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Functions with minimal number of critical points

Gutachter: **Prof.Dr. THOMAS BARTSCH**
Prof.Dr.Dr.h.c. ALBRECHT DOLD

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To A.L.M.I.

Contents

1 Introduction

This introduction presents that part of the critical points theory, which interacts with our research and the position of this work on the genealogical graph of this theory. We use the term graph, instead of the more familiar tree, to emphasize the relevance of the graph theory in our work.

In 1934 during their research in the calculus of variation, Lusternik and Schnirelmann [16] introduced a new numerical topological invariant (the Lusternik-Schnirelmann category), showing that it carried important information about both the existence of critical points and the cardinality of the critical set. In the sixties the Lusternik-Schnirelmann theory of critical points was extended to Hilbert manifold by Schwartz [27] and to Banach manifolds by Palais [23]. The motivation for considering critical point theory in the context of infinite dimensional manifolds comes from the immediate applicability of results to proving existence theorems in the calculus of variations. Then, in 1968, Takens [33] considered finite dimensional manifolds with boundary and functions which are constant regular and maximal on boundary. This article will play a central role in our work. A different direction of research was stimulated by the survey of I.M. James [13] from 1978. His comprehensive review put definitively the accent on the Lusternik-Schnirelmann category giving to homotopy theorists the invitation to introduce and develop many variations of the Lusternik-Schnirelmann category. These invariants became an outstanding research subject and later, in 1995, another survey was necessary for the updating (James, [14]). This development distracts the attention from the original problem. Here we present it in a simplified way leaving out details: let M be a smooth compact manifold and $f : M \to \mathbb{R}$ a smooth function. By $crit(f)$ we denote the number of critical points of f, i.e. the cardinality of the critical set K_f of f.

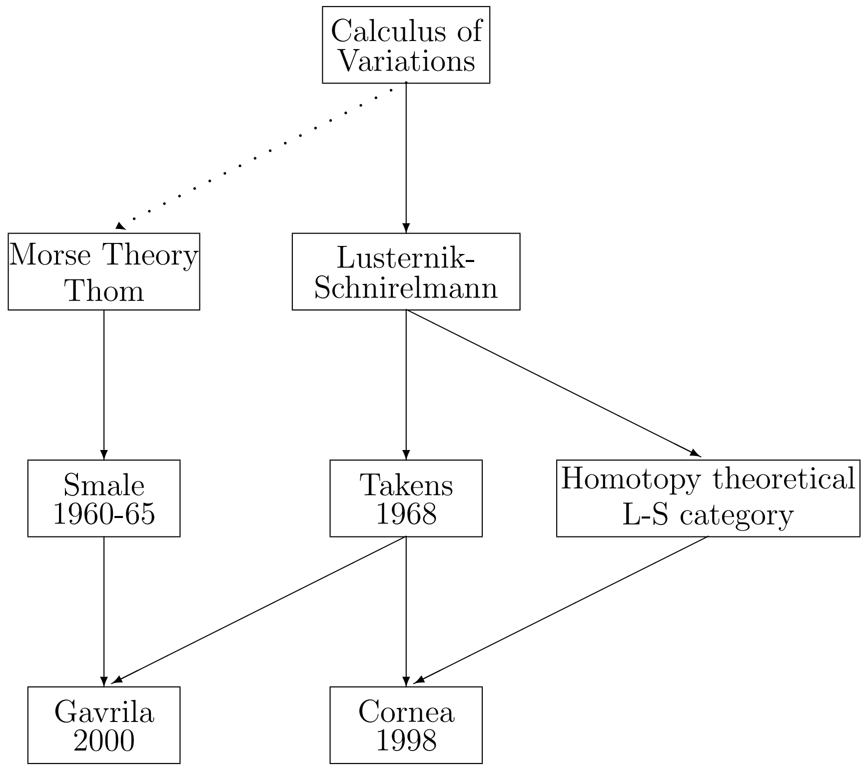
Assume f has a finite number of critical points. Than the first numerical question about the $crit(f)$ concerns its magnitude. An estimation of $crit(f)$ is given by the famous Lusternik-Schnirelmann theorem:

$$cat(M) \leq crit(f),$$

where $cat(M)$ is the L-S category. But is the L-S category $cat(M)$ of M the minimum of the set

$$\{crit(f) \mid f \in C^\infty(M)\}?$$

In order to find manifolds with this property Takens introduced in 1968 the notion of F. category. For closed manifolds, the F. category $F(M)$ of M is exactly the minimum of $\{crit(f) \mid f \in C^\infty(M)\}$. Instead of $F(M)$ we prefer the notation $crit(M)$ as in [5] and we call it simply the "*crit*". Information about this notion are



scarce and the list of open problems very long. The aim of this thesis is to analyze the general properties of *crit* and to compute it for some particular manifolds.

The paper of Takens was an isolated effort to characterize $crit(M)$ until 1998 when Cornea "destroyed" the tree structure of the numerical critical point theory (see the figure) introducing a cycle in the graph. He associated to a manifold with boundary W (with fairly general properties) an invariant of L-S category type, the cone-length $Cl(W)$ of W. Then using methods of surgery he constructed a function on W constant regular and maximal on ∂W with at most $Cl(W)+1$ critical points.

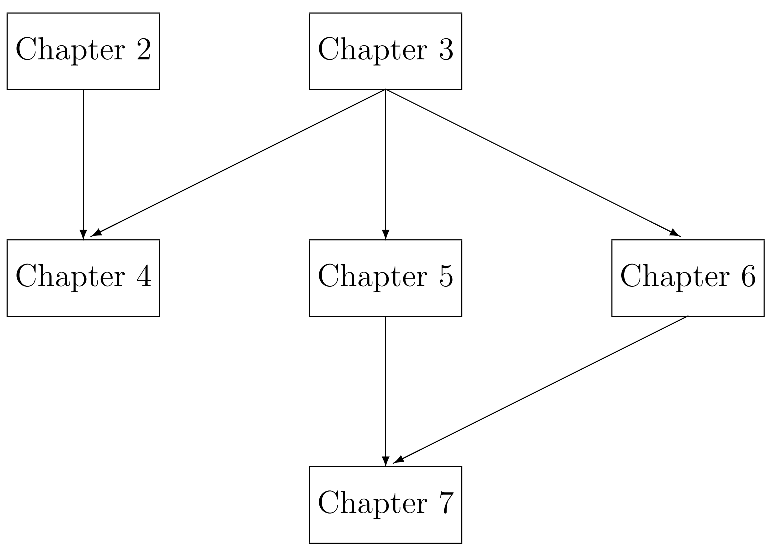
Similar problems for functions with non-degenerate critical points could be solved using the Morse theory (see Milnor [18] or Hajduk [10]). In this case the manifold has the homotopy type of a CW-complex with precisely one cell for each critical point. This is a better result, but the assumption of non-degeneracy is inconvenient since we are interested in the function rather than in the manifold. Inspired by Takens paper [33] and the Morse theory in the Smale setting [19], [30], [31] we return to the original question about the minimal number of critical points for a smooth function on a manifold and we analyze the *crit* for products of manifolds. For the Lusternik-Schnirelmann category *cat* we know that

$$cat(N \times Q) \leq cat(N) + cat(Q) - 1$$

for any connected manifolds N and Q. In this thesis we mainly analyze whether the previous inequality remains true for the *crit*. Using concepts from the graph theory we can give some positive answers. For instance if $\mathbb{S}^n$ is the n-dimensional sphere and the dimension of M is at most 7 than

$$crit(M \times \mathbb{S}^n) \leq crit(M) + 1.$$

In chapter 2 we introduce a generalization of the category in the sense of Lusternik-Schnirelmann. This new concept unifies many different notions: the L-S category, the strong category, the ball category, etc. Exploiting the advantages of the axiomatic definition of the generalized Lusternik-Schnirelmann category we prove that the ball category *bcat* (recall $cat(M) \leq bcat(M)$) is a lower bound for the number of the critical points of a function. This result is used in the chapter 4 to compute the *crit* of some closed manifolds. In chapter 3 the fusing lemma gives sufficient conditions whether we get from a triad function with two critical points a triad function with at most one critical points. The analogue of fusing lemma for non-degenerate critical points is the cancellation lemma (see [24]). Using the fusing lemma we prove a first proposition about the *crit* for the product of some class of manifolds. To extend this result, in chapter 5 we define and characterize different graphs attached to a function with a minimal number of points (a function f with $crit(f) = crit(M)$). Then we can apply (in chapter 7) the extending fusing lemma (from chapter 6) in order to obtain upper bounds of *crit* for the product of a bigger class of manifolds.



I want to especially express my gratitude to Professor Albrecht Dold for his generous support during my first year in Heidelberg. My appreciation goes to my PhD supervisor, Professor Thomas Bartsch for his providing me with several comments and important hints to simplify the presentation of the work. Special thanks are addressed to my friend Stefan Born for his comments and inciting discussions about "short trousers" and to Bjoern Grahn (Kupferstecher) who get a passion for L-S category and LaTeX, typing parts of this thesis. To all other people I thank collectively because otherwise the list would be so long that somebody could ask himself what my own contribution to this work is. Last but not least I was supported by the DAAD while this work was under way.

2 The Lusternik-Schnirelmann theorem for the ball category

In this chapter we try to find a category which is a better lower bound for the number of critical points of a function than the Lusternik-Schnirelmann category. The Lusternik-Schnirelmann category $cat(X)$ of a space X is the smallest number k such that there is a closed covering $\{X_1, \dots, X_k\}$ of X for which each X_i is contractible in X. The motivation for introducing this concept was the fact that it gives a lower bound for the number of critical points of a function. More precisely, in their study of the "calculus of variations in the large" Lusternik and Schnirelmann [16] proved the following:

Theorem 1. *If M is a closed smooth manifold and f is a smooth function on M then the number of critical points of f is at least $cat(M)$.*

This is the result we want to extend. The previous generalizations followed two main directions:

(a) the manifold M gets a more general structure. Schwartz [27] generalized the Lusternik-Schnirelmann theorem to Hilbert manifolds, Palais [23] extended it to Finsler manifolds and more recently Szulkin [32] considered C^1 manifolds.

(b) the class of differentiability of the function f (and implicitly of M) get larger and larger. Palais [23] already considered C^{2-} functions (C^1 functions whose derivative are locally Lipschitz), then the theory evolved very fast via the Clark's subdifferential and finally Corvellec, Degiovanni and Morzocchi [6], [7] considered even metric spaces in order to prove a theorem of Lusternik-Schnirelmann type for the "critical" points of a continuous function.

Our direction of research is different: we are looking for another (better than cat) lower bound for the number of critical points of f denoted by $crit(f)$. We prove for a closed smooth manifold M of dimension $n \geq 1$ and a smooth function f on M that the ball category $bcat(M)$ of M (and implicitly the strong category $scat(M)$ because $scat(M) \leq bcat(M)$) is a lower bound for $crit(f)$.

The ball category $bcat(M)$ and the strong category $scat(M)$ of M have a similar definition as the Lusternik-Schnierelmann category $cat(M)$ of M: it is the smallest integer k such that M can be covered with k open n-balls (or equivalently, with k open subsets homeomorphic to $\mathbb{R}^n$) respectively with open subsets of M which are contractible in themselves. Results about the ball category may be found in Singhof [29] or Montejano [21], details about the strong category from a new perspective in Clapp and Puppe [4].

To get a better lower bound of the $crit f$ we introduce an abstract notion, the generalized category of a topological space X. The generalized category $\lambda cat_X^{\mathcal{I}}$ of the space X must satisfy the property of non-triviality, monotonicity, sub-additivity and invariance under the space of mappings

$$\mathcal{I} \subseteq Def_{Id}(X) = \{\varphi \in C(X \times [0,1], X) \mid \varphi_0 = Id\}.$$

The space of mappings $\mathcal{I}$ in the definition of $\lambda cat_X^{\mathcal{I}}$ is different from a problem to the other and must be adapted to the properties of the space X (smoothness, non-compactness, etc) in order to obtain the category which gives the best information about the number of critical points for functions defined on X. For smooth manifolds, $\mathcal{I}$ can be chosen the space of diffeotopies, denoted $\mathfrak{Diff}$.

We sacrifice the generality and restrict our research to M a closed finite dimensional smooth manifold and to smooth function on M. The others possible extensions (for Finsler manifolds, manifolds with boundary, continuous functions, etc.) would take the volume of another thesis. Therefore unless otherwise stated, in this chapter a manifold M is assumed to be smooth and to have finite dimension. The main theorem about the number of critical points will have the following aspect:

Theorem 2. *Let M be a closed manifold and $f : M \to \mathbb{R}$ be a smooth function. Then the function f has at least* bcat(M) *critical points.*

1. The generalized category

To prove the Lusternik-Schnierelmann theorem for the ball category we introduce a general notion of category.

Definition 2.1. Let X be a topological space and $\mathcal{I} \subseteq Def_{Id}(X)$. A non-negative integer valued function $\lambda cat_X^{\mathcal{I}}$ defined on the power set of X is called a *generalized category relative* to $\mathcal{I}$ if it satisfies the following conditions:

($1_{\mathcal{I}}$.) (Non-triviality) $\lambda cat_X^{\mathcal{I}}(A) = 0$ iff $A = \emptyset$.

($2_{\mathcal{I}}$.) (Monotonicity) if $A \subseteq B \subseteq X$ then $\lambda cat_X^{\mathcal{I}}(A) \leq \lambda cat_X^{\mathcal{I}}(B)$.

($3_{\mathcal{I}}$.) (Sub-additivity) for each $A, B \subseteq X$ holds the inequality

$$\lambda cat_X^{\mathcal{I}}(A \cup B) \leq \lambda cat_X^{\mathcal{I}}(A) + \lambda cat_X^{\mathcal{I}}(B).$$

($4_{\mathcal{I}}$.) (Invariance under deformations) if $\varphi \in \mathcal{I}$ and $A \subseteq X$ then

$$\lambda cat_X^{\mathcal{I}}(A) \leq \lambda cat_X^{\mathcal{I}}(\varphi_1(A)).$$

The generalized categories we use are mostly well known, but there are also some new ones.

Example 2.2. 1. Let $A \subseteq X$. The *Lusternik-Schnirelmann category* $cat_X(A)$ of A in X is the smallest integer k such that A can be covered by k closed subsets of X each of which is contractible in X. Evidently $cat(X) = cat_X(X)$.

If $\mathcal{I} \subseteq Def_{Id}(X)$ then the Lusternik-Schnirelmann category is a generalized category relative to $\mathcal{I}$. The first three condition of the definition of the generalized category are trivialy satisfied. The last one $(4_{\mathcal{I}})$ is easy to verify but we nevertheless prove it because it always appears in the literature for A closed and φ a deformation of A in X. Let $\varphi_1(A) \subseteq B_1 \cup \ldots \cup B_k$, each B_i closed and contractible in X. If $A_i = \varphi_1^{-1}(B_i)$ then A_i is closed in X and $A \subseteq A_1 \cup \ldots \cup A_k$. Since $\varphi_t \mid A_i$ is a deformation of A_i into B_i and B_i is contractible in X, A_i is contractible in X. Therefore $cat_X(A) \leq cat_X(\varphi_1(A)) = k$.

Remark 2.3. If X is an ANR (for the definition of an ANR see Dold [8]) then the previous definition of the Lusternik- Schnirelmann category is equivalent to the one with open coverings instead of closed coverings. Both are generalized categories relative to $\mathcal{I}$; even the definition of Lusternik-Schnirelmann category with arbitrary coverings instead of closed coverings generates a generalized Lusternik-Schnirelmann category relative to $\mathcal{I}$.

2. If $\mathcal{I} \subseteq \mathfrak{Iso}(X)$ the space of isotopies of X then the strong category *scat* with the usual definition is a generalized category relative to $\mathcal{I}$.

3. If $\mathcal{I} \subseteq \mathfrak{Iso}(X)$ then the covering dimension dim_X defined for subsets of X is a generalized category relative to $\mathcal{I}$.

4. If $\mathcal{I} \subseteq Def_{Id}(X)$ and $\mathcal{J} \subseteq \mathfrak{Iso}(X)$ and p is a positive integer than $p \cdot cat$ and $p \cdot scat$ are generalized category relative to $\mathcal{I}$ respectively relative to $\mathcal{J}$.

The great advantage of the generalized category is that it gives a unitary handling for the different definitions of categories. For instance we do not need to differentiate between the definition for a categories with open coverings and with closed coverings.

2. Generalized categories for closed manifolds

Let M be a closed manifold. According to the differential structure of a manifold, the natural candidate for $\mathcal{I}$ is the subspace $\mathfrak{Diff}(M)$ of $\mathfrak{Iso}(M)$, consisting of the diffeotopies of M. A diffeotopy φ of M is smooth path in the group of diffeomorphisms $Diff(M)$, starting at the identity, i.e. a smooth application

$$\varphi : M \times [0,1] \to M,$$

$$\varphi(x,t) = \varphi_t(x)$$

such that

$$\begin{cases} \varphi_t \in Diff(M) \quad for\ each \quad t \in [0,1] \\ \varphi_0 = id_M. \end{cases}$$

Then a generalized category $\lambda cat_M^{\mathfrak{Diff}}$ relative to $\mathfrak{Diff}(M)$ satisfies the properties $(1_{\mathfrak{Diff}(M)})$, $(2_{\mathfrak{Diff}(M)})$, $(3_{\mathfrak{Diff}(M)})$ and

($4_{\mathfrak{Diff}(M)}$.) (Invariance under diffeotopies) if φ is a diffeotopy and $A \subseteq M$ then

$$\lambda cat_M^{\mathfrak{Diff}}(A) \leq \lambda cat_M^{\mathfrak{Diff}}(\varphi_1(A)).$$

Let $f : M \to \mathbb{R}$ be a smooth function on M. We want to find a lower bound for the number of critical point of f denoted by $crit(f)$.

For each $c \in \mathbb{R}$ let

$$M_c = f^{\leq c} = \{x \in M \mid f(x) \leq c\}$$

and for each positive integer $m \leq \lambda cat_M^{\mathfrak{Diff}}(M)$ define

$$c_m(f) = \inf\{c \in \mathbb{R} \mid \lambda cat_M^{\mathfrak{Diff}}(M_c) \geq m\}.$$

A part of the values c_m are critical values of f. To show this fact we need some results known in the literature as the non-critical interval theorem and the deformation lemma. The results are adapted from Palais [23] for our aim in the following:

Lemma 2.4. *Let M be a closed manifold, $f : M \to \mathbb{R}$ a smooth function and $c \in \mathbb{R}$.*

(1) If c is a regular value of f then for some $\varepsilon > 0$ there is a diffeotopy φ of M with $\varphi_1(M_{c+\varepsilon}) \subseteq M_{c-\varepsilon}$.

(2) If c is a isolated critical value of f then for any neighborhood U of the set of critical points of f in the level c there is $\varepsilon > 0$ and a diffeotopy φ of M with $\varphi_1(M_{c+\varepsilon} \setminus U) \subseteq M_{c-\varepsilon}$.

Proof. Palais proved in [23] for f a C^{2-} function on a complete C^2 Finsler manifold without boundary the existence of a isotopy with the qualities required by part (1) respectively by part (2) of the lemma. Now, f is a smooth function on M a closed smooth manifold and therefore we get a smooth (instead of a C^{1-}) pseudo-gradient vector field on M. This is the fundamental argument for the existence of the diffeotopy φ and the proof of this lemma may repeat verbatim the proof of Palais.♣

The function f, defined on the compact manifold M, possesses a minimal value denoted by $\min f$. The next theorem give us the first information about the critical values of f.

Theorem 3. *For each*

$$m = \lambda cat_M^{\mathfrak{Diff}}(f^{-1}(\min f)), \ldots, \lambda cat_M^{\mathfrak{Diff}}(M),$$

the value $c_m(f)$ is a critical value of f.

Proof. Suppose that $c_m(f) = c$ is not a critical value of f. By the first part of the lemma 2.4 for some $\varepsilon > 0$ there is a diffeotopy φ_t of M with $\varphi_1(M_{c+\varepsilon}) \subseteq M_{c-\varepsilon}$. Now applying first the monotonicity and then the invariance under the diffeotopy property from the definition of the generalized category we obtain:

$$\lambda cat_M^{\mathfrak{Diff}}(M_{c-\varepsilon}) \geq \lambda cat_M^{\mathfrak{Diff}}(\varphi_1(M_{c+\varepsilon})) \geq \lambda cat_M^{\mathfrak{Diff}}(M_{c+\varepsilon}) \geq m.$$

But by definition $c_m(f) = c \leq c - \varepsilon$, contradiction. ♣

The next theorem will show that the equality in the following sequence

$$c_{\lambda cat_M^{\mathfrak{Diff}}(f^{-1}(\min f))}(f) \leq \ldots \leq c_m(f) \leq c_{m+1}(f) \leq \ldots \leq c_{\lambda cat_M^{\mathfrak{Diff}}(M)}(f)$$

will be sometimes (depending on the generalized category that we use) compensated by having more critical points at that level. To count their multiplicity we introduce for any generalized category $\lambda cat_M^{\mathfrak{Diff}}$ its *multiplicity counter*, a positive integer or infinity valued function $n_{\lambda cat_M^{\mathfrak{Diff}}} : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$. For each $k \in \mathbb{N}$ we define:

$$n_{\lambda cat_M^{\mathfrak{Diff}}}(k) = \sup\{l \in \mathbb{N} \mid \inf_{card(P)<l} \sup_U \{k - \lambda cat_M^{\mathfrak{Diff}}(U) \mid U \ neighb.\ of\ P \subset M\} > 0\}.$$

The multiplicity counter is well defined: for each positive integer k we have $n_{\lambda cat_M^{\mathfrak{Diff}}}(k) \geq 1$. Indeed

$$1 \in \{l \in \mathbb{N} \mid \inf_{card(P)<l} \sup_U \{k - \lambda cat_M^{\mathfrak{Diff}}(U) \mid U \ neighborhood\ of\ P \subset M\} > 0\}$$

because the only set $P \subset X$ of cardinality $card(P) < 1$ is the empty set $\emptyset$ which is in the same time a neighborhood of itself hence by the property $(1_{\mathfrak{Diff}(M)})$ of $\lambda cat_M^{\mathfrak{Diff}}$ we have $k - \lambda cat_M^{\mathfrak{Diff}}(P) = k > 0$. It is relatively easy to determine the multiplicity counter for the categories of the example 2.2. If λcat is the "scaled" Lusternik-Schnirelmann category $p \cdot cat$ or the "scaled" strong category $p \cdot scat$ then $n_{\lambda cat}(k) = 1$ for $k \leq p$ and $n_{\lambda cat}(k) = \infty$ else. For the topological dimension dim_M we have:

$$n_{dim_M}(k) = \begin{cases} 1 & \text{if} \quad k \leq n \\ \infty & \text{else.} \end{cases}$$

Using the multiplicity counter we can prove a theorem about the multiplicity of critical points situated in the same level.

Theorem 4. *Let m, k be positive integers such that*

$$\lambda cat_M^{\mathfrak{Diff}}(f^{-1}(\min f)) \leq m + 1 \leq m + k \leq \lambda cat_M^{\mathfrak{Diff}}(M).$$

If

$$c_{m+1}(f) = c_{m+2}(f) = \ldots = c_{m+k}(f) = c$$

then

(a) *for each neighborhood U of the critical set K_c the generalized category $\lambda cat_M^{\mathfrak{Diff}}$ satisfies the inequality: $\lambda cat_M^{\mathfrak{Diff}}(U) \geq k$.*

(b) *at the level c there are at least $n_{\lambda cat_M^{\mathfrak{Diff}}}(k)$ critical points.*

(c) *for $k \geq 2$, if there is an open (or closed) ball $\overset{\circ}{\mathbb{B}} \subseteq M$ such that $\lambda cat_M^{\mathfrak{Diff}}(\overset{\circ}{\mathbb{B}}) = 1$ then the level c contains an infinite number of critical points.*

Proof. (a) By the second part of the lemma 2.4 there is $\varepsilon > 0$ and a diffeotopy φ of M such that $\varphi_1(M_{c+\varepsilon} \setminus U) \subseteq M_{c-\varepsilon}$. Applying first the invariance under diffeotopy of $\lambda cat_M^{\mathfrak{Diff}}$ and afterwards its monotonicity property we obtain:

$$\lambda cat_M^{\mathfrak{Diff}}(M_{c+\varepsilon} \setminus U) \leq \lambda cat_M^{\mathfrak{Diff}}(\varphi_1(M_{c-\varepsilon} \setminus U)) \leq \lambda cat_M^{\mathfrak{Diff}}(M_{c-\varepsilon}) \leq m.$$

Last inequality holds because $c - \varepsilon < c = c_{m+1}(f) = \inf\{a \in \mathbb{R} | \lambda cat_M^{\mathfrak{Diff}}(M_a) \geq m+1\}$. Similar arguments imply

$$m+k \leq \lambda cat_M^{\mathfrak{Diff}}(M_{c+\varepsilon}) \leq \lambda cat_M^{\mathfrak{Diff}}((M_{c+\varepsilon} \setminus U) \cup U) \leq \lambda cat_M^{\mathfrak{Diff}}((M_{c+\varepsilon} \setminus U) + \lambda cat_M^{\mathfrak{Diff}}(U).$$

Comparing the two previous inequalities it is obvious that $\lambda cat_M^{\mathfrak{Diff}}(U) \geq k$.

(b) By the theorem 3 the value c is a critical value of f hence $1 \leq card(K_c)$. We suppose that K_c contains less than $n_{\lambda cat_M^{\mathfrak{Diff}}}(k)$ critical points: $1 \leq card(K_c) < n_{\lambda cat_M^{\mathfrak{Diff}}}(k)$. On the other hand by the definition of the multiplicity counter $n_{\lambda cat_M^{\mathfrak{Diff}}}$ for each $1 \leq l \leq n_{\lambda cat_M^{\mathfrak{Diff}}}(k)$:

$$\inf_{card(P)<l} \sup_U \{k - \lambda cat_M^{\mathfrak{Diff}}(U) \mid U \ neighborhood\ of\ P \subset M\} > 0.$$

In particular for $l = card(K_c) + 1$ we get a neighborhood U of K_c such that $k - \lambda cat_M^{\mathfrak{Diff}}(U) > 0$ contradicting the first part of the theorem.

