Collection Harald Schröer

Mathematical essays by

Harald Schröer

compiled by

Gabriele Dörflinger

English Edition

Heidelberg 2013

Author:
Diplom-Mathematician
Harald Schröer
Am Limperichsberg 2
D-53639 Königswinter

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Chapter A.

Integral calculus

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01. Reducing 3-dimensional Integrals to 1-dimensional Integrals

First we view the following integrals:

$$\int_{\frac{-c}{2}}^{\frac{c}{2}}\int_{\frac{-b}{2}}^{\frac{b}{2}}\int_{\frac{-a}{2}}^{\frac{a}{2}} g(x_1,x_2,x_3)\,dx_1dx_2dx_3 = \int_0^1\int_0^1\int_0^1 f(x_1,x_2,x_3)\,dx_1dx_2dx_3$$
$$= \lim_{\mu\to\infty}\lim_{\lambda\to\infty}\int_0^1 f(x_1,\langle\lambda\cdot x_1\rangle,\langle\mu\cdot x_1\rangle)\,dx_1 \tag{1}$$

We used the transformation formula see Forster [2] (§§2 and 13) and Corollary 3.2.8(He [3] p.88):

We need the function

$$\vec{A}(x_1,x_2,x_3) = (cx_1 - \frac{c}{2}, bx_2 - \frac{b}{2}, ax_3 - \frac{a}{2})$$

with jacobian matrix

$$D\vec{A}(x_1,x_2,x_3) = \begin{pmatrix} c & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix}$$

and determinant

$$\det D\vec{A}(x_1,x_2,x_3) = c\cdot b\cdot a.$$

The function f is defined through:

$$f(x_1,x_2,x_3) := a\cdot b\cdot c\cdot g(cx_1 - \frac{c}{2}, bx_2 - \frac{b}{2}, ax_3 - \frac{a}{2})$$

$\langle\quad\rangle$ is the fractional part of a number.

We can generalize this idea to every 3-dimensional integral. We view convex combinations, see Barner [1] chapter 13.2 p.30. We construct the function

$$\vec{A}(x_1,x_2,x_3) = \begin{pmatrix} x_1\cdot a + (1-x_1)\cdot b \\ x_2\cdot c + (1-x_2)\cdot d \\ x_3\cdot e + (1-x_3)\cdot f \end{pmatrix}$$

from $[0,1]\times[0,1]\times[0,1]$ to $[a,b]\times[c,d]\times[e,f]$.

We calculate the jacobian matrix:

$$D\vec{A}(x_1,x_2,x_3) = \begin{pmatrix} (a-b) & 0 & 0 \\ 0 & (c-d) & 0 \\ 0 & 0 & (e-f) \end{pmatrix}$$

with $\det D\vec{A}(x_1, x_2, x_3) = (a - b) \cdot (c - d) \cdot (e - f)$.

Then we have the transformation:

$$\int_a^b \int_c^d \int_e^f g(x_1, x_2, x_3)\, dx_1 dx_2 dx_3 = \int_0^1 \int_0^1 \int_0^1 f(x_1, x_2, x_3)\, dx_1 dx_2 dx_3 \tag{2}$$

with

$$f(x_1, x_2, x_3) = |\det D\vec{A}(x_1, x_2, x_3)| \cdot g(\vec{A}(x_1, x_2, x_3))$$

The transformation to an 1-dimensional integral is the same as in equation (1).

For 2-dimensional integrals we need corollary 3.2.5 at He [3].

References

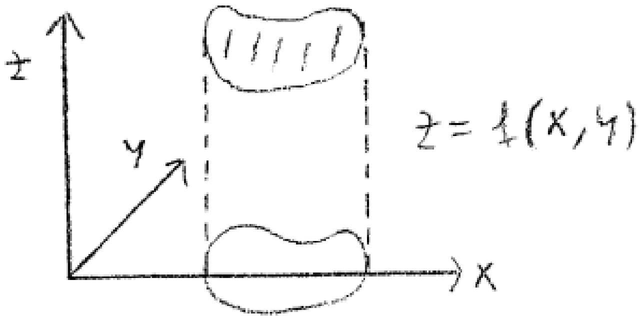
[1] Martin Barner, Friedrich Flohr “Analysis II” de Gruyter Verlag Berlin 1983

[2] Otto Forster “Analysis 3” 2.edition Vieweg Verlag Brunswick 1983

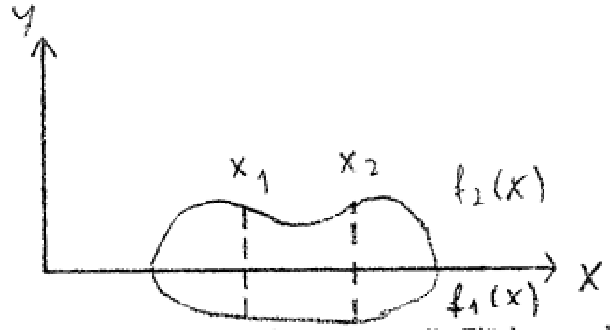
[3] Tian-Xiao He “Dimensionality reducing expansion of multivariate integration” Birkhäuser Verlag Boston 2001

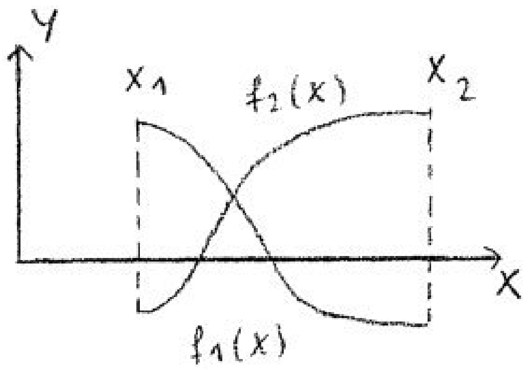
02. Surface area calculation

Let us view the surfaces as in the shown figures. Can we calculate the surface area? Which quantities are important to the calculation? Are there quantities from which the surface area is independent? Is it easy or difficult to calculate the surface area?



Here, two possibilities are shown:





The area is limited by $f_1(x), f_2(x)$ to y. (see figure)

For the surface area calculation we need a formula of the integral calculus. With Forster [1] §14 (14.7) p.142,143 the following is valid:

$f : T \rightarrow R$ is continuously differentiable

$T \subset R^{n-1}$
$M := \{(x_1, \ldots, x_n) \in T \times R \quad : \quad x_n = f(x_1, \ldots, x_{n-1})\}$
Thus, if the integral exists:

$$\mathrm{Vol}_{n-1}(M) = \int\limits_T \sqrt{1 + ||\nabla f(t)||^2}\, d^{n-1}t$$

$|| \cdot || =$ euclidean norm

Special case $n = 3$:
Now we insert $x_1 = x, x_2 = y, x_3 = z = f(x, y)$.

$f : T \to R$ is continuously differentiable.

$$T \subset R^2 \qquad (x, y) \in T \qquad t = (x, y)$$

$$M = \{(x, y, z) \in T \times R \quad : \quad z = f(x, y)\}$$

Then we obtain:

$$O = \mathrm{Vol}_2(M) = \int\limits_T \sqrt{1 + ||\nabla f(t)||^2}\, d^2t$$

$$\text{with} \qquad \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

The searched surface is a special case of this formula:

$$T = \{(x, y) \in R^2 \,|\quad x_1 \leq x \leq x_2 \quad f_1(x) \leq y \leq f_2(x)\}$$

With T, we can write the two dimensional integral in the following form:

$$O = \int\limits_{x_1}^{x_2} \int\limits_{f_1(x)}^{f_2(x)} \sqrt{1 + \left(\frac{\partial f(x, y)}{\partial x}\right)^2 + \left(\frac{\partial f(x, y)}{\partial y}\right)^2}\, dy\, dx$$

This is the searched formula of the surface area.

The formula is valid, if the integrals exist and, if $f(x, y)$ is continuously differentiable. That means $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous to (x, y).

References

[1] Otto Forster "Analysis 3" 2.edition 1983 Vieweg Verlag Brunswick

Chapter B.

Differential calculus

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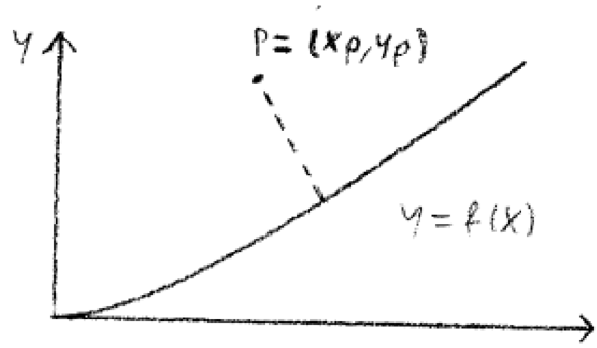
03. The minimal distance

1. The minimal distance of real functions on the plane:

One function $f(x)$ has a mimimal distance to the point $P = (x_p, y_p)$. We want to calculate this distance.(see fig.)



To distance D, it is valid:

$$D^2 = (y - y_p)^2 + (x - x_p)^2$$

Then we derivate D^2:

$$(D^2)' = 2 \cdot (y - y_p) \cdot y' + 2 \cdot (x - x_p)$$

Necessary condition of local extremes:

$$y' \cdot (y - y_p) + x - x_p = 0 \tag{1}$$

If we derivate D, then we get (with the chain rule) the same result.

Examples:

$y = p \cdot x^m$

$$pmx^{m-1} \cdot (px^m - y_p) + x - x_p = 0$$

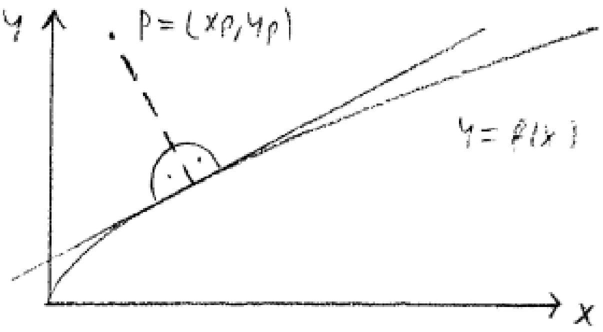
$y = \sin x$

$$\cos x \cdot (\sin x - y_p) + x - x_p = 0$$

$y = r \cdot a^x$

$$ra^x \ln a \cdot (ra^x - y_p) + x - x_p = 0$$

Now we want to answer the question, whether the minimal distance is orthogonal to the tangent of the function?



m is the slope of the perpendicular of the tangent. Then it must be shown $y' \cdot m = -1$. It is valid:

$$m = \frac{y - y_p}{x - x_p}$$

We obtain with (1):

$$y' = -\frac{x - x_p}{y - y_p}$$

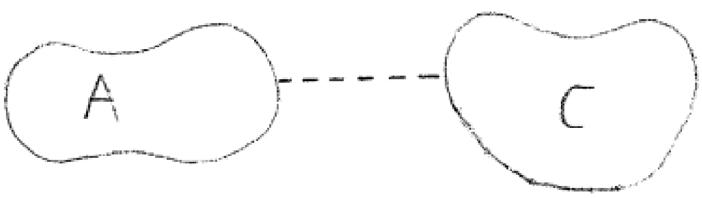
Check:

$$y' \cdot m = -\frac{x - x_p}{y - y_p} \cdot \frac{y - y_p}{x - x_p} = -1$$

The supposition is true.

2. Minimal distance between two-dimensional sets:

We look at the following figure:



Both areas are represented with $A = \vec{a}(p_1, p_2, t)$ and $C = \vec{c}(q_1, q_2, t)$.

p_1, p_2, q_1, q_2 are area parameters. t is the time or an additional parameter.

We obtain distance r:

$$r(p_1, p_2, q_1, q_2, t) = \sqrt{(\vec{a} - \vec{c})^2}$$

We introduce the following notations:

$$D_1 := \frac{d}{dp_1} \qquad D_2 := \frac{d}{dp_2} \qquad D_3 := \frac{d}{dq_1} \qquad D_4 := \frac{d}{dq_2}$$

The product rule of scalar products:

$$D(\vec{a}\cdot\vec{c}) = D\vec{a}\cdot\vec{c} + \vec{a}\cdot D\vec{c}$$

It follows:

$$D(\vec{a}^2) = 2\cdot\vec{a}\cdot D\vec{a}$$

We have:

$$r^2 = (\vec{a}-\vec{c})^2$$

r and, therefore, r^2 must be derivated:

$$D_1 r^2 = 2\cdot(\vec{a}-\vec{c})\cdot D_1\vec{a}$$

$$D_2 r^2 = 2\cdot(\vec{a}-\vec{c})\cdot D_2\vec{a}$$

$$D_3 r^2 = 2\cdot(\vec{a}-\vec{c})\cdot -D_3\vec{c}$$

$$D_4 r^2 = 2\cdot(\vec{a}-\vec{c})\cdot -D_4\vec{c}$$

Necessary criterion of local extremes:

$$D_i r^2 = 0 \qquad \text{for} \qquad i \in 1,2,3,4$$

It follows:

$$(\vec{a}-\vec{c})\cdot D_1\vec{a} = 0$$

$$(\vec{a}-\vec{c})\cdot D_2\vec{a} = 0$$

$$(\vec{a}-\vec{c})\cdot D_3\vec{c} = 0$$

$$(\vec{a}-\vec{c})\cdot D_4\vec{c} = 0$$

$D_1\vec{a}, D_2\vec{a}, D_3\vec{c}, D_4\vec{c}$ are tangent vectors (see Forster [1] §15 theorem 1 p.148.)

We get the following result:

The minimal distance is orthogonal to both sets.

The result is valid to sets with many finite parameters, as well. This can be proven in the same way.

In the case, $A \cap C = \emptyset$ exists a minimal distance. If the four equations haven't got a solution, then the minimal distance is a boundary minimum.

Special cases:

A is a curve, if either p_1 or p_2 vanish.

C is a curve, if either q_1 or q_2 vanish.

A is a point, if p_1 and p_2 vanish.

C is a point, if q_1 and q_2 vanish.

If t vanishes, then the sets are (temporally) invariant.

Special choice of parameters:

for curves: i can be 1 or 2.

$$\vec{a} = \begin{pmatrix} p_i \\ f_1(p_i) \\ g_1(p_i) \end{pmatrix} \qquad \vec{c} = \begin{pmatrix} q_i \\ f_2(q_i) \\ g_2(q_i) \end{pmatrix}$$

for areas:

$$\vec{a} = \begin{pmatrix} p_1 \\ p_2 \\ f_1(p_1, p_2) \end{pmatrix} \qquad \vec{c} = \begin{pmatrix} q_1 \\ q_2 \\ f_2(q_1, q_2) \end{pmatrix}$$

References

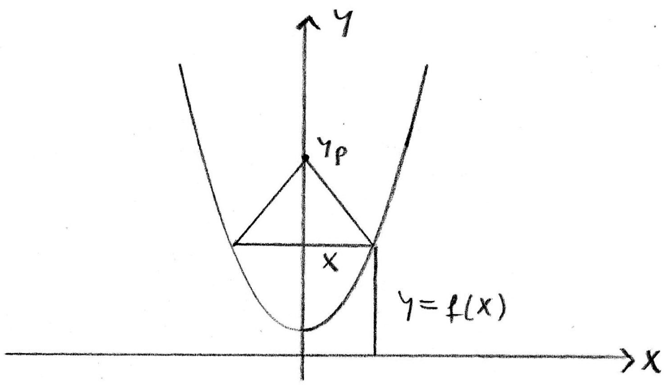
[1] Otto Forster “Analysis 3” 2.edition 1983 Vieweg Verlag Brunswick

04. Described Triangles,Trapeziums and Rectangles

Here we treat different extremum problems with side conditions. Here the side conditions are often constructed by general functions. If there are maxima or minima, this is dependent from the side condition function. This is true for the existence of extrema, as well. If we set the first derivative equal to zero, we get a necessary, but not a sufficient, condition to the existence of local extrema. Here we learn to know the forms of these necessary conditions. The second aim is to get the objective function in dependence from the functions. Here there is no discussion about local extremum, maximum, minimum. For this we must choose the side conditions functions at first. Then we can decide about this with the second or higher derivative.

1. Isosceles triangle with one function

We want to calculate the maximum area of the following isosceles triangle. (see fig.)



Let $y = f(x)$ be a axisymmetric function that means $f(x) = f(-x)$. We assume that the derivative $y' = f'(x)$ exists.

For the triangle's area F it is valid:

$$F = x \cdot (y_p - y) = xy_p - xy$$

The necessary condition of a local extremum must be:

$$0 = F' = y_p - y - xy'$$

The extremum condition can be written as:

$$y + xy' = y_p$$

This condition is valid to every axisymmetric function.

An example with $p > 0$ and m as even number:

$$y = px^m \qquad y' = pmx^{m-1}$$

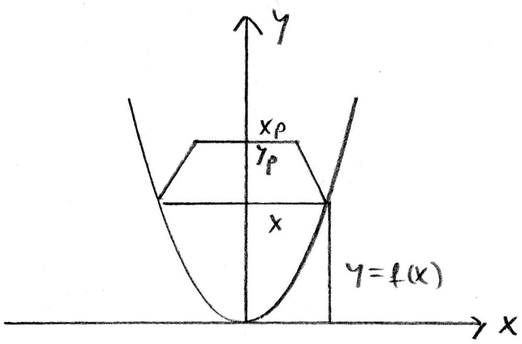
$$px^m + pmx^m = y_p \qquad p \cdot (m+1)x^m = y_p$$

It follows:

$$x = \sqrt[m]{\frac{y_p}{p \cdot (m+1)}}$$

2. Trapezium described into one function

We look at the trapezium in the following figure: (x_p, y_p are let invariant.)



We assume that $y = f(x)$ is differentiable, symmetric and convex. Here we search for the maximum area, too. To the area F it is valid:

$$F = \frac{(a+c) \cdot h}{2}$$

with the parallel sides a, c and the trapezium's height h With $c = 2x_p$, $a = 2x$ and $h = y_p - y$ it follows:

$$F = (y_p - y) \cdot (x_p + x)$$
$$F' = y_p - y - y' \cdot (x_p + x)$$

Necessary condition of local extrema:

$$F' = 0 \qquad \Rightarrow \qquad y' \cdot (x_p + x) = y_p - y$$

Special case isosceles triangle: $x_p = 0$

$$\Rightarrow \qquad xy' + y = y_p$$

Now we view the rectangle with $x_p = x$. Then we obtain the area:

$$F = 2x \cdot (y_p - y) \qquad F' = 2 \cdot (y_p - y) - 2xy'$$

We get as necessary condition $F' = 0$ to the rectangle:

$$xy' = y_p - y$$

Now we want to calulate the extremum perimeter. The perimeter is $U = a + c + 2 \cdot \sqrt{(y_p - y)^2 + (x - x_p)^2}$. If a und c are inserted, it follows:

$$U = 2 \cdot (x + x_p) + 2 \cdot \sqrt{(y_p - y)^2 + (x - x_p)^2}$$

chain rule:

$$U' = 2 + \frac{(-2) \cdot (y_p - y) \cdot y' + 2 \cdot (x - x_p)}{\sqrt{(y_p - y)^2 + (x - x_p)^2}}$$

With the necessary condition $U' = 0$ it becomes:

$$(y_p - y)^2 + (x - x_p)^2 = ((y_p - y) \cdot y' - x + x_p)^2$$

Special case isosceles triangle with $x_p = 0$:

$$(y_p - y)^2 + x^2 = ((y_p - y)y' - x)^2$$

The rectangle with $x_p = x$ must be calculated separately.

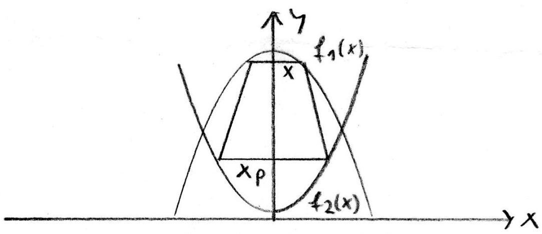
Perimeter of the rectangle:

$$U = 4x + 2 \cdot (y_p - y) \qquad U' = 4 - 2y'$$

$$U' = 0 \qquad \Rightarrow \qquad y' = f'(x) = 2$$

3. The trapezium described into two functions

We view the trapezium in the figure:



x is variable, x_p is invariant. We assume that the functions f_1, f_2 are axisymmetric and concave respectively convex. With the elementary area formula we obtain:

$$F = (x + x_p) \cdot (f_1(x) - f_2(x_p))$$

with the product rule:

$$F' = (x + x_p) \cdot f_1'(x) + f_1(x) - f_2(x_p)$$

$F' = 0$ is necessary condition of local extrema, with that it follows:

$$f_2(x_p) - f_1(x) = (x + x_p) \cdot f_1'(x)$$

Now we work with the extremum perimeter:

$$U = 2 \cdot (x_p + x) + 2 \cdot \sqrt{(x - x_p)^2 + (f_1(x) - f_2(x_p))^2}$$

U derived with the chain rule:

$$U' = 2 + 2 \cdot \frac{x - x_p + (f_1(x) - f_2(x_p)) \cdot f_1'(x)}{\sqrt{(x - x_p)^2 + (f_1(x) - f_2(x_p))^2}}$$

The necessary condition of local extrema $U' = 0$ leads to:

$$(x - x_p + f_1'(x) \cdot (f_1(x) - f_2(x_p)))^2 = (x - x_p)^2 + (f_1(x) - f_2(x_p))^2$$

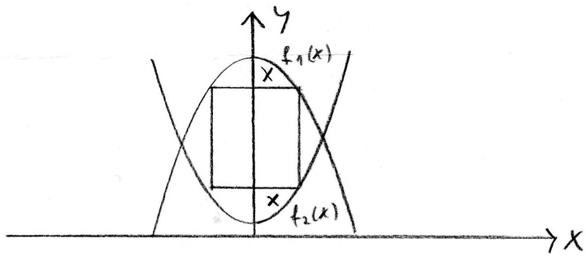
For the special case $x_p = 0$ we have a isosceles triangle. The condition of extremum area is:

$$f_2(0) - f_1(x) = x \cdot f_1'(x)$$

condition of extremum perimeter:

$$(x + f_1'(x) \cdot (f_1(x) - f_2(0)))^2 = x^2 + (f_1(x) - f_2(0))^2$$

The formulas of the rectangle must be derived separately. We view the figure:



The area of the rectangle is:

$$F = (y_1 - y_2) \cdot 2x$$

Derivative:

$$F' = 2 \cdot ((y_1' - y_2') \cdot x + y_1 - y_2)$$

With the necessary condition $F' = 0$ it follows:

$$x \cdot (y_1' - y_2') = y_2 - y_1$$

For $y_2 = k$ (constant) we get:

$$xy_1' = k - y_1$$

Now we determine the perimeter of the rectangle:

$$U = 4x + 2 \cdot (y_1 - y_2) \qquad U' = 4 + 2 \cdot (y_1' - y_2')$$

The necessary condition is $U' = 0$, it follows:

$$y_2' - y_1' = 2 \qquad y_1 \geq y_2$$

More extremum problems can be found at Schröer [1].

References

[1] Harald Schröer, "Special extreme value problems and extremum principles", Wissenschaft & Technik Verlag, Berlin, 2002

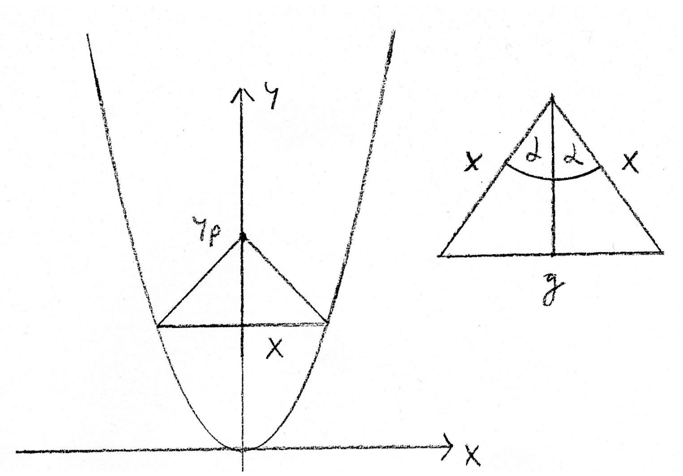
05. Inscribed pyramid and frustum of pyramid

Abstract: Here we treat two extremum problems with side conditions.If there are maxima or minima, this is dependent from the side condition function. This is true for the existence of extrema, as well. If we set the first derivative equal to zero, we get a necessary, but not a sufficient, condition to the existence of local extrema. Here we learn to know the forms of these necessary conditions. The second aim is to get the objective function in dependence from the functions. Here there is no discussion about local extremum, maximum, minimum. For this we must choose the side conditions functions at first. Then we can decide about this with the second or higher derivative.

Key words: Extremum - side condition - pyramid - frustum of pyramid - maximum - minimum

1. Pyramid with one function

Now, we view a pyramid with a regular n-gon as base like in the figure:



Then the following equations are valid:

$$\alpha = \frac{\pi}{n}$$

$$A = \frac{gx \cdot \cos\alpha}{2} \qquad \text{with} \qquad g = 2 \cdot \sin\alpha \cdot x$$

$$\Rightarrow \qquad A = x^2 \cdot \sin\alpha\cos\alpha$$

Then to the base G it follows:

$$G = n \cdot A = n \cdot x^2 \cdot \cos\alpha\sin\alpha$$

At last for the pyramid's volume V we obtain:

$$V = \frac{G \cdot h}{3} \qquad \text{with} \qquad h = y_p - y \qquad \text{and} \qquad \sin 2\alpha = 2 \cdot \sin\alpha\cos\alpha$$

it follows:

$$V = \frac{1}{6} \cdot nx^2 \cdot (y_p - y) \cdot \sin\left(\frac{2\pi}{n}\right)$$

With the product rule we get:

$$V' = \frac{1}{6} \cdot n \sin\left(\frac{2\pi}{n}\right) ((y_p - y) \cdot 2x - x^2 y')$$

The necessary criterion of local extrema $V' = 0$ leads to:

$$2 \cdot (y_p - y) - xy' = 0$$

To the cylinder the following is valid:

$$G = \pi x^2 \qquad V = G \cdot h = \pi x^2 \cdot (y_p - y)$$

The calculation is analogous. The criterion of local extremum remains unchanged.

2. Frustum of pyramid described with one function

For the two bases we have:

$$G_1 = ax^2 \qquad G_2 = ax_p^2 \qquad \text{with} \qquad a = \frac{1}{2} \cdot n \cdot \sin\left(\frac{2\pi}{n}\right)$$

To the frustum of pyramid's volume the known formula is valid:

$$V = \frac{h}{3} \cdot (G_1 + \sqrt{G_1 G_2} + G_2) \qquad \text{with} \qquad h = y_p - y$$

Insertion:

$$V = \frac{y_p - y}{3} \cdot (ax^2 + axx_p + ax_p^2)$$

Derivation:

$$V' = \frac{-y'}{3} \cdot a \cdot (x^2 + xx_p + x_p^2) + \frac{y_p - y}{3} \cdot a \cdot (2x + x_p)$$

Necessary condition of local extrema:

$$V' = 0$$

It follows:

$$(y_p - y) \cdot (2x + x_p) - y' \cdot (x^2 + x_p x + x_p^2) = 0$$

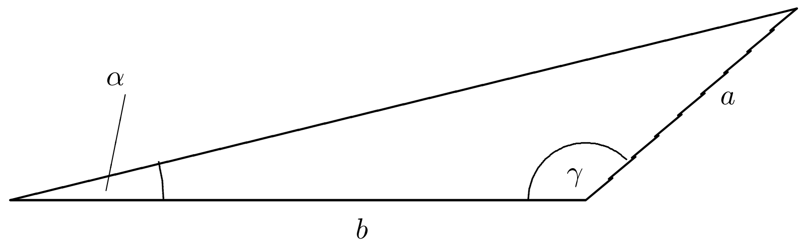
Further examples can be found at Schröer [1].

References

[1] Harald Schröer, "Special extreme value problems and extremum principles", Wissenschaft & Technik Verlag, Berlin 2002

06. Extremum Angle

We view the triangle in the following figure:



The function $\alpha = f(\gamma)$ is explained as:

$$\tan\alpha = \frac{a \cdot \sin\gamma}{b - a \cdot \cos\gamma} \qquad 0 \le \gamma \le 180° \qquad b \ge a \tag{1}$$

We want to find local extrema of α in dependence of γ. The tangent is a strictly increasing function in the interval $[0, 180°]$. It is enough to view $\tan\alpha$.

We calculate the derivation of $\tan\alpha$ to γ with the quotient rule:

$$\frac{d\tan\alpha}{d\gamma} = \frac{\cos\gamma \cdot a \cdot (b - a\cos\gamma) - \sin\gamma \cdot a \cdot a \cdot \sin\gamma}{(b - a\cos\gamma)^2}$$

It is valid that $\tan\alpha \ge 0$ for $\gamma \in [0, 180°]$. With the Rolle's theorem and a theorem about continuous function about the existence of local extrema for example see Forster [1], §11, theorem 2, p.67, we have got a local maximum in this interval.

Necessary condition of local extrema:

$$\frac{d\tan\alpha}{d\gamma} = 0$$

It follows:

$$ab\cos\gamma - a^2 \cdot (\sin^2\gamma + \cos^2\gamma) = 0$$

With $\sin^2\gamma + \cos^2\gamma = 1$ we obtain:

$$ab\cos\gamma = a^2$$

with that:

$$\cos\gamma = \frac{a}{b}$$

In the case $a \ll b$ is $\gamma \approx 90°$.

$$\sin\gamma = \sqrt{1 - \cos^2\gamma} = \sqrt{1 - \frac{a^2}{b^2}}$$

Insertion in (1):

$$\tan \alpha_{max} = \frac{\sqrt{1-\frac{a^2}{b^2}} \cdot a}{b - \frac{a^2}{b}} = \frac{\frac{a}{b} \cdot \sqrt{b^2 - a^2}}{\frac{1}{b} \cdot (b^2 - a^2)} = \frac{a}{\sqrt{b^2 - a^2}}$$

With that we have got:

$$\tan \alpha_{max} = \frac{a}{\sqrt{b^2 - a^2}}$$

and

$$\tan \alpha_{max} \approx \frac{a}{b} \qquad \text{for} \qquad a \ll b$$

We have found a representation of the maximum angle. The problem of determination the phase of planets is an application, see Voigt [3], chapter II.9.2, p.70,71. Further extremum problems can be found at Schröer [2].

References

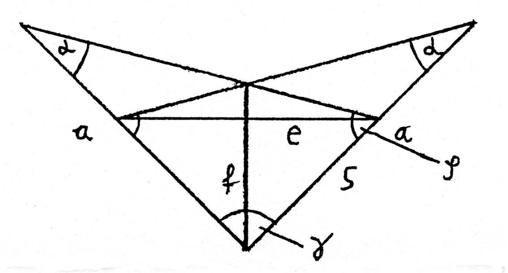
[1] Otto Forster, “Analysis 1”, Vieweg Verlag, Brunswick, 4.edition, 1984

[2] Harald Schröer “Special extreme value problems and extremum principles” Berlin, 2002

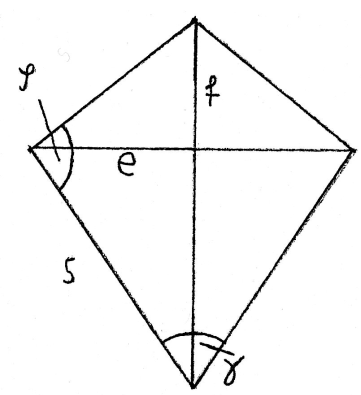
[3] Hans Heinrich Voigt, “Abriß der Astronomie”, BI, Mannheim, 4.edition, 1988

07. The foot-rule problem

Now we view the figure in the foot-rule:



The kite is interesting that is spaned to four sides of a foot-rule.



We search the maximum area of the kite with known a and α. γ is the changable variable.

Sine law:

$$\frac{f}{\sin\alpha} = \frac{a}{\sin(180° - \frac{\gamma}{2} - \alpha)} \qquad \Rightarrow \qquad f = \frac{a\sin\alpha}{\sin(180° - \frac{\gamma}{2} - \alpha)}$$

$$\frac{s}{\sin\alpha} = \frac{a}{\sin(180° - \gamma - \alpha)} \qquad \text{and} \qquad \frac{e}{2} = s \cdot \sin\frac{\gamma}{2}$$

$$\Rightarrow \qquad \frac{e}{2} = \frac{a\sin\alpha\sin\frac{\gamma}{2}}{\sin(180° - \gamma - \alpha)}$$

With the area's formula $F = \frac{ef}{2}$ we obtain:

$$F = \frac{a^2\sin^2\alpha\sin\frac{\gamma}{2}}{\sin(180° - \frac{\gamma}{2} - \alpha)\cdot\sin(180° - \gamma - \alpha)} \qquad \alpha = \text{const.}$$

with $\sin(180° - \beta) = \sin\beta$ it becomes:

$$F(\gamma) = \frac{a^2\sin^2\alpha\sin\frac{\gamma}{2}}{\sin(\frac{\gamma}{2} + \alpha)\cdot\sin(\gamma + \alpha)} \tag{1}$$

This area formula must be derived to γ. We calculate the denominator's derivation with the product rule and the chain rule:

$$\left(\sin\left(\frac{\gamma}{2}+\alpha\right)\cdot\sin(\gamma+\alpha)\right)' = \frac{1}{2}\cdot\cos\left(\frac{\gamma}{2}+\alpha\right)\sin(\gamma+\alpha)+\sin\left(\frac{\gamma}{2}+\alpha\right)\cos(\gamma+\alpha)$$

Counter: $\qquad (\sin\frac{\gamma}{2})' = \frac{1}{2}\cdot\cos\frac{\gamma}{2}$

Quotient rule:

$$\frac{F'(\gamma)}{a^2\sin^2\alpha} = \frac{1}{\sin^2\left(\frac{\gamma}{2}+\alpha\right)\cdot\sin^2(\gamma+\alpha)}\cdot\left(\frac{1}{2}\cdot\cos\frac{\gamma}{2}\cdot\sin\left(\frac{\gamma}{2}+\alpha\right)\cdot\sin(\gamma+\alpha)\right.$$
$$\left.-\left(\frac{1}{2}\cdot\cos\left(\frac{\gamma}{2}+\alpha\right)\cdot\sin(\gamma+\alpha)+\sin\left(\frac{\gamma}{2}+\alpha\right)\cdot\cos(\gamma+\alpha)\right)\cdot\sin\frac{\gamma}{2}\right)$$

The necessary condition of local extrema is $F'(\gamma)=0$.

$$\frac{1}{2}\cdot\cos\frac{\gamma}{2}\sin\left(\frac{\gamma}{2}+\alpha\right)\sin(\gamma+\alpha)-\left(\frac{1}{2}\cdot\cos\left(\frac{\gamma}{2}+\alpha\right)\sin(\gamma+\alpha)\right.$$
$$\left.+\sin\left(\frac{\gamma}{2}+\alpha\right)\cos(\gamma+\alpha)\right)\cdot\sin\frac{\gamma}{2}=0$$

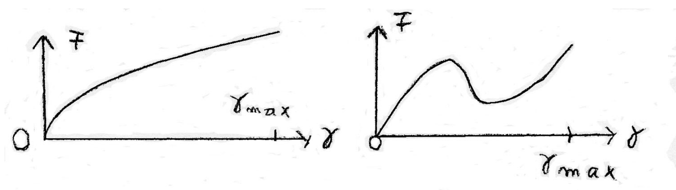
It is valid $\frac{\sin\beta}{\cos\beta}=\tan\beta$. We divide through $\cos\left(\frac{\gamma}{2}+\alpha\right)\cdot\cos(\gamma+\alpha)$. Then it follows:

$$\frac{1}{2}\cdot\cos\frac{\gamma}{2}\tan\left(\frac{\gamma}{2}+\alpha\right)\tan(\gamma+\alpha)-\left(\frac{1}{2}\cdot\tan(\gamma+\alpha)+\tan\left(\frac{\gamma}{2}+\alpha\right)\right)\cdot\sin\frac{\gamma}{2}=0$$

divided through $\sin\frac{\gamma}{2}\cdot\tan\left(\frac{\gamma}{2}+\alpha\right)\cdot\tan(\gamma+\alpha)$:

$$\frac{1}{2\cdot\tan\frac{\gamma}{2}}-\frac{1}{2\cdot\tan\left(\frac{\gamma}{2}+\alpha\right)}-\frac{1}{\tan(\gamma+\alpha)}=0 \qquad (2)$$

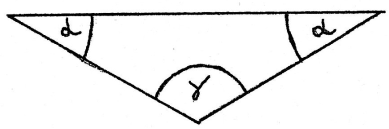
With (2) we must determine γ in dependence of α. This can be done by replacing the tangent-expressions through sine and cosine. In this case the application of the Newton's method is difficult.
To decide local maximum or local minimum, we must calculate the 2. derivation in the concrete case. Saddle points are possible, too. (see figure)



For a kite it must be:

$$360^\circ-\gamma-2\varphi<180^\circ \qquad \text{and} \qquad \varphi=180^\circ-\alpha-\gamma$$
$$\Rightarrow \qquad 360^\circ-\gamma-2\cdot(180^\circ-\alpha-\gamma)<180^\circ \qquad \Leftrightarrow \qquad 2\alpha+\gamma<180^\circ$$
$$\Rightarrow \qquad \gamma<180^\circ-2\alpha \qquad \Rightarrow \qquad \alpha<\frac{180^\circ-\gamma}{2}$$

In case $2\alpha+\gamma=180^\circ$ there is a isosceles triangle (a special kite).



Further unusual extremum problems can be found at Schröer [1].

References

[1] Harald Schröer "Special extreme value problems and extremum principles", Wissenschaft & Technik Verlag, Berlin, 2002

08. Formulas for logarithms as limits

Here we will give a representation of logarithms to any base.

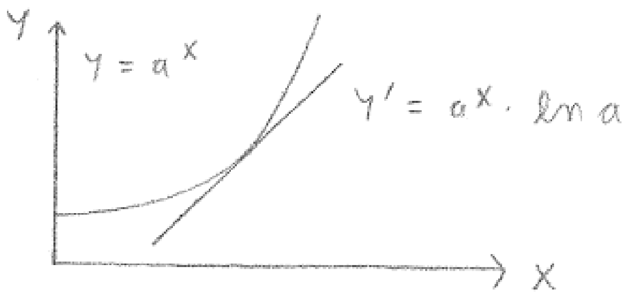
We begin with the equation:

$$a^{\log_a x} = x$$

This is equivalent to:

$$\log_a x = \frac{\ln x}{\ln a} \tag{1}$$

ln is the natural logarithm.



For the derivation of a^x is valid:

$$\frac{d}{dx}a^x = a^x \cdot \ln a$$

If we think at the definition of the derivation:

$$\begin{aligned} a^x \cdot \ln a &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= a^x \cdot \lim_{h\to 0} \frac{a^h - 1}{h} \end{aligned}$$

h is a null sequence. Thus we obtain:

$$\ln a = \lim_{h\to 0} \frac{a^h - 1}{h}$$

Now we use the formula (1) to get an expression of $\log_a x$:

$$\log_a x = \frac{\lim_{h\to 0} \frac{x^h-1}{h}}{\lim_{h\to 0} \frac{a^h-1}{h}} \tag{2}$$

$$= \lim_{h \to 0} \frac{x^h - 1}{a^h - 1}$$

For $a = 10$ we obtain a formula for the decimal logarithm of x:

$$\log_{10} x = \lim_{h \to 0} \frac{x^h - 1}{10^h - 1}$$

Now we want to derive a second formula for the logarithm of any base.

Now we use the other derivation's definition of a^x.

$$\begin{aligned} a^x \cdot \ln a &= \lim_{h \to 0} \frac{a^x - a^{x-h}}{h} \\ &= a^x \cdot \lim_{h \to 0} \frac{1 - a^{-h}}{h} \end{aligned}$$

Then we have the following expression:

$$\ln a = \lim_{h \to 0} \frac{1 - a^{-h}}{h}$$

Now we take equation (1) again:

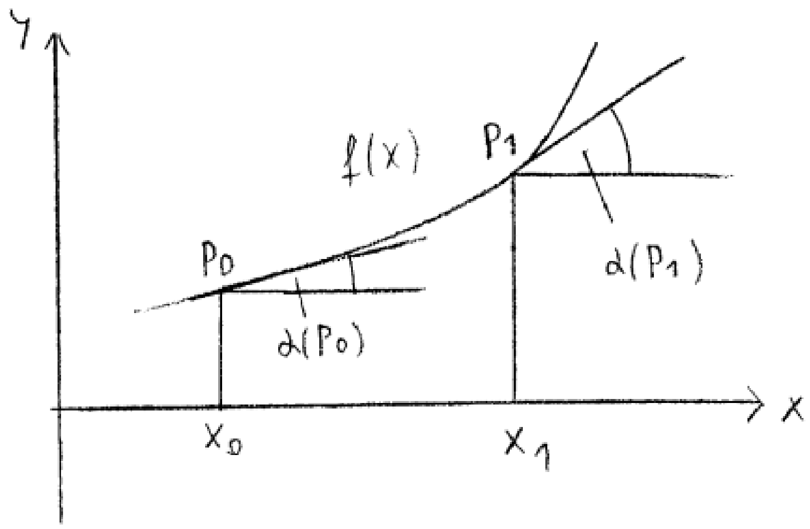
$$\log_a x = \lim_{h \to 0} \frac{1 - x^{-h}}{1 - a^{-h}} \tag{3}$$

Thus we have proved a second formula for logarithms.
$a = 10$ yields the decimal logarithm:

$$\log_{10} x = \lim_{h \to 0} \frac{1 - x^{-h}}{1 - 10^{-h}}$$

09. The circle of curvature

Here we want to deal with the curvature of a function in R^2. The function shall be **no** straight line. This means that the second derivation of the function is unequal to zero. We can take the following method:



We use the definition of the curvature in Bronstein [1] chapter 4.3.1.2 p.589:

$$k = \lim_{P_0 \to P_1} \frac{\alpha(P_1) - \alpha(P_0)}{\overset{\frown}{P_0 P_1}}$$

$\overset{\frown}{P_0 P_1}$ is the arc length between P_0 and P_1. Thus the curvature is the quotient of one angular difference and one arc length. If we insert for the arc length and for the angular difference, then we obtain: (f' = derivation of f)

$$k = \lim_{x \to x_0} \frac{\arctan|f'(x)| - \arctan|f'(x_0)|}{\int\limits_{x_0}^{x} \sqrt{1 + f'(t)^2}\, dt}$$

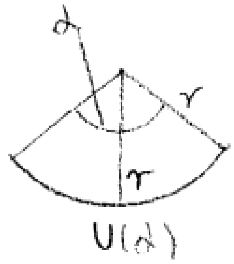
Here we use l'Hospital's rule:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \qquad\qquad g'(a) \neq 0$$

It must be $f(a) = g(a) = 0$, this is true in our case. Because of $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$ with this rule we get:

$$k = \lim_{x \to x_0} \frac{\frac{f''(x)}{1+f'(x)^2}}{\sqrt{1 + f'(x)^2}} = \frac{f''(x_0)}{(1 + f'(x_0)^2)^{\frac{3}{2}}}$$

Now we look at the following figure:



Then we can conclude to the radius of curvature r:

$$k := \frac{\alpha}{u(\alpha)} = \frac{\alpha}{\frac{\alpha}{2\pi} \cdot 2\pi r} = \frac{1}{r}$$

It follows for the radius of curvature:

$$r = \frac{(1 + f'(x_0)^2)^{\frac{3}{2}}}{f''(x_0)}$$

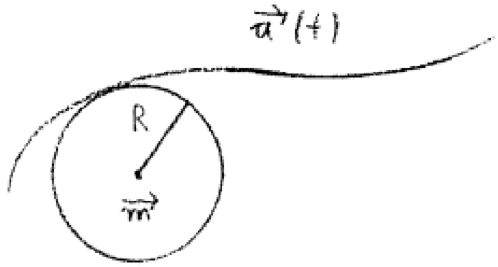
It is $f''(x_0) \neq 0$.

References

[1] I. N. Bronstein, K. A. Semendjajew "Taschenbuch der Mathematik" Teubner Verlagsgesellschaft Leipsic 1985 22.edition

10. Angular velocity and radius vector of any motion

We see a path $\vec{a}(t)$ in R^n. t is a given parameter, for example, the time.



We can construct to every path at any given place a circle of curvature.

$\vec{x}(t)$ = curvature vector $\qquad \vec{x}(t) \in R^n$
$x(t) := |\vec{x}(t)|$ = measure of curvature $\qquad |\cdot|$ = absolute value
With Barner [1] chapter 14.1 p.91 it is valid for the curvature vector:

$$\vec{x} = \frac{\ddot{\vec{a}}}{\dot{a}^2} - \frac{\dot{\vec{a}} \cdot \ddot{\vec{a}}}{\dot{a}^4} \cdot \dot{\vec{a}} \qquad \frac{d\vec{a}}{dt} =: \dot{\vec{a}} \qquad \dot{a} := |\dot{\vec{a}}| \tag{1}$$

and for measure of curvature:

$$|\vec{x}| = \frac{\sqrt{\dot{a}^2 \cdot \ddot{a}^2 - \left(\dot{\vec{a}} \cdot \ddot{\vec{a}}\right)^2}}{\dot{a}^3} \tag{2}$$

For the midpoint $\vec{m}$ of the circle of curvature the following formula is mentioned in Barner [1] chapter 14.1 p.89:

$$\vec{m} = \vec{a} + \frac{\vec{x}}{x^2} \tag{3}$$

The absolute value of the radius of curvature is:

$$R(t) = \frac{1}{x(t)}$$

We get for the radius vector:

$$\vec{R} = \vec{a} - \vec{m} \tag{4}$$

Now we introduce the angular velocity $\vec{w}$ in R^3. It is shown for example in Budo [2] §14 p.72:

$$\dot{\vec{a}} = \vec{w} \times \vec{R} \tag{5}$$

Now the problem is to determine $\vec{w}$. Because $\vec{w}$ is perpendicular to $\vec{R}$, it follows for the absolute values of the vectors:

$$\dot{a} = R \cdot w \qquad \Rightarrow \qquad w = \frac{\dot{a}}{R}$$

For the determination of $\vec{w}$ we need the vector operator method:

$\vec{R}$ perpendicular to $\vec{w}$ yields:

$$\vec{R} \cdot \vec{w} = 0$$

With equation (5) we construct:

$$\dot{\vec{a}} \times \vec{R} = (\vec{w} \times \vec{R}) \times \vec{R}$$

With the expansion theorem:

$$= (\vec{w} \cdot \vec{R}) \cdot \vec{R} - (\vec{R} \cdot \vec{R}) \cdot \vec{w}$$

Thus it follows:

$$\dot{\vec{a}} \times \vec{R} = -R^2 \cdot \vec{w}$$

or

$$\vec{R} \times \dot{\vec{a}} = R^2 \cdot \vec{w}$$

We transform:

$$\vec{w} = \frac{\vec{R} \times \dot{\vec{a}}}{R^2} \tag{6}$$

Thus the problem is solved completely.

References

[1] Martin Barner, Friedrich Flohr “Analysis II” de Gruyter Verlag Berlin 1983

[2] A Budo “Theoretische Mechanik” 10.edition VEB Deutscher Verlag der Wissenschaften Berlin 1980

11. Stability at systems of usual differential equations in virus dynamics

Summary In this paper we discuss different models of differential equation systems, that describe virus dynamics in different situations. The modeling of this situation is realized in [NM]. We inquire the stability of differential equations. We use theorems of the stability theory.

1. Introduction

In virus dynamics we examine which conditions are necessary for virus increasing or decreasing. This is important for the development of disease. Increasing or decreasing can be described well with differential equations. It is important to inquire for which conditions we must calculate with decreasing or a constant level or an unchecked increasing. For illustrating we use different models.
In chapter 2 we discuss a simple model of describing virus dynamics. Chapter 3 contains a model of the HIV–virus. In chapter 4 we treat the dynamics of the Hepatitis B–virus and in chapter 5 the dynamics of immune response. At last we have combined some basic theoretical theorems for stability analysis in chapter 6.

2. A basic model

For description of a first model of virus dynamics we view the dynamics of three different types of objects in a whole body, in a set of blood or tissue: we distinguish uninfected cells from infected cells and virus particles. We want to describe the temporal changing of this objects thus we view the *number* of these cells in a time interval $[0, T] \subset I\!R$. We define:

$$\begin{array}{ll} x = x(t) & \text{the number of uninfected cells,} \\ y = y(t) & \text{the number of infected cells, and} \\ v = v(t) & \text{the number of virus particles} \end{array}$$

A simple model that describes virus dynamics, is a usual differential equation system of the form, see [NM], equation (3.1):

$$\begin{aligned} \dot{x}(t) &= \lambda - dx(t) - \beta x(t)v(t) \\ \dot{y}(t) &= \beta x(t)v(t) - ay(t) \\ \dot{v}(t) &= ky(t) - uv(t) \end{aligned} \tag{2. 1}$$

We have unknown functions $x, y, v : [0, T] \to I\!R$ and the systems (2. 1) contains several constant positive values.

$\lambda =$ increasing rate of uninfected cells,
$d =$ dying rate of uninfected cells,
$a =$ dying rate of infected cells,

u = dying rate of virus particles,
k = increasing rate of virus particles cause of infected cells, and
β = increasing rate of the virus particles cause of reactions between virus particles and uninfected cells.

Typical values for these constants are for example $\lambda = 10^5$,$d = 0.1$, $a = 0.5$,$\beta = 2 \cdot 10^{-7}$, $k = 100$ and $u = 5$. Fixed points play a major role for the stability analysis. A fixed point $x^\star$ is a constant for all $t \in I\!R$. If $\dot{x} = f(x)$ is a differential equation, it is valid $\dot{x} = f(x^\star) = 0$.

First we look at the following fixed points:

$$x^\star = \frac{\lambda}{d} \qquad y^\star = 0, \qquad v^\star = 0. \tag{2. 2}$$

This simplifies the differential equation to:

$$\dot{x} = \lambda - dx =: \tilde{f}(x).$$

We need the differentiation for stability analysis:

$$D\tilde{f}(x) = -d.$$

We construct the characteristic polynomial with the unknown s:

$$\det\left(D\tilde{f}(x) - s\right) = -d - s = 0$$

With the zero respectively the characteristic number:

$$s = -d < 0$$

Because of theorem 6.1 in the appendix, the fixed point (2. 2) is stable.

For stability analysis of (2. 1) we need the fixed points

$$x^\star = \frac{au}{\beta k} \quad , \quad y^\star = \left(\frac{\beta\lambda k}{adu} - 1\right) \cdot \frac{du}{\beta k} \quad , \quad v^\star = \left(\frac{\beta\lambda k}{adu} - 1\right) \cdot \frac{d}{\beta}. \tag{2. 3}$$

With insertion in the equation (2. 1) we show, that the property of fixed points consists. As the system (2. 1) we define:

$$\tilde{f}(x, y, v) := (\lambda - dx - \beta xv, \beta xv - ay, ky - uv)$$

The linearization leads to the Jacobi matrix

$$D\tilde{f}(x, y, v) = \begin{pmatrix} (-d - \beta v) & 0 & -\beta x \\ \beta v & -a & \beta x \\ 0 & k & -u \end{pmatrix}.$$

Insertion of the fixed points:

$$D\tilde{f}(x^\star, y^\star, v^\star) = \begin{pmatrix} \frac{-\beta\lambda k}{au} & 0 & \frac{-au}{k} \\ d \cdot \left(\frac{\beta\lambda k}{adu} - 1\right) & -a & \frac{au}{k} \\ 0 & k & -u \end{pmatrix}$$

We calculate the characteristic polynomial with the Sarrus rule:
E = unit matrix

$$\det(D\tilde{f}(x^\star, y^\star, v^\star) - sE) = -s^3 + s^2 \cdot \left(-\frac{\beta\lambda k}{au} - a - u\right)$$

$$+s \cdot \left(-\frac{\beta\lambda k}{u} - \frac{\beta\lambda k}{a}\right) - \beta\lambda k + aud$$

Multiplication with -1 leads to the normal form

$$s^3 + a_1 s^2 + a_2 s + a_3 = 0.$$

with:

$$a_1 := \frac{\beta\lambda k}{au} + a + u \qquad a_2 := \frac{\beta\lambda k}{u} + \frac{\beta\lambda k}{a} \qquad a_3 := \beta\lambda k - aud$$

Because of the theorem 6.2 in the appendix this polynomial has zeros with negative real parts if and only if Δ_1, Δ_2 and Δ_3 are larger than zero. It is

$$\Delta_1 := a_1 \quad , \quad \Delta_2 := \det \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix} \quad , \quad \Delta_3 := a_3 \cdot \Delta_2.$$

Then we obtain

$$\Delta_1 = \frac{\beta\lambda k}{au} + a + u \quad ,$$

$$\Delta_2 = \left(\frac{\beta\lambda k}{au} + a + u\right) \cdot \left(\frac{\beta\lambda k}{u} + \frac{\beta\lambda k}{a}\right) - \beta\lambda k + aud \quad ,$$

$$\Delta_3 = (\beta\lambda k - aud) \cdot \Delta_2.$$

Thus the fixed point is stable if and only if the following is valid:

$$aud < \beta\lambda k \qquad \text{and} \qquad \Delta_2 > 0$$

$adu < \beta\lambda k$ is equivalent to $R_0 = \frac{\beta\lambda k}{adu} > 1$ see [NM] chapter 3 p.19.

3. Anti-viral drug models

We have treated the basic model and now we view the HIV–virus. We begin with "reverse transcriptase inhibitors". First we assume that the drug is 100 % effective. Then we can set $\beta = 0$(see equation (2.1)):

$$\begin{aligned} \dot{y} &= -ay \\ \dot{v} &= ky - uv \end{aligned} \qquad (3.\ 4)$$

This system contains the following values:

y = number of infected cells,
v = number of free virus particles,
a = dying rate of infected cells,
k = increasing rate of free virus particles because of infected cells, and
u = dying rate of free virus particles

With the function $\tilde{f}(y, v) := (-ay, ky - uv)$ we obtain the Jacobi matrix

$$D\tilde{f}(y, v) = \begin{pmatrix} -a & 0 \\ k & -u \end{pmatrix}.$$

Characteristic polynomial of linearized system: (E = unit matrix)

$$\det\left(D\tilde{f}(y, v) - sE\right) = (-a - s) \cdot (-u - s) = 0$$

With the eigenvalues $s_1 = -u < 0$ and $s_2 = -a < 0$ in case $u \neq a$ we have a stable node and in case $u = a$ a stable star. This system only contains stable fixed points.

3.1. HIV: Protease inhibitors

Now we turn to the following differential equation system, with some further values:

$$\begin{aligned} \dot{y} &= \beta x v - ay \\ \dot{v} &= -uv \\ \dot{w} &= ky - uw \end{aligned} \tag{3. 5}$$

It is:

x = number of uninfected cells,
y = number of infected cells,
v = number of virus particles,
w = number of uninfected virus particles,
β = increasing rate of infected cells because of reaction between uninfected cells and virus particles,
a = dying rate of infected cells,
u = dying rate of virus particles and uninfected virus particles, and
k = increasing rate of uninfected virus particles.

We construct:

$$\tilde{f}(y, v, w) := (\beta x v - ay, -uv, ky - uw)$$

Jacobi matrix:

$$D\tilde{f}(y, v, w) = \begin{pmatrix} -a & \beta x & 0 \\ 0 & -u & 0 \\ k & 0 & -u \end{pmatrix}$$

With the Sarrus rule we get the characteristic polynomial

$$\det(D\tilde{f}(y,v,w) - sE) = (-a-s)\cdot(-u-s)^2 = 0.$$

With the zeros:

$$s_1 = -a < 0 \qquad s_2 = s_3 = -u < 0$$

Thus there are only stable fixed points at this system.

Now we look at a more complicated system. This system contains uncomplete virus particles and latently infected cells, too:

$$\begin{aligned} \dot{x} &= \lambda - dx - \beta xv \\ \dot{y}_1 &= q_1\beta xv - a_1 y_1 + \alpha y_2 \\ \dot{y}_2 &= q_2\beta xv - a_2 y_2 - \alpha y_2 \\ \dot{y}_3 &= q_3\beta xv - a_3 y_3 \\ \dot{v} &= ky_1 - uv \end{aligned} \tag{3. 6}$$

with the new values:

y_1 = number of virus-producing cells,
y_2 = number of latently infected cells,
y_3 = number of cells with uncomplete virus,
q_1 = probability of virus-producing cells,
q_2 = probability of latently infected cells,
q_3 = probability of cells with uncomplete virus,
a_1 = dying rate of virus-producing cells,
a_2 = dying rate of latently infected cells,
a_3 = dying rate of cells with uncomplete virus, and
α = rate of latently infected cells become reactivated to turn into virus-producing cells.

To the values λ, d and β see equation (2. 1)

Typical values for the constants are for example $\lambda = 10^7$, $d = 0.1$, $a_1 = 0.5$, $a_2 = 0.01$,$a_3 = 0.008$,$\alpha = 0.4$, $\beta = 5\cdot 10^{-10}$, $q_1 = 0.55$, $q_2 = 0.05$, $q_3 = 0.4$, $k = 500$ and $u = 5$.

We define:

$$\tilde{f}(x,y_1,y_2,y_3,v) := \begin{pmatrix} \lambda - dx - \beta xv \\ q_1\beta xv - a_1 y_1 + \alpha y_2 \\ q_2\beta xv - a_2 y_2 - \alpha y_2 \\ q_3\beta xv - a_3 y_3 \\ ky_1 - uv \end{pmatrix}$$

Jacobi matrix:

$$D\tilde{f} = \begin{pmatrix} -d-\beta v & 0 & 0 & 0 & -\beta x \\ q_1\beta v & -a_1 & \alpha & 0 & q_1\beta x \\ q_2\beta v & 0 & (-a_2-\alpha) & 0 & q_2\beta x \\ q_3\beta v & 0 & 0 & -a_3 & q_3\beta x \\ 0 & k & 0 & 0 & -u \end{pmatrix}$$

We get the fixed points

$$x^\star = \frac{x_0}{R_0} \qquad \text{and} \qquad v^\star = (R_0 - 1)\cdot\frac{d}{\beta}$$

with

$$x_0 = \frac{\lambda}{d} \qquad \text{and} \qquad R_0 = \frac{\beta\lambda k}{a_1 d u}\cdot\left(q_1 + \frac{q_2\alpha}{\alpha+\alpha_2}\right)$$

through insertion in the system (3. 6) .
x_0 is the fixed point of uninfected cells before infection and R_0 the basic reproduction ratio.

The characteristic polynomial is calculated with:

$$\det(D\tilde{f}(x^\star, v^\star) - sE)$$

Multiplication of characteristic polynomial with -1 leads to the normal form

$$s^5 + a_1 s^4 + a_2 s^3 + a_3 s^2 + a_4 s + a_5 = 0.$$

We define

$$\Delta_1 := a_1 \quad , \quad \Delta_2 := \det\begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix} \quad , \quad \Delta_3 := \det\begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix} \quad ,$$

$$\Delta_4 := \det\begin{pmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 \\ a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & a_5 & a_4 \end{pmatrix} \quad , \quad \Delta_5 := a_5\cdot\Delta_4.$$

It follows with theorem 6.2 in the appendix:

The normed polynomial has solutions with negative real parts, if and only if Δ_1, Δ_2, Δ_3, Δ_4 and Δ_5 are all positive.
Thus the system is stable, if and only if Δ_1, Δ_2, Δ_3, Δ_4 and Δ_5 all greater than zero.

Let us consider an anti-viral-therapy, which prevents infection of new cells. That means $\beta = 0$. With system (3.6) we obtain:

$$\begin{aligned} \dot{y}_1 &= -a_1 y_1 + \alpha y_2 \\ \dot{y}_2 &= -a_2 y_2 - \alpha y_2 \\ \dot{y}_3 &= -a_3 y_3 \\ \dot{v} &= k y_1 - u v \end{aligned} \tag{3. 7}$$

We construct:

$$\tilde{f}(y_1, y_2, y_3, v) := \begin{pmatrix} -a_1 y_1 + \alpha y_2 \\ -a_2 y_2 - \alpha y_2 \\ -a_3 y_3 \\ k y_1 - uv \end{pmatrix}$$

Jacobi matrix:

$$D\tilde{f}(y_1, y_2, y_3, v) = \begin{pmatrix} -a_1 & \alpha & 0 & 0 \\ 0 & (-a_2 - \alpha) & 0 & 0 \\ 0 & 0 & -a_3 & 0 \\ k & 0 & 0 & -u \end{pmatrix}$$

We determine the characteristic polynomial. We develop with the 4. column:

$$\det(D\tilde{f}(y_1^\star, y_2^\star, y_3^\star, v^\star) - sE)$$

$$= (-u - s) \cdot (-a_1 - s) \cdot (-a_2 - \alpha - s) \cdot (-a_3 - s) = 0$$

We get

$$s_1 = -u < 0 \quad , \quad s_2 = -a_1 < 0 \quad , \quad s_3 = -a_2 - \alpha < 0 \quad , \quad s_4 = -a_3 < 0.$$

Thus this system has only stable fixed points.

4. Dynamics of hepatitis B virus

Now we turn to a model of dynamics of hepatitis B virus, through following differential equation system (vgl. Novak [NM] p.45):

$$\begin{aligned} \dot{y} &= \beta x v - a y \\ \dot{v} &= -uv. \end{aligned} \tag{4. 8}$$

To the values see (2. 1). We define:

$$\tilde{f}(y, v) := (\beta x v - a y, -uv)$$

functional matrix:

$$D\tilde{f}(y, v) = \begin{pmatrix} -a & \beta x \\ 0 & -u \end{pmatrix}$$

Characteristic polynomial:

$$\det(D\tilde{f}(y, v) - sE) = (-a - s) \cdot (-u - s)$$

The zeros are:

$$s_1 = -a < 0 \quad , \quad s_2 = -u < 0$$

In case $a \neq u$ we have a stable node. In the case $a = u$ there is a stable star.

If we set $\beta = 0$, then we obtain the special system

$$\begin{aligned} \dot{y} &= -ay \\ \dot{v} &= -uv. \end{aligned} \tag{4. 9}$$

This system has the same characteristic polynomial and thus the same stability properties.

5. Dynamics of immune responses

Now we take the immune response into consideration. We look at different plausible models. There are differences because of the CTL-reaction. We view a system with self-regulating CTL-reaction:

$$\begin{aligned} \dot{x} &= \lambda - dx - \beta xv \\ \dot{y} &= \beta xv - ay - pyz \\ \dot{v} &= ky - uv \\ \dot{z} &= c - bz \end{aligned} \tag{5. 10}$$

Let be:

x = number of uninfected cells,
y = number of infected cells,
v = number of free virus particles,
z = CTL-reaction, that eliminates infected cells.
λ = increasing rate of uninfected cells,
d = dying rate of uninfected cells,
β = increasing rate of virus particles because of the reactions between uninfected cells and virus particles
a = dying rate of infected cells,
p = dying rate of infected cells because of the reactions between infected cells and CTL-reaction,
k = increasing rate of free virus particles,
u = dying rate of free virus particles,
c = increasing rate of CTL-reaction, and
b = dying rate of CTL-reaction.

Typical values for the constants are for example $\lambda = 1$, $d = 0.01$, $a = 0.5$, $\beta = 0.005$, $k = 50$, $u = 5$, $p = 1$ and $b = 0.05$.

We construct:

$$\tilde{f}(x, y, v, z) := \begin{pmatrix} \lambda - dx - \beta xv \\ \beta xv - ay - pyz \\ ky - uv \\ c - bz \end{pmatrix}$$

Jacobi matrix:

$$D\tilde{f}(x,y,v,z) = \begin{pmatrix} (-d-\beta v) & 0 & -\beta x & 0 \\ \beta v & (-a-pz) & \beta x & -py \\ 0 & k & -u & 0 \\ 0 & 0 & 0 & -b \end{pmatrix}$$

Now we look at a system with nonlinear CTL-reaction:

$$\begin{aligned} \dot{x} &= \lambda - dx - \beta xv \\ \dot{y} &= \beta xv - ay - pyz \\ \dot{v} &= ky - uv \\ \dot{z} &= cyz - bz \end{aligned} \tag{5. 11}$$

We obtain:

$$\tilde{f}(x,y,v,z) := \begin{pmatrix} \lambda - dx - \beta xv \\ \beta xv - ay - pyz \\ ky - uv \\ cyz - bz \end{pmatrix}$$

Jacobi matrix:

$$D\tilde{f}(x,y,v,z) = \begin{pmatrix} (-d-\beta v) & 0 & -\beta x & 0 \\ \beta v & (-a-pz) & \beta x & -py \\ 0 & k & -u & 0 \\ 0 & cz & 0 & (cy-b) \end{pmatrix}$$

Now we consider a system with linear immune response:

$$\begin{aligned} \dot{x} &= \lambda - dx - \beta xv \\ \dot{y} &= \beta xv - ay - pyz \\ \dot{v} &= ky - uv \\ \dot{z} &= cy - bz \end{aligned} \tag{5. 12}$$

We define

$$\tilde{f}(x,y,v,z) := \begin{pmatrix} \lambda - dx - \beta xv \\ \beta xv - ay - pyz \\ ky - uv \\ cy - bz \end{pmatrix}.$$

Jacobi matrix:

$$D\tilde{f}(x,y,v,z) = \begin{pmatrix} (-d-\beta v) & 0 & -\beta x & 0 \\ \beta v & (-a-pz) & \beta x & -py \\ 0 & k & -u & 0 \\ 0 & c & 0 & -b \end{pmatrix}$$

For the characteristic polynomial of these 3 systems (as function from the fixed points)

$$\det(D\tilde{f}(x^\star, y^\star, v^\star, z^\star) - sE)$$

we get a polynomial

$$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0.$$

The system is stable if and only if Δ_1, Δ_2, Δ_3 and Δ_4 are all positive. It is

$$\Delta_1 := a_1 \quad , \quad \Delta_2 := \det \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix} \quad ,$$

$$\Delta_3 := \det \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{pmatrix} \quad , \quad \Delta_4 := a_4 \cdot \Delta_3.$$

Thus the criterion of stability is indicated.

Now we take a simplified approach. We neglect the numbers of uninfected and infected cells. We assume that the virus population is only controlled by immune responses. We view the differential equation system

$$\begin{aligned} \dot{v} &= v \cdot (r - pz) \\ \dot{z} &= c - bz. \end{aligned} \tag{5. 13}$$

The new constants r and p are increasing respectively dying rates. We construct:

$$\tilde{f}(v, z) := (v \cdot (r - pz), c - bz)$$

Jacobi matrix:

$$D\tilde{f}(v, z) = \begin{pmatrix} r - pz & -pv \\ 0 & -b \end{pmatrix}$$

Characteristic polynomial:

$$\det(D\tilde{f}(v, z) - sE) = (r - pz - s) \cdot (-b - s)$$

The zeros are:

$$s_1 = r - \frac{pc}{b} \quad , \quad s_2 = -b < 0$$

Thus we have a saddle point. There are stable and unstable areas. Because of the fixed point condition $z^\star = \frac{c}{b}$, that can be recognized directly from (5. 13), it follows:
If $r < \frac{pc}{b}$, then v decreases gradually to zero. The immune system controlls the virus.
If $r > \frac{pc}{b}$, then v increases more and more. The immune system cannot controll the virus.

Now we turn to the changed system

$$\begin{aligned} \dot{v} &= v \cdot (r - pz) \\ \dot{z} &= cv - bz. \end{aligned} \tag{5. 14}$$

We define:

$$\tilde{f}(v,z) := (v \cdot (r - pz), cv - bz)$$

Jacobi matrix:

$$D\tilde{f}(v,z) = \begin{pmatrix} r - pz & -pv \\ c & -b \end{pmatrix}$$

characteristic polynomial with fixed points $v^\star, z^\star$:

$$\det(D\tilde{f}(v^\star, z^\star) - sE) = s^2 + bs + cpv^\star = 0$$

We get the solutions

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4cpv^\star}}{2}.$$

With the insertion of

$$v^\star = \frac{rb}{cp} \qquad\qquad z^\star = \frac{r}{p}$$

in the differential equation (5. 14) we obtain the fixed point property.

Insertion of $v^\star$ in the characteristic polynomial leads to the solutions

$$s_1 < 0 \quad , \quad s_2 < 0.$$

Thus the system is stable with $v^\star = \frac{rb}{cp}$ and any given $z^\star$.

Now we change the system to

$$\begin{aligned} \dot{v} &= v \cdot (r - pz) \\ \dot{z} &= z \cdot (cv - b). \end{aligned} \tag{5. 15}$$

We construct:

$$\tilde{f}(v,z) := (v \cdot (r - pz), z \cdot (cv - b))$$

Jacobi matrix:

$$D\tilde{f}(v,z) = \begin{pmatrix} r - pz & -pv \\ cz & cv - b \end{pmatrix}$$

characteristic polynomial:

$$\det(D\tilde{f}(v^\star, z^\star) - sE) = s^2 + s \cdot (-r - cv^\star + b + pz^\star) + rcv^\star - rb + pcv^\star z^\star + pz^\star b$$

For the fixed points we get from the system (5. 15)

$$v^\star = \frac{b}{c} \qquad z^\star = \frac{r}{p}.$$

These inserted in the characteristic polynomial

$$s^2 + 2rb = 0.$$

The zeros are

$$s_{1,2} = \pm\sqrt{-2rb}.$$

The decision remains open, because it is no linear system.

6. Appendix

In this section we combine the main results that are necessary to the stability analysis.

Theorem 6.1 *Let be $x = x(t) : [0, T] \to I\!R^n$ and A a $n \times n$–matrix with real values. The system $\dot{x}(t) = Ax$ is stable,if and only if all eigenvalues of A have negative real parts.*

With Leipholz [L] chapter 1.3.2 p.36 the following theorem is valid:

Theorem 6.2 *Le be*

$$H = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \dots \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & \dots \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

the Hurwitz matrix to the polynomial $s^n + a_1 s^{n-1} + \dots a_{n-1} s + a_n$. Then the zeros of the polynomial have negative real parts if and only if the chief minor determinants of the matrix H are all positive. The chief minor determinants are the values

$$\Delta_1 = a_1 \quad , \quad \Delta_2 = \det \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix} \quad , \quad \Delta_3 = \det \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix} \quad ,$$

$$\dots, \Delta_{n-1} = \det \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \dots \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & \dots \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad , \quad \Delta_n = a_n \cdot \Delta_{n-1}.$$

References

[NM] Martin A. Novak and Robert M. May, *Virus dynamics* Oxford University Press 2000.

[L] Horst Leipholz, *Stability Theory. An Introduction to the Stability of Dynamic Systems and Rigid Bodies*, 2nd ed.,Teubner 1987.

Chapter C.

Numerics

12. Convergence measures

1. Convergence measures in the set of real numbers

1.1. Real sequences

First we treat here real sequences. We assume that the sequence a_n converges to the sequence b_n.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Then we define the convergence measure:

$$k(n) := \frac{|a_n - b_n|}{|a_{n+1} - b_{n+1}|}$$

$|\cdot|$ is the absolute value of a number. A special case is for example $b_n = b$. That means the sequence a_n converges to a constant.

k is dependent upon n. k is a measure of the **local** speed of convergence. The larger k is, the faster the convergence is.

For example we take the sequences $a_n = \frac{2}{n}$ and $b_n = \frac{1}{n}$. Then the convergence measure is:

$$k(n) = \frac{\frac{2}{n} - \frac{1}{n}}{\frac{2}{n+1} - \frac{1}{n+1}} = \frac{n+1}{n}$$

1.2. Differentiable functions

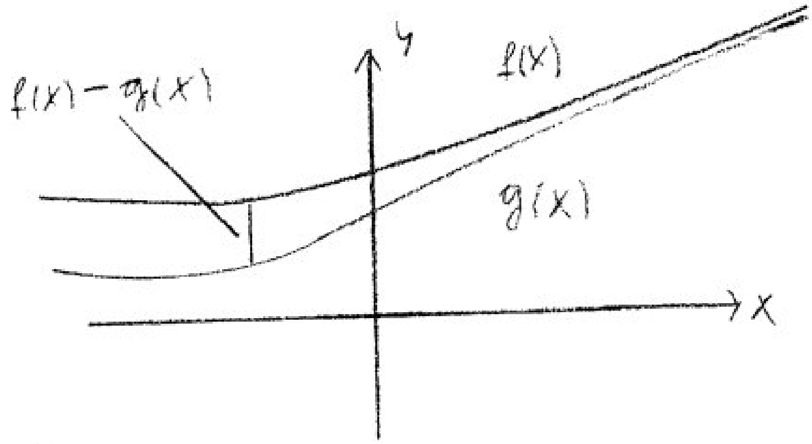
At this function we can define another convergence measure. We assume that f and g are real differentiable functions. Besides it shall be valid:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x)$$

or

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} g(x)$$

That means asymptotic approach of both functions. We look at the following figure:



Then it is:

$$k := \left|\frac{d}{dx}(f(x) - g(x))\right| = |f'(x) - g'(x)|$$

In the special case $g(x) = b =$ const. is b an asymptote parallel to the x-axis. For example we take $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$. We obtain the convergence measure:

$$k = \left|\frac{d}{dx}\left(\frac{1}{x} - \frac{1}{x^2}\right)\right| = \left|-\frac{1}{x^2} + \frac{2}{x^3}\right|$$

2. Convergence measures in metric spaces

Now we have the occasion to get to know these convergence notions in any given metric space.

Let M be any given set. A metric is a function

$$d: \; M \times M \longrightarrow R^+$$

with following properties: $\qquad x, y \in M$

$$d(x, y) = 0 \qquad \Leftrightarrow \qquad x = y$$

$$d(x, y) = d(y, x) \qquad \text{symmetry}$$

$$d(x, z) \leq d(x, y) + d(y, z) \qquad \text{triangle inequality}$$

The pair (M, d) is called metric space. d is the distance function.

2.1. Sequences

Now we view the both sequences $x_n, y_n \in M$. d is a metric. It's assumed:

$$\lim_{n\to\infty} d(x_n, y_n) = 0$$

This is equivalent to:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$$

The convergence measure can be defined as in the first case:

$$k(n) := \frac{d(x_n, y_n)}{d(x_{n+1}, y_{n+1})}$$

Here is valid, too that the convergence is faster the larger is k. k is here a measure of local convergence, too.

2.2. Differentiable functions

Now we look at differentiable functions in metric spaces.

$$d(x(t), y(t)) \qquad t \in I \subset R$$

shall be differentiable to t. Additional assumed is:

$$\lim_{t\to\infty} d(x(t), y(t)) = 0 \quad \Leftrightarrow \quad \lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t)$$

or:

$$\lim_{t\to-\infty} d(x(t), y(t)) = 0 \quad \Leftrightarrow \quad \lim_{t\to-\infty} x(t) = \lim_{t\to-\infty} y(t)$$

Now we define the convergence measure as in the second case:

$$k := \left| \frac{d}{dt} d(x(t), y(t)) \right|$$

With $d(x, y) = ||x - y||$ we have defined at the same time the convergence measures in normed space to every norm $|| \cdot ||$. The reason is that every norm is a metric, too.

Here the defined convergence measure for sequences is connected with the order of convergence that is mentioned in many books about numerical mathematics.

Chapter D.

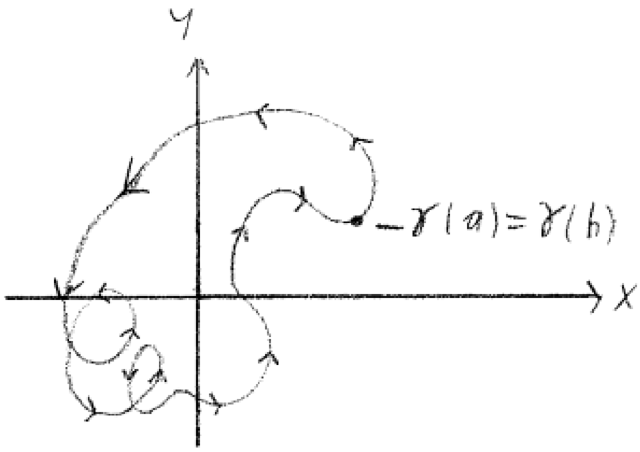
Complex analysis

13. The winding number

The problem is to determine the winding number of a closed path:

$$\gamma : [a,b] \longrightarrow U \subset R^2 \qquad a,b \in R$$

with $\gamma(a) = \gamma(b)$



γ shall be piecewise continuously differentiable. The set R^2 is isomorphic to the set of complex numbers C. We identify the curve γ with the complex path of integration $\beta = \gamma_1 + i \cdot \gamma_2$, with $i = \sqrt{-1}$. The further proceeding can be done with Lieb [1] chapter II §1 Def. 1.1 p.38, chapter IV §1 Def 1.2 p.101 and example 1 p.101.

$$\beta : [a,b] \longrightarrow C \qquad \beta(a) = \beta(b)$$

is a closed path of integration and shall be piecewise continuously differentiable.

Now we use the line integral in C:

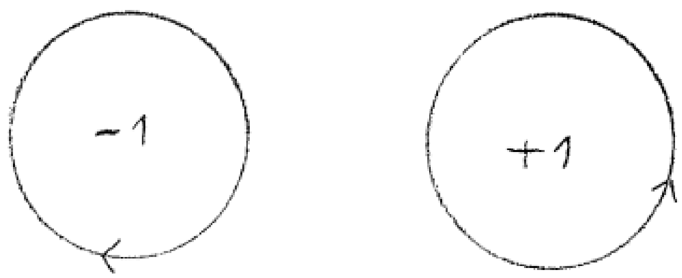
$$\int_\beta f(z)\,dz := \int_a^b f(\beta(t))\cdot\beta'(t)\,dt \qquad \beta' = \frac{d\beta}{dt}$$

How often the path of integration γ leads around the point $p = (p_1, p_2) \in R^2$? In C it's the point $z_0 = p_1 + i\cdot p_2 \notin \beta([a,b])$. Corresponding
$p \notin \gamma([a,b])$ is valid, too.

We get the searched winding number with Lieb [1] chapter IV §1 Def. 1.2 p.101:

$$n(\gamma, p) = n(\beta, z_0) = \frac{1}{2\pi i}\cdot\int_\beta \frac{dz}{z - z_0} \tag{1}$$

$n(\gamma, p)$ is a integer number see Lieb [1] chapter IV §1 theorem 1.2 p.103. If $n(\gamma, p)$ is positive, then we have the number of windings against the clockwise sense. If $n(\gamma, p)$ is negative, then the absolute value of this quantity is the number of the windings with the clockwise sense.



Now the path of integration and p respectively z_0 shall be known. We will change the formula (1) in a pure real form. With $\beta = \gamma_1 + i\cdot\gamma_2$,$\beta' = \gamma_1' + i\cdot\gamma_2'$ and $z_0 = p_1 + i\cdot p_2$ we obtain:

$$\begin{aligned}\frac{\beta'}{\beta - z_0} &= \frac{\gamma_1' + i\cdot\gamma_2'}{\gamma_1 + i\cdot\gamma_2 - p_1 - i\cdot p_2}\\ &= \frac{\gamma_1' + i\cdot\gamma_2'}{(\gamma_1 - p_1) + i\cdot(\gamma_2 - p_2)}\end{aligned}$$

Now we use the decomposition:

$$\frac{x_1 + i\cdot x_2}{y_1 + i\cdot y_2} = \frac{x_1y_1 + x_2y_2}{y_1^2 + y_2^2} + i\cdot\frac{x_2y_1 - x_1y_2}{y_1^2 + y_2^2}$$

Thus:

$$\frac{\beta'}{\beta - z_0} = \frac{\gamma_1'\cdot(\gamma_1 - p_1) + \gamma_2'\cdot(\gamma_2 - p_2)}{(\gamma_1 - p_1)^2 + (\gamma_2 - p_2)^2}$$

$$+i\cdot\frac{\gamma_2'\cdot(\gamma_1 - p_1) - \gamma_1'\cdot(\gamma_2 - p_2)}{(\gamma_1 - p_1)^2 + (\gamma_2 - p_2)^2} =: A + i\cdot B \tag{2}$$

Because of $\gamma_1(t), \gamma_2(t), p_1(t), p_2(t) \in R$, it follows $A(t), B(t) \in R$. It is a decomposition into real- and imaginary part. Because of equation (2) we have:

$$n(\gamma, p) = \frac{1}{2\pi i} \cdot \int_a^b (A + i \cdot B)\, dt = \frac{1}{2\pi i} \cdot \int_a^b A\, dt + \frac{1}{2\pi} \cdot \int_a^b B\, dt$$

From $i^2 = -1$ we follow $\frac{1}{i} = -i$ and:

$$n(\gamma, p) = \frac{1}{2\pi} \cdot \int_a^b B\, dt - i \cdot \frac{1}{2\pi} \cdot \int_a^b A\, dt$$

But $n(\gamma, p)$ is a integer number, thus:

$$\frac{1}{2\pi} \cdot \int_a^b A\, dt = 0$$

We get:

$$n(\gamma, p) = \frac{1}{2\pi} \cdot \int_a^b B\, dt$$

or inserted for B:

$$n(\gamma, p) = \frac{1}{2\pi} \cdot \int_a^b \frac{\gamma_2' \cdot (\gamma_1 - p_1) - \gamma_1' \cdot (\gamma_2 - p_2)}{(\gamma_1 - p_1)^2 + (\gamma_2 - p_2)^2}\, dt$$

Assumptions: $\gamma(a) = \gamma(b)$ (closed path) and γ must be piecewise continuously differentiable.

Thus we have a pure real representation of the winding number.

References

[1] Wolfgang Fischer, Ingo Lieb “Funktionentheorie” Vieweg Verlag 4.edition 1985 Brunswick

Chapter E.

Geometry

Contents

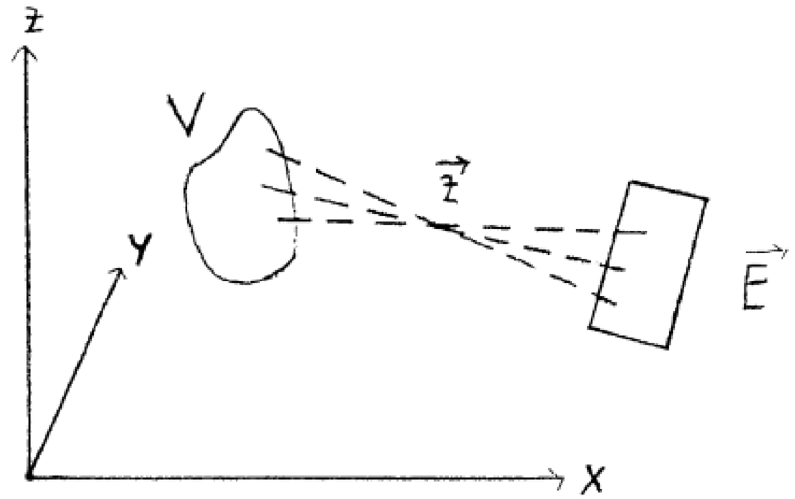
14. Projections on planes

1. Introduction

We have a set, a projection point and a plane. The aim of this work is to get a mathematical description of a projection on a plane. For the reader it may be interesting to get an image of the central projection and to see the mathematical form of the projection. This problem can be compared to the optical image on the retina in human beings and animals. I have not seen any derivation of this in any publication. With this essay , I hope to close any gaps in usable mathematical equations. If we have a representation of a figure in R^2, then the figure can be drawn in a coordinate system. The projections play a major role in mathematics, physics, biology and in science in general including architecture, engineering and descriptive geometry. Applications of projections in biology are, for example, in the book "Arctificial and biological vision systems" [1]. Applications in architecture, engineering and technology are, as example, in Salkowski [7], Graf [4],Rehbock [6] and Hohenberg [5].

