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Chapter 1

Introduction

In the study of complex systems, statistical analysis has always been an important tool. Due to the high number of degrees of freedom an exact calculation of the energy levels is only possible in a small number of systems, which have enough conserved quantities to ensure integrability. In all other cases one has to rely on some approximation scheme. In the statistical approach one refrains a priori from attempts to evaluate the eigenvalues of a system exactly and concentrates on their stochastic behavior. Random matrix theory has been introduced by Wigner into nuclear physics. He was the first to observe that the spectral fluctuations of complex nuclei coincide with the statistical properties of a Hamiltionian, whose interactions are completely random. The most prominent characteristic of this Wigner–Dyson statistics is the repulsion between adjacent levels, i. e. the vanishing of the nearest neighbor spacing distribution at the origin. Later Wigner–Dyson statistics turned out not to be restricted to nuclear physics and to systems with many degrees of freedom. In [BGS] it was for the first time conjectured that Wigner–Dyson statistics describe also the statistical properties of quantum systems with few degrees of freedom provided their classical dynamics is chaotic. This conjecture has been confirmed by overwhelming numerical evidence, although a rigorous proof is still lacking.

Berry and Tabor [BT] had given strong arguments to justify that the eigenvalues of a generic integrable quantum system with more than one degree of freedom should obey Poisson statistics. To Poissonian statistics one refers to the statistics of completely uncorrelated eigenvalues. Its most prominent feature is the exponential decrease of the nearest neighbor spacing distribution.

However, for many realistic systems none of these two limiting cases applies. Such a situation occurs for example in nuclear physics. Nuclei are capable to collective motion. A nucleus viewed as a piece of elastic matter can vibrate and also rotate if it is deformed. In a stochastic model these regular motions should be described by an Hamiltonian H_0 exhibiting Poisson statistics. Apart from these collective modes the nucleus can also have single–particle excitations which cause the eigenstates of an eigenvalue of $H^{(0)}$ to spread. This random part of the Hamiltonian should exhibit Wigner–Dyson statistics. The crucial quantity is the spreading width defined as the mean square of a perturbation matrix element on the scale of the mean level spacing due to $H^{(0)}$.

A second example of a system exhibiting intermediate statistics is the hydrogen atom in a magnetic field. The classical system is essentially integrable for a weak and a very strong magnetic field. It is chaotic for a intermediate value of the magnetic field. The level statistics of the quantum system exhibit a crossover from Poisson to Wigner–Dyson statistics. This is shown in figure 1.1. One can nicely observe that already for a small magnetic field, i. e. a small chaotic admixture level repulsion occurs.

As a third example we mention the spectral statistics of disordered solids. Suppose we

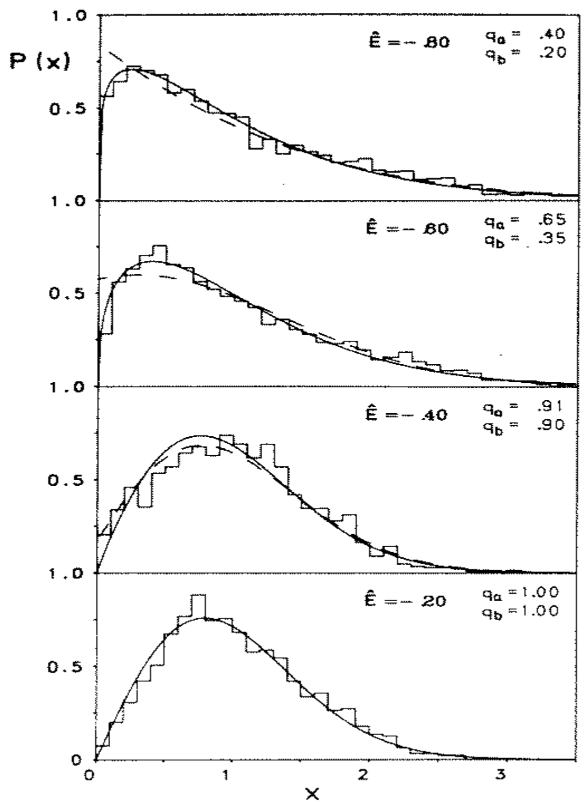

Figure 1.1: *Nearest neighbor spacing distribution versus the spacing x on the scale of the mean level spacing for the hydrogen atom in a strong magnetic field B in units of $B_0 = m_e^2 e^3 c/(2\pi h)^3$. The levels are taken from the vicinity of the scaled binding energy $\widehat{E} = E/B^{2/3}$. Solid and dashed lines are fits, except for the bottom figure which represents the GOE (taken from [WIN]).*

have a quantum particle in a two–dimensional random medium of finite size. For low disorder its wavefunction will delocalize over the whole system and eventually over the whole energy surface. In this regime one expects Wigner–Dyson statistics. In the opposite case of high disorder the wavefunction localizes in some region of the probe. In this regime one finds Poisson statistics, since the wavefunctions do not "communicate". Here, the decisive quantity is the Thouless energy, defined as the inverse of the classical diffusion time of the particle through the sample. Efetov [EFE] described the eigenvalue statistics of a disordered solid by a supersymmetric non–linear σ–model. Thereby, he proved that on energy scales much smaller than the Thouless energy the level fluctuations exhibit indeed Wigner–Dyson behavior. Later Altshuler and Shklovskii [ASH] derived corrections to Wigner–Dyson behavior.

Several stochastic models have been used to describe the transitions from Poisson regularity to chaos. We mention two of them: a block–diagonal Hamiltonian with one block that exhibits Poisson statistics and another one with Wigner–Dyson statistics. The system will show a transition from Wigner–Dyson statistics to Poisson statistics according to the ratio of the dimensions of the two blocks. Other frequently studied objects are random band matrices of dimension N. All matrix elements beyond a certain distance d from the diagonal are set to zero. Here, a crossover transition takes place from Poisson to Wigner–Dyson statistics as the ratio d/N increases from zero to unity.

Another natural approach to model a transition from Wigner–Dyson to Poisson statistics is a Hamiltonian which consists of a sum of two matrices,

$$H(t) = H^{(0)} + \sqrt{t} H^{(1)} \quad . \tag{1.1}$$

The $H^{(0)}$ is a diagonal matrix and exhibits Poisson statistics while $H^{(1)}$ is a random matrix and models the chaotic admixture. The Hamiltonian can be a model for various physical systems. In the case of nuclear physics, the parameter t is related to the spreading width. In disordered samples, it corresponds to the Thouless energy, while it is related to the magnetic field strength for the case of a hydrogen atom in a magnetic field.

The exact calculation of the eigenvalue correlation functions of Hamiltonian (1.1) has only been done for systems with broken time reversal invariance. The physically more interesting case of time reversal invariant ensembles has so far resisted to an exact analytical treatment. This discrepancy has had an analogue in classical random matrix theory. While the time reversal non–invariant ensemble was comparably simple, considerable difficulties had to be overcome in calculating the eigenvalue correlation functions for the time–reversal invariant ensemble. This work addresses the task of finding exact expressions for the eigenvalue correlation functions of the model (1.1) also for time–reversal invariant ensembles. We will derive integral expressions for the one–point and the two–point correlation function, which are exact for all values of the transition parameter.

Dyson [DYS2] connected the random Hamiltonian (1.1) in a beautiful picture with a stochastic process. He showed that the eigenvalues of the Hamiltonian (1.1) are moving stochastically in the same way as Brownian particles obeying the laws of two–dimensional electrodynamics. Starting from a given initial condition the particles move towards the chaotic equilibrium distribution described by one of the classical Gaussian random matrix ensembles. In this picture it becomes clear that the model (1.1) is just one special case of a diffusion process for the special case of a Poissonian initial condition. At this point random matrix theory becomes intertwined with other fields of physics. Dysons's Brownian motion is closely related to a class of completely integrable systems, the celebrated Calogero–Sutherland Hamiltonians. The field received a strong boost from a completely different direction, when Itzykson and Zuber calculated the famous group integral named after them in the context of large N expansion in $SU(N)$ gauge–field theory. This Itzykson–Zuber integral is the propagator of Dyson's Brownian motion for the case of a diffusion into the equilibrium of the Gaussian unitary ensemble. By means of this integral analytic expressions for the evolution of the eigenvalue correlators became available for the transition from a time–reversal invariant ensemble to an ensemble with broken time–reversal invariance. The "success" of the Itzykson–Zuber integral in different branches of physics makes it interesting to study this type of group integrals also for the other ensembles and on a rather general footing. This is another important goal of this work.

For the evaluation of the eigenvalue correlators the supersymmetric technique has proved to be quite powerful. Remarkably, an analogue to Dyson's Brownian motion exists in superspace [GUH4]. In the very same way as the joint–probability distribution of the eigenvalues is propagated by a group integral in ordinary space, the k–point correlators of this probability distribution are propagated by a group integral in superspace. The evaluation of these integrals in superspace is crucial for the derivation of the eigenvalue correlators of the random–matrix model. Thus, the general study of group integrals in superspace is another aim of this work.

The group integrals of the type of the Itzykson–Zuber integral are of the form

$$\int d\mu(U)\exp(i\mathrm{tr}\,U^\dagger xUk)\ , \tag{1.2}$$

where x and k are diagonal matrices or supermatrices and the integration domain is a group manifold in ordinary space or in superspace. As we have seen, there exist sufficient physical motivations for their study. However these integrals are also worth to be studied on their own right, since they are the generalization of vector Bessel functions to matrix space and, as we will see, they are just special cases of a much wider class of functions.

The work is organized as follows. In Chapter 2 we give an account on Dyson's concept of Brownian motion in matrix and supermatrix space. We particularly emphasize the distinguished rôle of the Itzykson–Zuber integral in matrix and supermatrix space. We show in detail why the group integrals (1.2) are the crucial quantity in the exact evaluation of the eigenvalue correlators of our random matrix model (1.1).
In the next chapter we derive an explicit parametrization of supergroups by generalizing ideas originally due to Gelfand. The main characteristic of these so–called Gelfand–Tzetlin coordinates is their recursive structure. The exhibit many beautiful features indicating that they are the natural coordinates of the group manifold.

However, for the explicit evaluation of the group integrals (1.2) a small but important modification of the original method is needed. By introducing the radial Gelfand–Tzetlin coordinates we find a recursion formula establishing a connection between the group integral in a matrix space of dimension N and a group integral in a matrix space of dimension $N-1$. This recursion formula is the main result of Chapter 4 and one of the most important results of this work. Through this recursion formula, though derived by group theoretical methods, it becomes obvious that the group integrals (1.2) are embedded in a much wider class of functions. We also derive explicit expressions for some groups in ordinary space.

In Chapter 5 we finally use the methods developed before, to evaluate the group integral of type (1.2) over the unitary orthosymplectic supergroups $UOSp(2/2)$ and $UOSp(4/4)$, which yield the eigenvalue correlators of our model (1.1). The results are summarized in Chapter 6.

Chapter 2

Brownian motion

In this introductory chapter we compile some basic facts about diffusion processes in matrix and supermatrix space. In doing so, we emphasize the special rôle of the unitary group both in matrix and in supermatrix space. After introducing some notations and conventions which we will use throughout in the sequel, we introduce Dyson's Brownian motion model. In this section we also summarize some basic result of the theory of Lie–groups essential for the understanding of the Harish–Chandra theorem, which we will state at the end of the section. In the last section we give an account on diffusion processes in supermatrix space.

2.1 Basics and notations

We consider $N \times N$ Hermitean matrices H whose elements $H_{nm},\ n, m = 1, \ldots, N$ are real, complex or quaternion variables. In other words, each element H_{nm} has β real components $H_{nm}^{(\alpha)},\ \alpha = 0, \ldots, (\beta - 1)$ with $\beta = 1, 2, 4$, respectively,

$$H_{nm} = \sum_{\alpha=0}^{\beta-1} H_{nm}^{(\alpha)} \tau^{(\alpha)} \ . \tag{2.1}$$

Here, we use the basis $\tau^{(\alpha)},\ \alpha = 0, \ldots, (\beta - 1)$. We have $\tau^{(0)} = 1$ for the real case with $\beta = 1$. For the complex case with $\beta = 2$, we have $\tau^{(0)} = 1$ and $\tau^{(1)} = i$. Finally, we choose

$$\begin{aligned} \tau^{(0)} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \qquad & \tau^{(1)} &= \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} , \\ \tau^{(2)} &= \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} , \qquad & \tau^{(3)} &= \begin{bmatrix} +i & 0 \\ 0 & -i \end{bmatrix} \end{aligned} \tag{2.2}$$

as a basis for the quaternions. By Hermitecity we always mean $H_{nm} = H_{mn}^*$. Therefore on the diagonal we have N independent variables $H_{nn} = H_{nn}^{(0)}$ and $\beta(N-1)N/2$ independent variables outside the diagonal. The classical Gaussian random matrix ensembles are defined via the infinitesimal probability related to the values $\beta = 1, 2, 4$

$$P_{N\beta}(H)d[H] = 2^{N(N-1)\beta/4} \left(\frac{\beta}{2\pi a^2}\right)^{N/2+N(N-1)\beta/4} \exp\left(-\frac{\beta}{2a^2}\mathrm{Tr}\, H^2\right) d[H] \quad . \tag{2.3}$$

The infinitesimal volume element we define as

$$d[H] \;=\; \prod_{n=1}^{N} dH_{nn}^{(0)} \prod_{n<m} \prod_{\alpha=0}^{\beta-1} dH_{nm}^{(\alpha)} \quad . \tag{2.4}$$

Furthermore, in order to treat the three cases on the same footing, we introduced in Eq. (2.3) the trace Tr and the determinant Det with Tr = tr and Det = det for $\beta = 1, 2$ and with

$$\mathrm{Tr}\, K \;=\; \frac{1}{2}\mathrm{tr}\, K \qquad \text{and} \qquad \mathrm{Det}\, K \;=\; \sqrt{\det K} \tag{2.5}$$

in the case $\beta = 4$ for a matrix K with quaternion entries.

For $\beta = 1$ H is real symmetric and one obtains the Gaussian orthogonal ensemble (GOE), which describes systems with time reversal invariance. The set of matrices, which diagonalize a real symmetric matrix form the orthogonal group $O(N) = U(N;1)$. The ensemble of Hermitean matrices (GUE) corresponds to $\beta = 2$. It describe systems with maximally broken time reversal invariance. The diagonalizing group is the unitary group $U(N) = U(N;2)$. Finally for $\beta = 4$ one obtains the Gaussian symplectic ensemble consisting of Hermitean self–dual matrices. These are diagonalized by the unitary symplectic group $USp(2N) = U(N;4)$. They describe systems with Kramers degeneracy. The notation $U(N;\beta)$ for the unitary group defined over the real $\beta = 1$, the complex $\beta = 2$ or the quaternionic field $\beta = 4$ is due to Gilmore [GIL]. It emphasizes the meaning of the parameter β as the dimension of the field, over which H is defined. One can think of a generalized concept of Hermitecity and built up H by a sum of β independent real symmetric matrices with the β diagonal elements merged into a single variable. In this sense one might also define Gaussian ensembles for other integer values of β. At this point we restrict ourselves to the classical random matrix ensembles. The volume of the groups $U(N;\beta)$ is given by

$$\mathrm{vol}\, U(N;\beta) \;=\; \prod_{n=1}^{N} \frac{2\pi^{\beta n/2}}{\Gamma(\beta n/2)} \;=\; \frac{2^N \pi^{\beta N(N+1)/4}}{\prod_{n=1}^{N}\Gamma(\beta n/2)} \; . \tag{2.6}$$

We use it to normalize the invariant measure $d\mu(U)$ of $U \in U(N;\beta)$ to unity,

$$\int d\mu(U) \;=\; 1 \; . \tag{2.7}$$

The N real eigenvalues x_n, $n = 1, \ldots, N$ of H are ordered in the diagonal matrix x. We have $x = \mathrm{diag}\,(x_1, \ldots, x_N)$ for $\beta = 1$ and $\beta = 2$ and $x = \mathrm{diag}\,(x_1, x_1, \ldots, x_N, x_N)$ for $\beta = 4$. For later purposes we also define the matrix $\hat{x} = \mathrm{diag}\,(x_1, \ldots, x_n)$ in all three cases. That means we have $x = \hat{x}$ for $\beta = 1, 2$ and $x = 1_2 \otimes \hat{x}$ for $\beta = 4$. Upon diagonalization

$$H \;=\; U^\dagger x U \; , \qquad \text{with} \qquad H_{nm} \;=\; U_n^\dagger x U_m \tag{2.8}$$

the infinitesimal probability Eq. (2.3) transforms to [MEH1]

$$P_{N\beta}(H) \;=\; C_N^{(\beta)}\, W(x) d[x] d\mu(U) \quad , \quad W(x) \;=\; |\Delta_N(x)|^\beta \exp\left(-\sum_{i=1}^{N} \frac{x_i^2}{2a^2}\right) \tag{2.9}$$

$d[x]$ denotes the product of all differentials dx_n and we defined Vandermonde's determinant

$$\Delta_N(x) = \prod_{i<j}^{N} (x_i - x_j) \quad . \tag{2.10}$$

The normalization constant

$$C_N^{(\beta)} = 2^{-N/2} \left(\frac{\beta}{a^2}\right)^{N/2+N(N-1)\beta/4} \frac{\pi^{-N-\beta N(N-1)/4}}{N!} \frac{\Gamma^N(\beta/2)}{\prod_{n=1}^N \Gamma(\beta n/2)} \tag{2.11}$$

one obtains from the constants given in Mehta's book [MEH1] and from Eqs. (2.6) and (2.3). In the sequel we fix the scale by setting the standard deviation $a = 1$.

2.2 Diffusion processes in matrix space

In 1962 [DYS2] Dyson was the first to observe that the eigenvalue distribution of a Gaussian random matrix ensemble (2.9) is the stationary limit of the time–dependent joint probability density of a diffusion process. This process describes the over-damped motion of N particles on a line subjected to the Coulomb law of two–dimensional electrodynamics [DYS1] with each particle confined by a harmonic potential. The corresponding Fokker–Planck equation reads

$$\frac{\partial}{\partial t} P_{N\beta}(x,t) = \sum_{i=1}^{N} \left[-\beta \frac{\partial}{\partial x_i} E(x_i) + \frac{\partial^2}{\partial x_i^2} \right] P_{N\beta}(x,t) \quad , \tag{2.12}$$

where the drift term is given by

$$E(x_i) = \sum_{j\neq i} \frac{1}{x_i - x_j} - x_i \quad . \tag{2.13}$$

In the Brownian motion picture β plays the rôle of an inverse temperature. The friction coefficient we have set to $(\beta a^2)^{-1} = \beta^{-1}$. A second important observation of Dyson was the following. When the eigenvalues x_i of H move according to (2.12), the independent entries of H move according to another, much simpler law,

$$\frac{\partial}{\partial t} P_{N\beta}(H,t) = \left[-\beta \operatorname{Tr} \left(\frac{\partial}{\partial H} H \right) + \Delta \right] P_{N\beta}(H,t) \quad . \tag{2.14}$$

Here we have defined the gradient and the Laplacian in matrix space

$$\left(\frac{\partial}{\partial H}\right)_{ij} = \frac{1+\delta_{ij}}{2} \frac{\partial}{\partial H_{ij}} \quad , \quad \Delta = \operatorname{Tr} \left(\frac{\partial}{\partial H}\right)^2 \quad . \tag{2.15}$$

Eq. (2.14) is the sum of $(N + N(N-1)\beta/2)$ Ornstein–Uhlenbeck processes [GAR]. Its Green's function for an arbitrary initial condition $P_N^{(0)}(H^{(0)})$ is known to be

$$P_{N\beta}(H,t|H^{(0)}) = C_N^{\beta} (1-e^{-2t})^{-N/2-N(N-1)\beta/4} \exp\left(-\frac{\beta}{2(1-e^{-2t})} \operatorname{Tr} \left(H - e^{-t} H^{(0)} \right)^2 \right) . \tag{2.16}$$

Here, $P_{N\beta}(H,t|H^{(0)})$ is the probability distribution of a Hamiltonian that interpolates between an arbitrary Hamiltonian $H^{(0)}$ and a random Hamiltonian $H^{(1)}$

$$H(t) = \gamma^{(0)}(t) H^{(0)} + \gamma^{(1)}(t) H^{(1)} \quad , \tag{2.17}$$

with

$$\gamma^{(0)}(t) = e^{-t} \ , \quad \gamma^{(1)}(t) = (1-e^{-2t})^{1/2} \quad . \tag{2.18}$$

Matrix ensembles of the type (2.17) are of prominent interest in quantum chaos and many particle physics, since they represent the generic system exhibiting intermediate spectral statistics. Much work has been devoted [GMGW] to calculate the k–point eigenvalue correlators of $H(t)$ for different types of initial conditions $H^{(0)}$. The dynamics of the transition depends on the choice of the transition functions $\gamma^{(i)}(t)$ [LH]. Both functions must be defined on the same interval $[t_0, t_1]$ and the quotient $\gamma^{(0)}(t)/\gamma^{(1)}(t)$ should tend to the limits zero for $t \to t_0$ and infinity for $t \to t_1$. Any choice of $\gamma^{(i)}(t)$ essentially different from Eq. (2.18) yields a propagator for a stochastic process, which differs from Dyson's model. However, all these processes propagate to the same equilibrium state of the classical Gaussian random matrix ensemble. If one is interested only in the equal time eigenvalue correlators of $H(t)$ for some fixed value of $\gamma^{(0)}(t)/\gamma^{(1)}(t)$, one chooses most conveniently

$$\gamma^{(0)}(t) = 1 \ , \quad \gamma^{(1)}(t) = \sqrt{t} \quad . \tag{2.19}$$

With this choice of $\gamma^{(i)}(t)$ the propagator becomes especially simple

$$P_{N\beta}(H,t|H^{(0)}) = (2)^{N(N-1)\beta/2} \left(\frac{\beta}{2\pi t}\right)^{N/2+N(N-1)\beta/4} \exp\left(-\frac{\beta}{2t}\mathrm{Tr}\left(H - H^{(0)}\right)^2\right) \quad . \tag{2.20}$$

This is the kernel of a pure diffusion in the space of $N \times N$ matrices, defined by the differential equation

$$\begin{aligned} 2\beta\frac{\partial}{\partial t}P_{N\beta}(H,t|H^{(0)}) &= \Delta_\beta P_{N\beta}(H,t|H^{(0)}) \ , \\ \lim_{t\to 0} P_{N\beta}(H,t|H^{(0)}) &= \delta(H - H^{(0)}) \ . \end{aligned} \tag{2.21}$$

The time dependent joint probability density is the convolution integral

$$P_{N\beta}(H,t) = \int P_{N\beta}(H,t|H^{(0)})P_N^{(0)}(H^{(0)})d[H^{(0)}] \quad . \tag{2.22}$$

Without loss of generality, we can assume $H^{(0)}$ to have a smaller symmetry than $H^{(1)}$. Therefore the basis rotation which diagonalizes $H^{(0)}$ can be absorbed in the measure of the random matrix $H^{(1)}$ and $H^{(0)}$ can be taken to be diagonal. The diffusion only takes place in the space of eigenvalues of H. We decompose $H(t)$ in angle–eigenvalue coordinates by writing $H(t) = U^{-1}xU$, where $U \in U(N;\beta)$ and we average over the group. We obtain

$$P_{N\beta}(x,t) = C_N^{(\beta)} \int \Gamma^{(\beta)}(x, H^{(0)}, t)P_N^{(0)}(H^{(0)})|\Delta_N^\beta(H^{(0)})|d[H^{(0)}] \quad . \tag{2.23}$$

The diffusion kernel $\Gamma^{(\beta)}(x, H^{(0)}, t)$ is given by

$$\begin{aligned} \Gamma^{(\beta)}(x, H^{(0)}, t) &= t^{-N/2-N(N-1)\beta/4} \\ &\quad \exp\left(\frac{-\beta}{2t}\left(\mathrm{Tr}\, x^2 + \mathrm{Tr}\,(H^{(0)})^2\right)\right)\Phi_N^{(\beta)}(-ix\beta/t, H^{(0)}) \qquad (2.24) \\ \Phi_N^{(\beta)}(-ix/t, H^{(0)}) &= \int_{U\in U(N;\beta)} \exp\left(\frac{1}{t}\mathrm{Tr}\, U^{-1}xUH^{(0)}\right)d\mu(U) \ . \qquad (2.25) \end{aligned}$$

The eigenvalue distribution $P_{N\beta}(x,t)$ obeys a Fokker–Planck equation, which is obtained by transforming the Laplacian (2.15) into angle–eigenvalue coordinates. Since $P_{N\beta}(x,t)$

depends only on the eigenvalues, only the radial part, i. e. the eigenvalue part contributes

$$\Delta_x P_{N\beta}(x,t) = 2\beta \frac{\partial}{\partial t} P_{N\beta}(x,t) \ , \tag{2.26}$$

with the radial part of the Laplacian being

$$\Delta_x = \sum_{n=1}^{N} \frac{1}{|\Delta_N(x)|^\beta} \frac{\partial}{\partial x_n} |\Delta_N(x)|^\beta \frac{\partial}{\partial x_n} \ = \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + \sum_{n<m} \frac{\beta}{x_n - x_m} \left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial x_m} \right) \ . \tag{2.27}$$

Thus, we arrived at another type of Brownian motion for the eigenvalues of our matrix ensemble (2.17) which is somewhat easier to treat than Dyson's model (2.12). The functions $\Phi_N^{(\beta)}(x,k)$ with two diagonal matrices $x = \mathrm{diag}\,(x_1,\ldots,x_N)$ and $k = \mathrm{diag}\,(k_1,\ldots,k_N)$ as arguments, for $\beta = 1,2$, and $x = 1_2 \otimes \hat{x}$, $k = 1_2 \otimes \hat{k}$ for $\beta = 4$ were introduced by Gelfand [GEL1] as *zonal spherical functions.* Sometimes they are also called matrix Bessel functions [GUH1]. These Bessel functions of matrix arguments are in general rather complicated functions with still many unknown properties. However their asymptotics were stated already in 1958 [HC2]

$$\lim_{\substack{x_n\to\infty \\ k_m\to\infty}} \Phi_N^{(\beta)}(x,k) \ \sim \frac{1}{N!} \frac{\det[\exp(ix_n k_m)]_{n,m=1,\ldots,N}}{|\Delta_N(x)\Delta_N(k)|^{\beta/2}} \ . \tag{2.28}$$

From this formula together with Eq. (2.24) one derives readily the initial condition for the diffusion kernel

$$\lim_{t\to 0} \Gamma^{(\beta)}(x,H^{(0)},t) \ = \frac{1}{N!} \ \frac{\det\left[\delta(x_n - H_m^{(0)})\right]_{nm}}{|\Delta(x)\Delta(H^{(0)})|^{\beta/2}} \ . \tag{2.29}$$

A special rôle is played by the unitary group. The matrix Bessel function of the unitary group, $\Phi_N^{(2)}(x,k)$, is the celebrated Itzykson–Zuber integral [IZ1]. Later it was discovered to be a special case of a more general integral formula due to Harish-Chandra [HC1]

$$\Phi_N^{(2)}(-ix,k) = \frac{1}{N!} \frac{\det\left[\exp(x_i k_j)\right]_{ij}}{\Delta_N(x)\Delta_N(k)} \ . \tag{2.30}$$

By means of the Itzykson–Zuber formula analytic results for many different transitions became available. With Eq. (2.30) Pandey and Mehta derived the eigenvalue correlators from Eq. (2.23) for the time–reversal invariance breaking transition GOE $\longrightarrow$ GUE [MP1, MP2]. More examples can be found in [GMGW]. There exist several explanations for the fact, that the unitary case is so much simpler than the other ones. The most sophisticated one is probably due to Duistermaat and Heckman [DH] within the framework of symplectic geometry. They interpret the Itzykson–Zuber integral as a partition function with the trace appearing in the exponential of Eq. (2.25) as Hamiltonian. Then they showed that this partition function reduces to its asymptotic value, see [SZA] for a review. We prefer a more simplistic approach to a better understanding of the peculiarity of the unitary case. To this end we compile some facts about Lie–groups and Lie–algebras. Then we state Harish-Chandra's theorem.

2.2.1 Lie–groups, Lie–algebras and symmetric spaces

We introduce the notation $\mathcal{H}_N^{(\beta)}$ for the Lie–algebras, which are related to the Lie–groups $U(N;\beta)$ via the exponential mapping

$$\mathcal{H}_N^{(\beta)} = \{G : \exp(iG) \in U(N;\beta)\} \quad . \tag{2.31}$$

That means $\mathcal{H}_N^{(2)}$ is the vector space of Hermitean $N \times N$ matrices provided with the Lie–bracket. $\mathcal{H}_N^{(1)}$ is the vector space of skew–Hermitean matrices and $\mathcal{H}_N^{(4)}$ is the vector space of all Hermitean matrices leaving invariant the symplectic metric. An important subspace of a Lie–algebra is the so–called Cartan subalgebra $\mathcal{H}_{0,N}^{(\beta)}$. It is defined as

$$\mathcal{H}_{0,N}^{(\beta)} = \left\{G_i, G_j \in \mathcal{H}_N^{(\beta)} : [G_i, G_j] = 0\right\} \quad . \tag{2.32}$$

That is the maximal subspace of commuting elements. Its dimension r is called the *rank* of the Lie–algebra. More explicitly we have

$$\begin{aligned}
G \in \mathcal{H}_{0,2r}^{(1)} &= \operatorname{diag}\left(iG_1\tau^{(1)}, \ldots, iG_r\tau^{(1)}\right) \\
G \in \mathcal{H}_{0,(2r+1)}^{(1)} &= \operatorname{diag}\left(0, iG_1\, \tau^{(1)}, \ldots, iG_r\, \tau^{(1)}\right) \\
G \in \mathcal{H}_{0,r}^{(2)} &= \operatorname{diag}\left(G_1, G_2 \ldots, G_r\right) \\
G \in \mathcal{H}_{0,2r}^{(4)} &= \operatorname{diag}\left(iG_1\, \tau^{(3)}, \ldots, iG_r\, \tau^{(3)}\right)
\end{aligned} \tag{2.33}$$

for the three groups of interest using the notation of (2.2). One can choose a basis for the algebra in terms of the Cartan subalgebra and generalized ladder operators E_α. If we write an arbitrary element $G \in \mathcal{H}_{0,N}^{(\beta)}$ as $G = \sum_{i=1}^r G_i, i = 1, \ldots, r$, then the Cartan subalgebra relates to the ladder operators via the commutation relation

$$[G_i, E_\alpha] = \alpha_i E_\alpha \quad , \quad [E_{-\alpha}, E_\alpha] = \sum_{i=1}^r \alpha_i G_i \quad . \tag{2.34}$$

The r dimensional vector α is called root vector and its components are called roots. For example, in case of the unitary group the roots are all possible differences of the matrix elements G_i. The root vectors can be mapped onto each other by reflections in hyperplanes orthogonal to a root vector. The set of all possible reflections and products of them forms the *Weyl group*. Another important notion of group theory is a *symmetric space*. A symmetric space is defined as a space of constant curvature, i. e. no points are singled out. It can be realized as the coset of a Lie group on one of its subgroups. The subgroup is the set of fixed points of an involutive automorphism[1]. The classical matrix ensembles defined in the previous section describe time evolution operators, which form a subset of the unitary group. For $\beta = 1, 2, 4$ these subsets are given by the symmetric spaces

$$\frac{U(N)}{O(N)} \quad , \quad \frac{U(N)}{1} \quad , \quad \frac{U(2N)}{USp(2N)} \quad . \tag{2.35}$$

[1] An involutive automorphism usually denoted by σ, is defined by the properties $\sigma \neq 1$ and $\sigma^2 = 1$.

For $\beta = 1$ the involutive automorphism is the complex conjugation and for $\beta = 4$ adjunction with the symplectic metric

$$U = g^T U g \quad , \quad U \in USp(2N) \; , \quad g = \tau^{(1)} \otimes 1_N \quad . \tag{2.36}$$

The geometrical object associated with the GUE is the unitary group, whereas with the GOE and GSE the symmetric spaces as defined in Eq. (2.35) are associated !

We can now state the Harish–Chandra theorem.

Theorem 2.1 *Let V be a compact semi-simple group and G, G' elements of its Cartan subalgebra $\mathcal{H}_0$, then*

$$\int_{U\in V} \exp\left(\operatorname{tr} U^{-1}GUG'\right) d\mu(U) = \frac{1}{|W|} \sum_{s\in W} \frac{1}{\pi(G)\pi(s(G'))} \exp\left(\operatorname{tr} s(G)G'\right) \quad , \tag{2.37}$$

where $\pi(G)$ is defined as the product of all positive roots of $\mathcal{H}_0$ and W is the Weyl reflection group of V with $|W|$ elements.

For $V = U(N)$ the Cartan subalgebra consists of diagonal matrices and the product over the positive roots the Vandermonde determinant and one recovers the Itzykson–Zuber formula. Regrettably, for $V = O(N)$ or $V = USp(2N)$ the diagonal matrices (with Kramers degeneracy in the symplectic case) do not belong to the algebra of the group. This means that for the GOE and GSE the matrix Bessel functions are not included in the Harish–Chandra theorem. This is the reason, why the GOE and GSE in almost all applications are much more difficult to treat as the GUE. This fact is a major motivation of this work.

2.2.2 Connection with Calogero–Sutherland models

The diffusion equations (2.27) and (2.12) are closely related to the so called Calogero–Sutherland Hamiltonians. It is known [RIS] that in one dimension a Fokker–Planck equation can be cast into an imaginary time Schrödinger equation by adjunction with the square root of the stationary probability density. In the case of the diffusion equation (2.12) , (2.13) we define

$$H_C = \frac{1}{\sqrt{W_{\text{eq}}(x)}} \sum_{i=1}^{N} \left[-\beta \frac{\partial}{\partial x_i} E(x_i) + \frac{\partial^2}{\partial x_i^2} \right] \sqrt{W_{\text{eq}}(x)} \quad , \tag{2.38}$$

where for the time being $W_{\text{eq}}(x) = W(x)$ as given by Eq. (2.9). We obtain an imaginary time Schrödinger equation

$$\frac{\partial}{\partial t} \psi^{(\beta)}(x,t) = -H_C \psi^{(\beta)}(x,t), \quad \psi^{(\beta)}(x,t) = \frac{1}{W_{\text{eq}}(x)} P(x,t) \tag{2.39}$$

with an Hermitean operator H_C

$$H_C = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{\beta(\beta-2)}{2} \sum_{i<j}^{N} \frac{1}{(x_i - x_j)^2} + \frac{1}{16} \sum_{i=1}^{N} x_i^2 + \text{ const.} \quad , \tag{2.40}$$

which describes the motion of N particles on the line under the influence of a square pairwise interaction. H_C is known as *Calogero Hamiltonian* [CAL]. For the pure diffusion (2.27) there is no harmonic binding term and the particles keep spreading apart for $t \to \infty$. Although no stationary distribution exists, one can associate with Eq. (2.27) the Hamiltonian

$$H_D = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{\beta(\beta-2)}{2} \sum_{i<j}^{N} \frac{1}{(x_i - x_j)^2} \tag{2.41}$$

by defining formally

$$W_{\mathrm{eq}}(x) = |\Delta_N(x)|^{-\beta/2} \quad , \quad \psi^{(\beta)}(x,t) = |\Delta_N(x)|^{\beta/2} P(x,t) \quad . \tag{2.42}$$

H_D is in contrast to H_C a scattering system with a continuous spectrum, the large time behavior is determined by the states near the ground state. In order to have a well defined thermodynamic limit one may confine the particles on a circle. Thus one arrives at

$$H_{CS} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{\beta(\beta-2)}{2} \sum_{i<j}^{N} \frac{(\pi/N)^2}{\sin^2[\pi(x_i - x_j)/N]} \quad . \tag{2.43}$$

This is the *Calogero Sutherland Hamiltonian*, which also can be derived directly from Dyson's circular ensembles [DYS2, MEH1]. In the thermodynamic limit the particle density $R_1(x)$ of the ground state is described by Wigner's semi-circle law. The mean particle level spacing $D = 1/R_1(0)$ scales as $D \propto 1/\sqrt{N}$. Therefore in the thermodynamic limit the harmonic confining term in (2.40) vanishes on the scale of the mean level spacing. On this "unfolded scale" the correlation functions become independent of the confinement mechanism. The three Hamiltonians H_C, H_D and H_{CS} are known to be integrable systems for arbitrary β [SU1], for a review see [SU2]. However, the three values $\beta = 1, 2, 4$ are distinguished, since they establish a connection to the random matrix ensembles. Indeed for these values of β they belong to a much wider class of integrable systems, which can be constructed by means of the root space of a simple Lie algebra or – still more general – a Kac–Moody algebra [OP]. This class comprises Hamiltonians which can be derived by an adjunction procedure like in Eq. (2.38) from a Laplace–Beltrami operator of a group acting in a symmetric space. This space has positive curvature for H_{CS} and zero curvature for H_D, H_C [2]. The matrix Bessel functions provide in a natural way a set of eigenfunctions $\psi_k^{(\beta)}(x)$ of H_D. For $\beta = 1, 2, 4$ we have

$$H_D \psi_k^{(\beta)}(x) = \mathrm{Tr}\, k^2 \psi_k^{(\beta)}(x) \quad , \quad \psi_k^{(\beta)}(x) = |\Delta_N(x)|^{\beta/2} \Phi_N^{(\beta)}(x,k) \quad . \tag{2.44}$$

Furthermore there also exist orthogonality and completeness relations,

$$\begin{aligned} \int d[k] |\Delta(k)|^\beta \, \psi_k^{(\beta)}(x) \psi_k^{(\beta)}(x') &= \left(D_N^{(\beta)} C_N^{(\beta)}\right)^2 \det[\delta(x_i - x_j')]_{ij} \\ \int d[x] \psi_k^{(\beta)}(x) \psi_{k'}^{(\beta)}(x) &= \left(D_N^{(\beta)} C_N^{(\beta)}\right)^2 \frac{\det[\delta(k_i - k_j')]_{ij}}{|\Delta(k')|^{\beta/2} |\Delta(k)|^{\beta/2}} \,, \end{aligned} \tag{2.45}$$

with the normalization constant

$$D_N^{(\beta)} = \frac{1}{(2\pi)^{n/2} \pi^{\beta N(N-1)/4}} \,. \tag{2.46}$$

[2] In [BEE, CAS] it was pointed out, that the DMPK equation for scattering matrices with broken time reversal symmetry corresponds to a Laplace–Beltrami operator in a symmetric space of negative curvature.

These well known relations will be rederived in Section (4.2.2). We can consider one set of variables as a set of reals labeling the eigenstates of H_D. To emphasize this, we have written k as an index, although it is still a continuous variable. We remark that for $\beta = 2$ the interaction vanishes. The statement of the Itzykson–Zuber formula is in this context simply that the eigenstates of the Laplacian are plane waves.

There exist already sets of eigenfunctions of the Hamiltonians H_C, H_D, H_{CS} for arbitrary β. Essentially, these solutions are a product of the ground state wave function and a symmetric polynomial in the coordinates of the N particles. In case of the Calogero–Sutherland Hamiltonian H_{CS} these polynomials are known as Jack polynomials [STA, FOR, HA]. In this approach the energy eigenvalues are labeled by a partition[3] of length N. In order to obtain an orthogonality condition like Eq. (2.45) one needs to sum over an infinite number of partitions. The crucial difference to the matrix Bessel functions is that the Jack polynomials are symmetric polynomials in one set of variables wheras the matrix Bessel functions are symmetric in two sets of variables. In *addition* they are symmetric under interchange of the two sets of variables. Due to this symmetry the eigenfunctions $\psi_k^{(\beta)}(x)$ related to the matrix Bessel function via Eq. (2.44) are, at least for H_D, the more natural eigenfunctions.

We can view the matrix Bessel function defined as the group integral (2.25) as a solution of the Hamiltonian H_D for the coupling parameters $\beta = 1, 2, 4$. Then one might pose the question: does such an integral solution exist also for other parameters β? The affirmative answer will be given in Section 4.4.

2.3 Diffusion in the space of supermatrices

We now turn to the matrix model (1.1) and (2.17). The problem is to calculate the eigenvalue correlation functions of an Hamiltonian

$$H = H^{(0)} + \sqrt{t} H^{(1)} \tag{2.47}$$

interpolating between a Gaussian random matrix $H^{(1)}$ and an arbitrarily distributed matrix $H^{(0)}$. That means in the generic case that $H^{(0)}$ breaks the invariance under orthogonal, unitary or unitary symplectic transformations of the probability density (2.3). It is the symmetry breaking term $H^{(0)}$, which causes unsurmountable problems for the method of orthogonal polynomials. This is the classical method of deriving eigenvalue correlators from the pure Gaussian ensemble, cf. [MEH1]. For a small chaotic admixture, i. e. in a perturbative expansion around the regular Poissonian limit, closed expressions were derived in [FKPT] and by a different method in [LES]. As already mentioned for non time–reversal invariant systems the Itzykson–Zuber integral is a powerful tool. Lenz [LEN] obtained an integral representation of the two–point correlator for the transition from Poisson regularity to chaos in the GUE case. Pandey derived an exact expression for the two–point correlation function without using the Itzykson–Zuber formula on the scale of the mean level spacing. For arbitrary level number Guhr derived an exact expression for the k–point correlation function [GUH4], by using the supersymmetric generalization of the Itzykson–Zuber formula [GUH1].

[3] A partition of an integer Λ is a decreasing set of integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$ such that $\sum_{i=1}^{N} \lambda_i = \Lambda$

Since one cannot take advantage of a Itzykson–Zuber formula, for the GOE and GSE much less analytic results are known. Alternative methods are needed. Nagao and Forrester adressed the model (2.47) with the method of Jack polynomials [NF]. The probably most promising approach is by supersymmetry. Since its introduction in condensed matter physics in 1983 [EFE], this has by now become a generally accepted method.[4]

Remarkably, there also exists a class of diffusion equations in superspace. As we will see, certain solutions of these diffusion equations can be interpreted as the generating functional of the k-point eigenvalue correlator, which is defined as

$$\begin{aligned} R_k(x_1,\ldots,x_k,t) &= \frac{1}{\pi^k}\int P_N^{(0)}(H^{(0)})P_{N\beta}(H,t|H^{(0)})d[H^{(1)}]d[H^{(0)}] \\ &\quad \prod_{p=1}^{k} \mathrm{Im}\ \mathrm{tr}\, \frac{1}{x_p - i\varepsilon - H(t)} \quad , \end{aligned} \tag{2.48}$$

where $P_{N\beta}(H,t|H^{(0)})$ is defined in Eq. (2.20). We use the definition $x^{\pm} = x \pm i\varepsilon$. For convenience we define another correlation function by including the real part

$$\begin{aligned} \widehat{R}_k(x_1,\ldots,x_k,t) &= \frac{1}{\pi^k}\int P_N^{(0)}(H^{(0)})d[H^{(0)}]P_{N\beta}(H,t|H^{(0)})d[H^{(1)}] \\ &\qquad\qquad \prod_{p=1}^{k} \mathrm{tr}\, \frac{1}{x_p^{\pm} - H(t)} \quad . \end{aligned} \tag{2.49}$$

One can convince oneself that the physically relevant correlations (2.48) can always be constructed as a linear combination of the functions $\widehat{R}_k(x_1,\ldots,x_k,t)$. We denote this procedure by the symbol $\Im$. We can rewrite Eq. (2.49) in terms of derivatives

$$\hat{R}_k(x_1,\ldots,x_k,t) = \frac{1}{(2\pi)^k}\frac{\partial^k}{\prod_{p=1}^{k}\partial J_p} Z_k(x+J,t)\bigg|_{J=0} \tag{2.50}$$

of a normalized generating function

$$\begin{aligned} Z_k(x+J,t) &= \int P_N^{(0)}(H^{(0)})d[H^{(0)}]P_{N\beta}(H,t|H^{(0)})d[H_\beta^{(1)}] \\ &\quad \prod_{p=1}^{k} \mathrm{detg}^{\,2}\left(x_p^{\pm}\otimes 1_2 + J_p \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix} - H(t)\otimes 1_2\right) \quad . \end{aligned} \tag{2.51}$$

Here the matrices 1_2 and $\mathrm{diag}\,(1,-1)$ are graded. The next steps in the evaluation of $Z_k(x+J,t)$ are:

- The graded determinant is written as a functional integral over a graded vector field $\zeta(i), i = 1,\ldots,N$. The specific form of the vectors has to be chosen according to the ensemble under consideration. For the orthogonal ensemble they are graded real $4k$ vectors as defined in [EFE] and for the symplectic ensemble they are “quaternionic“ like $4k \times 2$ vectors as defined in [WEG1].

[4] The model (2.47) has been investigated just recently by a variation of the supersymmetry method in [DK]

- The integral over the random matrix ensemble is performed and the quartic terms are removed by a Hubbard-Stratonovich transformation, i. e. the supersymmetric generalization of a Fourier transformation.