(c) Suppose $|K_c| < \infty$. There is some positive integer l such that $K_c = \{x_1, \ldots, x_l\}$. A well known theorem (see Hirsch [11]) states the existence of a diffeotopy $\varphi : M \times I \to M$ such that $\varphi_1(x_i) \in \overset{\circ}{\mathbb{B}}$ for each $i \in \overline{1,l}$. Let $U = \varphi_1^{-1}(\overset{\circ}{\mathbb{B}})$ be a neighborhood of K_c. By the property of invariance under diffeotopies of $\lambda cat_M^{\mathfrak{Diff}}$,

$$\lambda cat_M^{\mathfrak{Diff}}(U) \leq \lambda cat_M^{\mathfrak{Diff}}(\varphi_1(U)) = \lambda cat_M^{\mathfrak{Diff}}(\overset{\circ}{\mathbb{B}}) = 1$$

but from part (a) of this theorem $\lambda cat_M^{\mathfrak{Diff}}(U) \geq k \geq 2$. This contradiction confirms the fact that the number of critical points at the level c is infinite. ♣

The second part of the proposition suggests the property that a generalized category must satisfy such that either

1. all the critical values $c_m(f)$ are distinct for $\lambda cat_M^{\mathfrak{Diff}}(f^{-1}(\min f)) \leq m \leq \lambda cat_M^{\mathfrak{Diff}}(M)$,

   or

2. f has infinity many critical points.

Definition 2.5. A generalized category $\lambda cat_X^{\mathcal{I}}$ relative to $\mathcal{I}$ is called a *generalized Lusternik-Schnirelmann category* if it satisfies the following property:

($5_{\mathcal{I}}$.) for each point $x \in X$ there is a neighborhood U_x of x such that

$$\lambda cat_X^{\mathcal{I}}(U_x) \leq 1.$$

Remark 2.6. For generalized categories relative to $\mathfrak{Diff}$ the condition $(5_{\mathfrak{Diff}(M)})$ is fulfilled iff there exists an open set U with $\lambda cat_M^{\mathfrak{Diff}}(U) \leq 1$ or iff there is an open ball $\mathring{\mathbb{B}}$ with $\lambda cat_M^{\mathfrak{Diff}}(\mathring{\mathbb{B}}) = 1$. This remark is primary a consequence of the invariance under diffeotopies of $\lambda cat_M^{\mathfrak{Diff}}$.

From the last two theorems we obtain that the lower bound for $crit(f)$ is given by any generalized Lusternik-Schnirelmann category $\lambda cat_M^{\mathfrak{Diff}}$ relative to $\mathfrak{Diff}(M)$:

Theorem 5. *(Lusternik-Schnirelmann theorem for a generalized L-S category) Let M be a closed manifold and $f : M \to \mathbb{R}$ be a smooth function. For each generalized Lusternik-Schnirelmann category $\lambda cat_M^{\mathfrak{Diff}}$ relative to $\mathfrak{Diff}$ the function f has at least $\lambda cat_M^{\mathfrak{Diff}}(M)$ critical levels.*

In particular, the ball category is a generalized Lusternik-Schnirelmann category so we obtain from the previous theorem the theorem 2 as corollary.

Proof. We assume that the number $crit(f)$ of critical points of the function f is finite, otherwise the problem is trivial. The set $f^{-1}(\min f)$ consists only in critical points, so $f^{-1}(\min f)$ is finite. By the remark 2.6 there is an open ball $\mathring{\mathbb{B}}$ such that $\lambda cat_M^{\mathfrak{Diff}}(\mathring{\mathbb{B}}) = 1$. Therefore it follows as in proof of the part (c) of the previous theorem that $\lambda cat_M^{\mathfrak{Diff}}\,(f^{-1}(\min f)) \leq \lambda cat_M^{\mathfrak{Diff}}(\mathring{\mathbb{B}}) = 1$. On the other hand $f^{-1}(\min f)$ is not empty hence $\lambda cat_M^{\mathfrak{Diff}}\,(f^{-1}(\min f)) = 1$.

The $crit(f)$ is finite, hence by the part (c) of theorem 4 all the critical levels $c_m(f)$ are distinct for each m,

$$1 = \lambda cat_M^{\mathfrak{Diff}}\left(f^{-1}(\min f)\right) \leq m \leq \lambda cat_M^{\mathfrak{Diff}}(M).$$

Therefore the function f has $\lambda cat_M^{\mathfrak{Diff}}(M)$ distinct critical levels and $crit(f) \geq \lambda cat_M^{\mathfrak{Diff}}(M)$. ♣

For an alternative proof of the theorem 5 we can use the part (b) of the theorem 4 and the following remark:

Remark 2.7. For each generalized Lusternik-Schnirelmann category $\lambda cat_M^{\mathfrak{Diff}}$ relative to $\mathfrak{Diff}$ the multiplicity counter satisfies the inequality $n_{\lambda cat_M^{\mathfrak{Diff}}}(k) \geq k$. Let $P = \{x_1, x_2, ..., x_l | l \in \mathbb{N}\}$ be a subset $P \subset M$ with $card(P) < k$. By the property $(5_{\mathfrak{Diff}(M)})$ of the generalized Lusternik-Schnirelmann category $\lambda cat_M^{\mathfrak{Diff}}$ there are U_i neighborhoods of x_i such that $\lambda cat_M^{\mathfrak{Diff}}(U_i) = 1$. Let $U = \cup_{i=1}^{l} U_i$. Since $\lambda cat_M^{\mathfrak{Diff}}$ is sub-additive

$$\lambda cat_M^{\mathfrak{Diff}}(U) = \lambda cat_M^{\mathfrak{Diff}}(\bigcup_{i=1}^{l} U_i) \leq \sum_{i=1}^{l} \lambda cat_M^{\mathfrak{Diff}}(U_i) < k$$

hence

$$\inf_{card(P)<k} \sup_{U} \{k - \lambda cat_M^{\mathfrak{Diff}}(U) \mid U \text{ } neighborhood \text{ } of \text{ } P\} > 0$$

and by definition of the multiplicity counter $n_{\lambda cat_M^{\mathfrak{Diff}}}(k) \geq k$.

3. The best generalized Lusternik-Schnirelmann category $\lambda cat_M^{\mathfrak{Diff}}$

Let $\mathcal{I}_1, \mathcal{I}_2 \subseteq Def_{Id}(X)$. Then the set of generalized categories on the topological space X is partially ordered by the natural ordering:

$$\lambda_1 cat_X^{\mathcal{I}_1} \leq \lambda_2 cat_X^{\mathcal{I}_2}$$

iff for each $A \subseteq X$

$$\lambda_1 cat_X^{\mathcal{I}_1}(A) \leq \lambda_2 cat_X^{\mathcal{I}_2}(A).$$

For the closed manifolds the partial ordering restricted to generalized Lusternik-Schnirelmann categories relative to $\mathfrak{Diff}$ has a maximum. To characterize this maximum we introduce o new notion:

Definition 2.8. The disc category $dcat_X(A)$ of A in M is the smallest integer k such that A can be covered by k closed submanifolds of M, each of them diffeomorphic to the closed unit disc of dimension $dim(M)$.

The disc category is a generalized Lusternik-Schnirelmann category and also the best one:

Proposition 2.9. *Let M be a closed manifold. Then the disc category dcat is the maximal element of the partial ordered set*

$$\{\lambda cat_X^{\mathfrak{Diff}} : \mathcal{P}(M) \to \mathbb{Z}_+ | \ \lambda cat_M^{\mathfrak{Diff}} is\ a\ gen.\ L-S\ cat.\ rel.\ to\ \mathfrak{Diff}(M)\},$$

denoted by $Cat^{\mathfrak{Diff}}(M)$.

Proof. Consider $A \subseteq X$ and $\lambda cat_M^{\mathfrak{Diff}}$ a generalized Lusternik-Schnirelmann category relative to $\mathfrak{Diff}(M)$. Let $\{A_1, A_2, \ldots, A_{dcat_M(A)}\}$ be a minimal covering of A with closed discs. By the remark 2.6 concerning the property $(5_{\mathfrak{Diff}(M)})$ of the generalized Lusternik-Schnirelmann category there exists an open ball $\overset{\circ}{\mathbb{B}} \subset M$ such that $\lambda cat_M^{\mathfrak{Diff}}(\overset{\circ}{\mathbb{B}}) = 1$. Let $\mathbb{D} \subset \overset{\circ}{\mathbb{B}}$ be a closed disc. Then by the monotonicity and nontriaviality of the generalized Lusternik-Schnirelmann category $\lambda cat_M^{\mathfrak{Diff}}(\mathbb{D}) = 1$. For each $k \in \overline{1, dcat_M(A)}$ there is a diffeotopy $\varphi^k : M \times I \to M$ such that $\varphi_1^k(A_k) = \mathbb{D}$. By the invariance under diffeotopy:

$$\lambda cat_M^{\mathfrak{Diff}}(A_k) \leq \lambda cat_M^{\mathfrak{Diff}}(\varphi_1^k(A_k)) = \lambda cat_M^{\mathfrak{Diff}}(\mathbb{D}) = 1.$$

Then

$$\lambda cat_M^{\mathfrak{Diff}}(A) \leq \lambda cat_M^{\mathfrak{Diff}}\left(\bigcup_{k=1}^{dcat_M(A)} A_k\right) \leq \sum_{k=1}^{dcat_M(A)} \lambda cat_M^{\mathfrak{Diff}}(A_k) = dcat_M(A).$$

Hence the partial ordered set $Cat^{\mathfrak{Diff}}(M)$ has as maximum the disc category $dcat$.♣

At the end of this chapter we can give an answer to the question why our Lusternik-Schnirelmann theorem is better than the classical one. A priori we know that $cat(M) \leq bcat(M)$, but in the literature there are no examples of manifolds M with $cat(M) < bcat(M)$ (for a class of manifolds for which the equality holds see Singhof [29]). Nevertheless the Lusternik-Schnirelmann theorem for the ball category combined with our results about the upper bound for *crit* allows us to compute in the following chapters the *crit* for a fairly general class of manifolds. Without the Lusternik-Schnirelmann theorem for the ball category these computations would be valid only accepting that Ganea's conjecture (i.e. $cat(M \times \mathbb{S}^n) = cat(M) + 1$) is true.

Another application of the Lusternik-Schnirelmann theorem for the ball category concerns the number of open (or closed) balls which cover a closed manifold M of dimension n. By Takens [34] $crit(f) \leq n + 1$, hence using the theorem 2 we obtain the results by Luft [15], Osborne and Stern [22]: the manifold M can by covered by $n + 1$ open balls. This result can be radically improved in many situations using results about *crit* from this thesis instead of Takens' result.

For further generalizations of the theorem 5 we suggest to look for a space $\mathcal{I} \subset \mathfrak{Diff}(M)$ such that for some generalized category $\lambda cat^{\mathcal{I}}_M$ the multiplicity counter $n_{\lambda cat^{\mathcal{I}}_M}$ equals the identity $Id_{\mathbb{N}}$.

3 The fusing lemma

We start with an example which describes the topic of this chapter. Let T be a torus in $\mathbb{R}^3$, standing on the x, y plane. The height function z, denoted by h, is a Morse function with four critical points: one minimum, one maximum and two saddle points. On the other hand, as described in [16], there is a function $\tilde{h}$ on the torus with only three critical points: one minimum, one maximum and one monkey-saddle point. This means that two critical points (the saddle points) of h can be replaced by one critical point (the monkey-saddle point) of $\tilde{h}$. We say that the two saddle points *fuse* into the monkey-saddle point.

This phenomenon is the central concern of this chapter. We describe it in a general setting, at the moment only informally. Let $(W; V_0, V_1)$ be a triad and $f : W \to \mathbb{R}$ a smooth triad function with $crit(f)$ critical points. The question is whether it is possible to find a triad function $\tilde{f} : W \to \mathbb{R}$ with less that $crit(f)$ critical points such that $\tilde{f}$ equals f on a neighborhood of ∂W. If $\tilde{f}$ exists we say that the critical points of f *fuse*. The main result of this chapter, the fusing lemma, establishes sufficient conditions for the fusing of two critical points. Later, we consider a smooth function on a closed manifold, we restrict it to convenient triads and we fuse critical points in the triads. After that, we glue the triads and the new triad functions together obtaining a function on the initial manifold with less critical points.

In this chapter and in the remainder of the thesis everything will be considered from the smooth, equivalently C^∞, point of view; manifolds, submanifolds, triads, and functions will be C^∞. All the manifolds are connected, unless otherwise stated. We are interested in functions with minimal number of critical points therefore we will consider exclusively functions with isolated critical points. Such functions on closed manifolds have a finite number of critical points.

The first definition introduces a concept analogous to the Morse triad function.

Definition 3.1. A *triad function* on a smooth manifold triad $(W; V_0, V_1)$ is a smooth function $f : W \to [a, b]$ such that:

1. $f^{-1}(a) = V_0$ and $f^{-1}(b) = V_1$.

2. all the critical points of f are interior (i.e. lie in $W \setminus \partial W$).

In order to prove the fusing lemma, it is useful to list some elements of the critical point theory.

Definition 3.2. Let M be a smooth Riemannian manifold without boundary and let $f : M \to \mathbb{R}$ be differentiable at $p \in M$. Then $X_p \in T_p(M)$ is called a *pseudo-gradient vector* for f at p if

1. $\| X_p \| \leq 2 \| df_p \|$.
2. $\langle df_p, X_p \rangle \geq \| df_p \|^2$.

If f is smooth at each point of $S \subseteq M$ and X is a smooth vector field on S then X is called a smooth *pseudo-gradient vector field* for f on S if for each $p \in S$, X_p is a pseudo-gradient vector for f at p.

In this chapter we work (like in [19], [23]) with the general notion of pseudo-gradient vector field, but starting with chapter 5 we consider only negative gradient vector fields, in order to make the presentation simpler.

For triad functions we need pseudo-gradient vector fields that are transversal to the boundary. To construct such vector fields is easy. Let $f : W \to [0,1]$ be a triad function on the triad (W, V_0, V_1). We are looking for a pseudo-gradient vector field X on W such that for any $x \in \partial W$ the vector X_x (at x) is transversal (not tangent) to ∂W, adopting in this way the setting of the classical paper by Smale [31]. Let $\widehat{W}$ be the manifold obtained by gluing the triads $(V_0 \times [-1,0]; V_0 \times \{-1\}, V_0 \times \{0\})$ and $(V_1 \times [1,2]; V_1 \times \{1\}, V_1 \times \{2\})$ to W along the boundaries $V_0 \times \{0\} \approx V_0$ and $V_1 \times \{1\} \approx V_1$ respectively. Let $\widehat{f}$ be a triad function on $\widehat{W}$ such that $\widehat{f}|_W = f$ and let $\widehat{X}$ be a pseudo-gradient vector field of $\widehat{f}|_{\widehat{W} \setminus \partial \widehat{W}}$. The pseudo-gradient vector field X that we need for f is the restriction of $\widehat{X}$ on W. Consider the dynamical system

$$(*) \begin{cases} \dot{\phi}(x,t) & = & -\widehat{X}(\phi(x,t)) \\ \phi(x,t) & = & x \end{cases}$$

defined on $\widehat{W} \setminus \partial \widehat{W}$. The maximal solution of this dynamical system we denote by $\widehat{\alpha}$ and we call it the (maximal) *local flow associated* to $-\widehat{X}$. For each $x \in W$ let $[t_x, T_x]$ be the maximal interval such that $f(\widehat{\alpha}(x, t_x)) \in [0,1]$ and $f(\widehat{\alpha}(x, T_x)) \in [0,1]$. If $\widehat{\alpha}([t_x, T_x])$ contains no critical points then:

1. $T_x < +\infty$.
2. $f(\widehat{\alpha}(x, T_x)) = 0$.

The properties of t_x are similar to the properties of T_x. Let α be the restriction of $\widehat{\alpha}_x$ to the closed interval $[t_x, T_x] := J_x$ for each $x \in W$. If $x \in \partial W$ then $0 \in Bd(J_x) = J_x \setminus Int(J_x)$. The maps $\alpha_x : J_x \to W$ and sometimes the sets $\alpha_x(J_x)$ are called *trajectories* or *flow lines* of the vector field X. When x is fixed, $\alpha(x,t)$ will be written as $\alpha(t)$.

The next construction is characteristic for deformation theorems and represents an improvement of the classical Palais-Smale result. We use the following notation:

$$f^{\,Ra} = \{x \in M \mid \, f(x)\, R\, a\},$$

where $R \in \{\leq, \geq, =, <, >\}$. When there is no risk of confusion we use the simplified notation f^a instead of $f^{\leq a}$.

Theorem 6 (Deformation lemma). *Let $f : W \to [0,1]$ be a smooth function with only isolated critical points on the compact triad $(W;V_0,V_1)$ and $[a,b] \subset [0,1]$ with $K_f \cap f^{-1}(a,b] = \emptyset$.*

Then there exists a strong deformation retraction of $f^{\leq b}$ onto $f^{\leq a}$.

Proof. The proof is standard and very simple. Let $U = f^{\leq b} \setminus f^{\leq a}$ and fix $x \in U$. Since f has no critical points in U and since the critical points of f on $f^{-1}(a)$ are isolated the flow orbit $\alpha(x,t)$ converges towards a point $\pi^a(x) \in f^{-1}(a)$ as $t \to T_x^a$ (here T_x^a is defined with respect to $[a,b]$). This defines the retraction. It is well known and easy to see that the induced map $\pi^a : f^{\leq b} \to f^{\leq a}$ is continuous. ♣

The previous retraction we call the *projection along the trajectories* $\pi^a : f^{\leq b} \to f^{\leq a}$:

$$\pi^a(x) = \begin{cases} \alpha(x, T_x^a) & \text{if} \quad x \in f^{\leq b} \setminus f^{\leq a} \\ x & \text{if} \quad x \in f^{\leq a}. \end{cases}$$

For any critical point $x \in K_f$, we define the set of the trajectories going to x or coming out from x :

$$K_x = \{y \in W | \, x \in cl\, \alpha(y, (t_y, T_y))\}.$$

An equivalent expression of K_x is the following:

$$K_x = \{y \in W | \, \omega^*(y) = x \text{ or } \omega(y) = x\}.$$

The study of the topological properties of K_x is very important for fusing of critical points.

Lemma 3.3. *Let $(W;V_0,V_1)$ be a compact triad and $f : W \to [0,1]$ a triad function with exactly two critical points x_0 and x_1 and no connecting trajectories between the critical points ($K_{x_0} \cap K_{x_1} = \emptyset$). Then $\bar{K}_{x_0} \cap \bar{K}_{x_1} = \emptyset$ and K_{x_0}, K_{x_1} are closed.*

Proof. Since the case $c_0 = f(x_0) = f(x_1) = c_1$ is a simple consequence of the continuous dependence on the initial value of the ODE solutions, we assume that $c_0 < c_1$.

Assertion 1. *$x_0 \notin \bar{K}_{x_1}$ and $x_1 \notin \bar{K}_{x_0}$.*

Let U be an open neighborhood of x_0 such that $x_1 \notin U$ and $U \subseteq f^{-1}(0,c)$ where $c = \frac{f(x_0)+f(x_1)}{2}$. If $x_0 \in \bar{K}_{x_1}$ there exists a sequence $y_n \in K_{x_1} \cap U$ such that $y_n \to x_0$. From the deformation lemma we get the continuous projections along the trajectories $\pi_u^0 : f^{\geq c_0} \cap U \to f^{=c_0}$ and $\pi_d^0 : f^{\leq c_0} \cap U \to f^{=c_0}$, where the first projection is in the positive sense and the second in the negative one. Because for both projections $\lim_{n\to\infty} \pi(y_n) = \pi(\lim_{n\to\infty}(y_n)) = \pi(x_0)$ we can assume without loss of generality that $y_n \in f^{=c_0}$, $y_n \in K_{x_1}$. Any point y_n is on a trajectory coming from x_1 and let $z_n \in f^{=c}$ be the intersection point of that trajectory with $f^{=c}$. The set $f^{=c}$ is compact therefore $z_n \to z \in f^{=c}$, or a subsequence of z_n. Computing $\alpha(z,t_z) = \pi_d^1(z) = \lim_{n\to\infty} z_n = \lim_{n\to\infty} \alpha(z_n, t_{z_n}) = x_1$ and $\alpha(z,T_z) = \pi_u^0(z) = \lim_{n\to\infty} z_n = \lim_{n\to\infty} \alpha(z_n, t_{z_n}) = \lim_{n\to\infty} = x_0$. This contradicts the assumption that there are no connecting trajectories between x_0 and x_1.

Remark 3.4. Is 1. true for Finsler manifolds and Palais-Smale functions ?

ASSERTION 2. *$\bar{K}_{x_0} \cap K_{x_1} = \emptyset$ and $K_{x_0} \cap \bar{K}_{x_1} = \emptyset$.*

Suppose $\bar{K}_{x_0} \cap K_{x_1} \neq \emptyset$. Then there is $y \in K_{x_1}$ and a sequence $y_n \in K_{x_0}$ such that $y_n \to y$. The followings three cases occur:

2a. $f(x_0) > f(y)$.

2b. $f(x_o) < f(y)$.

2c. $f(x_0) = f(y)$.

2a. The projection $\pi_d^0(x) = \alpha(x, t_x)$ of $W \setminus f^{>c_0}$ onto $f^{=c_0}$ along the trajectories is continuous, where, remember, t_x is the arriving time in the negative sense. By projection of the sequence y_n on $f^{=c_0}$,

$$x_0 = \lim_{n\to\infty} \pi(y_n) = \pi(\lim_{n\to\infty} y_n) = \alpha(\lim_{n\to\infty} y_n, \lim_{n\to\infty} t_{y_n}) = \alpha(y, t_y) \in K_{x_1}.$$

This is absurd because there are not connecting trajectories between the critical points.

2b. After projection of the sequence y_n on $f^{=c_1}$ from up or down we can assume without loss of generality that $y_n \in f^{=c_1}$. Now $y \in K_{x_1} \cap f^{=c_1} = \{x_1\}$ hence $y_n \to x_1$, i.e. $x_1 \in \bar{K}_{x_0}$, a fact which contradicts (2a).

2c. This case presents three subcases:

1. there is a subsequence y_{n_k} such that $f(y_{n_k}) < f(y)$.
2. there is a subsequence y_{n_k} such that $f(y_{n_k}) > f(y)$.
3. there is a subsequence y_{n_k} such that $f(y_{n_k}) = f(y)$.

The subcase 2c.1 and 2c.2 are similar to case 2a and 2b respectively, and 2c.3 is trivial.

ASSERTION 3. *The sets K_{x_0}, K_{x_1} are closed and disjoint.*

Let $y \in \bar{K}_{x_0} \cap \bar{K}_{x_1}$. There are $y_n^0 \in K_{x_0}$, $y_n^1 \in K_{x_1}$ such that $y_n^0 \to y$ resp. $y_n^1 \to y$. Depending on the value of $f(y)$ relative to $f(x_0)$ and $f(x_1)$ it is possible to project one of the sequences on $f^{=c_0}$ or $f^{=c_1}$. For instance, if $f(x_0) > f(y)$ then the right projection is $\pi_d^0(x) = \alpha(x, t_x)$ of $W \setminus f^{>c_0}$ onto $f^{=c_0}$. Then $y \in K_{x_0}$, because $\alpha(y, t_y) = \pi(y) = \pi(\lim_{n\to\infty} y_n^0) = \lim_{n\to\infty} \pi(y_n^0) = \lim_{n\to\infty} \alpha(y_n^0, t_{y_n^0}) = x_0$. Hence $y \in \bar{K}_{x_0} \cap K_{x_1} = \emptyset$ therefore by 2., we have $\bar{K}_{x_0} \cap \bar{K}_{x_1} = \emptyset$.

To show that K_{x_0} and K_{x_1} are closed, it is sufficient, under the hypothesis $\bar{K}_{x_0} \cap \bar{K}_{x_1} = \emptyset$, to prove that $K_{x_0} \cup K_{x_1}$ is closed. Suppose $K_{x_0} \cup K_{x_1}$ is not closed. Then there is a sequence $y_n \in K_{x_0} \cup K_{x_1}$ and $y \notin K_{x_0} \cup K_{x_1}$ such that $\lim_{n\to\infty} y_n = y$. We can admit y, $y_n \in f^{=c_0}$ or y, $y_n \in f^{=c_1}$, because depending on the position of y, y_n in $f^{\leq c_0}$ or $f^{>c_1}$ we can project them onto $f^{=c_0}$ resp. $f^{=c_1}$, the hypothesis data remaining unaltered. The projection π_u^0 of $(f^{=c_1} \setminus \{x_1\}) \setminus f^{<c_0}$ onto $f^{=c_0}$ is by the deformation lemma continuous, so $\pi_u^0(y) = \lim_{n\to\infty} \pi_u^0(y) = x_0$, hence $y \in K_{x_0}$, contradiction. In conclusion K_{x_0} and K_{x_1} are disjoint closed sets $\clubsuit$.

Before proving the fusing lemma we answer a question that a careful reader will surely ask: why do we use a pseudo-gradient vector field instead of the more natural gradient vector field? We give here only two answers: first, the theory that we develop could work on Finsler manifolds. An analysis of this case is possible only if we use pseudo-gradient vector fields. Second, the sufficient condition of the fusing lemma ($K_{x_0} \cap K_{x_1} = \emptyset$) could be fulfilled by a pseudo-gradient vector field but not necessarily by the gradient vector field, so the area of applicability of the fusing lemma increases if we work with pseudo-gradient vector fields. Maybe the second argument is vulnerable: the existence of a pseudo-gradient vector field without connecting orbits could imply the existence of a Riemannian metric such that the associated gradient vector field has no connecting orbits.

For the fusing lemma we need a definition which is related to the cancellation of non-degenerate critical points.

