bounded by $ce^{\gamma t}$ for all $t > 0$. Combining this with Lemma 8.1, namely,

$$(8.32) \qquad 0 < c_0 \le F(e^{-\gamma t}\check{h}_j^i) = F(e^{-\gamma t}(h_j^i - \tilde{v}f'\delta_j^i + \psi_\alpha\nu^\alpha\delta_j^i)),$$

we infer that $e^{-\gamma t}\check{h}_j^i$ lies in a compact subset of Γ_+ for all $t > 0$. Hence we have constants $c, \tilde{c}_1, \tilde{c}_2, \bar{c}_1, \bar{c}_2 > 0$ (not depending on t_0 or T), such that for all times and especially in (t_0, x_0)

$$(8.33) \quad -ce^{\gamma t} \le \kappa_i \le ce^{\gamma t} \quad \wedge \quad \tilde{c}_1 e^{\gamma t} \le F \le \tilde{c}_2 e^{\gamma t} \quad \wedge \quad 0 < \bar{c}_1 g_{ij} \le F^{ij} \le \bar{c}_2 g_{ij}.$$

We again look at (8.24) multiplied by F^2 in (t_0, x_0). We assume that h_n^n is large and will show that it is a priori bounded. We have

$$(8.34) \qquad -F^{ii}h_{ii}^2 \le -\bar{c}_1(h_n^n)^2$$

and furthermore

$$(8.35) \qquad \begin{aligned} 0 \le & 2Fh_n^n + \frac{c}{h_n^n}F + \lambda|f'|^{\frac{3}{2}}F^{ij}g_{ij} + f''\tilde{v}^2 F^{ij}g_{ij} - \frac{\lambda}{2}F^{ij}h_{ik}h_j^k\tilde{v} \\ \le & \epsilon F^2 + c_\epsilon(h_n^n)^2 + \lambda|f'|^{\frac{3}{2}}F^{ij}g_{ij} + f''\tilde{v}^2 F^{ij}g_{ij} - \frac{\lambda}{2}\bar{c}_1\tilde{v}(h_n^n)^2, \end{aligned}$$

$\epsilon > 0$ small; we remember

$$(8.36) \qquad f'' \le -ce^{2\gamma t}.$$

If λ is sufficiently large and t_0 sufficiently large we get a contradiction. $\qquad \square$

Lemma 8.5.

$$(8.37) \qquad \sup_{M(t)}\max_i |\kappa_i u| \to 0 \quad t \to \infty.$$

Proof. We remember that

$$(8.38) \qquad F = F(\check{h}_i^j) = F(h_i^j - \tilde{v}f'\delta_i^j + \psi_\alpha\nu^\alpha\delta_i^j) = F(\kappa_i - \tilde{v}f' + \psi_\alpha\nu^\alpha),$$

where the κ_i are the eigenvalues of h_i^j, now numbered such that κ_n is the smallest one.

The function $\varphi = -uF$ satisfies the following parabolic equation, cf. Lemma 8.2 and Lemma 7.3,

$$
\begin{aligned}
(8.39) \quad \dot\varphi - F^{-2} F^{ij} \varphi_{ij} =\ & \frac{2u}{F^3} F^{ij} F_i F_j - \frac{u}{F} F^{ij} h_i^k h_{kj} - \frac{u}{F} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta \\
& - \frac{u}{F} f' \eta_{\alpha\beta} \nu^\alpha \nu^\beta F^{ij} g_{ij} - \frac{u}{F^2} f' F^k u_k F^{ij} g_{ij} + \frac{u}{F} \tilde{v}^2 f'' F^{ij} g_{ij} \\
& + \frac{u}{F} \psi_{\alpha\beta} \nu^\alpha \nu^\beta F^{ij} g_{ij} - \frac{u}{F^2} F^k \psi_\alpha x_k^\alpha F^{ij} g_{ij} - 2\tilde{v} \\
& - \frac{\tilde{v}^2}{F} f' F^{ij} g_{ij} + \frac{\tilde{v}}{F} \psi_\alpha \nu^\alpha F^{ij} g_{ij} + \frac{1}{F} F^{ij} \bar{h}_{ij} + \frac{2}{F^2} F^{ij} u_i F_j.
\end{aligned}
$$

For $t > 0$ we define $\tilde\varphi(t) = \inf_{S_0} \varphi(t, \cdot)$ and choose $x_t \in S_0$ such that

$$(8.40) \qquad \tilde\varphi(t) = \varphi(t, x_t),$$

then $\tilde\varphi$ is differentiable a.e. and we have

$$(8.41) \qquad \dot{\tilde\varphi}(t) = \dot\varphi(t, x_t)$$

for a.e. $t > 0$.

Let $t_0 > 0$ be sufficiently large, then combining (8.39) and (8.41) and using $\varphi_i = 0$ yields

$$
\begin{aligned}
(8.42) \quad \dot{\tilde\varphi}(t) \geq\ & -\frac{u}{F} F^{ij} h_i^k h_{kj} + u \frac{\tilde{v}^2}{F} f'' F^{ij} g_{ij} - \frac{\tilde{v}^2}{F} f' F^{ij} g_{ij} \\
& - 2\tilde{v} - \frac{c_0}{F} F^{ij} g_{ij}
\end{aligned}
$$

for a.e. $t > t_0$, where $c_0 = c_0(t_0)$ and the right side is evaluated at (t, x_t). Due to the assumptions on f we may furthermore assume that for all $t > t_0$ the following inequality holds in (t, x_t)

$$(8.43) \qquad \frac{u}{F} \tilde{v}^2 f'' F^{ij} g_{ij} - \frac{\tilde{v}^2}{F} f' F^{ij} g_{ij} \geq 2\tilde{v} - \frac{c_0}{F} F^{ij} g_{ij},$$

which leads to

$$(8.44) \qquad \dot{\tilde\varphi}(t) \geq -\frac{u}{F} F^{ij} h_i^k h_{kj} - 2\frac{c_0}{F} F^{ij} g_{ij}$$

for a.e. $t > t_0$ in view of (8.42); again the right side is evaluated at (t, x_t).

We assume that (8.37) is not true, then there are sequences $0 < t_k \to \infty$, $x_k \in S_0$ and a constant $c_1 > 0$ such that

$$(8.45) \qquad \sup_{M(t_k)} \max_i \kappa_i u = \kappa_n u|_{(t_k, x_k)} \to c_1,$$

which implies

$$
\begin{aligned}
(8.46) \quad \limsup_{k \to \infty} \tilde\varphi(t_k) &< F(-\frac{c_1}{2} + \tilde\gamma^{-1}, \tilde\gamma^{-1}, ..., \tilde\gamma^{-1}) \\
&< F(\tilde\gamma^{-1} - r, ..., \tilde\gamma^{-1} - r) \\
&=: c(r),
\end{aligned}
$$

for $r > 0$ sufficiently small and fixed from now on.

Next, we will show that, after increasing t_0 if necessary, there exists $\delta > 0$ such that the following implication holds for a.e. $t > t_0$

$$(8.47) \qquad \tilde\varphi(t) \leq c(r) \Rightarrow \dot{\tilde\varphi}(t) \geq \delta$$

in contradiction to (8.46).

For that purpose assume t_0 to be sufficiently large. Let $t > t_0$ be such that $\tilde{\varphi}$ is differentiable in t and $\tilde{\varphi}(t) \leq c(r)$, then it follows from (8.38) that we have in (t, x_t)

$$(8.48) \qquad |u|\kappa_n + \tilde{v}|f' u| + |u|\psi_\alpha \nu^\alpha \leq -r + \tilde{\gamma}^{-1},$$

i.e.

$$(8.49) \qquad \kappa_n \leq -\frac{r}{2|u|}.$$

Hence, we infer from (8.44)

$$(8.50) \qquad \dot{\tilde{\varphi}}(t) \geq \frac{r^2}{4F|u|} - 2\frac{c_0}{F}F^{ij}g_{ij}.$$

After a possibly further enlargement of t_0 we get a positive lower bound for the right side of the last inequality that does not depend on t, thus the desired $\delta > 0$, which completes the proof. □

Now we are able to prove a decay of $\|A\|$.

Lemma 8.6. *For any $0 < \lambda < \gamma$ there exists $c_\lambda > 0$ such that*

$$(8.51) \qquad \|A\|e^{\lambda t} \leq c_\lambda.$$

Proof. Define $\varphi = \frac{1}{2}\|A\|^2 e^{2\lambda t}$ with $0 < \lambda < \gamma$, then

$$(8.52) \quad e^{-2\lambda t}(\dot{\varphi} - \frac{1}{F^2}F^{ij}\varphi_{ij}) = -\frac{1}{F^2}F^{kl}h^i_{j;k}h^j_{i;l} + (\dot{h}^i_j - \frac{1}{F^2}F^{kl}h^i_{j;kl})h^j_i + \lambda\|A\|^2.$$

Let $0 < T < \infty$ be large, and $x_0 = x_0(t_0)$, with $0 < t_0 \leq T$, be a point in $M(t_0)$ such that

$$(8.53) \qquad \sup_{M_0} \varphi < \sup\{\sup_{M(t)} \varphi : 0 < t \leq T\} = \varphi(x_0).$$

From Lemma 8.5 we know that

$$(8.54) \qquad \sup_{M(t)} \|A\||u| \longrightarrow 0 \quad as \quad t \longrightarrow \infty$$

so that especially in view of the homogeneity of F

$$(8.55) \qquad 0 < c_1 < F^i(\check{\kappa}_i) \leq c_2 \quad \wedge \quad |F^{ij}(\check{\kappa}_i)| \leq ce^{-\gamma t}$$

(first and second derivatives of F considered as a function on Γ_+). In x_0 we have due to (8.52) and Lemma 8.3, after multiplication by F^2 and some straight-forward estimates,

$$
\begin{aligned}
0 \leq & -F^{kl}h^i_{j;k}h^j_{i;l} - 2F^{-1}F^i F_j h^j_i + 2Fh^{ir}h_{rj}h^j_i + F^{ij,rs}\check{h}_{ij;p}\check{h}_{rs;p}h^{pp} \\
& + c|f'|^{1+\epsilon}\|A\| + c|f'|^{\frac{1}{2}}\|A\|^2 + \tilde{v}^2 f''\|A\|^2 F^{ij}g_{ij} + \lambda F^2\|A\|^2 \\
(8.56) \quad \leq & -\frac{1}{2}F^{kl}h^i_{j;k}h^j_{i;l} + \tilde{v}^2 f''\|A\|^2 F^{ij}g_{ij} + cF\|A\|^2\|A\| + c|f'|^{1+\epsilon}\|A\| \\
& + c|f'|^{\frac{1}{2}}\|A\|^2 + \lambda F^2\|A\|^2.
\end{aligned}
$$

For the last inequality we used that in local coordinates (such that $g_{ij} = \delta_{ij}$, h_{ij} diagonal and F^{ij} diagonal)

$$(8.57) \qquad |F_i F_j| \leq c \sum_{i,k,l} |h_{kl;i}|^2 + c\|A\|^2|f'|^{\frac{1}{2}} + c|f'|^{2+\epsilon},$$

where we used Lemma 7.4, and where $0 < \epsilon < 1$ is arbitrary but fixed, so that

$$(8.58) \qquad F^{-1}F^i F_j h_i^j \leq c\frac{\|A\|}{F}\sum_{i,k,l}|h_{kl;i}|^2 + c\|A\|^2|f'|^{-\frac{1}{2}}\|A\| + c|f'|^{1+\epsilon}\|A\|,$$

where $\frac{\|A\|}{F} \to 0$ because of Lemma 8.5.