- Finally the remaining $\zeta(i)$ integrals are evaluated yielding [EFE, VWZ].

$$\begin{aligned} Z_k(x+J,t) &= 2^{k(k-2)} \int d[\sigma] \exp\left(-\frac{1}{t}\mathrm{trg}\,(\sigma - x \pm J)^2\right) \\ & \int d[H^{(0)}] P_N^{(0)}(H^{(0)}) \mathrm{detg}^{\,-\beta/2}\left(1_N \otimes \sigma^{\pm} - H^{(0)} \otimes 1_{4k}\right) \quad . \end{aligned} \tag{2.52}$$

The generating function is now represented by an integral over a graded $4k \times 4k$ matrix σ. The advantage of the representation (2.52) is that the matrix dimension of σ depends only on the order of the correlation function but is decoupled from the level number N. σ belongs to the ensemble of graded “real“, Hermitean supermatrices [WEG1], which have the form

$$\sigma = \begin{bmatrix} \sqrt{c}\sigma^{(R)} & \sigma^{(A)\dagger} \\ \sigma^{(A)} & \sqrt{-c}\sigma^{(HSd)} \end{bmatrix} \quad , \quad c \in \{1,-1\}. \tag{2.53}$$

For the GOE one has $c=1$ and for the GSE $c=-1$. Both ensembles in ordinary space are mapped onto the same supermatrix ensemble. Only the fermion–fermion blocks and boson–boson blocks are interchanged. The two commuting blocks $\sigma^{(R)}$ and $\sigma^{(HSd)}$ are real symmetric and Hermitean selfdual matrices respectively and $\sigma^{(A)}$ is of the form

$$\sigma^{(A)} = [\sigma_1^{(A)}, \ldots, \sigma_{k_1}^{(A)}] \,, \qquad \sigma_i^{(A)} = \begin{pmatrix} \sigma_{1i}^{(A)} \\ \sigma_{1i}^{(A)*} \\ \vdots \\ \sigma_{k_2 i}^{(A)} \\ \sigma_{k_2 i}^{(A)*} \end{pmatrix} \quad . \tag{2.54}$$

The entries $\sigma_{ij}^{(A)}$ are anticommuting numbers. The infinitesimal volume element is given by

$$d[\sigma] = \prod_{i=1}^{k_1}\prod_{j=1}^{k_2} d\sigma_{ij}^{(A)*} d\sigma_{ij}^{(A)} \prod_{i<j} d\sigma_{ij}^{(R)} \prod_{i=1}^{k_1} d\sigma_{ii}^{(R)} \prod_{i<j} d\sigma_{ij}^{(HSd)} \prod_{i=1}^{k_2} d\sigma_{ii}^{(HSd)} \,, \tag{2.55}$$

where $d\sigma_{ij}^{(HSd)}$ is the product of the differentials of all independent elements of the quaternion $\sigma_{ij}^{(HSd)}$. Moreover, in Eq. (2.52) we have defined the graded $4k \times 4k$ matrix

$$x \pm J = \mathrm{diag}\,(1_2 \otimes (x+J), 1_2 \otimes (x-J)) \quad , \tag{2.56}$$

where $x = \mathrm{diag}\,(x_1, \ldots, x_k)$ and $J = \mathrm{diag}\,(J_1, \ldots, J_k)$ are ordinary diagonal matrices. The generating function Eq. (2.52) obeys a diffusion equation in supermatrix space in analogy to the diffusion equation in ordinary matrix spaces as described in the previous section [GUH4]. Indeed the function

$$Q_k(\sigma,\rho,t) = 2^{k(k-2)} \exp\left(-\frac{1}{t}\mathrm{trg}\,(\sigma-\rho)^2\right) \tag{2.57}$$

with σ, ρ graded real, is the kernel of the diffusion equation

$$\Delta_\rho Q_k(\sigma,\rho,t) = \frac{\partial}{\partial t} Q_k(\sigma,\rho,t) \quad , \quad \lim_{t\to 0} Q_k(\sigma,\rho,t) = \delta(\sigma-\rho) \ , \tag{2.58}$$

where we have defined the delta–function for an anticommuting variable as $\delta(\zeta) = \sqrt{2\pi}\zeta$. It is normalized such that

$$\int \delta(\zeta) d\zeta \ = \ 1 \quad . \tag{2.59}$$

The supersymmetric counterpart of the Laplacian (2.15) reads

$$\Delta_\rho = \mathrm{trg} \left(\frac{\partial}{\partial \rho}\right)^2 \quad . \tag{2.60}$$

According to the block structure of ρ the diffusion coefficients are different.

$$\left(\frac{\partial}{\partial \rho^{(R)}}\right)_{ij} = \frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \rho_{ij}^{(R)}} \quad , \left(\frac{\partial}{\partial \rho^{(HSd)}}\right)_{ij} = \frac{1+\delta_{ij}}{4} \frac{\partial}{\partial \rho_{ij}^{(HSd)}} \quad , \left(\frac{\partial}{\partial \rho^{(A)}}\right)_{ij} = \frac{1}{4} \frac{\partial}{\partial \rho_{ij}^{(A)}} \ . \tag{2.61}$$

As in ordinary space the diffusion takes place in the radial space of the matrices, thus we can take the initial condition as a diagonal matrix. Then the solution reads

$$Z_k(r,t) = \int Q_k(\sigma,r,t) Z_k^{(0)}(\sigma) d[\sigma] \quad , \quad \lim_{t\to 0} Z_k(r,t) = Z_k^{(0)}(r) \quad . \tag{2.62}$$

With the replacement $\rho \longrightarrow x \pm J$ and the initial condition

$$Z_k^{(0)}(\sigma) = \int d[H^{(0)}] P_N^{(0)}(H^{(0)}) \mathrm{detg}^{\,-\beta/2} \left(\sigma^{\pm} \otimes 1_N - 1_{4k} \otimes H^{(0)}\right) \tag{2.63}$$

this is the generating function $Z_k(x+J,t)$ of Eq. (2.52). Conveniently, the entire dependence on the matrix $H^{(0)}$ has been absorbed in the initial condition, this allows us to treat with Eq. (2.52) different transitions on the same footing. The picture of a diffusion process in superspace is very instructive. In the same way as the joint probability density Eq. (2.20) evolves towards the chaotic equilibrium distribution Eq. (2.3) the generating function of the eigenvalue correlators evolves towards the generating function of the eigenvalue correlators of the random matrix ensembles. Indeed, Pandey and Mehta showed [MP1] that in the transition from a time reversal invariant ensemble towards time reversal symmetry breaking the transition of a GOE correlator to a GUE correlator preserves for arbitrary transition parameter the structure of a quaternionic determinant which is characteristic for both ensembles. This becomes immediately clear in the supersymmetric framework.

In order to evaluate further the generating function, one can in analogy to the ordinary case, cf. Eqs. (2.23) to (2.25), go to angle eigenvalue coordinates [GUH1]. The graded "real" Hermitean $4k \times 4k$ matrices are diagonalized by a set of supermatrices, which forms the so called unitary orthosymplectic group $UOSp(2k/2k)$ [BER]. After the transformation of variables

$$\begin{aligned} \sigma &= u^{-1} s u \qquad , \qquad u \in UOSp(2k/2k) \\ s &= \mathrm{diag}\,(\sqrt{c}\, s_{(2k)1}, \ldots, \sqrt{c}\, s_{11}, \sqrt{-c}\, 1_2 s_{12}, \ldots, \sqrt{-c}\, 1_2 s_{k2}) \end{aligned} \tag{2.64}$$

we find

$$
\begin{aligned}
Z_k(x+J,t) &= \int \Gamma_k(s,x+J,t)\, Z_k^{(0)}(s)\, \widetilde{B}_k^{(c)}(s) d[s] & (2.65)\\
\Gamma_k(s,x+J,t) &= 2^{k(k-2)} \exp\left(-\frac{1}{t}\left(\operatorname{trg} s^2 + \operatorname{trg}(x\pm J)^2\right)\right) \\
&\qquad\qquad \Phi_{(2k)(2k)}(-i2s/t, x\pm J) \quad . & (2.66)
\end{aligned}
$$

The nontrivial part of $\Gamma_k(s,x+J,t)$ is a group integral over the unitary orthosymplectic group

$$
\Phi_{(2k)(2k)}(-is/t, x\pm J) = \int_{u\in UOSp(2k/2k)} \exp\left(\frac{1}{t}u^{-1}su(x\pm J)\right) d\mu(u) \ . \tag{2.67}
$$

The Berezinian is given by [GUH4]

$$
\widetilde{B}^{(1)}_{(2k)k}(s) = \frac{|\Delta_{2k}(s_1)|\Delta_k^4(is_2)}{\prod_{i=1}^{2k}\prod_{j=1}^{k}(s_{i1}-is_{j2})^2} \quad , \quad \widetilde{B}^{(-1)}_{(2k)k}(s) = \frac{|\Delta_{2k}(is_1)|\Delta_k^4(s_2)}{\prod_{i=1}^{2k}\prod_{j=1}^{k}(is_{i1}-s_{j2})^2} \ . \tag{2.68}
$$

Notice the striking resemblance to Eq. (2.24) and (2.25). The function $\Phi_{(2k)(2k)}(s,r)$, with both s and r defined according to Eq. (2.64), can be considered as the supersymmetric generalization of the matrix Bessel functions in ordinary space. As in the ordinary case only the radial part of the Laplace operator contributes

$$
\Delta_s = \frac{1}{\widetilde{B}^{(c)}_{(2k)k}(s)}\left(\sum_{i=1}^{2k}\frac{\partial}{\partial s_{i1}}\widetilde{B}^{(c)}_{(2k)k}(s)\frac{\partial}{\partial s_{i1}} + \frac{1}{2}\sum_{i=1}^{k}\frac{\partial}{\partial s_{i2}}\widetilde{B}^{(c)}_{(2k)k}(s)\frac{\partial}{\partial s_{i2}}\right) \quad . \tag{2.69}
$$

The diffusion kernel $\Gamma_k(s,r)$ satisfies the diffusion equation in the space of the eigenvalues of both the supermatrix σ and the supermatrix ρ

$$
\Delta_r\Gamma_k(s,r,t) = \Delta_s\Gamma_k(s,r,t) = \frac{\partial}{\partial t}\Gamma_k(s,r,t) \quad , \tag{2.70}
$$

with the initial condition [GUH4]

$$
\lim_{t\to 0}\Gamma_k(s,r,t) = \frac{2^{k(k-2)}}{(2k)!k!}\frac{\det\left[\delta(s_{i1}-r_{j1})\right]_{i,j=1\ldots 2k}\det\left[\delta(s_{i2}-r_{j2})\right]_{i,j=1\ldots k}}{\sqrt{\widetilde{B}^{(c)}_{(2k)k}(s)\widetilde{B}^{(c)}_{(2k)k}(r)}} \tag{2.71}
$$

Inserting Eq. (2.71) into Eq. (2.65) recovers Eq. (2.62). Similar relations are given in [GW] for the space of complex matrices in ordinary and in superspace.

In Ref. [GUH1] the equations corresponding to Eq. (2.65) to (2.67) have been derived for the unitary ensemble. In that case one has to average over the unitary supergroup. The integral can be performed in a closed form. As in the ordinary space, the unitary group is distinguished also in superspace by the fact that an arbitrary graded diagonal matrix is contained in its Cartan subalgebra. This is obvious, since the Cartan subalgebra of $U(k_1/k_2)$ is just the direct sum of a Cartan subalgebra of $U(k_1)$ and $U(k_2)$. This leads to a direct supersymmetric generalization of the Itzykson–Zuber integral. By the same reason it is also clear, that the graded diagonal matrices s and $(x\pm J)$ do not belong to the Cartan subalgebra of $UOSp(2k/2k)$. In the supersymmetric formalism however the matrix dimension of the supermatrix is decoupled from the level number. Therefore one only needs to treat a $4k \times 4k$ matrix in order to derive the k–point correlator. Hence we are liberated from solving Eq. (2.67) for arbitrary dimension and can focus onto the one– and two–point correlators. Much of the remainder of this work is devoted to the solution of Eq. (2.67). We finish this chapter with two comments on the transformation into angle eigenvalue coordinates.

- The first comment is on the transformation properties of the measure. In going from the Cartesean coordinates to angle eigenvalue coordinates the measure is transformed as

$$d[\sigma] \;=\; \widetilde{B}^{(c)}_{(2k)k}(s)d\mu(u)d[s] \; . \tag{2.72}$$

  This is not completely correct. It has been pointed out that there arise additional terms to the integration measure. These are called Efetov–Wegner–Parisi–Sourlas terms. A complete mathematical treatment was given by Rothstein [ROT]. They occur when the integration domain is non compact and a commuting variable is shifted by nilpotents. In general they have to be added to preserve the symmetry of the measure. They ensure the normalization of the generating function in Eq. (2.65) at $J = 0$. The rôle of these terms in the context of diffusion equations has been thoroughly discussed in [GUH4, GW]. The terms represent stationary points of the diffusion process. We do not worry about them as long as

$$Z_k(r,t) = \int \Gamma_k(s,r,t) Z_k^{(0)}(r)\widetilde{B}^{(c)}_{(2k)k}(s)d[s] \tag{2.73}$$

  is the solution of the same diffusion equation

$$\Delta_r Z_k(r,t) = \frac{\partial}{\partial t} Z_k(r,t) \qquad , \qquad \lim_{t\to 0} Z_k(r,t) \;=\; Z_k^{(0)}(r) \tag{2.74}$$

  as $Z_k(r,t)$ defined in Eq. (2.62). By this condition $Z_k(r,t)$ is essentially determined.

- A second comment is on the generating function as it stands in Eq. (2.52). If one is interested in the correlations on the scale of the mean level spacing, one might use a saddle–point approximation to solve Eq. (2.52). Within the saddle–point approximation the model Eq. (2.48) has been treated by an expansion around the chaotic limit, i. e. for $1/t \to 0$ in [GWE] and later more carefully in [FGM]. To use the saddle–point approximation one has to evaluate the average over a retarded and an advanced Greens function to get a non–vanishing result for the two–point correlator. This leads to a non–compact symmetry in the boson–boson block. Then, in the case of the two–point function, the diagonalizing group is not $UOSp(4/4)$ but $UOSp(2,2/4)$, i. e. a group with non–compact degrees of freedom. This was pointed out in [SW]. This group is more difficult to treat than the compact one, due to convergence questions. It is a highly convenient feature of the graded eigenvalue method that the position of the imaginary increment of the eigenvalues appears only in the initial condition of the supersymmetric diffusion equation and does not influence the group manifold.

Chapter 3

Gelfand–Tzetlin coordinates in superspace

In this chapter we construct an explicit coordinate system for the unitary orthosymplectic supergroup. If one tackles the problem of calculating the supersymmetric matrix Bessel function (2.67) by a explicit parametrization of the group, one quickly realizes, that it is a hopeless enterprise, unless one uses the appropriate coordinate system. Gelfand [GT1] devised in 1950 a coordinate system for the unitary group, which shows nice features. For example, it provides a link to representation theory. The coordinates parametrizing the group can be arranged in a so–called Gelfand–Tzetlin pattern in the very same way as a set of integers or half–integers labeling the states of a representation. Thus the latter can be interpreted as the quantized version of the continuous parameters. A full–fledged theory of these so–called Gelfand–Tzetlin coordinates was given by Guillemin and Sternberg [GS1, GS2], see also [AFS]. Moreover they proved to be a useful tool for the calculation of group integrals. The Itzykson–Zuber integral in ordinary space [SHA] is easily derived in this coordinate system.

In [GUH3] the idea of the Gelfand–Tzetlin coordinates was generalized to the unitary supergroup. The originally more geometric construction by Gelfand was substituted by an algebraic method. The idea, however, remains the same: to parametrize a column of the unitary supermatrix by projecting a diagonal matrix onto a space orthogonal to this column. Again the supersymmetric Itzykson–Zuber integral is readily derived, once the coordinate system is established.

Our aim is to use the ideas of [GUH3] and [GT1] to construct a coordinate system of the unitary orthosymplectic groups $UOSp(k_1/2k_2)$. This group contains the symplectic group $USp(2k_2)$ and the orthogonal group $O(k_1)$ as subgroups. Thus, we also construct a coordinate system for these two groups in ordinary space.

The chapter is organized as follows: After a brief review of the supergroups we introduce the Gelfand–Tzetlin equations. Their solutions are presented and the Haar measure is derived. An extra section is devoted to the unitary symplectic group. In the final section the generalized Gelfand–Tzetlin patterns are derived and some aspects of group theory are discussed.

3.1 The classical supergroups

There exists a classification of the classical superalgebras similar to Cartan's classification of the Lie–algebras in ordinary space [KAC1, KAC1]. In principle to each classical superalgebra a supergroup is associated via the exponential mapping. However, the classification pattern of the supergroups is usually somewhat coarser [RIT]. If one omits those supergroups, which emerge from the exceptional superalgebras, one is left with only four different types of subgroups of the *general linear supergroup* $GL(k_1/k_2)$, defined as the group of invertible $(k_1+k_2)\times(k_1+k_2)$ matrices of the form

$$u = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{3.1}$$

where the diagonal $k_1 \times k_1$ and $k_2 \times k_2$ block matrices a and d have commuting elements and the off–diagonal blocks b, c anticommuting ones. The most important subgroups for applications in physics are the *unitary supergroup* $U(k_1/k_2)$ and the *orthosymplectic supergroup* $OSp(k_1/k_2)$ [1]. They are defined as

$$\begin{aligned} U(k_1/k_2) &= \left\{u \in GL(k_1/k_2) : u^\dagger u = 1\right\} \quad , \\ OSp(k_1/2k_2) &= \left\{u \in GL(k_1/2k_2) : u^\dagger g u = g\right\} \quad , \end{aligned} \tag{3.2}$$

where we have defined the "orthosymplectic metric"

$$g = \begin{bmatrix} 1_{k_1} & 0 \\ 0 & 1_{k_2} \otimes \tau^{(1)} \end{bmatrix} \quad . \tag{3.3}$$

The Hermitean conjugate we define as [RS]

$$u^\dagger = \begin{bmatrix} a^\dagger & c^\dagger \\ -b^\dagger & d^\dagger \end{bmatrix} \quad . \tag{3.4}$$

The compact form of $OSp(k_1/2k_2)$ is usually denoted by $UOSp(k_1/2k_2)$. Although in principle it might be possible to construct the Gelfand–Tzetlin coordinate system for non–compact groups as well, we restrict ourselves to the compact case.

The group elements act on a graded space, which we denote $L = {}^0L \oplus {}^1L$. It decomposes in a sum of an even ${}^0L$ and an odd ${}^1L$ subspace according to its transformation properties under the parity automorphism A [BER]. In a physical language this means that the elements of ${}^1L$ are vectors and the ones of ${}^0L$ pseudovectors. We define a basis $e_j \equiv e_{j1}, j = 1, \ldots k_1$ for ${}^0L$, and $e_{k_1+j} \equiv e_{j2}, j = 1, \ldots 2k_2$ for ${}^1L$ respectively. We denote the algebra of $UOSp(k_1/2k_2)$ by $\mathcal{H}_{k_1 2k_2}$. It is defined via the exponential mapping

$$\mathcal{H}_{k_1 2k_2} = \{h : \exp(ih) \in UOSp(k_1/2k_2)\} \quad . \tag{3.5}$$

Particularly important for the construction of the Gelfand–Tzetlin basis is the Cartan subalgebra, which we denote by $\mathcal{H}_{0,k_1 2k_2}$. It is defined as the direct sum of the Cartan subalgebras of the orthogonal and symplectic subgroups respectively [KAC1]. For k_1 even it reads in the defining matrix representation

$$h \in \mathcal{H}_{0,k_1 2k_2} = \mathrm{diag}\left(ih_{11}\tau^{(1)}, ih_{21}\tau^{(1)}, \ldots, ih_{\frac{k_1}{2}1}\tau^{(1)}, h_{12}\tau^{(3)}, \ldots, h_{k_2 2}\tau^{(3)}\right) \quad , \tag{3.6}$$

[1] The two other classical groups are associated with the so called strange superalgebras usually denoted $P(n)$ and $Q(n)$ [KAC1].

and for k_1 odd

$$h \in \mathcal{H}_{0,k_1 2k_2} = \mathrm{diag}\left(ih_{11}\tau^{(1)}, ih_{21}\tau^{(1)}, \ldots, ih_{\frac{k_1-1}{2}1}\tau^{(1)}, 0, h_{12}\tau^{(3)}, \ldots, h_{k_2 2}\tau^{(3)}\right) , \quad (3.7)$$

where we used the Pauli matrices as defined in Eq. (2.2). The Cartan subalgebra of $UOSp(k_1/2k_2)$ is just the direct sum of the Cartan subalgebra of $O(k_1)$ and $USp(2k_2)$. This is certainly not true for the entire algebra of $UOSp(k_1/2k_2)$.

3.2 The Gelfand–Tzetlin equations

The main idea of the Gelfand–Tzetlin coordinates is to make use of the group–chain structure that holds for the classical groups in ordinary space

$$U(N;\beta) = \frac{U(N;\beta)}{U(N-1;\beta)} \otimes \frac{U(N-1;\beta)}{U(N-2;\beta)} \otimes \cdots \frac{U(2;\beta)}{U(1;\beta)} \otimes U(1;\beta) \quad (3.8)$$

in the notation of Section 2.1. What now follows is a generalization of the steps in [GUH3] to the group chain of the supergroup $UOSp(k_1/2k_2)$. It looks as

$$\begin{aligned} UOSp(k_1/k_2) = & \frac{UOSp(k_1/2k_2)}{UOSp\left((k_1-1)/2k_2\right)\right)} \otimes \frac{UOSp\left((k_1-1)/2k_2\right)}{UOSp\left((k_1-2)/2k_2\right)\right)} \otimes \cdots \frac{UOSp(1/2k_2)}{USp(2k_2)} \otimes \\ & \frac{USp(2k_2)}{USp(2k_2-2)} \otimes \cdots \frac{USp(4)}{SU(2)} \otimes SU(2) \quad . \end{aligned} \quad (3.9)$$

We apply the method of projecting onto a smaller subspace to construct a coordinate system. To this end we write $u \in UOSp(k_1/2k_2)$ as $u = [u_1, u_2, \ldots, u_{k_1+2k_2}]$. Furthermore we denote by u_{ji} the entries of the normalized supervector u_i in the basis $e_{j1}, j = 1, \ldots k_1$ and $e_{j2}, j = 1, \ldots 2k_2$. The orthogonality condition requires the vectors u_i, $i \le k_1$ to be real

$$u_{ji} = u_{ji}^* \quad , \quad \text{for} \quad j \le k_1 \quad \text{and} \quad u_{(k_1+2j)i} = u_{(k_1+2j-1)i}^* \quad , \quad \text{for} \quad j \le k_2. \quad (3.10)$$

If we consider the first vector, it is parametrized by k_1 real commuting numbers u_{j1} and $2k_2$ complex anticommuting numbers, denoted α_j and $\alpha_j^* \equiv u_{(2j)i}$, $j \le k_2$. We define also $|\alpha_j|^2 = \alpha_j^*\alpha_j$. Moreover, we set k_1 even to compactify the notation. The supervector u_1 describes the coset space

$$u_1 \cong \frac{UOSp(k_1/2k_2)}{UOSp((k_1-1)/2k_2)} \quad , \quad (3.11)$$

which is – similar to ordinary spaces – isomorphic to the surface of the $(k_1/2k_2)$ dimensional sphere $S^{(k_1-1)|2k_2}$. Now we go from the Cartesian to a new set of coordinates for u_1 by projecting a fixed element h of the Cartan subalgebra on a space of superdimension $((k_1-1)/2k_2)$ orthogonal to u_1 :

$$h_p^{(1)}\, e_p^{(1)} = (1 - u_1 u_1^\dagger) h (1 - u_1 u_1^\dagger)\, e_p^{(1)} = (1 - u_1 u_1^\dagger) h\, e_p^{(1)} . \quad (3.12)$$

This is the Gelfand–Tzetlin eigenvalue equation. It generalizes [GUH3] for the unitary supergroup to $UOSp(k_1/2k_2)$. It is convenient to rotate the basis so that h becomes diagonal before solving (3.12)

$$e'_{(2i-1)1} = \frac{1}{\sqrt{2}}\left(e_{(2i-1)1} + ie_{(2i)1}\right) \quad ,$$

$$
\begin{aligned}
e'_{(2i)1} &= \frac{1}{\sqrt{2}}\left(ie_{(2i-1)1}+e_{(2i)1}\right) \quad , \qquad i=1\ldots k_1/2 \\
e'_{i2} &= e_{i2} \quad , \qquad i=k_1+1,\ldots k_1+2k_2\,.
\end{aligned} \tag{3.13}
$$

Notice that the bosonic entries of u'_1 are now complex numbers

$$
u'_{(2j)1} = i\,u'^{*}_{(2j-1)1} \equiv \frac{i}{\sqrt{2}}|v_j^{(1)}|\,e^{-i\vartheta_j^{(1)}} \quad , \quad j=1,\ldots,k_1/2\,. \tag{3.14}
$$

In order to solve the eigenvalue equation (3.12), we look at the characteristic function

$$
\begin{aligned}
z\left(h_p^{(1)}\right) &= \operatorname{detg}\left(\left(1-u_1u_1^\dagger\right)h-h_p^{(1)}\right) \\
&= \operatorname{detg}\left(h-h_p^{(1)}\right)\operatorname{detg}\left(1-(h-h_p^{(1)})^{-1}u_1u_1^\dagger h\right) \\
&= \operatorname{detg}\left(h-h_p^{(1)}\right)\operatorname{detg}\left(1-u_1^\dagger\frac{h}{h-h_p^{(1)}}u_1\right) \\
z\left(h_p^{(1)}\right) &= -h_p^{(1)}\operatorname{detg}\left(h-h_p^{(1)}\right)u_1^\dagger\frac{1}{h-h_p^{(1)}}u_1 \quad .
\end{aligned} \tag{3.15}
$$

The function $z(h_p^{(1)})$ behaves differently for the k_1 eigenvalues of the boson–boson block $h_p^{(1)} \equiv h_{p1}^{(1)}$, $p=1,\ldots,k_1$ and the $2k_2$ eigenvalues of the fermion–fermion block $h_{k_1+p}^{(1)} \equiv ih_{p2}^{(1)}$, $p=1,\ldots,2k_2$. Therefore the above equation has to be discussed in the limits

$$
z\left(h_p^{(1)}\right) \longrightarrow \begin{cases} 0 & \text{for } p=1,\ldots,k_1 \\ \infty & \text{for } p=k_1+1,\ldots,\ k_1+2k_2 \end{cases} \quad . \tag{3.16}
$$

Including the normalization condition $u_1^\dagger u_1=1$ we arrive at the following set of equations:

$$
1 = \sum_{p=1}^{k_1/2}|v_p^{(1)}|^2+\sum_{p=1}^{k_2}|\alpha_p^{(1)}|^2, \tag{3.17}
$$

$$
0 = (h_{p1}^{(1)})^2\left(\sum_{q=1}^{k_1/2}\frac{|v_q^{(1)}|^2}{(h_{q1})^2-(h_{p1}^{(1)})^2}+\sum_{q=1}^{k_2}\frac{|\alpha_q^{(1)}|^2}{(ih_{q2})^2-(h_{p1}^{(1)})^2}\right) \qquad p=1,\ldots,(k_1-1)\,, \tag{3.18}
$$

$$
\begin{aligned}
z_p &= (ih_{p2}^{(1)})^2\frac{\prod_{q=1}^{k_1/2}\left((h_{q1})^2-(ih_{p2}^{(1)})^2\right)}{\prod_{q=1}^{k_2}\left((ih_{q2})^2-(ih_{p2}^{(1)})^2\right)} \\
&\quad \left(\sum_{q=1}^{k_1/2}\frac{|v_q^{(1)}|^2}{(h_{q1})^2-(ih_{p2}^{(1)})^2}+\sum_{q=1}^{k_2}\frac{|\alpha_q^{(1)}|^2}{(ih_{q2})^2-(ih_{p2}^{(1)})^2}\right)\,, \\
&\qquad z_p\to\infty, \quad p=1,\ldots,2k_2\,.
\end{aligned} \tag{3.19}
$$

This is a system of equations in $(h_{p1}^{(1)})^2$ and $(ih_{p2}^{(1)})^2$ respectively. Moreover the second equation has a twofold degenerate solution at $h_{p1}^{(1)}=0$. Hence the projected matrix $h^{(1)} \equiv (1-u_1u_1^\dagger)h(1-u_1u_1^\dagger)$ is of the form (3.7) in a certain basis $e_j^{(1)}$ and belongs itself to the algebra of $UOSp((k_1-1)/2k_2)$. This is crucial for the recursion. Out of the (k_1+2k_2+1) equations in (3.17) to (3.19) only $(k_1/2+k_2)$ are independent. Thus a one to one correspondence is established between the moduli squared of the entries of the vector u'_1 and the eigenvalues of the supermatrix $h^{(1)}$. These we call bosonic eigenvalues, if they satisfy Eq. (3.18) and fermionic eigenvalues if they satisfy Eq. (3.19). With the substitutions $h_{q1}^{(j)} \rightarrow (h_{q1}^{(j)})^2$ and $ih_{q2}^{(j)} \rightarrow (ih_{q2}^{(j)})^2$, $j=1,2$, this set of independent equations is the same as the corresponding equations for the unitary supergroup. So we can directly read off its solutions from [GUH3]. They will be stated below.

3.2.1 Recursion

Now a recursion can be performed up to the k_1–th level (see [GT1, SHA, AFS] for the construction in ordinary space and [GUH3] for the one in superspace). The Cartan subalgebra looks differently depending on whether the bosonic dimension k_1 is even or odd, cf. Eq. (3.6) and (3.7). For this reason we have to deal with two different sets of equations in doing the recursion procedure. We refer to a level as an even level, if (k_1-n+1) is even, and as an odd level otherwise.

In the n–th step the vector u'_n is expanded in a set of $(k_1-n+1+2k_2)$ basis vectors ${e'_j}^{(n-1)}$, which span the subspace of L orthogonal to $[u_1,\ldots,u_{n-1}]$. This set splits into two disjoint subsets. The first subset contains (k_1-n+1) vectors ${e'_{j1}}^{(n-1)}$ spanning some subspace of ${}^0L$. The second one contains $2k_2$ basis vectors ${e'_{j2}}^{(n-1)}$ spanning ${}^1L$. The entries of u_n in this basis are complex numbers

$$
\begin{aligned}
{e'_{(2p)1}}^{(n-1)\dagger}u_n &= i\left({e'_{(2p-1)1}}^{(n-1)\dagger}u_n\right)^* \equiv && \frac{i}{\sqrt{2}}|v_p^{(n)}|e^{-i\vartheta_p^{(n)}}\,, \\
& && p \leq \frac{k_1-n+1}{2} \quad \text{if } (k_1-n+1) \text{ even}, \\
& && p \leq \frac{k_1-n}{2} \quad \text{if } (k_1-n+1) \text{ odd}, \\
{e'_{(2p)2}}^{(n-1)\dagger}u_n &= \left({e'_{(2p-1)1}}^{(n-1)\dagger}u_n\right)^* \equiv && \frac{1}{\sqrt{2}}\alpha_p^{(n)*} \qquad p \leq k_2\,.
\end{aligned}
\tag{3.20}
$$

For (k_1-n+1) odd, the remaining entry is parametrized by a real number and an integer $r \in \{0,1\}$ as

$$
{e'_{(k_1-n+1)1}}^{(n-1)\dagger}u_n \equiv (-1)^r v^{(n)}_{\frac{k_1-n}{2}+1} \quad . \tag{3.21}
$$

Moreover we will use the notation

$$
u'^{(n-1)}_n = \sum_{j=1}^{k_1+2k_2-n+1} e'^{(n-1)\dagger}_j u_n e'^{(n-1)}_j \quad . \tag{3.22}
$$

The projection of h onto this subspace

$$
h^{(n-1)} \equiv \left(1-\sum_{i=1}^{n-1} u_i u_i^\dagger\right) h \left(1-\sum_{i=1}^{n-1} u_i u_i^\dagger\right) \tag{3.23}
$$

belongs to the Cartan subalgebra of $UOSp\,((k_1-n)/2k_2)$. The new coordinates are obtained by projecting $h^{(n-1)}$ on the subspace orthogonal to u_n by

$$h_p^{(n)}\, e_p^{(n)} \;=\; (1-u_n u_n^\dagger)\, h^{(n-1)}\, e_p^{(n)} \quad . \tag{3.24}$$

For (k_1-n+1) even, this leads to a equation system as (3.17) to (3.19) reduced by $(n-1)/2$ unknown variables. For (k_1-n+1) odd, the equations have a slightly different form:

$$1 \;=\; \sum_{p=1}^{\frac{k_1-n}{2}} |v_p^{(n)}|^2 + |v^{(n)}_{\frac{k_1-n}{2}+1}|^2 + \sum_{p=1}^{k_2} |\alpha_p^{(n)}|^2 \,, \tag{3.25}$$

$$0 = h_{p1}^{(n)} \left(\sum_{q=1}^{\frac{k_1-n}{2}} \frac{(h_{p1}^{(n)})^2\, |v_q^{(n)}|^2}{(h_{q1}^{(n-1)})^2 - (h_{p1}^{(n)})^2} + |v^{(n)}_{\frac{k_1-n}{2}+1}|^2 + \sum_{q=1}^{k_2} \frac{(h_{p1}^{(n)})^2\, |\alpha_q^{(n)}|^2}{(ih_{q2}^{(n-1)})^2 - (h_{p1}^{(n)})^2} \right) ,$$

$$p = 1,\ldots,(k_1-n)\,, \tag{3.26}$$

$$z_p \;=\; -ih_{p2}^{(n)}\, \frac{\prod_{q=1}^{\frac{k_1-n}{2}} \left((h_{q1}^{(n-1)})^2 - (ih_{p2}^{(n)})^2 \right)}{\prod_{q=1}^{k_2} \left((ih_{q2}^{(n-1)})^2 - (ih_{p2}^{(n)})^2 \right)}$$

$$\left(\sum_{q=1}^{\frac{k_1-n}{2}} \frac{(h_{p1}^{(n)})^2\, |v_q^{(n)}|^2}{(h_{q1}^{(n-1)})^2 - (ih_{p2}^{(n)})^2} + |v_{\frac{k_1-n}{2}+1}|^2 + \sum_{q=1}^{k_2} \frac{(h_{p1}^{(n)})^2\, |\alpha_q^{(n)}|^2}{(ih_{q2}^{(n-1)})^2 - (ih_{p2}^{(n)})^2} \right) ,$$

$$z_p \to \infty \quad , \quad p = 1,\ldots,2k_2 \,. \tag{3.27}$$

The difference between Eq. (3.25) to (3.27) and the corresponding equations for the even levels (3.17) to (3.19) is due to the isolated entry (3.21), which has to be treated separately. It reflects the difference between the even orthogonal group $O(2N)$ and the odd orthogonal group $O(2N-1)$ in ordinary space.

The new basis vectors ${e'_j}^{(n)}$ are related to the basis vectors of the upper level by a $(k_1-n+2k_2)\times(k_1-n+1+2k_2)$ rectangular supermatrix $\widehat{b}'^{(n)}$. The moduli squared of its entries ${\widehat{b}'_{pm}}^{(n)}$ are determined by rewriting equation (3.24) and multiplying from the left hand side with ${e'_m}^{(n-1)\dagger}$

$${e'_m}^{(n-1)\dagger}\, h^{(n-1)}\, {e'_p}^{(n)} = h_p^{(n)}\, {e'_m}^{(n-1)\dagger}\, {e'_p}^{(n)} + {e'_m}^{(n-1)\dagger}\, u_n\, b_p^{(n)} \qquad , \tag{3.28}$$

where we defined $b_p^{(n)} \equiv u_n^\dagger\, h^{(n-1)} {e'_p}^{(n)}$. On the other hand we have

$${e'_m}^{(n-1)\dagger}\, h^{(n-1)}\, {e'_p}^{(n)} = h_m^{(n-1)}\, {e'_m}^{(n-1)}\, {e'_p}^{(n)} \quad , \tag{3.29}$$

yielding for the matrix elements of $\widehat{b}'^{(n)}$

$${\widehat{b}'_{pm}}^{(n)} = \frac{1}{h_m^{(n-1)} - h_p^{(n)}}\, {u'_{mn}}^{(n-1)}\, b_p^{(n)} \quad . \tag{3.30}$$

The modulus squared of $b_p^{(n)}$ is determined by the normalization of the rotated basis vectors ${e'_m}^{(n)\dagger}\,{e'_p}^{(n)} = \delta_{mp}$, i.e by the condition that the matrix $\widehat{b'}^{(n)}\,\widehat{b'}^{(n)\dagger}$ is unity in the $(k_1 - n + 2k_2)$ dimensional subspace orthogonal to $[u_1, \ldots, u_n]$. According to the block structure of the supermatrix $\widehat{b'}^{(n)}$ the vector $b^{(n)}$ has commuting and anticommuting elements. For $(k_1 - n + 1)$ even we define $|w_p^{(n)}|^2 = |b_{2p}^{(n)}|^2 = |b_{2p-1}^{(n)}|^2$, $p = 1, \ldots, (k_1 - n + 1)/2$ for the commuting and $|\beta_p^{(n)}|^2 = |b_{k_1-n+1+2p}^{(n)}|^2 = |b_{k_1-n+2p}^{(n)}|^2$, $p = 1, \ldots, k_2$ for the anticommuting elements. For $(k_1 - n + 1)$ odd we define $|w_p^{(n)}|^2$ and $|\beta_p^{(n)}|^2$ correspondingly.

Again there is a difference in the determining equations of $|w_p^{(n)}|^2$ and $|\beta_p^{(n)}|^2$ between the even and the odd levels of the recursion. For $(k_1 - n + 1)$ even we have

$$\frac{1}{|w_p^{(n)}|^2} = \sum_{m=1}^{(k_1-n+1)/2} \frac{(h_{m1}^{(n-1)})^2 + (h_{p1}^{(n)})^2}{\left((h_{m1}^{(n-1)})^2 - (h_{p1}^{(n)})^2\right)^2} |v_m^{(n)}|^2 + \sum_{m'=1}^{k_2} \frac{(ih_{m'2}^{(n-1)})^2 + (h_{p1}^{(n)})^2}{\left((ih_{m'2}^{(n-1)})^2 - (h_{p1}^{(n)})^2\right)^2} |\alpha_{m'}^{(n)}|^2 ,$$
$$p = 1, \ldots, \frac{k_1 - n - 1}{2} \quad . \tag{3.31}$$

For the remaining modulus squared we obtain

$$\frac{1}{|w_{\frac{k_1-n+1}{2}}^{(n)}|^2} = \sum_{m=1}^{(k_1-n+1)/2} \frac{1}{(h_{m1}^{(n-1)})^2} |v_m^{(n)}|^2 + \sum_{m'=1}^{k_2} \frac{1}{(ih_{m'2}^{(n-1)})^2} |\alpha_{m'}^{(n)}|^2 \quad . \tag{3.32}$$

The moduli squared of the anticommuting coordinates of $b^{(n)}$ fulfil formally a similar equation. However it is mathematically more precise to write it in the inverted form in order to avoid the appearance of purely nilpotent numbers in the denominator.

$$1 = |\beta_p^{(n)}|^2 \left(\sum_{m=1}^{(k_1-n+1)/2} \frac{(h_{m1}^{(n-1)})^2 + (ih_{p2}^{(n)})^2}{\left((h_{m1}^{(n-1)})^2 - (ih_{p2}^{(n)})^2\right)^2} |v_m^{(n)}|^2 + \sum_{m'=1}^{k_2} \frac{(ih_{m'2}^{(n-1)})^2 + (h_{p1}^{(n)})^2}{\left((ih_{m'2}^{(n-1)})^2 - (h_{p1}^{(n)})^2\right)^2} |\alpha_{m'}^{(n)}|^2 \right)$$
$$p = 1, \ldots, k_2. \tag{3.33}$$

The corresponding equations for the odd levels are obtained from Eqs. (3.31) and (3.33) as follows. In Eq. (3.31) the sum over m runs only to $(k_1 - n)/2$ and in addition the term $|v_{\frac{k_1-n}{2}+1}^{(n)}|^2/(h_{p1}^{(n)})^2$ is subtracted. The same happens in Eq. (3.33): the first sum runs only to $(k_1 - n)/2$ and the term $|v_{\frac{k_1-n}{2}+1}^{(n)}|^2/(ih_{p2}^{(n)})^2$ is subtracted. Eq. (3.32) does not exist for the odd levels.

Again the structure of Eqs. (3.31) to (3.33) is very similar to the case of the unitary supergroups. A sketch of its solution is given in Appendix A.1.

3.2.2 Solutions

Up to the k_1–th level both sets of equations (3.17) to (3.19) and (3.31) to (3.33) have to be solved for even and odd levels separately. As already mentioned above, for the even levels there is a one to one correspondence to the unitary case. This allows us to carry over the results for the moduli squared from [GUH3]

$$
|v_p^{(n)}|^2 = \frac{\prod_{q=1}^{\frac{k_1-n-1}{2}} \left((h_{p1}^{(n-1)})^2 - (h_{q1}^{(n)})^2\right) \prod_{q=1}^{k_2} \left((h_{p1}^{(n-1)})^2 - (ih_{q2}^{(n-1)})^2\right)}{\prod_{q=1,q\neq p}^{\frac{k_1-n+1}{2}} \left((h_{p1}^{(n-1)})^2 - (h_{q1}^{(n-1)})^2\right) \prod_{q=1}^{k_2} \left((h_{p1}^{(n-1)})^2 - (ih_{q2}^{(n)})^2\right)}
$$

$$
p = 1, \ldots, (k_1 - n + 1)/2 \quad , \quad (k_1 - n + 1)\ \text{even},
$$

$$
|\alpha_p^{(n)}|^2 = \left((ih_{p2}^{(n)})^2 - (ih_{p2}^{(n-1)})^2\right)
$$

$$
\frac{\prod_{q=1}^{\frac{k_1-n-1}{2}} \left((ih_{p2}^{(n-1)})^2 - (h_{q1}^{(n)})^2\right) \prod_{q=1,q\neq p}^{k_2} \left((ih_{p2}^{(n-1)})^2 - (ih_{q2}^{(n-1)})^2\right)}{\prod_{q=1}^{\frac{k_1-n+1}{2}} \left((ih_{p2}^{(n-1)})^2 - (h_{q1}^{(n-1)})^2\right) \prod_{q=1,q\neq p}^{k_2} \left((ih_{p2}^{(n-1)})^2 - (ih_{q2}^{(n)})^2\right)} ,
$$

$$
p = 1, \ldots, k_2 \quad . \tag{3.34}
$$

We have included the first level by setting $h = h^{(0)}$. In solving Eqs. (3.31) to (3.33) one cannot make direct use of the results of the unitary case, but an explicit calculation, given in Appendix A.1 yields

$$
|w_p^{(n)}|^2 = \frac{-\prod_{m=1}^{\frac{k_1-n+1}{2}} \left((h_{m1}^{(n-1)})^2 - (h_{p1}^{(n)})^2\right) \prod_{q=1}^{k_2} \left((h_{p1}^{(n)})^2 - (ih_{q2}^{(n)})^2\right)}{2(h_{p1}^{(n)})^2 \prod_{q=1,q\neq p}^{\frac{k_1-n-1}{2}} \left((h_{p1}^{(n)})^2 - (h_{q1}^{(n)})^2\right) \prod_{q=1}^{k_2} \left((h_{p1}^{(n)})^2 - (ih_{q2}^{(n-1)})^2\right)}
$$

$$
p = 1, \ldots, (k_1 - n - 1)/2 \quad ,
$$

$$
|w_{\frac{k_1-n+1}{2}}^{(n)}|^2 = \frac{\prod_{m=1}^{\frac{k_1-n+1}{2}} (h_{m1}^{(n-1)})^2 \prod_{q=1}^{k_2} (ih_{q2}^{(n)})^2}{\prod_{m=1}^{\frac{k_1-n-1}{2}} (h_{m1}^{(n)})^2 \prod_{q=1}^{k_2} (ih_{q2}^{(n-1)})^2} \quad ,
$$

$$
|\beta_p^{(n)}|^2 = \left((ih_{p2}^{(n)})^2 - (ih_{p2}^{(n-1)})^2\right)
$$

$$
\frac{\prod_{q=1,q\neq p}^{k_2} \left((ih_{p2}^{(n)})^2 - (ih_{q2}^{(n)})^2\right) \prod_{q=1}^{\frac{k_1-n+1}{2}} \left((ih_{p2}^{(n)})^2 - (h_{q1}^{(n)})^2\right)}{2(ih_{p2}^{(n)})^2 \prod_{q=1}^{\frac{k_1-n-1}{2}} \left((ih_{p2}^{(n)})^2 - (h_{q1}^{(n)})^2\right) \prod_{q=1,q\neq p}^{k_2} \left((ih_{p2}^{(n)})^2 - (ih_{q2}^{(n-1)})^2\right)} . \tag{3.35}
$$

Notice that the squares of the fermionic eigenvalues of the different levels $(ih_{p2}^{(n)})^2$ differ only by a nilpotent number. We took this into account by introducing the nilpotent Gelfand–Tzetlin parameters

$$
|\xi_p^{(n)}|^2 \equiv (ih_{p2}^{(n)})^2 - (ih_{p2}^{(n-1)})^2 \quad . \tag{3.36}
$$

We point out that it is a nontrivial fact, that the difference of the fermionic eigenvalues of two neighbouring levels can be expressed as the modulus squared of one anticommuting number. The solutions for the odd levels, i.e. for $(k_1 - n + 1)$ odd, are stated in Appendix A.2.

A comparison with the corresponding results for the unitary groups $U(k_1/2k_2)$ in Ref. [GUH3] shows, that the above results are in agreement with the following replacement in the Cartan subalgebras of $U((k_1 - n + 1))$ and $U((k_1 - n))$ defined as

$$
\begin{aligned}
h^{(n-1)} &= \mathrm{diag}\,(h_{11}^{(n-1)}, \ldots, h_{(k_1-n+1)1}^{(n-1)}, ih_{12}^{(n-1)}, \ldots, ih_{2k_2 2}^{(n-1)}) \quad , \\
h^{(n)} &= \mathrm{diag}\,(h_{11}^{(n)}, \ldots, h_{(k_1-n)1}^{(n)}, ih_{12}^{(n)}, \ldots, ih_{2k_2 2}^{(n)}) \quad . \qquad (3.37)
\end{aligned}
$$

If one maps $h^{(n-1)}$ and $h^{(n)}$ in Eq. (3.37) onto the Cartan subalgebra of $UOSp((k_1 - n + 1))$ and $UOSp((k_1 - n))$ as

$$
\begin{aligned}
h^{(n-1)} &\rightarrow \mathrm{diag}\,(ih_{11}^{(n-1)}\tau^{(3)}, \ldots, ih_{\frac{k_1-n+1}{2}1}^{(n-1)}\tau^{(3)}, ih_{12}^{(n-1)}\tau^{(3)}, \ldots, ih_{k_2 2}^{(n-1)}\tau^{(3)}) \\
h^{(n)} &\rightarrow \mathrm{diag}\,(ih_{11}^{(n)}\tau^{(3)}, \ldots, ih_{\frac{k_1-n-1}{2}1}^{(n-1)}\tau^{(3)}, 0, ih_{12}^{(n)}\tau^{(3)}, \ldots, ih_{k_2 2}^{(n)}\tau^{(3)}) \quad , \\
& \qquad (k_1 - n + 1) \quad \text{even} \quad . \qquad (3.38)
\end{aligned}
$$

the results of Eqs. (3.34) and (3.35) as well as the results of Appendix A.2 are recovered. We stress that this connection between the unitary supergroup and the unitary orthosymplectic one is natural but not a priori clear. It indicates a special relationship between the groups, which will become more apparent in the Gelfand–Tzetlin pattern given in Section 4.3. All results can be checked by inserting them in the defining equations and making manipulations similar to the ones used in Appendix A.1. Moreover we mention that from the solutions, stated in Eq. (3.34) and (3.35) and Appendix A.2 one derives the corresponding formulae for the group $SO(k_1)$ in ordinary space by setting all anticommuting numbers to zero.