Definition 3.5. Let f be a triad function on $(W; V_0, V_1)$ with the critical set $K_f = \{x_1, \dots, x_{crit(f)}\}$. We say that the critical points $x_1, \dots, x_k, k \geq 2$ *fuse* if there is a triad function $\tilde{f} : W \to \mathbb{R}$ such that

1. $K_{\tilde{f}} = (K_f \setminus \{x_1, \dots, x_k\}) \cup K$ with $card(K) \leq 1$.
2. $\tilde{f} = f$ in a neighborhood of ∂W.

In the fusing lemma we want to fuse a pair of critical points. The sufficient condition is given in terms of (pseudo-)gradient dynamics.

Lemma 3.6 (Fusing lemma). *Let $(W; V_0, V_1)$ be a compact connected triad and $f : W \to [0, 1]$ a triad function with exactly two critical points x_0 and x_1. Suppose there are no connecting trajectories of the pseudo-gradient vector field X between the critical points (i.e. $K_{x_0} \cap K_{x_1} = \emptyset$). Then x_0 and x_1 fuse and, in particular, $crit(W) \leq 1$.*

Proof. Assume like in the previous lemma that $c_0 \leq c_1$. We divide the proof of the fusing lemma into three parts. The first part is only a technical lemma which we include here for the completeness of the proof.

Assertion 1. *There is a smooth function $F : [0, 1] \times [0, 1] \to [0, 1]$ with the properties:*

(1_F) *For any x and y, $\frac{\partial F}{\partial x}(x, y) > 0$.*

(2_F) *For any x near 0 or 1 and any y, $F(x, y) = x$.*

(3_F) $F(c_0, 0) = F(c_1, 1) = c_1$.

Before starting the proof of this assertion we make a remark on the property (3_F) of the function F: instead of the value c_1 for $F(c_0, 0) = F(c_1, 1)$ we can consider any $c \in (0, 1)$ and the result remains true.

Let ϵ be a positive real such that $\epsilon \leq \frac{1}{4}\min\{c_0, 1-c_1\}$. We consider a smooth bump function $\varphi_1 : \mathbb{R} \to \mathbb{R}_+$ such that the support $supp\,(\varphi_1) \subseteq [\epsilon, c_0 - \epsilon]$. Let $\lambda_1 = \frac{c_1 - c_0}{\int_{\mathbb{R}} \varphi_1(t)dt}$ and let $h_1 : [0, c_0] \to \mathbb{R}_+$ be a function given by $h_1(s) = \int_0^s (1 + \lambda_1\varphi_1(t))\,dt$, for each $s \in [0, c_0]$. Note that

(1_{h_1}) h_1 equals the identity in a neighborhood of 0 and c_0 in $[0, c_0]$, respectively.

(2_{h_1}) $\frac{\partial h_1}{\partial s}(s) > 0$, for each $s \in [0, c_0]$.

Now, we consider a smooth bump function $\varphi_2 : \mathbb{R} \to \mathbb{R}_+$ such that the support $supp\,(\varphi_2) \subseteq [c_0 + \epsilon, 1 - \epsilon]$ and $supp\,(1 - \varphi_2) \subseteq [c_0 + \frac{1-c_1}{2}, 1 - \frac{1-c_1}{2}]$. Let $\lambda_2 = \frac{c_1 - c_0}{\int_{\mathbb{R}} \varphi_2(t)dt}$ and let $h_2 : [c_0, 1] \to \mathbb{R}_+$ be a function given by $h_2(s) = c_1 + \int_0^s (1 + \lambda_2\varphi_2(t))\,dt$, for each $s \in [c_0, 1]$. Note that

(1_{h_2}) h_2 equals the identity in a neighborhood of c_0 and c_1 in $[c_0, 1]$, respectively.

(2_{h_2}) $\frac{\partial h_2}{\partial s}(s) > 0$, for each $s \in [c_0, 1]$.

Let $h : [0, 1] \to \mathbb{R}_+$ be the function given by

$$h(s) = \begin{cases} h_1(s) & \text{if} \quad s \in [0, c_0], \\ h_2(s) & \text{if} \quad s \in (c_0, 1]. \end{cases}$$

The function h is smooth and $h(c_0) = c_1$. Using the function h we can get the function $F : [0, 1] \times [0, 1] \to [0, 1]$ by the rule:

$$F(x, y) = (1 - y)h(x) + y\,Id(x).$$

The function F has all the properties required by the assertion 1.

Assertion 2. *There is a smooth triad function $\tilde{f}$ on W such that $K_f = K_{\tilde{f}}$, f agrees with f near $V_0 \cup V_1$ and $\tilde{f}(x_0) = \tilde{f}(x_1)$.*

Let $p : W \setminus (K_{x_0} \cup K_{x_1}) \to V_0$ be the projection of $W \setminus (K_{x_0} \cup K_{x_1})$ on the manifold V_0 along the trajectories of the vector field X. The function p is smooth (cf. the deformation lemma). By 3.3.3 the sets K_{x_0} and K_{x_1} are closed and disjoint, hence there exists U_0 and U_1 closed neighborhoods of $K_{x_0} \cap V_0$ and $K_{x_1} \cap V_0$ respectively, such that $U_0 \cup U_1 = \emptyset$. The Theorem of Tietze (the C^∞ form) and the fact that the sets U_0, U_1 are G_δ sets are the simplest arguments for the existence of a smooth function $q : V_0 \to [0, 1]$ such that $q^{-1}(0) = U_0$ and $q^{-1}(1) = U_1$.

Let $\mu : W \to [0, 1]$ be the extension to W of the composed function $q \circ p$ such that μ is zero on K_{x_0} and one on K_{x_1}. Evidently μ is zero on K_{x_0}, one on K_{x_1} and μ is constant on each trajectory of X. Furthermore the function μ is smooth. First we show that μ is continuous.

Consider $y \in W \setminus (K_{x_0} \cup K_{x_1})$ and $(y_n)_{n \in \mathbb{N}}$ a sequence in W such that $y_n \to y$, for $n \to \infty$. The set $W \setminus (K_{x_0} \cup K_{x_1})$ is open therefore $y_n \in W \setminus (K_{x_0} \cup K_{x_1})$, for n big enough. By continuity of p on $W \setminus (K_{x_0} \cup K_{x_1})$, $p(y_n) \to p(y)$ and hence by continuity of q we get $\mu(y_n) = q \circ p(y_n) \to q \circ p(y) = \mu(y)$, for $n \to \infty$.

Consider, now, $y \in K_{x_0}$ (for $y \in K_{x_1}$ the reasoning is similar) and $(y_n)_{n\in\mathbb{N}} \in W \setminus K_{x_0}$ a sequence in W such that $y_n \to y$, for $n \to \infty$. Without loss of generality, assume that $y_n \notin K_{x_1}$ (K_{x_1} is closed and disjoint of K_{x_0}) and that $y, y_n \in f^{\leq c_1}$ for each $n \in \mathbb{N}$. Then by the compactness of V_0 we can assume that $p(y_n) \to z \in V_0$. Let π_d^0 be the projection of $W \setminus f^{>c_0}$ onto $f^{=c_0}$ and π_u^0 be the projection of $f^{\leq c_1} \setminus \{x_1\}$ onto $f^{=c_0}$. By continuity of π_d^0 and π_u^0 (see deformation lemma) we have $\pi_d^0(p(y_n)) \to \pi_d^0(z)$ and $\pi_u^0(y_n) \to \pi_u^0(y) = x_0$. But $p(y_n)$ and y_n are on the same trajectory, therefore $(\pi_d^0(p(y_n)))_{n\in\mathbb{N}} = (\pi_u^0(y_n))_{n\in\mathbb{N}}$. Their limits are equal: $\pi_d^0(z) = x_0$. Hence z is an element of K_{x_0} and by definition of q we obtain $q(z) = 0$. Straightforward $\mu(y_n) = q \circ p(y_n) \to q(z) = 0 = \mu(y)$, for $n \to \infty$.

Consider, as the last case to be analyzed, $y, y_n \in K_{x_0}$, for each $n \in \mathbb{N}$. Then continuity of μ at y is trivial because $0 = \mu(y_n) \to 0 = \mu(y)$, for $n \to \infty$.

For the smoothness of μ is sufficient to remark that p as composition of two smooth functions is smooth on $W \setminus (K_{x_0} \cup K_{x_1})$ which is a open neighborhood of $\mu^{-1}(V_0 \setminus p((U_0 \cup U_1) \setminus (K_0 \cup K_1)))$. The smoothness of μ near $K_0 \cup K_1$ is clear because μ equals zero in the neighborhood $\mu^{-1}(U_0)$ of K_0 and equals one in the neighborhood $\mu^{-1}(U_1)$ of K_1. The new triad function $\tilde{f} : W \to [0,1]$ is defined by $\tilde{f}(z) = F(f(z), \mu(z))$, where $F : [0,1] \times [0,1] \to [0,1]$ is the function constructed in the assertion 1.

The property (1_F) does not allow the existence of other critical points of $\tilde{f}$ than x_0 and x_1, hence $K_f = K_{\tilde{f}}$. By the property (2_F) the function $\tilde{f}$ agrees with f near $V_0 \cup V_1$. Furthermore the critical points x_0 and x_1 are in the same level of $\tilde{f}$:

$$\tilde{f}(x_0) = F(f(x_0), \mu(x_0)) = F(c_0, 0) = c_1 = F(c_1, 1) = F(f(x_1), \mu(x_1)) = \tilde{f}(x_1).$$

Assertion 3. *The critical points x_0 and x_1 fuse , i.e. there is a function $g : W \to [0,1]$ such that g agrees with f near $V_0 \cup V_1$ and g has only one critical point.*

The connectedness of W implies the existence of a path $\gamma : [0,1] \to W$ from x_0 to x_1. Let $\tilde{X}$ be a pseudo-gradient vector field of $\tilde{f}$. We project $\gamma([0,1])$ first downward and then upward on $\tilde{f}^{=c_1}$, along the trajectories of the pseudo-gradient vector field $\tilde{X}$. In this way we get a path $\gamma' : [0,1] \to W$ from x_0 to x_1 situated in the critical level c_1. Under this hypothesis, by Takens [33] it is possible to modify the function $\tilde{f}$ in a neighborhood of the path γ' such that the new function g has only one critical point. Here ends the proof of the assertion 3 and implicitly the proof of the fusing lemma. ♣

Remark 3.7. The core of the assertion 3 constitutes the smoothing lemma for a continuous function on $\mathbb{R}^n$ which is everywhere smooth, but at 0. This lemma has been used by Takens [33] for the construction of g.

Remark 3.8. It is possible to change the order of the proof by projecting first some path γ (from x_0 to x_1) near to x_0 onto $f^{=c_0}$ and near x_1 onto $f^{=c_1}$. These projections deliver more information about the structure of the critical levels, for instance that the critical levels are locally connected.

The following result is a cheap extension of the fusing lemma and therefore it deserves no extra name:

Lemma 3.9 (Fusing lemma). *Let $(W; V_0, V_1)$ be a compact connected triad and $f : W \to [0,1]$ a triad function with the critical set $K_f = A_0 \cup A_1$ such that all the points of A_0 are in the same critical level c_0 and all the points of A_1 are in the same critical level c_1. If for each $x_0 \in A_0$ and $x_1 \in A_1$ we have $K_{x_0} \cap K_{x_1} = \emptyset$ then $crit(W) \leq 1$.*

Proof. In fact the previous proof may be repeated verbatim. ♣

Corollary 3.10. *Let $f : W \to [a,b]$ be a triad function on the connected manifold W such that $f(K_f) = c \in (a,b)$. Then $f^{=c}$ is path-connected.*

Proof. It follows from the proof of assertion 3 of fusing lemma. ♣

Before starting another application of the fusing lemma we prove an easy extension of the proposition (2.9) by Takens from [33]. We need the following:

Definition 3.11. Two critical points of f, x_0 and x_1 are *f-equivalent*, $x_0 \sim_f x_1$ iff $f(x_0) = f(x_1) = c$ and there is a path from x_0 to x_1 contained in $f^{=c}$. The critical points x_0 and x_1 are *equivalent* in $f^{=c}$ iff x_0 and x_1 are in the same path-connected component of $f^{=c}$. We write $x_0 \sim_c x_1$. Both $\sim_f$ and $\sim_c$ are equivalence relations.

For the next proposition we recall that in this work all the considered functions have only isolated critical point. For details of the proof see the original paper by Takens [33].

Proposition 3.12. *Let $f : W \to [0,1]$ be a triad function with exactly one critical level $c \in (0,1)$. Then there is a triad function $\tilde{f} : W \to [0,1]$ with the properties:*

1. *$\tilde{f} = f$ on a neighborhood on ∂W.*
2. *$crit(\tilde{f}) \leq card(K_f/_{\sim_c})$.*

Proof. Let $\mathcal{K} = (K_f/_{\sim_c})$. For each $K \in \mathcal{K}$ by the definition of the equivalence relation $\sim_c$ there is a path $\gamma_K : [0,1] \to f^{=c}$ which goes through all the critical points of K (we recall that we analyze only functions with isolated critical points). We choose γ_K such that it is a piecewise smooth embedding. For each $K', K'' \in \mathcal{K}$, we have $\gamma_{K'}([0,1]) \cap \gamma_{K''}([0,1]) = \emptyset$, hence there is a family of disjoint open sets $\{U_K\}_{K\in\mathcal{K}}$ with the property that U_K is a neighborhood of $\gamma_K([0,1])$ and $\bar{U}_K \cap \partial W = \emptyset$, for each $K \in \mathcal{K}$. Next we choose a map $\Theta : W \to W$ with the properties:

1Θ. $\Theta \circ \gamma_K([0,1])$ is the point $\gamma_K(0)$ for each $K \in \mathcal{K}$.

2Θ. $\Theta|_{W \setminus \cup_{K\in\mathcal{K}} \gamma_K([0,1])}$ is a diffeomorphism onto $W \setminus \cup_{K\in\mathcal{K}}\{\gamma_K(0)\}$.

3Θ. Θ is the identity outside of $\cup_{K\in\mathcal{K}} U_K$.

The continuous function $g : W \to [0,1]$ given by

$$g(x) = \begin{cases} \lim_{y\to x} f \circ \Theta^{-1}|_{W\setminus \cup_{K\in\mathcal{K}}\{\gamma_K(0)\}}(y) & \text{if} \quad x \in \cup_{K\in\mathcal{K}}\{\gamma_K(0)\} \\ f\circ\Theta^{-1}(x) & \text{else,} \end{cases}$$

is C^∞ on $W\setminus \cup_{K\in\mathcal{K}}\{\gamma_K(0)\}$ and has no critical points in $W\setminus \cup_{K\in\mathcal{K}}\{\gamma_K(0)\}$. Furthermore $g = f$ on a neighborhood of ∂W. Applying successively for each $K \in \mathcal{K}$ the smoothing lemma by Takens [33] to g at the point $\{\gamma_K(0)\}$ for $C = \bar{U}_K$ we obtain a triad function $\tilde{f}$. The critical set $K_{\tilde{f}}$ of the function $\tilde{f}$ is included in $\cup_{K\in\mathcal{K}}\{\gamma_K(0)\}$. Hence

$$crit(K_{\tilde{f}}) \leq card(\mathcal{K}) = card(K_f/_{\sim_c}).$$

The function $\tilde{f}$ coincides with g in a neighborhood of ∂W and therefore it coincides with f in a neighborhood of ∂W.♣

Corollary 3.13 (Takens). *Let $f : M \to \mathbb{R}$ be a function on a closed manifold M with k path-connected critical levels. Then $crit(M) \leq k$.*

Proof. We apply the previous proposition for each critical level. After fusing, each critical level has at most one critical point, altogether at most k critical points. ♣

Definition 3.14. Let M be a manifold and $f : M \to \mathbb{R}$ a smooth function. We say that f is a *minimal* function if $crit(f) = crit(M)$.

Until now we have referred to a pseudo-gradient vector field in order to get the greatest generality of the fusing lemma. To avoid any complication for the applications we use the negative pseudo-gradient vector field (in some fixed Riemannian structure) of the function under discussion. Then the obstruction for the fusing of two critical points in the setting of the fusing lemma is the existence of a trajectory of the negative gradient vector field of f between the two critical points.

Let $c, d \in (a,b)$ and let $\varphi : [a,b] \to [a,b]$ be a diffeomorphism of $[a,b]$ such that φ equals the identity on a neighborhood of $\partial([a,b])$ and $\varphi(c) = d$. Consider $f : W \to [a,b]$ a triad function on $(W; V_0, V_1)$. By the expression *we translate the level c to the level d* we emphasize the existence of the triad function $f^t = \varphi \circ f : W \to [a,b]$ and its properties:

1. $K_{f^t} = K_f$.
2. $f^t = f$ on a neighborhood of $\partial W = V_0 \cup V_1$.
3. $f^t(f^{=c}) = d$.

Now we are prepared to obtain the first result concerning the number of critical points for products. This theorem is at the same time the first step of the induction for proving a more general result.

Theorem 7. *Let M be a closed manifold having a minimal function with exactly one local minimum or with exactly one local maximum and let $\mathbb{S}^n$ be the n dimensional sphere. Then*

$$crit(M \times \mathbb{S}^n) \leq crit(M) + 1.$$

Proof. We first prove the proposition for the case that the minimal function f has only one local minimum. The case with only one local maximum is analogue. Let $K_f = \{x_1, x_2, \dots, x_{critM}\}$ be the critical set of f and x_1 the unique local minimum. By a small modification of f it is possible to get a smooth function with only one critical point on each critical level, hence we can suppose that the critical values are in the following order: $f(x_1) < f(x_2) < \dots < f(x_{crit(M)})$. Moreover we can suppose that $f(x_i) = i$ for each $i \in \overline{1, crit(M)}$. Let $f_0 := f$ and we prove by induction on $k \in \mathbb{N}$, the existence of a smooth function $f_k : M \to \mathbb{R}$ with the properties:

1_k. $K_{f_k} = K_f$.

2_k. $f_k(x_l) = f_{k-1}(x_l)$ for each $l \neq k$ and $l \in \overline{1, crit(M)}$.

$2'_k$. $f_k(x_{k+1}) \in \mathbb{N}$ and $x_{k+1}, \dots, x_{crit(M)}$ are not local minimizers of f_k.

3_k. for each $1 < p \leq k$ the critical point x_p has a *predecessor*, i.e. there is a critical point x_q with $1 \leq q < p$ such that $f_k(x_p) = f_k(x_q) + 1$ and there is a trajectory of the negative gradient vector field of f_k going from x_p to x_q.

The critical set K_f has a poset structure: $x_i < x_j$ iff there is a $x \in M$ such that $\omega^*(x) = x_j$ and $\omega(x) = x_i$.

Let $k = 1$. Then $f_1 = f_0$; there is nothing to prove.

Let $1 \leq k < crit(M)$ and assume the existence of $f_k : M \to \mathbb{R}$ with the required properties. We denote by $\mathcal{F}_{k+1}$ the set of function $g : M \to \mathbb{R}$ with the properties:

1_g. $K_g = K_f$.

2_g. $g(x_l) = f_k(x_l)$ for each $l \neq k + 1$, $l \in \overline{1, crit(M)}$.

$2'_g$. $g(x_{k+1}) \in \mathbb{N}$ and $x_{k+1}, \dots, x_{crit(M)}$ are not local minimizers.

3_g. for each $1 < p \leq k$ the critical point x_p has a predecessor relative to g, i.e. there is a critical point x_q with $1 \leq q < p$ such that $g(x_p) = g(x_q) + 1$ and there is a trajectory of the negative gradient vector field of g going from x_p to x_q.

The function $f_k : M \to \mathbb{R}$ satisfies the above properties hence $\mathcal{F}_{k+1} \neq \emptyset$. Let

$$\underline{k} = \min\{g(x_{k+1}) | g \in \mathcal{F}_{k+1}\}$$

and let $g_{\underline{k}} \in \mathcal{F}_{k+1}$ such that $g_{\underline{k}}(x_{k+1}) = \underline{k}$.

Let $q \in \overline{1, k}$ be the largest integer with the property that $x_q < x_{k+1}$ and for any critical point $x_{q'} < x_{k+1}$ follows $g_{\underline{k}}(x_{q'}) \leq g_{\underline{k}}(x_q)$. The existence of q is assured by the second part of the property 2'$_g$. We state that x_q is a predecessor of x_{k+1} relative to the function $g_{\underline{k}}$. Suppose the contrary: $g_{\underline{k}}(x_{k+1}) - g_{\underline{k}}(x_q) = c_{k+1} - c_q > 1$. First we enumerate the properties of $g_{\underline{k}}$ that follow from the definition:

1. $g_{\underline{k}}$ is a minimal function on M.
2. $g_{\underline{k}}(K_{g_{\underline{k}}}) \subseteq \mathbb{Z}$.
3. $g_{\underline{k}}(x_l) \leq f_0(x_l)$ for each $l \in \overline{1,k}$.
4. $g_{\underline{k}}(x_l) = f_0(x_l)$ for each $l \in \overline{k+2, crit(M)}$.

For each $c \leq \underline{k}$ let W_c be the connected component of x_{k+1} in $g_{\underline{k}}^{\leq c_{k+1}+\frac{1}{2}=\underline{k}+\frac{1}{2}} \setminus g_{\underline{k}}^{<c}$.

Let

$$\bar{c} = \sup\{c \leq \underline{k} \mid \quad g_{\underline{k}}|_{W_c} \text{ has at least two critical distinct levels}\}$$

with other words $g_{\underline{k}}(W_c \cap K_{g_{\underline{k}}}) \setminus \{\underline{k}\} \neq \emptyset$. The choice of x_q implies $c_q = g_{\underline{k}}(x_q) \leq \bar{c}$; the properties 2 and 2' imply $\bar{c} \in \mathbb{Z}$ and $\bar{c} \leq c_{k+1} - 1$. But we can prove more about $\bar{c}$: $\bar{c} = c_{k+1} - 1$.

Suppose the contrary: $\bar{c} < c_{k+1} - 1$. The triad function $g_{\underline{k}}|_{W_{\bar{c}+\frac{1}{2}}}$ has by the definition of $\bar{c}$ only one critical level. In this level there is only one critical point x_{k+1} of $g_{\underline{k}}|_{W_{\bar{c}+\frac{1}{2}}}$, otherwise we can reduce by the fusing lemma the numbers of critical points of $g_{\underline{k}}|_{W_{\bar{c}+\frac{1}{2}}}$, contradicting the minimality of $g_{\underline{k}}(x_{k+1})$. In $W_{\bar{c}+\frac{1}{2}}$ we translate the (critical) level c_{k+1} to the level $\bar{c}+1$. Let $g_{\underline{k}}^t : W_{\bar{c}+\frac{1}{2}} \to \left[\bar{c}+\frac{1}{2}, c_{k+1}+\frac{1}{2}\right]$ be the function which realize this translation. Then the function $\tilde{g}_{\underline{k}} : M \to \mathbb{R}$ given by

$$\tilde{g}_{\underline{k}}(x) = \begin{cases} g_{\underline{k}}^t(x) & \text{if} \quad x \in W_{\bar{c}+\frac{1}{2}} \\ g_{\underline{k}}(x) & \text{else} \end{cases}$$

is an element of $\mathcal{F}_{k+1}$ with $\tilde{g}_{\underline{k}}(x_{k+1}) = \bar{c}+1 < c_{k+1} = \underline{k}$. This fact contradicts the definition of $\underline{k}$, hence $\bar{c} = c_{k+1} - 1$. Up to now we know that $c_q \leq \bar{c} = c_{k+1} - 1$.

Then $c_q < \bar{c}$ because we have supposed that $g_{\underline{k}}(x_{k+1}) - g_{\underline{k}}(x_q) = c_{k+1} - c_q > 1$. Here we would need the following version of the fusing lemma:

Lemma 3.15. *Let $(W; V_0, V_1)$ be a compact connected triad and $f : W \to \mathbb{R}$ a triad function with only two critical levels c_0 and c_1. If there is a critical points $x_0 \in f^{=c_0}$ such that $K_{x_0} \cap K_{x_1} = \emptyset$ for each $x_1 \in f^{=c_1}$ then $crit(W) \leq crit(f) - 1$.*

We prove this lemma in the context of the proposition. By the definition of $\bar{c}$ the function $g_{\underline{k}}|_{W_{\bar{c}-\frac{1}{2}}}$ has at least two critical levels. But $g_{\underline{k}}(W_{\bar{c}-\frac{1}{2}}) \subseteq \left[\bar{c}-\frac{1}{2}, c_{k+1}+\frac{1}{2}\right] = \left[\bar{c}-\frac{1}{2}, \bar{c}+\frac{3}{2}\right]$ and $g_{\underline{k}}(K_{g_{\underline{k}}}) \subseteq \mathbb{Z}$ hence $g_{\underline{k}}|_{W_{\bar{c}-\frac{1}{2}}}$ has exactly two critical levels: $\bar{c}$ and $\bar{c}+1 = c_{k+1}$. Let $W_{NW} = (W_{\bar{c}-\frac{1}{2}} \setminus W_{\bar{c}+\frac{1}{2}}) \cap g_{\underline{k}}^{\geq \bar{c}+\frac{1}{2}}$.

We want to prove that (a) $W_{\bar{c}+\frac{1}{2}}$ contains exactly one critical point x_{k+1} and that (b) there is a path in $\big(W_{\bar{c}-\frac{1}{2}} \setminus int(W_{NW})\big) \cap g_{\underline{k}}^{\leq c_{k+1}}$ from x_{k+1} to a critical point $x'_{\bar{c}}$ of $g_{\underline{k}}$ with $g_{\underline{k}}(x'_{\bar{c}}) = \bar{c}$.

(a) If $W_{\bar{c}+\frac{1}{2}}$ contains more than one critical point, by the fusing lemma applied to the function $g_{\underline{k}}|_{W_{\bar{c}+\frac{1}{2}}}$, having exactly one critical level c_{k+1}, we can reduce the number of critical points, contradicting the minimality of $g_{\underline{k}}$.

