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Abstract

We analyze a Hotelling location-then-price duopoly game under demand uncertainty with uniformly distributed consumers in a standard quadratic costs scenario. The novelty of our approach consists of assuming that firms' beliefs are represented by non-extreme-outcome-additive (neo-additive) capacities. We derive firms' subgame-perfect product design decisions under ambiguity. Furthermore, we investigate the influence of ambiguity and ambiguity attitude on equilibrium product differentiation and contrast our results with an environment of risky firms. We find that the impact of the degree of confidence or ambiguity is particularly significant when it comes to delivering accurate explanations for a wide range of phenomena related to observed product design behavior.

Keywords: Hotelling, Confidence, Optimism, Pessimism, Degree of Ambiguity, Choquet Expected Utility, Neo-additive Capacities, Product Differentiation

JEL classifications: C72, D43, D81, L13, R32

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1 Introduction

Product development is one of the most influential processes for the success of an enterprise, see for instance Brown and Eisenhardt (1995). Firms compete by creating products with new or different characteristics, amongst others, in order to enter new markets, to retain current customers or to attract new purchasers.

A well-known and widely studied model of product differentiation is the location-then-price duopoly game by Hotelling (1929).[1] In his original framework, Hotelling discussed a model with two firms and uniformly distributed consumers along a compact interval facing linear transportation costs. At the first stage of the game, firms choose simultaneously their locations on this interval. At the second stage, firms face price competition. Hotelling's result can be summarized in the so-called "Principle of Minimum Differentiation", implying that both firms' unique subgame-perfect locations are given by the midpoint of the support of consumer interval.

A vast literature deals with a multitude of extensions of Hotelling's model.[2] In particular, D'Aspremont et al. (1979) show that, under linear cost functions, the existence of a subgame-perfect Nash equilibrium (*henceforth SPNE*) is not guaranteed. As a resort to this complication, D'Aspremont et al. (1979) replaced Hotelling's original assumption of linear costs by quadratic costs. In the literature, Hotelling models with quadratic cost functions are frequently denoted by "AGT-models"[3]. Under this assumption, the existence of a unique SPNE is ensured for all parameter constellations. Contrary to Hotelling's original findings, firms' equilibrium product characteristics under quadratic costs are located at the boundaries of the support of the consumer interval. This finding can be subsumed under the so-called "Maximum Differentiation Principle". Hotelling models with diverse types of costs functions are discussed in Anderson (1988).

Another strand of literature focuses on relaxing the assumption of uniform consumer densities thereby questioning the robustness of the principle of minimum or maximum dif-

[1] The "location" in Hotelling's game is typically interpreted as a position in a geographical or product type space. In this paper, we focus in the following on the latter interpretation.

[2] See e.g. Gabszewicz and Thisse (1992) for a survey.

[3] AGT stands for D'Asprémont, Gabszewicz and Thisse

ferentiation. For instance, Anderson et al. (1997) consider general log-concave consumer densities. Neven (1986) finds that firms deviate from the "Maximum Differentiation Principle", by selecting product characteristics which are closer to the center of mass of the respective consumer taste distribution.

Since in most real-world situations, firms are confronted with uncertain consumer preferences, a part of the more recent literature analyzes the impact of demand uncertainty on equilibrium product differentiation. Balvers and Szerb (1996) consider a Hotelling framework incorporating random shocks on the quality of each firm's product, under the assumption that there is no price competition. Harter (1996) considers a Hotelling model with demand location uncertainty, where firms enter the market sequentially. Similar to Harter (1996), Casado-Izaga (2000)[4], Meagher and Zauner (2004) and Meagher and Zauner (2005) discuss extensions of Hotelling's model, where demand uncertainty is introduced by enabling the midpoint of the consumer interval to be probabilistic. Meagher and Zauner (2005) generalize Casado-Izaga (2000) by parametrizing the length of the support. They find that equilibrium differentiation increases in the size of the support. Meagher and Zauner (2004) restrict the support of midpoint of the consumer distribution to compact subsets of the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$, but allow for a broad class of density functions. Again, Meagher and Zauner *(henceforth MZ)* come to the conclusion that uncertainty constitutes a differentiation force, namely an increase in the variance of the underlying probability distribution over the midpoint leads to more pronounced equilibrium product differentiation.

All the contributions above imply that firms' beliefs are represented by a unique common prior. However, in reality, this assumption may be violated for several reasons. First of all, the assumption of a unique common probability distribution for both firms is more restrictive than it may seem to be at first glance, especially in situations, where both firms are ex-ante completely uninformed or incapable to rely on past experiences or observable data. Furthermore, critiques in favor of a unique common probability distribution may

[4]Casado-Izaga (2000) assume that consumers are uniformly distributed over the interval $[\Theta, \Theta + 1]$ where Θ is drawn from a uniform distribution $[0, 1]$. Consequently, the midpoint of the consumer interval follows implicitly a uniform distribution on $[\frac{1}{2}, \frac{3}{2}]$.

argue, that it is possible to apply the "principle of insufficient reason"[5] in case of missing information. However, Ellsberg (1961) indicate in his famous mind experiment that there are situations under ambiguity where a considerable share of individuals display preferences which are incompatible with probabilistic beliefs. Ellsberg's hypothesis is meanwhile confirmed by experimental evidence (Camerer and Weber, 1992). Ambiguity or Knightian Uncertainty refers to situations where probabilities are unknown or imperfectly known. Already Knight (1921), made the distinction between incalculable and calculable risk. Nevertheless, ambiguity is not captured by the classical theories of decision under uncertainty. Expected utility theory which is axiomatized in Von Neumann and Morgenstern (1944) refers to situations with known (objective) probabilities. In subjective expected utility (*henceforth SEU*) theory by Savage (1954), decision-makers' exhibit preferences which are compatible with a subjective probability measure.
By now several decision theoretic models of ambiguity have been developed. Prominent examples are the multiple prior model of Gilboa and Schmeidler (1989), the Choquet expected utility (*henceforth CEU*) model of Schmeidler (1989) and the smooth ambiguity model of Klibanoff et al. (2005).

Although ambiguity is prevalent in many real-world situations, there are almost no Hotelling models incorporating this type of uncertainty. To our knowledge, a recent paper by Król (2012) is the sole contribution on this topic. Król (2012) introduces complete ambiguity[6] into the framework of Meagher and Zauner (2004) and examines, amongst others, the influence of ambiguity attitude on firms' decisions if firms use the Arrow/Hurwicz α-maxmin criterion[7]. Król (2012) finds that ambiguity can be an agglomeration force if firms are sufficiently pessimistic.

The present paper studies the impact of the degree of ambiguity and ambiguity attitude on product differentiation. Inspired by the contributions of MZ and Król (2012),

[5]The "principle of insufficient reason" or "principle of indifference", amongst others, enunciated in the works of Pierre-Simon Laplace (see e.g. Laplace (1812)) states roughly that if we have no information about the occurrence of events and therefore no reason to believe that an event will occur preferentially compared to another, we may consider the events as equally likely.

[6]Complete ambiguity or ignorance refers to a situation where no probabilistic information at all is available.

[7]See Hurwicz (1951) and Arrow and Hurwicz (1972).

we develop a Hotelling model with demand location uncertainty by using the framework of Meagher and Zauner (2004) and Schmeidler's concept of CEU. More specifically, we assume that firms' beliefs are represented by neo-additive capacity introduced by Chateauneuf et al. (2007). Our framework provides additional analytical tools for understanding product differentiation under demand uncertainty. Besides firms' ambiguity attitude, we distinguish four sources of ambiguity and determine their influence on firms' decisions: (i) the variance of firms' prior beliefs, (ii) the degree of ambiguity, (iii) the size of the support of the uncertainty and (iv) the magnitude of the parameter of consumers' quadratic cost functions. In particular, (ii) offers plausible possible explanations for real-life phenomena. In fact, the models of Meagher and Zauner (2004)[8] and Król (2012) are special cases of our model. Hence, our model can be considered as a generalization of both models.

Our paper is organized as follows: In the following section, we describe in detail our model. Afterwards, we derive firms' (pure strategy) subgame-perfect product characteristic choices for the Hotelling game under ambiguity. Thereby, we assume that firms' beliefs are represented by neo-additive capacities. In section 4 of our paper, we carry out a comparative static analysis with respect to all model parameters and study implications for equilibrium product characteristics and Choquet expected profits. Section 5 deals with implications for possible applications of the Hotelling model under demand location uncertainty. In particular, we will reexamine the examples mentioned in Król (2012). Finally, section 6 concludes with a summary and a discussion of our findings.

[8]With a technical restriction. For more details see section 3, especially Remarks 3.1 and 3.3.

2 The Model

2.1 Basic Framework

The framework is based on the modified AGT-model of Meagher and Zauner (2004). There are two firms, $i = 1, 2$, interacting in a two-stage Hotelling duopoly. Each firm produces a homogeneous commodity at constant marginal production costs which are normalized to zero. At the first stage of the game, firms simultaneously select their product characteristics from the real line, $x_i \in \mathbb{R}$, where it is assumed w.l.o.g. (without loss of generality) that $x_1 \leq x_2$. At the second stage of the game, firms face price competition, setting prices, $p_i \in \mathbb{R}_+$, simultaneously.

Furthermore, there is a unit mass of consumers, each consumer being uniquely characterized by a specific taste, $s \in \mathbb{R}$, representing his/her ideal commodity. Consumer tastes are assumed to be uniformly distributed on an interval of the form $[M - \frac{1}{2}, M + \frac{1}{2}]$ where $M \in \mathbb{R}$. Note that in contrast to Hotelling (1929), the support of the consumer taste distribution is a strict subset of the product type space.[9] A customer whose taste is located at s and who consumes product i faces a disutility from not consuming his/her ideal product. The utility loss of the consumer depends on the squared distance between s and the product characteristic x_i, formally $t(s - x_i)^2$, where $t \in \mathbb{R}_{++}$.[10] In addition, the consumer has to pay the price of product i, p_i. Consequently, the consumer's total costs are $p_i + t(s - x_i)^2$. Moreover, we assume that every customer purchases one unit of the homogeneous good from the firm where total costs are lower.

In the certainty model, M and t are a fixed and exogenously given parameters known to both firms throughout the game. In the risk model of Meagher and Zauner (2004), M is unknown to both firms, whereas the scaling parameter t is normalized to 1. In the model of Król (2012) firms face ambiguity with respect to (t, M), resolving ambiguity with the Arrow/Hurwicz α-maxmin criterion. Similar to these models, we presume that

[9] For the purpose of comparability, firms shall face the same choice sets under certainty and uncertainty. Therefore, as in the paper by Anderson et al. (1997), firms may choose product characteristics from the real line.

[10] The parameter t allows for an up- or downward distortion of this quadratic disutility.

the realization of (t, M) denoted by $(\hat{t}, \hat{M})$ is revealed to both firms before the price competition.

Assumption 1. *Uncertainty is resolved at the second stage of the game. Hence, the realization* $(\hat{t}, \hat{M})$ *is revealed to both firms after the product design competition, but before the price game.*

Remark 2.1. Assumption 1 is based on the fact that firms can normally change prices easier than the characteristics of a product (Meagher and Zauner, 2004). For instance, if actual sales volumes differ from estimated sales volumes, a firm can usually adjust the price of the product.

Next, we assume that firms are not necessarily completely uninformed. There can be some probabilistic information which is represented by a common prior q on (t, M). We will refer to q as "reference probability distribution" or "reference prior". Similar to the risk case, we have to make several assumptions concerning the reference probability q.

Assumption 2. *The reference prior* q *on* (t, M) *satisfies the subsequent requirements:*

(R1) Expectation and variance of M *exist:* $\mathbb{E}_q[M] < \infty$ *and* $\mathbb{E}_q|M^2| < \infty$.

(R2) The expectation of M *is normalized to zero:* $\mathbb{E}_q[M] = 0$.

(R3) The distribution of M *has no atoms.*

(R4) The support of M *is the symmetric interval* $[-L, L] \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]$.

(R5) The support of t *is the interval* $[\underline{t}, 1]$ *where* $\underline{t} \in (0, 1]$.

(R6) The expectation of t *is normalized to* 1*:* $\mathbb{E}_q[t] = 1$.

(R7) The random variables t *and* M *are uncorrelated.*

where we have slightly abused notation denoting by $\mathbb{E}_q$ *an expectation w.r.t. (with respect to) the prior* q.

The random variable M enters quadratically into each firm's objective function for the first stage of the game (see equation (3.3) and Lemma 3.4). Therefore, firms' product characteristic choices are going to depend solely on the first and second moment of the distribution of M. Consequently, (R1) guarantees the existence of best response functions.

Moreover, taking (R1) and (R4) together, we can formulate the following lemma which is a very useful tool for the mathematical proofs of the comparative statics section.

Lemma 2.1. *The requirements* (R_1) *and* (R_4) *imply*

$$\mathbb{E}_q[M] \in [-L, L] \qquad and \qquad Var_q(M) \in \left[0, L^2\right]$$

Proof. The proof of the lemma is contained in the appendix.

The requirements (R2) and (R6) are due to reasons of symmetry and tractability. Requirement (R3) is purely technical in nature and could be replaced in order to allow for discrete or mixtures with discrete distributions. (R4) makes sure that the support of M is a compact subset of the interval $[-\frac{1}{2}, \frac{1}{2}]$, restricting the size of uncertainty to be relatively small. In addition, it assures that the extreme intervals for possible realizations of the consumer distribution $[-L - \frac{1}{2}, -L + \frac{1}{2}]$ and $[L - \frac{1}{2}, L + \frac{1}{2}]$ always have a non-empty intersection, and therefore overlap. Furthermore, (R4) and (R5) imply that the support of q is a subset of $[\underline{t}, 1] \times [-L, L]$. Lastly, (R7) ensures that we can disentangle the random variables t and M.

2.2 Introducing Ambiguity into the Game

In the following, we quickly discuss the three prominent decision theoretic models of ambiguity mentioned in the introduction, and illustrate why we make use of the CEU approach.