To estimate $F^{ij,rs}\check{h}_{ij;p}\check{h}_{rs;p}h_p^p$ we used [8, inequality (2.1.73)] and (8.55).

Now we have

$$(8.59) \qquad \begin{aligned} F &= |f'|F(\frac{\check{\kappa}_i}{|f'|}) = |f'|F(1,...,1) + |f'|(F(\frac{\check{\kappa}_i}{|f'|}) - F(1,...,1)) \\ &\leq n|f'| + |f'|c(t), \end{aligned}$$

where $0 < c(t) \to 0$, hence

$$(8.60) \qquad \begin{aligned} \tilde{v}^2 f''\|A\|^2 F^{ij}g_{ij} + \lambda F^2\|A\|^2 &\leq c\|A\|^2 - (\gamma - \lambda)n^2|f'|^2\|A\|^2 \\ &\quad + \lambda cc(t)|f'|^2\|A\|^2. \end{aligned}$$

Together with (8.56) we deduce that φ is a priori bounded from above. $\qquad\square$

In the next two theorems we prove the optimal decay of $\|Du\|$ and $\|A\|$ which finishes the C^2-estimates.

Theorem 8.7. *Let $\tilde{u} = ue^{\gamma t}$, then $\|D\tilde{u}\|$ is uniformly bounded during the evolution.*

Proof. Let $\varphi = \varphi(t)$ be defined by

$$(8.61) \qquad \varphi = \sup_{M(t)} \log \tilde{v}e^{2\gamma t}.$$

Then, in view of the maximum principle, we deduce from the evolution equation of $\tilde{v}$, cf. Lemma (4.2),

$$(8.62) \qquad \begin{aligned} \dot{\varphi} &\leq ce^{-\epsilon t} + F^{-2}(f''\|D\tilde{u}\|^2 F^{ij}g_{ij} + 2\gamma F^2\varphi) \\ &\leq ce^{-\epsilon t} + 2F^{-2}(f''F^{ij}g_{ij} + \gamma F^2)\varphi \\ &\leq ce^{-\epsilon t}(1+\varphi), \end{aligned}$$

where $\epsilon > 0$ small, i.e., φ is uniformly bounded. $\qquad\square$

Theorem 8.8. *The quantity $w = \frac{1}{2}\|A\|^2 e^{2\gamma t}$ is uniformly bounded during the evolution.*

Proof. Define $\varphi = \varphi(t)$ by

$$(8.63) \qquad \varphi = \sup_{M(t)} w.$$

We deduce from Lemma 8.3 that for a.e. $t \geq t_0$, $t_0 > 0$ large,

$$(8.64) \qquad \begin{aligned} \dot{\varphi} &= \frac{1}{F^2}F^{ij}\varphi_{ij} - \frac{1}{F^2}F^{kl}h_{j;k}^i h_{i;l}^j e^{2\gamma t} + (\dot{h}_j^i - \frac{1}{F^2}F^{kl}h_{j;kl}^i)h_i^j e^{2\gamma t} \\ &\quad + \gamma\|A\|^2 e^{2\gamma t} \\ &\leq -\frac{1}{2F^2}F^{kl}h_{j;k}^i h_{i;l}^j e^{2\gamma t} + F^{-3}(-2h^{ij}F_iF_j e^{2\gamma t} - Ff'''h^{ij}\tilde{u}_i\tilde{u}_j F^{ij}g_{ij}\tilde{v}) \\ &\quad + 2F^{-2}(nf''\tilde{v}^2\varphi + \gamma F^2\varphi) + ce^{-\epsilon t}(1+\varphi) \\ &\quad + F^{-1}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta\nu^\gamma x_j^\delta h^{ij}e^{2\gamma t}, \end{aligned}$$

where $\epsilon > 0$ is small.

For the last inequality we estimated the crucial term

$$(8.65) \qquad F^{ij,rs}\check{h}_{ij;p}\check{h}_{rs;q}h^{pq}$$

in the following way.

Since $F^{ij}g_{ij} \geq F(1,...,1)$ and

$$(8.66) \qquad F^{ij}(g_{kl})g_{ij} = F(1,...,1)$$

we deduce that the derivative vanishes in $\check{h}_{kl} = g_{kl}$

$$(8.67) \qquad F^{ij,rs}(g_{kl})g_{ij} = 0.$$

Hence

$$(8.68) \qquad F^{ij,rs}(\check{h}_{kl})g_{ij} = |u|(F^{ij,rs}(|u|\check{h}_{kl}) - F^{ij,rs}(\frac{1}{\tilde{\gamma}}g_{kl}))g_{ij}$$

which means by mean value theorem

$$(8.69) \qquad \|F^{ij,rs}(\check{h}_{kl})g_{ij}\| \leq c|u|^2.$$

Although the last inequality is good enough, we mention that its right side could be improved to $c|u|^{3-\epsilon}$, $\epsilon > 0$ arbitrary, cf. Lemma 8.6.

Furthermore, to estimate (8.65) we use

$$(8.70) \qquad \check{h}_{ij;p} = h_{ij;p} - \tilde{v}_p f' g_{ij} - \tilde{v}f'' u_p g_{ij} + \psi_{\alpha\beta}\nu^\alpha x_p^\beta g_{ij} + \psi_\alpha x_r^\alpha h_p^r g_{ij}$$

and

$$(8.71) \qquad \tilde{v}_p = \eta_{\alpha\beta}\nu^\alpha x_p^\beta - u_r h_p^r$$

So in view of Lemma 8.6 and Theorem 8.7 we have choosing coordinates such that (h_{ij}) diagonal and $g_{ij} = \delta_{ij}$

$$(8.72) \qquad \begin{aligned} |F^{ij,rs}\check{h}_{ij;p}\check{h}_{rs;q}h^{pq}| &\leq |F^{ij,rs}h_{ij;p}h_{rs;q}h^{pq}| \\ &+ 2|F^{ij,rs}h_{rs;p}g_{ij}(-\tilde{v}_q f' - \tilde{v}f'' u_q + \psi_{\alpha\beta}\nu^\alpha x_q^\beta + \psi_\alpha x_r^\alpha h_q^r)h^{pq}| \\ &+ |F^{ij,rs}g_{rs}g_{ij}\sum_p(-\tilde{v}_p f' - \tilde{v}f'' u_p + \psi_{\alpha\beta}\nu^\alpha x_p^\beta + \psi_\alpha x_r^\alpha h_p^r)^2 h_p^p| \\ &\leq c|u|\|DA\|^2\|A\| + c\|A\||u|(\|DA\| + 1). \end{aligned}$$

The second term of the right side of inequality (8.64) can be estimated as follows

$$(8.73) \qquad \begin{aligned} F^{-3}(-2h^{ij}F_iF_je^{2\gamma t} - Ff'''h^{ij}\tilde{u}_i\tilde{u}_jF^{ij}g_{ij}\tilde{v}) &\leq \\ F^{-3}(-2|f''|^2 + f'f''')h^{ij}\tilde{u}_i\tilde{u}_j\tilde{v}^2n^2 + cF^{-3}\|DA\|^2e^{2\gamma t} \\ + ce^{-\epsilon t}(1 + \varphi). \end{aligned}$$

Now, we observe that

$$(8.74) \qquad (f'' + \tilde{\gamma}|f'|^2)' = f''' + 2\tilde{\gamma}f'f'' = Cf',$$

where C is a bounded function in view of (1.9). Hence

$$(8.75) \qquad 2|f''|^2 - f'f''' = 2|f''|^2 + 2\tilde{\gamma}|f'|^2f'' - C|f'|^2,$$

i.e.,

$$(8.76) \qquad |2|f''|^2 - f'f'''| \leq c|f'|^2$$

because of (1.8) and we conclude that the left-hand side of (8.73) can be estimated from above by

$$(8.77) \qquad ce^{-\epsilon t}(1 + \varphi) + cF^{-2}\|DA\|^2 e^{\gamma t}.$$

Next, we estimate

$$(8.78) \qquad F^{-2}(nf''\tilde{v}^2 + \gamma F^2)\varphi \le ce^{-\epsilon t}\varphi$$

and finally

$$(8.79) \qquad F^{-1}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta h^{ij}e^{2\gamma t} \le ce^{-\epsilon t}(1+\varphi) + F^{-1}\bar{R}_{0i0j}h^{ij}e^{2\gamma t}\tilde{v}^2,$$

but

$$(8.80) \qquad \bar{R}_{0i0j} \le c|u|,$$

cf. proof of Lemma 4.6(iii).

Hence we deduce

$$(8.81) \qquad \dot{\varphi} \le ce^{-\epsilon t}(1+\varphi)$$

for some positive ϵ and for a.e. $t \ge t_0$, i.e. φ is bounded. $\qquad\square$

9. Higher order estimates–Asymptotic behaviour of the flow

In this section and the following two sections many proofs are identical to the proofs in [5]. For reasons of completeness and convenience for the reader we present them here.

Let us now introduce the following abbreviations

Definition 9.1. (i) For arbitrary tensors S, T denote by $S * T$ any linear combination of contractions of $S \otimes T$. The result can be a tensor or a function. Note that we do not distinguish between $S * T$ and $cS * T$, where c is a constant.