(b) We know that $W_{\bar{c}-\frac{1}{2}}$ is by definition a path connected manifold and $g_{\underline{k}}|_{W_{\bar{c}-\frac{1}{2}}}$ has exactly two critical levels: $\bar{c}$ and $\bar{c}+1 = c_{k+1}$. Let $x' \in K_{g_{\underline{k}}} \cap W_{\bar{c}-\frac{1}{2}} \cap g_{\underline{k}}^{=\bar{c}}$. Then there is a path γ from $\gamma(0) = x'$ to $\gamma(1) = x_{k+1}$ in $W_{\bar{c}-\frac{1}{2}}$. Assume that $\gamma([0,1]) \subseteq W_{\bar{c}-\frac{1}{2}} \cap \big(g_{\underline{k}}^{\leq c_{k+1}} \setminus g_{\underline{k}}^{<\bar{c}}\big)$, otherwise we project $\gamma([0,1]) \cap \complement\Big(W_{\bar{c}-\frac{1}{2}} \cap \big(g_{\underline{k}}^{\leq c_{k+1}} \setminus g_{\underline{k}}^{<\bar{c}}\big)\Big)$ along the trajectories of $-\nabla g_{\underline{k}}$ on $g_{\underline{k}}^{=c_{k+1}}$ and $g_{\underline{k}}^{=\bar{c}}$, getting a path with the required properties. Let $x'_{\bar{c}} = \gamma(t_0)$ where

$$0 \leq t_0 = \sup\{t \in [0,1] |\, \gamma(t) \in K_{g_{\underline{k}}} \cap W_{\bar{c}-\frac{1}{2}} \cap g_{\underline{k}}^{=\bar{c}}\}.$$

Without loss of generality we assume $t_0 = 0$. Then $\gamma([0,1]) \cap K_{g_{\underline{k}}} \cap W_{\bar{c}-\frac{1}{2}} \cap g_{\underline{k}}^{=\bar{c}} = \emptyset$. If we prove that $\gamma([0,1]) \cap K_{g_{\underline{k}}} \cap W_{NW} \cap g_{\underline{k}}^{=c_{k+1}} = \emptyset$, then we can project $\gamma([0,1]) \cap W_{NW}$ along the trajectories on the bottom boundary $W_{NW} \cap g_{\underline{k}}^{\bar{c}+\frac{1}{2}}$ of W_{NW}. In this way we get a path $\hat{\gamma} : [0,1] \to W_{\bar{c}-\frac{1}{2}} \setminus int(W_{NW}) \cap g_{\underline{k}}^{\leq c_{k+1}}$ and we would prove (b). It remains to prove that $\gamma([0,1]) \cap K_{g_{\underline{k}}} \cap W_{NW} \cap g_{\underline{k}}^{=c_{k+1}} = \emptyset$. We assume the contrary: there is $x'_{k+1} \in K_{g_{\underline{k}}} \cap W_{NW} \cap g_{\underline{k}}^{=c_{k+1}}$ and $t_1 \in (0,1)$ such that $\gamma(t_1) = x'_{k+1}$. Then $\gamma([t_1,1]) \cap W_{\bar{c}-\frac{1}{2}} \cap g_{\underline{k}}^{=\bar{c}} \neq \emptyset$, otherwise there is $0 < \varepsilon < 1$ such that the connected component $W_{\bar{c}+\varepsilon}$ of x_{k+1} contains two distinct critical points x_{k+1} and x'_{k+1} in the unique critical level of $g_{\underline{k}}|_{W_{\bar{c}+\varepsilon}}$, contradicting by the fusing lemma the minimality of $g_{\underline{k}}$. Moreover $\gamma([t_1,1]) \cap K_{g_{\underline{k}}} \cap W_{\bar{c}-\frac{1}{2}} \cap g_{\underline{k}}^{=\bar{c}} \neq \emptyset$ else we can move $\gamma|_{[t_1,1]}$ along the trajectories into $W_{\bar{c}+\varepsilon}$, so we arrive at the same contradiction like above. But $\gamma([t_1,1]) \cap K_{g_{\underline{k}}} \cap W_{\bar{c}-\frac{1}{2}} \cap g_{\underline{k}}^{=\bar{c}} \neq \emptyset$ contradicts the assumptions on γ, hence $\gamma([0,1]) \cap K_{g_{\underline{k}}} \cap W_{NW} \cap g_{\underline{k}}^{=c_{k+1}} = \emptyset$.

Now we return to the main proof. The function $g_{\underline{k}}|_{W_{NW}}$ has at most one critical level c_{k+1} because $g_{\underline{k}}(W_{NW}) = [c_{k+1} - \frac{1}{2}, c_{k+1} + \frac{1}{2}]$. We translate the level c_{k+1} to the level $c_{k+1} + \frac{3}{8}$. If $g_{\underline{k}}^t : W_{NW} \to \Big[c_{k+1} - \frac{1}{2}, c_{k+1} + \frac{1}{2}\Big]$ is the function that realizes this translation then let $\tilde{g} : W_{\bar{c}-\frac{1}{2}} \to \Big[c_{k+1} - \frac{1}{2}, c_{k+1} + \frac{1}{2}\Big]$ be the function given by:

$$\tilde{g}(x) = \begin{cases} g_{\underline{k}}^t(x) & \text{if} \quad x \in W_{NW} \\ g_{\underline{k}}(x) & \text{else.} \end{cases}$$

Let $W^{x_{k+1}}$ be the connected component of x_{k+1} in $W_{\bar{c}-\frac{1}{2}} \cap \tilde{g}^{\leq c_{k+1}+\frac{1}{4}}$. Then
A) by a) $x_{k+1} \in W^{x_{k+1}}$ and it is the unique critical point of $\tilde{g}|_{W^{x_{k+1}}}$ in the level c_{k+1};
B) by b) there is a path γ in $\big(W_{\bar{c}-\frac{1}{2}} \setminus int(W_{NW})\big) \cap g_{\underline{k}}^{\leq c_{k+1}}$ from x_{k+1} to a critical

point $x'_{\bar{c}}$ of $g_{\underline{k}}$ from the level $\bar{c}$. By the fact that $\tilde{g} = g_{\underline{k}}$ on $W_{\bar{c}-\frac{1}{2}} \setminus int(W_{NW})$ it follows that γ is contained in $W^{x_{k+1}}$ and implicitly $x'_{\bar{c}} \in W^{x_{k+1}}$. This means that $\tilde{g}|_{W^{x_{k+1}}}$ has at least one critical point in the level $\bar{c}$;
C) by the choice of x_q it follows $K^{g_{\underline{k}}}_{x_{k+1}} \cap K^{g_{\underline{k}}}_{x'} = \emptyset$ for each $x' \in K_{g_{\underline{k}}} \cap W^{x_{k+1}} \cap g_{\underline{k}}^{=\bar{c}} = K_{\tilde{g}} \cap W^{x_{k+1}} \cap \tilde{g}^{=\bar{c}}$. But $\tilde{g}$ agrees with $g_{\underline{k}}$ on a neighborhood of $K^{g_{\underline{k}}}_{x_{k+1}}$ hence $K^{\tilde{g}}_{x_{k+1}} \cap K^{\tilde{g}}_{x} = \emptyset$, for each $x' \in K_{\tilde{g}} \cap W^{x_{k+1}} \cap \tilde{g}^{=\bar{c}}$.

Using A), B) and C) we can by the fusing lemma reduce the number of critical points of $\tilde{g}|_{W^{x_{k+1}}}$, contradicting the minimality of $g_{\underline{k}}$. Hence $\bar{c} = c_q$ and therefore we get $f_{k+1} = g_{\underline{k}}$, ending the induction.

Next we get a function on $M \times \mathbb{S}^n$ with at most $crit(M)+1$ critical levels, having critical points that are in the same path-connected component of a critical level. The critical points in the same path connected component of the same critical level fuse by the fusing lemma. For ease of the notation we put $f = f_{crit(M)}$. Let $g : \mathbb{S}^n \to \mathbb{R}$ be the height function on the sphere such that for the south pole x_S and north pole x_N, $g(x_S) = 0$ and $g(x_N) = 1$ respectively. The function $F : M \times \mathbb{S}^n \to \mathbb{R}$ given by $F(x, y) = f(x) + g(y)$ has the critical set $K_F = \{(x, y)|x \in K_f \text{ and } y \in K_g\}$. Let x_p be a critical point and x_q one of its predecessors. Then there is a trajectory of $-\nabla f$ going from x_p to x_q. Reparametrizing it we get a path $\alpha : [0, 1] \to M$ such that $\alpha(0) = x_q$ and $\alpha(1) = x_p$. Let $\beta : [0, 1] \to \mathbb{S}^n$ be a reparametrized trajectory from x_N to x_S of the negative gradient vector field of g. It is easy to reparametrize the path β such that $f(\alpha(t)) + g(\beta(1-t)) = f(x_q) + 1 = f(x_p) =: k$. This means that the critical points (x_q, x_N) and (x_p, x_S) of F are connected by the path $t \mapsto (\alpha(t), \beta(1-t)) \in M \times \mathbb{S}^n$, which lies in the critical level $F^{=k}$ for any $t \in [0, 1]$. In other words $(x_q, x_N) \sim_k (x_p, x_S)$. Applying the proposition 3.12 in the triad $(F^{\leq k+\frac{1}{2}} \setminus F^{<k-\frac{1}{2}}; F^{=k-\frac{1}{2}}, F^{=k+\frac{1}{2}})$ to $F_k = F|_{F^{\leq k+\frac{1}{2}} \setminus F^{<k-\frac{1}{2}}}$ we obtain a new triad function $\tilde{F}_k : F^{\leq k+\frac{1}{2}} \setminus F^{<k-\frac{1}{2}} \to [k - \frac{1}{2}, k + \frac{1}{2}]$ with at least one critical point less than F_k; suppose that the critical point of F_k that become regular for $\tilde{F}_k$ is (x_p, x_S). In conclusion we substitute here two critical critical points, (x_p, x_S) and (x_q, x_N) by one critical point (x_q, x_N), where x_q is a predecessor of x_p.

It only remains to count the number of cancellable critical points. As we have constructed the function f above, for any $2 \leq q \leq crit(M)$, the critical point $x_p \in K_f$ has a predecessor, therefore the number of cancellable critical points is at least $crit(M) - 1$.

Computing:

$$crit(M \times \mathbb{S}^n) \leq 2crit(M) - (crit(M) - 1) = crit(M) + 1.$$

This means that the function $\tilde{F}$ that we obtain from the triad functions $\tilde{F}_k$'s has at most $crit(M) + 1$ critical points.

For the case of a minimal function with one local maximum we apply the previous construction to $-f$.♣

The last remark of this chapter concerns the minimal functions from the previous theorem:

Remark 3.16. If M has a minimal function with two or more local minima or has a minimal function with two or more local maxima then $bcat(M) < crit(M)$. The proposition 5.25 is a generalization of this remark and the reader is referred to it for the proof.

4 Applications of the fusing lemma and examples

Definition 4.1. Let $(W; V_0, V_1)$ be a triad. We define the *crit* of W as:

$$crit(W) = \inf \{crit(f) \mid f : W \to \mathbb{R} \text{ a triad function } \}.$$

In the remainder we focus only on compact manifolds hence for us $crit(W) < +\infty$. One of the most important triads in the differential topology is the h-cobordism. The triad $(W; V_0, V_1)$ is an h-cobordism if W is connected and both V_0 and V_1 are deformation retracts of W. Using the fusing lemma we give an estimation of the *crit* for h-cobordisms.

Proposition 4.2. *Let $(W; V_0, V_1)$ be a nontrivial h-cobordism of dimension $n \geq 6$. Then $1 \leq crit(W) \leq 2$.*

Remark 4.3. For $n \geq 6$ the group

$$\Pi := \pi_1(W) \cong \pi_1(V_0) \cong \pi_1(V_1)$$

is not trivial, otherwise the h-cobordism theorem asserts that W is necessarily trivial: $W \approx V_0 \times [0, 1]$.

Proof. The first inequality is immediate: if $crit(W) = 0$ then there is a triad function with no critical points. By the non-critical interval theorem (see for instance Theorem 3.4 of [19]) the cobordism $(W; V_0, V_1)$ is trivial $W \approx V_0 \times [0, 1]$.

For the second inequality, let $\tau(W, V_0)$ be the Whitehead torsion of the pair (W, V_0). First we construct a h-cobordism $(W'; V_0, V_1')$ with the property that $\tau(W', V_0) = \tau(W, V_0)$ and then a triad function for it with only two critical points. By the existence theorem of Stallings (see Rourke and Sanderson [24]) it is possible to construct a h-cobordism of any given torsion in such a way that $(W'; V_0, V_1')$ is obtained from $V_0 \times [0, 1]$ by attaching handles of index 2 and 3. This construction corresponds to a Morse triad function $f : W' \to [0, 1]$ with only non-degenerate critical points of index 2 and 3, respectively. With the classical techniques developed by Morse (see Milnor [19]) it is possible to bring all the critical points of index 2 in the level 1/4 and the critical points of index 3 in the level 3/4. The manifold V_0 is a deformation retract of the connected manifold W, and both V_0 and V_1' are deformation retracts of the manifold W' hence

$$\pi_0(V_0) \cong \pi_0(V_1') \cong \pi_0(W') = 0.$$

The manifold V_0 being connected, all the hypothesis of the fusing lemma for the triad $(f^{\leq 1/2}; V_0, f^{=1/2})$ are satisfied, so we can fuse the critical points of index 2 into one critical point. Then $crit(f^{\leq 1/2}) \leq 1$. The same procedure for the critical points of index 3 in the triad $(f^{\geq 1/2}, f^{=1/2}, V_1')$ implies $crit(f^{\geq 1/2}) \leq 1$, hence $crit(W') \leq 2$.

Now we turn to the initial h-cobordism $(W; V_0, V_1)$. By the uniqueness theorem of h-cobordism W' is diffeomorphic to W under a diffeomorphism which preserves V_0, because their triads have the same torsion:

$$\tau(W, V_0) = \tau(W', V_0).$$

But the *crit* is invariant under diffeomorphism, hence $crit(W) = crit(W') \leq 2$. ♣

Remark 4.4. The above theorem proven, it follows by the s-cobordism theorem that for each h-cobordism $(W; V_0, V_1)$ with $n \geq 6$ the $crit(W) \geq 1$ iff $\tau(W, V_0) \neq 0$.

Recall that by Lusternik-Schnirelmann theorem for a triad $(W; V_0, V_1)$ it holds:

$$\max \{cat((W, V_0)), cat((W, V_1))\} \leq crit(W).$$

We continue the study of the *crit* for triads giving an example where the previous inequality is strict. This consists in an h-cobordism $(W; V_0, V_1)$ with the upper boundary V_1 diffeomorphic to the lower boundary V_0, with $cat(W, V_0) = cat(W, V_1) = 0$ nevertheless it does not admit a triad function with no critical points.

Corollary 4.5. *There exists a h-cobordism $(W; V_0, V_1)$ such that V_0 is diffeomorphic to V_1 and $crit(W) \geq 1$.*

Proof. Let V_0 be a closed manifold of even dimension $n \geq 6$ such that the Whitehead group of $\pi_1(V_0)$ is finite commutative and not trivial (for concrete examples see Milnor [20]). For some element $0 \neq \tau \in Wh(\pi_1(V_0))$ of the Whitehead group we get like in the previous proposition a h-cobordism $(W'; V_0, V_1')$ such that $\tau(W, V_0) = \tau_0$. Then we can construct the double $(W; V_0, V_1)$ of the h-cobordism $(W'; V_0, V_1')$ obtained by pasting together two copies of W' along the boundary V_1'. Obviously $(W; V_0, V_1)$ is an h-cobordism, where V_0 and V_1 are two copies of the same manifold, hence they are diffeomorphic. The torsion $\tau(W, V_0)$ is equal to

$$\tau(W, V_0) + (-1)^n \overline{\tau}(W, V_0) = \tau(W, V_0) + \overline{\tau}(W, V_0)$$

and by commutativity of $Wh(\pi_1(V_0))$ is equal to $2\tau(W, V_0) \neq 0$. By the remark after the proposition 4.2 the non-triviality of the torsion $\tau(W, V_0)$ implies that $crit(W) \geq 1$. ♣

A practical application of the theorem 7 is the computation of the *crit* for some product of manifolds.

Proposition 4.6. *Let $N = \mathbb{S}^{p_1} \times \ldots \times \mathbb{S}^{p_n}$ be a product of $n \geq 1$ spheres. Then $crit(N) = n + 1$.*

Proof. The proof is by induction on n. For $n = 1$ the result is obviously true. Suppose it is true for $n - 1$ and let $N = \mathbb{S}^{p_1} \times \ldots \times \mathbb{S}^{p_n}$. We know that $cat(\mathbb{S}^{p_1} \times \ldots \times \mathbb{S}^{p_{n-1}}) = n$. The induction hypothesis implies that $cat(\mathbb{S}^{p_1} \times \ldots \times \mathbb{S}^{p_{n-1}}) = crit(\mathbb{S}^{p_1} \times$

$\ldots \times \mathbb{S}^{p_{n-1}}$). Then by the remark 3.16 each minimal function on $\mathbb{S}^{p_1} \times \ldots \times \mathbb{S}^{p_{n-1}}$ has exactly one local minimum, hence we can apply the *crit* inequality:

$$crit(\mathbb{S}^{p_1} \times \ldots \times \mathbb{S}^{p_n}) \leq crit(\mathbb{S}^{p_1} \times \ldots \times \mathbb{S}^{p_{n-1}}) + 1 = n + 1.$$

On the other side by the Lusternik-Schnirelmann theorem

$$n + 1 = cat(\mathbb{S}^{p_1} \times \ldots \times \mathbb{S}^{p_n}) \leq crit(\mathbb{S}^{p_1} \times \ldots \times \mathbb{S}^{p_n})$$

hence $crit(\mathbb{S}^{p_1} \times \ldots \times \mathbb{S}^{p_n}) = n + 1.$♣

Example 4.7. Two famous products of manifolds are $L(7,1) \times \mathbb{S}^4$ and $L(7,2) \times \mathbb{S}^4$ (the symbol $L(r,s)$ represents the lens space of dimension 3 of type (r,s), where the Poincaré group is cyclic of order r). Milnor [17] proved that $L(7,1) \times \mathbb{S}^4$ and $L(7,2) \times \mathbb{S}^4$ are h-cobordant but not diffeomorphic, hence $crit(L(7,1) \times \mathbb{S}^4)$ is not a priori equal to $crit(L(7,2) \times \mathbb{S}^4)$. Let $p = 1$ or $p = 2$.

To get a lower bound for $crit(L(7,p) \times \mathbb{S}^4)$ we use the Lusternik-Schnirelmann theorem:

$$cat(L(7,p) \times \mathbb{S}^4) \leq crit(L(7,p) \times \mathbb{S}^4).$$

The Lusternik-Schnirelmann category of $L(7,p)$ is computed, for instance, by Fadell [9] and is maximal:

$$cat(L(7,p)) = \dim(L(7,p)) + 1 = 4.$$

Moreover, $cat(L(7,p))$ verifies the inequality $\dim(L(7,p)) \leq 2cat(L(7,p)) - 5$, therefore we can here apply an improvement from [25] of the Singhof's theorem [29] about the Ganea conjecture. By this result

$$cat(L(7,p) \times \mathbb{S}^4) = cat(L(7,p)) + 1 = 5$$

hence $crit(L(7,p) \times \mathbb{S}^4) \geq 5$.

To get an upper bound for $crit(L(7,p) \times \mathbb{S}^4)$ we use the *crit* inequality. We can apply the proposition 7 since $cat(L(7,p)) = crit(L(7,p))$ implies by the remark 3.16 that each minimal function on $L(7,p)$ has only one local minimum. Then

$$crit(L(7,p) \times \mathbb{S}^4) \leq crit(L(7,p)) + 1 = cat(L(7,p)) + 1 = 5.$$

The lower and the upper bound that we find for $crit(L(7,p) \times \mathbb{S}^4)$ are equal, in other words

$$crit(L(7,1) \times \mathbb{S}^4) = crit(L(7,2) \times \mathbb{S}^4) = 5.$$♣

Remark 4.8. The results by Takens [33] can not be applied for this example, because $L(7,p) \times \mathbb{S}^4$ is not simply connected either for $p = 1$ or for $p = 2$.

We generalize the above example for all closed manifolds M with $cat(M) = crit(M)$. The results of Singhof [29] or Rudyak [25] are in this case useless because we have no information about the dimension of M. In particular the manifolds of small category and big dimension do not verify the hypothesis of the quoted results. Instead of these we use our result, the generalized Lusternik-Schnirelmann theorem for the ball category.

Proposition 4.9. *Let M^m be a closed manifold such that $cat(M) = crit(M)$. If n is a positive integer and $\mathbb{S}^n$ is the sphere of dimension n, then*

$$crit(M \times \mathbb{S}^n) = crit(M) + 1.$$

Proof. Recall that $cat(M)+1 \leq bcat(M \times \mathbb{S}^n)$ for each closed manifold M (from Montejano [21]). Indeed, let $bcat(M \times \mathbb{S}^n) = r$. Then there is a cover $\{B_1, \ldots, B_r\}$ of $M \times \mathbb{S}^n$, where B_i is a $(m+n)$-ball for each $i \in \overline{1,r}$. We can assume that B_1 is so small that $B_1 \cap (M \times \{a\}) = \emptyset$ for some $a \in \mathbb{S}^n$. Otherwise we replace $\{B_1, \ldots, B_r\}$ by $\{h(B_1), \ldots, h(B_r)\}$, where $h : M \times \mathbb{S}^n \to M \times \mathbb{S}^n$ is a homeomorphism such that $h(B_1) \cap (M \times \{a\}) = \emptyset$. Then $\{B_2 \cap (M \times \{a\}), \ldots, B_r \cap (M \times \{a\})\}$ is a covering of $M \times \{a\}$ by contractible sets. Hence

$$cat(M) + 1 = cat(M \times \{a\}) + 1 \leq bcat(M \times \mathbb{S}^n).$$

After this *intermezzo* we pass to the main proof. By the generalized Lusternik-Schnirelmann theorem for the ball category, that we proved in the chapter 2:

$$bcat(M \times \mathbb{S}^n) \leq crit(M \times \mathbb{S}^n)$$

therefore

$$crit(M) + 1 = cat(M) + 1 \leq bcat(M \times \mathbb{S}^n) \leq crit(M \times \mathbb{S}^n).$$

Since $cat(M) = crit(M)$ we can apply the inequality from proposition 7 for the *crit*:

$$crit(M \times \mathbb{S}^n) \leq crit(M) + 1.$$

Combining the two inequalities, we obtain $crit(M \times \mathbb{S}^n) = crit(M) + 1$. ♣

In this context let $(W; V_0, V_1)$ be a triad such that W and V_0 are connected manifolds. If this triad admits a triad function with only one critical point, a natural question is whether the Conley Index of this unique critical point and the connectedness of the lower boundary V_0 delivers us information about the connectedness of the upper boundary V_1.

In the next examples we construct two triads $(W; V_0, V_1)$ and $(W'; V_0', V_1')$ with the following properties:

1. V_0 and V_0' are connected,
2. W admits a minimal triad function with only one critical point $x_{1,2}$ and W' admits a minimal triad function with only one critical point $x_{1,2}'$ such that $CH(x_{1,2}) \cong CH(x_{1,2}')$.

Unfortunately V_1 is connected but V_1' not.

Example 4.10. Let $\mathbb{T}$ be the 2-dimensional torus, $\mathbb{T} \approx \mathbb{S}^1 \times \mathbb{S}^1$. Consider $\mathbb{D}_1$ and $\mathbb{D}_2$ two closed discs disjoint embedded into $\mathbb{T}$. The manifold $W \setminus (Int\,(\mathbb{D}_1) \cup Int\,(\mathbb{D}_2))$ has the boundary consisting of two disjoint manifolds V_0 and V_1, both diffeomorphic to the 1-dimensional sphere $\mathbb{S}^1$. There are different ways to construct on $(W; V_0, V_1)$ a triad function with only one critical point:

a) With a monkey-saddle point on $I \times I$ taking into account the equivalence relation which generates the torus thorough the factorization of $I \times I$.

b) By constructing a nice filling of the triad $(W; V_0, V_1)$. Then by Takens [33] it is possible to construct a corresponding triad function with only one critical point.

c) Using the fusing lemma for a convenient Morse function on the torus.

Here we present the third method: let $f : \mathbb{T} \to [0, 3]$ be the height function on the torus (see Milnor [18]) and x_1, x_2 be its critical points of index 1 such that $1 = f(x_1) < f(x_2) = 2$. The dimension of the unstable and stable manifold for x_1 and x_2 permit us by tranversality arguments to perturb f slightly to a triad function $\tilde{f} : W \to [0, 3]$ with the properties:

a) $K_{\tilde{f}} = K_f$.

b) $\tilde{f} = f$ in a neighborhood of ∂W.

c) there is no connecting trajectory between x_1 and x_2, relative to $-\nabla \tilde{f}$.

Then by c) the triad function $\tilde{f}$ satisfies the hypothesis of the fusing lemma. Using it, we get a triad function on $(W; V_0, V_1)$ with only one critical point $x_{1,2}$. The Conley-Index $CH(x_{1,2})$ of the point $x_{1,2}$ is isomorphic to $H_*(W, V_0)$. Hence

$$CH_i(x_{1,2}) \cong H_i(W, V_0) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if} \quad i = 1 \\ 0 & \text{else,} \end{cases}$$

because W is the result of attaching two handles of index 1 to V_0, therefore W/V_0 is homotopy equivalent to a wedge of spheres $\mathbb{S}^1 \vee \mathbb{S}^1$. In this example V_1 is connected.