The multiple prior model (MP) was pioneered by Gilboa and Schmeidler (1989). The key idea of this approach is that, in choice situations featuring ambiguity, individuals dispose of insufficient information to form a unique prior. Instead of that, individual beliefs are captured by a set coexisting priors. Multiple prior decision rules are related to non-probabilistic classical decision rules in economics. Gilboa and Schmeidler (1989) axiomatize minimum expected utility (*henceforth MEU*) which corresponds to the decision rule by Wald (1950) (maxmin criterion). A decision-maker with MEU preferences maximizes the expected value of and underlying objective function with respect to the worst-case scenario, i.e. the prior which yields the lowest expected value. In contrast to that, a decision-maker utilizing the optimistic maxmax criterion, bases his expected value on the best-case scenario. A generalization of the maxmin functional which takes both criteria into account is the α-MEU model. This model is related to the Arrow/Hurwicz α-maxmin criterion mentioned in the introduction.[11] A decision-maker exhibiting α-MEU preferences maximizes a convex combination of the lowest and the highest expected value regarding an underlying objective. The parameter of the convex combination, α, can be interpreted as the ambiguity attitude of the decision-maker. Ghirardato et al. (2004) (*henceforth GMM*) provide an axiomatization of α-MEU preferences. However, Eichberger et al. (2011) show that in case of finite state spaces any α-MEU preferences which satisfy GMM's axioms must be either maxmin or maxmax.

The second approach we want to mention is the smooth ambiguity model by Klibanoff et al. (2005). In this setting, uncertainty of an individual with respect to first-order probabilities is captured by a (subjective) second-order probability distribution. Ambiguity attitude is modeled by limiting the reduction of compound lotteries with regard to first and second order probabilities.

The third prominent approach known as CEU theory was introduced and axiomatized by Schmeidler (1989). In this model, decision-makers' beliefs are represented by nonadditive probabilities (or capacities). A capacity is a normalized and monotonic set function.

Definition 2.1 (compare Schmeidler (1989))**.** *Let Ω be a finite or infinite non-empty set of states of the world and let Σ be an algebra of events defined on it. A capacity is a real-valued function $\nu : \Sigma \to \mathbb{R}$ such that*

(1) $\nu(\varnothing) = 0$ and $\nu(\Omega) = 1$ (normalization)

(2) $E, F \in \Sigma$ and $E \subseteq F$ implies $\nu(E) \leq \nu(F)$ (monotonicity).

Hence, a capacity can be seen as a generalization of a probability measure which not necessarily satisfies the property of σ-additivity. The expectation w.r.t. a nonadditive measure is formed by using a Choquet integral (see Choquet (1955)). The CEU theory

[11]Król (2012) uses basically the multiple prior approach by assuming that firms' common prior set is the set of all possible probability distributions on a given state space.

has a solid axiomatic foundation. Moreover, it is more general than the multiple prior approach due to the wide spectrum of possible capacities. However, for a certain class of capacities, CEU is equivalent to MEU and α-MEU respectively (see e.g. Chateauneuf et al. (2007)). In this case, both approaches yield the same decision rule. In the present paper, ambiguity is modelled with neo-additive capacities which are axiomatized in Chateauneuf et al. (2007).

Assumption 3. *Each firm's belief is represented by a neo-additive capacity as defined in Definition 2.2.*

The neo-additive capacity represents firms' ex-ante uncertainty of (t, M). We refer to the definition of a neo-additive capacity from Eichberger et al. (2009), page 359:

Definition 2.2. *Let q be a probability distribution on $\Omega = [\underline{t}, 1] \times [-L, L]$ satisfying Assumption 2. Then, for real numbers α and δ define a neo-additive capacity ν by $\nu(\varnothing) = 0$, $\nu(\Omega) = 1$, $\nu(A) = \delta\alpha + (1-\delta)q(A)$, where $A \in \Sigma$ is a nonempty and strict subset of Ω.*

Remark 2.2. From our point of view, neo-additive capacities display several nice features. The parameter δ can be interpreted as a measure of ambiguity or of firms' confidence in the common reference prior q. Hence, one can interpret our model as a setting where firms exhibit uncertainty with respect to their prior beliefs due to imprecise or doubtful information.

Moreover, the parameter α describes firms' attitude towards ambiguity. The higher α, the more pessimistic firm managers are. Consequently, neo-additive capacities allow for a clear separation between the degree of ambiguity and firms' ambiguity attitude which is, as we want to argue in this paper, essential for many economic applications. Furthermore, it is worthwhile to mention some special cases of neo-additive capacities which are relevant in this paper. For $\delta = 0$, one obtains SEU of Savage (1954). For $\delta = 1$ and $\alpha \in [0, 1]$ one obtains α-MEU including the special case MEU for $\alpha = 1$.

The rationale speaking for the introduction of neo-additive capacities lies in in the fact that firms might not completely trust the information available to them at the time of making their product choice. There are multiple reasons why this might be the case, e.g.

firms introducing newly innovated products into the market could have data on similar products that are already established in the market but have no data on the new good. It seems plausible that firms use this data to predict the market outcome, still firms cannot account for short-term trends in consumer tastes. Furthermore, data reliability is closely tied to the comparability of the reference product with the newly innovated product. The more heterogeneous both products are, the less plausible it seems to rely on available data of the reference product. In contrast to an α-MEU approach, neo-additive capacities allow for a model of partial information, where firms have a certain stock of data available whose reliability might be questionable up to a certain degree. Interpreted in this way, the α-MEU approach refers to a situation where firms have ex-ante no information about the distribution of consumer tastes or completely distrust information available at the time of making their product design choices. Moreover, neo-additive capacities allow for an additional interpretative component for a multitude of possible applications of the Hotelling model by adding an additional explanatory source for increasing or decreasing product differentiation under ambiguity.

3 Solving the Game

In this section, we determine equilibrium product differentiation under ambiguity by backward induction. Consequently, in a first step, we solve the price subgame at the second stage where the midpoint M of the consumer distribution and the cost parameter t are fix and known to both firms. By using equilibrium prices, we afterwards identify firms' optimal product design decisions at the first stage of the game.

3.1 The Price Subgame

According to Assumption 1, uncertainty is resolved at the second stage of the game, i.e. the realization $(\hat{t}, \hat{M})$ is known to both firms.

If firms do not differentiate their products, $x_1 = x_2$, they face standard Bertrand competition. At the Bertrand equilibrium, each firm charges a price of zero.

Otherwise, if $x_1 < x_2$, firms' price reaction functions depend parametrically on the distance of the midpoint of firms' product characteristics, $\bar{x} := \frac{x_1+x_2}{2}$, to the midpoint of the support of the consumer taste distribution, $\hat{M}$. If the distance is sufficient small, there exists a consumer who is indifferent between buying from firm 1 or firm 2, $p_1+\hat{t}(x_1-\xi)^2 = p_2+\hat{t}(x_2-\xi)^2$. Hence, the taste of the indifferent consumer, $\xi \in [\hat{M}-\frac{1}{2}, \hat{M}+\frac{1}{2}]$, is uniquely determined by

$$\xi(x_1, x_2, p_1, p_2) = \bar{x} + \frac{p_2 - p_1}{2\hat{t}\Delta_x} \tag{3.1}$$

where $\Delta_x = x_2 - x_1 > 0$ denotes firms' product differentiation.

All consumers whose tastes are located to the left of ξ purchase product 1. Therefore, the aggregate demand for the products are $D_1(\xi) = \xi - \hat{M} + \frac{1}{2}$ and $D_2(\xi) = 1 - D_1(\xi)$. Hence, one of the firms covers the whole market if the indifferent consumer is located at one of the boundaries of the support of the consumer taste distribution, $[\hat{M} - \frac{1}{2}, \hat{M} + \frac{1}{2}]$. Since production costs are normalized to zero, firm i's profit function is

$$\pi_i(p_i, \xi) = p_i D_i(\xi) \tag{3.2}$$

Solving the first-order condition of equation (3.2) yields the following price equilibrium.

Lemma 3.1 (Interior price equilibrium)**.** *If $x_1 < x_2$ and $(\hat{M} - \bar{x}) \in [-\frac{3}{2}, \frac{3}{2}]$, then firms charge the subsequent equilibrium prices*

$$p_1^* = \frac{2}{3}\hat{t}\Delta_x \left(\bar{x} - \hat{M} + \frac{3}{2}\right) \quad \textit{and} \quad p_2^* = \frac{2}{3}\hat{t}\Delta_x \left(-\bar{x} + \hat{M} + \frac{3}{2}\right)$$

Proof. The proof of the lemma is contained in the appendix. Compare also Anderson et al. (1997), page 107 and Meagher and Zauner (2004), page 203.

At the interior price equilibrium, the taste of the indifferent consumer is located at $\overline{\xi} := \xi(x_1, x_2, p_1^*, p_2^*) = \frac{2\hat{M}+\bar{x}}{3}$. However, the indifferent consumer exists only if his/her taste is contained in the consumer interval, i.e. if $(\hat{M} - \bar{x}) \in [-\frac{3}{2}, \frac{3}{2}]$. Otherwise, the product characteristic of one firm is so far away from consumer tastes that the other firm can guarantee to obtain the whole market demand (Anderson et al., 1997). We need to consider two cases: If we have $(\hat{M} - \bar{x}) < -\frac{3}{2}$, firm 1's product characteristic is closer to any consumer taste in $[\hat{M} - \frac{1}{2}, \hat{M} + \frac{1}{2}]$ than the product characteristic of firm 2. Therefore, firm 1 will charge a price subject to (3.1) such that it can get the whole consumer demand. If so, firm 2 will get a profit of zero, no matter what price it may charge. Similarly, for $(\hat{M} - \bar{x}) > \frac{3}{2}$, firm 2 will charge a price subject to (3.1), such that the demand for product 1 is zero, no matter what price firm 1 may charge. In total, we obtain two boundary price equilibria given by Lemma (3.2) below. In each of these equilibria one of the two firms becomes a monopolist.

Lemma 3.2 (Boundary price equilibria). *If $x_1 < x_2$ and $(\hat{M} - \bar{x}) \notin [-\frac{3}{2}, \frac{3}{2}]$, then firms charge the subsequent equilibrium prices*

$$p_1^* = 2\hat{t}\Delta_x \left(\bar{x} - \hat{M} - \frac{1}{2}\right) \quad \text{and} \quad p_2^* = 0 \quad \text{if} \quad (\hat{M} - \bar{x}) < -\frac{3}{2}$$

and

$$p_2^* = 2\hat{t}\Delta_x \left(\hat{M} - \bar{x} - \frac{1}{2}\right) \quad \text{and} \quad p_1^* = 0 \quad \text{if} \quad (\hat{M} - \bar{x}) > \frac{3}{2}.$$

Proof. The proof of the lemma is contained in the appendix. Compare also Anderson et al. (1997), page 107 and Meagher and Zauner (2004), page 203.

3.2 Product Design Competition

As shown in the previous section, the equilibrium for the price subgame is unique for a fixed pair of product characteristics, (x_1, x_2). By using equilibrium prices from Lemma 3.1 and 3.2, we obtain firms' second stage profits, $\Pi_i(x_i, x_j, \hat{t}, \hat{M}) := \pi_i(p_i^*, \xi(x_i, x_j, p_i^*, p_j^*))$[12], at the realization $(\hat{t}, \hat{M})$ depending on firms' product characteristics:

$$\Pi_i(x_i, x_j, \hat{t}, \hat{M}) = \begin{cases} \hat{t}\,\Delta_x \left[1 + 2\,(-1)^i (\hat{M} - \bar{x})\right] & \text{for} \quad (-1)^i\,(\hat{M} - \bar{x}) > \frac{3}{2} \\ \hat{t}\,\Delta_x \left[3(-1)^i + 2(\hat{M} - \bar{x})\right]^2 /18 & \text{for} \quad (\hat{M} - \bar{x}) \in [-\frac{3}{2}, \frac{3}{2}] \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

[12]Note that the profit functions are well-behaved in the sense that they are compatible with the case that firms do not differentiate: $\lim\limits_{x_i \to x_j} \Pi_i(x_i, x_j, \hat{t}, \hat{M}) = 0$.

where $\bar{x} := \frac{x_1+x_2}{2}$ and $\Delta_x := x_2 - x_1$ as before.

In the following, we want to elaborate on each firm's objective function at the first stage of the game. In order to do so, we make use of the fact that the second piece of (3.3) is monotonic in $(\hat{t}, \hat{M})$ as specified in Lemma 3.3 below.

Lemma 3.3. *If* $(\hat{M} - \bar{x}) \in [-\frac{3}{2}, \frac{3}{2}]$*, then firm* i*'s profit function* $\Pi_i(x_1, x_2, \hat{t}, \hat{M})$ *is strictly increasing in* $\hat{t}$*, strictly decreasing in* $\hat{M}$ *for* $i = 1$ *and strictly increasing for* $i = 2$ *provided that* $x_1 < x_2$*.*

Proof. The proof of the lemma is contained in the appendix.