3.2.3 Invariant measure

In the n–th level the invariant measure of $UOSp((k_1 - n + 1)/2k_2)$ decomposes as

$$
d\mu(u_n)\, d\mu(u) \quad , \quad u \in UOSp((k_1 - n)/2k_2) \quad . \qquad (3.39)
$$

To go from the measure of the coset in Cartesian coordinates $d\mu(u_n) = \delta(1 - u_n^\dagger u_n) \prod_i du_{in}$ to the Gelfand–Tzetlin coordinates it is more convenient to evaluate the invariant length $du_n^\dagger du_n$ rather than to calculate the Berezinian directly. For $(k_1 - n + 1)$ even, the invariant length element reads, cf. Eq. (3.14)

$$
du_n^\dagger du_n = \sum_{m=1}^{\frac{k_1-n+1}{2}} \frac{1}{4|v_m^{(n)}|^2} \left(d|v_m^{(n)}|^2 \right)^2 + \sum_{m=1}^{\frac{k_1-n+1}{2}} |v_m^{(n)}|^2 (d\vartheta_m^{(n)})^2 + \sum_{m'=1}^{k_2} d(\alpha_{m'}^{(n)})^* d\alpha_{m'}^{(n)} \; . \qquad (3.40)
$$

It is a highly welcome feature of the Gelfand–Tzetlin coordinates in ordinary space that the metric remains diagonal. This holds also in superspace. After calculating the differentials from Eq. (3.34) and making use of relation (3.30) one arrives at

$$
\begin{aligned}
du_n^\dagger du_n &= \sum_{1=1}^{\frac{k_1-n-1}{2}} \left(\frac{1}{2|w_m^{(n)}|^2} (dh_m^{(n)})^2 + |v_m^{(n)}|^2 (d\vartheta_m^{(n)})^2 \right) + \\
& \sum_{m'=1}^{k_2} \sqrt{\frac{|\alpha_{m'}^{(n)}|^2}{2(ih_{m'2}^{(n)})^2 |\beta_{m'}^{(n)}|^2}}\, d(\xi_{m'}^{(n)})^* d\xi_{m'}^{(n)} \quad . \qquad (3.41)
\end{aligned}
$$

A sketch of the steps leading to Eq. (3.41) is given in Appendix A.3. Conveniently, the metric $g^{(n)}$ is diagonal in the commuting differentials $dh_m^{(n)}$, $d\vartheta_m^{(n)}$ and $d|\xi_{m'}^{(n)}|^2$. The determinant of $g^{(n)}$ can be read off from Eq. (3.41)

$$\begin{aligned}\mathrm{detg}\ g^{(n)} &= \prod_{1=1}^{\frac{k_1-n-1}{2}} \frac{|v_m^{(n)}|^2}{2|w_m^{(n)}|^2} \prod_{m'=1}^{k_2} \frac{8(ih_{m'2}^{(n)})^2|\beta_{m'}^{(n)}|^2}{|\alpha_{m'}^{(n)}|^2} \\ &= 4^{k_2} \frac{\prod_{p=1}^{\frac{k_1-n+1}{2}} \prod_{q=1}^{k_2} \left((h_{p1}^{(n-1)})^2 - (ih_{q2}^{(n-1)})^2\right)^2}{\prod_{p,q\neq p}^{\frac{k_1-n+1}{2}} \left((h_{p1}^{(n-1)})^2 - (h_{q1}^{(n-1)})^2\right) \prod_{p,q\neq p}^{k_2} \left((h_{p2}^{(n-1)})^2 - (h_{q2}^{(n-1)})^2\right)} \\ &\quad \frac{\prod_{p,q\neq p}^{\frac{k_1-n-1}{2}} \left((h_{p1}^{(n)})^2 - (h_{q1}^{(n)})^2\right) \prod_{p,q\neq p}^{k_2} \left((h_{p2}^{(n)})^2 - (h_{q2}^{(n)})^2\right)}{\prod_{p=1}^{\frac{k_1-n-1}{2}} \prod_{q=1}^{k_2} \left((h_{p1}^{(n)})^2 - (ih_{q2}^{(n)})^2\right)^2} \prod_{p}^{\frac{k_1-n-1}{2}} (h_{p1}^{(n)})^2 \quad . \end{aligned} \tag{3.42}$$

The invariant measure is

$$d\mu(u_n) = 2^{k_2} \prod_{p}^{\frac{k_1-n-1}{2}} h_{p1}^{(n)} \frac{B_{\frac{k_1-n-1}{2}k_2}\left((h^{(n)})^2\right)}{B_{\frac{k_1-n+1}{2}k_2}\left((h^{(n-1)})^2\right)} d[h_1^{(n)}]d[\vartheta^{(n)}]d[\xi^{(n)}]$$
$$n \leq k_1, \quad (k_1-n+1) \text{ even,} \tag{3.43}$$

where we introduced the function

$$B_{nm}(s) = \frac{\prod_{p>q}^{n}(h_{p1}-h_{q1}) \prod_{p>q}^{m}(ih_{p2}-ih_{q2})}{\prod_p^n \prod_q^m (h_{p1}-ih_{q2})} \ . \tag{3.44}$$

It can be viewed as the supersymmetric generalization of the Vandermonde determinant. Furthermore we used the notation

$$d[h_1^{(n)}] = \prod_{p=1}^{\frac{k_1-n+1}{2}} dh_{p1}^{(n)} \ , \quad d[\vartheta^{(n)}] = \prod_{p=1}^{\frac{k_1-n+1}{2}} d\vartheta_p^{(n)} \quad \text{and} \quad d[\xi^{(n)}] = \prod_{p=1}^{k_2} d\xi_p^{(n)*} d\xi_p^{(n)} \quad . \tag{3.45}$$

For the odd levels we obtain

$$d\mu(u_n) = 2^{k_2} \frac{\prod_{p=1}^{k_2} ih_{p2}^{(n)} ih_{p2}^{(n-1)}}{\prod_{p=1}^{\frac{k_1-n}{2}} h_{p1}^{(n-1)}} \frac{B_{\frac{k_1-n}{2}k_2}\left((h^{(n)})^2\right)}{B_{\frac{k_1-n}{2}k_2}\left((h^{(n-1)})^2\right)} d[h_1^{(n)}]d[\vartheta^{(n)}]d[\xi^{(n)}]$$
$$n \leq k_1, \quad (k_1-n+1) \text{ odd.} \tag{3.46}$$

Now we can write down the Haar measure of $u \in UOSp(k_1/2k_2)$ in the following factorized form

$$d\mu(u) = 2^{k_1 k_2} \frac{\Delta_{k_2}\left((ih_2^{(k_1)})^2\right)}{B_{\frac{k_1}{2}k_2}\left((h)^2\right)} \prod_{i=1}^{k_1} \prod_{p=1}^{k_2} ih_{p2}^{(i)} d[h_1^{(i)}]d[\vartheta^{(i)}]d[\xi^{(i)}]d\mu(u') \ , \tag{3.47}$$

with $u' \in USp(2k_2)$. The measure of the orthogonal group in ordinary space can be obtained from the invariant length (3.40) by setting all anticommuting variables to zero.

With regard to Eqs. (3.43) and (3.46) we notice a remarkable feature of the Gelfand–Tzetlin coordinates. The measure factorizes in each level. We have

$$d\mu(u_n) \;=\; d\mu(h^{(n)}, h^{(n-1)}) \;=\; \mu(h^{(n-1)}) d\mu(h^{(n)}) \quad . \tag{3.48}$$

In multiplying the coset measures of the different levels parts of the adjactent levels cancel each other. This yields the simple expression (3.47). The Gelfand–Tzetlin coordinates of the unitary group in ordinary and in superspace [GUH3] as well as the ones of the orthogonal group in ordinary space have the corresponding feature.

3.3 The unitary symplectic group

After the k_1-th step of the recursion no more anticommuting variables appear. Now the task is to parametrize the compact group $USp(2k_2)$ in ordinary space. This has a value in its own right. We achieve it by making use of the isomorphism $USp(2k_2) \cong U(k_2;4)$. The parametrization of the group of unitary matrices over the quaternionic field can be performed analogously to the group $U(k_2, 2)$ over the complex field. Therefore we can take advantage of the results in Refs. [GT1, SHA]. We write $U \in U(k_2, q)$ as $U = [U_1, \ldots, U_{k_2}]$, where the normalized vectors U_i have quaternionic coordinates. Since we are dealing now with a group in ordinary space we use capital letters in order to denote vectors. The Cartan subalgebra is of the form $h_2^{(k_1)} = \mathrm{diag}\,(h_{12}^{(k_1)}\tau^{(3)}, \ldots, h_{k_2 2}^{(k_1)}\tau^{(3)})$. Now the Gelfand–Tzetlin equation reads in the first level

$$(1 - U_1 U_1^\dagger) i h_2^{(k_1)} (1 - U_1 U_1^\dagger) E_n^{(1)} = i h_{n2}^{(k_1+1)} E_n^{(1)} \tag{3.49}$$

Since the operator on the left hand side is not Hermitean selfdual, Eq. (3.49) has not a unique solution, see e. g. [MEH2]. Nevertheless, if we multiply Eq. (3.49) on both sides with $1_{k_2} \otimes \tau^{(3)}$ from the right, we get a well defined eigenvalue equation for a selfdual matrix, which is known to have k_2 scalar eigenvalues $\tau^{(3)} i h_{p2}^{(k_1+1)}$ which we also denote by $i h_{p2}^{(k_1+1)}$. Now we can make the same steps as the ones which led to Eq. (3.15) and the equation reduces to the well known Gelfand–Tzetlin equation of the unitary group $U(k_2; c)$ [GT1, SHA].

$$\begin{aligned} 1 &= \sum_{n=1}^{k_2} |U_{n1}|^2 \quad , \\ 0 &= \sum_{m=1}^{k_2} \frac{|U_{m1}|^2}{i h_{m2}^{(k_1)} - i h_{p2}^{(k_1+1)}} \; , \qquad p = 1, \ldots, k_2 - 1 \; . \end{aligned} \tag{3.50}$$

Thus a one to one correspondence is established between the $(k_2 - 1)$ eigenvalues $i h_{p2}^{(1)}$ and the moduli squared of the quaternionic entries

$$|U_{n1}|^2 = \mathrm{Tr}\, U_{j1}^\dagger U_{j1} \quad . \tag{3.51}$$

All formulae derived in [GT1] for the unitary group can now be adopted to the unitary symplectic one.

The single but important difference arises for the Haar measure. If we decompose the entries of U_{n1} as $U_{n1} = |U_{n1}|\widehat{U}_{n1}$, with $\widehat{U}_{n1}$ a unimodular quaternion, the invariant length element is

$$\mathrm{Tr}\, dU_1^\dagger\, dU_1 = \sum_{n=1}^{k_2} \left(|U_{n1}|^2 d\widehat{U}_{n1}^\dagger d\widehat{U}_{n1} + \frac{1}{4|U_{n1}|^2}(d|U_{n1}|^2)^2 \right) \quad . \tag{3.52}$$

Using the parametrization of the unimodular quaternion

$$\widehat{U}_{n1} = \begin{bmatrix} \cos\psi_n^{(1)} \exp(-i\gamma_{n1}^{(1)}) & -\sin\psi_n^{(1)} \exp(i\gamma_{n2}^{(1)}) \\ \sin\psi_n^{(1)} \exp(-i\gamma_{n2}^{(1)}) & \cos\psi_n^{(1)} \exp(i\gamma_{n1}^{(1)}) \end{bmatrix} \quad , \tag{3.53}$$

allows us to write the invariant length as

$$\mathrm{Tr}\, dU_1^\dagger dU_1 = \sum_{n=1}^{N} \left(\frac{1}{4|U_{nN}|^2}(d|U_{nN}|^2)^2 + \sum_{i=1}^{2} |U_{nN}|^2 (d\gamma_n^{(i)})^2 + |U_{nN}|^2 (d\cos\psi_n)^2 \right) \quad . \tag{3.54}$$

Eqs. (3.50) were solved in [GT1]. Using the result

$$|U_{n1}|^2 = \frac{\prod_{m=1}^{k_2-1}(ih_{n2}^{(k_1)} - ih_{m2}^{(k_1+1)})}{\prod_{m\neq n}(ih_{n2}^{(k_1)} - ih_{m2}^{(k_1)})} \tag{3.55}$$

and

$$\sum_{n=1}^{k_2} \frac{1}{4|U_{nN}|^2}(d|U_{nN}|^2)^2 = \sum_{n=1}^{k_2-1} \frac{\prod_{m\neq n}^{k_2-1}(ih_{n2}^{(k_1+1)} - ih_{m2}^{(k_1+1)})}{4\prod_{m=1}^{k_2}(ih_{m2}^{(k_1)} - ih_{n2}^{(k_1+1)})} \left(d(ih_{n2}^{(k_1+1)})\right)^2 \quad , \tag{3.56}$$

we find for the determinant of the metric $g^{(k_1+1)}$ expressed in the new coordinates

$$\det g^{(k_1+1)} = \frac{\Delta_{k_2-1}^2(ih^{(k_1+1)})}{4^{k_2-1}\Delta_{k_2}^6(ih^{(k_1)})} \prod_{n,m}(ih_{n2}^{(k_1)} - ih_{m2}^{(k_1+1)})^2 \,. \tag{3.57}$$

This yields the measure of the coset in the first level

$$d\mu(U_1) = \frac{\Delta_{k_2-1}(ih^{(k_1+1)})}{2^{k_2-1}\Delta_{k_2}^3(ih^{(k_1)})} \prod_{n,m}(ih_{n2}^{(k_1)} - ih_{m2}^{(k_1+1)}) \prod_{n=1}^{k_2} \left(d(ih_{n2}^{(k_1+1)}) d(\cos\psi_n^{(1)}) d\gamma_{n1}^{(1)} d\gamma_{n1}^{(2)} \right) \quad . \tag{3.58}$$

It is quite remarkable that the factorization property (3.48) does not hold for the unitary symplectic group. This is a peculiarity of the Gelfand–Tzetlin parametrization of the unitary–symplectic group.

3.4 Matrix elements

An arbitrary column of the orthosymplectic supermatrix can now be expressed in the Gelfand–Tzetlin parametrization. In the primed basis we have

$$u_p' = \widehat{b}'^{(1)T} \widehat{b}'^{(2)T} \ldots \widehat{b}'^{(n-1)T} u_p'^{(n-1)} \,, \tag{3.59}$$

where $b^{(n)}$ and the scalar products are defined in Eq. (3.29) and Eq. (3.20). Up to now we only have constructed a unitary representation of $UOSp(k_1/2k_2)$. To obtain an orthosymplectic representation we have to assure that when the matrix $u' = \left[u'_1, \ldots, u'_{k_1+k_2}\right]$ is rotated back into the unprimed basis, the vectors u_j, $j \leq k_1$ become real. We discuss only the case $(k_1 - n + 1)$ even. To achieve our goal we recall that so far the vector $b^{(n)}$ entering in the projection matrix in Eq. (3.30) has been determined only up to a phase. There is an ambiguity in choosing the phase of $b^{(n)}$. The Gelfand–Tzetlin coordinates parametrize the vector u_n only up to some phases associated with the action of the Cartan subgroup of $UOSp\left((k_1 - n + 1)/2k_2\right)$. Therefore the projection matrix $\widehat{b}^{(n)}$ is invariant under the action of the Cartan subgroup of $UOSp\left((k_1 - n + 1)/2k_2\right)$ as well. Thus we are allowed to multiply $\widehat{b}^{(n)}$ with an arbitrary element of the Cartan subgroup without changing its projection properties. We set $b^{(n)}_{2p} = ib^{(n)}_{2p-1}$, $p \leq (k_1 - n - 1)/2$, $b^{(n)}_{k_1-n} = i|w_{\frac{k_1-n+1}{2}}|$ in the commuting sector and $b^{(n)}_{k_1-n+1+2p} = -b^{(n)*}_{k_1-n+2p}$, $p = 1, \ldots, k_2$ in the anticommuting one. The remaining phases we fix to be zero. With this choice of phases and after undoing the basis rotation the columns as well as the rows of $\widehat{b}^{(n)T}$ fulfill the reality condition (3.10). Also the vectors $u_n^{(n-1)}$ become real. An explicit form of the real matrices $\widehat{b}^{(n)}$ is given in Appendix A.4.

3.5 Gelfand-Tzetlin pattern

In representation theory the Gelfand Tzetlin scheme is constructed from the following observation [BAR]. An irreducible representation of a Lie group $U(N;\beta)$ is defined by an ordered set of integers or half integers called highest weight. This irreducible representation can be decomposed in irreducible representations of $U(N-1;\beta)$. In the decomposition each irreducible representation of $U(N-1;\beta)$ occurs never or exactly once. Only those irreducible representations appear, whose highest weights satisfy certain betweenness conditions depending on the group under consideration. Following the group chain (3.8) to the end, one has labelled all states of the irreducible representation of $U(N;\beta)$ by a set of integers or half integers, arranged in a Gelfand–Tzetlin pattern.

The analogue for the coordinates is as follows. We consider the adjoint group action on an element h of the Cartan Subalgebra $\mathcal{O}_N = U^\dagger h U$, $U \in U(N;\beta)$. This subset of the complete algebra is usually called *orbit*. We can map the $U(N;\beta)$ orbit labelled by an ordered set of eigenvalues $h_i > h_{i+1}$ onto many different $U(N-1;\beta)$ orbits by projecting $\mathcal{O}_N$ onto a $(N-1)$ dimensional subspace. But only those $U(N-1;\beta)$ orbits $\mathcal{O}_{N-1}$ can be reached, whose eigenvalues interlace two neighboring eigenvalues of $\mathcal{O}_N$. This is the so called minimax principle for selfadjoint operators [DS]. The Gelfand–Tzetlin method uses the eigenvalues of the projected matrix as coordinates of the coset $U(N;\beta)/U(N-1;\beta)$. However, h is a fixed point of the action of the Cartan subgroup $\exp\left(ig_0\right)$, $g_0 \in \mathcal{H}^{(\beta)}_{0,N}$. Therefore the coset $U(N;\beta)/U(N-1;\beta)$ is parametrized by the eigenvalues of $\mathcal{O}_{N-1}$ only up to equivalence classes with respect to the action of the Cartan subgroup of $U(N;\beta)$, parametrized by g_0. In this way the set of parameters describing the coset is split into two parts: One part consists of the eigenvalues of $\mathcal{O}_{N-1}$, the other one of the independent elements of g_0. Guillemin and Sternberg [GS1] introduced the concept of complete integrability by interpreting the entries of h as action and the elements of g_0 as angle coordinates of a generalized mechanical system. This theory applies only to the groups

$U(N;\beta)$, $\beta = 1,2$ but not to the unitary symplectic group. This can be considered as the reason for the relatively complicated expression of the measure for $U(N;4)$.

The Gelfand–Tzetlin pattern can be extracted from the positive definiteness of the moduli squared of the bosonic matrix elements $|v_i^{(n)}|^2$. If one restricts oneself to the subgroup, which consists of the direct product $O(k_1) \otimes USp(2k_2)$ the pattern of the $SO(k_1)$ and $USp(2k_2)$ are rederived which are well known from representation theory [BAR]. Nevertheless we state them here in a somewhat different form, which emphasizes the relation to the pattern of the unitary group $U(k)$. This is the famous triangle [GN]

$$\begin{array}{ccccccccc}
\underline{h_1^{(0)}} & & \underline{h_2^{(0)}} & & \cdots & & \underline{h_k^{(0)}} & & \underline{h_{k+1}^{(0)}} \\
& h_1^{(1)} & & h_2^{(1)} & \cdots & h_{k-2}^{(1)} & & h_k^{(1)} & \\
& & & & \vdots & & & & \\
& & & h_1^{(k-1)} & & h_2^{(k-1)} & & & \\
& & & & h_1^{(k)} & & & &
\end{array} \tag{3.60}$$

with the boundary conditions

$$h_{i+1}^{(j-1)} \leq h_i^{(j)} \leq h_i^{(j-1)} \quad . \tag{3.61}$$

The first row labels the orbit, by means of which the parametrization was performed. We underlined them in order to distinguish them from the coordinates of the group. From this pattern the pattern of the orthogonal group can be derived by the substitution rule (3.38), i. e. by assigning to the Cartan subalgebra of the unitary group $U(k)$ the corresponding one of the orthogonal group $O(k)$. We restrict ourselves to the case of even k. The pattern (3.60) becomes

$$\begin{array}{c}
\underline{h_1^{(0)}} \;\; \underline{h_2^{(0)}} \;\; \cdots \;\; \underline{h_{k+1}^{(0)}} \;\; -\underline{h_{k+1}^{(0)}} \;\; \cdots \;\; -\underline{h_2^{(0)}} \;\; -\underline{h_1^{(0)}} \\
h_1^{(1)} \;\; h_2^{(1)} \;\; \cdots \;\; h_k^{(1)} \;\; 0 \;\; -\underline{h_k^{(1)}} \;\; \cdots \;\; -\underline{h_2^{(1)}} \;\; -\underline{h_1^{(1)}} \\
\cdots \;\; \cdots \;\; \cdots \;\; \cdots \;\; \cdots \;\; \cdots \\
\cdots \;\; \cdots \;\; \cdots \;\; \cdots \;\; \cdots \\
h_1^{(2k-2)} \;\; h_2^{(2k-2)} \;\; -\underline{h_2^{(2k-2)}} \;\; -\underline{h_1^{(2k-2)}} \\
h_1^{(2k-1)} \;\; 0 \;\; -\underline{h_1^{(2k-1)}} \\
h_1^{(2k)} \;\; -\underline{h_1^{(2k)}} \\
0
\end{array} \tag{3.62}$$

with the boundary conditions

$$\begin{array}{ccc}
h_{i+1}^{(j-1)} & \leq h_i^{(j)} \leq & h_i^{(j-1)} \\
|h_j^{(2k-2j+2)}| & \leq & h_j^{(2k-2j+1)}
\end{array} \quad . \tag{3.63}$$

We notice the symmetry along the middle axis. The parameter space of the $O(2k+2)$ is already covered by one half of the triangle. The other half can be neglected. Indeed by restricting ourselves to the left half of the triangle, the patterns appear in their usual form, as they are known from representation theory [GT2]. This symmetry reflects the time reversal invariance of the orthogonal ensembles. By construction the pattern of the unitary symplectic group $USp(2k)$ coincides with the one of the unitary group $U(k)$.

To the two patterns of the $O(k_1) \otimes USp(2k_2)$ subgroup one may add the anticommuting Gelfand–Tzetlin coordinates arranged in a rectangular pattern. In this way one obtains a generalized Gelfand–Tzetlin pattern for the unitary orthosymplectic supergroup $UOSp(k_1/2k_2)$.

$$
\begin{array}{c}
\underline{h_1^{(0)}} \quad \underline{h_2^{(0)}} \quad \cdots \quad \underline{h_{k_1/2}^{(0)}} \quad -\underline{h_{k_1/2}^{(0)}} \quad \cdots \quad -\underline{h_2^{(0)}} \quad -\underline{h_1^{(0)}} \\
h_1^{(1)} \quad h_2^{(1)} \quad \cdots \quad h_{k_1/2-1}^{(1)} \quad 0 \quad -\underline{h_{k_1/2-1}^{(1)}} \quad \cdots \quad -\underline{h_2^{(1)}} \quad -\underline{h_1^{(1)}} \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
h_1^{(k_1-4)} \quad h_2^{(k_1-4)} \quad -\underline{h_2^{(k_1-4)}} \quad -\underline{h_1^{(k_1-4)}} \\
h_1^{(k_1-3)} \quad 0 \quad -\underline{h_1^{(k_1-3)}} \\
h_1^{(k_1-2)} \quad -\underline{h_1^{(k_1-2)}} \\
0 \\
|\xi_1^{(1)}|^2 \quad |\xi_2^{(1)}|^2 \quad \cdots \quad |\xi_{k_2}^{(1)}|^2 \\
|\xi_1^{(2)}|^2 \quad |\xi_2^{(2)}|^2 \quad \cdots \quad |\xi_{k_2}^{(2)}|^2 \\
\vdots \qquad\qquad\qquad \vdots \\
|\xi_1^{(k_1)}|^2 \quad |\xi_2^{(k_1)}|^2 \quad \cdots \quad |\xi_{k_2}^{(k_1)}|^2 \\
h_1^{(k_1)} \qquad h_2^{(k_1)} \qquad \cdots \qquad h_{k_2-1}^{(k_1)} \qquad h_{k_2}^{(k_1)} \\
h_1^{(k_1+1)} \quad h_2^{(k_1+1)} \quad \cdots \quad h_{k_2-2}^{(k_1+1)} \quad h_{k_2-1}^{(k_1+1)} \\
\vdots \\
h_1^{(k_1+k_2-1)} \quad h_2^{(k_1+k_2-1)} \\
h_1^{(k_1+k_2)}
\end{array}
\tag{3.64}
$$

with the boundary conditions

$$
\begin{array}{rcl}
h_{(i+1)1}^{(m-1)} & \leq h_{i1}^{(m)} \leq & h_{i1}^{(m-1)} \\
h_{(i+1)2}^{(k_1+l)} & \leq h_{i2}^{(k_1+l+1)} \leq & h_{i2}^{(k_1+l)} \\
-h_{j1}^{(k_1-2j-1)} & \leq h_{j1}^{(k_1-2j)} \leq & h_{j1}^{(k_1-2j-1)} ,
\end{array}
$$

where $1 \leq j \leq k_1/2-1$, $1 \leq m \leq k_1-2$ and $0 \leq l \leq k_2-1$.

In [GUH5] it was pointed out that the unitary supergroup $U(1/1)$ can be represented by supersymmetric Wigner functions. They are matrix elements of irreducible representations acting on a Hilbert space. These functions are eigenfunctions to a supersymmetric Casimir operator. The eigenvalues are moduli squared of anticommuting numbers. Hence, there exists a representation of the supergroup $U(1/1)$ labeled by an anticommuting variable. Therefore one might give the supersymmetric Gelfand–Tzetlin pattern (3.64) a similar interpretation as in ordinary space. The two triangles label the basis of an irreducible representation of the product $O(k_1) \otimes USp(2k_2)$, whereas the remaining coset $UOSp(k_1/2k_2)/(O(k_1) \otimes USp(2k_2))$ is represented by a block of anticommuting variables. It is an intriguing question, which rôle the anticommuting numbers play in this representation and whether the theory of Guillemin and Sternberg has an analogue for supergroups.

3.6 Summary of Chapter 3 and outlook

We constructed a coordinate system for the unitary orthosymplectic group $UOSp(k_1/2k_2)$. To this end we generalized a method invented by Gelfand for the unitary group in ordinary space and later generalized by Guhr to the unitary supergroup. By projecting an element of the Cartan subalgebra onto a space orthogonal to a column of the unitary orthosymplectic matrix a set of eigenvalue equations is obtained. This set of equations establishes a one to one correspondence between the moduli squared of the elements of the column and the eigenvalues of the projected matrix. In this way the set of coordinates parametrizing the column is split in at one hand the Gelfand–Tzetlin coordinates and on the other hand additional angles. The principal characteristic of the coordinates is their recursive structure. The group was parametrized coset by coset according to the group chain structure of Eq. (3.9). The invariant Haar–measure of the group was derived in the Gelfand–Tzetlin parametrization. The measure of each column factorizes into terms containing coordinates of only one level. This yields a remarkably simple structure of the group measure.

We also constructed paramatrizations of the orthogonal and the unitary symplectic group, which appear as subgroups of the unitary orthosymplectic one. The latter is distinguished from the other groups by the fact, that the measure does not show the factorization property mentioned above.

The Gelfand–Tzetlin coordinates can be arranged in a supersymmetric Gelfand–Tzetlin pattern. An outstanding feature of the pattern is the appearance of moduli squared of anticommuting variables. An interpretation of these anticommuting variables as eigenvalues of a set of invariant operators is likely to exist. It is an iteresting task to clarify this rôle of the anticommuting numbers in representation theory.

The Gelfand–Tzetlin coordinates might also serve as an important tool for the evaluation of group integrals by an explicit parametrization. Specifically they might be useful for the calculation of the generalization of the Harish–Chandra integral to the unitary orthosymplectic group $UOSp(k_1/2k_2)$. This is given by

$$I(g,h) \ = \ \int_{u\in UOSp(k_1/2k_2)} \exp\left(i\mathrm{trg}\, u^{-1}hug\right) d\mu(u) \ , \tag{3.65}$$

where h and g are elements of the Cartan subalgebra. But also the parametrization of the orthogonal group in ordinary space seems to be well suited for a certain class of group integrals in ordinary space. Work in this direction is in progress.

So far the Gelfand–Tzetlin coordinates were only constructed for compact groups. But there is no apparent obstacle to construct them also for non–compact groups. It might be interesting to see if such a construction is indeed possible and how the non–compactness of some parameters is reflected in the corresponding Gelfand–Tzetlin pattern.

Chapter 4

Matrix Bessel functions

The Gelfand–Tzetlin coordinates parametrize the group in terms of the associated algebra. As pointed out in the previous chapter, they are the most natural coordinates for the group manifold. However, the matrix Bessel functions as defined in Eq. (2.25) have two diagonal matrices as argument, which in general do not belong to the algebra. Thus, a connection between the parameters of the group manifold and the arguments of the matrix Bessel function can in general not be established by the Gelfand–Tzetlin coordinates. However, it turns out that we can construct another coordinate system by taking advantage of the recursive structure of the Gelfand–Tzetlin method. This system establishes the desired link between the group manifold and the arguments of the matrix Bessel functions by means of a recursion formula. This formula will be stated and derived in Section 4.3.
Surprisingly this recursion formula has a meaning beyond group theory. It represents an eigenfunction of the radial Laplace operator (2.27) for arbitrary β. This will be proved in Section 4.4. Yet another surprise is that closed expressions can be derived for $\beta = 4$ and strong evidence exists that this is also possible for all even β. This will be discussed in Section 4.5.
In the first two sections we recall some elementary facts of vector and matrix Bessel functions emphasizing the analogies between them.

4.1 Vector Bessel functions revisited

First we compile some well known results for the vector case [GUH6]. In a real, d dimensional space with $d = 2, 3, 4, \ldots$, we consider a position vector $\vec{r} = (x_1, \ldots, x_d)$ and a wave vector $\vec{k} = (k_1, \ldots, k_d)$. The plane wave $\exp\left(i\vec{k}\cdot\vec{r}\right)$ satisfies the wave equation

$$\Delta \exp\left(i\vec{k}\cdot\vec{r}\right) \;=\; -\vec{k}^2 \exp\left(i\vec{k}\cdot\vec{r}\right) \tag{4.1}$$

where we define the Laplacean as in the physics literature,

$$\Delta \;=\; \frac{\partial^2}{\partial \vec{r}^2} \;=\; \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} \;. \tag{4.2}$$

The zero–th order Bessel function in this space is the angular average of the plane wave,

$$\chi^{(d)}(kr) \;=\; \int d\Omega \, \exp\left(i\vec{k}\cdot\vec{r}\right) \;, \tag{4.3}$$

over the solid angle Ω, defining the orientation of either $\vec{r}$ or $\vec{k}$. In our context, it is advantageous to take Ω as the solid angle of $\vec{k}$. Obviously, only the relative angle between $\vec{r}$ and $\vec{k}$ matters and $\chi^{(d)}(kr)$ can only depend on the product of the lengths $r = |\vec{r}|$ and $k = |\vec{k}|$ of the two vectors. We normalize the measure $d\Omega$ with the volume $2\pi^{d/2}/\Gamma(d/2)$ of the unit sphere, i.e. we have

$$\int d\Omega \;=\; 1 \;. \tag{4.4}$$

Thus, by construction, we also have

$$\chi^{(d)}(0) \;=\; 1 \;. \tag{4.5}$$

It is convenient to view $\vec{r}$ as the azimuthal direction of the coordinate system in which we measure Ω. Thus, in these spherical coordinates, one finds $\vec{k} \cdot \vec{r} = kr\cos\vartheta$ where ϑ is the azimuthal angle. The measure $d\Omega$ contains $\sin^{d-2}\vartheta$ and one has

$$\begin{aligned} \chi^{(d)}(kr) &= \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \int_0^{\pi} \exp\left(ikr\cos\vartheta\right) \sin^{d-2}\vartheta d\vartheta \\ &= 2^{(d-2)/2}\Gamma(d/2)\, \frac{J_{(d-2)/2}(kr)}{(kr)^{(d-2)/2}} \end{aligned} \tag{4.6}$$

where $J_\nu(z)$ is the standard Bessel function [AS] of order ν. The functions (4.6) are often referred to as zonal functions.

There is a remarkable difference for the functions $\chi^{(d)}(kr)$ if one compares even and odd dimensions. For example, one has in $d = 2$ dimensions $\chi^{(2)}(kr) = J_0(kr)$ and in $d = 3$ dimensions $\chi^{(3)}(kr) = (\pi/2)^{1/2} J_{1/2}(kr)/(kr)^{1/2} = j_0(kr)$ with the spherical Bessel function $j_0(z)$ of zeroth order [AS]. In $d = 2$ dimensions, $J_0(z)$ is a complicated infinite series in the argument z, in $d = 3$ dimensions, however, $j_0(z)$ is the simple ratio $j_0(z) = \sin z/z$. One easily sees how this generalizes. Upon introducing $\xi = \cos\vartheta$ as integration variable in Eq. (4.6), one finds the representation

$$\chi^{(d)}(kr) \;=\; \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \int_{-1}^{+1} \exp\left(ikr\xi\right) \left(1-\xi^2\right)^{(d-3)/2} d\xi \;. \tag{4.7}$$

In dimensions $d \geq 3$, this can be cast into the form

$$\chi^{(d)}(kr) \;=\; \frac{2\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \sum_{\mu=0}^{\infty} \begin{pmatrix} (d-3)/2 \\ \mu \end{pmatrix} \frac{\partial^{2\mu}}{\partial(kr)^{2\mu}} \frac{\sin kr}{kr} \;. \tag{4.8}$$

For even d, the exponent $(d-3)/2$ is a fraction $-1/2, +1/2, +3/2, \ldots$, and the function $(1-\xi^2)^{(d-3)/2}$ in the integrand in Eq. (4.7) is an *infinite* power series. This yields, for $d = 4, 6, 8, \ldots$, the complicated power series (4.8) involving an infinite number of inverse powers of kr. However, if d is odd, the exponent $(d-3)/2$ is an integer $0, 1, 2, \ldots$, and the function $(1-\xi^2)^{(d-3)/2}$ is a *finite* polynomial of order $(d-3)/2$ in ξ^2. Thus, $\chi^{(d)}(kr)$ acquires a comparatively simple structure, because it only contains a finite number of inverse powers of kr. Formally, this means that for odd d all binomial coefficients for $\mu > (d-3)/2$ are zero.

The differential equation for the functions $\chi^{(d)}(kr)$ is easily obtained by averaging Eq. (4.1) over the solid angle Ω of $\vec{k}$, i.e. by integrating both sides,

$$\Delta \int d\Omega \exp\left(i\vec{k}\cdot\vec{r}\right) \;=\; -\vec{k}^2 \int d\Omega \exp\left(i\vec{k}\cdot\vec{r}\right) \;. \tag{4.9}$$

We notice that the Laplacean Δ commutes with the integral, because the former is in the space of the position vector, the latter in the space of the wave vector. Moreover, the integral trivially commutes with $\vec{k}^2 = k^2$. Hence, one arrives at

$$\Delta_r \chi^{(d)}(kr) \;=\; -k^2\,\chi^{(d)}(kr)\;. \tag{4.10}$$

Since $\chi^{(d)}(kr)$ depends exclusively on radial variables, we replaced the full Laplacean Δ with its radial part

$$\Delta_r \;=\; \frac{1}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1}\frac{\partial}{\partial r} \;=\; \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}\;. \tag{4.11}$$

In general, there are two fundamental solutions $\chi_+^{(d)}(kr)$ and $\chi_-^{(d)}(kr)$ of the differential equation (4.10) which behave as $\exp(\pm ikr)\,/\,(kr)^{(d-1)/2}$ for large arguments kr. Thus, to obtain the full solutions, one can make the Hankel ansatz

$$\chi_\pm^{(d)}(kr) \;=\; \frac{\exp(\pm ikr)}{(kr)^{(d-1)/2}}\,w_\pm^{(d)}(kr)\;. \tag{4.12}$$

Here, $w_\pm^{(d)}(kr)$ is a function with the property $w_\pm^{(d)}(kr) \to 1$ for $kr \to \infty$. The differential equation follows easily from Eq. (4.11) and is given by

$$\left(\frac{\partial^2}{\partial r^2} \pm i2k\frac{\partial}{\partial r} - \frac{d-1}{2}\left(\frac{d-1}{2}-1\right)\frac{1}{r^2}\right) w_\pm^{(d)}(kr) \;=\; 0\;. \tag{4.13}$$

For $d \geq 3$, one uses the ansatz as an asymptotic power series

$$w_\pm^{(d)}(kr) \;=\; \sum_{\mu=0}^{\infty} \frac{a_\mu}{(\pm kr)^\mu} \tag{4.14}$$

which yields a recursion for the coefficients

$$a_{\mu+1} \;=\; \frac{1}{i2(\mu+1)}\left(\mu(\mu+1) - \frac{d-1}{2}\left(\frac{d-1}{2}-1\right)\right) a_\mu\;, \tag{4.15}$$

with the starting value $a_0 = 1$. A special situation occurs when the integer running index μ reaches the critical value $\mu_c = (d-3)/2$. If d is odd, μ_c is integer and the recursion terminates at $\mu = \mu_c$, i.e. one has $a_\mu = 0,\ \mu > \mu_c$. Thus, the asymptotic series becomes a *finite* polynomial in inverse powers of kr. However, if d is even, μ_c is half–odd integer and the series cannot terminate, it is always *infinite*. This explains the different structure of the Bessel functions in even and odd dimensional spaces from the viewpoint of the differential equation.

4.2 Matrix Bessel functions in ordinary space

In this section we compile some more facts about matrix Bessel functions in addition to the properties already stated in Section 2.2. Specifically the matrix Bessel function of the orthogonal group is a widely studied object in the field of multivariate statistics [MUI]. A classical article on the topic of special functions of matrix arguments is by Hertz [HER]. The matrix Bessel function of the unitary symplectic group has attracted so far less attention.

4.2.1 Integral definition and differential equation

As in the case of vector Bessel functions, we start in the matrix case with the plane wave. For two matrices H and K with the same symmetries $H^\dagger = H$ and $K^\dagger = K$, we introduce the matrix plane wave as $\exp(i\mathrm{Tr}\, HK)$ where the trace is the proper scalar product in the matrix space. The matrix plane wave has the property

$$\frac{1}{(2\pi)^N \pi^{\beta N(N-1)/2}} \int d[H] \exp(i\mathrm{Tr}\, HK) \;=\; \delta(K) \tag{4.16}$$

where $\delta(K)$ is the product of the δ distributions of all independent variables. The matrix plane wave is eigenfunction of the Laplacean in matrix space as defined in Eq. (2.15)

$$\Delta \exp(i\mathrm{Tr}\, HK) \;=\; -\mathrm{Tr}\, K^2 \exp(i\mathrm{Tr}\, HK) \ . \tag{4.17}$$

Analogously to vector Bessel functions, we define the matrix Bessel functions as the angular average

$$\Phi_N^{(\beta)}(x,k) \;=\; \int_{U\in U(N;\beta)} d\mu(U) \exp(i\mathrm{Tr}\, HK) \ . \tag{4.18}$$

The diagonal matrix k contains the eigenvalues of K which is diagonalized by a matrix V such that $K = V^\dagger k V$. Due to the invariance of the measure $d\mu(U)$, the matrix V is absorbed and the functions $\Phi_N^{(\beta)}(x,k)$ depend on the eigenvalue x and k only,

$$\Phi_N^{(\beta)}(x,k) \;=\; \int_{U\in U(N;\beta)} d\mu(U) \exp(i\mathrm{Tr}\, U^\dagger x U k) \ . \tag{4.19}$$

Thus, in the scalar product $\mathrm{Tr}\, HK$, solely the relative angles between H and K matter. The matrix Bessel functions are symmetric in the arguments,

$$\Phi_N^{(\beta)}(x,k) \;=\; \Phi_N^{(\beta)}(k,x) \tag{4.20}$$

and normalized to unity,

$$\Phi_N^{(\beta)}(x,0) \;=\; 1 \qquad \text{and} \qquad \Phi_N^{(\beta)}(0,k) \;=\; 1 \ . \tag{4.21}$$

due to Eq. (2.7) As in the vector case, the differential equation is obtained by averaging Eq. (4.17) over the relative angles,

$$\Delta \int_{V\in U(N;\beta)} d\mu(V) \exp(i\mathrm{Tr}\, HK) \;=\; -\mathrm{Tr}\, K^2 \int_{V\in U(N;\beta)} d\mu(V) \exp(i\mathrm{Tr}\, HK) \ . \tag{4.22}$$

Again, the Laplacean Δ commutes with the integral, because the former is in the space of the matrix H, the latter over the diagonalizing matrix V of K. The integral also commutes with $\mathrm{Tr}\, K^2 = \mathrm{Tr}\, k^2$. Due to the symmetry between H and K, the integral is obviously identical to the definition (4.19) and we find

$$\Delta_x \Phi_N^{(\beta)}(x,k) \;=\; -\mathrm{Tr}\, k^2\, \Phi_N^{(\beta)}(x,k) \ . \tag{4.23}$$

Since the matrix Bessel function $\Phi_N^{(\beta)}(x,k)$ depends only on the radial variables, the full Laplacean reduces to its radial part Δ_x as stated in Eq. (2.27). We notice that these steps are fully parallel to the corresponding discussion in Sec. 4.1. Importantly, due to the symmetry (4.20), the functions $\Phi_N^{(\beta)}(x,k)$ must also solve the differential equation in the k_n, $n=1,\ldots,N$ which results from Eq. (4.23) by exchanging x and k. Obviously, this a very restrictive requirement.

Comparing the radial operator (4.11) in the vector case and the radial operator (2.27), we see that it is β that corresponds to the *spatial* dimension d or, more precisely, to $d-1$. The rôle played by the *matrix* dimension N is a different one. To illustrate this, we study the simplest non–trivial case $N=2$. We find straightforwardly from the differential equation (4.23)

$$\Phi_2^{(\beta)}(x,k) \;=\; \exp\left(i\frac{(x_1+x_2)(k_1+k_2)}{2}\right)\chi^{(\beta+1)}\left(\frac{(x_1-x_2)(k_1-k_2)}{2}\right) \tag{4.24}$$

where $\chi^{(d)}$ is the vector Bessel function in d dimensions as defined in Eq. (4.3). This functions appears in the solution, because the differences x_1-x_2 and k_1-k_2 directly correspond to the lengths $|\vec{r}|$ and $|\vec{k}|$. In higher matrix dimensions N, this simple correspondence is lost. However, we will see in great detail that the features of the functions $\Phi_N^{(\beta)}(x,k)$, in particular whether or not explicit solutions can be constructed, are stronger influenced by β than by N.

Another point in this context deserves to be underlined. In the vector case, the differential equation (4.10) and the solution (4.6) were constructed for integer dimensions d. However, both equations are also well defined for any real and positive d. Similarly, we observe in the matrix case that the differential equation (4.23) was derived for the cases $\beta=1,2,4$. However, neither itself nor its solution (4.24) for $N=2$ are confined to these cases $\beta=1,2,4$, they are valid for any real and positive β. Thus, the cases $\beta=1,2,4$ which correspond to a matrix model, i.e. to the defining integral (4.18) of the matrix Bessel functions, are only special cases of a much more general problem, namely finding the solutions of the differential equation (4.23) for every integer N and for *arbitrary* real values of β. We will return to this in Sec. 4.4.

4.2.2 Fourier–Bessel analysis

One can do a Fourier–analysis on the curved space of the eigenvalues of x,k [HC2]. We write the Fourier transform of a function $f(H)$ as

$$F(K) \;=\; D_N^{(\beta)}\int d[H]\exp\left(i\mathrm{Tr}\,HK\right)f(H) \tag{4.25}$$

where the matrices H and K have the same symmetries. If we choose a symmetric normalization,

$$D_N^{(\beta)} \;=\; \frac{1}{(2\pi)^{N/2}\pi^{\beta N(N-1)/4}}\ , \tag{4.26}$$

we can write the inverse transform as

$$f(H) \;=\; D_N^{(\beta)}\int d[K]\exp\left(-i\mathrm{Tr}\,KH\right)F(K)\ . \tag{4.27}$$

We notice that, according to Eq. (4.16), the Fourier transform of the constant $D_N^{(\beta)}$ is the δ distribution $\delta(K)$ and vice versa.

If f is an invariant function such that $f(H) = f(x)$, its Fourier transform turns out to be invariant as well, $F(K) = F(k)$. Introducing eigenvalue–angle coordinates, we easily find

$$F(k) \;=\; D_N^{(\beta)} C_N^{(\beta)} \int d[x]\, |\Delta_N(x)|^\beta\, \Phi_N^{(\beta)}(x,k)\, f(x) \tag{4.28}$$

for the Fourier transform and

$$f(x) \;=\; D_N^{(\beta)} C_N^{(\beta)} \int d[k]\, |\Delta_N(k)|^\beta\, \Phi_N^{(\beta)*}(k,x)\, F(k) \tag{4.29}$$

for its inverse. We now insert the transform (4.28) into the inverse (4.29) and conclude that

$$\begin{aligned}\left(D_N^{(\beta)} C_N^{(\beta)}\right)^2 \int d[k]\, |\Delta_N(k)|^\beta\, \Phi_N^{(\beta)}(x,k)\, \Phi_N^{(\beta)*}(k,y) \\ = \frac{\det[\delta(x_n - y_m)]_{n,m=1,\ldots,N}}{|\Delta_N(x)\Delta_N(y)|^{\beta/2}} \ .\end{aligned} \tag{4.30}$$

These are the orthogonality and completeness relations given in Section 2.2.2.