(ii) The symbol A represents the second fundamental of the hypersurfaces $M(t)$ in N, $\tilde{A} = Ae^{\gamma t}$ is the scaled version, and $D^m A$ resp. $D^m\tilde{A}$ represent the covariant derivative of order m.

(iii) For $m \in \mathbb{N}$ denote by $\tilde{O}_m$ a tensor expression defined on $M(t)$ that satisfies the pointwise estimate

$$(9.1) \qquad \|\tilde{O}_m\| \le c_m(1 + \|\tilde{A}\|_m)^{p_m}$$

and

$$(9.2) \qquad \|D\tilde{O}_m\| \le c_m(1 + \|\tilde{A}\|_m)^{p_m}(1 + \|D^{m+1}\tilde{A}\|),$$

where $c_m, p_m > 0$ are constants and

$$(9.3) \qquad \|\tilde{A}\|_m = \sum_{|\alpha| \le m}\|D^\alpha\tilde{A}\|.$$

(iv) For arbitrary $m \in \mathbb{N}$ denote by O_m a tensor expression defined on $M(t)$ that satisfies

$$(9.4) \qquad D^k O_m = \tilde{O}_{m+k} \quad \forall k \in \mathbb{N}.$$

(v) By the symbol O we denote a tensor expression such that $DO = O_0$.

Remark 9.2.

$$(9.5) \qquad D^k O_m = O_{m+k} \quad \forall(k,m) \in \mathbb{N} \times \mathbb{N}.$$

Lemma 9.3. *We have*

$$(9.6) \qquad D(uf') = e^{-2\gamma t}O$$

especially

$$(9.7) \qquad D^m(uf') = e^{-2\gamma t}O_{m-2}, \quad m \geq 2.$$

Proof. Differentiating and adding a zero yields

$$(9.8) \qquad D_i(uf') = u_i f'(1 - \tilde{\gamma}f'u) + uu_i(\tilde{\gamma}|f'|^2 + f'')$$

from which we deduce the claim in view of (1.8), (1.9) and (1.10).

$\square$

Lemma 9.4. *We have*

$$(9.9) \qquad D(u\check{h}_{kl}) = e^{-2\gamma t}O_0 + e^{-2\gamma t}D\tilde{A}O.$$

Proof. Differentiating yields ($g_{ij} = \delta_{ij}$, $h_{ij} = $ diagonal)

$$(9.10) \qquad \begin{aligned} D_i(u\check{h}_{kl}) =& u_i\check{h}_{kl} + u(h_{kl;i} - \eta_{\alpha\beta}\nu^\alpha x_i^\beta f' g_{kl} \\ &+ (\psi_\alpha\nu^\alpha)_i g_{kl}) + uu_i\kappa_i f' g_{kl} - u\tilde{v}f'' u_i g_{kl} \end{aligned}$$

and now we focus on the last term and write there

$$(9.11) \qquad f'' = (f'' + \tilde{\gamma}|f'|^2) - \tilde{\gamma}|f'|^2.$$

Then all terms can be estimated obviously except for

$$(9.12) \qquad u_i\check{h}_{kl} + \tilde{\gamma}|f'|^2 u\tilde{v}u_i g_{kl},$$

for which we use (1.31).

$\square$

Corollary 9.5. *We have*

$$(9.13) \qquad D^m(u\check{h}_{kl}) = e^{-2\gamma t}O_{m-1} + e^{-2\gamma t}D^m\tilde{A} * O.$$

Definition 9.6. We denote by $\mathcal{D}^m F$ the derivatives of order m of F with respect to $\check{h}_j^i$.

Lemma 9.7. *We have*

$$(9.14) \qquad D^m\mathcal{D}F = e^{-2\gamma t}O_{m-1} + e^{-2\gamma t}D^m\tilde{A} * \mathcal{D}^2 F(|u|\check{h}_{kl}) * O,$$

$$(9.15) \qquad |F^{ij}(\check{h}_{kl})g_{ij} - F^{ij}(g_{kl})g_{ij})| \leq ce^{-2\gamma t},$$

$$(9.16) \qquad DF = \mathcal{D}F * DA + e^{-\gamma t}\mathcal{D}F * O_0 + e^{\gamma t}\mathcal{D}F * O,$$

and

$$(9.17) \qquad D^m F = \mathcal{D}F * D^m A + e^{-\gamma t}O_{m-1} + e^{\gamma t}O_{m-2},$$

for $m \geq 2$.

Proof. To prove (9.14) we write

$$(9.18) \qquad \mathcal{D}F(\check{h}_{kl}) = \mathcal{D}F(|u|\check{h}_{kl})$$

and infer

$$(9.19) \qquad D\mathcal{D}F(\check{h}_{kl}) = \mathcal{D}^2 F(|u|\check{h}_{kl})D(|u|\check{h}_{kl}),$$

hence the desired result follows in view of (9.13) and the fact that

$$(9.20) \qquad \|\mathcal{D}^m F(|u|\check{h}_{kl})\|$$

is bounded for all $m \in \mathbb{N}$.

(9.15) is proved by applying the mean value theorem.

(9.16) follows by

$$(9.21) \qquad F_k = F^{ij} h_{ij;k} + F^{ij} g_{ij} (-\tilde{v}_k f' - \tilde{v} f'' u_k + \psi_{\alpha\beta} \nu^\alpha x_k^\beta + \psi_\alpha x_r^\alpha h_k^r).$$

To prove (9.17) we differentiate (9.16) and get

$$
\begin{aligned}
(9.22) \qquad D^2 F =& D\mathcal{D}F * DA + \mathcal{D}F * D^2 A + e^{-\gamma t} D\mathcal{D}F * O_0 \\
& + e^{-\gamma t} \mathcal{D}F * O_1 + e^{\gamma t} D\mathcal{D}F * O + e^{\gamma t} \mathcal{D}F * O_0 \\
=& \mathcal{D}F * D^2 A + e^{-\gamma t} O_1 + e^{\gamma t} O_0,
\end{aligned}
$$

from which the claim follows easily. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Now we want to write the evolution equation for $\tilde{h}_l^k$ in the form

$$
\begin{aligned}
(9.23) \qquad \dot{\tilde{h}}_i^j - F^{-2} F^{kl} \tilde{h}_{i;kl}^j =& F^{-3} D\tilde{A} * DA * O_0 + F^{-2} D\tilde{A} * O_0 + F^{-1} O_0 \\
& + F^{-3} D\tilde{A} * DA * O.
\end{aligned}
$$

To check this we consider all the terms in (8.6) separately and start with

$$(9.24) \qquad \boxed{(-2F^{-3} F^k F_l - F^{-2} F^{ij} g_{ij} f''' u^k u_l \tilde{v}) e^{\gamma t}}.$$

We have

$$
\begin{aligned}
(9.25) \qquad F_k =& F^{rs} h_{rs;k} - \eta_{\alpha\beta} \nu^\alpha x_k^\beta f' F^{rs} g_{rs} - \eta_\alpha x_i^\alpha h_k^i f' F^{rs} g_{rs} - \tilde{v} f'' u_k F^{rs} g_{rs} \\
& + \psi_{\alpha\beta} \nu^\alpha x_k^\beta F^{rs} g_{rs} + \psi_\alpha x_r^\alpha h_k^r F^{rs} g_{rs} \\
=& A_1 + A_2 + A_3 + A_4 + A_5 + A_6,
\end{aligned}
$$

hence

$$
\begin{aligned}
(9.26) \qquad (-2F^{-3} A_4 A_4 & - F^{-2} F^{ij} g_{ij} f''' u^k u_l \tilde{v}) e^{\gamma t} \\
&= F^{-3} (-2|f''|^2 + f' f''') \tilde{v}^2 (F^{ij} g_{ij})^2 u_l u^k e^{\gamma t} \\
&\quad - F^{-2} F^{rs} h_{rs} F^{ij} g_{ij} f''' u^k u_l \tilde{v} e^{\gamma t} \\
&\quad - F^{-2} (F^{ij} g_{ij})^2 \psi_\alpha \nu^\alpha f''' u^k u_l \tilde{v} e^{\gamma t} \\
&= F^{-1} O_0,
\end{aligned}
$$

where we observed that

$$(9.27) \qquad \varphi = -2|f''|^2 + f' f''' = (f'' + \tilde{\gamma}|f'|^2)' f' - 2f''(f'' + \tilde{\gamma}|f'|^2).$$

In view of the assumptions on f the spatial derivatives of φ can be estimated by

$$(9.28) \qquad \|D^m \varphi\| \le c_m (1 + \|\tilde{u}\|_{m-1})^{p_{m-1}} (1 + \|D^m \tilde{u}\|) e^{2\gamma t} \quad \forall m \in \mathbb{N}^*$$

for some suitable $p_{m-1} \in \mathbb{N}$. Furthermore, we have

$$(9.29) \qquad -2F^{-3} A_1 A_1 e^{\gamma t} = F^{-3} O_0 * D\tilde{A} * DA.$$

All remaining terms are estimated as follows

$$(9.30) \qquad -2F^{-3} e^{\gamma t} \sum_{(i,j) \notin \{(1,1),(4,4)\}} A_i A_j = F^{-2} D\tilde{A} * O_0 + F^{-2} O_0$$

hence

$$(9.31) \qquad (-2F^{-3}F^k F_l - F^{-2}F^{ij}g_{ij}f'''u^k u_l \tilde{v})e^{\gamma t} = \\ F^{-3}D\tilde{A} * DA * O_0 + F^{-2}D\tilde{A} * O_0 + F^{-1}O_0.$$

Now, there are some quiet easy estimates, namely

$$(9.32) \qquad \boxed{2F^{-1}h^{kr}h_{rl}e^{\gamma t}} = F^{-2}O_0$$

$$\boxed{-F^{-2}e^{\gamma t}F^{ij}h_{aj}h_i^a h_l^k} = F^{-2}O_0$$

$$\boxed{2F^{-2}g^{pk}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}x_r^\alpha x_p^\beta x_i^\gamma x_l^\delta h_j^r e^{\gamma t}} = F^{-2}O_0$$

$$\boxed{F^{-2}g^{pk}F^{ij}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x_p^\beta x_i^\gamma x_l^\delta x_j^\epsilon e^{\gamma t}} = F^{-1}O_0.$$