Now, turning to the first stage of the game, according to the assumptions 1 and 2, the actual realization $(\hat{t}, \hat{M})$ of the random variable (t, M) is unknown. Furthermore, firms might not have sufficient probabilistic information. Therefore, ex-ante, firms consider the Choquet expected value of (3.3), $\mathbb{CEU}[\Pi_i(x_i, x_j, \hat{t}, \hat{M})]$, w.r.t. a neo-additive capacity in accordance with Assumption 3 and Definition 2.2. Following Lemma 3.1 in Chateauneuf et al. (2007), page 541, we obtain the following representation of firm i's Choquet expected profit at the first stage of the game:

$$\begin{aligned}\mathbb{CEU}[\Pi_i(x_1, x_2, t, M)] = \int \Pi_i(x_i, x_j, \hat{t}, \hat{M}) d\nu &= (1-\delta)\mathbb{E}_q[\Pi_i(x_i, x_j, t, M)] \\ &+ \delta[(1-\alpha)\max\{\Pi_i(x_i, x_j, \hat{t}, \hat{M}) : (\hat{t}, \hat{M}) \in \operatorname{supp}(t, M)\} \\ &+ \alpha \min\{\Pi_i(x_i, x_j, \hat{t}, \hat{M}) : (\hat{t}, \hat{M}) \in \operatorname{supp}(t, M)\}]\end{aligned} \tag{3.4}$$

Remark 3.1. These Choquet expected profits allow for a nice interpretation, namely that they generalize Hotelling models which have been treated in the literature before. For $\delta = 0$ and a constant scaling factor $t = 1$, we obtain the model of Meagher and Zauner (2004) with a normalized mean of M. For $\delta = 1$, we obtain the α-MEU model of Król (2012). Thus, we can consider these cases as extreme cases of the capacity model. For $0 < \delta < 1$, we obtain a convex combination of the risk case and the α-MEU model.

As a next step, we consider the α-MEU part of equation (3.4). Making use of Lemma 3.3, we obtain for $(\hat{M}-\bar{x}) \in [-\frac{3}{2}, \frac{3}{2}]$ the following explicit functional relationships

$$\begin{aligned}
\max\{\Pi_1(x_i, x_j, \hat{t}, \hat{M}) : (\hat{t}, \hat{M}) \in \operatorname{supp}(t, M)\} &= \Pi_1(x_1, x_2, 1, -L) \\
\min\{\Pi_1(x_i, x_j, \hat{t}, \hat{M}) : (\hat{t}, \hat{M}) \in \operatorname{supp}(t, M)\}] &= \Pi_1(x_1, x_2, \underline{t}, L) \\
\max\{\Pi_2(x_i, x_j, \hat{t}, \hat{M}) : (\hat{t}, \hat{M}) \in \operatorname{supp}(t, M)\} &= \Pi_2(x_1, x_2, 1, L) \\
\min\{\Pi_2(x_i, x_j, \hat{t}, \hat{M}) : (\hat{t}, \hat{M}) \in \operatorname{supp}(t, M)\}] &= \Pi_2(x_1, x_2, \underline{t}, -L)
\end{aligned} \tag{3.5}$$

Remark 3.2. One can interpret this result as follows. Firm 1's best-case scenario is when the midpoint of the consumer interval takes as realization the lower support boundary $-L$. This is true, since we assumed w.l.o.g. that firm 1 is the firm whose product characteristic is left of firm 2's product characteristic. Therefore, it is more convenient for firm 1 if the consumer distribution is located closer to its own product characteristic. Similarly, firm 1's worst-case scenario is when the midpoint of the consumer interval takes as realization the upper support boundary L. For firm 2 the reverse result holds: Realizations of M which are more to the left are detrimental and realizations to the right are beneficial. The best possible case is if $\hat{M} = L$ and the worst possible case materializes if $\hat{M} = -L$. Concerning the cost parameter, best- and worst-cases are the same for both firms. The best-case arises for a high cost parameter $t = 1$, the worst-case arises for $t = \underline{t}$.

The first term of firm i's Choquet expected profit equals the (usual) expectation of its profit function with respect to the reference prior, $\mathbb{E}_q[\Pi_i(x_1, x_1, t, M)]$. In order to elaborate on this part, we need the following Lemma, which can be considered as an analogue to the global competition lemma in Meagher and Zauner (2004).[13]

Lemma 3.4 (Global competition). *Under assumptions 1,2 and 3, at any pure strategy SPNE for the Hotelling game with ambiguous demand location uncertainty, the support* $[-L, L]$ *of* M *is contained in* $[\bar{x}-\frac{3}{2}, \bar{x}+\frac{3}{2}]$, *formally* $[-L, L] \subset \left[\bar{x}-\frac{3}{2}, \bar{x}+\frac{3}{2}\right]$. *An equivalent*

[13]For the Hotelling model under certainty, Anderson et al. (1997) point out a similar property in footnote 8.

and more intuitive formulation of this finding is given by

$$L - \frac{3}{2} < \bar{x} < -L + \frac{3}{2} \tag{3.6}$$

Proof. The proof of the lemma is contained in the appendix.

Lemma 3.4 proves very useful when it comes to determining firms' subgame-perfect product characteristic choices. In fact, due to Lemma 3.2, one could expect, that there are equilibria where for some realizations of uncertainty, one or the other firm can monopolize the market. However, according to the global competition lemma, firm i's objective function at the first stage of the game is given by the Choquet expected value of the second piece of (3.3).

Furthermore, the global competition lemma implies that $\mathbb{E}_q[\Pi_i(x_i, x_j, t, M)]$ depends solely on the entries of the mean vector $\mathbb{E}_q[(t, M)] = (\mu_t, \mu_M)$ as well as the entries of the variance-covariance matrix

$$Cov_q(t, M) = \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_M^2 \end{pmatrix}.$$

The following lemma gives $\mathbb{E}_q[\Pi_i(x_i, x_j, t, M)]$ in an explicit mathematical form.

Lemma 3.5. *If $x_1 \leq x_2$ w.l.o.g., then, under assumptions 1,2 and 3, at any pure strategy SPNE for the Hotelling game under uncertainty, firms choose product characteristics, (x_1^*, x_2^*), such that firm i's expected profit w.r.t. the reference prior q is*

$$\begin{aligned}\mathbb{E}_q[\Pi_i(x_i^*, x_j^*, t, M)] &= \mu_t \int_{-L}^{L} (-1)^j \, \frac{2}{9} \, (x_j^* - x_i^*) \, \left(\bar{x}^* - \left(M + \frac{3}{2}(-1)^i \right) \right)^2 f(M) dM \\ &= \frac{(-1)^j}{18} \, \mu_t \, (x_j^* - x_i^*) \Big\{ (2\bar{x}^* - 3(-1)^i)^2 \\ &\quad - 4\mu_M (2\bar{x}^* - 3(-1)^i) + 4(\mu_M + \sigma_M^2) \Big\}\end{aligned} \tag{3.7}$$

where $\bar{x}^* = x_i^* + x_j^*$.

Proof. The proof of the lemma is contained in the appendix.

Now, that we specified firms' objective functions for the first stage of the game, we can derive firms' subgame-perfect product characteristics. Firm 1's best reply, $R_1^*(\hat{x}_2)$, given the product characteristic choice of firm 2, $\hat{x}_2$, is

$$R_1^*(\hat{x}_2) := \arg\max_{x_1 \in \mathbb{R}} \Big\{ (1-\delta)\mathbb{E}_q[\Pi_1(x_1, \hat{x}_2, t, M)] + \delta\big[(1-\alpha)\Pi_1(x_1, \hat{x}_2, 1, -L) + \alpha\Pi_1(x_1, \hat{x}_2, \underline{t}, L)\big] \Big\}$$

and firm 2's best reply, $R_2^*(\hat{x}_1)$, given the product choice of firm 1, $\hat{x}_1$, is

$$R_2^*(\hat{x}_1) := \arg\max_{x_2 \in \mathbb{R}} \Big\{ (1-\delta)\mathbb{E}_q[\Pi_2(\hat{x}_1, x_2, t, M)] + \delta\big[(1-\alpha)\Pi_2(\hat{x}_1, x_2, 1, L) + \alpha\Pi_2(\hat{x}_1, x_2, \underline{t}, -L)\big] \Big\}.$$

Hence, the following system of equations describes firms' mutual best replies:

$$R_1^*(x_2^*) = x_1^* \qquad \text{and} \qquad R_2^*(x_1^*) = x_2^* \tag{3.8}$$

By solving (3.8), one obtain firms' subgame-perfect product characteristic choices as specified in Proposition 3.1.

Proposition 3.1 (Equilibrium differentiation). *Under Assumptions 1,2 and 3, there is a unique pure strategy SPNE for the Hotelling game under ambiguity. The equilibrium differentiation, $\Delta_x^* := x_2^* - x_1^*$, is*

$$\Delta_x^* = \frac{4\,\delta\,(\alpha\,\underline{t} + (1-\alpha))\,L^2 + 12\,\delta\,((1-\alpha) - \alpha\,\underline{t})\,L + (4 - 4\,\delta)\,\sigma^2 + 9\,(\alpha\,\delta\,\underline{t} - \alpha\,\delta + 1)}{4\,\delta\,(\alpha\,\underline{t} + (\alpha - 1))\,L - 6\,(\alpha\,\delta\,\underline{t} + \alpha\,\delta - 1)}$$

and firms' Choquet expected equilibrium profits are

$$\mathbb{CEU}[\Pi_i^*] = \frac{\left(4\,\delta\,L^2\,(\alpha\,\underline{t} - \alpha + 1) - 12\,\delta\,L\,(\alpha\,\underline{t} - \alpha + 1) + 4\,\sigma^2\,(1-\delta) + 9\,(\alpha\,\delta\,\underline{t} - \alpha\,\delta + 1)\right)^2}{36\,\left(2\,\delta\,L\,(-\alpha\,\underline{t} - \alpha + 1) + 3\,(\alpha\,\delta\,\underline{t} - \alpha\,\delta + 1)\right)}$$

where $\Pi_i^* := \Pi_i(x_1^*, x_2^*, t, M)$.

Proof. Firms' equilibrium product characteristics and the proof of the proposition is contained in the appendix.

Remark 3.3. It is worthwhile to highlight and discuss some special cases of this equilibrium. Setting $\delta = 1$, which corresponds to a situation under complete ambiguity (or without any confidence into the reference prior q), one obtain the equilibrium of Król (2012) in its full generality. Setting $\delta = 0$, we obtain the equilibrium in Meagher and Zauner (2004) with the slight difference, that we allow explicitly for a normalized probability which may not necessarily be objectively given and is therefore subjective in nature. The normalization $\mathbb{E}_q[M] = 0$ ensures symmetry and is, in our view, not a strong restriction. We can interpret this assumption in the following way: Both firms determine the expected midpoint of the consumer interval and align all possible product designs symmetrically around this mean. If the mean is nonzero, firms can transform the set of all product characteristics to be centered around zero. After determining their product characteristic choices in the normalized setting, firms may retransform their product characteristic decision into the non-normalized product space and obtain the optimal product design. For consumer distributions with nonzero mean, there are no solutions in closed-form for firms' subgame-perfect product characteristic choices. Nevertheless, it is plausible to argue, that both firms will shift their subgame-perfect locations into the direction of this mean.

More special cases arise, when firms are purely pessimistic $\alpha = 1$ or purely optimistic $\alpha = 0$. The following corollaries describes the pure strategy SPNE under global pessimism and optimism respectively.

Corollary 3.1 (Equilibrium differentiation with global pessimism)**.** *If firms are purely pessimistic, $\alpha = 1$, then the equilibrium differentiation is*

$$\Delta_x^* = \frac{4\,\delta\,L^2 + 12\,\delta\,L + (4 - 4\,\delta)\;\sigma^2 + 9}{4\,\delta\,L + 6}$$

and firms' Choquet expected equilibrium profits are

$$\mathbb{CEU}[\Pi_i^*] = \frac{\left(4\,\delta\,L^2 + 12\,\delta\,L - 4\,\delta\,\sigma^2 + 4\,\sigma^2 + 9\right)^2}{36\;(2\,\delta\,L + 3)}$$

Proof. Consider proposition 3.1 for $\alpha = 1$. $\square$

Corollary 3.2 (Equilibrium differentiation with global optimism)**.** *If firms are purely optimistic, $\alpha = 0$, then the equilibrium differentiation is*

$$\Delta_x^* = -\frac{4\,\delta\,\underline{t}\,L^2 - 12\,\delta\,\underline{t}\,L + 9\,\delta\,\underline{t} + (4 - 4\,\delta)\;\sigma^2 - 9\,\delta + 9}{4\,\delta\,\underline{t}\,L - 6\,\delta\,\underline{t} + 6\,\delta - 6}$$

and firms' Choquet expected equilibrium profits are

$$\mathbb{CEU}[\Pi_i] = \frac{\left(4\,\delta\,\underline{t}\,L^2 - 12\,\delta\,\underline{t}\,L + 9\,\delta\,\underline{t} - 4\,\delta\,\sigma^2 + 4\,\sigma^2 - 9\,\delta + 9\right)^2}{36\;\left(-2\,\delta\,\underline{t}\,L + 3\,\delta\,\underline{t} - 3\,\delta + 3\right)}$$

Proof. Consider proposition 3.1 for $\alpha = 0$. □

4 Comparative Statics

The aforementioned Hotelling model under ambiguity yields interesting comparative static results. In this section, we are going to discuss and interpret the basic properties of firms' product characteristic choices w.r.t. changes in the model parameters. Similar to Król (2012), the following proposition examines c.p. *(ceteris paribus)* variations in the global ambiguity attitude α. Afterwards, in Proposition 4.2 - Proposition 4.3 of this paragraph, we want to investigate the influence of four different sources of ambiguity on equilibrium product differentiation. Finally, we conclude this section with a discussion of the meaning of the degree of ambiguity and the support of the uncertainty.

Proposition 4.1 (Variation in firms' ambiguity attitude α)**.** *Under Assumption 1,2 and 3, at any SPNE for the Hotelling game under ambiguity, an increase in the degree of pessimism, α, leads to the following effects:*

$$\frac{\partial x_1^*}{\partial \alpha} \geq 0 \qquad \textit{and} \qquad \frac{\partial x_2^*}{\partial \alpha} \leq 0$$

Proof. The proof of the proposition is contained in the appendix.

Remark 4.1. Firms' equilibrium differentiation, Δ_x^*, is nonincreasing in α and strictly decreasing in α for $\delta > 0$. In case of full confidence (or without ambiguity), $\delta = 0$, equilibrium product characteristics are independent of α. In this case, firms will not deviate from their initial product characteristics and equilibrium differentiation remains unchanged.