2. The problem

Let us look at the following problem:



Let's view the central projection of set $V \subset R^3$ and from point $\vec{z}$ on plane $\vec{E}$. We see the projection on the plane $\vec{E}$. $\vec{E}$ figures as a "screen" behind $\vec{z}$. This situation can be compared with the figure of a lens of an eye, if the distance between V and $\vec{z}$ is very great compared with the distance between $\vec{z}$ and $\vec{E}$.

3. The projection

Parametrization of the plane:

$$\vec{E} := \vec{p} + \lambda \cdot \vec{v} + \mu \cdot \vec{w} \qquad \lambda, \mu \in R \quad \vec{p}, \vec{v}, \vec{w} \in R^3$$

Assumption: $\vec{v}$ and $\vec{w}$ are orthogonal.
A dependence on time $t \in R$ is possible.

$$\vec{p} = \vec{p}(t) \quad \vec{v} = \vec{v}(t) \quad \vec{w} = \vec{w}(t) \quad \vec{z} = \vec{z}(t)$$

parametrization of V:

$$V = \vec{a}(b_1, b_2, b_3, c_1, \ldots, c_n, t) = \vec{a}(\vec{b}, \vec{c}, t) \qquad \vec{b} = (b_1, b_2, b_3) \in U \subset R^3$$

$$\vec{c} = (c_1, \ldots, c_n) \in Q \subset R^n$$

b_1, b_2, b_3 are parameters of the area.
$c_1, \ldots, c_n$ are arbitrary parameters.

Projecting line (see figure):

$$\vec{s} := \vec{a} + \varepsilon \cdot (\vec{z} - \vec{a}) \qquad \varepsilon \in R$$

Figure on the plane:

intersection of the projecting line and the plane (see figure):

$$\vec{a} + \varepsilon \cdot (\vec{z} - \vec{a}) = \vec{p} + \lambda \cdot \vec{v} + \mu \cdot \vec{w}$$

The following results:

$$\lambda(\vec{b}, \vec{c}, t), \mu(\vec{b}, \vec{c}, t), \varepsilon(\vec{b}, \vec{c}, t)$$

, because $\vec{a}$ is a function of $\vec{b}, \vec{c}$ and t.

The figure has the following coordinates:

$$\vec{R}(\vec{b}, \vec{c}, t) := \vec{a} + \varepsilon(\vec{b}, \vec{c}, t) \cdot (\vec{z} - \vec{a}) \tag{1}$$

$$= \vec{p} + \lambda(\vec{b}, \vec{c}, t) \cdot \vec{v} + \mu(\vec{b}, \vec{c}, t) \cdot \vec{w}$$

Shape of the figure on the plane:

$$\vec{f}(\vec{b}, \vec{c}, t) := \vec{R}(\vec{b}, \vec{c}, t) - \vec{p}$$

The point $\vec{p}$ is the origin of the coordinate system on the plane.

This is a figure in R^3. The aim is to obtain a figure in R^2.

With the rotation of the coordinate system, see e.g. Bronstein [2] chapter 2.6.5.2.3 p.216,217 we can obtain a figure in R^2.

A possible translation of the geometrical midpoint can be useful.

Eventually, the rotation in R^2 e.g. see Bronstein [2] chapter 2.6.5.1.3 p.212,213 can be necessary.

These 3 operations could be (dependent on the figure) affected in another order.

The aim of these operations is to get a simple representation of the figure.

4. To the assumption, that $\vec{v}$ and $\vec{w}$ are orthogonal:

The form of the pictures can be easily determined, if $\vec{v}$ and $\vec{w}$ are orthogonal. If $\vec{v}$ and $\vec{w}$ are not orthogonal, it would be favorable to make the directional vectors orthogonal. But the plane cannot change **itself**.

Let's use a normal vektor:

$$\vec{n} := \vec{v} \times \vec{w}$$

and then:

$$\vec{q} := \vec{n} \times \vec{v}$$

Then $\vec{q}$ is orthogonal to $\vec{n}$ and $\vec{v}$. $\vec{q}$ is on the plane, because $\vec{q}$ is orthogonal to $\vec{n}$.

$\vec{q}$ can be calculated with $\vec{q} \cdot \vec{n} = 0$ and $\vec{q} \cdot \vec{v} = 0$, as well.(Scalar product)

Then $\vec{q}$ is determined except for the absolute value. The absolute value can be chosen.

The new orthogonal directional vectors are $\vec{v}$ and $\vec{q}$.

If $\vec{v}$ and $\vec{w}$ are not orthogonal, the method must be done first.

5. The size of the picture:

Definition of the distance vector:

$$\vec{r}(\lambda, \mu) := \vec{z} - \vec{E}(\lambda, \mu) = \vec{z} - \vec{p} - \lambda \cdot \vec{v} - \mu \cdot \vec{w}$$

$$r(\lambda, \mu) = |\vec{r}(\lambda, \mu)|$$

The following minimal distance becomes interesting:

$$d := \min\{r(\lambda, \mu) \,|\, (\lambda, \mu) \in R^2\} = r(\lambda_0, \mu_0)$$

The determination of λ_0, μ_0 with $\text{grad}[r(\lambda, \mu)] = 0$ is more difficult. Let's use a simple method:

Let us first calculate the normal form of the plane with $\vec{n} := \vec{v} \times \vec{w}$, e.g. see Fischer [3] chapter 0.5.5 p.25. We change the parametric form of the plane into the normal form.

Then we can determine the minimal distance d, e.g. see Fischer [3] chapter 0.4.6 p.21.

If plane E has a normal form

$$n_1 x + n_2 y + n_3 z = e \qquad e \in R \quad \vec{n} = (n_1, n_2, n_3)$$

, thus the distance d with $\vec{z} = (z_1, z_2, z_3)$:

$$d = \frac{|n_1 z_1 + n_2 z_2 + n_3 z_3 - e|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

If $\vec{p}, \vec{v}, \vec{w}, \vec{z}$ are functions of t, then r and d are functions of t, as well.

It is valid:

picture size $\sim d(t)$ (intercept theorem), if $\vec{p}, \vec{v}, \vec{w}, V$ are constant and $\vec{z}$ is a function of t.

References

[1] "Arctificial and biological vision systems" Springer Verlag Berlin 1992

[2] Bronstein, Semendjajew "Taschenbuch der Mathematik" 22.edition 1985 Leipzig

[3] Gerd Fischer "Lineare Algebra" 8.edition Vieweg Verlag Braunschweig 1984

[4] Ulrich Graf "Darstellende Geometrie" 8.edition Heidelberg 1964

[5] F. Hohenberg "Konstruktive Geometrie für Techniker" Wien 1956

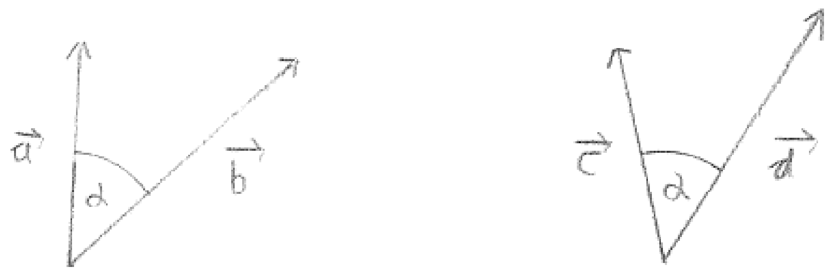
[6] F.Rehbock "Darstellende Geometrie" 3.edition Heidelberg 1969

D. Geometry

[7] E. Salkowski "Grundzüge der darstellende Geometrie" 9.edition Leipzig 1963

15. Rotations

Here we view an interesting rotation problem of vectors:



$\vec{a}$ and $\vec{b}$ are rotated in the plane into the vectors $\vec{c}$ and $\vec{d}$. $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in R^3$

At rotation, the absolute value of the vectors and the angle between the vectors remains unchanged:

$$|\vec{a}| = |\vec{c}| \qquad |\vec{b}| = |\vec{d}|$$

$$\alpha := \angle(\vec{a}, \vec{b}) = \angle(\vec{c}, \vec{d})$$

It follows for the absolute value of the vector product:

$$|\vec{a} \times \vec{b}| = |\vec{c} \times \vec{d}| \tag{1}$$

because of:

$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin\alpha \qquad |\vec{c} \times \vec{d}| = |\vec{c}| \cdot |\vec{d}| \cdot \sin\alpha$$

From the figure it's clear that the vectors $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$ show in the same direction. With the equation (1) we can conclude:

$$\vec{a} \times \vec{b} = \vec{c} \times \vec{d} \tag{2}$$

Now we deal with the problem to determine the fourth vector if three vectors are known. We assume that $\vec{a}, \vec{b}, \vec{c}$ are known and $\vec{d}$ must be calculated. The simple solving of equation (2) to $\vec{d}$ leads to no logical solution. To calculate $\vec{d}$ we must use another method. With $|\vec{a}| = |\vec{c}|$ and $|\vec{b}| = |\vec{d}|$ and $\alpha = \angle(\vec{a}, \vec{b}) = \angle(\vec{c}, \vec{d})$ we get:

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\alpha = |\vec{c}| \cdot |\vec{d}| \cdot \cos\alpha = \vec{c} \cdot \vec{d}$$

Thus we have the following result about scalar products:

$$\vec{a} \cdot \vec{b} = \vec{c} \cdot \vec{d} \tag{3}$$

We calculate with equation (2):

$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{c} \times \vec{d}) \times \vec{c}$$

Now we use the expansion theorem. (For example see Bartsch [1] chapter 7.3.2 p.275:)

$$= (\vec{c} \cdot \vec{c}) \cdot \vec{d} - (\vec{d} \cdot \vec{c}) \cdot \vec{c}$$

With equation (3):

$$= \vec{a}^2 \cdot \vec{d} - (\vec{a} \cdot \vec{b}) \cdot \vec{c}$$

We get:

$$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a}^2 \cdot \vec{d} - (\vec{a} \cdot \vec{b}) \cdot \vec{c}$$

Now we can solve to $\vec{d}$:

$$\vec{d} = \frac{(\vec{a} \times \vec{b}) \times \vec{c} + (\vec{a} \cdot \vec{b}) \cdot \vec{c}}{\vec{a}^2}$$

With the expansion theorem:

$$= \frac{(\vec{a} \cdot \vec{c}) \cdot \vec{b} - (\vec{b} \cdot \vec{c}) \cdot \vec{a} + (\vec{a} \cdot \vec{b}) \cdot \vec{c}}{\vec{a}^2}$$

$\vec{c}, \vec{a}, \vec{b}$ must lie **in one plane**. That means that the vectors must be linearly dependent. From

$$k_1 \cdot \vec{a} + k_2 \cdot \vec{b} + k_3 \cdot \vec{c} = 0$$

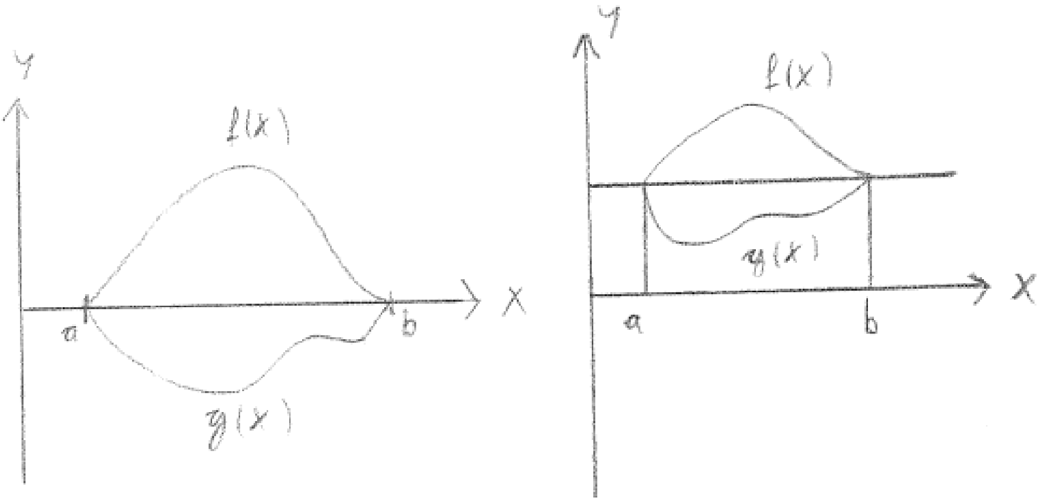
with $k_1, k_2, k_3 \in R$ doesn't follow $(k_1, k_2, k_3) = (0, 0, 0)$. $\vec{a}, \vec{b}, \vec{c}$ must be carefully chosen with each other. If this isn't done, it can be that $|\vec{b}| \neq |\vec{d}|$. But the absolute values of $\vec{b}$ and $\vec{d}$ must be equal.

References

[1] Hans-Jochen Bartsch "Taschenbuch mathematischer Formeln" Verlag Harri Deutsch Frankfort on the Main 1986 7. till 9.edition

16. The barycenter

We look at the following areas:



Our aim is to determine the barycenter of these areas. We view a general area $A \subset R^2$. With $u = (x_1, x_2) = (x, y) \in A \subset R^2$, the mass $m(A)$ and the density $\varphi(u)$ for the position $s(A) = (s_1, s_2) \in A$ of the barycenter is valid:

$$s_i(A) = \frac{1}{m(A)} \int\limits_A \varphi(u) \cdot x_i \, d^2u$$

It is $i \in 1, 2$.

Now we turn to the special areas above, then we can conclude with the figures:

$$f(a) = g(a) \qquad f(b) = g(b)$$

We assume that the density $\varphi = c$ is constant. Then we have:

$$m(A) = F(A) \cdot c \tag{1}$$

$F(A)$ is the area or the 2-dimensional volume of A. Now we calculate the x-coordinate of the barycenter:

$$x_s = \frac{1}{m(A)} \cdot \int\limits_a^b \int\limits_{g(x)}^{f(x)} c \cdot x \, dydx$$

Now we insert equation (1) for $m(A)$:

$$x_s = \frac{1}{F(A)} \cdot \int\limits_a^b x \int\limits_{g(x)}^{f(x)} dydx = \frac{1}{F(A)} \cdot \int\limits_a^b x \cdot [y]_{g(x)}^{f(x)} \, dx$$

We calculate:

$$x_s = \frac{1}{F(A)} \cdot \int_a^b x \cdot (f(x) - g(x))\,dx$$

In the last equation we insert:

$$F(A) = \int_a^b (f(x) - g(x))\,dx \tag{2}$$

We obtain:

$$x_s = \frac{\int_a^b x \cdot (f(x) - g(x))\,dx}{\int_a^b (f(x) - g(x))\,dx} \tag{3}$$

Thus the x-coordinate is calculated. Now we deal with the y-coordinate of the barycenter. We use again the general formula of the barycenter:

$$y_s = \frac{1}{m(A)} \cdot \int_a^b \int_{g(x)}^{f(x)} c \cdot y\,dydx$$

With equation (1) again:

$$y_s = \frac{1}{F(A)} \cdot \int_a^b \int_{g(x)}^{f(x)} y\,dydx = \frac{1}{F(A)} \cdot \int_a^b \left[\frac{y^2}{2}\right]_{g(x)}^{f(x)} dx$$

This leads to:

$$y_s = \frac{1}{2F(A)} \cdot \int_a^b (f(x)^2 - g(x)^2)\,dx$$

Now we use again equation (2) and we get at last:

$$y_s = \frac{\int_a^b (f(x)^2 - g(x)^2)\,dx}{2 \cdot \int_a^b (f(x) - g(x))\,dx} \tag{4}$$

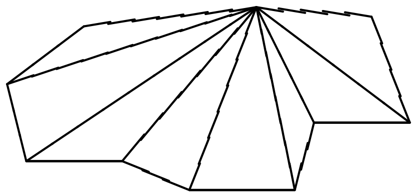
With the equations (3) and (4) the coordinates of the barycenter of the area A can be determined. The formulae (3) and (4) are in Bartsch [1] chapter 10.11 p.437 third section without proof.

References

[1] "Taschenbuch der Mathematik" Hans-Jochen Bartsch Verlag Harri Deutsch 7.-9. edition Frankfort on the Main 1986

17. The diagonals in n-gon

Every point of a general n-gon is connected with $n-3$ diagonals. With that we have $n \cdot (n-3)$ diagonals, see figure.

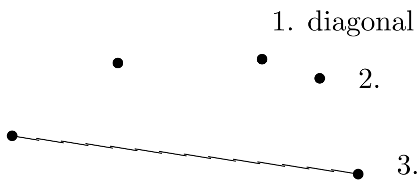


Every diagonal has got two points of the n-gon. With that every diagonal is counted twice. The number of diagonals is at least:

$$z_D = \frac{n \cdot (n-3)}{2}$$

Now we view a regular n-gon:

We want to determine the number of intersected diagonals at the m. diagonal, see figure.



On the one side from the m. diagonal there are m points. On the other side there are $n-m-2$ points.

With that the m. diagonal intersects $m \cdot (n-m-2)$ diagonals in the regular n-gon.

The maximum:

Now we calculate the maximun of $m \cdot (n-m-2) = mn - m^2 - 2m$.

Derivation to m:

$$f'(m) = n - 2m - 2$$

Criterion of local extremum:

$$0 = f'(m) = n - 2 - 2m \qquad f''(m) = -2$$

It follows:

$$m = \frac{n-2}{2}$$

It is a maximum. We will determine this maximum number of the even n-gon. Later we will work with the uneven n-gon.
At the even n-gon $\frac{n-2}{2}$ is an integer number. This number can be inserted instead of m in the formula $m \cdot (n - m - 2)$.

Insertion:

$$\frac{n-2}{2} \cdot \left(n - \frac{n-2}{2} - 2\right) = \frac{n-2}{2} \cdot \frac{n-2}{2} = \frac{(n-2)^2}{4}$$

In the even regular n-gon a diagonal can intersect at the most $\frac{(n-2)^2}{4}$ diagonals.

In the uneven n-gon $\frac{n-2}{2}$ is no integer number. Because of this, the insertions must be done to $\frac{n-2}{2} + \frac{1}{2} = \frac{n-1}{2}$ and $\frac{n-2}{2} - \frac{1}{2} = \frac{n-3}{2}$.

Insertion of $\frac{n-1}{2}$ instead of m in $m \cdot (n - m - 2)$:

$$\frac{n-1}{2} \cdot \left(n - \frac{n-1}{2} - 2\right) = \frac{n-1}{2} \cdot \frac{2n-n+1-4}{2} = \frac{n-1}{2} \cdot \frac{n-3}{2}$$

Insertion of $\frac{n-3}{2}$:

$$\frac{n-3}{2} \cdot \left(n - \frac{n-3}{2} - 2\right) = \frac{n-3}{2} \cdot \frac{2n-n+3-4}{2} = \frac{n-3}{2} \cdot \frac{n-1}{2}$$

Both maxima are equal. The diagonals can be intersected in the uneven regular n-gon at the most $\frac{(n-3)\cdot(n-1)}{4}$ diagonals.

Number estimation of intersection points of diagonals in the regular n-gon:

These conclusions can be seen at [1]. The connecting lines between 4 points yield intersection points of the diagonals. Intersection points of several diagonals are possible. We have the result:

intersection points' number of diagonals in the regular n-gon

$$\leq \binom{n}{4} = \frac{n!}{4! \cdot (n-4)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3)}{24} \leq \frac{n^4}{24}$$

On this way we have got an estimation.

Further infomations about the n-gon can be found at Schröer [2].

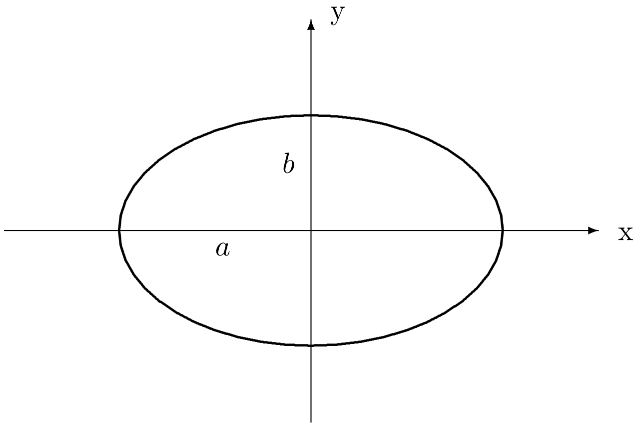
References

[1] "The number of triangles formed by intersecting diagonals of a regular polygon"

[2] Harald Schröer "The n-gon with exercises and solutions" Wissenschaft & Technik Verlag Berlin 2002

18. A special ellipse

We view an usual ellipse:



We need the canonical equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

a and b are the semimajor and the semiminor axis. We transform to y^2:

$$y^2 = b^2 \cdot \left(1 - \frac{x^2}{a^2}\right) = \frac{b^2}{a^2} \cdot (a^2 - x^2)$$

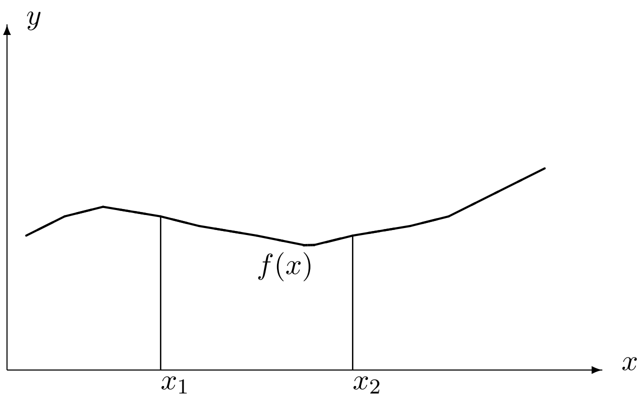
It follows:

$$y = \pm\frac{b}{a} \cdot \sqrt{a^2 - x^2}$$

We derive with the chain rule:

$$y'(x) = \frac{b}{a} \cdot \frac{-x}{\sqrt{a^2 - x^2}}$$

Now we look at the following figure:



The arc length U_s can be expressed generally:

$$U_s = \int\limits_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\, dx$$

Thus the whole perimeter of the ellipse is:

$$U = 4 \cdot \int\limits_0^a \sqrt{1 + y'(x)^2}\, dx \tag{1}$$

$$= 4 \cdot \int\limits_0^a \sqrt{1 + \frac{b^2}{a^2} \cdot \frac{x^2}{a^2 - x^2}}\, dx$$

Now we introduce the linear eccentricity e and the numerical eccentricity ε. We have the relations $e^2 = a^2 - b^2$ and $\varepsilon = \frac{e}{a}$, then we obtain:

$$\varepsilon = \frac{\sqrt{a^2 - b^2}}{a} \qquad \Rightarrow \qquad \varepsilon^2 a^2 = a^2 - b^2$$

At last:

$$b^2 = a^2 - \varepsilon^2 a^2$$

In the case of $\varepsilon = 1$ follows $b = 0$. The case $\varepsilon = 1$ yields with equation (1) the perimeter:

$$U = 4 \cdot \int\limits_0^a \sqrt{1 + 0}\, dx = 4 \cdot \int\limits_0^a 1\, dx = 4 \cdot a$$

In the case $b = 0$ we can see this result too. Then the ellipse looks like a line segment.

In general, we have for the ellipse's perimeter the following formula (see Bartsch [1] chapter 7.5.1 p. 302):

$$U = 2 \cdot \pi \cdot a \cdot w$$

with:

$$w = 1 - \left(\frac{1}{2}\right)^2 \cdot \varepsilon^2 - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \cdot \frac{\varepsilon^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \cdot \frac{\varepsilon^6}{5} - \cdots$$

For $\varepsilon = 1$ we have calculated $U = 4a$. Thus we can conclude for $\varepsilon = 1$:

$$4a = U = 2\pi a \cdot w$$

Then we follow:

$$w = \frac{2}{\pi} \qquad \text{for} \qquad \varepsilon = 1$$

Finally we get the remarkable equation:

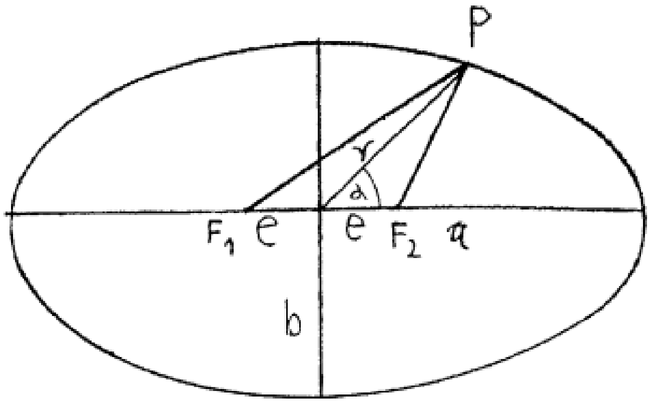
$$\frac{2}{\pi} = 1 - \left(\frac{1}{2}\right)^2 - \frac{1}{3} \cdot \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 - \frac{1}{5} \cdot \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 - \cdots$$

References

[1] Hans-Jochen Bartsch “Taschenbuch mathematischer Formeln” Verlag Harri Deutsch Frankfort on the Main 1986 7.-9.edition

19. The semiaxes-angular equation of the ellipse

We view the following ellipse:



a is the semimajor axis and b the semiminor axis. $e = \sqrt{a^2 - b^2}$ shall be the linear eccentricity. In the case $a = b$ is r in fact the radius of the circle (a special case of the ellipse). We want to get the angular dependence of r. For the ellipse is valid for the distance sum and the both focal points F_1, F_2:

$$\overline{PF_1} + \overline{PF_2} = 2a$$

We insert twice the cosine law:

$$2a = \sqrt{e^2 + r^2 - 2er\cos\alpha}$$
$$+\sqrt{e^2 + r^2 - 2er\cos(180° - \alpha)}$$

With $\cos(180° - \alpha) = -\cos\alpha$:

$$2a = \sqrt{e^2 + r^2 - 2er\cos\alpha}$$
$$+\sqrt{e^2 + r^2 + 2er\cos\alpha}$$

Squared:

$$4a^2 = 2e^2 + 2r^2$$
$$+2 \cdot \sqrt{e^4 + 2e^2r^2 + r^4 - 4e^2r^2\cos^2\alpha}$$

We solve to the root and then we square again:

$$(2a^2 - e^2 - r^2)^2$$
$$= e^4 + 2e^2r^2 + r^4 - 4e^2r^2\cos^2\alpha$$

Multiplying out:

$$4a^4 + e^4 + r^4 - 4a^2e^2 - 4a^2r^2 + 2e^2r^2$$
$$= e^4 + 2e^2r^2 + r^4 - 4e^2r^2\cos^2\alpha$$

Comprised:

$$4a^4 - 4a^2e^2 - 4a^2r^2 = -4e^2r^2\cos^2\alpha$$

Divided through $4a^2$:

$$a^2 - e^2 - r^2 = -\frac{e^2}{a^2}\cdot r^2\cos^2\alpha$$

We introduce $a^2 - e^2 = b^2$ and the numerical eccentricity $\varepsilon = \frac{e}{a}$:

$$b^2 - r^2 = -\varepsilon^2 r^2\cos^2\alpha$$

Transformed:

$$b^2 = r^2\cdot(1-\varepsilon^2\cos^2\alpha)$$

Solving to r:

$$r = \frac{b}{\sqrt{1-\varepsilon^2\cos^2\alpha}} \tag{1}$$

Thus we have r as function of the numerical eccentricity, the semiaxes and the angle α. Now we want to derive a formula which contains only the semiaxes and the angle α. We insert

$$\varepsilon = \frac{e}{a} = \frac{\sqrt{a^2-b^2}}{a}$$

in the equation (1):

$$r = \sqrt{\frac{b^2}{1-\frac{a^2-b^2}{a^2}\cdot\cos^2\alpha}}$$

Extension and simplification:

$$r = \frac{ab}{\sqrt{a^2\cdot(1-\cos^2\alpha)+b^2\cos^2\alpha}}$$

Because of $\sin^2\alpha + \cos^2\alpha = 1$:

$$r = \frac{ab}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}}$$

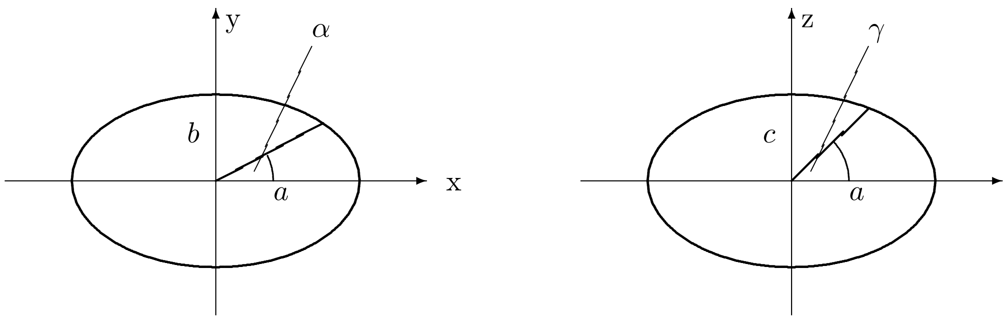
This is a semiaxes-angular equation, that can be transformed to:

$$a^2\sin^2\alpha + b^2\cos^2\alpha = \frac{a^2b^2}{r^2} \tag{2}$$

20. The "radius" of ellipsoid and the surface of the revolution ellipsoid

1. The "radius" of ellipsoid

We view the following figures:



$r =$ "radius" of ellipsoid

$a, b, c =$ semiaxes of ellipsoid

r is not a real radius. Only in the case of equal semiaxes r is the sphere's radius. It is interesting to consider r at ellipsoid, too. At ellipsoid r is not constant. r depends from the coordinates x, y, z. The first equation follows with Pythagoras theorem:

$$r^2 = x^2 + y^2 + z^2 \tag{1}$$

We obtain the second equation (for example with Bronstein [1] chapter 2.6.6.2 p. 233):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{2}$$

To the angles we get with the figure the relations:

$$\tan\alpha = \frac{y}{x} \qquad \tan\gamma = \frac{z}{x}$$

We transform equation (2) to z^2 and we insert at z^2 in equation (1):

$$r^2 = x^2 + y^2 + c^2 \cdot \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

Now we have one equation for r that is only dependent from two coordinates and the semiaxes. Solving to equation (2) to y^2 and insertion in equation (1) yields:

$$r^2 = x^2 + b^2 \cdot \left(1 - \frac{x^2}{a^2} - \frac{z^2}{c^2}\right) + z^2$$

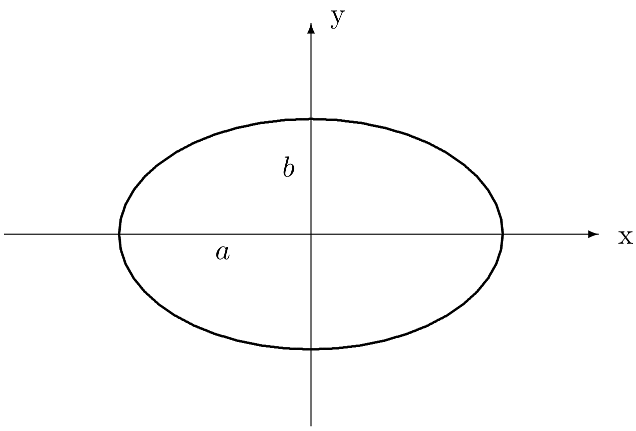
Rewriting from equation (2) to x^2 and insertion in equation (1) leads to:

$$r^2 = a^2 \cdot \left(1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) + y^2 + z^2$$

It is much more difficult to get an expression of r with only 2 angles and the 3 semiaxes. Courageous readers can get down to this problem.

2. The surface of the revolution ellipsoid

We look at an ellipse:



We need the canonical equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \Rightarrow \qquad y^2 = b^2 \cdot \left(1 - \frac{x^2}{a^2}\right)$$

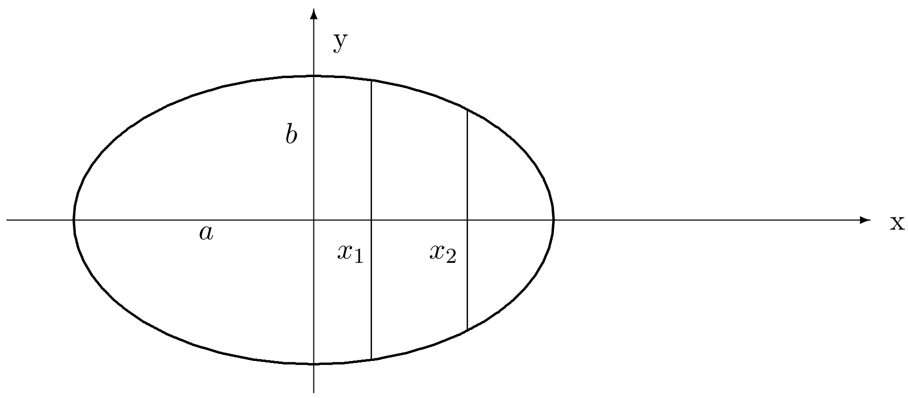
Solving to y:

$$y = \pm\frac{b}{a} \cdot \sqrt{a^2 - x^2}$$

We derive:

$$y' := \frac{dy}{dx} = \pm\frac{b}{a} \cdot \frac{-x}{\sqrt{a^2 - x^2}}$$

Now we view the revolution ellipsoid in the following figure:



For the lateral area of the revolution body is valid:

$$M(x_1, x_2) = 2\pi \cdot \int\limits_{x_1}^{x_2} y \cdot \sqrt{1 + y'^2}\, dx$$

It is $-a \leq x_1 \leq x_2 \leq a$. Now we insert for the revolution ellipsoid:

$$M(x_1, x_2) = 2\pi \cdot \frac{b}{a} \cdot \int\limits_{x_1}^{x_2} \sqrt{a^2 - x^2} \cdot \sqrt{1 + \frac{b^2 x^2}{a^2 \cdot (a^2 - x^2)}}\, dx$$

$$= 2\pi \cdot \frac{b}{a} \cdot \int\limits_{x_1}^{x_2} \sqrt{a^2 - x^2 + \frac{b^2}{a^2} \cdot x^2}\, dx$$

Then we achieve:

$$M(x_1, x_2) = 2\pi \cdot \frac{b}{a} \cdot \int\limits_{x_1}^{x_2} \sqrt{a^2 - \left(1 - \frac{b^2}{a^2}\right) \cdot x^2}\, dx \tag{3}$$

We transform this term further to: $\qquad a > b > 0$

$$M(x_1, x_2) = 2\pi \cdot \frac{b}{a} \cdot \sqrt{1 - \frac{b^2}{a^2}} \cdot \int\limits_{x_1}^{x_2} \sqrt{\frac{a^2}{1 - \frac{b^2}{a^2}} - x^2}\, dx \tag{4}$$

The integration can be done for example with Gröbner [2] chapter 236 p.52 Nr. 5f or Bronstein [1] chapter 1.1.3.3 p.44 Nr. 157.

$$\int \sqrt{R^2 - x^2}\, dx = \frac{x}{2} \cdot \sqrt{R^2 - x^2} + \frac{R^2}{2} \cdot \arcsin \frac{x}{|R|} + c$$

c is the integration constant. This integral can be proved with differentiation. It is valid $a > b > 0$. We insert for the revolution ellipsoid:

$$M(x_1, x_2) = 2\pi \cdot \frac{b}{a} \cdot \sqrt{1 - \frac{b^2}{a^2}} \cdot \left[\frac{x}{2} \cdot \sqrt{\frac{a^2}{1 - \frac{b^2}{a^2}} - x^2}\right.$$

$$\left. + \frac{\frac{1}{2} \cdot a^2}{1 - \frac{b^2}{a^2}} \cdot \arcsin\left(\frac{x}{a} \cdot \sqrt{1 - \frac{b^2}{a^2}}\right)\right]_{x_1}^{x_2}$$

Now let be $x_2 = h$ and $x_1 = 0$:

$$M(0, h) = 2\pi \cdot \frac{b}{a} \cdot \sqrt{1 - \frac{b^2}{a^2}} \cdot \left(\frac{h}{2} \cdot \sqrt{\frac{a^2}{1 - \frac{b^2}{a^2}} - h^2}\right.$$

$$\left. + \frac{a^2}{2 \cdot \left(1 - \frac{b^2}{a^2}\right)} \cdot \arcsin\left(\frac{h}{a} \cdot \sqrt{1 - \frac{b^2}{a^2}}\right)\right)$$

Now we simplify further for $h = a$, then we get the half surface:

$$M(0,a) = 2\pi \cdot \frac{b}{a} \cdot \sqrt{1 - \frac{b^2}{a^2}} \cdot \left(\frac{a}{2} \cdot \sqrt{\frac{a^2}{1 - \frac{b^2}{a^2}} - a^2} \right.$$

$$\left. + \frac{a^2}{2 \cdot \left(1 - \frac{b^2}{a^2}\right)} \cdot \arcsin\left(\sqrt{1 - \frac{b^2}{a^2}}\right)\right)$$

We make a intermediate calculation:

$$\sqrt{\frac{a^2}{1 - \frac{b^2}{a^2}} - a^2} = \sqrt{\frac{a^4}{a^2 - b^2} - a^2} = \sqrt{\frac{a^4 - a^4 + b^2 a^2}{a^2 - b^2}} = \frac{ba}{\sqrt{a^2 - b^2}}$$

With this intermediate calculation we get:

$$M(0,a) = 2\pi \cdot \frac{b}{a^2} \cdot \sqrt{a^2 - b^2} \cdot \left(\frac{ba^2}{\sqrt{a^2 - b^2}} \cdot \frac{1}{2} + \frac{a^4}{2 \cdot (a^2 - b^2)} \cdot \arcsin\left(\sqrt{1 - \frac{b^2}{a^2}}\right)\right)$$

At last we obtain:

$$M(0,a) = 2\pi \cdot \left(\frac{b^2}{2} + \frac{ba^2}{2 \cdot \sqrt{a^2 - b^2}} \cdot \arcsin\left(\sqrt{1 - \frac{b^2}{a^2}}\right)\right)$$

This formula is valid for $a > b > 0$. We get the whole surface O with:

$$O = 2 \cdot M(0,a) = M(-a,a)$$

$$= 4\pi \cdot \left(\frac{b^2}{2} + \frac{ba^2}{2 \cdot \sqrt{a^2 - b^2}} \cdot \arcsin\left(\sqrt{1 - \frac{b^2}{a^2}}\right)\right)$$

Now we examine the special case $r = a = b$ (sphere). This case can't be calculated with the next to the last formula. We must begin with equation (3). We insert in this formula:

$$M(0,r) = 2\pi \cdot \int_0^r r \, dx = 2\pi [rx]_0^r = 2\pi r^2$$

$M(0,r)$ is the surface of the half sphere.

References

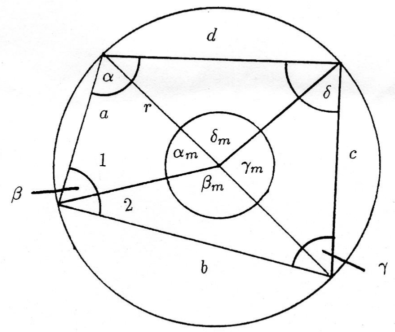
[1] I.N. Bronstein, K.A. Semendjajew "Taschenbuch der Mathematik" Teubner Verlagsgesellschaft Leipsic 1985 22.edition

[2] Wolfgang Gröbner and Nikolaus Hofreiter "Integraltafel Erster Teil" 5.edition 1975 Springer Verlag Vienna

21. The fourth side and the area of a inscribed tetragon

Abstract: We want to derive two unknown area formulas and we calculate the fourth side, if 3 sides are given. The assumption is that the midpoint of the circle is in the tetragon.

Key words: Inscribed tetragon - diagonal - side - area - circumradius - angle



We add the areas of the 4 isosceles triangles with the bases a, b, c, d, at which we use the half product of base and height (see figure).
For the area F we obtain:

$$F = \frac{r}{2} \cdot \left(a \cdot \cos \frac{\alpha_m}{2} + b \cdot \cos \frac{\beta_m}{2} + c \cdot \cos \frac{\gamma_m}{2} + d \cdot \cos \frac{\delta_m}{2} \right)$$

With the figure we have:

$$\sin \frac{\alpha_m}{2} = \frac{a}{2r} \qquad \sin \frac{\beta_m}{2} = \frac{b}{2r} \qquad \sin \frac{\gamma_m}{2} = \frac{c}{2r} \qquad \sin \frac{\delta_m}{2} = \frac{d}{2r}$$

With $\sin^2 \varphi + \cos^2 \varphi = 1$ we follow:

$$\cos \frac{\alpha_m}{2} = \frac{\sqrt{4r^2 - a^2}}{2r} \qquad \cos \frac{\beta_m}{2} = \frac{\sqrt{4r^2 - b^2}}{2r}$$

$$\cos \frac{\gamma_m}{2} = \frac{\sqrt{4r^2 - c^2}}{2r} \qquad \cos \frac{\delta_m}{2} = \frac{\sqrt{4r^2 - d^2}}{2r}$$

If we insert these expressions in the first area formula, then we yield:

$$F = \frac{1}{4} \cdot \left(a \cdot \sqrt{4r^2 - a^2} + b \cdot \sqrt{4r^2 - b^2} + c \cdot \sqrt{4r^2 - c^2} + d \cdot \sqrt{4r^2 - d^2} \right) \tag{1}$$

Thus we have the area as function from radius and the four sides. Now we will see how we can express one side through the other three sides.

In the inscribed tetragon is the sum of the corresponding angles equal to 180°. Thus we have $\alpha + \gamma = 180°$. Because of the 4 isosceles triangles we recognize:

$$\beta_1 = \alpha_2 \qquad \delta_1 = \gamma_2$$

We can conclude:

$$\alpha = \alpha_1 + \beta_1 \qquad \gamma = \gamma_1 + \delta_1$$

and we get at last:

$$\alpha_1 + \beta_1 + \gamma_1 + \delta_1 = 180° \tag{2}$$

With the 4 isosceles triangles we follow:

$$\alpha_2 + \beta_2 + \gamma_2 + \delta_2 = 180° \tag{3}$$

With the figure it is:

$$\cos\beta_1 = \frac{a}{2r} \qquad \cos\gamma_1 = \frac{b}{2r} \qquad \cos\delta_1 = \frac{c}{2r}$$

and with equation (2):

$$\frac{d}{2r} = \cos\alpha_1 = \cos(180° - \beta_1 - \gamma_1 - \delta_1)$$

Thus:

$$d = 2r \cdot \cos\left(180° - \arccos\frac{a}{2r} - \arccos\frac{b}{2r} - \arccos\frac{c}{2r}\right)$$

Thus the determination of the side d with the radius and the sides a, b, c is possible. Because of this, the area is a function of radius and three sides.

With the heights of the triangles perpendicular to the radii we get with the area formula of triangles:

$$F = \frac{r^2}{2} \cdot (\sin\alpha_m + \sin\beta_m + \sin\gamma_m + \sin\delta_m)$$

Because of the isosceles triangles we have $\alpha_m = 180° - 2\alpha_2$. By reason of $\sin(180° - \varphi) = \sin\varphi$ we can follow:

$$\sin\alpha_m = \sin(180° - 2\alpha_2) = \sin 2\alpha_2$$

With this insertion we obtain:

$$F = \frac{r^2}{2} \cdot (\sin 2\alpha_2 + \sin 2\beta_2 + \sin 2\gamma_2 + \sin 2\delta_2) \tag{4}$$

With equation (3) we can determine the fourth angle with three of these angles. The area is a function of radius and three of these angles.

The following assumption is valid: The midpoint of the circle must be in the tetragon. If the assumption is not valid, then we can use a formula at Bartsch [1] chapter 6.5 p.231 for the circumradius. To solve this equation to the fourth side, we need a numerical method(for example Newton's method). A more general area formula can be found in Bronstein [2] chapter 2.6.1 p.193.
Further trigonometric problems that can be found scarcely in the literature are at Schröer [3].

References

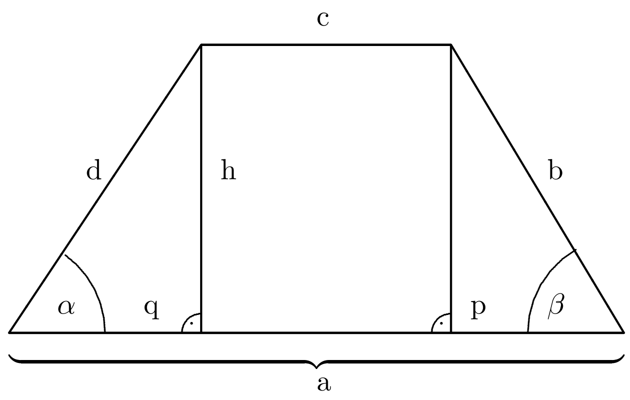
[1] Hans-Jochen Bartsch "Taschenbuch mathematischer Formeln" 7.-9.edition 1986 VEB Fachbuchverlag Leipsic

[2] I.N. Bronstein, K.A. Semendjajew "Taschenbuch der Mathematik" Teubner Verlagsgesellschaft Leipsic 1985 22.edition

[3] Harald Schröer "Trigonometric problem cases well solved" Wissenschaft & Technik Verlag Berlin 2001

22. The case of 4 faces at a trapezoid

The faces a, b, c and d in a trapezoid are known.



α and β are acute angles. Then we yield the following equations:

$$a - c = p + q \qquad q^2 + h^2 = d^2 \qquad p^2 + h^2 = b^2$$

Putting equal to h^2:

$$d^2 - q^2 = b^2 - p^2$$

or:

$$b^2 - d^2 = p^2 - q^2$$

Division through $a - c = p + q$:

$$\frac{b^2 - d^2}{a - c} = \frac{p^2 - q^2}{p + q} = p - q$$

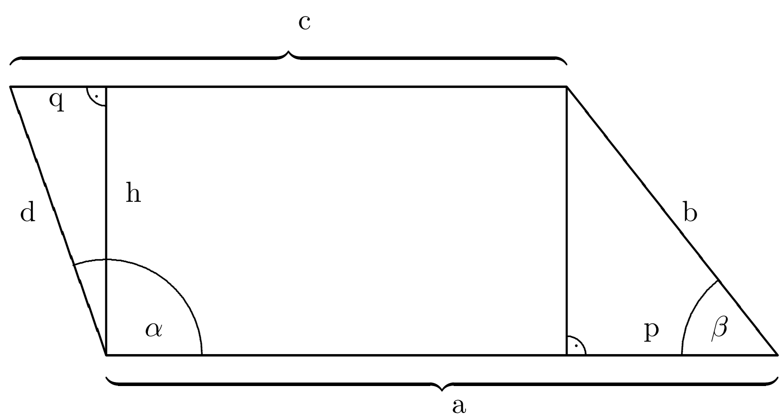
To this equation we add $p + q = a - c$. Then q vanishes and we obtain:

$$2p = a - c + \frac{b^2 - d^2}{a - c}$$

or:

$$p = \frac{(a - c)^2 + b^2 - d^2}{2 \cdot (a - c)}$$

Finally we calculate $q = a - c - p$. Now we view the case that α is an obtuse angle and β is an acute angle.



We can see the following equations:

$$p = a - (c - q) \qquad h^2 = d^2 - q^2 \qquad h^2 = b^2 - p^2$$

Equating:

$$d^2 - q^2 = b^2 - p^2$$

Thus:

$$p^2 - q^2 = b^2 - d^2$$

Division through $p - q = a - c$:

$$\frac{b^2 - d^2}{a - c} = \frac{p^2 - q^2}{p - q} = p + q$$

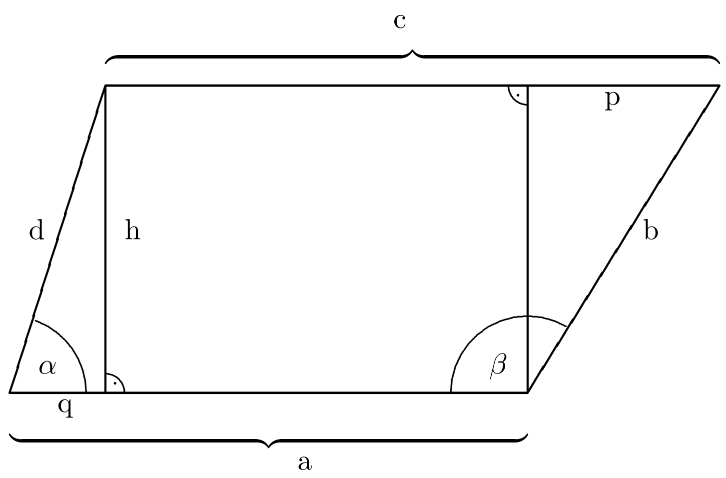
We add the equation $p - q = a - c$ and we get:

$$2p = a - c + \frac{b^2 - d^2}{a - c}$$

or:

$$p = \frac{(a - c)^2 + b^2 - d^2}{2 \cdot (a - c)}$$

With $q = p - a + c$ the calculation is finished. Now we can work with the case that β is an obtuse angle and α is an acute angle.



We recognize the 3 equations:

$$p = c - (a - q) \qquad h^2 = b^2 - p^2 \qquad h^2 = d^2 - q^2$$

Equating:

$$b^2 - p^2 = d^2 - q^2$$

or:

$$b^2 - d^2 = p^2 - q^2$$

Division through $p - q = c - a$:

$$\frac{b^2 - d^2}{c - a} = \frac{p^2 - q^2}{p - q} = p + q$$

If we add $p - q = c - a$ to this equation, then we yield:

$$2p = c - a + \frac{b^2 - d^2}{c - a}$$

or:

$$p = \frac{(c - a)^2 + b^2 - d^2}{2 \cdot (c - a)}$$

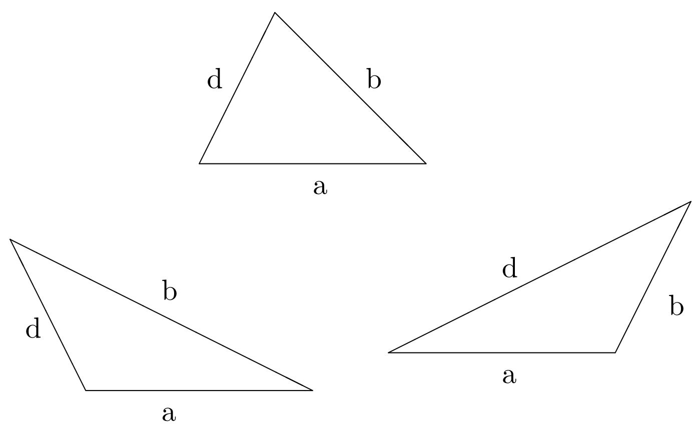
Finally it follows q from $q = p - c + a$.

In all three cases we find the height with $h^2 = b^2 - p^2 = d^2 - q^2$.

Sums, differences, products and quotients of rational numbers are rational numbers. In addition we conclude the following theorem:

> If all four faces of a trapezoid are rational numbers, then p and q are rational too.
> h must not be rational.

In case $c = 0$ follows the same assertion for a general triangle as special case. The transformations of p, q and h are valid to the following general triangles:



23. Geodesic lines on different surfaces and the maximum distance on a spherical surface

1. Geodesic lines

Here we treat a problem of differential geometry. A line that connects two different points on a surface with minimum perimeter is called geodesic line or geodesic. Now we indicate, how to calculate the geodesic lines on a known surface in R^3.

The general equation of geodesic lines is in Laugwitz [6], chapter II, task 4.13, p.41:

$$\ddot{u}^i + \sum_{j,k=1}^{2} \Gamma^i_{jk} \dot{u}^j \dot{u}^k = \lambda \cdot \dot{u}^i \qquad \text{with} \qquad \lambda = -\frac{d^2t}{ds^2} \cdot \left(\frac{ds}{dt}\right)^2$$

The Christoffel symbols of second kind Γ^i_{jk} are explained by the equation (4.9) in Laugwitz [6]:

$$\Gamma^r_{ik} = \sum_{l=1}^{2} \frac{g^{lr}}{2} \left(\frac{\partial g_{il}}{\partial u^k} - \frac{\partial g_{ik}}{\partial u^l} + \frac{\partial g_{lk}}{\partial u^i} \right)$$

The quantity g^{il} is introduced with equation (4.6) in Laugwitz [6]:

$$\sum_{l=1}^{2} g^{il} g_{lk} = \delta^i_k \qquad g^{il} = g^{li} \qquad \delta^i_k = \begin{cases} 1 & : \quad i = k \\ 0 & : \quad i \neq k \end{cases}$$

The metric fundamental element g_{ik} is defined with equation (3.16) in Laugwitz [6]:

$$g_{ik} = x_i x_k$$

With $x_i = \frac{\partial x}{\partial u^i}$ because of equation (3.2) in Laugwitz [6], x is the surface patch because of equation (3.1). In general the geodesic lines can be calculated on a known surface in this way. An interesting part result is the Clairaut theorem on a surface of rotation see Arnol'd [1], chapter 4.3.4.3, p.94 and figure 66 or Strubecker [8], equation (38.6). At Köhnlein [5] triangles on surfaces of rotation are treated.

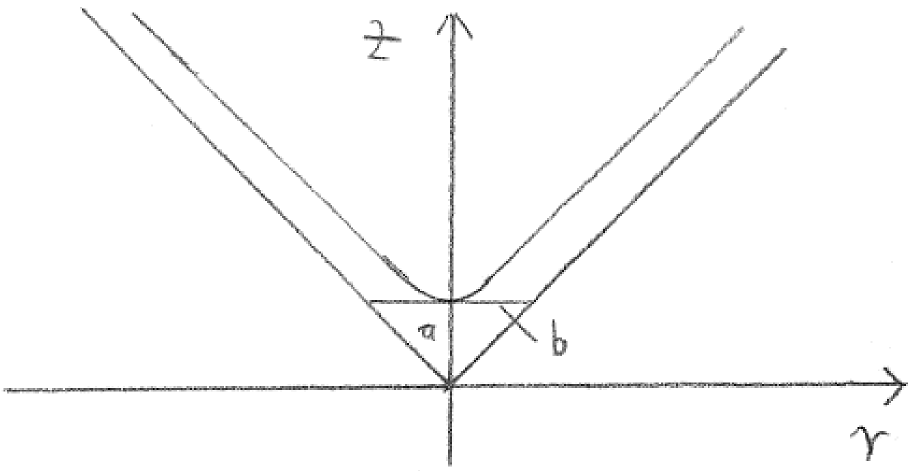
Strubecker [8] (chapter III 38, example 5, p.227–232) works with goedesic lines on paraboloids of revolution.
We find geodesic lines on ellipsoids of revolution in Strubecker [8], chapter III 38, example 6, p.232,233 together with an elliptic integral that must be determined numerically perhaps with Hofreiter [4], Nr.244, 1)-3), p.81,82.
The problem of determination of geodesics on ellipsoids of revolution plays a major role in geodesy, because the earth itself can be described as ellipsoid of revolution. In geodesy there is the "second geodesic fundamental task": Determination of the distance of two points, that have known terrestrial longitude and latitude. In Schödlbauer [7] different methods of distance calculation are presented for short, middle and long distances. In this book also ellipsoidal triangles are calculated. This is also done in Heck [3], chapter 6.6, p.212-214 with sides all smaller than $\frac{1}{10}$ of the earth radius. Similar formulas can

be found partly at Großmann [2], §38.2, p.87.

At hyperboloid of revolution we have the equation:

$$\frac{z^2}{a^2} - \frac{r^2}{b^2} = 1$$

see figure:



We get:

$$z^2 = a^2 \cdot \left(1 + \frac{r^2}{b^2}\right)$$

This leads to:

$$z(r) = \frac{a}{b} \cdot \sqrt{b^2 + r^2}$$

Further we obtain:

$$\frac{\partial z(r)}{\partial r} = \frac{a}{b} \cdot \frac{r}{\sqrt{b^2 + r^2}}$$

If we insert this in the equation (38.8) at Strubecker [8] p. 223, then we get the polar equation of geodesics on a hyperboloid of revolution:

$$(\varphi - \varphi_0) = k \cdot \int \frac{1}{r} \cdot \sqrt{\frac{1 + \frac{a^2 r^2}{b^2 (b^2 + r^2)}}{r^2 - k^2}}\, dr$$

2. The maximum distance on a spherical surface

Here we want to calculate the place with the greatest distance on the earth from towns as for example Boston, New York, Washington, Paris, London, Berlin, Hamburg, Copenhagen, Stockholm, Helsinki.

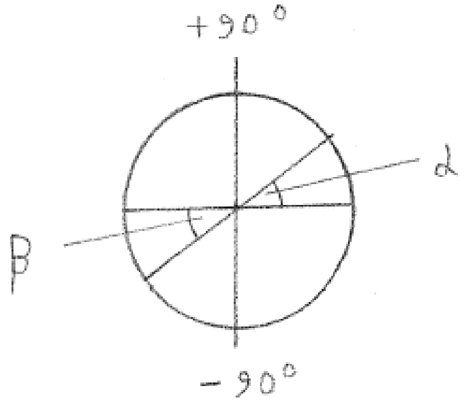
We have the following quantities:

R = radius of the (earth-) ball

α = latitude angle of the starting place $\qquad \alpha \in [-90°, 90°]$

φ = longitude of the starting place $\qquad \varphi \in [-180°, 180°]$

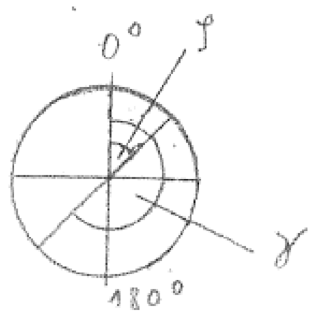
Degree is the unit of longitude and latitude angle. The longitude is counted west of the zero meridian as positive and east as negative. Often, atlas' have the negative signs left out in the eastern direction.



Now, we will search for the longitude and latitude angle with the maximum distance from the starting place.

β, γ = latitude angle, longitude of the place that has the maximum distance from the starting place.

For β we obtain $\beta = -\alpha$ with the first figure.



We get γ with the second figure:

$$\gamma° = \begin{cases} \varphi° - 180° & \text{if} \quad \varphi° \geq 0 \\ 180° + \varphi° & \text{if} \quad \varphi° \leq 0 \end{cases}$$

maximum distance $= \pi \cdot R$

(γ, β) determines the place with maximum distance from (φ, α).

References

[1] Vladimir Arnol'd "Mathematische Methoden der klassischen Mechanik" Birkhäuser Verlag Basel 1988

[2] Walter Großmann "Geodätische Rechnungen und Abbildungen" Verlag Konrad Wittwer Stuttgart 1964

[3] Bernhard Heck "Rechenverfahren und Auswertemodelle der Landesvermessung: klassische und moderne Methoden" Karlsruhe Wichmann Verlag 1987

[4] Wolfgang Gröbner und Nikolaus Hofreiter "Integraltafel Erster Teil" 5.edition 1975 Springer Verlag Vienna

[5] Walter Köhnlein "Untersuchungen über große geodätische Dreiecke auf geschlossenen Rotationsflächen unter besonderer Berücksichtigung des Rotationsellipsoiden" dissertation, Bayerische Akademie der Wissenschaften, Munich 1962

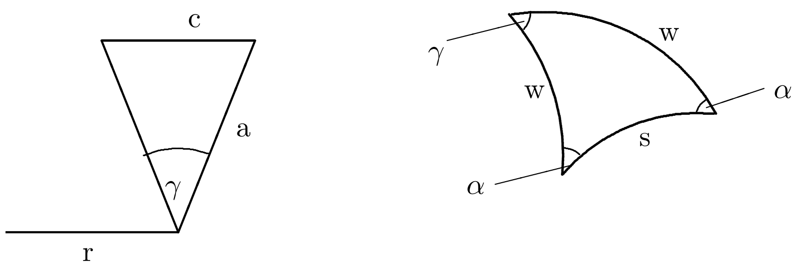
[6] Detlef Laugwitz "Differentialgeometrie" 3.edition 1977 Teubner Verlag Stuttgart

[7] Albert Schödlbauer "Rechenformeln und Rechenbeispiele zur Landesvermessung" Wichmann-Skripten Heft 2 Teil 1 Herbert Wichmann Verlag Karlsruhe 1981

[8] Karl Strubecker "Differentialgeometrie" volume 2 de Gruyter Verlag Berlin

24. The solid angle through the isosceles triangle

We want to determine the solid angle Ω through the isosceles triangle. The method is similar to Schröer [1] (chapter 5 and 6). We view the following triangles. The spherical triangle is on the unit sphere.



With the distance r and the sides a, c we get the following equation:

$$\sin\frac{\gamma}{2} = \frac{c}{2a} \tag{1}$$

We calculate the side w:

$$\tan w = \frac{a}{r}$$

The third side in spherical triangle can be presented with the cosine law for sides:

$$\cos s = \cos^2 w + \sin^2 w \cdot \cos\gamma$$

We use the spherical law of sines to determine the other angles:

$$\frac{\sin\alpha}{\sin w} = \frac{\sin\gamma}{\sin s}$$

Now we turn to the spherical excess ε and to the solid angle:

$$\Omega = \varepsilon = \gamma + 2\alpha - \pi$$

If $r \to 0$, we have the maximum solid angle:

$$\Omega \to \frac{\gamma}{2\pi} \cdot 2\pi = \gamma = 2 \cdot \arcsin\left(\frac{c}{2a}\right)$$

For $a, c \ll r$ it is valid with equation (1):

$$\Omega \approx \frac{ca}{2r^2} \cdot \cos\frac{\gamma}{2} = \frac{ca}{2r^2} \cdot \sqrt{1 - \left(\frac{c}{2a}\right)^2}$$

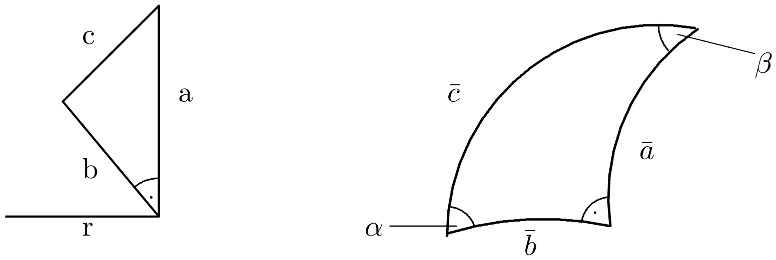
$\Phi = I \cdot \Omega$ with I as luminous intensity (radiant intensity) is the luminous flux (radiant flux or radiant power) in vacuum.

References

[1] Harald Schröer "Luminous Flux and Illumination",english and german edition, Wissenschaft und Technik Verlag Berlin 2001

25. The solid angle through the right-angled triangle

We determine the solid angle Ω through the right-angled triangle. The method is similar to Schröer [1] (chapter 5 and 6). We view the following triangles:



a, b, c = sides of the right-angled triangle,
$\bar{a}, \bar{b}, \bar{c}$ = sides of the right-angled spherical triangle on the unit sphere,
r is the distance.

In both triangles we follow the equations:

$$\tan \bar{a} = \frac{a}{r} \qquad \tan \bar{b} = \frac{b}{r}$$

$$\tan \alpha = \frac{\tan \bar{a}}{\sin \bar{b}} \qquad \tan \beta = \frac{\tan \bar{b}}{\sin \bar{a}}$$

The both last equations are in [2](p.189, theorem 5). Now we calculate the spherical excess ε and the solid angle:

$$\Omega = \varepsilon = \alpha + \beta + \frac{\pi}{2} - \pi = \alpha + \beta - \frac{\pi}{2}$$

If $r \to 0$, we have the maximum solid angle:

$$\Omega \to \frac{\pi}{2 \cdot 2\pi} \cdot 2\pi = \frac{\pi}{2}$$

For $a, b \ll r$ it is valid:

$$\Omega \approx \frac{a \cdot b}{2 \cdot r^2}$$

$\Phi = I \cdot \Omega$ with I as luminous intensity (radiant intensity) is the luminous flux (radiant flux or radiant power) in vacuum.

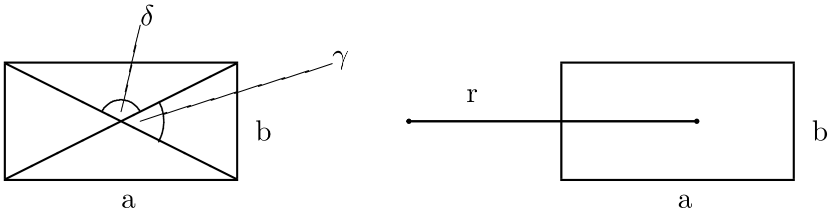
References

[1] Harald Schröer "Luminous Flux and Illumination",english and german edition, Wissenschaft und Technik Verlag Berlin 2001

[2] "dtv-Atlas zur Mathematik", volume 1, third edition, Deutscher Taschenbuch Verlag Munich 1978

26. The solid angle through the vertical rectangle

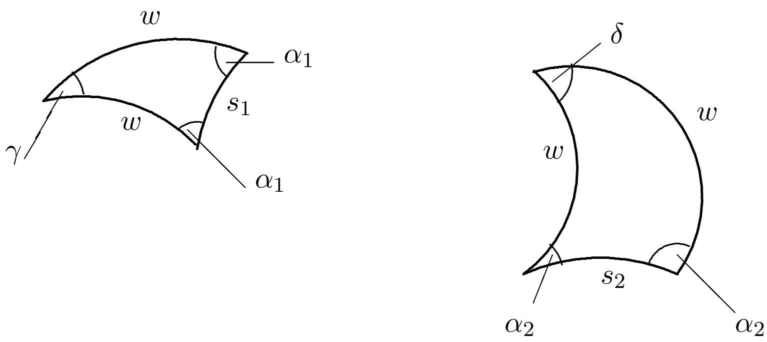
We want to determine the solid angle Ω through the vertical rectangle. The method is similar to Schröer [1] (chapter 5 and 6). We view the following rectangle:



With the distance r and the sides a, b we get the following equations:

$$\tan\frac{\gamma}{2} = \frac{b}{a} \qquad \tan\frac{\delta}{2} = \frac{a}{b}$$

We look at the spherical triangles on the unit sphere:



We get the side w with:

$$\tan w = \frac{\sqrt{a^2 + b^2}}{2r}$$

With the cosine law for sides we calculate the other sides:

$$\cos s_1 = \cos^2 w + \sin^2 w \cdot \cos\gamma$$

$$\cos s_2 = \cos^2 w + \sin^2 w \cdot \cos\delta$$

We use the spherical law of sines to determine the other angles:

$$\frac{\sin\alpha_1}{\sin w} = \frac{\sin\gamma}{\sin s_1}$$

$$\frac{\sin\alpha_2}{\sin w} = \frac{\sin\delta}{\sin s_2}$$

If ε is the spherical excess, then it is valid:

$$\varepsilon_1 = \gamma + 2\alpha_1 - \pi \qquad \varepsilon_2 = \delta + 2\alpha_2 - \pi$$

For the solid angle we have:

$$\Omega = 2 \cdot (\varepsilon_1 + \varepsilon_2)$$

The maximum solid angle can be 2π, if $r \to 0$.

In the case $a, b \ll r$ it is valid:

$$\Omega \approx \frac{a \cdot b}{r^2}$$

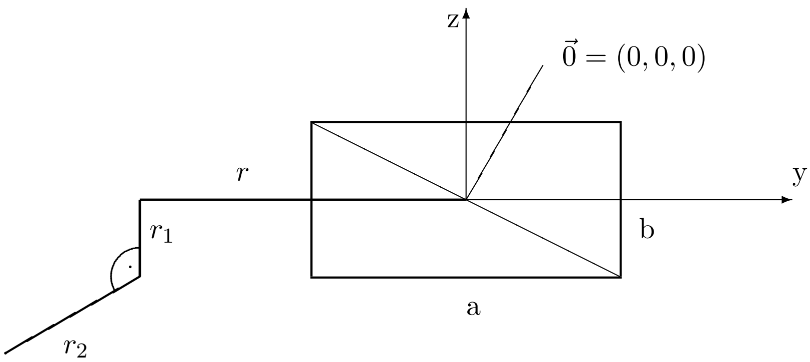
$\Phi = I \cdot \Omega$ with I as luminous intensity (radiant intensity) is the luminous flux (radiant flux or radiant power) in vacuum.

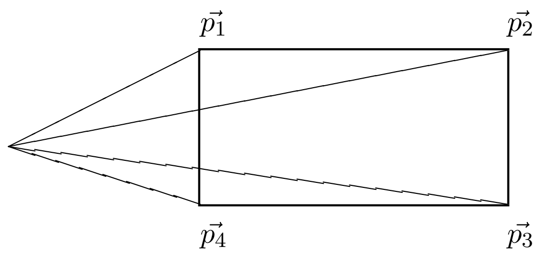
References

[1] Harald Schröer "Luminous Flux and Illumination", english and german edition, Wissenschaft und Technik Verlag Berlin 2001

27. The solid angle through the inclined rectangle

We want to calculate the solid angle through the inclined rectangle. The method is similar to Schröer [1] (chapter 5). The rectangle is in the origin.





We introduce the distance vector $\vec{l} = (r, r_2, r_1)$. We construct the difference vectors:

$$\vec{p_1} = \begin{pmatrix} 0 \\ \frac{-a}{2} \\ \frac{b}{2} \end{pmatrix} - \vec{l} \qquad \vec{p_2} = \begin{pmatrix} 0 \\ \frac{a}{2} \\ \frac{b}{2} \end{pmatrix} - \vec{l}$$

$$\vec{p_3} = \begin{pmatrix} 0 \\ \frac{a}{2} \\ \frac{-b}{2} \end{pmatrix} - \vec{l} \qquad \vec{p_4} = \begin{pmatrix} 0 \\ \frac{-a}{2} \\ \frac{-b}{2} \end{pmatrix} - \vec{l}$$

Now we determine the sides of both spherical triangles on the unit sphere:

$$\cos\alpha_{12} := \cos\angle(\vec{p_1}, \vec{p_2}) = \frac{\vec{p_1} \cdot \vec{p_2}}{|\vec{p_1}| \cdot |\vec{p_2}|}$$

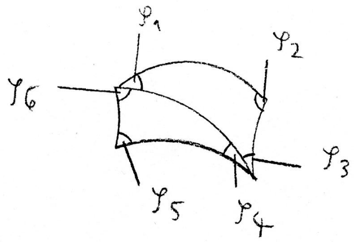
$$\cos\alpha_{23} := \cos\angle(\vec{p_2}, \vec{p_3}) = \frac{\vec{p_2} \cdot \vec{p_3}}{|\vec{p_2}| \cdot |\vec{p_3}|}$$

$$\cos\alpha_{13} := \cos\angle(\vec{p_1}, \vec{p_3}) = \frac{\vec{p_1} \cdot \vec{p_3}}{|\vec{p_1}| \cdot |\vec{p_3}|}$$

$$\cos\alpha_{34} := \cos\angle(\vec{p_3}, \vec{p_4}) = \frac{\vec{p_3} \cdot \vec{p_4}}{|\vec{p_3}| \cdot |\vec{p_4}|}$$

$$\cos\alpha_{14} := \cos\angle(\vec{p}_1, \vec{p}_4) = \frac{\vec{p}_1 \cdot \vec{p}_4}{|\vec{p}_1| \cdot |\vec{p}_4|}$$

We view the following figure:



We need the cosine law for sides to get two angles:

$$\begin{aligned}
\cos\alpha_{13} &= \cos\alpha_{12}\cos\alpha_{23} + \sin\alpha_{12}\sin\alpha_{23}\cos\varphi_2 \\
\Rightarrow \qquad \cos\varphi_2 &= \frac{\cos\alpha_{13} - \cos\alpha_{12}\cos\alpha_{23}}{\sin\alpha_{12}\sin\alpha_{23}} \\
\cos\alpha_{13} &= \cos\alpha_{34}\cos\alpha_{14} + \sin\alpha_{34}\sin\alpha_{14}\cos\varphi_5 \\
\Rightarrow \qquad \cos\varphi_5 &= \frac{\cos\alpha_{13} - \cos\alpha_{34}\cos\alpha_{14}}{\sin\alpha_{34}\sin\alpha_{14}}
\end{aligned}$$

Now we use the spherical law of sines in the first triangle:

$$\frac{\sin\varphi_1}{\sin\alpha_{23}} = \frac{\sin\varphi_2}{\sin\alpha_{13}} = \frac{\sin\varphi_3}{\sin\alpha_{12}}$$

In the second triangle:

$$\frac{\sin\varphi_4}{\sin\alpha_{14}} = \frac{\sin\varphi_5}{\sin\alpha_{13}} = \frac{\sin\varphi_6}{\sin\alpha_{34}}$$

We obtain the spherical excess in radian measure:

$$\varepsilon_1 = \varphi_1 + \varphi_2 + \varphi_3 - \pi \qquad\qquad \varepsilon_2 = \varphi_4 + \varphi_5 + \varphi_6 - \pi$$

For the solid angle we get $\Omega = \varepsilon_1 + \varepsilon_2$.

If $a, b \ll r$, then we have the approximation:

$$\Omega \approx \frac{ab \cdot \cos\alpha}{r^2 + r_1^2 + r_2^2} \qquad \text{with} \qquad \tan\alpha = \frac{\sqrt{r_1^2 + r_2^2}}{r}$$

$\Phi = I \cdot \Omega$ with I as luminous intensity (radiant intensity) is the luminous flux (radiant flux or radiant power) through the inclined rectangle in vacuum.

References

[1] Harald Schröer "Luminous Flux and Illumination",english and german edition, Wissenschaft und Technik Verlag Berlin 2001

Chapter F.

Stochastics

Contents

28. Linear and nonlinear regression with series of measurements

First method

We view the following series of measurements:

$$\begin{array}{cc} x_{11} & y_{21} \\ x_{12} & y_{22} \\ x_{13} & y_{23} \\ \vdots & \vdots \\ x_{1i} & y_{2i} \\ \vdots & \vdots \\ x_{1n} & y_{2n} \end{array}$$

To every $x_{1i} \in R$ a $y_{2i} \in R \;\; i \in \{1, \ldots, n\}$ is mesured.
We assume that one approximation $y = f(x)$ exists so that
$y_{2i} \approx f(x_{1i})$ for $i \in \{1, \ldots, n\}$.

We make the approximation more exact or we **adapt** $f(x)$ at the series of measurements.

We define: $y_{1i} := f(x_{1i}) \quad i \in \{1, \ldots, n\}$

Now we look at the values $y_{11}, \ldots, y_{1n}$ and $y_{21}, \ldots, y_{2n}$. We will determine with linear regression the function $g(y) = my + b$ so that $g(y_{1i}) \approx y_{2i} \;\; i \in \{1, \ldots, n\}$ and

$$\sum_{i=1}^{n} (g(y_{1i}) - y_{2i})^2$$

is a minimun.

Next, we look at the following series of measurements:

$$x_1, \ldots, x_n \quad \text{and} \quad y_1, \ldots, y_n$$

Then, known formulas of linear regression are valid, see e.g. Bronstein [1], chapter 5.2.4.1, p.692.

$$\bar{x} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i \qquad \bar{y} = \frac{1}{n} \cdot \sum_{i=1}^{n} y_i$$

$$s_x^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \bar{x})^2 \qquad s_y^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (y_i - \bar{y})^2$$

$$m_{xy} = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \bar{x}) \cdot (y_i - \bar{y})$$

$$r = \frac{m_{xy}}{s_x \cdot s_y}$$

$$m = b_{Y|X} = r \cdot \frac{s_y}{s_x} = \frac{m_{xy}}{s_x^2}$$

$$b = \bar{y} - m \cdot \bar{x}$$

with
$\bar{x}, \bar{y}$ = means
s_x^2, s_y^2 = sample variances
m_{xy} = empirical covariance
r = empirical correlation coefficient
m = slope of the regression line
b = line segment of the regression line

The regression line can be written as:

$$y = mx + b$$

These formulas can be used with the series $y_{11}, \ldots, y_{1n}$ and $y_{21}, \ldots, y_{2n}$.

$$\bar{y}_1 = \frac{1}{n} \cdot \sum_{i=1}^{n} y_{1i} \quad \bar{y}_2 = \frac{1}{n} \cdot \sum_{i=1}^{n} y_{2i}$$

$$S_1^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (y_{1i} - \bar{y}_1)^2 \quad S_2^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (y_{2i} - \bar{y}_2)^2$$

$$m_{12} = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (y_{1i} - \bar{y}_1) \cdot (y_{2i} - \bar{y}_2)$$

$$m = \frac{m_{12}}{S_1^2} \qquad b = \bar{y}_2 - m\bar{y}_1$$

$$r = \frac{m_{12}}{S_1 \cdot S_2} \quad \text{(correlation)}$$

The searched function is $g(y) = my + b$.

corrected function:

$$y = m \cdot f(x) + b$$

Thereby $f(x)$ is adapted as a series of measurements.

r is the correlation of g.

This method is mentioned shortly in Bronstein [1], chapter 7.1.5.1.2, p.788 on "linearization". Later, we see a second method of linearization.

Application to vectors:

This method can be generalised as vectors. The $2n$ vectors are measured as follows:

$$\begin{array}{cc} x_{11} & y_{21} \\ \vdots & \vdots \\ x_{1i} & y_{2i} \\ \vdots & \vdots \\ x_{1n} & y_{2n} \end{array}$$

with

$$x_{1i} =: \begin{pmatrix} x_{1i1} \\ \vdots \\ x_{1im} \end{pmatrix} \in R^m \qquad i \in \{1, \dots, n\}$$

$$y_{2i} =: \begin{pmatrix} y_{2i1} \\ \vdots \\ y_{2im} \end{pmatrix} \in R^m \qquad i \in \{1, \dots, n\}$$

We assume that one approximation

$$f : U \longrightarrow V \quad U, V \subset R^m \quad f =: \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$$

with $y_{2i} \approx f(x_{1i})$ for $i \in \{1, \dots, n\}$ exist.

Now we define:

$$f(x_{1i}) =: y_{1i} =: \begin{pmatrix} y_{1i1} \\ \vdots \\ y_{1im} \end{pmatrix} \in R^m$$

$$\text{for} \quad i \in \{1, \dots, n\}$$

Now it is possible to use the linear regression **componentwise**.

y_{1ij} and y_{2ij} can be opposed. $\quad j \in \{1, \dots, m\}$

$$\begin{array}{cc} y_{11j} & y_{21j} \\ \vdots & \vdots \\ y_{1nj} & y_{2nj} \end{array}$$

$$\text{Let be} \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \in V \subset R^m.$$

We search m functions of g_j.

$$g_j(y_j) = m_j y_j + b_j$$

and $g_j(y_{1ij}) \approx y_{2ij}$ for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. We must determine m linear functions of g_j.

Application of the linear regression:

$$\bar{y}_{1j} = \frac{1}{n} \cdot \sum_{i=1}^{n} y_{1ij} \qquad \bar{y}_{2j} = \frac{1}{n} \cdot \sum_{i=1}^{n} y_{2ij}$$

$$S_{1j}^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (y_{1ij} - \bar{y}_{1j})^2 \qquad S_{2j}^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (y_{2ij} - \bar{y}_{2j})^2$$

$$m_{12j} = \frac{1}{n-1} \sum_{i=1}^{n} (y_{1ij} - \bar{y}_{1j}) \cdot (y_{2ij} - \bar{y}_{2j})$$

Then we have:

$$g_j(y_j) = m_j y_j + b_j \quad j \in \{1, \dots, m\}$$

with

$$m_j = \frac{m_{12j}}{S_{1j}^2} \qquad b_j = \bar{y}_{2j} - m_j \cdot \bar{y}_{1j}$$

$$r_j = \frac{m_{12j}}{S_{1j} \cdot S_{2j}} \qquad \text{(correlation of } g_j\text{)}$$

Adaptation of the series of measurements:

$$G(y) := \begin{pmatrix} g_1(y_1) \\ \vdots \\ g_m(y_m) \end{pmatrix}$$

corrected function:

$$y = G(f(x)) = \begin{pmatrix} g_1(f_1(x)) \\ \vdots \\ g_m(f_m(x)) \end{pmatrix}$$

$$= \begin{pmatrix} m_1 f_1(x) + b_1 \\ \vdots \\ m_m f_m(x) + b_m \end{pmatrix}$$

$$\text{with} \qquad f =: \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$$

r_j is an assertion about the quality of g_j.

It is unknown, if $r = \frac{1}{m}\sum_{i=1}^{m} r_j$ is a correlation of G?

If the corrected function is not exact enough, the corrected function can be used as f. This is a second correction with G_2. The twice corrected function is:

$$y = G_2(G_1(f(x)))$$

This function can be inserted again until

$$y = G_r(G_{r-1}(\ldots G_1(f(x))\ldots))$$

has the wanted exactness.

A proof of this method:

$||\cdot||$ = euclidean norm

$$\sum_{i=1}^{n} ||y_{2i} - y_{1i}||^2 = \sum_{j=1}^{m}\sum_{i=1}^{n}(y_{2ij} - y_{1ij})^2$$

$$\bar{y}_{1i} := \begin{pmatrix} \bar{y}_{1i1} \\ \vdots \\ \bar{y}_{1im} \end{pmatrix} := \begin{pmatrix} m_1 y_{1i1} + b_1 \\ \vdots \\ m_m y_{1im} + b_m \end{pmatrix}$$

$$\sum_{i=1}^{n} ||y_{2i} - \bar{y}_{1i}||^2 = \sum_{j=1}^{m}\sum_{i=1}^{n}(y_{2ij} - \bar{y}_{1ij})^2$$

It can be recognized, because of the method of least squares, see e.g. Bronstein [1], chapter 7.1.5.1, p.787,788 :

$$\sum_{i=1}^{n}(y_{2ij} - \bar{y}_{1ij})^2 < \sum_{i=1}^{n}(y_{2ij} - y_{1ij})^2 \tag{1}$$

if

$$\begin{pmatrix} y_{11j} \\ \vdots \\ y_{1nj} \end{pmatrix} \neq \begin{pmatrix} y_{21j} \\ \vdots \\ y_{2nj} \end{pmatrix}$$

If the vectors are equal, there is a sign of equality at (1).

We conclude:

$$\sum_{j=1}^{m}\sum_{i=1}^{n}(y_{2ij} - \bar{y}_{1ij})^2 < \sum_{j=1}^{m}\sum_{i=1}^{n}(y_{2ij} - y_{1ij})^2$$

if

$$\begin{pmatrix} y_{11j} \\ \vdots \\ y_{1nj} \end{pmatrix} \neq \begin{pmatrix} y_{21j} \\ \vdots \\ y_{2nj} \end{pmatrix} \qquad \text{for one } j \in \{1, \ldots, m\}$$

In case

$$\begin{pmatrix} y_{11j} \\ \vdots \\ y_{1nj} \end{pmatrix} = \begin{pmatrix} y_{21j} \\ \vdots \\ y_{2nj} \end{pmatrix} \qquad \text{for all } j \in \{1, \ldots, m\}$$

an adaptation is not necessary. q.e.d.

Second method

We view again the following series of measurements:

$$\begin{array}{cc} x_{11} & y_{21} \\ x_{12} & y_{22} \\ x_{13} & y_{23} \\ \vdots & \vdots \\ x_{1i} & y_{2i} \\ \vdots & \vdots \\ x_{1n} & y_{2n} \end{array}$$

To every $x_{1i} \in R$ is mesured a $y_{2i} \in R$. $i \in \{1, \ldots, n\}$
We assume again that one approximation $y = f(x)$ exist so that $y_{2i} \approx f(x_{1i})$ for $i \in \{1, \ldots, n\}$.
Now we use another method to **adapt** the function $f(x)$ at the series of measurements.

We define: $y_{1i} := f(x_{1i}) \quad x_{2i} := f^{-1}(y_{2i})$
$i \in \{1, \ldots, n\}$
It is assumed that f possesses an inverse function f^{-1}.