4.2.3 Alternative integral representation

We now give a useful integral representation for the matrix Bessel functions $\Phi_N^{(\beta)}(x,k)$ in the cases $\beta = 1,2,4$. Although we can hardly believe that this representation is completely new, we could not find it in the literature. We can write

$$\Phi_N^{(\beta)}(x,k) \;=\; A_N^{(\beta)} \int d[T] \exp\left(i \mathrm{Tr}\, T\right)\, \mathrm{Det}^{\,-\beta/2}\left(x \otimes \hat{k} - T \otimes 1_N\right) \tag{4.31}$$

where 1_N is the $N \times N$ unit matrix. The normalization constant is given by

$$A_N^{(\beta)} \;=\; \frac{i^{\beta N^2/2} \pi^{\beta N(N-1)/4}}{\beta^{N+\beta N(N-1)/2}} \prod_{n=1}^{N} \Gamma(\beta n/2) \ . \tag{4.32}$$

The matrix T in Eq. (4.31) is real symmetric, Hermitean or Hermitean self–dual, respectively, for $\beta = 1,2,4$. The measure $d[T]$ is Cartesian and given by Eq. (2.4). All independent variables in T are integrated over the entire real axis. To ensure convergence, the diagonal elements of T have to be given a proper imaginary increment. We notice that x and T are $N \times N$ matrices for $\beta = 1,2$ and $2N \times 2N$ for $\beta = 4$ with twofold degenerated eigenvalues. The matrix $\hat{k}$ is, in all three cases of β, just the $N \times N$ matrix $\hat{k} = \mathrm{diag}\,(k_1, k_2, \ldots, k_N)$, as defined in Section 2.1. The integral representation (4.31) becomes especially convenient, if either x or $\hat{k}$ is highly degenerate. For definiteness consider the case that x takes only two different values

$$x \;=\; \mathrm{diag}\,(x_2 \otimes 1_{N-M}, x_1 \otimes 1_N) \qquad , \quad M < N \quad . \tag{4.33}$$

The integral representation (4.31) becomes

$$\begin{aligned}\Phi_N^{(\beta)}(x,k) = B_N^{(\beta)} (x_2 - x_1)^{-MN\beta/2+M} \exp\left(i\, x_1\, \mathrm{Tr}\, k\right) \\ \int d[t] \exp\left((x_2 - x_1) \sum_{n=1}^{M} t\right) |\Delta_M(t)|^\beta \prod_{i=1}^{N} \prod_{j=1}^{M} (k_i - t_j)^{-\beta/2} \quad ,\end{aligned} \tag{4.34}$$

where $d[t] = \prod_{n=1}^{M} dt_n$. Thus it reduces to an M–fold integral over the real axis. Again the t_n have an imaginary increment, which ensures convergence.

In the general case the integral representation (4.31) leads to an integral equation for the matrix Bessel functions,

$$\Phi_N^{(\beta)}(x,k) \;=\; B_N^{(\beta)}\,\mathrm{Det}^{\,1-\beta/2}x \int d[t]|\Delta_N(t)|^\beta \frac{\Phi_N^{(\beta)}(x,t)}{\prod_{n,m}^{N}(k_m-t_n)^{\beta/2}}\;, \tag{4.35}$$

where the normalization constant reads

$$B_N^{(\beta)} \;=\; \frac{i^{\beta N^2/2}\Gamma^N(\beta/2)}{(2\pi)^N N!}\;. \tag{4.36}$$

The t_n in Eq. (4.35) have a proper imaginary increment and their domain of integration is the real axis. Due to the symmetry relation (4.20), the variables x and k can be interchanged in Eqs. (4.31) and (4.35). A derivation of these results is given in Appendix B.1.

4.3 Recursion formula in ordinary space

The matrix Bessel functions, defined in Eq. (4.19),

$$\Phi_N^{(\beta)}(x,k) \;=\; \int d\mu(U)\,\exp(i\mathrm{Tr}\,U^\dagger xUk) \tag{4.37}$$

depend on the radial space of the eigenvalues x and k. We emphasize that the radial spaces do not lie in the manifolds covered by the groups $U(N;\beta)$. However, we will show that the group integral (4.37) can be exactly mapped onto a recursive structure which acts exclusively in the radial space. This remarkable feature is the main result of this section.

The matrix Bessel functions $\Phi_N^{(\beta)}(x,k)$ can be calculated iteratively by means of

Theorem 4.1 (Recursion Formula) *Let $\Phi_N^{(\beta)}(x,k)$ be defined as in Eq. (4.37), then it can be written as*

$$\Phi_N^{(\beta)}(x,k) \;=\; \int d\mu(x,x')\,\exp\left(i(\mathrm{Tr}\,x-\mathrm{Tr}\,x')k_N\right)\,\Phi_{N-1}^{(\beta)}(x',\widetilde{k}) \tag{4.38}$$

where $\Phi_{N-1}^{(\beta)}(x',\widetilde{k})$ is the group integral (4.37) over $U(N-1;\beta)$. We have introduced the diagonal matrix $\widetilde{k} = \mathrm{diag}\,(k_1,\ldots,k_{N-1})$ for $\beta = 1,2$ and $\widetilde{k} = \mathrm{diag}\,(k_1,k_1,\ldots,k_{N-1},k_{N-1})$ for $\beta = 4$ such that $k = \mathrm{diag}\,(\widetilde{k},k_N)$ for $\beta = 1,2$ and $k = \mathrm{diag}\,(\widetilde{k},k_N,k_N)$ for $\beta = 4$. The invariant measure is given by

$$\begin{aligned} d\mu(x,x') \;&=\; G_N^{(\beta)}\frac{\Delta_{N-1}(x')}{\Delta_N^{\beta-1}(x)}\prod_{n,m}|x_n-x'_m|^{(\beta-2)/2}\,d[x'] \quad, \\ G_N^{(\beta)} \;&=\; \frac{2^{N-1}\Gamma(N\beta/2)}{\Gamma^N(\beta/2)} \quad. \end{aligned} \tag{4.39}$$

The domain of integration is compact and given by

$$x_n \;\geq\; x'_n \;\geq\; x_{n+1}\;, \qquad n = 1,\ldots,(N-1)\;. \tag{4.40}$$

This formula and its derivation are due to Guhr [GK]. Importantly, the $N-1$ integration variables $x'_n,\ n=1,\ldots,N-1$, ordered in the diagonal matrix $x'=\operatorname{diag}(x'_1,\ldots,x'_{N-1})$ for $\beta=1,2$ and $x'=\operatorname{diag}(x'_1,x'_1,\ldots,x'_{N-1},x'_{N-1})$ for $\beta=4$ are arguments of $\Phi^{(\beta)}_{N-1}(x',\widetilde{k})$. Moreover, we notice that their further appearance in the exponential is a simple one due to the trace.

The coordinates x' are constructed in the spirit of, but in general they are different from, the Gelfand–Tzetlin coordinates as discussed in Chapter 3. To clearly distinguish these two sets of coordinates from each other, we refer to the latter as *angular* Gelfand–Tzetlin coordinates and to the variables x' as *radial* Gelfand–Tzetlin coordinates. The difference is at first sight minor, but of crucial importance. In the angular case, x is in the Cartan subalgebra belonging to $U(N;\beta)$. In the radial case, however, x is in the *radial* space of the eigenvalues of the real–symmetric, Hermitean or Hermitean self–dual matrix H, which are the arguments of the functions (4.37). While the angular Gelfand–Tzetlin coordinates never leave the group space, the radial ones establish an exact and unique relation between the group and the radial space. The radial Gelfand–Tzetlin coordinates re–parametrize the sphere that is described by the N^{th} column U_N of the matrix $U\in U(N;\beta)$. The recursion formula (4.38) can only be constructed in the radial coordinates x', but not in the angular ones. The radial and the angular Gelfand–Tzetlin coordinates are, in general, different. They happen to coincide for $\beta=2$, i.e. for the unitary group $U(N)$. Remarkably for the unitary-symplectic group the radial Gelfand–Tzetlin coordinates coincide with the angular ones as well. Indeed in Section 3.3 we mapped the angular Gelfand-Tzetlin eigenvalue equation onto a radial eigenvalue equation to obtain a unique solution.

The invariant measure $d\mu(x,x')$ is, apart from phase angles, the invariant measure $d\mu(U_N)$ on the sphere in question, expressed in the radial coordinates x'. It only contains algebraic functions. The domain of integration reflects a "betweenness condition" for the radial Gelfand–Tzetlin coordinates. It coincides with the "betweenness" condition of the unitary group as depicted in the pattern (3.60). This is why the Vandermonde determinants in the measure (4.39) do not have an absolute value sign.

The general recursion formula (4.38) states an iterative way for constructing the matrix Bessel function $\Phi^{(\beta)}_N(x,k)$ for arbitrary N from the matrix Bessel function $\Phi^{(\beta)}_2(x,k)$ for $N=2$ which can usually be obtained trivially. We remark that the recursion formula allows one to express the matrix Bessel functions in the form

$$
\begin{aligned}
\Phi^{(\beta)}_N(x,k) \quad &= \quad \int \prod_{n=1}^{N-1} d\mu(x^{(n-1)},x^{(n)}) \\
&\qquad \exp\Big(i(\operatorname{Tr}x^{(n-1)}-\operatorname{Tr}x^{(n)})k_{N-n+1}\Big)\,\exp\Big(ix_1^{(N-1)}k_1\Big) \qquad (4.41)
\end{aligned}
$$

where we have introduced the radial Gelfand–Tzetlin coordinates $x^{(n)}_m,\ m=1,\ldots,N-n$ on $N-1$ levels $n=1,\ldots,(N-1)$. We define $x^{(0)}=x$ and $x^{(1)}=x'$.

4.3.1 Derivation

We introduce a matrix $V=\operatorname{diag}(\widetilde{V},V_0)$ with $\widetilde{V}\in U(N-1;\beta)$ and $V_0\in U(1;\beta)$ such that $V\in U(N-1;\beta)\otimes U(1;\beta)\subset U(N;\beta)$ and multiply the right hand side of the definition (4.37) with

$$1 = \int d\mu(V) = \int d\mu(V_0) \int d\mu(\widetilde{V}) . \tag{4.42}$$

The invariance of the Haar measure $d\mu(U)$ allows us to replace U with $UV^\dagger$ and to write

$$\Phi_N^{(\beta)}(x,k) = \int d\mu(V_0) \int d\mu(\widetilde{V}) \int d\mu(U) \exp(i\mathrm{Tr}\, U^\dagger x U V^\dagger k V) . \tag{4.43}$$

We collect the first $N-1$ columns U_n of U in the $N \times (N-1)$ rectangular matrix B such that $B = [U_1\, U_2 \cdots U_{N-1}]$ and $U = [B\, U_N]$. We notice that

$$\begin{aligned} B^\dagger B &= 1_{N-1} \quad , \\ BB^\dagger &= \sum_{n=1}^{N-1} U_n U_n^\dagger = 1_N - U_N U_N^\dagger . \end{aligned} \tag{4.44}$$

As already stated in Sec. 2.1, the elements of a vector or a matrix are scalar for $\beta = 1, 2$ and quaternion for $\beta = 4$. In this sense, we also write 1_N as the unit matrix for $\beta = 4$ because its elements are $\tau^{(0)}$. By defining the $(N-1) \times (N-1)$ square matrices $\widetilde{H} = B^\dagger x B$ and $\widetilde{K} = \widetilde{V}^\dagger \widetilde{k} \widetilde{V}$ we may rewrite the trace in Eq. (4.43) as

$$\mathrm{Tr}\, U^\dagger x U V^\dagger k V = \mathrm{Tr}\, \widetilde{H}\widetilde{K} + H_{NN} k_N \tag{4.45}$$

with $H_{NN} = U_N^\dagger x U_N$ according to Eq. (2.8). We notice that V_0 has dropped out. Since the first term of the right hand side of Eq. (4.45) depends only on the first $(N-1)$ columns U_n collected in B and the second term depends only on U_N, we use the decomposition

$$d\mu(U) = d\mu(B)\, d\mu(U_N) \tag{4.46}$$

of the measure to cast Eq. (4.43) into the form

$$\Phi_N^{(\beta)}(x,k) = \int d\mu(U_N) \exp(iH_{NN}k_N) \int d\mu(\widetilde{V}) \int d\mu(B) \exp(i\mathrm{Tr}\, \widetilde{H}\widetilde{K}) \tag{4.47}$$

where we have already done the trivial integration over V_0.

The difficulty to overcome lies in the decomposition (4.46). While $d\mu(U_N)$ is simply the invariant measure on the sphere described by U_N, the measure $d\mu(B)$ is rather complicated. Pictorially speaking, the degrees of freedom in $d\mu(B)$ have always to know that they are locally orthogonal to U_N. Thus, $d\mu(B)$ depends on U_N. Luckily, there is one distinct set of coordinates that is perfectly suited to this situation. It is the system of the radial Gelfand–Tzetlin coordinates. We construct it by applying the methods of Chapter 3. The $N \times N$ matrix $(1_N - U_N U_N^\dagger)$ is a projector onto the $(N-1) \times (N-1)$ space obtained from the original $N \times N$ space by slicing off the vector U_N. We now project the *radial coordinates* x onto this space and study the spectrum. The defining equation reads

$$(1_N - U_N U_N^\dagger)\, x\, (1_N - U_N U_N^\dagger)\, E_n' = x_n' E_n' \qquad n = 1, \ldots, N-1 . \tag{4.48}$$

By the steps which led to Eq. (3.15) this equation can be cast into the form

$$0 = -x_n' \,\mathrm{Det}\, (x - x_n')\ \mathrm{Tr}\, U_N^\dagger \frac{1_N}{x - x_n'} U_N . \tag{4.49}$$

Together with the normalization $\mathrm{Tr}\, U_N^\dagger U_N = 1$, this yields the N equations

$$\begin{aligned} 1 &= \mathrm{Tr}\, U_N^\dagger U_N = \sum_{n=1}^{N} \sum_{\alpha=0}^{\beta-1} U_{nN}^{(\alpha)2} \\ 0 &= \mathrm{Tr}\, U_N^\dagger \frac{1_N}{x - x'_n} U_N = \sum_{m=1}^{N} \sum_{\alpha=0}^{\beta-1} \frac{U_{mN}^{(\alpha)2}}{x_m - x'_n} , \qquad n = 1, \ldots, N-1 . \end{aligned} \tag{4.50}$$

In these formulae, the trace Tr is only needed in the symplectic case. We notice that the equations for the variables x' depend on the variables x as parameters. We emphasize once more that x in these equations is in the radial space and, in general, not in the Cartan subalgebra of $U(N;\beta)$.

At this point, it is not clear yet why the introduction of the radial Gelfand–Tzetlin coordinates is at all helpful. The great advantage will reveal itself when we express the matrix $\widetilde{H}$ and the matrix element H_{NN} in the trace (4.47) in these coordinates. To this end, we first multiply Eq. (4.48) from the right with $E_n'^\dagger$ and sum over n,

$$(1_N - U_N U_N^\dagger)\, x\, (1_N - U_N U_N^\dagger) = \sum_{n=1}^{N-1} x'_n\, E'_n E_n'^\dagger \tag{4.51}$$

where we used the completeness relation

$$\sum_{n=1}^{N-1} E'_n E_n'^\dagger + U_N U_N^\dagger = 1_N . \tag{4.52}$$

Taking the trace of the spectral expansion (4.51) we find immediately

$$\mathrm{Tr}\, x - \mathrm{Tr}\, x' = \mathrm{Tr}\, U_N^\dagger x U_N = H_{NN} . \tag{4.53}$$

This is a remarkably simple result. An analogous expression exists for the NN matrix element of the unitary group in the theory of angular Gelfand–Tzetlin coordinates for the unitary group [GT1, SHA]. Here we have shown that Eq. (4.53) is a general feature in every radial space.

We now turn to the $(N-1) \times (N-1)$ matrix $\widetilde{H}$. Its $N-1$ eigenvalues y_n, $n = 1, \ldots, N-1$ are determined by the characteristic equation

$$\begin{aligned} 0 = \mathrm{Det}\left(\widetilde{H} - y_n\right) &= \mathrm{Det}\left(B^\dagger x B - y_n\right) \\ &= -\frac{1}{y_n}\,\mathrm{Det}\left(B B^\dagger x - y_n\right) \\ &= -\frac{1}{y_n}\,\mathrm{Det}\left((1_N - U_N U_N^\dagger)x - y_n\right) \end{aligned} \tag{4.54}$$

where we used Eq. (4.44) and re–expressed a $(N-1)\times(N-1)$ determinant as a $N \times N$ determinant. The comparison of Eq. (4.54) with Eq. (4.49) shows that, most advantageously, we have $y_n \equiv x'_n$, $n = 1, \ldots, N-1$. Thus we may write

$$\widetilde{H} = B^\dagger x B = \widetilde{U}^\dagger x' \widetilde{U} \tag{4.55}$$

by introducing the $(N-1) \times (N-1)$ *square* matrix $\widetilde{U}$ which diagonalizes $\widetilde{H}$. Obviously, $\widetilde{U}$ must be a complicated function of the $N \times (N-1)$ *rectangular* matrix B, i.e. of the columns U_n, $n = 1, \ldots, N-1$. However, all we need to know is that $\widetilde{U}$ must be in the group $U(N-1;\beta)$ because, by construction, $\widetilde{H}$ has the symmetry $\widetilde{H}^\dagger = \widetilde{H}$.

Collecting everything, we cast Eq. (4.47) into the form

$$\begin{aligned}\Phi_N^{(\beta)}(x,k) &= \int d\mu(x,x') \exp\left(i(\mathrm{Tr}\, x - \mathrm{Tr}\, x')k_N\right) \\ &\qquad \int d\mu(\widetilde{V}) \int d\mu(B) \exp(i\mathrm{Tr}\, \widetilde{U}^\dagger x' \widetilde{U} \widetilde{V}^\dagger \widetilde{k} \widetilde{V}) \ . \end{aligned} \tag{4.56}$$

We may now use the invariance of the Haar measure $d\mu(\widetilde{V})$ to absorb $\widetilde{U}$ such that

$$\begin{aligned}\Phi_N^{(\beta)}(x,k) &= \int d\mu(x,x') \exp\left(i(\mathrm{Tr}\, x - \mathrm{Tr}\, x')k_N\right) \\ &\qquad \int d\mu(\widetilde{V}) \exp(i\mathrm{Tr}\, x' \widetilde{V}^\dagger \widetilde{k} \widetilde{V}) \int d\mu(B) \ . \end{aligned} \tag{4.57}$$

Thus, the integration over B is trivial and yields unity due to our normalization. The remaining integration over $\widetilde{V}$ gives precisely the matrix Bessel function $\Phi_{N-1}^{(\beta)}(x', \widetilde{k})$. This completes the derivation of the first part of theorem 4.1. The introduction of the matrix $V = \mathrm{diag}\,(\widetilde{V}, V_0)$ was not strictly necessary. Alternatively, one could have shown that the measure $d\mu(B)$ can be identified with $d\mu(\widetilde{U})$ and have done the corresponding integral. However, the introduction of V makes this part of the derivation more transparent.

For $\beta = 4$ the measure entering the recursion formula has already been derived in Section 3.3. For $\beta = 1, 2$ the derivation is analogous. The normalization constant is obtained from results in Gilmore's book [GIL]. It ensures normalization to unity according to Eq. (2.7). The three cases are summarized in Eq. (4.39).

4.4 Radial functions for arbitrary β

We now turn to the question stated at the end of Section 2.2.2. We saw that for the values $\beta = 1, 2, 4$ the matrix Bessel functions can be considered as integral solutions of the differential equation

$$\Delta_x \Phi_N^{(\beta)}(x,k) = -\sum_{n=1}^{N} k_n^2\, \Phi_N^{(\beta)}(x,k) \quad , \quad \Phi_N^{(\beta)}(0,k) \ = \ 1 \ . \tag{4.58}$$

The operator

$$\Delta_x \ = \ \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + \sum_{n<m} \frac{\beta}{x_n - x_m}\left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial x_m}\right) \tag{4.59}$$

is – through the adjunction operation (2.41) – related to the Hamiltonian H_D defined in Eq. (2.41). The definition of matrix Bessel functions as a group integral (4.19) confines β to the values $\beta = 1, 2, 4$. Discussing the simplest case $N = 2$, we already saw in Sec. 4.2.1 that $\Phi_2^{(\beta)}$ is well defined for *arbitrary* values of β. This was a simple consequence of the explicit form (4.24) which expresses $\Phi_2^{(\beta)}$ in terms of the Bessel function $\chi^{(\beta+1)}$. The latter is known to be well defined for arbitrary β. It seems natural that this carries over to arbitrary N.

We now pose the question. Do integral solutions of Eq. (4.58) exist, which generalize the matrix Bessel functions to arbitrary β?

They should have the same symmetries as the matrix Bessel functions, i. e. they should also be solutions of the differential equation

$$\Delta_k \Phi_N^{(\beta)}(x,k) = -\sum_{n=1}^{N} x_n^2\, \Phi_N^{(\beta)}(x,k) \quad , \quad \Phi_N^{(\beta)}(x,0) \;=\; 1 \; . \tag{4.60}$$

Moreover we require that the solutions are symmetric in the argument

$$\Phi_N^{(\beta)}(x,k) \;=\; \Phi_N^{(\beta)}(k,x) \quad . \tag{4.61}$$

In the sequel, we want to refer to the functions $\Phi_N^{(\beta)}(x,k)$ for arbitrary β as *radial functions* while we reserve the term *matrix Bessel functions* to the cases $\beta = 1,2,4$ where the direct connection to matrices and groups exists.

4.4.1 Recursive solution

The recursion formula (4.38) was derived using group theoretical methods. Nevertheless the recursion integral itself does not refer to any group properties. However it depends on the parameter β analytically. Therefore the integral (4.38) is also meaningful for arbitrary $\beta > 0$. The statement of the following theorem is that this integral is also a solution of the differential equations (4.58) and (4.60).

Theorem 4.2 *Let $\Phi_{N-1}^{(\beta)}(x',\widetilde{k})$ be the solution of the differential equation (4.59) for $N-1$, then an integral solution of Eq. (4.59) for N is given by*

$$\Phi_N^{(\beta)}(x,k) \;=\; \int d\mu(x,x')\, \exp\left(i\left(\sum_{n=1}^{N} x - \sum_{n=1}^{N-1} x'\right) k_N\right)\, \Phi_{N-1}^{(\beta)}(x',\widetilde{k}) \quad , \tag{4.62}$$

where x' is the set of integration variables x'_n, $n = 1,\ldots,(N-1)$. The integration measure

$$d\mu(x,x') \;=\; G_N^{(\beta)}\, \frac{\Delta_{N-1}(x')}{\Delta_N^{\beta-1}(x)} \prod_{n,m} |x_n - x'_m|^{(\beta-2)/2}\, d[x'] \; . \tag{4.63}$$

is the continuation of Eq. (4.39) to arbitrary β. The normalization constant

$$G_N^{(\beta)} \;=\; 2^{N-1}\frac{\Gamma(N\beta/2)}{\Gamma^N(\beta/2)} \tag{4.64}$$

is the continuation of the constant in Eq. (4.39). As in the cases $\beta = 1,2,4$, the inequalities

$$x_n \;\geq\; x'_n \;\geq\; x_{n+1} \; , \qquad n = 1,\ldots,(N-1) \tag{4.65}$$

define the domain of integration.

The keystone for the proof is the following

Proposition 4.1 *Let $f(x')$ be a symmetric and analytic function in the $(N-1)$ variables x_i, $i = 1, \ldots, (N-1)$, further let Δ_x be defined as in Eq. (4.59) then we have*

$$\Delta_x \int d\mu(x,x') \exp\left(i\left(\sum_{n=1}^{N} x_n - \sum_{n=1}^{N-1} x'_n\right) k_N\right) f(x') = \tag{4.66}$$

$$-k_N^2 \int d\mu(x,x') \exp\left(i\left(\sum_{n=1}^{N} x_n - \sum_{n=1}^{N-1} x'_n\right) k_N\right) f(x') +$$

$$\int d\mu(x,x') \exp\left(i\left(\sum_{n=1}^{N} x_n - \sum_{n=1}^{N-1} x'_n\right) k_N\right) \Delta_{x'} f(x') \tag{4.67}$$

where the measure $d\mu(x,x')$ is given in Eq. (4.63). The integration domain is defined by the inequalities (4.65).

A sketch of the proof is given in Appendix B.2. Setting in Eq. (4.66)

$$f(x') = \Phi_{N-1}^{(\beta)}(x', \widetilde{k}) \tag{4.68}$$

yields

$$\Delta_x \Phi_N^{(\beta)}(x,k) = -k_N^2 \Phi_N^{(\beta)}(x,k)$$

$$+ \int d\mu(x,x') \exp\left(i\left(\sum_{n=1}^{N} x_n - \sum_{n=1}^{N-1} x'_n\right) k_N\right) \Delta_{x'} \Phi_{N-1}^{(\beta)}(x', \widetilde{k}) \ , \tag{4.69}$$

This equation establishes a not immediately obvious, but nevertheless natural connection between, on the one hand, the action of the Laplacean Δ_x in the N variables x_n on the radial function in N dimensions, i.e. on the recursion integral (4.62), and, on the other hand, the recursion integral over the Laplacean $\Delta_{x'}$ in the $N-1$ variables x'_n acting on the radial function in $N-1$ dimensions. There is a compensation term which is just $-k_N^2 \Phi_N^{(\beta)}(x,k)$. Thus, we can prove the eigenvalue equation (4.58) by induction: assuming that it is correct for $N-1$, identity (4.69) implies Eq. (4.58) for N. The induction starts with $N = 2$ where the eigenvalue equation (4.58) is clearly valid for arbitrary β as shown in Sec. 4.2.1 by deriving the explicit solution (4.24). The symmetry relation (4.61) is non–trivial. In the matrix cases $\beta = 1, 2, 4$, it is obvious from the integral definitions (4.18) and (4.19). For arbitrary β, we cannot use this argument, we only have the recursion (4.62). In App. B.3, we prove the symmetry relation (4.61) by an explicit change of variables and derive the normalization constant $G_N^{(\beta)}$. This completes the proof. ‡

Eq. (4.69) is just one example of a relation between operators acting on $\Phi_N^{(\beta)}(x,k)$ and operators acting under the integral on $\Phi_N^{(\beta)}(x', \tilde{k})$. It turns out that these relations are crucial in deriving explicit results from the recursion formula. This holds especially in the supersymmetric case to be discussed later. We mention another examples

$$\sum_{m=1}^{N} \frac{\partial}{\partial x_m} \Phi_N^{(\beta)}(x,k) \quad = \quad ik_N \, \Phi_N^{(\beta)}(x,k) \, + \int d\mu(x,x') \exp\left(i\left(\sum_{n=1}^{N} x_n - \sum_{n=1}^{N-1} x'_n\right) k_N\right)$$

$$\left(\sum_{i=1}^{N-1} \frac{\partial}{\partial x'_m}\right) \Phi_{N-1}^{(\beta)}(x', \widetilde{k}) \quad . \tag{4.70}$$

Thus the radial function is an eigenfunction not only of Δ_x but we also have

$$\sum_{m=1}^{N} \frac{\partial}{\partial x_m} \Phi_N^{(\beta)}(x,k) = i \sum_{m=1}^{N} k_m \Phi_N^{(\beta)}(x,k) \tag{4.71}$$

and more generally

$$\left(\sum_{m=1}^{N} \frac{\partial}{\partial x_i}\right)^K \Phi_N^{(\beta)}(x,k) = \left(i \sum_{m=1}^{N} k_m\right)^K \Phi_N^{(\beta)}(x,k) \quad . \tag{4.72}$$

The proof of Eq. (4.70) is along the same lines as the proof of Proposition 4.1.

Finally we comment on the definition domain of β. We have seen in Section 4.2.1 that for $N = 2$ the matrix Bessel function is well defined for arbitrary complex β. This should also be true for our recursion formula (4.62). However, for $\beta \leq 0$ non–integrable singularities arise at the boundaries in the integral in Eq. (4.62). At the same time the normalization constant becomes zero for $\beta = 0, -2, -4, \ldots$ compensating the singularities of the integral. This makes the recursion formula for $\beta \leq 0$ not ill–defined but it gets more difficult to treat. Therefore we restricted us in the discussion to positive values of β.

4.4.2 Hankel ansatz

In the spirit of Eq. (4.12) for the vector case, we make a Hankel ansatz for our radial functions for arbitrary β. Since the sum over the k_n^2 on the right hand side of the eigenvalue equation (4.60) is invariant under all permutations of the k_n or, equivalently, their indices n, we can label a set of solutions $\Phi_{N,\omega}^{(\beta)}(x,k)$ by an element ω of the permutation group $\mathcal{S}^N$ of N objects. For these solutions, we make the ansatz

$$\Phi_{N,\omega}^{(\beta)}(x,k) = \frac{\exp\left(i \sum_{n=1}^{N} x_n k_{\omega(n)}\right)}{|\Delta_N(x)\Delta_N(k)|^{\beta/2}} W_{N,\omega}^{(\beta)}(x,k) \ , \tag{4.73}$$

where $\omega(k)$ is the diagonal matrix constructed from k by permuting the k_n, or the indices n. The normalized full solution $\Phi_N^{(\beta)}(x,k)$ satisfying the constraint (4.61), is then given as the linear combination

$$\Phi_N^{(\beta)}(x,k) = \frac{\text{const}}{N!} \sum_{\omega \in \mathcal{S}^N} (-1)^{\pi(\omega)} \Phi_{N,\omega}^{(\beta)}(x,k) \tag{4.74}$$

of the functions (4.73). Here, $\pi(\omega)$ is the parity of the permutation.

We find for the function $W_{N,\omega}^{(\beta)}(x,k)$ the differential equation

$$L_{x,\omega(k)} W_{N,\omega}^{(\beta)}(x,k) = 0 \tag{4.75}$$

where the operator is given by

$$L_{x,\omega(k)} = \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + i2 \sum_{n=1}^{N} k_{\omega(n)} \frac{\partial}{\partial x_n} - \beta\left(\frac{\beta}{2} - 1\right) \sum_{n<m} \frac{1}{(x_n - x_m)^2} \ . \tag{4.76}$$

This differential equation generalizes Eq. (4.13) to the matrix case for $\beta = 1, 2, 4$ and, furthermore, the latter to general radial function for arbitrary β.

Again, due to the symmetry (4.61), the differential equation (4.75) must also hold if x and $\omega(k)$ are interchanged. It is the last term of the operator $L_{x,\omega(k)}$ that makes the differential equation (4.75) so difficult. This shows that the case $\beta = 2$ corresponding to unitary matrices $U \in U(N)$ is special: the last term vanishes and we simply have $W_{N,\omega}^{(2)}(x, k) = 1$. For arbitrary β, it is obvious from the differential operator that $W_{N,\omega}^{(\beta)}(x, k) \to 1$ if $|x_n - x_m| \to \infty$ for all pairs $n < m$. Once more, this must also be true if $|k_n - k_m| \to \infty$. Thus, we expect that $W_{N,\omega}^{(\beta)}(x, k)$ is some kind of asymptotic series, generalizing Eq. (4.14) in the vector case.

Hence, we conclude that the leading contribution in an asymptotic expansion of the functions (4.73) is given by

$$\Phi_{N,\omega}^{(\beta)}(x, k) \ \sim \ \frac{\exp\left(i \sum_{n=1}^{N} x_n k_{\omega(n)}\right)}{|\Delta_N(x)\Delta_N(k)|^{\beta/2}} \ . \tag{4.77}$$

According to Eq. (4.74), this means that

$$\Phi_N^{(\beta)}(x, k) \ \sim \ \frac{\det[\exp(ix_n k_m)]_{n,m=1,\ldots,N}}{\Delta_N^{\beta/2}(x)\Delta_N^{\beta/2}(k)} \tag{4.78}$$

is the asymptotic behavior of the radial functions $\Phi_N^{(\beta)}(x, k)$ if the differences $|x_n - x_m|$ and $|k_n - k_m|$ are large for all pairs $n < m$. This generalizes the well known asymptotic behavior of the matrix Bessel functions, cf. Eq. (2.28) to arbitrary β.

The functions $W_{N,\omega}^{(\beta)}(x, k)$ are translation invariant, i.e. they depend only on the differences $(x_n - x_m)$. We show this in Appendix. B.5. Due to the symmetry, this argument carries also over to k and $W_{N,\omega}^{(\beta)}(x, k)$ depends only on the differences $(k_n - k_m)$ as well. Moreover, the symmetry implies that it depends only on the products $(k_{\omega(n)} - k_{\omega(m)})(x_n - x_m)$.

Collecting all these pieces of information, we make the ansatz

$$W_{N,\omega}^{(\beta)}(x, k) \ = \ \sum_{\{\mu\}} \frac{a_{\mu_{12}\mu_{13}\cdots\mu_{(N-1)N}}}{\prod_{n<m}\Big((k_{\omega(n)} - k_{\omega(m)})(x_n - x_m)\Big)^{\mu_{nm}}} \tag{4.79}$$

with coefficients $a_{\mu_{12}\mu_{13}\cdots\mu_{(N-1)N}}$ that depend on $N(N-1)/2$ integer indices μ_{nm}, as many as there are differences. The summation is over the set of these indices. The presence of the k_n makes it very difficult to solve Eq. (4.75) with the ansatz (4.79). In the vector case, one easily sees that the differential equation (4.13) in r can be transformed into an equation in the dimensionless variables kr such that k does not appear anymore. This leads to the simple recursion (4.15) for the coefficients. Here, in the matrix case, the k_n cannot easily be absorbed and the recursion formulae for the coefficients will depend on the k_n in a non–trivial way. However, in some simple cases, it is possible to solve them.

4.5 Integrals over the unitary-symplectic group

The difficulties in extracting more explicit expressions from the recursion formula 4.38 are due to the measure (4.39). We notice that in the case that β is not even, it becomes a non–polynomial expression in the integration variables x'_n. Particularly for $\beta = 1$, i. e. for the integration over the orthogonal group we have

$$\begin{aligned}\Phi_N^{(1)}(x,k) &= G_N^{(1)} \int \left(\frac{\Delta_{N-1}(x')}{\Delta_N^{\beta-1}(x)} \frac{1}{\sqrt{-\prod_{n,m}(x_n - x'_m)}} \right. \\ &\qquad \left. \exp\left(i \left(\sum_{n=1}^{N} x - \sum_{n=1}^{N-1} x' \right) k_N \right) \Phi_{N-1}^{(1)}(x', \tilde{k}) \right) d[x'] \ . \end{aligned} \tag{4.80}$$

We notice that annoying square roots appear. They – in general – inhibit a further evaluation. To our knowledge the only generic case which allows for an evaluation in a closed form is the smallest non–trivial case $N = 2$, cf. Eq. (4.24). In general only an infinite series in a special set of symmetric functions called *zonal functions* is available [MUI]. Things change completely for even $\beta > 0$. In this case the measure is a polynomial in the integration variables and – in principle – all integrations can be performed in a closed form. This reflects the correspondence of the matrix Bessel functions for even β to the vector Bessel functions in odd dimensions and *vice versa* as stated in Section 4.2.1.

Although all integral involved are elementary, up to now it has only been possible for $N = 3$ to evaluate the matrix Bessel function for the unitary symplectic group with the recursion formula (4.38). We consider now

$$\Phi_N^{(4)}(-ix,k) = \int_{U \in USp(2N)} \exp\left(\mathrm{Tr}\, u^{-1} x u k \right) d\mu(U) \quad , \tag{4.81}$$

x and k are diagonal matrices with Kramers degeneracy.

$$x = \mathrm{diag}\,(x_1 1_2, \ldots, x_N 1_2) \quad , \quad k = \mathrm{diag}\,(k_1 1_2, \ldots, k_N 1_2) \quad . \tag{4.82}$$

Starting point of the recursion is the smallest non–trivial case $N = 2$. We obtain after an elementary calculation

$$\Phi_2^{(4)}(-ix,k) = G_2^{(4)} \sum_{\omega \in \mathcal{S}^2} \left(\frac{1}{\Delta_2^2(x)\, \Delta_2^2(\omega(k))} - \frac{2}{\Delta_2^3(x)\, \Delta_2^3(\omega(k))} \right) \exp(\mathrm{Tr}\, x\omega(k)) \quad . \tag{4.83}$$

The sum runs over all elements of the permutation group $\mathcal{S}^N$ and the volume of the group is normalized to unity. After inserting Eq. (4.83) into the recursion formula, we find for $USp(6)$, the next step in the recursion,

$$\begin{aligned}\Phi_3^{(4)}(-ix,k) &= G_3^{(4)} G_2^{(4)} \sum_{\omega \in \mathcal{S}^2} \int_{x_2}^{x_1} dx'_1 \int_{x_3}^{x_2} dx'_2 \frac{\prod_{i=1}^{3} \prod_{j=1}^{2} (x_i - x'_j)}{\Delta_3^3(x) \Delta_2^2(\omega(\tilde{k}))} \\ &\qquad \exp\left((\mathrm{Tr}\, x - \mathrm{Tr}\, x') k_3 + \mathrm{Tr}\, x\omega(\tilde{k}) \right) \left(\frac{1}{\Delta_2(x')} - \frac{2}{\Delta_2^2(x') \Delta_2(\omega(\tilde{k}))} \right) \quad . \end{aligned} \tag{4.84}$$

Although the integrand is finite everywhere, inclusively in $x'_1 = x'_2 = x_2$, the denominators $\Delta_2(x')$ and $\Delta_2^2(x')$ raise a technical difficulty. The key to remove them is an integration by parts in the second term of the round bracket in Eq. (4.84). By writing

$$\frac{2}{\Delta_2^2(x')} = - \left(\frac{\partial}{\partial x'_1} - \frac{\partial}{\partial x'_2} \right) \frac{1}{\Delta_2(x')} \tag{4.85}$$

and observing that the product $\prod_{i=1}^{3}\prod_{j=1}^{2}(x_i - x'_j)$ annihilates all boundary terms, we can perform an integration by parts and arrive at

$$\Phi_3^{(4)}(-ix,k) = G_3^{(4)}G_2^{(4)} \sum_{\omega\in\mathcal{S}^2} \frac{1}{\Delta_3^3(x)\Delta_2^3(\omega(\tilde{k}))}$$
$$\int_{x_2}^{x_1} dx'_1 \int_{x_3}^{x_2} dx'_2 \sum_{i=1}^{3}\prod_{\substack{j=1\\ j\neq i}}^{3}(x_j - x'_1)(x_j - x'_2)\ \exp\left((\mathrm{Tr}\, x - \mathrm{Tr}\, x')k_3 + \mathrm{Tr}\, x\omega(\tilde{k})\right)\ , \tag{4.86}$$

where no denominator is left. Due to the permutation symmetry of the original integral, in the further evaluation of Eq. (4.86) we can restrict ourselves to the unity element e of the permutation group. Thus we need only consider the limits $x'_i \to x_i$, $i = 1,2$ at performing a series of integration by parts. After collecting orders in k we find

$$\begin{aligned}\Phi_{3,e}^{(4)}(-ix,k) \quad = \quad & G_3^{(4)}G_2^{(4)}\frac{1}{\Delta_3^3(x)\Delta_3^3(k)}\left(-\Delta_3(x)\Delta_3(k) + 2\sum_{i<j}^{3}\left(\frac{\Delta_3(x)\Delta_3(k)}{(x_i - x_j)(k_i - k_j)}\right) - \right.\\ & \left. 4\sum_{i<j}^{3}(x_i - x_j)(k_i - k_j) + 12\right)\exp\left(\mathrm{Tr}\, xk\right) \quad .\end{aligned} \tag{4.87}$$

By introducing the composite variable

$$z_{\omega(ij)} = (x_i - x_j)(k_{\omega(i)} - k_{\omega(j)}) \quad i,j = 1,\ldots,3, \quad \omega\in\mathcal{S}^3, \tag{4.88}$$

we can express $\Phi_3^{(4)}(-ix,k)$ compactly as

$$\Phi_3^{(4)}(-ix,k) = G_3^{(4)}G_2^{(4)} \sum_{\omega\in\mathcal{S}^3} \frac{1}{\Delta_3^3(x)\Delta_3^3(\omega(k))}\left(4 + \prod_{i<j}^{3}(2 - z_{\omega(ij)})\right)\exp\left(\mathrm{Tr}\, x\omega(k)\right)\ . \tag{4.89}$$

Crucial in the derivation was the identity (4.85). Regrettably, it is not clear, how to generalize it to higher matrix dimension. One must find an operator, which annihilates through integrations by parts all terms containing integration variables in the denominator. An alternative way of constructing the polynomial part of $\Phi_N^{(\beta)}$ for arbitrary β is the Hankel ansatz described in the previous section. In Appendix B.4 we derive $\Phi_4^{(4)}(-ix,k)$ in this way. Up to a normalization constant it is given by

$$\begin{aligned}\Phi_4^{(4)}(-ix,k) = \sum_{\omega\in\mathcal{S}^4}\frac{1}{\Delta_4^3(x)\Delta_4^3(\omega(k))}&\left(\prod_{i<j}(2 - z_{\omega(ij)}) + \right.\\ \frac{1}{2}\sum_{l<m<n}\prod_{\substack{i<j\\ \neq lm\\ \neq ln\\ \neq mn}}(2 - z_{\omega(ij)}) + \frac{1}{4}\sum_{\substack{l<m\\ k<n}}&\left.\prod_{\substack{i<j,\neq lk,\neq ln\\ \neq mk,\neq mn,\neq kn}}(2 - z_{\omega(ij)})\right)\exp\left(\mathrm{Tr}\, x\omega(k)\right)\ .\end{aligned} \tag{4.90}$$

Comparing this result with Eq. (4.89) we notice that the composite variables z_{ij} enter only linearly in the polynomial part of $\Phi_N^{(4)}(-ix,k)$, $N = 2,3,4$. This is what we expected.

Also the spherical Bessel function $j_1(z)$, the counterpart of $\Phi_N^{(4)}(x,k)$ in the vector case, cf. Eq. (4.6) and (4.24) has a polynomial part linear in z. We conjecture that this analogy between the spherical Bessel function $j_{(\beta/2-1)}(z)$ and the matrix Bessel functions $\Phi_N^{(\beta)}(x,k)$ holds for all N and all even β.

4.6 Summary of Chapter 4 and outlook

We derived an integral representation for matrix Bessel functions by means of a special coordinate system. This coordinate system is in general related to but different from the Gelfand–Tzetlin coordinates as constructed in the previous section. We applied the Gelfand–Tzetlin method of projecting onto a subspace orthogonal to a column of the unitary, orthogonal or unitary–symplectic matrix. This time we did not project an element of the algebra but we projected one of the two diagonal matrices appearing as arguments of the matrix Bessel function $\Phi_N^{(\beta)}(x,k)$. This lead to a recursion formula for $\Phi_N^{(\beta)}(x,k)$. It establishes a connection between $\Phi_N^{(\beta)}(x,k)$ and $\Phi_{N-1}^{(\beta)}(x,k)$ through a $(N-1)$–fold integral. The measure entering in the integral is an algebraic function. One can say that these *radial Gelfand–Tzetlin* coordinates are not the natural coordinates of the *group*, but they are the natural coordinates of the matrix Bessel functions.

Though derived by group theoretical methods the recursion formula does not refer to any group properties at all. Instead, the parameter β appears in a natural way as continuous parameter. This happens in a way similar to the index ν of the Bessel functions J_ν in vector analysis. By analytic continuation of β in the recursion formula we obtained solutions of a Calogero–Sutherland type Hamiltonian H_D, as defined in Eq. (2.41) for arbitrary β. Importantly these solutions have the same symmetries as the matrix Bessel functions. It is an intriguing problem to find a geometric and group theoretical interpretation of the recursion integral for arbitrary β. We conjecture, that the parameter β is related to the deformation parameter of quantum groups.

For generic β a further evaluation of the recursion formula seems not to be possible due to the complexity of the measure. An exception are even integers. In this case the measure becomes particularly simple and the integral can in principle be performed in a closed form. We derived closed expressions for the smallest non–trivial even integer $\beta = 4$ and for $N \leq 4$. It was crucial in the evaluation to find appropriate operators acting under the integral which transformed the originally rational function in the integration variables into a product of an exponential and a polynomial. This, in turn, was elementary. A generalization of these results to arbitrary matrix dimension N remains as a challenging task. Work in this direction is in progress.

The recursion formula can be regarded as an integral solution of the Hamiltonian H_D which describes a scattering system of N particles on the line. In the Calogero–Sutherland model the particles are enclosed on the circle. It is yet another interesting question whether an integral solution in the spirit of our recursion formula exists also for this case.

Chapter 5

Supersymmetric matrix Bessel functions

In this chapter we treat the matrix Bessel function of the unitary orthosymplectic group $UOSp(k_1/2k_2)$ with the supersymmetric generalization of the recursion formula. The integration over supergroups involves integrals over anticommuting variables. These can in principle always be performed. However, for groups of higher dimension the vast amount of anticommuting variables is a serious obstacle for the supersymmetric method. It turns out that the supersymmetric recursion formula is an appropriate tool in this case. The $UOSp(k_1/2k_2)$ contains the orthogonal group $O(k_1)$ as subgroup. As we have seen the matrix Bessel function of the orthogonal group can in general not be expressed in a closed form. But all other integrals can be performed yielding rather compact expressions.

In the introductory section we compile some facts of supersymmetric matrix Bessel functions in addition to those already given in Section 2.3. In the next section we state the supersymmetric version of the recursion formula and sketch its derivation. The latter is not essentially different from the one in ordinary space.

In the next two sections we derive explicit expressions for the supersymmetric matrix Bessel functions of the groups $UOSp(2/2)$ and $UOSp(4/4)$, which are, as outlined in the introductory chapter, important in matrix models.