The results of Proposition 4.1 are related to the findings in Król (2012) stating that a higher degree of pessimism leads to less product differentiation. This finding extends to our model, with the difference that the magnitude of the effect is weakened the more confidence firms have in the reference prior q. In case of full confidence (or without ambiguity), firms' attitude towards ambiguity becomes irrelevant for their product differentiation choices. The intuition about this result is the following: For a high degree of pessimism α, each firm puts a larger weight on the maxmin criterion than on the max-max criterion. Therefore, the worst-case scenario becomes increasingly important. The worst-case of firm 1 is that the expectation of M is equal to L. As the expectation moves to the right, and firm 1 considers this expectation as relevant, firm 1 has an incentive to choose a product characteristic right to its initial characteristics. Similarly, for firm 2, the worst-case scenario corresponds to left boundary of the support $-L$. Since firm 2 puts increasingly more weight to this worst-case, firm 2 has an incentive to move to the left. All in all, equilibrium differentiation decreases.
To sum up, contrary to the risk models of MZ (Meagher and Zauner (2004) and Meagher and Zauner (2005)), ambiguity is not per se a differentiation force. What matters is ambiguity attitude of both firms. We call this attitude the degree of global optimism or pessimism, since we consider a market where both firms exhibit the same ambiguity attitude. Hence, attitude towards ambiguity becomes a global characteristic of the market and could be as well interpreted as 'market sentiment'.

Next, starting with the variance of the reference prior σ^2, we examine c.p. changes in different sources of ambiguity.

Proposition 4.2 (Variation in the variance σ^2). *If $0 \leq \delta < 1$, then, under Assumptions 1,2 and 3, at any SPNE for the Hotelling game under ambiguity, we have:*

$$\frac{\partial x_1^*}{\partial \sigma^2} < 0 \qquad \textit{and} \qquad \frac{\partial x_2^*}{\partial \sigma^2} > 0$$

Proof. The proof of the proposition is contained in the appendix.

Remark 4.2. Firm 1 (resp. firm 2) chooses a product characteristic left (resp. right) from its initial characteristic as the variance of M increases. Consequently, equilibrium differentiation, Δ_x^*, increases in σ^2. However, for the case of ignorance (complete ambiguity or complete distrust into the reference prior), $\delta = 1$, equilibrium product characteristics remain unchanged.

This result is in line with the result for the risk case of Meagher and Zauner (2004) where a higher variance of the underlying distribution for the midpoint of the consumer interval leads to higher competitive differentiation. Uncertainty as measured by the variance of the underlying distribution constitutes a differentiation force. According to Meagher and Zauner (2004) the intuition here is that, in the Hotelling game, firms are confronted with two effects. If a firm chooses a product characteristic more away from that of its competitor, at given prices, it looses market share (*demand effect*). At the same time, however, more product differentiation weakens price competition and leads to higher equilibrium prices (*price effect*). Due to quadratic cost functions, the price effect dominates the demand effect. If firms face uncertainty with respect to the distribution of the consumer interval, the negative effect of loosing market share in some realizations of uncertainty is not so dramatic as in the certainty case, since there are some other realizations of the random variable M, where the firm is better located with respect to the location of the consumer interval than before. Consequently, an increasing variance of the underlying probability distribution strengthens the dominance of the price effect. Therefore, equilibrium differentiation is even more excessive than under certainty. Of course, the same interpretation applies for the capacity model as long as $0 \leq \delta < 1$ with the sole difference that the effect of a c.p. increase in σ^2 is weaker the less confident firms are in the reference prior q.

The following proposition examines c.p. variations in the magnitude of quadratic transportation costs.

Proposition 4.3 (Variation in the magnitude of the cost parameter $\underline{t}$)**.** *If* $0 < \alpha \leq 1$ *and* $0 < \delta \leq 1$*, then, under Assumption 1,2 and 3, at any SPNE for the Hotelling game under ambiguity, we have*

$$\frac{\partial x_1^*}{\partial \underline{t}} > 0 \qquad \textit{and} \qquad \frac{\partial x_2^*}{\partial \underline{t}} < 0$$

Proof. The proof of the proposition is contained in the appendix.

Remark 4.3. In case of full pessimism or full confidence firms do not relocate. As a consequence firms' equilibrium differentiation will decrease for $0 < \alpha \leq 1$ and $0 < \delta \leq 1$ and remain unchanged for $\delta = 0$ or $\alpha = 0$.

The result of Proposition 4.3 is quite similar to the statement in Król (2012). Comparing the Hotelling model with a standard symmetric Bertrand competition, we can observe the following important difference. In the standard Bertrand scenario, firms offer homogeneous products. The only Nash equilibrium in pure strategies is that firms set prices equal to marginal costs, implying zero profits for both firms. In a Bertrand world with heterogeneous products this finding is no longer true. By introducing transportation costs, the Hotelling framework adds an additional product characteristic to a homogeneous and symmetric Bertrand competition, thereby rendering products per se more heterogeneous. It is therefore intuitive that a higher transportation cost weakens competition between firms. In the Hotelling model there are two countervailing incentives at work that determine firms' product design choices. One is that firms want to locate in the center of the Hotelling interval in order to obtain a higher market share. The second is that firms want to differentiate their products more, in order to weaken price competition. Now, if price competition is weakened by a higher transportation cost, it is plausible that firms have an incentive to reduce product differentiation in order to obtain a higher market share.

The subsequent proposition explores a c.p. increase in firms' confidence level δ.

Proposition 4.4 (Variation in the confidence level δ)**.** *Under Assumption 1,2 and 3, at any SPNE for the Hotelling game under ambiguity, we have*

$$\frac{\partial x_1^*}{\partial \delta} = \begin{cases} < 0 & \textit{for} \quad 0 \leq \alpha < \alpha^* \\ = 0 & \textit{for} \quad \alpha = \alpha^* \\ > 0 & \textit{for} \quad 1 \geq \alpha > \alpha^* \end{cases}$$

where $\alpha^* \in [0,1]$ is a cutoff-value with $\alpha^* = \alpha^*(\delta, \underline{t}, \sigma^2, L)$. Similarly we obtain for x_2^*

$$\frac{\partial x_2^*}{\partial \delta} = \begin{cases} > 0 & \textit{for} \quad 0 \leq \alpha < \alpha^* \\ = 0 & \textit{for} \quad \alpha = \alpha^* \\ < 0 & \textit{for} \quad 1 \geq \alpha > \alpha^* \end{cases}$$

for the same cutoff-value α^*. Taking these results together we obtain for Δ^*

$$\frac{\partial \Delta^*}{\partial \delta} = \begin{cases} > 0 & \textit{for} \quad 0 \leq \alpha < \alpha^* \\ = 0 & \textit{for} \quad \alpha = \alpha^* \\ < 0 & \textit{for} \quad 1 \geq \alpha > \alpha^* \end{cases}$$

Proof. The proof of the proposition is contained in the appendix.

The finding in Proposition 4.4 can be summarized in the following way: If each firm's attitude to ambiguity is sufficiently optimistic, a lower confidence into the reference prior will increase equilibrium differentiation. For sufficiently pessimistic firms a lower confidence into the reference prior will decrease equilibrium differentiation. Furthermore, there is an intermediate value of global pessimism α^*, depending on the size of the support L and the cost parameter $\underline{t}$, such that firms' equilibrium differentiation will remain unchanged no matter which global confidence level firms might assign to the reference probability distribution for the midpoint M.

Proposition 4.5 (Variation in the size of the support L). *If $0 < \delta \leq 1$, then, under Assumption 1,2 and 3, at any SPNE for the Hotelling game under ambiguity, we have:*

$$\frac{\partial x_1^*}{\partial L} = \begin{cases} < 0 & \textit{for} \quad 0 \leq \alpha < \hat{\alpha} \\ = 0 & \textit{for} \quad \alpha = \hat{\alpha} \\ > 0 & \textit{for} \quad 1 \geq \alpha > \hat{\alpha} \end{cases}$$

where $\hat{\alpha} \in [0, 1]$ is a cutoff-value with $\hat{\alpha} = \hat{\alpha}(\delta, \underline{t}, \sigma^2)$. Similarly we obtain for x_2^*

$$\frac{\partial x_2^*}{\partial L} = \begin{cases} > 0 & \textit{for} \quad 0 \leq \alpha < \hat{\alpha} \\ = 0 & \textit{for} \quad \alpha = \hat{\alpha} \\ < 0 & \textit{for} \quad 1 \geq \alpha > \hat{\alpha} \end{cases}$$

for the same cutoff-value $\hat{\alpha}$. Taking these results together we obtain for Δ^*

$$\frac{\partial \Delta^*}{\partial L} = \begin{cases} > 0 & \textit{for} \quad 0 \leq \alpha < \hat{\alpha} \\ = 0 & \textit{for} \quad \alpha = \hat{\alpha} \\ < 0 & \textit{for} \quad 1 \geq \alpha > \hat{\alpha} \end{cases}$$

Proof. The proof of the proposition is contained in the appendix.

Remark 4.4. In case of full confidence $\delta = 0$ firms do not relocate and competitive differentiation remains unchanged.

As we can see, an increasing support for M leads to similar comparative static results as changes in the confidence level δ. If firms are sufficiently pessimistic, an increase in the support will foster decreasing product differentiation. For an intermediate value of pessimism firms do not relocate. If firms are sufficiently optimistic, an increase in L yields higher equilibrium differentiation.

Support and Degree of Ambiguity

The degree of ambiguity (or of firms' confidence in the reference prior) plays a central role in this paper. For this reason, we discuss in the following its meaning in conjunction with the support of the uncertainty. As proposition 4.5 shows, we can replicate similar comparative static results as in Król (2012) by varying the length L of the support of the midpoint M. Even though similar product differentiation choices might be generated by variations in the size of the support L as compared to variations in the confidence level δ, it indispensable to notice meaningful differences between the two sources of ambiguity. First of all, variations in L and δ might go in similar directions, nevertheless the magnitude of both effects is different. In fact, both effects are interrelated. An increase in the support has a stronger effect on equilibrium differentiation if firms' confidence in the reference prior is low. In case of full confidence, changes in the support do not affect firms' product design decisions. Secondly, there is a clear difference between both sources of uncertainty concerning economic applications. The support of M consists of all possible midpoint realizations of the consumer interval. Before firms make their design choices, they anticipate all possible demand realizations and summarize them in the support interval $[-L, L]$. An increase in the support interval would mean that firms allow ex-ante for a larger variety of feasible demand realizations. In our view it is plausible to argue that in many economic applications the size of the support L is an exogenously fixed variable. What would it actually mean if L was an endogenous variable? It would mean that firms adjust their views on possible demand realizations in the light of higher or lower uncertainties by including or excluding certain market demand scenarios. Furthermore, this would imply that firms were ex-ante wrongly informed or had not precise information about lower and upper bounds of market demand in face of uncertainty. We do not want to argue that such a scenario is completely implausible, our point is that the interpretation of support variations is closely tied to firms' wrong perception of possible demand realizations.

In contrast to the previous interpretation, c.p. variations in the confidence level δ depart on the assumption of an exogenously fixed support length. Firms know possible upper and lower bounds of demand and consider demand uncertainty defined on a fixed support. The reference prior q might reflect firms' ex-ante information about the market environment, e.g. firms might have observable data or can pursue market research to estimate an underlying probability distribution for market demand. Under the assumption that firm managers are sufficiently pessimistic, increasing product differentiation might have different reasons. One explanation could be that firms become more optimistic, meaning that due to a change in the market environment firms adjust their ambiguity attitudes to account for this new situation. On the other hand, it might be the case that firms obtain more reliable data on market outcomes, therefore increasing their confidence in the data available but do not adjust their attitude towards ambiguity. In such a scenario, a higher confidence into the reference prior weakens the impact of pessimism on product differentiation choices.

5 Examples and applications

In this section, we apply our model to a variety of real-life examples. At first, we discuss sports betting regarding horse racing and football games. The second application refers to financial markets, or to be more precise to the mutual funds market. Furthermore, we want to mention that similar cases were already discussed in Król (2012). The purpose of this section consists of providing the reader with additional insight into the mechanics of the capacity approach. In particular, we want discuss implications of confidence and pessimism for the interpretation of these examples. One reason why the aforementioned applications are so apt to be discussed in a Hotelling framework, is given by the fact that in these markets exists a relatively clear measure of firms' product differentiation. We will discuss these measures in the respective subsections. Moreover, consumer preferences are often fluctuating depending on partially unobserved factors, e.g. individual subjective evaluations. Due to firms' imperfect probabilistic information regarding market demand, it is plausible that ambiguity is prevalent in those markets.

5.1 Sports betting

In case of sports betting, the odd of a bookmaker represents his/her product characteristic. Furthermore, the preferences of a bettor over odds are determined by subjective probability estimations of the particular sporting event. Since bookmakers usually want to make a profit regardless of the result of the sporting event, one can assume that bookmakers are worst-case-oriented.[14] If bookmakers were rather optimistic, they would constantly offer odds exceeding the expectation of the underlying distribution and eventually run the risk of bankruptcy.

Horse racing

Smith et al. (2009) examine horse racing data from the UK. The authors provide evidence for an increased fluctuation and divergence of betting exchange prices shortly before the race. This can be interpreted as a higher degree of ambiguity with respect to bettors' preferences. At the same time, bookmakers' odds are getting increasingly similar.[15] Supposing that horse racing bookmakers exhibit a sufficiently high degree of pessimism, our model provides a possible explanation for this observation. Recall that, given that firms are sufficiently pessimistic, an increase of ambiguity leads to decreasing product differentiation. The intuition here lies in the fact that pessimistic firms place more weight on worst-case scenarios. Moreover, the worst-case scenario for the firm whose product characteristic is located on the left hand side corresponds to the realization $M = L$. Similarly, the worst-case scenario for the firm on the right hand side corresponds to the realization $M = -L$. If firms become more pessimistic, the firm which is left selects a product characteristic right from its initial characteristic. Hence, the higher firms' pessimism, the lower their product differentiation. The strength of this effect increases with increasing ambiguity, δ, since firms' confidence in their prior belief determines the influence of the worst-case scenario on their decision process.