The second possibility is to view the values $x_{11}, \ldots x_{1n}$ and $x_{21}, \ldots, x_{2n}$. We determine with linear regression a linear function $g(x) = mx + b$ so that $g(x_{1i}) \approx x_{2i} \quad i \in \{1, \ldots, n\}$ and

$$\sum_{i=1}^{n} \left(g(x_{1i}) - x_{2i}\right)^2$$

is a minimum. This method is called linearization.

We again use the known formulas of linear regression.

Then, it becomes:

$$\bar{x}_1 = \frac{1}{n} \cdot \sum_{i=1}^{n} x_{1i} \qquad \bar{x}_2 = \frac{1}{n} \cdot \sum_{i=1}^{n} x_{2i}$$

$$S_1^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_{1i} - \bar{x}_1)^2 \qquad S_2^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_{2i} - \bar{x}_2)^2$$

$$m_{12} = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_{1i} - \bar{x}_1) \cdot (x_{2i} - \bar{x}_2)$$

$$m = \frac{m_{12}}{S_1^2} \qquad b = \bar{x}_2 - m \cdot \bar{x}_1$$

$$r = \frac{m_{12}}{S_1 \cdot S_2}$$

The searched function is:

$$g(x) = mx + b$$

The function $f(x)$ is adapted at the series of measurements, if we insert $g(x)$ instead of x in $f(x)$. The corrected function is:

$$y = f(mx + b)$$

r is an assertion about the quality of

$$g(x) = mx + b.$$

Thus, the second method of linearization is put in front.

Application to vectors:

This method can also be used by vectors. The following $2n$ vectors are measured:

$$\begin{array}{cc} x_{11} & y_{21} \\ \vdots & \vdots \\ x_{1i} & y_{2i} \\ \vdots & \vdots \\ x_{1n} & y_{2n} \end{array}$$

with

$$x_{1i} =: \begin{pmatrix} x_{1i1} \\ \vdots \\ x_{1im} \end{pmatrix} \in R^m \quad i \in \{1, \ldots, n\}$$

$$y_{2i} =: \begin{pmatrix} y_{2i1} \\ \vdots \\ y_{2im} \end{pmatrix} \in R^m \quad i \in \{1, \ldots, n\}$$

Again we assume, that one approximation
$f : U \longrightarrow V \quad U, V \subset R^m$

with $y_{2i} \approx f(x_{1i})$ for $i \in \{1, \ldots, n\}$ exists. $f : U \longrightarrow V$ must be bijective, otherwise there is no inverse function.

The following quantities will be defined:

$$f(x_{1i}) =: y_{1i} =: \begin{pmatrix} y_{1i1} \\ \vdots \\ y_{1im} \end{pmatrix} \in R^m \quad i \in \{1, \ldots, n\}$$

$$f^{-1}(y_{2i}) =: x_{2i} =: \begin{pmatrix} x_{2i1} \\ \vdots \\ x_{2im} \end{pmatrix} \in R^m \quad i \in \{1, \ldots, n\}$$

It is favorable to use the linear regression **componentwise**.

x_{1ij} and x_{2ij} can be opposed. $\quad j \in \{1, \ldots, m\}$

$$\begin{matrix} x_{11j} & & x_{21j} \\ \vdots & & \vdots \\ x_{1nj} & & x_{2nj} \end{matrix}$$

Let

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in U \subset R^m$$

Then m functions of g_j with $g_j(x_j) = m_j x_j + b_j$ and $g_j(x_{1ij}) \approx x_{2ij}$ for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ are searched.

m functions of g_j must be determined.

Application of the linear regression:

$$\bar{x}_{1j} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_{1ij} \qquad \bar{x}_{2j} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_{2ij}$$

$$S_{1j}^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_{1ij} - \bar{x}_{1j})^2 \quad S_{2j}^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_{2ij} - \bar{x}_{2j})^2$$

$$m_{12j} = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_{1ij} - \bar{x}_{1j}) \cdot (x_{2ij} - \bar{x}_{2j})$$

We obtain:

$$g_j(x_j) = m_j x_j + b_j \quad j \in \{1, \ldots, m\}$$

with

$$m_j = \frac{m_{12j}}{S_{1j}^2} \qquad b_j = \bar{x}_{2j} - m_j \cdot \bar{x}_{1j}$$

$$r_j = \frac{m_{12j}}{S_{1j} \cdot S_{2j}} \quad \text{is the correlation of } g_j.$$

Adaptation to the series of measurements:

$$G(x) := \begin{pmatrix} g_1(x_1) \\ \vdots \\ g_m(x_m) \end{pmatrix}$$

corrected function:

$$\begin{aligned} y &= f(G(x)) = f(g_1(x_1), \ldots, g_m(x_m)) \\ &= f(m_1 x_1 + b_1, \ldots, m_m x_m + b_m) \end{aligned}$$

r_j is an assertion about the quality of g_j. It is unknown, if $r := \frac{1}{m} \cdot \sum_{j=1}^{m} r_j$ is a correlation of G?

If the corrected function is not exact enough, the corrected function can be used as f. There is a second correction with G_2. The twice corrected function is:

$$y = f(G_1(G_2(x)))$$

This function can be inserted again until

$$y = f(G_1(G_2(\ldots(G_r(x))\ldots)))$$

has the wanted exactness.

A proof of this method:

$|| \cdot ||$ = euclidean norm

$$\sum_{i=1}^{n} ||x_{1i} - x_{2i}||^2 = \sum_{j=1}^{m} \sum_{i=1}^{n} (x_{1ij} - x_{2ij})^2$$

$$\bar{x}_{2i} := \begin{pmatrix} \bar{x}_{2i1} \\ \vdots \\ \bar{x}_{2im} \end{pmatrix} := \begin{pmatrix} \frac{x_{2i1} - b_1}{m_1} \\ \vdots \\ \frac{x_{2im} - b_m}{m_m} \end{pmatrix}$$

$$\sum_{i=1}^{n} ||x_{1i} - \bar{x}_{2i}||^2 = \sum_{j=1}^{m} \sum_{i=1}^{n} (x_{1ij} - \bar{x}_{2ij})^2$$

It can be seen, because of the method of least squares, e.g. Bronstein [1], chapter 7.1.5.1, p.787,788 :

$$\sum_{i=1}^{n}(x_{1ij}-\bar{x}_{2ij})^2 < \sum_{i=1}^{n}(x_{1ij}-x_{2ij})^2 \tag{2}$$

$$\text{if} \qquad \begin{pmatrix} x_{11j} \\ \vdots \\ x_{1nj} \end{pmatrix} \neq \begin{pmatrix} x_{21j} \\ \vdots \\ x_{2nj} \end{pmatrix}$$

If the vectors are equal, there is a sign of equality at (2).

It follows:

$$\sum_{j=1}^{m}\sum_{i=1}^{n}(x_{1ij}-\bar{x}_{2ij})^2 < \sum_{j=1}^{m}\sum_{i=1}^{n}(x_{1ij}-x_{2ij})^2$$

$$\text{if} \qquad \begin{pmatrix} x_{11j} \\ \vdots \\ x_{1nj} \end{pmatrix} \neq \begin{pmatrix} x_{21j} \\ \vdots \\ x_{2nj} \end{pmatrix} \qquad \text{for one } j \in \{1,\ldots,m\}$$

$$\text{In case} \qquad \begin{pmatrix} x_{11j} \\ \vdots \\ x_{1nj} \end{pmatrix} = \begin{pmatrix} x_{21j} \\ \vdots \\ x_{2nj} \end{pmatrix} \qquad \text{for all } j \in \{1,\ldots,m\}$$

an adaptation is not necessary. q.e.d.

At present we have described two methods that refine the approximative law $y = f(x)$. These methods can be used, if the form of the law is changed insignificantly. In the nature of science this is important, for the following reason: In known law, there are often variables that have a certain meaning. Perhaps, this is valid to other sciences as well. Numerical mathematics developes methods that can totally change the form of the wanted law.

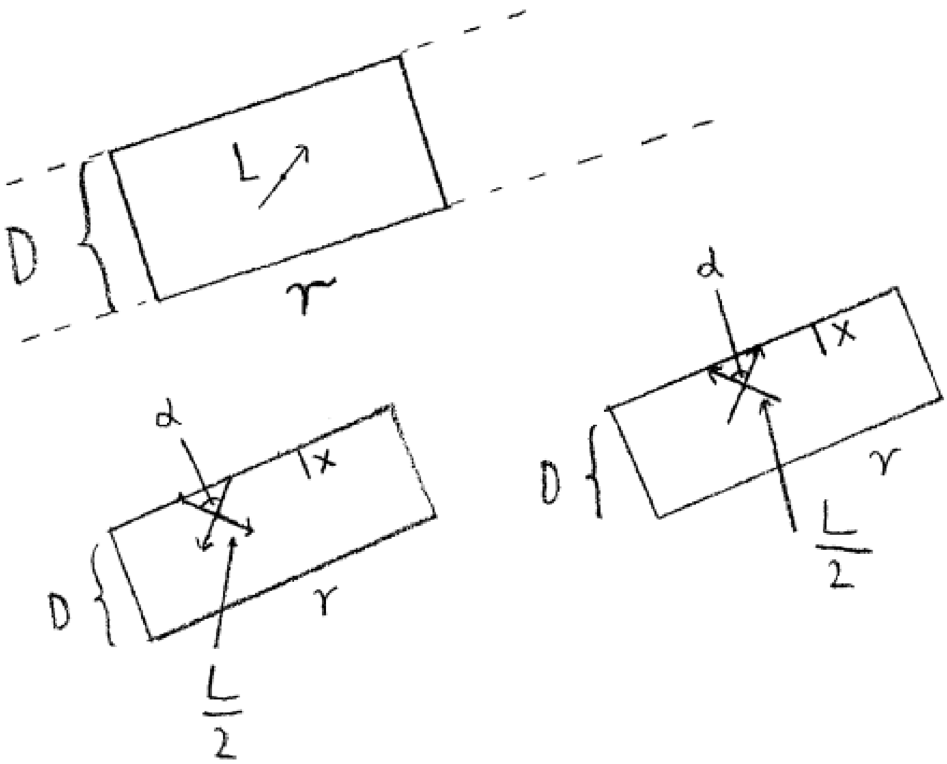
Another problem is in finding the first approximation $y = f(x)$. We can e.g. use the methods in Bronstein [1], chapter 7.1.5, p.786-790 (method of the least squares, method of the choosed point, averaging method). These methods can be used for refinements, e.g. more mesured points or more parameters are possibilities.

References

[1] Bronstein, Semendjajew "Taschenbuch der Mathematik", 22.edition, 1985, Teubner Verlag, Leipsic

29. Buffon's needle problem

We accidentally let a needle fall with the length L on a table. On the table are drawn parallel lines with distance D. We assume $L \leq D$.



We denote:

F=$r \cdot D$ = area of the limited surface patch
x = distance of the needle's geometric midpoint to one parallel (see figure). With the figure we recognize:

$$\cos\frac{\alpha}{2} = \frac{x}{\left(\frac{L}{2}\right)} = \frac{2x}{L}$$

We transform this equation to:

$$\alpha = 2 \cdot \arccos\left(\frac{2x}{L}\right) = \alpha(x)$$

In distance x the needle has a "rotation probability"of $\frac{2\alpha}{2\pi} = \frac{\alpha}{\pi}$, if the needle intersects to one parallel.

Now we introduce the probability w that the needle intersects one parallel:

$$w = \frac{2r \cdot \int\limits_0^{\frac{L}{2}} \frac{2\alpha(x)}{2\pi}\,dx}{F} = \frac{2r \cdot \int\limits_0^{\frac{L}{2}} \frac{2\arccos\left(\frac{2x}{L}\right)}{\pi}\,dx}{r \cdot D}$$

finally:

$$w = \frac{4}{\pi D} \cdot \int\limits_0^{\frac{L}{2}} \arccos\left(\frac{2x}{L}\right)\, dx$$

Now we use the integral:

$$\int \arccos\frac{x}{a}\, dx = x \cdot \arccos\frac{x}{a} - \sqrt{a^2 - x^2}$$

This can be proved by differentiation.

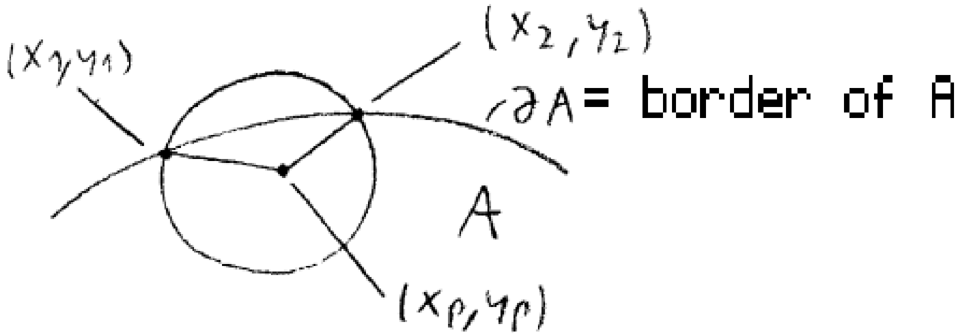
Let it be that $a = \frac{L}{2}$, then we get:

$$w = \frac{4}{\pi D} \cdot \left[x \cdot \arccos\left(\frac{2x}{L}\right) - \sqrt{\frac{L^2}{4} - x^2} \right]_0^{\frac{L}{2}}$$

After evaluation at boundary of the interval:

$$w = \frac{2 \cdot L}{\pi \cdot D}$$

Now we generalize the Buffon's needle problem.



We view a general convex set $A \subset R^2$. That means that between two points of this set the whole connecting line segment belongs to this set too. Let it be that $(x_p, y_p) \in A$, then the angle α can be expressed with:

$$\cos\alpha = \frac{\begin{pmatrix} x_1 - x_p \\ y_1 - y_p \end{pmatrix} \cdot \begin{pmatrix} x_2 - x_p \\ y_2 - y_p \end{pmatrix}}{\left|\begin{pmatrix} x_1 - x_p \\ y_1 - y_p \end{pmatrix}\right| \cdot \left|\begin{pmatrix} x_2 - x_p \\ y_2 - y_p \end{pmatrix}\right|}$$

(x_1, y_1) and (x_2, y_2) are vectors from the intersection between the set's A boundary and the solution set of $(y - y_p)^2 + (x - x_p)^2 = \frac{L^2}{4}$. Only these two solutions may exist. Thus the probability that this needle intersects or touchs the boundary can be written:

$$w = \frac{1}{\pi F} \cdot \int\limits_A \alpha\, dy_p dx_p$$

We have to solve a two-dimensional integral. α must be in circular measure. The area of the set A is denoted with F. We must think that x_1, y_1, x_2, y_2 are functions of x_p and y_p, which must be inserted before the integration.

Chapter G.

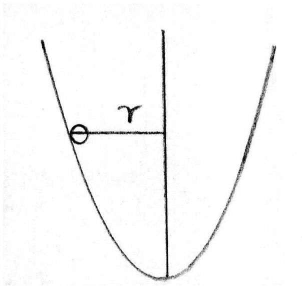
Mechanics

Contents

30. Orbits with constant velocity on bodies of rotation shells

1. The frictionless case

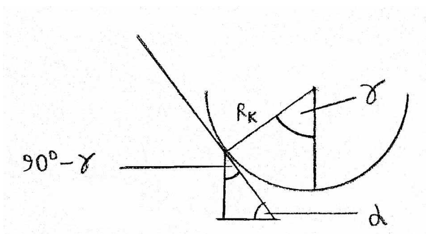
In a body of rotation is a ball.



A shell rotates, then a ball on the shell moves to a higher position. There are several questions. Is the ball in balance in this situation or does the ball fall down? Is there a free play? Are there areas in which the ball cannot stay? It's clear that the rotation velocity of the shell plays a major role in this situation. What is with other quantities? Which dependences are from this quantities? Are there quantities from which the situation is independent? The situation of a rotating shell is physical equivalent to a circling ball on a motionless shell. In all chapters the radius of the ball is very small compared to the measurements of the body of rotation. We want to determine the dependence from r and v (velocity) and the angular velocity w, too. Gravitation and centrifugal force must be in balance. The centrifugal force often acts in everyday life. Everybody knows of the centrifugal force for example at a round-about or driving a curve.

1.1. shell of the ball

At first the shell of the body of rotation shall be a shell of the ball. The angle of inclination of the inclined plane is α.

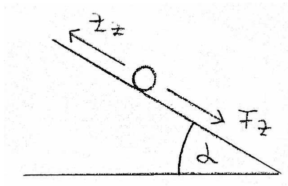


m = mass of the small interior ball

R_k = radius of the ball's shell (interior radius of the shell)

g = gravitation acceleration

$$\alpha = 90° - (90° - \gamma) = \gamma \qquad \text{with that} \qquad \alpha = \gamma$$

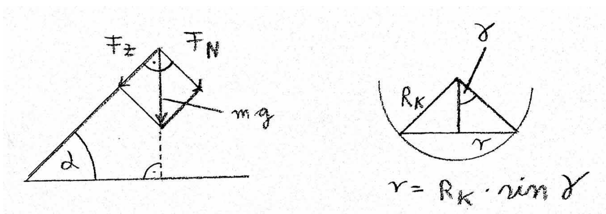


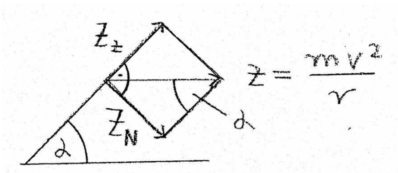
F_z = falling force

Z_z = centrifugal force against the falling force

$$F_z = mg \sin \alpha \qquad \text{and} \qquad Z_z = \frac{mv^2}{r} \cdot \cos \alpha$$

see the following figures:





F_N, Z_N are normal forces.

It follows:

$$Z_z = Z \cdot \cos \alpha \qquad Z_N = Z \cdot \sin \alpha$$

we obtain:

$$Z_z = \frac{mv^2}{r} \cdot \cos \alpha \qquad \alpha = \gamma$$

$$r = R_k \cdot \sin \alpha$$

insertion:

$$Z_z = \frac{mv^2}{R_k \cdot \sin \gamma} \cdot \cos \gamma$$

For F_z we get because of $\alpha = \gamma$:

$$F_z = mg \cdot \sin \gamma$$

For a stable orbit on the shell it must be $F_z = Z_z$ (balance), with that:

$$mg\sin\gamma = \frac{mv^2}{R_k \cdot \sin\gamma} \cdot \cos\gamma$$

transformed:

$$v^2 = g \cdot R_K \cdot \frac{\sin^2\gamma}{\cos\gamma} \qquad \text{with} \qquad \frac{\sin\gamma}{\cos\gamma} = \tan\gamma$$

We get:

$$v(\gamma) = \sqrt{g \cdot R_k \cdot \sin\gamma \cdot \tan\gamma}$$

$$w = \frac{v}{r} \qquad w = \text{angular velocity}$$

$$w(\gamma) = \frac{\sqrt{gR_k \sin\gamma \tan\gamma}}{R_k \sin\gamma} = \sqrt{\frac{gR_k \sin\gamma \tan\gamma}{R_k^2 \cdot \sin^2\gamma}}$$

$$= \sqrt{\frac{g\tan\gamma}{R_k \sin\gamma}} = \sqrt{\frac{g}{R_k \cdot \cos\gamma}}$$

$$\text{time of rotation:} \qquad U_t = \frac{2\pi}{w}$$

approximation:

For $\gamma \ll 90°$ it is valid $\sin\gamma \approx \tan\gamma$

It follows:

$$v \approx \sqrt{g \cdot R_k \cdot \sin^2\gamma} = \sin\gamma \cdot \sqrt{g \cdot R_K}$$

For γ near 90° we have $\sin\gamma \approx 1$:

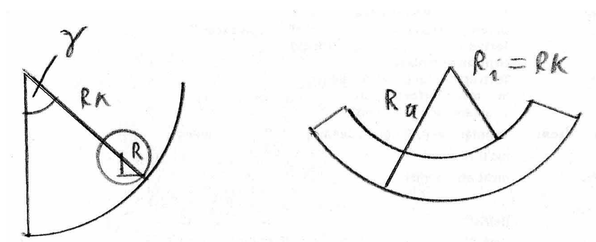
$$v \approx \sqrt{g \cdot R_K \cdot \tan\gamma}$$

If the barycenter is the midpoint of the ball, we can do the following.

R = radius of the small ball

$R_k = R_i$ =radius of the shell of the ball (interior radius)

R_a = exterior radius



With the figure we obtain the relation:

$$r_m = r \cdot \frac{\sin\gamma \cdot R_k - \sin\gamma \cdot R}{\sin\gamma \cdot R_k} \tag{1}$$

With this we get:

$$r_m = r \cdot \frac{R_k - R}{R_k}$$

r_m of a small ball is less smaller than r.

If we insert r_m instead of r into the equations, we get the angular velocity and the velocity at the barycenter. We assume that the small ball has a (local) constant density. With the barycenter's correction we obtain the barycenter's velocity:

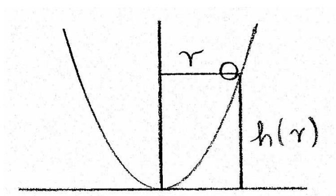
$$v_m = \sqrt{g \cdot r_m \cdot \tan\gamma} \qquad \text{with} \qquad r_m = R_k \cdot \sin\gamma \cdot \left(1 - \frac{R}{R_k}\right)$$

We get the velocity at the point of contact:

$$v = v_m \cdot \frac{R_k \cdot \sin\gamma}{r_m}$$

1.2. general body of rotation

Now we look at the shell of a general body of rotation.



$h(r)$ is the function of the body of rotation.

$$h'(r) = s = \tan\alpha = \text{slope}$$

α is the angle of inclination. We obtain as to the ball's shell:

$$F_z = mg \cdot \sin\alpha \qquad Z_z = \frac{mv^2}{r} \cdot \cos\alpha$$

It is valid:

$$\cos\alpha = \frac{1}{\sqrt{1 + \tan^2\alpha}} \qquad \frac{\sin\alpha}{\cos\alpha} = \tan\alpha$$

With this it follows:

$$\sin\alpha = \frac{\tan\alpha}{\sqrt{1 + \tan^2\alpha}}$$

Now we insert s and we get:

$$F_z = \frac{mgs}{\sqrt{1 + s^2}} \qquad Z_z = \frac{mv^2}{r\sqrt{1 + s^2}}$$

For a stable orbit it must be again $F_z = Z_z$ (balance), with that:

$$\frac{mv^2}{r\sqrt{1+s^2}} = \frac{mgs}{\sqrt{1+s^2}} \qquad \Rightarrow \qquad v^2 = g \cdot r \cdot s$$

With that it follows:

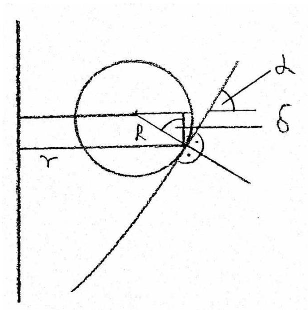
$$v = +\sqrt{g \cdot r \cdot s}$$

For the angular velocity we get:

$$w = \frac{v}{r} = \sqrt{\frac{grs}{r^2}} = \sqrt{\frac{g \cdot s}{r}}$$

Because of $s = h'(r)$ a transformation from v or w to r isn't possible in general. This must be done with a given function $h(r)$.

If the barycenter is in the midpoint of the ball, we can do the following.



R is the radius of the ball.

For the angle δ it is valid: $\delta = 180° - 90° - (90° - \alpha) = \alpha$ With the figure we obtain:

$$r_m = r \cdot \frac{r - R \cdot \sin\delta}{r}$$

With $\tan\alpha = s$ and $\sin\delta = \frac{\tan\delta}{\sqrt{1+\tan^2\delta}}$ we conclude:

$$r_m = r \cdot \frac{r - \frac{R \cdot s}{\sqrt{1+s^2}}}{r} \tag{2}$$

with $s = h'(r)$

With a small ball r_m is less smaller than r.

If we insert r_m instead of r into the equations, we get the angular velocity and the velocity at the barycenter of the ball. We assume that the small ball has a (local) constant density. This is valid to all chapters. The exact calculation with barycenter we get the barycenter's velocity (small ball):

$$v_m = \sqrt{g \cdot r_m \cdot h'(r)}$$

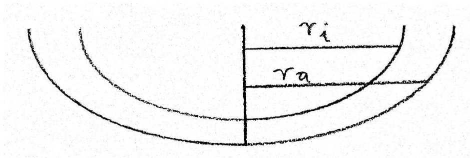
Angular velocity:

$$w = \frac{v_m}{r_m} = \sqrt{\frac{g \cdot h'(r)}{r_m}}$$

With

$$v = v_m \cdot \frac{r}{r_m}$$

we obtain the velocity at the point of contact.

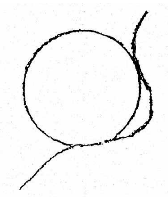


The rotation body's function $h(r)$ is always the interior one. Wir define:

$r = r_i =$ interior radius

$r_a =$ exterior radius

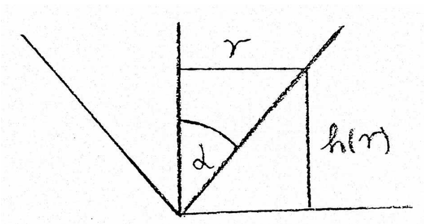
$h(r)$ must have only **one** touching point with the ball. With that no "troughs" with two touching points see figure:



From the general body of rotation we can derive different special cases.

1.3. Cone of rotation

Now we view a cone of rotation:



$\alpha =$ apex angle of the cone

Because of the figure we recognize:

$$h(r) = \frac{r}{\tan\alpha} \qquad \Rightarrow \qquad s = h'(r) = \frac{1}{\tan\alpha}$$

If we insert in the general formula $v = \sqrt{grs}$ for $h(r)$ and s, we get the velocity:

$$v = \sqrt{\frac{g \cdot r}{\tan\alpha}}$$

Transformed to r:

$$r = \frac{v^2 \cdot \tan\alpha}{g}$$

We obtain the angular velocity:

$$w = \frac{v}{r} = \sqrt{\frac{gr}{\tan\alpha \cdot r^2}} = \sqrt{\frac{g}{r \cdot \tan\alpha}}$$

Transformed to r again:

$$r = \frac{g}{w^2 \cdot \tan\alpha}$$

See Sommerfeld [5], §14.1, p.73.

Because of the using of equation (2) we calculate now $\frac{s}{\sqrt{1+s^2}}$.

$$\frac{s}{\sqrt{1+s^2}} = \frac{1}{\tan\alpha \cdot \sqrt{1 + \frac{1}{\tan^2\alpha}}} = \frac{1}{\sqrt{\tan^2\alpha + 1}} = \cos\alpha$$

With equation (2) it follows:

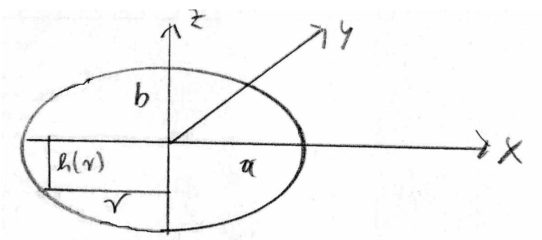
$$r_m = r \cdot \frac{r - R \cdot \cos\alpha}{r}$$

The consideration of barycenter leads to:

$$v_m = \sqrt{\frac{g \cdot r_m}{\tan\alpha}} \qquad w = \sqrt{\frac{g}{r_m \cdot \tan\alpha}} \qquad v = v_m \cdot \frac{r}{r_m}$$

1.4. Ellipsoid of revolution

We now treat the ellipsoid of revolution. The semi-axis a is on the x-axis and on the y-axis, the semi-axis b on the z-axis. The z-axis shall be axis of rotation. The semi-axes a and b can be arbitrary.



The canonical equation of the ellipse is:

$$\frac{r^2}{a^2} + \frac{h^2}{b^2} = 1$$

Transformed:

$$h^2 = b^2 \cdot \left(1 - \frac{r^2}{a^2}\right) = b^2 \cdot \frac{a^2 - r^2}{a^2}$$

With this we follow:

$$h(r) = \pm\frac{b}{a} \cdot \sqrt{a^2 - r^2}$$

Differentiation to r:

$$s = h'(r) = \frac{b}{a} \cdot \frac{1}{2} \cdot \frac{-2r}{\pm\sqrt{a^2 - r^2}} = \pm\frac{b}{a} \cdot \frac{r}{\sqrt{a^2 - r^2}}$$

With $v = +\sqrt{grs}$ follows:

$$v = +\sqrt{g \cdot r \cdot \frac{b}{a} \cdot \frac{r}{\sqrt{a^2 - r^2}}}$$

at last:

$$v = r \cdot \sqrt{\frac{b \cdot g}{a \cdot \sqrt{a^2 - r^2}}} \tag{3}$$

The angular velocity w can be written as:

$$w = \frac{v}{r} = \sqrt{\frac{bg}{a \cdot \sqrt{a^2 - r^2}}} \tag{4}$$

Approximation for $r \ll a$:

$$v \approx \frac{r}{a} \cdot \sqrt{bg} \qquad w = \frac{v}{r} \approx \frac{1}{a} \cdot \sqrt{bg}$$

Now we must search the relation between r and r_m. In the case of the ellipsoid of revolution we get for the expression:

$$\frac{s}{\sqrt{1+s^2}} = \frac{br}{a \cdot \sqrt{a^2 - r^2}} \cdot \frac{1}{\sqrt{1 + \frac{b^2r^2}{a^2 \cdot (a^2 - r^2)}}}$$

$$= \frac{br}{a \cdot \sqrt{a^2 - r^2 + \frac{b^2r^2}{a^2}}} = \frac{br}{\sqrt{a^4 - r^2a^2 + b^2r^2}}$$

We insert this term in equation (2):

$$r_m = r \cdot \frac{r - \frac{brR}{\sqrt{a^4 - r^2a^2 + b^2r^2}}}{r}$$

$$r_m = r\left(1 - \frac{bR}{\sqrt{a^4 - r^2a^2 + b^2r^2}}\right) \tag{5}$$

With barycenter's correction we get:

$$v_m = r_m \cdot \sqrt{\frac{b \cdot g}{a \cdot \sqrt{a^2 - r^2}}} \qquad v = v_m \cdot \frac{r}{r_m}$$

The angular velocity is the same.

1.5. A ball as special case of the ellipsoid of revolution:

It is possible to treat the ball as special case of the ellipsoid of revolution with the ball's radius $R_k = a = b$, too. With specialization of the formulas (3) and (4) we get:

$$v = r \cdot \sqrt{\frac{g}{\sqrt{R_k^2 - r^2}}} \qquad w = \sqrt{\frac{g}{\sqrt{R_k^2 - r^2}}}$$

Approximation for $r \ll R_k$:

$$v \approx r \cdot \sqrt{\frac{g}{R_k}} \qquad w \approx \sqrt{\frac{g}{R_k}}$$

Specialization of (5):

$$r_m = r \cdot \left(1 - \frac{R}{R_k}\right)$$

A calculation with barycenter yields:

$$v_m = r_m \cdot \sqrt{\frac{g}{\sqrt{R_k^2 - r^2}}} \qquad v = v_m \cdot \frac{r}{r_m}$$

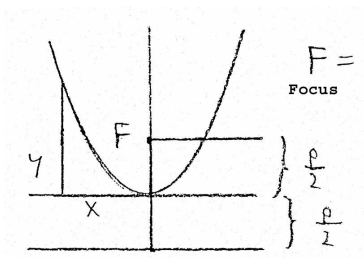
The angular velocity remains unchanged.

1.6. Paraboloid

We now turn to the paraboloid of revolution. For the parabola it is valid:

$$x^2 = 2py$$

We look at the following figure:



Notations:

focal length $f = \frac{p}{2}$ $\qquad x = r$, $y = h$

$$r^2 = 2h \cdot 2f = 4hf$$

With this we obtain:

$$h = \frac{r^2}{4f} \qquad s = h'(r) = \frac{r}{2f}$$

With $v = +\sqrt{grs}$ follows:

$$v = \sqrt{gr \cdot \frac{r}{2f}} = r \cdot \sqrt{\frac{g}{2f}}$$

For the angular velocity:

$$w = \frac{v}{r} = \sqrt{\frac{g}{2f}}$$

w is independent of r.

We transform the velocity equation to:

$$r = v \cdot \sqrt{\frac{2f}{g}}$$

To the relation between r and r_m:

$$\frac{s}{\sqrt{1+s^2}} = \frac{r}{2f \cdot \sqrt{1+\frac{r^2}{4f^2}}} = \frac{r}{\sqrt{4f^2+r^2}}$$

With equation (2) we get:

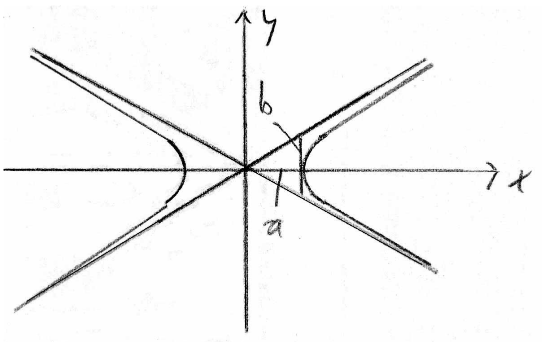
$$r_m = r \cdot \left(1 - \frac{R}{\sqrt{4f^2+r^2}}\right)$$

A calculation with the small ball's barycenter leads to:

$$v_m = r_m \cdot \sqrt{\frac{g}{2f}} \qquad v = v_m \cdot \frac{r}{r_m}$$

1.7. Hyperboloid

We now view the orbit on the hyperboloid of revolution's shell:

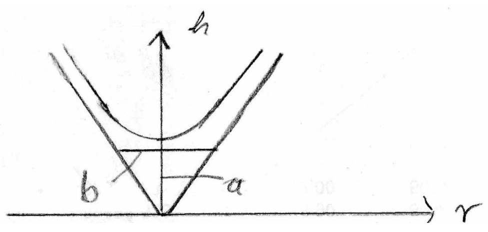


Canonical equation of the hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Wise are the notations:

$y = r$, $x = h$



We insert in the canonical equation:

$$\frac{h^2}{a^2} - \frac{r^2}{b^2} = 1$$

Transformed:

$$h^2 = a^2 \cdot \left(1 + \frac{r^2}{b^2}\right) = \frac{a^2}{b^2} \cdot (b^2 + r^2)$$

with that:

$$h = \frac{a}{b} \cdot \sqrt{b^2 + r^2}$$

The slope s can be calculated as:

$$s = h'(r) = \frac{a}{b} \cdot \frac{1}{2} \cdot \frac{2r}{\sqrt{b^2 + r^2}} = \frac{a}{b} \cdot \frac{r}{\sqrt{b^2 + r^2}}$$

With $v = +\sqrt{grs}$ follows:

$$v = \sqrt{gr \cdot \frac{a}{b} \cdot \frac{r}{\sqrt{b^2 + r^2}}} = r \cdot \sqrt{\frac{ag}{b \cdot \sqrt{b^2 + r^2}}} \tag{6}$$

One approximation for $a, b \ll r$:

$$v \approx r \cdot \sqrt{\frac{ag}{br}} = \sqrt{\frac{arg}{b}}$$

$$\text{asymptote:} \qquad h'(r) \approx \frac{a}{b}$$

angular velocity:

$$w = \frac{v}{r} = \sqrt{\frac{ag}{b \cdot \sqrt{b^2 + r^2}}}$$

for $a, b \ll r$:

$$w \approx \sqrt{\frac{ag}{br}}$$

Now we determine the ball's midpoint velocity:

$$\frac{s}{\sqrt{1 + s^2}} = \frac{ar}{b \cdot \sqrt{b^2 + r^2}} \cdot \frac{1}{\sqrt{1 + \frac{a^2 r^2}{b^2 \cdot (b^2 + r^2)}}}$$

$$= \frac{ar}{b \cdot \sqrt{b^2 + r^2 + \frac{a^2 r^2}{b^2}}} = \frac{ra}{\sqrt{b^4 + r^2 b^2 + a^2 r^2}}$$

From (2) we conclude:

$$r_m = r \cdot \frac{r - \frac{Rra}{\sqrt{b^4+r^2b^2+r^2a^2}}}{r}$$

at last:

$$r_m = r \cdot \left(1 - \frac{Ra}{\sqrt{b^4 + r^2b^2 + r^2a^2}}\right)$$

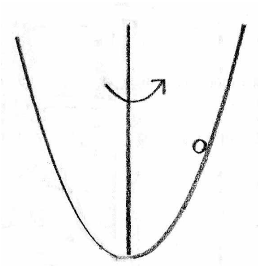
With this barycenter's correction we get:

$$v_m = r_m \cdot \sqrt{\frac{a \cdot g}{b \cdot \sqrt{b^2 + r^2}}} \qquad v = v_m \cdot \frac{r}{r_m}$$

The angular velocity is the same.

Generalizations:

a): Instead shell of rotation bodies we can view an arm that has got the form of the rotation body.



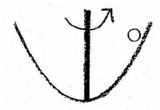
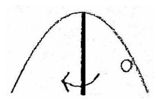
This arm rotates and the body is in balance between gravitation and centrifugal force. If the body glides frictionless, the same equations are valid. (We especially achieve this with rotation bodies as balls, cylinder.)

b): The frictionless case in medium can be treated in the same way, if we insert instead of g, $\frac{g \cdot (\varphi_K - \varphi_F)}{\varphi_K}$ $= g \cdot \left(1 - \frac{\varphi_F}{\varphi_K}\right)$ (see for example Budo [2] §16 p.85) in the equations. φ_F is the medium's density (liquid or gas) and φ_K is the body's density that is in balance (not the density of the rotation body's shell).

reasons:

There are only gravitation and centrifugal force in this balance case. The centrifugal force is independent from medium, in the orbit case the centrifugal force is only dependent from r and v, we can see it in the derivation of the centrifugal force, but not from medium. The gravitation acceleration changes from g to $g \cdot \frac{\varphi_K - \varphi_F}{\varphi_K}$. Only with the gravitation there is a change, because of this we can insert $g \cdot \left(1 - \frac{\varphi_F}{\varphi_K}\right)$ instead of g.

In the case $\varphi_F > \varphi_K$ the rotation body's shell (respectively the arm) must be turned round.

instead of  then 

Then the same equations are valid again.

2. The friction case

We introduce the following notations:

m = mass of the ball
R = radius of the ball
g = gravitation acceleration
F_R = friction force
μ = general friction coefficient
μ_H = static friction coefficient
μ_G = sliding friction coefficient
μ' = rolling friction coefficient
α = angle of inclination of the plane

We explain the factor δ in the following way:

$$\delta := \begin{cases} \frac{5}{7} & \text{if} \quad \frac{\mu'}{R} < \mu_H \text{ (rolling)} \\ 1 & \text{if} \quad \mu_H < \frac{\mu'}{R} \text{ (sliding)} \\ \frac{5}{7} \text{ or } 1 & \text{if} \quad \frac{\mu'}{R} = \mu_H \neq 0 \text{ (decision is open)} \\ 1 & \text{if} \quad 0 = \frac{\mu'}{R} = \mu_H \text{ (frictionless)} \end{cases}$$

See Assmann [1], volume 1, chapter 11.10, p. 265.

The acceleration b on the inclined plane is:
$b = \delta g \cdot \sin\alpha$, the velocity v and the distance s are $v = \delta g \sin\alpha \cdot t$ and $s = \frac{1}{2} \cdot \delta g \cdot t^2 \cdot \sin\alpha$.

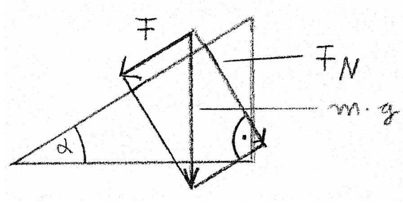
If a ball is rolling on the inclined plane, it is $\delta = \frac{5}{7}$ see for example Budo [2], §57, p.302, equation (8). The moment of inertia J of a ball is $J = \frac{2}{5} \cdot mR^2$. The general formula for rolling on the inclined plane can be written as:

$$b = \frac{mg\sin\alpha}{m + \frac{J}{R^2}} \tag{7}$$

One derivation can be found at * (at the end of the chapter) or we can see this with Budo [2], §57, p.302, equations (5)-(7). With the moment of inertia of the ball we get:

$$b = \frac{mg\sin\alpha}{m + \frac{2}{5} \cdot m} = \frac{5}{7} \cdot g\sin\alpha \tag{8}$$

Then we get in the case of rolling $\delta = \frac{5}{7}$.



The forces are (see fig.):

$$F = mg \sin \alpha \qquad F_N = mg \cos \alpha$$

friction force:

$$F_R = \mu \cdot F_N = mg\mu \cos \alpha$$

In the friction case is:

$$\mu = \begin{cases} \frac{\mu'}{R} & \text{if} \quad \mu_H > \frac{\mu'}{R} \text{ (rolling)} \\ \mu_G & \text{if} \quad \frac{\mu'}{R} > \mu_H \text{ (sliding)} \\ \mu_G \text{ or } \frac{\mu'}{R} & \text{if} \quad \mu_H = \frac{\mu'}{R} \text{ (decision is open)} \end{cases}$$

See Assmann [1], volume 1, chapter 11.10, p.265. In the case of friction is:

$$F = mg \cdot (\delta \sin \alpha - \mu \cos \alpha) \qquad b = g \cdot (\delta \sin \alpha - \mu \cos \alpha)$$

$$v = gt \cdot (\delta \sin \alpha - \mu \cos \alpha) \qquad s = \frac{1}{2} \cdot gt^2 (\delta \sin \alpha - \mu \cos \alpha)$$

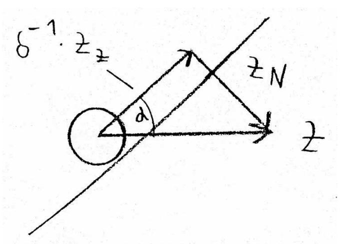
If we consider orbits on rotation bodies shells and Z is the centrifugal force, the inequality of stable orbits are see Sommerfeld [5], volume 1, §14.1, p.73:

$$Z_z - F_z \leq Z_R + F_R \qquad F_z - Z_z \leq Z_R + F_R$$

with:

$$F_z = \delta mg \sin \alpha \qquad F_R = mg \cos \alpha \cdot \mu$$

To the centrifugal forces (see fig.):



$Z = \frac{mv^2}{r}$ = centrifugal force

$$Z_N = Z \cdot \sin \alpha \qquad Z_z = \delta \cdot \frac{mv^2}{r} \cdot \cos \alpha$$

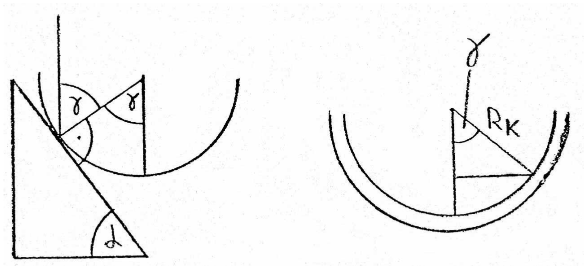
Z_R= friction force of the centrifugal force

$$Z_R = \mu \cdot Z_N = \mu \cdot \frac{mv^2}{r} \cdot \sin \alpha$$

2.1. Orbits on the ball's shell

Now we view orbits on the ball's shell with friction.

R_K = ball's radius (interior radius of ball's shell)
γ = height angle of the ball's shell
$\alpha = \gamma$



The limit velocities of stable orbits can be obtained with the following equations:

$$Z_z - F_z = Z_R + F_R \tag{9}$$

$$F_z - Z_z = Z_R + F_R \tag{10}$$

It is $r = \sin\gamma \cdot R_K$ and $\alpha = \gamma$. With (9) it follows:

$$\delta \cdot \frac{mv_{max}^2 \cos\gamma}{\sin\gamma \cdot R_K} - \delta mg \sin\gamma = \mu \frac{mv_{max}^2 \sin\gamma}{\sin\gamma \cdot R_K} + mg\cos\gamma \cdot \mu$$

with $\tan\gamma = \frac{\sin\gamma}{\cos\gamma}$ we get:

$$\frac{\delta mv_{max}^2}{\tan\gamma \cdot R_K} - \delta mg\sin\gamma = \frac{\mu mv_{max}^2}{R_K} + mg\cos\gamma \cdot \mu$$

ordered:

$$v_{max}^2 \cdot \left(\frac{\delta m}{\tan\gamma \cdot R_K} - \frac{\mu m}{R_K}\right) = mg\cos\gamma \cdot \mu + \delta mg\sin\gamma$$

Solving:

$$v_{max}^2 = \frac{g\sin\gamma \cdot \left(\delta + \frac{\mu}{\tan\gamma}\right)}{\left(\frac{\delta}{\tan\gamma} - \mu\right) \cdot \frac{1}{R_K}}$$

at last:

$$v_{max} = \sqrt{\frac{gR_K \sin\gamma \cdot (\tan\gamma \cdot \delta + \mu)}{\delta - \mu\tan\gamma}}$$

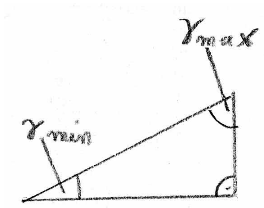
With equation (10) we can get with the same method an expression of the minimum velocity:

$$v_{min} = \sqrt{\frac{gR_K \sin\gamma \cdot (\delta\tan\gamma - \mu)}{\delta + \tan\gamma \cdot \mu}}$$

In general it is:

$$\text{angular velocity} = w = \frac{v}{r} = \frac{v}{\sin\gamma \cdot R_K}$$

It exists one γ_{max} and one γ_{min}, which can be determined in the following way with the maximum and minimum velocity.



$v_{max} = \infty$ if $\delta - \mu \tan\gamma = 0$, it follows:

$$\tan\gamma_{max} = \frac{\delta}{\mu}$$

$v_{min} = 0$ if $\delta \tan\gamma - \mu = 0$ from this we get:

$$\tan\gamma_{min} = \frac{\mu}{\delta}$$

We obtain in addition:

$$\gamma_{max} + \gamma_{min} = 90^\circ$$

With that the ball can move only in the interval $[\gamma_{min}, \gamma_{max}]$ in a stable orbit. Outside from this interval no stable orbits with constant height angle exist.

If we insert zero for μ we get from the maximum velocity and the minimum velocity the same expression

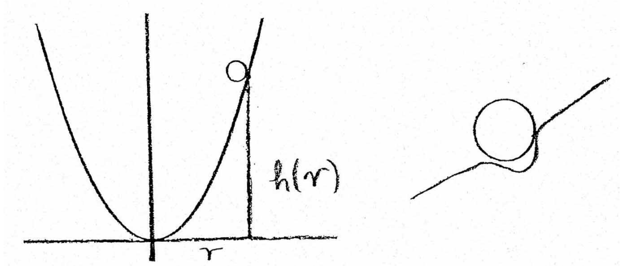
$$v = \sqrt{gR_K \sin\gamma \tan\gamma}$$

for the frictionless case. This is the formula of chapter 1.

The relation between r and r_m is the same as in chapter 1 for the frictionless case.

2.2. The orbit on the general rotation body's shell

$h(r)$ = function of the rotation body see figure, $s = h'(r)$.
$h(r)$ has got only **one** touching point with the ball. With that there is no "trough".



We always mention the function of the rotation body the one of the interior shell.
r = interior radius

The following is valid:

$$s = \tan\alpha \qquad \cos\alpha = \frac{1}{\sqrt{1+s^2}} \qquad \sin\alpha = \frac{s}{\sqrt{1+s^2}}$$

It follows:

$$F_z = \delta mg \cdot \frac{s}{\sqrt{1+s^2}} \qquad F_R = \frac{mg\mu}{\sqrt{1+s^2}}$$

$$Z_z = \delta \cdot \frac{mv^2}{r \cdot \sqrt{1+s^2}} \qquad Z_R = \frac{\mu m v^2 s}{r \cdot \sqrt{1+s^2}}$$

Now we again use the equation (9).

$$\frac{\delta m v_{max}^2}{r \cdot \sqrt{1+s^2}} - \frac{\delta mgs}{\sqrt{1+s^2}} = \frac{\mu m v_{max}^2 s}{r \cdot \sqrt{1+s^2}} + \frac{mg\mu}{\sqrt{1+s^2}}$$

simplified:

$$\frac{v_{max}^2 \cdot \delta}{r} - \frac{v_{max}^2 \cdot \mu s}{r} = g\mu + \delta g s$$

$$\frac{v_{max}^2}{r} \cdot (\delta - \mu s) = g \cdot (\mu + \delta s)$$

at last:

$$v_{max} = \sqrt{\frac{rg \cdot (\delta s + \mu)}{\delta - \mu s}}$$

On the same way we can derive a formula of the minimum velocity with the equation (10). This formula is:

$$v_{min} = \sqrt{\frac{rg \cdot (\delta s - \mu)}{\mu s + \delta}}$$

We here have one s_{max} and one s_{min} that can be calculated with the equations of maximum velocity and minimum velocity.

$v_{max} = \infty$ if $\delta - \mu s = 0$ it follows:

$$s_{max} = \frac{\delta}{\mu} \tag{11}$$

$v_{min} = 0$ if $\delta s - \mu = 0$ we obtain:

$$s_{min} = \frac{\mu}{\delta} \tag{12}$$

We find:

$$s_{min} \cdot s_{max} = 1$$

In the frictionless case with $\mu = 0$ we get:

$$v_{max} = v_{min} = \sqrt{grs}$$

This formula is the same as the one in chapter 1.

The relation between r and r_m is the same as in chapter 1.

angular velocity $= w = \frac{v}{r}$

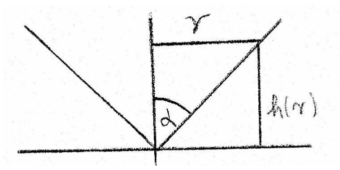
We can derive from the general body of rotation different special cases.

2.3. The cone of revolution

In this case we have:

$$h(r) = \frac{r}{\tan\alpha} \qquad s = h'(r) = \frac{1}{\tan\alpha}$$

$\alpha =$ apex angle of the cone



We insert in the formula of the maximum velocity of the general rotation body:

$$v_{max} = \sqrt{\frac{rg\cdot(\delta s+\mu)}{\delta-\mu s}} = \sqrt{\frac{rg\cdot\left(\frac{\delta}{\tan\alpha}+\mu\right)}{\delta-\frac{\mu}{\tan\alpha}}}$$

at last:

$$v_{max} = \sqrt{\frac{rg\cdot(\delta+\tan\alpha\cdot\mu)}{\delta\tan\alpha-\mu}}$$

In the same way we get from the minimum velocity formula of the general rotation body the minimum velocity of the cone of revolution:

$$v_{min} = \sqrt{\frac{rg\cdot(\delta-\mu\tan\alpha)}{\mu+\delta\tan\alpha}}$$

From the general limit conditions $s_{max} = \frac{\delta}{\mu}$ and $s_{min} = \frac{\mu}{\delta}$ we follow $\frac{1}{\tan\alpha_{max}} = \frac{\delta}{\mu}$ and $\frac{1}{\tan\alpha_{min}} = \frac{\mu}{\delta}$, on this way we obtain:

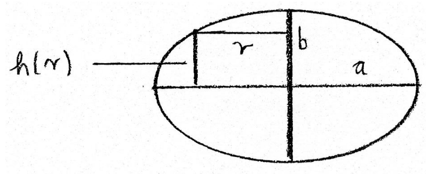
$$\tan\alpha_{max} = \frac{\mu}{\delta} \qquad \tan\alpha_{min} = \frac{\delta}{\mu}$$

At the cone of revolution there are no r_{min},r_{max} but it exists α_{min} and α_{max}. If we insert $\mu = 0$ in both velocity formulas, we get the formula $v = \sqrt{\frac{rg}{\tan\alpha}}$ for the frictionless case. We know this formula from the chapter before. The relation between r and r_m is the same as in the chapter before with the frictionless case.

angular velocity $= w = \frac{v}{r}$

2.4. The ellipsoid of revolution

We look at the figure:



From the canonical equation of the ellipse with the semi-axis a, b we get (see the chapter before, too): $b =$ axis of revolution

$$h(r) = \pm\frac{b}{a} \cdot \sqrt{a^2 - r^2} \qquad s = h'(r) = \pm\frac{b}{a} \cdot \frac{r}{\sqrt{a^2 - r^2}}$$

We again use the formula of the maximum velocity of the general rotation body:

$$v_{max} = \sqrt{\frac{rg \cdot (\delta s + \mu)}{\delta - \mu s}} = \sqrt{\frac{rg \cdot \left(\delta \cdot \frac{b}{a} \cdot \frac{r}{\sqrt{a^2-r^2}} + \mu\right)}{\delta - \frac{b}{a} \cdot \frac{\mu r}{\sqrt{a^2-r^2}}}}$$

at last:

$$v_{max} = \sqrt{\frac{rg \cdot (\delta br + \mu a \cdot \sqrt{a^2 - r^2})}{\delta a \cdot \sqrt{a^2 - r^2} - b\mu r}}$$

In the same way we can derive from the formula of the minimum velocity of the general rotation body the following equation:

$$v_{min} = \sqrt{\frac{rg \cdot (\delta br - \mu a \cdot \sqrt{a^2 - r^2})}{\mu br + \delta a \cdot \sqrt{a^2 - r^2}}}$$

With the general limit conditions $s_{max} = \frac{\delta}{\mu}$ and $s_{min} = \frac{\mu}{\delta}$ we get the special conditions of the ellipsoid of revolution:

$$\frac{\delta}{\mu} = \frac{b}{a} \cdot \frac{r_{max}}{\sqrt{a^2 - r_{max}^2}} \qquad \frac{\mu}{\delta} = \frac{b}{a} \cdot \frac{r_{min}}{\sqrt{a^2 - r_{min}^2}}$$

Now, we transform to r_{max}:

$$\delta^2 a^2 \cdot (a^2 - r_{max}^2) = \mu^2 b^2 r_{max}^2$$

$$\delta^2 a^4 - \delta^2 a^2 r_{max}^2 = \mu^2 b^2 r_{max}^2$$

$$r_{max}^2 = \frac{\delta^2 a^4}{\mu^2 b^2 + \delta^2 a^2}$$

with that:

$$r_{max} = \frac{\delta a^2}{\sqrt{\mu^2 b^2 + \delta^2 a^2}} < a$$

The other limit condition can be tranformed to r_{min} in the same way:

$$r_{min} = \frac{a^2\mu}{\sqrt{b^2\delta^2 + a^2\mu^2}} < a$$

For $\mu \ll 1$ it follows now:

$$r_{min} \approx \frac{a^2\mu}{b\delta} \qquad r_{max} \approx a$$

Insertion of $\mu = 0$ in the formulas of maximum and minimum velocity leads to the known formula for the frictionless case (see chapter 1):

$$v = r \cdot \sqrt{\frac{gb}{a \cdot \sqrt{a^2 - r^2}}}$$

The equation between r and r_m is the same as in chapter 1.

angular velocity $= w = \frac{v}{r}$

2.5. Special case ball $(a = b = R_K)$

With specializing the equations of the ellipsoid of revolution we obtain the equations of the ball in dependence from r and not from the height angle. With $a = b = R_K$ we get:

$$v_{max} = \sqrt{\frac{rg \cdot (\delta r + \mu \cdot \sqrt{R_K^2 - r^2})}{\delta \cdot \sqrt{R_K^2 - r^2} - \mu r}}$$

$$v_{min} = \sqrt{\frac{rg \cdot (\delta r - \mu \cdot \sqrt{R_K^2 - r^2})}{\mu r + \delta \cdot \sqrt{R_K^2 - r^2}}}$$

$$r_{max} = \frac{\delta \cdot R_K}{\sqrt{\mu^2 + \delta^2}} \qquad r_{min} = \frac{R_K \cdot \mu}{\sqrt{\delta^2 + \mu^2}}$$

For $\mu \ll 1$ is $r_{min} \approx \frac{R_K \cdot \mu}{\delta}$ and $r_{max} \approx R_K$. For the frictionless case $\mu = 0$ we get with the velocity formulas the known formula

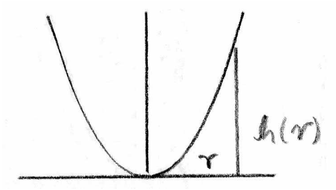
$$v = r \cdot \sqrt{\frac{g}{\sqrt{R_K^2 - r^2}}}$$

from chapter 1 again.
The relation between r and r_m is the same as in chapter 1.

angular velocity $= w = \frac{v}{r}$

2.6. The paraboloid

Now we use the equations of the general rotation body to the paraboloid of revolution. f is the focal distance of the paraboloid.



For the paraboloid is valid see chapter 1:

$$h(r) = \frac{r^2}{4f} \qquad s = h'(r) = \frac{r}{2f}$$

Now we insert the parabola function to the formula of the maximum velocity of the general rotation body.

$$v_{max} = \sqrt{\frac{rg \cdot (\delta s + \mu)}{\delta - \mu s}} = \sqrt{\frac{rg \cdot \left(\delta \cdot \frac{r}{2f} + \mu\right)}{\delta - \frac{\mu r}{2f}}}$$

With that it follows:

$$v_{max} = \sqrt{\frac{rg \cdot (\delta r + 2f\mu)}{2f\delta - \mu r}}$$

In the same way we get from the minimum velocity formula the minimum velocity of the paraboloid:

$$v_{min} = \sqrt{\frac{rg \cdot (\delta r - 2f\mu)}{\mu r + 2f\delta}}$$

From the general limit condition $s_{max} = \frac{\delta}{\mu}$ and $s_{min} = \frac{\mu}{\delta}$ we obtain with insertion to s:

$$\frac{r_{max}}{2f} = \frac{\delta}{\mu} \qquad \text{with that:} \qquad r_{max} = 2 \cdot \frac{\delta f}{\mu}$$

and

$$\frac{r_{min}}{2f} = \frac{\mu}{\delta} \qquad \text{with that:} \qquad r_{min} = 2 \cdot \frac{f\mu}{\delta}$$

In the case $\mu = 0$ we get with the help of the velocity formulas the known formula for the frictionless case (chapter 1):

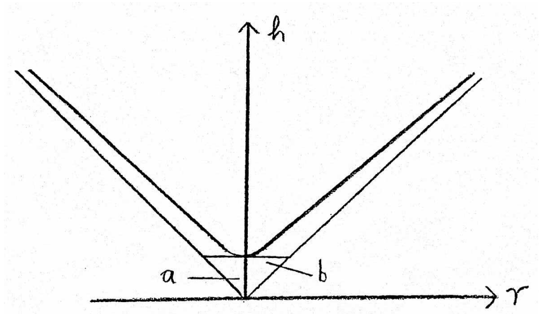
$$v = r \cdot \sqrt{\frac{g}{2f}}$$

The equation between r and r_m is the same as in chapter 1.

angular velocity $= w = \frac{v}{r}$

2.7. The hyperboloid

Under consideration of the figure and the canonical equation of the hyperbola we obtain as in chapter 1: $a, b =$ semi axes



$$h(r) = \frac{a}{b} \cdot \sqrt{b^2 + r^2} \qquad s = h'(r) = \frac{a}{b} \cdot \frac{r}{\sqrt{b^2 + r^2}}$$

If we insert the hyperbola equation in the formula of the maximum velocity (general rotation body), we get:

$$v_{max} = \sqrt{\frac{rg \cdot (\delta s + \mu)}{\delta - \mu s}} = \sqrt{\frac{rg \cdot \left(\delta \cdot \frac{a}{b} \cdot \frac{r}{\sqrt{b^2+r^2}} + \mu\right)}{\delta - \frac{\mu r a}{b \cdot \sqrt{b^2+r^2}}}}$$

with that:

$$v_{max} = \sqrt{\frac{rg \cdot (\delta a r + \mu b \cdot \sqrt{b^2 + r^2})}{\delta b \cdot \sqrt{b^2 + r^2} - \mu r a}}$$

In the same way we can conclude with the minimum velocity formula of the rotation body:

$$v_{min} = \sqrt{\frac{rg \cdot (\delta a r - \mu b \sqrt{b^2 + r^2})}{\mu a r + \delta b \sqrt{b^2 + r^2}}}$$

If $b \ll r$, we can follow:

$$s = h'(r) = \frac{a}{b} \cdot \frac{r}{\sqrt{b^2 + r^2}} \approx \frac{a}{b}$$

Now it follows for $b \ll r$:

$$v_{max} \approx \sqrt{\frac{rg \cdot (\delta \cdot \frac{a}{b} + \mu)}{\delta - \frac{\mu a}{b}}} = \sqrt{\frac{rg \cdot (\delta a + \mu b)}{\delta b - \mu a}}$$

With the minimum velocity formula we can calculate in the same way in the case $b \ll r$:

$$v_{min} \approx \sqrt{\frac{rg \cdot (\delta a - \mu b)}{\mu a + \delta b}}$$

If we insert the formula of s into the general limit conditions $s_{max} = \frac{\delta}{\mu}$ and $s_{min} = \frac{\mu}{\delta}$, we now get:

$$\frac{a}{b} \cdot \frac{r_{max}}{\sqrt{b^2 + r_{max}^2}} = \frac{\delta}{\mu} \qquad \frac{a}{b} \cdot \frac{r_{min}}{\sqrt{b^2 + r_{min}^2}} = \frac{\mu}{\delta}$$

We transform the first equation:

$$a^2\mu^2 r_{max}^2 = \delta^2 \cdot (b^2 + r_{max}^2) \cdot b^2$$

$$a^2\mu^2 r_{max}^2 = \delta^2 b^4 + \delta^2 b^2 r_{max}^2$$

$$r_{max}^2 = \frac{\delta^2 b^4}{a^2\mu^2 - \delta^2 b^2}$$

at last:

$$r_{max} = \frac{\delta b^2}{\sqrt{a^2\mu^2 - \delta^2 b^2}}$$

From the equation of r_{min} with a similar calculation we obtain:

$$r_{min} = \frac{\mu b^2}{\sqrt{a^2\delta^2 - \mu^2 b^2}}$$

As approximation for $\mu \ll \delta \leq 1$:

$$r_{min} \approx \frac{\mu b^2}{a\delta} \qquad r_{max} \approx b$$

For $\mu = 0$ the velocity formulas lead to the known formula of chapter 1 for the frictionless case:

$$v = r \cdot \sqrt{\frac{ga}{b \cdot \sqrt{b^2 + r^2}}}$$

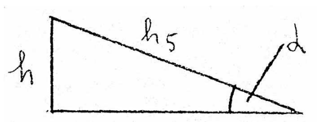
For r and r_m the same relation as in chapter 1 is valid.

angular velocity $= w = \frac{v}{r}$

to *: The derivation of formula (7):

Notations:

kinetic energy $= E_{kin} = \frac{mv^2}{2}$
potential energy $= E_{pot} = mgh$ with $h = h_s \sin\alpha$ see fig.
rotational energy $= E_{rot} = \frac{Jw^2}{2}$ $\qquad J =$ moment of inertia



law of conservation of energy:

$$\frac{mv^2}{2} + \frac{Jw^2}{2} = mgh$$

We assume that the body is only rolling not gliding. Then it is valid:
angular velociy $= w = \frac{v}{R}$ $\quad R =$ radius of the rolling body

$$\frac{mv^2}{2} + \frac{Jv^2}{2R^2} = mgh_s \sin\alpha$$

With transformation we get:

$$v = \sqrt{\frac{2mgh_s \sin\alpha}{m + \frac{J}{R^2}}}$$

Because the rolling body has the constant acceleration $b = g \sin\alpha$, there is a uniform accelerated motion. For this motion it is valid: $v^2 = 2h_s \cdot b$ transformed to $b = \frac{v^2}{2h_s}$. Now we insert the equation of v:

$$b = \frac{2mgh_s \sin\alpha}{2h_s \cdot \left(m + \frac{J}{R^2}\right)} = \frac{mg \sin\alpha}{m + \frac{J}{R^2}}$$

Then we have the wanted formula.

3. Inversions

3.1. Inversions to the frictionless case

The shell of the ball:

We take the notation of chapter 1.1.

angular velocity:

$$w = \sqrt{\frac{g}{R_k \cdot \cos\gamma}} \qquad \Rightarrow \qquad w^2 = \frac{g}{R_k \cdot \cos\gamma}$$

transformed to $\cos\gamma$:

$$\cos\gamma = \frac{g}{R_k \cdot w^2}$$

velocity formula:

$$v^2 = g \cdot R_k \cdot \sin\gamma \cdot \tan\gamma$$

with

$$\sin^2\gamma + \cos^2\gamma = 1 \qquad \sin\gamma = \sqrt{1 - \cos^2\gamma} \qquad \tan\gamma = \frac{\sin\gamma}{\cos\gamma}$$

It follows:

$$v^2 = gR_k \cdot \sqrt{1 - \cos^2\gamma} \cdot \frac{\sqrt{1 - \cos^2\gamma}}{\cos\gamma} = gR_k \cdot \frac{1 - \cos^2\gamma}{\cos\gamma}$$

multiplying out:

$$v^2 \cos\gamma = gR_k - gR_k \cos^2\gamma$$

At last we get an quadratic equation:

$$\cos^2\gamma + \frac{v^2}{gR_k} \cdot \cos\gamma - 1 = 0$$

Solution of the quadratic equation:

$$\cos\gamma = +\sqrt{1 + \left(\frac{v^2}{2gR_k}\right)^2} - \frac{v^2}{2gR_k}$$

Only the root with the positive sign makes sense, with a negative sign is $\cos\gamma < 0$ and in the interval $[0, 90°]$ there is no solution .

Ellipsoid of revolution:

We take the symbols of chapter 1.4. The problem is to obtain r as function of the velocity v. Here we begin with the velocity equation (3) in chapter 1:

$$v = r \cdot \sqrt{\frac{b \cdot g}{a \cdot \sqrt{a^2 - r^2}}}$$

It follows:

$$\frac{v^2}{r^2} = \frac{bg}{a \cdot \sqrt{a^2 - r^2}}$$

transformed:

$$\frac{v^4 a^2}{r^4} \cdot (a^2 - r^2) = b^2 \cdot g^2$$

We change this expression to a polynomial:

$$v^4 a^4 - v^4 a^2 \cdot r^2 = b^2 g^2 \cdot r^4$$

Normed form:

$$r^4 + \frac{v^4 a^2}{b^2 g^2} \cdot r^2 - \frac{v^4 a^4}{b^2 g^2} = 0$$

This expression is an quadratic equation to r^2, with that:

$$r^2 = +\sqrt{\frac{v^4 a^4}{b^2 g^2} + \left(\frac{v^4 a^2}{2b^2 g^2}\right)^2} - \frac{v^4 a^2}{2b^2 g^2} \tag{13}$$

Now we derive a formula of r in dependence to the angular velocity w. We begin with the equation (4) in chapter 1 of the angular velocity.

$$w^2 = \frac{bg}{a \cdot \sqrt{a^2 - r^2}}$$

transformed:

$$\sqrt{a^2 - r^2} = \frac{bg}{aw^2}$$

Solving to r:

$$r^2 = a^2 - \left(\frac{bg}{aw^2}\right)^2$$

$$r = \sqrt{a^2 - \left(\frac{bg}{aw^2}\right)^2} \tag{14}$$

ball as special case of the ellipsoid of revolution:

It is also possible to view the ball as special case of the ellipsoid of revolution with the ball radius $R_k = a = b$. Specializing of (13) and (14):

$$r(v)^2 = +\sqrt{\frac{v^4 R_k^2}{g^2} + \left(\frac{v^4}{2g^2}\right)^2} - \frac{v^4}{2g^2}$$

$$r = \sqrt{R_k^2 - \left(\frac{g}{w^2}\right)^2}$$

Hyperboloid:

We take the notations of chapter 1.7. Now we try to get the radius r with the velocity v, equation (6) of chapter 1:

$$v = \sqrt{gr \cdot \frac{a}{b} \cdot \frac{r}{\sqrt{b^2 + r^2}}} = r \cdot \sqrt{\frac{ag}{b \cdot \sqrt{b^2 + r^2}}}$$

transformed:

$$\frac{v^2}{r^2} = \frac{ag}{b \cdot \sqrt{b^2 + r^2}}$$

$$v^4 b^2 \cdot (b^2 + r^2) = a^2 g^2 r^4$$

multiplying out:

$$v^4 b^4 + v^4 b^2 r^2 = a^2 g^2 r^4$$

Normed form:

$$r^4 - \frac{v^4 b^2}{a^2 g^2} \cdot r^2 - \frac{v^4 b^4}{a^2 g^2} = 0$$

This expression is an quadratic equation of r^2.

$$r^2 = +\sqrt{\frac{v^4 b^4}{a^2 g^2} + \left(\frac{v^4 b^2}{2a^2 g^2}\right)^2} + \frac{v^4 b^2}{2a^2 g^2}$$

Determination of r with the angular velocity w:

$$w = \sqrt{\frac{ag}{b \cdot \sqrt{b^2 + r^2}}}$$

Transformation:

$$\sqrt{b^2 + r^2} = \frac{ag}{bw^2}$$

solved to r:

$$r = \sqrt{\left(\frac{ag}{bw^2}\right)^2 - b^2}$$

3.2. Inversions of the friction case

We take the notation of chapter 2.1 - 2.7. If we solve the formula of the maximum velocity to the radius r or to the height angle γ, we get formulas of r_{min} respectively γ_{min}. One solving of the minimum velocity formula to r or γ leads to an expression of r_{max} and γ_{max}. At the cone of revolution we get:

$$r_{min} = \frac{(\delta \tan\alpha - \mu) \cdot v^2}{g \cdot (\delta + \tan\alpha \cdot \mu)} \qquad r_{max} = \frac{(\mu + \delta \tan\alpha) \cdot v^2}{g \cdot (\delta - \mu \tan\alpha)}$$

At the paraboloid the solving to r is more complicated. The solving leads with the help of an quadratic equation of r to the following expressions:

$$r_{min} = +\sqrt{\left(\frac{2f\mu g + \mu v^2}{2\delta g}\right)^2 + \frac{2fv^2}{g}} - \frac{2f\mu g + \mu v^2}{2\delta g}$$

$$r_{max} = +\sqrt{\left(\frac{2f\mu g + \mu v^2}{2g\delta}\right)^2 + \frac{2fv^2}{g}} + \frac{2f\mu g + \mu v^2}{2g\delta}$$

We must choose the positive root, because with a negative root the solutions are negativ, too.

At the ellipsoid and the hyberboloid we can do such solvings to r, too. We then get polynomials of degree 4 in r, which can still be solve exactly. The solving of the velocity formula with height angle γ to γ leads to a polynomial of degree 4 in $\tan\gamma$.

The formulas of velocities and angular velocities at orbits on the rotation body's shell can be written as a function of h (at the ellipsoid with the height angle γ, too). At all special rotation bodies $r = f(h)$ can be inserted. At the general rotation body we must derive the case with h again. (From these equation we can perhaps prove the special formulas again.) For the insertions of $v, w = f(r)$ to $v, w = f(h)$ are valid:

$x = r, y = h$

$$\text{for ellipsoid:} \quad \frac{r^2}{a^2} + \frac{h^2}{b^2} = 1 \qquad (a \leq b) \text{ or } (a \geq b)$$

$$\text{for balls:} \quad r^2 + h^2 = R_K^2$$

$$\text{for hyperboloids:} \quad \frac{h^2}{a^2} - \frac{r^2}{b^2} = 1$$

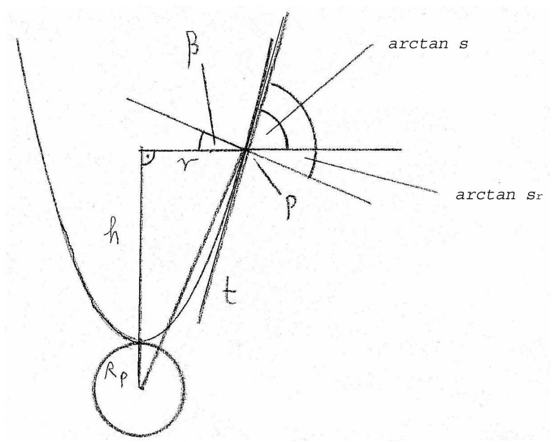
$$\text{for paraboloids:} \quad r^2 = 4hf$$

$$\text{for cones of revolution:} \quad h = r \cdot \tan\alpha$$

4. Stable orbits on macro rotation body's shell in the frictionless case

4.1. The general rotation body's shell

A planet stays or moves uniformly in vacuum. The planet can be seen as a ball. Vertical to the planet's surface there is the axis of the general rotation body. The planet doesn't rotate at all.



R_p = radius of the planet
g = gravitational acceleration in P
m_p = mass of the planet
G = gravitational constant
v = velocity
m = mass of the ball
t = tangent
$h(r)$ = rotation body's function
$s = h'(r)$
$h(r)$ has got only **one** touching point with the ball.

$$g = \frac{Gm_p}{(h + R_p)^2 + r^2}$$

centrifugal force $= Z = \frac{mv^2}{r}$

Now we must calculate s_r. It is valid $s_r = f(s, h, r)$.

$$\beta = 90^\circ - \arctan \frac{h + R_p}{r}$$

We have for s_r:

$$\arctan s_r = \arctan s + \beta = \arctan s + 90^\circ - \arctan \frac{h + R_p}{r}$$

because of $\tan(90^\circ - a) = \frac{1}{\tan a}$ it follows:

$$\arctan s + \arctan \frac{r}{h + R_p} = \arctan s_r$$

We can find a simple relation by using the addition theorem of the tangent.

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

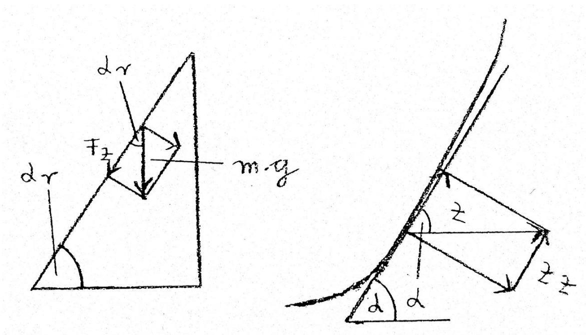
With this we get:

$$s_r = \frac{s + \frac{r}{h+R_p}}{1 - \frac{sr}{h+R_p}} = \frac{r + s\cdot(h+R_p)}{h+R_p-sr}$$

In the case $h, r \ll R_p$ is $s_r \approx s$.

condition of orbits: $s_r > 0$ is equivalent to $0 < r + s\cdot(h+R_p)$ and $h + R_p - sr > 0$. $s_r < 0$ is not possible here. We follow with the both conditions:

$$s > \frac{-r}{h+R_p} \qquad \text{and} \qquad \frac{h+R_p}{r} > s$$



F_z = falling force $\quad$ Z_z = centrifugal force against the falling force

$$F_z = mg\cdot\sin\alpha_r \qquad Z_z = Z\cdot\cos\alpha$$

$$\tan\alpha_r = s_r \qquad \tan\alpha = s \qquad \sin\alpha = \cos\alpha\cdot\tan\alpha$$

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} \qquad \Rightarrow \qquad \cos\alpha = \frac{1}{\sqrt{1+s^2}}$$

$$\sin\alpha_r = \frac{s_r}{\sqrt{1+s_r^2}}$$

This leads to:

$$F_z = \frac{mgs_r}{\sqrt{1+s_r^2}}$$

centrifugal force$=Z = \frac{mv^2}{r}$ we obtain:

$$Z_z = \frac{mv^2}{r\cdot\sqrt{1+s^2}}$$

In the orbit we need $Z_z = F_z$, with that:

$$\frac{mgs_r}{\sqrt{1+s_r^2}} = \frac{mv^2}{r\cdot\sqrt{1+s^2}}$$

At last it follows:

$$v = \sqrt{grs_r \cdot \sqrt{\frac{1+s^2}{1+s_r^2}}}$$

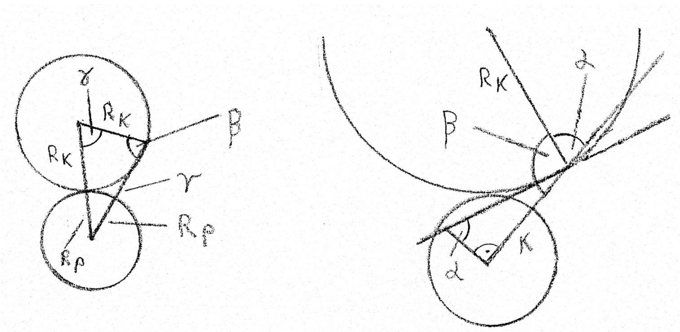
$$w = \frac{v}{r} = \sqrt{\frac{gs_r}{r} \cdot \sqrt{\frac{1+s^2}{1+s_r^2}}}$$

The relation between r and r_m is the same.

In the general equations we can insert in s the formulas for s at cone of revolution, ellipsoid of revolution, ball, paraboloid of revolution, hyperboloid of revolution. Then we get the equations of the macro case in the frictionless case.

4.2. The macro ball's shell with height angle

Now we treat the macro ball shell:
R_p = radius of the planet
m_p = mass of the planet
R_K = radius of the macro ball
m = mass of the ball
γ = height angle
α = inclination angle to the gravitation
K = gravitational force
G = gravitational constant



Law of cosines:

$$r^2 = (R_K + R_p)^2 + R_K^2 - 2R_K \cdot (R_K + R_p) \cdot \cos\gamma$$

In the frictionless case in the homogeneous field it was shown $\alpha = \gamma$. In the gravitational field the relation $\alpha = f(\gamma)$ is more complicated than the gravitational acceleration g:

$$g = \frac{Gm_p}{r^2} = \frac{Gm_p}{(R_K + R_p)^2 + R_K^2 - 2R_K \cdot (R_p + R_K) \cdot \cos\gamma} \tag{15}$$

Once again the law of cosines:

$$(R_K + R_p)^2 = r^2 + R_K^2 - 2rR_K \cos\beta$$

transformed and for r inserted:

$$\cos\beta = \frac{r^2 + R_K^2 - (R_K + R_p)^2}{2rR_K}$$

$$= \frac{R_K - (R_K + R_p)\cdot\cos\gamma}{\sqrt{(R_K + R_p)^2 + R_K^2 - 2R_K\cdot(R_K + R_p)\cdot\cos\gamma}}$$

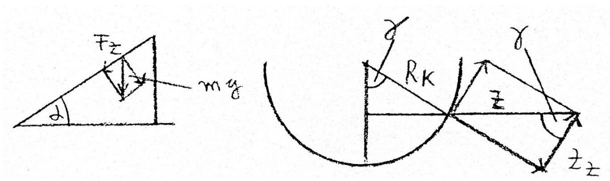
With the figure we see:

$$\beta = 180^\circ - \alpha \qquad \Rightarrow \qquad \cos\beta = \cos(180^\circ - \alpha) = -\cos\alpha$$

It follows:

$$\cos\alpha = \frac{(R_K + R_p)\cdot\cos\gamma - R_K}{\sqrt{(R_K + R_p)^2 + R_K^2 - 2R_K\cdot(R_p + R_K)\cdot\cos\gamma}} \tag{16}$$

F_z = falling force
Z_z = centrifugal force against the falling force



See figure.

$$F_z = mg\sin\alpha \qquad Z_z = Z\cdot\cos\gamma$$

$$Z = \frac{mv^2}{R_K\cdot\sin\gamma}$$

With $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ we get:

$$Z_z = \frac{mv^2}{R_K\cdot\tan\gamma}$$

An orbit on the macro ball shell needs $F_z = Z_z$, with that:

$$mg\sin\alpha = \frac{mv^2}{R_K\cdot\tan\gamma}$$

It follows:

$$v = \sqrt{gR_K\sin\alpha\tan\gamma} \tag{17}$$

w = angular velocity

$$w = \frac{v}{R_K\cdot\sin\gamma} = \sqrt{\frac{gR_K\sin\alpha\tan\gamma}{R_K^2\cdot\sin^2\gamma}} = \sqrt{\frac{g\sin\alpha\sin\gamma}{R_K\sin^2\gamma\cos\gamma}}$$

at last:

$$w = \sqrt{\frac{g\sin\alpha}{R_K\sin\gamma\cos\gamma}} \tag{18}$$

In equation (17) and (18) for g and α equation (15) respectively (16) must be inserted. The equation between r and r_m remains unchanged. The generalization a) at the end of chapter 1 is valid to the macro rotation body, too.

5. Stable orbits on macro rotation body's shell with friction

5.1. The general rotation body's shell with friction

We take the notation of the frictionless case (chapter 4.1).

R_p = radius of the planet
g = gravitational acceleration in P
m_p = mass of the planet
G = gravitational constant
v = velocity
m = mass of the ball
$h(r)$ = rotation body's function
$s = h'(r)$
$h(r)$ has got only **one** touching point.

$$g = \frac{Gm_p}{(h + R_p)^2 + r^2}$$

It is valid see chapter 4.1:

$$F_z = \frac{mgs_r\delta}{\sqrt{1+s_r^2}} \qquad Z_z = \frac{mv^2\delta}{r \cdot \sqrt{1+s^2}}$$

δ is there because of the friction case. (This is explained later)

F_R = friction force of the gravitation
Z_R = friction force of the centrifugal force

See the figures in chapter 4.1:

$$F_R = mg\cos\alpha_r \cdot \mu$$

$$Z_R = Z\sin\alpha \cdot \mu \qquad Z = \frac{mv^2}{r}$$

μ is a general friction coefficient. This coefficient is explained with:

$$\mu = \begin{cases} \frac{\mu'}{R} & \text{if} \quad \mu_H > \frac{\mu'}{R}\,\text{(rolling)} \\ \mu_G & \text{if} \quad \frac{\mu'}{R} > \mu_H\,\text{(sliding)} \\ \mu_G\,\text{or}\,\frac{\mu'}{R} & \text{if} \quad \mu_H = \frac{\mu'}{R}\,\text{(decision is open)} \end{cases}$$

and

$$\delta := \begin{cases} \frac{5}{7} & \text{if} \quad \frac{\mu'}{R} < \mu_H\,\text{(rolling)} \\ 1 & \text{if} \quad \mu_H < \frac{\mu'}{R}\,\text{(sliding)} \\ \frac{5}{7}\,\text{or}\,1 & \text{if} \quad \frac{\mu'}{R} = \mu_H \neq 0\,\text{(decision is open)} \\ 1 & \text{if} \quad 0 = \frac{\mu'}{R} = \mu_H\,\text{(frictionless)} \end{cases}$$

See Assmann [1], edition 1, chapter 11.10, p. 265.

It is:

μ_G = friction coefficient of sliding
μ_H = static friction coefficient
μ' = friction coefficient of rolling
R =Radius of the ball that is on the rotation body's shell. To the reason of δ and μ see chapter 2 at the beginning.

With chapter 4.1 it is: $\tan\alpha_r = s_r$ and $\tan\alpha = s$ we conclude:

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} \qquad \sin\alpha = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}$$

It follows:

$$F_R = \frac{mg\mu}{\sqrt{1+s_r^2}} \qquad Z_R = \frac{mv^2\mu s}{r\cdot\sqrt{1+s^2}}$$

Stability inequality: $|\cdot|$ = absolute value

$$|F_z - Z_z| \leq F_R + Z_R$$

limit cases:

$$Z_z - F_z = F_R + Z_R \tag{19}$$

$$F_z - Z_z = F_R + Z_R \tag{20}$$

With (1) it follows:

$$\frac{mv_{max}^2\cdot\delta}{r\cdot\sqrt{1+s^2}} - \frac{mgs_r\cdot\delta}{\sqrt{1+s_r^2}} = \frac{mg\mu}{\sqrt{1+s_r^2}} + \frac{mv_{max}^2\cdot\mu s}{r\cdot\sqrt{1+s^2}}$$

simplified:

$$\frac{v_{max}^2}{r\cdot\sqrt{1+s^2}}\cdot(\delta-\mu s) = \frac{g}{\sqrt{1+s_r^2}}\cdot(\mu+s_r\cdot\delta)$$

transformed:

$$v_{max}^2 = rg\cdot\sqrt{\frac{1+s^2}{1+s_r^2}}\cdot\frac{\mu+s_r\cdot\delta}{\delta-\mu\cdot s}$$

at last:

$$v_{max} = \sqrt{rg\cdot\sqrt{\frac{1+s^2}{1+s_r^2}}\cdot\frac{\mu+s_r\cdot\delta}{\delta-\mu s}} \tag{21}$$

In the same way we get with (20) an expression of the minimum velocity:

$$v_{min} = \sqrt{rg\cdot\sqrt{\frac{1+s^2}{1+s_r^2}}\cdot\frac{s_r\delta-\mu}{\mu s+\delta}} \tag{22}$$

In the case $r \ll R_p$ is $s_r \approx s$ see chapter 4.1. In this case the equations (21) and (22) lead to the formulas of the homogeneous field (chapter 2.2).