Example 4.11. Let S_2 be an orientable surface of genus 2 obtained as the topological sum of two tori $\mathbb{T}_1$ and $\mathbb{T}_2$. The surface S_2 has two humps in the superior part and two humps in the inferior part. By a diffeomorphism we can transform the two-humped parts into dromedary humps. Then the height function on this surface is a Morse function $f' : S_2 \to [0, 3]$ with exactly 6 critical points: 1 minimum in the level zero, 2 critical points of index one in the level one, 2 critical points of index one in the level two and 1 maximum in the level three. The manifold $W' = f'^{-1}([1/2, 3/2])$ has the boundary consisting of two disjoint manifolds $V_0' = f'^{-1}(1/2) \approx \mathbb{S}^1$ and $V_1' = f'^{-1}(3/2) \approx \mathbb{S}^1 \sqcup \mathbb{S}^1 \sqcup \mathbb{S}^1$. The triad function $f'|_{W'}$ has 2 critical points of index one in the critical level 1. By the fusing lemma (by proposition 3.13, too, because the critical level is connected) it is possible to fuse the two critical points to one point $x_{1,2}'$. The Conley index of $x_{1,2}'$ is isomorphic to $H_*(W', V_0')$ hence:

$$CH_i(x_{1,2}') \cong H_i(W', V_0') \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if} \quad i = 1 \\ 0 & \text{else.} \end{cases}$$

The last isomorphism is a consequence of the fact that W' is the result of the attaching of two handles, both of index one to V_0', therefore W'/V_0' is homotopic equivalent to a wedge of spheres $\mathbb{S}^1 \vee \mathbb{S}^1$. In this example the upper boundary of the triad V_1' is not connected.

The area of applicability of the fusing lemma is larger that the area of applicability of the proposition 3.13 by Takens. The following example of a 2-dimensional manifold confirms this claim.

Example 4.12. Let $(W'; V_0', V_1')$ be the triad from previous example and let $f' : W' \to [1/2, 3/2]$ be the correspondent triad function. The function f' has two critical points x_1 and x_2, both of index one and both situated in the critical level 1. In this case it is possible to apply the proposition by Takens, because the critical level is connected, but a small perturbation can change this favorable situation in such a way that the critical points lie in two different levels. Nevertheless, the fusing lemma preserves its applicability in this new situation, too. Let $\varepsilon > 0$ such that $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) = \emptyset$ and $\bar{B}(x_2, \varepsilon) \subseteq {f'}^{-1}(1/2, 3/2)$. Consider a bump function $\phi : W \to [0, 1]$ such that $\phi = 1$ on a neighborhood U of x_2 and $\phi = 0$ outside $B(x_2, \varepsilon)$. We can find $\lambda > 0$ with the property that for each $\lambda \in (-\lambda_0, \lambda_0) \setminus \{0\}$ the triad function $f_\lambda = f' + \lambda\,\phi$ has the values in $[1/2, 3/2]$, $f_\lambda(x_1) \neq f_\lambda(x_2)$ and $K_{f_\lambda} = K_{f'}$. For details see the lemma 2.8 of Milnor [19]. For any $\lambda \in (-\lambda_0, \lambda_0) \setminus \{0\}$ the function f_λ is a triad function with two critical points x_1 and x_2 situated in different critical levels, hence the proposition by Takens is not applicable. But all the conditions of the fusing lemma are satisfied. First W is connected and second there are no connecting trajectories between x_1 and x_2: in a neighborhood of K_{x_1} we have $f_\lambda = f'$, therefore the correspondent gradient vector fields are equal in this neighborhood. If follows that the set consisting of trajectories going from or to x_1 relative to the gradient vector field of f_λ coincide with the set consisting of trajectories going from or to x_1 relative to the gradient vector field of f', hence $K_1^{f_\lambda} \cap K_2^{f_\lambda} = \emptyset$.
Applying the fusing lemma we get for each $\lambda \in (-\lambda_0, \lambda_0) \setminus \{0\}$ a triad function $\tilde{f}_\lambda : W \to [0, 2]$ with exactly one critical point.

Another application of the fusing lemma is the interesting from our point of view part of the Proposition 2.9 by Takens [33] which states that the *crit* of a closed manifold is at most the dimension of the manifold plus one. To prove this fact Takens asserts: “By Smale every compact connected manifold M^n with $\partial M^n = \emptyset$ resp. $\partial M^n \neq \emptyset$, admits a function g with $(n+1)$ resp. (n) connected critical levels.” Without references I was not able to find this result in the published work of Smale. On the other hand it is more plausible that the presupposed function g has *at most* $(n + 1)$ resp. *at most* (n) connected critical levels. These are the reasons why we present here a proof of proposition by Takens without using the result by Smale, but the fusing lemma.

Proposition 4.13. *Let M be a closed manifold. Then*

$$crit(M) \leq dim(M) + 1.$$

Proof. First we consider the case $dim(M) = n \geq 3$. As an old theorem of the Morse theory asserts (see Milnor [19]), there exists a *nice* Morse function $f : M \to \mathbb{R}$ on M. The function f has by definition two important properties:

1. all the critical points of index i are in the level i, for each $i \in \overline{0, n}$
2. f has exactly one local minimum and exactly one local maximum.

Then $f^{=1/2}$ is diffeomorphic to the sphere $\mathbb{S}^{n-1}$ and the same for $f^{=n-1/2}$. The manifold M splits with the help of the nice Morse function f in triads. For each $i \in \overline{0, n}$, let $(W_i; V_i, V_{i+1})$ be the triad such that:

$$W_i = f^{-1}\Big(\Big[i - \frac{1}{2}, i + \frac{1}{2}\Big]\Big),\ V_i = f^{=i-\frac{1}{2}} \text{ and } V_{i+1} = f^{=i+\frac{1}{2}}.$$

All the boundaries V_i are connected for $i \in \overline{0, n}$; first we show this fact by induction for $0 \leq i \leq [\frac{n}{2}] + 1$. The boundaries $V_0 = \emptyset$ and $V_1 \approx \mathbb{S}^{n-1}$ are connected. Assume V_i is connected for $i \leq [\frac{n}{2}]$. The long exact sequence in homology for the pair (W_i, V_{i+1}),

$$\to H_1(W_i, V_{i+1}) \to H_0(V_{i+1}) \to H_0(W_i) \to H_0(W_i, V_{i+1}) \to 0$$

is by Poincaré Duality (see Browder [2] for the properties of the Poincaré triads) and the five lemma isomorphic to

$$\to H^{n-1}(W_i, V_i) \to H^n(V_i) \to H^n(W_i) \to H^n(W_i, V_i) \to 0.$$

The manifold W_i is obtained from V_i by attaching handles of index i. The manifold V_i is connected and the attached handles, too, hence W_i is connected and $H_0(W_i) = \mathbb{Z}$. Because $i \leq [\frac{n}{2}]$ and $n \geq 3$ we attach to V_i neither handles of index $n - 1$ nor of index n in order to obtain W_i, therefore $H^{n-1}(W_i, V_i) = 0$ and $H^n(W_i, V_i) = 0$, respectively. From the exact sequence it follows $H_0(V_{i+1}) \cong H_0(W_i) = \mathbb{Z}$, therefore V_{i+1} is connected for each $0 \leq i \leq [\frac{n}{2}] + 1$. To prove that V_{i+1} is connected for $[\frac{n}{2}] + 1 \leq i \leq n$ it suffices to apply the same reasoning to the function $n - f$.

For each $i \in \overline{0, n}$ the connected manifold W_i satisfies the hypothesis of the fusing lemma hence it admits a triad function $f_i : W_i \to \big[i - \frac{1}{2}, i + \frac{1}{2}\big]$ such that:

1. f_i is equal to $f|_{W_i}$ in a neighborhood of $\partial W_i = V_i \cup V_{i+1}$,
2. f_i has at most one critical point.

Now we get the required function $\tilde{f} : M \to \mathbb{R}$ defined by $\tilde{f}(x) = f_i(x)$ for each $x \in W_i$; it is smooth and has at most $dim(M) + 1$ critical points.

The case $n = 1$ is immediate and the case $n = 2$ is consequence of the fact that $V_1 \approx V_2 \approx \mathbb{S}^1$, hence connected. ♣

Using an argument from the proof of the fusing lemma (a path from x_1 to x_2, two critical points in the same critical level, can be deformed to a path in the critical level) we get a stronger result than the result attributed by Takens to Smale:

Proposition 4.14. *Every nice Morse function on a closed connected manifold M^n has all its critical levels connected. In particular, there is a Morse function on M^n with* at most $(n+1)$ *critical levels, all connected.*

Proof. Let $f : M \to \mathbb{R}$ be a nice Morse function. For $i = 0$ and $i = n$ the critical levels $f^{=i}$ are obviously connected. For $i \in \overline{1, n-1}$ let $(W_i; V_i, V_{i+1})$ be the triad defined in the previous proposition. Let x_1 and x_2 be two points in the critical level $f^{=i}$. By the connectedness of W_i (proof of proposition 4.13) there is a path in W_i from x_1 to x_2. Then we deform this path to a path in $f^{=i}$ from x_1 to x_2, like in the assertion 3 of fusing lemma. Hence the critical level $f^{=i}$ is connected and so all the critical levels of f are connected.♣

We are now going to introduce the concept of a Morse decomposition of an invariant set S in a flow on the manifold M, using the paper by Salamon [26]. In the concrete case that we analyze in this chapter S is the set of critical points K_f of some smooth function $f : M \to \mathbb{R}$ with the connecting trajectories relative to the flow generated by the negative gradient vector field of f.

Consider M a manifold and $\Phi : M \times \mathbb{R} \to M$ a flow on M. A set $S \subset M$ is said to the *invariant* if $\Phi(S, \mathbb{R}) = S$. The ω-limit sets of a set $Y \subset M$ are given by

$$\omega(Y) = \cap_{t>0} cl(\Phi(Y, [t, \infty))),$$

$$\omega^*(Y) = \cap_{t>0} cl(\Phi(Y, (-\infty, -t])).$$

Now we have all the data for the definition of a Morse decomposition.

Definition 4.15 (Morse decomposition). Let $S \subset M$ be a compact, invariant set. Then a finite collection $\{M(\pi) | \pi \in P\}$ of compact invariant, pairwise disjoint sets in S is said to be a *Morse decomposition* if there exists a bijection $\Pi : \{1, 2, \ldots, n\} \to P$ (an ordering) such that for every $\gamma \in S \setminus \cup_{\pi \in P} M(\pi)$ there exists indices $i, j \in \{1, \ldots, n\}$ such that $i < j$ and

$$\omega(\gamma) \subset M(\Pi(i)), \omega^*(\gamma) \subset M(\Pi(j)).$$

Every ordering of P with this property is said to be *admissible*. The sets $M(\pi)$ are called *Morse sets*.

A finite collection $\{M(\pi) | \pi \in P\}$ of S can have more than one admissible ordering on P. If S is a compact, invariant set in M and $\{M(\pi) | \pi \in P\}$ a Morse decomposition of S, then we define a minimal partial order on P. For $\pi', \pi'' \in P$ we have

$$\pi' < \pi''$$

if $\pi' \neq \pi''$ and π' comes before π'' in every admissible ordering of P. This defines a partial order on P. Clearly, any ordering of P is admissible if and only if it extends the minimal partial order on P. A subset $I \subset P$ is said to be an *interval* if

$$\pi', \pi'' \in I, \pi \in P, \pi' < \pi < \pi'' \Rightarrow \pi \in I.$$

Now we present a particular situation which is important for the process of fusing critical points. Let $f : M \to \mathbb{R}$ be a smooth function on a closed manifold M. Assume that f has only a finite number of critical points: $K_f = \{x_\pi | \pi \in P\}$. Consider $-\nabla_f$ the negative gradient vector field on M and Φ_f the associated (global) flow. The invariant set $S = K_f \cup \{\textit{connecting trajectories}\} = M$ is compact. A Morse decomposition of M is easy to get: first we order the critical values. We choose an ordering Π such that

$$f(x_{\Pi(i)}) < f(x_{\Pi(j)}) \Rightarrow i < j \text{ for } i, j \in \{1, \dots, n\}.$$

So we obtain a finite collection $\{x_{\Pi(i)} | i \in \overline{1, crit(f)}\}$ of compact invariants sets in S with an ordering Π. We denote the minimal partial order associated to this Morse decomposition by $(K_f, <_{-\nabla_f})$.

Definition 4.16. The partial order $(K_f, <_{-\nabla_f})$ is called the *partial order associated* to $-\nabla_f$.

Now we have all the notions we need for the following:

Proposition 4.17. *Let $f : M \to \mathbb{R}$ be a minimal function on a simply connected closed manifold M having exactly one local minimum and exactly one local maximum. Then $(K_f, <_{-\nabla_f})$ is a totally ordered Morse decomposition.*

Proof. Let $\{x_1, x_2, \dots, x_n\}$ be the critical set K_f of f. Consider $c \in f(M) \setminus f(K)$ a regular value. We compute the zero homology group of $f^{=c}$ in order to show that this set is connected. By the deformation lemma it exists $\varepsilon > 0$ such that $f^{\leq c+\varepsilon} \cap f^{\geq c-\varepsilon}$ is diffeomorphic to $f^{=c} \times [0, 1]$. The reduced Mayer-Vietoris sequence of $(f^{\geq c-\varepsilon}, f^{\leq c+\varepsilon})$ has the following form in its inferior part:

$$\begin{aligned} \to \tilde{H}_1(f^{\geq c-\varepsilon} \cup f^{\leq c+\varepsilon}) \to \tilde{H}_0(f^{\geq c-\varepsilon} \cap f^{\leq c+\varepsilon}) &\to \tilde{H}_0(f^{\geq c-\varepsilon}) \oplus \tilde{H}_0(f^{\leq c+\varepsilon}) \to \\ &\to \tilde{H}_0(f^{\geq c-\varepsilon} \cup f^{\leq c+\varepsilon}) \to 0. \end{aligned}$$

By the hypothesis the reduced homology group $\tilde{H}_1(f^{\geq c-\varepsilon} \cup f^{\leq c+\varepsilon}) = \tilde{H}_1(M) = 0$ and $\tilde{H}_0(f^{\geq c-\varepsilon} \cup f^{\leq c+\varepsilon}) = 0$. The set $f^{\geq c-\varepsilon}$ is connected, because every point of $f^{\geq c-\varepsilon}$ can be linked by trajectories with the unique maximum, similarly $f^{\leq c+\varepsilon}$ is connected. Hence $\tilde{H}_0(f^{\geq c-\varepsilon}) = 0$ and $\tilde{H}_0(f^{\leq c+\varepsilon}) = 0$. From the exact sequence it follows that $\tilde{H}_0(f^{\geq c-\varepsilon} \cap f^{\leq c+\varepsilon}) = 0$ and from the diffeomorphism $f^{\leq c+\varepsilon} \cap f^{\geq c-\varepsilon} \approx f^{=c} \times [0, 1]$ it follows $\tilde{H}_0(f^{=c}) = 0$, therefore $f^{=c}$ is connected.

Now we prove that f has n distinct critical levels. Suppose the existence of $i \neq j \in \overline{1, n}$ such that $f(x_i) = f(x_j)$. Without loss of generality we can suppose that $f^{-1}\big(f(x_i)\big) \cap K_f = \{x_i, x_j\}$. Else we can modify f to another minimal function $\tilde{f}$ such that $\tilde{f}$ contains only two critical points x_i and x_j in the level $\tilde{f}(x_i)$ and $\tilde{f}$ has exactly one local minimum and exactly one local maximum. Let $\varepsilon > 0$ such that $f^{\leq f(x_i)+\varepsilon} \setminus f^{<f(x_i)-\varepsilon} \cap K_f = \{x_i, x_j\}$. Then the manifold $f^{\leq f(x_i)+\varepsilon} \setminus f^{<f(x_i)-\varepsilon}$ is connected because $f^{=f(x_i)+\varepsilon}$ is connected. By the fusing lemma x_i and x_j fuse

in the connected triad $(f^{\le f(x_i)+\varepsilon} \setminus f^{<f(x_i)-\varepsilon}; f^{=f(x_i)-\varepsilon}, f^{=f(x_i)+\varepsilon})$, contradicting the minimality of f. Therefore all the critical points of f lie in different critical levels.

In the last part of this proof we show that $(K_f, <_{-\nabla_f})$ is totally ordered. Let $f(x_i)$ and $f(x_j)$ be two consecutive critical values of f. Suppose $f(x_i) > f(x_j)$. Then for some $\varepsilon > 0$ the triad $(f^{f(x_i)+\varepsilon} \setminus f^{<f(x_j)-\varepsilon}; f^{=f(x_j)-\varepsilon}, f^{=f(x_i)+\varepsilon})$ is connected and contains only two critical points: x_i and x_j. There is a connecting trajectory from x_i to x_j, else the critical points x_i and x_j fuse, contradicting the minimality of f. The existence of the trajectory implies $i > j$. Concluding, the ordering of the critical values of f induce an ordering of K_f and implicitly the Morse decomposition $\{x_i \mid i \in \overline{1, crit(f)}\}$ is totally ordered. ♣

Now we give another application of the fusing lemma.

Proposition 4.18. *Let M be a closed connected manifold. There exists a smooth function $f_0 : M \to \mathbb{R}$ with at most $dim(M) + 1$ critical points such $(K_{f_0}, <_{-\nabla_{f_0}})$ is totally ordered.*

Proof. Let $A \subset \mathbb{N}$ be the set of positive integers k with the property that M admits a smooth function $f : M \to [0, k]$ with $k + 1$ critical points $x_0, x_1, \dots, x_k$, such that $f(x_i) = i$ and the corresponding triads $(W_i; V_i, V_{i+1})$ are all connected, for each $i \in \overline{0, k}$. A function with these properties is called unofficially *no-name* function.

The set A is not empty because by the proof of the proposition 4.13, for $k = dim(M)$ there is a no-name function on M. Let k_0 be the smallest element of A and $f_0 : M \to \mathbb{R}$ a no-name function with $k_0 + 1$ critical points. Then $(K_{f_0}, <_{-\nabla_{f_0}})$ is totally ordered. Otherwise we get a no-name function g on M with k_0 critical points in the following way: let i be a natural integer such that $x_i \not<_{-\nabla_{f_0}} x_{i+1}$. By the properties of the no-name functions the manifold $W_i \cup W_{i+1}$ is connected. Furthermore, there is no trajectory between x_i and x_{i+1} because $f_0(x_i) = i$, $f_0(x_{i+1}) = i + 1$ and $x_i \not<_{-\nabla_{f_0}} x_{i+1}$. Therefore we can apply the fusing lemma for $f_0 |_{W_i \cup W_{i+1}}$. Hence there is a function $f_{0,i} : W_i \cup W_{i+1} \to [i - 1/2, i + 3/2]$ with the properties:

1. $f_{0,i}$ is equal to f_0 in a neighborhood of $\partial(W_i \cup W_{i+1}) = V_i \cup V_{i+2}$.

2. $f_{0,i}$ has exactly one critical point, situated in the level $i + 1$.

Let $\phi : [i - 1/2, i + 3/2] \to [i - 1/2, i + 1/2]$ a smooth function such that

1ϕ. $\phi = id$ close to $i - 1/2$ and $\phi = id - 1$ close to $i + 3/2$.

2ϕ. $\phi(i + 1) = i$.

3ϕ. $\phi' > 0$.

Now define the function $g : M \to \mathbb{R}$ by

$$g(x) = \begin{cases} f_0(x) & \text{if} \quad x \in f^{\le i-1/2} \\ \phi(f_{0,i}(x)) & \text{if} \quad x \in W_i \cup W_{i+1} \\ f_0(x) - 1 & \text{if} \quad x \in f^{\ge i+3/2}. \end{cases}$$

The smooth function g is a no-name function with k_0 critical points, contradiction.♣

Using the function f_0 from the previous proposition and repeating verbatim the last part of the proof of theorem 7 we can give an estimation for the *crit* of some special product manifolds:

Proposition 4.19. *Let M be a closed manifold of dimension m and let $\mathbb{S}^n$ be the n dimensional sphere. Then $crit(M \times \mathbb{S}^n) \leq m + 2$.*

5 The graphs of a nice minimal function

This chapter is the direct consequence of the chapter about the fusing lemma, where we have proven the *crit* inequality for manifolds having a minimal function with only one local minimum or only one local maximum. In this chapter we analyze the graph structure of the set

$$S = K_f \cup \{\text{connecting trajectories}\}.$$

Let $f : M \to \mathbb{R}$ be a smooth function on a closed manifold M with $crit(M)$ critical points i.e. a minimal function. The *G-graph* of f denoted by $\vec{G}(f) = (V(G(f)), \vec{E}(G(f)))$ is an oriented graph obtained in the following manner:

$$V(G(f)) = K_f$$

and

$$\begin{aligned} \vec{E}(G(f)) \; = \{\overrightarrow{xy} \mid \; & x, y \in V(G(f)) \text{ and there is } z \in M \text{ such that} \\ & \omega^*(z) = x \text{ and } \omega(z) = y\} \end{aligned}$$

where ω, ω^* are defined relative to the negative gradient vector field $-\nabla f$ of f, in some fixed Riemannian structure. If we write $G(f)$ then we mean the graph that we obtain from $\vec{G}(f)$ if we forget the orientation of the edges. Obviously we have the following characterization:

- x is a local minimum of f iff x is a sink vertex of $\vec{G}(f)$
- x is a local maximum of f iff x is a source vertex of $\vec{G}(f)$.

Definition 5.1. We denote by $m(f)$ the number of local minima of f and by $Min(f)$ the set of local minima of f.

In this chapter we analyze the case $m(f) \geq 2$ because the case $m(f) = 1$ is the subject of chapter 3.

Proposition 5.2. *The G-graph of f is connected, if M is connected.*

Proof. Suppose the existence of G_1, G_2 not empty sub-graphs of $G = G(f)$ such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \emptyset$. Let $M_1 = \{x \in M | \omega(x) \in V(G_1)\}$ and $M_2 = \{x \in M | \omega(x) \in V(G_2)\}$. Then $\emptyset \neq V(G_1) \subseteq M_1$, $\emptyset \neq V(G_2) \subseteq M_2$ and $M_1 \cup M_2 = M$, $M_1 \cap M_2 = \emptyset$. On the other hand there is no trajectory from $V(G_1)$ to $V(G_2)$ or from $V(G_2)$ to $V(G_1)$, therefore $M_1 = \{x \in M | \omega(x) \in V(G_1) \text{ and } \omega^*(x) \in V(G_1)\}$ and $M_2 = \{x \in M | \omega(x) \in V(G_2) \text{ and } \omega^*(x) \in V(G_2)\}$. It follows that M_1 and M_2 are closed, contradiction with the connectedness of M. ♣

Definition 5.3. Let $f : M \to \mathbb{R}$ be a smooth function on M. If the critical points $x_q, x_p \in K_f$ satisfy both conditions:

a) $\overrightarrow{x_p x_q} \in \vec{E}(G(f))$;

b) $f(x_p) = f(x_q) + 1$,

we say that x_q is a *predecessor* of x_p (relative to f).

We borrow a name from Morse theory to identify the functions with the following property:

Definition 5.4. Let $f : M \to \mathbb{R}$ be a function on M such that for each critical point $x \in K_f$ one of the following statements is true:

a) x is a local minimum of f and $f(x) = 0$.

b) x has a predecessor.

Then f is called a *nice* function.

The existence of a nice minimal function on each closed manifold is the subject of the next paragraph. Here we make some preparations for this purpose looking for a minimal function f_M on M having a Morse decomposition which is convenient for the next steps. This function f_M we obtain in a classical way by local modification of f (see pag. 17 of [19]) such that $K_f = K_{f_M}$ and all the critical points lie in different levels. Furthermore, after composing with a diffeomorphism we can assume that

$$f_M(K_{f_M}) \subseteq \mathbb{Z} \cap [1, crit(M)].$$

Let $\{x_1, \ldots, x_{m(f)}\}$ be a ordering of $Min(f_M)$ such that

$$i > j \text{ implies } f_M(x_i) > f_M(x_j) \text{ for } i, j \in \overline{1, m(f)}$$

and let $\{x_{m(f)+1}, \ldots, x_{crit(M)}\}$ be a ordering of $K_{f_M} \setminus Min(f_M)$ such that

$$i > j \text{ implies } f_M(x_i) > f_M(x_j) \text{ for } i, j \in \overline{m(f) + 1, crit(M)}.$$

Then $\{x_1, x_2, \ldots, x_{crit(M)}\}$ is a Morse decomposition of M, because the existence of $z \in M$ such that

$$\omega^*(z) = x_i \text{ and } \omega(z) = x_j$$

implies $i > j$. We prove previous assertion: $\omega^*(z) = x_i$ and $\omega(z) = x_j$ implies $f_M(x_i) > f_M(x_j)$ and x_i is not a local minimum. If x_j is a local minimum then $i > m(f) \geq j$, else $i, j > m(f)$ and the definition of the ordering for $K_{f_M} \setminus Min(f_M)$ implies again $i > j$. We prefer this Morse decomposition because all the local minima here are ordered from 1 to $m(f)$.

The G-graph of f_M has the property that

$$\overrightarrow{x_i x_j} \in \vec{E}(G(f_M)) \text{ implies } i > j.$$

The F-graphs

Our goal is to construct a sequence of functions $f_k : M \to \mathbb{R}$ for $1 \leq k \leq crit(M)$ such that $\vec{G}(f_k) = \vec{G}(f_M)$ and for each $1 \leq p \leq k$, one of the two following statements is fulfilled:

a) x_p is a local minimum and $f_k(x_p) = 0$.

b) there is a natural $1 \leq q \leq p$ such that $\overrightarrow{x_p x_q} \in \vec{E}(G(f_k))$ and $f_k(x_p) = f_k(x_q) + 1$.

Proof. We prove by induction the existence of a function f_k with the above properties and the supplementary condition:

$$\begin{array}{rclcl} f_k(x_i) & < & f_M(x_i) & \text{for} & i \leq k \\ f_k(x_i) & = & f_M(x_i) & \text{for} & i > k. \end{array}$$

Let $\varphi_1 : [1, crit(M)] \to [0, crit(M)]$ be a diffeomorphism such that φ_1 agrees with the identity on $\left[\frac{3}{2}, crit(M)\right]$. Let $f_1 = \varphi_1 \circ f_M$ where f_M is the function defined above. Then f_1 satisfies all the required properties. Let $K_{f_1} = \{x_1, \ldots, x_{crit(M)}\}$ be the critical set of f_1 with the ordering given by the Morse decomposition described above. With this ordering x_1 is the global minimum of f_1 and $f_1(x_1) = 0$.