[14] For more details see Król (2012).

[15] As pointed out by Król (2012), one can verify that the differences between bookmakers' odds are decreasing in the corresponding time period by using price comparison websites.

Football games

Bookmakers' odds on football games exhibit an interesting feature. Typically, whenever a rather strong team plays against a rather weak team there is little divergence between bookmakers' odds in favor of a victory of the strong team. In contrast, the odds for a victory of the weak team are more volatile. In fact, odds become less volatile when the perceived relative strength of both teams is fairly similar. This observation can be verified by comparing bookmakers' match day odds. In the examples below, we analyzed the odds of ten bookmakers.

Bookmaker	**Odd in favor of Hannover 96**	**Odd in favor of Bayern Munich**
bet365	10	1.25
Sportingbet	11	1.222
Tipico	13	1.25
bwin	9.5	1.22
Interwetten	9	1.27
Bet-at-home	11.5	1.24
Betsson	12.5	1.19
mybet	14	1.22
Betvictor	10.5	1.25
Unibet	12	1.25
Estimated Variances	**2.5667**	**0.0005**

Table 1: Example for the constellation of a strong team versus a weak team

Bookmaker	**Odd in favor of SC Freiburg**	**Odd in favor of FC Augsburg**
bet365	3.1	2.25
Sportingbet	3	2.3
Tipico	3.1	2.35
bwin	2.85	2.3
Interwetten	2.75	2.4
Bet-at-home	3	2.3
Betsson	3.1	2.27
mybet	3.2	2.3
Betvictor	3.125	2.3
Unibet	2.95	2.35
Estimated Variances	**0.0189**	**0.017**

Table 2: Example for the constellation of two balanced teams

Remark 5.1. Both examples are games from match day 22 on February 23, 2014 of the German Bundesliga.[16] The first example refers to a game of Bayern Munich, the strong team, versus Hannover 96 representing the weaker team. Stated odds are to be considered as multiplication factors of the placed bet in case of winning the bet. For instance, suppose one puts a bet of €1 in favor of a victory of Hannover 96 at bet365. In case Hannover 96 wins, the bettor receives €10. The second game, SC Freiburg versus FC Augsburg, is more balanced in terms of relative strength. Estimators used in our examples are

$$\bar{x} = \sum_{i=1}^{n} x_i \quad \text{for the mean and } s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \text{ for the variance.}$$

If bookmakers agreed on a unique prior over the outcome of the game, this phenomenon would be inexplicable. Again, the explanation might lie in bookmakers' confidence in their prior beliefs. Assume bookmakers' choose their odds such that a bettor will always loose a fraction of his money if he bets on both teams, then odds on one team become a function of the odds of the other team.[17] In the situation described above, it is obviously very likely that the strong team wins. Hence, bookmakers' can be confident that the bulk of bettors will bet on the very strong team. This leads to two effects. Firstly, in order to avoid bankruptcy, bookmakers' need to choose odds close to one for a win of the strong team. Secondly, since bookmakers' face little ambiguity over bettors' preferences, they can differentiate their odds for the weak team. This result is in line with our model.

5.2 Mutual funds

Król (2012) provides the example of the managed mutual funds' market. In this context, one can interpret a position in the product space as a portfolio's position ranging from safe investments to risky portfolios. Król (2012) shows, based on data about the daily returns of the fifteen most popular actively managed US mutual funds, that, after the financial crisis 2008, fund managers tend to differentiate their products less. Król (2012) argues that, before the crisis, financial firms' did not consider the post-crisis range of

[16]Data was collected online on February 19th, 2014 at 3:30 pm from the respective websites of the bookmakers.

[17]See Król (2012).

investor behavior as possible.[18] For this reason, the crisis forced firms to revise their beliefs. Furthermore, the author interprets conservative stress test simulations following the crisis as a signal sent out to the competitors that a firm uses a worst-case-based approach for decision making. This is exactly the point where we want to add to the debate. For instance, consider government-imposed stress tests after the crisis. If one interprets such stress tests as signals, then the strategy of a firm is independent of its type. Since each type sends the same signal, no new information is revealed to the other firms. In our view it is debatable whether stress test simulations induced a shift in fund managers' ambiguity attitude towards a more pessimistic preference approach, or whether exactly those fund managers knew more clearly that investors would prefer more secure assets after the crisis. If so, a possible explanation for lower post-crisis product differentiation is that firms were less uncertain about investor preferences. In our view, it is not implausible that fund managers ambiguity attitude remained relatively stable even though government stress tests were imposed. Furthermore, due to market research and historical data[19] it is likely that fund managers know the whole range of possible individual investor behaviors.[20] However, investor preferences are highly fluctuating, since they depend on investors' subjective evaluations of the fund's performance which itself is based on numerous observed and unobserved factors as recent stock market developments or individual future expectations. At the end and shortly after the financial crisis, firms' were highly confident in terms of investor preferences, since it was self-evident that post-crisis, the majority of investors would prefer assets which were rather safe. Again, this finding is in line with our model.

[18] In particular, the shift of consumer preferences toward safe investments due to decreasing stock prices during the crisis.

[19] Financial firms' can rely on past data of various historical economic crises including stock market crashes (e.g. the Great Depression in the 1930s), bubbles (e.g. dot-com bubble in 2000) and financial crises (e.g. Asian financial crisis in 1997).

[20] This would induce that variations in the support of the midpoint of the consumer distribution cannot account for the observation of decreasing product differentiation.

6 Conclusion

We presented an extension of Hotelling's model incorporating ambiguity in the form of demand location uncertainty as well as uncertainty with respect to the intensity of transportation costs. Ambiguity was introduced by representing firms' beliefs with neo-additive capacities. We analyzed firms' optimal product characteristic choices and found a unique SPNE in pure strategies for the Hotelling game under ambiguity.

Our model incorporates a variety of different sources of uncertainty. First of all, there is the variance σ^2 of the reference probability q. As in the standard risk case of Meagher and Zauner (2004), a higher variance implies that firms increase product differentiation. Thus, if the measure of uncertainty is given by the variance of the underlying reference probability, it can be considered as differentiation force.

Secondly, there is the length of the support interval of M. The larger the support of M, the larger the number of demand realizations that firms consider as possible market outcomes. Hence, the length of the support interval might be interpreted as an additional measure of uncertainty. As our results show, the effects on an increasing support are strongly related to firms' ambiguity attitude α and degree of ambiguity (confidence) δ. The effect of an increase in uncertainty can go in adverse directions. If firms are rather pessimistic, a larger support results in lower equilibrium differentiation, if firms are rather optimistic, a larger support engenders opposite results. All in all, uncertainty as measured by the support length can be - depending on parameters - a differentiation or agglomeration force.

A third measure of uncertainty is given by the confidence parameter δ reflecting firms uncertainty on observables. Interpreted in this way, rising uncertainty is tied to lower data reliability yielding lower confidence levels in the reference probability q. Again, similar to the case of support variations, this can trigger off opposing effects. When firms are pessimistic enough, equilibrium differentiation is going down, when firms a rather optimistic product differentiation is going to increase. One can also argue the other way round. For a given confidence level, increasing pessimism yields lower equilibrium differentiation, whereas an increase in optimism increases equilibrium differentiation.

Finally the last source of uncertainty lies in the transportation cost parameter t. As the support interval $[\underline{t}, 1]$ for t increases, firms' equilibrium differentiation remains the same in case of full optimism and full confidence and decreases in all other cases. Thus, excluding these boundary cases, we can say that uncertainty with respect to the transportation cost parameter constitutes an agglomeration force.

As we can see from the preceding line of arguments, one should be very cautious when it comes to drawing conclusions from real-world applications of Hotelling models under uncertainty. In our view, it is indispensable to clearly identify the driving factors of an observed increase or decrease in product differentiation since the interpretation and conclusions from observed firm behavior might change in the light of different sources of uncertainty. In particular, it seems worthwhile for policymakers to disentangle the effect of confidence and ambiguity attitude on product differentiation, since it might really matter for official regulatory procedures whether observed product differentiation choices are to be attributed to perceived changes in data-reliability or whether firms feature more or less optimistic behavioral patterns.

Appendix

Proof of Lemma 2.1. The support of M is restricted on the interval $[-L, L] \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$. Mean and variance of M exists. For the mean we can perform the following line of estimates.

$$\mathbb{E}_q[M] = \int\limits_{\mathbb{R}} M d\mathbb{P} \leq \int\limits_{\mathbb{R}} L d\mathbb{P} = L \int\limits_{\mathbb{R}} 1 d\mathbb{P} = L$$

and

$$\mathbb{E}_q[M] = \int\limits_{\mathbb{R}} M d\mathbb{P} \geq \int\limits_{\mathbb{R}} -L d\mathbb{P} = -L \int\limits_{\mathbb{R}} 1 d\mathbb{P} = -L$$

Similarly, for the second moment of M we get

$$E[M^2] = \int\limits_{\mathbb{R}} M^2 d\mathbb{P} \leq \int\limits_{\mathbb{R}} L^2 d\mathbb{P} = L^2 \quad \text{and} \quad \mathbb{E}_q[M^2] = \int\limits_{\mathbb{R}} M^2 d\mathbb{P} \geq 0$$

and for the variance σ^2 we have

$$\sigma_M^2 = \mathbb{E}_q[M^2] - \mathbb{E}_q[M]^2 \leq \mathbb{E}_q[M]^2 \leq L^2 \quad \text{and} \quad \sigma_M^2 \geq 0$$

□

Proof of Lemma 3.1. According to Assumption 1, at the second stage of the game, the true midpoint $\hat{M}$ and cost intensity $\hat{t}$ is revealed. Hence, firms know that consumers are uniformly distributed along $[\hat{M} - \frac{1}{2}, \hat{M} + \frac{1}{2}]$ and face a disutility of $\hat{t}(x - x_i)^2$. Now, suppose $(\hat{M} - \bar{x}) \in [-\frac{3}{2}, \frac{3}{2}]$ where $\bar{x} = \frac{x_1+x_2}{2}$ as before, then inserting firms' demand functions (compare section 3.1) into firms' profit functions given by (3.2) yields:

$$\pi_1(p_1, p_2, x_1, x_2, \hat{t}, \hat{M}) = p_1 \left(-\hat{M} + \frac{x_1 + x_2}{2} + \frac{p_2 - p_1}{2\hat{t}(x_1 - x_2)} + \frac{1}{2} \right)$$
$$\pi_2(p_1, p_2, x_1, x_2, \hat{t}, \hat{M}) = p_2 \left(\hat{M} - \left(\frac{x_1 + x_2}{2} + \frac{p_2 - p_1}{2\hat{t}(x_1 - x_2)} \right) + \frac{1}{2} \right)$$

Note that firms' profit functions are strictly concave in their own price. Consequently, firms' unique best response function are determined by the first-order condition of their profit maximization problems:

$$\frac{\partial \pi_1}{\partial p_1} := -\hat{M} + \frac{p_2 - p_1}{2\hat{t}(x_2 - x_1)} - \frac{p_1}{2\hat{t}(x_2 - x_1)} + \frac{x_2 + x_1}{2} + \frac{1}{2} \stackrel{!}{=} 0$$
$$\frac{\partial \pi_2}{\partial p_2} := \hat{M} - \frac{p_2}{2\hat{t}(x_2 - x_1)} + \frac{p_1 - p_2}{2\hat{t}(x_2 - x_1)} - \frac{x_2 + x_1}{2} + \frac{1}{2} \stackrel{!}{=} 0$$

By solving the equation system $\left(\frac{\partial \pi_1}{\partial p_1}, \frac{\partial \pi_2}{\partial p_2}\right) = (0, 0)$, one obtain the price equilibrium in Lemma 3.1. □

Proof of Lemma 3.2. Let $(\hat{M} - \bar{x}) \notin [-\frac{3}{2}, \frac{3}{2}]$, i.e. either $(\hat{M} - \bar{x}) < -\frac{3}{2}$ or $(\hat{M} - \bar{x}) > \frac{3}{2}$. Whenever the condition $(\hat{M} - \bar{x}) < -\frac{3}{2}$ is satisfied, firm 1 is closer to the consumer interval than firm 2 for any consumer $x \in [\hat{M} - \frac{1}{2}, \hat{M} + \frac{1}{2}]$. In this case, firm 1 can set a price, such that it obtains the whole consumer demand, even if firm 2 charges a zero price, formally from equation (3.1) one obtain the condition:

$$\xi(x_1, x_2, p_1, p_2) \stackrel{!}{=} \hat{M} + \frac{1}{2}$$

Solving this equation for p_1 yields firm 1's best reply:

$$p_1(p_2) = 2\hat{t}(x_1 - x_2)\left[\bar{x} - \hat{M} - \frac{1}{2}\right] + p_2$$

Given this strategy of firm 1, firm 2's best-reply is given by the set $\mathbb{R}_+$. Let us now consider the case, that firm 2 plays the price $p_2 = 0$. In this case, firm 1's best reply is given by $p_1(0) = 2\hat{t}(x_1 - x_2)\left[\bar{x} - \hat{M} - \frac{1}{2}\right]$. As shown before, $p_2 = 0$ is a best reply to $p_1 = 2\hat{t}(x_1 - x_2)\left[\bar{x} - \hat{M} - \frac{1}{2}\right]$, therefore

$$(p_1^*, p_2^*) = \left(2\hat{t}(x_1 - x_2)\left[\bar{x} - \hat{M} - \frac{1}{2}\right], 0\right)$$

is a price equilibrium. If firm 2 plays a strictly positive price $\hat{p}_2 > 0$, then, firm 1's best reply is given by $p_1(\hat{p}_2) = 2(x_1 - x_2)\left[\hat{M} + \frac{1}{2} - \frac{x_1+x_2}{2}\right] + \hat{p}_2$. If firm 1 sticks to this strategy, firm 2's has an incentive to lower its price, thereby allowing firm 2 to obtain a positive share of consumer demand. Therefore an equilibrium with $\hat{p}_2 > 0$ does not exist. The case $(\hat{M} - \bar{x}) > \frac{3}{2}$ can be solved along the same line of arguments.