With $\mu = 0$ we obtain with (21) and (22) the formula for the frictionless case (chapter 4.1).

$v_{max} = \infty$ if $\delta - \mu s = 0$ see equation (21) it follows:

$$s_{max} = \frac{\delta}{\mu}$$

$v_{min} = 0$, if $s_r \cdot \delta - \mu = 0$ see equation (22).

$$\Rightarrow \qquad s_{r\,min} = \frac{\mu}{\delta}$$

In chapter 4.1 is was derived:

$$s_r = \frac{r + s \cdot (h + R_p)}{h + R_p - sr}$$

s_r increases with s, if $h + R_p - s \cdot r > 0$, it exists one s_{min}. We transform to s:

$$s_r \cdot h + s_r \cdot R_p - sr \cdot s_r = r + sh + sR_p$$

Solving to s:

$$s_r \cdot h + s_r \cdot R_p - r = s \cdot (r \cdot s_r + h + R_p)$$

$$\Rightarrow \qquad s = \frac{s_r \cdot (h + R_p) - r}{s_r \cdot r + h + R_p}$$

Insertion for $s_{r\,min}$:

$$s_{min} = \frac{\frac{\mu}{\delta} \cdot (h + R_p) - r}{\frac{\mu}{\delta} \cdot r + h + R_p}$$

The relations between r to r_m and to the angular velocity w are the same.

In the general equation s can be inserted the formulas of s at cone of revolution, ellipsoid of revolution, ball, paraboloid of revolution, hyperboloid of revolution. Then we get the equation of the analogous macro case.

5.2. Stable orbits on macro ball's shell (case of friction)

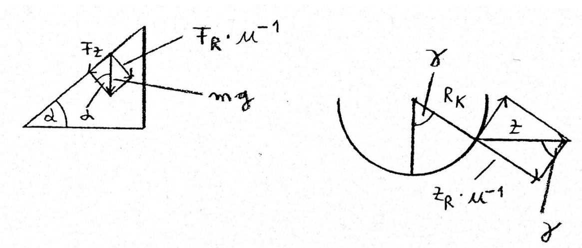
Symbols are taken from chapter 4.2 (macro ball frictionless case).

R_p = radius of the planet
m_p = mass of the planet
R_K = radius of the macro ball
m = mass of the ball
γ = height angle
α = inclination angle to the gravitation
K = gravitational force
G = gravitational constant

With chapter 4.2 it is valid:

$$F_z = mg\delta \sin\alpha \qquad Z_z = \frac{mv^2\delta}{R_K \cdot \tan\gamma}$$

δ is there because of the friction case. δ is explained in chapter 2 and in chapter 5.1.
F_R = friction force of the gravitation
Z_R = friction force of the centrifugal force



With chapter 4.2 we also get:(To μ see chapter 2 or chapter 5.1)

$$F_R = mg\cos\alpha \cdot \mu \qquad Z_R = Z\sin\gamma \cdot \mu$$

with:

$$Z = \frac{mv^2}{R_K \cdot \sin\gamma}$$

stability inequality :

$$|F_z - Z_z| \leq F_R + Z_R$$

limit cases:

$$Z_z - F_z = F_R + Z_R \tag{23}$$

$$F_z - Z_z = F_R + Z_R \tag{24}$$

With (23) it follows:

$$\frac{mv_{max}^2 \cdot \delta}{R_K \tan\gamma} - \delta\sin\alpha \cdot mg = mg\cos\alpha \cdot \mu + \frac{mv_{max}^2 \cdot \mu}{R_K}$$

simplified:

$$\frac{v_{max}^2}{R_K} \cdot \left(\frac{\delta}{\tan\gamma} - \mu\right) = g \cdot (\cos\alpha \cdot \mu + \sin\alpha \cdot \delta)$$

transformed to v_{max}:

$$v_{max} = \sqrt{\frac{gR_K \cdot (\cos\alpha \cdot \mu + \sin\alpha \cdot \delta)}{\frac{\delta}{\tan\gamma} - \mu}} \tag{25}$$

In the same way we get with (24) the following expression:

$$v_{min} = \sqrt{\frac{gR_K \cdot (\sin\alpha \cdot \delta - \cos\alpha \cdot \mu)}{\mu + \frac{\delta}{\tan\gamma}}} \tag{26}$$

In the homogeneous case if $R_K \ll R_p$ is $\alpha \approx \gamma$. Then we obtain:

$$v_{max} \approx \sqrt{\frac{gR_K \cdot (\cos\gamma \cdot \mu + \sin\gamma \cdot \delta)}{\frac{\delta}{\tan\gamma} - \mu}}$$

with $\sin\gamma = \cos\gamma \cdot \tan\gamma$:

$$v_{max} \approx \sqrt{\frac{gR_K \sin\gamma \cdot (\mu + \tan\gamma \cdot \delta)}{\delta - \mu\tan\gamma}}$$

In the same way we can view equation (26) with the using of $\sin\gamma = \cos\gamma\tan\gamma$, it follows:

$$v_{min} \approx \sqrt{\frac{gR_K \sin\gamma \cdot (\tan\gamma \cdot \delta - \mu)}{\mu\tan\gamma + \delta}}$$

These are the formulas, that were derived in chapter 2.1 (homogeneous case). For $\mu = 0$ we conclude with (25) and (26):

$$v = \sqrt{gR_K \sin\alpha\tan\gamma}$$

This is the formula (3) of the frictionless case in chapter 4.

$v_{max} = \infty$, if $\frac{\delta}{\tan\gamma} - \mu = 0$, see equation (25) with that:

$$\tan\gamma_{max} = \frac{\delta}{\mu}$$

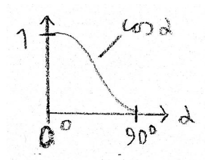
$v_{min} = 0$, if $\sin\alpha \cdot \delta - \cos\alpha \cdot \mu = 0$ with equation (26), with that it follows with $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$:

$$\tan\alpha_{min} = \frac{\mu}{\delta}$$

With chapter 4 equation (16) is:

$$\cos\alpha = \frac{(R_K + R_p) \cdot \cos\gamma - R_K}{\sqrt{(R_K + R_p)^2 + R_K^2 - 2R_K \cdot (R_p + R_K) \cdot \cos\gamma}}$$

$\cos\alpha$ decreases, if γ increases.



With that α increases, if γ increases. It exist one γ_{min}. For easier writing we introduce some symbols:

$$\cos\alpha = \frac{A\cos\gamma - R_K}{\sqrt{B - C\cos\gamma}}$$

with:

$$A := R_p + R_K \qquad B := (R_K + R_p)^2 + R_K^2 \qquad C := 2R_K \cdot (R_p + R_K)$$

Now we transform to $\cos\gamma$:

$$\cos^2\alpha \cdot (B - C\cos\gamma) = (A\cos\gamma - R_K)^2$$

$$\cos^2\alpha \cdot B - R_K^2 = A^2\cos^2\gamma + (C\cos^2\alpha - 2AR_K)\cdot\cos\gamma$$

$$\frac{B\cos^2\alpha - R_K^2}{A^2} = \cos^2\gamma + \frac{\cos^2\alpha \cdot C - 2AR_K}{A^2}\cdot\cos\gamma$$

At last it follows:

$$\cos\gamma_{1,2} = \pm\sqrt{\frac{B\cos^2\alpha - R_K^2}{A^2} + \frac{(C\cos^2\alpha - 2AR_K)^2}{4A^4}} - \frac{C\cos^2\alpha - 2AR_K}{2A^2}$$

Additional condition: $\cos\gamma \geq 0 \qquad (0 \leq \gamma \leq 90°)$

With this formula we can determine γ_{min} with α_{min}. The equations between r and r_m and to the angular velocity w don't change.

References

[1] Bruno Assmann “Technische Mechanik” volume 1 Oldenbourg Verlag 12.edition Munich 1991

[2] A Budo “Theoretische Mechanik” 10.edition VEB Deutscher Verlag der Wissenschaften Berlin 1980

[3] Otto Forster “Analysis 1” 4.edition 1984 Vieweg Verlag Brunswick

[4] Otto Forster “Analysis 2” 5.edition 1984 Vieweg Verlag Brunswick

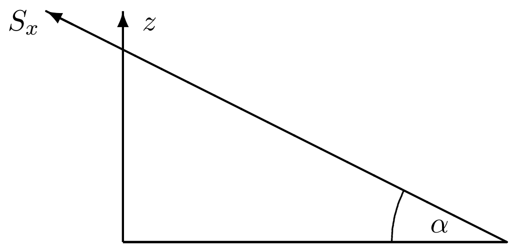
[5] Arnold Sommerfeld “Mechanik” volume 1 Verlag Harri Deutsch 8.edition Frankfort on the Main 1977

31. Trajectories of balls on the inclined plane

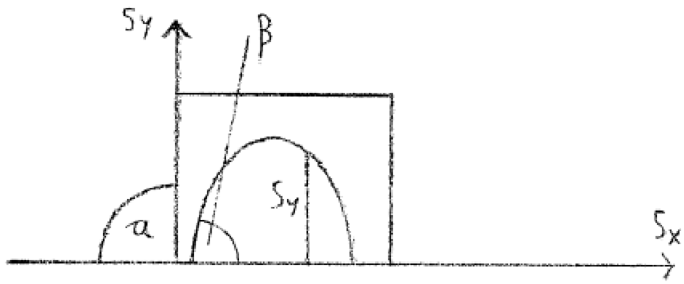
1. The frictionless case

Throws are known phenomenons. We can call upon throws at sporting events for example. The aim is to get a maximum throwing distance. Another example, is the trajectory of projection of a canonball fired from a canon.
We know trajectories of projection as a part of mechanics. Trajectories are determined by throwing angle and initial velocity. This is valid in vacuum and in a medium e.g. air. We can view such trajectories of projection on the inclined plane as well. We will see that the inclined throw in the homogeneous field is a special case of the throw on the inclined plane. Here the trajectories of projection are dependent upon throwing angle and initial velocity, also.
First, we will treat the frictionless case. Thereafter, it will be easier to understand the friction case (chapter 2). In the frictionless case, a ball **slides** on the inclined plane.

a = angle of inclination of the inclined plane
β = throw angle on the inclined plane $\quad -90° \leq \beta \leq 90°$
s = distance
v = velocity
v_0 = initial velocity
t = time



On the inclined plane, there is the falling acceleration $g \cdot \sin a$.

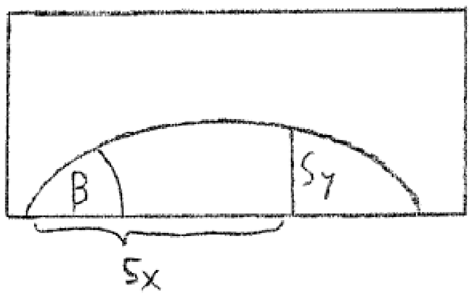


Formulation:

$$s_y = \sin\beta \cdot v_0 t - \frac{1}{2} \cdot g \sin a \cdot t^2 \tag{1}$$

$$v_y = \frac{ds_y}{dt} = \sin\beta \cdot v_0 - g \sin a \cdot t \tag{2}$$

$$s_x = \cos\beta \cdot v_0 t \tag{3}$$



Equating (2) to zero and solved to t, then we get the time of climb t_s for $\beta > 0$.

$$0 = v_y = \sin\beta \cdot v_0 - g \sin a \cdot t_s$$

it follows:

$$t_s = \frac{v_0}{g} \cdot \frac{\sin\beta}{\sin a}$$

t_s inserted in (1) then we yield the maximum altitude of the throw for $\beta > 0$.

$$h_{max} = s_y(t_s) = \frac{\sin\beta \cdot v_0^2 \sin\beta}{g \sin a} - \frac{1}{2} \cdot \frac{g \sin a \cdot v_0^2 \sin^2\beta}{g^2 \cdot \sin^2 a} = \frac{v_0^2 \sin^2\beta}{g \sin a} - \frac{1}{2} \cdot \frac{v_0^2 \sin^2\beta}{g \sin a}$$

thus

$$h_{max} = \frac{v_0^2 \sin^2\beta}{2g \sin a}$$

Equating of (1) to zero and solved to t, then we obtain the throwing time t_w for $\beta > 0$.

$$0 = s_y = \sin\beta \cdot v_0 t_w - \frac{1}{2} \cdot g \sin a \cdot t_w^2$$

Transformation:

$$t_w = 2 \cdot \frac{v_0}{g} \cdot \frac{\sin\beta}{\sin a}$$

That means $t_w = 2 \cdot t_s$.

t_w inserted in equation (3), then we get the throwing range w for $\beta > 0$.

$$w = s_x(t_w) = \cos\beta \cdot v_0 t_w = \frac{\cos\beta \cdot v_0 \cdot 2v_0 \sin\beta}{g \sin a} = 2 \cdot \frac{v_0^2}{g} \cdot \frac{\sin\beta \cos\beta}{\sin a}$$

at last we yield:

$$w = \frac{v_0^2}{g} \cdot \frac{\sin(2\beta)}{\sin a}$$

because of $\sin(2\beta) = 2\sin\beta\cos\beta$

equation (3) solved for t:

$$t = \frac{s_x}{\cos\beta \cdot v_0}$$

and inserted in (1), then we have the equation of the trajectory:

$$s_y = \frac{\sin\beta \cdot v_0 s_x}{v_0 \cos\beta} - \frac{g \sin a \cdot s_x^2}{2\cos^2\beta \cdot v_0^2}$$

with $\frac{\sin\beta}{\cos\beta} = \tan\beta$

$$s_y = s_x \cdot \tan\beta - \frac{1}{2} \cdot s_x^2 \cdot \frac{g}{v_0^2} \cdot \frac{\sin a}{\cos^2\beta}$$

therefore, a parabola

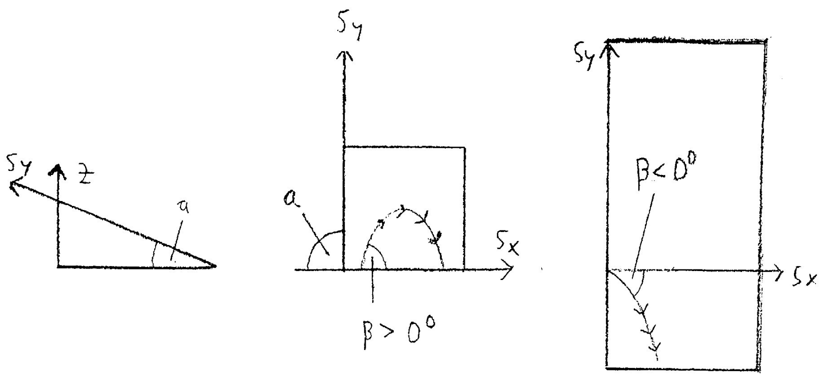
2. The case of friction

Now we take into consideration friction. This consideration is only valid for small relative velocities.

s = distance
v = velocity
b = acceleration
v_0 = initial velocity
a = angle of inclination of the inclined plane
β = throwing angle on the inclined plane $\quad -90^\circ \leq \beta \leq 90^\circ$
$\mu = \frac{\mu'}{R}$ is the rolling friction with rolling friction coefficient μ' and the ball's radius R. (see Assmann [1] chapter 11.10 p.265)
μ_H = static friction coefficient

$$\delta := \begin{cases} \frac{5}{7} & \text{if} \quad \mu' > 0 \quad \text{(rolling)} \\ 1 & \text{if} \quad \mu' = 0 \quad \text{(frictionless)} \end{cases}$$

for the ball see Budo [2] §57 p.302 equations (6) - (8)
t = time



In general: $z = s_y \cdot \sin a$ is valid

The acceleration b on the inclined plane is:
$b = \delta g \cdot \sin a$, the velocity v and the distance s are $v = \delta g \sin a \cdot t$ and $s = \frac{1}{2} \cdot \delta g \cdot t^2 \cdot \sin a$.

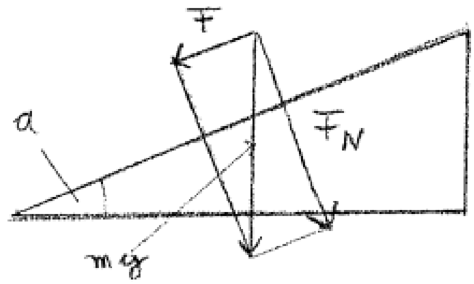
If a ball is rolling on the inclined plane, it is $\delta = \frac{5}{7}$ e.g. Budo [2] §57 p.302 equation (8). The moment of inertia J of a ball is $J = \frac{2}{5} \cdot mR^2$. The general formula for rolling on the inclined plane can be written:

$$b = \frac{mg \sin a}{m + \frac{J}{R^2}} \tag{4}$$

One derivation can be found at *, at the end of the chapter or we can see this with Budo [2] §57 p.302 equations (5)-(7). The ball's moment of inertia insertion leads to:

$$b = \frac{mg \sin a}{m + \frac{2}{5} \cdot m} = \frac{5}{7} \cdot g \sin a \tag{5}$$

Then we have in the case of rolling $\delta = \frac{5}{7}$.



For the forces we obtain (see fig.):

$$F = mg \sin a \qquad F_N = mg \cos a$$

friction force:

$$F_R = \mu \cdot F_N = mg\mu \cos a$$

In the case of friction:

$$F = mg \cdot (\delta \sin a - \mu \cos a) \qquad b = g \cdot (\delta \sin a - \mu \cos a) \tag{6}$$

$$v = gt \cdot (\delta \sin a - \mu \cos a) \qquad s = \frac{1}{2} \cdot gt^2 (\delta \sin a - \mu \cos a)$$

Formulation:

$$v_y = \sin \beta \cdot (v_0 - \mu \cos a \cdot gt) - g \cdot (\delta \sin a - \mu \cos a) \cdot t \tag{7}$$

$$s_y(t) = \int_0^t v_y(\tau)\, d\tau = \sin \beta \cdot (v_0 t - \mu \cos a \cdot g \cdot \frac{t^2}{2}) - \frac{g}{2} \cdot (\delta \sin a - \mu \cos a) \cdot t^2 \tag{8}$$

$$b_y(t) = \frac{d}{dt}\, v_y(t) = -\mu \cos a \cdot g \sin \beta - g \cdot (\delta \sin a - \mu \cos a)$$

For the horizontal component we yield:

$$v_x = \cos \beta \cdot (v_0 - \mu \cos a \cdot gt)$$

$$s_x(t) = \int\limits_0^t v_x(\tau)\, d\tau$$

it follows:

$$s_x = \cos \beta \cdot (v_0 t - \frac{\mu}{2} \cdot \cos a \cdot gt^2) \tag{9}$$

Equating (7) to zero and solved for t, then we get the time of climb t_s for $\beta > 0$.

$$0 = v_y = \sin \beta \cdot v_0 - \sin \beta \cdot \mu \cos a \cdot gt_s - g\delta \sin a \cdot t_s + g\mu \cos a \cdot t_s$$

Transformation:

$$t_s = \frac{\sin \beta \cdot v_0}{\sin \beta \cdot \mu \cos a \cdot g + g\delta \sin a - g\mu \cos a}$$

at last:

$$t_s = \frac{\sin \beta \cdot v_0}{\mu \cos a \cdot g \cdot (\sin \beta - 1) + g\delta \sin a}$$

If we insert t_s in t in (8), we obtain the maximum altitude $h_{max} := s_y(t_s)$.

Equating (8) to zero and solved for t, then we get the throwing time t_w for $\beta > 0$.

$$0 = s_y = \sin \beta \cdot v_0 t_w - \mu \cos a \cdot g \cdot \frac{t_w^2}{2} \cdot \sin \beta - \frac{g}{2} \cdot (\delta \sin a - \mu \cos a) \cdot t_w^2$$

Solving:

$$t_w = \frac{2 \cdot \sin \beta \cdot v_0}{\mu \cos a \cdot g \sin \beta + g \cdot (\delta \sin a - \mu \cos a)} = 2 \cdot t_s$$

Inserted t_w in t in (9), then we get the throwing range $w = s_x(t_w)$ for $\beta > 0$. Then it is $s_y(t_w) = 0$.

In the case $\delta \sin a - \mu \cos a \leq 0$ there is no free falling.

$$\Leftrightarrow \qquad \delta \sin a \leq \mu \cos a$$

$$\Leftrightarrow \qquad \tan a \leq \frac{\mu}{\delta} \qquad \text{because of} \qquad \tan a = \frac{\sin a}{\cos a}$$

In this case we have:

$$v_y = \sin \beta \cdot (v_0 - \mu \cos a \cdot gt) \tag{10}$$

$$s_y(t) = \int\limits_0^t v_y(\tau)\, d\tau = \sin \beta \cdot \left(v_0 t - \mu \cos a \cdot g \cdot \frac{t^2}{2} \right) \tag{11}$$

Equating (10) to zero and solved for t, then we yield the time t_E. After this time, the ball stands still.

$$0 = \sin\beta \cdot (v_0 - \mu\cos a \cdot g t_E)$$

transformed:

$$t_E = \frac{v_0}{\mu\cos a \cdot g}$$

Inserted t_E in t in (11), then we get the altitude at the end h_E:

$$h_E = s_y(t_E) = \sin\beta \cdot \left(v_0 \cdot \frac{v_0}{\mu\cos a \cdot g} - \frac{1}{2} \cdot \mu\cos a \cdot g \cdot \frac{v_0^2}{\mu^2\cos^2 a \cdot g^2} \right)$$

$$= \frac{\sin\beta \cdot v_0^2}{\mu\cos a \cdot g} - \frac{1}{2} \cdot \frac{v_0^2 \sin\beta}{\mu\cos a \cdot g}$$

We obtain:

$$h_E = \frac{1}{2} \cdot \frac{\sin\beta \cdot v_0^2}{\mu\cos a \cdot g}$$

Now we calculate the height of ascent in the special case $\beta = 90°$ at $\tan a > \frac{\mu}{\delta}$. Then equation (5) changes to:

$$s_y = v_0 t - \mu\cos a \cdot g \cdot \frac{t^2}{2} - \frac{g}{2} \cdot (\delta\sin a - \mu\cos a) \cdot t^2$$

The time of climb can be written as:

$$t_s = \frac{v_0}{g\delta\sin a}$$

Insertion:

$$h_{max} = s_y(t_s) = \frac{v_0^2}{g\delta\sin a} - \frac{\mu\cos a \cdot g v_0^2}{2g^2\delta^2\sin^2 a} - \frac{g\delta\sin a \cdot v_0^2}{2g^2\delta^2\sin^2 a} + \frac{g\mu\cos a \cdot v_0^2}{2g^2\delta^2\sin^2 a}$$

$$= \frac{v_0^2}{g\delta\sin a} - \frac{\mu\cos a \cdot v_0^2}{2g\delta^2\sin^2 a} - \frac{v_0^2}{2g\delta\sin a} + \frac{\mu\cos a \cdot v_0^2}{2g\delta^2\sin^2 a}$$

at last it follows:

$$h_{max} = \frac{v_0^2}{2g\delta\sin a}$$

In the case $\beta \leq 0$ there is no t_s and t_w.

More difficult is the treatment in consideration of a medium (gas, liquid).

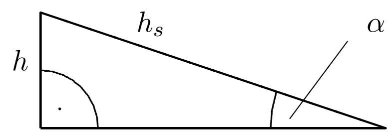
Some remarks to the validity of these equations:

The sliding friction must be locked out, as in Budo [2] §57 p.302 equation (10) at the ball, it is valid $\tan a \leq \frac{7}{2} \cdot \mu_H$. That means only at a small angle of inclination is the sliding friction locked out. The relative velocity must be small, also. Lastly, the ball and the inclined plane must be elastic. This is true for steel balls and the steel underlayer.

to *: The derivation of formula (4):

Notations:

kinetic energy = $E_{kin} = \frac{mv^2}{2}$
potential energy = $E_{pot} = mgh$ with $h = h_s \sin a$ see fig.
rotational energy = $E_{rot} = \frac{Jw^2}{2}$ $\qquad J$ = moment of inertia



law of conservation of energy:

$$\frac{mv^2}{2} + \frac{Jw^2}{2} = mgh$$

We assume that the body is only rolling not gliding. Then it is valid:
angular velocity = $w = \frac{v}{R}$ $\quad R$ = radius of the rolling body

$$\frac{mv^2}{2} + \frac{Jv^2}{2R^2} = mgh_s \sin a$$

With transformation we yield:

$$v = \sqrt{\frac{2mgh_s \sin a}{m + \frac{J}{R^2}}}$$

Because the rolling body has the constant acceleration $b = g \sin a$, there is a uniform accelerated motion. For this motion it is valid: $v^2 = 2h_s \cdot b$ transformed to $b = \frac{v^2}{2h_s}$. Now we insert the equation of v:

$$b = \frac{2mgh_s \sin a}{2h_s \cdot \left(m + \frac{J}{R^2}\right)} = \frac{mg \sin a}{m + \frac{J}{R^2}}$$

Then we have the wanted formula.

References

[1] Bruno Assmann “Technische Mechanik” Band 1 Oldenbourg Verlag 12.edition Munich 1991

[2] A Budo “Theoretische Mechanik” 10.edition VEB Deutscher Verlag der Wissenschaften Berlin 1980

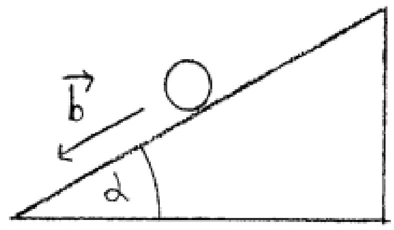
32. A ball's motion with artificial acceleration on an inclined plane

Let be:

m = mass of the ball
R = radius of the ball
α = angle of inclination of the plane
b_k = artificial acceleration
F_z = force that is acting on the ball (on the inclined plane)
$b_z = \frac{F_z}{m}$ = affiliated acceleration
$\mu = \frac{\mu'}{R}$ is the rolling friction with the rolling friction coefficient μ' and the radius R of the ball. (see Assmann [1] chapter 11.10 p.265)
μ_H = static friction coefficient

$$\delta := \begin{cases} \frac{5}{7} & \text{if} \quad \mu' > 0 \quad \text{(rolling)} \\ 1 & \text{if} \quad \mu' = 0 \quad \text{(frictionless)} \end{cases}$$

for the ball see Budo [2] §57 p.302 equations (6) - (8)
t = time



The acceleration b on the inclined plane is:
$b = \delta g \cdot \sin\alpha$, the velocity v and the distance s are $v = \delta g \sin\alpha \cdot t$ and $s = \frac{1}{2} \cdot \delta g \cdot t^2 \cdot \sin\alpha$.

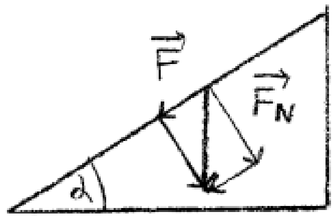
If a ball is rolling on an inclined plane, it is $\delta = \frac{5}{7}$, e.g. Budo [2] §57 p.302 equation (8). The moment of inertia J of a ball is $J = \frac{2}{5} \cdot mR^2$. The general formula for rolling on the inclined plane:

$$b = \frac{mg\sin\alpha}{m + \frac{J}{R^2}} \tag{1}$$

One derivation can be found at *, at the end of the text or see Budo [2] §57 p.302 equations (5)-(7). The insertion of the ball's moment of inertia leads to:

$$b = \frac{mg\sin\alpha}{m + \frac{2}{5} \cdot m} = \frac{5}{7} \cdot g\sin\alpha \tag{2}$$

Thus $\delta = \frac{5}{7}$.



For the different forces we obtain (see fig.):

$$F = mg\sin\alpha \qquad F_N = mg\cos\alpha$$

friction force:

$$F_R = \mu \cdot F_N = mg\mu\cos\alpha$$

In the case of friction:

$$F = mg \cdot (\delta\sin\alpha - \mu\cos\alpha) \qquad b = g \cdot (\delta\sin\alpha - \mu\cos\alpha)$$

$$v = gt \cdot (\delta\sin\alpha - \mu\cos\alpha) \qquad s = \frac{1}{2} \cdot gt^2 \cdot (\delta\sin\alpha - \mu\cos\alpha)$$

Thus the force F_z can be written:

$$F_z = m\delta \cdot (g\sin\alpha + b_k) - mg\mu\cos\alpha$$

We obtain the acceleration b_z with:

$$b_z = \frac{F_z}{m} = \delta \cdot (g\sin\alpha + b_k) - g\mu\cos\alpha$$

The artificial acceleration b_k vanishes at time t_1. For the velocity we get:

$$v_z = \begin{cases} b_z \cdot t & : \quad t \leq t_1 \\ b_z \cdot t_1 + g \cdot (\delta\sin\alpha - \mu\cos\alpha) \cdot (t - t_1) & : \quad t \geq t_1 \end{cases}$$

For the distance we yield:

$$s_z = \frac{1}{2} \cdot b_z t^2 \qquad \text{for} \qquad t \leq t_1$$

$\bar{s}_z$ = distance of time t_1 to time t

$$\bar{s}_z(t) = \int_{t_1}^{t} v_z(\tau)\,d\tau = \int_{t_1}^{t} (b_z t_1 + g \cdot (\delta\sin\alpha - \mu\cos\alpha) \cdot (\tau - t_1))\,d\tau$$

$$= \left[b_z t_1 \tau + g \cdot (\delta\sin\alpha - \mu\cos\alpha) \left(\frac{\tau^2}{2} - t_1\tau \right) \right]_{t_1}^{t}$$

$$= b_z t_1 t + g \cdot (\delta\sin\alpha - \mu\cos\alpha) \cdot \left(\frac{t^2}{2} - t_1 t \right) - b_z t_1^2 - g \cdot (\delta\sin\alpha - \mu\cos\alpha) \cdot \left(\frac{t_1^2}{2} - t_1^2 \right)$$

$$= b_z t_1 \cdot (t - t_1) + g \cdot (\delta \sin\alpha - \mu \cos\alpha) \cdot \left(\frac{t^2}{2} + \frac{t_1^2}{2} - t_1 t \right)$$

thus:

$$s_z = \begin{cases} \frac{1}{2} \cdot b_z t^2 & : \quad t \leq t_1 \\ \frac{1}{2} \cdot b_z t_1^2 + \bar{s}_z(t) & : \quad t \geq t_1 \end{cases}$$

If we view the formula of $v_z(t)$ in the case of $t \geq t_1$, the velocity $v_z(t)$ reduces in the case of $\delta \sin\alpha - \mu \cos\alpha < 0$ until $v_z(t)$ is zero at time t_2.

$$0 = v_z(t_2) = b_z t_1 + g \cdot (\delta \sin\alpha - \mu \cos\alpha) \cdot (t_2 - t_1)$$

It follows:

$$t_2 - t_1 = \frac{b_z \cdot t_1}{g \cdot (\mu \cos\alpha - \delta \sin\alpha)}$$

or

$$t_2 = t_1 \cdot \left(1 + \frac{b_z}{g \cdot (\mu \cos\alpha - \delta \sin\alpha)} \right)$$

Special case $\alpha = 0$:

$$t_2 - t_1 = \frac{b_z \cdot t_1}{g \cdot \mu}$$

or

$$t_2 = t_1 \cdot \left(1 + \frac{b_z}{g \cdot \mu} \right)$$

At $v_z(t)$ we recognize a uniform motion at $t \geq t_1$ in the case of $\delta \sin\alpha - \mu \cos\alpha = 0$ with the velocity $v_z = b_z \cdot t_1$. This is equivalent to:

$$\delta \sin\alpha = \mu \cos\alpha$$

or

$$\tan\alpha = \frac{\mu}{\delta}$$

because of $\frac{\sin\alpha}{\cos\alpha} = \tan\alpha$.

More difficult is the treatment in consideration of a medium (gas, liquid).

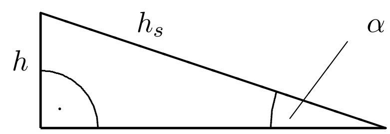
To the validity of these equations here are some remarks:

The sliding friction must be locked out, with Budo [2] §57 p.302 equation (10) at the ball $\tan\alpha \leq \frac{7}{2} \cdot \mu_H$ is valid. Meaning, that only at a small angle of inclination can the sliding friction be locked out. The relative velocity must be small, as well. Lastly the ball and the inclined plane must be elastic. This is true for steel balls on a steel plane.

to *: The derivation of formula (1):

Notations:

kinetic energy = $E_{kin} = \frac{mv^2}{2}$
potential energy = $E_{pot} = mgh$ with $h = h_s \sin\alpha$ see fig.
rotational energy = $E_{rot} = \frac{Jw^2}{2}$ $\qquad J$ = moment of inertia



law of conservation of energy:

$$\frac{mv^2}{2} + \frac{Jw^2}{2} = mgh$$

We assume the body is only rolling, not gliding. Thus, it is valid:
angular velocity = $w = \frac{v}{R}$ $\qquad R$ = radius of the rolling body

$$\frac{mv^2}{2} + \frac{Jv^2}{2R^2} = mgh_s \sin\alpha$$

With the transformation we yield:

$$v = \sqrt{\frac{2mgh_s \sin\alpha}{m + \frac{J}{R^2}}}$$

Because the rolling body has the constant acceleration $b = g\sin\alpha$, there is a uniform accelerated motion. For this motion, it is valid: $v^2 = 2h_s \cdot b$ transforms to $b = \frac{v^2}{2h_s}$. Now we insert the equation of v:

$$b = \frac{2mgh_s \sin\alpha}{2h_s \cdot \left(m + \frac{J}{R^2}\right)} = \frac{mg\sin\alpha}{m + \frac{J}{R^2}}$$

Thus, the wanted formula.

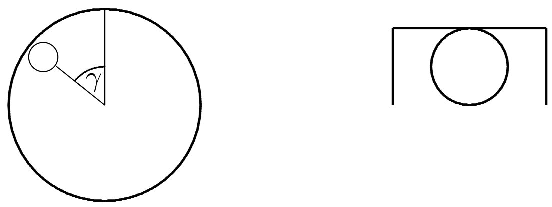
References

[1] Bruno Assmann “Technische Mechanik” Band 1 Oldenbourg Verlag 12.edition Munich 1991

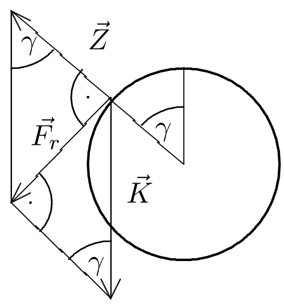
[2] A Budo “Theoretische Mechanik” 10.edition VEB Deutscher Verlag der Wissenschaften Berlin 1980

33. The vertical loop

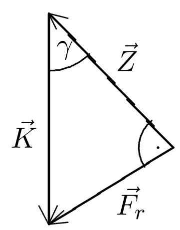
We view the vertical loop of the earth's surface with a small ball as shown in the figure.



The figure shows the orbit of the loop. The small ball cannot escape the loop. Imagine the lower part of the figure as a course for marbles. We need energy to place the ball at a position in the lower part of the loop. For a position in the upper part of the loop a minimum velocity is necessary. Otherwise the ball falls down.



If we look at the figure, we recognize two forces, the gravity $K = mg$ with m as mass of the ball and the earth's acceleration g. The other force is the centrifugal force $Z = \frac{mv^2}{R}$ with radius R of the loop and the velocity of the ball v.



The resultant force $\vec{F}_r = \vec{K} + \vec{Z}$ must be tangential to the figure, if the ball moves with minimum velocity. The following equation is recognizable at the figure.

$$\frac{Z}{K} = \cos\gamma \qquad \text{it follows} \qquad \frac{v_{min}^2}{gR} = \cos\gamma$$

with γ as height angle.

One transformation leads to:

$$v_{min} = \sqrt{g \cdot R \cdot \cos\gamma}$$

At the highest point: $(\gamma = 0)$

$$v_{min} = \sqrt{g \cdot R}$$

At the highest point we can equate $Z = K$. Then we get the same result.

Now we use a second method to get the formula of minimum velocity.

We assume an arbitrary ball's velocity v. It is (see fig.)

$$F_r^2 = K^2 + Z^2 - 2KZ \cdot \cos\gamma$$

It follows:

$$F_r = \sqrt{m^2g^2 + \frac{m^2v^4}{R^2} - \frac{2m^2gv^2\cos\gamma}{R}}$$

With the figure we can see (law of cosines):

$$K^2 = F_r^2 + Z^2 - 2F_rZ \cdot \cos\beta$$

Transformation and insertion:

$$\cos\beta = \frac{F_r^2 + Z^2 - K^2}{2F_rZ} = \frac{K^2 + Z^2 - 2KZ\cos\gamma + Z^2 - K^2}{2F_rZ}$$

$$= \frac{2Z^2 - 2KZ\cos\gamma}{2ZF_r} = \frac{Z - K\cos\gamma}{F_r}$$

Insertion for Z and K:

$$\cos\beta = \frac{\frac{mv^2}{R} - mg\cos\gamma}{\sqrt{\frac{m^2v^4}{R^2} + m^2g^2 - \frac{2m^2gv^2\cos\gamma}{R}}}$$

$\vec{F}_r$ is tangential, if $\beta = 90°$. It follows $\cos\beta = 0$, with that:

$$0 = \frac{mv_{min}^2}{R} - mg\cos\gamma$$

at last, we obtain:

$$v_{min} = \sqrt{g \cdot R \cdot \cos\gamma}$$

The result is proved twice.

At last we introduce the radius r of the small ball. If we take the barycenter of the small ball into consideration, then we must insert $R - r$ instead of R in the equations. We assume that the small ball has a (local) constant density. Further results can be found at Schröer [1].

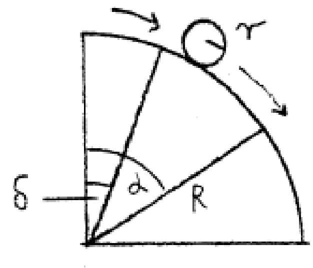
References

[1] Harald Schröer "The loop" Wissenschaft & Technik Verlag Berlin 2002

34. Falling from a ball

We treat the frictionless case:

δ = start angle
R = radius of the big ball
r = radius of the small ball, that moves on the big ball
v = velocity of the small ball
v_0 = start velocity of the small ball
m = mass of the small ball
g = gravitational acceleration



In the frictionless case, a ball is sliding.

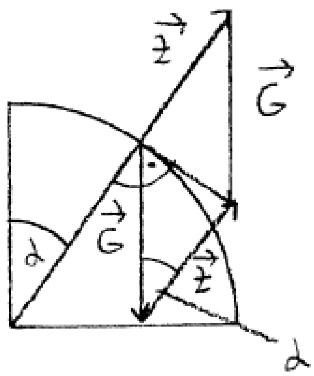
Law of conservation of energy:

$$\frac{mv^2}{2} + mgR\cos\alpha = mgR\cos\delta + \frac{mv_0^2}{2} \tag{1}$$

G = gravitational force
Z = centrifugal force

$$G = mg \qquad Z = \frac{mv^2}{R}$$

limit condition:



$$\cos\alpha = \frac{Z}{G} = \frac{v^2}{Rg}$$

We transform equation (1) to v:

$$v^2 = \frac{mgR \cdot (\cos\delta - \cos\alpha) + \frac{mv_0^2}{2}}{\frac{m}{2}}$$

with that:

$$v^2 = 2gR \cdot (\cos\delta - \cos\alpha) + v_0^2$$

We obtain:

$$\cos\alpha = \frac{v^2}{Rg} = \frac{2gR \cdot (\cos\delta - \cos\alpha) + v_0^2}{Rg}$$

$$\cos\alpha = 2 \cdot (\cos\delta - \cos\alpha) + \frac{v_0^2}{Rg}$$

$$3\cos\alpha = 2\cos\delta + \frac{v_0^2}{Rg}$$

at last:

$$\cos\alpha = \frac{2}{3} \cdot \cos\delta + \frac{v_0^2}{3Rg}$$

If α is bigger, then the small ball flies away from the big ball.

Special cases:

$$\cos\alpha = \frac{2}{3} \cdot \cos\delta \qquad \text{at} \quad v_0 = 0$$

$$\cos\alpha = \frac{2}{3} + \frac{v_0^2}{3Rg} \qquad \text{at} \quad \delta = 0$$

$\cos\alpha = \frac{2}{3}$ at $v_0 = 0$ and $\delta = 0$

These formulas are valid for a body that moves down on a cylinder, the cylinder has the same cross-section.

If we take in consideration the barycenter of the small ball, we must insert $R+r$ instead of R in all equations. We assume that the ball has a constant density.

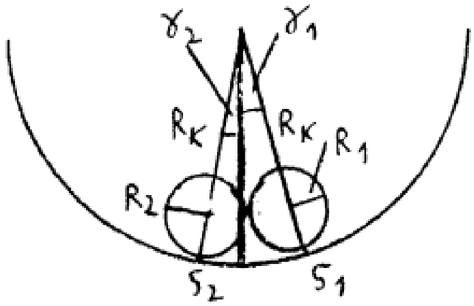
35. Motionless balls

1. The frictionless case

1.1. Two balls in a spherical shell

We want to occupy ourselves with the balance of two balls on a spherical shell. We have the following questions: Is this balance stable, unstable, or indifferent? What position do the two balls have? Which quantities determine the balance?

Two balls with mass m_1, m_2 and radii R_1, R_2 are on a spherical shell with the radius R_K, see fig.



$$\gamma_1 + \gamma_2 = \gamma_g$$

$$R_1 + R_2 \leq R_K$$

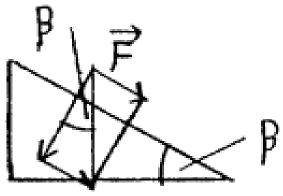
We don't take friction into consideration. At first, we assume a vacuum. The problem is in determining the angles γ_1, γ_2.

$$\gamma_1 = \angle FMS_1 \qquad \gamma_2 = \angle FMS_2$$

Then, the force on an inclined plane with the angle of inclination β is:

$$F = mg \sin \beta$$

see fig.

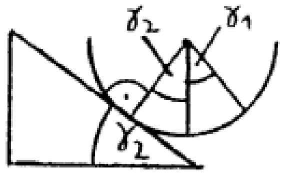


m = moved mass g = earth's acceleration

On the spherical shell the forces are:

$$F_1 = m_1 g \sin\gamma_1 \qquad F_2 = m_2 g \sin\gamma_2$$

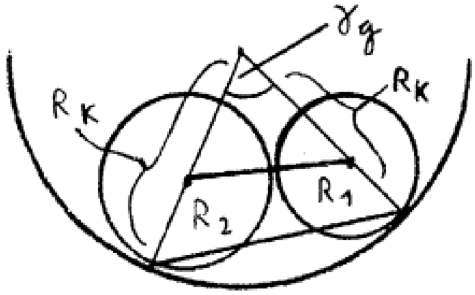
Here γ_1, γ_2 are the angles of inclination, see fig.



The spherical shell is balanced, thus $F_1 = F_2$. Equating, we get:

$$m_1 \cdot \sin\gamma_1 = m_2 \cdot \sin\gamma_2$$

We view the fig.



We need the cosine law to determine γ_g:

$$(R_1 + R_2)^2 = (R_K - R_1)^2 + (R_K - R_2)^2 - 2 \cdot (R_K - R_1) \cdot (R_K - R_2) \cdot \cos\gamma_g$$

transformed:

$$\cos\gamma_g = \frac{(R_K - R_1)^2 + (R_K - R_2)^2 - (R_1 + R_2)^2}{2 \cdot (R_K - R_1) \cdot (R_K - R_2)}$$

In this way, γ_g is known:

$$\gamma_g = \gamma_1 + \gamma_2 \tag{1}$$

$$m_1 \cdot \sin\gamma_1 = m_2 \cdot \sin\gamma_2 \tag{2}$$

(1) can be inserted with the addition theorem $\sin(a - b) = \sin a \cos b - \sin b \cos a$.

$$m_1 \sin\gamma_1 = m_2 \sin(\gamma_g - \gamma_1) = m_2 \cdot (\sin\gamma_g \cos\gamma_1 - \sin\gamma_1 \cos\gamma_g)$$

Insertion of $\cos\gamma_1 = \sqrt{1-\sin^2\gamma_1}$ and transformation:

$$m_1 \sin\gamma_1 + m_2 \sin\gamma_1 \cos\gamma_g = m_2 \sin\gamma_g \cdot \sqrt{1-\sin^2\gamma_1}$$

squared:

$$\sin^2\gamma_1 \cdot (m_1 + m_2\cos\gamma_g)^2 = m_2^2 \sin^2\gamma_g - m_2^2 \sin^2\gamma_g \sin^2\gamma_1$$

solved for $\sin\gamma_1$:

$$\sin\gamma_1 = \sqrt{\frac{m_2^2 \sin^2\gamma_g}{(m_1 + m_2\cos\gamma_g)^2 + m_2^2\sin^2\gamma_g}} = \frac{m_2\sin\gamma_g}{\sqrt{(m_1 + m_2\cos\gamma_g)^2 + m_2^2\sin^2\gamma_g}}$$

and $\gamma_2 = \gamma_g - \gamma_1$

with medium:

φ_F = density of the liquid (gas)
φ_{Ki} = density of the ball i $\qquad i \in \{1,2\}$
$a_i := 1 - \frac{\varphi_F}{\varphi_{Ki}}$
(see Budo [2] §16 p.85)

In a medium (liquid or gas), the forces can be represented as:

$$F_1 = a_1 g m_1 \sin\gamma_1 \qquad F_2 = a_2 g m_2 \sin\gamma_2$$

The equations (1) and (2) change to ($F_1 = F_2$):

$$a_1 m_1 \sin\gamma_1 = a_2 m_2 \sin\gamma_2$$

$$\gamma_g = \gamma_1 + \gamma_2$$

We set $M_i := a_i \cdot m_i$ with $i \in \{1,2\}$ and, thus, we transform in the same way as in a vacuum. We obtain:

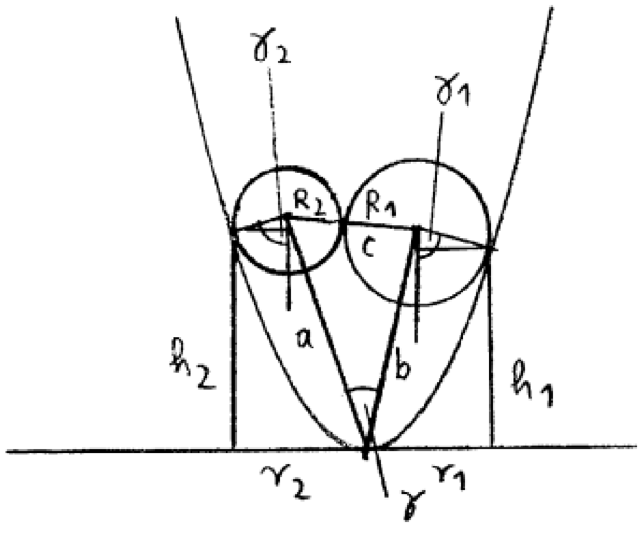
$$\sin\gamma_1 = \frac{a_2 m_2 \sin\gamma_g}{\sqrt{(a_1 m_1 + a_2 m_2\cos\gamma_g)^2 + a_2^2 m_2^2 \sin^2\gamma_g}}$$

and $\gamma_2 = \gamma_g - \gamma_1$

1.2. 2 motionless balls in a body of revolution

So far, we have viewed the frictionless case on a spherical shell. Now we replace the spherical shell with the shell of a body of revolution and work the same problem.

$h(r)$ = revolution body's function
$s = h'(r)$
R_1, R_2 = radii of the balls
m_1, m_2 = mass of the balls
Two balls lie on the shell of a body of revolution



cosine law:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \qquad \Leftrightarrow \qquad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

With the figure, we recognize:

$$a^2 = (h_2 + R_2\cos\gamma_2)^2 + (r_2 - R_2\sin\gamma_2)^2$$

$$b^2 = (h_1 + R_1\cos\gamma_1)^2 + (r_1 - R_1\sin\gamma_1)^2$$

$$c = R_1 + R_2$$

Positions of the midpoints (x_1, y_1) and (x_2, y_2):

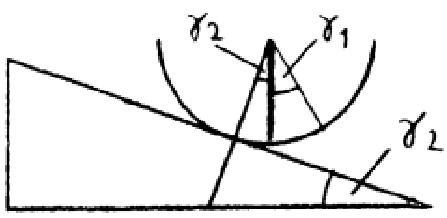
$$x_1 = r_1 - R_1\sin\gamma_1 \qquad x_2 = r_2 - R_2\sin\gamma_2$$

$$y_1 = h_1 + R_1\cos\gamma_1 \qquad y_2 = h_2 + R_2\cos\gamma_2$$

(r_1, h_1) and (r_2, h_2) are the coordinates of the touching points between ball and body of revolution.

$$h_1 = h(r_1) \qquad h_2 = h(r_2) \qquad s_1 = h'(r_1) \qquad s_2 = h'(r_2)$$

γ_1, γ_2 are pitch angles, as well, see figure.



it is valid:

$$s_1 = \tan\alpha_1 \qquad s_2 = \tan\alpha_2$$

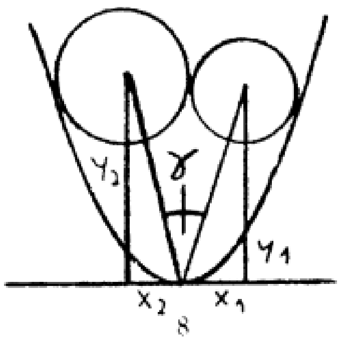
$$\sin\alpha = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} \qquad \cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}}$$

it follows:

$$\sin\gamma_1 = \frac{s_1}{\sqrt{1+s_1^2}} \qquad \sin\gamma_2 = \frac{s_2}{\sqrt{1+s_2^2}}$$

$$\cos\gamma_1 = \frac{1}{\sqrt{1+s_1^2}} \qquad \cos\gamma_2 = \frac{1}{\sqrt{1+s_2^2}}$$

With the following figure we recognize one equation:

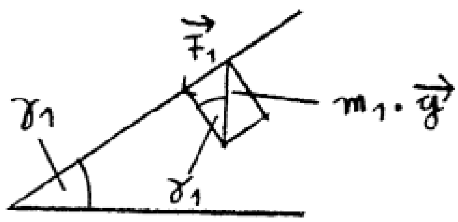


$$\gamma = \arctan\frac{x_1}{y_1} + \arctan\frac{x_2}{y_2}$$

Falling force components:

$$F_1 = m_1 g \sin\gamma_1 \qquad F_2 = m_2 g \sin\gamma_2$$

see figure:



It must be $F_1 = F_2$.

$$\Rightarrow \qquad m_1 g \cdot \frac{s_1}{\sqrt{1+s_1^2}} = m_2 g \cdot \frac{s_2}{\sqrt{1+s_2^2}}$$

We yield 2 equations:

$$\frac{m_1 s_1}{\sqrt{1+s_1^2}} = \frac{m_2 s_2}{\sqrt{1+s_2^2}} \tag{3}$$

$$\gamma = \arctan\frac{x_1}{y_1} + \arctan\frac{x_2}{y_2} \tag{4}$$

The equations of $\gamma, x_1, x_2, y_1, y_2$ must be inserted in equation (4). Then the equations (3) and (4) are 2 equations of the unknowns r_1, r_2.
We note that the unknowns r_1, r_2 are in the equations of $\cos\gamma$, and that $\sin\gamma_1$, $\sin\gamma_2$, $\cos\gamma_1$, $\cos\gamma_2$ must be inserted, as well. Solving r_1 and r_2 is complicated.

In a medium the forces only change to:

$$F_1 = m_1 a_1 g \cdot \frac{s_1}{\sqrt{1+s_1^2}} \qquad F_2 = m_2 a_2 g \cdot \frac{s_2}{\sqrt{1+s_2^2}}$$

with $a_1 = 1 - \frac{\varphi_F}{\varphi_{K1}}$, $a_2 = 1 - \frac{\varphi_F}{\varphi_{K2}}$

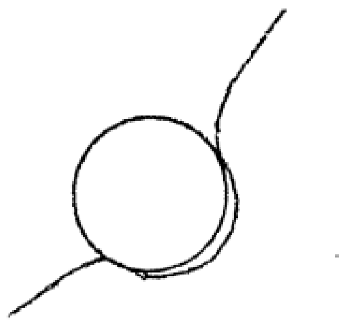
φ_F = density of the medium (liquid, gas)
φ_{K1} = density of the first ball
φ_{K2} = density of the second ball
see Budo [2] §16 p.85

Equation (3) changes to:

$$\frac{m_1 a_1 s_1}{\sqrt{1+s_1^2}} = \frac{m_2 a_2 s_2}{\sqrt{1+s_2^2}}$$

Equation (4) is the same.

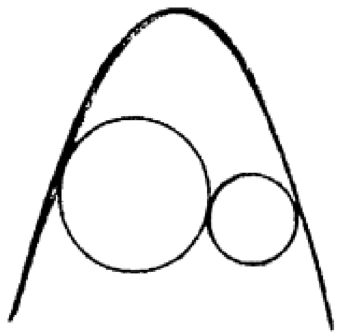
$h(r)$ must be so, that the two balls have only **one** touching point.



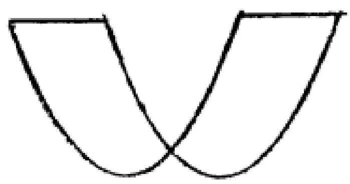
Thus no throughs, where a ball settles down.

If we insert for $h(r)$ the equations of an ellipse, parabola, hyperbola, or cone, we get the equations of revolution ellipsoid shells, revolution parabola shells, revolution hyperbola shells, and revulution cone shells.

We assume that the densities of both balls are greater than the density of the medium. If we have the reverse case, than the body of revolution must be turned round. see fig.



This inversion is valid to the friction case, as well (chapter 2). Instead of the revolution body shell, we can view a lateral area of a general cylinder with the form of a revolution body. see fig.



Then we have the same equations of motionless revolution bodies of the same type, thus the same moment of inertia. This is valid to the later treated friction case, as well.
In the case of a spherical shell, it is a circular cylinder. In the case of a parabola shell, it is a parabola cylinder.

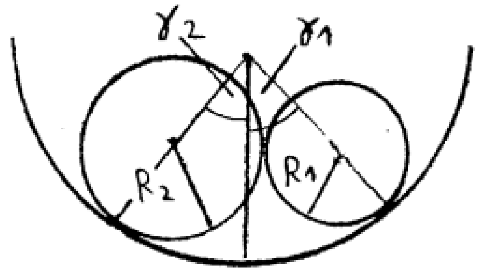
In the frictionless case, both balls have a stable balance.

2. The friction case:

2.1. Motionless balls on a spherical shell with friction

Now we view the friction case. We will see that there is a free play because of the friction forces - in contrast to the frictionless case.

The notation is the same as in chapter 1.1.



$m_1, m_2 =$ mass of the balls
$R_1, R_2 =$ radii of the balls
$R_K =$ radius of the spherical shell

It is valid as in the frictionless case, see chapter 1.1:

$$\cos\gamma_g = \frac{(R_K - R_2)^2 + (R_K - R_1)^2 - (R_1 + R_2)^2}{2 \cdot (R_K - R_2) \cdot (R_K - R_1)} \tag{5}$$

$$\gamma_g = \gamma_1 + \gamma_2 \tag{6}$$

In the friction case, there is free play because of friction forces F_{R1} and F_{R2}.

$F_{R1}, F_{R2} =$ friction case of the balls with the radii R_1 and R_2.

Here γ_1, γ_2 are pitch angles of the balls as in chapter 1.1. The forces can be written as:

$$F_1 = \delta_1 m_1 g \sin\gamma_1 \qquad F_2 = \delta_2 m_2 g \sin\gamma_2 \tag{7}$$

$$F_{R1} = m_1 g \cos\gamma_1 \cdot \mu_1 \qquad F_{R2} = m_2 g \cos\gamma_2 \cdot \mu_2$$

with $\quad i \in \{1, 2\}$

$$\delta_i := \begin{cases} \frac{5}{7} & \text{if} \quad \frac{\mu_i'}{R_i} < \mu_{Hi} \,\text{(rolling)} \\ 1 & \text{if} \quad \mu_{Hi} < \frac{\mu_i'}{R_i} \,\text{(sliding)} \\ \frac{5}{7} \text{ or } 1 & \text{if} \quad \frac{\mu_i'}{R_i} = \mu_{Hi} \neq 0 \,\text{(decision remains open)} \\ 1 & \text{if} \quad 0 = \frac{\mu_i'}{R_i} = \mu_{Hi} \,\text{(frictionless)} \end{cases}$$

see Assmann [1] edition 1 chapter 11.10 p. 265

$\mu_i' =$ rolling friction coefficient of the ball i with radius R_i
$\mu_{Hi} =$ static friction coefficient of the ball i
$\mu_{Gi} =$ sliding friction coefficient of the ball i

and

$$\mu_i = \begin{cases} \frac{\mu_i'}{R_i} & \text{if} \quad \mu_{Hi} > \frac{\mu_i'}{R_i} \,\text{(rolling)} \\ \mu_{Gi} & \text{if} \quad \frac{\mu_i'}{R_i} > \mu_{Hi} \,\text{(sliding)} \\ \mu_{Gi} \text{ or } \frac{\mu_i'}{R_i} & \text{if} \quad \mu_{Hi} = \frac{\mu_i'}{R_i} \,\text{(decision remains open)} \end{cases}$$

see Assmann [1] edition 1 chapter 11.10 p.265

The acceleration b on the inclined plane is:
$b = \delta g \cdot \sin\alpha$, the velocity v and the distance s are $v = \delta g \sin\alpha \cdot t$ and $s = \frac{1}{2} \cdot \delta g \cdot t^2 \cdot \sin\alpha$.

If a ball is rolling on the inclined plane, it is $\delta = \frac{5}{7}$ i.e. Budo [2] §57 p.302 Gl. (8). The moment of inertia J of a ball is $J = \frac{2}{5} \cdot mR^2$. The general formula for rolling on the inclined plane can be written:

$$b = \frac{mg\sin\alpha}{m + \frac{J}{R^2}} \tag{8}$$

One derivation can be found at * at the end of the chapter or we can see this with Budo [2] §57 p.302 Gl. (5)-(7). The ball's moment of inertia insertion leads to:

$$b = \frac{mg\sin\alpha}{m + \frac{2}{5} \cdot m} = \frac{5}{7} \cdot g\sin\alpha \tag{9}$$

Then we have, in the case of rolling, $\delta = \frac{5}{7}$.

stability inequality: $\quad | \cdot | =$ value in R

$$|F_1 - F_2| \leq F_{R1} + F_{R2}$$

limit cases:

$$F_1 - F_2 = F_{R1} + F_{R2} \qquad F_2 - F_1 = F_{R1} + F_{R2} \tag{10}$$

With (1) we can determine γ_g and then with (2),(3),(4) and lastly γ_1 and γ_2 are searched. We get subintervals of $[0, 90°]$ for γ_1 and γ_2.

In a medium with the factors:

$$a_i = 1 - \frac{\varphi_F}{\varphi_{Ki}} \qquad \text{see Budo [2] §16 p.85}$$

with

φ_{Ki} = density of the ball i with radius R_i $\qquad i \in \{1, 2\}$
φ_F = density of the medium (liquid, gas)

only equation (7) changes to:

$$F_1 = \delta_1 m_1 a_1 g \sin\gamma_1 \qquad F_{R1} = m_1 a_1 g \cos\gamma_1 \cdot \mu_1$$

$$F_2 = \delta_2 m_2 a_2 g \sin\gamma_2 \qquad F_{R2} = m_2 a_2 g \cos\gamma_2 \cdot \mu_2$$

The equations (5),(6) and (8) remain unchanged. The calculation is the same.

2.2. 2 motionless balls on a body of revolution with friction

Now we treat 2 motionless balls on a general revolution body shell with friction. The notations are the same as in chapter 1.2.

$h(r)$ = revolution body function
$s = h'(r)$
R_1, R_2 = radii of the balls
m_1, m_2 = mass of the balls

We make the same assumptions about the revolution body function $h(r)$ as in the frictionless case in chapter 1.2. δ_i and μ_i for $i \in \{1,2\}$ are defined as in chapter 2.1.

$$\delta_i := \begin{cases} \frac{5}{7} & \text{if} \quad \frac{\mu_i'}{R_i} < \mu_{Hi} \,\text{(rolling)} \\ 1 & \text{if} \quad \mu_{Hi} < \frac{\mu_i'}{R_i} \,\text{(sliding)} \\ \frac{5}{7} \,\text{or}\, 1 & \text{if} \quad \frac{\mu_i'}{R_i} = \mu_{Hi} \neq 0 \,\text{(decision remains open)} \\ 1 & \text{if} \quad 0 = \frac{\mu_i'}{R_i} = \mu_{Hi} \,\text{(frictionless)} \end{cases}$$

μ_i' = rolling friction coefficient of the ball i with radius R_i
μ_{Hi} = static friction coefficient of the ball i
μ_{Gi} = sliding friction coefficient of the ball i

and

$$\mu_i = \begin{cases} \frac{\mu_i'}{R_i} & \text{if} \quad \mu_{Hi} > \frac{\mu_i'}{R_i} \,\text{(rolling)} \\ \mu_{Gi} & \text{if} \quad \frac{\mu_i'}{R_i} > \mu_{Hi} \,\text{(sliding)} \\ \mu_{Gi} \,\text{or}\, \frac{\mu_i'}{R_i} & \text{if} \quad \mu_{Hi} = \frac{\mu_i'}{R_i} \,\text{(decision remains open)} \end{cases}$$

see Assmann [1] edition 1 chapter 11.10 p.265

It is valid for $i \in \{1,2\}$:

$$F_i = m_i g \sin\gamma_i \qquad F_{R2} = m_i g \cos\gamma_i \cdot \mu_i$$

We know from chapter 1.2:

$$\sin\gamma_i = \frac{s_i}{\sqrt{1+s_i^2}} \qquad \cos\gamma_i = \frac{1}{\sqrt{1+s_i^2}}$$

with

$$s_i = h'(r_i)$$

It follows:

$$F_i = \frac{m_i g s_i \cdot \delta_i}{\sqrt{1+s_i^2}} \qquad F_R = \frac{m_i g \cdot \mu_i}{\sqrt{1+s_i^2}}$$

Stability inequality: $\quad |\cdot|$ = value in R

$$|F_1 - F_2| \leq F_{R1} + F_{R2}$$

limit cases:

$$F_1 - F_2 = F_{R1} + F_{R2}$$
$$F_2 - F_1 = F_{R1} + F_{R2}$$

Both of these equations must be used instead of (3) from chapter 1. All the other equations are the same as in the frictionless case in chapter 1.2. The unknowns r_1, r_2 are searched for these equations. Because of the stability inequality we get subintervals of r_1, r_2.

In a medium (liquid, gas) the forces have the factors $a_1 = 1 - \frac{\varphi_F}{\varphi_{K1}}$ and $a_2 = 1 - \frac{\varphi_F}{\varphi_{K2}}$ see Budo [2] §16 p.85. φ_F is the density of the liquid and φ_{K1}, φ_{K2} are the densities of both balls. Then we obtain the forces:

$$F_i = \frac{m_i g s_i a_i \cdot \delta_i}{\sqrt{1+s_i^2}}$$
$$F_{Ri} = \frac{m_i g a_i \cdot \mu_i}{\sqrt{1+s_i^2}}$$

All the other equations remain unchanged.

We get the equations of revolution conic sections (ellipsoid, paraboloid, hyperboloid, cone) if we insert for $h(r)$ the equation of an ellipse, parabola, hyperbola, or cone.

Both balls are in stable balance in a determined area. If we move the balls out of the area, then the balls move to their previous position. However, in this determined area, both balls can be shifted arbitrary. That means an indifferent balance in this determined area.

to *: The derivation of acceleration at chapter's begin:

Notations:

kinetic energy = $E_{kin} = \frac{mv^2}{2}$
potential energy = $E_{pot} = mgh$ with $h = h_s \sin\alpha$ see fig.
rotational energy = $E_{rot} = \frac{Jw^2}{2}$ J = moment of inertia

law of conservation of energy:

$$\frac{mv^2}{2} + \frac{Jw^2}{2} = mgh$$

We assume that the body is only rolling not gliding. Then it is valid:
angular velociy $= w = \frac{v}{R}$ $R =$ radius of the rolling body

$$\frac{mv^2}{2} + \frac{Jv^2}{2R^2} = mgh_s \sin\alpha$$

With transformation we yield:

$$v = \sqrt{\frac{2mgh_s \sin\alpha}{m + \frac{J}{R^2}}}$$

Because the rolling body has the constant acceleration $b = g\sin\alpha$, there is a uniform accelerated motion. For this motion it is valid: $v^2 = 2h_s \cdot b$ transformed to $b = \frac{v^2}{2h_s}$ Now we insert the equation of v:

$$b = \frac{2mgh_s \sin\alpha}{2h_s \cdot \left(m + \frac{J}{R^2}\right)} = \frac{mg\sin\alpha}{m + \frac{J}{R^2}}$$

Then we have the wanted formula.

References

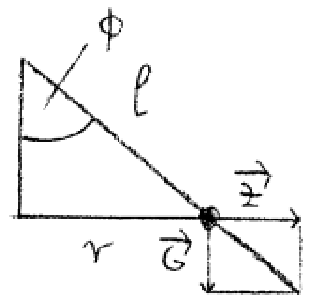
[1] Bruno Assmann "Technische Mechanik" Band 1 Oldenbourg Verlag 12.edition Munich 1991

[2] A Budo "Theoretische Mechanik" 10.edition VEB Deutscher Verlag der Wissenschaften Berlin 1980

36. The swinging body and the rotating disk

1. The swinging body

We view a swinging body as in the figure:



G=mg = gravitation with the earth's acceleration g
m = mass of the swinging body
Z=$\frac{mv^2}{r}$ = centrifugal force with the velocity v and radius r

We only achieve a stationary state, if the resultant of G and Z is in the same direction as the pendulum. l is the pendulum's length and ϕ is the belonging angle. Resultant force:

$$F = \sqrt{G^2 + Z^2} = m \cdot \sqrt{g^2 + \frac{v^4}{r^2}}$$

The condition of stationary state means:

$$\tan\phi = \frac{Z}{G} = \frac{v^2}{gr}$$

It is valid $r = l \sin\phi$. We obtain:

$$\tan\phi = \frac{v^2}{gl\sin\phi}$$

transformed:

$$v = \sqrt{gl\sin\phi\tan\phi}$$

Inversion:

$$v^2 = g \cdot l \cdot \sin\phi \cdot \tan\phi$$

with

$$\sin^2\phi + \cos^2\phi = 1 \qquad \sin\phi = \sqrt{1-\cos^2\phi} \qquad \tan\phi = \frac{\sin\phi}{\cos\phi}$$

it follows:

$$v^2 = gl \cdot \sqrt{1-\cos^2\phi} \cdot \frac{\sqrt{1-\cos^2\phi}}{\cos\phi} = gl \cdot \frac{1-\cos^2\phi}{\cos\phi}$$

multiplied:

$$v^2 \cos\phi = gl - gl\cos^2\phi$$

at last we get a quadratic equation:

$$\cos^2\phi + \frac{v^2}{gl} \cdot \cos\phi - 1 = 0$$

Solution of the quadratic equation:

$$\cos\phi = +\sqrt{1 + \left(\frac{v^2}{2gl}\right)^2} - \frac{v^2}{2gl}$$

Only the root with the positive sign makes sense, there is no solution with $\cos\phi < 0$ in the interval $[0, 90°]$.

We yield the angular velocity with $r = l\sin\phi$:

$$w = \frac{v}{r} = \sqrt{\frac{gl\sin\phi\tan\phi}{l^2\sin^2\phi}} = \sqrt{\frac{g}{l\cos\phi}} \qquad \text{with} \qquad \sin\phi = \cos\phi \cdot \tan\phi$$

It is valid for the period of revolution:

$$T = \frac{2\pi}{w} = \frac{2\pi}{\sqrt{\frac{g}{l\cos\phi}}} = 2\pi \cdot \sqrt{\frac{l\cos\phi}{g}}$$

For $\phi \ll 90°$ it follows with $\cos\phi \approx 1$ the approximation:

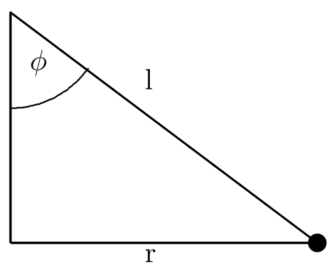
$$T \approx 2\pi\sqrt{\frac{l}{g}}$$

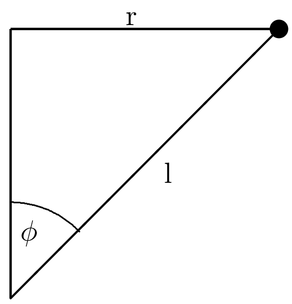
The frictionless case in a medium can be treated in the same way. Instead of inserting g, we insert $\frac{g\cdot(\varphi_K - \varphi_F)}{\varphi_K}$ $= g \cdot \left(1 - \frac{\varphi_F}{\varphi_K}\right)$ (e.g. Budo [1] §16 p.85) into the equations. φ_F is the medium's density (liquid or gas) and φ_K is the body's density.

reasons:

In this balance case, there are only gravitation and centrifugal force. The centrifugal force is independent from the medium, in the orbit case the centrifugal force is only dependent upon r and v. We can see it in the derivation of the centrifugal force, but not from the medium. The gravitational acceleration changes from g to $g \cdot \frac{\varphi_K - \varphi_F}{\varphi_K}$. Only with the gravitation is there a change, because of this we can insert $g \cdot \left(1 - \frac{\varphi_F}{\varphi_K}\right)$ instead of g.

In the case $\varphi_F > \varphi_K$ the apparatus must be turned round.

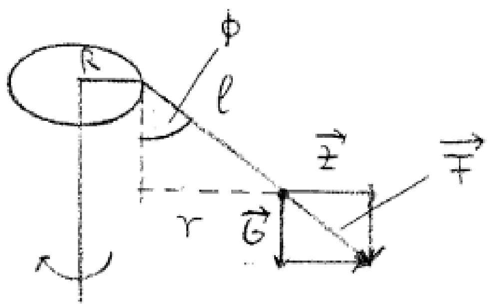
instead of 

then 

Then the same equations are valid again.

2. The rotating disk and the swinging body

We look at the following figure: A (swinging) body hangs on a rotating disk.



G = gravitation with the earth's acceleration g
m = mass of the swinging body
Z = centrifugal force
r = radius
w = angular velocity

We have the following equations (using the same notations):

$$\tan\phi = \frac{Z}{G} \qquad G = mg \qquad Z = mrw^2 = m \cdot (R + l\sin\phi) \cdot w^2$$

With the tangent equation we obtain:

$$\tan\phi = \frac{(R + l\sin\phi) \cdot w^2}{g}$$

Transformed to the angular velocity:

$$w = \sqrt{\frac{g\tan\phi}{R + l\sin\phi}}$$

We get the following relationship to velocity:

$$v = w \cdot r = w \cdot (R + l \sin \phi)$$

$$\text{angle of deviation} = \phi = \angle(\vec{G}, \vec{G} + \vec{Z}) = \angle(\vec{G}, \vec{F}) \qquad \vec{F} = \vec{G} + \vec{Z}$$

$\vec{F}$ is the resultant force, it is valid:

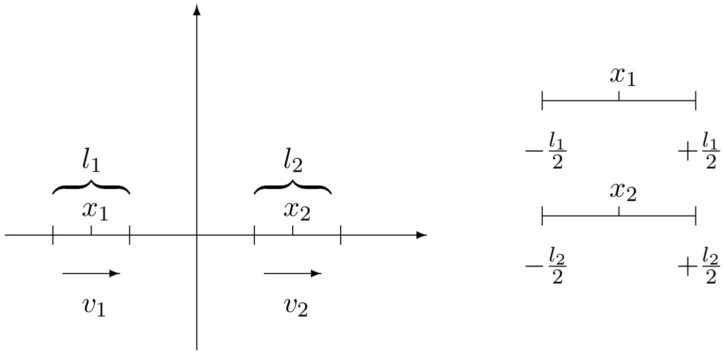
$$F = \sqrt{G^2 + Z^2} = m \cdot \sqrt{g^2 + (R + l \sin \phi)^2 \cdot w^4}$$

References

[1] A Budo "Theoretische Mechanik" 10.edition VEB Deutscher Verlag der Wissenschaften Berlin 1980

37. The general overtaking

We view the following figures:



Two trains with lengths l_1 and l_2 move in the coordinate system on the x-axis with the velocities v_1 and v_2. The velocities shall be functions of the time t. l_1 and l_2 must not be the real length. Combinations with distances are possible for example $l_1 = l_{r1} + 2d_1$ and $l_2 = l_{r2} + 2d_2$ (both sides equal distance) or $l_1 = l_{r1} + d_{11} + d_{12}$ and $l_2 = l_{r2} + d_{21} + d_{22}$ (different distances) with l_r as the real length of the train.

First we want to calculate the observation times, these are the times in which an observer can see the other train from a vertical viewpoint. We introduce for l_1,l_2 a coordinate l with zero point at centre of the train. It is valid:

$$\begin{aligned} &\text{for} \quad l_1: \qquad && -\frac{l_1}{2} \le l \le \frac{l_1}{2} \\ &\text{for} \quad l_2: \qquad && -\frac{l_2}{2} \le l \le \frac{l_2}{2} \end{aligned}$$

At time $t = 0$ the centres of the trains shall be at starting-points x_1 and x_2.

Observation time for l_1:

At time t_1 the following coordinates must be equal:

$$x_1 + l + \int_0^{t_1} v_1(t)\,dt = x_2 - \frac{l_2}{2} + \int_0^{t_1} v_2(t)\,dt \tag{1}$$

at t_2:

$$x_1 + l + \int_0^{t_2} v_1(t)\,dt = x_2 + \frac{l_2}{2} + \int_0^{t_2} v_2(t)\,dt \tag{2}$$

We get the observation time $t_b = |t_2 - t_1|$. One, several or no solutions of t_1 and t_2 are possible. The solutions can be positive or negative. Then we must view exactly the positions of both train to decide when the observation time begins and ends respectively.

Observation time for l_2:

for t_1:

$$x_2 + l + \int_0^{t_1} v_2(t)\,dt = x_1 - \frac{l_1}{2} + \int_0^{t_1} v_1(t)\,dt \tag{3}$$

for t_2:

$$x_2 + l + \int_0^{t_2} v_2(t)\,dt = x_1 + \frac{l_1}{2} + \int_0^{t_2} v_1(t)\,dt \tag{4}$$

The observation time is again $t_b = |t_2 - t_1|$. One or no or several solutions are possible here too. At complex velocity functions the positions of the trains must be viewed.

The overtaking time:

Now we deal with the time that is necessary to the whole overtaking. The calculations of the grenz-times are secured with the following conditions.

for t_2:

$$x_1 - \frac{l_1}{2} + \int_0^{t_2} v_1(t)\,dt = x_2 + \frac{l_2}{2} + \int_0^{t_2} v_2(t)\,dt \tag{5}$$

for t_1:

$$x_1 + \frac{l_1}{2} + \int_0^{t_1} v_1(t)\,dt = x_2 - \frac{l_2}{2} + \int_0^{t_1} v_2(t)\,dt \tag{6}$$

We yield the overtaking time $t_w = |t_2 - t_1|$. For t_1 and t_2 one or no or several solutions are possible. Especially in case of several solutions we must take into considerations the positions of the trains.

The overtaking way is the shift of the train midpoint's position x_m to times t_1 and t_2.

at t_2:

$$x_{m1} = x_1 + \int_0^{t_2} v_1(t)\,dt \qquad\qquad x_{m2} = x_2 + \int_0^{t_2} v_2(t)\,dt$$

at t_1:

$$x_{m1} = x_1 + \int_0^{t_1} v_1(t)\,dt \qquad\qquad x_{m2} = x_2 + \int_0^{t_1} v_2(t)\,dt$$

Then we obtain the overtaking ways:

$$s_1 = \left| \int_0^{t_2} v_1(t)\,dt - \int_0^{t_1} v_1(t)\,dt \right| = \left| \int_{t_1}^{t_2} v_1(t)\,dt \right| \tag{7}$$

$$s_2 = \left| \int_0^{t_2} v_2(t)\, dt - \int_0^{t_1} v_2(t)\, dt \right| = \left| \int_{t_1}^{t_2} v_2(t)\, dt \right| \tag{8}$$

Now we view the special case of temporal constant velocities.

Observation time for l_1:

With the equations (1) and (2) we get after integration for t_1, t_2:

$$t_1 = \frac{x_2 - x_1 - \frac{l_2}{2} - l}{v_1 - v_2} \qquad t_2 = \frac{x_2 - x_1 + \frac{l_2}{2} - l}{v_1 - v_2}$$

and then:

$$t_b = \frac{l_2}{|v_1 - v_2|}$$

Observation time for l_2:

Integration and solving of the equations (3) and (4) to t_1 and t_2:

$$t_2 = \frac{x_2 - x_1 + l - \frac{l_1}{2}}{v_1 - v_2} \qquad t_1 = \frac{x_2 - x_1 + l + \frac{l_1}{2}}{v_1 - v_2}$$

we yield:

$$t_b = \left| \frac{-l_1}{v_1 - v_2} \right| = \frac{l_1}{|v_1 - v_2|}$$

Now we calculate the overtaking time. Integration and solving of the equations (5) and (6):

$$t_2 = \frac{x_1 - x_2 - \frac{l_1}{2} - \frac{l_2}{2}}{v_2 - v_1} \tag{9}$$

$$t_1 = \frac{x_1 - x_2 + \frac{l_1}{2} + \frac{l_2}{2}}{v_2 - v_1} \tag{10}$$

To obtain the overtaking time t_w we calculate the difference again:

$$t_w = \left| \frac{-(l_1 + l_2)}{v_2 - v_1} \right| = \frac{l_1 + l_2}{|v_2 - v_1|}$$

Integration of (7) and (8) and insertion of (9) and (10) leads to the overtaking ways:

$$s_1 = \left| -\frac{v_1 \cdot (l_1 + l_2)}{v_2 - v_1} \right| = (l_1 + l_2) \cdot \left| \frac{v_1}{v_2 - v_1} \right|$$

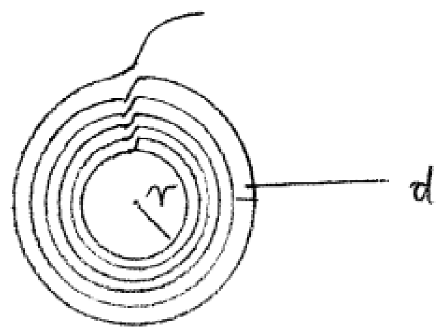
$$s_2 = \left| -\frac{v_2 \cdot (l_1 + l_2)}{v_2 - v_1} \right| = (l_1 + l_2) \cdot \left| \frac{v_2}{v_2 - v_1} \right|$$

The general equations are valid, if $v_1, v_2 \ll c$ (light speed). If $v_1 + v_2 > \frac{c}{10}$ we must calculate relativisticly.

38. The roll

1. A roll:

We view a model of a roll like in the figure:



We introduce:

d:= thickness of the cord
n:= number of rotations
U:= length of the cord
First we assume that the cord has the length:

$$U = 2\pi \cdot (r + d) + 2\pi \cdot (r + 2d) + \cdots + 2\pi \cdot (r + nd)$$

With the sum of powers we get:

$$U = 2\pi \cdot \left(nr + \frac{n \cdot (n+1) \cdot d}{2} \right) \tag{1}$$

or:

$$U = \pi n \cdot (2r + d \cdot (n+1)) \tag{2}$$

Now we solve the equation (1) to n:

$$\frac{U}{2\pi} = \frac{2rn + dn^2 + nd}{2}$$

Transformed:

$$\frac{dn^2}{2} + \left(\frac{d}{2} + r \right) \cdot n = \frac{U}{2\pi}$$

Normal form:

$$n^2 + \left(1 + \frac{2r}{d} \right) \cdot n = \frac{U}{d\pi}$$

This is a quadratic equation in n. We use the known solution formula:

$$n = +\sqrt{\frac{U}{d\pi} + \left(\frac{1}{2} + \frac{r}{d} \right)^2} - \frac{1}{2} - \frac{r}{d}$$

Now we transform the equation (2) to d:

$$d = \frac{\frac{U}{\pi n} - 2r}{n+1} = \frac{U - 2\pi r n}{\pi n \cdot (n+1)}$$

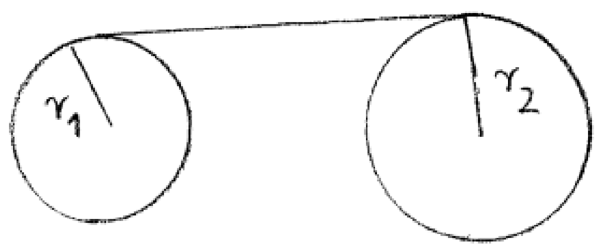
At last we solve the equation (2) to r:

$$2r = \frac{U}{\pi n} - d \cdot (n+1)$$

$$r = \frac{U - \pi d n \cdot (n+1)}{2\pi n}$$

2. Two rolls:

Now we view the spooling of a cord from one roll to another roll. We can think for example of a film projector.



r_1, r_2 = radii of the rolls without cord
d = thickness of the cord (film-strip)
n_1, n_2 = number of rotations of the rolls $\qquad n_1, n_2 \in R^+$
t = time

With equation (2) we have for the first roll:

$$U_1 = \pi n \cdot (2r_1 + d \cdot (n+1)) \tag{3}$$

In regard to the rotation of the first roll the relation $n_1 = \frac{\varphi_1}{2\pi}$ is valid. φ_1 is the covered angle of the first roll in circular measure. If we know the angular acceleration α_1 of the first roll, then it is:

$$\omega_1(t) = \int \alpha_1(t)\, dt + c_1$$

ω_1 = angular velocity of the first roll
c_i = integration constants

$$\varphi_1(t) = \int \omega_1(t)\, dt + c_2$$

In the special case of uniform rotation $\varphi_1 = \omega_1 \cdot t + c_2$ is valid. Now we determine the diameter D_1 of the first roll with cord:

$$D_1 = 2nd + 2r_1$$

The cord or the film-strip has a certain length U_g. This length is known. Thus we get $U_2 = U_g - U_1$. With the equation (3) and $n_1 = \frac{\varphi_1}{2\pi}$ we obtain:

$$U_2 = U_g - \pi \cdot \frac{\varphi_1}{2\pi} \cdot \left(2r_1 + d \cdot \left(\frac{\varphi_1}{2\pi} + 1\right)\right)$$

On the other hand we can write U_2 because of equation (2):

$$U_2 = \pi n_2 \cdot (2r_2 + d \cdot (n_2 + 1))$$

Now we solve this equation to n_2. This can be done in the same way as solving from U to n in the first part:

$$n_2 = +\sqrt{\frac{U_2}{d\pi} + \left(\frac{1}{2} + \frac{r_2}{d}\right)^2} - \frac{1}{2} - \frac{r_2}{d}$$

Thus we have:

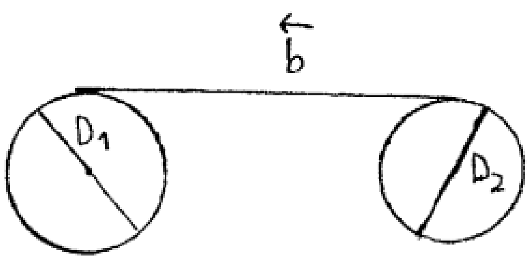
$$D_2 = 2n_2 d + 2r_2$$

D_2 is the diameter of the second roll with cord.