5.1 Matrix Bessel functions in superspace

Many of the properties of matrix Bessel functions in ordinary space carry over to superspace. For the reasons stated in Sections 2.3 and 3.1, we restrict ourselves to the discussion of the supersymmetric matrix Bessel function of the unitary orthosymplectic group $UOSp(k_1/2k_2)$. They are defined as

$$\Phi_{k_1 2k_2}(r,s) = \int_{u\in UOSp(k_1/2k_2)} \exp\left(i\mathrm{trg}\, u^{-1}sur\right) d\mu(u) \ . \tag{5.1}$$

The diagonal supermatrices $r = \mathrm{diag}\,(\sqrt{c}r_1, \sqrt{-c}r_2)$ and $s = \mathrm{diag}\,(\sqrt{c}s_1, \sqrt{-c}s_2)$ a twofold degenerate in either the fermion–fermion or the boson–boson block

$$r_1 = \mathrm{diag}\,(r_{k_1 1}, r_{(k_1-1)1}, \ldots, r_{11}) \quad , \quad r_2 = \mathrm{diag}\,(r_{12}1_2, \ldots, r_{k_2 2}1_2) \tag{5.2}$$

$$s_1 = \mathrm{diag}\,(s_{k_1 1}, s_{(k_1-1)1} \ldots, s_{11}) \quad , \quad s_2 = \mathrm{diag}\,(s_{12}1_2, \ldots, s_{k_2 2}1_2)\ . \tag{5.3}$$

We recall that the non–trivial part of the k-point diffusion kernel defined in Eqs. (2.66) and (2.67) belongs to this class of functions. They are eigenfunctions of the radial part of the Laplace operator acting in the space of "real" Hermitean matrices as defined in Eqs. (2.53) and (2.54). In the definition (2.53) of the"real" Hermitean matrices a parameter c enters, which yields for $c = 1$ the real symmetric and for $c = -1$ the Hermitean selfdual matrix as boson–boson block. The eigenfunctions of the Laplace operator in the space of "real" Hermitean supermatrices Δ_σ, defined in Eqs. (2.60) are the plane waves

$$\Delta_\sigma \exp(i\mathrm{trg}\,\sigma\rho) = -\mathrm{trg}\,\rho^2 \exp(i\mathrm{trg}\,\sigma\rho) \quad . \tag{5.4}$$

As in Eq. (4.19) the matrix Bessel functions are obtained by averaging over the angular coordinates, i. e. over the diagonalizing group . The Laplacian commutes with the average. Thus we arrive at the differential equation for $\Phi_{k_1 2k_2}(r, s)$

$$\Delta_s \Phi_{k_1 2k_2}(r, s) = -\mathrm{trg}\, r^2 \Phi_{k_1 2k_2}(r, s) \qquad , \tag{5.5}$$

where we have defined the radial part of Δ_σ generalizing Eq. (2.69)

$$\Delta_s = \frac{1}{\widetilde{B}^{(c)}_{k_1 k_2}(s)} \left(\sum_{i=1}^{k_1} \frac{\partial}{\partial s_{i1}} \widetilde{B}^{(c)}_{k_1 k_2}(s) \frac{\partial}{\partial s_{i1}} + \frac{1}{2} \sum_{i=1}^{k_2} \frac{\partial}{\partial s_{i2}} \widetilde{B}^{(c)}_{k_1 k_2}(s) \frac{\partial}{\partial s_{i2}} \right) \quad . \tag{5.6}$$

The Berezinian is given by [GUH4]

$$\widetilde{B}^{(1)}_{k_1 k_2}(s) = \frac{|\Delta_{k_1}(s_1)| \Delta^4_{k_2}(is_2)}{\prod_{i=1}^{k_1} \prod_{j=1}^{k_2} (s_{i1} - is_{j2})^2} \quad , \quad \widetilde{B}^{(-1)}_{k_1 k_2}(s) = \frac{|\Delta_{k_1}(is_1)| \Delta^4_{k_2}(s_2)}{\prod_{i=1}^{k_1} \prod_{j=1}^{k_2} (is_{i1} - s_{j2})^2}\ . \tag{5.7}$$

Notice that Δ_s depends on c only by a factor $\sqrt{c}$. Thus to simplify the notation we set $c = 1$ and omit the index c. With regard to the initial condition there is a comment in order. The initial conditions (4.21) do not carry over to the supersymmetric case. The reason comes from the fact that the volume of some supergroups is zero [BER] resulting in the vanishing of $\Phi_{k_1 k_2}(0, s)$ for certain values of k_1 and k_2. This is in discrepancy to the normalization of the plane waves (5.4) to unity at the origin. The reason of this contradiction was already discussed at the end of Section 2.3: in going from Cartesian to angle eigenvalue coordinates one has to add additional terms to the measure to preserve the symmetries of the original integral.

Thus a normalization problem arises. To solve it, we use the following strategy. First we evaluate the matrix Bessel functions without taking care of the normalization. We just add a normalization constant $\widehat{G}_{k_1 2k_2}$ to the integral definition. We determine its value afterwards by comparing the asymptotics of the matrix Bessel function for large arguments with the Gaussian integral. The latter yields normalized δ functions for large arguments.

5.2 Radial Gelfand–Tzetlin coordinates for orthosymplectic groups

A recursion formula like Theorem 4.1 also exists in superspace.

Theorem 5.1 (Supersymmetric recursion formula) *Let $\Phi_{k_1 2k_2}(s,r)$ be defined as a group integral as in Eq. (5.1). It has two diagonal matrices defined as in Eq. (5.3) as argument. Then it can be written as*

$$\Phi_{k_1 2k_2}(s,r) \ = \widehat{G}_{k_1 2k_2} \int d\mu(s,s') \exp\left(i(\operatorname{trg} s - \operatorname{trg} s')r_{k_1 1}\right) \Phi_{(k_1-1)2k_2}(s',\widetilde{r}) \quad , \tag{5.8}$$

where $\Phi^{(\beta)}_{(k_1-1)2k_2}(s',\widetilde{r})$ is the group integral (5.1) over UOSp $((k_1-1)/2k_2))$ and $\widehat{G}_{k_1 2k_2}$ is a normalization constant. We have introduced the diagonal matrix

$$\widetilde{r} = \operatorname{diag}\left(r_{(k_1-1)1},\ldots,r_{11},ir_2\right) = \operatorname{diag}\left(\widetilde{r}_1,i\widetilde{r}_2\right) \tag{5.9}$$

such that $r = \operatorname{diag}(r_{k_1 1},\widetilde{r})$ and the diagonal matrix

$$s' = \operatorname{diag}\left(s'_{(k_1-1)1},\ldots,s'_{11},is'_2\right) = \operatorname{diag}\left(s'_1,is'_2\right) \quad . \tag{5.10}$$

The invariant measure is given by

$$\begin{aligned}
d\mu(s,s') &= 2^{k_2+1}\mu_B(s_1,s'_1)\mu_F(s_2,s'_2)\mu_{BF}(s,s')d[\xi']d[s'_1] \\
\mu_B(s_1,s'_1) &= \frac{\Delta_{k_1}(s'_1)}{\sqrt{-\prod_{p=1}^{k_1}\prod_{q=1}^{k_1-1}(s_{p1}-s'_{q1})}} \\
\mu_F(s_2,s'_2) &= \frac{\Delta^4_{k_2}(is'_2)}{\prod_{p=1}^{k_2}\prod_{q=1}^{k_2}(is_{p2}-is'_{q2})^2} \\
\mu_{BF}(s,s') &= \frac{\prod_{p=1}^{k_1}\prod_{l=1}^{k_2}\prod_{q=1}^{k_1-1}(is'_{l2}-s_{p1})(is_{l2}-s'_{q1})}{\prod_{p=1}^{k_1-1}\prod_{l=1}^{k_2}(is'_{l2}-s'_{p1})^2} \ .
\end{aligned} \tag{5.11}$$

We define the differentials

$$d[\xi'] = \prod_{p=1}^{k_2} d\xi'^*_p d\xi'_p \quad , \quad d[s'_1] = \prod_{n=1}^{k_1-1} ds'_{n1} \ . \tag{5.12}$$

The domain of integration for the bosonic variables is compact and given by

$$s_{n1} \ \geq \ s'_{n1} \ \geq \ s_{(n+1)1} \ , \qquad n = 1,\ldots,(k_1-1) \ . \tag{5.13}$$

The fermionic variables is'_{p2} are related to Grassmann variables ξ'_p and ξ'^*_p through

$$|\xi'_p|^2 \ = is'_{p2} - is_{p2} \quad . \tag{5.14}$$

Notice the difference of definition (5.14) to the corresponding one (3.36) for the *angular* coordinates. We splitted the Jacobian in three parts. One of them, $\mu_B(s_1,s'_1)$, depends only on bosonic eigenvalues and one, $\mu_F(s_1,s'_1)$, only on fermionic eigenvalues, i. e. only on Grassmann variables. The third part mixes commuting and anticommuting integration variables. A comparison of the measure (5.11) with the measure in the angular Gelfand–Tzetlin coordinates (3.43), (3.46) and (3.47) reveals the difference between the two parametrizations. In the angular Gelfand–Tzetlin coordinates the measure factorizes according to Eq. (3.48). Upon multiplication, parts of the measures of adjacent levels cancel each other. In the radial coordinates the measure is a product of differences of all possible combinations of eigenvalues of the two matrices s and s'.

As in ordinary space, cf. Eq. (4.41) we can write

$$\Phi_{k_1 2k_2}(s,r) = \int \prod_{n=1}^{k_1-1} d\mu(s^{(n-1)}, s^{(n)})$$
$$\exp\Big(i(\mathrm{trg}\, s^{(n-1)} - \mathrm{trg}\, s^{(n)}) r_{(k_1-n+1)1}\Big) \exp\Big(i s_{11}^{(k_1-1)} r_{11}\Big) \Phi_{k_2}^{(4)}(-i2 s_2^{(k_1-1)}, r_2)\ . \quad (5.15)$$

We have set $s = s^{(0)}$ and $s' = s^{(1)}$. The product of the measures is more complicated than the corresponding expression in the *angular* Gelfand–Tzetlin coordinates in Eq. (3.48). No cancelations take place between the adjacent levels. But the radial Gelfand–Tzetlin coordinates have the valuable property that the Grassmann variables only appear as moduli squared in the integrand. Thus, the number of integrals over anticommuting variables is reduced by the half. Moreover the exponential is a simple function of the integration variables. This is another important advantage of the radial Gelfand–Tzetlin coordinates. We stress that the radial Gelfand–Tzetlin coordinates are the natural coordinates of the matrix Bessel functions, because the coordinates emerge from the expression itself and they are not introduced by hand.

5.2.1 Derivation

All crucial steps needed for the derivation of the supersymmetric recursion formula (5.8) carry over from the ordinary recursion formula (4.38) in Sec. 4.3. We order the columns of the matrix $u \in UOSp(k_1/2k_2)$ in the form $u = [u_{k_1}\, u_{k_1-1} \cdots u_1\, u_{k_1+1} \cdots u_{k_1+k_2}]$. We also introduce a rectangular matrix $b = [u_{k_1-1} \cdots u_1\, u_{k_1+1} \cdots u_{k_1+k_2}]$ such that $u = [u_{k_1}\, b]$. Analogously to the ordinary case, we have

$$\begin{aligned} b^\dagger b &= 1_{(k_1-1)2k_2} \\ b b^\dagger &= \sum_{p=1}^{k_1-1} u_p u_p^\dagger + \sum_{p=1}^{k_2} u_p u_p^\dagger = 1_{k_1 2k_2} - u_{k_1} u_{k_1}^\dagger\ . \end{aligned} \quad (5.16)$$

We define the square matrix $\widetilde{\sigma} = b^\dagger s b$ and rewrite the trace in the exponent as

$$\mathrm{trg}\, u^\dagger s u r = \mathrm{trg}\, \widetilde{\sigma}\widetilde{r} + \sigma_{k_1k_1} r_{k_11} \quad , \quad (5.17)$$

with $\sigma_{k_1k_1} = u_{k_1}^\dagger s u_{k_1}$. Similarly to the ordinary case, the first term of the right hand side of Eq. (5.17) depends only on the first $k_1 - 1 + k_2$ columns u_p collected in b and the second term depends only on u_{k_2}. Thus, it is useful to decompose the invariant measure,

$$d\mu(u) = d\mu(b)\, d\mu(u_{k_1})\ , \quad (5.18)$$

and to write Eq. (5.1) in the form

$$\Phi_{k_1 2k_2}(s,r) = \int d\mu(u_{k_1}) \exp(i\sigma_{k_1k_1} r_{k_11}) \int d\mu(b) \exp(i\mathrm{trg}\, \widetilde{\sigma}\widetilde{r})\ . \quad (5.19)$$

Since the coordinates b are locally orthogonal to u_{k_1}, the measure $d\mu(b)$ also depends on u_{k_1}.

We now generalize the radial Gelfand–Tzetlin coordinates introduced in Sec. 4.3 for the ordinary spaces to the superspace. Naturally, the projector reads $(1_{k_1 2k_2} - u_{k_1} u_{k_1}^\dagger)$ and we have the defining equation

$$(1_{k_1 2k_2} - u_{k_1} u_{k_1}^\dagger)\, s\, (1_{k_1 2k_2} - u_{k_1} u_{k_1}^\dagger)\, e'_p \;=\; s'_p\, e'_p \qquad p = 1, \ldots, k_1 - 1, k_1 + 1, \ldots, k_1 + k_2 \tag{5.20}$$

for the $(k_1 - 1 + k_2)$ radial Gelfand–Tzetlin coordinates s'_p and the corresponding vectors e'_p as eigenvalues and eigenvectors of the matrix $(1_{k_1 2k_2} - u_{k_1} u_{k_1}^\dagger)\, s\, (1_{k_1 2k_2} - u_{k_1} u_{k_1}^\dagger)$ which has the generalized rank $k_1 - 1 + k_2$. Due to $u_{k_1}^\dagger e'_p = 0$, we find

$$(1_{k_1 2k_2} - u_{k_1} u_{k_1}^\dagger)\, s\, e'_p \;=\; s'_p\, e'_p \qquad p = 1, \ldots, k_1 - 1, k_1 + 1, \ldots, k_2\ . \tag{5.21}$$

The eigenvalues s'_p are calculated from the characteristic function

$$\begin{aligned} z(s'_p) \;&=\; \operatorname{detg}\left((1_{k_1 2k_2} - u_{k_1} u_{k_1}^\dagger)s - s'_p\right) \\ &=\; -s'_p \operatorname{detg}\left(s - s'_p\right) u_{k_1}^\dagger \frac{1_{k_1 2k_2}}{s - s'_p} u_{k_1} \end{aligned} \tag{5.22}$$

which has to be discussed in the limits

$$z(s'_p) \;\longrightarrow\; \begin{cases} 0 & \text{for } p = 1, \ldots, k_1 - 1 \\ \infty & \text{for } p = k_1 + 1, \ldots, k_1 + k_2 \end{cases} \ . \tag{5.23}$$

Thus, together with the normalization $u_{k_1}^\dagger u_{k_1} = 1$, these are $k_1 + k_2$ equations for the elements of u_{k_1}.

The two parts of the integral (5.19) have to be expressed in terms of the radial Gelfand–Tzetlin coordinates s'_p. In a calculation fully analogous to the ordinary case, we find

$$\sigma_{k_1 k_1} \;=\; \operatorname{trg} s \;-\; \operatorname{trg} s'\ . \tag{5.24}$$

The eigenvalues $t_p,\ p = 1, \ldots, k_1 - 1, k_1 + 1, \ldots, k_1 + k_2$ of $\widetilde{\sigma}$ obtain from the characteristic function

$$w(t_p) \;=\; \operatorname{detg}\,(\widetilde{\sigma} - t_p) \;=\; -\frac{1}{t_p} \operatorname{detg}\left((1_{k_1 2k_2} - u_{k_1} u_{k_1}^\dagger)s - t_p\right) \ . \tag{5.25}$$

Comparison with Eq. (5.23) shows that the characteristic functions $w(t_p)$ and $z(s'_p)$ are, apart from the non–zero factor $-t_p$, identical. This implies $t_p \equiv s'_p,\ p = 1, \ldots, k_1 - 1, k_1 + 1, \ldots, k_1 + k_2$. Thus, by introducing the square matrix $\widetilde{u}$ which diagonalizes $\widetilde{\sigma}$, we may write

$$\widetilde{\sigma} \;=\; b^\dagger s b \;=\; \widetilde{u}^\dagger s' \widetilde{u}\ . \tag{5.26}$$

By construction, $\widetilde{u}$ must be in the group $UOSp(k_1 - 1/2k_2)$, because σ and $\widetilde{\sigma}$ share the same symmetries.

These intermediate results allow us to transform Eq. (5.19) into

$$\Phi_{k_1 2k_2}(s, r) \;=\; \int d\mu(s, s') \exp(i(\operatorname{trg} s - \operatorname{trg} s') r_{k_1 1}) \int d\mu(b) \exp(i \operatorname{trg} \widetilde{u}^\dagger s' \widetilde{u} \widetilde{r}) \tag{5.27}$$

where $d\mu(s', s)$ is, apart from phase angles, the invariant measure $d\mu(u_{k_1})$, expressed in the radial Gelfand–Tzetlin coordinates s'. To do the integration over b, we view, for the moment, the vector u_{k_1} as fixed and observe that the measure $d\mu(b)$ is the invariant measure of the group $UOSp(k_1-1/2k_2)$ under the constraint that b is locally orthogonal to u_{k_1}. The matrix $\widetilde{u} \in UOSp(k_1-1/2k_2)$ is constructed from b under the same constraint. Thus, since b and $\widetilde{u}$ cover the same manifold, the integral over b in Eq. (5.27) must yield the matrix Bessel function $\Phi_{(k_1-1)2k_2}(s',\widetilde{r})$ and we arrive at the supersymmetric recursion formula (5.8). We remark that the in the last step the derivation differs from the derivation in ordinary space. Since the group volume of supergroups is ill–defined we could not introduce unity as in ordinary space, cf. (4.42). However, this is just a minor problem. The invariance of the measure is the crucial property we need for the proof and this holds both in superspace and in ordinary space.

In order to evaluate the invariant measure, we have to solve the system of equations (5.22). It is very similar to the system (3.17) to (3.19) of the angular Gelfand–Tzetlin equation.

$$
\begin{aligned}
1 &= \sum_{p=1}^{k_1} |v_p^{(1)}|^2 + \sum_{p=1}^{k_2} |\alpha_p^{(1)}|^2, && (5.28)\\
0 &= \sum_{q=1}^{k_1} \frac{|v_q^{(1)}|^2}{s_{q1}-s'_{p1}} + \sum_{q=1}^{k_2} \frac{|\alpha_q^{(1)}|^2}{is_{q2}-s'_{p1}}, \quad p=1,\ldots,k_1-1\ , && (5.29)\\
z_p &= is_{p2}^{(1)} \frac{\prod_{q=1}^{k_1}(s_{q1}-is'_{p2})}{\prod_{q=1}^{k_2}(is_{q2}-is'_{p2})^2} \left(\sum_{q=1}^{k_1} \frac{|v_q^{(1)}|^2}{s_{q1}-is'_{p2}} + \sum_{q=1}^{k_2} \frac{|\alpha_q^{(1)}|^2}{is_{q2}-is'_{p2}} \right)\ , \\
& \qquad z_p \to \infty, \quad p=1,\ldots,k_2\ . && (5.30)
\end{aligned}
$$

In Appendix C.1 we sketch the solution of this equation system for small dimensions. Inspired by these solutions one can conjecture the general solutions and verify them by plugging them directly in Eq. (5.30). We state the expressions for $|v_p^{(1)}|^2 = |u_{pk_1}|^2$ with $p=1,\ldots,k_1$ and $|\alpha_p^{(1)}|^2 = |u_{(k_1+2p)k_1}|^2 + |u_{(k_1+2p-1)k_1}|^2$ with $p=1,\ldots,k_2$.

$$
\begin{aligned}
|v_p^{(1)}|^2 &= \frac{\prod_{q=1}^{k_1-1}(s_{p1}-s'_{q1})\prod_{q=1}^{k_2}(s_{p1}-is_{q2})^2}{\prod_{q=1}^{k_2}(s_{p1}-is'_{q2})^2\prod_{q=1,q\neq p}^{k_1}(s_{p1}-s_{q1})}\ , \quad p=1,\ldots,k_1\ ,\\
|\alpha_p^{(1)}|^2 &= 2\,(is'_{p2}-is_{p2})\frac{\prod_{q=1}^{k_1-1}(is_{p2}-s'_{q1})\prod_{q=1,q\neq p}^{k_2}(is_{p2}-is_{q2})^2}{\prod_{q=1,q\neq p}^{k_2}(is_{p2}-is'_{q2})^2\prod_{q=1}^{k_1}(is_{p2}-s_{q1})}\ ,\\
& \qquad p=1,\ldots,k_2\ . && (5.31)
\end{aligned}
$$

These expressions are similar to the ones derived in [GUH3] for unitary matrices. All products in (5.31) involving fermionic eigenvalues are squared. This reflects the degeneracy of s in the fermion-fermion block. As in the case of the angular Gelfand Tzetlin coordinates, we have introduced a new anticommuting variable by defining

$$
|\xi'_p|^2 = is'_{p2} - is_{p2}\ . \qquad (5.32)
$$

Now the invariant measure can be calculated in the same way as for the angular Gelfand–Tzetlin coordinates as outlined in Section 3.2.3 and appendix A.3. The result is summarized in Eqs. (5.11).

5.3 Low dimensional example

We apply the recursion formula (5.1) to the matrix Bessel function of $UOSp(2/2)$. In this case it reads explicitly

$$\Phi_{22}(-is,r) = \widehat{G}_{22} \int d\mu(s,s') \exp((\operatorname{trg} s - \operatorname{trg} s')r_{21})\ \Phi_{12}(-is',\widetilde{r}) \quad . \tag{5.33}$$

The function $\Phi_{12}(-is',r)$ can readily be evaluated as

$$\Phi_{12}(-is',\widetilde{r}) = \widehat{G}_{12}\Big(1 - 2(r_{11} - ir_{12})(s'_{11} - is'_{12})\Big)\ \exp(-2ir_{12}is_{12}). \tag{5.34}$$

The measure of the coset $UOSp(2/2)/UOSp(1/2)$ can be read off from formula (5.11),

$$d\mu(s,s') = \frac{(is_{12} - s'_{11})\prod_{n=1}^{2}(is'_{12} - s_{n1})}{\sqrt{-\prod_{n=1}^{2}(s'_{11} - s_{n1})}\,(is'_{12} - s'_{11})^2} ds'_{11} d\xi_1'^{*} d\xi_1' \ . \tag{5.35}$$

Performing the Grassmann integration and collecting terms yields

$$\begin{aligned}\Phi_{22}(-is,r) \quad = \quad & \widehat{G}_{22}\ \exp\left(r_{21}(s_{11}+s_{21}) - 2is_{12}ir_{12}\right) \int_{s_{21}}^{s_{11}} \mu_B(s,s')ds'_{11} \prod_{q=1}^{2}(is_{12} - s_{q1}) \\ & \left(4\prod_{j=1}^{2}(ir_{12} - r_{j1}) - 2(ir_{12} - r_{21})\sum_{q=1}^{2}\frac{1}{is_{12} - s_{q1}} + 2M_{11}(s'_1, s_1)\right) \\ & \qquad \exp\left(s'_{11}(r_{11} - r_{21})\right) \quad ,\end{aligned} \tag{5.36}$$

where we have introduced the operator

$$M_{mj}(s_1,s'_1) = \frac{1}{(is_{m2} - s'_{j1})}\left(\frac{1}{2}\sum_{n=1}^{k_1}\frac{1}{is_{m2} - s_{n1}} - \frac{1}{is_{m2} - s'_{j1}} - \sum_{\substack{n=1\\ n\neq j}}^{k_1}\frac{1}{s'_{j1} - s'_{n1}} - \frac{\partial}{\partial s'_{j1}}\right) \ . \tag{5.37}$$

At first sight it is not clear now, how to proceed further. As we have seen in Sections 4.5 and 4.2 in the general case it is impossible to perform the integration over the orthogonal group even for the simplest case of $O(2)$. On the other hand one can argue as follows: It is always possible to parametrize the group element $u \in UOSp(2/2)$ in a non–canonical coset parametrization in the spirit of an Euler parametrization in ordinary space. Inserting this parametrization into the defining equation of the matrix Bessel function (5.1) one can expand the trace in all Grassmann variables. The expansion coefficients are polynomials in the commuting integration variables and – more important – in the matrix elements of s and r. The invariant measure can be expanded in the Grassmann variables as well. It does not depend at all on r and s. Although this procedure becomes rapidly out of hand even for small groups, it is clear that the outcome of this expansion will be polynomial in the eigenvalues of s and r. In other words: eigenvalues can only appear in the denominator by an integration over commuting variables and never by a Grassmann integration. Therefore, before performing any integral over commuting variables, there must exist a form of $\Phi_{22}(-is,r)$, which is polynomial in the eigenvalues of s and r.

To remove the denominators we use the following identity:

Lemma 5.1 *Let $f(s'_1)$ be an analytic, symmetric function in $s'_{1i}, i = 1, \dots k_1$. Furthermore define the operator*

$$L_m(s) = \sum_{j=1}^{k_1} \frac{1}{is_{m2} - s_{j1}} \frac{\partial}{\partial s_{j1}} \quad . \tag{5.38}$$

Then

$$\begin{aligned} L_m(s) \int_{s_{21}}^{s_{11}} \dots \int_{s_{k_1 1}}^{s_{(k_1-1)1}} \mu_B(s,s')\, f(s'_1)\, d[s'_1] = \\ - \int_{s_{21}}^{s_{11}} \dots \int_{s_{1k_1}}^{s_{(k_1-1)1}} \mu_B(s,s') \sum_{j=1}^{k_1-1} M_{mj}(s_1, s'_1)\, f(s'_1)\, d[s'_1] \quad . \end{aligned} \tag{5.39}$$

<u>*Derivation:*</u>
The derivation is similar to the calculation in Appendix B.2. First we rewrite the integral in terms of θ–functions

$$l.\ h.\ s. \ = \ L_m(s) \int \mu_B(s,s')\, f(s'_1)\, d[s'_1] \prod_{k \le l} \theta(s_{k1} - s'_{l1}) \prod_{l<n} \theta(s'_{l1} - s_{n1}) \quad . \tag{5.40}$$

Now the integration domain is the real axis for all integration variables. The action of $L_m(s)$ on the integral yields:

$$\begin{aligned} l.\ h.\ s. \ = \ & \int \Bigg(\mu_B(s,s') \sum_{i=1}^{k_1} \sum_{j=1}^{k_1-1} \frac{1}{2} \frac{-1}{(is_{12} - s_{i1})(s_{i1} - s'_{j1})} f(s'_1) \\ & \prod_{k \le l} \theta(s_{k1} - s'_{l1}) \prod_{l<n} \theta(s'_{l1} - s_{n1}) \Bigg) \ d[s'_1] \\ & + \int \mu_B(s,s') \sum_{i=1}^{k_1} \frac{1}{is_{m2} - s_{i1}} \frac{\partial}{\partial s_{i1}} \prod_{k \le l} \theta(s_{k1} - s'_{l1}) \prod_{l<n} \theta(s'_{l1} - s_{n1})\, d[s'_1] \,. \end{aligned} \tag{5.41}$$

After a decomposition to partial fractions of the first term in Eq. (5.41) we find

$$\begin{aligned} l.\ h.\ s. \ = \ & \int \Bigg(\mu_B(s,s') \sum_{i=1}^{k_1} \sum_{j=1}^{k_1-1} \frac{1}{2} \frac{-1}{(is_{12} - s_{i1})(is_{12} - s'_{j1})}\, f(s'_1) - \\ & \Delta_{k_1}(s'_1)\, f(s'_1) \sum_{j=1}^{k_1-1} \frac{1}{is_{12} - s'_{j1}} \frac{\partial}{\partial s'_{j1}} \frac{1}{\sqrt{-\prod_{i=1}^{k_1}(s_{i1} - s'_{j1})}} \Bigg) \\ & \prod_{k \le l} \theta(s_{k1} - s'_{l1}) \prod_{l<n} \theta(s'_{l1} - s_{n1})\, d[s'_1] \\ & + \int \mu_B(s,s')\, f(s') \sum_{i=1}^{k_1} \frac{1}{is_{m2} - s_{i1}} \frac{\partial}{\partial s_{i1}} \prod_{k \le l} \theta(s_{k1} - s'_{l1}) \prod_{l<n} \theta(s'_{l1} - s_{n1})\, d[s'_1] \,. \end{aligned} \tag{5.42}$$

Performing an integration by parts yields

$$
\begin{aligned}
l.\ h.\ s. \quad = \quad & \int \mu_B(s,s') \sum_{j=1}^{k_1-1} (-M_{mj}(s_1,s_1')) \prod_{k\leq l} \theta(s_{k1}-s_{l1}') \prod_{l<n} \theta(s_{l1}'-s_{n1})\, d[s_1'] + \\
& \int \mu_B(s,s')\, f(s') \left(\sum_{i=1}^{k_1} \frac{1}{is_{m2}-s_{i1}} \frac{\partial}{\partial s_{i1}} + \sum_{j=1}^{k_1-1} \frac{1}{is_{m2}-s_{j1}'} \frac{\partial}{\partial s_{j1}'} \right) \\
& \prod_{k\leq l} \theta(s_{k1}-s_{l1}') \prod_{l<n} \theta(s_{l1}'-s_{n1})\, d[s_1'] \ .
\end{aligned}
\tag{5.43}
$$

The derivatives of the θ–functions yields δ–functions. Upon integration of the δ–distribution the two terms in the last integral cancel each other. Hence the last term vanishes identically.‡

Setting $f(s_1') = \exp(-s_{11}'(r_{11}-r_{21}))$ in Lemma 5.1 and inserting Eq. (5.39) into Eq. (5.36) we arrive at the following expression for $\Phi_{22}(-is,r)$:

$$
\begin{aligned}
\Phi_{22}(-is,r) = \widehat{G}_{22} \exp(-2is_{12}ir_{12}) & \left(4 \prod_{j=1}^{2} (ir_{12}-r_{j1})(is_{12}-s_{j1}) - \right. \\
& \left. 2 \sum_{q=1}^{2} (is_{12}-s_{q1}) \left(ir_{12} - r_{21} - r_{11} - \frac{\partial}{\partial s_{q1}} \right) \right) \Phi_2^{(1)}(-is_1, r_1) \ ,
\end{aligned}
\tag{5.44}
$$

where $\Phi_2^{(1)}(s_1,r_1)$ is the matrix Bessel function of the orthogonal group $O(2)$ in ordinary space as defined in Eq. (4.19). Although this can already be taken as result, we make some further manipulations by using the representation (4.24) of $\Phi_2(s_1,ir_1)$ to underline the symmetry between r and s

$$
\begin{aligned}
& \Phi_{22}(-is,r) = G_{22} \exp\left(\mathrm{trg}\,(rs) - \frac{z}{2} \right) \\
& \left(4 \prod_{j=1}^{2} (ir_{21}-r_{1j})(is_{21}-s_{1j}) - \sum_{q=1}^{2} (is_{21}-s_{1q}) \sum_{p=1}^{2} (ir_{21}-r_{1p}) - z\frac{d}{dz} \right) 2\pi I_0(z/2) \ ,
\end{aligned}
\tag{5.45}
$$

where we have introduced $z = (s_{11}-s_{21})(r_{11}-r_{12})$ and the modified Bessel function $I_0(z)$.

5.3.1 Alternative derivations of $\Phi_{22}(s,r)$

Lemma (5.1) was crucial in the derivation of $\Phi_{22}(-is,r)$. By means of this lemma the denominator problem was overcome in one step. We now want to look to this problem from a different point of view. It provides both insight into the functioning of the radial Gelfand–Tzetlin coordinates and an alternative though pedestrian method of removing the denominators. To this end we rederive Eq.(5.44) in two other ways.

First we use a different parametrization of the coset $UOSp(2/2)/UOSp(1/2)$ by writing the first column of $u \in UOSp(2/2)$ as

$$u_1 = \begin{pmatrix} \sqrt{1-|\alpha|^2}\cos\vartheta \\ \sqrt{1-|\alpha|^2}\sin\vartheta \\ \frac{1}{\sqrt{2}}\alpha \\ \frac{1}{\sqrt{2}}\alpha^* \end{pmatrix} \quad . \tag{5.46}$$

This is a canonical way to parametrize the supersphere $S^{1|2}$ which is isomorphic to the coset $UOSp(2/2)/UOSp(1/2)$. It coincides with the *angular* Gelfand-Tzetlin coordinates as described in Chapter 3, for the value $(h_{11}^2 - ih_{12}^2) = 1$, cf. Eq. (3.34). The invariant measure is in these coordinates simply $d\mu(u_1) = d\alpha^* d\alpha d\vartheta$. Thus one obtains directly the volume $V(S^{1|2}) = 0$, see for example [ZIR1]. This is not at all obvious in the parametrization of the measure by *radial* Gelfand–Tzetlin coordinates (5.35). There, one has to perform the Grassmann integration and apply Lemma 5.1 to achieve this goal.

In order to use the recursion formula in the parametrization (5.46) one has to solve the Gelfand–Tzetlin equations (5.28) to (5.30) for the eigenvalues. The unique solution of the bosonic equation (5.29) is

$$s_{11}' = a_0 + \frac{\prod_{i=1}^2 (s_{i1} - a_0)}{is_{12} - a_0}|\alpha|^2 \quad , \quad a_0 = \frac{s_{11}+s_{21}}{2} - \frac{s_{11}-s_{12}}{2}\cos 2\vartheta \quad . \tag{5.47}$$

The fermionic equation yields

$$is_{12}' = is_{12} + \frac{\prod_{i=1}^2 (s_{i1} - is_{12})}{is_{12} - a_0}|\alpha|^2 \quad . \tag{5.48}$$

After inserting Eqs. (5.47) and (5.48) and the above measure $d\mu(u_1)$ into the recursion formula (5.8), the Grassmann integration can be performed. Remarkably, we arrive at the "denominator–free" expression

$$\begin{aligned} \Phi_{22}(s,r) &= \widehat{G}_{22} \int_0^{2\pi} d\vartheta \exp\left(\mathrm{trg}\,(rs) - z/2 + \frac{z}{2}\cos 2\vartheta\right) \\ &\Bigg[\left(\prod_{i=1}^2 (r_{i1} - ir_{12})(s_{i1} - is_{12}) + \frac{1}{2}\sum_{\substack{i=1\\j=1}}^2 (s_{j1} - is_{12})(r_{i1} - ir_{12}) \right) \\ &-\frac{1}{2}\left(ir_{12} - \frac{1}{2}(r_{11}+r_{21})\right)(s_{11} - s_{21})\cos 2\vartheta - \\ &\frac{z}{8}(ir_{12} - r_{11})(s_{11} - s_{21})\sin^2 2\vartheta \Bigg] \quad . \end{aligned} \tag{5.49}$$

To make contact with Eq. (5.45) one has to realize, that in Eq. (5.49) an additional total derivative appears in the integrand. This becomes obvious if one adds and subtracts $z/4\cos 2\vartheta$ in the squared bracket of Eq. (5.49)

$$\begin{aligned} \Phi_{22}(s,r) &= \widehat{G}_{22} \int_0^{2\pi} d\vartheta \exp\left(\mathrm{trg}\,(rs) - z/2 + \frac{z}{2}\cos 2\vartheta\right) \\ &\left(\prod_{i=1}^2 (r_{i1} - ir_{12})(s_{i1} - is_{12}) + \frac{1}{2}\sum_{\substack{i=1\\j=1}}^2 (s_{j1} - is_{12})(r_{i1} - ir_{12}) - \frac{z}{4}\cos 2\vartheta \right) \end{aligned}$$

$$+\frac{ir_{12}-r_{11}}{r_{11}-r_{21}}\int_0^{2\pi} d\vartheta \frac{\partial^2}{\partial(2\vartheta)^2}\exp\left(\mathrm{trg}\,(rs)-z/2+\frac{z}{2}\cos 2\vartheta\right) \ . \tag{5.50}$$

While the first integral reproduces Eq. (5.45), the second vanishes identically. In general in performing Grassmann integrations one has to take care of boundary contributions [BER, ROT]. These contributions can appear whenever even coordinates are shifted by nilpotents and the function one integrates does not have compact support [BER]. However in our case the base space is always given by a n-dimensional sphere, i. e. by a compact manifold *without boundary*. Thus in a properly chosen coordinate system no boundary terms should appear. With regard to Eq. (5.50) this means: the fact, that the last term in Eq. (5.50) vanishes is a direct consequence of the compactness of the circle and of the analyticity of the function, we integrate. However, by the radial Gelfand–Tzetlin coordinates only the moduli squared of the vector u_1 are determined. Therefore not the whole sphere, but only a 2^{n+1}-ant of it is covered by Eq. (5.31). In our case not the circle but only a quarter of it is parametrized. This is allowed since the matrix Bessel functions depend only on the moduli squared $|u_{i1}|^2$. However one has to ensure, that the introduction of these artificial boundaries does not alter the result. To this end we use the integration formula

Lemma 5.2 *Let $s_{21} < s'_{11} < s_{11}$ be real and ξ', ξ'^* anticommuting. Furthermore define*

$$f(s'_{11},\xi,\xi^*) = f_0(s'_{11}) + f_1(s'_{11})|\xi|^2 \quad , \tag{5.51}$$

with two analytic functions $f_0(s'_{11}), f_1(s'_{11})$. Then the integral

$$I = \int_{s_{21}}^{s_{11}} ds'_{11} d\xi^* d\xi f(s'_{11},\xi,\xi^*) \tag{5.52}$$

transforms under a shift of s'_{11} by nilpotents

$$s'_{11} = y + g(y)|\xi|^2 \tag{5.53}$$

as follows:

$$I = \int_{s_{21}}^{s_{11}} dy d\xi^* d\xi \frac{\partial s'_{11}}{\partial y} f\left(y(s'_{11}),\xi,\xi^*\right) - \left[f_0(s_{11})g(s_{11}) - f_0(s_{21})g(s_{21})\right] \ . \tag{5.54}$$

The proof is by direct calculation. The second term in Eq. (5.54) is often referred to as boundary term. It can be viewed as the integral of a total derivative (an exact one–form) which has to be added to the integration measure for functions with non compact support [ROT]. For functions of an arbitrary number of commuting and anticommuting arguments a similar integral formula holds with additional boundary terms [BER]. In going from the "canonical coordinates" $(\vartheta, \alpha, \alpha^*)$ to the radial ones $(s'_{11}, \xi'_1, \xi'^*_1)$ in principle boundary terms can arise, since the bosonic Gelfand–Tzetlin eigenvalue (5.47) contains nilpotents. However, the crucial quantity is $g(y)$ in Lemma 5.2, which in our case is

$$g(s'_{11}) = \frac{\prod_{i=1}^2 (s_{i1} - s'_{11})}{is_{12} - s'_{11}} \ . \tag{5.55}$$

So $g(s'_{11})$ causes the boundary term to vanish at s_{11} and s_{21}. It is the product structure of the left hand side of Eq. (5.31) which always guarantees the vanishing of the boundary terms, when one goes from the Cartesian set of coordinates to the radial Gelfand–Tzetlin coordinates. Therefore one might think of the denominators, arising in Eqs. (5.36), (5.37) as belonging to total derivatives of functions, which vanish at the boundaries. Keeping this in mind we derive Eq. (5.45) in yet another way. We expand the product

$$\prod_{q=1}^{k_1} (is_{m2} - s_{q1}) = \sum_{n=0}^{k_1} \frac{1}{n!} (is_{m2} - s'_{j1})^n \frac{\partial^n}{\partial (s'_{j1})^n} \prod_{q=1}^{k_1} \left(s'_{j1} - s_{q1}\right) \quad , \tag{5.56}$$

and insert it into the integral

$$\begin{aligned} &\int_{s_{21}}^{s_{11}} \cdots \int_{k_1}^{k_1-1} \mu_B(s_1, s'_1) K_{mj}(s_1, s'_1) f(s'_1) d[s'_1] = \\ &\quad \int_{s_{21}}^{s_{11}} \cdots \int_{k_1}^{k_1-1} \mu_B(s_1, s'_1) \prod_{n=1}^{k_1} (is_{m2} - s_{n1}) M_{mj}(s_1, s'_1) \, f(s'_1) \, d[s'_1] \quad . \end{aligned} \tag{5.57}$$

Now we can remove the term proportional to $(is_{m2} - s'_{j1})^{-2}$ in the integrand by an integration by parts. Through the expansion (5.56) the vanishing of the boundary terms is assured. We arrive at

$$\begin{aligned} K_{mj}(s_1, s'_1) \quad = \quad & -\sum_{n=2}^{k_1} \frac{1}{n!} (is_{m2} - s'_{j1})^{n-2} \frac{\partial^n}{\partial (s'_{j1})^n} \prod_{q=1}^{k_1} \left(s'_{j1} - s_{q1}\right) + \\ & \frac{\prod_{q=1}^{k_1} (is_{m2} - s_{q1}) - \prod_{q=1}^{k_1} \left(s'_{j1} - s_{q1}\right)}{is_{m2} - s'_{j1}} \\ & \left(\frac{1}{2} \sum_{q=1}^{k_1} \frac{1}{is_{m2} - s_{q1}} - \frac{1}{2} \sum_{q=1}^{k_1} \frac{1}{s'_{j1} - s_{q1}} - \sum_{\substack{q=1 \\ q \neq j}}^{k_1} \frac{1}{s'_{j1} - s'_{q1}} - \frac{\partial}{\partial s'_{j1}} \right) \quad . \end{aligned} \tag{5.58}$$

We notice, that in the new operator $K_{mj}(s_1, s'_1)$ all denominators of the type $(is_{m2} - s'_{j1})^{-1}$ have disappeared. For $k_1 = 2$ we calculate

$$K_{11} = -\left(is_{12} + s'_{11} - s_{11} - s_{21}\right) \frac{\partial}{\partial s'_{q1}} \quad , \tag{5.59}$$

which can be inserted into Eq. (5.36) by using the definition (5.57). Finally the result (5.45) is reproduced by the substitution, cf. Eq. (5.47)

$$s'_{11} = \frac{s_{11} + s_{21}}{2} - \frac{s_{11} - s_{12}}{2} \cos 2\vartheta \quad . \tag{5.60}$$

Clearly the result of this procedure is summarized in Lemma 5.1.

Finally some remarks are in order:

First, one might conclude from the above discussion, that the radial Gelfand–Tzetlin coordinates are less adapted to the problem than the "canonical" parametrization (5.46), because in the latter no denominators appear. We stress, that this is not true. Certainly the denominators appear due to the shift of the bosonic variable by nilpotents in Eq. (5.47). However, the difficulty in deriving Eq. (5.45) is to identify the different parts of the integrand after the Grassmann integration. Some of them belong to total derivatives

and this problem exists in both parametrizations.
Second, we emphasize, that the appearance of total derivatives in the integrand is not a peculiarity of supersymmetry. Already in Section 4.5, when we were dealing with the matrix Bessel function in ordinary space we had to overcome the same problem. We conclude that the appearance of these total derivatives is an intrinsic property of the recursion formula. However, a geometrical interpretation of this phenomenon is hard to find.
Third, we want to comment on the last way of deriving Eq. (5.45). At first sight it seems to be a somehow artificial way. But we emphasize that for groups of higher order such a series of integration by parts is the only feasible way to remove the denominators and the total derivatives from the integrand, *without* identities like Lemma 5.1. Indeed all results derived in the following section were originally derived in this way. The way of derivation we sketch now was found afterwards.