□

Proof of Lemma 3.3. Lemma 3.4 implies that firms' second-stage profits at $(\hat{t}, \hat{M})$ equal the second piece of (3.3):

$$\begin{aligned}
\Pi_1 &= \frac{1}{18}\hat{t}(x_2 - x_1)[-3 + 2(\hat{M} - \bar{x})]^2 \\
\Pi_2 &= \frac{1}{18}\hat{t}(x_2 - x_1)[3 + 2(\hat{M} - \bar{x})]^2
\end{aligned}$$

Both profit functions are continuously differentiable with respect to $\hat{t}$ and $\hat{M}$. Differentiation with respect to $\hat{t}$ yields

$$\begin{aligned}
\frac{\partial \Pi_1}{\partial \hat{t}} &= \frac{2}{9}(x_2 - x_1)\left[\frac{x_1 + x_2}{2} - \hat{M} + \frac{3}{2}\right]^2 > 0 \\
\frac{\partial \Pi_2}{\partial \hat{t}} &= -\frac{2}{9}(x_1 - x_2)\left[\frac{x_1 + x_2}{2} - \hat{M} - \frac{3}{2}\right]^2 > 0
\end{aligned}$$

Differentiation with respect to $\hat{M}$ yields

$$\begin{aligned}
\frac{\partial \Pi_1^*}{\partial \hat{M}} &= \underbrace{-\frac{4}{9}\hat{t}(x_2 - x_1)}_{<0}\underbrace{\left[\frac{x_1 + x_2}{2} - \hat{M} + \frac{3}{2}\right]}_{>0} < 0 \\
\frac{\partial \Pi_2^*}{\partial \hat{M}} &= \underbrace{\frac{4}{9}\hat{t}(x_1 - x_2)}_{<0}\underbrace{\left[\frac{x_1 + x_2}{2} - \hat{M} - \frac{3}{2}\right]}_{<0} > 0
\end{aligned}$$

□

Proof of Lemma 3.4. The proof of the lemma follows exactly the same line of arguments as in the proof of Lemma 3.1 in Król (2012), page 602 with a slight modification in case 3. There are three different cases to be considered.

(1) Case 1 refers to a situation where either firm 1 or firm 2 can monopolize the market for certain realizations of the midpoint M. If firm 1 can monopolize the market for certain realizations of M we can conclude that firm 1 will monopolize the market if $\hat{M} = -L$, since w.lo.g. firm 1 is the firm left of firm 2. Similarly, we can conclude that firm 2 can monopolize the market for $\hat{M} = L$. This is finding is impossible. If firm 1 monopolizes the market for the realization $\hat{M} = -L$, we have by Lemma 3.1, equation (3.2) that $\frac{x_1+x_2}{2} - \frac{3}{2} > -L$. If firm 2 monopolizes the market, we have by (3.2) that $\frac{x_1+x_2}{2} + \frac{3}{2} < L$. Thus, we must have that $L + \frac{x_1+x_2}{2} > \frac{3}{2}$ and $L - \frac{x_1+x_2}{2} > \frac{3}{2}$ holds at the same time implying $\left|\frac{x_1+x_2}{2}\right| < L - \frac{3}{2}$. This is a contradiction since L is assumed to be smaller than $\frac{1}{2}$.

(2) Case 2 describes a scenario where one of the two firms can monopolize the market for each realization $\hat{M}$ of uncertainty. If firm j is a monopolist, the other firm can deviate from its original location in order to obtain a positive market share and therefore make

strictly positive profits. Król (2012) suggests the location $x_{-j} = -x_j$.

(3) Case 3 refers to a situation where w.l.o.g. firm 1 can monopolize the market for some realizations of uncertainty, in particular the realization $\hat{M} = -L$ and for the remaining realizations, in particular the realization $\hat{M} = L$ there exists a competitive equilibrium. Consider now the profit function of firm 2 in case of a competitive equilibrium[21] :

$$\frac{\partial \Pi_2}{\partial x_2}(x_1, x_2, L, t) = \frac{t\ (2\,L - 3\,x_2 + x_1 + 3)\ (2\,L - x_2 - x_1 + 3)}{18}$$

We want to show that

$$\frac{\partial \Pi_2}{\partial x_2}(x_1, x_2, L, t) < 0.$$

We determine the sign of both brackets. Consider the expression in within the second bracket first. We have

$$2L + 3 - x_1 - x_2 > 0 \quad \Leftrightarrow \quad 2L + 3 > x_1 + x_2 \quad \Leftrightarrow \quad L + \frac{3}{2} > \bar{x}$$

The last condition corresponds to the requirement for a competitive solution in case that the midpoint $M = L$ realizes. Therefore it must be by assumption positive. The second bracket is negative. The monopolistic outcome for the midpoint realization $M = -L$ requires $L + \bar{x} > \frac{3}{2}$. Solving this inequality for x_2 we obtain $x_2 > 3 - 2L - x_1$. With the help of this inequality we can conduct an estimation for the expression in the first bracket:

$$3 + 2L + x_1 - 3x_2 < 8L - 6 + 4x_1 < 8L - 8 < 0$$

The last inequality follows from the fact that $L < \frac{1}{2}$ and $x_1 < 0$. Thus we proved that

$$\frac{\partial \Pi_2}{\partial x_2}(x_1, x_2, L, t) < 0.$$

[21] We consider the profit function of firm 2, Król (2012) considers the profit function of firm 1

This finding shows that firm 2 has an incentive to move leftwards in order to reduce both firms' product differentiation. This finding shows, that a strict competitive solution does not exist under the above stated parameter restrictions. Since we consider a symmetric scenario, a similar argument holds for a scenario where firm 2 becomes a monopolist for $\hat{M} = L$ and for $\hat{M} = -L$ there is a competitive solution.

$\square$

Proof of Lemma 3.5. The first part of firms' Choquet expected profit is

$$E_q[\Pi_i(x_1, x_2, t, M)] = \int_{-L}^{L} (-1)^j \, \frac{2}{9} \, t \, (x_j - x_i) \left(\frac{x_i + x_j}{2} - \left(M + \frac{3}{2}(-1)^i \right) \right)^2 f(M) dM$$

This expectation is of the form

$$\mathbb{E}_q[g_i(t) \, h_i(M)]$$

with real-valued Borel-measurable functions g_i and h_i for $i = 1, 2$. We define

$$g_i(t) = t \qquad \text{and} \qquad h_i(M) = (-1)^j \, \frac{2}{9} \, t \, (x_j - x_i) \left(\frac{x_i + x_j}{2} - \left(M + \frac{3}{2}(-1)^i \right) \right)^2 .$$

By Assumption 2, $(R7)$, t and M are uncorrelated. By Lemma 5.20 in Meintrup and Schäffler (2005), page 131, we obtain that also $g_i(t)$ and $h_i(M)$ are uncorrelated. Thus, we can conclude

$$\mathbb{E}_q[\Pi_i^*(x_1, x_2, t, M)] = E_q[g_i(t) \, h_i(M)] = \mathbb{E}_q[g_i(t)] \, \mathbb{E}_q[h_i(M)] = \mu_t \, \mathbb{E}_q[h_i(M)]$$

In the following, we can rely on the results in Meagher and Zauner (2004) page 205, since $\mathbb{E}_q[h_i(M)]$ is equal to firm i's expected profit function in the risk case. Thus

$$\mathbb{E}_q[\Pi_i(x_1, x_2, t, M)] = t_\mu \int_{-L}^{L} (-1)^j \frac{2}{9} (x_j - x_i) \left(\frac{x_i + x_j}{2} - \left(M + \frac{3}{2}(-1)^i \right) \right)^2 f(M) dM$$

$$= \frac{(-1)^j}{18} t_\mu (x_j - x_i) \{ (x_i + x_j - 3(-1)^i)^2$$

$$- 4\mu_M (x_i + x_j - 3(-1)^i) + 4(\mu_M + \sigma_M^2) \}$$

$\square$

Proof of Proposition 3.1. We derive expected CEU profits at the first stage of the game. By doing so, we obtain for firm 1:

$$\begin{aligned} &\mathbb{CEU}[\Pi_1(x_1, x_2, \alpha, \delta, \underline{t}, \sigma^2, L)] \\ &:= \delta \left(\frac{2\,(1-\alpha)\,(x_2 - x_1)\,\left(L + \frac{x_2+x_1}{2} + \frac{3}{2}\right)^2}{9} + \frac{2\,\alpha\,\underline{t}\,(x_2 - x_1)\,\left(-L + \frac{x_2+x_1}{2} + \frac{3}{2}\right)^2}{9} \right) \\ &+ \frac{(1-\delta)\,(x_2 - x_1)\,\left((x_2 + x_1 + 3)^2 + 4\,\sigma^2\right)}{18} \end{aligned} \tag{A.1}$$

for firm 2 we get:

$$\begin{aligned} &\mathbb{CEU}[\Pi_2(x_1, x_2, \alpha, \delta, \underline{t}, \sigma^2, L)] \\ &:= \delta \left(\frac{2\,\alpha\,\underline{t}\,(x_2 - x_1)\,\left(L + \frac{x_2+x_1}{2} - \frac{3}{2}\right)^2}{9} + \frac{2\,(1-\alpha)\,(x_2 - x_1)\,\left(-L + \frac{x_2+x_1}{2} - \frac{3}{2}\right)^2}{9} \right) \\ &+ \frac{(1-\delta)\,(x_2 - x_1)\,\left((x_2 + x_1 - 3)^2 + 4\,\sigma\right)}{18} \end{aligned} \tag{A.2}$$

Taking the derivative of (A.1) with respect to x_1 yields

$$\begin{aligned}
&\frac{\partial \mathbb{CEU}[\Pi_1(x_1, x_2, \alpha, \delta, \underline{t}, \sigma^2, L)]}{\partial x_1} := \\
&-\frac{2\delta\ (1-\alpha)\ \left(L + \frac{x_2+x_1}{2} + \frac{3}{2}\right)^2}{9} + \frac{2\delta\ (1-\alpha)\ (x_2 - x_1)\ \left(L + \frac{x_2+x_1}{2} + \frac{3}{2}\right)}{9} \\
&+ \frac{2\delta\,\alpha\,\underline{t}\ (x_2 - x_1)\ \left(-L + \frac{x_2+x_1}{2} + \frac{3}{2}\right)}{9} - \frac{2\delta\,\alpha\,\underline{t}\left(-L + \frac{x_2+x_1}{2} + \frac{3}{2}\right)^2}{9} \\
&- \frac{(1-\delta)\ \left((x_2 + x_1 + 3)^2 + 4\,\sigma\right)}{18} + \frac{(1-\delta)\ (x_2 - x_1)\ (x_2 + x_1 + 3)}{9}
\end{aligned} \tag{A.3}$$

Similarly, we take the derivative of (A.2) with respect to x_2

$$\begin{aligned}
&\frac{\partial \mathbb{CEU}[\Pi_2(x_1, x_2, \alpha, \delta, \underline{t}, \sigma^2, L)]}{\partial x_2} := \\
&\frac{2\delta\,\alpha\,\underline{t}\left(L + \frac{x_2+x_1}{2} - \frac{3}{2}\right)^2}{9} + \frac{2\delta\,\alpha\,\underline{t}\ (x_2 - x_1)\ \left(L + \frac{x_2+x_1}{2} - \frac{3}{2}\right)}{9} \\
&+ \frac{2\delta\ (1-\alpha)\ (x_2 - x_1)\ \left(-L + \frac{x_2+x_1}{2} - \frac{3}{2}\right)}{9} + \frac{2\delta\ (1-\alpha)\ \left(-L + \frac{x_2+x_1}{2} - \frac{3}{2}\right)^2}{9} \\
&+ \frac{(1-\delta)\ \left((x_2 + x_1 - 3)^2 + 4\,\sigma\right)}{18} + \frac{(1-\delta)\ (x_2 - x_1)\ (x_2 + x_1 - 3)}{9}
\end{aligned} \tag{A.4}$$

Now, we solve the following system of equations

$$\begin{aligned}
\frac{\partial \mathbb{CEU}[\Pi_1(x_1, x_2, \alpha, \delta, \underline{t}, \sigma^2, L)]}{\partial x_2} &= 0 \\
\frac{\partial \mathbb{CEU}[\Pi_2(x_1, x_2, \alpha, \delta, \underline{t}, \sigma^2, L)]}{\partial x_2} &= 0
\end{aligned} \tag{A.5}$$

and obtain three solution pairs. The first solution pair (x_1^*, x_2^*) is given by:

$$x_1^* = \frac{4\,\delta\,(\,\alpha\,\underline{t} + (1-\,\alpha))\ L^2 + 12\,\delta\,((1-\,\alpha) - \,\alpha\,\underline{t})\ L + (4 - 4\,\delta)\ \sigma^2 + 9\,(\,\alpha\,\delta\,\underline{t} - \,\alpha\,\delta + 1)}{8\,\delta\,(\,\alpha\,\underline{t} + (\,\alpha - 1))\ L - 12\,(\,\alpha\,\delta\,\underline{t} + \,\alpha\,\delta - 1)}$$

$$x_2^* = -\frac{4\,\delta\,(\,\alpha\,\underline{t} + (1-\,\alpha))\ L^2 + 12\,\delta\,((1-\,\alpha) - \,\alpha\,\underline{t})\ L + (4 - 4\,\delta)\ \sigma^2 + 9\,(\,\alpha\,\delta\,\underline{t} - \,\alpha\,\delta + 1)}{8\,\delta\,(\,\alpha\,\underline{t} + (\,\alpha - 1))\ L - 12\,(\,\alpha\,\delta\,\underline{t} + \,\alpha\,\delta - 1)}$$