The cord is pulled with **one** acceleration b from the first roll.

$$b = \frac{D_1(t) \cdot \alpha_1(t)}{2} = \frac{D_2(t) \cdot \alpha_2(t)}{2}$$

See the following figure:



It follows to angular acceleration of the second roll:

$$\alpha_2 = \frac{D_1 \cdot \alpha_1}{D_2}$$

To angular velocity of the second roll it is valid:
(c_i = integration constants)

$$\omega_2(t) = \int \alpha_2(t)\, dt + c_3$$

For the covered angle of the second roll we obtain:

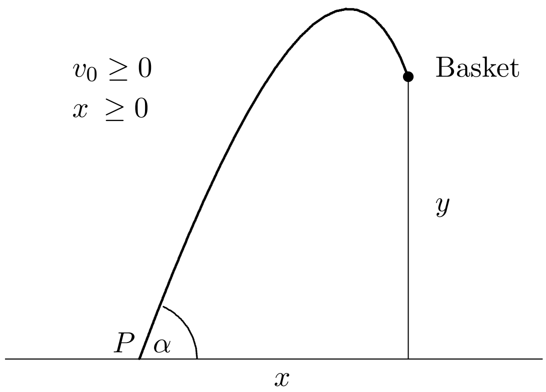
$$\varphi_2(t) = \int \omega_2(t)\,dt + c_4$$

Here the extension of n_1, n_2 to the set of positive real numbers is useful. Otherwise we have difficulties with the integrals. It's clear that these equations describe the motion of both of the rolls only approximately. The equations are more accurate when the thickness of the cord d is smaller.

39. The basketball problem

In basketball there is the problem of hitting a basket at a determined distance and a determined height. The quantities that can be changed during the inclined throw are the initial velocity and the angle of throw. At first we neglect the air resistance.

The basket has from the point of throw P the distance x and the height y. The point P shall be the origin $(x = 0, y = 0)$ of the coordinate system.



a = angle of throw $0° \leq a \leq 90°$
v_0 = initial velocity
g = earth acceleration

Now, we need the known equation of the inclined throw:

$$y = x \cdot \tan a - \frac{gx^2}{2v_0^2 \cos^2 a}$$

We solve to v_0:

$$\frac{gx^2}{2v_0^2 \cos^2 a} = x \tan a - y$$

$$v_0 = \frac{x}{\cos a} \cdot \sqrt{\frac{g}{2 \cdot (x \tan a - y)}} \tag{1}$$

Thus we have the initial velocity $v_0(a)$ as a function from the angle of throw.

Now, we want to calculate the angle of throw with consideration of the initial velocity. We insert at the equation of the inclined throw $\cos^2 a = \frac{1}{1+\tan^2 a}$ and we obtain:

$$y = x \tan a - \frac{gx^2 \cdot (1 + \tan^2 a)}{2v_0^2}$$

Transformed step by step:

$$\frac{2v_0^2 y}{gx^2} = \frac{2v_0^2 \tan a}{gx} - 1 - \tan^2 a$$

Then, we get the following quadratic equation for $\tan a$:

$$\tan^2 a - \frac{2v_0^2 \tan a}{gx} = -\frac{2yv_0^2}{gx^2} - 1$$

Now, we use the known quadratic formula:

$$\tan a_{1,2} = \pm\sqrt{\left(\frac{v_0^2}{gx}\right)^2 - \frac{2yv_0^2}{gx^2} - 1} + \frac{v_0^2}{gx} \tag{2}$$

Thus, two possible solutions for $a(v_0)$ exist. We have obtained the angle of throw with consideration of the initial velocity. With the equations (2) and (1), we can find the angle of throw to fit the initial velocity and vice versa.

Now, we treat the same problem in a medium (gas) with constant density. For trajectories of projection, it is valid (see Budo [1] §16 p.85 equation (22)):

$$m\dot{v}_x = -F(v) \cdot \frac{v_x}{v}$$

$$m\dot{v}_y = -mg - F(v) \cdot \frac{v_y}{v}$$

with:

v = absolute value of the velocity

$$v(t) = |(v_x(t), v_y(t))| := \sqrt{v_x(t)^2 + v_y(t)^2}$$

t = time
m = mass of the ball
$F(v)$ = retarding force in a medium
The point above v means the differential qoutient to time.

Now, we will occupy ourselves with the case $F(v) = kv$. k is a determined constant. This assumption is actualized at small velocities and small bodies. Then, both differential equations simplify to:

$$m\dot{v}_x = -kv_x$$

$$m\dot{v}_y = -mg - kv_y$$

If we insert $\dot{x} = v_x$ and $\dot{y} = v_y$, we get (see Heuser [2] chapter 5 p.78):

$$x(t) = \frac{mv_0 \cos a}{k} \cdot \left(1 - e^{-\frac{kt}{m}}\right) \tag{3}$$

$$y(t) = \frac{m}{k} \cdot \left(v_0 \sin a + \frac{mg}{k}\right) \cdot \left(1 - e^{-\frac{kt}{m}}\right) - \frac{mgt}{k} \tag{4}$$

the solution of both differential equations.

Again, we have the problem of calculating the dependence between the angle of throw a and the initial velocity v_0. We transform the equation (3):

$$\frac{xk}{mv_0 \cos a} = 1 - e^{-\frac{kt}{m}}$$

$$\Rightarrow \qquad e^{-\frac{kt}{m}} = 1 - \frac{xk}{mv_0 \cos a}$$

finally:

$$t = -\frac{m}{k} \cdot \ln\left(1 - \frac{xk}{mv_0 \cos a}\right)$$

Now, we insert the expressions of t and $1 - e^{-\frac{kt}{m}}$ in the equation (4):

$$y = \frac{m}{k}\left(v_0 \sin a + \frac{mg}{k}\right) \cdot \frac{xk}{mv_0 \cos a} + \frac{m^2 g}{k^2} \cdot \ln\left(1 - \frac{xk}{mv_0 \cos a}\right)$$

With $\tan a = \frac{\sin a}{\cos a}$ we obtain:

$$y = x \tan a + \frac{mgx}{kv_0 \cos a} + \frac{gm^2}{k^2} \cdot \ln\left(1 - \frac{xk}{mv_0 \cos a}\right)$$

The solving of v_0 or a is not possible. Here, we must use numerical methods (for example Newton's approximation formula). These methods lead to approximate solutions.

For very exact calculations, we must insert instead of g the acceleration $g \cdot \left(1 - \frac{\varphi}{\varphi_K}\right)$ (see Budo [1] §16 p.85). It is:

φ = density of the medium (gas)
φ_K = density of the body (ball)

For large velocities, the function $F(v) = Cv^2$ is often used. Then C is a determined constant. This case is treated at Timmermann [5], at Parker [6] and at Kooy [7]. At Budo [1] §16 p.85,86, at Kamke [3] p.624 Nr.9-17 and at Lohr [4] p.197-205, it is indicated what can be done for this case.

References

[1] A. Budo "Theoretische Mechanik" 10.edition VEB Deutscher Verlag der Wissenschaften Berlin 1980

[2] Harro Heuser "Gewöhnliche Differentialgleichungen" 2.edition B.G. Teubner Stuttgart 1991

[3] E.Kamke "Differentialgleichungen - Lösungsmethoden und Lösungen" Volume I 10.edition B.G. Teubner Stuttgart 1983

[4] E.Lohr "Mechanik der Festkörper" Walter de Gruyter Berlin 1952

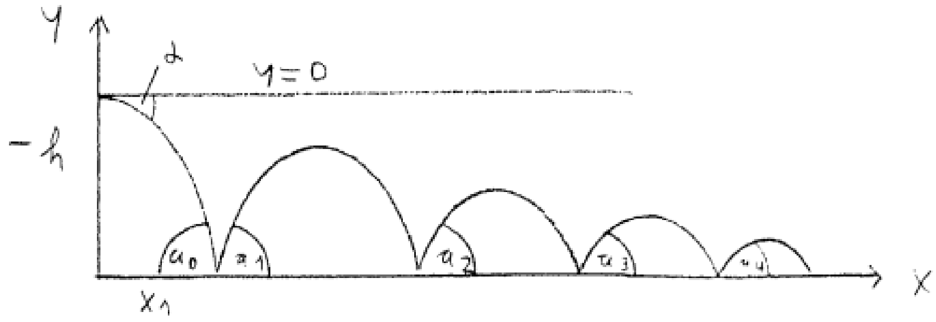
[5] P.Timmermann, J. van der Weele "On the rise and fall of a ball with linear or quadratic drag" Am. J. Phys. 67 (6), June 1999 p.538-546

[6] G.W.Parker "Projectile motion with air resistance quadratic in the speed" Am. J. Phys. 45, p.606-610 (1977)

[7] J.M.J. Kooy and J.W.H.Uytenbogaart, "Ballistics of the Future" (Technical University H.Stam, Haarlem, 1946), p.118-131

40. Successive trajectories of projection

If a body falls to the floor, it is reflected and describes a trajectory of projection. A new reflection is happening, and we view a smaller trajectory of projection. We look at the following figure:



It is remarkable that the angles $a_0, a_1, a_2, a_3, \cdots, a_n, \cdots$ are all equal. The body starts at the altitude h over the floor. x_1 is the x-coordinate at the first incidence on the floor. We insert in the known equation of the inclined projection:

$$-h = x_1 \tan\alpha - \frac{gx_1^2}{2v_A^2 \cdot \cos^2\alpha}$$

Let be:
α = angle of throw
g = earth's acceleration
v_A = initial velocity

Now we transform to x_1:

$$x_1^2 - \frac{2v_A^2 \cdot \tan\alpha \cos^2\alpha}{g} \cdot x_1 = \frac{2hv_A^2 \cos^2\alpha}{g}$$

This is a quadratic equation. We use the known solution formula:

$$x_1 = +\sqrt{\frac{2hv_A^2 \cos^2\alpha}{g} + \frac{v_A^4 \tan^2\alpha \cos^4\alpha}{g^2}} + \frac{v_A^2 \tan\alpha \cos^2\alpha}{g}$$

Now we use $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$:

$$x_1 = +\sqrt{\frac{2hv_A^2 \cos^2\alpha}{g} + \frac{v_A^4 \sin^2\alpha \cos^2\alpha}{g^2}} + \frac{v_A^2 \sin\alpha \cos\alpha}{g}$$

and with $\frac{\sin 2\alpha}{2} = \sin\alpha \cdot \cos\alpha$:

$$x_1 = +\sqrt{\frac{2hv_A^2 \cos^2\alpha}{g} + \frac{v_A^4 \sin^2(2\alpha)}{4g^2}} + \frac{v_A^2 \sin 2\alpha}{2g} \tag{1}$$

Thus x_1 is known. To determine a_0 we derive the equation of projection:

$$y = x\tan\alpha - \frac{gx^2}{2v_A^2\cos^2\alpha}$$

$$y' = \frac{dy}{dx} = \tan\alpha - \frac{gx}{v_A^2\cos^2\alpha}$$

x_1 inserted:

$$y'(x_1) = \tan\alpha - \frac{gx_1}{v_A^2\cos^2\alpha}$$

It is $y'(x_1) < 0$ thus $-y'(x_1) = \tan a_0$.

Now we determine the first bounce velocity v_0 with the energy theorem:

$$\frac{m_1 v_0^2}{2} = \frac{m_1 v_A^2}{2} + m_1 gh \qquad h > 0$$

m_1 is the mass of the body.

$$\Rightarrow \qquad v_0^2 = v_A^2 + 2gh$$

Now we need the time t_A till to the first bounce. We use the temporal equation of the inclined projection:

$$-h = v_A t_A \sin\alpha - \frac{gt_A^2}{2}$$

It follows:

$$t_A^2 - \frac{2v_A\sin\alpha \cdot t_A}{g} = \frac{2h}{g}$$

Thus a quadratic equation for t_A. We use the known solution formula:

$$t_A = +\sqrt{\frac{2h}{g} + \frac{v_A^2\sin^2\alpha}{g^2}} + \frac{v_A\sin\alpha}{g} \tag{2}$$

To calculate the velocity v_1 after the first collision, we need the velocity formula for the partial elastic collision. The second body is the floor with the velocity zero before the collision. Then we obtain with Kuchling [1] chapter 7.3.5 p.121 formula M 7.40:

$$v_1 = v_0 \cdot \left|\frac{m_1 - km_2}{m_1 + m_2}\right| = v_0 \cdot a$$

m_1 = mass of the body $\qquad |\cdot|$ = absolute value
m_2 = mass of the floor
k = collision number $\qquad k \in [0, 1]$

The collision number can be determined with a method in Kuchling [1] chapter 7.3.5 p.122.
At $k = 1$ there is a elastic collision. At $k = 0$ we have an inelastic collision. In this case the body is swallowed from the floor. Then there are no reflections and no further trajectories of projection. Because of this we assume $k > 0$. Then the velocity is reduced with a constant factor:

$$a := \left|\frac{m_1 - km_2}{m_1 + m_2}\right| \tag{3}$$

At the second collision the initial velocity of the projection reduces again with the factor a. ($v_2 = a \cdot v_1 = a^2 \cdot v_0$)

For the initial velocity v_n after the n. collision respectively for the n. trajectory of projection we obtain:

$$v_n = a \cdot v_{n-1} = a^n \cdot v_0$$

Thus we can view the n. trajectory of projection. The angle of throw a_0 remains the same.

Temporal equation of projection:

$$\bar{y} = v_n \cdot t \sin a_0 - \frac{gt^2}{2}$$

Equation of projection:

$$\bar{y} = x \cdot \tan a_0 - \frac{gx^2}{2v_n^2 \cos^2 a_0}$$

Duration of ascent:

$$t_{sn} = \frac{v_n \cdot \sin a_0}{g}$$

Throwing time:

$$t_{wn} = \frac{2v_n \cdot \sin a_0}{g} = \frac{2v_0 \cdot \sin a_0}{g} \cdot a^n$$

Height of ascent:

$$h_{sn} = \frac{v_n^2 \cdot \sin^2 a_0}{2g} = \frac{v_0^2 \cdot \sin^2 a_0}{2g} \cdot a^{2n}$$

In the case of perpendicular projection it is $\sin a_0 = 1$, then we get for the height of ascent with equation (3):

$$h_{sn} = \frac{v^2}{2g} \cdot \left| \frac{m_1 - km_2}{m_1 + m_2} \right|^{2n}$$

In the case of elastic collision ($k = 1$) the height of ascent is:

$$h_s = \frac{v_0^2 \cdot \sin^2 a_0}{2g} \cdot \left| \frac{m_1 - m_2}{m_1 + m_2} \right|^{2n}$$

The range of throw can be written as:

$$s_{wn} = \frac{v_n^2 \cdot \sin 2a_0}{g} = \frac{v_0^2 \cdot \sin 2a_0}{g} \cdot a^{2n} \tag{4}$$

Now we calculate the duration t_G of the whole process:

$$t_G = t_A + \sum_{i=1}^{\infty} t_{wi}$$

$$= t_A + \frac{2v_0 \sin a_0}{g} \cdot \sum_{i=1}^{\infty} a^i$$

This is a geometirc series, for $k \in (0,1]$ we obtain:

$$t_G = t_A + \frac{2v_0 \sin a_0}{g} \cdot \frac{a}{1-a}$$

a and t_A can be calculated with the equations (3) and (2).

Now we determine the whole way:

$$s_{wG} = x_1 + \sum_{i=1}^{\infty} s_{wi}$$

$$= x_1 + \frac{v_0^2 \sin 2a_0}{g} \cdot \sum_{i=1}^{\infty} a^{2i}$$

Here we use again the geometric series for $k \in (0,1]$:

$$s_{wG} = x_1 + \frac{v_0^2 \sin 2a_0}{g} \cdot \frac{a^2}{1-a^2}$$

To determine x_1 and a the equation (1) and (3) are useful. With the geometric sums

$$\sum_{i=1}^{n} a^i = \frac{1-a^{n+1}}{1-a} - 1$$

$$\sum_{i=1}^{n} a^{2i} = \frac{1-a^{2\cdot(n+1)}}{1-a^2} - 1$$

we can calculate at last the sums:

$$\sum_{i=1}^{n} t_{wi} \qquad \text{and} \qquad \sum_{i=1}^{n} s_{wi}$$

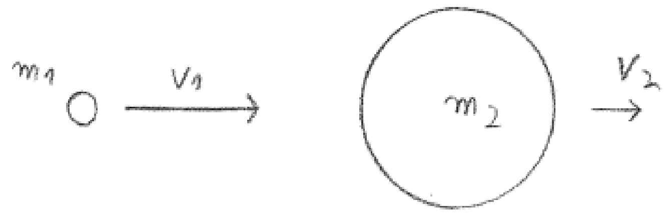
References

[1] Horst Kuchling “Taschenbuch der Physik” 1979 Verlag Harri Deutsch Frankfort on the Main

41. Special case of the elastic collision and the ballistic pendulum

1. Special case of the elastic collision

We want to investigate some interesting cases at the elastic collision. We know the formulas of the elastic collision with v as velocity before the collision and u the velocity after the collision.



Two masses m_1 and m_2 collide. In general you can say the following for the velocities after the collision:

$$u_1 = \frac{2m_2v_2 + v_1 \cdot (m_1 - m_2)}{m_1 + m_2}$$

$$u_2 = \frac{2m_1v_1 + v_2 \cdot (m_2 - m_1)}{m_1 + m_2}$$

Now we view the case $m_1 \ll m_2$ and $v_1 \gg v_2$ as in the figure:

$$u_1 \approx \frac{2m_2v_2 - m_2v_1}{m_2} = 2v_2 - v_1 \approx -v_1$$

The last result represents for example the reflection of a particle on a wall. Under the same conditions we get for u_2:

$$u_2 \approx \frac{2m_1v_1 + m_2v_2}{m_2} = \frac{2m_1v_1}{m_2} + v_2$$

If $v_2 = 0$ then it follows:

$$u_2 \approx \frac{2m_1v_1}{m_2}$$

Now we investigate the case $m_1 \gg m_2$ and $|v_2| \gg v_1$ with $v_2 < 0$ (the second mass flies against the first mass). Then we get:

$$u_1 \approx \frac{2m_2v_2 + m_1v_1}{m_1} = 2v_2 \cdot \frac{m_2}{m_1} + v_1$$

In the case $v_1 = 0$ the term simplifies to:

$$u_1 \approx 2 \cdot v_2 \cdot \frac{m_2}{m_1}$$

The term for u_2 can be written as:

$$u_2 \approx \frac{2m_1v_1 - m_1v_2}{m_1} = 2v_1 - v_2$$

At $v_1 = 0$ it follows $u_2 \approx -v_2$.

In both cases the small mass is reflected with practically equal velocity in the opposite direction. At the large mass ratio of mass and the own velocity before the collision are important.

Now we view the case $m_1 \gg m_2$ and $v_1 \gg v_2$:

$$u_1 \approx \frac{v_1 \cdot (m_1 - m_2)}{m_1 + m_2} \approx v_1$$

$$u_2 \approx \frac{2m_1v_1 - m_1v_2}{m_1} = 2v_1 - v_2 \approx 2v_1$$

Now we are going to look at the case $m_1 \ll m_2$ and $v_1 \ll v_2$:

$$u_1 \approx \frac{2m_2v_2 - m_2v_1}{m_2} = 2v_2 - v_1 \approx 2v_2$$

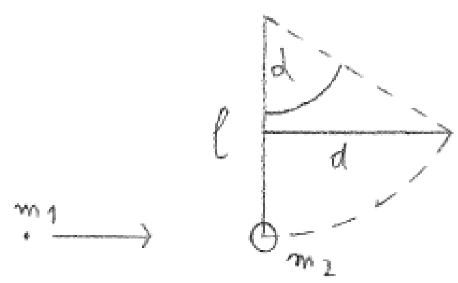
$$u_2 \approx \frac{v_2 \cdot (m_2 - m_1)}{m_1 + m_2} \approx v_2$$

In the last two cases we get the following result:

The large mass changes its velocity insignificantly. The small mass obtains twice as much velocity than the velocity of the large mass before the collision.

2. The ballistic pendulum

We view the following pendulum:



A bullet with the mass m_1 is fired from a gun or a pistol. This mass collides inelastic with the mass m_2 at the pendulum. The pendulum deflects at the angle α. The aim is to determine the bullet's velocity of the mass m_1. l is the length of the pendulum. The bullet with the mass m_1 is shot with the velocity v_1. It collides with the motionless mass m_2 inelastic. You can say that $v_2 = 0$. After the collision both masses have the velocity:

$$u = \frac{m_1 v_1}{m_1 + m_2} \tag{1}$$

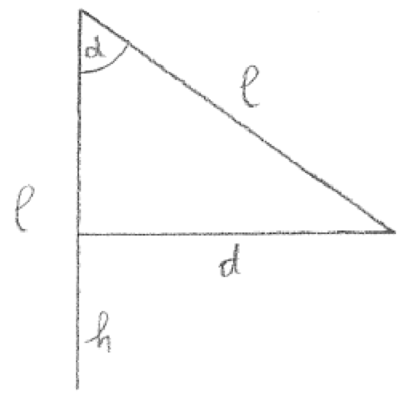
The relation between the velocity u and the reached altitude can be determined with the energy theorem. With $M = m_1 + m_2$ we have:

$$\frac{Mu^2}{2} = Mgh$$

It follows:

$$u = \sqrt{2gh}$$

g = earth's acceleration



According to the figure is $h = l \cdot (1 - \cos\alpha)$. It follows:

$$u = \sqrt{2gl \cdot (1 - \cos\alpha)}$$

Equation (1) transformed:

$$v_1 = \frac{u \cdot (m_1 + m_2)}{m_1} \tag{2}$$

We insert u:

$$v_1 = \frac{(m_1 + m_2) \cdot \sqrt{2gl \cdot (1 - \cos\alpha)}}{m_1} \tag{3}$$

or

$$v_1 = \frac{(m_1 + m_2) \cdot \sqrt{2gh}}{m_1}$$

With that we have an expression for the bullet's velocity v_1. α or h must be measured. If the deflection d is measured, we can calculate with $d = l \cdot \sin\alpha$ the angle α.
Now we solve the equation (3) to $\cos\alpha$:

$$\frac{m_1^2 v_1^2}{(m_1+m_2)^2} = 2gl \cdot (1-\cos\alpha)$$

$$\Rightarrow \qquad \cos\alpha = 1 - \frac{m_1^2 v_1^2}{2gl \cdot (m_1+m_2)^2}$$

With that it is possible to determine vice versa from the bullet's velocity v_1 the angle α and then the deflection d, too.

Now we are going to look at the usual case $m_2 \gg m_1$. If α is in circular measure, then we can use the Taylor series of cosine:

$$1 - \cos\alpha = \frac{\alpha^2}{2!} - \frac{\alpha^4}{4!} + \frac{\alpha^6}{6!} - + \cdots$$

$n! = 1 \cdot 2 \cdot 3 \ldots \cdot n$

In the case $\alpha \ll \frac{\pi}{2}$ we get:

$$1 - \cos\alpha \approx \frac{\alpha^2}{2} \approx \frac{d^2}{2l^2}$$

Inserted in equation (3) at $1 - \cos\alpha$:

$$v_1 \approx d \cdot \sqrt{\frac{g}{l}} \cdot \frac{m_1+m_2}{m_1}$$

transformed to d:

$$d \approx \frac{m_1 v_1}{m_1+m_2} \cdot \sqrt{\frac{l}{g}}$$

In this case we can simplify both last terms with:

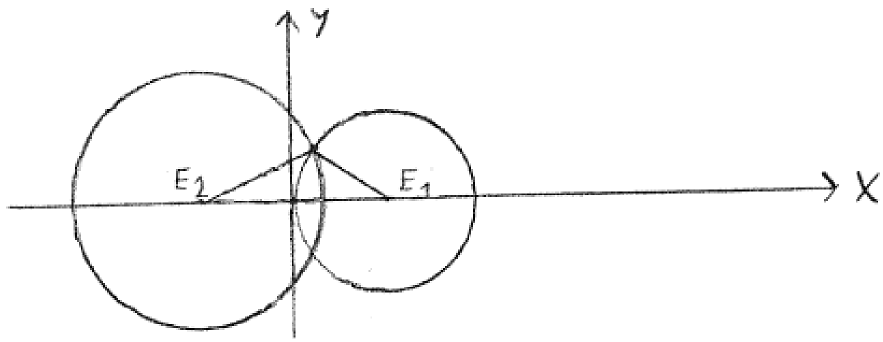
$$\frac{m_1+m_2}{m_1} \approx \frac{m_2}{m_1}$$

42. Interference of two wave sources

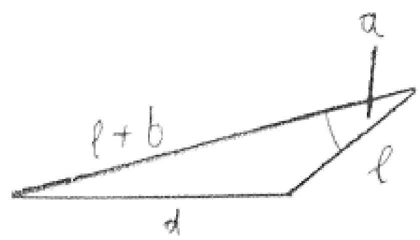
Interference is a known phenomenon of all waves, for example, acoustic, water, and light waves. Optical interferences occur at Newton's rings, slits, double slits, and thin sheaths. Here, we will treat transverse waves. Electromagnetic waves and water waves are examples of transverse waves. Acoustic waves are longitudinal waves that are not treated here.

We study the interference area of two transverse wave sources.

d = distance of the two wave sources
b = phase difference
λ = wave length of the emited waves



First, we will treat a simple wave, the sinusoidal wave. At lines of interference it must be $b = \frac{2n-1}{2} \cdot \lambda \quad n \in N$.



We calculate, at first, the angle a with the law of cosines:

$$d^2 = (l+b)^2 + l^2 - 2 \cdot (l+b) \cdot l \cos a$$

Solved:

$$\cos a = \frac{(l+b)^2 + l^2 - d^2}{2 \cdot (l+b) \cdot l} = \frac{2l^2 + 2bl + b^2 - d^2}{2l^2 + 2bl} = 1 + \frac{b^2 - d^2}{2l^2 + 2bl}$$

Therefore, $a \in [0°, 180°]$, it follows $\cos a \leq 1$. Thus:

$$1 + \frac{b^2 - d^2}{2l^2 + 2bl} \leq 1 \qquad \Leftrightarrow \qquad \frac{b^2 - d^2}{2l^2 + 2bl} \leq 0$$

Therefore, $l, b \geq 0$ thus $2l^2 + 2bl \geq 0$ thus $b^2 - d^2 \leq 0$ or $b \leq d$.
This is a condition of phase difference lines or in the special case of interference lines.
In the interference case, we have:
$b = \frac{\lambda}{2}, \frac{3\lambda}{2}, \ldots, \frac{2n_{max}-1}{2} \cdot \lambda \leq d$ thus:

$$\frac{2n_{max} - 1}{2} \cdot \lambda \leq d \quad \Leftrightarrow \quad 2n_{max} - 1 \leq \frac{2d}{\lambda} \quad \Leftrightarrow \quad 2n_{max} \leq \frac{2d}{\lambda} + 1$$

$$n_{max} \leq \frac{d}{\lambda} + \frac{1}{2} \qquad n_{max} \in N_0$$

is the condition of the number of interference lines.

Now we derive the interference equation with the general phase difference $b \geq 0$. There are two circles, we use the equation of a circle:

$$(l + b)^2 = y^2 + \left(x + \frac{d}{2}\right)^2 \tag{1}$$

$$l^2 = y^2 + \left(x - \frac{d}{2}\right)^2 \tag{2}$$

We subtract equation (2) from equation (1):

$$2lb + b^2 = 2xd \qquad \Leftrightarrow \qquad l = \frac{2xd - b^2}{2b}$$

Insertion of l at (2) and transformation to y:

$$y^2 = \left(\frac{2xd - b^2}{2b}\right)^2 - \left(x - \frac{d}{2}\right)^2$$

simpified:

$$y^2 = \left(\frac{xd}{b} - \frac{b}{2}\right)^2 - \left(x - \frac{d}{2}\right)^2 \tag{3}$$

With this coordinate equation, we can determine lines with phase difference b. If we insert $b = \frac{2n-1}{2} \cdot \lambda \quad n \in N$ for the interference case, we get the interference equation:

$$y^2 = \left(\frac{2xd}{(2n-1) \cdot \lambda} - \frac{2n-1}{4} \cdot \lambda\right)^2 - \left(x - \frac{d}{2}\right)^2 \qquad 1 \leq n \leq n_{max}$$

Another case is the maximum superposition, then it is $b = n \cdot \lambda \quad n \in N_0$.

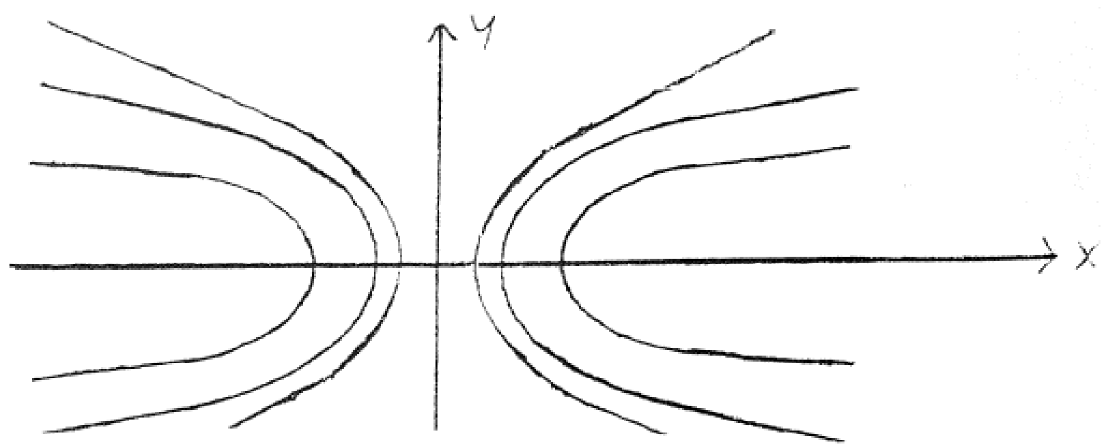
Approximations of (3):

$$y^2 \approx \left(\frac{xd}{b}\right)^2 - \left(x - \frac{d}{2}\right)^2 \qquad x, d \gg b$$

or in the interference case:

$$y^2 \approx \left(\frac{2xd}{(2n-1)\cdot\lambda}\right)^2 - \left(x - \frac{d}{2}\right)^2 \qquad x, d \gg \lambda$$

Further special cases can be calculated.



The figure shows interference lines.

The differentiation of (3) is done with the chain rule:

$$y' = \frac{\left(\frac{xd}{b} - \frac{b}{2}\right)\cdot\frac{d}{b} - \left(x - \frac{d}{2}\right)}{\sqrt{\left(\frac{xd}{b} - \frac{b}{2}\right)^2 - \left(x - \frac{d}{2}\right)^2}}$$

for $x \gg d, b$:

$$y' \approx \frac{x\cdot\left(\frac{d^2}{b^2} - 1\right)}{x\cdot\sqrt{\frac{d^2}{b^2} - 1}} = \sqrt{\frac{d^2}{b^2} - 1}$$

For the angle α we have:

$$\tan\alpha = \frac{y}{x} = \frac{\sqrt{\left(\frac{xd}{b} - \frac{b}{2}\right)^2 - \left(x - \frac{d}{2}\right)^2}}{x}$$

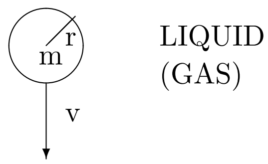
In the case $x \gg d, b$ it follows:

$$\tan\alpha \approx \frac{x\cdot\sqrt{\frac{d^2}{b^2} - 1}}{x} = \sqrt{\frac{d^2}{b^2} - 1} \qquad (4)$$

That means exceeding the limit to tangent and asymptote.

43. Falling ball in a viscous liquid (gas)

Let us view a falling ball in a viscous liquid(gas). First, one differential equation of the falling ball in a viscous liquid (gas) is developed. Then, a solution of this differential equation is obtained through the separation of variables. The solution is the velocity function. With integration, the depth (height) is obtained.



quantities:

m= mass of the ball
r= radius of the ball
v= velocity of the ball
b= acceleration of the ball
s= depth (height) of the ball
φ_K= density of the ball
φ= density of the liquid (gas)
η= dynamic viscosity of the liquid (gas)
c_w= drag coefficient
t= time
g= acceleration of free fall (due to gravity)
λ= mean free path (gas)

$$a := 1 - \frac{\varphi}{\varphi_K}$$

$F(v)$= resistance in a medium

a) liquids:

$$F(v) = \alpha_1 v^2 + \beta_1 v \tag{1}$$

α_1 and β_1 are functions from $r, \varphi_K, \varphi, \eta$ and c_w. With Budo [1] §92 p.525 (92.28), §94 p.535 (94.6) ,Kuchling [3] chapter 10.3.1. p.165 (M 10.20) and Timmermann [5] this formula is valid for $\frac{\varphi r v}{\eta} < 1$.

b) gas:

$$F(v) = \alpha_1 v^2 + \beta_1 v \tag{2}$$

α_1 and β_1 are functions from $r, \varphi_K, \varphi, \eta, c_w$ and λ. This is valid for $\frac{\lambda}{r} < 1$, see Budo [1] §92 p.525 between (92.28) and (92.29), see Kuchling [3] chapter 10.3.1. p.165 (M 10.20) and Timmermann [5].

equation of motion:

With Budo [1] §16 p.83 (16.3) and p.85 (16.21), it is valid:

$$m\dot{v} = mag - F(v) \qquad v(0) = 0$$

an initial-value problem

This initial-value problem can be treated through the seperation of variables.

$$F(v) = \alpha_1 v^2 + \beta_1 v \qquad \alpha_1, \beta_1 > 0$$

$$m\dot{v} = mag - \alpha_1 v^2 - \beta_1 v$$

$$\dot{v} = A - Bv - Cv^2 \qquad v(0) = 0$$

with

$$A := ag \qquad B := \frac{\beta_1}{m} \qquad C := \frac{\alpha_1}{m} \qquad B, C > 0$$

If $a = 1 - \frac{\varphi}{\varphi_K} > 0$ then $A > 0$
(motion downward)

If $a < 0$ then $A < 0$
(motion upwards)

limits of velocity:

$$0 = \dot{v} = A - Bv - Cv^2$$

$$0 = Cv^2 + Bv - A$$

$$0 = v^2 + \frac{B}{C}v - \frac{A}{C}$$

$$v_{1,2} = \pm\sqrt{\frac{A}{C} + \left(\frac{B}{2C}\right)^2} - \frac{B}{2C}$$

$$= -\frac{B}{2C} \pm \sqrt{\frac{B^2}{4C^2} + \frac{4AC}{4C^2}}$$

$$= \frac{1}{2C} \cdot \left(-B \pm \sqrt{B^2 + 4AC}\right)$$

$$v_1 = \frac{1}{2C} \cdot \left(-B + \sqrt{B^2 + 4AC}\right) > 0$$

if $A > 0$

If $A < 0$, then $v_1 < 0$.

$$v_2 = \frac{1}{2C}\left(-B - \sqrt{B^2 + 4AC}\right) < 0$$

Factoring:

$$A - Bv - Cv^2 = -C(v - v_1)(v - v_2)$$

partial fraction decomposition:

$$\frac{1}{A - Bv - Cv^2} = \frac{\alpha}{v - v_1} + \frac{\beta}{v - v_2}$$

$$1 = -C\alpha(v - v_2) - C\beta(v - v_1)$$

$$v = v_1 \quad \Rightarrow \quad 1 = -C\alpha(v_1 - v_2) \quad \alpha = \frac{-1}{C(v_1 - v_2)}$$

$$v = v_2 \quad \Rightarrow \quad 1 = -C\beta(v_2 - v_1) \quad \beta = \frac{1}{C(v_1 - v_2)}$$

it follows:

$$\frac{1}{\dot{v}} = \frac{1}{A - Bv - Cv^2} = \frac{-1}{C(v_1 - v_2)(v - v_1)} + \frac{1}{C(v_1 - v_2)(v - v_2)}$$

$$= \frac{-1}{C(v_1 - v_2)}\left(\frac{1}{v - v_1} - \frac{1}{v - v_2}\right)$$

Integration: (seperation of variables)

$$\int dt = \int \frac{dv}{A - Bv - Cv^2} = \int \frac{-1}{C(v_1 - v_2)} \cdot \left(\frac{1}{v - v_1} - \frac{1}{v - v_2}\right) dv$$

$$= \frac{-1}{C(v_1 - v_2)} \cdot (\ln|v - v_1| - \ln|v - v_2|) = t + d_1$$

d_1, d_2 and d are integration constants.

$$\ln|v - v_2| - \ln|v - v_1| = C(v_1 - v_2)(t + d_1)$$

$$\ln\left|\frac{v - v_2}{v - v_1}\right| = C(v_1 - v_2) \cdot t + d_2$$

$$\frac{v - v_2}{v - v_1} = de^{C(v_1 - v_2)\cdot t} \tag{3}$$

$$v - v_2 = vde^{C(v_1 - v_2)\cdot t} - v_1 de^{C(v_1 - v_2)\cdot t}$$

$$v \cdot \left(1 - de^{C(v_1 - v_2)\cdot t}\right) = v_2 - v_1 de^{C(v_1 - v_2)\cdot t}$$

$$v = \frac{v_2 - v_1 de^{C(v_1 - v_2)\cdot t}}{1 - de^{C(v_1 - v_2)\cdot t}}$$

with $v(t = 0) = 0$ and (3) we get $\frac{v_2}{v_1} = d$

$$v = \frac{v_2 - v_1 \frac{v_2}{v_1} e^{C(v_1 - v_2)\cdot t}}{1 - \frac{v_2}{v_1} e^{C(v_1 - v_2)\cdot t}}$$

$$v_2 < 0$$

$$v = \frac{v_2 \cdot \left(1 - e^{C(v_1 - v_2)\cdot t}\right)}{1 + \frac{|v_2|}{v_1} e^{C(v_1 - v_2)\cdot t}}$$

$$v = \frac{v_2 \left(e^{-C(v_1 - v_2)\cdot t} - 1\right)}{e^{-C(v_1 - v_2)\cdot t} + \frac{|v_2|}{v_1}}$$

end velocity: $\quad v_2 < 0$

$$\lim_{t\to\infty} v(t) = \frac{v_2(0-1)}{0 + \frac{|v_2|}{v_1}} = \frac{-v_2}{|v_2|} \cdot v_1 = v_1$$

$$v_1 = \frac{1}{2C}\left(-B + \sqrt{B^2 + 4AC}\right)$$

$$v = \frac{E(e^{Ft} - 1)}{e^{Ft} + G}$$

with

$$F := -C(v_1 - v_2) \qquad E := v_2 \qquad G := \frac{|v_2|}{v_1}$$

$$b(t) = \dot{v}(t) = E \cdot \frac{Fe^{Ft}(e^{Ft} + G) - (e^{Ft} - 1)Fe^{Ft}}{(e^{Ft} + G)^2}$$

$$= E \cdot \frac{Fe^{Ft}(G+1)}{(e^{Ft} + G)^2}$$

$$= v_2 \frac{-C(v_1 - v_2)e^{-C(v_1 - v_2)\cdot t} \cdot \left(\frac{|v_2|}{v_1} + 1\right)}{\left(e^{-C(v_1 - v_2)\cdot t} + \frac{|v_2|}{v_1}\right)^2}$$

$$1) \qquad \varphi_K > \varphi \qquad \Rightarrow \qquad a > 0 \qquad \Rightarrow \qquad A > 0$$

$$v_2 < 0 \qquad v_1 > 0 \qquad C > 0$$

$$\Rightarrow \quad v_1 - v_2 > 0 \quad \Rightarrow \quad -C(v_1 - v_2) < 0 \quad \Rightarrow v_2 \cdot (-C) \cdot (v_1 - v_2) > 0$$

$$\Rightarrow \quad \dot{v}(t) > 0$$

$\Rightarrow \quad v(t)$ is a strictly increasing function.

$$2) \qquad \varphi_K < \varphi \qquad \Rightarrow \qquad a < 0 \qquad \Rightarrow \qquad A < 0$$

$$v_2 < 0 \qquad v_1 < 0$$

$$a) \qquad |v_2| > |v_1|$$

$$\Rightarrow \quad \frac{|v_2|}{v_1} + 1 < 0 \quad \text{and} \quad v_1 - v_2 > 0$$

$$\Rightarrow \quad -Cv_2(v_1 - v_2) \cdot \left(\frac{|v_2|}{v_1} + 1\right) < 0 \quad \Rightarrow \quad \dot{v}(t) < 0$$

$\Rightarrow \quad v(t)$ is a strictly decreasing function.

$$b) \qquad |v_2| < |v_1|$$

This case doesn't occur (see formulas v_1 and v_2).

Because $v(0) = 0$, it follows $v(t) > 0$ in case 1 ($a > 0$, motion downward) and in case 2 ($a < 0$, motion upwards) $v(t) < 0$.

Integration:

$$s(t) = \int v(t)\,dt = \int \frac{E(e^{Ft}-1)}{e^{Ft}+G}\ dt$$

see Gröbner [2] volume 1 Nr.311 p.107 2)

$$e^{Ft} = y \qquad \text{thus} \qquad t = \frac{1}{F}\ln y$$

$$\frac{dt}{dy} = \frac{1}{Fy} \qquad y(t) = e^{Ft} \quad y(0) = 1$$

$$s(t) = \int\limits_{y(0)}^{y(t)} \frac{E(y-1)\,dy}{(y+G)Fy} = \frac{E}{F}\cdot\int\limits_{y(0)}^{y(t)} \frac{(y-1)\,dy}{y^2+Gy}$$

partial fraction decomposition:

$$\frac{1}{y^2+Gy} = \frac{\delta}{y} + \frac{\varepsilon}{y+G}$$

$$1 = \delta(y+G) + \varepsilon y = y(\delta+\varepsilon) + \delta G$$

$$1 = \delta G \qquad \delta + \varepsilon = 0$$

$$\Rightarrow \delta = \frac{1}{G} \qquad \varepsilon = -\frac{1}{G}$$

thus:

$$\frac{1}{y^2+Gy} = \frac{1}{Gy} - \frac{1}{G(y+G)}$$

test:

$$\frac{1}{Gy} - \frac{1}{G(y+G)} = \frac{y+G-y}{Gy(y+G)} = \frac{G}{Gy(y+G)}$$

$$= \frac{1}{y(y+G)} = \frac{1}{y^2+Gy}$$

thus:

$$\frac{F}{E}\cdot s(t) = \int\limits_{y(0)}^{y(t)} \left(\frac{y-1}{Gy} - \frac{y-1}{G(y+G)}\right)\ dy$$

see Gröbner [2] Nr.12 p.7 4c):

$$= \left[\frac{y}{G} - \frac{\ln|y|}{G} - \frac{y-1}{G} - \frac{-1\cdot G^2 + (-1)\cdot G}{G^2} \cdot \ln|G(y+G)|\right]_{y(0)}^{y(t)}$$

$$= \left[\frac{y - \ln|y|}{G} - \frac{y-1}{G} + \frac{G+1}{G} \cdot \ln|G(y+G)|\right]_{y(0)}^{y(t)}$$

test using differentiation:

$$\frac{d}{dy}\ \frac{E}{F} \cdot \left(\frac{y}{G} - \frac{\ln|y|}{G} - \frac{y-1}{G} + \frac{G+1}{G} \ln|G(y+G)|\right)$$

$$= \frac{E}{F} \cdot \left(\frac{1}{G} - \frac{1}{Gy} - \frac{1}{G} + \frac{G+1}{G} \cdot \frac{G}{G(y+G)}\right)$$

$$= \frac{E}{F} \cdot \left(\frac{1}{G} - \frac{1}{Gy} - \frac{1}{G} + \frac{G+1}{G(y+G)}\right)$$

$$= \frac{E}{F} \cdot \left(\frac{y-1}{Gy} + \frac{-y-G+G+1}{G(y+G)}\right)$$

$$= \frac{E}{F} \cdot \left(\frac{y-1}{Gy} + \frac{1-y}{G(y+G)}\right) = \frac{E}{F} \cdot \left(\frac{y-1}{Gy} - \frac{y-1}{G(y+G)}\right)$$

The antiderivative is confirmed.

$$y(t) = e^{Ft} \qquad y(0) = 1$$

thus:

$$\frac{F}{E} \cdot s(t) = \frac{e^{Ft} - Ft}{G} - \frac{e^{Ft}-1}{G} + \frac{G+1}{G} \cdot \ln|G(e^{Ft}+G)|$$
$$-\frac{1}{G} - \frac{G+1}{G} \cdot \ln|G(1+G)|$$
$$= \frac{e^{Ft} - Ft}{G} - \frac{e^{Ft}-1}{G} - \frac{1}{G} + \frac{G+1}{G} \ln\left|\frac{e^{Ft}+G}{1+G}\right|$$

we conclude:

$$s(t) = \frac{E}{F}\left(\frac{e^{Ft} - Ft}{G} - \frac{e^{Ft}-1}{G} - \frac{1}{G} + \frac{G+1}{G} \cdot \ln\left|\frac{e^{Ft}+G}{1+G}\right|\right)$$

$$s(t) = \frac{E}{GF}\left(e^{Ft} - Ft - e^{Ft} + 1 - 1 + (G+1) \cdot \ln\left|\frac{e^{Ft}+G}{1+G}\right|\right)$$

$$s(t) = \frac{E}{FG}\left((G+1) \cdot \ln\left|\frac{e^{Ft}+G}{1+G}\right| - Ft\right)$$

$$F = -C(v_1 - v_2) \qquad G = \frac{|v_2|}{v_1} \qquad E = v_2$$

$$s(t) = \frac{v_2 \cdot v_1}{-C(v_1-v_2)\cdot|v_2|} \cdot \left(\left(\frac{|v_2|}{v_1} + 1\right) \cdot \ln\left|\frac{e^{-C(v_1-v_2)\cdot t} + \frac{|v_2|}{v_1}}{1 + \frac{|v_2|}{v_1}}\right| + C(v_1-v_2)\cdot t\right)$$

$$v_2 < 0$$

$$= \frac{v_1}{C(v_1 - v_2)} \cdot \left(\left(\frac{|v_2|}{v_1} + 1 \right) \cdot \ln \left| \frac{e^{-C(v_1 - v_2)\cdot t} + \frac{|v_2|}{v_1}}{1 + \frac{|v_2|}{v_1}} \right| + C(v_1 - v_2) \cdot t \right)$$

thus $s(t = 0) = 0$

s must not be too large, because of the pressure (gas) and the density (liquid). The free fall in air with pressure change is treated at Shea [4].

References

[1] Budo, A: Theoretische Mechanik 10.edition (1980) Deutscher Verlag der Wissenschaften

[2] Gröbner, W and Hofreiter, N: Integraltafel 1.Teil 5.edition (1975) Springer Verlag Vienna

[3] Kuchling, H: Taschenbuch der Physik (1979) Verlag Harri Deutsch Frankfort on the Main

[4] P.Mohazzabi, J.H.Shea: High-altitude free fall, Am.J.Phys. 64 (10) p.1242-1246 (1996)

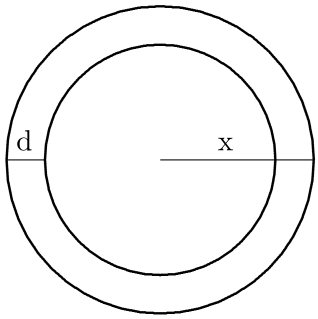
[5] P.Timmermann, J. van der Weele: On the rise and fall of a ball with linear or quadratic drag, Am.J.Phys. 67 (6), June 1999 p.538-546

44. Hollow ball suspending in a liquid

Abstract: We search a relation between radius and wall thickness of a hollow ball that is suspended in a liquid.

Key words: Radius - wall thickness - hollow ball - liquid - gas - medium

We view the following hollow ball in a liquid:



The wall thickness d of the hollow ball is given. Which radius x has the hollow ball for suspending in the liquid?

In [2] an example is calculated. The Archimedes' law is used. We will generalize this calculation here. It is:
$V = \frac{4}{3} \cdot \pi x^3$ = volume of the ball
$V'' = \frac{4}{3} \cdot \pi (x-d)^3$ = volume of the interior ball
$V' = V - V''$ = volume of the hollow ball

Now we turn to the weights. The interior of the ball contains gas with the density φ_G. The hollow ball has the density φ_K. The weight is explained through:
$G' = g \cdot \varphi_K \cdot V'$ $\quad$ g = earth's acceleration
$G'' = g \cdot \varphi_G \cdot V''$

The total weight of the ball is the sum $G = G' + G''$. φ_F shall be the density of the liquid. Now we use Archimedes's law to the suspended ball:

$$g \cdot V \cdot \varphi_F = G = G' + G'' = (V' \cdot \varphi_K + V'' \cdot \varphi_G) \cdot g$$

With the volume formulas

$$V = \frac{4}{3} \cdot \pi \cdot x^3$$

$$V'' = \frac{4}{3} \cdot \pi \cdot (x-d)^3$$

$$V' = \frac{4}{3} \cdot \pi \cdot \left(x^3 - (x-d)^3\right)$$

we get:

$$\varphi_F \cdot x^3 = \varphi_K \cdot \left(x^3 - (x-d)^3\right) + \varphi_G \cdot (x-d)^3$$

Now we solve d:

$$(\varphi_F - \varphi_K) \cdot x^3 = (\varphi_G - \varphi_K) \cdot (x-d)^3$$

$$(x-d)^3 = \frac{\varphi_K - \varphi_F}{\varphi_K - \varphi_G} \cdot x^3$$

It follows:

$$x - d = x \cdot \sqrt[3]{\frac{\varphi_K - \varphi_F}{\varphi_K - \varphi_G}}$$

$$d = x \cdot \left(1 - \sqrt[3]{\frac{\varphi_K - \varphi_F}{\varphi_K - \varphi_G}}\right) \tag{1}$$

It is now possible to solve d and x.

One example:

A hollow ball is suspended in water. In the ball is air with the density 0.0012928 $\frac{\text{kg}}{\text{dm}^3}$. The hollow ball consists of aluminium with the density 2.702 $\frac{\text{kg}}{\text{dm}^3}$. The density of water is 0.9982 $\frac{\text{kg}}{\text{dm}^3}$ at 20 degree Celsius. What is the radius of the hollow ball, if the wall thickness is d=0.015m?

density φ_K of aluminium $= 2.702$ $\frac{\text{kg}}{\text{dm}^3}$
density φ_F of water $= 0.9982$ $\frac{\text{kg}}{\text{dm}^3}$ at 20 degree Celsius
density φ_G of air $= 0.0012928$ $\frac{\text{kg}}{\text{dm}^3}$

We search the radius x of the hollow ball.

We use the equation (1):

$$d = x \cdot \left(1 - \sqrt[3]{\frac{\varphi_K - \varphi_F}{\varphi_K - \varphi_G}}\right)$$

Transformation:

$$x = \frac{d}{1 - \sqrt[3]{\frac{\varphi_K - \varphi_F}{\varphi_K - \varphi_G}}}$$

We get $x = 0.105$m.

Further such examples can be found at Schröer [1].

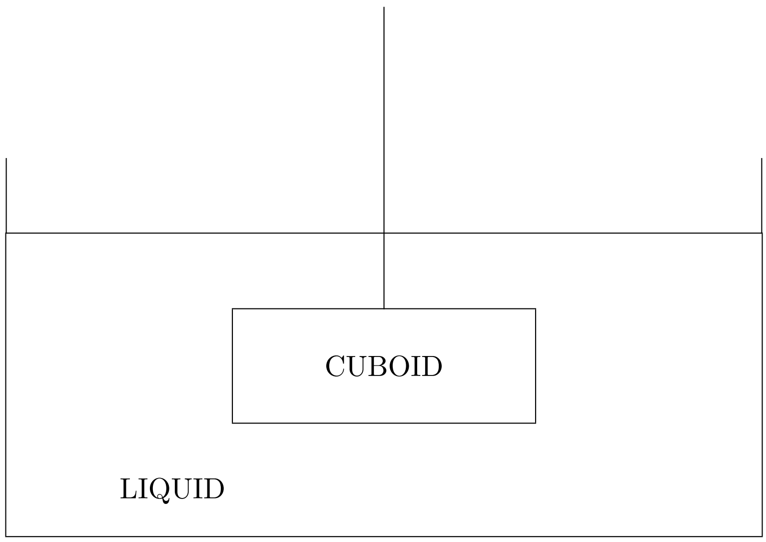
References

[1] Harald Schröer "The floating ball and the suspended hollow ball", Wissenschaft & Technik Verlag, Berlin 2002

[2] http://www.uni-flensburg.de/mathe/zero/veranst/anwmod/kap1/Links/anw_kap1_Ink_3.html

45. Acceleration in liquids and gas'

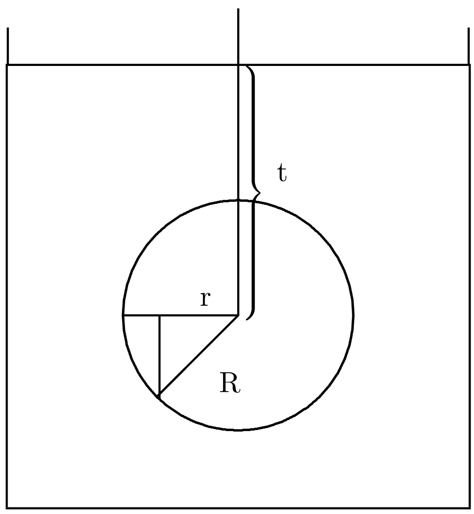
We dip a cuboid in a liquid. The cuboid has the mass m and the volume V.



γ and φ are the specific weight and density of the body. The specific weight and the density of the liquid are denoted with γ_F and φ_F respectively. The force on the cuboid in the liquid can be written as:

$$F = V \cdot (\gamma - \gamma_F)$$

Now we view a ball in the same liquid. The question is, if the same formula is valid for a ball, too?



R is the ball's radius. We can calculate the buoyant force with integration over the pressure in the liquid:

$$F_\uparrow = 2\pi\gamma_F \cdot \int_0^R r \cdot (t + \sqrt{R^2 - r^2})\, dr$$

$$= 2\pi\gamma_F \cdot \int_0^R tr\, dr + 2\pi\gamma_F \cdot \int_0^R r \cdot \sqrt{R^2 - r^2}\, dr$$

is the buoyant force in the depth t.

$$\int r \cdot \sqrt{R^2 - r^2}\, dr = -\frac{1}{3} \cdot (R^2 - r^2)^{\frac{3}{2}}$$

follows with simple differentiation with the chain rule. Thus we obtain:

$$F_\uparrow = 2\pi\gamma_F \cdot \left[\frac{tr^2}{2}\right]_0^R + 2\pi\gamma_F \cdot \left[-\frac{1}{3} \cdot (R^2 - r^2)^{\frac{3}{2}}\right]_0^R = 2\pi\gamma_F \cdot \left(\frac{tR^2}{2} + \frac{R^3}{3}\right)$$

finally:

$$F_\uparrow = \pi R^2 t \cdot \gamma_F + \frac{2}{3} \cdot \pi R^3 \cdot \gamma_F$$

The force that acts below can be expressed as:

$$F_\downarrow = \frac{4}{3} \cdot \pi R^3 \cdot \gamma + \pi R^2 t \cdot \gamma_F - \frac{2}{3} \cdot \pi R^3 \cdot \gamma_F$$

We yield the difference of both forces:

$$F_\downarrow - F_\uparrow = \frac{4}{3} \cdot \pi R^3 \cdot \gamma - \frac{4}{3} \cdot \pi R^3 \cdot \gamma_F = V \cdot (\gamma - \gamma_F)$$

We find the same result with a ball. Is this result valid for all bodies in a liquid? This question can be answered with "yes". A proof is in Forster [1] §15 (15.5) p.157,158.

We look at an arbitrary body in a liquid. The acceleration in a liquid is the force divided through the mass:

$$b = \frac{V \cdot (\gamma - \gamma_F)}{m}$$

We take into consideration the relations $\gamma = g \cdot \varphi$ and $\gamma_F = g \cdot \varphi_F$, with g as earth acceleration:

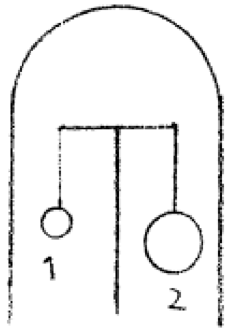
$$b = \frac{V \cdot g \cdot (\varphi - \varphi_F)}{m}$$

With $\frac{m}{V} = \varphi$ we obtain:

$$b = g \cdot \left(1 - \frac{\varphi_F}{\varphi}\right)$$

Now we have a term that is only dependent from densities and earth acceleration. These considerations can be extended to gases, if we insert the density of gas in φ_F.

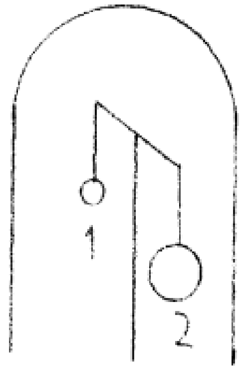
Now we view two masses m_1 and m_2 under a shade. The body 2 shall be greater than the body 1. First the air is in the shade.



The masses are chosen that the weight F is the same. The balance is in equilibrium. Then the following equation is valid:

$$m_1 g \cdot \left(1 - \frac{\varphi_G}{\varphi_1}\right) = F = m_2 g \cdot \left(1 - \frac{\varphi_G}{\varphi_2}\right) \tag{1}$$

φ_G is the density of the air (gas). φ_1 and φ_2 shall be the densities of the masses m_1 and m_2.



Now the air is exhausted. The balance isn't in equilibrium. For the weight forces in vacuum F_1 and F_2 we get:

$$F_1 = m_1 g = \frac{F}{1 - \frac{\varphi_G}{\varphi_1}} \qquad F_2 = m_2 g = \frac{F}{1 - \frac{\varphi_G}{\varphi_2}}$$

The larger body with mass m_2 posess a smaller density because of (1). Thus we have $\varphi_2 < \varphi_1$. But it follows $F_2 > F_1$. The balance bows in favor of the larger body.

If we construct $m_1 = \frac{F_1}{g}$ and $m_2 = \frac{F_2}{g}$ it follows $m_2 > m_1$, too. To the volumes we yield the following equations:

$$V_1 = \frac{m_1}{\varphi_1} = \frac{F}{g \cdot (\varphi_1 - \varphi_G)}$$

$$V_2 = \frac{m_2}{\varphi_2} = \frac{F}{g \cdot (\varphi_2 - \varphi_G)}$$

With $\varphi_2 < \varphi_1$ we get $V_2 > V_1$. If two mass are at the balance in equilibrium, then it follows: A smaller density leads to a greater volume and vice versa. The greater volume or the smaller density of the body 2 are the cause for the experimental result. Usually it is reasoned that the body with greater volume experiences a greater lift force F_A in air than the other body. In fact this result follows from the equation $F_A = V \cdot \varphi_G \cdot g$, with the specific weight $\varphi_G \cdot g$ of the gas (Archimedes' law). But the cause is the greater volume respectively the smaller density.

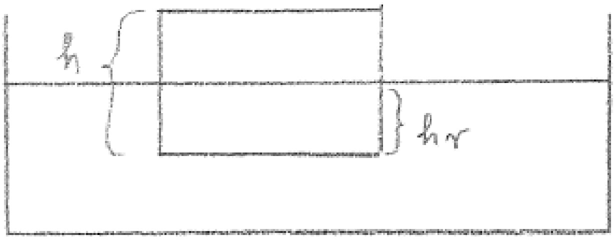
These considerations are valid for liquids too, if the liquid's density is smaller than the densities of both bodies. We must insert the liquid's density in φ_G.

References

[1] Otto Forster "Analysis 3" 2.edition 1983 Vieweg Verlag Brunswick

46. Swimming cuboid

A cuboid swims in a liquid. The height h shall be not larger than $\frac{1}{10}$ of the breadth respectively the length of the cuboid. The density φ of the cuboid must be smaller than the density φ_F of the liquid. Besides we assume that $\frac{\varphi}{\varphi_F} \in (0, 0.2) \cup (0.8, 1)$.



h_r is the height under the liquid's surface. First we want to determine this height. The force of gravity in the liquid is with the basal plane A of the cuboid:

$$F_{\downarrow} = A \cdot h \cdot \gamma$$

γ is the specific weight of the cuboid. With p as pressure we obtain for the buoyant force:

$$F_{\uparrow} = p \cdot A = \gamma_F \cdot h_r \cdot A$$

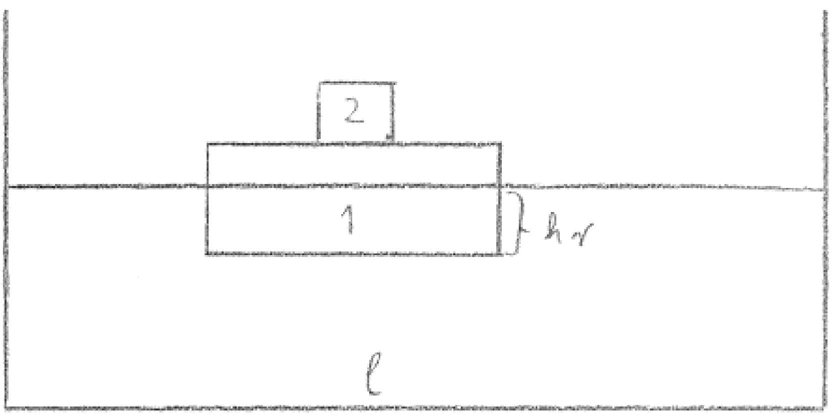
γ_F is the specific weight of the liquid. At swimming the force of gravity and the buoyant force must be equal:

$$\gamma \cdot h \cdot A = \gamma_F \cdot h_r \cdot A$$

If we take into considerations the relations $\gamma = \varphi \cdot g$ and $\gamma_F = \varphi_F \cdot g$ with the earth acceleration g, we see finally:

$$\frac{h_r}{h} = \frac{\gamma}{\gamma_F} = \frac{\varphi}{\varphi_F} \tag{1}$$

Now we view both of these cuboids 1 and 2 as in the figure:



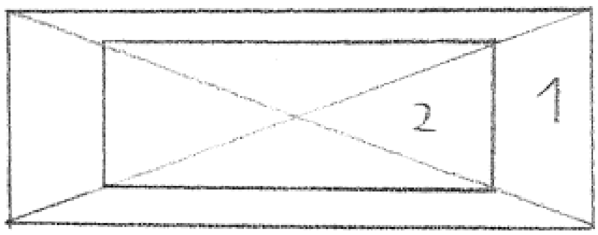
The cuboid 1 swims in the liquid. Then it is valid $\varphi_1 < \varphi_F$.

a_i, b_i, h_i = length, breadth, height of the i. cuboid $\qquad i \in 1, 2$
l, b = length, breadth of liquid's bin

Assumed is here:

$$l, b \gg a_1, b_1 \gg a_2, b_2, h_2 \qquad h_1 \leq \frac{1}{10} \cdot \text{minimum}\{a_1, b_1\}$$

The second cuboid is in the middle of the first cuboid. The edges of both cuboids shall be parallel, see figure.



Here, both of the cuboids do not tip over. With the densities φ_1 and φ_2 of both cuboids we can view the forces. At swimming force of gravity and buoyant force are equal:

$$g \cdot (\varphi_1 a_1 b_1 h_1 + \varphi_2 a_2 b_2 h_2) = F_{\downarrow} = F_{\uparrow} = g \varphi_F a_1 b_1 h_r$$

We transform to:

$$h_r = \frac{\varphi_1 a_1 b_1 h_1 + \varphi_2 a_2 b_2 h_2}{\varphi_F a_1 b_1} = \frac{\varphi_1}{\varphi_F} \cdot h_1 + \frac{\varphi_2 a_2 b_2}{\varphi_F a_1 b_1} \cdot h_2 \tag{2}$$

Thus we have an expression of h_r. The mass of cuboid 2 is in the counter of the second term. Apparently the body 2 must not be a cuboid. But the body 2 must be in the middle and the presumptions about the lengths must be true. Now we look at the displaced volume V_s of the liquid:

$$V_s = a_1 b_1 h_r \qquad \text{for} \qquad h_r \leq h_1$$

$$V_s = (h_r - h_1) \cdot a_2 b_2 + a_1 b_1 h_1 \qquad \text{for} \qquad h_r \geq h_1$$

If we insert h_r of equation (2), then we obtain:

$$V_s = \frac{\varphi_1 a_1 b_1 h_1}{\varphi_F} + \frac{\varphi_2}{\varphi_F} \cdot a_2 b_2 h_2 \qquad \text{for} \quad h_r \leq h_1$$

$$V_s = \left(\left(\frac{\varphi_1}{\varphi_F} - 1 \right) \cdot h_1 + \frac{\varphi_2 a_2 b_2 h_2}{\varphi_F a_1 b_1} \right) \cdot a_2 b_2 + a_1 b_1 h_1$$
$$\text{for} \qquad h_r \geq h_1$$

In general we have the first case. The second case occurs rarely. These formulas are only valid, if the body 2 is a cuboid.

With V_s we can conclude to the height's difference in the bin. Then we have:

$$\Delta r = \frac{V_s}{l \cdot b}$$

These considerations are valid for $\frac{\varphi_1}{\varphi_F} \in (0, 0.2) \cup (0.8, 1)$. Besides φ_2 must not be extremly great.

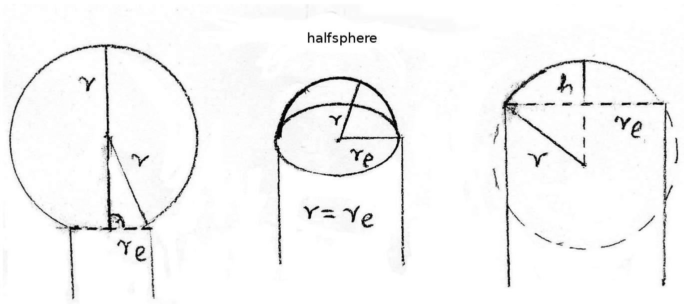
47. Soap Bubbles

Abstract: First we view a soap bubble with opening. Then we describe a system of two soap bubbles. Finally we treat a connected system of n soap bubbles.

Key words: soap bubble - pressure - liquid - gas - medium - ball

1. The simple Soap Bubble

We view a soap bubble that is connected with a tap.



σ is the surface tension and r the soap bubble's radius. r_e shall be the circular opening's radius of the tap. First we want to determine the soap bubble's pressure p. The soap bubble will have the shape of a spherical segment approximately. The force acting on the soap bubble is denoted with F.

The spherical segment's surface can be written:

$$A = m \cdot \pi r^2 \qquad 0 < m \leq 4$$

It is valid, if $\Delta r \ll r$:

$$F = p \cdot A$$

We have the following work:

$$\Delta W = F \cdot \Delta r$$

It follows:

$$\Delta W = F \cdot \Delta r = p \cdot A \cdot \Delta r = p \cdot m\pi r^2 \cdot \Delta r$$

The same work must be done for the surface's increase:

$$\Delta W = \sigma \cdot \Delta A = \sigma m\pi \cdot [(r + \Delta r)^2 - r^2]$$

$$= \sigma m\pi(2r\Delta r + \Delta r^2)$$

If $\Delta r \ll r$, then we get:

$$\Delta W = 2m\sigma\pi r\Delta r$$

With equating we obtain:

$$pm\pi r^2 \cdot \Delta r = 2m\sigma\pi r \cdot \Delta r$$

Then we have for the soap bubble's pressure:

$$p = \frac{2\sigma}{r}$$

A soap bubble consists of two boundary layers. Thus the whole pressure is:

$$p = \frac{2\sigma}{r} + \frac{2\sigma}{r + \Delta r}$$

If $\Delta r \ll r$, we get:

$$p \approx \frac{4\sigma}{r}$$

This pressure formula will be used. To this derivation see Höfling [1] chapter 2.12.5, p.286.

We want to see the movement of a deflating soap bubble, at open tap. We have for the spherical segment's surface:

$$O = 2\pi r h = 2\pi r \cdot (r \pm \sqrt{r^2 - r_e^2}) \tag{1}$$

In case of positive root's sign it is clear that with smaller r the surface decreases. Now it will be shown in case of negative root's sign the surface decreases, if r increases, see the figures at the beginning. We need the differentiation of the surface. We use the product- and the chain rule:

$$\begin{aligned} O'(r) &= 2\pi \left(r - \sqrt{r^2 - r_e^2} + r \cdot \left(1 - \frac{r}{\sqrt{r^2 - r_e^2}} \right) \right) \\ &= 2\pi \cdot \left(2r - \frac{2r^2 - r_e^2}{\sqrt{r^2 - r_e^2}} \right) \end{aligned}$$

For $r_e > 0$ it is valid :

$$0 < r_e^4$$

It follows:

$$4r^4 - 4r^2 r_e^2 < 4r^4 - 4r^2 r_e^2 + r_e^4$$

Factor put before the brackets:

$$4r^2 \cdot (r^2 - r_e^2) < 4r^4 - 4r^2 r_e^2 + r_e^4$$

Square root:

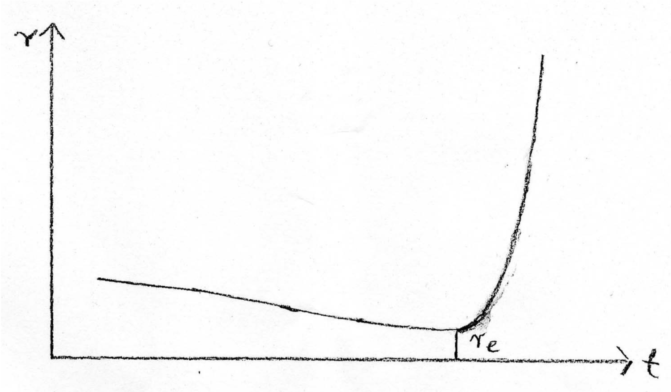
$$2r \cdot \sqrt{r^2 - r_e^2} < 2r^2 - r_e^2$$

This leads to:

$$2r < \frac{2r^2 - r_e^2}{\sqrt{r^2 - r_e^2}}$$

Thus it is proved that $O'(r) < 0$. We conclude with the monotonicity theorem: The surface decreases if r increases. It's valid for the negative root sign.

We view the temporal deflating soap bubble. We have the following figure, see Jackson [2]:



At Jackson [2] the time $t(r)$ is calculated. A differential equation with separated variables is solved. A spherical segment model is used. We follow that $O(r(t))$ decreases with t.

What happens with the surface, if $r \to \infty$?

Then we have:

$$O(r) = 2\pi r(r - \sqrt{r^2 - r_e^2})$$

With the differentiation we conclude for $h \ll a$:

$$\sqrt{a} - \sqrt{a-h} \approx \frac{h}{2\sqrt{a}}$$

Thus for $r_e \ll r$:

$$r - \sqrt{r^2 - r_e^2} \approx \frac{r_e^2}{2\sqrt{r^2}} = \frac{r_e^2}{2r}$$

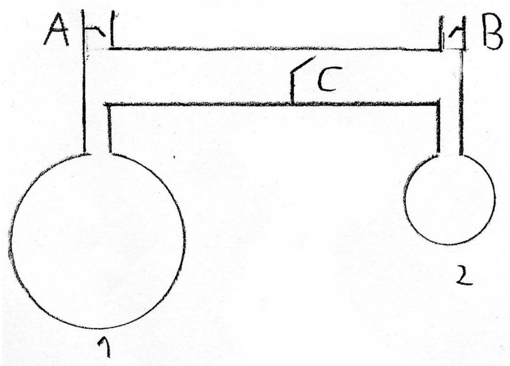
Then we get:

$$O(r) \approx 2\pi r \cdot \frac{r_e^2}{2r} = \pi \cdot r_e^2$$

The surface reduces to the circular area of the opening.

2. A System of 2 Soap Bubbles

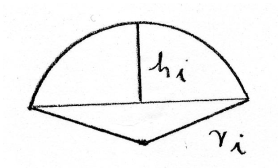
Now we look at two soap bubbles as in the figure with the taps A, B and C.



First the tap C is closed. Both soap bubbles are generated. The taps A and B are open. Then A and B will be closed and the tap C will be opened, see Höfling [1] chapter 2.12.5, p.286.

We define:

$\sigma_i, p_i, r_i, r_{ei}, h_i$ = surface tension, pressure, radius, opening's radius and height of the i. soap bubble, i=1 or 2.



Assumption:

$$\frac{\sigma_1}{r_1} < \frac{\sigma_2}{r_2} \qquad\qquad r_i > r_{ei}$$

Then soap bubble 2 deflates, and soap bubble 1 expands. The force to a soap bubble can be written as:

$$F(h) = O(h) \cdot p(r(h)) = 2\pi r h \cdot \frac{4\sigma}{r} = 8\pi\sigma h$$

Now we use the energy theorem (potential energy). Only the situation at the end is interesting for us. Thus the kinetic energy is zero. The potential energy by gravity ($m \cdot g \cdot \Delta h$) can be neglected because of the small liquid's and gas mass. We get:

$$W_R + 8\pi\sigma_1 \cdot \int_{h_1}^{h_3} h\,dh = 8\pi\sigma_2 \cdot \int_0^{h_2} h\,dh$$

The friction work is denoted with W_R. h_3 is the new height of soap bubble 1. After the integration:

$$W_R + 4\pi\sigma_1 \cdot (h_3^2 - h_1^2) = 4\pi\sigma_2 h_2^2$$

We solve to h_3:

$$h_3^2 - h_1^2 = \frac{4\pi\sigma_2 h_2^2 - W_R}{4\pi\sigma_1}$$

At last:

$$h_3 = \sqrt{\frac{\sigma_2 h_2^2}{\sigma_1} - \frac{W_R}{4\pi\sigma_1} + h_1^2}$$

We get the radius with equation (1):

$$h_i = r_i \pm \sqrt{r_i^2 - r_{ei}^2}$$

Squared:

$$(h_i - r_i)^2 = r_i^2 - r_{ei}^2$$

It follows:

$$-2h_i r_i + h_i^2 = -r_{ei}^2$$

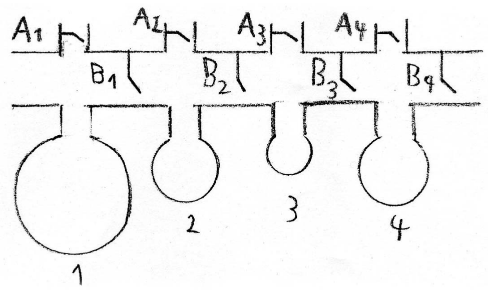
Finally we get:

$$r_i = \frac{h_i^2 + r_{ei}^2}{2h_i} = \frac{h_i}{2} + \frac{r_{ei}^2}{2h_i}$$

In this model the viscosity belongs to W_R. A special case is $\sigma_1 = \sigma_2 = \sigma$.

3. A system with n Soap Bubbles

The variables are analogous for $i \in 1,...,n+1$. Now we view a system like in the figure:



First the taps $B_1,...,B_n$ are closed. The taps $A_1,...,A_n$ are open. n soap bubbles are generated. Then $A_1,...,A_n$ will be closed and the taps $B_1,...,B_n$ will be opened.

Assumption:

$$\frac{\sigma_1}{r_1} < \frac{\sigma_i}{r_i} \qquad \text{for} \qquad i \in 2,...,n$$

Then soap bubble 1 expands and the other soap bubbles deflate. We generalize the calculation. We search the new height h_{n+1} of soap bubble 1. We use the energy theorem as before:

$$W_R + 8\pi\sigma_1 \cdot \int_{h_1}^{h_{n+1}} h\,dh = 8\pi \cdot \sum_{i=2}^{n} \sigma_i \cdot \int_{0}^{h_i} h\,dh$$

Integration:

$$W_R + 4\pi\sigma_1 \cdot (h_{n+1}^2 - h_1^2) = 4\pi \cdot \sum_{i=2}^{n} \sigma_i h_i^2$$

Solving:

$$h_{n+1}^2 - h_1^2 = \frac{4\pi \cdot \sum_{i=2}^{n} \sigma_i h_i^2 - W_R}{4\pi\sigma_1}$$

Finally we obtain:

$$h_{n+1} = \sqrt{\frac{1}{\sigma_1} \cdot \sum_{i=2}^{n} \sigma_i h_i^2 - \frac{W_R}{4\pi\sigma_1} + h_1^2}$$

We get the radius as before from:

$$r_i = \frac{h_i^2 + r_{ei}^2}{2h_i} = \frac{h_i}{2} + \frac{r_{ei}^2}{2h_i} \qquad \text{for} \qquad i \in 1, ..., n+1$$

A special case is $\sigma_1 = \sigma_2 = ... = \sigma_n = \sigma$. We remark that n is limited because of the life duration of soap bubbles.
We mention further literature to soap bubbles. Soap bubbles at low temperatures were treated by Grosse [3]. The possible mathematical solutions of soap bubbles are presented at Ferus [4]. Oscillations of soap bubbles can be found at Kornek [5].

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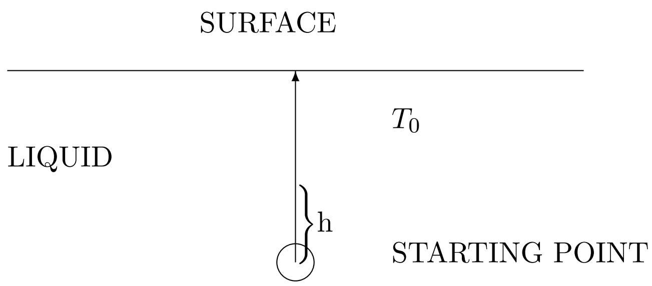
48. Ascent of a gas pocket in a liquid

Everybody has seen how air bubbles ascend in water. An interested observer remarks that the air bubble's diameter increases during the ascent, therefore the following questions. Why do air bubbles ascend and not descend? Why does an air bubble's diameter increase during ascent? How fast does an air bubble move? How long does it take till an air bubble reaches the surface? Now, we will occupy ourselves with the motion of an air pocket in a liquid.

We introduce the following quantities:

r = radius of the gas pocket
o = surface tension of the liquid
p = pressure
g = acceleration of gravity (earth's acceleration)
φ_F = density of the liquid
φ_G = density of the gas
T_0 = depth of the air pocket at the beginning in the liquid
h = height that the air pocket has ascended
see the figure:



h is measured from bottom to top.

The pressure in a gas pocket can be written using Kuchling [4] chapter 11.1.2 p.169,170:

$$p = \frac{2o}{r}$$

The gravity pressure in a liquid can be described, for example, using Kuchling [4] chapter 8.1.2 p.146:

$$p = g \cdot (T_o - h) \cdot \varphi_F$$

The pressures must be equal, thus:

$$\frac{2o}{r} = g \cdot (T_0 - h) \cdot \varphi_F$$

It follows:

$$r = \frac{2o}{g \cdot (T_0 - h) \cdot \varphi_F}$$

Here, we recognize that the radius of the gas pocket decreases with the depth $T_0 - h$. Thus, with the ascent, the gas pocket becomes larger.

The mass m of the gas pocket must be known, then:

$$\varphi_G = \frac{m}{\frac{4}{3} \cdot \pi r^3}$$

We obtain, if we insert for r :

$$\varphi_G = \frac{3mg^3 \cdot (T_0 - h)^3 \cdot \varphi_F^3}{4\pi \cdot 8o^3} = \frac{3mg^3 \cdot (T_0 - h)^3 \cdot \varphi_F^3}{32 \cdot \pi o^3}$$

Here we can answer the question: Why does a gas pocket move upwards in the liquid? The density of the gas is in general much smaller than the density of the liquid.

We use the energy conservation law $E_{ges} = E_{pot} + E_{kin}$. It is:

E_{ges} = total energy
E_{pot} = potential energy
$E_{kin} = \frac{mv^2}{2}$ = kinetic energy
We have the pontential energy difference $E_{ges} - E_{pot} = m \cdot g \cdot h$. It follows:

$$m \cdot g \cdot h = \frac{mv^2}{2} \qquad \text{or} \qquad h = \frac{v^2}{2g}$$

We yield the falling acceleration b in a liquid using Budo [1] §16 p.85:

$$b = \left(\frac{\varphi_F}{\varphi_G} - 1\right) \cdot g$$

Now we can write the differential equation of the movement:

$$\dot{v} = \left(\frac{\varphi_F}{\varphi_G} - 1\right) \cdot g = \frac{\varphi_F}{\varphi_G} \cdot g - g$$

The dot above v means differentiation with respect to time.

If we insert the expressions of h and φ_G, we get:

$$\dot{v} = \frac{32 \cdot \pi o^3}{3 \cdot mg^2 \cdot \left(T_0 - \frac{v^2}{2g}\right)^3 \cdot \varphi_F^2} - g = F(v) \tag{1}$$

and $v(0) = 0$ as the initial value problem. This differential equation, respectively, the initial value problem, must now be solved.

Separation of variables:

$$\frac{dv}{dt} = \dot{v} = F(v)$$

Integration:(i.e. Forster [2] §11 Satz 1 p.111-113)

$$\int_0^t dt - c_1 = \int_0^v \frac{dv}{F(v)} = \int_0^v \frac{dv}{\frac{k}{\left(T_0 - \frac{v^2}{2g}\right)^3} - g} = \int_0^v \frac{\left(T_0 - \frac{v^2}{2g}\right)^3 dv}{k - g \cdot \left(T_0 - \frac{v^2}{2g}\right)^3}$$

$$\text{with} \qquad k = \frac{32 \cdot \pi o^3}{3 \cdot m g^2 \varphi_F^2}$$

We divide the polynomials:

$$\frac{\left(T_0 - \frac{v^2}{2g}\right)^3}{k - g \cdot \left(T_0 - \frac{v^2}{2g}\right)^3} = -\frac{1}{g} + \frac{k}{g \cdot \left(k - g \cdot \left(T_0 - \frac{v^2}{2g}\right)^3\right)}$$

Thus, the integration yields:

$$\int_0^v \frac{\left(T_0 - \frac{v^2}{2g}\right)^3 dv}{k - g \cdot \left(T_0 - \frac{v^2}{2g}\right)^3} = -\frac{v}{g} + \int_0^v \frac{k\, dv}{g \cdot \left(k - g \cdot \left(T_0 - \frac{v^2}{2g}\right)^3\right)}$$

We must calculate the zeros of the denominator of the last integral. If we know the zeros, we can do an exact integration. Because of these zeros, which can be simple or multiple, an approximation with numerical methods using the Simpson rule makes no sense. We must factorize the denominator and than we can calculate the zeros with respect to v. We use a special factorizing that can be found in many mathematical collections of formulas i.e. Sieber [5] p.3.

$$a^3 - b^3 = (a - b) \cdot (a^2 + ab + b^2)$$

$a^2 + ab + b^2 = 0$ is a quadratic equation with respect to a, we obtain a as:

$$a_{1,2} = \pm\sqrt{-b^2 + \frac{b^2}{4}} - \frac{b}{2} = \pm i \cdot b \cdot \frac{\sqrt{3}}{2} - \frac{b}{2} \qquad \text{with} \qquad i = \sqrt{-1}$$

Then we can do a further factorizing:

$$a^2 + ab + b^2 = \left(a - \frac{i\sqrt{3}b}{2} + \frac{b}{2}\right) \cdot \left(a + \frac{i\sqrt{3}b}{2} + \frac{b}{2}\right)$$

Together we get:

$$a^3 - b^3 = (a - b) \cdot \left(a - \frac{i\sqrt{3}b}{2} + \frac{b}{2}\right) \cdot \left(a + \frac{i\sqrt{3}b}{2} + \frac{b}{2}\right)$$

This factorizing can be used on the denominator of the integral:

$$k - g \cdot \left(T_0 - \frac{v^2}{2g}\right)^3 = \left(\sqrt[3]{k} - \sqrt[3]{g} \cdot \left(T_0 - \frac{v^2}{2g}\right)\right)$$
$$\cdot \left(\sqrt[3]{k} - \frac{i\sqrt{3}\sqrt[3]{g}}{2} \cdot \left(T_0 - \frac{v^2}{2g}\right) + \frac{\sqrt[3]{g}}{2} \cdot \left(T_0 - \frac{v^2}{2g}\right)\right)$$
$$\cdot \left(\sqrt[3]{k} + \frac{i\sqrt{3}\sqrt[3]{g}}{2} \cdot \left(T_0 - \frac{v^2}{2g}\right) + \frac{\sqrt[3]{g}}{2} \cdot \left(T_0 - \frac{v^2}{2g}\right)\right)$$

We yield the following zeros:

$$v_{N1,2} = \pm\sqrt{2g \cdot \left(T_0 - \sqrt[3]{\frac{k}{g}}\right)}$$

$$v_{N3,4} = \pm\sqrt{\frac{k^{\frac{1}{3}} - \frac{i\sqrt{3}g^{\frac{1}{3}}T_0}{2} + \frac{g^{\frac{1}{3}}T_0}{2}}{\frac{g^{-\frac{2}{3}}}{4} - \frac{i\sqrt{3}g^{-\frac{2}{3}}}{4}}}$$

$$v_{N5,6} = \pm\sqrt{\frac{k^{\frac{1}{3}} + \frac{i\sqrt{3}g^{\frac{1}{3}}T_0}{2} + \frac{g^{\frac{1}{3}}T_0}{2}}{\frac{i\sqrt{3}g^{-\frac{2}{3}}}{4} + \frac{g^{-\frac{2}{3}}}{4}}}$$

The zeros can be simple zeros or multiple zeros.