Furthermore, we have

$$(9.33) \qquad \boxed{F^{-1}e^{\gamma t}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_l^\beta \nu^\gamma x_r^\delta g^{rk}} = -F^{-1}\bar{R}_{\alpha\beta\gamma\delta}\eta^\alpha \tilde{v}x_l^\beta \nu^\gamma x_r^\delta e^{\gamma t} \\ - F^{-1}\bar{R}_{i\beta\gamma\delta}u^i x_l^\beta \eta^\gamma x_r^\delta g^{rk}e^{\gamma t} + F^{-1}\bar{R}_{i\beta j\delta}\tilde{v}^{-2}u^i x_l^\beta u^j x_r^\delta g^{rk}e^{\gamma t} \\ = F^{-2}O,$$

$$(9.34) \qquad \boxed{F^{-2}e^{\gamma t}g^{pk}F^{ij,rs}\check{h}_{ij;p}\check{h}_{rs;l}} = F^{-2}e^{\gamma t}g^{pk}\check{h}_{ij;p}D_l F^{ij} \\ = F^{-2}D\tilde{A} * O + F^{-3}D\tilde{A} * DA * O + F^{-3}D\tilde{A} * O_0 + F^{-2}O_0$$

and

$$(9.35) \qquad \boxed{F^{-2}e^{\gamma t}F^{ij}g_{ij}(\psi_\alpha x_r^\alpha h^{rk}{}_{;l} + f'u^r h_{l;r}^k)} = F^{-2}D\tilde{A} * O_0,$$

so that only the following term is left

$$(9.36) \qquad \boxed{F^{-2}e^{\gamma t}F^{ij}g_{ij}f''\tilde{v}^2 h_l^k + \gamma h_l^k e^{\gamma t}} = F^{-2}e^{\gamma t}(F^{ij}g_{ij}f''\tilde{v}^2 h_l^k + \gamma F^2 h_l^k).$$

There holds

$$(9.37) \qquad F^2 = (F^{ij}h_{ij})^2 - 2F^{ij}h_{ij}\tilde{v}f'F^{rs}g_{rs} + 2F^{ij}h_{ij}\psi_\alpha \nu^\alpha F^{rs}g_{rs} \\ + \tilde{v}^2|f'|^2(F^{ij}g_{ij})^2 - 2\psi_\alpha \nu^\alpha \tilde{v}f'(F^{ij}g_{ij})^2 + (\psi_\alpha \nu^\alpha)^2(F^{ij}g_{ij})^2$$

and

$$(9.38) \qquad f''\tilde{v}^2 h_l^k + \gamma h_l^k \tilde{v}^2|f'|^2 F^{ij}g_{ij} = \tilde{v}^2 h_l^k(f'' + \gamma|f'|^2 n) \\ + \tilde{v}^2 h_l^k \gamma|f'|^2(F^{ij}g_{ij} - n),$$

so that we infer

$$(9.39) \qquad F^{-2}e^{\gamma t}F^{ij}g_{ij}f''\tilde{v}^2 h_l^k + \gamma h_l^k e^{\gamma t} = F^{-2}O_0.$$

Using the fact that

$$(9.40) \qquad \dot{g}_{ij} = -2F^{-1}h_{ij} = F^{-2}O_0$$

(9.23) is proved.

Differentiating (9.23) covariantly with respect to a spatial variable we deduce

(9.41)
$$\frac{D}{dt}(D\tilde{A}) - F^{-2}F^{ij}(D\tilde{A})_{ij} = F^{-1}O_0 + F^{-3}D^2\tilde{A} * DAO_0$$
$$+ F^{-2}O_0 * D^2\tilde{A} + F^{-4}D\tilde{A} * DA * DA * O_0 + F^{-3}O_0 * D\tilde{A} * DA$$
$$+ F^{-2}D\tilde{A} * O_0 + F^{-3}D\tilde{A} * DA * DO_0 + F^{-2}D\tilde{A} * DO_0 + F^{-1}DO_0.$$

And using induction we conclude for $m \in \mathbb{N}^*$

(9.42)
$$\frac{D}{dt}(D^{m+1}\tilde{A}) - F^{-2}F^{ij}(D^{m+1}\tilde{A})_{ij} = F^{-1}O_m$$
$$+ \Theta F^{-3}D^{m+1}\tilde{A} * D^{m+1}A * O_0 + F^{-3}D^{m+2}\tilde{A} * DA * O_0$$
$$+ F^{-2}D^{m+1}\tilde{A} * O_m + F^{-2}D^{m+2}\tilde{A} * O_0$$
$$+ F^{-2}DO_m,$$

where $\Theta = 1$ if $m = 1$ and $\Theta = 0$ else.

We are now going to prove uniform bounds for $\frac{1}{2}\|D^{m+1}\tilde{A}\|$ for all $m \in \mathbb{N}$. First we observe that

(9.43)
$$\frac{D}{dt}(\frac{1}{2}\|D\tilde{A}\|^2) - F^{-2}F^{ij}\frac{1}{2}(\|D\tilde{A}\|^2)_{ij} = -F^{-2}F^{ij}(D\tilde{A})_i(D\tilde{A})_j$$
$$+ F^{-1}O_0 * D\tilde{A} + F^{-3}D^2\tilde{A} * DA * O_0 * D\tilde{A} + F^{-2}O_0 * D^2\tilde{A} * D\tilde{A}$$
$$+ F^{-4}D\tilde{A} * DA * DA * O_0 * D\tilde{A} + F^{-3}O_0 * D\tilde{A} * DA * D\tilde{A}$$
$$+ F^{-2}D\tilde{A} * O_0 * D\tilde{A} + F^{-3}D\tilde{A} * DA * DO_0 * D\tilde{A}$$
$$+ F^{-2}D\tilde{A} * DO_0 * D\tilde{A} + F^{-1}DO_0 * D\tilde{A}.$$

Furthermore we have for $m \in \mathbb{N}^*$

(9.44)
$$\frac{D}{dt}(\frac{1}{2}\|D^{m+1}\tilde{A}\|^2) - F^{-2}F^{ij}\frac{1}{2}(\|D^{m+1}\tilde{A}\|^2)_{ij} =$$
$$- F^{-2}F^{ij}(D^{m+1}\tilde{A})_i(D^{m+1}\tilde{A})_j + F^{-1}O_m * D^{m+1}\tilde{A}$$
$$+ \Theta F^{-3}D^{m+1}\tilde{A} * D^{m+1}A * O_0 * D^{m+1}\tilde{A}$$
$$+ F^{-3}D^{m+2}\tilde{A} * DA * O_0 * D^{m+1}\tilde{A} + F^{-2}D^{m+1}\tilde{A} * O_m * D^{m+1}\tilde{A}$$
$$+ F^{-2}D^{m+2}\tilde{A} * O_0 * D^{m+1}\tilde{A} + F^{-2}DO_m * D^{m+1}\tilde{A}.$$

Theorem 9.8. *The quantities $\frac{1}{2}\|D^m\tilde{A}\|^2$ are uniformly bounded during the evolution for all $m \in \mathbb{N}^*$*

Proof. We prove the theorem recursively by estimating

(9.45)
$$\varphi = \log\frac{1}{2}\|D^{m+1}\tilde{A}\|^2 + \mu\frac{1}{2}\|D^m\tilde{A}\|^2 + \lambda e^{-\gamma t},$$

where μ is a small positive constant and $\lambda >> 1$ large.

We shall only treat the case $m = 0$.

Fix $0 < T < \infty$, T very large, and suppose that

(9.46)
$$0 < \sup_{[0,T]} \sup_{M(t)} \varphi = \varphi(t_0, \xi_0)$$

for large $0 < t_0 \leq T$.

Applying the maximum principle we deduce

$$
\begin{aligned}
0 \leq & \frac{1}{\|D\tilde{A}\|^2}(\frac{D}{dt}\|D\tilde{A}\|^2 - F^{-2}F^{ij}\|D\tilde{A}\|_{ij}^2) + \mu\tilde{A}(\dot{\tilde{A}} - F^{-2}F^{ij}\tilde{A}_{ij}) \\
& + F^{-2}F^{ij}\tilde{A}_i\tilde{A}_j(-\mu + \mu^2\tilde{A}^2) - \lambda\gamma e^{-\gamma t} \\
\text{(9.47)} \quad \leq & -\frac{1}{2}\frac{1}{\|D\tilde{A}\|^2}F^{-2}F^{ij}(D\tilde{A})_i(D\tilde{A})_j - \frac{\lambda}{2}\gamma e^{-\gamma t} + cF^{-4}\|D\tilde{A}\|^2 \\
& + cF^{-2}\|D\tilde{A}\| + F^{-2}F^{ij}\tilde{A}_i\tilde{A}_j(-\mu + \mu^2\tilde{A}^2) \\
< & 0,
\end{aligned}
$$

here we assumed that $\|D\tilde{A}\|$ is larger than a sufficiently large positive constant that does not depend on t_0, T.

Thus φ is a priori bounded.

The proof for $m \geq 1$ is similar. $\qquad\square$

10. CONVERGENCE OF $\tilde{u}$ AND THE BEHAVIOUR OF DERIVATIVES IN t

Lemma 10.1. $\tilde{u}$ converges in $C^m(S_0)$ for any $m \in \mathbb{N}$, if t tends to infinity, and hence $D^m\tilde{A}$ converges.

Proof. $\tilde{u}$ satisfies the evolution equation

$$
\text{(10.1)} \qquad \dot{\tilde{u}} = \frac{\tilde{v}e^{\gamma t}}{F}(1 - \gamma u f' F^{ij}g_{ij} + \frac{\gamma u}{\tilde{v}}F^{ij}h_{ij} + \frac{\gamma u}{\tilde{v}}\psi_\alpha\nu^\alpha F^{ij}g_{ij}).
$$

Using (9.15) and the already known exponential decays we deduce

$$
\text{(10.2)} \qquad\qquad |\dot{\tilde{u}}| \leq ce^{-2\gamma t},
$$

hence $\tilde{u}$ converges uniformly. Due to Theorem 9.8 $D^m\tilde{u}$ is uniformly bounded, hence $\tilde{u}$ converges in $C^m(S_0)$.