The second pair of solutions (x_1^{**}, x_2^{**}) is given by:

$$\begin{aligned}
x_1^{**} =&(2\alpha\delta(\underline{t}-1)+2)^{-1}\Bigg(\alpha^2\delta^2\left(4L^2\left(\underline{t}^2+6\underline{t}+1\right)-12L\left(\underline{t}^2-1\right)+9(\underline{t}-1)^2\right) \\
&-2\alpha\delta\Big(2L^2(\delta(5\underline{t}+3)+\underline{t}-1)+6L(\delta(-\underline{t})+\delta+\underline{t}+1) \\
&-(\underline{t}-1)(2(\delta-1)\sigma^2+9)\Big)+8\delta^2L^2-4\delta L^2+12\delta L+4\delta\sigma^2-4\sigma^2+9\Bigg)^{\frac{1}{2}} \\
&+\alpha\delta(2L(\underline{t}+1)-3\underline{t}+3)-2\delta L-3
\end{aligned}$$

and

$$\begin{aligned}
x_2^{**} =& \Big(2\alpha\delta(\underline{t}-1)+2\Big)^{-1}\Bigg\{\Bigg(\alpha^2\delta^2(4L^2(\underline{t}^2+6\underline{t}+1)-12L(\underline{t}^2-1)+9(\underline{t}-1)^2) \\
&-2\alpha\delta(2L^2(\delta(5\underline{t}+3)+\underline{t}-1)+6L(\delta(-\underline{t})+\delta+\underline{t}+1) \\
&-(\underline{t}-1)(2(\delta-1)\sigma^2+9))+8\delta^2L^2-4\delta L^2+12\delta L+4\delta\sigma^2-4\sigma^2+9\Bigg)^{\frac{1}{2}} \\
&-\alpha\delta(2L(\underline{t}+1)-3\underline{t}+3)+2\delta L+3\Bigg\}
\end{aligned}$$

Finally, the last pair of solutions (x_1^{***}, x_2^{***}) is given by:

$$\begin{aligned}
x_1^{***} =& \Bigg(\Bigg(\alpha^2\delta^2\left(4L^2\left(\underline{t}^2+6\underline{t}+1\right)-12L\left(\underline{t}^2-1\right)+9(\underline{t}-1)^2\right)-2\alpha\delta(2L^2(\delta(5\underline{t}+3)+\underline{t}-1) \\
&+6L(\delta(-\underline{t})+\delta+\underline{t}+1)-(\underline{t}-1)(2(\delta-1)\sigma^2+9))+8\delta^2L^2-4\delta L^2+12\delta L+4\delta\sigma^2 \\
&-4\sigma^2+9\Bigg)^{\frac{1}{2}}-\alpha\delta(2L(\underline{t}+1)-3\underline{t}+3)+2\delta L+3\Bigg)\cdot(2\alpha\delta(\underline{t}-1)+2)^{-1}
\end{aligned}$$

and

$$x_2^{***} = -\Bigg(\Big(\alpha^2\delta^2(4L^2(\underline{t}^2 + 6\underline{t} + 1) - 12L(\underline{t}^2 - 1) + 9(\underline{t} - 1)^2) - 2\alpha\delta(2L^2(\delta(5\underline{t} + 3) + \underline{t} - 1)$$
$$+ 6L(\delta(-\underline{t}) + \delta + \underline{t} + 1) - (\underline{t} - 1)(2(\delta - 1)\sigma^2 + 9)) + 8\delta^2L^2 - 4\delta L^2 + 12\delta L + 4\delta\sigma^2$$
$$- 4\sigma^2 + 9\Big)^{\frac{1}{2}} + \alpha\delta(2L(\underline{t} + 1) - 3\underline{t} + 3) - 2\delta L - 3\Bigg)(2\alpha\delta(\underline{t} - 1) + 2)^{-1}$$

The solution pairs (x_1^{**}, x_2^{**}) and (x_1^{***}, x_2^{***}) do not fulfill the global competition condition according to Lemma 3.4:

$$L - \frac{3}{2} < \bar{x} < -L + \frac{3}{2}$$

We obtain equilibrium profits by inserting the derived equilibrium locations into (A.1) and (A.2). After several steps of algebra, we get

$$\mathbb{CEU}[\Pi_i] = \frac{(4\,\delta\,L^2\,(\,\alpha\,\underline{t} -\ \alpha +\ 1) - 12\,\delta\,L\,(\,\alpha\,\underline{t} -\ \alpha + 1) + 4\,\sigma^2\,(1 -\ \delta) + 9\,(\alpha\,\delta\,\underline{t} -\ \alpha\,\delta + 1))^2}{36\ (2\,\delta\,L\,(-\,\alpha\,\underline{t} - \alpha + 1) + 3\,(\,\alpha\,\delta\,\underline{t} -\ \alpha\,\delta + 1))}$$

The competitive differentiation is given by

$$\begin{aligned}\Delta_x^* &= x_2^* - x_1^* = 2x_2^* \\ &= \frac{4\,\delta\,(\,\alpha\,\underline{t} + (1 -\ \alpha))\ L^2 + 12\,\delta\,((1 -\ \alpha) -\ \alpha\,\underline{t})\ L + (4 - 4\,\delta)\ \sigma^2 + 9\,(\,\alpha\,\delta\,\underline{t} -\ \alpha\,\delta + 1)}{4\,\delta\,(\,\alpha\,\underline{t} + (\,\alpha - 1))\ L - 6\,(\,\alpha\,\delta\,\underline{t} +\ \alpha\,\delta - 1)}.\end{aligned}$$

□

Before starting with the proofs of the comparative static analysis, we want to point out that for many of the estimations performed in the subsequent five proofs, we make use of the following intrinsic parameter restrictions:

- upper and lower support boundaries for M: $0 < L \leq \frac{1}{2}$
- upper and lower bound of the confidence parameter: $0 \leq \delta \leq 1$
- upper and lower bound of ambiguity attitude: $0 \leq \alpha \leq 1$

- upper and lower bound of the variance of M: $0 \leq \sigma^2 \leq L^2 \leq \frac{1}{4}$

- upper and lower bound of the transportation cost parameter: $0 < \underline{t} \leq 1$

Proof of Proposition 4.1. The derivative of x_1^* with respect to α is given by

$$\frac{\delta(-8\,\delta\,\underline{t}\,L^3 + 6\,\delta\,\underline{t}\,L^2 - 6\,\underline{t}\,L^2 - 6\,\delta\,L^2 + 6\,L^2 + 4\,\delta\,\sigma^2\,\underline{t}\,L - 4\,\sigma^2\,\underline{t}\,L)}{2\,(2\,\alpha\,\delta\,\underline{t}\,L + 2\,\alpha\,\delta\,L - 2\,\delta\,L - 3\,\alpha\,\delta\,\underline{t} + 3\,\alpha\,\delta - 3)^2}$$
$$+ \frac{\delta(9\,\delta\,\underline{t}\,L + 9\,\underline{t}\,L + 4\,\delta\,\sigma^2\,L - 4\,\sigma^2\,L - 9\,\delta\,L + 9\,L - 6\,\delta\,\sigma^2\,\underline{t} + 6\,\sigma^2\,\underline{t} + 6\,\delta\,\sigma^2 - 6\,\sigma^2)}{2\,(2\,\alpha\,\delta\,\underline{t}\,L + 2\,\alpha\,\delta\,L - 2\,\delta\,L - 3\,\alpha\,\delta\,\underline{t} + 3\,\alpha\,\delta - 3)^2}$$

The denominator is positive. Therefore, the sign of the derivative is determined by its numerator or to be more precise by the numerator divided by δ. We analyze the sign of this expression in several steps. First of all, we simplify this expression in the following way

$$\begin{aligned}
&-8\delta\underline{t}L^3 + 6\delta\underline{t}L^2 - 6\underline{t}L^2 - 6\delta L^2 + 6L^2 + 4\delta\sigma^2\underline{t}L - 4\sigma^2\underline{t}L \\
&+ 9\delta\underline{t}L + 9\underline{t}L + 4\delta\sigma^2 L - 4\sigma^2 L - 9\delta L + 9L - 6\delta\sigma^2\underline{t} + 6\sigma^2\underline{t} + 6\delta\sigma^2 - 6\sigma^2 \\
&= -8\delta\underline{t}L^3 + 6L^2[\delta\underline{t} - \underline{t} - \delta + 1] + 4\sigma^2 L[\delta\underline{t} - \underline{t} + \delta - 1] + 9L[\delta\underline{t} + \underline{t} - \delta + 1] \\
&+ 6\sigma^2[-\delta\underline{t} + \underline{t} + \delta - 1] \\
&= -8\delta\underline{t}L^2 + (6L^2 - 6\sigma^2)[1 - \delta\underline{t} - \underline{t} - \delta] + 4\sigma^2 L[\delta\underline{t} - \underline{t} + \delta - 1] + 9L[\delta\underline{t} + \underline{t} - \delta + 1]
\end{aligned}$$

We have $\sigma^2 \leq L^2$, thus we can conclude that $6L^2 - 6\sigma^2 \geq 0$. The expression $1 + \delta\underline{t} - \underline{t} - \delta]$ is non-negative. This we can prove in a few small steps. Inserting $\underline{t} = 0$ we obtain $1 - \delta \geq 0$. Inserting $\underline{t} = 1$ we obtain the value 0. Taking the derivative with respect to $\underline{t}$, we get $\delta - 1$ which means that the function is a constant for $\delta = 0$ and strictly decreasing for $0 < \delta \leq 1$. Inserting $\delta = 0$ we obtain the value 1 which is non-negative. By applying the mean value theorem for continuous functions for the case $0 < \delta \leq 1$, we obtain that $1 + \delta\underline{t} - \underline{t} - \delta \geq 0$. To sum up, we have that

$$(6L^2 - 6\sigma^2)[1 - \delta\underline{t} - \underline{t} - \delta] \geq 0$$

Using this result we can perform the following line of estimations

$$
\begin{aligned}
& -8\delta \underline{t} L^2 + (6L^2 - 6\sigma^2)[1 - \delta \underline{t} - \underline{t} - \delta] + 4\sigma^2 L[\delta \underline{t} - \underline{t} + \delta - 1] + 9L[\delta \underline{t} + \underline{t} - \delta + 1] \\
& \geq -8\delta \underline{t} L^3 + \delta \underline{t}[4\sigma^2 L + 9L] + [1 + \underline{t} - \delta][9L - 4\sigma^2 L] \\
& \geq -8\delta \underline{t} + 9\delta \underline{t} L + [1 + \underline{t} - \delta][9L - L] \\
& = \delta \underline{t} L + 8\underline{t} L \\
& > 0 \quad \text{since} \quad L > 0,\ \underline{t} > 0
\end{aligned}
$$

This proves that $\frac{\partial x_1^*}{\partial \alpha} > 0$ and $\frac{\partial x_2^*}{\partial \alpha} = -\frac{\partial x_1^*}{\partial \alpha} < 0$. □

Proof of Proposition 4.2. The derivative of x_1^* with respect to σ^2 is given by

$$
\frac{\partial x_1^*}{\partial \sigma^2} = \frac{4 - 4\,\delta}{(8\,\alpha\,\delta\,\underline{t} + (8\,\alpha - 8)\ \delta)\ L - 12\,\alpha\,\delta\,\underline{t} + 12\,\alpha\,\delta - 12}
$$

The numerator is non-negative since $4 - 4\delta \geq$ for $0 \leq \delta \geq 1$ and strictly positive for $0 \leq \delta < 1$. For the denominator we can conduct the following line of estimations:

$$
\begin{aligned}
& (8\,\alpha\,\delta\,\underline{t} + (8\,\alpha - 8)\ \delta)\ L - 12\,\alpha\,\delta\,\underline{t} + 12\,\alpha\,\delta - 12 \\
& = 8\delta L(\alpha(\underline{t} + 1) - 1) + 12(\alpha\delta(1 - \underline{t}) - 1) \\
& \leq 8\delta L(\alpha(\underline{t} + 1 - 1) + 12(\alpha\delta(1 - \underline{t}) - 1) \\
& \leq 8\alpha\delta \underline{t} L + 12(\alpha\delta(1 - \underline{t}) - 1) \\
& \leq 4\alpha\delta \underline{t} + 12\alpha\delta - 12\alpha\delta \underline{t} - 12 \\
& \leq -8\alpha\delta \underline{t} + 12\alpha\delta - 12 \\
& \leq -8\alpha\delta \underline{t} \\
& \leq 0
\end{aligned}
$$

Thus, $\frac{\partial x_1^*}{\partial \sigma^2} \leq 0$ and $\frac{\partial x_2^*}{\partial \sigma^2} = -\frac{\partial x_1^*}{\partial \sigma^2} \geq 0$. For $\delta = 1$ both x_1^* and x_2^* are independent of σ^2. Therefore $\frac{\partial x_2^*}{\partial \sigma^2} = \frac{\partial x_1^*}{\partial \sigma^2} = 0$. □

Proof of Proposition 4.3. We have

$$\frac{\partial x_1^*}{\partial \underline{t}} = \frac{\alpha\,\delta\,(2\,L-3)\,(4\,\alpha\,\delta\,L^2 - 4\,\delta\,L^2 + 6\,\alpha\,\delta\,L - 3\,\delta\,L - 3\,L + 2\,\delta\,\sigma^2 - 2\,\sigma^2)}{2\,(2\,\alpha\,\delta\,\underline{t}\,L + 2\,\alpha\,\delta\,L - 2\,\delta\,L - 3\,\alpha\,\delta\,\underline{t} + 3\,\alpha\,\delta - 3)^2}$$

It is obvious that the denominator is positive. Turning to the numerator we can see that $\alpha\delta(2L-3)$ is negative, since $L \leq \frac{1}{2}$. Thus, the remaining part to analyze is given by the expression

$$4\,\alpha\,\delta\,L^2 - 4\,\delta\,L^2 + 6\,\alpha\,\delta\,L - 3\,\delta\,L - 3\,L + 2\,\delta\,\sigma^2 - 2\,\sigma^2 \tag{A.6}$$