The integral

$$\int_0^v \frac{k\,dv}{g \cdot \left(k - g \cdot \left(T_0 - \frac{v^2}{2g}\right)^3\right)}$$

can be determined using Gröbner [3] chapter 11 Nr.14 or Nr.15 p.5,6.

At Nr.14, there is a partial fraction decomposition and then an integration - once with pure complex and once without complex numbers.
At Nr.15, an integral is chosen. The coefficients must be determined. Here, a complex calculation and a calculation without complex numbers is possible.

The 6 zeros $v_{N1}, \ldots, v_{N6}$ can be (as a function of k, T_0 and g) simple zeros or multiple zeros. Thus, the methods of Nr.14 respectively Nr.15 are generally not favorable. It is better to work with concrete values of g, T_0, k. Then it can be decided clearly, if the zeros are simple or multiple.

We must take into consideration the calculation of complex logarithms in Gröbner [3] chapter 11 Nr.9a p.2.

After calculation of the integral, the integration constant c_1 must be chosen that $v(t = 0) = 0$.

Thus we have:

$$t(v) = -\frac{v}{g} + \int_0^v \frac{k\,dv}{g \cdot \left(k - g \cdot \left(T_0 - \frac{v^2}{2g}\right)^3\right)} + c_1$$

The function $v(t)$ can only be approximately determined numerically.

A big problem is the calculation of $h(t)$ respectively $b(t)$ because of the relationships:

$$h(t) = \int v(t)\,dt \qquad b(t) = \dot{v}$$

The function $v(t)$ isn't known exactly or can only be approximately determined. If we want to calculate $h(t)$ and $b(t)$, as well, then we must use a numerical method, that approximately determines $v(t), b(t)$, and $h(t)$ at the same time. To solve this problem, literature on numerical mathematics must be searched for.

Here, we get the momentary depth:

$$T = T_0 - h(t)$$

Thus, the motion is determined completly. The determination of the temporal motion is, as we have seen, a complicated problem. We have treated a mathematical model with inviscid fluid and spherical bubbles. With viscosity the situation can be different, see Tuteja [6] or Miyagi [7].

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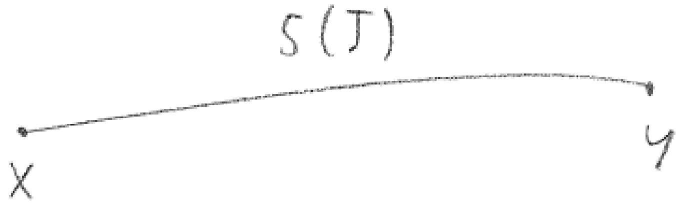
Chapter H.

Electricity

49. The minimum resistance

We view two points $x, y \in R^n$ with $n = 2, 3$. If an electrical current flows from x to y , then the way of minimum resistance is chosen. We have a resistance function $q(\bar{x}) \in R$ with $\bar{x} \in R^n$. One example is the lightning in the earth's atmosphere.



$s(\tau) \in R^n$ shall be a connected path with $s(a) = x$ and $s(b) = y$. x and y and $q(\bar{x})$ are known. Then the total resistance R can be written in the form:

$$R(s, x, y) = \int\limits_{s[x,y]} q(s)\, ds = \int\limits_a^b q(s(\tau)) \cdot \left| \frac{d}{d\tau}\, s(\tau) \right| \, d\tau$$

It is a line integral of first kind, see Bronstein [2] chapter 3.1.8.2 p.319 and Bartsch [1] chapter 10.8 p.421.

Now we search the path s with minimum R. We define:

$$F(s, s', \tau) := q(s(\tau)) \cdot \left| \frac{d}{d\tau}\, s(\tau) \right|$$

We use the Euler differential equations of variational calculus see Forster [3] §9 p.92:

$$\frac{d}{d\tau} \frac{\partial F}{\partial s_i'}\, (s, s', \tau) - \frac{\partial F}{\partial s_i}\, (s, s', \tau) = 0 \tag{1}$$

for $i = 1, \ldots, n$, thus n equations $\qquad n = 2, 3$
and $\quad s(a) = x \quad s(b) = y$
It is an usual differential equation system of second order with 2 initial values.

We assume that s and F are twice continuous differentiable.
Now we insert F in the differential equation system: $\qquad$ grad := gradient

$$\frac{d}{d\tau}\left[\mathrm{grad}_{s'}|s'| \cdot q(s)\right] - (\mathrm{grad}_s q(s)) \cdot |s'| = 0$$

It is:

$$\frac{d}{d\tau}\, q(s) = \langle \mathrm{grad}_s q(s), s' \rangle$$

$$\text{with} \qquad \langle\ \ ,\ \ \rangle = \text{scalar product in } R^n$$

with the chain rule in Forster [3] §6 p.50. Now we apply the product rule to the system:

$$\frac{d}{d\tau}[\mathrm{grad}_{s'}|s'|] \cdot q(s) + \mathrm{grad}_{s'}|s'| \cdot \langle \mathrm{grad}_s q(s), s' \rangle - (\mathrm{grad}_s q(s)) \cdot |s'| = 0$$

Now we consider:

$$\mathrm{grad}_{s'}|s'| = \frac{s'}{|s'|} \qquad \text{according to Forster [3] (5.4) S.37}$$

With the product rule we obtain:

$$\frac{d}{d\tau}\,\frac{s'}{|s'|} = \frac{s''}{|s'|} - \frac{1}{2} \cdot \frac{s'}{|s'|^3} \cdot \frac{d}{d\tau}\,(|s'|^2)$$

with:

$$\frac{d}{d\tau}(|s'|^2) = 2 \cdot \langle s', s'' \rangle$$

For the system we get:

$$\left[\frac{s''}{|s'|} - \frac{s'}{|s'|^3} \cdot \langle s', s'' \rangle\right] \cdot q(s) + \frac{s'}{|s'|} \cdot \langle \mathrm{grad}_s q(s), s' \rangle - (\mathrm{grad}_s q(s)) \cdot |s'| = 0 \qquad (2)$$

The solving of s'' is possible. With that we have an explicit differential equation system of second order. We can change this system in $2n$ differential equations of first order, for example according to Forster [3] §10 p.99. Also this system is explicit. An exact solution of this system of first order is perhaps only possible with series development for example according to Kamke [4] part A §2 (6.3) p.38.

In dependence from the form of $q(s)$ also other methods can be used.

From the system (2) and the initial conditions we can get one solution or several solutions or even a family of solutions. In extreme cases there can be no solutions. In the case of several solutions we look for the solution with the smallest $R(s, x, y)$. In the case of a family of solution $R(s, x, y)$ must be minimized. If there are no constraints at the family of parameters, the minimizing is done with the gradient setting equal to zero and the hessian matrix. If there are only equality constraints, the method with the Lagrange multipliers can be used. Are there inequalities at these constraints, we must use methods of non linear optimization.

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Chapter I.

Optics

Contents

50. Conic section mirrors

1. Conic section mirrors in general

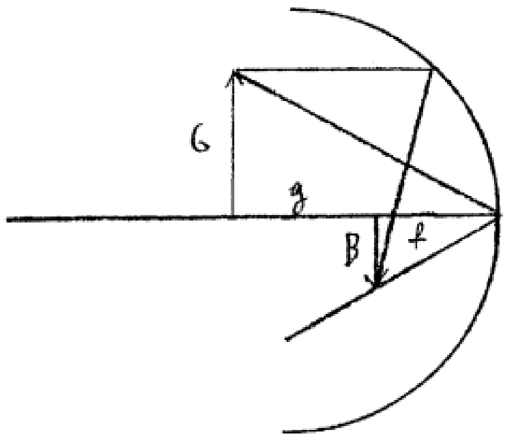
Here we want to search for the reason why at all conic section mirrors the imaging equations

$$\frac{1}{f} = \frac{1}{g} + \frac{1}{\bar{b}} \qquad\qquad \frac{G}{B} = \frac{g}{\bar{b}}$$

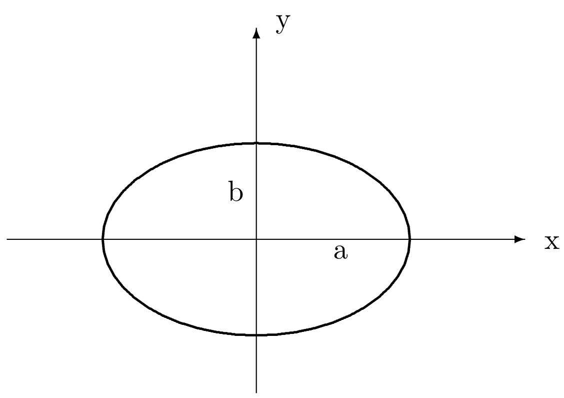
are valid. It is:

f = focal length of the mirror
g = object distance
$\bar{b}$ = image distance
G = object size
B = image size

See the following figure:



We know that at spherical mirrors these imaging equations are valid. Now we look at the ellipse:



$a, b =$ major and minor semiaxis

We can write the canonical equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

It follows:

$$y = b \cdot \sqrt{1 - \frac{x^2}{a^2}} = \frac{b}{a} \cdot \sqrt{a^2 - x^2}$$

We derive:

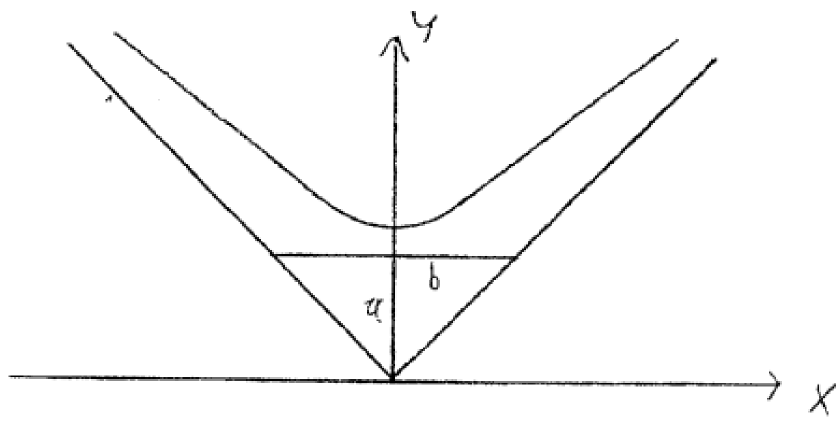
$$\frac{dy}{dx} = \frac{b}{a} \cdot \frac{-x}{\pm\sqrt{a^2 - x^2}}$$

In the case $a \gg x$ we obtain:

$$\frac{dy}{dx} \approx \pm\frac{b}{a^2} \cdot x$$

Here it is interesting that the differentiation is proportional to x. This is valid for the special case circle ($a = b = r$), too.

Now we deal with the hyperbola:



$a, b =$ semiaxes of the hyperbola

Here we have the canonical equation:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

We get:

$$y = a \cdot \sqrt{1 + \frac{x^2}{b^2}} = \frac{a}{b} \cdot \sqrt{b^2 + x^2}$$

We derive:

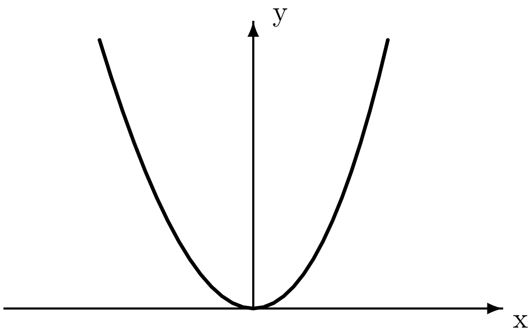
$$\frac{dy}{dx} = \frac{a}{b} \cdot \frac{x}{\sqrt{b^2 + x^2}}$$

In the case $b \gg x$ we follow:

$$\frac{dy}{dx} \approx \frac{a}{b^2} \cdot x$$

The differentiation is proportional to x.

At last we view the parabola:



The equation of the parabola is $y = mx^2$ with $m \in R$. Thus we get:

$$\frac{dy}{dx} = 2mx$$

Here the differentiation is proportional to x, too.

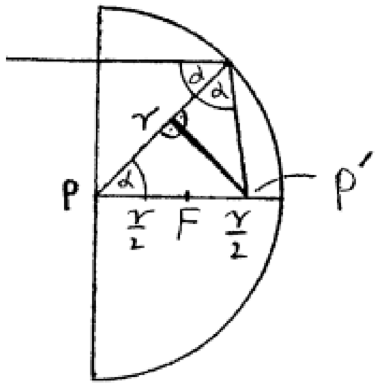
At all conic sections the differentiation are proportional to x in small sections. That means that the imaging equations are valid in the case $G, B \ll a, b, r$. Thus these imaging equations are valid for spherical mirrors, parabolic mirrors and mirrors with the form of revolution ellipsoids respectively revolution hyperboloids.

2. The deviation from focal ray

We view a spherical concave mirror with the radius r as in the following figure:

F = FOCAL POINT

$X = \overline{PP'}$



From the optics it is known that a axial parallel ray after the reflection goes approximately through the focal point. The problem is to determine this very small deviation. This distance to the focal point shall be expressed with the angle α. Because of the reflection law and the equality of alternate angles we have a isosceles triangle. Then we can write:

$$x = \frac{r}{2 \cdot \cos\alpha}$$

The searched deviation is then:

$$D = x - \frac{r}{2} = \frac{r}{2 \cdot \cos\alpha} - \frac{r}{2}$$

or:

$$D = \frac{r}{2} \cdot \left(\frac{1}{\cos\alpha} - 1\right)$$

D is the distance between the focal point and the intersection point of the ray at the optic axis.

The maximum x can be equal to r. If we insert in the first equation, then we obtain:

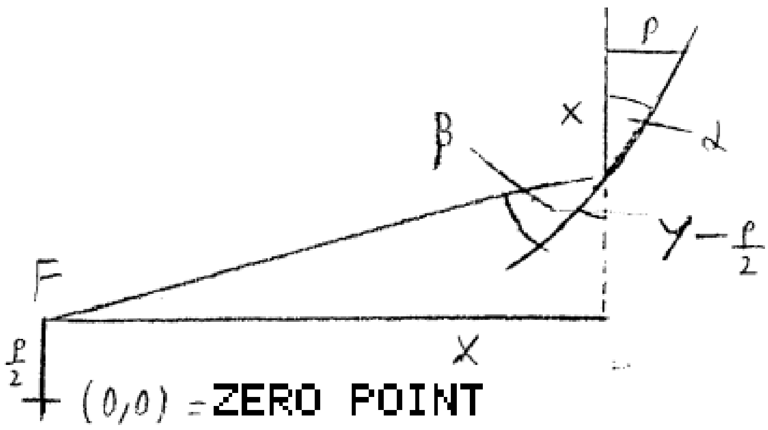
$$r = \frac{r}{2 \cdot \cos\alpha_{max}}$$

It follows:

$$\frac{1}{2} = \cos\alpha_{max} \qquad \alpha_{max} = 60^\circ$$

3. Proof that at the parabolic mirror axial parallel rays are reflected into the focal point.

We view the following figure:



We prove that axial parallel rays after the reflection go through the focal point F. With the reflection law we can conclude $\beta = \alpha$. Thus we get the following equation:

$$\tan 2\alpha = \frac{x}{y - \frac{p}{2}}$$

We use the vertex equation $x^2 = 2py$. We insert this equation in the equation before at y:

$$\tan 2\alpha = \frac{x}{\frac{x^2}{2p} - \frac{p}{2}} = \frac{2px}{x^2 - p^2}$$

With the vertex equation follows $y = \frac{x^2}{2p}$ and $\frac{dy}{dx} = \frac{x}{p}$.

With the figure (pitch angle) we recognize $\tan \alpha = \frac{p}{x}$. Now we use the addition theorem of tangent:

$$\tan 2\alpha = \frac{2 \cdot \tan \alpha}{1 - \tan^2 \alpha}$$

Insertion:

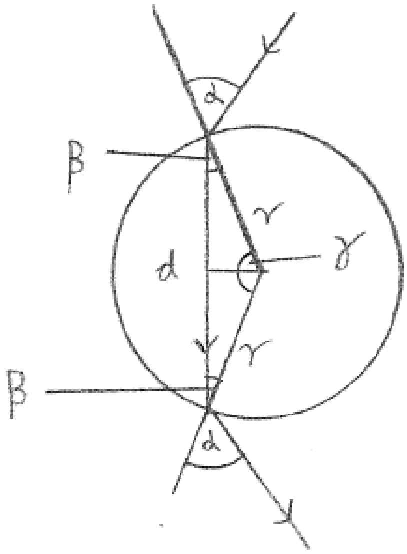
$$\tan 2\alpha = \frac{2 \cdot \frac{p}{x}}{1 - \frac{p^2}{x^2}} = \frac{2px}{x^2 - p^2}$$

Both representations for $\tan 2\alpha$ are equal. Thus at the parabolic mirror the axial parallel rays go after the reflection through the focal point F.

51. Refraction at a glass ball, refraction of rays at a plane-parallel plate and the visual shift through a liquid

1. Refraction at a glass ball

Here we view the path of rays through glass balls. The environment shall be vacuum. We look at the figure:



r = radius of the glass ball
α = angle of entry
n = refractive index of the glass ball

With the figure we recognize:

$$d = 2r \cdot \cos\beta = 2r \cdot \sqrt{1 - \sin^2\beta}$$

We can write the refraction law in the form:

$$\frac{\sin\alpha}{n} = \sin\beta$$

We insert the refraction law:

$$d = 2r \cdot \sqrt{1 - \frac{\sin^2\alpha}{n^2}}$$

or:

$$d = \frac{2r}{n} \cdot \sqrt{n^2 - \sin^2\alpha} \qquad\qquad n > 1 \geq \sin\alpha$$

In the figure we see $\frac{\gamma}{2} + \beta = 90°$ with both right-angled triangles. With the refraction law we obtain:

$$n = \frac{\sin\alpha}{\sin\beta} = \frac{\sin\alpha}{\sin\left(90° - \frac{\gamma}{2}\right)}$$

or:

$$\cos\frac{\gamma}{2} = \frac{\sin\alpha}{n}$$

For the transmission time of the ray through the glass ball we get:

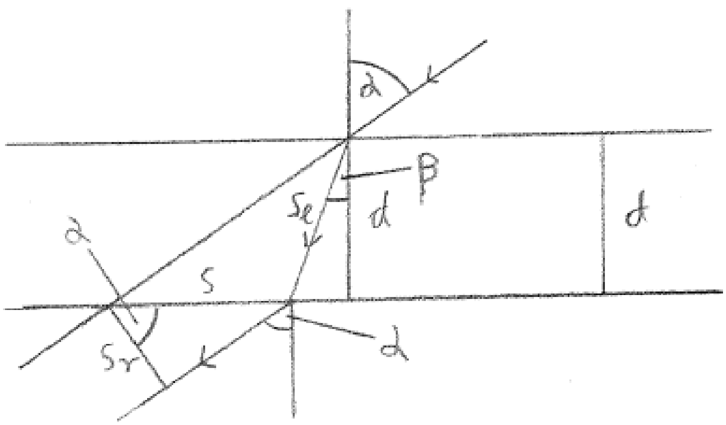
$$t = \frac{d \cdot n}{c}$$

If we replace the vacuum with a gas, we must insert $\frac{n_{Glas}}{n_{Gas}}$ instead of n. These both refractive indices must be relative to the vacuum.

These equations can be used for liquids drops, too. But these drops must have the form of a ball.

2. Refraction of rays at a plane-parallel plate

We look at the path of the ray through a glass plate as it is shown in the figure. Outside the glass plate shall be vacuum.



n = refractive index of the plate
d = thickness of the plate
α = angle of entry
β = refraction angle

Then the refraction law is valid:

$$\frac{\sin\alpha}{\sin\beta} = n$$

We search the parallel shift s_r of the ray. With the figure we conclude:

$$s = d \cdot (\tan\alpha - \tan\beta)$$

or:

$$s = d \cdot \left(\tan\alpha - \frac{\sin\beta}{\cos\beta}\right)$$

To the searched quantity s_r there is the following relation:

$$s_r = s \cdot \cos\alpha$$

If we use $\sin\alpha = \cos\alpha \cdot \tan\alpha$ and $\sin^2\beta + \cos^2\beta = 1$, then we obtain:

$$s_r = d \cdot \left(\sin\alpha - \frac{\cos\alpha \sin\beta}{\sqrt{1-\sin^2\beta}}\right)$$

Now we insert the refraction law at $\sin\beta$:

$$s_r = d \cdot \left(\sin\alpha - \frac{\cos\alpha \cdot \frac{\sin\alpha}{n}}{\sqrt{1-\frac{\sin^2\alpha}{n^2}}}\right)$$

At last we get:

$$s_r = d \cdot \sin\alpha \cdot \left(1 - \frac{\cos\alpha}{\sqrt{n^2-\sin^2\alpha}}\right)$$

The refractive index is always larger than 1. With that it follows $n > \sin\alpha$. We can determine the transmission length s_l with:

$$s_l = \frac{d}{\cos\beta} = \frac{d}{\sqrt{1-\frac{\sin^2\alpha}{n^2}}}$$

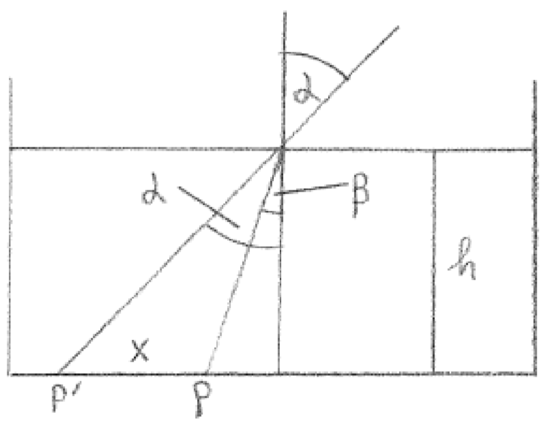
We calculate for the transmission time:

$$t = \frac{s_l \cdot n}{c}$$

If we replace the vacuum through a gas with the refractive index n_G, we insert the quotient $\frac{n_P}{n_G}$ instead of n. Here n_P is the refractive index of the plate relative to the vacuum.

3. Visual shift through a liquid

We view the following figure:



A receptacle contains a liquid with the height h. On the floor, there is a small pearl P. If we look at the angle α on the liquid's surface, then the pearl is visually shifted from P to P'. We have the law of refraction:

$$\frac{\sin\alpha}{\sin\beta} = n \qquad \text{or} \qquad \sin\beta = \frac{\sin\alpha}{n}$$

n = refractive index of the liquid
x = visual shift

With the figure for the visual shift we can conclude:

$$x = h \cdot (\tan\alpha - \tan\beta)$$

With

$$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{\sin\beta}{\sqrt{1-\sin^2\beta}} \qquad \text{because of} \qquad \sin^2\beta + \cos^2\beta = 1$$

we obtain:

$$\tan\beta = \frac{\sin\alpha}{n \cdot \sqrt{\frac{n^2-\sin^2\alpha}{n^2}}} = \frac{\sin\alpha}{\sqrt{n^2-\sin^2\alpha}}$$

With $\tan\alpha = \frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}$ we get the following expression of the visual shift:

$$x = h \cdot \sin\alpha \cdot \left(\frac{1}{\sqrt{1-\sin^2\alpha}} - \frac{1}{\sqrt{n^2-\sin^2\alpha}}\right) \tag{1}$$

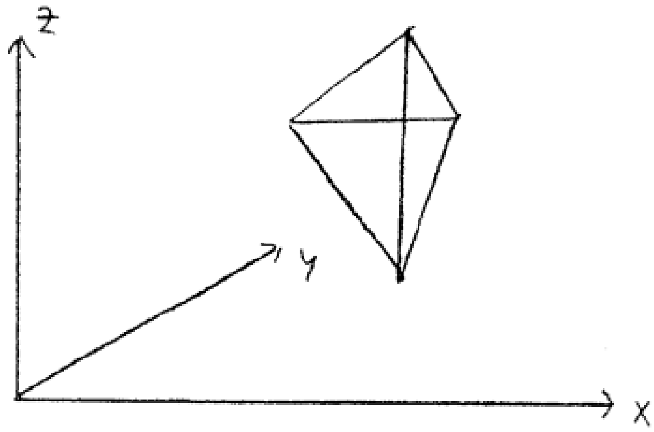
We can generalize this problem. The vacuum can be replaced with any gas. If n_G is the refractive index of the gas and n_F the refractive index of the liquid, then the law of refraction can be written as:

$$\frac{\sin\alpha}{\sin\beta} = \frac{n_F}{n_G}$$

We must insert $\frac{n_F}{n_G}$ instead of n in equation (1).

52. Similar figures of triangle in the eye

We want to ascertain which conditions are necessary for similar figures of triangle in the eye. We look at the following figure:



We view two triangles with the vectors $\vec{p}_{11}, \vec{p}_{12}, \vec{p}_{13} \in R^3$ and $\vec{p}_{21}, \vec{p}_{22}, \vec{p}_{23} \in R^3$. Centre of stretching shall be the circumcentre $\vec{p}_r$ of both triangles. Then we have:

r_1 = circumradius of the first triangle
r_2 = circumradius of the second triangle

To determine $\vec{p}_r$ we use the following 3 equations:

$$(\vec{p}_r - \vec{p}_{11})^2 = (\vec{p}_r - \vec{p}_{12})^2 = (\vec{p}_r - \vec{p}_{13})^2 = r_1^2$$

With these 3 equations $\vec{p}_r$ is calculated. We introduce the ratio of stretching $\bar{a} = \frac{r_2}{r_1}$. Now we occupy with the points $\vec{p}_{21}, \vec{p}_{22}, \vec{p}_{23}$ of the second triangle:

$$(\vec{p}_r - \vec{p}_{1k}) \cdot \bar{a} = \vec{p}_r - \vec{p}_{2k} \qquad k \in 1, 2, 3$$

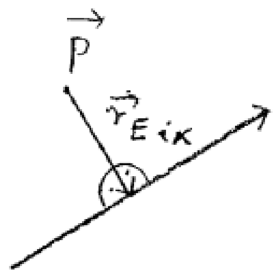
With this collinearity condition we have 9 equations to three vectors of the second triangle. Now we represent the side vectors of both triangles:

$$\vec{u}_{11} = \vec{p}_{11} - \vec{p}_{12} \qquad \vec{u}_{21} = \vec{p}_{21} - \vec{p}_{22}$$

$$\vec{u}_{12} = \vec{p}_{12} - \vec{p}_{13} \qquad \vec{u}_{22} = \vec{p}_{22} - \vec{p}_{23}$$

$$\vec{u}_{13} = \vec{p}_{13} - \vec{p}_{11} \qquad \vec{u}_{23} = \vec{p}_{23} - \vec{p}_{21}$$

Now we give the position $\vec{p} \in R^3$ of the view point (eye). We look at the following figure:



For $\vec{r}_E$ we use a formula that is derived in the appendix.

$$\vec{r}_{Eik} = \vec{p}_{ik} + \frac{\vec{u}_{ik} \cdot (\vec{p} - \vec{p}_{ik})}{|\vec{u}_{ik}|^2} \cdot \vec{u}_{ik} - \vec{p} \tag{1}$$

$|\cdot|$ = euclidean value of a vector in R^3

Now we define the object distance and the object size:

$$|\vec{r}_{Eik}| =: g_{ik} \qquad \text{(object distance)}$$

$$|\vec{u}_{ik}| =: G_{ik} \qquad \text{(object size)}$$

In the eye the image distance b is given. The eye lens can change its focal length f. We obtain the focal length using:

$$\frac{1}{f} = \frac{1}{g_{ik}} + \frac{1}{b}$$

Imaging equation:

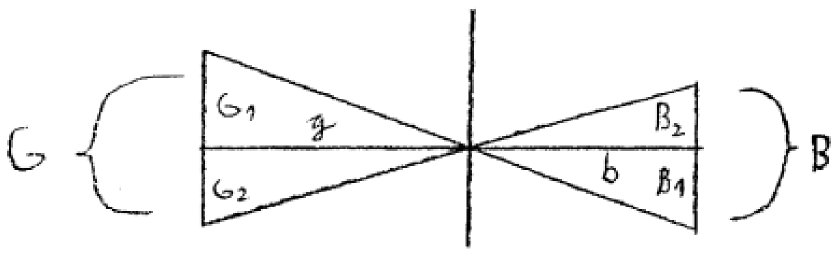
$$B_{ik} = \frac{G_{ik} \cdot b}{g_{ik}} \tag{2}$$

Only in the case

$$\frac{B_{21}}{B_{11}} = \frac{B_{22}}{B_{12}} = \frac{B_{23}}{B_{13}}$$

we have a similar figure in the eye. This is not always so.

$$G_1 + G_2 = G$$
$$B_1 + B_2 = B$$



In general we get a case as in this figure. Because of the equations

$$B_1 = \frac{G_1 \cdot b}{g} \qquad\qquad B_2 = \frac{G_2 \cdot b}{g}$$

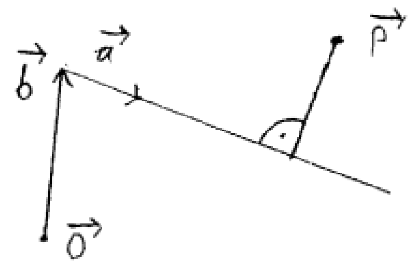
and

$$B = B_1 + B_2 = (G_1 + G_2) \cdot \frac{b}{g} = \frac{G \cdot b}{g}$$

we can insert G directly in equation (2).

Appendix:

Here we want to prove the formula (1). The problem is to determine the minimum distance between the point $\vec{p}$ and a straight line.



The straight line has the form:

$$\vec{v} = \vec{b} + \lambda \cdot \vec{a} \qquad\qquad \lambda \in R \qquad \vec{b}, \vec{a} \in R^3$$

Distance vector:

$$\vec{r} = \vec{v} - \vec{p} = \vec{b} + \lambda \cdot \vec{a} - \vec{p}$$

To absolute value of the distance we obtain:

$$r = \sqrt{(\vec{b} + \lambda \cdot \vec{a} - \vec{p})^2}$$

We differentiate r with the chain rule in λ:

$$\frac{dr}{d\lambda} = \frac{(\vec{b} + \lambda \cdot \vec{a} - \vec{p}) \cdot \vec{a}}{\sqrt{(\vec{b} + \lambda \cdot \vec{a} - \vec{p})^2}}$$

The necessary condition of a local minimum:

$$\frac{dr}{d\lambda} = 0$$

Thus it follows:

$$(\vec{b} + \lambda_{min} \cdot \vec{a} - \vec{p}) \cdot \vec{a} = 0$$

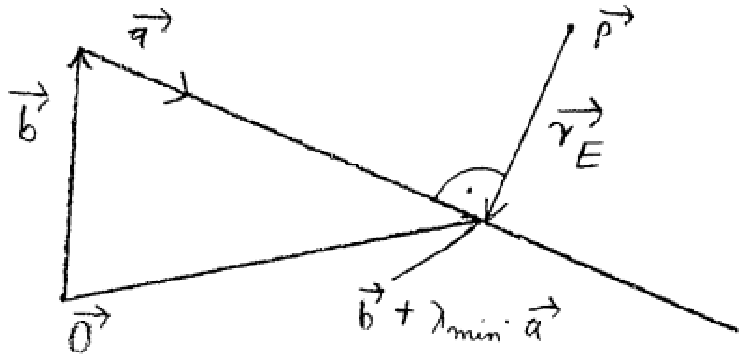
We transform to λ_{min}:

$$\lambda_{min} \cdot \vec{a}^2 = \vec{a} \cdot \vec{p} - \vec{a} \cdot \vec{b}$$

Thus we get the unique result:

$$\lambda_{min} = \frac{\vec{a} \cdot (\vec{p} - \vec{b})}{\vec{a}^2} \tag{3}$$

We see the following figure:



It is clear that a minimum distance exists. λ_{min} is unique, determined with equation (3). Thus at λ_{min} is a minimum distance. Now we calculate the minimum distance itself:

$$\vec{r}_E = \vec{b} + \lambda_{min} \cdot \vec{a} - \vec{p}$$

Insertion of equation (3) at λ_{min}:

$$\vec{r}_E = \vec{b} + \frac{\vec{a} \cdot (\vec{p} - \vec{b})}{\vec{a}^2} \cdot \vec{a} - \vec{p} \tag{4}$$

Now it is shown that $\vec{r}_E$ is perpendicular to $\vec{a}$:

$$\vec{r}_E \cdot \vec{a} = \vec{a} \cdot \vec{b} + \frac{\vec{a} \cdot (\vec{p} - \vec{b})}{\vec{a}^2} \cdot \vec{a}^2 - \vec{a} \cdot \vec{p}$$

$$= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{p} - \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{p} = 0$$

Thus $\vec{r}_E$ is perpendicular to $\vec{a}$. With the figure we can see that a minimum distance exists. This minimum distance can be calculated with formula (4). Formula (1) is an application of formula (4).

Chapter J.

Photometry (Radiation)

Contents

53. The luminous flux through the inclined rectangle in medium

Abstract: We calculate the luminous flux through a rectangle in a medium with constant absorption coefficient. The first aim is to get a integral representation. This integral is reduced to an integral over an intervall. To physical radiation dimensions analogous equations are valid as to photometric radiation dimensions.

Key words: Optical engineering - geometrical optics - photometry - radiation - radiometry

1. Introduction

We view a light source **Q** with the luminous intensity I and a inclined rectangle with the sides a und b in medium (gas). Density and absorption coefficient m are constant. The gas shall be pure air. The absorption is treated in [2]. The integration is similar to the inclined circle in [5]. With the transformation formula [3] we get integrals over the unit square. This two-dimensional integral can be reduced to an integral over an intervall (see [4]). Then we can use Simpson's rule. The deviation is proportional to n^{-4} see [1]. n is the number of steps.

2. Calculation

We view the following figure:

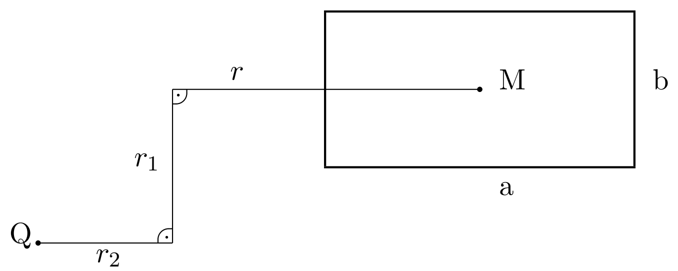


Fig.1. Light source Q with inclined rectangle, the distance r and the shifts r_1 and r_2. r is the distance to the rectangle's geometrical midpoint M.

We construct the distance vector $(r, r_2 + x_1, r_1 + x_2)$ with $x_2 \in [-\frac{b}{2}, \frac{b}{2}]$ and $x_1 \in [-\frac{a}{2}, \frac{a}{2}]$. We obtain the illumination:

$$E(x_1, x_2) = \frac{I \cdot \exp\left(-ml(x_1, x_2)\right)}{(l(x_1, x_2))^2} \tag{1}$$

with

$$l(x_1, x_2) = \sqrt{(r_1 + x_2)^2 + (r_2 + x_1)^2 + r^2}.$$

The angle of inclination β at the point (x_1, x_2) can be written:

$$\cos\beta(x_1, x_2) = \frac{r}{l(x_1, x_2)} \quad . \tag{2}$$

We define a function:

$$f(x_1, x_2) := E(x_1, x_2) \cdot \cos\beta(x_1, x_2) \quad .$$

We conclude for the luminous flux Φ through the rectangle:

$$\Phi = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x_1, x_2)\, dx_1 dx_2 \quad . \tag{3}$$

In special case $r_1 = 0$ and $r_2 = 0$ it is valid:

$$\Phi = 4 \cdot \int_{0}^{\frac{b}{2}} \int_{0}^{\frac{a}{2}} f(x_1, x_2)\, dx_1 dx_2 \tag{4}$$

Now we use the transformation formula to equation (3):

$$\Phi = \int_0^1 \int_0^1 g(x_1, x_2)\, dx_1 dx_2$$

with the function

$$\vec{A}(x_1, x_2) = (bx_1 - \frac{b}{2}, ax_2 - \frac{a}{2})$$

and

$$g(x_1, x_2) = ab \cdot f(bx_1 - \frac{b}{2}, ax_2 - \frac{a}{2}).$$

It is valid:

$$\int_0^1 \int_0^1 g(x_1, x_2)\, dx_1 dx_2 = \lim_{N\to\infty} \int_0^1 g(x_1, \langle Nx_1 \rangle)\, dx_1$$

$\langle Nx_1 \rangle$ is the fractional part of $N \cdot x_1$, and N is a natural number. Then we obtain:

$$\Phi = \lim_{N\to\infty} \int_0^1 g(x_1, \langle Nx_1 \rangle)\, dx_1.$$

The table shows the evaluation.

Table 1. Luminous flux for a=2m,b=1.5m,I=1cd,r_1=0.5m and m= 0.00003 m^{-1} in dependence from r. Let be r_2=0.

r[m]	Φ[Lumen]
1	1.78
1.2	1.42
1.4	1.14
1.6	0.930
1.8	0.769
2	0.644
2.2	0.545
2.4	0.467
2.6	0.404
2.8	0.353
3	0.311
3.2	0.275
3.4	0.246
3.6	0.220
3.8	0.199
4	0.180
4.2	0.164
4.4	0.150
4.6	0.137
4.8	0.127
5	0.117

For x_1, x_2, a and $b \ll r$ we have the approximation formula:

$$\Phi \approx \frac{Iab \cdot \exp\left(-m \cdot \sqrt{r^2 + r_1^2 + r_2^2}\right) \cdot \cos\alpha}{r^2 + r_1^2 + r_2^2}$$

with the angle of inclination α. It is valid:

$$\tan\alpha = \frac{\sqrt{r_1^2 + r_2^2}}{r}\,.$$

3. Conclusions

We got the luminous flux through the inclined rectangle with an one-dimensional integral. For $r \gg a, b$ the deviation between exact evaluation and approximation is small. The deviation becomes greater if not $r \gg a, b$. We get more accuracy if $N = 10^k$ and k is a natural number. k is chosen that for every step of Simpson's rule $\langle N x_1 \rangle = 0$. Then the calculated values are independent from N.
These equations are valid for the analogous physical dimensions radiant intensity, irradiance and radiant flux or radiant power.

References

[1] Barner M, Flohr F: Analysis I, p.414. de Gruyter, Berlin 1983

[2] Fabelinskii,I: Molecular Scattering of Light, New York 1968

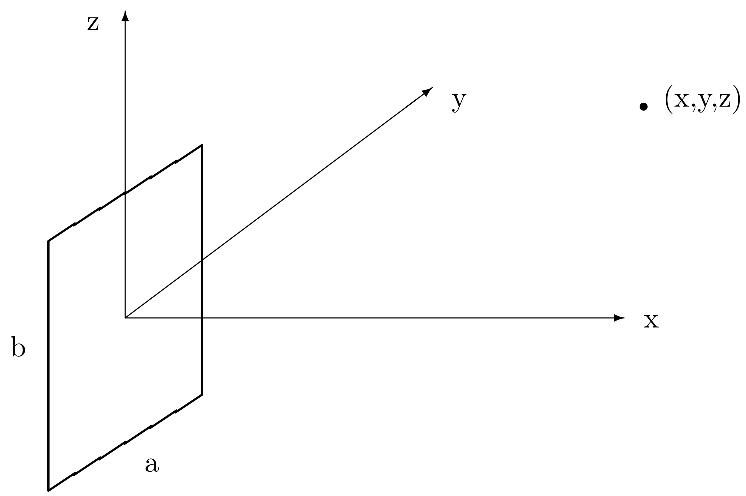
[3] Forster, O: Analysis 3, pp.14-17. Vieweg Verlag, Brunswick 1983

[4] He T: Dimensionality reducing expansion of multivariate integration, p.86. Birkhäuser, Boston 2001

[5] Schröer,H: Luminous Flux and Illumination, pp.29-62. Wissenschaft & Technik Verlag, Berlin 2001

54. Rectangle as radiator — the illumination (irradiance) in vacuum

The rectangle is in the origin. a and b are the sides. We determine the illumination(irradiance). The method is similar to Schröer [2] (chapter 7). We view the following rectangle:



We introduce the luminous intensity(radiant intensity) I. The luminous intensity density(radiant intensity density) is:

$$w = \frac{I}{ab}$$

We define $x_1 \in [\frac{-a}{2}, \frac{a}{2}]$ and $x_2 \in [\frac{-b}{2}, \frac{b}{2}]$.

We have the distance:

$$l(x_1, x_2) := \sqrt{x^2 + (y - x_1)^2 + (z - x_2)^2}$$

We assume $x > 0$.

Then we can construct the illumination(irradiance) in vacuum:

$$\vec{E}(x, y, z) = \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-a}{2}}^{\frac{a}{2}} \frac{I}{ab} \cdot \frac{\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix}}{\left| \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix} \right|^3} \, dx_1 dx_2 \tag{1}$$

Now we construct an approximation for $a, b \ll x$:

$$\vec{E} \approx \frac{w \cdot ab}{(x^2 + y^2 + z^2)^{1.5}} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{E} \approx \frac{I}{(x^2+y^2+z^2)^{1.5}} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \qquad |\vec{E}| \approx \frac{I}{r^2} \qquad \text{for} \qquad \sqrt{x^2+y^2+z^2} =: r \gg a, b$$

We view the special case $y = z = 0$:

Because of symmetry it follows:

$$\vec{E}(x,y,z) = (E_1(x,y,z), 0, 0)$$

$$E_1(x,y,z) = \frac{Ix}{ab} \cdot \int\limits_{\frac{-b}{2}}^{\frac{b}{2}} \int\limits_{\frac{-a}{2}}^{\frac{a}{2}} \frac{dx_1 dx_2}{(x^2+x_1^2+x_2^2)^{1.5}}$$

For the integration see Bronstein [1] p.47 number 206 with $c := x^2 + x_2^2$:

$$\int\limits_{\frac{-a}{2}}^{\frac{a}{2}} \frac{dx_1}{(x_1^2+c)^{1.5}} = \left[\frac{x_1}{c \cdot \sqrt{x_1^2+c}} \right]_{x_1=\frac{-a}{2}}^{x_1=\frac{a}{2}} = \frac{a}{c \cdot \sqrt{\frac{a^2}{4}+c}}$$

We get:

$$E_1(x,0,0) = \frac{Ix}{b} \cdot \int\limits_{\frac{-b}{2}}^{\frac{b}{2}} \frac{dx_2}{(x^2+x_2^2) \cdot \sqrt{\frac{a^2}{4}+x^2+x_2^2}}$$

We obtain with “Mathematica”:

$$= \frac{2I}{ba} \cdot \left[\arctan \left(\frac{\frac{a}{2} \cdot x_2}{x \cdot \sqrt{\frac{a^2}{4}+x^2+x_2^2}} \right) \right]_{x_2=\frac{-b}{2}}^{x_2=\frac{b}{2}}$$

We conclude because of $-\arctan(-s) = \arctan s$:

$$E_1(x,0,0) = \frac{4I}{ba} \cdot \arctan \left(\frac{ab}{4x \cdot \sqrt{\frac{a^2+b^2}{4}+x^2}} \right)$$

For $x \gg a, b$ we have the approximation:

$$E_1(x,0,0) \approx \frac{I}{x^2}$$

In the case $y, z \neq 0$ with Bronstein [1] number 242,p.49 and number 250,p.50 $\vec{E}(x,y,z) = (E_1, E_2, E_3)$ can be reduced to one-dimensional integrals. These integrals can be treated with numerical methods.

$$E_1(x,y,z) = \frac{I}{ab} \cdot \int\limits_{\frac{-b}{2}}^{\frac{b}{2}} \int\limits_{\frac{-a}{2}}^{\frac{a}{2}} \frac{x \, dx_1 dx_2}{(l(x_1,x_2))^3}$$

with $l(x_1, x_2) = \sqrt{x_1^2 - 2yx_1 + x^2 + y^2 + (z - x_2)^2}$

$$= \frac{I}{ab} \cdot \int\limits_{\frac{-b}{2}}^{\frac{b}{2}} \left[\frac{2 \cdot (2x_1 - 2y) \cdot x}{\Delta \cdot l(x_1, x_2)} \right]_{x_1 = \frac{-a}{2}}^{x_1 = \frac{a}{2}} \, dx_2$$

with $\Delta := 4 \cdot (x^2 + y^2 + (z - x_2)^2) - 4y^2$

$$E_2(x, y, z) = \frac{I}{ab} \cdot \int\limits_{\frac{-b}{2}}^{\frac{b}{2}} \int\limits_{\frac{-a}{2}}^{\frac{a}{2}} \frac{(y - x_1)\, dx_1 dx_2}{(l(x_1, x_2))^3}$$

$$= \frac{I}{ab} \cdot \int\limits_{\frac{-b}{2}}^{\frac{b}{2}} H(x_2)\, dx_2$$

with $\quad H(x_2) := \left[\frac{2y \cdot (2x_1 - 2y) - 4yx_1 - 4 \cdot (x^2 + y^2 + (z - x_2)^2)}{\Delta \cdot l(x_1, x_2)} \right]_{x_1 = \frac{-a}{2}}^{x_1 = \frac{a}{2}}$

$$E_3(x, y, z) = \frac{I}{ab} \cdot \int\limits_{\frac{-b}{2}}^{\frac{b}{2}} \int\limits_{\frac{-a}{2}}^{\frac{a}{2}} \frac{(z - x_2)\, dx_1 dx_2}{(l(x_1, x_2))^3}$$

$$= \frac{I}{ab} \cdot \int\limits_{\frac{-b}{2}}^{\frac{b}{2}} \left[\frac{2 \cdot (2x_1 - 2y) \cdot (z - x_2)}{\Delta \cdot l(x_1, x_2)} \right]_{x_1 = \frac{-a}{2}}^{x_1 = \frac{a}{2}} \, dx_2$$

The absolute value is:

$$E(x, y, z) = \sqrt{\sum_{i=1}^{3} (E_i(x, y, z))^2}$$

To solve these integrals we can use the Simpson's rule.

References

[1] Bronstein, Semendjajew "Taschenbuch der Mathematik", 22.edition Teubner Verlag Leipsic 1985

[2] Harald Schröer "Luminous Flux and Illumination", english and german edition, Wissenschaft und Technik Verlag Berlin 2001

55. Rectangle as radiator — the illumination in medium

Abstract: We calculate the illumination of a rectangle in a medium with constant absorption coefficient. The first aim is to get a integral representation. This integral is reduced to an integral over an interval.

Key words: Optical engineering - geometrical optics - photometry - radiation - radiometry

1. Introduction

We view a rectangle light source with the luminous intensity I in medium (gas). a and b are the sides. Density and absorption coefficient m are constant. We assume the medium is in the whole room. The gas shall be pure air. The absorption is treated in [2]. The integration is similar to the inclined circle in [5]. With the transformation formula [3] we get integrals over the unit square. These two-dimensional integrals can be reduced to integrals over intervals (see [4]). Then we can use Simpson's rule. The deviation is proportional to n^{-4} see [1]. n is the number of steps.

2. Calculation

We view the following figure:

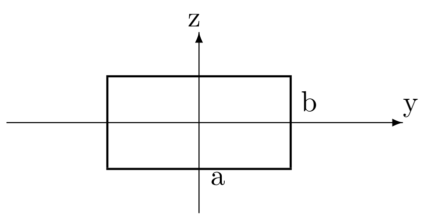


Fig.1. Rectange light source with the sides a and b. The geometrical midpoint is in the origin.

We introduce the luminous intensity density:

$$w = \frac{I}{ab}$$

E is the illumination. The illumination at point (x, y, z) with $x > 0$ can be described as

$$\vec{E} = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \vec{f}(x_1, x_2)\, dx_1 x_2 \tag{1}$$

with

$$\vec{f}(x_1, x_2) := w \cdot \frac{\exp\left[-m \cdot l(x_1, x_2)\right]}{[l(x_1, x_2)]^3} \cdot \begin{pmatrix} x \\ y - x_1 \\ z - x_2 \end{pmatrix}$$

and

$$l(x_1, x_2) = \sqrt{x^2 + (y - x_1)^2 + (z - x_2)^2}.$$

Now we use the transformation formula:

$$\vec{E}(x, y, z) = \int_0^1 \int_0^1 \vec{g}(x_1, x_2)\, dx_1 dx_2 \tag{2}$$

with the function

$$\vec{A}(x_1, x_2) = (bx_1 - \frac{b}{2}, ax_2 - \frac{a}{2})$$

and

$$\vec{g}(x_1, x_2) = ab \cdot \vec{f}(bx_1 - \frac{b}{2}, ax_2 - \frac{a}{2}).$$

It is valid:

$$\int_0^1 \int_0^1 g(x_1, x_2)\, dx_1 dx_2 = \lim_{N \to \infty} \int_0^1 g(x_1, \langle N x_1 \rangle)\, dx_1$$

for every continuous function g. $\langle N x_1 \rangle$ is the fractional part of $N \cdot x_1$ and N is a natural number.

We introduce $\vec{E}$ =(E_1, E_2, E_3) and $\vec{g} = (g_1, g_2, g_3)$ and get one-dimensional integrals. We have for $i \in 1, 2.3$:

$$E_i(x, y, z) = \lim_{N \to \infty} \int_0^1 g_i(x_1, \langle N x_1 \rangle)\, dx_1.$$

The table shows the evaluation.

Table 1. illumination for $a = 2$m, $b = 1.5$m, $I = 1$cd and $m = 0.00003$ m^{-1} in dependence from x. Let be $y = z = 0$.

x[m]	E[Lux]
4	0.0561
4.2	0.0514
4.4	0.0472
4.6	0.0435
4.8	0.0402
5	0.0373
5.2	0.0347
5.4	0.0323
5.6	0.0302
5.8	0.0282

6	0.0265
6.2	0.0249
6.4	0.0234
6.6	0.0220
6.8	0.0208
7	0.0197
7.2	0.0186
7.4	0.0177
7.6	0.0168
7.8	0.0160
8	0.0152

For $r >> a, b$ we have the approximation formula:

$$E = \frac{I \cdot \exp(-mr)}{r^2}$$

with $r = \sqrt{x^2 + y^2 + z^2}$.

3. Conclusions

We got the illumination of rectangle light source with an one-dimensional integral. For $r >> a, b$ the deviation between exact evaluation and approximation formula is small. The deviation becomes greater if not $r >> a, b$. We get more accuracy if $N = 10^k$ and k is a natural number. k is chosen that for every step of Simpson's rule $\langle N x_1 \rangle = 0$. Then the calculated values are independent from N.
These equation are valid for the analogous physical dimensions radiant intensity and irradiance for visual wavelengths.

References

[1] Barner M, Flohr F: Analysis I, p.414. de Gruyter, Berlin 1983

[2] Fabelinskii I: Molecular Scattering of Light, New York 1968

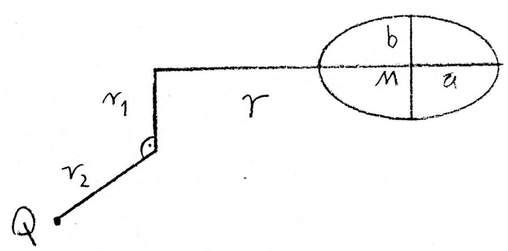
[3] Forster O: Analysis 3, pp.14-17. Vieweg Verlag, Brunswick 1983

[4] He T: Dimensionality reducing expansion of multivariate integration, p.86. Birkhäuser, Boston 2001

[5] Schröer H: Luminous flux and illumination, pp.29-32,75-80. Wissenschaft & Technik Verlag, Berlin 2001

56. Luminous flux (radiant power, radiant flux) and solid angle through the inclined ellipse

We want to determine the luminous flux(radiant power,radiant flux) and the solid angle Ω through the inclined ellipse. The method is similar to Schröer [3] (chapter 7 and 9). We view the following ellipse:



a and b are the semiaxes. r is the distance. We introduce the medium's absorption coefficient m and luminous intensity(radiant intensity) I of the point light source Q. The angle of inclination of the ellipse is:

$$\tan\alpha = \frac{\sqrt{r_1^2 + r_2^2}}{r}$$

M is the geometrical midpoint of the ellipse.

We use the canonical equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \Rightarrow \qquad \frac{y}{b} = \sqrt{1 - \frac{x^2}{a^2}}$$

$$h(x) := y = \frac{b}{a} \cdot \sqrt{a^2 - x^2}$$

We define $x_1 \in [-a, a]$ and $x_2 \in [-h(x_1), h(x_1)]$. We construct the distance vector:

$$\vec{l} = (r, x_1 + r_2, r_1 + x_2)$$

It follows:

$$l(x_1, x_2) = \sqrt{r^2 + (x_1 + r_2)^2 + (r_1 + x_2)^2}$$

Now we turn to the illumination(irradiance):

$$E(x_1, x_2) = \frac{I \cdot e^{-m \cdot l(x_1, x_2)}}{(l(x_1, x_2))^2}$$

We need the angle of inclination at (x_1, x_2):

$$\cos\beta(x_1, x_2) = \frac{r}{l(x_1, x_2)}$$

We define:

$$f(x_1, x_2) := E(x_1, x_2) \cdot \cos\beta(x_1, x_2)$$

We construct the luminous flux(radiant power,radiant flux) through the ellipse:

$$\Phi = \int_{-a}^{a} \int_{-h(x)}^{h(x)} f(x_1, x_2)\, dx_2 dx_1 \tag{1}$$

We see the approximation for $a, b \ll r$:

$$\Phi \approx \frac{I \cdot e^{-m \cdot \sqrt{r^2 + r_1^2 + r_2^2}} \cdot \pi ab \cdot \cos\alpha}{r^2 + r_1^2 + r_2^2} \tag{2}$$

with:

$$\tan\alpha = \frac{\sqrt{r_1^2 + r_2^2}}{r}$$

With $m = 0$ and without I we get the solid angle through the ellipse.

We find a program to calculate (1) at Robinson [2].

We reduce the solid angle to an one-dimensional integral:

Let be $m = 0$ and we calculate the interior integral of (1):

$$\int_{-h(x_1)}^{h(x_1)} \frac{dx_2}{(l(x_1, x_2))^3} = \left[\frac{2 \cdot (2x_2 + 2r_1)}{\Delta \cdot l(x_1, x_2)}\right]_{x_2 = -h(x_1)}^{x_2 = h(x_1)} =: Z(x_1)$$

with

$$l(x_1, x_2)^2 = x_2^2 + 2r_1 x_2 + r_1^2 + (x_1 + r_2)^2 + r^2$$

and

$$\Delta := 4 \cdot (r_1^2 + (x_1 + r_2)^2 + r^2) - 4r_1^2$$

For the integration see Bronstein [1] number 242 p.49.

The solid angle can be written as:

$$\Omega = r \cdot \int_{-a}^{a} Z(x_1)\, dx_1 \tag{3}$$

We can use numerical methods for example the Simpson's rule.

References

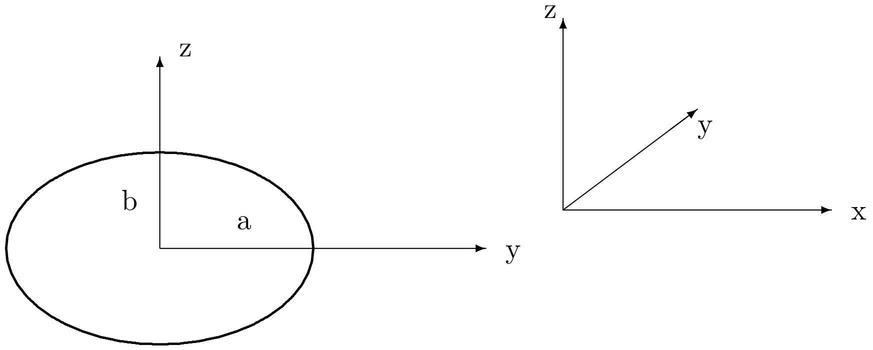
[1] Bronstein, Semendjajew "Taschenbuch der Mathematik", 22.edition Teubner Verlag Leipsic 1985

[2] Robinson, de Doncker, "Algorithm 45, Automatic computation of improper integrals over a bounded or unbounded planar region", Computing, 27 (1981) 3, p.253-284

[3] Harald Schröer "Luminous Flux and Illumination",english and german edition, Wissenschaft und Technik Verlag Berlin 2001

57. Ellipse as radiator — the illumination (irradiance)

The ellipse is in the origin. a and b are the semiaxes. We want to determine the illumination(irradiance). The method is similar to Schröer [3] (chapter 7). We view the following ellipse:



We introduce the absorption coefficient m and the luminous intensity(radiant intensity) I. The luminous intensity density(radiant intensity density) is:

$$w = \frac{I}{\pi ab}$$

We define $x_1 \in [-a, a]$ and $x_2 \in [-h(x_1), h(x_1)]$.

We use the canonical equation:

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \qquad \Rightarrow \qquad x_2^2 = b^2 \cdot \left(1 - \frac{x_1^2}{a^2}\right)$$

It follows:

$$h(x_1) := x_2 = \frac{b}{a} \cdot \sqrt{a^2 - x_1^2}$$

We have the distance:

$$l(x_1, x_2) := \sqrt{x^2 + (y - x_1)^2 + (z - x_2)^2}$$

We assume $x > 0$.

Then we can construct the illumination(irradiance)in vacuum:

$$\vec{E}(x, y, z) = \int_{-a}^{a} \int_{-h(x_1)}^{h(x_1)} \frac{I}{\pi ab} \cdot \frac{\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix}}{\left| \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix} \right|^3} \, dx_2 dx_1 \tag{1}$$

The illumination(irradiance) in medium can be written:

$$\vec{E}(x,y,z) = \frac{I}{\pi ab} \cdot \int_{-a}^{a} \int_{-h(x_1)}^{h(x_1)} \frac{e^{-m \cdot l(x_1,x_2)}}{(l(x_1,x_2))^3} \cdot \begin{pmatrix} x \\ y - x_1 \\ z - x_2 \end{pmatrix} dx_2 dx_1 \tag{2}$$

Now we construct an approximation for $a, b \ll x$:

$$\vec{E} \approx \frac{w \cdot \pi \cdot ab \cdot e^{-m \cdot \sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)^{1.5}} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{E} \approx \frac{I \cdot e^{-m \cdot \sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)^{1.5}} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \qquad |\vec{E}| \approx \frac{I \cdot e^{-mr}}{r^2} \qquad \text{for} \qquad \sqrt{x^2+y^2+z^2} =: r \gg a, b$$

We find a program to calculate (2) at Robinson [2].

Now we view the special case $y = z = 0$ in vacuum, that means $m = 0$.

$$\text{symmetry} \qquad \Rightarrow \qquad \vec{E}(x,y,z) = (E_1(x,y,z), 0, 0)$$

$$E_1(x,y,z) = \frac{Ix}{\pi ab} \cdot \int_{-a}^{a} \int_{-h(x_1)}^{h(x_1)} \frac{dx_2 dx_1}{(x^2 + x_1^2 + x_2^2)^{1.5}}$$

For the integration see Bronstein [1] p.47 number 206:

$$E_1(x,y,z) = \frac{Ix}{\pi ab} \cdot \int_{-a}^{a} \left[\frac{x_2}{c \cdot \sqrt{x_2^2 + c}} \right]_{x_2=-h(x_1)}^{x_2=h(x_1)} dx_1$$

with $c := x^2 + x_1^2$

$$= \frac{2Ix}{\pi ab} \cdot \int_{-a}^{a} \frac{h(x_1)\, dx_1}{(x^2 + x_1^2) \cdot \sqrt{(h(x_1))^2 + x^2 + x_1^2}}$$

$$= \frac{2Ix}{\pi a^2} \cdot \int_{-a}^{a} \frac{\sqrt{a^2 - x_1^2}\, dx_1}{(x^2 + x_1^2) \cdot \sqrt{b^2 \cdot \left(1 - \frac{x_1^2}{a^2}\right) + x^2 + x_1^2}}$$

The calculation leads to elliptic integrals of first and third kind.

We can use numerical methods for example the Simpson's rule.

In the case $y, z \neq 0$ and $m = 0$ with Bronstein [1] number 242,p.49 and number 250,p.50 $\vec{E}(x,y,z)$ can be reduced to one-dimensional integrals. These integrals can be treated with numerical methods.

References

[1] Bronstein, Semendjajew "Taschenbuch der Mathematik", 22.edition Teubner Verlag Leipsic 1985

[2] Robinson, de Doncker, "Algorithm 45, Automatic computation of improper integrals over a bounded or unbounded planar region", Computing, 27 (1981) 3, p.253-284

[3] Harald Schröer "Luminous Flux and Illumination",english and german edition, Wissenschaft und Technik Verlag Berlin 2001

Chapter K.

Theory of relativity and microphysics

58. Velocity and temperature

Abstract: In this paper the velocity is determined as function of the temperature. The classic and the relativistic case is treated.

Key words: Velocity - temperature - kinetic energy - relativistic statistics

We use the known formula of relativistic kinetic energy:

$$E_{kin} = mc^2 \cdot \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) \tag{1}$$

v is the velocity, m the mass of a particle and c is the light velocity in vacuum.

We transform this formula to v:

$$\frac{E_{kin}}{mc^2} + 1 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

It follows:

$$\sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\frac{E_{kin}}{mc^2} + 1}$$

Then we get:

$$\frac{v^2}{c^2} = 1 - \left(\frac{1}{\frac{E_{kin}}{mc^2} + 1} \right)^2$$

Solving to v:

$$v = c \cdot \sqrt{1 - \left(\frac{1}{\frac{E_{kin}}{mc^2} + 1} \right)^2}$$

At last:

$$v = c \cdot \sqrt{1 - \left(\frac{mc^2}{E_{kin} + mc^2} \right)^2} \tag{2}$$

In the classical case we simple have:

$$v = \sqrt{\frac{2 \cdot E_{kin}}{m}}$$

In thermodynamics (see Höfling [1], part 1, chapter 3.1.6, p.336) every particle has the average kinetic energy:

$$E_{kin} = \frac{3}{2} \cdot kT$$

k is the Boltzmann constant, and T is the temperature in Kelvin.

It follows:

$$v = \sqrt{\frac{3kT}{m}} \tag{3}$$

Both formulas are valid, if the temperature is very small compared to 10^8 K.

In case of high temperatures in Neugebauer [2] chapter 3.2.2 p.85 a relation $E_{kin}(T)$ can be found with Bessel functions. If we insert the function $E_{kin}(T)$ in the equation (2), then we get the velocity $v(E_{kin}(T))$ - as function from the temperature.

Further relativistic problems that can be found scarcely in the literature are in Schröer [3].

References

[1] O. Höfling "Physik", volume II, Dümmler Verlag, Bonn 1981

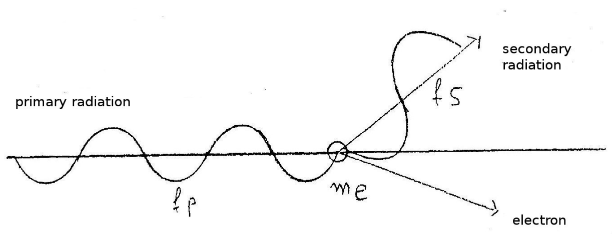
[2] Gernot Neugebauer "Relativistische Thermodynamik", Akademie-Verlag, Berlin 1980

[3] Harald Schröer "Particular problems of special relativity theory", Wissenschaft & Technik Verlag, Berlin 2002

59. Compton effect — deflection of the electron

If electromagnetic radiation hits motionless electrons, then the wavelength of the radiation increases. This scattering is called Compton effect. We introduce the following values:

λ_p, λ_s = primary wavelength respectively secondary wavelength of the radiation
f_p, f_s = primary frequency respectively scondary frequency of the radiation
m_e = mass of the electron
c = light velocity
h = Planck's constant
ϑ = scattering angle



In many books of physics (for example in Höfling [1] chapter 8.1.5 p.725-729) a derivation of the wavelength's shift is given. This formula can be written as:

$$\Delta\lambda := \lambda_s - \lambda_p = \frac{h}{m_e \cdot c} \cdot (1 - \cos\vartheta)$$

We obtain with the wavelength - frequency - relation:

$$\Delta\lambda = \frac{c}{f_s} - \frac{c}{f_p} = \frac{h}{m_e c} \cdot (1 - \cos\vartheta) \tag{1}$$

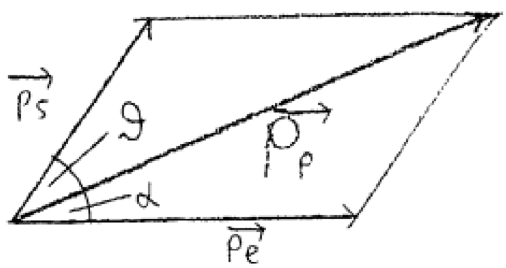
We solve this equation to f_s:

$$f_s = \frac{c}{\frac{h}{m_e c} \cdot (1 - \cos\vartheta) + \frac{c}{f_p}}$$

Extension:

$$f_s = \frac{m_e f_p c^2}{h \cdot (1 - \cos\vartheta) \cdot f_p + m_e c^2} \tag{2}$$

Now we view the following figure:



The aim is to calculate the deflection angle α of the electron. We use the impulse conservation law:

$$\vec{p_e} = \vec{p_p} - \vec{p_s}$$

The indices e,p,s represent electron, primary radiation and secondary radiation. We can it express with the cosine law, too:

$$p_e^2 = p_p^2 + p_s^2 - 2p_p p_s \cdot \cos\vartheta \tag{3}$$

We use the cosine law once again:

$$p_s^2 = p_p^2 + p_e^2 - 2p_p p_e \cdot \cos\alpha \tag{4}$$

We transform the last equation to $\cos\alpha$ and we insert for p_e the equation (3):

$$\cos\alpha = \frac{p_p^2 + p_e^2 - p_s^2}{2p_e p_p} = \frac{p_p - p_s \cdot \cos\vartheta}{\sqrt{p_p^2 + p_s^2 - 2p_p p_s \cdot \cos\vartheta}}$$

We can write the photon momentums as:

$$p_p = \frac{h \cdot f_p}{c} \qquad p_s = \frac{h \cdot f_s}{c}$$

An insertion in the last term from the equation of $\cos\alpha$ leads to:

$$\cos\alpha = \frac{f_p - f_s \cdot \cos\vartheta}{\sqrt{f_p^2 + f_s^2 - 2f_p f_s \cdot \cos\vartheta}} \tag{5}$$

Equation (1) transformed:

$$1 - \cos\vartheta = \frac{\left(\frac{c}{f_s} - \frac{c}{f_p}\right) \cdot m_e c}{h}$$

At last:

$$\cos\vartheta = 1 - \frac{m_e c^2}{h} \cdot \left(\frac{1}{f_s} - \frac{1}{f_p}\right) \tag{6}$$

Now we can insert this expression in the equation (5) for $\cos\vartheta$:

$$\cos\alpha = \frac{f_p - f_s \cdot \left(1 - \frac{m_e c^2}{h} \cdot \left(\frac{1}{f_s} - \frac{1}{f_p}\right)\right)}{\sqrt{f_p^2 + f_s^2 - 2f_p f_s \cdot \left(1 - \frac{m_e c^2}{h} \cdot \left(\frac{1}{f_s} - \frac{1}{f_p}\right)\right)}}$$

Thus we have the deflection angle as function of both frequencies. Both frequencies can be measured. We can determine with the last equation the deflection angle. We yield the scattering angle ϑ with equation (6).

References

[1] Oskar Höfling “Physik” Band II Teil 3 Dümmler Verlag 12.edition 1979 Bonn

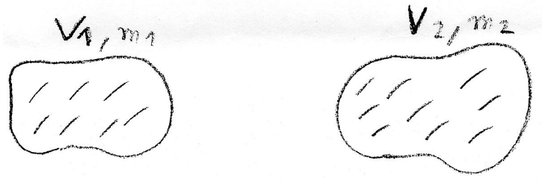
Chapter L.

Astronomy

60. Gravitational acceleration at revolution ellipsoid, ball and general revolution solid

Abstract: The gravitational accelerations on planets, which have the form of revolution ellipsoid, ball and general revolution solid, are calculated.

Key words: Gravitational acceleration - gravitation - ball - revolution ellipsoid - revolution solid - planet



m_1, m_2 are two masses with volumina V_1, V_2.

m_1 generates a gravitational acceleration $\vec{g}_1$, and m_2 has a gravitational acceleration $\vec{g}_2$. If G is the gravitational constant and φ_1, φ_2 the densities of m_1 and m_2, then the accelerations can be expressed as:

$$\vec{g}_1(\vec{r},t) = \int\limits_{V_1(t)} \frac{G\varphi_1(\vec{x},t)}{|\vec{x}-\vec{r}|^2} \cdot \frac{\vec{x}-\vec{r}}{|\vec{x}-\vec{r}|}\, d\vec{x} \tag{1}$$

$$\vec{g}_2(\vec{r},t) = \int\limits_{V_2(t)} \frac{G\varphi_2(\vec{x},t)}{|\vec{x}-\vec{r}|^2} \cdot \frac{\vec{x}-\vec{r}}{|\vec{x}-\vec{r}|}\, d\vec{x} \tag{2}$$

t is the time. We have also included the case temporal changing fields. Now we turn to the force:

$\vec{K}_{1,2}$ = force, with that m_2 is picked up by m_1.
$\vec{K}_{2,1}$ = force, with that m_1 is picked up by m_2.
These forces can be presented with:

$$\vec{K}_{1,2}(t) = \int\limits_{V_2(t)} \vec{g}_1(\vec{x},t) \cdot \varphi_2(\vec{x},t)\, d\vec{x} \tag{3}$$

$$\vec{K}_{2,1}(t) = \int\limits_{V_1(t)} \vec{g}_2(\vec{x},t) \cdot \varphi_1(\vec{x},t)\, d\vec{x} \tag{4}$$

Besides the 3. Newton axiom must be valid:

$$\vec{K}_{1,2}(t) = -\vec{K}_{2,1}(t) \tag{5}$$

But we must assume at the three last equations, that m_1 and m_2 are motionless. If these masses move, we have in general with $\vec{v}$ as velocity:

$$\vec{K} = \frac{d}{dt}(m(t) \cdot \vec{v}(t)) = m(t) \cdot \dot{\vec{v}} + \dot{m} \cdot \vec{v}$$

Then the calculation will become more complicated.

Important special cases are, if both masses are motionless and $\vec{g}_1$ is local constant in the area V_2 repectively $\vec{g}_2$ in the area V_1. Then we obtain:

$$\vec{K}_{1,2} \approx m_2(t) \cdot \vec{g}_1(\vec{x}, t) \tag{6}$$

$$\vec{K}_{2,1} \approx m_1(t) \cdot \vec{g}_2(\vec{x}, t) \tag{7}$$

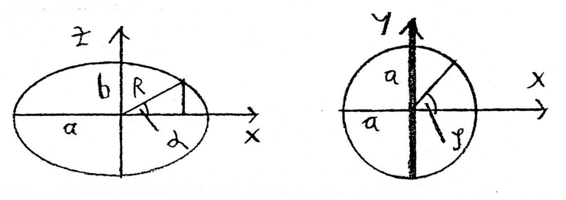
In our case the masses are constant. The term $\dot{m} \cdot \vec{v}$ must not be taken in consideration. We can insert in the equations of the gravitation theory instead off $\frac{G}{|\vec{x}-\vec{r}|^2}$ another suitable function $k(\vec{x} - \vec{r})$. With that the gravitational acceleration can be modified. This can be used in the celestial mechanics, if the masses are considered as mass points.
At last we take into consideration that at changes of gravitational fields and at not spherical bodies gravitational waves are radiated. The gravitational waves move with light velocity c. Then instead of t we must insert:

$$t' = t - \frac{|\vec{x} - \vec{r}|}{c}$$

With that we get $\varphi_1(\vec{x}, t')$ and $\varphi_2(\vec{x}, t')$ instead of $\varphi_1(\vec{x}, t)$ and $\varphi_2(\vec{x}, t)$. The densities φ_1 and φ_2 remain of course functions of $\vec{x}$ and t.
Now we treat different special cases of revolution solids.

1. Revolution ellipsoid

We assume that the midpoint of the revolution ellipsoid is in the origin:



α = latitude
φ = longitude
a, b = major and minor semiaxis $\qquad a \geq b$

a, b can be functions of the time t. Now we must calculate R as function of a, b and of the latitude α. We use the following equation:

$$x \cdot \tan \alpha = z \tag{8}$$

and the canonical equation:

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{9}$$

For R we have the equation $R^2 = x^2 + z^2$. Equating of the first both equations leads to:

$$b \cdot \sqrt{1 - \frac{x^2}{a^2}} = x \cdot \tan\alpha$$

or:

$$\left(\frac{b}{a}\right)^2 \cdot (a^2 - x^2) = x^2 \cdot \tan^2\alpha$$

Now we solve to x^2:

$$x^2 = \frac{b^2}{\tan^2\alpha + \frac{b^2}{a^2}} = \frac{b^2 a^2}{a^2\tan^2\alpha + b^2}$$

With $R^2 = x^2 + z^2$ and $z = x \cdot \tan\alpha$ we obtain:

$$R^2 = x^2 \cdot (1 + \tan^2\alpha) = \frac{a^2 b^2}{a^2\tan^2\alpha + b^2} \cdot \left(1 + \tan^2\alpha\right)$$

It is valid:

$$1 + \tan^2\alpha = \frac{1}{\cos^2\alpha}$$

Then we get:

$$R^2 = \frac{a^2 b^2}{a^2\tan^2\alpha \cdot \cos^2\alpha + b^2 \cdot \cos^2\alpha}$$

At last the relation $\tan\alpha \cdot \cos\alpha = \sin\alpha$ leads to:

$$R = \frac{ab}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}}$$

With $r = R \cdot \cos\alpha$ we get:

$$r = \frac{\cos\alpha \cdot ab}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}}$$

For $\vec{X}$ we have the expression:

$$\vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ \tan\alpha \end{pmatrix}$$

We must insert for r the equation before.

Now we want to write b in another form. We use the linear eccentricity e and the numerical eccentricity ε:

$$e^2 := a^2 - b^2 \qquad \varepsilon^2 := \frac{e^2}{a^2}$$

Now we solve:

$$b^2 = a^2 - e^2 = a^2 - a^2\varepsilon^2 = a^2 \cdot (1 - \varepsilon^2)$$

It follows:

$$b = a \cdot \sqrt{1 - \varepsilon^2} \qquad \varepsilon \leq 1$$

If we write b in the form $b = a \cdot k$, then $k = \sqrt{1 - \varepsilon^2}$. Then we can introduce $\bar{a} \in [0, a]$ and $\bar{b} = \bar{a} \cdot k \in [0, b]$.

$(\bar{a}, \alpha, \varphi)$ is a suitable nature coordinate system for revolution ellipsoids. Now it follows with $\bar{a}$ und $\bar{b} = \bar{a} \cdot k$:

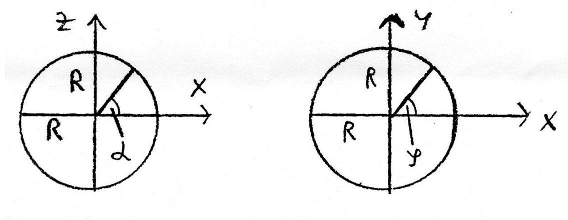
$$\vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{\cos\alpha \cdot \bar{a} \cdot \bar{a} \cdot k}{\sqrt{\bar{a}^2 \sin^2\alpha + \bar{a}^2 k^2 \cos^2\alpha}} \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ \tan\alpha \end{pmatrix}$$

$$= \frac{\cos\alpha \cdot \bar{a} \cdot k}{\sqrt{\sin^2\alpha + k^2 \cos^2\alpha}} \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ \tan\alpha \end{pmatrix} =: \vec{\phi}_1(\bar{a}, \alpha, \varphi)$$

$\vec{\phi}_1$ is the relation $(\bar{a}, \alpha, \varphi) \longrightarrow (x, y, z)$. We define
$\vec{y}_1 := (\bar{a}, \alpha, \varphi)$.

2. Ball

The midpoint of the ball shall be in the origin.



We can introduce spherical coordinates:
α = latitude $\quad -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$
φ = longitude $\quad -\pi \leq \varphi \leq \pi$

The ball's radius shall be $R(t)$. t is the time again. Now we take $r \in [0, R]$:

$$\vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \cdot \begin{pmatrix} \cos\alpha \cos\varphi \\ \cos\alpha \sin\varphi \\ \sin\alpha \end{pmatrix} =: \vec{\phi}_2(r, \alpha, \varphi)$$

To the last but one vector see for example Bartsch [1], chapter 7.2.1, p.265. $\vec{\phi}_2$ is the relation between (r, α, φ) and (x, y, z).

Now we can determine at revolution ellipsoid and at ball the gravitational acceleration

$\vec{g}$. $\vec{P}$ belongs to surface. We use the general transformation formula (see for example Forster [2] §13 theorem 2 p.120) of the moredimensional integral calculus:

$$\vec{g}_i(\vec{P}) = \int_{V_i} \frac{G\varphi_i(\vec{X},t)}{|\vec{X}-\vec{P}|^2} \cdot \frac{\vec{X}-\vec{P}}{|\vec{X}-\vec{P}|}\, d\vec{X}$$

$$= \int_{U_i} \frac{G \cdot (\vec{\phi}_i(\vec{y}_i) - \vec{P})}{|\vec{\phi}_i(\vec{y}_i) - \vec{P}|^3} \cdot \varphi_i(\vec{\phi}_i(\vec{y}_i), t) \cdot |\det D\vec{\phi}_i(\vec{y}_i)|\, d\vec{y}_i$$

$D\vec{\phi}_i$ is the Jacobi-matrix of $\vec{\phi}_i$ and $V_i = \vec{\phi}_i(U_i)$. $i = 1$ is for the revolution ellipsoid and $i = 2$ for the ball. Besides $\vec{y}_1 = (\bar{a}, \alpha, \varphi)$ and $\vec{y}_2 = (r, \alpha, \varphi)$ are defined. If we denote the function under the last integral sign with $\vec{h}_i(\vec{y}_i, t)$, we obtain with $(\bar{a}, \alpha, \varphi) \in U_1 = [0, a] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, 2\pi]$ for the revolution ellipsoid on the surface:

$$\vec{g}_1(\vec{P}) = \int_0^a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \vec{h}_1(\bar{a}, \alpha, \varphi, t)\, d\varphi d\alpha d\bar{a} \qquad \text{with} \qquad \vec{P} := \vec{\phi}_1(a, \alpha, \varphi)$$

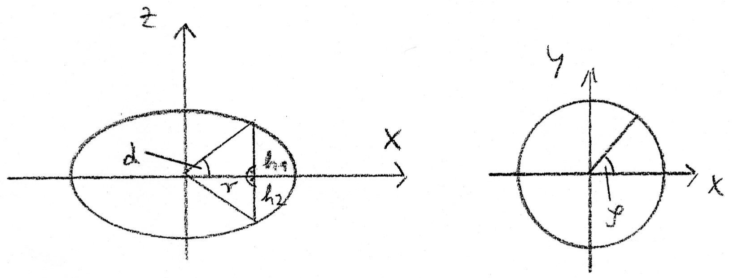
For the ball we get with $(r, \alpha, \varphi) \in U_2 = [0, R] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\pi, \pi]$ on the surface:

$$\vec{g}_2(\vec{P}) = \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} \vec{h}_2(r, \alpha, \varphi, t)\, d\varphi d\alpha dr \qquad \text{with} \qquad \vec{P} := \vec{\phi}_2(R, \alpha, \varphi)$$

In both cases we have 3-dimensional integrals. Of course we can use the integral of the revolution ellipsoid for the case of the ball, too. But the second integral is easier to calculate.

3. The natural coordinate system of the general revolution solid

Now we want to introduce natural coordinates for the general revolution solid:



α = latitude
φ = longitude

We represent the vector $\vec{X}$ with:

$$\vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ h_1(r) \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ r\tan\alpha \end{pmatrix} = r\cdot\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ \tan\alpha \end{pmatrix} =: \vec{\phi}_3(r,\alpha,\varphi)$$

for $z \geq 0$, $\alpha \geq 0$ and $r \in [0, r_0]$

$$\vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ -h_2(r) \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ r\tan\alpha \end{pmatrix} = r\cdot\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ \tan\alpha \end{pmatrix} =: \vec{\phi}_4(r,\alpha,\varphi)$$

for $z \leq 0$, $\alpha \leq 0$ and $r \in [0, r_0]$

r_0 is the equator radius of the revolution solid. (r, α, φ) are the wanted natural coordinates of the revolution solid. Then we can express the gravitational acceleration with the general transformation formula (see for example Forster [2], §13, theorem 2, p.120):

$$\vec{g}(\vec{P}) = \sum_{i=3}^{4} \int\limits_{V_i} \frac{G\varphi_i(\vec{X},t)}{|\vec{X}-\vec{P}|^2} \cdot \frac{\vec{X}-\vec{P}}{|\vec{X}-\vec{P}|}\, d\vec{X}$$

$$= \sum_{i=3}^{4} \int\limits_{U_i} \frac{G\cdot(\vec{\phi}_i(\vec{y}_i)-\vec{P})}{|\vec{\phi}_i(\vec{y}_i)-\vec{P}|^3} \cdot \varphi_i(\vec{\phi}_i(\vec{y}_i),t) \cdot |\det D\vec{\phi}_i(\vec{y}_i)|\, d\vec{y}_i$$

It is $V_i = \vec{\phi}_i(U_i)$. $i = 3$ is for the north part and $i = 4$ for the south part of the revolution solid. Besides $\vec{y}_3 = (r, \alpha, \varphi)$ and $\vec{y}_4 = (r, \alpha, \varphi)$ are defined. If we denote the function under the last integral sign with $\vec{h}_i(\vec{y}_i, t)$, we obtain with $(r, \alpha, \varphi) \in U_3 = [0, r_0] \times [0, \frac{\pi}{2}] \times [0, 2\pi]$ respectively $U_4 = [0, r_0] \times [-\frac{\pi}{2}, 0] \times [0, 2\pi]$:

$$\vec{g}(\vec{P}) = \int_0^{r_0}\int_0^{\frac{\pi}{2}}\int_0^{2\pi} \vec{h}_3(r,\alpha,\varphi,t)\, d\varphi d\alpha dr + \int_0^{r_0}\int_{-\frac{\pi}{2}}^{0}\int_0^{2\pi} \vec{h}_4(r,\alpha,\varphi,t)\, d\varphi d\alpha dr$$

With $\vec{P} := \vec{\phi}_3(r_0, \alpha, \varphi)$ if $\alpha \geq 0$ and $\vec{P} := \vec{\phi}_4(r_0, \alpha, \varphi)$ if $\alpha \leq 0$.