The convergence of $D^m\tilde{A}$ follows from Theorem 9.8 and the convergence of $\tilde{h}_{ij}$, which in turn can be deduced from

$$
\text{(10.3)} \qquad\qquad h_{ij}\tilde{v} = -u_{ij} + \bar{h}_{ij}.
$$

$\qquad\square$

Combining the equations (9.23), (9.41) and (9.42) we immediately conclude

Lemma 10.2. $\|\frac{D}{dt}D^m\tilde{A}\|$ and $\|\frac{D}{dt}D^mA\|$ decay by the order $e^{-\gamma t}$ for any $m \in \mathbb{N}$.

Corollary 10.3. $\frac{D}{dt}D^mAe^{\gamma t}$ converges, if t tends to infinity.

Proof. Applying the product rule we obtain

$$
\text{(10.4)} \qquad\qquad \frac{D}{dt}D^m\tilde{A} = \frac{D}{dt}D^mAe^{\gamma t} + \gamma D^m\tilde{A},
$$

hence the result, since the left-hand side converges to zero and $D^m\tilde{A}$ converges. $\quad\square$

Corollary 10.4. *We have*

$$
\text{(10.5)} \qquad\qquad \|D^mF^{-1}\| \leq c_mF^{-1} \quad \forall m \in \mathbb{N}.
$$

Proof. Use (9.17). $\qquad\square$

In the next Lemmas we prove some auxiliary estimates.

Lemma 10.5. *The following estimates are valid*

$$(10.6) \qquad \qquad \|D\dot{u}\| \leq ce^{-\gamma t},$$

$$(10.7) \qquad \qquad \|\frac{d}{dt}F^{-1}\| \leq cF^{-1},$$

and

$$(10.8) \qquad \qquad |\dot{\tilde{v}}| \leq ce^{-2\gamma t}.$$

Proof. "(10.6)" The estimate follows immediately from

$$(10.9) \qquad \qquad \dot{u} = \frac{\tilde{v}}{F},$$

in view of Corollary 10.4.

"(10.7)" Differentiating with respect to t we obtain

$$(10.10) \quad \begin{aligned} \frac{d}{dt}F^{-1} = &-F^{-2}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f^{'}F^{ij}g_{ij} - \tilde{v}f^{''}\dot{u}F^{ij}g_{ij} + \frac{d}{dt}(\psi_\alpha\nu^\alpha)F^{ij}g_{ij}) \\ &+ F^{ij}(-\tilde{v}f^{'} + \psi_\alpha\nu^\alpha)\dot{g}_{ij} \end{aligned}$$

and the result follows from (10.8) and the known estimates for $|\dot{u}|$ and F.

"(10.8)" We differentiate the relation $\eta_\alpha\nu^\alpha$ to get

$$(10.11) \quad \begin{aligned} \dot{\tilde{v}} &= \eta_{\alpha\beta}\nu^\alpha\dot{x}^\beta + \eta_\alpha\dot{\nu}^\alpha \\ &= -\eta_{\alpha\beta}\nu^\alpha\nu^\beta F^{-1} + (F^{-1})_k u^k, \end{aligned}$$

cf. (8.2), yielding the estimate for $|\dot{\tilde{v}}|$, in view of Corollary 10.4. $\qquad \square$

Lemma 10.6.

$$(10.12) \qquad \qquad \|F^{ij,kl}(\check{h}_{rs})g_{ij}\| \leq ce^{-3\gamma t},$$

$$(10.13) \qquad \qquad \|F^{ij,kl}(|u|\check{h}_{rs})g_{ij}\| \leq ce^{-2\gamma t},$$

$$(10.14) \qquad \qquad \|\frac{D}{dt}(u\check{h}_{kl})\| \leq ce^{-2\gamma t},$$

$$(10.15) \qquad \qquad \|\frac{D}{dt}F^{ij,kl}(\check{h}_{rs})g_{ij}\| \leq ce^{-3\gamma t},$$

$$(10.16) \qquad \qquad \|\frac{D}{dt}F^{ij,kl}(\check{h}_{rs})\| \leq ce^{-\gamma t}$$

$$(10.17) \qquad \qquad \|\frac{D}{dt}F^{ij}\| \leq ce^{-2\gamma t},$$

$$(10.18) \qquad \qquad \|\frac{D}{dt}D\check{h}_{kl}\| + \|\frac{D}{dt}DF\| \leq ce^{\gamma t}.$$

Proof. "(10.12)" Use Theorem 8.8, (8.67) and (8.68).

"(10.13)" Obvious.

"(10.14)" We have

$$\frac{D}{dt}(u\check{h}_{kl}) = \dot{u}\check{h}_{kl} + u\dot{\check{h}}_{kl}$$

(10.19)
$$= \frac{\tilde{v}}{F}\check{h}_{kl} + u\dot{h}_{kl} - u\dot{\tilde{v}}f'g_{kl} - u\tilde{v}f''\frac{\tilde{v}}{F}g_{kl} - u\tilde{v}f'\dot{g}_{kl}$$
$$+ u\frac{D}{dt}(\psi_\alpha\nu^\alpha g_{kl}),$$

hence in view of (10.8)

$$\text{(10.20)} \qquad \|\frac{D}{dt}(u\check{h}_{kl})\| \le ce^{-2\gamma t} + n|\frac{\tilde{v}^2}{F}(-f' - uf'')| \le ce^{-2\gamma t}.$$

Here, concerning the summand

$$\text{(10.21)} \qquad \frac{\tilde{v}}{F}\check{h}_{kl} - u\tilde{v}f''\frac{\tilde{v}}{F}g_{kl},$$

we use

$$\text{(10.22)} \qquad |-f' - uf''| \le |-f' + u\tilde{\gamma}|f'|^2| + c|u|,$$

which follows from (1.8), and then (1.31).

"(10.15), (10.16)" We have

$$\text{(10.23)} \qquad \frac{D}{dt}F^{ij,kl}(\check{h}_{rs}) = \frac{D}{dt}(|u|F^{ij,kl}(|u|\check{h}_{rs}))$$

which implies the claim together with (10.13) und (10.14).

"(10.17)" Use (10.14) and $F^{ij}(\check{h}_{rs}) = F^{ij}(|u|\check{h}_{rs})$.

"(10.18)" Obvious in view of (10.6) and (10.8). $\qquad\qquad\square$

Lemma 10.7. *We have*

$$\text{(10.24)} \qquad |\ddot{\tilde{v}}| + \|D\dot{\tilde{v}}\| \le ce^{-2\gamma t}$$

and $\dot{\tilde{v}}e^{2\gamma t}$ and $\ddot{\tilde{v}}e^{2\gamma t}$ converge, if t goes to infinity.

Proof. Differentiating (10.11) covariantly with respect to x we infer the estimate for $\|D\dot{\tilde{v}}\|$. A direct computation and easy check of each of the (many) appearing terms yield the convergence of $\dot{\tilde{v}}e^{2\gamma t}$ and $\ddot{\tilde{v}}e^{2\gamma t}$, especially the lemma is proved. $\quad\square$

Finally let us estimate $\ddot{\check{h}}_i^j$ and $\dddot{\check{h}}_i^j$.

Lemma 10.8. *$\ddot{\check{h}}_i^j$ and $\dddot{\check{h}}_i^j$ decay like $e^{-\gamma t}$.*

Proof. The estimate for $\ddot{\check{h}}_i^j$ follows immediately by differentiating equation (8.6) covariantly with respect to t and by applying the above lemmata as well as Theorem 9.8.

Now we estimate $\dddot{\check{h}}_i^j$. We have

$$\text{(10.25)} \qquad \dddot{\check{h}}_l^k = e^{\gamma t}\dddot{h}_l^k + 2\gamma e^{\gamma t}\ddot{h}_l^k + \gamma^2 e^{\gamma t}h_l^k.$$

Now we insert (8.6) and the equation which results from (8.6) after covariant differentiation with respect to t into (10.25).

Then many of the appearing terms decay like $e^{-\gamma t}$ obviously. To see the decay of the remaining terms, namely

$$
\begin{aligned}
&\frac{D}{dt}(F^{-3}F^k F_l)e^{\gamma t} + \frac{D}{dt}(F^{-2}g^{pk}F^{ij,rs}\check{h}_{ij;p}\check{h}_{rs;l})e^{\gamma t} \\
&\quad - \frac{D}{dt}(F^{-2}F^{ij}g_{ij}u_l u^k \tilde{v}f''')e^{\gamma t} + \frac{D}{dt}(F^{-2}f''\tilde{v}^2 h_l^k F^{ij}g_{ij})e^{\gamma t} \\
&\quad - 4\gamma F^{-3}F^k F_l e^{\gamma t} + 2\gamma F^{-2}g^{pk}F^{ij,rs}\check{h}_{ij;p}\check{h}_{rs;l}e^{\gamma t} \\
&\quad - 2\gamma F^{-2}u_l u^k \tilde{v}f''' F^{ij}g_{ij}e^{\gamma t} \\
&\quad + 2\gamma e^{\gamma t}F^{-2}f''\tilde{v}^2 h_l^k F^{ij}g_{ij} + \gamma^2 e^{\gamma t}h_l^k \\
&= S_1 + \ldots + S_9,
\end{aligned}
$$

(10.26)

we use the technique developed in (9.23) et seq., confer also the proof of Theorem 8.8, to rearrange terms. In this way we see the claimed decay of $S_5 + S_7$ and $\frac{1}{2}S_8 + S_9$. The summand $S_1 + S_3$ can be handled similar. The summand S_2 decays as it should due to Lemma 10.6. S_6 is obvious. To estimate $S_4 + \frac{1}{2}S_8$ we differentiate in S_4 by product rule and use (8.6) to substitute $\dot{h}_l^k$. Then a little bit rearranging terms leads to the desired estimate. □

From Corollary 10.3, Lemma 10.8 and (10.25) we infer

Corollary 10.9. *The tensor $\ddot{h}_i^j e^{\gamma t}$ converges, if t tends to infinity.*

The claims in Theorem 1.2 are now almost all proved with the exception of two. In order to prove the remaining claims we need:

Lemma 10.10. *The function $\varphi = e^{\tilde{\gamma}f}u^{-1}$ converges to $-\tilde{\gamma}\sqrt{m}$ in $C^\infty(S_0)$, if t tends to infinity.*

Proof. φ converges to $-\tilde{\gamma}\sqrt{m}$ in view of (1.7). Hence, we only have to show that

(10.27) $$\|D^m\varphi\| \le c_m \quad \forall m \in \mathbb{N}^*,$$

which will be achieved by induction.