Now we modify f_1 to a minimal function whose local minima are all in the zero level. Let x_2 be a local minimum of f_1 (here we analyze only the case $m(f) \geq 2$, the case $m(f) = 1$ is the subject of chapter 3). There is an $\epsilon > 0$ such that the connected component W_1 of x_2 in $f_1^{c_2+\epsilon}$ (e.g. $\epsilon = \frac{1}{2}$) contains only one critical point: x_2. By gluing the manifold $\partial W_1 \times I$ along the boundaries $\partial W_1 \times \{0\}$ and $\partial W_1 \times \{1\}$ between $f_1^{\geq c_2+\epsilon}$ and $f_1^{\leq c_2+\epsilon}$ respectively, we get a manifold M' diffeomorphic to M. On this manifold we get a new function f_2 in a natural way:

$$f_2(x) = \begin{cases} (c_2 + \epsilon) - p_2(x) f_1(x_2) & \text{if} \quad x \in \partial W_1 \times I \\ f_1(x) - f_1(x_2) & \text{if} \quad x \in W_1 \\ f_1(x) & \text{else} \end{cases}$$

where $p_2 : \partial W_1 \times I \to I$ is the projection of $\partial W_1 \times I$ on the second component I. The function f_2 is minimal, $G(f_2) = G(f_1) = G(f_M)$ and supplementary the points x_1 and x_2 are in the zero level. By repeating this construction for all the local minima x_i of f, $2 \leq i \leq m(f)$ we get a minimal function $f_{m(f)}$ with all the local minima in the level zero. The position of all the other critical points remains unchanged, thus they satisfy the inequalities imposed by the chosen Morse decomposition for f_M:

$$0 = f_{m(f)}(x_1) = \ldots = f_{m(f)}(x_{m(f)}) < f_{m(f)}(x_{m(f)+1}) < \ldots < f_{m(f)}(x_{crit(M)}).$$

Assume the existence of f_k for some $k \geq m(f)$. Since $\overrightarrow{G}(f_k) = \overrightarrow{G}(f_M)$ the critical point x_{k+1} is not a local minimum. Let q be the smallest integer $1 \leq q \leq k$ such that

$\overrightarrow{x_{k+1}x_q} \in \vec{E}(f_k)$ and

for each $1 \leq q' \leq k$ such that $\overrightarrow{x_{k+1}x_{q'}} \in \vec{E}(G(f_k))$ it follows $f_k(x_{q'}) \leq f_k(x_q)$.

Let W_{k+1} be the connected component of x_{k+1} in $f_k^{\leq c_{k+1}+\frac{1}{2}} \setminus f_k^{<c_q+\frac{1}{2}}$. We claim that the critical point x_{k+1} is the unique critical point contained in W_{k+1}. We'll prove this by contradiction. A very detailed proof can be find in the proof of theorem 7 and mainly consists in lemma 3.15. Here we only sketch it: let x_i with $i \neq k+1$ be a critical point situated in the highest critical level c_i of $f_k|_{W_{k+1}}$. Then $c_i = f_k(x_i) < f_k(x_{k+1}) = c_{k+1}$ by the supplementary conditions imposed on f_k. Without loss of generality we can assume that x_i and x_{k+1} are in the same connected component W_i of $f_k^{\leq c_{k+1}+\frac{1}{2}} \setminus f_k^{<c_i-\frac{1}{2}}$, else we can translate in the connected component of x_i in $f_k^{\leq c_{k+1}+\frac{7}{8}} \setminus f_k^{<c_i-\frac{1}{2}}$ the level c_i into the level $c_{k+1}+\frac{3}{4}$. We check that all the conditions of the fussing lemma 3.9 are fulfilled by $f_k|_{W_i}$: W_i is connected and $f_k|_{W_i}$ has exactly two critical levels c_{k+1} and c_i. The level c_{k+1} contains only one critical point x_{k+1}. The definition of q and $c_i < c_{k+1}$ do not allow the existence of a trajectory going from x_{k+1} to any critical point in the level c_i. Thus by the fusing lemma 3.9 we can reduce the number of critical points of f_k in this triad at least by one. This fact contradicts the minimality of f_k hence x_{k+1} is the unique critical point in W_{k+1}.

Let $\varphi_{k+1} : [f_k(x_q) - \frac{1}{2}, f_k(x_{k+1}) + \frac{1}{2}] \to [f_k(x_q) - \frac{1}{2}, f_k(x_{k+1}) + \frac{1}{2}]$ be a diffeomorphism such that φ_{k+1} agrees with the identity on a neighborhood of $\partial([f_k(x_q) - \frac{1}{2}, f_k(x_{k+1}) + \frac{1}{2}])$ and $\varphi_{k+1}(f_k(x_{k+1})) = \varphi_{k+1}(f_k(x_q)) + 1$. The existence of φ_{k+1} is assured by the inequality $f_k(x_{k+1}) - f_k(x_q) \geq 1$. The triad function $\varphi_{k+1} \circ f_k|_{W_{k+1}}$ agrees with $f_k|_{W_{k+1}}$ in a neighborhood of ∂W_{k+1}, hence the function $f_{k+1} : M \to \mathbb{R}$ given by:

$$f_{k+1}(x) = \begin{cases} \varphi_{k+1} \circ f_k|_{W_{k+1}} & \text{if} \quad x \in W_{k+1} \\ f_k(x) & \text{else,} \end{cases}$$

is smooth. We have $f_{k+1}(x_{k+1}) = f_{k+1}(x_q) + 1$ and the modification of f_k to f_{k+1} does not change the structure of the G-graph: $\vec{G}(f_{k+1}) = \vec{G}(f_k)$. The supplementary condition is satisfied because

$$f_{k+1}(x_{k+1}) = f_k(x_q) + 1 \overset{q \leq k}{<} f_M(x_q) + 1 \leq f_M(x_{k+1})$$

and for any $i \neq k+1$

$$f_{k+1}(x_i) = f_k(x_i).$$

Therefore f_{k+1} satisfies all the required properties. ♣

The function $f_{crit(M)}$ has the property that each of its critical points is a local minimum in the zero level or it has a predecessor, hence $f_{crit(M)}$ is a nice minimal function. So we have proven:

Proposition 5.5. *Every closed manifold possesses a nice minimal function.*

Now we define the first type of sub-graph of a G-graph that we need for our study:

Definition 5.6. Let $g : M \to \mathbb{R}$ be a nice function. An oriented sub-graph $\vec{F} \subseteq \vec{G}(g)$ is a *F-graph* of g iff

$$V(F) = K_g$$

and for each $x \in K_g \setminus Min(g)$

$$\{y \mid \overrightarrow{xy} \in \vec{E}(F)\} \subseteq \{y \mid y \text{ a predecessor of } x\}$$

and

$$card\big(\{y \mid \overrightarrow{xy} \in \vec{E}(F)\}\big) = 1.$$

In other words any F-graph $\vec{F}$ of g has the same vertex set as $\vec{G}(g)$ and each vertex $x \in V(F)$ which is not a sink point of $\vec{G}(g)$ is incident with exactly one edge of $\vec{F}$ beginning at x. Moreover the ending vertex of this edge must be a predecessor of x relative to g. About the existence of F-graphs we can prove:

Proposition 5.7. *Each nice minimal function on a closed manifold has an F-graph.*

Proof. Let $g : M \to \mathbb{R}$ be a nice minimal function and let $V(\vec{G}(g)) = \{x_1, \dots , x_{crit(M)}\}$. Then we define the oriented graph $\vec{F}(g)$ with the vertices:

$$V(\vec{F}(g)) = K_g$$

and the edges:

$\overrightarrow{x_i x_j} \in E(\vec{F}(g))$ iff $\overrightarrow{x_i x_j} \in E(\vec{G}(g))$ and j is the smallest integer such that x_j is a predecessor of x_i relative to g. Obviously $\vec{F}$ is an F-graph of g.♣

Proposition 5.8. *Each F-graph $\vec{F}$ of a nice minimal function $g : M \to \mathbb{R}$ is an oriented forest with $m(g)$ trees.*

Proof. Without loss of generality we take an ordering on $V(\vec{G}(g))$ such that x_i is a sink point for each $i \in \overline{1, m(g)}$. Let $F(x_i)$ be the connected component of x_i in $\vec{F}$ for each $1 \leq i \leq m(g)$.

Suppose that $F(x_i) \cap F(x_j) \neq \emptyset$ for some $1 \leq i \neq j \leq m(g)$. Then there is a path P in $\vec{F}$ from x_i to x_j. The path inherits an orientation from $\vec{F}$. Then x_i and x_j are sink points of $\vec{P}$, therefore there is a path $x_i' x x_j' \subseteq \vec{P}$ such that

$$\overrightarrow{xx_i'} \in E(\vec{P}) \subseteq \vec{F} \text{ and } \overrightarrow{xx_j'} \in E(\vec{P}).$$

This means that x is incident with two distinct edges of $\vec{F}$ starting at x, contradiction with the definition of F-graph. Hence

$$F(x_i) \cap F(x_j) = \emptyset \quad \text{for each } 1 \leq i \neq j \leq m(g).$$

On the other hand each $x \in V(\vec{F}) \setminus Min(g)$ is the initial vertex of a path having as terminal vertex a sink point. We can construct such a path in the following way: x is not a sink point then by the definition of F-graph there is an unique x_{p_1} such that $\overrightarrow{xx_{p_1}} \in E(\vec{F})$. If x_{p_1} is a sink point we get a path xx_{p_1} with the required properties, otherwise we continue in the same way until we arrive to a sink point x_{p_i}. Then $x \in F(x_{p_i})$. Furthermore $F = \cup_{i=1}^{m(g)} F(x_i)$ so $\vec{F}$ consists of $m(g)$ connected graphs. All this graphs are trees by the definition of F-graph.♣

With these notions we re-formulate the main result of chapter 3:

Proposition 5.9. *Let M be a closed manifold. If an F-graph of a nice minimal function on M is a tree then*

$$crit(M \times \mathbb{S}^n) \leq crit(M) + 1$$

for each positive integer n.

The MP-graphs

Let g be a nice minimal function on M and let $\vec{F}$ be an F-graph of g.

Definition 5.10. An oriented tree $\overrightarrow{MP}$ such that

$$\vec{F} \subseteq \overrightarrow{MP} \subseteq \vec{G}(g)$$

is called a *MP-graph* of g containing the F-graph $\vec{F}$.

We are looking for an MP-graph $\overrightarrow{MP}$ of g containing $\vec{F}$. With graph-theoretical arguments we prove the general result:

Proposition 5.11. *Let G be a connected graph and $F \subseteq G$ be a forest such that $V(F) = V(G)$. Then there is a tree $MP \subseteq G$ such that $F \subseteq MP \subseteq G$.*

Proof. Let F' be a forest having the minimal number $n \in \mathbb{N}$ of disjoint trees $T_1, T_2, \dots, T_n$ such that

$$F \subseteq F' = T_1 \cup T_2 \cup \ldots \cup T_n \subseteq G.$$

Obviously n is at most equal to the number of trees of F. If $n \neq 1$ then there is a vertex $y \in V(G) - V(T_1) = V(F') - V(T_1) = \cup_{i=2}^{n} V(T_i)$ that is adjacent in G to a vertex $z \in V(T_1)$, otherwise G is not connected. Let $2 \leq i \leq n$ such that $y \in T_i$. Then $T_1' = T_1 \cup T_i \cup yz$ is a tree and the forest $T_1' \cup (\cup_{j \in \{2,\dots,n\} \setminus \{i\}} T_j)$ has at most $n-1$ trees, contradicting the minimality of n.♣

The name of the MP-graphs comes from the geometrical-intuitive version of the previous proposition. If $x_1 \in T_1$ and $x_2 \in T_2$ are two local minima of g then they "produce" by a minimax theorem a critical point x_{12} of mountain pass type, and the edges $\overrightarrow{x_{12}x_1} \in \vec{G}(g)$, $\overrightarrow{x_{12}x_2} \in \vec{G}(g)$. One of these edges we must add, in the previous proof, to $\vec{F}$ in order to get a forest with at most $n-1$ trees.

Using the previous proposition we get a MP-graph $\overrightarrow{MP}$ of g containing $\vec{F}$. The MP-graph $\overrightarrow{MP}$ inherits the orientation from $\vec{G}$. So we have proven:

Corollary 5.12. *For each F-graph $\vec{F}$ of g there is a MP-graph of g containing $\vec{F}$.*

With the convention that $\max \emptyset = 0$ we give the following:

Definition 5.13. Let $\vec{G}$ be an oriented graph and $x \in V(\vec{G})$. The *height* of x is the nonnegative integer

$$\begin{aligned} h(x) = \quad \max\{i-1 \mid \ & \exists \text{ a path } y_1\, y_2 \ldots y_i \text{ such that} \\ & y_i = x \text{ and } \overrightarrow{y_k y_{k-1}} \in \vec{G} \text{ for each } 2 \le k \le i\}. \end{aligned}$$

The *length* of an edge $\overrightarrow{xx'} \in \vec{G}$ is the integer

$$l(\overrightarrow{xx'}) = h(x) - h(x').$$

A path has distinct vertices hence $h(x) \le |\vec{G}| - 1$, for each $x \in V(\vec{G})$. On the other hand a sink point has height zero, therefore

$$0 \le h(x) \le |\vec{G}| - 1 \qquad \text{for each } x \in V(\vec{G}).$$

Remark 5.14. For each edge $\overrightarrow{xx'} \in \vec{G}$ we have $l(\overrightarrow{xx'}) \ge 1$.

Proof. Let $i-1$ be the height of x'. There is a path $y_1 y_2 \ldots y_i$ such that $y_i = x'$ and $\overrightarrow{y_k y_{k-1}} \in \vec{G}$ for each $2 \le k \le i$. Then $y_1 y_2 \ldots y_i y_{i+1}$ is a path such that $y_i = x'$, $y_{i+1} = x$ and $\overrightarrow{y_k y_{k-1}} \in \vec{G}$ for each $2 \le k \le i+1$, hence $h(x) \ge i = h(x') + 1$.♣

Proposition 5.15. *Let g be a nice minimal function on M and $\vec{G}(g)$ its G-graph. If $x \in V(\vec{G}(g))$ then the height $h(x) = g(x)$.*

Proof. First we prove that $g(x) \le h(x)$. Let $\vec{F}$ be a F-graph of the nice minimal function g. The vertex x of $\vec{G}$ is at the same time a vertex of some tree in the forest $\vec{F}$. Hence there is a path $y_1\, y_2 \ldots y_i = x$ in that tree such that $\overrightarrow{y_k y_{k-1}} \in \vec{F}$ for each $2 \le k \le i$ and y_1 is a sink point. The path is determined by $\vec{F}$ uniquely because each vertex which is not a sink point is incident with an unique edge in $\vec{F}$ beginning with itself. We have the relations:

$$g(y_k) = g(y_{k-1}) + 1 \text{ and } g(y_1) = 0$$

for $2 \le k \le i$. It follows $g(x) \le h(x)$.

Now we prove that $g(x) \ge h(x)$. Let $y_1\, y_2 \ldots y_i = x$ be a path with $i = h(x) + 1$ such that $\overrightarrow{y_k y_{k-1}} \in \vec{G}(g)$ for each $2 \le k \le i$. Then y_1 is a minimum. Since g is a nice minimal function $g(y_1) = 0$ and $g(y_k) - g(y_{k-1}) \ge 1$. Summing all the inequalities for $2 \le k \le i$ we obtain $g(x) - g(y_1) \ge i - 1$ hence $g(x) \ge h(x)$.♣

The J-graphs

From now on we use the simplified notation for a graph, e.g. instead of $(V(G), E(G))$ we use G. For other notions about graphs see the appendix or the book by Bollobas [1].

We recapitulate what we have done until now: we have started with a nice minimal function g. To this nice minimal function we have associated an oriented graph $\vec{G}(g)$, the G-graph of g, uniquely determined by g. Then we have proven the existence of an F-graph, an oriented sub-graph of $\vec{G}(g)$ consisting of $m(g)$ trees. The nice minimal function g can have more that one F-graph, therefore we have proven that each F-graph $\vec{F}$ of g is contained in a spanning tree $\overrightarrow{MP}$ of the G-graph $\vec{G}(g)$, spanning tree that we have called a MP-graph of g containing $\vec{F}$. But the tree $\overrightarrow{MP}$ is not good enough for our purpose because not each edge of $\overrightarrow{MP}$ necessarily satisfies the following condition:

Definition 5.16. [1] Let J and H be sub-graphs of the oriented graph $\vec{G}$. We say that the edge $\overrightarrow{zz'} \in \vec{J}$ is *J-fragmentable in* H iff there is a path P from z to z' in $H - zz'$ such that for each $\overrightarrow{uu'} \in \vec{P}$:

Ja) $\overrightarrow{uu'} \in \vec{J}$ and $h(u) - h(u') = h(z) - h(z')$.

or

Jb) $h(u) - h(u') \leq h(z) - h(z') - 1$.

If $H = G$ we say that zz' is *J-fragmentable* omitting "in G". A graph J having all its edges not J-fragmentable is an *unfragmentable* graph.

For a nice minimal function $g : M \to \mathbb{R}$ and its G-graph $\vec{G}(g)$ the conditions Ja) and Jb) get by the proposition 5.15 the following form:

Ja) $\overrightarrow{uu'} \in \vec{J}$ and $g(u) - g(u') = g(z) - g(z')$.

or

Jb) $g(u) - g(u') \leq g(z) - g(z') - 1$.

Example 5.17. Let $J \subseteq G(g)$ be a tree. Then each edge of length one of J is not J-fragmentable.

Proof. Let $\overrightarrow{zz'}$ be an edge in $\vec{J}$ of length one. Suppose $\overrightarrow{zz'}$ is fragmentable. Then there is a path $P \subseteq G(g) - zz'$ such that for any $\overrightarrow{uu'} \in P$:

Ja) $\overrightarrow{uu'} \in \vec{J}$ and $g(u) - g(u') = 1$

or

Jb) $g(u) - g(u') \leq 0$.

But by the example 5.14 the graph $G(g)$ has no edges of length zero, hence the assertion Jb) does not occur. In this situation $P \subseteq J$. Furthermore $P \cup zz'$ is a cycle in J. But J is a tree, contradiction.♣

[1] We use the vector sign only when the orientation is requested.

Remark 5.18. With the notation of definition 5.16, if $\overrightarrow{zz'} \in \vec{J}$ is J-fragmentable in $H \subseteq G(g)$ then each $\overrightarrow{uu'} \in \vec{P}$ with $g(u) - g(u') = g(z) - g(z')$ is J-fragmentable in H.

Definition 5.19. A MP-graph $\vec{J}$ of g containing $\vec{F}$ having only J-unfragmentable edges is called a *J-graph* of g containing $\vec{F}$.

Our intention is to prove the existence of a J-graph of g for each F-graph $\vec{F}$. For this aim we prove by induction on i :

Proposition 5.20. *For each $1 \leq j \leq m(g)$ there is a tree $J_j \subseteq G(g)$ such that*

1. $F \subseteq J_j \subseteq G(g)$.
2. *The number of J_j-fragmentable edges of J_j in $\vec{G}(g)$ is at most $m(g) - j$.*

For $j = m(g)$ we obtain $J_{m(g)}$ an unfragmentable tree spanning $G(g)$.

Proof. Let $\overrightarrow{MP}$ be an MP-graph of g containing $\vec{F}$. Then $J_1 = MP$ is a tree and $F \subseteq J_1$. The number of its edges of length greater than 1 is at most $m(g)-1$. All the other edges are by the example 5.17 not J_1-fragmentable, therefore the condition 2 of the proposition is fulfilled, too.

Let $1 \leq j \leq m(g) - 1$ be a positive integer and J_j be a tree such that $F \subseteq J_j \subseteq G(g)$. At the moment we do not need all the induction hypothesis, but only part 1. Let $\overrightarrow{zz'}$ be a J_j-fragmentable edge (of J_j in $G(g)$). By definition there is a path P in $\vec{G}(g) - \overrightarrow{zz'}$ from z to z' and for each $\overrightarrow{uu'} \in \vec{P}$:

Ja) $uu' \in J_j$ and $g(u) - g(u') = g(z) - g(z')$

or

Jb) $g(u) - g(u') \leq g(z) - g(z') - 1$.

The length of $\overrightarrow{zz'} \in \vec{J_j}$ is at least 2 (see example 5.17) and the length of the edges of F is one hence if $g(u) - g(u') = g(z) - g(z')$ then $uu' \notin F$. The condition Ja) get a new form:

Ja') $uu' \in J_j - F$ and $g(u) - g(u') = g(z) - g(z')$

Let J'_j and J''_j be the two connected components of $J_j - zz'$ such that $z \in J'_j$ and $z' \in J''_j$. The path P does not contain the edge zz' hence the graph $(P \cup J_j) - zz' = P \cup J'_j \cup J''_j$. Furthermore it is connected. Let $V(P) = \{z_1 = z, z_2, \ldots , z_k = z'\}$ be the vertex set of P and I' resp. I'' be the set of indices i such that $z_i \in J'_j$ resp. $z_i \in J''_j$. Obviously $1 \in I'$ and $k \in I''$. Let i' be the largest element of I' and let i'' be the smallest element of I'' greater then i'. Then $z_{i'}z_{i''} \neq zz'$. The edge $z_{i'}z_{i''}$ is not an element of J_j else J_j contains a circuit made by zz', elements of J''_j, $z_{i''}z_{i'}$

and elements of J'_j, contradicting the fact that J_j is a tree. But $z_{i'}z_{i''} \in P$ hence it must satisfies at least the condition (b):

$$g(z_{i'}) - g(z_{i''}) \leq g(z) - g(z') - 1.$$

Let $L = J'_j \cup J''_j \cup z_{i'}z_{i''}$. The edge $z_{i'}z_{i''}$ is a bridge for L hence L has no circuit, i.e. is a tree. Furthermore $F \subseteq J'_j \cup J''_j \subseteq L \subseteq G(g)$. If $z_{i'}z_{i''} \in L$ is not L-fragmentable then we put $J_{j+1} = L$, else we repeat the above reasoning for L. After each such process the length of the new edge is at least one unit smaller than the length of the anterior fragmentable edge, therefore after at most $g(z) - g(z') - 1$ times we obtain an unfragmentable edge.

Now we assume that J_j satisfies the second condition of the induction hypothesis and has at most $m(g) - j$ J_j-fragmentable edges. The new graph J_{j+1} is a tree and $F \subseteq J_{j+1}$. It remains to count the number of J_{j+1}-unfragmentable edges. The new edge is by construction J_{j+1}-unfragmentable. We prove that the J_j-unfragmentable edges of J_j are J_{j+1}-unfragmentable. Let $\overrightarrow{vv'} \in \vec{J}_j$ be a J_j-unfragmentable edge. If $\overrightarrow{vv'}$ is J_{j+1}-fragmentable there is a path P_{j+1} from v to v' in $G(g) - vv'$. The path must contain the edge $\vec{J}_{j+1} - \vec{J}_j = \overrightarrow{ww'}$ and

$$g(v) - g(v') = g(w) - g(w')$$

else $\overrightarrow{vv'}$ is J_j-fragmentable edge. By the remark 5.18 $\overrightarrow{ww'}$ is J_{j+1}-fragmentable, contradiction. The number of J_{j+1}-fragmentable edges for J_{j+1} is at most $m(g) - j - 1$, because we replace a fragmentable edge by an edge which is not fragmentable and all the unfragmentable edges remain unfragmentable.♣

The tree $\vec{J}_{m(g)} \subseteq \vec{G}(g)$ is a J-graph of g containing F. Its orientation is inherited from $\vec{G}(g)$. So we have proven:

Corollary 5.21. *For each F-graph $\vec{F}$ of g there is a J-graph of g containing $\vec{F}$.*

Definition 5.22. Let $\vec{J}$ be an oriented graph. We say that $\vec{J}$ satisfies the (“ $\subseteq$ ”) condition if for $\overrightarrow{zz'} \in \vec{J}$ and $\overrightarrow{uu'} \in \vec{J}$ the property

$$\big(h(u'), h(u)\big) \cap \big(h(z'), h(z)\big) \neq \emptyset$$

implies

$$\begin{gathered} h(u) - h(u') = 1 \text{ or } h(z) - h(z') = 1 \text{ or} \\ (h(u) = h(z) \text{ and } h(u') = h(z')). \end{gathered}$$

We end this paragraph with the following:

Example 5.23. Each graph with all the edges of length one satisfies the condition (“ $\subseteq$ ”). A graph that does not satisfy the condition (“ $\subseteq$ ”) is the graph $\vec{J} = (V(J), \vec{E}(J))$, where

$$V(J) = \{1, 2, 3, 4\}$$
$$\vec{E}(J) = \{\overrightarrow{21}, \overrightarrow{32}, \overrightarrow{31}, \overrightarrow{43}, \overrightarrow{42}\}.$$

Remark 5.24. The previous example is very simple but the graph $\vec{J}$ is not a tree hence it cannot be the J-graph of a nice minimal function. The graph $\vec{J'} = (V(J'), \vec{E}(J'))$ where

$$V(J') = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$\vec{E}(J') = \{\overrightarrow{41}, \overrightarrow{74}, \overrightarrow{52}, \overrightarrow{85}, \overrightarrow{63}, \overrightarrow{96}, \overrightarrow{109}, \overrightarrow{81}, \overrightarrow{105}\}$$

is a tree that does not satisfy the condition (" $\subseteq$ ").

If a J-graph of a nice minimal function on M satisfies the condition (" $\subseteq$ ") then we can prove the *crit* inequality:

$$crit(M \times \mathbb{S}^n) \leq crit(M) + 1,$$

for each positive integer n. The proof of this inequality is the topic of the chapter 7.

Completing chapter 3

This last paragraph is the natural continuation of the topic described in chapter 3. In chapter 3 we missed the notions from this chapter, especially the graph theoretical language.