With that we have a sum of 3-dimensional integrals.

Such gravitational accelerations are used at Schröer [3] chapter 5, to calculate the total acceleration and the visual vertical.

References

[1] Hans-Jochen Bartsch “Taschenbuch mathematischer Formeln”, Verlag Harri Deutsch, 7.-9. edition, Frankfort on the Main 1986

[2] Otto Forster “Analysis 3”, Vieweg Verlag, 2. edition, Brunswick 1983

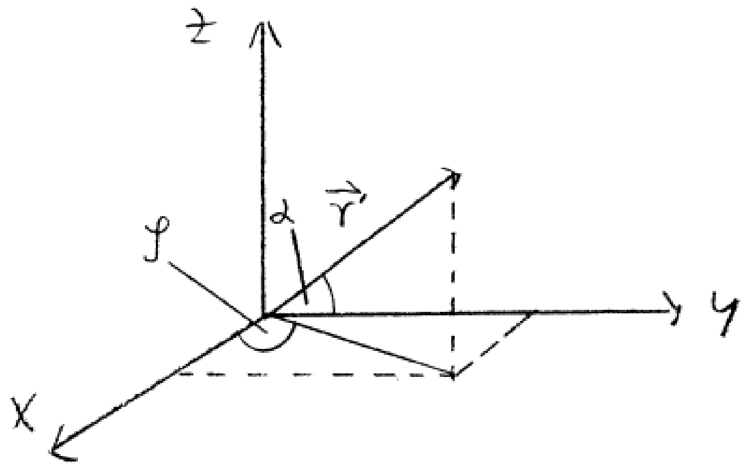
[3] Harald Schröer “Theory of orientation - The visual vertical”, Wissenschaft & Technik Verlag, Berlin 2002

61. Motions on a rotating planet

Here we want to describe the motion of a falling body on a rotating planet.

1. In general (spherical coordinates)

The planet rotates with an arbitrary changable angular velocity $\vec{w}(t)$ and has a translational acceleration $\vec{a}_{tr}(t)$. t is the time.



$\vec{r} = (x, y, z)$ = localized vector $\qquad r = |\vec{r}|$
φ = longitude
α = latitude

The origin shall be the geometrical midpoint of the planet. We introduce spherical coordinates e.g. Budo [1], §3, p.13:

$$x = r\sin(90^\circ - \alpha)\cos\varphi = r\cos\alpha\cos\varphi$$
$$y = r\sin(90^\circ - \alpha)\sin\varphi = r\cos\alpha\sin\varphi$$
$$z = r\cos(90^\circ - \alpha) = r\sin\alpha$$

Now we turn to the gravitational force. For complicated gravitational fields the form $\vec{F}(\vec{r}, \alpha, \varphi, t)$ or even $\vec{F}(\vec{r}, \dot{\vec{r}}, \ddot{\vec{r}}, \alpha, \dot{\alpha}, \ddot{\alpha}, \varphi, \dot{\varphi}, \ddot{\varphi}, t)$ is possible. Important simple is the central field with:

$$\vec{F} = -f(|\vec{r}|) \cdot \frac{\vec{r}}{|\vec{r}|}$$

Very often Newton's gravitation law is valid:

$$\vec{F} = -\frac{G \cdot M \cdot m}{|\vec{r}|^2} \cdot \frac{\vec{r}}{|\vec{r}|}$$

G = gravitational constant
$M = M(t)$ = mass of the planet

m = mass of the test piece $\qquad$ m is constant.

For the motion in a rotating system we have see Budo [1], §14, p.74, equation (16):

$$m\ddot{\vec{r}} = \vec{F} - m\vec{a}_{tr} - m \cdot (\vec{w} \times (\vec{w} \times \vec{r})) - m \cdot (\dot{\vec{w}} \times \vec{r}) - 2m \cdot (\vec{w} \times \dot{\vec{r}}) \tag{1}$$

If we insert the vector

$$\vec{r} = r \cdot \begin{pmatrix} \cos\alpha\cos\varphi \\ \cos\alpha\sin\varphi \\ \sin\alpha \end{pmatrix}$$

into the equation (1), we obtain 3 differential equations of r, α and φ.

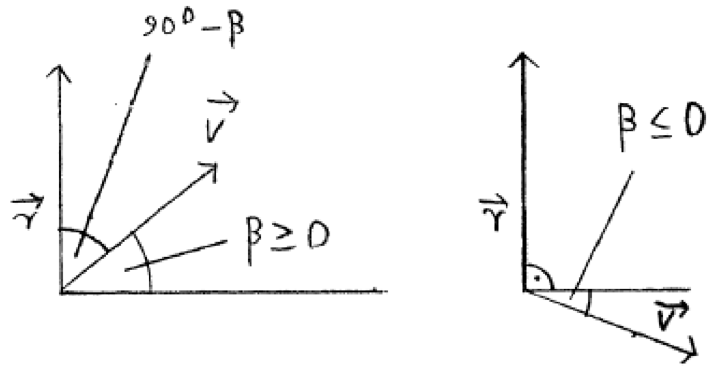
initial conditions:

$$\vec{r}(t_0) = \vec{r}_0 \qquad \vec{v}(t_0) = \vec{v}_0$$

The height can be presented with $r(t) - R_p$, R_p is the planet's radius.

throwing angle β:

$$\vec{v} = \dot{\vec{r}}$$



$\angle(\vec{r}, \vec{v}) = 90° - \beta \qquad \beta = -(\angle(\vec{r}, \vec{v}) - 90°) = 90° - \angle(\vec{r}, \vec{v})$

With that:

$$\sin\beta(t_0) = \cos(90° - \beta(t_0)) = \frac{\vec{r}_0 \cdot \vec{v}_0}{|\vec{r}_0| \cdot |\vec{v}_0|}$$

General pitch angle $\beta(t)$:

$$\sin\beta(t) = \frac{\vec{r} \cdot \vec{v}}{|\vec{r}| \cdot |\vec{v}|}$$

The differential equation system (1) can be witten as:

$$\vec{a} = \dot{\vec{r}} \tag{2}$$

$$m\dot{\vec{a}} = \vec{F} - m\vec{a}_{tr} - m \cdot (\vec{w} \times (\vec{w} \times \vec{r})) - m \cdot (\dot{\vec{w}} \times \vec{r}) - 2m \cdot (\vec{w} \times \vec{a})$$

We get 6 differential equation of first order with the unknowns $\vec{a}(t), \vec{r}(t) \in R^3$. This differential equation is not linear because of $\vec{F}$. For an exact solution the only possibility is the expansion into a series see Kamke [4], A, §2, (6.3), p.38. This is valid to all central fields. For the general gravitational field it is possible, when the differential equation system is explicit solvable to $\dot{\vec{r}}$ and $\ddot{\vec{r}}$. If the system is not explicit solvable and $\vec{w}(t)$ is not temporal constant, then no exact solution can be calculated. But with numerical methods approximate solutions can be determined.

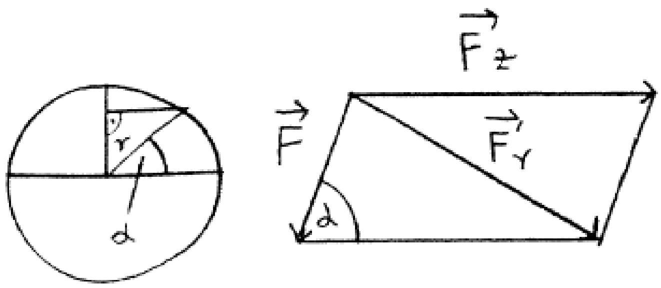
Instead of a planet sphere we can choose a general body of rotation. Even general bodies are possible see Schröer [5], chapter 10.

2. Motion in a local coordinate system on planet's surface:

In this case we assume $\vec{a}_{tr} = 0$. M and m are constant. Further the gravitation shall be a central field, with that:

$$\vec{F} = -m \cdot f(|\vec{r}|) \cdot \frac{\vec{r}}{|\vec{r}|}$$

The angular velocity has the form $\vec{w}(t) = (0, 0, w(t))$.



Now we introduce the centrifugal force $\vec{F}_z$ in this case:

$$F_z = mrw^2 \cdot \cos\alpha$$

resultant force:

$$\vec{F}_r = \vec{F} + \vec{F}_z$$

To the resultant acceleration we can write:

$$b_r(r, \alpha, w) = \frac{F_r}{m} = \frac{\sqrt{F^2 + F_z^2 - 2FF_z \cos\alpha}}{m}$$

$$= \sqrt{(f(r))^2 + r^2w^4 \cos^2\alpha - 2f(r)rw^2 \cos^2\alpha}$$

with $r = R_p + z$. We use R_p as planet's radius and z as height. With Budo [1], §24, p.119,120 and equation (1) we get the differential equation system:

$$\ddot{\vec{r}} = \vec{b}_r(r, \alpha, w) + 2 \cdot \dot{\vec{r}} \times \vec{w} - \dot{\vec{w}} \times \vec{r} \tag{3}$$

with the vectors:

$$\vec{b}_r = \begin{pmatrix} 0 \\ 0 \\ -b_r \end{pmatrix} \qquad \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \dot{\vec{r}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

$$\vec{w} = \begin{pmatrix} -w\cos\alpha \\ 0 \\ w\sin\alpha \end{pmatrix} \qquad \dot{\vec{w}} = \begin{pmatrix} -\dot{w}\cos\alpha \\ 0 \\ \dot{w}\sin\alpha \end{pmatrix}$$

The meaning of the coordinates:

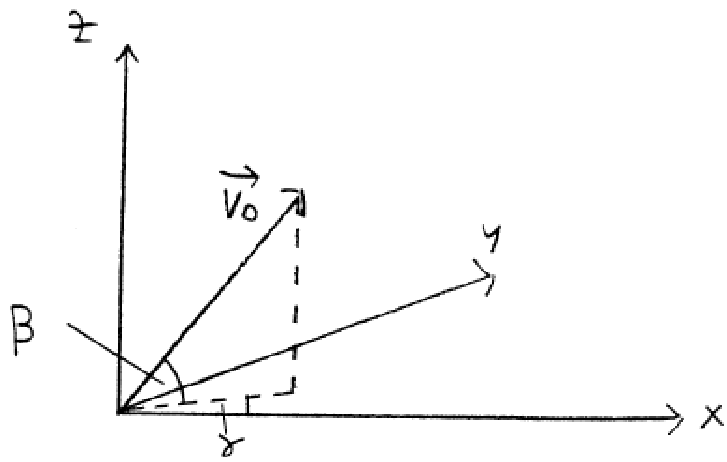
x = north-south-direction (south positive)
y = west-east-direction (east positive)
z = height (upwards positive)

The initial conditions with the direction angle γ and throwing angle β are:

$$\vec{r}(t_0) = \vec{r}_0 \qquad \vec{v}(t_0) = \vec{v}_0$$

We introduce spherical coordinates:

$$\vec{v}_0 = |\vec{v}_0| \cdot \begin{pmatrix} \cos\beta\cos\gamma \\ \cos\beta\sin\gamma \\ \sin\beta \end{pmatrix} \qquad \begin{matrix} \gamma \in [0, 360°[\\ \beta \in [-90°, 90°] \end{matrix}$$



$\vec{r}(t) = (x(t), y(t), z(t))$ presents the solution of the differential equation system. Insertion of the vectors leads to the following differential equation system:

$$\ddot{x} = 2\dot{y}w\sin\alpha + y\dot{w}\sin\alpha$$

$$\ddot{y} = -2\dot{z}w\cos\alpha - 2\dot{x}w\sin\alpha - x\dot{w}\sin\alpha - z\dot{w}\cos\alpha$$

$$\ddot{z} = 2\dot{y}w\cos\alpha - b_r + y\dot{w}\cos\alpha$$

We have an inhomogeneous linear differential equation system of second order. $b_r = b_r(r, \alpha, w)$ and α shall be constants.

Transformation:

$$\dot{x} = a_1$$

$$\dot{y} = a_2$$

$$\dot{z} = a_3$$

$$\dot{a}_1 = 2a_2 w \sin\alpha + y\dot{w}\sin\alpha$$

$$\dot{a}_2 = -2a_3 w \cos\alpha - 2a_1 w \sin\alpha - x\dot{w}\sin\alpha - z\dot{w}\cos\alpha$$

$$\dot{a}_3 = 2a_2 w \cos\alpha - b_r + y\dot{w}\cos\alpha$$

Then we have an inhomogeneous linear differential equation system of first order. If w is not constant, the only exact method is the expansion of series see Kamke [4], A, §2, (6.3), p.38. The alternative is a numerical calculation. If we know a fundamental system of the belonging linear homogeneous system, we can determine the solution of the inhomogeneous system with variation of constant, see Forster [2], §12, p.128, theorem 4. If w is constant, we have a system with constant coefficients. In this case the calculation of the exact solution is possible. This is done in Greiner [3], I.2, p.8-17.
We see that there are possibilities to solve this problem.

References

[1] A Budo "Theoretische Mechanik", 10.edition, VEB Deutscher Verlag der Wissenschaften, Berlin, 1980

[2] Otto Forster "Analysis 2", 5.edition, 1984, Vieweg Verlag, Brunswick

[3] Walter Greiner "Mechanik", part 2, Verlag Harri Deutsch, 1989

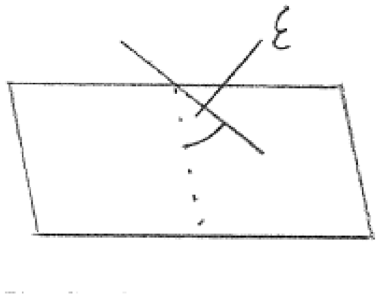
[4] Erich Kamke "Differentialgleichungen: Lösungsmethoden und Lösungen", volume 1, 10.edition, Teubner Verlag, Stuttgart, 1983

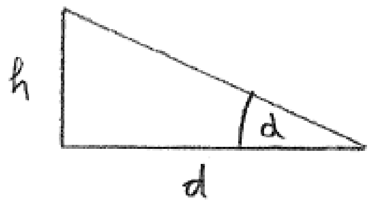
[5] Harald Schröer "Theory of orientation", Wissenschaft und Technik Verlag, Berlin, 2002

62. Projection of angles and trajectories of projection on inclined surface

1. Projections of angles

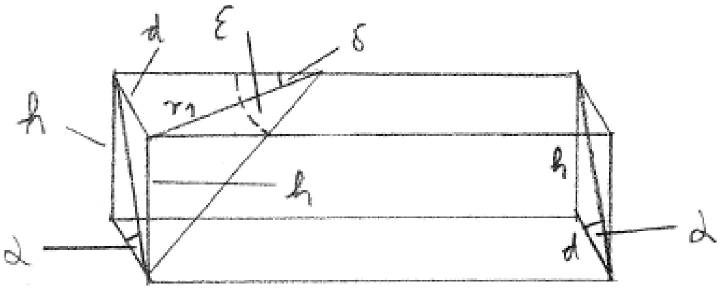
Here we want to look at the projection of angles. This is a theme that is mentioned scarcely in the mathematical literature. The reader gets an impression of how angles can be changed with projections.





A straight line turns round about an angle δ seeing from the edge. The plane is inclined with the angle α:

ε is the respective angle of inclination at angular displacement δ. We can look at the following figure:



Now we calculate ε. We have:

$$\tan\varepsilon = \frac{h}{r_1}$$

h and r_1 must be replaced:

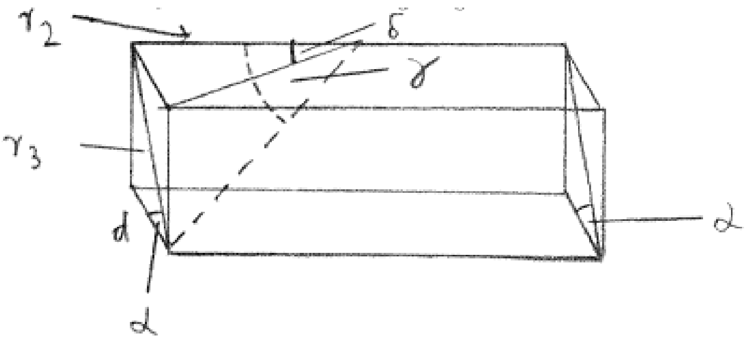
$$h = d \cdot \tan\alpha \qquad r_1 = \frac{d}{\sin\delta}$$

Then we obtain with insertion:

$$\tan\varepsilon = \tan\alpha \cdot \sin\delta \tag{1}$$

Special case: $\varepsilon = \alpha$ at $\delta = 90°$

Now we turn to the angle γ on the inclined plane. γ is the projection of δ on the inclined plane.



From the figure we take:

$$\tan\gamma = \frac{r_3}{r_2}$$

We replace r_3 and r_2 through:

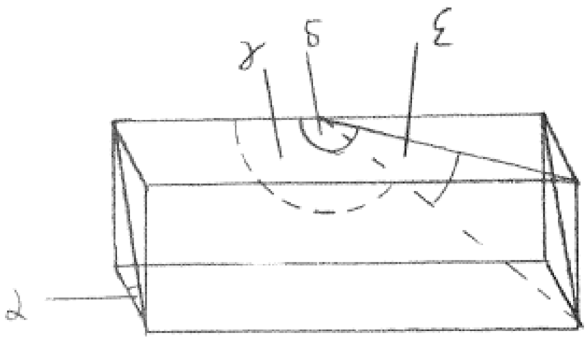
$$r_3 = \frac{d}{\cos\alpha} \qquad r_2 = \frac{d}{\tan\delta}$$

We get:

$$\tan\gamma = \frac{\tan\delta}{\cos\alpha} \tag{2}$$

Special case: $\gamma = \delta$ for $\alpha = 0$

The extension in $0 \le \delta \le 180°$:



With the figure and equation (1) is:

$$\tan\varepsilon = \tan\alpha \cdot \sin(180° - \delta)$$

With $\sin(180° - \delta) = \sin\delta$ we get again equation (1).

For the angle γ we can follow with equation (2):

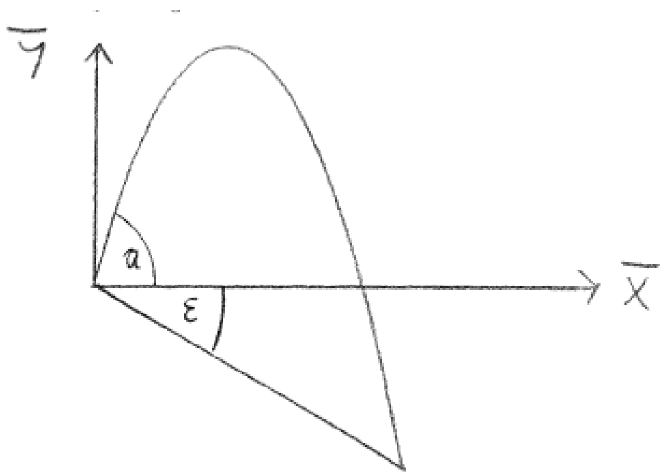
$$\tan(180° - \gamma) = \frac{\tan(180° - \delta)}{\cos\alpha}$$

With $\tan(180° - a) = -\tan a$ we obtain again equation (2).

With that the equations (1) and (2) are valid for $\delta \in [0, 180°]$.

2. One application: Trajectories of projection on inclined surface

We can use the information from section 1 to trajectories of projections on planet with inclined surface. Because the gravitational field is locally homogeneous the trajectories of projections are inclined parabolas. First we introduce a coordinate system $\bar{x}$ and $\bar{y}$. $\bar{y}$ is parallel with gravitational direction $\vec{g}$. $\bar{x}$ is perpendicular to it.



v = initial velocity
t = time
a = throwing angle

Equations of trajectory of projection:

$$\bar{x} = vt \cdot \cos a \tag{3}$$

$$\bar{y} = vt \cdot \sin a - \frac{gt^2}{2} \tag{4}$$

$$\bar{y} = \bar{x} \cdot \tan a - \frac{gx^2}{2v^2\cos^2 a} \tag{5}$$

Duration of ascent:

$$\bar{t}_s = \frac{v \cdot \sin a}{g}$$

Height of ascent:

$$\bar{h} = \frac{v^2 \cdot \sin^2 a}{2g}$$

Through differentiation to the time we get the components of velocity:

$$\frac{d\bar{x}}{dt} = \bar{v}_{\bar{x}} = v \cdot \cos a$$

$$\frac{d\bar{y}}{dt} = \bar{v}_{\bar{y}} = v \cdot \sin a - gt$$

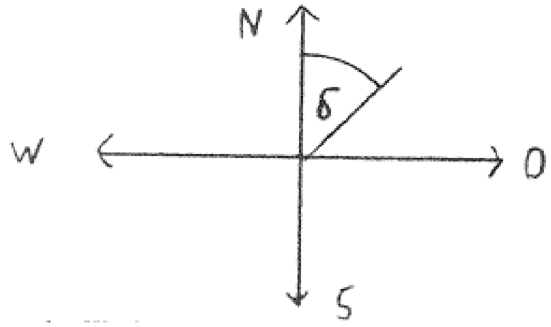
Then we can calculate the orbit velocity $v_B = \sqrt{\bar{v}_{\bar{x}}^2 + \bar{v}_{\bar{y}}^2}$. With $\sin^2 a + \cos^2 a = 1$ we obtain:

$$v_B = \sqrt{v^2 - 2g \cdot \left(vt \cdot \sin a - \frac{gt^2}{2}\right)}$$

Because the plane is inclined the throwing range and the throwing time must be determined with another method. α is again the inclination angle of the plane. Because of equation (1) it is valid:

$$-\tan\varepsilon = \tan(-\varepsilon) = \tan\alpha \cdot \sin(\delta - 90°) = -\tan\alpha \cdot \cos\delta$$

With that we have $\tan\varepsilon = \tan\alpha \cdot \cos\delta$.



N = NORTH
O = EAST
S = SOUTH
W = WEST

To determine the throwing time t_w:

$$\sin a \cdot v \cdot t_w - \frac{gt_w^2}{2} = \tan\varepsilon \cdot \cos a \cdot v \cdot t_w$$

We can divide through t_w.

$$\sin a \cdot v - \frac{gt_w}{2} = \tan\varepsilon \cdot \cos a \cdot v$$

Solving to t_w:

$$t_w = 2 \cdot \frac{\sin a \cdot v - \tan\varepsilon \cdot \cos a \cdot v}{g}$$

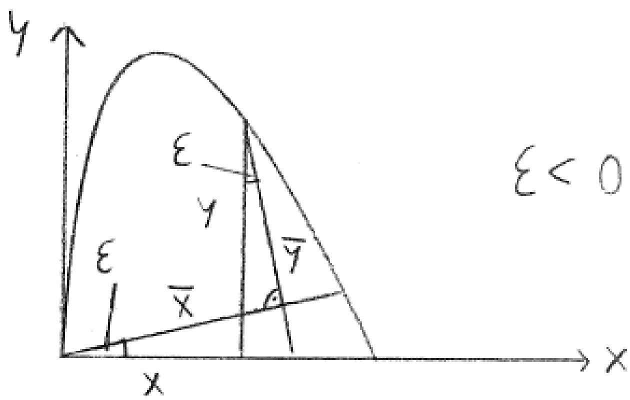
If we insert the throwing time in the equation (3), we get the throwing range $\bar{w}$:

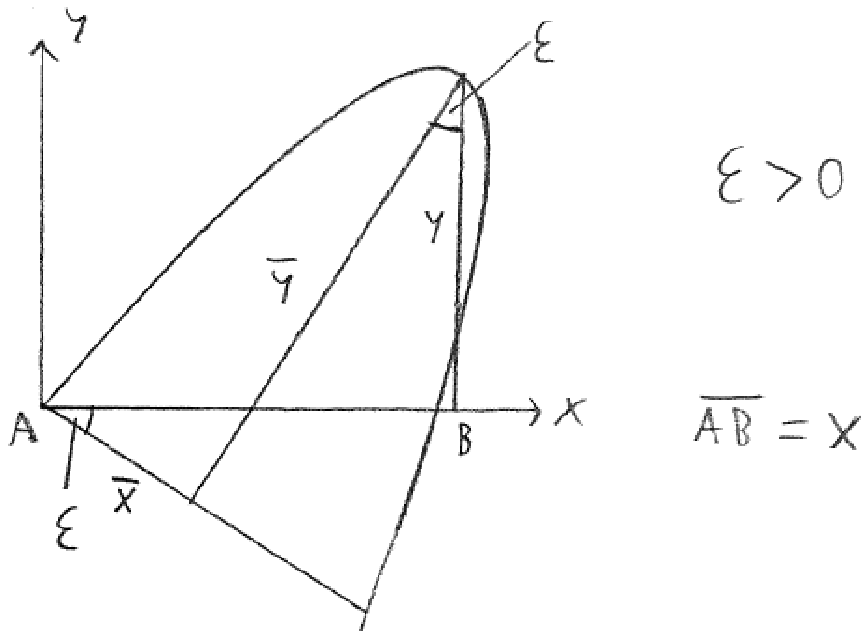
$$\bar{w} = \cos a \cdot v \cdot t_w$$

We can present $\bar{y}$ to time t_w with:

$$\bar{y}_w = \sin a \cdot v \cdot t_w - \frac{gt_w^2}{2}$$

To get the real x and y we must turn the throwing parabola to the angle ε. y is perpendicular and x is parallel to the surface.





From these figures we can determine the equation:

$$y = \cos\varepsilon \cdot (\bar{y} - \bar{x} \cdot \tan\varepsilon) \tag{6}$$

$$y = \cos\varepsilon \cdot \left(\sin a \cdot vt - \frac{gt^2}{2} - \tan\varepsilon \cos a \cdot vt\right) \tag{7}$$

With differentiation:

$$\frac{dy}{dt} = v_y = \cos\varepsilon \cdot (\sin a \cdot v - gt - \tan\varepsilon \cos a \cdot v)$$

With the last figures we also conclude:

$$x = \frac{\bar{x}}{\cos\varepsilon} + y \cdot \tan\varepsilon \tag{8}$$

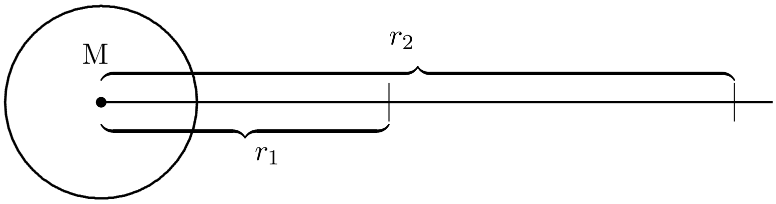
It is $v_x = \frac{dx}{dt}$. We get the orbit velocity with $v_B = \sqrt{v_x^2 + v_y^2}$. The real throwing angle is equal to $a - \varepsilon$. With calculating the maximum of y (equation (7)) with the differential calculus we can determine the duration of ascent and height of ascent relative to the coordinates x and y. The throwing range relative to x we get through insertion of t_w for t in equation (8). We still have not talk about the absolute value of the angle α on a planet. The inclination angle α is calculated in Schröer [1] at constant and non constant rotation time (angular velocity). On earth at 50 degree north or south latitude is α about one hundredth degree.

References

[1] Harald Schröer "Theory of orientation", Wissenschaft und Technik Verlag, Berlin, 2002

63. Frequency shift of the radiation in gravitational field

The radiation's frequency changes on the way through the gravitational field. First we look at a planet:



The work between the distances r_1 and r_2 in gravitational field of a planet with mass M is:

$$W = \int_{r_1}^{r_2} g(r) \cdot m(r)\, dr$$

With:

$g(r)$ = gravitational acceleration in distance r
$m(r)$ = mass of a photon in distance r

For the energy of a photon it is valid $W = h \cdot f$, with photon's frequency $f(r)$ and Planck's quantum h. Because of the frequency's variation in gravitational field with the distance, the frequency is a function of r. We get the photon's mass with the equation $m(r) \cdot c^2 = h \cdot f(r)$ with c as light speed e.g. Hammer [2] chapter 8.2.2.2 p.189. Thus we obtain:

$$m(r) = \frac{h \cdot f(r)}{c^2}$$

The frequency's shift is expressed with the following equation:

$$h \cdot (f(r_2) - f(r_1)) = -\int_{r_1}^{r_2} g(r) \cdot m(r)\, dr \tag{1}$$

With differentiation with respect to r follows the differential equation:

$$h \cdot f'(r) = -g(r) \cdot m(r)$$

with initial value $f_1 = f(r_1)$ that is given. If we insert the photon's mass, we get:

$$f'(r) = -\frac{g(r)}{c^2} \cdot f(r) \tag{2}$$

Thus we have an initial value problem with a homogeneous linear differential equation of first order. For the solution we can write e.g. Forster [1] §11 theorem 2 p.114:

$$f(r) = f(r_1) \cdot \exp\left(-\frac{1}{c^2} \cdot \int_{r_1}^{r} g(\bar{r})\, d\bar{r} \right)$$

In general it is $g(r) = \frac{GM}{r^2}$ because of Newton's law of gravitation. G is the gravitational constant. Thus we obtain:

$$f(r) = f(r_1) \cdot \exp\left(-\frac{1}{c^2}\int_{r_1}^{r} \frac{GM}{\bar{r}^2}\, d\bar{r}\right)$$

$$= f(r_1) \cdot \exp\left(\frac{GM}{c^2} \cdot \left(\frac{1}{r} - \frac{1}{r_1}\right)\right)$$

Then we have:

$f(r) \leq f(r_1)$ if $r \geq r_1$
$f(r) \geq f(r_1)$ if $r \leq r_1$

The first case means energy loss for the radiation respectively smaller frequency. In the second case the radiation get more energy therefore the frequency increases.

Now we want to derive an approximation in the Newton case. We assume that the frequency shift Δf is very small compared to the frequency $f(r_1)$. Then the photon's mass $m = \frac{hf(r_1)}{c^2}$ is approximate constant. Now the work in gravitational field can be written as:

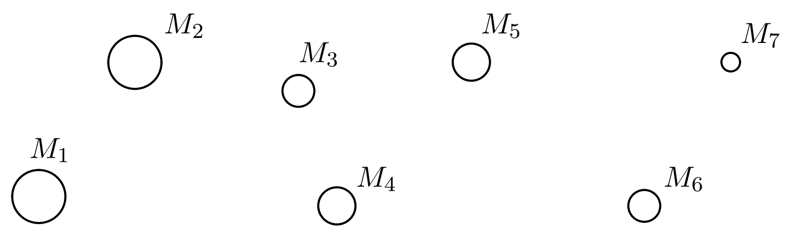
$$\Delta W \approx GMm \cdot \left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Besides we have $\Delta W = h \cdot \Delta f$, then we find:

$$\Delta f \approx \frac{GMf(r_1)}{c^2} \cdot \left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Then it is $f(r_2) = f(r_1) - \Delta f$.

Now we have gravitational fields of n planets with masses $M_1, \ldots, M_n$.



First we assume that the frequency shift is very small. Then the photon's mass is constant again. The masses $M_1, \ldots, M_n$ are on positions $\vec{x}_1, \ldots, \vec{x}_n \in R^3$. Then in Newton's case a photon that moves from $\vec{r}_1$ to $\vec{r}_2 \in R^3$ wins or loses the energy:

$$\Delta W \approx Gm \cdot \sum_{i=1}^{n} M_i \cdot \left(\frac{1}{|\vec{r}_1 - \vec{x}_i|} - \frac{1}{|\vec{r}_2 - \vec{x}_i|}\right)$$

$|\cdot|$ is the absolute value of a vector. With $\Delta W = h\cdot\Delta f$ and the photon's mass $m \approx \frac{h\cdot f(\vec{r}_1)}{c^2}$ we get:

$$\Delta f \approx \frac{Gf(\vec{r}_1)}{c^2}\cdot\sum_{i=1}^{n} M_i\cdot\left(\frac{1}{|\vec{r}_1-\vec{x}_i|}-\frac{1}{|\vec{r}_2-\vec{x}_i|}\right)$$

Now we view the case with large frequency shift. Then the photon's mass is not constant. For this case we have the following integral equation:

$$h\cdot(f(\vec{r}_2)-f(\vec{r}_1)) = -\sum_{i=1}^{n}\int_{\vec{r}_1}^{\vec{r}_2}\vec{g}_i(\vec{r}-\vec{x}_i)\cdot m(\vec{r})\,d\vec{r}$$

$$= -\int_{t_1}^{t_2}\sum_{i=1}^{n}\vec{g}_i(\vec{r}-\vec{x}_i)\cdot m(\vec{r})\cdot\dot{\vec{r}}\,dt$$

$\vec{r}$ is a function of t. Besides we have $\vec{r}_1 = \vec{r}(t_1)$ and $\vec{r}_2 = \vec{r}(t_2)$ and the gravitational accelerations $\vec{g}_1$, ...,$\vec{g}_n$ of the masses M_1, ... , M_n. This is a line integral of second kind. Differentiation with respect to t leads to the differential equation:

$$-\sum_{i=1}^{n}\vec{g}_i(\vec{r}-\vec{x}_i)\cdot m(\vec{r})\cdot\dot{\vec{r}} = h\cdot\mathrm{grad}(f(\vec{r}))\cdot\dot{\vec{r}}$$

If we insert $m(\vec{r}) = \frac{h\cdot f(\vec{r})}{c^2}$, we obtain the partial differential equation:

$$0 = \mathrm{grad}(f(\vec{r}))\cdot\dot{\vec{r}} + \frac{f(\vec{r})}{c^2}\cdot\sum_{i=1}^{n}\vec{g}_i(\vec{r}-\vec{x}_i)\cdot\dot{\vec{r}} \tag{3}$$

We have a linear partial differential equation. It is not easy to solve this equation. We won't solve it here. For the solution see Kamke [3] D. §1 chapter 3.10, 5.8, 6.9. These chapters are surveys of solutions of partial differential equations.

References

[1] Otto Forster "Analysis 2" 5.edition 1984 Vieweg Verlag Brunswick

[2] Karl Hammer "Grundkurs der Physik" Teil 2 3.edition Oldenbourg Verlag Munich 1987

[3] Erich Kamke "Differentialgleichungen - Lösungsmethoden und Lösungen" Volume II Teubner Verlag Stuttgart 6.edition 1979

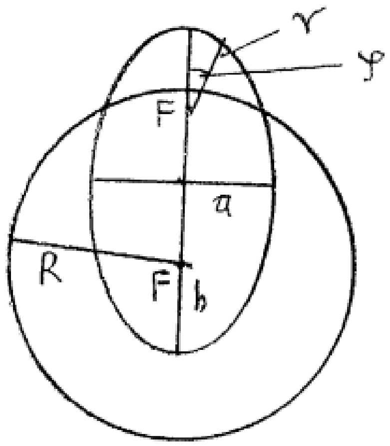
64. The local sector of ellipse

If we view a trajectory of projection on the earth's surface in homogeneous field, then we can find out that these trajectories of projection are parabolas.
In fact in the motion of celestial bodies there are other orbits, too, for example the ellipse. A ball that is thrown with a small initial velocity describes a parabola in homogeneneous field that means with constant gravitational acceleration. If we would do this experiment in a height of 1000 km, then the ball describes until the arrival on the earth's surface a sector of an ellipse.

Now it's obvious to suppose that a very small sector of an ellipse is similar to a parabola. We will occupy with this theme.

F = FOCAL POINT

a,b = SEMIAXES



We view an ellipse and a circle with the radius R. R can be the earth's radius or a planet's radius. We consider the difference $h = r - R$ with the conic section equation:

$$h(\varphi) = r - R = \frac{p}{1 + e\cos\varphi} - R$$

φ = angle in circular measure (true anomaly)
e = numerical eccentricity
p = parameter

See for example Voigt [1] chapter II.2.2 a p.33. The derivation is with the reciprocal rule $\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$:

$$h'(\varphi) = \frac{pe\sin\varphi}{(1 + e\cos\varphi)^2}$$

At $\varphi \ll 1$ we obtain as approximation:

$$h'(\varphi) \approx \frac{pe \cdot \varphi}{(1 + e)^2}$$

This is valid because of $\sin\varphi \approx \varphi$ and $\cos\varphi \approx 1$ for $\varphi \ll 1$. Now we integrate this approximation:

$$h(\varphi) \approx \frac{pe}{(1 + e)^2} \cdot \int \varphi \, d\varphi = \frac{pe\varphi^2}{2 \cdot (1 + e)^2} + c$$

c = integration constant

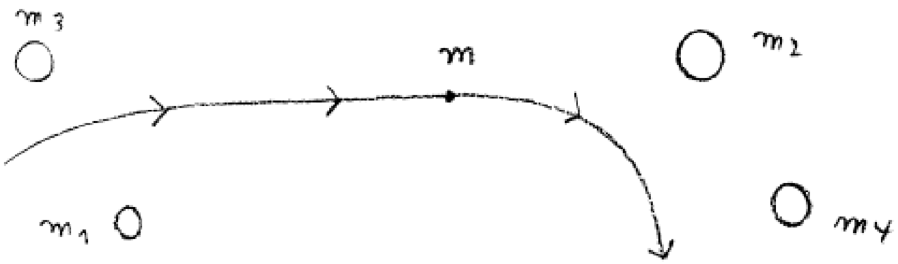
Thus it is $h(\varphi) \approx k\varphi^2 + c$. This is a parabola equation. We recognize that a small sector of an ellipse is similar to a parabola. This is the mathematical reason why a trajectory of an ellipse in a very small sector (homogeneous field $h \ll R$) converges against a parabola.

References

[1] Hans Heinrich Voigt "Abriß der Astronomie" 4.edition BI Mannheim 1988

65. The motion in gravitational and electromagnetic fields

Here we give a general description of the motion of a sample.



n bodies with mass $m_i(t)$ and charge $Q_i(t)$ are moving in a coordinate system in R^3 in medium. t is time. The motions $\vec{r}_1(t), \ldots, \vec{r}_n(t) \in R^3$ are known. The sample has the mass $m(t)$ and charge $Q(t)$. We are looking for the motion $\vec{r}(t)$ of the sample. We assume that the sample has a small mass and a small charge compared to the other n bodies. Thus the pertubing influence through the sample is not important. Thus:

$$m_i(t) \gg m(t) \qquad \text{and} \qquad Q_i(t) \gg Q(t) \qquad \text{for all} \quad i \in 1, \ldots, n$$

Now we introduce different forces:

$\vec{F}_g$ = gravitational force
$\vec{F}_E$ = electric force
$\vec{F}_m$ = magnetic force
$\vec{F}_b$ = retarding force of the medium

$\vec{F}_{bi}, \vec{F}_{gi}, \vec{F}_{Ei}, \vec{F}_{mi}$ = retarding force (of the medium for example of the atmosphere) / gravitational force / electric force / magnetic force of the i-th body.

$\varphi_i, \vec{g}_i, \vec{E}_i, \vec{B}_i$ = density (of the medium for example of the atmosphere) / gravitational acceleration / electric field intensity / magnetic flux density of the i-th body.

$\vec{F}_p$ = force that is produced by the sample itself - for example if the sample is a rocket.

$\vec{F}_{ba}, \vec{F}_{ga}, \vec{F}_{Ea}, \vec{F}_{ma}$ = retarding force (of the medium - for example interstellar matter) / gravitational force / electric force / magnetic force, that is given in R^3 without influence of the n bodies and the sample.

φ_K = density of the sample (If the density in the sample is not spatially constant, then φ_K is the average density that can be time-dependent.)

Now we turn to the notations used:

$\vec{y} := (m, m_1, \ldots, m_n, Q, Q_1, \ldots, Q_n, \vec{r}, \dot{\vec{r}}, \ddot{\vec{r}}, \vec{r}_1, \dot{\vec{r}}_1, \ddot{\vec{r}}_1, \ldots, \vec{r}_i, \dot{\vec{r}}_i, \ddot{\vec{r}}_i, \ldots, \vec{r}_n, \dot{\vec{r}}_n, \ddot{\vec{r}}_n, \vec{F}_{g1}, \vec{F}_{E1},$

$\vec{F}_{m1}$,$\vec{F}_{b1}$,$\vec{g}_1$,$\vec{E}_1$,$\vec{B}_1$,φ_1, ...,$\vec{F}_{gi}$,$\vec{F}_{Ei}$,$\vec{F}_{mi}$,$\vec{F}_{bi}$,$\vec{g}_i$, $\vec{E}_i$, $\vec{B}_i$, φ_i, ..., $\vec{F}_{gn}$,$\vec{F}_{En}$, $\vec{F}_{mn}$, $\vec{F}_{bn}$,$\vec{g}_n$,$\vec{E}_n$,$\vec{B}_n$,φ_n, $\vec{F}_{ga}$,$\vec{F}_{Ea}$,$\vec{F}_{ma}$,$\vec{F}_{ba}$,$\vec{g}_a$,$\vec{E}_a$, $\vec{B}_a$,φ_a,φ_K)

$\vec{y}_i := (m, Q, m_i, Q_i, \vec{r}, \dot{\vec{r}}, \ddot{\vec{r}}, \vec{r}_i, \dot{\vec{r}}_i, \ddot{\vec{r}}_i, \vec{g}_i, \vec{E}_i, \vec{B}_i, \varphi_i)$

$\vec{y}_a := (m, Q, \vec{r}, \dot{\vec{r}}, \ddot{\vec{r}}, \vec{g}_a, \vec{E}_a, \vec{B}_a, \varphi_a)$

Thus we can have recursive relations that should be avoided in practical cases of course. Now the dependences are:

$$m(t,\vec{y}) \qquad Q(t,\vec{y}) \qquad \varphi_K(t,\vec{y})$$

$$\vec{F}_{gi}(t,\vec{y}_i) \qquad \vec{F}_{Ei}(t,\vec{y}_i) \qquad \vec{F}_{mi}(t,\vec{y}_i) \qquad \vec{F}_{bi}(t,\vec{y}_i)$$

$$\vec{g}_i(t,\vec{y}_i) \qquad \vec{E}_i(t,\vec{y}_i) \qquad \vec{B}_i(t,\vec{y}_i) \qquad \varphi_i(t,\vec{y}_i)$$

$$\vec{F}_{ga}(t,\vec{y}_a) \qquad \vec{F}_{Ea}(t,\vec{y}_a) \qquad \vec{F}_{ma}(t,\vec{y}_a) \qquad \vec{F}_{ba}(t,\vec{y}_a)$$

$$\vec{g}_a(t,\vec{y}_a) \qquad \vec{E}_a(t,\vec{y}_a) \qquad \vec{B}_a(t,\vec{y}_a) \qquad \varphi_a(t,\vec{y}_a)$$

Usually we have:

$$\vec{F}_g = m \cdot \vec{g} \qquad \vec{F}_E = Q \cdot \vec{E} \qquad \vec{F}_m = Q \cdot (\dot{\vec{r}} \times \vec{B})$$

$$\vec{F}_b = |\vec{F}_b(|\dot{\vec{r}}|)| \cdot \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} = F_b(|\dot{\vec{r}}|) \cdot \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$$

The retarding force function F_b is often written in the following form:

$$F_b(|\dot{\vec{r}}|) = -C \cdot |\dot{\vec{r}}|^2 \qquad \text{or} \qquad F_b = -K \cdot |\dot{\vec{r}}|$$

C and K are certain constants see Budo [1] §16 p.83.

$\vec{g}, \vec{E}, \vec{B}, \varphi$ = gravitational acceleration/ electric field intensity/magnetic flux density/ density of the medium

$$\vec{F}_g = \vec{F}_{ga} + \sum_{i=1}^{n} \vec{F}_{gi}$$

$$\vec{F}_E = \vec{F}_{Ea} + \sum_{i=1}^{n} \vec{F}_{Ei}$$

$$\vec{F}_m = \vec{F}_{ma} + \sum_{i=1}^{n} \vec{F}_{mi}$$

$$\vec{F}_b = \vec{F}_{ba} + \sum_{i=1}^{n} \vec{F}_{bi}$$

$$\varphi = \varphi_a + \sum_{i=1}^{n} \varphi_i$$

We introduce the quantity $a = 1 - \frac{\varphi}{\varphi_K}$. Then the total force on the sample without the force from the sample itself is according to Budo [1] §16 p.83-85:

$$\vec{F}_{ges} = a \cdot (\vec{F}_g + \vec{F}_E + \vec{F}_m) + \vec{F}_b$$

If we finally combine $\vec{F}_{ges}$ with the force $\vec{F}_p$ from the sample itself, then we get the differential equation of the sample's motion:

$$\frac{d}{dt}(m(t) \cdot \dot{\vec{r}}) = \vec{F}_{ges}(t) + a(t) \cdot \vec{F}_p(t) \tag{1}$$

with initial values:

$$\vec{r}(t_0) = \vec{r}_0 \qquad \text{or} \qquad \dot{\vec{r}}(t_0) = \vec{v}_0 \qquad \text{or} \qquad \ddot{\vec{r}}(t_0) = \vec{b}_0$$

velocity $= \dot{\vec{r}}$ acceleration $= \ddot{\vec{r}}$

We can use this equation, if $|\dot{\vec{r}}|$ is smaller than $\frac{c}{10}$. c is the velocity of light in vacuum. The relativistic force can be used, too. Perhaps we can get more accurate results with this force than with the classical force. The relativistic force is with Sandhas [2] chapter 27 p.67:

$$\vec{F} = \frac{1}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} \cdot \frac{d}{dt}\left(\frac{m \cdot \vec{v}}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}\right)$$

Thus we get:

$$\frac{1}{\sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}}} \cdot \frac{d}{dt}\left(\frac{m(t) \cdot \dot{\vec{r}}}{\sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}}}\right) = \vec{F}_{ges}(t) + a(t) \cdot \vec{F}_p(t) \tag{2}$$

with initial values:

$$\vec{r}(t_0) = \vec{r}_0 \qquad \text{or} \qquad \dot{\vec{r}}(t_0) = \vec{v}_0 \qquad \text{or} \qquad \ddot{\vec{r}}(t_0) = \vec{b}_0$$

At velocities that are larger than $\frac{c}{10}$ we need the general theory of relativity because of the accelerated frames. We do not consider this case here.

The retarding force $\vec{F}_b$ in medium can have a very complex form at supersonic velocity and still larger velocities. Then we must take many effects (for example supersonic bang) of the medium into consideration.

In the case $\vec{F}_b = 0$ and $a = 1$ the motion is in the vacuum. $\vec{F}_E = 0$ and $\vec{F}_m = 0$ lead to the motion in gravitational fields. In the case $\vec{F}_p = 0$ the sample is without own drive.

With $\vec{F}_{ges} + a \cdot \vec{F}_p = 0$ follows the most general force equilibrium that is possible here. The area of $\vec{F}_g = 0$ is called Roche limit or Roche area in the astronomy. Analogously we set $\vec{F}_E$ and $\vec{F}_m$ to zero. Then we get analogous areas for the electric and for the magnetic forces.

References

[1] A Budo "Theoretische Mechanik" 10.edition VEB Deutscher Verlag der Wissenschaften Berlin 1980

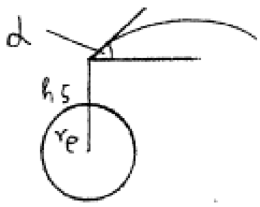
[2] Werner Sandhas "Theoretische Elektrodynamik - Kurzfassung" Universität Bonn Physikalisches Institut Sommersemester 1982 ISSN-0172-8733

66. Trajectories in the gravitational field of a planet

We consider the trajectory of a thrown body whose initial velocity is large enough to reach the outer space. Here the homogeneous field at the surface of a planet (earth) cannot be used any more, we need to use the gravitational field instead.

We explain the following quantities:

h_s = initial height
α = angle of throw
v_0 = initial velocity
h = height of throw at the time t
r_e = planet's radius (earth's radius)
m_e = planet's mass (earth's mass)
G = gravitational constant
$R := r_e + h_s$
$r := r_e + h$
r, R are distances from the planet's midpoint.



A body with the mass m that moves from R to r has to do the work:

$$W_{pot} = Gm_e m \cdot \left(\frac{1}{r_e + h_s} - \frac{1}{r_e + h} \right)$$

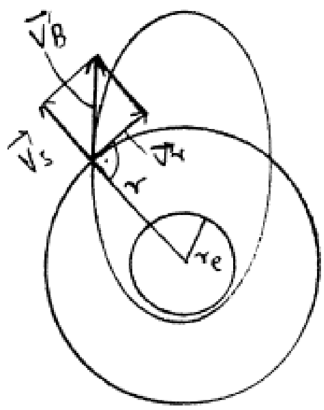
If r is smaller than R, then this work adds to its kinetic energy. The energy theorem of the trajectory of projection can be written as follows:

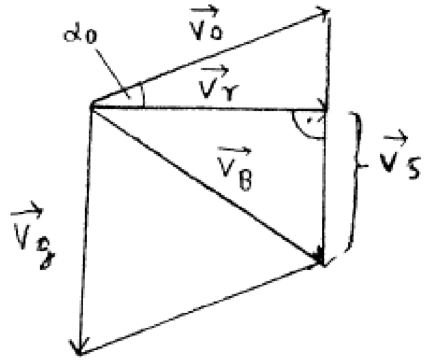
$$\frac{mv_0^2}{2} = W_{pot} + \frac{mv_B^2}{2}$$

v_B is the orbital velocity with the height h. We obtain by substituting:

$$v_B^2 = v_0^2 - 2Gm_e \cdot \left(\frac{1}{r_e + h_s} - \frac{1}{r_e + h} \right) \tag{1}$$

Now we look at the following two figures:





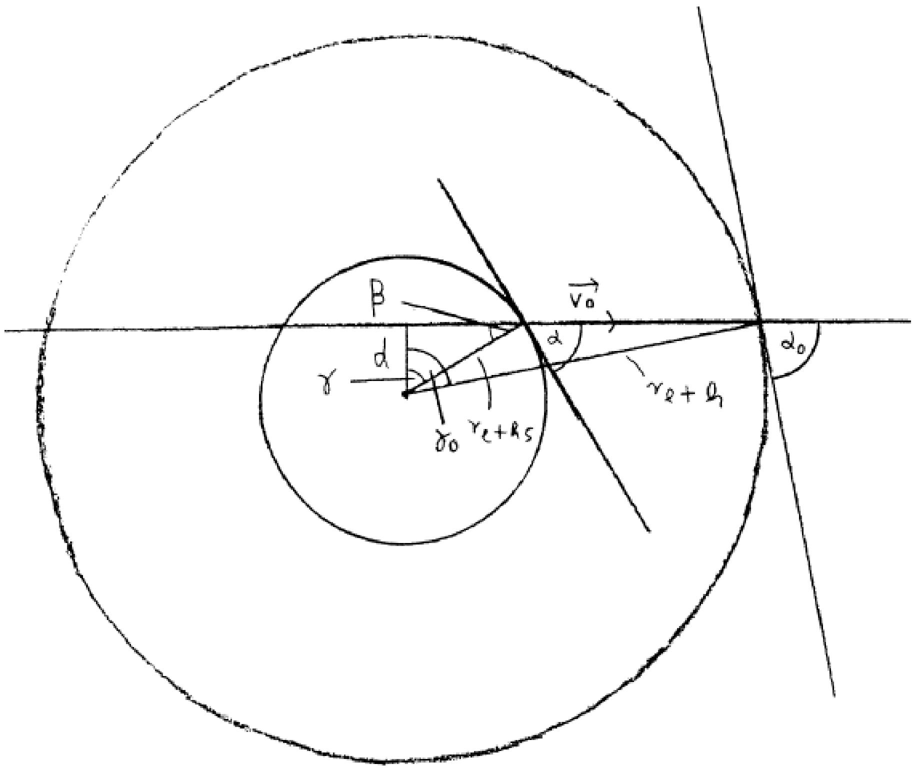
We can infer:

$$\vec{v}_B = \vec{v}_0 + \vec{v}_g = \vec{v}_r + \vec{v}_s$$

The orbital velocity is the vector sum of the initial velocity $\vec{v}_0$ and the gravitational component $\vec{v}_g$. Besides $\vec{v}_B$ can be decomposed in a perpendicular velocity $\vec{v}_s$ and a tangential velocity $\vec{v}_r$. With the second figure it is $\angle(\vec{v}_r, \vec{v}_0) = \alpha_0$ and $v_r = v_0 \cdot \cos\alpha_0$. It follows:

$$v_B^2 = \cos^2\alpha_0 \cdot v_0^2 + v_s^2 \tag{2}$$

Now we replace α_0 with an expression that depends from the angle of throw α.



The tangent is in the direction of $\vec{v}_r$. $\vec{v}_0$ indicates the direction of the perpendicular to d. It is valid $\gamma + \beta = 90°$ and $\beta + 90° + \alpha = 180°$. Then we obtain:

$$\gamma = 90° - (180° - 90° - \alpha) = \alpha$$

Analogously we get $\gamma_0 = \alpha_0$. With the figure we deduce:

$$\frac{d}{r_e + h_s} = \cos\gamma = \cos\alpha \qquad \Rightarrow \qquad d = \cos\alpha \cdot (r_e + h_s)$$

and:

$$\frac{d}{r_e + h} = \cos\gamma_0 = \cos\alpha_0$$

Now we substitute for d:

$$\cos\alpha_0 = \frac{\cos\alpha \cdot (r_e + h_s)}{r_e + h} \tag{3}$$

We solve equation (2) for v_s and we insert expression (3) for $\cos\alpha_0$ to reach:

$$v_s = \sqrt{v_B^2 - v_0^2 \cos^2\alpha \cdot \left(\frac{r_e + h_s}{r_e + h}\right)^2}$$

If we want to solve for t, then we must calculate a solution of the differential equation $v_s = \frac{dh}{dt}$. With the separation of variables (see for example Forster [3] §11 theorem 1 p.112) we can transform the equation to:

$$t = \int \frac{dh}{v_s(h)} - c \qquad ,$$

where c is an integration constant. If we substitute the expression for v_s in this equation and then the term (1) for v_B, we get finally:

$$c + t = \int \frac{dh}{\sqrt{v_0^2 - 2Gm_e \cdot \left(\frac{1}{r_e+h_s} - \frac{1}{r_e+h}\right) - \left(v_0 \cos\alpha \cdot \frac{r_e+h_s}{r_e+h}\right)^2}} \tag{4}$$

Before we integrate the equation, we will determine the throwing range. We have got:

$$v_r = v_0 \cdot \cos\alpha_0 = v_0 \cos\alpha \cdot \frac{r_e + h_s}{r_e + h}$$

Now it is possible to calculate the angular velocity w from the planet's midpoint.

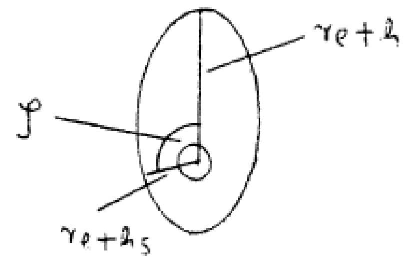
$$w = \frac{v_r}{r_e + h}$$

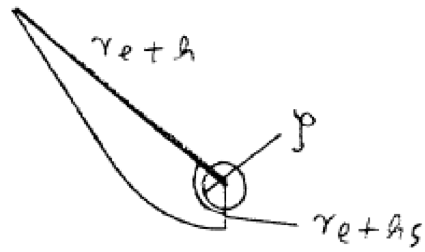
With

$$\varphi = \int_0^t w \, dt$$

we obtain the angle in circular measure as seen from the planet's midpoint. $\gamma = \frac{180°}{\pi} \cdot \varphi$ is this angle in degrees (see figure):

ELLIPSE HYPERBOLA





$r_e + h$ and φ are the polar coordinates of the trajectory. If we insert the expression for w together with the term for v_r in the integral for φ, then we obtain:

$$\varphi = v_0 \cos\alpha \cdot (r_e + h_s) \cdot \int_0^t \frac{dt}{(r_e + h(t))^2} \tag{5}$$

There is a relation between $h(t) = 0$ and the throwing time t_w. The presumption is that one h_{max} exists. This is not always the case. If t_w exists, then we obtain the throwing range w on the planet with:

$$w = v_0 \cos\alpha \cdot (r_e + h_s)^2 \cdot \int_0^{t_w} \frac{dt}{(r_e + h(t))^2}$$

At $h_s = 0$ we get the throwing range on the surface.

Now we calculate the integral (4) for t. This integral can be written as:

$$t - c_3 = \int \frac{(r_e + h) \cdot \sqrt{r_e + h_s}\, dh}{\sqrt{v_0^2 \cdot (r_e + h)^2 \cdot (r_e + h_s) - 2Gm_e \cdot (h - h_s) \cdot (r_e + h) - v_0^2 \cos^2\alpha \cdot (r_e + h_s)^3}}$$

with an integration constant c_3.

We define:

$$\begin{aligned} a &:= v_0^2 \cdot (r_e + h_s) - 2Gm_e \\ 2b &:= 2v_0^2 r_e \cdot (r_e + h_s) - 2Gm_e \cdot (r_e - h_s) \\ c &:= v_0^2 r_e^2 \cdot (r_e + h_s) + 2Gm_e h_s r_e - v_0^2 \cos^2\alpha \cdot (r_e + h_s)^3 \end{aligned}$$

it follows:

$$\begin{aligned} t - c_3 &= \int \frac{(r_e + h) \cdot \sqrt{r_e + h_s}}{\sqrt{ah^2 + 2bh + c}}\, dh \qquad (6) \\ &= \int \frac{r_e \cdot \sqrt{r_e + h_s}\, dh}{\sqrt{ah^2 + 2bh + c}} + \int \frac{h \cdot \sqrt{r_e + h_s}\, dh}{\sqrt{ah^2 + 2bh + c}} \end{aligned}$$

At integration it is important to distinguish, if $a > 0$,$a = 0$ or $a < 0$. Now we take a closer look at a.

$$a := v_0^2 \cdot (r_e + h_s) - 2Gm_e > 0$$

It follows:

$$v_0 > \sqrt{\frac{2Gm_e}{r_e + h_s}} \qquad \text{(hyperbola)}$$

$$a = 0 \qquad \Rightarrow \qquad v_0 = \sqrt{\frac{2Gm_e}{r_e + h_s}} \qquad \text{(parabola)}$$

$$a < 0 \qquad \Rightarrow \qquad v_0 < \sqrt{\frac{2Gm_e}{r_e + h_s}} \qquad \text{(ellipse)}$$

See Budo [2] §21.2. p.106 equation (17).

$$\sqrt{\frac{2Gm_e}{r_e + h_s}}$$

is the escape velocity of the planet with mass m_e. This velocity is also called the velocity of recession. At hyperbola and parabola $h(t)$ is unique. In the case of ellipse h_{max} exists, to a determined h can be attached several times.

1. The hyperbola

We would use for the calculation of equation (6) an integral from Ryshik [6] edition 1 p.115 Nr.2.261 and p.117 Nr.2.264. But we prefer integrals from Gröbner [4] chapter 231 p.37 Nr.8a and 7b. We take the integral decomposition of equation (6). The integral 8a yields:

$$\int \frac{r_e \cdot \sqrt{r_e + h_s}\, dh}{\sqrt{ah^2 + 2bh + c}} = \frac{r_e \cdot \sqrt{r_e + h_s}}{\sqrt{a}} \cdot \ln\left(c_1 \cdot \left(\frac{ah + b}{\sqrt{a}} + \sqrt{ah^2 + 2bh + c}\right)\right)$$

$a > 0$ is true. $\qquad c_1 =$ integration constant.

Now to the integral 7b:

$$\int \frac{h \cdot \sqrt{r_e + h_s}\, dh}{\sqrt{ah^2 + 2bh + c}} = \frac{\sqrt{r_e + h_s}}{a} \cdot \sqrt{ah^2 + 2bh + c} - \frac{b}{a} \cdot \int \frac{\sqrt{r_e + h_s}\, dh}{\sqrt{ah^2 + 2bh + c}}$$

The sum of both integrals yields:

$$t = \frac{\sqrt{r_e + h_s}}{a} \cdot \sqrt{ah^2 + 2bh + c} + \left(r_e - \frac{b}{a}\right) \cdot \int \frac{\sqrt{r_e + h_s}\, dh}{\sqrt{ah^2 + 2bh + c}} + c_3$$

$$= \frac{\sqrt{r_e + h_s}}{a} \cdot \sqrt{ah^2 + 2bh + c} + c_2 + \left(r_e - \frac{b}{a}\right) \cdot \frac{\sqrt{r_e + h_s}}{\sqrt{a}} \cdot \ln\left(c_1 \cdot \left(\frac{ah + b}{\sqrt{a}} + \sqrt{ah^2 + 2bh + c}\right)\right)$$

$c_1, c_2, c_3 =$ integration constants
The choice of the integration constants c_1, c_2 can be done through $t(h_s) = 0$.

Now we turn again to equation (5) of φ for the cases of hyperbola, parabola and ellipse:

$$\varphi = v_0 \cos\alpha \cdot (r_e + h_s) \cdot \int_0^t \frac{dt}{(r_e + h(t))^2}$$

Now we use the substitution formula:

$$h = f(t) \qquad f(0) = h(0) = h_s \qquad f(t) = h(t) = h$$

With the inverse mapping theorem:

$$\frac{dh}{dt} = \frac{1}{\frac{d}{dh}(f^{-1}(h))} = \frac{1}{\frac{d}{dh}t(h)}$$

$$\frac{d}{dh}t(h) =: T(h) \qquad \Rightarrow \qquad \frac{dh}{dt} = \frac{1}{T(h)}$$

The integral transforms into:

$$\varphi = v_0 \cos\alpha \cdot (r_e + h_s) \cdot \int\limits_{h_s}^{h} \frac{T(h)\,dh}{(r_e + h)^2} \tag{7}$$

This equation is valid for hyperbola, parabola and ellipse. With equation (6) we have:

$$T(h) = \frac{(r_e + h) \cdot \sqrt{r_e + h_s}}{\sqrt{ah^2 + 2bh + c}}$$

For all conic sections follow:

$$\varphi = v_0 \cos\alpha \cdot (r_e + h_s)^{\frac{3}{2}} \cdot \int\limits_{h_s}^{h} \frac{dh}{(r_e + h) \cdot \sqrt{ah^2 + 2bh + c}}$$

$a \neq 0$ holds in the cases of hyperbola and ellipse. Then the integration can be done with Gröbner [4] chapter 231 p.38 Nr. 10a -10d. At the parabola $a = 0$ holds. Here the integration is possible with Gröbner [4] chapter 212 p.28 Nr.9a -9c.

2. The parabola

In case of the parabola the integral (6) for t simplifies because of $a = 0$:
c_3 = integration constant

$$t - c_3 = \int \frac{r_e \cdot \sqrt{r_e + h_s}}{\sqrt{2bh + c}}\,dh + \int \frac{h \cdot \sqrt{r_e + h_s}}{\sqrt{2bh + c}}\,dh$$

with Bronstein [1] chapter 1.3.3.3. p.42 Nr. 124,125:

$$= \sqrt{r_e + h_s} \cdot \frac{2 \cdot \sqrt{2bh + c}}{2b} \cdot r_e + \frac{2 \cdot (2bh - 2c)}{3 \cdot 4b^2} \cdot \sqrt{2bh + c} \cdot \sqrt{r_e + h_s}$$

$$= \frac{\sqrt{r_e + h_s} \cdot \sqrt{2bh + c}}{b} \cdot \left(r_e + \frac{bh - c}{3b} \right)$$

c_3 is determined through $t(h_s) = 0$.

3. The ellipse

In this case $a < 0$ holds. We integrate the equation (6) using Gröbner [4] chapter 231 p.37 Nr. 7b and 8b. $b^2 - ac > 0$ must hold.

$$\frac{t}{\sqrt{r_e + h_s}} = \int \frac{r_e \, dh}{\sqrt{ah^2 + 2bh + c}} + \int \frac{h \, dh}{\sqrt{ah^2 + 2bh + c}} + c_3$$

with integral 7b:

$$= \left(r_e - \frac{b}{a} \right) \cdot \int \frac{dh}{\sqrt{ah^2 + 2bh + c}} + \frac{\sqrt{ah^2 + 2bh + c}}{a} + c_4$$

integral 8b:

$$= \left(r_e - \frac{b}{a} \right) \cdot \frac{-1}{\sqrt{-a}} \cdot \arcsin \left(\frac{ah + b}{\sqrt{b^2 - ac}} \right) + \frac{\sqrt{ah^2 + 2bh + c}}{a} + c_5$$

c_i = integration constants.

With that, $t(h)$ is known. c_5 can be chosen using $t(h_s) = 0$. With this we have solved the problem completly. In Budo [2] §21.3 p.107 ff. and §45.2 S.244 ff. there is another solution of the problem. In Kamke [5] Kap. C.9 p.627 Nr.9.26 the corresponding three dimensional vector differential equation is solved for every central force.

References

[1] I.N. Bronstein, K.A. Semendjajew "Taschenbuch der Mathematik" Teubner Verlagsgesellschaft Leipsic 1985 22.edition

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[3] Otto Forster "Analysis 2" 5.edition 1984 Vieweg Verlag Brunswick

[4] Wolfgang Gröbner und Nikolaus Hofreiter "Integraltafel Erster Teil" 5.edition 1975 Springer Verlag Vienna

[5] Erich Kamke "Differentialgleichungen: Lösungsmethoden und Lösungen" Band 1 10.edition Teubner Verlag Stuttgart 1983

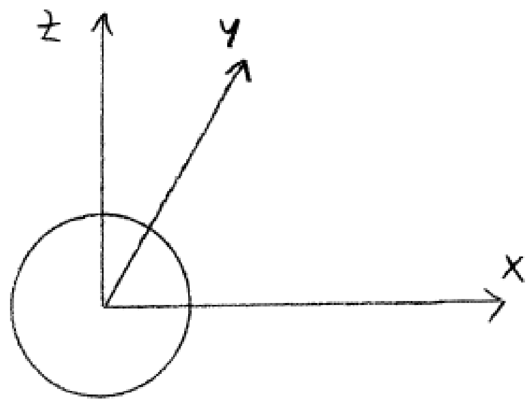
[6] I.Ryshik und I.Gradstein "Summen-, Produkt- und Integraltafeln - Tables of series, products and integrals" Band 1 Verlag Harri Deutsch Thun 1981

67. Rotating disk, orbits in hollow balls and swinging body in gravitational field — a trial

We introduce the following quantities:

$\vec{w}_p$ = angular velocity of the planet
U_t = rotation time of the planet
R = radius of the planet
m_1 = mass of the planet
α = latitude
G = gravitational constant
h_a = height above the surface of the planet

$\vec{w}_p$ shall be constant. First we assume that $|\vec{w}_p|$ is very small. Else we must take into consideration the Coriolis force of the planet. This will be done later. $|\cdot|$ is the absolute value of a vector. We look at a coordinate system of this planet:



The gravitational acceleration can be written as:

$$\vec{g}(\vec{X}) = -\frac{Gm_1}{|\vec{X}|^2} \cdot \frac{\vec{X}}{|\vec{X}|} \qquad \vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The centrifugal acceleration can be presented as:

$$\vec{b}_z(\vec{X}) = \sqrt{x^2+y^2} \cdot \vec{w}_p^2 \cdot \frac{1}{\sqrt{x^2+y^2}} \cdot \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \vec{w}_p^2 \cdot \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

It is $\vec{w}_p = (0, 0, |\vec{w}_p|)$ with $w_p = \frac{2\pi}{U_t}$.

Now we explain the spherical coordinates r, α_1, α of $\vec{X}$:

$r = |\vec{X}| = R + h_a$
$\alpha_1 =$ longitude
$\alpha =$ latitude $\qquad -90° \leq \alpha \leq 90°$

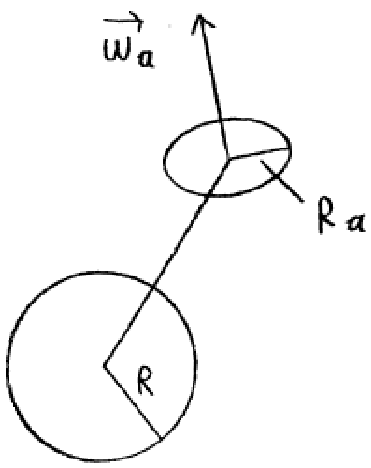
$90° - \alpha$ is the polar angle. The position is determined by R, h_a, α_1 and α. This leads to following spherical coordinates:

$$\vec{X} = r \cdot \begin{pmatrix} \sin(90° - \alpha)\cos\alpha_1 \\ \sin(90° - \alpha)\sin\alpha_1 \\ \cos(90° - \alpha) \end{pmatrix} = r \cdot \begin{pmatrix} \cos\alpha\cos\alpha_1 \\ \cos\alpha\sin\alpha_1 \\ \sin\alpha \end{pmatrix} \tag{1}$$

see Forster [3], §3, (3.6), p.33 und Bronstein [1], chapter 4.2.2.2, p.564

1. Rotating disk (round-about)

$R_a =$ radius of the disk
$\vec{w}_a =$ angular velocity of the disk



$\vec{w}_a$ shall be constant. Now we turn to the known spherical coordinates $|\vec{w}_a|$,α_2,α_3 of $\vec{w}_a$:

$\alpha_2 =$ longitude angle
$\alpha_3 =$ latitude angle $\qquad -90° \leq \alpha_3 \leq 90°$

Then we obtain with equation (1):

$$\vec{w}_a := |\vec{w}_a| \cdot \begin{pmatrix} \cos\alpha_3\cos\alpha_2 \\ \cos\alpha_3\sin\alpha_2 \\ \sin\alpha_3 \end{pmatrix} \tag{2}$$

Now we view the quantities:

r_a = distance of the disk's axis $\vec{w}_a$, This can be at the most equal to R_a.
α_3 = latitude angle (inclination angle) of the disk and of $\vec{w}_a$
α_2 = longitude angle of $\vec{w}_a$
α_4 = longitude angle of the disk

We consider α_3 in the interval $[0, 90°]$. It is valid for $0 \leq \alpha_4 \leq 360°$, see appendix:

$$\sin\alpha_5 = \sin\alpha_3 \cdot \sin\alpha_4$$

$$\tan\alpha_6 = \cos\alpha_3 \cdot \tan\alpha_4$$

α_5 is the latitude and α_6 the longitude of $\vec{r}_a$. Now we have with equation (1):

$$\vec{r}_a(\alpha_3, \alpha_4, r_a) = r_a \cdot \begin{pmatrix} \sin(90° - \alpha_5)\cos\alpha_6 \\ \sin(90° - \alpha_5)\sin\alpha_6 \\ \cos(90° - \alpha_5) \end{pmatrix} = r_a \cdot \begin{pmatrix} \cos\alpha_5\cos\alpha_6 \\ \cos\alpha_5\sin\alpha_6 \\ \sin\alpha_5 \end{pmatrix}$$

It is $-90° \leq \alpha_5 \leq 90°$.

$\vec{r}_a$ and $\vec{w}_a$ can be changed with a method in Schröer [4], chapter 7 into the surface coordinate system.

Now we calculate the centrifugal acceleration $\vec{z}_1$ on the disk in distance r_a:

$$\vec{z}_1 = |\vec{r}_a| \cdot \vec{w}_a^2 \cdot \frac{\vec{r}_a}{|\vec{r}_a|} = \vec{r}_a \cdot \vec{w}_a^2$$

The total acceleration $\vec{b}_{ges}$ at the place $\vec{X} + \vec{r}_a$ can be expressed as:

$$\vec{b}_{ges}(\vec{X} + \vec{r}_a) = \vec{z}_1 + \vec{g}(\vec{X} + \vec{r}_a) + \vec{b}_z(\vec{X} + \vec{r}_a)$$

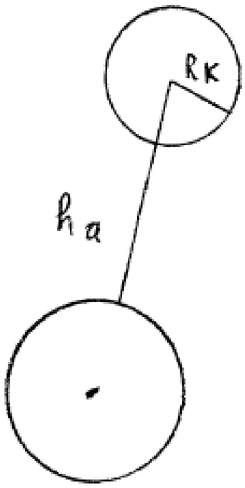
$\beta = \angle(\vec{b}_{ges}, \vec{g})$ is the deviation angle. Here we use the scalar product:

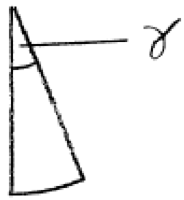
$$\cos\beta = \frac{\vec{b}_{ges}(\vec{X} + \vec{r}_a) \cdot \vec{g}(\vec{X} + \vec{r}_a)}{|\vec{b}_{ges}(\vec{X} + \vec{r}_a)| \cdot |\vec{g}(\vec{X} + \vec{r}_a)|}$$

2. Orbit in a hollow ball

We introduce following notations:

R_k = hollow ball's radius
γ = inclination angle of the small ball's orbit in the big hollow ball
$\vec{w}_a$ = angular velocity of the hollow ball, this quantity is constant.
Assumption: The radius of the small ball is very small compared to R_k. We can construct with equation (1) spherical coordinates:





$$\vec{w}_a = |\vec{w}_a| \cdot \begin{pmatrix} \sin(90° - \alpha_3)\cos\alpha_2 \\ \sin(90° - \alpha_3)\sin\alpha_2 \\ \cos(90° - \alpha_3) \end{pmatrix} = |\vec{w}_a| \cdot \begin{pmatrix} \cos\alpha_3\cos\alpha_2 \\ \cos\alpha_3\sin\alpha_2 \\ \sin\alpha_3 \end{pmatrix} \tag{3}$$

with:
$\alpha_3 =$ latitude angle of $\vec{w}_a$ $\quad -90° \leq \alpha_3 \leq 90°$
$\alpha_2 =$ longitude angle of $\vec{w}_a$

The angular velocity $\vec{w}_b$ of the small ball in the big hollow ball can also be decomposed in spherical coordinates with equation (1):

$$\vec{w}_b = |\vec{w}_b| \cdot \begin{pmatrix} \sin(90° - \gamma)\cos\alpha_4 \\ \sin(90° - \gamma)\sin\alpha_4 \\ \cos(90° - \gamma) \end{pmatrix} = |\vec{w}_b| \cdot \begin{pmatrix} \cos\gamma\cos\alpha_4 \\ \cos\gamma\sin\alpha_4 \\ \sin\gamma \end{pmatrix}$$

with:
$\gamma =$ inclination angle of the ball's orbit (of $\vec{w}_b$) $\quad -90° \leq \gamma \leq 90°$
$\alpha_4 =$ longitude angle of $\vec{w}_b$ (the ball's orbit)
γ and α_4 are valid relative the coordinate system of the planet's midpoint.

Now we work with accelerations:

$\vec{z}_1 =$ centrifugal acceleration of the rotating big hollow ball ($\vec{w}_a$).
$\vec{z}_c =$ Coriolis acceleration of the big hollow ball ($\vec{w}_a$)
$\vec{z}_2 =$ centrifugal acceleration of the small ball ($\vec{w}_b$)

There are no further accelerations because $\vec{w}_a$ is constant.

Spherical coordinates of $\vec{r}_a$ with equation (1):

$$\vec{r}_a = R_k \cdot \begin{pmatrix} \sin(90^\circ - \alpha_5)\cos\alpha_6 \\ \sin(90^\circ - \alpha_5)\sin\alpha_6 \\ \cos(90^\circ - \alpha_5) \end{pmatrix} = R_k \cdot \begin{pmatrix} \cos\alpha_5\cos\alpha_6 \\ \cos\alpha_5\sin\alpha_6 \\ \sin\alpha_5 \end{pmatrix}$$

with:

$R_k = |\vec{r}_a|$
$\alpha_5 =$ height angle of $\vec{r}_a$ $\qquad -90^\circ \leq \alpha_5 \leq 90^\circ$
$\alpha_6 =$ longitude angle of $\vec{r}_a$

$\vec{r}_a$ is the motion of the small ball in the big hollow ball. $\vec{w}_a$,$\vec{w}_b$ and $\vec{r}_a$ can be transformed with the method in Schröer [4], chapter 7 into the surface coordinate system.

With Budo [2], §14, equation (16), p.74 it is valid:

$$\vec{z}_1 = -\vec{w}_a \times (\vec{w}_a \times \vec{r}_a) \qquad \vec{z}_2 = -\vec{w}_b \times (\vec{w}_b \times \vec{r}_a)$$

$$\vec{z}_c = -2 \cdot (\vec{w}_a \times \vec{v})$$

$\vec{v} =$ velocity of the small ball in the big hollow ball

$$\vec{v} = \vec{w}_b \times \vec{r}_a \qquad \text{see Budo [2], §14, equation (8), p.72}$$

The total acceleration $\vec{b}_{ges}$ can be presented as, see Budo [2], §14, equation (16), p.74:

$$\vec{b}_{ges}(\vec{X} + \vec{r}_a) = \vec{g}(\vec{X} + \vec{r}_a) + \vec{b}_z(\vec{X} + \vec{r}_a) + \vec{z}_1 + \vec{z}_c + \vec{z}_2$$

We obtain the deviation angle $\beta = \angle(\vec{g}(\vec{X} + \vec{r}_a), \vec{b}_{ges}(\vec{X} + \vec{r}_a))$ with the scalar product:

$$\cos\beta = \frac{\vec{g}(\vec{X} + \vec{r}_a) \cdot \vec{b}_{ges}(\vec{X} + \vec{r}_a)}{|\vec{g}(\vec{X} + \vec{r}_a)| \cdot |\vec{b}_{ges}(\vec{X} + \vec{r}_a)|}$$

The special case $\vec{w}_a = 0$ is the inclined macro loop.

The angle $\beta_2 = \angle(\vec{r}_a, \vec{b}_{ges})$ plays a major role. With the scalar product we get:

$$\cos\beta_2 = \frac{\vec{r}_a \cdot \vec{b}_{ges}(\vec{X} + \vec{r}_a)}{|\vec{r}_a| \cdot |\vec{b}_{ges}(\vec{X} + \vec{r}_a)|}$$

1) $0 \leq \beta_2 \leq 90^\circ$, the small ball remains at the shell of the big hollow ball.

2) $90^\circ \leq \beta_2 \leq 180^\circ$, the small ball falls within the big hollow ball.

With

$$1) \qquad \Leftrightarrow \qquad 0 = \cos 90^\circ \leq \frac{\vec{r}_a \cdot \vec{b}_{ges}}{|\vec{r}_a| \cdot |\vec{b}_{ges}|} \leq \cos 0^\circ = 1$$

and

$$2) \qquad \Leftrightarrow \qquad 0 = \cos 90^\circ \geq \frac{\vec{r}_a \cdot \vec{b}_{ges}}{|\vec{r}_a| \cdot |\vec{b}_{ges}|} \geq \cos 180^\circ = -1$$

concrete conditions of remain or falling can be derived.

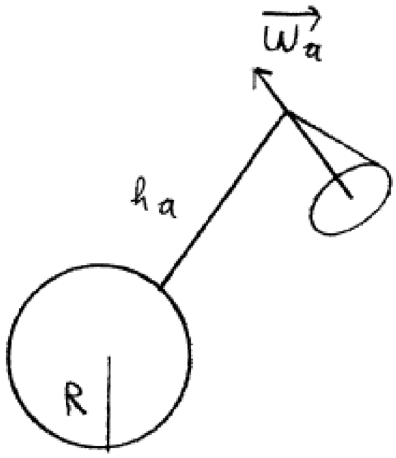
3. The swinging body

We decompose the angular velocity $\vec{w}_a$ of the body's circular motion (without gravitation) into spherical coordinates. This angular velocity shall be constant. With equation (1):

$$\vec{w}_a = |\vec{w}_a| \cdot \begin{pmatrix} \sin(90^\circ - \alpha_3)\cos\alpha_2 \\ \sin(90^\circ - \alpha_3)\sin\alpha_2 \\ \cos(90^\circ - \alpha_3) \end{pmatrix} = |\vec{w}_a| \cdot \begin{pmatrix} \cos\alpha_3\cos\alpha_2 \\ \cos\alpha_3\sin\alpha_2 \\ \sin\alpha_3 \end{pmatrix}$$

with the quantities:

α_3 = latitude angle of $\vec{w}_a$ $\quad -90^\circ \leq \alpha_3 \leq 90^\circ$
α_2 = longitude angle of $\vec{w}_a$
l = length of the rope that is fastened to the body. l is known. $\quad l = |\vec{l}_a|$
$\vec{l}_a$ = motion of the swinging body



We introduce spherical coordinates with equation (1):

$$\vec{l}_a = l \cdot \begin{pmatrix} \sin(90^\circ - \alpha_5)\cos\alpha_6 \\ \sin(90^\circ - \alpha_5)\sin\alpha_6 \\ \cos(90^\circ - \alpha_5) \end{pmatrix} = l \cdot \begin{pmatrix} \cos\alpha_5\cos\alpha_6 \\ \cos\alpha_5\sin\alpha_6 \\ \sin\alpha_5 \end{pmatrix} \tag{4}$$

with following quantities:

α_6 = longitude angle of $\vec{l}_a$
α_5 = latitude angle of $\vec{l}_a$
α_5 and α_6 are relative to the midpoint of the planet's coordinate system. $\vec{w}_a$ and $\vec{l}_a$ can be transformed into the surface coordinate system with the method in Schröer [4], Kapitel 7.

Now we turn to the centrifugal acceleration $\vec{z}_1$ of the swinging body:

$$\vec{z}_1 = -\vec{w}_a \times (\vec{w}_a \times \vec{l}_a) \qquad \text{see Budo [2], §14, equation (16), p.74}$$

We write the total acceleration with the same equation in Budo [2]:

$$\vec{b}_{ges}(\vec{X}+\vec{l}_a) = \vec{g}(\vec{X}+\vec{l}_a) + \vec{b}_z(\vec{X}+\vec{l}_a) + \vec{z}_1 \tag{5}$$

We calculate the deviation angle $\beta = \angle(\vec{b}_{ges}, \vec{g})$ with the scalar product:

$$\cos\beta = \frac{\vec{b}_{ges}(\vec{X}+\vec{l}_a)\cdot\vec{g}(\vec{X}+\vec{l}_a)}{|\vec{b}_{ges}(\vec{X}+\vec{l}_a)|\cdot|\vec{g}(\vec{X}+\vec{l}_a)|}$$

$\vec{b}_{ges}$ must have the same direction as $\vec{l}_a$:

$$\frac{\vec{b}_{ges}(\vec{X}+\vec{l}_a)}{|\vec{b}_{ges}(\vec{X}+\vec{l}_a)|} = \frac{\vec{l}_a}{|\vec{l}_a|} \tag{6}$$

With the equation (5) and (6) the motion $\vec{l}_a$ of the swinging body is described. l is known. α_6 and α_5 must be determined with (5) and (6).
The angle $\gamma = \angle(\vec{l}_a, \vec{w}_a)$ of the swinging body:

$$\cos\gamma = \frac{\vec{l}_a\cdot\vec{w}_a}{|\vec{l}_a|\cdot|\vec{w}_a|}$$

In special case $\frac{\vec{w}_a}{|\vec{w}_a|} = \frac{\vec{X}}{|\vec{X}|}$ is $\vec{w}_a$ perpendicular to planet's surface. The swinging body moves in a circle.