5.4 The $\Phi_{k_14}(s,r)$ series

The starting point of the recursion is the matrix Bessel function over the unitary symplectic group $\Phi_2^{(4)}(-is_2,r_2)$, which already has been calculated in Eq. (4.83). Since the subgroup $O(1)$ of $UOSp(1/4)$ is trivial, no commuting integral has to be performed to derive $\Phi_{14}(-is,r)$. Plugging the measure (5.11) into the recursion formula (5.8) and performing the Grassmann integrations yields in a straightforward calculation:

$$\begin{aligned}\Phi_{14}(-is,r) \quad = \quad & \widehat{G}_{14}\exp\left(\operatorname{trg} rs\right)\left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)}+\frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\right)\\ & \Big(2(is_{21}-s_{11})(ir_{21}-r_{11})-1\Big)\Big(2(is_{22}-s_{11})(ir_{22}-r_{11})-1\Big)-\\ & \widehat{G}_{14}\exp\left(\operatorname{trg}\left(rs\right)\right)\frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}+\quad(ir_{12}\longleftrightarrow ir_{22})\quad .\end{aligned}\tag{5.61}$$

The term $(ir_{12}\longleftrightarrow ir_{22})$ accounts to the permutation group $\mathcal{S}^2$ in Eq. (4.83). Anticipating that the structure of $\Phi_{14}(-is,r)$ will persist in all levels up to $\Phi_{44}(-is,r)$, we point out that $\Phi_{14}(-is,r)$ essentially consists of two parts. A comparison with Eqs. (5.34), and (4.83) shows, that the first part of $\Phi_{14}(-is,r)$ is a product of an exponential with three other terms. The first one,

$$\left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)}+\frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\right)\quad ,\tag{5.62}$$

stems from the integral over the $USp(4)$ subgroup. The other two terms can be identified with the matrix Bessel functions

$$\Phi_{12}(-is,r)\quad ,\quad s=\operatorname{diag}\left(s_{11},is_{12},is_{12}\right)\quad ,\quad r=\operatorname{diag}\left(r_{11},ir_{12},ir_{12}\right)\tag{5.63}$$

and

$$\Phi_{12}(-is,r)\quad ,\quad s=\operatorname{diag}\left(s_{11},is_{22},is_{22}\right)\quad ,\quad r=\operatorname{diag}\left(r_{11},ir_{22},ir_{22}\right)\quad .\tag{5.64}$$

The second part can be considered as a correction term, which destroys this product structure of $\Phi_{14}(-is,r)$. It would have allowed us to identify the different parts of the product with the integrations over the corresponding subsets of the group. That is to say that $\Phi_2^{(4)}(-is_2, r_2)$ arises from the integration over the $USp(4)$ subgroup, the $O(1)$ integration yields unity and the other two factors arise from the integration over the coset

$$\frac{UOSp(1/4)}{USp(4) \otimes O(1)} \quad . \tag{5.65}$$

In the next step one has one integration over a commuting variable which in general cannot be performed. After the Grassmann integration, we are left with a considerable amount of terms. In order to arrange them in a convenient way we introduce the following notation for the product of two operators $D_1(s)D_2(s)$ acting on a function $f(s)$

$$[D_1^{\rightarrow}(s)\, D_2(s)]\, f(s) = D_1(s)\, D_2(s)\, f(s) - (D_1(s)\, D_2(s))\; f(s) \quad . \tag{5.66}$$

This means, an operator with an arrow only acts on the terms outside the squared bracket. With this notation we can write

$$\begin{aligned}
\Phi_{24}(-is, r) = \widehat{G}_{24} \exp\left(\mathrm{tr}\,(r_2 s_2) + r_{21}(s_{11} + s_{21})\right) \int_{s_{12}}^{s_{11}} d\mu_B(s_1, s_1') \Bigg[\prod_{i=1}^{2}\prod_{j=1}^{2}(is_{i2} - s_{j1}) \\
\left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)} + \frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\right) \\
\left(4\prod_{i=1}^{2}(ir_{12} - r_{i1}) + 2\sum_{k=1}^{2}\frac{r_{11} - ir_{12}}{is_{12} - s_{k1}} + 2M_{11}^{\rightarrow}(s_1, s_1')\right) \\
\left(4\prod_{i=1}^{2}(ir_{22} - r_{i1}) + 2\sum_{k=1}^{2}\frac{r_{11} - ir_{22}}{is_{22} - s_{k1}} + 2M_{21}(s_1, s_1')\right) + \\
\left(\frac{1}{\Delta_2^2(ir_2)\Delta_3^2(is_2)} + \frac{1}{\Delta_2^3(ir_2)\Delta_2^4(is_2)}\right)\left(\frac{4}{is_{12} - s_{11}'}M_{21}(s_1, s_1') - \frac{2}{is_{22} - s_{11}'}M_{11}(s_1, s_1')\right) \\
+ \frac{2}{\Delta_2^3(ir_2)\Delta_2^4(is_2)}\left(2\mathrm{tr}\, r_1 - \mathrm{tr}\, ir_2 + \sum_{i=1}^{2}\frac{1}{is_{22} - s_{i1}}\right) M_{11}(s_1, s_1') \\
- \frac{2}{\Delta_2^3(ir_2)\Delta_2^4(is_2)}\left(2\mathrm{tr}\, r_1 - \mathrm{tr}\, ir_2 + \sum_{i=1}^{2}\frac{1}{is_{12} - s_{i1}}\right) M_{21}(s_1, s_1') \\
- \frac{4}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\sum_{k=1}^{2}\prod_{j=1}^{2}\frac{r_{11} - ir_{j2}}{s_{k1} - is_{j2}}\Bigg] \exp\left(s_{11}'(r_{11} - r_{21})\right) \; + (ir_{12} \longleftrightarrow ir_{22}) \quad .
\end{aligned} \tag{5.67}$$

In Eq. (5.67), the product $M_{11}^{\rightarrow}(s_1, s_1')M_{21}(s_1, s_1')$ appears. This indicates that an identity exists which is similar to Lemma 5.1. This identity should map a product of operators $L_1(s)L_2(s)$ acting on the integral onto a product of operators $M_{11}(s_1, s_1')M_{21}(s_1, s_1')$ acting under the integral. Neither the outer operators, $L_m(s)$, nor the inner ones, $M_{mj}(s)$, commute. Hence the desired identity is non–trivial. It is given by the

Lemma 5.3 *We have the same conditions as in Lemma 5.1, furthermore we define*

$$[L_m^{\rightarrow}(s)L_l(s)] = \sum_{n=1}^{k_1}\sum_{q=1}^{k_1}\frac{1}{(is_{m2} - s_{n1})(is_{l2} - s_{q1})}\frac{\partial^2}{\partial s_{n1}\partial s_{q1}} \quad . \tag{5.68}$$

Then it holds that

$$[L_m^{\rightarrow}(s)L_l(s)]\int_{s_{21}}^{s_{11}}\cdots\int_{s_{k_1 1}}^{s_{(k_1-1)1}} \mu_B(s, s')\, d[s_1']\, f(s_1') =$$

$$
\int_{s_{21}}^{s_{11}} \cdots \int_{s_{1k_1}}^{s_{(k_1-1)1}} \mu_B(s,s') \left[\sum_{j=1}^{k_1-1} \sum_{k=1}^{k_1-1} M^{\rightarrow}_{mj}(s_1,s_1') M_{lk}(s_1,s_1') - \right.
$$
$$
\frac{1}{is_{l2}-is_{m2}} \sum_{j=1}^{k_1-1} \left(\frac{1}{is_{m2}-s_{j1}'} M_{lj}(s_1,s_1') - \frac{1}{is_{l2}-s_{j1}'} M_{mj}(s_1,s_1') \right)
$$
$$
\left. -\frac{1}{2} \sum_{k\neq j}^{k_1-1} \frac{1}{(is_{m2}-s_{k1}')(is_{m2}-s_{j1}')(is_{l2}-s_{k1}')(is_{l2}-s_{j1}')} \right] f(s_1')\, d[s_1'] \quad . \tag{5.69}
$$

The derivation goes along the same lines as the derivation of Lemma 5.1 and by using Lemma 5.1. With Lemma 5.3 and Lemma 5.1 the denominator problem again is overcome in one step and after some further manipulations we arrive at the result.

$$
\Phi_{24}(-is,r) = 2\pi \widehat{G}_{24} \exp\left(\mathrm{trg}\,(rs) - \frac{z}{2}\right) \left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)} + \frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)} \right)
$$
$$
\left[\left(4 \prod_{i=1}^{2} (r_{i1}-ir_{12})(s_{i1}-is_{12}) - \sum_{\substack{i=1\\ j=1}}^{2} (s_{j1}-is_{12})(r_{i1}-ir_{12}) - z\frac{\partial^{\rightarrow}}{\partial z} \right) \right.
$$
$$
\left. \left(4 \prod_{i=1}^{2} (r_{i1}-ir_{22})(s_{i1}-is_{22}) - \sum_{\substack{i=1\\ j=1}}^{2} (s_{j1}-is_{22})(r_{i1}-ir_{22}) - z\frac{\partial}{\partial z} \right) \right] I_0(z/2)
$$
$$
-2\pi \widehat{G}_{24} \exp\left(\mathrm{trg}\,(rs) - \frac{z}{2}\right) \frac{2}{\Delta_2^3(ir_2)\Delta_2^3(is_2)} \sum_{\substack{i=1\\ k=1}}^{2} \prod_{j=1}^{2} (s_{i1}-is_{j2})(r_{k1}-ir_{j2}) I_0(z/2)
$$
$$
-2\pi \widehat{G}_{24} \exp\left(\mathrm{trg}\,(rs) - \frac{z}{2}\right) \frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)} \left((\mathrm{trg}\, s)(\mathrm{trg}\, r) - 1\right) z \frac{\partial}{\partial z} I_0(z/2)
$$
$$
+ \qquad (ir_{12} \longleftrightarrow ir_{22}) \quad . \tag{5.70}
$$

As in Section 5.3 we defined the composite variable $z = (s_{11}-s_{12})(r_{11}-r_{12})$. A comparison with Eqs. (5.44) and (5.61) reveals the similarity of the structure of $\Phi_{24}(-is,r)$ and $\Phi_{14}(-is,r)$. It also decomposes into two parts. The first part is a product, whose component can be assigned to the integrations over the different submanifolds of the group in the same way as in the case of $\Phi_{14}(-is,r)$. The other one might be interpreted as a correction term due to the non–commutativity of the operators L_m in Eq. (5.69).

This structure of $\Phi_{k_1 4}(-is,r)$ is likely to hold also for arbitrary k_1. However for $k_1 > 2$ it was up to now not possible to treat the general case. Fortunately, the matrix Bessel function, appearing in the supersymmetric diffusion kernel of the two–point function (2.66), has a twofold degeneracy in one matrix argument, cf. Eqs. (2.67) and (2.56). In this case it is possible to carry on the recursion up to $\Phi_{44}(-is,r)$. From now on we restrict ourselves to this case. In the case, that one of the two matrices has an additional degeneracy, one might think of achieving some simplification by applying the projection procedure onto the degenerate matrix. This results in a simplification of the invariant measure. However, it turned out to be better to apply the projection onto the non-degenerate matrix, i. e. to use the measure as it stands in Eq. (5.11). Hence we consider $\Phi_{34}(-is,r)$ for the case that

$$
r_1 = \mathrm{diag}\,(r_{21}, r_{11}, r_{11}) \quad . \tag{5.71}
$$

After performing the Grassmann integral one can arrange the terms in a way similar to Eq. (5.67). The complete expression is stated in Appendix C.2. We then notice that we can use Lemma 5.3 and find after some further manipulations

$$
\begin{aligned}
\Phi_{34}(-is,r) &= 4\,\widehat{G}_{34}\exp\left(\operatorname{tr} r_2 s_2\right)(r_{11}-ir_{12})(r_{11}-ir_{22})\left[\left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)}+\frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\right)\right.\\
&\left(4\prod_{k=1}^{2}(r_{k1}-ir_{12})\prod_{k=1}^{3}(s_{k1}-is_{12})+2\sum_{k=1}^{3}\prod_{j\neq k}^{3}(s_{j1}-is_{12})\left(r_{11}+r_{21}-ir_{12}-\frac{\partial^{\rightarrow}}{\partial s_{k1}}\right)\right)\\
&\left(4\prod_{k=1}^{2}(r_{k1}-ir_{22})\prod_{k=1}^{3}(s_{k1}-is_{22})+2\sum_{k=1}^{3}\prod_{j\neq k}^{3}(s_{j1}-is_{22})\left(r_{11}+r_{21}-ir_{22}-\frac{\partial}{\partial s_{k1}}\right)\right)\\
&-\frac{4}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\sum_{i=1}^{3}\prod_{\substack{j=1\\ j\neq i}}^{3}(s_{j1}-is_{12})(s_{j1}-is_{22})\\
&\left(r_{11}^2+r_{21}^2+r_{11}r_{21}-(r_{11}+r_{21})(ir_{12}+ir_{22})+ir_{12}ir_{22}-(r_{11}+r_{21}-ir_{12}-ir_{22})\frac{\partial}{\partial s_{i1}}\right)\\
&\left.-\frac{2}{\Delta_2^3(ir_2)\Delta_2^4(is_2)}\sum_{i,k}^{3}\prod_{\substack{j=1\\ j\neq i}}^{3}\prod_{\substack{l=1\\ l\neq k}}^{3}(s_{j1}-is_{12})(s_{l1}-is_{22})\left(\frac{\partial}{\partial s_{i1}}-\frac{\partial}{\partial s_{k1}}\right)\right]\Phi_3^{(1)}(-is_1,r_1)\\
&\qquad\qquad\qquad\qquad+(ir_{12}\longleftrightarrow ir_{22})\quad ,
\end{aligned}
\tag{5.72}
$$

where $\Phi_3^{(1)}(s_1,r_1)$ is the matrix Bessel function of the orthogonal group. We notice, that the structure of $\Phi_{14}(-is,r)$ and $\Phi_{24}(-is,r)$ reappears in $\Phi_{34}(-is,r)$.

We now turn to $\Phi_{44}(-is,r)$. Again we consider the case, that the matrix r is degenerate.

$$
r_1 = \operatorname{diag}(r_{21},r_{21},r_{11},r_{11}) \quad . \tag{5.73}
$$

In evaluating $\Phi_{44}(-is,r)$ the main problem is to choose a convenient representation for the matrix Bessel function $\Phi_3^{(1)}(-is_1',\tilde{r}_1)$, which enters in the recursion formula . The representation, we derived in Section 4.2.3 turns out to be very convenient for our purposes. Due to the degeneracy in $\tilde{r}_1$ the original threefold integral in Eq. (4.32) can be reduced to a integral over only one variable according to (4.34). By the same token $\Phi_4^{(1)}(-is_1,r_1)$ can be represented by a twofold integral. Explicit expressions and more details are given in Appendix C.2. After plugging in Eq. (5.70) into the recursion formula and performing the Grassmann integration one can arrange the terms in a similar way as in the case of $\Phi_{34}(-is,r)$. We then notice, that Lemma 5.1 and Lemma 5.3 do not suffice alone. More identities are needed. The first one is given by

Lemma 5.4 *We have the same conditions as in Lemma 5.1, we define the operator*

$$
\widetilde{L}_m(s)=\sum_{q=1}^{k_1}\frac{1}{is_{m2}-s_{q1}}\frac{\partial^2}{\partial s_{q1}^2}+\frac{1}{2}\sum_{q\neq n}\frac{1}{(is_{m2}-s_{q1})(s_{q1}-s_{n1})}\left(\frac{\partial}{\partial s_{q1}}-\frac{\partial}{\partial s_{n1}}\right)\quad . \tag{5.74}
$$

Then we have

$$
\widetilde{L}_m(s)\int_{s_{21}}^{s_{11}}\cdots\int_{s_{k_1 1}}^{s_{(k_1-1)1}}\mu_B(s,s')d[s_1']f(s_1') =
$$

$$\int_{s_{21}}^{s_{11}} \cdots \int_{s_{1k_1}}^{s_{(k_1-1)1}} \mu_B(s,s') \left[\sum_{j=1} M_{mj}^{\rightarrow}(s_1,s_1') \frac{\partial}{\partial s_{j1}'} \right] f(s_1')\, d[s_1'] \,. \tag{5.75}$$

The proof goes along the same lines as the proof of Lemma 5.1 and the proof of theorem 4.1 in Appendix B.2. Thus, we have yet another prescription how to transform an operator $\widetilde{L}_m(s)$ acting onto an integral into an operator acting under the integral. However, we need one more such transformation rule to derive $\Phi_{44}(s,r)$. It prescribes how the product $\left[L_m(s)^{\rightarrow}\widetilde{L}_l(s)\right]$ transforms into operators acting under the integral. This rule is stated in Appendix C.2. Finally we arrive at

$$\begin{aligned}
\Phi_{44}(-is,r) &= 4\,\widehat{G}_{44}\ \exp\left(-\mathrm{tr}\,(r_2 s_2)\right) \prod_{i,j}(r_{1i}-ir_{2j}) \left[\left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)} + \frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)} \right) \right. \\
&\left(8\prod_{i=1}^{2}(r_{i1}-ir_{12})\prod_{j=1}^{4}(s_{j1}-is_{12}) + 4\sum_{i=1}^{4}\prod_{j\neq i}^{4}(s_{j1}-is_{12})\left(r_{11}+r_{21}-ir_{12}-\frac{\partial^{\rightarrow}}{\partial s_{i1}})\right)\right) \\
&\left(8\prod_{i=1}^{2}(r_{i1}-ir_{22})\prod_{j=1}^{4}(s_{j1}-is_{22}) + 4\sum_{i=1}^{4}\prod_{j\neq i}^{4}(s_{j1}-is_{22})(r_{11}+r_{21}-ir_{22}-\frac{\partial}{\partial s_{1i}})\right) \\
&-\frac{16}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\sum_{i}\prod_{j\neq i}^{4}(s_{j1}-is_{12})(s_{j1}-is_{22}) \\
&\left(r_{11}^2+r_{21}^2+r_{11}r_{21}-(ir_{12}+ir_{22})(r_{11}+r_{21})+ir_{12}ir_{22}+\mathrm{trg}\,(r/2)\frac{\partial}{\partial s_{i1}}\right) \\
&\left. -\frac{8}{\Delta_2^3(ir_2)\Delta_2^4(is_2)}\sum_{\substack{i=1\\ j=1}}^{4}\prod_{l\neq i}^{4}(s_{l1}-is_{12})\prod_{l\neq j}^{4}(s_{l1}-is_{22})\left(\frac{\partial}{\partial s_{i1}}-\frac{\partial}{\partial s_{j1}}\right)\right]\Phi_4^{(1)}(-is_1,r_1) \\
&+ \qquad (ir_{12}\longleftrightarrow ir_{22})\,.
\end{aligned} \tag{5.76}$$

We stress that in the derivation of Eq. (5.76) we extensively used certain properties of the matrix Bessel functions $\Phi_3^{(1)}(s_1,r_1)$ and $\Phi_4^{(1)}(s_1,r_1)$, which hold only for the case that one matrix has an additional degeneracy, cf. Eqs. (C.18) and (C.24). It is an open question, if in the general case more identities as Lemma 5.1 and Lemmata 5.3, 5.4 and the identity C.1 are needed.

5.4.1 Asymptotics

It is gratifying to see that our results for the matrix Bessel functions $\Phi_{k_1 2k_2}(-is,r)$, $k_1 \leq 4$, $k_2 \leq 2$ all have the correct asymptotic behavior for large arguments. We find from the expressions in Eqs. (5.33), (5.44) and in Eqs. (5.61), (5.70), (5.72) and (5.76)

$$\begin{aligned}
\lim_{\substack{s\to\infty\\ r\to\infty}} \Phi_{k_1 2k_2}(-is,r) &= 2^{k_1 k_2}\widehat{G}_{k_1 2k_2}\frac{\prod_{l=1}^{k_1}\prod_{m=1}^{k_2}(s_{l1}-is_{m2})(r_{l1}-ir_{m2})}{\Delta_{k_2}^2(is_2)\Delta_{k_2}^2(ir_2)} \\
&\det\left[\exp(2s_{i2}r_{j2})\right]_{i,j=1\ldots k_2}\ \lim_{\substack{s_1\to\infty\\ r_1\to\infty}}\Phi_{k_1}^{(1)}(-is_1,r_1) \quad .
\end{aligned} \tag{5.77}$$

In the degenerate case each degenerate eigenvalue contributes according to its multiplicity. As stated already in Section 2.2 the asymptotics of the matrix Bessel functions of the orthogonal group is given by

$$\lim_{t\to 0}\Phi^{(1)}_{k_1}(-is_1/t,r_1) \;=\widehat{C}^{(k_1)}t^{(k_1-1)k_1/4}\frac{\det[\exp(s_{n1}r_{m1}/t)]_{n,m=1,\ldots,k_1}}{|\Delta_{k_1}(s_1)\Delta_{k_1}(r_1)|^{1/2}}\ . \tag{5.78}$$

The constant $\widehat{C}^{(k_1)}$ can be found in Muirhead's book [MUI]

$$\widehat{C}^{(k_1)} = \frac{\Gamma(k_1/2)}{k_1!}\,\pi^{k_1^2/2-k_1/4}\quad . \tag{5.79}$$

Thus we have

$$\begin{aligned}\lim_{t\to 0}\Phi_{k_12k_2}(-is/t,r) &= 2^{k_1k_2}t^{1/4\left((k_1-2k_2)^2+(k_1-2k_2)\right)}\widehat{C}^{(k_1)}\widehat{G}_{k_12k_2}\\ &\frac{\det[\exp(s_{n1}r_{m1}/t)]_{n,m=1,\ldots,k_1}\det\left[\exp(2s_{i2}r_{j2}/t)\right]_{i,j=1\ldots k_2}}{\sqrt{\widetilde{B}_{k_1k_2}(s)\widetilde{B}_{k_1k_2}(r)}}\quad .\end{aligned} \tag{5.80}$$

On the other hand we derive from the Gaussian integral the generalized diffusion kernel in the curved space of the eigenvalues, cf. Eq. (2.66).

$$\begin{aligned}\Gamma_{k_1k_2}(s,r,t) \;&=\; \left(\frac{\pi}{2t}\right)^{-1/4\left((k_1-2k_2)^2+(k_1-2k_2)\right)}2^{k_2^2-k_2-k_1/2}\\ &\exp\left(-\frac{1}{t}\left(\operatorname{trg}s^2\;+\;\operatorname{trg}r^2\right)\right)\Phi_{k_12k_2}(-is/t,r)\end{aligned} \tag{5.81}$$

This kernel has, in generalization of Eq. (2.71) the asymptotic behavior

$$\begin{aligned}\lim_{t\to 0}\Gamma_{k_1k_2}(s,r,t) \;&=\; \left(\frac{\pi}{2}\right)^{-1/4\left((k_1-2k_2)^2+(k_1-2k_2)\right)}\frac{2^{k_2^2-k_2-k_1/2}}{k_1!\;k_2!}\\ &\frac{\det\left[\delta(s_{i1}-r_{j1})\right]_{i,j=1\ldots k_1}\det\left[\delta(s_{i2}-r_{j2})\right]_{i,j=1\ldots k_2}}{\sqrt{\widetilde{B}_{k_1k_2}(s)\widetilde{B}_{k_1k_2}(r)}}\quad .\end{aligned} \tag{5.82}$$

Comparing Eq. (5.80) with Eq. (5.82) determines the normalization constant $\widehat{G}_{k_1k_2}$.

$$\widehat{G}_{k_12k_2} = \frac{1}{k_2!\Gamma(k_1/2)}\pi^{-\frac{1}{4}\left((k_1-2k_2)^2+2k_1^2-2k_2\right)}\,2^{3k_2(k_2-k_1)+k_1^2/4-5k_2/2-k_1/2}\quad . \tag{5.83}$$

Particularly we have verified that the diffusion kernel of the one–point function and of the two–point function of Dyson's Brownian motion

$$\begin{aligned}\Gamma_k(s,r,t) \;&\equiv\; \Gamma_{(2k)k}(s,r,t)\\ &=\; 2^{k(k-2)}\exp\left(-\frac{1}{t}\left(\operatorname{trg}s^2+\operatorname{trg}r^2\right)\right)\Phi_{(2k)(2k)}(-i2s/t,r)\quad ,\end{aligned} \tag{5.84}$$

yields indeed the initial condition (2.66).

5.5 Summary of Chapter 5 and outlook

The recursion formula derived in Chapter 4 has an analogue in superspace. By means of the recursion formula we derived explicit expressions for the matrix Bessel functions of the unitary orthosymplectic groups $\Phi_{22}(-is, r)$ and $\Phi_{44}(-is, r)$, which yield the diffusion kernel of the one–point function and the two–point function of Dyson's Brownian motion.

One major advantage of the radial Gelfand–Tzetlin coordinates in superspace is that the Grassmann variables only appear as moduli squared. Thus, the number of Grassmann integrals is a priori reduced by the half. This is an extremely welcome feature of this parametrization. The evaluation of the Grassmann integrals was performed level by level. With respect to the recursion formula this was the natural way to proceed. It has the advantage that one has to perform only a reduced number of integrals, more precisely k_2, over anticommuting variables in each level. Thus the resulting expressions are large but feasible. The paricular advantage of this way to proceed is that the structure of the matrix Bessel functions is not or very little, influenced by the matrix dimension. Afterwards we have seen that the structure of $\Phi_{44}(-is, r)$ was already apparent in $\Phi_{14}(-is, r)$. The matrix Bessel functions in ordinary space showed a similar feature. There, the structure of the Matrix Bessel functions is much more influenced by the group parameter β than by the matrix dimension. However, as in ordinary space it remains as a demanding task to find the real structure of these matrix Bessel function for arbitrary matrix dimension. Our results may serve as a guideline how a general solutions looks like.

The main difficulty to be overcome was the appearance of total derivatives in the integral over the commuting variables after performing the Grassmann integration. These total derivatives occured already in ordinary space. This lead us to the conclusion that their occurrence is an intrinsic property of the recursion formula. They were removed by a set of identities between operators acting onto the integral in the recursion formula (5.8) and operators acting under the integral. The nature of these operators, particularly their relation to the Laplacian (5.6) has not become clear so far. We stress that the understanding of these operator identities is crucial for the evaluation of matrix Bessel functions for arbitrary matrix dimension. Moreover, similar operator identities may be useful in ordinary space to evaluate the matrix Bessel function of the unitary symplectic group.

The radial Gelfand–Tzetlin coordinates are as in ordinary space the natural coordinate system for the matrix Bessel functions in superspace. This parametrization represents the appropriate tool for the recursive integration of Grassmann variables. Once the particular features of this parametrization are better understood, they may allow for the evaluation of higher dimensional group integral and thereby allow for the calculation of higher order k–point functions of random matrix models with the supersymmetric method.

Chapter 6

Applications

In this chapter we apply the results of Chapter 5 to our matrix model (2.47). It is both a summary of the results of the last chapter as well as an outlook. We give integral expressions for the level density and the two–point correlation function of the random matrix model Eq. (1.1) which are exact for all transition parameters and all N.

6.1 Level density

We restrict ourselves to the transition towards the GOE and suppress the index c. The corresponding formulae for the transition towards the GSE are derived accordingly. In Section 5.3 we derived the matrix Bessel function

$$\Phi_{22}(-is,r) = \frac{(2\pi)^{-1/2}}{4}\exp\left(\mathrm{trg}\,(rs) - \frac{z}{2}\right)$$
$$\left(4\prod_{j=1}^{2}(ir_{12}-r_{j1})(is_{12}-s_{j1}) - \sum_{q=1}^{2}(is_{12}-s_{q1})\sum_{p=1}^{2}(ir_{12}-r_{p1}) - z\frac{d}{dz}\right) I_0(z/2)\ . \tag{6.1}$$

Where $I_0(z)$ is the modified Bessel function. With the replacement $r \longrightarrow (x \pm J)$ and $s \longrightarrow s/t$ we obtain the diffusion kernel for the level density

$$\Gamma(s,x+J,t) = (2\pi)^{-1/2}\frac{J_1}{2t}$$
$$\exp\left(-\frac{1}{t}(s_{11}-x_1-J_1)^2 - \frac{1}{t}(s_{21}-x_1-J_1)^2 + \frac{2}{t}(is_{12}-x_1+J_1)^2\right)$$
$$\left(-2\frac{J_1}{t}\prod_{j=1}^{2}(is_{12}-s_{j1}) + \sum_{q=1}^{2}(is_{12}-s_{q1})\right)\quad . \tag{6.2}$$

This yields after inserting into Eq. (2.65) and with Eq. (2.50) the level density

$$\begin{aligned}\widehat{R}_1(x_1,t) \quad = \quad & \frac{(2\pi)^{-3/2}}{t}\int\exp\left(-\frac{1}{t}(s_{11}-x_1)^2 - \frac{1}{t}(s_{21}-x_1)^2 + \frac{1}{t}(is_{12}-x_1)^2\right)\\ & \sum_{q=1}^{2}(is_{12}-s_{q1})\widetilde{B}_{21}(s)\ Z_1^{(0)}(s)\,d[s]\quad ,\end{aligned} \tag{6.3}$$

where the Berezinian was defined in Eq. (2.68). This result holds exactly for an arbitrary initial condition and for arbitrary N. For the initial condition we derive from Eq. (2.63) for the transition to the GOE

$$Z_1^{(0)}(s) = \int d[H^{(0)}]P(H^{(0)}) \prod_{n=1}^{N} \frac{\prod_{j=1}^{k}(is_{j2} - H_{nn}^{(0)})}{\prod_{j=1}^{2k}(s_{j1} + i\varepsilon - H_{nn}^{(0)})^{1/2}} \ . \tag{6.4}$$

In the limit $t \to \infty$ the stationary distribution of classical Gaussian random matrix theory is recovered. This can be seen by re–writing Eq. (6.2) for the rescaled energy $\widetilde{x}_1 = x_1/t$ and the rescaled source variable $\widetilde{J}_1 = J_1/t$, see also [GUH4]. In this limit the average over the initial condition yields unity and we arrive at an integral representation of the one–point correlation function of the classical orthogonal Gaussian ensemble

$$\begin{aligned} R_1(x_1) = (2\pi)^{-3/2}\Im \int \exp\left(-(s_{11} - x_1)^2 - (s_{21} - x_1)^2 + (is_{12} - x_1)^2\right) \\ \frac{|s_{11} - s_{21}|}{(is_{12} - s_{11})(is_{12} - s_{21})} \left(\frac{1}{is_{12} - s_{11}} + \frac{1}{is_{12} - s_{21}}\right) \frac{(is_{12})^N}{(s_{11} + i\varepsilon)^{N/2}(s_{21} + i\varepsilon)^{N/2}} \, d[s] \end{aligned} \tag{6.5}$$

where the symbol $\Im$ denotes the imaginary part. Eq. (6.5) is equivalent to the classical expressions for the one–point functions as derived by Mehta and Gaudin [MEH1].

Finally we state an integral expression for the one–point function for the case of Poissonian initial conditions. The Poisson ensemble is characterized by the absence of correlations between the diagonal elements of $H^{(0)}$. All higher correlation functions factorize into a product of one–point functions. This is most easily done by defining

$$P(H^{(0)})d[H^{(0)}] = \prod_{n=1}^{N} p^{(0)}\left(H_{nn}^{(0)}\right) \prod_{n<m} \delta\left(H_{mn}^{(0)}\right) \ . \tag{6.6}$$

Inserting this distribution into Eq. (6.4) we derive

$$Z_1^{(0)}(s) = \left(\int dz p(z) \frac{\prod_{j=1}^{k}(is_{j2} - z)}{\prod_{j=1}^{2k}(s_{j1} + i\varepsilon - z)^{1/2}}\right)^N \ , \tag{6.7}$$

as initial condition for the diffusion towards orthogonal chaos. Inserting this initial condition into Eq. (6.3) yields the level density of a transition ensemble between Poisson regularity and GOE chaos in terms of a fourfold integral. A further analysis seems to be possible and is highly desirable. In a first step one might evaluate Eq. (6.3) with the initial condition (6.7) on the scale of the mean level spacing and verify that it yields unity. Another interesting problem is to choose a deterministic initial condition, i. e. a fixed matrix $H^{(0)}$ which takes only two different values, a and $-a$. With increasing t the gap between the two eigenvalues closes. Brézin and Hikami [BH1] investigated the critical behavior of the level density at the closure of the gap by using the to Eq. (6.3) corresponding integral representation of the unitary ensemble [GUH4]. The same analysis should be possible with Eq. (6.3) in the GOE and GSE case.

6.2 Two–point function

In Section 5.4 we derived the matrix Bessel function $\Phi_{44}(-is,r)$. This yields an integral expression for the two–point function. The derivation of the corresponding formulae for the GSE being straightforward, we restrict ourselves again to the diffusion towards the orthogonal chaos.

$$\begin{aligned}
\Phi_{44}(-is,r) &= 4\,\widehat{G}_{44}\ \exp\left(\mathrm{tr}\,(r_2 s_2)\right)\prod_{i,j}(r_{i1}-ir_{j2})\left[\left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)}+\frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\right)\right.\\
&\left(8\prod_{i=1}^{2}(r_{i1}-ir_{12})\prod_{j=1}^{4}(s_{j1}-is_{12})+4\sum_{i=1}^{4}\prod_{j\neq i}^{4}(s_{j1}-is_{12})\left(r_{11}+r_{21}-ir_{12}-\frac{\partial^{\rightarrow}}{\partial s_{i1}})\right)\right)\\
&\left(8\prod_{i=1}^{2}(r_{i1}-ir_{22})\prod_{j=1}^{4}(s_{j1}-is_{22})+4\sum_{i=1}^{4}\prod_{j\neq i}^{4}(s_{j1}-is_{22})(r_{11}+r_{21}-ir_{22}-\frac{\partial}{\partial s_{i1}})\right)\\
&-\frac{16}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\sum_{i=1}^{4}\prod_{j\neq i}^{4}(s_{j1}-is_{12})(s_{j1}-is_{22})\\
&\left(r_{11}^2+r_{21}^2+r_{11}r_{12}-(ir_{21}+ir_{22})(r_{11}+r_{21})+ir_{12}ir_{22}+\mathrm{trg}\,(r/2)\frac{\partial}{\partial s_{i1}}\right)\\
&\left.-\frac{8}{\Delta_2^3(ir_2)\Delta_2^4(is_2)}\sum_{\substack{i=1\\ j=1}}^{4}\prod_{l\neq i}^{4}(s_{l1}-is_{12})\prod_{k\neq j}^{4}(s_{k1}-is_{12})\left(\frac{\partial}{\partial s_{i1}}-\frac{\partial}{\partial s_{j1}}\right)\right]\Phi_4(s_1,r_1)\\
&\qquad + \qquad (ir_{12}\longleftrightarrow ir_{22}) \qquad , \qquad\qquad (6.8)
\end{aligned}$$

where the arrow is to be understood according to Eq. (5.61) in chapter 5. The term $(ir_{12}\longleftrightarrow ir_{22})$ accounts to the permutation group $\mathcal{S}^2$ acting on ir_2 in the fermion–fermion block. With the replacement $r\longrightarrow(x\pm J)$ and $s\longrightarrow s/t$ we obtain the diffusion kernel for the two–point correlation function

$$\begin{aligned}
\Gamma_k(s,x+J,t) &= \exp\left(-\frac{1}{t}\left(\mathrm{trg}\,s^2+\mathrm{trg}\,(x\pm J)^2\right)\right)\\
&\qquad \Phi_{44}(-2is/t,x\pm J) \quad . \qquad (6.9)
\end{aligned}$$

A considerable simplification is achieved through the derivative with respect to the source terms. We find for the two point correlation function

$$\begin{aligned}
\widehat{R}_2(x_1,x_2,t) &= 2^8\widehat{G}_{44}\,t^{-4}\pi^{-2}\int\left(\widetilde{B}_{42}(s)Z_2^{(0)}(s)\frac{\prod_{k=1}^4(is_{12}-s_{k1})(is_{22}-s_{k1})}{(is_{12}-is_{22})^2}\right.\\
&\exp\left(-\mathrm{tr}\,s_1^2/t-2x_1^2/t-2x_2^2/t+2(is_{12}-x_1)^2/t+2(is_{22}-x_2)^2/t\right)\\
&\sum_{k,j}^{4}\left[\frac{1}{(is_{12}-s_{k1})(is_{22}-s_{j1})}\left(x_1-t\,\frac{\partial}{\partial s_{j1}}\right)\left(x_2-t\,\frac{\partial}{\partial s_{k1}}\right)+\right.\\
&\frac{t}{(x_1-x_2)(is_{12}-is_{22})(is_{12}-s_{k1})(is_{22}-s_{j1})}\left(x_1-t\,\frac{\partial}{\partial s_{j1}}\right)\left(x_2-t\,\frac{\partial}{\partial s_{k1}}\right)-\\
&\left.\left.\frac{1}{2}\frac{t^3}{(x_1-x_2)(is_{12}-s_{k1})(is_{22}-s_{j1})(is_{12}-is_{22})^2}\left(\frac{\partial}{\partial s_{j1}}-\frac{\partial}{\partial s_{k1}}\right)\right]\Phi_4^{(1)}(-2is_1/t,x)\right)\ d[s]\\
&\qquad + \qquad (x_1\longleftrightarrow x_2) \quad , \qquad\qquad (6.10)
\end{aligned}$$

where we introduced

$$x = \mathrm{diag}\,(x_1, x_1, x_2, x_2) \qquad \text{and} \qquad d[s] = \prod_{j=1}^{4} ds_{j1} ds_{12} ds_{22} \quad . \tag{6.11}$$

The last line indicates that the integral with x_1 and x_2 interchanged has to be added. This yields yet another simplification, since all terms in Eq. (6.10) antisymmetric under interchange of x_1 and x_2 drop out. We arrive at the remarkably compact expression

$$\begin{aligned} R_2(x_1, x_2, t) = 2^8 \widehat{G}_{44}\, t^{-4} \pi^{-2}\, \Im \int \Bigg(& \sqrt{\widetilde{B}_{42}(s)} \sqrt{|\Delta_4(s_1)|} Z_2^{(0)}(s) \\ & \exp\left(-\mathrm{tr}\, s_1^2/t - 2x_1^2/t - 2x_2^2/t + 2(is_{12} - x_1)^2/t + 2(is_{22} - x_2)^2/t\right) \\ & \sum_{k,j}^{4} \left[\frac{1}{(is_{12} - s_{k1})(is_{22} - s_{j1})} \left(x_1 - t\frac{\partial}{\partial s_{j1}} \right) \left(x_2 - t\frac{\partial}{\partial s_{k1}} \right) \right] \Phi_4^{(1)}(-2is_1/t, x) \Bigg)\, d[s] \end{aligned} \tag{6.12}$$

The symbol $\Im$ denotes a certain linear combination of $\widehat{R}_2(x_1, x_2, t)$ as explained in Section 2.3. The normalization constant is derived from Eq. (5.83)

$$\widehat{G}_{44} = 2(2\pi)^{-4} \tag{6.13}$$

This result yields an exact expression for the two–point function of Dyson's Brownian motion for every initial condition. Plugging in the initial condition Eq. (6.7) yields an exact integral representation of the two–point function for our random matrix model Eq. (1.1).

Crucial for a further evaluation is to find a convenient representation for the matrix Bessel function of the orthogonal group $\Phi_4^{(1)}(-2is_1/t, x)$, which enters in the integral representation of the two–point function. We derived in Section 4.2.3 a representation in terms of a twofold integral

$$\begin{aligned} \Phi_1^{(1)}(s_1, x) = {} & B_4^{(1)} (x_2 - x_1)^{-2} \exp\left(i\, x_1\, \mathrm{Tr}\, s_1\right) \\ & \int dt_1 dt_2 \exp\left((x_2 - x_1)(t_1 + t_2)\right) |t_1 - t_2| \prod_{n=1}^{4} \prod_{j=1}^{2} (s_{n1} - t_j)^{-1/2} \quad , \end{aligned} \tag{6.14}$$

where $B_4^{(1)}$ is the normalization constant given in Eq. (4.36). As in the case of the one point–function, an integral representation of the two–point correlation function of the classical Gaussian random matrix ensemble is obtained by setting the elements of the diagonal matrix $H^{(0)}$ to zero. It is well known that the k–point correlation functions of the Gaussian random matrix ensembles decompose into a quaternionic determinant of a self–dual matrix. That means the two–point function can be written schematically as

$$R_2(x_1, x_2) = S(x_1, x_1)\, S(x_2, x_2) - S(x_1, x_2)\, S(x_2, x_1) + JS(x_1, x_2)\, DS(x_2, x_1) \ . \tag{6.15}$$

The exact expressions for the functions $S(x, y)$, $JS(x, y)$, $DS(x, y)$ can be found in [MEH1, GMGW].

It is of greatest interest to see how this structure of a quaternionic determinant comes about in our integral representation (6.12). In the case of the GUE it turned out that the structure of a quaternionic determinant appeared already in the diffusion kernel $\Gamma_k(s, x)$

[GUH1, GUH4]. This followed as a direct consequence of the supersymmetric Itzykson–Zuber integral. In that case the originally fourfold integral representation of the two–point function decomposed into a sum of products of twofold integrals *before* performing the eigenvalue integrals. Thereby, with the two–point function one had readily derived the k–point function due to the structure of a quaternionic determinant of the k–point correlator. This means, that a coupling to a deterministic matrix $H^{(0)}$ does not destroy the quaternionic structure of the correlation functions. Only averaging over $H^{(0)}$ destroys this structure.

It would be extremely convenient if a similar property also exists for our sixfold integral representation (6.12). In that case it should decompose into a sum of products of threefold integrals *without* performing any eigenvalue integration. Certainly, the clue of any further analysis of Eq. (6.12) is to find the most convenient representation of $\Phi_4^{(1)}(s_1, x)$. However, in our opinion it is extremely probable that such a decomposition exists. The property that a coupling of a random matrix to a deterministic one does not alter the structure of a quaternionic determinant should not be restricted to the GUE but also apply to the GOE and GSE.

6.3 Resumee

We derived integral expressions of the one–point function, Eq. (6.3) and the two–point function Eq. (6.12) of Dyson‘s Brownian motion model. With the initial condition (6.7) we get an integral representation of the one and two–point correlation functions of the random matrix model (1.1) for arbitrary transition parameter and arbitrary matrix dimension.

However, we stress that the methods we used to derived these expressions have a value on their own right. Specifically the recursion formula yields new insights in the complicated structure of matrix Bessel functions. These functions occur in a much wider range in physics and mathematics than random matrix theory. It came as a great surprise that there exist closed expressions for group integrals beyond the Harish–Chandra case. The calculation of the group integrals over the unitary symplectic group for arbitrary matrix dimension seems to us as important as the further analysis of the integral expression of the two–point function Eq. (6.12).

Also the fact that the recursion formula represents a generalization of the matrix Bessel functions to arbitrary group index β opens another wide field for further investigations.