We have

(10.28) $$\varphi_i = \tilde{\gamma}e^{\tilde{\gamma}f}f' u_i u^{-1} - e^{\tilde{\gamma}f}u^{-2}u_i = \varphi(\tilde{\gamma}f' u - 1)u^{-1}u_i.$$

Now, we observe that

(10.29) $$u^{-1}u_i = \tilde{u}^{-1}\tilde{u}_i$$

and $f' u$ have uniformly bounded C^m-norms in view of Lemma 10.1 and Lemma 9.3.

The proof of the lemma is then completed by a simple induction argument. □

When we formulated Theorem 1.2 (iii) and (iv) we did not use the current notation where we distinguish quantities related to $\check{g}_{\alpha\beta}$ in contrast to those related to $\bar{g}_{\alpha\beta}$ by the superscript $\check{\ }$.

In the following two lemmas we reformulate Theorem 1.2 (iii) and (iv) using the current notation.

Lemma 10.11. *Let $(\check{g}_{ij})$ be the induced metric of the leaves $M(t)$ of the IFCF, then the rescaled metric*

(10.30) $$e^{\frac{2}{n}t}\check{g}_{ij}$$

converges in $C^\infty(S_0)$ to

(10.31)
$$(\tilde{\gamma}^2 m)^{\frac{1}{\tilde{\gamma}}}(-\tilde{u})^{\frac{2}{\tilde{\gamma}}}\bar{\sigma}_{ij},$$

where we are slightly ambiguous by using the same symbol to denote $\tilde{u}(t, \cdot)$ and $\lim \tilde{u}(t, \cdot)$.

Proof. There holds

(10.32)
$$\breve{g}_{ij} = e^{2f}e^{2\psi}(-u_iu_j + \sigma_{ij}(u, x)).$$

Thus, it suffices to prove that

(10.33)
$$e^{2f}e^{\frac{2}{n}t} \to (\tilde{\gamma}^2 m)^{\frac{1}{\tilde{\gamma}}}(-\tilde{u})^{\frac{2}{\tilde{\gamma}}}$$

in $C^\infty(S_0)$. But this is evident in view of the preceding lemma, since

(10.34)
$$e^{2f}e^{\frac{2}{n}t} = (-e^{\tilde{\gamma}f}u^{-1})^{\frac{2}{\tilde{\gamma}}}(-\tilde{u})^{\frac{2}{\tilde{\gamma}}}.$$

$\square$

Lemma 10.12. *The leaves $M(t)$ of the IFCF get more umbilical, if t tends to infinity, namely*

(10.35)
$$\breve{F}^{-1}|\breve{h}_i^j - \frac{1}{n}\breve{H}\delta_i^j| \le ce^{-2\gamma t}.$$

In case $n + \omega - 4 > 0$, we even get a better estimate, namely

(10.36)
$$|\breve{h}_i^j - \frac{1}{n}\breve{H}\delta_i^j| \le ce^{-\frac{1}{2n}(n+\omega-4)t}.$$

Proof. Denote by $\breve{h}_{ij}, \breve{\nu}$, etc., the geometric quantities of the hypersurfaces $M(t)$ with respect to the original metric $(\breve{g}_{\alpha\beta})$ in N, then

(10.37)
$$e^{\tilde{\psi}}\breve{h}_i^j = h_i^j + \tilde{\psi}_\alpha\nu^\alpha\delta_i^j, \quad \breve{F} = e^{-\tilde{\psi}}F$$

and hence,

(10.38)
$$\breve{F}^{-1}|\breve{h}_i^j - \frac{1}{n}\breve{H}\delta_i^j| = F^{-1}|h_i^j - \frac{1}{n}H\delta_i^j| \le ce^{-2\gamma t}.$$

In case $n + \omega - 4 > 0$, we even get a better estimate, namely

(10.39) $$|\breve{h}_i^j - \frac{1}{n}\breve{H}\delta_i^j| = e^{-\psi}e^{-f}e^{-\frac{1}{n}t}|h_i^j - \frac{1}{n}H\delta_i^j|e^{\gamma t}e^{(\frac{1}{n}-\gamma)t} \le ce^{-\frac{1}{2n}(n+\omega-4)t}$$

in view of (10.33). $\square$

11. Transition from big crunch to big bang

We shall define a new spacetime $\hat{N}$ by reflection and time reversal such that the IFCF in the old spacetime transforms to an IFCF in the new one.

By switching the light cone we obtain a new spacetime $\hat{N}$. If we extend F, which is defined in the positive cone $\Gamma_+ \subset \mathbb{R}^n$, to $\Gamma_+ \cup (-\Gamma_+)$ by

(11.1)
$$F(\kappa_i) = -F(-\kappa_i)$$

for $(\kappa_i) \in -\Gamma_+$ the flow equation in N is independent of the time orientation, and we can write it as

(11.2)
$$\dot{x} = -\breve{F}^{-1}\breve{\nu} = -(-\breve{F})^{-1}(-\breve{\nu}) =: -\hat{F}^{-1}\hat{\nu},$$

where the normal vector $\hat{\nu} = -\breve{\nu}$ is past directed in $\hat{N}$ and the curvature $\hat{F} = -\breve{F}$ negative.

Introducing a new time function $\hat{x}^0 = -x^0$ and formally new coordinates $(\hat{x}^\alpha)$ by setting

(11.3) $$\hat{x}^0 = -x^0, \quad \hat{x}^i = x^i,$$

we define a spacetime $\hat{N}$ having the same metric as N–only expressed in the new coordinate system–such that the flow equation has the form

(11.4) $$\dot{\hat{x}} = -\hat{F}^{-1}\hat{\nu},$$

where $M(t) = \operatorname{graph} \hat{u}(t)$, $\hat{u} = -u$, and

(11.5) $$(\hat{\nu}^\alpha) = -\tilde{v}e^{-\tilde{\psi}}(1, \hat{u}^i)$$

in the new coordinates, since

(11.6) $$\hat{\nu}^0 = -\breve{\nu}^0 \frac{\partial \hat{x}^0}{\partial x^0} = \breve{\nu}^0$$

and

(11.7) $$\hat{\nu}^i = -\breve{\nu}^i.$$

The singularity in $\hat{x}^0 = 0$ is now a past singularity, and can be referred to as a big bang singularity.

The union $N \cup \hat{N}$ is a smooth manifold, topologically a product $(-a, a) \times S_0$– we are well aware that formally the singularity $\{0\} \times S_0$ is not part of the union; equipped with the respective metrics and time orientations it is a spacetime which has a (metric) singularity in $x^0 = 0$. The time function

(11.8) $$\hat{x}^0 = \begin{cases} x^0, & \text{in } N \\ -x^0, & \text{in } \hat{N} \end{cases}$$

is smooth across the singularity and future directed.

$N \cup \hat{N}$ can be regarded as a cyclic universe with a contracting part $N = \{\hat{x}^0 < 0\}$ and an expanding part $\hat{N} = \{\hat{x}^0 > 0\}$ which are joined at the singularity $\{\hat{x}^0 = 0\}$.

We shall show that the inverse F-curvature flow, properly rescaled, defines a natural C^3-diffeomorphism across the singularity and with respect to this diffeomorphism we speak of a transition from big crunch to big bang.

The inverse F-curvature flows in N and $\hat{N}$ can be uniformly expressed in the form

(11.9) $$\dot{\hat{x}} = -\hat{F}^{-1}\hat{\nu},$$

where (11.9) represents the original flow in N, if $\hat{x}^0 < 0$, and the flow in (11.4), if $\hat{x}^0 > 0$.

Let us now introduce a new flow parameter

(11.10) $$s = \begin{cases} -\gamma^{-1}e^{-\gamma t}, & \text{for the flow in } N \\ \gamma^{-1}e^{-\gamma t}, & \text{for the flow in } \hat{N} \end{cases}$$

and define the flow $y = y(s)$ by $y(s) = \hat{x}(t)$. $y = y(s, \xi)$ is then defined in $[-\gamma^{-1}, \gamma^{-1}] \times S_0$, smooth in $\{s \neq 0\}$, and satisfies the evolution equation

(11.11) $$y' := \frac{d}{ds}y = \begin{cases} -\hat{F}^{-1}\hat{\nu}e^{\gamma t}, & s < 0 \\ \hat{F}^{-1}\hat{\nu}e^{\gamma t}, & s > 0. \end{cases}$$

Theorem 11.1. *The flow $y = y(s, \xi)$ is of class C^3 in $(-\gamma^{-1}, \gamma^{-1}) \times S_0$ and defines a natural diffeomorphism across the singularity. The flow parameter s can be used as a new time function.*

The flow y is certainly continuous across the singularity, and also future directed, i.e., it runs into the singularity, if $s < 0$, and moves away from it, if $s > 0$. The continuous differentiability of $y = y(s, \xi)$ with respect to s and ξ up to order three will be proved in a series of lemmata.

As in the previous sections we again view the hypersurfaces as embeddings with respect to the ambient metric

(11.12)
$$d\bar{s}^2 = -(dx^0)^2 + \sigma_{ij}(x^0, x)dx^i dx^j.$$

The flow equation for $s < 0$ can therefore be written as

(11.13)
$$y^{'} = -F^{-1}\nu e^{\gamma t}.$$

To prove that y is of class C^3 in $(-\gamma^{-1}, \gamma^{-1}) \times S_0$ we must show that $y^{'}$, y_i, $y_i^{'}$, $y^{''}$, y_{ij}, $y^{'''}$, $y_{ij}^{'}$, $y_i^{''}$, y_{ijk} (and derivatives obtained by commuting the order of differentiation) are continuous in $\{0\} \times S_0$, which means that we must show that for each of these derivatives the limits $\lim_{s\uparrow 0}$, $\lim_{s\downarrow 0}$ (uniformly with respect to the space variables ξ^i) exist and are the same.

Due to

(11.14)
$$y^0(s) = x^0(t), \quad y^i(s) = x^i(t) \quad \forall s < 0,$$

and

(11.15)
$$y^0(s) = -x^0(t), \quad y^i(s) = x^i(t) \quad \forall s > 0$$

we will consider the 0-component and the i-component of each of the above derivatives separately and calculate their limits as $s \uparrow 0$ and $s \downarrow 0$. Since in each case the limit $s \uparrow 0$ has the same value or the same value up to a sign as the limit $s \downarrow 0$ (provided one of them exists) it is sufficient to have a look at the limit $s \uparrow 0$ and prove its existence or that it is in addition zero respectively.