Proposition 5.25. *If a closed manifold M has a nice minimal function $f : M \to \mathbb{R}$ such that a J-graph $\vec{J}$ of f is not an interval than for any generalized Lusternik-Schnirelmann category λcat relative to $\mathfrak{Diff}(M)$ we get the strictly inequality $\lambda cat(M) < crit(M)$.*

Proof. Recall that $\vec{J}$ is a tree. If $\vec{J}$ is not an interval then there are two distinct vertices $x', x'' \in \vec{J}$ having a common predecessor x (i.e. $\overrightarrow{x'x} \in \vec{J}, \overrightarrow{x''x} \in \vec{J}$) or there is a vertex x having two predecessors $x', x'' \in \vec{J}$ (i.e. $\overrightarrow{xx'} \in \vec{J}, \overrightarrow{xx''} \in \vec{J}$). Then $h(x') = h(x'')$ and implicitly $f(x') = f(x'') = c$. This means that the function f has at the level c at least two critical points. Hence f has at most $crit(M) - 1$ distinct critical levels. But the Lusternik-Schnirelmann theorem for the generalized L-S category λcat (cf. theorem 5) implies the existence of $\lambda cat(M)$ distinct critical levels for f (f has a finite number of critical points as all the functions what we consider in this thesis). Therefore $\lambda cat(M) \leq crit(M) - 1$.♣

In the theorem 7 we have seen that the rules of cancelling critical points change into rules for cancelling vertices of trees. That is the reason why the proof of the following theorem deals with trees instead of critical points.

Theorem 8. *Let M be a closed manifold like in the theorem 7 and $N = \mathbb{S}^{p_1} \times \ldots \times \mathbb{S}^{p_{n-1}}$ be a product of $n - 1 \geq 1$ spheres. Then*

$$crit(M \times N) \leq crit(M) + n - 1.$$

Proof. Let $f : M \to \mathbb{R}$ be a nice minimal function on M with exactly one local minimum and let $\vec{F}_f$ be an F-graph of f. Then by proposition 5.8 $\vec{F}_f$ is a tree. Let $V(\vec{F}_f) = \{x_1, x_2, \dots, x_m\}$ where $m = crit(M)$ and let x_1 be the unique sink point of $\vec{F}_f$.

Let $g : N \to \mathbb{R}$ be a nice minimal function on N. Then $cat(N) = crit(N)$ implies by proposition 5.25 that each J-graph of g is an interval. Therefore each F-graph of g is an interval, too. We denote by $\vec{F}_g$ an F-graph of g (all the F-graphs of g are isomorphic). Let $V(\vec{F}_g) = \{y_1, y_2, \dots, y_n\}$ where y_n is the unique sink point of $\vec{F}_g$.

Now we pass to a graph theoretical problem. On the set

$$K_\infty = \{x_1, x_2, \dots, x_m\} \times \{y_1, y_2, \dots, y_n, y_{n+1}, \dots\} \supseteq V(\vec{F}_f) \times V(\vec{F}_g)$$

we define the equivalence relation: $(x', y_p) \sim (x'', y_q)$ iff one of the followings assertions are true:

1. $p = q$ and $x' = x''$.
2. $p < q$ and there is a path $x_{r_0} x_{r_1} \dots x_{r_{q-p}}$ in $\vec{F}_f$ such that $x_{r_0} = x'$, $x_{r_{q-p}} = x''$ and $\overrightarrow{x_{r_{i-1}} x_{r_i}} \in \vec{F}_f$ for each $i \in \overline{1, q-p}$.
3. $p > q$ and there is a path $x_{r_0} x_{r_1} \dots x_{r_{p-q}}$ in $\vec{F}_f$ such that $x_{r_0} = x''$, $x_{r_{p-q}} = x'$ and $\overrightarrow{x_{r_{i-1}} x_{r_i}} \in \vec{F}_f$ for each $i \in \overline{1, p-q}$.

We want to prove by induction on n that $card(K_n/_\sim) = m + n - 1$, where $K_n = \{x_1, x_2, \dots, x_m\} \times \{y_1, y_2, \dots, y_n\} = V(\vec{F}_f) \times V(\vec{F}_g)$.

For $n = 1$ the equality is trivial. Recall that for $n = 2$ we have shown that $card(K_2/_\sim) \le m + 1$ in the proof of theorem 7. Suppose $card(K_n/_\sim) = m + n - 1$. Let $z \in K_{n+1} \setminus K_n$. Then z has the form (x_p, y_{n+1}) where $p \in \overline{1, m}$. If $2 \le p \le m$ then x_p has an predecessor x_q in $\vec{F}_f$. Hence the points (x_p, y_{n+1}) and $(x_q, y_n) \in K_n$ are equivalent therefore in the same equivalence class. The point (x_1, y_{n+1}) is the unique point of $K_{n+1} \setminus K_n$ which is not equivalent to a point of K_n and therefore the unique point not in a class of $K_n/_\sim$. We have:

$$card(K_{n+1}/_\sim) = card(K_n/_\sim) + 1 = m + n.$$

We return to the topological aspect of the proof. Let $F : M \times N \to \mathbb{R}$ be the function given by $F(x, y) = f(x) + g(y)$ for each $x \in M$, $y \in N$. The critical set of F is $K_F = K_f \times K_g = K_n$. By corollary 3.13 there is a function on $M \times N$ with at most $card(K_F/_{\sim_F})$ critical points. On the other hand $\sim \subseteq \sim_F$, because $(x', y_p) \sim (x'', y_q)$ implies the existence of a path in the level $F((x', y_p)) = F((x'', y_q))$ from (x', y_p) to (x'', y_q). Hence

$$card(K_F/_{\sim_F}) \le card(K_F/_\sim) = m + n - 1.$$

Therefore $crit(M \times N) \le m + n - 1.$♣

6 Extended fusing lemma

Let $f : W \to \mathbb{R}$ be a triad function with the critical set K_f. As usual we assume that K_f is finite. In a previous chapter we have defined the set of trajectories going to or from the critical point x as

$$K_x = \{y \in W \,|\, \omega^*(y) = x \text{ or } \omega(y) = x\}.$$

Here we define for a set of critical points $A \subseteq K_f$ the set of the trajectories going to or from A as

$$K_A = \cup_{x \in A} K_x.$$

On the critical set we introduce an equivalence relation:

Definition 6.1. Two critical points of f, x_1 and x_2 are *f-equivalent*, $x_1 \sim_f x_2$ iff $f(x_1) = f(x_2) = c$ and there is a path from x_1 to x_2 contained in $f^{=c}$.

Let $A \subseteq K_f$ be a set of critical points containing x_1 and x_2. We say that x_1 and x_2 are *f-equivalent relative to A* iff $f(x_1) = f(x_2) = c$ and there is a path $\gamma_{x_1 x_2} : [0,1] \to W$ from x_1 to x_2 contained in $f^{=c}$ such that $\gamma_{x_1 x_2}([0,1]) \cap K_{K_f} \subseteq K_A$. In this chapter when we write $A/\sim_f$ or $\sim_f|_A$ we refer to $\sim_f$ relative to A.

Remark 6.2. By Takens (corollary 3.13) a triad function $f : W \to \mathbb{R}$ can be modified to a function $\tilde{f} : W \to \mathbb{R}$ with at most $card(K_f/\sim_f)$ critical points.

The following proposition is the natural extension of the fusing lemma for triad functions with more than two critical points. Indeed for $l = 2$, $A_0 = \{x_0\}$, $A_1 = \{x_1\}$ and $A_3 = \emptyset$ the proposition consists in a weaker form of the fusing lemma. This particular case of the following proposition is not exactly the fusing lemma because here we have an additional condition imposed on the path γ: $\gamma([0,1]) \cap K_{K_f} = \{x_0, x_1\}$. In the fusing lemma we do not have this condition, the existence of γ being sufficient to prove the lemma. This is the reason why the proof of the fusing lemma is more complicated than the proof of the following proposition:

Proposition 6.3 (Extended fusing lemma). *Let $f : W \to [0,1]$ be a triad function and $\{A_0, A_1, \ldots, A_{l+1}\}$ be a partition of the critical set K_f. Suppose that for the negative gradient vector field on W, the set of trajectories going to or from A_i is disjoint from the set of trajectories going to or from A_j, for each distinct i and j, $i, j = 0, \ldots, l+1$ (i.e. $K_{A_i} \cap K_{A_j} = \emptyset$ for each $i \neq j$, $i, j = 0, \ldots, l+1$).*

If there exists a path $\gamma : [0,1] \to W$ and $x_h \in A_h$ for $h = 0, \ldots, l$ such that $\gamma([0,1]) \cap K_{K_f} = \{x_h \mid h = 0, \ldots, l\}$, then there is a triad function $\tilde{f} : W \to [0,1]$ with the following properties:

a) $K_{\tilde{f}} = K_f$.

b) $\tilde{f} = f$ *in a neighborhood of* ∂W.

c) $card(A_0 \cup \ldots \cup A_l / \sim_{\tilde{f}}) \leq \sum_{h=0}^{l} card(A_h / \sim_f) - l$ *and*

$card(A_{l+1} / \sim_{\tilde{f}}) \leq card(A_{l+1} / \sim_f)$.

Proof. We divide the proof in three assertions.

Assertion 1. Suppose that $f(K_f) \subseteq \left[\frac{7}{16}, \frac{9}{16}\right]$. Then the extended fusing lemma is true for $l = 1$. We prove in addition that: $\sim_f|_{A_2} \subseteq \sim_{\tilde{f}}|_{A_2}$.

Proof. We construct the function $\tilde{f}$ similarly to the construction of $\tilde{f}$ in the assertion 2 of the fusing lemma.

Let $K_0 = K_{A_0}$ and $K_1 = K_{A_1 \cup A_2}$. For K_0 and K_1 we construct the function $\mu : W \to [0,1]$ like in the assertion 2 of the fusing lemma. The function $F : [0,1] \times [0,1] \to [0,1]$ we define here is only a bit different from its correspondent in the fusing lemma.

So, let $F : [0,1] \times [0,1] \to [0,1]$ be a smooth function such that:

aF) $\frac{\partial F}{\partial x}(x,y) > 0$ for each x and y in $[0,1]$.

bF) F_y is equal to the identity on $\left[0, \frac{1}{16}\right] \cup \left[\frac{15}{16}, 1\right]$ for each $y \in [0,1]$.

cF) $F_0(f(x_0)) = F_1(f(x_1))$ and $F_0'(x) = 1$ for each $x \in \left[\frac{1}{4}, \frac{3}{4}\right]$.

dF) F_1 equals the identity on $[0,1]$.

The construction of F can be derived from assertion 1 of the fusing lemma. Now define a new function $\tilde{f} : W \to [0,1]$ by $\tilde{f}(z) = F(f(z), \mu(z))$ for each $z \in W$. The smooth function $\tilde{f}$ has the following properties:

a$\tilde{f}$) $\tilde{f}$ has the same critical set as f, $K_{\tilde{f}} = K_f$.

b$\tilde{f}$) $\tilde{f} = f$ in a neighborhood of ∂W.

c$\tilde{f}$) $\tilde{f}(x_0) = \tilde{f}(x_1)$ and $\tilde{f}$ equals f plus a constant in the intersection of some neighborhood of K_0 with W', where $W' = f^{-1}\left(\left[\frac{1}{4}, \frac{3}{4}\right]\right)$.

d$\tilde{f}$) $\tilde{f}$ agrees with f in a neighborhood of K_1.

Each of the above properties of $\tilde{f}$ corresponds to the condition on F denoted by the same letter a, b, c or d. We insist only on c$\tilde{f}$), all the other properties being obvious. By definition, the function μ is zero on a neighborhood U_0 of K_0. At the same time $F_0' = 1$ on $\left[\frac{1}{4}, \frac{3}{4}\right]$ therefore there is $d \in \mathbb{R}$ such that $F_0(x) = x + d$ for each $x \in \left[\frac{1}{4}, \frac{3}{4}\right]$. Hence

$$\tilde{f}(z) = f(z) + d \text{ for each } z \in U_0 \text{ with } f(z) \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

The last two properties c$\tilde{f}$) and d$\tilde{f}$) are very important and they make the significant difference between this proposition and the fusing lemma. Here $K_{A_i}^{\tilde{f}} \cap W' = K_{A_i}^{f} \cap W'$ for $i = 0, 1, 2$. For $i = 1$ and $i = 2$, we have even $K_{A_i}^{\tilde{f}} = K_{A_i}^{f}$.

Let $\gamma : [0,1] \to W$ be the path from x_0 to x_1 with $\gamma([0,1]) \cap K_{K_f} = \{x_0, x_1\}$. Assume $\gamma([0,1]) \subseteq \left[\frac{7}{16}, \frac{9}{16}\right]$, else we project it from up and from down on the levels $f^{=\frac{9}{16}}$ and $f^{=\frac{7}{16}}$ respectively, getting a path in $\left[\frac{7}{16}, \frac{9}{16}\right]$ which does not intersect K_{K_f}, with the exception of x_0 and x_1. The property c$\tilde{f}$) and property d$\tilde{f}$) imply that $\gamma([0,1]) \cap K_{K_{\tilde{f}}} = \{x_0, x_1\}$. This property is necessary in order to project $\gamma([0,1])$ on the level $a = \tilde{f}(x_0) = \tilde{f}(x_1)$ such that $x_0 \sim_{\tilde{f}} x_1$ relative to $A_0 \cup A_1$. We can project the path γ directly on the the level a along the flow lines or stepwise using iteration of the projections from one critical level to the next one. We give the details of the projection with iterations. The projection follows in two steps: first we project the part of γ which is above the level a on $\tilde{f}^{=a}$ and then the part of γ which is under the level a on $\tilde{f}^{=a}$. Let $\{c_1, c_2, \ldots, c_{s+1}\}$ be the set of critical values of $\tilde{f}$ above a such that

$$c_1 \geq c_2 \geq \cdots \geq c_s \geq c_{s+1} = a.$$

The first deformation lemma asserts the existence of a strong deformation retraction π_1 of W onto $\tilde{f}^{\leq c_1}$. Then $\pi_1(\gamma) \subseteq \tilde{f}^{\leq c_1}$. For the next step of this iterative projection we need a stronger deformation lemma [3] :

Lemma 6.4 (Second deformation lemma). *Let $f : W \to [0,1]$ be a triad function with only isolated critical points. Suppose that $K_f \cap (a,b) = \emptyset$. Then $f^{\leq a}$ is a strong deformation retract of $f^{\leq b} \setminus K_b$.*

According to the second deformation lemma there is a strong deformation retraction π_2 from $\tilde{f}^{\leq c_1} \setminus K_{c_1}$ onto $\tilde{f}^{\leq c_2}$ hence $\pi_2 \circ \pi_1(\gamma) \subseteq \tilde{f}^{\leq c_2}$. Repeating this process for all the critical levels above a we get a iteration of γ such that $\gamma' = \pi_{s+1} \circ \pi_s \circ \cdots \circ \pi_1(\gamma) \subseteq \tilde{f}^{\leq a}$. If we do stepwise the analogous projection of γ' on each critical level below the level a we get a path γ'' contained in the level $\tilde{f}^{=a}$.

In order to prove that $\tilde{f}$ satisfies the condition c) of the proposition for $l = 1$ it remains to show that: $x \sim_f y$ relative to A_i implies $x \sim_{\tilde{f}} y$ relative to A_i for each $i \in \overline{0,2}$. Let i be fixed and $x \sim_f y$ relative to A_i. Then by the definition of the equivalence relation $\sim_f$ there exists a path $\gamma_{xy} : [0,1] \to W$ in the critical level of x and y such that $\gamma_{xy}([0,1]) \cap K_{K_f} \subseteq K_{A_i}$. Without loss of generality we assume that $\tilde{f} \circ \gamma_{xy}([0,1]) \subseteq \left[\frac{7}{16}, \frac{9}{16}\right]$. Then by the property c$\tilde{f}$) if $i = 0$ or by the property d$\tilde{f}$)

if $i = 1$ or $i = 2$ we have:

$$\gamma_{xy}([0,1]) \cap K^{\tilde{f}}_{K_{\tilde{f}}} = \gamma_{xy}([0,1]) \cap K^{\tilde{f}}_{K_{\tilde{f}}} \cap W' = \gamma_{xy}([0,1]) \cap K^{f}_{K_f} \cap W'.$$

By the definition of $x \sim_f y$ relative to A_i we get

$$\gamma_{xy}([0,1]) \cap K^{f}_{K_f} \cap W' \subseteq K^{f}_{A_i} \cap W'.$$

From the previous relation it follows $\gamma_{xy}([0,1]) \cap K^{\tilde{f}}_{K_{\tilde{f}}} \subseteq K^{f}_{A_i} \cap W'$. Then for each $z \in \gamma_{xy}([0,1]) \cap K_{K_{\tilde{f}}}$ we have $z \in K_{A_i}$ and therefore $\mu(z) = \mu(x) = \mu(y)$. But $z \in \gamma_{xy}([0,1])$ hence $f(z) = f(x) = f(y)$. Therefore $F(f(z), \mu(z)) = F(f(x), \mu(x)) = F(f(y), \mu(y))$ i.e. $\tilde{f}(z) = \tilde{f}(x) = \tilde{f}(y) := \tilde{c}$. This means that the path γ_{xy} intersects $K_{K_{\tilde{f}}}$ in the level $\tilde{f}^{=\tilde{c}}$. Using the second deformation lemma we can project continuously the path γ_{xy} on $\tilde{f}^{=\tilde{c}}$. We get a path $\tilde{\gamma}_{xy} : [0,1] \to W$ from x to y contained in $\tilde{f}^{=\tilde{c}}$ such that $\tilde{\gamma}_{xy}([0,1]) \cap K_{K_{\tilde{f}}} = \gamma_{xy}([0,1]) \cap K_{K_{\tilde{f}}} \subseteq K^{\tilde{f}}_{A_i}$. This means that $x \sim_{\tilde{f}} y$ relative to A_i.

In conclusion, we have $x_0 \sim_{\tilde{f}} x_1$ relative to $A_0 \cup A_1$, and $x \sim_f y$ relative to A_i implies $x \sim_{\tilde{f}} y$ relative to A_i for each $i \in \overline{0,2}$. Therefore:

$$\begin{aligned} card(A_0 \cup A_1 / \sim_{\tilde{f}}) &\leq card(A_0 / \sim_{\tilde{f}}) + card(A_1 / \sim_{\tilde{f}}) - 1 \leq \\ &\leq card(A_0 / \sim_f) + card(A_1 / \sim_f) - 1, \end{aligned}$$

and $\sim_f|_{A_2} \subseteq \sim_{\tilde{f}}|_{A_2}$.
Here we must make a remark which is useful for the others assertions:

Remark 6.5. If there is a critical subset $A \subseteq K_f$ such that $\gamma_{xy}([0,1]) \cap K_{K_f} \subseteq K_A$ then by definition $x \sim_f y$ relative to A. If in addition $K_A \cap K_{K_f \setminus A} = \emptyset$ from the proof of assertion 1 it follows that $x \sim_{\tilde{f}} y$ relative to A.

Assertion 2. The proposition is true for $l = 1$. In addition $\sim_f|_{A_2} \subseteq \sim_{\tilde{f}}|_{A_2}$.

Proof. We want to apply the result of the previous assertion, therefore we need a function as in assertion 1. Let f be a triad function as in the proposition. Then there is $0 < \varepsilon < 1/4$ such that $f(K_f) \subseteq [2\varepsilon, 1 - 2\varepsilon]$. Let δ be a real number such that $2\varepsilon < \delta < 1/2$.

Then there is a function $f_\delta : W \to [0,1]$ such that:

1δ**)** $K_{f_\delta} = K_f$.

2δ**)** $f = f_\delta$ on a neighborhood of ∂W.

3δ**)** $x \sim_f y$ relative to A_i implies $x \sim_{f_\delta} y$ relative to A_i, for each $i \in \overline{0,2}$.

4δ**)** $f_\delta(K_{f_\delta}) \subseteq [\delta, 1 - \delta]$.

In order to construct f_δ we need a smooth function $\phi : [0,1] \to [0,1]$ with the properties:

1ϕ) ϕ is a diffeomorphism.

2ϕ) $\phi = id$ on $[0,\varepsilon] \cup [1-\varepsilon, 1]$.

3ϕ) ϕ' equals a constant on the set $[2\varepsilon, 1-2\varepsilon]$.

4ϕ) $\phi([2\varepsilon, 1-2\varepsilon]) \subseteq [\delta, 1-\delta]$.

It is easy to check that the function $f_\delta = \phi \circ f$ has the desired properties 1δ), 2δ), 3δ) and 4δ).

For $\delta = \frac{7}{16}$ the function f_δ satisfies the condition of the assertion 1 hence there is a function $\tilde{f}_\delta : W \to [0,1]$ such that:

$$card(A_0 \cup A_1/\sim_{\tilde{f}_\delta}) \;\leq\; card(A_0/\sim_{f_\delta}) + card(A_1/\sim_{f_\delta}) - 1. \tag{6.1}$$

By the property 3δ) we have:

$$\sim_f|_{A_i} \subseteq \sim_{f_\delta}|_{A_i} \;\text{ for each } i \in \overline{0,2}, \tag{6.2}$$

hence the inequality 6.1 get the form that we required:

$$card(A_0 \cup A_1/\sim_{\tilde{f}_\delta}) \;\leq\; card(A_0/\sim_f) + card(A_1/\sim_f) - 1.$$

The supplementary statement is proved, too, because by the assertion 1 for f_δ we have $\sim_{f_\delta}|_{A_2} \subseteq \sim_{\tilde{f}_\delta}|_{A_2}$ and by 6.2 we have $\sim_f|_{A_2} \subseteq \sim_{f_\delta}|_{A_2}$ hence $\sim_f|_{A_2} \subseteq \sim_{\tilde{f}_\delta}|_{A_2}$.

Assertion 3. The proposition is true for each $l \geq 1$.

Proof. From the path γ we can easily obtain, eventually after a reordering of $\{A_0, \ldots, A_l\}$, for each $h \in \overline{1,l}$ a path $\gamma_h : [0,1] \to W$ such that $\gamma_h([0,1]) \cap K_{K_f} = \{z_h, z'_h\}$ with $z_h \in A_0 \cup \ldots \cup A_{h-1}$ and $z'_h \in A_h$. By the assertion 2 the proposition is true for $l = 1$. We apply this particular case to the partition $\{A_0 \cup \ldots \cup A_{h-1}, A_h, A_{h+1} \cup \ldots \cup A_{l+1}\}$ and the path γ_h for $h \in \overline{1,l}$ and we obtain the function $f_h : [0,1] \to \mathbb{R}$ such that

$$\begin{aligned} card(A_0 \cup \ldots \cup A_h/\sim_{f_h}) \;&\leq\; card(A_0 \cup \ldots \cup A_{h-1}/\sim_{f_{h-1}}) + \\ &+\; card(A_h/\sim_{f_{h-1}}) - 1, \end{aligned}$$

where we use the notation $f_0 := f$. When we pass from f_0 to f_h we do not alter the properties of γ_h that we need in order to apply the assertion 2. By the supplementary statement of the assertion 2 it follows:

$$\sim_{f_{h-1}}|_{A_{h+1}\cup\ldots\cup A_{l+1}} \subseteq \sim_{f_h}|_{A_{h+1}\cup\ldots\cup A_{l+1}} \tag{6.3}$$

for each $h \in \overline{1,l}$.

Summing the previous inequalities for $h \in \overline{1,l}$ we obtain:

$$card(A_0 \cup \ldots \cup A_l/\sim_{f_l}) \;\leq\; \sum_{h=1}^{l} card(A_h/\sim_{f_{h-1}}) + card(A_0/\sim_{f_0}) - l.$$

Let h be an integer between 1 and $l+1$. Then for each $1 \leq k < h$ the inclusion 6.3 implies

$$\sim f_{k-1} \mid_{A_{k+1} \cup \ldots \cup A_{l+1}} \subseteq \sim f_k \mid_{A_{k+1} \cup \ldots \cup A_{l+1}} .$$

Using the remark 6.5 after the assertion 1 we obtain $\sim_{f_{k-1}}|_{A_h} \subseteq \sim_{f_k}|_{A_h}$ for each $1 \leq k < h$. Hence

$$\sim_{f_0}| \; A_h \subseteq \sim_{f_1}| \; A_h \subseteq \; \ldots \subseteq \sim_{f_{h-1}}| \; A_h$$

and

$$card(A_h / \sim_{f_{h-1}}) \leq card(A_h / \sim_{f_{h-2}}) \leq \ldots \leq card(A_h / \sim_{f_0}).$$

In conclusion:

$$card(A_0 \cup \ldots \cup A_l / \sim_{f_l}) \quad \leq \quad \sum_{h=0}^{l} card(A_h / \sim_{f_0}) - l.$$

Furthermore $card(A_{l+1} / \sim_{f_l}) \leq card(A_{l+1} / \sim_{f_0})$. Hence the function $\tilde{f} = f_l$ satisfies all the requirements of the proposition. ♣

Corollary 6.6. *Under the hypothesis of proposition* 6.3 *there is a triad function* $\tilde{f} : W \to [0,1]$ *with the following properties:*

a) $K_{\tilde{f}} = K_f$.

b) $\tilde{f} = f$ *in a neighborhood of* ∂W.

c) $crit(\tilde{f}) \leq \sum_{h=0}^{l+1} card(A_h / \sim_f) - l$.

Proof. It follows from the remark 6.2 using the inequalities obtained in previous proposition.♣

7 The *crit* inequality for ("$\subseteq$") condition

We start this chapter with the following:

Lemma 7.1. *Let $\{K_i\}_{i\in I}$ be a partition of a finite set K and $\{B_j\}_{j\in J}$ be a family of subsets of K with the properties:*

Sa) $\cup_{j\in J} B_j \subseteq K$.

Sb) *for each $j \in J$ there is $i \in I$ such that $B_j \subseteq K_i$.*

Sc) *if there is $\{j_1, \dots, j_k\} \in J^k$ with $k \geq 2, j_l \neq j_m$ for each $1 \leq l \neq m \leq k$ such that*

$$B_{j_1} \cap B_{j_2} \neq \emptyset, B_{j_2} \cap B_{j_3} \neq \emptyset, \dots, B_{j_k} \cap B_{j_1} \neq \emptyset$$

then $card(B_{j_1} \cap \ldots \cap B_{j_k}) = 1$. With these assumptions:

$$\sum_{i\in I} \big(card(K_i) - 1\big) \geq \sum_{j\in J} \big(card(B_j) - 1\big).$$

Proof. First we prove the lemma by induction on $n = card(J) - card(I)$, under the same hypothesis but

Sa') $\cup_{j\in J} B_j = K$.

instead of Sa). Let $\pi : J \to I$ be a map such that $B_j \subseteq K_{\pi(j)}$ for each $j \in J$. The existence of the map π follows from Sb) and π is surjective because of Sa').