Appendix A

Calculations to Chapter 3

A.1 Solution of Equations (3.31) to (3.33)

We consider equation (3.31). If we plug in the solutions for $|v_m^{(n)}|^2$ and $|\alpha_m^{(n)}|^2$ given in (3.34) the right hand side can be expanded in a sum of monomials in the nilpotent Gelfand–Tzetlin variables $|\xi_q^{(n)}|^2$, $q = 1, \ldots, k_2$. Since each of the $|\xi_q^{(n)}|^2$ only appears linearly, the rank of the monomials cannot exceed k_2. Formally we can rewrite Eq. (3.31) as follows:

$$\frac{1}{|w_p^{(n)}|^2} = \sum_{r=0}^{k_2} M^{(r)} \tag{A.1}$$

Where $M^{(r)}$ is the nilpotent part of $1/|w_p^{(n)}|^2$, which consists of monomials in $|\xi_q^{(n)}|^2$ of rank r. Explicitly we have for $M^{(r)}$, $r = 1, \ldots, k_2$,

$$\begin{aligned} M^{(r)} \quad = \quad & \sum_{j_1 \leq j_2 \leq \ldots \leq j_r}^{k_2} \sum_{m}^{\frac{k_1-n+1}{2}} \frac{\prod_{q=1, q\neq p}^{\frac{k_1-n-1}{2}} \left((h_{m1}^{(n-1)})^2 - (h_{q1}^{(n)})^2 \right)}{\prod_{q=1, q\neq m}^{\frac{k_1-n+1}{2}} \left((h_{m1}^{(n-1)})^2 - (h_{q1}^{(n-1)})^2 \right)} \frac{(h_{m1}^{(n-1)})^2 + (h_{p1}^{(n)})^2}{(h_{m1}^{(n-1)})^2 - (h_{p1}^{(n)})^2} \\ & \frac{\prod_{i=1}^{r} |\xi_{j_i}|^2}{\prod_{i=1}^{r} \left((h_{m1}^{(n-1)})^2 - (ih_{j_i 2}^{(n-1)})^2 \right)} \quad + \\ & \sum_{\substack{j_2 \leq \ldots \leq j_r \\ j_1 \neq j_i}}^{k_2} \frac{\prod_{q=1, q\neq p}^{\frac{k_1-n-1}{2}} \left((ih_{j_1 2}^{(n-1)})^2 - (h_{q1}^{(n)})^2 \right)}{\prod_{q=1}^{\frac{k_1-n+1}{2}} \left((ih_{j_1 2}^{(n-1)})^2 - (h_{q1}^{(n-1)})^2 \right)} \frac{(ih_{j_1 2}^{(n-1)})^2 + (h_{p1}^{(n)})^2}{(ih_{j_1 2}^{(n-1)})^2 - (h_{p1}^{(n)})^2} \\ & \frac{\prod_{i=1}^{r} |\xi_{j_i}|^2}{\prod_{i=2}^{r} \left((ih_{j_1 2}^{(n-1)})^2 - (ih_{j_i 2}^{(n-1)})^2 \right)} \quad . \end{aligned} \tag{A.2}$$

The sum over m is the Laplace expansion of a determinant. For its evaluation we use the formula

$$\begin{aligned} \frac{1}{\prod_{i=1}^{r} \left((h_{m1}^{(n-1)})^2 - (ih_{i2}^{(n-1)})^2 \right)} \quad = \quad & \sum_{i=1}^{r} \frac{1}{(h_{m1}^{(n-1)})^2 - (ih_{i2}^{(n-1)})^2} \\ & \frac{1}{\prod_{i' \neq i}^{r} \left((ih_{i2}^{(n-1)})^2 - (ih_{i'2}^{(n-1)})^2 \right)} \end{aligned} \tag{A.3}$$

which is well known from complex analysis. After symmetrizing the second sum in the indices, j_i, $i = 1, \ldots, r$, we arrive at the following expression for $M^{(r)}$.

$$
\begin{aligned}
M^{(r)} \quad = \quad & \sum_{j_1 \le j_2 \le \ldots \le j_r}^{k_2} \sum_{i=1}^{r} \frac{1}{\prod_{i' \neq i}^{r} \left((ih_{j_i 2}^{(n-1)})^2 - (ih_{j_{i'} 2}^{(n-1)})^2 \right)} \\
& \sum_{m}^{\frac{k_1-n+1}{2}} \frac{\prod_{q=1, q\neq p}^{\frac{k_1-n-1}{2}} \left((h_{m1}^{(n-1)})^2 - (h_{q1}^{(n)})^2 \right)}{\prod_{q=1, q\neq m}^{\frac{k_1-n+1}{2}} \left((h_{m1}^{(n-1)})^2 - (h_{q1}^{(n-1)})^2 \right)} \\
& \left[\frac{(h_{m1}^{(n-1)})^2 + (h_{p1}^{(n)})^2}{\left((h_{p1}^{(n)})^2 - (ih_{j_i 2}^{(n-1)})^2 \right) \left((h_{m1}^{(n-1)})^2 - (h_{p1}^{(n)})^2 \right)} - \right. \\
& \left. \frac{(h_{m1}^{(n-1)})^2 + (h_{p1}^{(n)})^2}{\left((h_{p1}^{(n)})^2 - (ih_{j_i 2}^{(n-1)})^2 \right) \left((h_{m1}^{(n-1)})^2 - (ih_{j_i 2}^{(n-1)})^2 \right)} \right] \prod_{i=1}^{r} |\xi_{j_i}|^2 \\
+ \quad & \sum_{j_1 \le j_2 \le \ldots \le j_r}^{k_2} \sum_{i=1}^{r} \frac{1}{\prod_{i' \neq i}^{r} \left((ih_{j_i 2}^{(n-1)})^2 - (ih_{j_{i'} 2}^{(n-1)})^2 \right)} \\
& \frac{\prod_{q=1, q\neq p}^{\frac{k_1-n-1}{2}} \left((ih_{j_1 2}^{(n-1)})^2 - (h_{q1}^{(n)})^2 \right)}{\prod_{q=1}^{\frac{k_1-n+1}{2}} \left((ih_{j_1 2}^{(n-1)})^2 - (h_{q1}^{(n-1)})^2 \right)} \frac{(ih_{j_1 2}^{(n-1)})^2 + (h_{p1}^{(n)})^2}{(ih_{j_1 2}^{(n-1)})^2 - (h_{p1}^{(n)})^2} \prod_{i=1}^{r} |\xi_{j_i}|^2 \quad .
\end{aligned}
\tag{A.4}
$$

Now the determinant can be evaluated, using the translational invariance of the Vandermonde determinant $\Delta_N(x) = \Delta_N(x-c)$. Whereas the second term in the squared bracket cancels completely, the second sum the first term in the squared bracket yields

$$
\begin{aligned}
M^{(r)} \quad = \quad & \sum_{j_1 \le j_2 \le \ldots \le j_r}^{k_2} \sum_{i=1}^{r} \frac{1}{\prod_{i' \neq i}^{r} \left((ih_{j_i 2}^{(n-1)})^2 - (ih_{j_{i'} 2}^{(n-1)})^2 \right)} \\
& \frac{2(h_{p1}^{(n)})^2 \prod_{q=1, q\neq p}^{\frac{k_1-n-1}{2}} \left((h_{q1}^{(n)})^2 - (h_{p1}^{(n)})^2 \right)}{\left((h_{p1}^{(n)})^2 - (ih_{j_i 2}^{(n-1)})^2 \right) \prod_{m=1}^{\frac{k_1-n+1}{2}} \left((h_{m1}^{(n-1)})^2 - (h_{p1}^{(n)})^2 \right)} \prod_{i=1}^{r} |\xi_{j_i}|^2 \quad .
\end{aligned}
\tag{A.5}
$$

Using again identity (A.3) and summing over r gives

$$
\begin{aligned}
\frac{1}{|w_p^{(n)}|^2} \quad = \quad & \frac{2(h_{p1}^{(n)})^2 \prod_{q=1, q\neq p}^{\frac{k_1-n-1}{2}} \left((h_{q1}^{(n)})^2 - (h_{p1}^{(n)})^2 \right)}{\prod_{m=1}^{\frac{k_1-n+1}{2}} \left((h_{m1}^{(n-1)})^2 - (h_{p1}^{(n)})^2 \right)} \\
& \left(\sum_{r=0}^{k_2} \sum_{j_1 \le j_2 \le \ldots \le j_r}^{k_2} \prod_{i=1}^{r} \frac{|\xi_{j_i}|^2}{(h_{p1}^{(n)})^2 - (ih_{j_i 2}^{(n-1)})^2} \right) \quad .
\end{aligned}
\tag{A.6}
$$

Observing that the double sum in (A.6) amounts to

$$\prod_{q=1}^{k_2}\left(1+\frac{|\xi_q^{(n)}|^2}{(h_{p1}^{(n)})^2-(ih_{q2}^{(n-1)})^2}\right) \quad , \tag{A.7}$$

and using the definition (3.36) of $|\xi_q^{(n)}|^2$ we arrive at the final result

$$|w_p^{(n)}|^2 = \frac{\prod_{m=1}^{\frac{k_1-n+1}{2}}\left((h_{m1}^{(n-1)})^2-(h_{p1}^{(n)})^2\right)\prod_{q=1}^{k_2}\left((h_{p1}^{(n)})^2-(ih_{q2}^{(n)})^2\right)}{2(h_{p1}^{(n)})^2\prod_{q=1,q\neq p}^{\frac{k_1-n-1}{2}}\left((h_{p1}^{(n)})^2-(h_{q1}^{(n)})^2\right)\prod_{q=1}^{k_2}\left((h_{p1}^{(n)})^2-(ih_{q2}^{(n-1)})^2\right)} \quad . \tag{A.8}$$

Equation (3.32) and the corresponding equation for the odd levels are evaluated similarly, yielding the results stated in Section 3.4. Equation (3.33) has to be treated differently due to the Grassmann singularities, occurring on the left hand side. After plugging in the expressions Eq. (3.34) into (3.33) we have

$$\begin{aligned} 1 = {} & |\beta_p^{(n)}|^2 \\ & \left(\sum_{m}^{\frac{k_1-n+1}{2}}\frac{(h_{m1}^{(n-1)})^2+(ih_{p2}^{(n)})^2}{\left((h_{m1}^{(n-1)})^2-(ih_{p2}^{(n)})^2\right)^2}|v_m^{(n)}|^2+\sum_{\substack{m'=1\\ m'\neq p}}^{k_2}\frac{(ih_{m'2}^{(n-1)})^2+(ih_{p2}^{(n)})^2}{\left((ih_{m'2}^{(n-1)})^2-(ih_{p2}^{(n)})^2\right)^2}|\alpha_{m'}^{(n)}|^2\right. \\ & +\frac{\prod_{q=1}^{\frac{k_1-n-1}{2}}\left((ih_{p2}^{(n-1)})^2-(h_{q1}^{(n)})^2\right)\prod_{q=1,q\neq p}^{k_2}\left((ih_{p2}^{(n-1)})^2-(ih_{q2}^{(n-1)})^2\right)\Big)}{\prod_{q=1,q\neq p}^{k_2}\left((ih_{p2}^{(n-1)})^2-(ih_{q2}^{(n)})^2\right)\prod_{q=1}^{\frac{k_1-n+1}{2}}\left((ih_{p2}^{(n-1)})^2-(h_{q1}^{(n-1)})^2\right)} \\ & \left.\frac{(ih_{p2}^{(n-1)})^2+(ih_{p2}^{(n)})^2}{|\xi_p{}^{(n)}|^2}\right) \quad . \end{aligned} \tag{A.9}$$

In order to cancel the singularity, $|\beta_p^{(n)}|^2$ has to be expanded as $c_p^{(n)}\,|\xi_p^{(n)}|^2$. The expansion coefficient $c_p^{(n)}$ now contains a nonvanishing body and therefore its inverse is well defined. Dividing both sides by $c_p^{(n)}$ and ordering the right hand side by powers of $|\xi_p^{(n)}|^2$ one arrives at

$$\begin{aligned} \frac{1}{c_p^{(n)}} = {} & \\ & 2(ih_{p2}^{(n-1)})^2\frac{\prod_{q=1}^{\frac{k_1-n-1}{2}}\left((ih_{p2}^{(n-1)})^2-(h_{q1}^{(n)})^2\right)\prod_{q=1,q\neq p}^{k_2}\left((ih_{p2}^{(n-1)})^2-(ih_{q2}^{(n-1)})^2\right)\Big)}{\prod_{q=1,q\neq p}^{k_2}\left((ih_{p2}^{(n-1)})^2-(ih_{q2}^{(n)})^2\right)\prod_{q=1}^{\frac{k_1-n+1}{2}}\left((ih_{p2}^{(n-1)})^2-(h_{q1}^{(n-1)})^2\right)}+ \\ & \left(\sum_{m}^{\frac{k_1-n+1}{2}}\frac{(h_{m1}^{(n-1)})^2+(ih_{p2}^{(n)})^2}{\left((h_{m1}^{(n-1)})^2-(ih_{p2}^{(n)})^2\right)^2}|v_m^{(n)}|^2+\sum_{\substack{m'=1\\ m'\neq p}}^{k_2}\frac{(ih_{m'2}^{(n-1)})^2+(ih_{p2}^{(n)})^2}{\left((ih_{m'2}^{(n-1)})^2-(ih_{p2}^{(n)})^2\right)^2}|\alpha_{m'}^{(n)}|^2-\right. \\ & \left.\frac{\prod_{q=1}^{\frac{k_1-n-1}{2}}\left((ih_{p2}^{(n-1)})^2-(h_{q1}^{(n)})^2\right)\prod_{q=1,q\neq p}^{k_2}\left((ih_{p2}^{(n-1)})^2-(ih_{q2}^{(n-1)})^2\right)\Big)}{\prod_{q=1,q\neq p}^{k_2}\left((ih_{p2}^{(n-1)})^2-(ih_{q2}^{(n)})^2\right)\prod_{q=1}^{\frac{k_1-n+1}{2}}\left((ih_{p2}^{(n-1)})^2-(h_{q1}^{(n-1)})^2\right)}\right)|\xi_p^{(n)}|^2 \quad . \end{aligned} \tag{A.10}$$

Since $c_p^{(n)}$ and therefore also $1/c_p^{(n)}$ are of order zero in $|\xi_p^{(n)}|^2$ the whole term in round brackets can be neglected. Indeed it can be shown by manipulations similar to those which led to Eq. (A.8), that this term leads just to a shift of $(ih_{p2}^{(n-1)})^2 \to (ih_{p2}^{(n)})^2$ in the resulting expression for $c_p^{(n)}$. This does not affect $|\beta_p^{(n)}|^2$. One arrives at

$$|\beta_p^{(n)}|^2 = \left((ih_{p2}^{(n)})^2 - (ih_{p2}^{(n-1)})^2\right) \frac{\prod_{q=1,q\neq p}^{k_2}\left((ih_{p2}^{(n)})^2 - (ih_{q2}^{(n)})^2\right)\prod_{q=1}^{\frac{k_1-n+1}{2}}\left((ih_{p2}^{(n)})^2 - (h_{q1}^{(n-1)})^2\right)}{2(ih_{p2}^{(n)})^2\prod_{q=1}^{\frac{k_1-n-1}{2}}\left((ih_{p2}^{(n)})^2 - (h_{q1}^{(n)})^2\right)\prod_{q=1,q\neq p}^{k_2}\left((ih_{p2}^{(n)})^2 - (ih_{q2}^{(n-1)})^2\right)} . \tag{A.11}$$

The equations for the odd levels are treated in the same way.

A.2 Solutions for the odd levels

We state the odd solutions of the Gelfand–Tzetlin equations, i .e. the solutions of (3.25) to (3.27)

$$|v_p^{(n)}|^2 = \frac{\prod_{q=1}^{\frac{k_1-n}{2}}\left((h_{p1}^{(n-1)})^2 - (h_{q1}^{(n)})^2\right)\prod_{q=1}^{k_2}\left((h_{p1}^{(n-1)})^2 - (ih_{q2}^{(n-1)})^2\right)}{(h_{p1}^{(n-1)})^2\prod_{q=1,q\neq p}^{\frac{k_1-n}{2}}\left((h_{p1}^{(n-1)})^2 - (h_{q1}^{(n-1)})^2\right)\prod_{q=1}^{k_2}\left((h_{p1}^{(n-1)})^2 - (ih_{q2}^{(n)})^2\right)}$$

$$p = 1, \ldots, (k_1-n)/2 ,$$

$$|v_{\frac{k_1-n}{2}+1}^{(n)}|^2 = \frac{\prod_{q=1}^{\frac{k_1-n}{2}}(h_{q1}^{(n)})^2\prod_{q=1}^{k_2}(ih_{q2}^{(n-1)})^2}{\prod_{q=1}^{\frac{k_1-n}{2}}(h_{q1}^{(n-1)})^2\prod_{q=1}^{k_2}(ih_{q2}^{(n)})^2}$$

$$|\alpha_p^{(n)}|^2 = \left((ih_{p2}^{(n)})^2 - (ih_{p2}^{(n-1)})^2\right) \frac{\prod_{q=1}^{\frac{k_1-n}{2}}\left((ih_{p2}^{(n-1)})^2 - (h_{q1}^{(n)})^2\right)\prod_{q=1,q\neq p}^{k_2}\left((ih_{p2}^{(n-1)})^2 - (ih_{q2}^{(n-1)})^2)\right)}{(ih_{p2}^{(n-1)})^2\prod_{q=1}^{\frac{k_1-n}{2}}\left((ih_{p2}^{(n-1)})^2 - (h_{q1}^{(n-1)})^2\right)\prod_{q=1,q\neq p}^{k_2}\left((ih_{p2}^{(n-1)})^2 - (ih_{q2}^{(n)})^2\right)},$$

$$p = 1, \ldots, k_2 \quad , \tag{A.12}$$

and of the Eqs. (3.31) to (3.33) in the case that $k_1 - n + 1$ is odd

$$|w_p^{(n)}|^2 = \frac{1}{2}\frac{\prod_{m=1}^{\frac{k_1-n}{2}}\left((h_{m1}^{(n-1)})^2 - (h_{p1}^{(n)})^2\right)\prod_{q=1}^{k_2}\left((h_{p1}^{(n)})^2 - (ih_{q2}^{(n)})^2\right)}{\prod_{q=1,q\neq p}^{\frac{k_1-n}{2}}\left((h_{p1}^{(n)})^2 - (h_{q1}^{(n)})^2\right)\prod_{q=1}^{k_2}\left((h_{p1}^{(n)})^2 - (ih_{q2}^{(n-1)})^2\right)}$$

$$p = 1, \ldots, (k_1-n)/2$$

$$|\beta_p^{(n)}|^2 = \frac{1}{2}\left((ih_{p2}^{(n)})^2 - (ih_{p2}^{(n-1)})^2\right) \frac{\prod_{q=1,q\neq p}^{k_2}\left((ih_{p2}^{(n)})^2 - (ih_{q2}^{(n)})^2\right)\prod_{q=1}^{\frac{k_1-n}{2}}\left((ih_{p2}^{(n)})^2 - (h_{q1}^{(n-1)})^2\right)}{\prod_{q=1}^{\frac{k_1-n}{2}}\left((ih_{p2}^{(n)})^2 - (h_{q1}^{(n)})^2\right)\prod_{q=1,q\neq p}^{k_2}\left((ih_{p2}^{(n)})^2 - (ih_{q2}^{(n-1)})^2\right)} .$$

$$p = 1, \ldots, k_2 \tag{A.13}$$

A.3 Derivation of Equation (3.41)

The derivation of Eq. (3.41) follows the lines of the calculation for the unitary case [GUH3]. Restricting ourselves to the first level and assuming that k_1 is even, we first calculate the differentials

$$\begin{aligned}
d|v_p^{(n)}|^2 &= \sum_{q=1}^{k_1/2-1} \frac{|v_p^{(1)}|^2}{(h_{p1})^2-(h_{q1}^{(1)})^2} 2h_{q1}^{(1)} dh_{q1}^{(1)} + \sum_{q=1}^{k_2} \frac{|v_p^{(1)}|^2}{(h_{p1})-(ih_{q2}^{(1)})^2} \left(\xi_q^{(1)} d\xi_q^{(1)*} - \xi_q^{(1)*} d\xi_q^{(1)}\right) \ , \\
d\alpha_p^{(1)} &= \sum_{q=1}^{k_1/2-1} \frac{\alpha_p^{(1)}}{2\left((ih_{p2})^2-(h_{q1}^{(1)})^2\right)} 2h_{q1}^{(1)} dh_{q1}^{(1)} + \frac{\alpha_p^{(1)}}{(ih_{p2}^{(1)})^2-(ih_{p2})^2} \xi_p^{(1)*} d\xi_p^{(1)} + \\
& \quad \sum_{q=1,q\neq p}^{k_2} \frac{\alpha_p^{(1)}}{2\left((ih_{p2})^2-(ih_{q2}^{(1)})^2\right)} \left(\xi_q^{(1)} d\xi_q^{(1)*} - \xi_q^{(1)*} d\xi_q^{(1)}\right) \ , \qquad \text{(A.14)}
\end{aligned}$$

which gives inserted into Eq. (3.40) for the invariant length element squared after a straightforward calculation

$$\begin{aligned}
du_1^\dagger du_1 &= \frac{1}{4} \sum_{p,q=1}^{k_1/2-1} u_1^\dagger \left(\frac{1}{h-h_{p1}^{(1)}} - \frac{1}{h+h_{p1}^{(1)}}\right) \left(\frac{1}{h-h_{q1}^{(1)}} - \frac{1}{h+h_{q1}^{(1)}}\right) u_1 dh_{p1}^{(1)} dh_{q1}^{(1)} + \\
& \quad \frac{1}{4} \sum_{p,q=1}^{k_2} \frac{1}{ih_{p2}^{(1)} ih_{q2}^{(1)}} u_1^\dagger \left(\frac{1}{h-ih_{p2}^{(1)}} - \frac{1}{h+ih_{p2}^{(1)}}\right) \left(\frac{1}{h-ih_{q2}^{(1)}} - \frac{1}{h+ih_{q2}^{(1)}}\right) u_1 \\
& \quad \left(\xi_p^{(1)} d\xi_p^{(1)*} - \xi_p^{(1)*} d\xi_p^{(1)}\right) \left(\xi_q^{(1)} d\xi_q^{(1)*} - \xi_q^{(1)*} d\xi_q^{(1)}\right) + \\
& \quad \sum_{p=1}^{k_1/2} |v_p^{(1)}|^2 \left(d\theta_p^{(1)}\right)^2 + \frac{1}{2} \sum_{p=1}^{k_2} \frac{|\alpha_p|^2}{(ih_{p2})^2-(ih_{p2}^{(1)})^2} d\xi_p^{(1)*} d\xi_p^{(1)} + \\
& \quad \frac{1}{2} \sum_{p=1}^{k_1/2-1} \sum_{q=1}^{k_2} \frac{1}{ih_{q2}^{(1)}} u_1^\dagger \left(\frac{1}{h-h_{p1}^{(1)}} - \frac{1}{h+h_{p1}^{(1)}}\right) \left(\frac{1}{h-ih_{q2}^{(1)}} - \frac{1}{h+ih_{q2}^{(1)}}\right) u_1 \\
& \quad dh_{p1}^{(1)} \left(\xi_q^{(1)} d\xi_q^{(1)*} - \xi_q^{(1)*} d\xi_q^{(1)}\right) \ . \qquad \text{(A.15)}
\end{aligned}$$

This can be enormously simplified by observing that the bilinear forms in the above expression are related to the orthonormality relation of the eigenvectors. We relabel the eigenvalues as $h_{j1}^{(1)} \longrightarrow h_{(2j-1)1}^{(1)}$, $-h_{j1}^{(1)} \longrightarrow h_{(2j)1}^{(1)}$, $j=1,\ldots,k_1/2-1$ as well as $ih_{j2}^{(1)} \longrightarrow ih_{(2j-1)2}^{(1)}$ and $-ih_{j2}^{(1)} \longrightarrow ih_{(2j)2}^{(1)}$, $j=1,\ldots,k_2$. Then we find from the orthogonality of the projection matrix $\widehat{b}$ and from Eq. (3.30)

$$\begin{aligned}
\delta_{pq} &= e_p^{(1)\dagger} e_q^{(1)} \\
&= |b_p^{(1)}|^2 u_1^\dagger \frac{1}{\left(h-h_p^{(1)}\right)\left(h-h_q^{(1)}\right)} u_1 \ , \quad p,q=1,\ldots,k_1+k_2-1 \qquad \text{(A.16)}
\end{aligned}$$

which yields together with Eq. (3.31)

$$du_1^\dagger du_1 = \frac{1}{4} \sum_{p=1}^{k_1-1} \frac{\left(dh_{p1}^{(1)}\right)^2}{|w_p^{(1)}|^2} + \sum_{p=1}^{k_1} |v_p^{(1)}|^2 \left(d\theta_p^{(1)}\right)^2 +$$

$$\frac{1}{2}\sum_{p=1}^{k_2}\left(\frac{(ih_{p2})^2-(ih_{p2}^{(1)})^2}{(ih_{p2}^{(1)})^2|\beta_p^{(1)}|^2}+\frac{|\alpha_p^{(1)}|^2}{(ih_{p2})^2-(ih_{p2}^{(1)})^2}\right)d\xi_p^{(1)*}d\xi_p^{(1)}\ ,\tag{A.17}$$

where $|\alpha_p^{(1)}|^2$ and $|\beta_p^{(1)}|^2$ have to be inserted as they stand in Eq. (3.34) and (3.35). A further simplification is obtained by expanding in $|\xi_p^{(1)}|^2$ and reordering terms,

$$\begin{aligned}
\frac{(ih_{p2})^2-(ih_{p2}^{(1)})^2}{(ih_{p2}^{(1)})^2|\beta_p^{(1)}|^2}+\frac{|\alpha_p^{(1)}|^2}{(ih_{p2})^2-(ih_{p2}^{(1)})^2} &= \frac{|\alpha_p^{(1)}|^2}{(ih_{p2})^2-(ih_{p2}^{(1)})^2}\left(1+\frac{\left((ih_{p2})^2-(ih_{p2}^{(1)})^2\right)^2}{|\alpha_p^{(1)}|^2|\beta_p^{(1)}|^2(ih_{p2}^{(1)})^2}\right)\\
&= \frac{|\alpha_p^{(1)}|^2}{(ih_{p2})^2-(ih_{p2}^{(1)})^2}\left(1+\left(1+|\xi_p^{(1)}|^2\gamma_p^{(1)}(h)\right)\right)\\
&= 2\frac{|\alpha_p^{(1)}|^2}{(ih_{p2})^2-(ih_{p2}^{(1)})^2}\sqrt{1+|\xi_p^{(1)}|^2\gamma_p^{(1)}(h)}\\
&= 2\frac{|\alpha_p^{(1)}|^2}{(ih_{p2})^2-(ih_{p2}^{(1)})^2}\sqrt{\frac{\left((ih_{p2})^2-(ih_{p2}^{(1)})^2\right)^2}{|\alpha_p^{(1)}|^2|\beta_p^{(1)}|^2(ih_{p2}^{(1)})^2}}\\
&= 2\sqrt{\frac{|\alpha_p^{(1)}|^2}{|\beta_p^{(1)}|^2(ih_{p2}^{(1)})^2}}\ ,
\end{aligned}\tag{A.18}$$

where $\gamma_p^{(1)}(h)$ is defined through the expansion. Thus, we find the desired result (3.41).

A.4 Real form of the projection matrices

In this appendix we state an explicit form of the projection matrix $\widehat{b}^{(n)}$. We restrict ourselves to the case $n\le k_1$, (k_1-n+1) even. The rectangular $(k_1-n+1+k_2)\times(k_1-n+k_2)$ matrix $\widehat{b}^{(n)T}$ can schematically be written as

$$\widehat{b}^{(n)}=\begin{bmatrix}\tilde{b}_{11}^{(n)} & \tilde{b}_{1}^{(n)} & \tilde{b}_{12}^{(n)}\\ \tilde{b}_{21}^{(n)} & \tilde{b}_{2}^{(n)} & \tilde{b}_{22}^{(n)}\end{bmatrix}\ .\tag{A.19}$$

Here, $\tilde{b}_{11}^{(n)}$ is a $\frac{k_1-n+1}{2}\times\frac{k_1-n-1}{2}$ matrix with entries

$$(\tilde{b}_{11}^{(n)})_{ij}=\sqrt{2}\frac{|v_i^{(n)}||w_j^{(n)}|}{(h_{i1}^{(n-1)})^2-(h_{j1}^{(n)})^2}\begin{bmatrix}h_{j1}^{(n)}\cos\vartheta_i^{(n)} & h_{j1}^{(n-1)}\sin\vartheta_i^{(n)}\\ -h_{j1}^{(n)}\sin\vartheta_i^{(n)} & h_{j1}^{(n-1)}\cos\vartheta_i^{(n)}\end{bmatrix}\ .\tag{A.20}$$

Furthermore $\tilde{b}_{12}^{(n)}$ is a $\frac{k_1-n+1}{2} \times k_2$ matrix with the entries

$$(\tilde{b}_{12}^{(n)})_{ij} = \frac{|v_i^{(n)}|}{(h_{i1}^{(n-1)})^2 - (ih_{j2}^{(n)})^2} \begin{bmatrix} \beta_j^{(n)*}\left(ih_{j2}^{(n)}\cos\vartheta_i^{(n)} + h_{j1}^{(n-1)} i\sin\vartheta_i^{(n)}\right) & -i\beta_j^{(n)*}\left(h_{i1}^{(n-1)}\cos\vartheta_i^{(n)} + ih_{j2}^{(n)} i\sin\vartheta_i^{(n)}\right) \\ -\beta_j^{(n)}\left(ih_{j2}^{(n)}\cos\vartheta_i^{(n)} - h_{j1}^{(n-1)} i\sin\vartheta_i^{(n)}\right) & -i\beta_j^{(n)}\left(h_{i1}^{(n-1)}\cos\vartheta_i^{(n)} - ih_{j2}^{(n)} i\sin\vartheta_i^{(n)}\right) \end{bmatrix} . \tag{A.21}$$

Moreover, $\tilde{b}_{21}^{(n)}$ is a $k_2 \times \frac{k_1-n+1}{2}$ matrix with the entries

$$(\tilde{b}_{21}^{(n)})_{ij} = \frac{|w_j^{(n)}|}{(ih_{i2}^{(n-1)})^2 - (h_{j1}^{(n)})^2} \begin{bmatrix} \alpha_i^{(n)} h_{j1}^{(n)} & i\alpha_i^{(n)} ih_{i2}^{(n-1)} \\ \alpha_i^{(n)*} h_{j1}^{(n)} & -i\alpha_i^{(n)*} ih_{i2}^{(n-1)} \end{bmatrix} , \tag{A.22}$$

and $\tilde{b}_{22}^{(n)}$ is a $k_2 \times \frac{k_1-n+1}{2}$ matrix with the entries

$$(\tilde{b}_{22}^{(n)})_{ij} = \sqrt{2} \begin{bmatrix} \frac{\alpha_i^{(n)}\beta_j^{(n)*}}{ih_{i2}^{(n-1)} - ih_{j2}^{(n)}} & \frac{\alpha_i^{(n)}\beta_j^{(n)}}{ih_{i2}^{(n-1)} + ih_{j2}^{(n)}} \\ \frac{-\alpha_i^{(n)*}\beta_j^{(n)*}}{ih_{i2}^{(n-1)} + ih_{j2}^{(n)}} & \frac{-\alpha_i^{(n)*}\beta_j^{(n)}}{ih_{i2}^{(n-1)} - ih_{j2}^{(n)}} \end{bmatrix} . \tag{A.23}$$

Finally the entries of $\tilde{b}_1^{(n)}$ and $\tilde{b}_2^{(n)}$ are

$$\begin{aligned} (\tilde{b}_1^{(n)})_i &= \sqrt{2}\frac{|v_i^{(n)}||w_{\frac{k_1-n+1}{2}}^{(n)}|}{h_{i1}^{(n-1)}} \begin{bmatrix} \sin\vartheta_i^{(n)} \\ \cos\vartheta_i^{(n)} \end{bmatrix} \qquad i = 1, \dots \frac{k_1-n+1}{2} , \\ (\tilde{b}_2^{(n)})_i &= \frac{1}{\sqrt{2}}\frac{|w_{\frac{k_1-n+1}{2}}^{(n)}|}{h_{i1}^{(n-1)}} \begin{bmatrix} i\alpha_i^{(n)} \\ -i\alpha_i^{(n)*} \end{bmatrix} \qquad i = 1, \dots, k_2. \end{aligned} \tag{A.24}$$

We notice that all elements of $\widehat{b}^{(n)}$ are real quantities.

Appendix B

Calculations to Chapter 3

B.1 Derivation of an alternative integral representation for matrix Bessel functions

We rewrite the invariant measure of $U \in U(N;\beta)$ using δ–distributions. The invariance simply means that all columns $U_n,\ n = 1, \ldots, N$ are orthonormal, $\mathrm{Tr}\, U_n^\dagger U_m = \delta_{nm}$. The trace Tr is only needed for $\beta = 4$, because the entries of U are quaternions in this case. Thus, we may write

$$d\mu(U) \ = \ M_N^{(\beta)}\, d[U] \prod_{n=1}^{N} \delta\left(\mathrm{Tr}\, U_n^\dagger U_n - 1\right) \prod_{n<m}^{N} \delta\left(\mathrm{Tr}\, U_n^\dagger U_m\right) \tag{B.1}$$

where $d[U]$ is the Cartesian measure of all entries of U and the integration is for all variables over the entire real axis. The constant $M_N^{(\beta)}$ will be determined later. Ullah [ULL] used such forms for the measure to work out certain probability density functions. The bilinear forms in the δ distributions have β components for $n \neq m$,

$$U_n^\dagger U_m \ = \ \sum_{\alpha=0}^{\beta-1} \left[U_n^\dagger U_m\right]^{(\alpha)} \tau^{(\alpha)} \ . \tag{B.2}$$

We notice that $\left[U_n^\dagger U_n\right]^{(\alpha)} = 0$ for $\alpha > 0$ in the case $n = m$, because the length of every vector is real. Thus, because of Eq. (B.2), the δ distributions in the measure (B.1) have to be products of δ distributions for every non–zero component $\left[U_n^\dagger U_m\right]^{(\alpha)}$. We now introduce Fourier representations

$$\begin{aligned} \delta\left(\left[U_n^\dagger U_m\right]^{(\alpha)}\right) \ &= \ \frac{1}{\pi}\int_{-\infty}^{+\infty} dT_{nm}^{(\alpha)} \exp\left(-i2\left[U_n^\dagger U_m\right]^{(\alpha)} T_{nm}^{(\alpha)}\right) \\ \delta\left(\left[U_n^\dagger U_n\right]^{(0)} - 1\right) \ &= \ \frac{1}{2\pi}\int_{-\infty}^{+\infty} dT_{nn}^{(0)} \exp\left(-i\left(\left[U_n^\dagger U_n\right]^{(0)} - 1\right) T_{nm}^{(0)}\right) \end{aligned} \tag{B.3}$$

for $n \neq m$ and $n = m$, respectively. The Fourier variables form the elements

$$T_{nm} \ = \ \sum_{\alpha=0}^{\beta-1} T_{nm}^{(\alpha)} \tau^{(\alpha)} \ . \tag{B.4}$$

of a matrix T which is real–symmetric, Hermitean or Hermitean self–dual according to $\beta = 1, 2, 4$. We notice that the diagonal elements $T_{nn} = T_{nn}^{(0)}$ are always real.

$$\begin{aligned}
\delta\left(\mathrm{Tr}\, U_n^\dagger U_m\right) &= \frac{1}{\pi^\beta}\int d^\beta T_{nm} \exp\Big(-i\mathrm{Tr}\, U_n^\dagger (T_{nm}\otimes 1_N)U_m \\
&\qquad\qquad -i\mathrm{Tr}\, U_m^\dagger (T_{nm}^*\otimes 1_N)U_n\Big) \\
\delta\left(\mathrm{Tr}\, U_n^\dagger U_n - 1\right) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dT_{nn} \exp\left(i\mathrm{Tr}\, T_{nn} - i\mathrm{Tr}\, U_n^\dagger (T_{nn}\otimes 1_N)U_n\right)
\end{aligned} \tag{B.5}$$

for $n \neq m$ and $n = m$, as above. Just as the trace Tr, the direct product is only needed in the case $\beta = 4$. We order the columns U_n, $n = 1, \ldots, N$ of the matrix U in a vector $\vec{U} = (U_1, U_2, \ldots, U_N)^T$ with N^2 elements. For $\beta = 1, 2$, the elements are scalars, for $\beta = 4$, they are quaternions. Collecting everything, we can rewrite the measure (B.1) in the form

$$d\mu(U) = \frac{M_N^{(\beta)} d[U]}{(2\pi)^N \pi^{\beta N(N-1)/2}} \int d[T] \exp\left(i\mathrm{Tr}\, T - i\mathrm{Tr}\, \vec{U}^\dagger (T\otimes 1_N)\vec{U}\right) . \tag{B.6}$$

To use this in the integral (4.19) for the matrix Bessel functions $\Phi_N^{(\beta)}(x,k)$, we also take advantage of the relation

$$\mathrm{Tr}\, U^{-1} x U k = \mathrm{Tr}\, \vec{U}^\dagger (x\otimes \hat{k})\vec{U} \tag{B.7}$$

which allows us to write

$$\begin{aligned}
\Phi_N^{(\beta)}(x,k) &= \frac{M_N^{(\beta)}}{(2\pi)^N \pi^{\beta N(N-1)/2}} \int d[U] \int d[T] \exp\left(i\mathrm{Tr}\,(T - i\varepsilon)\right) \\
&\qquad \exp\left(i\mathrm{Tr}\, \vec{U}^\dagger (x\otimes \hat{k} - (T - i\varepsilon)\otimes 1_N)\vec{U}\right) \\
&= \frac{M_N^{(\beta)} i^{\beta N^2} \pi^{\beta N/2}}{(2\pi)^N} \int d[T] \exp\left(i\mathrm{Tr}\,(T - i\varepsilon)\right) \\
&\qquad \mathrm{Det}^{-\beta/2}\left(x\otimes \hat{k} - (T - i\varepsilon)\otimes 1_N\right) .
\end{aligned} \tag{B.8}$$

Thus, the integration over U can be done as a Gaussian one and gives the result (4.31). To ensure the convergence of the Gaussian integrals over $\vec{U}$ we added to T an imaginary increment in the diagonal.

Formula (B.8) yields immediately the integral equation (4.35). Upon making the change of variables

$$T = x^{1/2} T' x^{1/2} , \qquad \text{implying} \qquad d[T] = \mathrm{Det}^{\,1+\beta(N-1)/2} x\, d[T'] , \tag{B.9}$$

we bring x into the exponential function and remove it from the determinant. We diagonalize $T' = V'^{-1} t' V'$ and find

$$\mathrm{Det}^{-\beta/2}\left(x\otimes \hat{k} - T\otimes 1_N\right) = \mathrm{Det}^{-\beta N/2} x \prod_{n,m} (k_m - t'_n)^{-\beta/2} . \tag{B.10}$$

The integral over V' is then just the integral definition (4.19) of the matrix Bessel function $\Phi_N^{(\beta)}(x,t')$ and we arrive at Eq. (4.35).

In order to derive formula (4.34) we extract a factor $\exp(x_2 \operatorname{Tr} k)$ from the integral Eq. (4.19). Only the reduced matrix enters now into the group integration

$$x = \operatorname{diag}\left((x_1 - x_2) \otimes 1_M, 0 \otimes N\right) \quad . \tag{B.11}$$

The non–trivial part only goes over the coset $U(N;\beta)/U(M-N;\beta)$. Therefore we are allowed to replace the product over N by a product over M in Eq. (B.1). The next steps are analogous to the general case.

The normalization constants remain to be derived. Conveniently, they nicely relate to a special form of Selberg's integral which is given in Eq. (17.5.2) of Mehta's book [MEH1],

$$\begin{aligned} J_N &= \int d[t] |\Delta_N(t)|^{2\gamma} \prod_{n=1}^{N} (a_1 + it_n)^{-b_1} (a_2 - it_n)^{-b_2} \\ &= \frac{(2\pi)^N}{(a_1 + a_2)^{(b_1+b_2)N - \gamma N(N-1) - N}} \\ &\qquad \prod_{n=0}^{N-1} \frac{\Gamma(1 + (n+1)\gamma)\Gamma(b_1 + b_2 - (N+n-1)\gamma - 1)}{\Gamma(1+\gamma)\Gamma(b_1 - n\gamma)\Gamma(b_2 - n\gamma)} \ . \end{aligned} \tag{B.12}$$

We now put $x = 0$ or $k = 0$ and have $\Phi_N^{(\beta)}(0,k) = 1$ or $\Phi_N^{(\beta)}(x,0) = 1$ on the left hand side of Eq. (B.8). We diagonalize $T = V^{-1}tV$ and use the invariance of the integral. Employing the measure in (2.9) and the constant $C_N^{(\beta)}$ given in Eq. (2.11), we find the condition

$$1 = \frac{M_N^{(\beta)} C_N^{(\beta)} \pi^{\beta N/2}}{(2\pi)^N} \int d[t] |\Delta_N(t)|^{\beta} \prod_{n=1}^{N} \frac{\exp(it_n)}{(it_n)^{\beta N/2}} \ . \tag{B.13}$$

We map this onto Selberg's integral (B.12) by setting $\gamma = \beta/2$, $b_1 = \beta N/2$ and $a_2 = b_2$, by using

$$\lim_{a_2 \to \infty} \frac{a_2^{a_2}}{(a_2 - it_n)^{a_2}} = \exp(it_n) \tag{B.14}$$

and by considering $a_2^{Na_2} J_N$ in the limits $a_1 \to 0$ and $a_2 \to \infty$. With the help of some standard asymptotic formulae for the Γ function, we obtain $M_N^{(\beta)}$ and, eventually, the constants $A_N^{(\beta)}$ and $B_N^{(\beta)}$ in Eqs. (4.32) and (4.36).

B.2 Proof of Theorem 4.1

First, in order to compactify the notation we define

$$d\widetilde{\mu}(x,x') = d\mu(x,x') \exp\left(i \left(\sum_{n=1}^{N} x_n - \sum_{n=1}^{N-1} x'_n\right) k_N\right) \quad , \tag{B.15}$$

where the measure given in Eq. (4.63). In order to prove the theorem we first write the integral in terms of θ–functions. Then the left hand side of Eq. (4.66) reads

$$l.\ h.\ s. = \Delta_x \int \widetilde{\mu}(x,x') f(x') \prod_{i \le j} \theta(x_i - x'_j) \prod_{j<k} \theta(x'_j - x_k) d[x'] \ , \tag{B.16}$$

where now the integration domain is the real axis for all variables. Now we can directly calculate the action of the operator Δ_x onto the integral. We find

$$\Delta_x \int \widetilde{\mu}(x,x') f(x') \prod_{i\leq j} \theta(x_i - x'_j) \prod_{j<k} \theta(x'_j - x_k) d[x'] =$$
$$\int f(x') \prod_{i\leq j} \theta(x_i - x'_j) \prod_{j<k} \theta(x'_j - x_k) \left(\Delta_{x'}^{(-)} + \beta \sum_{n\neq m} \frac{1}{(x'_n - x'_m)^2} - k_N^2 \right) \widetilde{\mu}(x,x') d[x']$$
$$+ \int f(x') \widetilde{\mu}(x,x') \Delta_x \prod_{i\leq j} \theta(x_i - x'_j) \prod_{j<k} \theta(x'_j - x_k) d[x']$$
$$+2 \int f(x') \sum_{n=1}^{N} \frac{\partial}{\partial x_n} \widetilde{\mu}(x,x') \frac{\partial}{\partial x_n} \prod_{i\leq j} \theta(x_i - x'_j) \prod_{j<k} \theta(x'_j - x_k) d[x'] \quad , \tag{B.17}$$

where we define the operator

$$\Delta_{x'}^{(-)} = \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} - \sum_{n<m} \frac{\beta}{x_n - x_m} \left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial x_m} \right) . \tag{B.18}$$

By a series of integrations by parts, the operator $\Delta_{x'}^{(-)}$ acting on $\widetilde{\mu}(x,x')$ is transformed to $\Delta_{x'}$ acting only on $f(x')$. At taking the derivative of the θ–functions, we notice that only adjactent levels contribute, because otherwise terms like $\theta(x_i - x_j)$ with $i > j$ arise which annihilate the integral due to the chosen ordering. Therefore, we can write

$$\frac{\partial}{\partial x_n} \prod_{i\leq j} \theta(x_i - x'_j) \prod_{j<k} \theta(x'_j - x_k) =$$
$$\prod(\theta_{\neq nn'} \, , _{\neq(n-1)'n}) \Big(\delta(x_n - x'_n)\theta(x'_{n-1} - x_n) - \delta(x'_{n-1} - x_n)\theta(x_n - x'_n) \Big) , \tag{B.19}$$

where $\prod(\theta_{\neq nn'}, \theta_{\neq(n-1)'n})$ denotes the product on the left hand side of Eq. (B.19) without the two factors $\theta(x'_{n-1} - x_n)\theta(x_n - x'_n)$. Importantly, this product is symmetric in x'_{n-1} and x'_n. The second derivatives yield

$$\frac{\partial}{\partial x_n} \prod_{i\leq j} \theta(x_i - x'_j) \prod_{j<k} \theta(x'_j - x_k) = \prod(\theta_{\neq nn'}, \theta_{\neq(n-1)'n})$$
$$\Big(\delta'(x_n - x'_n)\theta(x'_{n-1} - x_n) + \delta'(x'_{n-1} - x_n)\theta(x_n - x'_n) + \delta(x'_{n-1} - x_n)\delta(x_n - x'_n) \Big) \quad . \tag{B.20}$$

The last term vanishes upon integration, since it is symmetric in x'_{n-1} and x'_n, whereas the rest of the integrand is antisymmetric due to the Vandermonde determinant $\Delta(x')$ in the measure (4.63). Differentiation with respect to x'_n yields

$$\frac{\partial}{\partial x'_n} \prod_{i\leq j} \theta(x_i - x'_j) \prod_{j<k} \theta(x'_j - x_k) =$$
$$\prod(\theta_{\neq n'(n+1)}, \theta_{\neq nn'}) \Big(\delta(x'_n - x_{n+1})\theta(x_n - x'_n) - \delta(x_n - x'_n)\theta(x'_n - x_{n+1}) \Big) . \tag{B.21}$$

Integration by parts of the first term of the right hand side of Eq. (B.17) yields

$$\Delta_x \int \widetilde{\mu}(x,x') f(x') \prod_{i\le j} \theta(x_i - x'_j) \prod_{j<k} \theta(x'_j - x_k) d[x'] =$$
$$\int \widetilde{\mu}(x,x') \Delta_{x'} f(x') d[x'] - k_N^2 \int \widetilde{\mu}(x,x') f(x') d[x']$$
$$+2 \int f(x') \sum_{n=1}^{N-1} \Bigg(\prod (\theta_{\neq n'(n+1)}, \theta_{\neq nn'})$$
$$\Big(\delta(x_n - x'_n)\theta(x'_n - x_{n+1}) + \delta(x'_n - x_{n+1})\theta(x_n - x'_n)\Big)$$
$$\left(\frac{\partial}{\partial x_n} + \frac{\partial}{\partial x'_n} + \frac{1}{2} \sum_{m\neq n} \frac{\beta}{x_n - x_m} + -\frac{1}{2} \sum_{m\neq n} \frac{\beta}{x'_n - x'_m} \right)\Bigg) \widetilde{\mu}(x,x') d[x'] \ . \quad \text{(B.22)}$$

After inserting in Eq. (B.22) $\widetilde{\mu}(x,x')$ as it stands in Eq. (B.15) and Eq. (4.63) we find in a straightforward calculation

$$\Delta_x \int \widetilde{\mu}(x,x') f(x') d[x'] =$$
$$\int \widetilde{\mu}(x,x') \Delta_{x'} f(x') d[x'] \ - \ k_N^2 \int \widetilde{\mu}(x,x') f(x') d[x']$$
$$+2 \int f(x') \sum_{n=1}^{N-1} \Bigg(\prod (\theta_{\neq n'(n+1)}, \theta_{\neq nn'}) \Big(g(x'_n; y_n; x, x') - g(x'_n; x_n; x, x') \Big)$$
$$\Big(\delta(x_n - x'_n)\theta(x'_n - x_{n+1}) + \delta(x'_n - x_{n+1})\theta(x_n - x'_n)\Big)\Bigg) \widetilde{\mu}(x,x') d[x'] \quad , \quad \text{(B.23)}$$

with

$$\begin{aligned} g(x_n; x'_n; x, x') &= (\beta/2 - 1) \left(\sum_{m=1}^{N-1} \frac{1}{x_n - x'_m} - \sum_{m\neq n} \frac{1}{x_n - x_m} \right) \\ g(x'_n; x_n; x, x') &= (\beta/2 - 1) \left(\sum_{m\neq n} \frac{1}{x'_n - x'_m} - \sum_{m=1}^{N} \frac{1}{x'_n - x_m} \right) \quad . \end{aligned} \quad \text{(B.24)}$$

We now can perform the integration of the δ–distributions in Eq. (B.23). We notice that the difference $(g(x'_n; x_n; x, x') - g(x'_n; x_n; x, x'))$ vanishes linearly, whenever x'_n approaches one of the boundaries of its integration domain. Thus the second integral in Eq. (B.23) yields zero as long as the measure diverges less than $(x_n - x'_n)^{-1}$ when x'_n approaches x_n. This is always the case for $\beta > 0$. Eq. (4.70) in Section 4.4 is derived in the same way.‡

B.3 Symmetry of the radial functions for arbitrary β and calculation of the normalization constant $G_N^{(\beta)}$

Applying the recursion formula (4.62) to all $(N-1)$ levels, we can extend Eq. (4.41) to arbitrary β and write

$$\begin{aligned}\Phi_N^{(\beta)}(x,k) &= \int \prod_{n=1}^{N-1} d\mu(x^{(n)},x^{(n-1)}) \exp\left(ix_1^{(N-1)}k_1\right) \\ &\qquad \exp\left(i\left(\sum_{m=1}^{N-n+1} x_m^{(n-1)} - \sum_{m=1}^{N-n} x_m^{(n)}\right)k_{N-n+1}\right)\end{aligned} \tag{B.25}$$

where $x^{(0)} = x$. We change in Eq. (B.25) on the n–th level the variables $x_m^{(n)}, m = 1,\ldots,(N-n)$ to $r_m^{(n)}$ setting

$$\frac{\prod_{l=1}^{N-n-1}(x_m^{(n-1)} - x_l^{(n)})}{\prod_{l\neq m}(x_m^{(n-1)} - x_l^{(n-1)})} = r_m^{(n)} \tag{B.26}$$

for $n = 1,\ldots,(N-1)$. These are, on the n^{th} level, $(N-n+1)$ equations for making a change of $N-n$ variables. However, one has

$$\sum_{m=1}^{N-n+1} r_m^{(n)} = 1 \tag{B.27}$$

on all levels which eliminates one of the $(N-n+1)$ equations.

The original domains of integration are $x_m^{(n-1)} \geq x_m^{(n)} \geq x_{m+1}^{(n-1)}$. These boundaries transform to a positive definiteness condition for the new variable $r_m^{(n)}$. The Jakobi determinant of the variable transformation (B.26) is readily derived from the results of Gelfand [GT1, SHA], cf. Eqs. (3.55) and (3.56). It is given by

$$\det\left[\frac{\partial(x_m^{(n)})}{\partial(r_m^{(n)})}\right]_{n,m=1,\ldots,(N-1)} = \frac{\Delta_N(x^{(n-1)})}{\Delta_{N-1}(x^{(n)})} \tag{B.28}$$

The full invariant $\beta-$ dependent measure (4.63) can now be written in terms of $r_m^{(n)}$ as

$$d\mu(r^{(n)}) = \prod_{m=1}^{N-n+1} \sqrt{r_m^{(n)}}^{\,\beta-2}\, \delta\left(1 - \sum_{m=1}^{(N-n+1)} r_m^{(n)}\right) d[r^{(n)}] \quad , \tag{B.29}$$

where $d[r^{(n)}] = \prod_{m=1}^{N-n+1} dr_m^{(n)}$. In Eq. (B.29) it is obvious, that the measure is independent from x and k and therefore trivially symmetric in x and k. This allows us to write

$$d\mu(x^{(n)},x^{(n-1)}) = d\mu(r^{(n)}) = d\mu(k^{(n)},k^{(n-1)}) \tag{B.30}$$

The coordinates $r'_n = r_n^{(1)}$, $n = 1,\ldots,N$ on the first level of the recursion can be interpreted as the moduli squared of the coordinates on the unit sphere in the complex

N–dimensional space. Thus, in order to evaluate the normalization constant, it is natural to use the following type of hyperspherical coordinates

$$\begin{aligned} \sqrt{r'_n} &= \cos\vartheta_n \prod_{\nu=1}^{n-1} \sin\vartheta_\nu \ , \qquad n = 1, \ldots, (N-1) \ , \\ \sqrt{r'_N} &= \sin\vartheta_{N-1} \prod_{\nu=1}^{N-2} \sin\vartheta_\nu \end{aligned} \tag{B.31}$$

where the positive semidefiniteness of the r'_n restricts the domain of integration to $0 \leq \vartheta_n < \pi/2$, $n = 1, \ldots, (N-1)$. Thus, we integrate over a $(2^N)^{\text{th}}$ segment of the unit sphere. The measure (B.29) becomes

$$d\mu(r') = \prod_{n=1}^{N-1} \sin^{2(N-n)-1}\vartheta_n \cos\vartheta_n \, d\vartheta_n \quad . \tag{B.32}$$

It is, apart from the phase angles, the measure on the unit sphere. Collecting everything, we have

$$\begin{aligned} 1 &= \int d\mu(x', x) = G_N^{(\beta)} \int \prod_{n=1}^{N} \sqrt{r'_n}^{\,\beta-2} d\mu(r') \\ &= G_N^{(\beta)} \prod_{n=1}^{N-1} \int_0^{\pi/2} \sin^{(N-n)\beta-1}\vartheta_n \cos^{\beta-1}\vartheta_n \, d\vartheta_n \\ &= G_N^{(\beta)} \prod_{n=1}^{N-1} \frac{\Gamma((N-n)\beta/2)\Gamma(\beta/2)}{2\Gamma((N-n+1)\beta/2)} = G_N^{(\beta)} \frac{\Gamma^N(\beta/2)}{2^{N-1}\Gamma(N\beta/2)} \end{aligned} \tag{B.33}$$

where the integral over ϑ_n is just Euler's integral of the first kind.