We want to establish that expression (A.6) is smaller or equal than zero.

$$\begin{aligned}
&4\,\alpha\,\delta\,L^2 - 4\,\delta\,L^2 + 6\,\alpha\,\delta\,L - 3\,\delta\,L - 3\,L + 2\,\delta\,\sigma^2 - 2\,\sigma^2 \\
&< 4\delta L^2 - 4\delta L^2 + 6\alpha\delta L - 6\delta L + 2\sigma^2 - 2\sigma^2 \\
&= 6\alpha\delta L - 6\delta L \\
&\leq 6\delta L - 6\delta L \\
&= 0
\end{aligned}$$

Taking all the results of this proof together we obtain $\frac{\partial x_1^*}{\partial \underline{t}} > 0$ and $\frac{\partial x_2^*}{\partial \underline{t}} = -\frac{\partial x_1^*}{\partial \underline{t}} < 0$. $\square$

Proof of Proposition 4.4. The derivative of x_1^* with respect to δ is given by

$$-\frac{6\,\alpha\,\underline{t}\,L^2 - 6\,\alpha\,L^2 + 6\,L^2 + 4\,\alpha\,\sigma^2\,\underline{t}\,L - 9\,\alpha\,\underline{t}\,L + 4\,\alpha\,\sigma^2\,L - 4\,\sigma^2\,L - 9\,\alpha\,L + 9\,L - 6\,\alpha\,\sigma^2\,\underline{t} + 6\,\alpha\,\sigma^2 - 6\,\sigma^2}{2\,(2\,\alpha\,\delta\,\underline{t}\,L + 2\,\alpha\,\delta\,L - 2\,\delta\,L - 3\,\alpha\,\delta\,\underline{t} + 3\,\alpha\,\delta - 3)^2}$$

It is straightforward to see that the denominator is positive. Thus, turning to the numerator we obtain

$$\begin{aligned}
&-(6\,\alpha\,\underline{t}\,L^2 - 6\,\alpha\,L^2 + 6\,L^2 + 4\,\alpha\,\sigma^2\,\underline{t}\,L - 9\,\alpha\,\underline{t}\,L \\
&+ 4\,\alpha\,\sigma^2\,L - 4\,\sigma^2\,L - 9\,\alpha\,L + 9\,L - 6\,\alpha\,\sigma^2\,\underline{t} + 6\,\alpha\,\sigma^2 - 6\,\sigma^2) \\
&= -(6L^2(\alpha\underline{t} - \alpha + 1) + 4\sigma^2 L(\alpha - 1 + \alpha\underline{t}) + 9L(-\alpha\underline{t} - \alpha + 1) + 6\sigma^2(-\alpha\underline{t} + \alpha - 1)) \\
&= -((\alpha\underline{t} - \alpha + 1)(6L^2 - 6\sigma^2) + (\alpha\underline{t} + \alpha - 1)(4\sigma^2 L - 9L)) \\
&= 6(\alpha\underline{t} - \alpha + 1)(\sigma^2 - L^2) + L(\alpha\underline{t} + \alpha - 1)(9 - 4\sigma^2)
\end{aligned}$$

As a next step, we evaluate the numerator for $\alpha = 1$. We obtain

$$6\underline{t}(\sigma^2 - L^2) + L\underline{t}(9 - 4\sigma^2). \tag{A.7}$$

In the following we want to establish that expression (A.7) is larger than zero.

$$\begin{aligned} 6\underline{t}(\sigma^2 - L^2) + L\underline{t}(9 - 4\sigma^2) &= \underline{t}(6\sigma^2 - 6L^2 + 9L - 4L\sigma^2) \\ &> \underline{t}[6\sigma^2 - 6L + 9L - 4L\sigma^2) \\ &= \underline{t}(3L + 6\sigma^2 - 4L\sigma^2) \\ &\geq \underline{t}(3L + 6\sigma^2 - 2\sigma^2) \\ &= \underline{t}(3L - 4\sigma^2) > 0 \end{aligned}$$

Taking these results together, we obtain that $\frac{\partial x_1^*}{\partial \delta}$ is positive for $\alpha = 1$. Similarly, we evaluate the numerator for $\alpha = 0$ and obtain

$$6(\sigma^2 - L^2) - L(9 - 4\sigma^2) \tag{A.8}$$

In the next step, we want to establish that expression (A.8) is smaller than zero.

$$\begin{aligned} 6(\sigma^2 - L^2) - L(9 - 4\sigma^2) &= 6\sigma^2 - 6L^2 - 9L + 4L\sigma^2 \\ &< 6L^2 - 6L^2 - 9L + 4L^3 \\ &\leq -9L + 4L \\ &= -5L \\ &< 0 \end{aligned}$$

The numerator of $\frac{\partial x_1^*}{\partial \delta}$ is a continuous function in α. Furthermore we have that $\frac{\partial x_1^*}{\partial \delta} < 0$ for $\alpha = 0$ and $\frac{\partial x_1^*}{\partial \delta} > 0$ for $\alpha = 1$. By using the intermediate value theorem for continuous real-valued functions, we obtain that there is an $\alpha^* \in (0, 1)$ such that

$$\frac{\partial x_1^*}{\partial \delta} = \frac{\partial x_2^*}{\partial \delta} = 0.$$

What remains to be shown is that α^* is unique. Taking the derivative of the numerator we obtain

$$6(\underline{t}-1)(\sigma^2-L^2)+L(\underline{t}+1)(9-4\sigma^2)$$

As we can see, the numerator can be strictly increasing or strictly decreasing in α, since it is independent of α and nonzero. We want to show that the numerator of the derivative is strictly increasing in α. We can establish the following estimations

$$\begin{aligned}
&6(\underline{t}-1)(\sigma^2-L^2)+L(\underline{t}+1)(9-4\sigma^2)\\
&=6(\underline{t}\sigma^2-\underline{t}L^2-\sigma^2+L^2)+L(9\underline{t}-4\underline{t}\sigma^2+9-4\sigma^2)\\
&=6\underline{t}\sigma^2-6\underline{t}L^2-6\sigma^2+6L^2+9\underline{t}L-4\underline{t}\sigma^2L+9L-4\sigma^2L\\
&\geq 6\underline{t}\sigma^2-6\underline{t}\sigma^2-6\underline{t}L^2+6\underline{t}L^2-4\sigma^2L-4\sigma^2L+9\underline{t}L+9L\\
&>-8\sigma^2L+9L\\
&\geq -2L+9L\\
&=7L\\
&>0
\end{aligned}$$

This establishes that the numerator has for every parameter constellation a unique zero $\alpha^*\in(0,1)$, where $\frac{\partial x_1^*}{\partial\delta}<0$ for all $0\leq\alpha<\alpha^*$, $\frac{\partial x_1^*}{\partial\delta}=0$ for $\alpha=\alpha^*$ and $\frac{\partial x_1^*}{\partial\delta}>0$ for all $1\geq\alpha>\alpha^*$. Since $x_2^*=-x_1^*$ we obtain the postulated result for x_2^* without reexamining the respective derivative. $\square$

Proof of Proposition 4.5. The derivative of ${x_1}^*$ with respect to L is given by

$$\begin{aligned}
&(\,\delta(\,4\,\alpha^2\,\delta\,\underline{t}^2\,L^2-4\,\alpha^2\,\delta\,L^2+8\,\alpha\,\delta\,L^2-4\,\delta\,L^2-12\,\alpha^2\,\delta\,\underline{t}^2\,L+24\,\alpha^2\,\delta\,\underline{t}\,L\\
&-12\,\alpha\,\delta\,\underline{t}\,L-12\,\alpha\,\underline{t}\,L-12\,\alpha^2\,\delta\,L+12\,\alpha\,\delta\,L+12\,\alpha\,L-12\,L+9\,\alpha^2\,\delta\,\underline{t}^2\\
&+4\,\alpha\,\delta\,\sigma^2\,\underline{t}-4\,\alpha\,\sigma^2\,\underline{t}-9\,\alpha\,\delta\,\underline{t}+9\,\alpha\,\underline{t}+4\,\alpha\,\delta\,\sigma^2-4\,\delta\,\sigma^2-4\,\alpha\,\sigma^2+4\,\sigma^2-9\,\alpha^2\,\delta+9\,\alpha\,\delta+9\,\alpha-9\,)\,)\\
&\cdot(2(2\,\alpha\,\delta\,\underline{t}\,L+2\,\alpha\,\delta\,L-2\,\delta\,L-3\,\alpha\,\delta\,\underline{t}+3\,\alpha\,\delta-3)^2)^{-1}
\end{aligned}$$

As we can see, the denominator is positive. Therefore the sign of the derivative solely depends on the numerator. Since $\delta\geq 0$ it is sufficient to consider the sign of numerator divided by δ. We denote this expression with $(*)$. Inserting $\alpha = 0$ into expression $(*)$ yields

$$\begin{aligned}
&\delta\,\left(-4\,\delta\,L^2 - 12\,L - 4\,\delta\,\sigma^2 + 4\,\sigma^2 - 9\right)\\
&\leq \delta[-12L + 4\sigma^2 - 9]\\
&\leq \delta[-12L - 8]\\
&= -4\delta(3L + 2)\\
&< 0
\end{aligned}$$

This shows that the derivative is strictly negative for $\alpha = 0$. Similarly, inserting $\alpha = 1$ into $(*)$ we obtain

$$\delta\,\underline{t}\left(4\,\delta\,\underline{t}\,L^2 - 12\,\delta\,\underline{t}\,L + 12\,\delta\,L - 12\,L + 9\,\delta\,\underline{t} + 4\,\delta\,\sigma^2\ - 4\,\sigma^2\ - 9\,\delta\ + 9\right) \tag{A.9}$$

We want to show that expression (A.9) is strictly positive. This we show in the following way

$$\begin{aligned}
&\delta\,\underline{t}\left(4\,\delta\,\underline{t}\,L^2 - 12\,\delta\,\underline{t}\,L + 12\,\delta\,L - 12\,L + 9\,\delta\,\underline{t} + 4\,\delta\,\sigma^2\ - 4\,\sigma^2\ - 9\,\delta\ + 9\right)\\
&= \delta\,\underline{t}\,[4\delta\underline{t}L^2 + 12L[-\delta\underline{t} + \delta - 1] + 9[\delta\underline{t} - \delta + 1] + 4\sigma^2[\delta - 1]\,]\\
&= \delta\,\underline{t}\,[4\delta\underline{t}L^2 + [1 + \delta\underline{t} - \delta]\,\underbrace{[9 - 12L]}_{\geq 9-6=3} + 4\sigma^2[\delta - 1]\,]\\
&\geq \delta\,\underline{t}\,[4\delta\underline{t}L^2 + 3(1 + \delta\underline{t} - \delta] + 4\sigma^2[\delta - 1]\,]\\
&= \delta\,\underline{t}\,[4\delta\underline{t}L^2 + 3(1 - \delta) + 3\delta\underline{t} + 4\sigma^2(\delta - 1)]\\
&= \delta\,\underline{t}\,[4\delta\underline{t}L^2 + 3\delta\underline{t} + (1 - \delta)\,\underbrace{[3 - 4\sigma^2]}_{\geq 3-4\frac{1}{4}=2}]\\
&= \delta\,\underline{t}\,[4\delta\underline{t}L^2 + 3\delta\underline{t} + 2(1 - \delta)]\\
&> 0 \qquad (\underline{t} > 0,\ L > 0)
\end{aligned}$$

Now we know that $\frac{\partial x_1^*}{\partial L} < 0$ for $\alpha = 0$ and $\frac{\partial x_1^*}{\partial L} > 0$ for $\alpha = 1$. The derivative is continuous. By the intermediate value theorem for continuous functions we obtain that there is $\hat{\alpha} \in (0,1)$ such that $\frac{\partial x_1^*}{\partial L} = 0$ for $\alpha = \hat{\alpha}$. What remains to be shown is that $\hat{\alpha}$ is unique. In this case, we know that x_1^* is strictly decreasing in L for values of α smaller that $\hat{\alpha}$, constant for $\alpha = \hat{\alpha}$ and increasing for $1 \geq \alpha > \hat{\alpha}$. Solving expression $(*)$ for α we know that we find at least one zero, the zero $\hat{\alpha}$ in the interval $[0, 1]$. Since $(*)$ is a quadratic function in α, we can conclude that it has one more root $\hat{\hat{\alpha}}$. This root cannot be located in the interval $[0, 1]$ as well. This we can show by making use of a proof by contradiction. Assume w.l.o.g. that $\hat{\hat{\alpha}}$ was in the interval $[0, 1]$ as well and that $\hat{\alpha} < \hat{\hat{\alpha}}$. We can distinguish two cases. Case 1 is that the quadratic function has a global maximum and case 2 is that the quadratic function has a global minimum. Since we can find both roots in the interval $[0, 1]$ the global maximum or alternatively the global minimum are also located in this interval. Assume now that we have a quadratic function with a global maximum. In this case we have that $(*)$ is smaller zero for $\alpha < \hat{\alpha}$, equal to zero for $\alpha \in \{\hat{\alpha}, \hat{\hat{\alpha}}\}$ and smaller zero for $\alpha \in (\hat{\hat{\alpha}}, 1]$. The last statement contradicts that $(*)$ is larger zero for $\alpha = 1$ what we already showed above. For a global minimum a similar line of arguments holds. Since both roots are located in the interval $[0, 1]$, we can deduce that the minimum is located in this interval as well. In this case we have that $(*)$ is larger than zero for $\alpha < \hat{\alpha}$, equal to zero for $\alpha \in \{\hat{\alpha}, \hat{\hat{\alpha}}\}$ and again larger zero for $\alpha \in (\hat{\hat{\alpha}}, 1]$. The first statement contradicts that $(*)$ is smaller zero for $\alpha = 0$. To sum up, we have only one root in $[0, 1]$. $\square$

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