Now we also determine the Coriolis acceleration caused through rotation of the planet.

$$\vec{b}_c = -2\cdot(\vec{w}_p\times\vec{v}) \qquad \text{see Budo [2], §14, equation (16) p.74}$$

$$\vec{v} = \vec{w}_p\times(\vec{X}+\vec{r}_a) \qquad \text{in the cases 1) and 2)}$$

and

$$\vec{v} = \vec{w}_p\times(\vec{X}+\vec{l}_a) \qquad \text{in case 3)}$$

see Budo [2], §14, equation (8), p.72.

It follows:

$$\vec{b}_c(\vec{X}+\vec{r}_a) = -2\cdot[\vec{w}_p\times(\vec{w}_p\times(\vec{X}+\vec{r}_a))] \qquad \text{in cases 1) and 2)}$$

$$\vec{b}_c(\vec{X}+\vec{l}_a) = -2\cdot[\vec{w}_p\times(\vec{w}_p\times(\vec{X}+\vec{l}_a))] \qquad \text{in case 3)}$$

We calculate the total accelerations see Budo [2], §14, equation (16), p.74.

Then it is valid in case 1):

$$\vec{b}_{ges}(\vec{X}+\vec{r}_a) = \vec{z}_1 + \vec{b}_c + \vec{g}(\vec{X}+\vec{r}_a) + \vec{b}_z(\vec{X}+\vec{r}_a)$$

in case 2):

$$\vec{b}_{ges}(\vec{X}+\vec{r}_a) = \vec{g}(\vec{X}+\vec{r}_a) + \vec{b}_z(\vec{X}+\vec{r}_a) + \vec{z}_1 + \vec{z}_c + \vec{z}_2 + \vec{b}_c$$

in case 3) with equation (5):

$$\vec{b}_{ges}(\vec{X}+\vec{l}_a) = \vec{g}(\vec{X}+\vec{l}_a) + \vec{b}_z(\vec{X}+\vec{l}_a) + \vec{z}_1 + \vec{b}_c \tag{7}$$

All other equations don't change. $\vec{w}_p$ is constant. $|\vec{w}_p|$ must not be very small. If $|\vec{w}_p|$ is very small, the calculation can be done in all cases without $\vec{b}_z$ and $\vec{b}_c$.

We collect the cases once again:

case 1): rotating disk (round-about)
case 2): orbits in a hollow ball
case 3): swinging body

If the angular velocities are not temporal constant, then there are further accelerations. If we calculate the temporal motion, materials properties (at round-about and hollow ball friction, at swinging body elasticity properties of the rope) must be taken into consideration. This will not be done here.
The accelerations, caused through other planets, fixed stars, satellites and other masses (on earth besides sun and moon), can be presented through $\vec{b}_r(t)$. But this additional term is on earth strong temporal dependent.

On earth it is: $\odot$ = sun E= earth M=moon

$$\frac{Gm_\odot}{r_{E\odot}^2} + \frac{Gm_M}{r_{EM}^2} \approx 6\cdot 10^{-3}\frac{\mathrm{m}}{\mathrm{s}^2} + 4\cdot 10^{-5}\frac{\mathrm{m}}{\mathrm{s}^2}$$

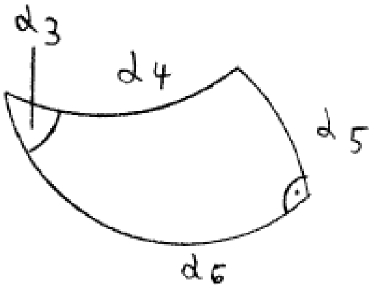
This is negligible with respect to $g = 9.81\frac{\mathrm{m}}{\mathrm{s}^2}$. This correction term is only necessary at accurate calculations. But $\vec{b}_r(t)$ can be not negligible if for example two planets are not far away from each other.

Appendix

Here the theme is the derivation of $\sin\alpha_5 = \sin\alpha_3 \cdot \sin\alpha_4$ and
$\tan\alpha_6 = \cos\alpha_3 \cdot \tan\alpha_4$ for $0° \leq \alpha_4 \leq 360°$.

$\alpha_5 =$ latitude of $\vec{r}_a$ $\qquad -90° \leq \alpha_5 \leq 90°$
$\alpha_6 =$ longitude of $\vec{r}_a$

First we consider the case $0° \leq \alpha_4 \leq 90°$. Then it is $\alpha_5 \geq 0°$.

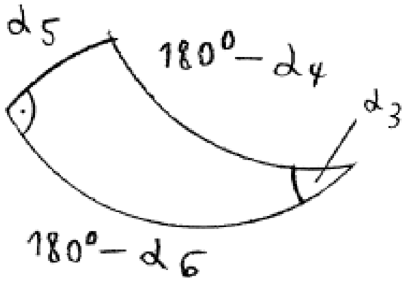


With Bronstein [1], chapter 2.6.4.3.2, p.209, equation (2.87) and (2.95):

$$\sin\alpha_5 = \sin\alpha_4 \cdot \sin\alpha_3$$

$$\cos\alpha_3 = \tan\alpha_6 \cdot \cot\alpha_4 = \frac{\tan\alpha_6}{\tan\alpha_4} \qquad \Rightarrow \qquad \tan\alpha_6 = \cos\alpha_3 \cdot \tan\alpha_4$$

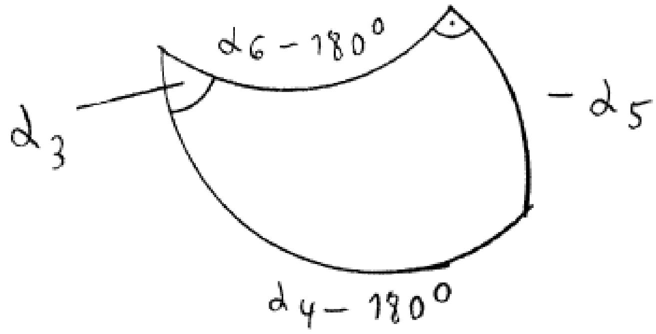
Now to the case $90° \leq \alpha_4 \leq 180°$, with $\alpha_5 \geq 0°$:



$\sin\alpha_5 = \sin(180° - \alpha_4) \cdot \sin\alpha_3 = \sin\alpha_4 \cdot \sin\alpha_3$

$$\cos\alpha_3 = \frac{\tan(180° - \alpha_6)}{\tan(180° - \alpha_4)} = \frac{-\tan\alpha_6}{-\tan\alpha_4} \qquad \Rightarrow \qquad \tan\alpha_6 = \cos\alpha_3 \cdot \tan\alpha_4$$

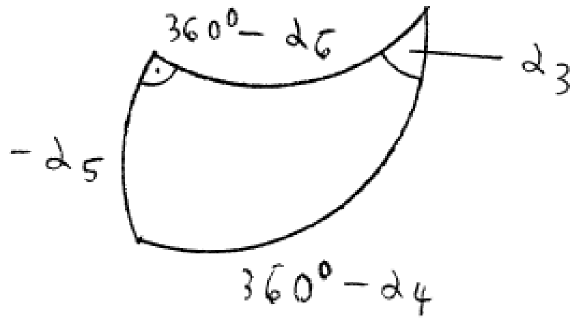
The case $180° \leq \alpha_4 \leq 270°$: $\qquad \alpha_5 \leq 0°$



$-\sin\alpha_5 = \sin(-\alpha_5) = \sin\alpha_3 \cdot \sin(\alpha_4 - 180°) = -\sin\alpha_3 \cdot \sin\alpha_4$
$\Rightarrow \qquad \sin\alpha_5 = \sin\alpha_3 \cdot \sin\alpha_4$

$$\cos\alpha_3 = \frac{\tan(\alpha_6 - 180°)}{\tan(\alpha_4 - 180°)} = \frac{\tan\alpha_6}{\tan\alpha_4} \qquad \Rightarrow \qquad \tan\alpha_6 = \cos\alpha_3 \cdot \tan\alpha_4$$

Finally to the case $270° \leq \alpha_4 \leq 360°$: $\qquad \alpha_5 \leq 0°$



$-\sin\alpha_5 = \sin(-\alpha_5) = \sin\alpha_3 \cdot \sin(360° - \alpha_4) = -\sin\alpha_3 \cdot \sin\alpha_4$
$\Rightarrow \qquad \sin\alpha_5 = \sin\alpha_3 \cdot \sin\alpha_4$

$$\cos\alpha_3 = \frac{\tan(360° - \alpha_6)}{\tan(360° - \alpha_4)} = \frac{-\tan\alpha_6}{-\tan\alpha_4} \qquad \Rightarrow \qquad \tan\alpha_6 = \cos\alpha_3 \cdot \tan\alpha_4$$

With that the assertion is proved.

References

[1] I.N. Bronstein, K.A. Semendjajew “Taschenbuch der Mathematik”, Teubner Verlagsgesellschaft, Leipsic, 1985, 22.edition

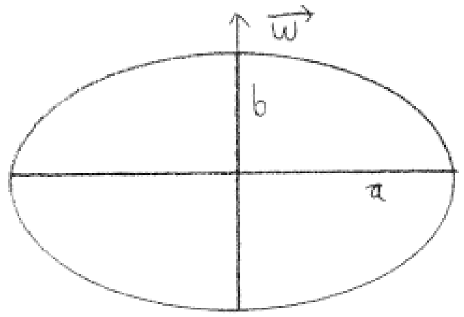
[2] A Budo “Theoretische Mechanik” 10.edition, VEB Deutscher Verlag der Wissenschaften, Berlin, 1980

[3] Otto Forster “Analysis 3”, 2.edition, 1983, Vieweg Verlag, Brunswick

[4] Harald Schröer “Theory of orientation - the visual vertical”, Verlag Wissenschaft und Technik, Berlin, 2002

68. The ellipsoid of revolution's form in consequence of the centrifugal force

We view the following ellipsoid of revolution:



We introduce:

a = major semiaxis
b = minor semiaxis
$\vec{w}$ = ellipsoid of revolution's angular velocity
T = ellipsoid of revolution's rotation time
t = time
We assume w as temporal constant. Then we have the relation $w = \frac{2\pi}{T}$.

Now we look at the following values:

centrifugal acceleration = $\vec{b}_z = aw^2$
gravitational acceleration = $\vec{g}$

The gravitational acceleration is known with the law of gravitation:

$$g = \frac{G \cdot M}{a^2}$$

G = constant of gravitation
M is the ellipsoid of revolution's mass with:

$$M(t) = \int\limits_{V(t)} \varphi(\vec{x}, t)\, d\vec{x}$$

with:

$\varphi(\vec{x}, t)$ = ellipsoid of revolution's density
(The density must be rotationally symmetric $\varphi(\vec{x}, t) = \varphi(|\vec{x}|, t)$)

Origin of the coordinate system = geometrical midpoint of the ellipsoid of revolution
$V(t)$ = ellipsoid of revolution's volumen
$|\cdot|$ = absolute value of a vector

Now we define the following value:

$$c := \frac{|\vec{b}_z|}{|\vec{g}|} = \frac{aw^2 \cdot a^2}{GM} = \frac{a^3 w^2}{GM} \tag{1}$$

This value plays a major role to the ellipsoid of revolution's form see Müller [1].

With Müller [1] chapter 3.4.2.4 p.99 the following relations:

A is the oblateness

$$A = \frac{a-b}{a} \tag{2}$$

$$A = f(c) \qquad A = kc \qquad \text{for} \qquad A, c \ll 1 \tag{3}$$

k= 0.5 (Roche model)
k= 1.25 (homogeneous model)

At $A, c \ll 1$ it follows:

$$1 - \frac{b}{a} = A = k \cdot \frac{w^2 a^3}{GM}$$

We find:

$$b = (1 - A) \cdot a = \left(1 - \frac{kw^2 a^3}{GM}\right) \cdot a$$

So we know the ellipsoid of revolution's form for $A, c \ll 1$.

The real value of k of the planets in our solar system are not totally equal to the ideal value of the Roche model respectively the homogeneous model. For the earth it is k =0.98, for the Mars $k = 1.11$, for Jupiter $k = 0.68$ and for Saturn $k = 0.58$ see Müller [1] chapter 3.4.2.4 p.100.

Because of this we must determine at the planets k with the oblateness A and c. The rotation times are known. Then we obtain k. We have the possibility to calculate the minor semiaxis b and the oblateness for other rotation times, if $A, c \ll 1$.

Problem: What happens if A, c aren't small?

Then we must use the general formula of gravitational acceleration.

$$\vec{g}(\vec{r}, t) = \int\limits_{V(t)} \frac{G\varphi(\vec{x}, t)}{|\vec{r} - \vec{x}|^3} \cdot (\vec{r} - \vec{x}) \, d\vec{x} \tag{4}$$

Then the equations (1) - (4) must be inserted into one another. Besides $A = f(c)$ must be known.

References

[1] B. Müller “Grundzüge der Astronomie” 1973 Teubner Verlag Leipsic

69. The apparent brightness of planets

Abstract: First the apparent brightness of planets are determined generally and then at inclined circular orbits.

Key words: Brightness - phase - phase angle - circular orbit - planet - magnitude

1. Foundations

Through a telescope, we can see the planets Mercury and Venus in different phases, similar to the moon. To determine the apparent brightness, the phase and the distance are necessary. Mercury and Venus do not have the maximum apparent brightness at minimum distance to earth. Then through a telescope we can see both planets as very thin crescents. Both planets achieve the maximum apparent brightness, if Venus and Mercury can be seen through a telescope as thick crescents. We want to inquire into the apparent brightness of a planet in dependence of its motion. We introduce the following quantities:

$\vec{r}_s(t)$ = position of a **self-luminous** body S (fixed star)

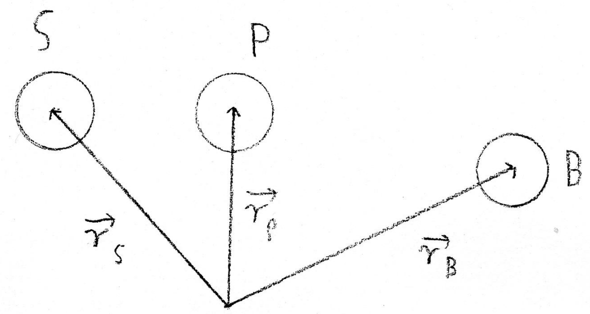
$\vec{r}_p(t)$ = position of a irradiated body P (planet), P is not self-luminous.

$\vec{r}_B(t)$ = position of another body B (planet) with an observer. B isn't self-luminous.

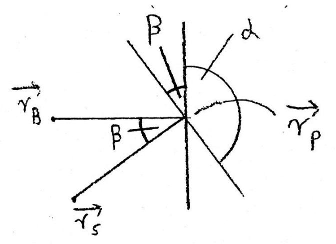
$\vec{r}_s, \vec{r}_B, \vec{r}_p \in R^3 \qquad t = \text{time}$

Presumption: P and B posess **no** atmospheres.

All 3 bodies are balls with a small radius compared to the distance of the 3 bodies.



The angle of reflection (phase angle):



α = phase angle of P at B

$$\alpha = 180^\circ - \beta \qquad \cos\alpha = \cos(180^\circ - \beta) = -\cos\beta$$

The angle β can be described with the scalar product:

$$\cos\beta = \frac{(\vec{r}_B - \vec{r}_p)\cdot(\vec{r}_s - \vec{r}_p)}{|\vec{r}_p - \vec{r}_B|\cdot|\vec{r}_s - \vec{r}_p|}$$

It follows:

$$\cos\alpha = \frac{(\vec{r}_p - \vec{r}_B)\cdot(\vec{r}_s - \vec{r}_p)}{|\vec{r}_p - \vec{r}_B|\cdot|\vec{r}_s - \vec{r}_p|} \tag{1}$$

Now we explain the following quantities:

R_s = radius of the ball S (fixed star)

R_p = radius of the ball P (planet)

R_B = radius of the ball B (planet with observer)

R_a = radius of the crystalline lens of the observer

The problem is to determine the brightness of the ball P seen from B.

$\Phi_{s,p}$ = luminous flux coming from S through P.

I = luminous intensity of S

$\Omega_{s,p}$ = solid angle of P seeing from S.

With Voigt [9] chapter V.1.1 p.177 we can write approximately the solid angle through:

$$\Omega_{s,p} = \frac{\pi R_p^2}{|\vec{r}_s - \vec{r}_p|^2} \qquad R_p \ll |\vec{r}_s - \vec{r}_p|$$

$$\Phi_{s,p} = I \cdot \Omega_{s,p}$$

see Kuchling [6] chapter 27.2.4. p.387.

The luminous flux Φ_α with consideration to the phase, see Montenbruck [7] chapter VI.3.2 p.112:

$$\Phi_\alpha = \frac{1+\cos(\pi-\alpha)}{2} \cdot \Phi_{s,p} = \frac{1-\cos\alpha}{2} \cdot \Phi_{s,p} \qquad \alpha \text{ in radian measure}$$

We need:

$a =$ albedo of the ball P

For the illumination E that gets the observer on the ball B we obtain with Kuchling [6] chapter 27.2.7. p.389:

$$E = \frac{\Phi_\alpha \cdot a}{|\vec{r}_p - \vec{r}_B|^2}$$

The luminous flux through the observer is, see Kuchling [6] (O 27.21) p.389:

$$\Phi_B = \pi \cdot R_a^2 \cdot E$$

Φ_B and E can be seen as measure of the brightness. If we insert then we obtain:

$$E = \frac{(1-\cos\alpha) \cdot \Phi_{s,p} \cdot a}{2 \cdot |\vec{r}_p - \vec{r}_B|^2} = \frac{\pi \cdot R_p^2 \cdot I \cdot a \cdot (1-\cos\alpha)}{2 \cdot |\vec{r}_s - \vec{r}_p|^2 \cdot |\vec{r}_p - \vec{r}_B|^2} \tag{2}$$

These are the formulas of the brightness with the presumption that the whole space is vacuum. With that there is no absorption. Now we take the absorption into consideration. Now the whole space is filled with a medium. m is the absorption coefficient. Then the following formulas change:

$$\Phi_{s,p} = \Omega_{s,p} \cdot I \cdot e(\vec{r}_s, \vec{r}_p, t)$$

and

$$E = \frac{\Phi_\alpha \cdot a}{|\vec{r}_p - \vec{r}_B|^2} \cdot e(\vec{r}_p, \vec{r}_B, t)$$

$e(\ldots)$ are attenuation factors. In case of constant absorption coefficient m we have:

$$e(\vec{r}_s, \vec{r}_p, t) = e^{-m \cdot |\vec{r}_s - \vec{r}_p|}$$

$$e(\vec{r}_p, \vec{r}_B, t) = e^{-m \cdot |\vec{r}_p - \vec{r}_B|}$$

In the case of different absorption coefficient:

$$e(\vec{x}, \vec{y}, t) = e^{-F(\vec{x}, \vec{y}, t)}$$

with

$$F(\vec{x}, \vec{y}, t) := \int_0^1 m(\vec{s}(\vec{x}, \vec{y}, \tau), t) \cdot |\vec{x} - \vec{y}| \, d\tau$$

with

$$\vec{s}(\vec{x}, \vec{y}, \tau) = \tau\vec{y} + (1 - \tau) \cdot \vec{x} \qquad \tau \in [0, 1]$$

and

$$\vec{s}(\tau = 0) = \vec{x} \qquad \vec{s}(\tau = 1) = \vec{y}$$

$m(\vec{x}, t)$ is an absorption function with a dependence upon $\vec{x} \in R^3$ and t. The formula of non uniform absorption is valid only for one extreme thin medium for example the outer space. This formula is valid approximate. The lightrays are refracted in the non homogeneous medium. It is difficult to take the change of the direction into consideration. In the case of very thin media as the extreme thin matter density of the outer space there is no appreciable change of the direction.
The formula with non uniform absorption is a line integral of the first type, see Bronstein [3] chapter 3.1.8.2 p.319.

If we insert them just as in the vacuum case, then we get:

$$E = \frac{\pi R_p^2 Ia \cdot (1 - \cos\alpha) \cdot e(\vec{r}_s, \vec{r}_p, t) \cdot e(\vec{r}_p, \vec{r}_B, t)}{2 \cdot |\vec{r}_s - \vec{r}_p|^2 \cdot |\vec{r}_p - \vec{r}_B|^2}$$

or

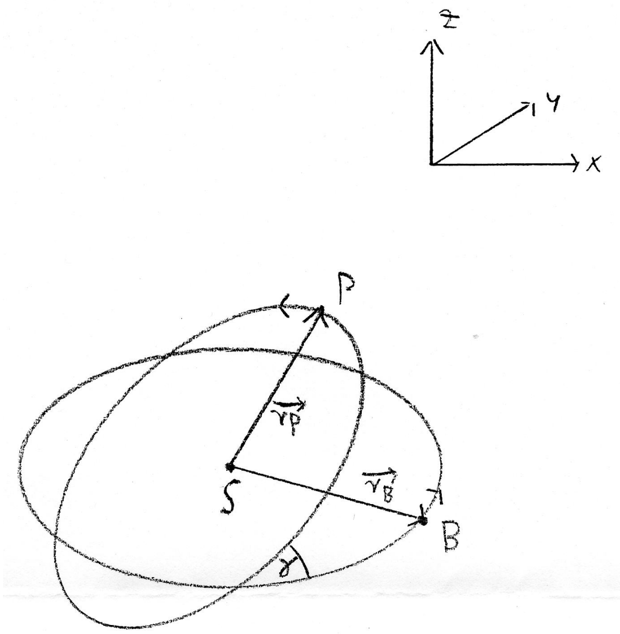
$$\Phi_B = \frac{\pi^2 R_p^2 R_a^2 \cdot Ia \cdot (1 - \cos\alpha) \cdot e(\vec{r}_s, \vec{r}_p, t) \cdot e(\vec{r}_p, \vec{r}_B, t)}{2 \cdot |\vec{r}_s - \vec{r}_p|^2 \cdot |\vec{r}_p - \vec{r}_B|^2}$$

In the case of absorption, the equations are valid only to light in the **visual** wavelength range. Already in the neighbour ranges of ultraviolet and infrared there are additional much emission and much scattering.

Plane polar coordinates can be used for $\vec{r}_s, \vec{r}_p, \vec{r}_B$ to the motion in circular orbits on the plane. This can be found at Schröer [8]. We need spherical coordinates for inclined circular orbits.

2. The apparent brightness of planets on inclined orbits

We look at an example that is similar to those of sun, earth and Venus. The sun is represented as fixed star S, the earth as observer planet B and Venus as irradiated planet P. We assume circular orbits. We neglect the absorption.



The orbit radii r_p and r_B, the start angles δ_p and δ_B and the masses m_P and m_B of the planets P (Venus) and B (earth) are known. The fixed star S (sun) is in the origin ($\vec{r}_s = \vec{0}$). m_s shall be the mass of the fixed star (sun). Then it is easy to determine the angular velocity of both planets. The angular velocities follow from the equality of gravitation force and centripetal force at circular motion:

$$\frac{G \cdot m_s m_p}{r_p^2} = m_p \cdot r_p w_p^2 \qquad \text{it follows:} \qquad w_p = \sqrt{\frac{G \cdot m_s}{r_p^3}}$$

$$\frac{G \cdot m_s m_B}{r_B^2} = m_B \cdot r_B w_B^2 \qquad \text{it follows:} \qquad w_B = \sqrt{\frac{G \cdot m_s}{r_B^3}}$$

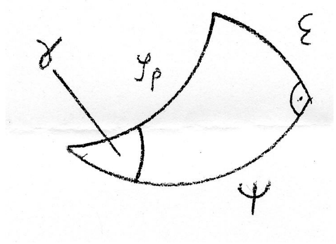
G is the gravitational constant. The circular orbit of the irradiated planet P shall be inclined relatively to the circular orbit of the observer's planet B. This inclination angle is denoted with γ and shall be constant. This model will describe the motion of earth and Venus.

The planet B with observer circles in a plane with the orbit vector:

$$\vec{r}_B(t) = r_B \cdot \begin{pmatrix} \cos(\delta_B + w_B \cdot t) \\ \sin(\delta_B + w_B \cdot t) \\ 0 \end{pmatrix} = r_B \cdot \begin{pmatrix} \cos\varphi_B \\ \sin\varphi_B \\ 0 \end{pmatrix}$$

Further it is valid $\varphi_p = \delta_p + w_p \cdot t$.

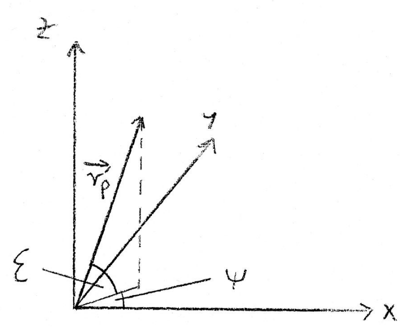
We view the following rectangular spherical triangle:



Then the following equations are valid:

$$\sin\varepsilon = \sin\gamma \cdot \sin\varphi_p \tag{3}$$
$$\tan\psi = \cos\gamma \cdot \tan\varphi_p \tag{4}$$

The derivation of both equations is in the appendix. Now we introduce spherical coordinates, see the following figure:



The vector $\vec{r}_p$ can be written as, see Bartsch [2], chapter 7.2.1, p.264,265:

$$\vec{r}_p(t) = r_p \cdot \begin{pmatrix} \cos\varepsilon \cdot \cos\psi \\ \cos\varepsilon \cdot \sin\psi \\ \sin\varepsilon \end{pmatrix}$$

The absolute value r_p of $\vec{r}_p$ is given.

From the first chapter we know the formulas:

$$\cos\alpha = \frac{(\vec{r}_p - \vec{r}_B) \cdot (\vec{r}_s - \vec{r}_p)}{|\vec{r}_p - \vec{r}_B| \cdot |\vec{r}_s - \vec{r}_p|}$$

$$E = \frac{\pi \cdot R_p^2 \cdot I \cdot a \cdot (1 - \cos\alpha)}{2 \cdot |\vec{r}_s - \vec{r}_p|^2 \cdot |\vec{r}_p - \vec{r}_B|^2}$$

$$\Phi_B = \pi \cdot R_a^2 \cdot E$$

Here it is $\vec{r}_s = \vec{0}$.

$$\cos\alpha = \frac{(\vec{r}_B - \vec{r}_p) \cdot \vec{r}_p}{|\vec{r}_B - \vec{r}_p| \cdot r_p}$$

$$E = \frac{\pi \cdot R_p^2 \cdot I \cdot a \cdot (1 - \cos\alpha)}{2r_p^2 \cdot |\vec{r}_B - \vec{r}_p|^2}$$

Now we must determine the numerator and the denominator of $\cos\alpha$.

We calculate the numerator of $\cos\alpha$:

$$(\vec{r}_B - \vec{r}_p) \cdot \vec{r}_p = \vec{r}_B \cdot \vec{r}_p - \vec{r}_p^{\,2}$$

Because of the scalar product property $\vec{r}_p^{\,2} = |\vec{r}_p|^2$:

$$= r_B r_p \cdot \begin{pmatrix} \cos\varphi_B \\ \sin\varphi_B \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\varepsilon\cos\psi \\ \cos\varepsilon\sin\psi \\ \sin\varepsilon \end{pmatrix} - r_p^2$$

$$= r_B r_p \cdot \begin{pmatrix} \cos\varphi_B \\ \sin\varphi_B \end{pmatrix} \cdot \begin{pmatrix} \cos\varepsilon\cos\psi \\ \cos\varepsilon\sin\psi \end{pmatrix} - r_p^2$$

$$= r_p \cdot \left[r_B \cdot \begin{pmatrix} \cos\varphi_B \\ \sin\varphi_B \end{pmatrix} \cdot \begin{pmatrix} \cos\varepsilon\cos\psi \\ \cos\varepsilon\sin\psi \end{pmatrix} - r_p \right]$$

Now we construct the denominator of $\cos\alpha$:

$$|\vec{r}_B - \vec{r}_p| \cdot r_p = \sqrt{\vec{r}_B^{\,2} - 2 \cdot \vec{r}_B \cdot \vec{r}_p + \vec{r}_p^{\,2}} \cdot r_p$$

We use the scalar product property $\vec{r}^{\,2} = |\vec{r}|^2$ for $\vec{r} \in R^3$:

$$= r_p \cdot \left[r_B^2 + r_p^2 - 2r_B r_p \cdot \begin{pmatrix} \cos\varphi_B \\ \sin\varphi_B \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\varepsilon\cos\psi \\ \cos\varepsilon\sin\psi \\ \sin\varepsilon \end{pmatrix} \right]^{\frac{1}{2}}$$

$$= r_p \cdot \left[r_B^2 + r_p^2 - 2r_B r_p \cdot \begin{pmatrix} \cos\varphi_B \\ \sin\varphi_B \end{pmatrix} \cdot \begin{pmatrix} \cos\varepsilon\cos\psi \\ \cos\varepsilon\sin\psi \end{pmatrix} \right]^{\frac{1}{2}}$$

At last we get:

$$\cos\alpha = \frac{\left(r_B \cdot \begin{pmatrix} \cos\varphi_B \\ \sin\varphi_B \end{pmatrix} \cdot \begin{pmatrix} \cos\varepsilon\cos\psi \\ \cos\varepsilon\sin\psi \end{pmatrix} - r_p \right) \cdot r_p}{r_p \cdot \left[r_B^2 + r_p^2 - 2r_B r_p \cdot \begin{pmatrix} \cos\varphi_B \\ \sin\varphi_B \end{pmatrix} \cdot \begin{pmatrix} \cos\varepsilon\cos\psi \\ \cos\varepsilon\sin\psi \end{pmatrix} \right]^{\frac{1}{2}}}$$

In the denominator of the illumination E is the square of the denominator of $\cos\alpha$. We obtain:

$$E = \frac{\pi \cdot R_p^2 \cdot I \cdot a \cdot (1 - \cos\alpha)}{2r_p^2 \cdot \left[r_B^2 + r_p^2 - 2r_B r_p \cdot \begin{pmatrix} \cos\varphi_B \\ \sin\varphi_B \end{pmatrix} \cdot \begin{pmatrix} \cos\varepsilon\cos\psi \\ \cos\varepsilon\sin\psi \end{pmatrix} \right]}$$

For the luminous flux:

$$\Phi_B = \pi \cdot R_a^2 \cdot E$$

We have the opportunity to treat the case of absorption, too. The absorption coefficient m shall be spatial but not temporal changable. The medium must be extreme thin (for example interplanetary space). At a dense medium we have at non constant m a greater change of the radiation's direction. In the case of absorption the illumination is, see chapter 1:

$$E = \frac{\pi \cdot R_p^2 \cdot I \cdot a \cdot (1-\cos\alpha) \cdot e(\vec{r}_p, \vec{r}_B) \cdot e(\vec{r}_s, \vec{r}_p)}{2r_p^2 \cdot \left[r_B^2 + r_p^2 - 2r_B r_p \cdot \begin{pmatrix} \cos\varphi_B \\ \sin\varphi_B \end{pmatrix} \cdot \begin{pmatrix} \cos\varepsilon\cos\psi \\ \cos\varepsilon\sin\psi \end{pmatrix} \right]}$$

It is:

$$e(\vec{r}_p, \vec{r}_B) = e^{-F(\vec{r}_p, \vec{r}_B)}$$

with:

$$F(\vec{r}_p, \vec{r}_B) = \int_0^1 m(\vec{s}(\vec{r}_p, \vec{r}_B, \tau)) \cdot |\vec{r}_p - \vec{r}_B| \, d\tau$$

at which:

$$\vec{s}(\vec{r}_p, \vec{r}_B, \tau) = \tau \cdot \vec{r}_B + (1-\tau) \cdot \vec{r}_p \qquad \tau \in [0,1]$$

Now we take $\vec{r}_s = \vec{0}$ into consideration:

$$e(\vec{r}_s, \vec{r}_p) = e(\vec{0}, \vec{r}_p) = e^{-F(\vec{0}, \vec{r}_p)}$$

with:

$$F(\vec{0}, \vec{r}_p) = \int_0^1 m(\vec{s}(\vec{0}, \vec{r}_p, \tau)) \cdot r_p \, d\tau$$

at which:

$$\vec{s}(\vec{0}, \vec{r}_p, \tau) = \tau \cdot \vec{r}_p \qquad \tau \in [0,1]$$

If the absorption coefficient m is constant, then it follows for the illumination :

$$E = \frac{\pi \cdot R_p^2 \cdot I \cdot a \cdot (1-\cos\alpha) \cdot e^{-m\cdot|\vec{r}_p - \vec{r}_B|} \cdot e^{-m\cdot r_p}}{2r_p^2 \cdot \left[r_B^2 + r_p^2 - 2r_p r_B \cdot \begin{pmatrix} \cos\varphi_B \\ \sin\varphi_B \end{pmatrix} \cdot \begin{pmatrix} \cos\varepsilon\cos\psi \\ \cos\varepsilon\sin\psi \end{pmatrix} \right]}$$

with:

$$|\vec{r}_p - \vec{r}_B| = \left[r_B^2 + r_p^2 - 2r_B r_p \cdot \begin{pmatrix} \cos\varphi_B \\ \sin\varphi_B \end{pmatrix} \cdot \begin{pmatrix} \cos\varepsilon\cos\psi \\ \cos\varepsilon\sin\psi \end{pmatrix} \right]^{\frac{1}{2}}$$

With M as apparent brightness, E as illumination and Φ_B as luminous flux it is valid (see Voigt [9] chapter IV.1.1, p.139 or Wendker [10] chapter 4.1.2, p.78 equation (4-1)) for the conversion:

$$M_1 - M_2 = -2.5 \cdot \lg\left(\frac{\Phi_{B1}}{\Phi_{B2}}\right) = -2.5 \cdot \lg\left(\frac{E_1}{E_2}\right)$$

We can found at Montenbruck [7] chapter VI.5, p.119 special formulas for the apparent brightness of Mercury, Venus, Mars, Jupiter, Saturn, Uranus, Neptune and Pluto. One difficulty is Saturn's ring. But this ring can be taken into consideration, too.

Now we want to calculate the maximum and minimum illumination. We need the differentiation of E. First we work with the vacuum case. We introduce the symbol k.

$$k := r_B r_p \cdot \begin{pmatrix} \cos\varphi_B \\ \sin\varphi_B \end{pmatrix} \cdot \begin{pmatrix} \cos\varepsilon\cos\psi \\ \cos\varepsilon\sin\psi \end{pmatrix}$$

The symbol h can be introduced:

$$h := \cos\alpha$$

We constuct the temporal differentiation of k:

$$\dot{k} = r_B r_p \cdot \left[\dot{\varphi}_B \cdot \begin{pmatrix} -\sin\varphi_B \\ \cos\varphi_B \end{pmatrix} \cdot \begin{pmatrix} \cos\varepsilon\cos\psi \\ \cos\varepsilon\sin\psi \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} \cos\varphi_B \\ \sin\varphi_B \end{pmatrix} \cdot \frac{d}{dt}\left(\cos\varepsilon \cdot \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} \right) \right]$$

with:

$$\frac{d}{dt}\left(\cos\varepsilon \cdot \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} \right) = -\dot{\varepsilon}\sin\epsilon \cdot \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} + \dot{\psi}\cos\varepsilon \cdot \begin{pmatrix} -\sin\psi \\ \cos\psi \end{pmatrix}$$

With the equations (3) and (4):

$$\varepsilon = \arcsin(\sin\gamma \cdot \sin\varphi_p)$$

$$\psi = \arctan(\cos\gamma \cdot \tan\varphi_p)$$

With the chain rule we obtain:

$$\dot{\varepsilon} = \frac{\dot{\varphi}_p \cdot \sin\gamma \cdot \cos\varphi_p}{\sqrt{1 - \sin^2\gamma \sin^2\varphi_p}} \tag{5}$$

$$\dot{\psi} = \frac{\dot{\varphi}_p \cdot \cos\gamma \cdot (\tan^2\varphi_p + 1)}{1 + \cos^2\gamma\tan^2\varphi_p} \tag{6}$$

With that k is completly derived.

Now we can write the illumination as:

$$E = \frac{\pi \cdot R_p^2 \cdot I \cdot a \cdot (1 - \cos\alpha)}{2r_p^2 \cdot (r_B^2 + r_p^2 - 2k)}$$

Now we construct the differentiation of E with the quotient rule:

$$\dot{E} = \frac{\pi R_p^2 I a}{2r_p^2} \cdot \frac{\sin\alpha \cdot \dot{\alpha} \cdot (r_B^2 + r_p^2 - 2k) + 2 \cdot (1 - \cos\alpha) \cdot \dot{k}}{(r_B^2 + r_p^2 - 2k)^2}$$

Now we derive α:

$$\cos\alpha = h$$

It follows:

$$\alpha = \arccos h$$

We get with the chain rule:

$$\dot{\alpha} = \frac{-\dot{h}}{\sqrt{1-h^2}}$$

with:

$$h = \frac{k - r_p^2}{r_p \cdot \sqrt{r_p^2 + r_B^2 - 2k}}$$

quotient rule:

$$\dot{h} = \frac{1}{r_p \cdot (r_p^2 + r_B^2 - 2k)} \cdot \left(\dot{k} \cdot \sqrt{r_p^2 + r_B^2 - 2k} - (k - r_p^2) \cdot \frac{-\dot{k}}{\sqrt{r_p^2 + r_B^2 - 2k}} \right)$$

It is:

$$\varphi_B = \delta_B + w_B \cdot t$$

$$\varphi_p = \delta_p + w_p \cdot t$$

It follows:

$$\dot{\varphi}_B = w_B \qquad \dot{\varphi}_p = w_p$$

With that the illumination E is completly derived in the vacuum case.

The necessary extremum condition of the illumination is $\dot{E} = 0$. With that:

$$\sin\alpha \cdot \dot{\alpha} \cdot (r_B^2 + r_p^2 - 2k) + 2 \cdot (1 - \cos\alpha) \cdot \dot{k} = 0$$

If we solve this equation to t, we get t-values with $E(t)$ as local maximum or local minimum or saddle point. If we have really an extremum, we must decide with Rolle's theorem or with the second derivation of E. Possibly we even need higher derivations see Barner [1] chapter 8.4 p.295.

Now we work with the differentiation of the illumination in the case of absorption: $(\vec{r}_s = \vec{0})$

$$E = \frac{\pi R_p^2 I a \cdot (1 - \cos\alpha) \cdot e(\vec{r}_p, \vec{r}_B) \cdot e(\vec{0}, \vec{r}_p)}{2r_p^2 \cdot (r_B^2 + r_p^2 - 2k)}$$

The absorption coefficient m is spatial not constant. We apply the quotient rule:

$$\dot{E} = \frac{\pi R_p^2 I a}{2r_p^2 \cdot (r_B^2 + r_p^2 - 2k)^2} \cdot \Big[\frac{d}{dt} \Big((1 - \cos\alpha) \cdot e(\vec{r}_p, \vec{r}_B) \cdot e(\vec{0}, \vec{r}_p) \Big) \cdot (r_B^2 + r_p^2 - 2k)$$

$$+ 2\dot{k} \cdot (1 - \cos\alpha) \cdot e(\vec{r}_p, \vec{r}_B) \cdot e(\vec{0}, \vec{r}_p) \Big]$$

with:

$$\frac{d}{dt} \Big((1 - \cos\alpha) \cdot e(\vec{r}_p, \vec{r}_B) \cdot e(\vec{0}, \vec{r}_p) \Big) = \sin\alpha \cdot \dot{\alpha} \cdot e(\vec{r}_p, \vec{r}_B) \cdot e(\vec{0}, \vec{r}_p) + (1 - \cos\alpha) \cdot \frac{d}{dt} \Big[e(\vec{r}_p, \vec{r}_B) \cdot e(\vec{0}, \vec{r}_p) \Big]$$

at which:

$$\frac{d}{dt}\left[e(\vec{r}_p,\vec{r}_B)\cdot e(\vec{0},\vec{r}_p)\right]=\frac{d}{dt}\,e(\vec{r}_p,\vec{r}_B)\cdot e(\vec{0},\vec{r}_p)+e(\vec{r}_p,\vec{r}_B)\cdot\frac{d}{dt}\,e(\vec{0},\vec{r}_p)$$

We derive:

$$\frac{d}{dt}\,e(\vec{r}_p,\vec{r}_B)=-e^{-F_1(\vec{r}_p,\vec{r}_B)}\cdot\dot{F}_1$$

The F-Funktion is indexed, because this function will be needed in different ways.

$\dot{F}_1$ is explained through:

$$\dot{F}_1=\frac{d}{dt}\left[\int_0^1 m(\vec{s}_1(\vec{r}_p,\vec{r}_B,\tau))\cdot|\vec{r}_p-\vec{r}_B|\,d\tau\right]$$

$\vec{s}$ is indexed, because it will be needed in different ways.

We assume that m is continuous in R^3 and the integrand is continuous differentiable with respect to t. Then we can exchange differentiation and integration, see for example Forster [4], §9, theorem 2, p.84, for weaker assumptions see Forster [5] §11, theorem 2 p.99:

$$\dot{F}_1=\int_0^1\frac{d}{dt}\,[m(\vec{s}_1(\vec{r}_p,\vec{r}_B,\tau))\cdot|\vec{r}_p-\vec{r}_B|]\,d\tau$$

The integrand is equal to:

$$\frac{d}{dt}\,m(\vec{s}_1(\vec{r}_p,\vec{r}_B,\tau))\cdot|\vec{r}_p-\vec{r}_B|+m(\vec{s}_1(\vec{r}_p,\vec{r}_B,\tau))\cdot\frac{d}{dt}\,|\vec{r}_p-\vec{r}_B|$$

with:

$$\frac{d}{dt}\,|\vec{r}_p-\vec{r}_B|=\frac{d}{dt}\,(\vec{r}_p^{\,2}+\vec{r}_B^{\,2}-2\cdot\vec{r}_p\cdot\vec{r}_B)^{\frac{1}{2}}$$

$$=\frac{d}{dt}\,(r_p^2+r_B^2-2k)^{\frac{1}{2}}=\frac{-\dot{k}}{\sqrt{r_p^2+r_B^2-2k}}$$

and:

$$\frac{d}{dt}\,m(\vec{s}_1(\vec{r}_p,\vec{r}_B,\tau))=\operatorname{grad}m(\vec{s}_1(\vec{r}_p,\vec{r}_B,\tau))\cdot\dot{\vec{s}}_1\qquad\operatorname{grad}m=\left(\frac{\partial m}{\partial x},\frac{\partial m}{\partial y},\frac{\partial m}{\partial z}\right)$$

with:

$$\dot{\vec{s}}_1=\tau\dot{\vec{r}}_B+(1-\tau)\dot{\vec{r}}_p\qquad\qquad\tau\in\,[0,1]$$

It is:

$$\dot{\vec{r}}_B=r_B\cdot\frac{d}{dt}\begin{pmatrix}\cos\varphi_B\\ \sin\varphi_B\\ 0\end{pmatrix}=r_B\cdot\dot{\varphi}_B\cdot\begin{pmatrix}-\sin\varphi_B\\ \cos\varphi_B\\ 0\end{pmatrix}$$

with:

$$\dot{\varphi}_B=w_B$$

and:

$$\dot{\vec{r}}_p = r_p \cdot \frac{d}{dt} \begin{pmatrix} \cos\varepsilon \cos\psi \\ \cos\varepsilon \sin\psi \\ \sin\varepsilon \end{pmatrix}$$

$$= r_p \cdot \begin{pmatrix} -\dot{\varepsilon} \sin\varepsilon \cos\psi - \dot{\psi} \cos\varepsilon \sin\psi \\ -\dot{\varepsilon} \sin\varepsilon \sin\psi + \dot{\psi} \cos\varepsilon \cos\psi \\ \dot{\varepsilon} \cos\varepsilon \end{pmatrix}$$

with the equations (5) and (6) of this chapter:

$$\dot{\varepsilon} = \frac{\dot{\varphi}_p \cdot \sin\gamma \cdot \cos\varphi_p}{\sqrt{1 - \sin^2\gamma \cdot \sin^2\varphi_p}}$$

$$\dot{\psi} = \frac{\dot{\varphi}_p \cdot \cos\gamma \cdot (\tan^2\varphi_p + 1)}{1 + \cos^2\gamma \cdot \tan^2\varphi_p}$$

and:

$$\dot{\varphi}_p = w_p$$

Now we derive the second e-term:

$$\frac{d}{dt} e(\vec{0}, \vec{r}_p) = -e^{-F_2(\vec{0}, \vec{r}_p)} \cdot \dot{F}_2$$

Now it is clear why the F-function and $\vec{s}$ must be indexed.

$$\dot{F}_2 = \frac{d}{dt} \left[\int_0^1 m(\vec{s}_2(\vec{0}, \vec{r}_p, \tau)) \cdot |\vec{r}_p| \, d\tau \right]$$

m shall be continuous in R^3. We assume that the integrand is continuous differentiable in t. Then it is valid, see Forster [4], §9, theorem 2, p.84 (for weaker assumptions see Forster [5] §11, theorem 2, p.99):

$$\dot{F}_2 = r_p \cdot \int_0^1 \frac{d}{dt} m(\vec{s}_2(\vec{0}, \vec{r}_p, \tau)) \, d\tau$$

with:

$$\frac{d}{dt} m(\vec{s}_2(\vec{0}, \vec{r}_p, \tau)) = \operatorname{grad} m(\vec{s}_2(\vec{0}, \vec{r}_p, \tau)) \cdot \dot{\vec{s}}_2$$

at which:

$$\dot{\vec{s}}_2 = \tau \cdot \dot{\vec{r}}_p \qquad \tau \in [0, 1]$$

$\dot{\vec{r}}_p$ was explained before.

With that the illumination E is completly derived.

Necessary condition of local extrema is $\dot{E} = 0$. With that it follows:

$$\frac{d}{dt}((1 - \cos\alpha) \cdot e(\vec{r}_p, \vec{r}_B) \cdot e(\vec{0}, \vec{r}_p)) \cdot (r_B^2 + r_p^2 - 2k)$$

$$+2\dot{k} \cdot (1 - \cos\alpha) \cdot e(\vec{r}_p, \vec{r}_B) \cdot e(\vec{0}, \vec{r}_p) = 0$$

Now we view the special case with constant absorption coefficient m. Then it is $\operatorname{grad} m = \vec{0}$. We obtain:

$$\dot{F}_1 = m \cdot \int_0^1 \frac{d}{dt} |\vec{r}_p - \vec{r}_B| \, d\tau$$

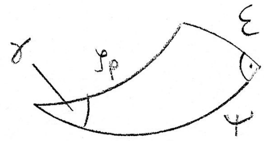
$$\dot{F}_2 = 0$$

3. Appendix

Here we make the derivation of $\sin\varepsilon = \sin\gamma \cdot \sin\varphi_p$ and
$\tan\psi = \cos\gamma \cdot \tan\varphi_p$ for $0° \leq \varphi_p \leq 360°$.

ε = latitude angle $\qquad -90° \leq \varepsilon \leq 90°$
ψ = longitude angle

First we view the case $0° \leq \varphi_p \leq 90°$. It is $\varepsilon \geq 0°$.

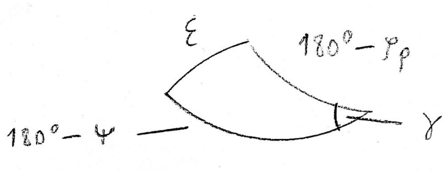


With Bronstein [3], chapter 2.6.4.3.2, p.209, equation (2.87) and (2.95):

$$\sin\varepsilon = \sin\varphi_p \cdot \sin\gamma$$

$$\cos\gamma = \tan\psi \cdot \cot\varphi_p = \frac{\tan\psi}{\tan\varphi_p} \qquad \Rightarrow \qquad \tan\psi = \cos\gamma \cdot \tan\varphi_p$$

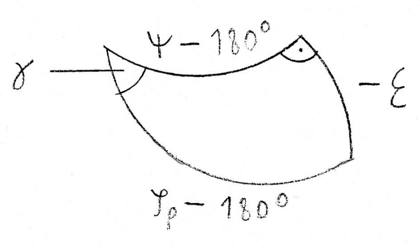
Now to the case $90° \leq \varphi_p \leq 180°$, with $\varepsilon \geq 0°$:



$\sin\varepsilon = \sin(180° - \varphi_p) \cdot \sin\gamma = \sin\varphi_p \cdot \sin\gamma$

$$\cos\gamma = \frac{\tan(180° - \psi)}{\tan(180° - \varphi_p)} = \frac{-\tan\psi}{-\tan\varphi_p} \qquad \Rightarrow \qquad \tan\psi = \cos\gamma \cdot \tan\varphi_p$$

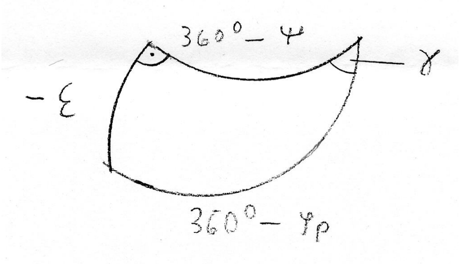
The case $180° \leq \varphi_p \leq 270°$: $\qquad \varepsilon \leq 0°$



$-\sin\varepsilon = \sin(-\varepsilon) = \sin\gamma \cdot \sin(\varphi_p - 180°) = -\sin\gamma \cdot \sin\varphi_p$
$\Rightarrow \qquad \sin\varepsilon = \sin\gamma \cdot \sin\varphi_p$

$$\cos\gamma = \frac{\tan(\psi - 180°)}{\tan(\varphi_p - 180°)} = \frac{\tan\psi}{\tan\varphi_p} \qquad \Rightarrow \qquad \tan\psi = \cos\gamma \cdot \tan\varphi_p$$

Finally to the case $270° \le \varphi_p \le 360°$: $\qquad \varepsilon \le 0°$



$-\sin\varepsilon = \sin(-\varepsilon) = \sin\gamma \cdot \sin(360° - \varphi_p) = -\sin\gamma \cdot \sin\varphi_p$
$\Rightarrow \qquad \sin\varepsilon = \sin\gamma \cdot \sin\varphi_p$

$$\cos\gamma = \frac{\tan(360° - \psi)}{\tan(360° - \varphi_p)} = \frac{-\tan\psi}{-\tan\varphi_p} \qquad \Rightarrow \qquad \tan\psi = \cos\gamma \cdot \tan\varphi_p$$

With that the assertion is proved.

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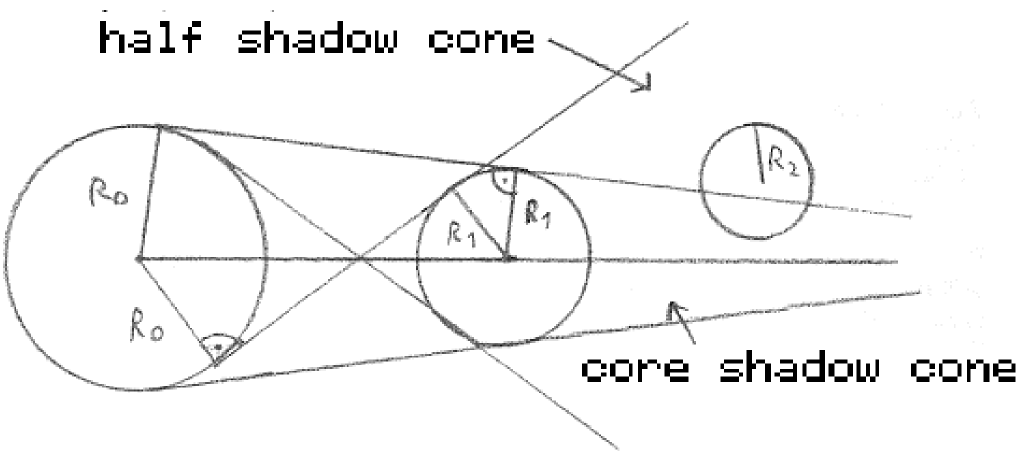
70. Eclipses, a non conventional representation

We want to view the constellations at eclipses more accurately.

1. Spaces of eclipse

We look at three balls with the positions $\vec{p}_0, \vec{p}_1, \vec{p}_2 \in R^3$. The vectors are relative to the midpoints of the ball. The positions shall be dependent from the time t.

R_0, R_1, R_2 = radii of the balls
The ball 0 shall be self beaming. (light source, fixed star)



The figure shows the core shadow cone and the half shadow cone. Now we introduce the following vectors:

$$\vec{r}_1 := \vec{p}_1 - \vec{p}_0 \qquad \vec{r}_2 := \vec{p}_2 - \vec{p}_0$$

$r_i = |\vec{r}_i|$ is the absolute value of the vector $\vec{r}_i$. At core shadow cone we have the apex angles w_{ik}:

$$\sin w_{ik} = \frac{R_0}{|\vec{r}_i + \tau \cdot (\vec{p}_i - \vec{p}_0)|} = \frac{R_i}{|\tau \cdot (\vec{p}_i - \vec{p}_0)|}$$

From this equation we get a τ for the core shadow eclipse. The apex of the core shadow cone can be written as:

$$\vec{s}_{ik} := \vec{p}_i + \tau \cdot (\vec{p}_i - \vec{p}_0)$$

For half shadows eclipses we look at the following double cone with the apex angle w_{ih}:

$$\sin w_{ih} = \frac{R_0}{|\tau \cdot \vec{r}_i|} = \frac{R_i}{|\vec{r}_i - \tau \cdot \vec{r}_i|}$$

From this we get τ for the half shadow cone. We get the cone apex with $\vec{s}_{ih} := \vec{p}_0 + \tau \cdot \vec{r}_i$. With that the apex of both cones are known. We define $\cos w_{ih} =: c_{ih}$ and $\cos w_{ik} =: c_{ik}$. Now we can write the cone equations for example see Köhler [1] chapter 10.2 p.10.3 equations (2) and (3), first for the core shadow cone:

$$(\vec{x} - \vec{s}_{ik}) \cdot \frac{-\vec{r}_i}{|\vec{r}_i|} = |\vec{x} - \vec{s}_{ik}| \cdot c_{ik} \qquad i \in 1,2 \tag{1}$$

For the half shadow cone:

$$\left[(\vec{x} - \vec{s}_{ih}) \cdot \frac{\vec{r}_i}{|\vec{r}_i|}\right]^2 = c_{ih}^2 \cdot (\vec{x} - \vec{s}_{ih})^2 \qquad i \in 1,2 \tag{2}$$

The equations (1) and (2) are valid to the lateral area of a cone. For the full cone a smaller apex angle is also possible. From $w' < w$ follows $\cos w' > \cos w$, with that we obtain with equation (1) to the core shadow cone:

$$(\vec{x} - \vec{s}_{ik}) \cdot \frac{-\vec{r}_i}{|\vec{r}_i|} \geq |\vec{x} - \vec{s}_{ik}| \cdot c_{ik} \qquad i \in 1,2 \tag{3}$$

For the half shadow cone:

$$\left[(\vec{x} - \vec{s}_{ih}) \cdot \frac{\vec{r}_i}{|\vec{r}_i|}\right]^2 \geq c_{ih}^2 \cdot (\vec{x} - \vec{s}_{ih})^2 \qquad i \in 1,2 \tag{4}$$

Now we need the ball's equation:

$$(\vec{x} - \vec{p}_i)^2 = R_i^2 \qquad i \in 0,1,2 \tag{5}$$

See Köhler [1] chapter 9.1.2 p.9.2.

With $\vec{y}_{ik}$ we denote the intersection set of the solution sets from equations (1) and (5). $\vec{y}_{ih}$ shall be the intersection set of the solution sets of (2) and (5). $\vec{y}_{ik}$ and $\vec{y}_{ih}$ are circles on the surface of the ball $\vec{p}_i$, with equal distance to $\vec{p}_0$.

Now we introduce the vectors $\vec{a}_{ik} := \vec{y}_{ik} - \vec{p}_0$ and $\vec{a}_{ih} := \vec{y}_{ih} - \vec{p}_0$. With the absolute values of these new vectors we can form:

$$|\vec{x} - \vec{p}_0| \geq a_{ih} \tag{6}$$
$$|\vec{x} - \vec{p}_0| \geq a_{ik} \tag{7}$$

We get from equation (5):

$$(\vec{x} - \vec{p}_i)^2 \geq R_i^2 \qquad i \in 0,1,2 \tag{8}$$

The intersection of solutions from inequations (3),(7) and (8) is the core shadow space M_{ik} of the ball $\vec{p}_i$. Further the half shadow space M_{ih} of the ball $\vec{p}_i$ is the intersection of solutions from the inequations (4), (6) and (8).

$M_{ik}(O_j)$:= core shadow space of the ball i on the ball j

$M_{ih}(O_j)$:= half shadow space of the ball i on the ball j

It must be $i \neq j$, with $i, j \in 1, 2$.

$M_{ik}(O_j)$ is the intersection of M_{ik} and the solution set of equation (5) with j as index. Further we can calculate $M_{ih}(O_j)$ as intersection of M_{ih} and the solution set of (5) with index j. $M_{ik}(O_j)$ is in the set $M_{ih}(O_j)$, we can recognize this from the figure.

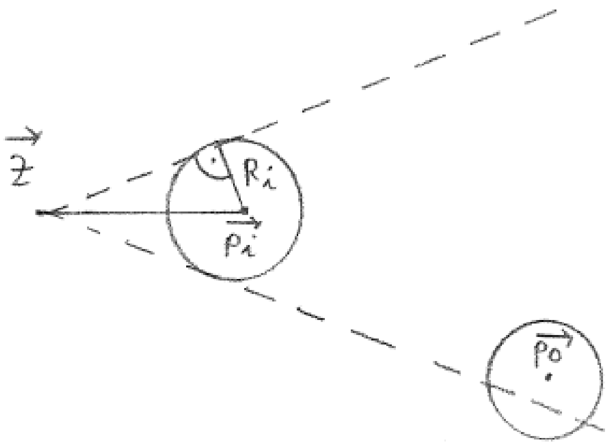
2. projection pictures

Now we turn to the projection pictures.

$\vec{z}$ = point, from that the pictures are seen. This point can also be changed temporal.

2.1. picture of the light source (corresponds to a solar eclipse)

$\vec{z}$ is the apex of a cone. The ball i is in this cone.



apex angle w_i:

$$\sin w_i = \frac{R_i}{|\vec{p}_i - \vec{z}|} \qquad c_i := \cos w_i$$

$$\vec{e}_i := \frac{\vec{p}_i - \vec{z}}{|\vec{p}_i - \vec{z}|}$$

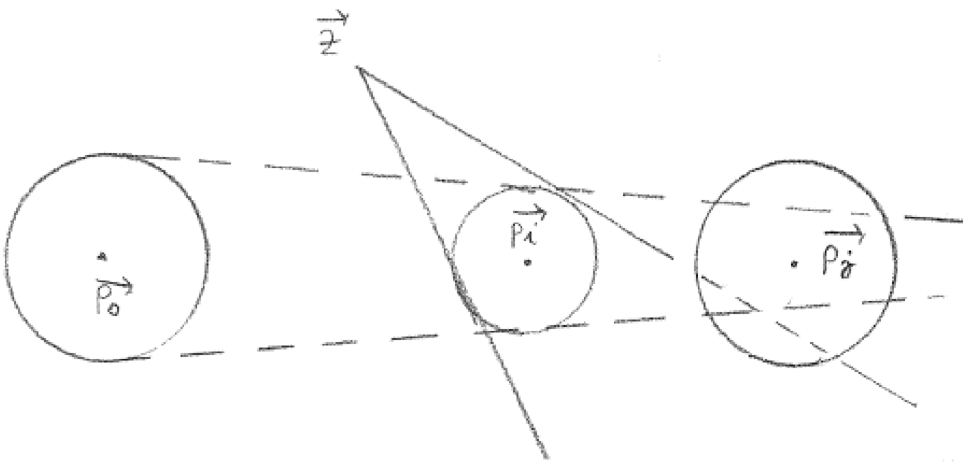
With Köhler [1] chapter 10.2 equation (2) and equation (3) in this text we can represent the space without the cone in the following way:

$$(\vec{x} - \vec{z}) \cdot \vec{e}_i \leq |\vec{x} - \vec{z}| \cdot c_i \qquad i \in 1, 2 \tag{9}$$

With N we denote the intersection of the solution from (9) for $i = 1$ and 2 and the inequation $(\vec{x}-\vec{p}_0)^2 \leq R_0^2$. That means a combination of 2 or 3 inequations in dependence if the light source is covered from one ball or 2 balls. This intersection can also be a function of time. Then the searched picture is the projection of N on an arbitrary plane behind the projection point $\vec{z}$. With that form and size of the picture are determined. This problem will be dealt with in general in the publication "Projections on planes" [2]. The observer's position $\vec{z}$ shall be on the surface of the ball i, then $\vec{z}$ must be in the solution set of equation (5) for i.

2.2. picture of an eclipsed ball (corresponds lunar eclipse)

We look at the following figure:



Because of the inequation (3) the solution set of

$$(\vec{x}-\vec{s}_{ik}) \cdot \frac{-\vec{r}_i}{|\vec{r}_i|} \leq |\vec{x}-\vec{s}_{ik}| \cdot c_{ik} \qquad i \in 1,2 \tag{10}$$

is equal to the space without core shadow cone. The space without half shadow cone can be represented because of the inequation (4) as solution set from:

$$\left[(\vec{x}-\vec{s}_{ih}) \cdot \frac{\vec{r}_i}{|\vec{r}_i|}\right]^2 \leq c_{ih}^2 \cdot (\vec{x}-\vec{s}_{ih})^2 \tag{11}$$

We introduce N_{ijk} respectively N_{ijh} as parts of the j. ball, that remain visible from the projection point $\vec{z}$. The lighted parts are considered as sets. The j. ball is perhaps covered partly by the i. ball. The index k symbolizes the core shadow and h the half shadow.

N_{ijh} is equal to the solution set of $(\vec{x}-\vec{p}_j)^2 \leq R_j^2$ combined with the inequations (11) and (9).
N_{ijk} is the solution set of $(\vec{x}-\vec{p}_j)^2 \leq R_j^2$ combined with the inequations (10) and (9).

N_{ijh} and N_{ijk} can also be dependent from the time.

Then we obtain the searched picture as projection of N_{ijh} respectively N_{ijk} with the projection point $\vec{z}$ on an arbitrary plane. For this we again refer to the publication "Projections on planes" [2]. The observer's position $\vec{z}$ shall be on the surface of the i. ball, then $\vec{z}$ must be in the solution set of equation (5) for i. With this picture is also determined, whether it is a partial or total eclipse or a half shadow eclipse or a core shadow eclipse.

Perhaps in all cases a generalization is possible with $R_i = R_{pi} + d_i . R_{pi}$ is the proper planet's radius and d_i is the thickness of the (dense) atmosphere of the i. ball. This generalization is only important for planets with very dense atmospheres. One example is the Venus. There core shadow and half shadow are relative to R_{pi}. The shadows shall be relative to R_i, then we have no exact bounds any more. Then we have often, because of the atmosphere, a continual transition. In the case of no atmosphere it is of course $R_i = R_{pi}$.

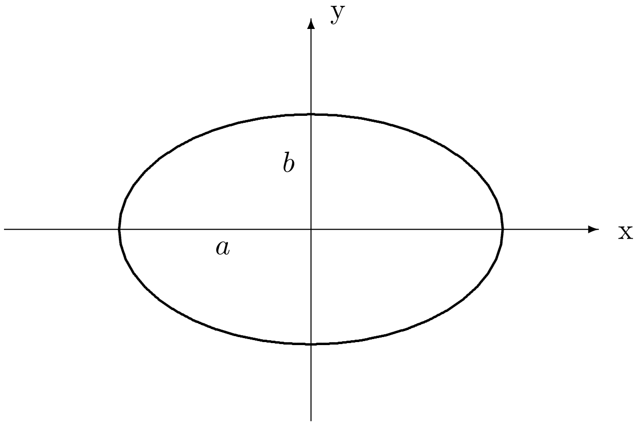
References

[1] Joachim Köhler, Rolf Höwelmann, Hardt Krämer "Analytische Geometrie und Abbildungsgeometrie in vektorieller Darstellung" Verlag Moritz Diesterweg 6.edition 1975 Frankfort on the Main

[1] Harald Schröer "Projections on planes" 2001

71. Geographical latitude and geocentric latitude

We view an ellipse with the semimajor axis a and the semiminor axis b.



We use the canonical equation of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

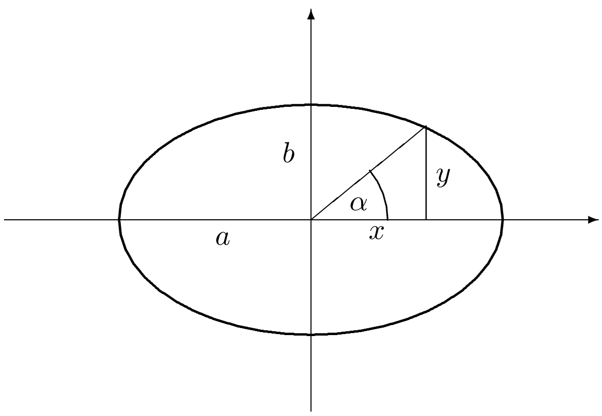
We transform to y:

$$y = b \cdot \pm\sqrt{1 - \frac{x^2}{a^2}} = \pm\frac{b}{a} \cdot \sqrt{a^2 - x^2}$$

We can calculate the differentiation with the chain rule:

$$y' = \frac{dy}{dx} = \pm\frac{b}{a} \cdot \frac{-x}{\sqrt{a^2 - x^2}}$$

For the relation between the angle α and the coordinates x and y we see the following figure:



We recognize:

$$x \cdot \tan\alpha = y \qquad\qquad \alpha \in [0°, 360°]$$

We insert in the canonical equation:

$$\frac{x^2}{a^2} + \frac{x^2 \tan^2 \alpha}{b^2} = 1$$

To one denominator:

$$\frac{x^2 b^2 + a^2 x^2 \tan^2 \alpha}{a^2 b^2} = 1$$

Now we solve this equation to x:

$$x^2 = \frac{a^2 b^2}{b^2 + a^2 \tan^2 \alpha}$$

or:

$$x = \frac{ab}{\sqrt{b^2 + a^2 \tan^2 \alpha}}$$

Now we insert this term into the differentiation equation:

$$y'(\alpha) = \frac{\pm b \cdot \frac{-ab}{\sqrt{b^2+a^2\tan^2\alpha}}}{a \cdot \sqrt{a^2 - \frac{a^2b^2}{b^2+a^2\tan^2\alpha}}}$$

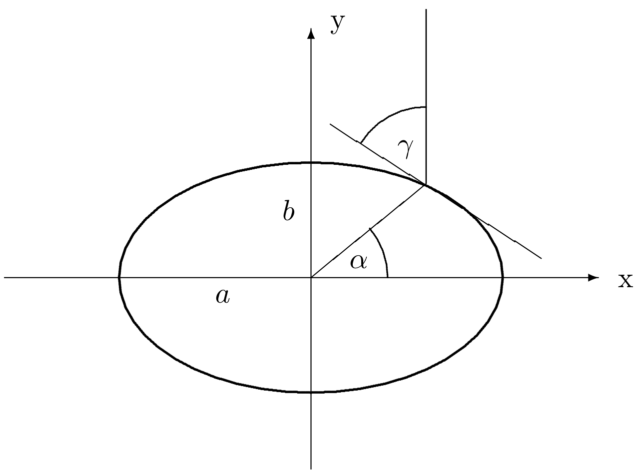
$$= \mp \frac{\frac{b^2}{\sqrt{b^2+a^2\tan^2\alpha}}}{\sqrt{\frac{a^2b^2+a^4\tan^2\alpha-a^2b^2}{b^2+a^2\tan^2\alpha}}}$$

$$= \frac{\mp b^2}{\sqrt{a^4 \tan^2 \alpha}}$$

Thus we obtain finally:

$$y'(\alpha) = \mp \left(\frac{b}{a}\right)^2 \cdot \tan^{-1} \alpha$$

Now we work with the problem to find a relation between the geographical latitude γ and the geocentric latitude α.



With the figure we recognize:

$$\frac{-1}{y'(\alpha)} = \tan\gamma \qquad\qquad \alpha, \gamma \in [-90°, +90°]$$

If we insert $y'(\alpha)$, this leads to:

$$b^2 \cdot \tan\gamma = -a^2 \cdot \tan\alpha$$

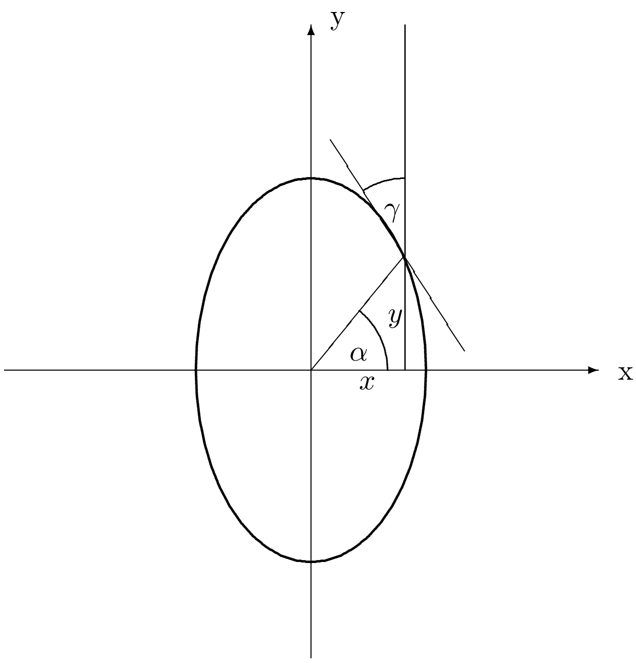
Then we get the expressions:

$$\tan\gamma = \frac{-a^2}{b^2} \cdot \tan\alpha \qquad\qquad \tan\alpha = \frac{-b^2}{a^2} \cdot \tan\gamma$$

Now we introduce the flattening $A = \frac{a-b}{a} = 1 - \frac{b}{a}$. Then we follow $\frac{b}{a} = 1 - A$. This insertion yields:

$$\tan\alpha = -(1-A)^2 \cdot \tan\gamma \qquad \text{or} \qquad \tan\gamma = \frac{-\tan\alpha}{(1-A)^2}$$

Now we view the general revolution solid like in the following figure:



The general revolution solid is described with $y = f(x)$. Here we have the relation:

$$\frac{i(x)}{\tan\gamma} = f'(x) \tag{1}$$

with:

$$i(x) := \begin{cases} -1 & : \quad x \geq 0 \\ 1 & : \quad x < 0 \end{cases}$$

We can express the geocentric latitude α with the following equation:

$$\tan\alpha = \frac{\pm f(x)}{|x|} \qquad\qquad |\cdot| = \text{absolute value} \tag{2}$$

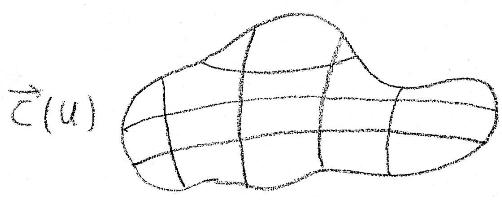
The function $y = f(x)$ must be known of course. In concrete cases, if γ is known, we must determine with equation (1) x. This inserted in equation (2) yields α. If α is known, then we calculate with equation (2) x and finally we insert this x in equation (1).
An unusual case is, if we know only a latitude α or γ. The problem is to determine the coordinates x and y with the known function f: In the case of known α we can calculate x with the equation (2). In the case of known γ we must use the equation (1) to get x. In both cases we obtain y with $y = f(x)$.
Equation (1) is for the conversion between x and γ and equation (2) is used for the conversion between x and α.

72. An observer on a irregular body

Abstract: An observer is on a planet or planetoid that looks like a irregular body. Normal vector, tangent vector, height and depth on a general surface are explained. The observer's position is described. The pitch angle is calculated.

Key words: Observer - planet - irregular body - normal vector - tangent vector - pitch angle - height - depth - planetoid

We view a total irregular body (planet or planetoid) with an observer. The body is parametrized through the function $\vec{c}: U \longrightarrow R^3$ with $U \subset R^4$ and $\vec{c}(k_1, k_2, k_3, t)$. k_1 and k_2 are surface parameters and k_3 is a height parameter. t is the time:

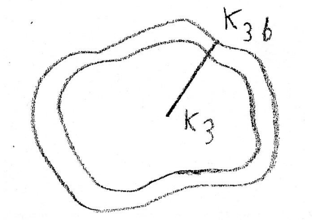


Now the body shall rotate on some kind. This can be done by a further function $\vec{d}: V \longrightarrow R^3$ with $V \subset R^4$ and $\vec{d}(\vec{c}(k_1, k_2, k_3, t), t)$. With this rotation the observer has the position:

$$\vec{p}(t) = \vec{d}(\vec{c}(k_1, k_2, k_3, t), t)$$

The expression of $\vec{c}$ has a temporal changing that has to do with the form of the body but not with the rotation. In the expression of $\vec{d}$ is a temporal changing that has only to do with the rotation.

Now k_{3b} shall be the value of k_3 on the surface.



Then we introduce the normal vector:

$$\vec{n}(k_1, k_2, k_{3b}, t) = \frac{\partial}{\partial k_1}\vec{p}(k_1, k_2, k_{3b}, t) \times \frac{\partial}{\partial k_2}\vec{p}(k_1, k_2, k_{3b}, t)$$

$\frac{\partial}{\partial k_1}\vec{p}$ and $\frac{\partial}{\partial k_2}\vec{p}$ are tangent vectors see Forster [1], §15, theorem 1, p.148.

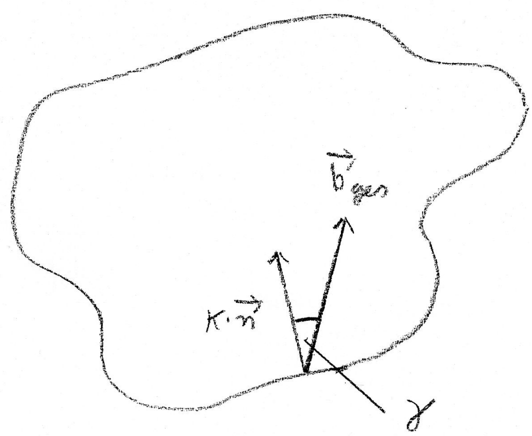
Which normal vector is directed **inwards** relative to the surface? t is known and k_{3b} is fixed. Then we have the following equation:

$$\vec{p}(k_1, k_2, k_3, t) = \vec{p}(k_1, k_2, k_{3b}, t) + k \cdot \vec{n}(k_1, k_2, k_{3b}, t)$$

If a solution (k, k_1, k_2, k_3) exists for all k with $0 \leq k \leq k_a \in R^+$ or $0 \geq k \geq k_a \in R^-$, then $k \cdot \vec{n}(k_1, k_2, k_{3b}, t)$ is one inwards directed normal vector.

pitch angle:

We look at the following figure:



If we take the normal vector $\vec{n} = \vec{n}(k_1, k_2, k_{3b}, t)$ and the total acceleration $\vec{b}_{ges}$, then we get with the scalar product for the pitch angle γ:

$$\cos \gamma = \frac{k \cdot \vec{n} \cdot \vec{b}_{ges}}{|k \cdot \vec{n}| \cdot |\vec{b}_{ges}|}$$

The height (or depth):

Now we determine the height (or depth) of $\vec{p}_a := \vec{d}(\vec{c}(k_{1a}, k_{2a}, k_{3a}, t), t)$. k_{1a}, k_{2a}, k_{3a} and t are fixed.

First we must examine, if $\vec{p}_a$ is over or under the surface. We have the following equation:

$$\vec{p}_a = \vec{d}(\vec{c}(k_1, k_2, k_3, t), t)$$

If one solution (k_1, k_2, k_3) of this equations exists, then $\vec{p}_a$ is not over the surface. If no solution (k_1, k_2, k_3) exists, then $\vec{p}_a$ is over the surface. Then it is clear, if it is a height or a depth.

The surface O can be written as:

$$O := \{\vec{d}(\vec{c}(k_1, k_2, k_{3b}, t), t) \in R^3 \qquad \text{with} \qquad (k_1, k_2) \in \bar{U} \subset R^2\}$$

k_{3b} is the surface parameter again. k_{3b} depends on the body itself. Now the distance is interesting:

$$r := |\vec{p}_a - \vec{d}(\vec{c}(k_1, k_2, k_{3b}, t), t)|$$

The minimum of $r(k_1, k_2)$ with $(k_1, k_2) \in \bar{U} \subset R^2$ is the height (depth) over (under) the surface.

With that we have a problem to determine a minimum under constraints. From the restriction to the set $\bar{U}$ it follows that the constraints are in general inequalities. To solve this problem we must view in literature of "Non linear optimization".

In rare cases $\bar{U}$ can be characterized only through equations. Then the constraints are equations. With that we have a minimum problem with equations as constraints. For this case we have in analysis the method with the Lagrange multipliers.

Then we get a new function $r_{min}(t)$, that can be examined over time intervals to maxima and minima. For the determination of these extrema (if this function is known) the differential calculus of one variable is sufficient.
These values are used at Schröer [2] chapter 5 to determine the total acceleration and the visual vertical.

References

[1] Otto Forster "Analysis 3", Vieweg Verlag, 2. edition, Brunswick 1983

[2] Harald Schröer "Theory of orientation - The visual vertical", Wissenschaft & Technik Verlag, Berlin 2002

73. The duration of a day on a planet

We want to determine, the time between two sunrises or sunsets on a planet. The sun is always a fixed star. This fixed star is circled by a planet. We assume circular orbits. This condition is approximately filled in our solar system with exception of Mercury and Pluto. We explain the following values:

$\vec{w}_p$ = angular velocity with which the planet rotates around its own axis.

$\vec{w}$ = angular velocity with which the planet circles around its fixed star (sun).

$$w_p = |\vec{w}_p| \qquad w = |\vec{w}|$$

T_p = rotation time of the planet in seconds

T = orbital period of the planet around its fixed star (sun) in seconds

These relationships are valid:

$$T_p = \frac{2\pi}{w_p} \qquad T = \frac{2\pi}{w}$$

or

$$\frac{2\pi}{T_p} = w_p \qquad \frac{2\pi}{T} = w$$

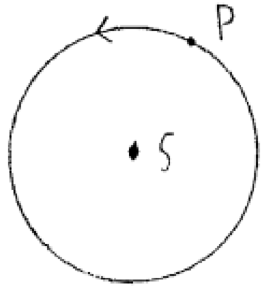
First we must determine the direction of rotation:

$w_p > 0$ (Rotation of the planet from west to east, mathematical positive direction, counterclockwise)

$w_p < 0$ (Rotation of the planet from east to west, mathematical negative direction, clockwise)

$w > 0$ (Circulation of the planet counterclockwise, mathematical positive direction)

$w < 0$ (Circulation of the planet clockwise, mathematical negative direction)



P = position of the planet (or of the satellite)
S = position of the fixed star (or of the planet)

The orbital period T and the rotation time T_p are known. Additionally $\vec{w}_p$ and $\vec{w}$ shall be parallel and constant. Thus, we have $\vec{w}_p$ and $\vec{w}$. Now, we can define a **day**:

Duration of a day:= time that is needed until S is seen from P in the same direction from a certain point on the planet's surface.

Let us investigate the different cases. These cases can be imagined with models:

$w_p > 0$ and $w = 0$
apparent motion of S from east to west

$w_p = 0$ and $w > 0$
apparent motion of S from west to east

Thus we can conclude the following results:

$$w_p - w > 0$$

apparent motion of S from east to west ($w_p > w$) seen from P

$$w - w_p > 0$$

apparent motion of S from west to east ($w > w_p$) seen from P

$$w - w_p = 0 \qquad w = w_p$$

S apparently doesn't move as seen from P, a bound rotation

We obtain the duration of a day T_w with:

$$T_w = \frac{2\pi}{|w_p - w|}$$

In the case of a bound rotation, T_w is infinite.

Of all planets in our solar system, it is valid $w, w_p > 0$. Only at Venus is $w_p < 0$.

These concepts can be used with planets and their satellites, as well. In this case, $\vec{w}_p$ is the angular velocity with which the satellite rotates around its axis, and $\vec{w}$ is the angular velocity with which the satellite circles around its planet. We must assume approximately circular orbits.

These views cannot be used with Mercury and Pluto, because their orbits are ellipses with large eccentricity. To Mercury, see Voigt [1] chapter II.9.3 p.74. But with all the other planets of our solar system, we can determine with this method the duration of one day.

References

[1] Hans Heinrich Voigt "Abriß der Astronomie" 4.edition 1988 BI-Verlag Mannheim

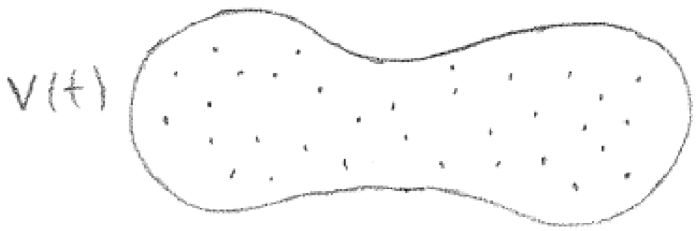
74. The number of new originated objects in space and the average distance

1. The number of new originated objects in space

The model that is presented here is important in calculating the number of new originated objects in astronomy, for example, open star clusters, globular star clusters, or associations. We know the average life time of these objects (see Voigt [1] chapter VIII.8 p.355).

At time, t, there are $N(t)$ objects in the volume $V(t)$ with the life time $T(t)$.

Then a new object in V every $\frac{T(t)}{N(t)}$ time unit (for example centuries, millenniums, hours) originates.



With that $\frac{N(t)}{T(t)}$ objects per time unit originates. Now we turn to the number of the new originated objects. We view the time between t_1 and t_2:

$$N(t_1 \leq t \leq t_2) = \int_{t_1}^{t_2} \frac{N(t)}{T(t)}\, dt$$

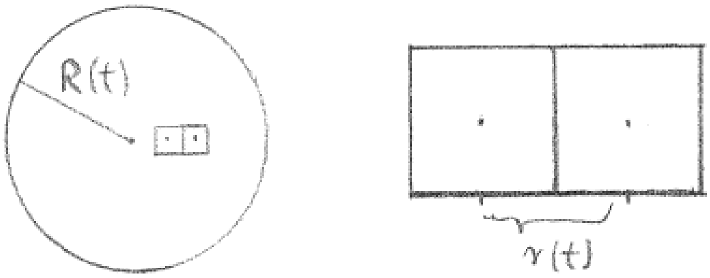
$N(t_1 \leq t \leq t_2)$ = number of objects that originate between t_1 and t_2.

The accuracy of these formulas is greater the larger $N(t)$ is.

2. The average distance

Here, the viewed model is important in physics and astronomy. In physics, for example, the model can be used for particles. In astronomy, typical objects are stars, star clusters, nebulas, galaxies, planetoids, comets, meteors.

$n(t)$ particles are distributed **uniformly** in the volume $V(t)$. t is the time.



$r(t)$ = average distance between two particles = edge length of a cube

We obtain the average distance:

$$r(t) = \left(\frac{V(t)}{n(t)}\right)^{\frac{1}{3}}$$

Now we treat the special case of the ball:

$$V(t) = \frac{4}{3} \cdot \pi R^3(t)$$

We solve for the average distance:

$$r(t) = \left(\frac{4\pi}{3n(t)}\right)^{\frac{1}{3}} \cdot R(t)$$

Now we introduce the particle number density $m(t) = \frac{n(t)}{V(t)}$. It follows:

$$r(t) = \left(\frac{1}{m(t)}\right)^{\frac{1}{3}}$$

The accuracy of these formulas is greater the larger $n(t)$ is.

References

[1] Hans Heinrich Voigt "Abriß der Astronomie" BI Mannheim 1975 2.edition

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