It remains to be shown that the change of variables (B.26) leads to the identity

$$\begin{aligned} &\sum_{n=1}^{N-1} \left(\sum_{m=1}^{N-n+1} x_m^{(n-1)} - \sum_{m=1}^{N-n} x_m^{(n)} \right) k_{N-n+1} + x_1^{(N-1)} k_1 \\ &\quad = \sum_{n=1}^{N-1} \left(\sum_{m=1}^{N-n+1} k_m^{(n-1)} - \sum_{m=1}^{N-n} k_m^{(n)} \right) x_{N-n+1} + k_1^{(N-1)} x_1 \ . \end{aligned} \tag{B.34}$$

Since the symmetry relation (4.61) holds for $\beta = 1, 2, 4$, we know that Eq. (B.34) must be true in these cases. However, as Eq. (B.34) does not involve β at all, it must also be valid for arbitrary β. We notice that this line of arguing cannot be spoiled by any other contribution to the argument of the exponential functions, because all other terms in the integrand are purely algebraic. This completes the proof of the symmetry relation (4.61) for arbitrary β.

B.4 Derivation of $\Phi_4^{(4)}(x,k)$ by an Hankel ansatz

We perform the derivation for $\Phi_4^{(4)}(-ix,k)$ in order to avoid inconvenient factors i. First we observe that the operator $L_{x,\omega(k)}$ defined in Eq. (4.76) splits into two parts. One part does not change the order in k,

$$\widetilde{\Delta}_{x,\omega(k)} = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - 4 \sum_{n<m} \frac{1}{(x_n - x_m)^2} \quad . \tag{B.35}$$

The other one raises the order in k by one

$$\Lambda_{x,\omega(k)} = 2 \sum_{i=1}^{N} k_{\omega(n)} \frac{\partial}{\partial x_i} \quad . \tag{B.36}$$

Since we can restrict ourselves to one element of the permutation group, in the sequel we discuss only the identity permutation. The symmetry of x and k together with the result for $\Phi_3^{(4)}$ suggests an expansion in the composite variable z_{ij} as defined in Eq. (4.88). To this end we define the elementary symmetric functions

$$e_\nu(z) = \sum_{i_1 j_1 < i_2 j_2 < \ldots < i_\nu j_\nu} \prod_{k=1}^{\nu} z_{i_k j_k} \quad , \tag{B.37}$$

with the the following ordering of the composite index $\{i_k j_k\}$, $i_k < j_k$. We say $\{i_k j_k\} < \{i_l j_l\}$ if $i_k < i_l$ or $i_k = i_l$ and $j_k < j_l$. All indices run to N. The highest order elementary symmetric function is of order $N(N-1)/2$ and is given by $\Delta(x)\Delta(k)$. The asymptotic formula (4.77) yields the leading term for large arguments. It is the starting point for a recursion in powers of z^{-1}.

$$W_N^{(4)}(z) = \sum_{\nu=0}^{N(N-1)/2} p_\nu(z^{-1}) \quad , \tag{B.38}$$

where $p_\nu(z)$ is a symmetric function of order ν in x_i and k_i. We investigate the action of the two operators defined in Eq. (B.35),(B.36) and find

$$\Lambda_{x,k} e_\nu(z^{-1}) = -2 \sum_{n<m}^{N} \frac{1}{(x_n - x_m)^2} e_{\nu-1}(z_{\neq nm}^{-1}) \tag{B.39}$$

$$\begin{aligned} \widetilde{\Delta}_{x,k} e_\nu(z^{-1}) &= -4 \sum_{n<m}^{N} \frac{1}{(x_n - x_m)^2} e_\nu(z_{\neq nm}^{-1}) \\ &\quad -2 \sum_{\substack{n<m \\ k\neq n \\ k\neq m}}^{N} \frac{1}{(x_n - x_m)^2}\, z_{nk}^{-1}\, z_{mk}^{-1}\, e_{\nu-2}(z^{-1}_{\substack{\neq nm \\ \neq nk \\ \neq mk}}) \quad . \end{aligned} \tag{B.40}$$

The function $e_\nu(z_{\neq nm})$ is the elementary symmetric function $e_\nu(z)$ with all terms containing z_{nm} omitted. For $\nu = 0, 1, 2$ we simply have $p_\nu(z^{-1}) = (-2)^\nu e_\nu(z^{-1})$. For $\nu \geq 3$ the last term in Eq. (B.40) causes the appearance of correction terms to the elementary symmetric functions. It arises due to the mixed derivatives, which have to be taken into account in the action of $\widetilde{\Delta}_{x,k}$ onto $e_\nu(z^{-1})$ for $\nu \geq 3$. Because of this term the Hankel Ansatz seems to be of limited use, since one has to construct a set of correction terms, which become for higher order of N increasingly complicated. Up to now the construction was only possible for $N = 4$. In order to construct the correction terms we define a new set of symmetric functions as follows

$$f_\nu(z^{-1}) \;=\; \sum_{k<l<m}^{N} z_{kl}^{-1} z_{km}^{-1} z_{lm}^{-1} e_{\nu-3}(z^{-1}_{\substack{\neq kl\\ \neq km\\ \neq lm}}) \quad . \tag{B.41}$$

Again we have to investigate the action of $\Lambda_{x,k}$ and $\widetilde{\Delta}_{x,k}$ on $f_\nu(z^{-1})$. We find

$$\widetilde{\Delta}_{x,k}\, f_3(z^{-1}) = -4 \sum_{n<m}^{N} \frac{1}{(x_n - x_m)^2}\, f_3(z^{-1}_{\neq nm}) \tag{B.42}$$

and

$$\Lambda_{x,k}\, f_3(z^{-1}) = -2 \sum_{\substack{n<m\\ k\neq n\\ k\neq m}}^{N} \frac{1}{(x_n - x_m)^2}\, z_{nk}^{-1}\, z_{mk}^{-1} \quad , \tag{B.43}$$

thus $f_3(z^{-1})$ is the desired “correction term“. We have

$$p_3(z^{-1}) = -2^3 \left(e_3(z^{-1}) + \frac{1}{2}\, f_3(z^{-1}) \right) \quad . \tag{B.44}$$

Favoritely, due to Eq. (B.42) in the next step the “correction term“ itself has not to be corrected. We find

$$p_4(z^{-1}) = 2^4 \left(e_4(z^{-1}) + \frac{1}{2}\, f_4(z^{-1}) \right) \quad . \tag{B.45}$$

Up to now these results are valid for arbitrary N. The action of $\widetilde{\Delta}_{x,k}$ onto the symmetric function $f_4(z^{-1})$ is not as simple as Eq. (B.42). After a series of manipulations we arrive at

$$\widetilde{\Delta}_{x,k}\, f_4(z^{-1}) = -4 \sum_{n<m}^{N} \frac{1}{(x_n - x_m)^2}\, f_4(z^{-1}_{\neq nm}) \;-\; 2 \sum_{\substack{n<m\\ k\neq n\\ k\neq m}}^{N} \frac{1}{(x_n - x_m)^2}\, z_{nk}^{-1}\, z_{mk}^{-1}\, f_2(z^{-1}_{\substack{\neq nm\\ \neq nk\\ \neq mk}}) \quad . \tag{B.46}$$

The contribution (B.40) has to be added to this expression stemming from the action of $\widetilde{\Delta}_{x,k}$ onto $e_4(z^{-1})$. On the other hand we calculate

$$\Lambda_{x,k}\, f_5(z^{-1}) = -2 \sum_{n<m}^{N} \frac{1}{(x_n - x_m)^2}\, f_4(z^{-1}_{\neq nm}) - 2 \sum_{\substack{n<m\\ k\neq n\\ k\neq m}}^{N} \frac{1}{(x_n - x_m)^2}\, z_{nk}^{-1}\, z_{mk}^{-1}\, e_2(z^{-1}_{\substack{\neq nm\\ \neq nk\\ \neq mk}}) \,. \tag{B.47}$$

Therefore we have to look for yet another correction term to compensate the second term in Eq. (B.46). We define

$$f_5'(z^{-1}) \;=\; \sum_{i_1<i_2<i_3<i_4} \prod_{r<j} z_{i_r i_j}^{-1} \sum_{r<j} z_{i_r i_j} \;=\; \sum_{\substack{j<k\\ l<m}} z_{jl}^{-1} z_{jm}^{-1} z_{kl}^{-1} z_{km}^{-1} z_{lm}^{-1} \quad . \tag{B.48}$$

$\Lambda_{x,k}\, f_5'(z^{-1})$ yields exactly the second term of Eq. (B.46). Pushing forward this procedure becomes more complicated step by step. There seems to be no obvious way of constructing the additional terms. Apparently for higher orders the correction terms also involve an increasing amount of indices. Nevertheless for $N = 4$ we are already at the end of the recursion. Then the general expression

$$p_5(z^{-1}) = -2^5 \left(e_5(z^{-1}) + \frac{1}{2} f_5(z^{-1}) + \frac{1}{4} f_5'(z^{-1}) \right) \tag{B.49}$$

reduces to

$$p_5(z^{-1}) = -72\, e_5(z^{-1}) \quad . \tag{B.50}$$

The last step can readily be done, since the action of $\widetilde{\Delta}_{x,k}$ onto $e_5(z^{-1})$ is already known by Eq. (B.40). Thus we arrive at

$$p_6(z^{-1}) = 288\, e_6(z^{-1}) \quad . \tag{B.51}$$

Importantly, we have

$$\widetilde{\Delta}_{x,k}\, e_6(z^{-1}) = \widetilde{\Delta}_{x,k} \frac{1}{\Delta_4(x)\Delta_4(k)} = 0 \quad . \tag{B.52}$$

That means, the sequence finishes after the 6–th step. Collecting everything and observing that, for $N = 4$, $f_5(z) = 2e_5(z)$ and $f_6(z) = 4e_6(z)$, we get

$$W_4^{(4)}(x,k) = \sum_{\nu=1}^{6} (-2)^{\nu} e_{\nu}(z^{-1}) + \sum_{\nu=3}^{6} (-2)^{\nu-1} f_{\nu}(z^{-1}) - 8\, e_5(z^{-1}) + 96\, e_6(z^{-1}) \; . \tag{B.53}$$

This can be rewritten more compactly in the spirit of Eq. (4.89) as

$$\begin{aligned} W_4^{(4)}(x,k) \;=\; & \frac{1}{\Delta_4(x)\Delta_4(k)} \Bigg(\prod_{i<j} (2 - z_{ij}) + \\ & \frac{1}{2} \sum_{l<m<n} \prod_{\substack{i<j \\ \neq lm \\ \neq ln \\ \neq mn}} (2 - z_{ij}) + \frac{1}{4} \sum_{\substack{l<m \\ k<n}} \prod_{\substack{i<j \\ \neq mk}} \prod_{\substack{\neq lk \\ \neq mn}} \prod_{\substack{\neq ln \\ \neq kn}} (2 - z_{ij}) \Bigg) \; . \end{aligned} \tag{B.54}$$

This form indicates a general structure for $W_N^{(4)}(x,k)$. the leading term is always the generating function of the elementary symmetric functions in z. To this term are added combinations of other symmetric functions, where certain combinations of indices are cut out.

B.5 Translation invariance of $W_{N,\omega}^{(\beta)}(x,k)$

We shift every x_n in the the recursion formula (4.62) for arbitrary β by a constant $\bar{x}$ and obtain

$$\begin{aligned} \Phi_N^{(\beta)}(x+\bar{x},k) \;=\; & \int d\mu(x', x+\bar{x})\, \exp\left(i \left(\sum_{n=1}^{N} x + N\bar{x} - \sum_{n=1}^{N-1} x' \right) k_N \right) \\ & \Phi_{N-1}^{(\beta)}(x', \widetilde{k}) \end{aligned} \tag{B.55}$$

with $x_n + \bar{x} \le x'_n \le x_{n+1} + \bar{x}$ as the domains of integration. The change of variables $x'_n \longrightarrow x'_n + \bar{x}$ removes $\bar{x}$ from the measure given in Eq. (4.63) and the domains of integration. We find

$$\begin{aligned}\Phi_N^{(\beta)}(x+\bar{x},k) \;&=\; \exp\left(i\bar{x}k_N\right) \int d\mu(x',x)\, \exp\left(i\left(\sum_{n=1}^{N} x - \sum_{n=1}^{N-1} x'\right) k_N\right) \\ &\qquad\qquad \Phi_{N-1}^{(\beta)}(x'+\bar{x},\tilde{k})\ .\end{aligned} \tag{B.56}$$

Now we want to employ an induction. We assume that the radial functions for arbitrary β have the property

$$\Phi_N^{(\beta)}(x+\bar{x},k) \;=\; \exp\left(i\bar{x}\sum_{n=1}^{N} k_n\right) \Phi_N^{(\beta)}(x,k)\ . \tag{B.57}$$

If this is correct for $N-1$, formula (B.56) implies that it is also true for N. The induction starts with $N=2$ where the correctness of Eq. (B.57) is immediately obvious from the explicit solution (4.24) for arbitrary β. Thus, Eq. (B.57) is valid for all N.

Since the k_n are arbitrary and since the sum over all k_n is invariant under the permutations $\omega(k)$, the property (B.57) must also be true for every function $\Phi_{N,\omega}^{(\beta)}(x,k)$ with $\omega \in S_N$. We compare this with the expression

$$\Phi_{N,\omega}^{(\beta)}(x+\bar{x},k) \;=\; \exp\left(i\bar{x}\sum_{n=1}^{N} k_n\right) \frac{\exp\left(i\sum_{n=1}^{N} x_n k_{\omega(n)}\right)}{\left(\Delta_N(x)\Delta_N(k)\right)^{\beta/2}}\, W_{N,\omega}^{(\beta)}(x+\bar{x},k) \tag{B.58}$$

which results from the Hankel ansatz (4.73). Hence, we conclude that we necessarily have

$$W_{N,\omega}^{(\beta)}(x+\bar{x},k) \;=\; W_{N,\omega}^{(\beta)}(x,k)\ . \tag{B.59}$$

This is the translational invariance.

Appendix C

Calculations to Chapter 4

C.1 Radial Gelfand–Tzetlin coordinates for the unitary orthosymplectic group $UOSp(k_1/2k_2)$

In this appendix we calculate the moduli squared of an orthogonal $(k_1/2k_2)$–dimensional unit supervector in radial Gelfand–Tzetlin coordinates. We expect that they are products of differences of eigenvalues. The smallest group, which allows for a construction of differences both in the bosonic as well as in the fermionic sector is the group $UOSp(2/4)$. The set of solutions of the Gelfand-Tzetlin equations (5.30) involves one bosonic and two fermionic eigenvalues. The eigenvalue equation reads

$$
\begin{aligned}
1 &= \sum_{p=1}^{2} \left(|v_p^{(1)}|^2 + |\alpha_p^{(1)}|^2 \right) , && \text{(C.1)} \\
0 &= \sum_{q=1}^{2} \left(\frac{|v_q^{(1)}|^2}{s_{q1} - s_1^{(1)}} + \frac{|\alpha_q^{(1)}|^2}{is_{q2} - s_1^{(1)}} \right) \quad , && \text{(C.2)} \\
z &= is_2^{(1)} \prod_{q=1}^{2} \frac{(s_{q1} - is_2^{(1)})}{(is_{q2} - is_2^{(1)})^2} \sum_{q=1}^{2} \left(\frac{|v_q^{(1)}|^2}{s_{q1} - is_2^{(1)}} + \frac{|\alpha_q^{(1)}|^2}{is_{q2} - is_2^{(1)}} \right) \quad , \quad z \to \infty . && \text{(C.3)}
\end{aligned}
$$

The bosonic equation (C.2) has a unique solution $s_1^{(1)} \equiv s'_{11}$. Taking s'_{11} as new parameter, Eq. (C.1) and Eq. (C.2) can be solved for the commuting moduli squared

$$
|v_p^{(1)}|^2 = \frac{s_{p1} - s'_{11}}{s_{p1} - s_{q1}} \left(1 - \sum_{k=1}^{2} \frac{is_{k2} - s_{q1}}{is_{k2} - s'_{11}} |\alpha_k^{(1)}|^2 \right) \quad p = 1, 2 \quad . \tag{C.4}
$$

These relations are plugged in Eq. (C.3) and we obtain

$$
z = is_2^{(1)} (s'_{11} - is_2^{(1)}) \prod_{q=1}^{2} \frac{(s_{q1} - is_2^{(1)})}{(is_{q2} - is_2^{(1)})^2} \left(1 + \sum_{k=1}^{2} \frac{c_k}{is_{k2} - is_2^{(1)}} |\alpha_k^{(1)}|^2 \right) \quad , \quad z \to \infty \ , \tag{C.5}
$$

where we have defined the commuting c–numbers

$$
c_k = \frac{\prod_{q=1}^{2} (is_{k2} - s_{q1})}{is_{k2} - s'_{11}} \quad k = 1, 2 \ . \tag{C.6}
$$

It remains to determine the set of solutions of the “fermionic“ eigenvalue equation (C.5). To this end both sides are inverted

$$0 = \prod_{q=1}^{2}(is_{q2} - is_2^{(1)})^2 \left(1 - \sum_{k=1}^{2} \frac{c_k}{is_{k2} - is_2^{(1)}}|\alpha_k^{(1)}|^2 + 2\prod_{k=1}^{2} \frac{c_k}{is_{k2} - is_2^{(1)}}|\alpha_k^{(1)}|^2\right) \quad . \tag{C.7}$$

Now one can take the square root on both sides

$$0 = \prod_{q=1}^{2}(is_{q2} - is_2^{(1)}) \left(1 - \frac{1}{2}\sum_{k=1}^{2} \frac{c_k}{is_{k2} - is_2^{(1)}}|\alpha_k^{(1)}|^2 + \frac{3}{4}\prod_{k=1}^{2} \frac{c_k}{is_{k2} - is_2^{(1)}}|\alpha_k^{(1)}|^2\right) \quad . \tag{C.8}$$

The most general form of the fermionic eigenvalue is

$$is_2^{(1)} = a_0 + \sum_{k=1}^{2} a_k|\alpha_k^{(1)}|^2 + a_{12}\prod_{k=1}^{2}|\alpha_k^{(1)}|^2 \quad . \tag{C.9}$$

After inserting this ansatz in Eq. (C.8) we obtain two sets of solutions for the coefficients a_{i0}, a_{i12} and a_{ij} $i = 1,2, j = 1,2$

$$\begin{aligned} is'_{12} &= is_{12} + \left(c_1 + \frac{c_1c_2}{is_{12} - is_{22}}|\alpha_2^{(1)}|^2\right)\frac{|\alpha_1^{(1)}|^2}{2} \\ is'_{22} &= is_{22} + \left(c_2 + \frac{c_1c_2}{is_{22} - is_{12}}|\alpha_1^{(1)}|^2\right)\frac{|\alpha_2^{(1)}|^2}{2} \quad . \end{aligned} \tag{C.10}$$

Remarkably, we have $a_{12} = a_{21} = 0$. This allows us to write the nilpotent part of is'_{k2} as the modulus squared of a new anticommuting coordinate.

$$is'_{k2} = is_{k2} + |\xi'_k|^2 \quad . \tag{C.11}$$

Solving now the equations (C.10) for $|\alpha_i^{(1)}|^2$, and inserting the results in Eq. (C.4) finally yields

$$\begin{aligned} |v_p^{(n)}|^2 &= \frac{(s_{p1} - s_{11}^{(1)})\prod_{q=1}^{2}(s_{p1} - is_{q2})^2}{(s_{p1} - s_{q1})\prod_{q=1}^{2}(s_{p1} - is_{q2}^{(1)})^2} \\ |\alpha_p^{(n)}|^2 &= 2\,(is_{p2}^{(1)} - is_{p2})\frac{(is_{p2} - s_{11}^{(1)})(is_{p2} - is_{q2})^2}{(is_{p2} - is_{q2}^{(1)})^2\prod_{q=1}^{2}(is_{p2} - s_{q1})} \quad , \\ & \qquad p,q = 1,2 \quad q \neq p \; . \end{aligned} \tag{C.12}$$

The structure of Eq. (C.12) indicates the form of the solutions for groups of higher order as they were stated in Eq. (5.31). They are checked by inserting them directly into the Gelfand Tzetlin equations (5.30). Then the manipulations to be performed are similar to the ones, which we applied in Appendix A.1. Therefore we do not reproduce them here.

C.2 Derivation of supersymmetric matrix Bessel functions for groups of higher order

We sketch the derivation of the two supersymmetric matrix Bessel functions $\Phi_{34}(-is, r)$ and $\Phi_{44}(-is, r)$ for the case, that one matrix has an additional degeneracy according to

Eq. (5.71) and (5.73). For $\Phi_{34}(-is, r)$ the recursion formula reads

$$\Phi_{34}(-is,r) = \widehat{G}_{34} \int d\mu(s,s') \, \exp\left((\mathrm{trg}\, s - \mathrm{trg}\, s')r_{21}\right)\Phi_{24}(-is',\widetilde{r}) \quad . \tag{C.13}$$

We introduce the notation

$$S_{ij} = (s_{i1} - is_{j2}) \quad , \quad R_{ij} = (r_{i1} - ir_{j2}) \quad . \tag{C.14}$$

Due to the degeneracy $\Phi_{24}(-is, \widetilde{r})$ is much simpler than the general case (5.70). The integral over the $O(2)$ subgroup is trivial. After performing the Grassmann integrals one arrives at an expression similar to Eq. (5.67)

$$\begin{aligned}
\Phi_{34}(-is,r) = {} & 4\,\widehat{G}_{34} \exp\left(\mathrm{tr}\, r_2 s_2 + r_{21}(s_{11}+s_{21})\right) \int d\mu_B(s_1,s_1') \prod_{i=1}^{2} R_{1i} \prod_{j=1}^{3} S_{ji} \\
& \Bigg[\left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)} + \frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\right) \left(4\prod_{i=1}^{2} R_{i1} - 2\sum_{k=1}^{3} R_{11}S_{k1}^{-1} + 2\sum_{j=1}^{2} M_{1j}^{\rightarrow}(s_1,s_1')\right) \\
& \qquad \left(4\prod_{i=1}^{2} R_{i2} - 2\sum_{k=1}^{3} R_{12}S_{k2}^{-1} + 2\sum_{j=1}^{2} M_{2j}(s_1,s_1')\right) + \\
& \left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^3(is_2)} + \frac{1}{\Delta_2^3(ir_2)\Delta_2^4(is_2)}\right) \sum_{j=1}^{2} \left(\frac{4}{is_{12}-s_{j1}'} M_{2j}(s_1,s_1') - \frac{4}{is_{22}-s_{j1}'} M_{1j}(s_1,s_1')\right) \\
& - \left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^3(is_2)} + \frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\right) \prod_{\substack{k=1 \\ j=1}}^{2} \frac{2}{is_{k2}-s_{j1}'} + \\
& \frac{2}{\Delta_2^3(ir_2)\Delta_2^4(is_2)} \left(\mathrm{trg}\, r + r_{21} - \sum_{i=1}^{2} S_{i2}^{-1}\right) \sum_{j=1}^{2} M_{1j}(s_1,s_1') - \\
& \frac{2}{\Delta_2^3(ir_2)\Delta_2^4(is_2)} \left(\mathrm{trg}\, r + r_{21} - \sum_{i=1}^{2} S_{i1}^{-1}\right) \sum_{j=1}^{2} M_{2j}(s_1,s_1') \\
& - \frac{4}{\Delta_2^3(ir_2)\Delta_2^3(is_2)} \sum_{k=1}^{3}\prod_{j=1}^{2} R_{1j}S_{kj}^{-1} \Bigg] \exp\left((s_{21}'+s_{11}')(r_{11}-r_{21})\right) \; + (ir_{12} \longleftrightarrow ir_{22}) \quad .
\end{aligned} \tag{C.15}$$

We realize that lemma 5.3 and lemma 5.1 are sufficient to remove the denominators. this happens in the same way as for $\Phi_{24}(s,r)$. A single sum $\sum_{j=1}^{2} M_{1j}(s_1,s_1')$ transforms according to lemma 5.1. Moreover we observe that the fourth and fifth line of Eq. (C.15) together with the product $\sum_{j=1}^{2} M_{1j}^{\rightarrow}(s_1,s_1')\sum_{k=1}^{2} M_{2k}(s_1,s_1')$ yield exactly the integrand of lemma 5.3. Thus, it can be transformed according to lemma 5.3. After rearranging terms we arrive at the result as stated in Eq. (5.72).

We now turn to $\Phi_{44}(s,r)$. We state again the recursion formula

$$\Phi_{44}(-is,r) = \widehat{G}_{44} \int d\mu(s,s') \, \exp\left((\mathrm{tr}\, s - \mathrm{tr}\, s')\, r_{21}\right) \Phi_{34}(-is',\widetilde{r}) \tag{C.16}$$

with degenerate $\widetilde{r} = \mathrm{diag}\,(r_{21}, r_{11}, r_{11})$ according to Eq. (5.71). It proves to be useful to take advantage of the representation (4.34) for $\Phi_3^{(1)}(-is_1', \widetilde{r}_1)$. Up to a normalization we have

$$\Phi_3^{(1)}(-is'_1,\widetilde{r}_1) \propto (r_{21}-r_{11})^{-1/2}\exp\left(r_{11}\mathrm{tr}\, s'_1\right) \int d[t]\exp\left(i(r_{21}-r_{11})t\right)\prod_{i=1}^{3}(s'_{i1}-it)^{-1/2} \quad . \tag{C.17}$$

Hence it follows the useful identity

$$\frac{\partial}{\partial s'_{i1}}\frac{\partial}{\partial s'_{j1}}\exp\left(-r_{11}\mathrm{tr}\, s'_1\right)\Phi_3^{(1)}(-is'_1,\widetilde{r}_1) = \frac{1}{2}\frac{1}{s'_{i1}-s'_{j1}}\left(\frac{\partial}{\partial s'_{i1}}-\frac{\partial}{\partial s'_{j1}}\right)\exp\left(-r_{11}\mathrm{tr}\, s'_1\right)\Phi_3^{(1)}(-is'_1,\widetilde{r}_1) \quad . \tag{C.18}$$

We stress that this relation, which is crucial in the derivation, only holds, because of the degeneracy in the matrix $\widetilde{r}_1$. With Eq. (C.17) and another identity,

$$\sum_{i=1}^{3}\frac{\partial}{\partial s'_{i1}}\exp\left(-r_{11}\mathrm{tr}\, s'_1\right)\Phi_3^{(1)}(-is'_1,\widetilde{r}_1) = (r_{21}-r_{11})\exp\left(-r_{11}\mathrm{tr}\, s'_1\right)\Phi_3^{(1)}(-is'_1,\widetilde{r}_1) \quad , \tag{C.19}$$

we are able to arrange the terms emerging from the Grassmann integration in a way similar to the former cases. We obtain

$$\begin{aligned}
\Phi_{44}(-is,r) = {} & 4\,\widehat{G}_{44}\exp\left(\mathrm{tr}\, r_2 s_2 + r_{21}\mathrm{tr}\, s_1\right)\int d\mu_B(s_1,s'_1)\prod_{i=1}^{2}R_{1i}\prod_{j=1}^{4}S_{ji} \\
& \Bigg[\left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)}+\frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\right) \\
& \left(8R_{11}R_{21}^2 - 4R_{11}R_{21}\sum_{k=1}^{4}S_{k1}^{-1} + 4\sum_{j=1}^{3}\left(R_{11}-\frac{\partial^{\rightarrow}}{\partial s'_{j1}}\right)M_{1j}^{\rightarrow}(s_1,s'_1)\right) \\
& \left(8R_{12}R_{22}^2 - 4R_{12}R_{22}\sum_{k=1}^{4}S_{k2}^{-1} + 4\sum_{j=1}^{3}\left(R_{12}-\frac{\partial^{\rightarrow}}{\partial s'_{j1}}\right)M_{2j}(s_1,s'_1)\right) \\
& +\frac{16}{\Delta_2^3(ir_2)\Delta_2^4(is_2)}\sum_{j=1}^{3}M_{1j}^{\rightarrow}(s_1,s'_1)\left(\frac{1}{2}R_{21}R_{22}\left(\mathrm{trg}\, r-\sum_{i=1}^{4}S_{i1}^{-1}\right)+\right. \\
& \left.\left(r_{11}-r_{21}-\frac{\partial}{\partial s'_{j1}}\right)\left(R_{12}R_{22}+R_{21}R_{22}+R_{11}R_{21}+\frac{1}{2}(R_{12}+R_{22})\sum_{i=1}^{4}S_{i1}^{-1}\right)\right) \\
& -\frac{16}{\Delta_2^3(ir_2)\Delta_2^4(is_2)}\sum_{j=1}^{3}M_{2j}^{\rightarrow}(s_1,s'_1)\left(\frac{1}{2}R_{21}R_{22}\left(\mathrm{trg}\, r-\sum_{i=1}^{4}S_{i2}^{-1}\right)+\right. \\
& \left.\left(r_{11}-r_{21}-\frac{\partial}{\partial s'_{j1}}\right)\left(R_{11}R_{21}+R_{21}R_{22}+R_{12}R_{22}+\frac{1}{2}(R_{12}+R_{22})\sum_{i=1}^{4}S_{i2}^{-1}\right)\right) \\
& -\frac{16}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\sum_{k=1}^{4}\prod_{i,j}^{2}R_{ij}S_{kj}^{-1}\Bigg]\exp\left(-r_{21}\mathrm{tr}\, s'_1\right)\Phi_3^{(1)}(-is'_1,\widetilde{r}_1) \\
& + \mathcal{C}(s,r) + \ (ir_{12}\longleftrightarrow ir_{22}) \quad .
\end{aligned} \tag{C.20}$$

Again all operators with an arrow are understood to act only onto the term outside the squared bracket, i. e. onto $\exp\left(-r_{21}\mathrm{tr}\, s'_1\right)\Phi_3^{(1)}(-is'_1,\widetilde{r}_1)$. Under the notion $\mathcal{C}(s,r)$

we summarized terms, which are expected to arise due to non–commutativity of some operators acting on the integral and some operators acting under the integral. The last two lines in Eq. (5.69) in Lemma 5.3 are examples of such terms

$$
\begin{aligned}
\mathcal{C}(s,r) = {}& 4\,\widehat{G}_{44}\exp\left(\mathrm{tr}\,r_2s_2 + r_{21}\mathrm{tr}\,s_1\right)\int d\mu_B(s_1,s_1')\prod_{i=1}^{2}R_{1i}\prod_{j=1}^{4}S_{ji} \\
& \Bigg[\left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^3(is_2)} + \frac{1}{\Delta_2^3(ir_2)\Delta_2^4(is_2)}\right) \\
& \sum_{j=1}^{3}\left(R_{21}R_{22} + (R_{21}+R_{22})(r_{11}-r_{21}) - (R_{21}+R_{22})\frac{\partial^{\rightarrow}}{\partial s_{j1}'}\right) \\
& \qquad\qquad \left(\frac{16}{is_{12}-s_{j1}'}M_{2j}(s_1,s_1') - \frac{16}{is_{22}-s_{j1}'}M_{1j}(s_1,s_1')\right) \\
& -\left(\frac{1}{\Delta_2^2(ir_2)\Delta_3^2(is_2)} + \frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\right) \\
& \quad \prod_{k=1}^{2}\prod_{j=1}^{3}\left(R_{21}R_{22} + (R_{21}+R_{22})(r_{11}-r_{21}) - (R_{21}+R_{22})\frac{\partial^{\rightarrow}}{\partial s_{j1}'}\right)\frac{8}{is_{k2}-s_{j1}'} \\
& -\left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)} + \frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\right) \\
& \qquad\qquad \sum_{j,k}^{3}\left((r_{11}-r_{21}) - \frac{\partial^{\rightarrow}}{\partial s_{j1}'}\right)\left((r_{11}-r_{21}) - \frac{\partial^{\rightarrow}}{\partial s_{k1}'}\right)M_{1j}^{\rightarrow}M_{2k} \\
& -\frac{8}{\Delta_2^2(ir_2)\Delta_2^3(is_2)}\sum_{j=1}^{3}\left((r_{11}-r_{21}) - \frac{\partial^{\rightarrow}}{\partial s_{j1}'}\right)\left(\sum_{k=1}^{4}S_{k2}^{-1}M_{j1}^{\rightarrow} - \sum_{k=1}^{4}S_{k1}^{-1}M_{j2}\right) \\
& +\frac{8}{\Delta_2^2(ir_2)\Delta_2^3(is_2)}\left(\sum_{i\neq j}M_{j1}M_{j2}^{\rightarrow}\left(\frac{\partial}{\partial s_{i1}'} - \frac{\partial}{\partial s_{j1}'}\right)\right)\Bigg]\exp\left(-r_{21}\mathrm{tr}\,s_1'\right)\Phi_3^{(1)}(-is_1',\widetilde{r}_1)\ .
\end{aligned}
\tag{C.21}
$$

In order to evaluate Eqs. (C.20) and (C.21) we need some more properties of the matrix Bessel function $\Phi_4^{(1)}(-is_1,r_1)$. We investigate the action of $\widetilde{L}_k$ on $\Phi_4^{(1)}(-is_1,r_1)$ using again the representation (4.34)

$$
\begin{aligned}
\Phi_4^{(1)}(-is_1,r_1) \propto {}& (r_{21}-r_{11})^{-2}\exp\left(r_{11}\mathrm{tr}\,s_1\right) \\
& \int\left(\exp\left(i(r_{21}-r_{11})(t_1+t_2)\right)\prod_{i=1}^{4}\prod_{n=1}^{2}(s_{i1}-it_n)^{-1/2}\right)|t_1-t_2|dt_1dt_2
\end{aligned}
\tag{C.22}
$$

After a straightforward calculation involving an integration by parts we find

$$
\begin{aligned}
\widetilde{L}_k\exp\left(-r_{21}\mathrm{tr}\,s_1\right)\Phi_4^{(1)}(-is_1,r_1) \quad = \quad & \sum_{i=1}^{4}\frac{1}{is_{k2}-s_{i1}}\left((r_{11}-r_{21})^2 + (r_{11}-r_{21})\frac{\partial}{\partial s_{i1}}\right) \\
& \exp\left(-r_{21}\mathrm{tr}\,s_1\right)\Phi_4^{(1)}(-is_1,r_1)
\end{aligned}
\tag{C.23}
$$

Now Eqs. (C.20) and (C.21) can be enormously simplified by the observation that

$$
\left((r_{11}-r_{21})L_k - \widetilde{L}_k\right)\exp\left(-r_{21}\mathrm{tr}\,s_1\right)\Phi_4^{(1)}(-is_1,r_1) = 0 \qquad , \tag{C.24}
$$

which follows directly from Eq. (C.23). We find for Eq. (C.20)

$$
\begin{aligned}
\Phi_{44}(-is,r) = {} & 4\,\widehat{G}_{44}\exp\left(\operatorname{tr} r_2 s_2 + r_{21}\operatorname{tr} s_1\right)\int d\mu_B(s_1,s_1')\prod_{i=1}^{2}R_{1i}\prod_{k=1}^{4}S_{ki} \\
& \Bigg[\left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)}+\frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\right) \\
& \Bigg(R_{22}R_{21}\left(8R_{12}R_{22}-4R_{12}\sum_{k=1}^{4}S_{k2}^{-1}\right)\left(8R_{11}R_{21}-4R_{11}\sum_{k=1}^{4}S_{k1}^{-1}+4\sum_{j=1}^{3}M_{1j}^{\rightarrow}(s_1,s_1')\right)+ \\
& R_{22}R_{21}\left(8R_{11}R_{21}-4R_{11}\sum_{k=1}^{4}S_{k1}^{-1}\right)\left(8R_{12}R_{22}-4R_{12}\sum_{k=1}^{4}S_{k2}^{-1}+4\sum_{j=1}^{3}M_{2j}(s_1,s_1')\right) \\
& \qquad +\sum_{j,i}R_{21}\left(r_{11}-r_{21}-\frac{\partial^{\rightarrow}}{\partial s_{j1}'}\right)M_{1i}^{\rightarrow}(s_1,s_1')M_{2j}(s_1,s_1') \\
& \qquad +\sum_{j,i}R_{22}\left(r_{11}-r_{21}-\frac{\partial^{\rightarrow}}{\partial s_{i1}'}\right)M_{1i}^{\rightarrow}(s_1,s_1')M_{2j}(s_1,s_1') \\
& \qquad +\sum_{j,i}R_{21}R_{22}M_{1j}^{\rightarrow}(s_1,s_1')M_{2i}(s_1,s_1')\Bigg) \\
& +\frac{8}{\Delta_2^3(ir_2)\Delta_2^4(is_2)}R_{21}R_{22}\left(\operatorname{trg} r-\sum_{i=1}^{4}S_{i1}^{-1}\right)\sum_{j=1}^{3}\left(M_{1j}(s_1,s_1')-M_{2j}(s_1,s_1')\right) \\
& -\frac{16}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\sum_{k=1}^{4}\prod_{i,j}^{2}R_{ij}S_{kj}^{-1}\Bigg]\exp\left(-r_{21}\operatorname{tr} s_1'\right)\Phi_3^{(1)}(-is_1',\tilde{r}_1) \\
& \qquad + \mathcal{C}(s,r) \ + \ (ir_{12}\longleftrightarrow ir_{22}) \quad .
\end{aligned}
\tag{C.25}
$$

The terms contained in Eq. (C.21) simplify, too. We arrive at

$$
\begin{aligned}
\mathcal{C}(s,r) = {} & 4\,\widehat{G}_{44}\exp\left(\operatorname{tr} r_2 s_2 + r_{21}\operatorname{tr} s_1\right)\int d\mu_B(s_1,s_1')\prod_{i=1}^{2}R_{1i}\prod_{j=1}^{4}S_{ji} \\
& \Bigg[\left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^3(is_2)}+\frac{1}{\Delta_2^3(ir_2)\Delta_2^4(is_2)}\right) \\
& \sum_{j=1}^{3}\left(R_{21}R_{22}+(R_{21}+R_{22})(r_{11}-r_{21})-(R_{21}+R_{22})\frac{\partial^{\rightarrow}}{\partial s_{j1}'}\right) \\
& \qquad\qquad \left(\frac{16}{is_{12}-s_{j1}'}M_{2j}(s_1,s_1')-\frac{16}{is_{22}-s_{j1}'}M_{1j}(s_1,s_1')\right) \\
& -\left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)}+\frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\right) \\
& \qquad \prod_{\substack{k=1\\ j=1}}^{2}\left(R_{21}R_{22}+(R_{21}+R_{22})(r_{11}-r_{21})-(R_{21}+R_{22})\frac{\partial^{\rightarrow}}{\partial s_{j1}'}\right)\frac{8}{is_{k2}-s_{j1}'} \\
& +\frac{8}{\Delta_2^2(ir_2)\Delta_2^3(is_2)}\left(\sum_{i\neq j}M_{j1}^{\rightarrow}(s_1,s_1')M_{j2}(s_1,s_1')\left(\frac{\partial}{\partial s_{i1}'}-\frac{\partial}{\partial s_{j1}'}\right)\right)\Bigg] \\
& \qquad\qquad \exp\left(-r_{21}\operatorname{tr} s_1'\right)\Phi_3^{(1)}(-is_1',\tilde{r}_1)\ .
\end{aligned}
\tag{C.26}
$$

In order to further simplify the expressions we can now invoke a symmetry argument between the eigenvalues r_{11} and r_{21} respectively. Since the product $R_{11}R_{12}$ appears as prefactor in front of the integral (C.25) in the final result also $R_{21}R_{22}$ must appear as prefactor, due to the symmetry in r_{11} and r_{21}. Thus all terms in Eqs. (C.25) and (C.26) which do not contain $R_{21}R_{22}$ as a factor must yield zero. The remaining terms which are proportional to $R_{21}R_{22}$ can be treated again with Lemma 5.1 and Lemma 5.3. However, we show that the other terms indeed vanish. To this end we need an additional identity to treat the operator product

$$\sum_{j=1}^{2} \frac{\partial^{\rightarrow}}{\partial s'_{j1}} M_{1j}^{\rightarrow}(s_1, s'_1) \sum_{k=1}^{2} M_{2k}(s_1, s'_1) \qquad . \tag{C.27}$$

It is provided by the

Lemma C.1 *We have the conditions of Lemma 5.1, furthermore we define*

$$\begin{aligned} L_m^{\rightarrow}(s)\widetilde{L}_n(s) \;=\; & \sum_{i,j}^{k_1} \frac{1}{(is_{m2}-s_{i1})(is_{n2}-s_{j1})} \frac{\partial^3}{\partial s_{i1}\partial s_{j1}^2} + \\ & \frac{1}{2}\sum_{i,j}^{k_1} \frac{1}{(is_{m2}-s_{i1})(is_{n2}-s_{j1})} \frac{\partial}{\partial s_{i1}} \sum_{k\neq j}^{k_1} \frac{1}{s_{j1}-s_{k1}} \left(\frac{\partial}{\partial s_{j1}} - \frac{\partial}{\partial s_{k1}}\right) \quad . \end{aligned} \tag{C.28}$$

Then it holds that

$$\begin{aligned} & L_m^{\rightarrow}(s)\widetilde{L}_n(s) \int_{s_{21}}^{s_{11}} \cdots \int_{s_{k_1 1}}^{s_{(k_1-1)1}} \mu_B(s,s') d[s'_1] f(s'_1) = \\ & \int_{s_{21}}^{s_{11}} \cdots \int_{s_{1k_1}}^{s_{(k_1-1)1}} \Bigg[\sum_{j=1}^{k_1-1} \sum_{i=1}^{k_1-1} M_{mi}^{\rightarrow}(s_1,s'_1) \frac{\partial^{\rightarrow}}{\partial s'_{j1}} M_{nj}(s_1,s'_1) f(s'_1) - \\ & \frac{1}{is_{n2}-is_{m2}} \sum_{i=1}^{k_1-1} \left(\frac{1}{is_{m2}-s'_{i1}} \frac{\partial^{\rightarrow}}{\partial s'_{i1}} M_{ni}(s_1,s'_1) - \frac{1}{is_{n2}-s'_{i1}} \frac{\partial^{\rightarrow}}{\partial s'_{i1}} M_{mi}(s_1,s'_1) \right) \\ & -\frac{1}{2} \sum_{k\neq l}^{k_1-1} \frac{1}{(is_{m2}-s'_{k1})(is_{n2}-s'_{k1})(s'_{k1}-s'_{l1})^2} \frac{\partial^{\rightarrow}}{\partial s_{k1}} \\ & +\frac{1}{2} \sum_{k\neq l}^{k_1-1} \frac{1}{(is_{m2}-s'_{k1})(is_{n2}-s'_{l1})(s'_{k1}-s'_{l1})^2} \frac{\partial^{\rightarrow}}{\partial s_{k1}} \Bigg] f(s'_1) \mu_B(s,s') d[s'_1] \ . \end{aligned} \tag{C.29}$$

The proof is along the same lines as the proof of Lemma 5.1. We notice that the different use of the arrow in Eq. (C.28). The operator $L_m^{\rightarrow}(s)$ acts also on a part of $\widetilde{L}_n(s)$. This is not consistent with the definition in Eq. (5.66). However, we use the arrow in order to avoid yet another notation. Now we are in the position to translate the left hand side of Eq. (C.30) into an expression in terms of $\Phi_4^{(1)}(-is, r)$. After some further manipulations involving the identities in Eqs. (C.24), (C.18) and (C.19) we arrive at

$$\begin{aligned} \Phi_{44}(-is, r) = & \, \widehat{G}_{44} \exp\left(\operatorname{tr} r_2 s_2 + r_{21} \operatorname{tr} s_1\right) \prod_{i,j}^{2} R_{ji} \prod_{k=1}^{4} S_{ki} \\ & \left[\left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)} + \frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)} \right) \right. \end{aligned}$$

$$
\begin{aligned}
&\left(8R_{12}R_{22} - 4R_{12}\sum_{k=1}^{4} S_{k2}^{-1} - 4L_2(s)\right)\left(8R_{11}R_{21} - 4R_{11}\sum_{k=1}^{4} S_{k1}^{-1} - 4L_1(s)\right) \\
&- \frac{8}{\Delta_2^3(ir_2)\Delta_2^4(is_2)}\left(\operatorname{trg} r - \sum_{i=1}^{4} S_{i1}^{-1}\right)(L_1(s) - L_2(s)) \\
&- \left.\frac{16}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\sum_{k=1}^{4}\prod_{j=1}^{2} R_{1j}S_{kj}^{-1}\right] \exp\left(-r_{21}\mathrm{tr}\, s_1'\right)\Phi_3^{(1)}(s_1', \tilde{r}_1) \\
&\qquad\qquad (ir_{12} \longleftrightarrow ir_{22}) \ .
\end{aligned}
\tag{C.30}
$$

After rearranging terms this yields the result for $\Phi_{44}(-is, r)$ as stated in Eq. (5.76).

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