Lemma 11.2. *y is of class C^1 in $(-\gamma^{-1}, \gamma^{-1}) \times S_0$.*

Proof. $y^{'}$ is continuous across the singularity if

(11.16)
$$\lim_{s\uparrow 0} \frac{d}{ds} y^0, \lim_{s\uparrow 0} y_i^j \quad \text{exist,}$$

and if

(11.17)
$$\lim_{s\uparrow 0} \frac{d}{ds} y^i = \lim_{s\uparrow 0} y_i^0 = 0.$$

Only the limit $\lim_{s\uparrow 0} y_i^j$ is not obvious, but one easily checks that x_i^j is a 'Cauchy sequence' as $t \to \infty$ since its derivative with respect to t can be estimated by $ce^{-\gamma t}$, hence $\lim_{s\uparrow 0} y_i^j$ exists as well.

Remark 11.3. The limit relations for $\langle D^m y, \frac{\partial}{\partial x^0} \rangle$ and $\langle D^m y, \frac{\partial}{\partial x^i} \rangle$, where $D^m y$ stands for covariant derivatives of order m of y with respect to s or ξ^i are identical to those for $\langle D^m y, -\nu \rangle$ and $\langle D^m y, x_i \rangle$ because ν converges to $-\frac{\partial}{\partial x^0}$, if $s \uparrow 0$. We want to point out that we have chosen local coordinates in S_0 which are given by the limit of the embedding vector x so that we also have $x_i \to \frac{\partial}{\partial x^i}$.

$\square$

Let us examine the second derivatives

Lemma 11.4. y is of class C^2 in $(-\gamma^{-1}, \gamma^{-1}) \times S_0$.

Proof. "y_i'": The normal component of y_i' has to converge and the tangential components have to converge to zero as $s \uparrow 0$. For $s < 0$ we have

$$(11.18) \qquad\qquad y' = -F^{-1}e^{\gamma t}\nu$$

and

$$(11.19) \qquad\qquad y_i' = F^{-2}F_i e^{\gamma t}\nu - F^{-1}e^{\gamma t}\nu_i.$$

The normal component is therefore equal to
$$(11.20)$$
$$-F^{-2}e^{\gamma t}(F^{kl}h_{kl;i} - F^{kl}g_{kl}\tilde{v}_i f' - F^{kl}g_{kl}\tilde{v}f''u_i + F^{kl}g_{kl}\psi_{\alpha\beta}x_i^\beta\nu^\alpha + F^{kl}g_{kl}\psi_\alpha x_r^\alpha h_i^r)$$

which converges to

$$(11.21) \qquad\qquad \lim n(Fu)^{-2}f''u^2\tilde{u}_i.$$

The tangential components are equal to

$$(11.22) \qquad\qquad -F^{-1}e^{\gamma t}h_{ik}$$

which converge to zero.

"y_{ij}": The Gaußformula yields

$$(11.23) \qquad\qquad y_{ij} = h_{ij}\nu$$

which converges to zero as it should.

"y''": Here, the normal component has to converge to zero, while the tangential ones have to converge.

We get for $s < 0$

$$(11.24) \qquad \begin{aligned} y'' &= -\frac{D}{dt}(F^{-1}\nu)e^{2\gamma t} - F^{-1}\nu\gamma e^{2\gamma t} \\ &= -F^{-1}\dot{\nu}e^{2\gamma t} + F^{-2}\nu\dot{F}e^{2\gamma t} - F^{-1}\nu\gamma e^{2\gamma t}. \end{aligned}$$

The normal component is equal to

$$(11.25) \qquad \begin{aligned} &-F^{-2}e^{2\gamma t}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f'F^{ij}g_{ij} - \tilde{v}f''\dot{u}F^{ij}g_{ij} \\ &\qquad + \psi_{\alpha\beta}\nu^\alpha\dot{x}^\beta F^{ij}g_{ij} + \psi_\alpha\dot{\nu}^\alpha F^{ij}g_{ij} - \gamma F) \\ &\qquad - F^{-2}e^{2\gamma t}(-\tilde{v}f' + \psi_\alpha\nu^\alpha)F^{ij}\dot{g}_{ij}. \end{aligned}$$

$F^{-2}e^{2\gamma t}$ converges, all terms converge to zero with the possible exception of

$$(11.26) \qquad -F^{ij}g_{ij}\tilde{v}f''\dot{u} - \gamma F = -F^{-1}(F^{ij}g_{ij}\tilde{v}^2 f'' + \gamma F^2),$$

which however converges to zero, too.

The tangential components are equal to

$$(11.27) \qquad \begin{aligned} F^{-1}D_k(F^{-1})e^{2\gamma t} = &-F^{-3}e^{2\gamma t}(F^{ij}h_{ij;k} - \tilde{v}_k f'F^{ij}g_{ij} \\ &- \tilde{v}f''u_k F^{ij}g_{ij} + \psi_{\alpha\beta}\nu^\alpha x_k^\beta F^{ij}g_{ij} + \psi_\alpha x_r^\alpha h_k^r F^{ij}g_{ij}), \end{aligned}$$

which converge to

$$(11.28) \qquad\qquad \lim -\tilde{\gamma}n(Fu)^{-3}(f'u)^2\tilde{u}\tilde{u}_k.$$

$\square$

Lemma 11.5. *y is of class C^3 in $(-\gamma^{-1}, \gamma^{-1}) \times S_0$.*

Proof. "y_{ijk}": Now, the normal component has to converge to zero, while the tangential ones should converge. Again we look at $s < 0$ and get

$$(11.29) \qquad\qquad y_{ij} = h_{ij}\nu,$$

$$(11.30) \qquad\qquad y_{ijk} = h_{ijk}\nu + h_{ij}\nu_k.$$

Hence, y_{ijk} converges to zero.

"y'_{ij}": The normal component has to converge, while the tangential ones should converge to zero.

Using the Ricci identities and Lemma 4.6 (iii) it can be easily checked that, instead of y'_{ij}, we may look at $\frac{D}{ds}(y_{ij})$.

From (11.29) we deduce

$$(11.31) \qquad\qquad \frac{D}{ds}y_{ij} = \dot{h}_{ij}\nu e^{\gamma t} + h_{ij}\dot{\nu}e^{\gamma t},$$

and conclude further that the normal component converges in view of Corollary 10.3 and the tangential ones converge to zero, since $\dot{\nu}$ vanishes in the limit.

"y''_i": The normal component has to converge to zero and the tangential ones have to converge.

From (11.24) we infer

$$
\begin{aligned}
(11.32) \quad y'' = & -F^{-3}e^{2\gamma t}F^{ij}(h_{ij;}{}^k - \tilde{v}^k f' g_{ij} - \tilde{v}f'' u^k g_{ij} + (\psi_\alpha\nu^\alpha)^k g_{ij})x_k \\
& + F^{-2}e^{2\gamma t}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f' F^{ij}g_{ij} + \frac{D}{dt}(\psi_\alpha\nu^\alpha)F^{ij}g_{ij})\nu \\
& + F^{-3}e^{2\gamma t}(-\tilde{v}^2 F^{ij}g_{ij}[f'' + \gamma F^{ij}g_{ij}|f'|^2] - \gamma[(F^{ij}h_{ij})^2 \\
& + (\psi_\alpha\nu^\alpha)^2(F^{ij}g_{ij})^2 - 2\tilde{v}f' F^{ij}h_{ij}F^{ij}g_{ij} \\
& + 2\psi_\alpha\nu^\alpha F^{ij}h_{ij}F^{ij}g_{ij} - 2\tilde{v}f'\psi_\alpha\nu^\alpha(F^{ij}g_{ij})^2])\nu \\
& + 2F^{-3}e^{2\gamma t}(\tilde{v}f' F^{ij}h_{ij} - \psi_\alpha\nu^\alpha F^{ij}h_{ij})\nu
\end{aligned}
$$

and thus

$$
\begin{aligned}
(11.33) \quad y''_l = & -(F^{-3}e^{2\gamma t}F^{ij}(h_{ij;}{}^k - \tilde{v}^k f' g_{ij} - \tilde{v}f'' u^k g_{ij} + (\psi_\alpha\nu^\alpha)^k)g_{ij})_l x_k \\
& - F^{-3}e^{2\gamma t}F^{ij}(h_{ij;}{}^k - \tilde{v}^k f' g_{ij} - \tilde{v}f'' u^k g_{ij} + (\psi_\alpha\nu^\alpha)^k g_{ij})h_{kl}\nu \\
& + (F^{-2}e^{2\gamma t}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f' F^{ij}g_{ij} + \frac{D}{dt}(\psi_\alpha\nu^\alpha)F^{ij}g_{ij}))_l \nu \\
& + F^{-2}e^{2\gamma t}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f' F^{ij}g_{ij} + \frac{D}{dt}(\psi_\alpha\nu^\alpha)F^{ij}g_{ij})\nu_l \\
& + (F^{-3}e^{2\gamma t}(-\tilde{v}^2 F^{ij}g_{ij}[f'' + \gamma F^{ij}g_{ij}|f'|^2] - \gamma[(F^{ij}h_{ij})^2 \\
& + (\psi_\alpha\nu^\alpha)^2(F^{ij}g_{ij})^2 - 2\tilde{v}f' F^{ij}h_{ij}F^{ij}g_{ij} + 2\psi_\alpha\nu^\alpha F^{ij}h_{ij}F^{ij}g_{ij} \\
& - 2\tilde{v}f'\psi_\alpha\nu^\alpha(F^{ij}g_{ij})^2]))_l \nu + F^{-3}e^{2\gamma t}(-\tilde{v}^2 F^{ij}g_{ij}[f'' + \gamma F^{ij}g_{ij}|f'|^2] \\
& - \gamma[(F^{ij}h_{ij})^2 + (\psi_\alpha\nu^\alpha)^2(F^{ij}g_{ij})^2 - 2\tilde{v}f' F^{ij}h_{ij}F^{ij}g_{ij} \\
& + 2\psi_\alpha\nu^\alpha F^{ij}h_{ij}F^{ij}g_{ij} - 2\tilde{v}f'\psi_\alpha\nu^\alpha(F^{ij}g_{ij})^2])\nu_l \\
& + (2F^{-3}e^{2\gamma t}(\tilde{v}f' F^{ij}h_{ij} - \psi_\alpha\nu^\alpha F^{ij}h_{ij}))_l \nu \\
& + 2F^{-3}e^{2\gamma t}(\tilde{v}f' F^{ij}h_{ij} - \psi_\alpha\nu^\alpha F^{ij}h_{ij})\nu_l.
\end{aligned}
$$

Therefore, the normal component converges to zero, while the tangential ones converge.

"y''''": Differentiating the equation (11.32) we get

$$
\begin{aligned}
y''' = {} & 3F^{-4}e^{3\gamma t}\dot{F}F^{ij}(h_{ij;}{}^{k} - \tilde{v}^{k}f'g_{ij} - \tilde{v}f''u^{k}g_{ij} + (\psi_{\alpha}\nu^{\alpha})^{k}g_{ij})x_{k} \\
& - 2\gamma F^{-3}e^{3\gamma t}F^{ij}(h_{ij;}{}^{k} - \tilde{v}^{k}f'g_{ij} - \tilde{v}f''u^{k}g_{ij} + (\psi_{\alpha}\nu^{\alpha})^{k}g_{ij})x_{k} \\
& - F^{-3}e^{3\gamma t}\frac{D}{dt}(F^{ij}(h_{ij;}{}^{k} - \tilde{v}^{k}f'g_{ij} - \tilde{v}f''u^{k}g_{ij} + (\psi_{\alpha}\nu^{\alpha})^{k}g_{ij}))x_{k} \\
& - F^{-3}e^{3\gamma t}F^{ij}(h_{ij;}{}^{k} - \tilde{v}^{k}f'g_{ij} - \tilde{v}f''u^{k}g_{ij} + (\psi_{\alpha}\nu^{\alpha})^{k}g_{ij})\dot{x}_{k} \\
& - 2F^{-3}e^{3\gamma t}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f'F^{ij}g_{ij} + \frac{D}{dt}(\psi_{\alpha}\nu^{\alpha})F^{ij}g_{ij})\nu \\
& + 2\gamma F^{-2}e^{3\gamma t}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f'F^{ij}g_{ij} + \frac{D}{dt}(\psi_{\alpha}\nu^{\alpha})F^{ij}g_{ij})\nu \\
& + F^{-2}e^{3\gamma t}\frac{D}{dt}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f'F^{ij}g_{ij} + \frac{D}{dt}(\psi_{\alpha}\nu^{\alpha})F^{ij}g_{ij})\nu \\
& + F^{-2}e^{3\gamma t}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f'F^{ij}g_{ij} + \frac{D}{dt}(\psi_{\alpha}\nu^{\alpha})F^{ij}g_{ij})\dot{\nu} \\
& - 3F^{-4}e^{3\gamma t}(-\tilde{v}^{2}F^{ij}g_{ij}[f'' + \gamma F^{ij}g_{ij}|f'|^{2}] - \gamma[(F^{ij}h_{ij})^{2} \\
& + (\psi_{\alpha}\nu^{\alpha})^{2}(F^{ij}g_{ij})^{2} - 2\tilde{v}f'F^{ij}h_{ij}F^{ij}g_{ij} \\
& + 2\psi_{\alpha}\nu^{\alpha}F^{ij}h_{ij}F^{ij}g_{ij} - 2\tilde{v}f'\psi_{\alpha}\nu^{\alpha}(F^{ij}g_{ij})^{2}])\nu \\
& + 2\gamma F^{-3}e^{3\gamma t}(-\tilde{v}^{2}F^{ij}g_{ij}[f'' + \gamma F^{ij}g_{ij}|f'|^{2}] - \gamma[(F^{ij}h_{ij})^{2} \\
& + (\psi_{\alpha}\nu^{\alpha})^{2}(F^{ij}g_{ij})^{2} - 2\tilde{v}f'F^{ij}h_{ij}F^{ij}g_{ij} \\
& + 2\psi_{\alpha}\nu^{\alpha}F^{ij}h_{ij}F^{ij}g_{ij} - 2\tilde{v}f'\psi_{\alpha}\nu^{\alpha}(F^{ij}g_{ij})^{2}])\nu \\
& + F^{-3}e^{3\gamma t}\frac{D}{dt}(-\tilde{v}^{2}F^{ij}g_{ij}[f'' + \gamma F^{ij}g_{ij}|f'|^{2}] - \gamma[(F^{ij}h_{ij})^{2} \\
& + (\psi_{\alpha}\nu^{\alpha})^{2}(F^{ij}g_{ij})^{2} - 2\tilde{v}f'F^{ij}h_{ij}F^{ij}g_{ij} \\
& + 2\psi_{\alpha}\nu^{\alpha}F^{ij}h_{ij}F^{ij}g_{ij} - 2\tilde{v}f'\psi_{\alpha}\nu^{\alpha}(F^{ij}g_{ij})^{2}])\nu \\
& + F^{-3}e^{3\gamma t}(-\tilde{v}^{2}F^{ij}g_{ij}[f'' + \gamma F^{ij}g_{ij}|f'|^{2}] - \gamma[(F^{ij}h_{ij})^{2} \\
& + (\psi_{\alpha}\nu^{\alpha})^{2}(F^{ij}g_{ij})^{2} - 2\tilde{v}f'F^{ij}h_{ij}F^{ij}g_{ij} \\
& + 2\psi_{\alpha}\nu^{\alpha}F^{ij}h_{ij}F^{ij}g_{ij} - 2\tilde{v}f'\psi_{\alpha}\nu^{\alpha}(F^{ij}g_{ij})^{2}])\dot{\nu} \\
& - 6F^{-4}e^{3\gamma t}(\tilde{v}f'F^{ij}h_{ij} - \psi_{\alpha}\nu^{\alpha}F^{ij}h_{ij})\nu \\
& + 4\gamma F^{-3}e^{3\gamma t}(\tilde{v}f'F^{ij}h_{ij} - \psi_{\alpha}\nu^{\alpha}F^{ij}h_{ij})\nu \\
& + 2F^{-3}e^{3\gamma t}\frac{D}{dt}(\tilde{v}f'F^{ij}h_{ij} - \psi_{\alpha}\nu^{\alpha}F^{ij}h_{ij})\nu \\
& + 2F^{-3}e^{3\gamma t}(\tilde{v}f'F^{ij}h_{ij} - \psi_{\alpha}\nu^{\alpha}F^{ij}h_{ij})\dot{\nu}.
\end{aligned}
$$

(11.34)

We remark that

(11.35) $$\dot{x}_{k} = F^{-2}F_{k}\nu - F^{-1}\nu_{k}$$

and

(11.36) $$\dot{u}_{k} = F^{-1}\tilde{v}_{k} - F^{-2}\tilde{v}F_{k}$$

and that in the following especially the results of Lemma 10.5, Lemma 10.8 and Corollary 10.9 will be used.

Let us consider the normal component of y''' first, which has to converge. We will present here only how to handle the following term, the other terms are easier.

$$(11.37) \quad \begin{aligned} \frac{D}{dt}[f'' + \gamma F^{ij}g_{ij}|f'|^2] =& \frac{D}{dt}[f'' + \tilde{\gamma}|f'|^2] + \frac{D}{dt}(F^{ij}g_{ij} - n)\gamma|f'|^2 \\ &+ 2(F^{ij}g_{ij} - n)\gamma f' f'' \dot{u} \\ \equiv& I_1 + I_2 + I_3. \end{aligned}$$

I_1 converges due to assumption (1.13), and the convergence of I_3 is obvious. For I_2 we use

$$(11.38) \quad \frac{D}{dt}F^{ij}g_{ij} = F^{ij,rs}(|u|\check{h}_{ij})g_{ij}\frac{D}{dt}(|u|\check{h}_{rs}) + F^{ij}\dot{g}_{ij}$$

together with (8.2), (10.13) and (10.14).

Now we consider the tangential component of y''', i.e. we prove

$$(11.39) \quad \langle y''', x_l \rangle \to 0.$$

The crucial terms are

$$(11.40) \quad \begin{aligned} 3F^{-4}e^{3\gamma t}(F^{ij}g_{ij})^2\tilde{v}^2(f'')^2\dot{u}u^k + 2\gamma F^{-3}e^{3\gamma t}\tilde{v}f'' u^k F^{ij}g_{ij} \\ + F^{-3}e^{3\gamma t}\tilde{v}f''' u^k \dot{u}F^{ij}g_{ij} + F^{-5}e^{3\gamma t}(F^{ij}g_{ij})^2\tilde{v}^3|f'|^2 u^k \end{aligned}$$

and can be rearranged to yield

$$(11.41) \quad F^{-5}e^{3\gamma t}n^2\tilde{v}u^k(4f''(f'' + \tilde{\gamma}|f'|^2) - f'(f'' + \tilde{\gamma}|f'|^2)').$$

Hence the tangential components tend to zero.

The remaining mixed derivatives of y which are obtained by commuting the order of differentiation in the derivatives we already treated are also continuous across the singularity in view of the Ricci identities and Lemma 4.6 (iii). □

REFERENCES

[1] Gerhardt, C.: Closed Weingarten hypersurfaces in Riemannian manifolds, J. Diff. Geom. 43 (1996), 612-641.

[2] Gerhardt, C.: Hypersurfaces of prescribed curvature in Lorentzian manifolds, Indiana Univ. Math. J. 49 (2000), 1125-1153, arXiv:math.DG/0409457.

[3] Gerhardt, C.: Hypersurfaces of prescribed mean curvature in Lorentzian manifolds, Math. Z. 235 (2000), 83-97.

[4] Gerhardt, C.: The inverse mean curvature flow in cosmological spacetimes, Adv. Theor. Math. Phys. 12, 1183 - 1207 (2008), arXiv:math.DG/0403097.

[5] Gerhardt, C.: The inverse mean curvature flow in ARW spaces - transition from big crunch to big bang, 2004, arXiv:math.DG/0403485, 39 pages.

[6] Gerhardt, C.: Analysis II, International Series in Analysis, International Press, Somerville, MA, 2006, 395 pages.

[7] Gerhardt, C.: On the CMC foliation of future ends of spacetime, Pacific J. Math. 226 (2006), 297-308.

[8] Gerhardt, C.: Curvature problems, Series in Geometry and Topology, International Press, Somerville, MA, 2006, 323 pages.

[9] Kröner, H: Diplomarbeit. Der inverse mittlere Krümmungsfluß in Lorentzmannigfaltigkeiten. Heidelberg 2007.