Let $n = 0$. Then the mapping π is bijective and $B_j = K_{\pi(j)}$ for each $j \in J$. The inequality is trivially satisfied. Now we treat the case $n+1 \geq 1$. If for each $j_1, j_2 \in J$ with $j_1 \neq j_2$ we have $B_{j_1} \cap B_{j_2} = \emptyset$ then $\{B_j\}_{j\in J}$ is a partition of K (exactly $\{K_i\}_{i\in I}$) and the inequality is trivially satisfied.

If there is some $j_1, j_2 \in J$ with $j_1 \neq j_2$ and $B_{j_1} \cap B_{j_2} \neq \emptyset$ then by the property Sc) for $\{j_1, j_2\}$ the cardinality $card(B_{j_1} \cap B_{j_2}) = 1$. Thus:

$$\sum_{j\in J} \big(card(B_j) - 1\big) = \sum_{j\in J\setminus\{j_1,j_2\}} \big(card(B_j) - 1\big) + card(B_{j_1}) - 1 + card(B_{j_2}) - 1 =$$

$$= \sum_{j\in J\setminus\{j_1,j_2\}} \big(card(B_j) - 1\big) + card(B_{j_1}) + card(B_{j_2}) - card(B_{j_1} \cap B_{j_2}) - 1 =$$

$$= \sum_{j\in J\setminus\{j_1,j_2\}} \big(card(B_j) - 1\big) + card(B_{j_1} \cup B_{j_2}) - 1 =$$

$$= \sum_{j\in (J\setminus\{j_1,j_2\})\cup\{j_{1,2}\}=J'} \big(card(B'_j) - 1\big).$$

where

$B'_j = B_j$ for $j \neq j_1$ and $j \neq j_2$

and

$B'_j = B_{j_1} \cup B_{j_2}$ for $j = j_{1,2}$.

Obviously $\{B'_j\}_{j\in J'}$ satisfies the conditions Sa'), Sb) and Sc). Hence, by the induction hypothesis:

$$\sum_{j\in(J\setminus\{j_1,j_2\})\cup\{j_{1,2}\}=J'} \big(card(B'_j) - 1\big) \leq \sum_{i\in I} \big(card(K_i) - 1\big)$$

hence we obtain the desired result:

$$\sum_{i\in I} \big(card(K_i) - 1\big) \geq \sum_{j\in J} \big(card(B_j) - 1\big).$$

Now we return to the original hypothesis and we apply the previous result to $\{K'_i\}_{i\in I'}$ a partition of a finite set K' and $\{B'_j\}_{j\in J}$, where $K' = K \cap (\cup_{j\in J} B_j)$, and $i \in I'$ iff $K'_i = K_i \cap (\cup_{j\in J} B_j) \neq \emptyset$. Then

$$\sum_{i\in I} \big(card(K_i) - 1\big) \geq \sum_{i\in I'} \big(card(K'_i) - 1\big) \geq \sum_{j\in J} \big(card(B_j) - 1\big).$$

and so we have proven the lemma under the general hypothesis. ♣

Let f be a nice minimal function on M with $crit(M)$ critical points. Suppose that a J-graph of f satisfies the condition (" $\subseteq$ "). We want to construct a function on $M \times \mathbb{S}^n$ with at most $crit(M)+1$ critical points. How can we make this construction?

Let $g : \mathbb{S}^n \to \mathbb{R}$ be the height function on $\mathbb{S}^n$ such that for y_S the south pole, $g(y_S) = 0$ and for y_N the north pole, $g(y_N) = 1$. Let $F_0 : M \times \mathbb{S}^n \to \mathbb{R}$ be the function given by $F_0(x, y) = f(x) + g(y)$, for each $x \in M, y \in \mathbb{S}^n$. Now we want to compute how many critical points fuse and how many critical points will be left out after fusing.

We use this J-graph of the function f in order to get some critical points of F_0 which could be fused by the fusing lemma or by the extended fusing lemma. The function F_0 has $2crit(M)$ critical points because $K_{F_0} = K_f \times K_g$.

To make the ideas clear we prove the *crit* inequality first for a particular case.

Definition 7.2. We say that an oriented J-graph $\vec{J}$ satisfies the strong (" $\subseteq$ ") condition if:

$$h(z) - h(z') = 1 \text{ for each } \overrightarrow{zz'} \in \vec{J}\ .$$

This is obviously a strong form of (" $\subseteq$ ") condition.

Example 7.3. Let $\vec{J}$ be an oriented graph such that $V(\vec{J}) = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $E(\vec{J}) = \{\overrightarrow{61}, \overrightarrow{62}, \overrightarrow{63}, \overrightarrow{73}, \overrightarrow{74}, \overrightarrow{75}\}$. Then $\vec{J}$ satisfies the strong (" $\subseteq$ ") condition.

Proposition 7.4. *If there is a J-graph $\vec{J}(f)$ of f which satisfies the strong (" $\subseteq$ ") condition then*

$$crit(M \times \mathbb{S}^n) \leq crit(M) + 1.$$

Proof. The function F_0 has $2crit(M)$ critical points. We use here the J-graph $\vec{J}(f)$ of f in order to find critical points of F_0 which can be fused by the corollary 3.13 in the following form adapted for our aim:

Proposition 7.5. *Let $f : N \to \mathbb{R}$ be a smooth function on the closed manifold N. Then there is a function $\tilde{f} : N \to \mathbb{R}$ with at most*

$$card\{< x >| \, x \in K_f\}$$

critical points where $< x >$ represents the equivalence class of x relative to the relation $\sim_f$.

For the function F_0 we cannot identify the equivalence classes of the equivalence relation $\sim_{F_0}$, but we can find sufficiently many critical points situated in the same equivalence class such that the maximal number of equivalence classes of $\sim_{F_0}$ is at most $crit(M) + 1$. We split the proof in two parts.

Part 1. Let x be a vertex of $\vec{J}(f)$ which is not a local minimum and let

$$\{x_1, \ldots , x_l\} := \{x' \in V | \overrightarrow{xx'} \in \vec{J}(f)\}.$$

Now we prove that the set $\{(x, y_S), (x_1, y_N), \ldots , (x_l, y_N)\}$ of critical points of F_0, is contained in the same equivalence class of $\sim_{F_0}$.

Let $1 \leq k \leq l$. By the strong (" $\subseteq$ ") condition on f we have $f(x) = f(x_k) + 1$, hence (x, y_S) and (x_k, y_N) lie both in the same level $F_0(x, y_S) = f(x) = f(x_k) + 1$. Furthermore there is a path from (x, y_S) to (x_k, y_N) situated in the level $F_0(x, y_S)$. We describe the construction of this path in detail. The edge $\overrightarrow{xx_k}$ is in $\vec{J}(f) \subseteq \vec{G}(f)$ hence there is a reparametrized trajectory $\alpha : [0, 1] \to M$ from x to x_k. Let $\beta : [0, 1] \to \mathbb{S}^n$ be the reparametrized path describing a trajectory on the sphere from y_N to y_S relative to the negative gradient vector field of the height function on the n-dimensional sphere. After an eventual reparametrization of β the path $\gamma : [0, 1] \to M \times \mathbb{S}^n$ defined by $\gamma(t) = (\alpha(t), \beta(1-t))$ for each $t \in [0, 1]$, remains all the time in the level $F(x, y_S)$. Then $(x, y_S) \sim_{F_0} (x_k, y_N)$ because $\gamma(0) = (x, y_s)$, $\gamma(1) = (x_k, y_N)$ and $F_0(\gamma([0, 1])) \subseteq F_0((x, y_S))$. Therefore $\{(x, y_S), (x_1, y_N), \ldots , (x_l, y_N)\} \subseteq < (x, y_S) >$.

Part 2. We want to apply the lemma 7.1 in order to get an estimate of $card\{< z >| \, z \in K_{F_0}\}$.

For this aim we need the following :

Definition 7.6. A vertex $x \in \vec{G}$ is called a *MP-point* if there are $x' \neq x'' \in V(G)$ such that $\overrightarrow{xx'} \in \vec{E}(G)$ and $\overrightarrow{xx''} \in \vec{E}(G)$ (x', x'' need not be predecessors of x). The set of all MP-points of $\vec{G}$ is denoted by $MP(\vec{G})$.

and we need some notations: let $I = \{< z >| \ z \in K_{F_0}\}$ and let $K_i = i$ for each $i \in I$. Then $K = \cup_{i\in I} K_i = K_{F_0}$. Our intention is to prove that $card(I) \leq crit(M) + 1$. Let

$$J_1 = \{(x, y_S) \mid x \text{ is a MP-point of } J(f)\}$$

$$J_2 = \{(x, y_S) \mid x \text{ is neither a MP-point of } J(f) \text{ nor in } Min(f)\}.$$

For $j = (x, y_S) \in J_1 \cup J_2$ we define:

$$B_j = \{j\} \cup \{(x', y_N) \mid \overrightarrow{xx'} \in \vec{J}(f)\}.$$

Let $J_3 = K_{F_0} \setminus \cup_{j\in J_1\cup J_2} B_j$, $J = J_1 \cup J_2 \cup J_3$ and for $j \in J_3$ we define $B_j = \{j\}$. We check if $\{B_j\}_{j\in J}$ satisfies the condition of the lemma 7.1 relative to $\{K_i\}_{i\in I}$:

Sa) obvious.

Sb) from part 1. it follows $B_j \subseteq < j >$.

To prove **Sc)** we make first the following remark: if for some $j \in J$ the set B_j contains a point of the form (x, y_S) then $j = (x, y_S)$ and j is the unique point of the form $(\cdot, y_S)$ in B_j. Let $(j_1, \dots, j_k) \in J^k$ with $k \leq 2, j_l \neq j_m$ for each $1 \leq l \neq m \leq k$ such that

$$B_{j_1} \cap B_{j_2} \neq \emptyset, B_{j_2} \cap B_{j_3} \neq \emptyset, \dots, B_{j_k} \cap B_{j_1} \neq \emptyset$$

and suppose $card(B_{j_1} \cap \dots \cap B_{j_k}) > 1$. According to the previous remark $B_{j_1} \cap \dots \cap B_{j_k}$ contains only points of the form $(\cdot, y_S)$. Let $j_1 = (x_1, y_S)$, $j_2 = (x_2, y_S)$ and $(x'_1, y_N), (x'_2, y_N) \in B_{j_1} \cap \dots \cap B_{j_k}$ such that $x'_1 \neq x'_2$. All the vertices x_1, x_2, x'_1, x'_2 are distinct from each other: by hypothesis $x_1 \neq x_2$ and $x'_1 \neq x'_2$; by the definition of B_{j_1} we have $x_1 \neq x'_1$ and $x_2 \neq x'_2$; analogously $x_2 \neq x'_1$, $x_2 \neq x'_2$. From the assumption $(x'_1, y_N), (x'_2, y_N) \in B_{j_1} \cap \dots \cap B_{j_k}$ it follows :

$$(x'_1, y_N), j_1, (x'_2, y_N) \in B_{j_1}$$

$$(x'_2, y_N), j_2, (x'_1, y_N) \in B_{j_2},$$

thus $\overrightarrow{x_1x'_1} \in \vec{J}(f)$, $\overrightarrow{x_1x'_2} \in \vec{J}(f)$, respectively $\overrightarrow{x_2x'_2} \in \vec{J}(f)$ and $\overrightarrow{x_2x'_1} \in \vec{J}(f)$. With the vertices $x'_1, x_1, x'_2, x_2, x_1$ and the edges $\overrightarrow{x'_1x_1}, \overrightarrow{x_1x'_2}, \overrightarrow{x'_2x_2}, \overrightarrow{x_2x'_1}$ we get a cycle in $J(f)$. This contradicts the fact $J(f)$ is a tree, hence $x'_1 = x'_2$ and $card(B_{j_1} \cap \dots \cap B_{j_k}) \leq 1$. In a similar manner it is possible to prove that $card(B_{j_1} \cap \dots \cap B_{j_k}) \geq 1$.

Applying the lemma 7.1 we obtain:

$$\begin{aligned} \sum_{i\in I} \big(card(K_i) - 1\big) \ &\geq \ \sum_{j\in J} \big(card(B_j) - 1\big) = \\ = \sum_{j\in J_1} \big(card(B_j) - 1\big) \ &+ \ \sum_{j\in J_2} \big(card(B_j) - 1\big) + \sum_{j\in J_3} \big(card(B_j) - 1\big). \end{aligned}$$

If $j \in J_1$ then $card(B_j) - 1 = card\{x' \in K_f \mid \overrightarrow{xx'} \in \vec{J}(f)\}$.
If $j \in J_2$ then $card(B_j) - 1 = 1$ because j has only one predecessor.
If $j \in J_3$ then $card(B_j) - 1 = 0$.

Summing we obtain

$$\sum_{i \in I} \big(card(K_i) - 1\big) \geq \sum_{j \in J_1} card\{x' \in K_f \mid \overrightarrow{xx'} \in \vec{J}(f)\} \tag{7.1}$$

$$+ \big(crit(M) - m(f) - card(MP(J(f)))\big). \tag{7.2}$$

But by the proposition 8.4 from appendix

$$\sum_{j \in J_1} card\{x' \in K_f \mid \overrightarrow{xx'} \in \vec{J}(f)\} - card(MP(J(f))) \geq m(f) - 1. \tag{7.3}$$

Concluding from 7.1 and 7.3:

$$\sum_{i \in I} \big(card(K_i) - 1\big) \geq crit(M) - 1.$$

On the other hand

$$\begin{aligned} \sum_{i \in I} \big(card(K_i) - 1\big) &= \sum_{i \in I} card(K_i) - card(I) = \\ &= card(K) - card(I) = 2crit(M) - card(I). \end{aligned}$$

Hence $card(I) \leq crit(M) + 1$ and the proposition 7.5 implies the existence of a function on $M \times \mathbb{S}^n$ with $crit(M) + 1$ critical points. ♣

Up to now we have considered a nice minimal function which satisfies the strong (“ $\subseteq$ ”) condition. We consider now a nice minimal function satisfying only the (“ $\subseteq$ ”) condition. In the former case the critical points of F_0 that fuse are in the same equivalence class of $\sim_{F_0}$ from the beginning. In the latter we must modify F_0 to a function F such that the candidates for the fusing will lie in the same equivalence class of the relation $\sim_F$. The modification of F_0 is based on the extended fusing lemma.

We can pass to the main result of this chapter:

Proposition 7.7. *Let M be a closed manifold and $f : M \to \mathbb{R}$ a nice minimal function. If there is a J-graph $\vec{J}(f)$ of f satisfying the (“ $\subseteq$ ”) condition then*

$$crit(M \times \mathbb{S}^n) \leq crit(M) + 1.$$

Before starting proving the proposition we recall the following:

Definition 7.8. We say that $\vec{J}(f)$ satisfies the (“ $\subseteq$ ”) condition if for $\overrightarrow{zz'} \in \vec{J}(f)$ and $\overrightarrow{uu'} \in \vec{J}(f)$ the property

$$\big(f(u'), f(u)\big) \cap \big(f(z'), f(z)\big) \neq \emptyset$$

implies

$$f(u) - f(u') = 1 \text{ or } f(z) - f(z') = 1 \text{ or}$$
$$(f(u) = f(z) \text{ and } f(u') = f(z')).$$

Proof. The proof has five parts. Let $\vec{F}(f)$ be an F-graph of f contained in $\vec{J}(f)$.

1. *An estimate for* $K_{F_0}/\sim_{F_0}$. Let $F_0 : M \to \mathbb{R}$ be the function obtained from f and g as before.

Let J_1 and J_2 be as in the proof of proposition 7.4. For $j = (x, y_S) \in J_1 \cup J_2$ we define

$$B'_j = \{j\} \cup \{(x', y_N) \mid \overrightarrow{xx'} \in \vec{F}(f)\}.$$

Then x' is by definition of an F-graph a predecessor of x.

Let $J_3 = K_{F_0} \setminus \cup_{j \in J_1 \cup J_2} B'_j$ and for $j \in J_3$ we define $B'_j = \{j\}$. We analyze the sets B'_j for $j \in J_1 \cup J_2$. The points j and (x', y_N) are equivalent: $j \sim_{F_0} (x', y_N)$ because x' is the predecessor of $pr_1(j) = x$ in the G-graph $\vec{G}(f)$ of f. Then there is $i \in I$ such that $B'_j \subseteq K_i$. Furthermore, according to the definition of an F-graph, $card(B'_j) = 2$ for each $j \in J_1 \cup J_2$. The set B'_j for $j \in J_3$ consists of only one point hence $card(B'_j) = 1$. Let $J = J_1 \cup J_2 \cup J_3$.

Concluding $\{B'_j\}_{j \in J}$ is a sub-partition of $K_{F_0}/\sim_{F_0}$, therefore

$$card(K_{F_0}/\sim_{F_0}) \leq card(J). \tag{7.4}$$

Computing $card(J)$ we obtain:

$$\begin{aligned} card(J) &= card(K_{F_0}) - card(J_1) - card(J_2) = \\ &= 2crit(M) - card\,MP(J(f)) - \big(crit(M) - cardMP(J(f)) - m(f)\big) = \\ &= crit(M) + m(f). \end{aligned}$$

From (7.4) and the previous equality we can obtain after fusing a function with at most $card(K_{F_0}/\sim_{F_0}) \leq crit(M) + m(f)$ critical points. To get the other $m(f) - 1$ critical points which must be canceled we use the extended fusing lemma, being careful not to destroy the equivalence relation between the critical points that we already counted as points that fuse.

We group the points that we already counted together with a symmetric relation E_0 defined on K_{F_0}. Let $(x_1, y_1), (x_2, y_2) \in V(\vec{G}(F_0))$. Then $(x_1, y_1)E_0(x_2, y_2)$ iff

$$\overrightarrow{x_1x_2} \in \vec{F}(f),\ y_1 = y_S \text{ and } y_2 = y_N$$
$$\text{or}$$
$$\overrightarrow{x_2x_1} \in \vec{F}(f),\ y_1 = y_N \text{ and } y_2 = y_S.$$

Note that $(x_1, y_1)E_0(x_2, y_2)$ iff there is $j \in J_1 \cup J_2$ such that the points (x_1, y_1) and (x_2, y_2) are in B'_j. This means that we have just counted the points (x_1, y_1) and (x_2, y_2) as points that can fuse. When we modify the function F_0 to F in order to get other points that fuse we do it in such a way that

$$(x_1, y_1)E_0(x_2, y_2) \text{ implies } (x_1, y_1) \sim_F (x_2, y_2).$$

2. *A partition of K_{F_0}.*

In order to apply the extended fusing lemma we need a partition of the critical set K_{F_0}. One way to get it is using the equivalence classes of an equivalence relation. For this reason we define a relation as follows:

Definition 7.9. Let $\vec{G}(g)$ be the G-graph of some function g. Let E be a symmetric relation on $V(\vec{G}(g))$ and let $u, v \in V(\vec{G}(g))$.
Then $u\mathcal{R}v$ iff there is a sequence of vertices $z_1, \dots, z_k \in V(\vec{G}(g))$ such that

$$u = z_1 R z_2 R \dots R z_k = v$$

where $R \in \{=, <, >, E\}$. Recall $v < u$ iff $\overrightarrow{uv} \in \vec{G}(g)$ and $u < v$ iff $\overrightarrow{vu} \in \vec{G}(g)$. When there is the risk of confusion we use the notation $<_g$.

It is easy to check that $\mathcal{R}$ is an equivalence relation on $V(\vec{G}(g))$.
Let $\{\overrightarrow{x_p x'_p} \mid p \in \overline{1, m(f)-1}\} = E\big(\vec{J}(f) - \vec{F}(f)\big)$ be the set of all edges of $\vec{J}(f) - \vec{F}(f)$. Every element of this set is an edge of the J-graph $\vec{J}$ therefore J-unfragmentable in $\vec{G}(f)$. Without loss of generality we can assume that:

$$f(x_1) \le f(x_2) \le \dots \le f(x_{m(f)-1}).$$

Now we split the manifold $M \times \mathbb{S}^n$ in triads. There is $\epsilon > 0$ such that

$$\begin{aligned} & K_{F_0} \cap F_0^{-1}\big(\big[F_0(x'_p, y_N) - \epsilon, F_0(x_p, y_S) + \epsilon\big]\big) = \\ = \; & K_{F_0} \cap F_0^{-1}\big(\big[F_0(x'_p, y_N), F_0(x_p, y_S)\big]\big), \end{aligned}$$

for each $p \in \overline{1, m(f)-1}$. We introduce the notation:

$$W_p = F_0^{-1}\left(\left[F_0(x'_p, y_N) - \frac{\epsilon}{2}, F_0(x_p, y_S) + \frac{\epsilon}{2}\right]\right).$$

For each $p \in \overline{1, m(f)-1}$ let $l(p)$ be the largest integer such that

$$f(x_p) = f(x_{p+1}) = \dots = f(x_{l(p)}).$$

3. *The case L1).* We analyze first the case L1) when $p = l(p)$ for each $p \in \overline{1, m(f)-1}$. In the triad W_p we can apply the extended fusing lemma to the function $F_0 \mid_{W_p}$. For this aim we define a partition of $K_{F_0} \cap W_p$, based on the equivalence relation $\mathcal{R}_p = \mathcal{R}_0 \mid_{K_{F_0} \cap W_p}$, where $\mathcal{R}_0$ is the relation obtained for $R \in \{=, <, >, E_0\}$.

Let

$$\begin{aligned} A_1^p &= \{(x, y) \mid (x, y)\mathcal{R}_p(x_p, y_S)\}, \\ A_2^p &= \{(x, y) \mid (x, y)\mathcal{R}_p(x'_p, y_N)\}, \\ A_3^p &= K_{F_0} \cap W_p \setminus (A_1^p \cup A_2^p). \end{aligned}$$

In order to apply the extended fusing lemma we must verify that:

1) $\{A_1^p, A_2^p, A_3^p\}$ is a partition of $K_{F_0} \cap W_p$,

2) the set of trajectories going to or from A_i^p and the set of trajectories going to or from A_j^p are disjoint for each $i \neq j \in \{1, 2, 3\}$ and

3) there is a path $\gamma_p : [0, 1] \to W_p$ from $(x_p, y_S) \in A_1^p$ to $(x'_p, y_N) \in A_2^p$, which does not intersect any trajectory going to or from a critical point in W_p.

We prove first the assertions:

1) and 2). Obviously $A_1^p \cap A_3^p = A_2^p \cap A_3^p = \emptyset$. On the other hand we have the implication: the existence of a trajectory between some $u \in A_1^p$ and some $v \in A_2^p$ implies $u < v$ or $u > v$, implies $u\mathcal{R}_p v$. Hence $A_1^p = A_2^p$ and $(x'_p, y_N) \in A_1^p$, implicitly $(x_p, y_S)\mathcal{R}_p(x'_p, y_N)$. Then by the definition of $\mathcal{R}_0$ there is a sequence of vertices $z_1, \dots, z_k \in K_{F_0} \cap W_p$ such that

$$(x_p, y_S) = z_1 R z_2 R \dots R z_k = (x'_p, y_N)$$

where $R \in \{=, <, >, E_0\}$. Then there is a sequence $pr_1(z_1), \dots, pr_2(z_k) \in K_f$ with the properties:

(Ra) $pr_1(z_i) \in f^{-1}[f(x'_p), f(x_p)]$ for each $1 \leq i \leq k$.

(Rb) $x_p = pr_1(z_1) R_f pr_1(z_2) R_f \dots R_f pr_2(z_k) = x'_p$ where $R_f \in \{=, <_f, >_f\}$.

(Rc) $\mid f(pr_1(z_i)) - f(pr_1(z_{i-1})) \mid \leq f(x_p) - f(x'_p) - 1$ for each $2 \leq i \leq k$.

From (Rb) it follows that there is a path P in $V(\vec{G}(f))$ from x_p to x'_p passing through the vertices $x_p = pr_1(z_1), \dots, pr_1(z_k) = x'_p$. Therefore $\overrightarrow{x_p x'_p} \in \vec{J}(f)$ is by the property (Rc) J-fragmentable in $\vec{G}(f)$ (by (Ra) even in $\vec{G}\big(f \mid_{[f(x'_p), f(x_p)]}\big)$). Contradiction with the hypothesis that $\vec{J}(f)$ as J-graph of f has all its edges J-unfragmentable, so $A_1^p \cap A_2^p = \emptyset$ and there is no trajectory between some $u \in A_1^p$ and some $v \in A_2^p$.

We make two remarks serving as hints for proving the assertions (Ra), (Rb), (Rc):

R1) If $z \in K_{F_0} \cap W_p$ then $F_0(z) = f(pr_1(z)) + \kappa$, where $\kappa = 0$ or 1, and

$$f(x'_p) + 1 = F_0(x'_p, y_N) \leq F_0(z) \leq F_0(x_p, y_S) = f(x_p).$$

R2) The second remark is the following:

Proposition 7.10. *Let $\vec{G}(F_0)$ be the G-graph of $F_0 = f + g$ and let $z, z' \in V(\vec{G}(F_0))$. Then $z' < z$ in $\vec{G}(F_0)$ iff*

$$\begin{gathered}(pr_1(z') < pr_1(z) \ \ in \ \vec{G}(f) \quad and \quad pr_2(z') \leq pr_2(z) \ \ in \ \vec{G}(g)) \\ or \\ (pr_1(z') \leq pr_1(z) \ \ in \ \vec{G}(f) \quad and \quad pr_2(z') < pr_2(z) \ \ in \ \vec{G}(g)).\end{gathered}$$

3) We continue with the third condition for the extended fusing lemma, the existence of the path γ_p. We know that $\overrightarrow{x_p x'_p} \in \overrightarrow{J}(f)$, hence there is a trajectory from x_p to x'_p relative to the negative gradient vector field of f. Let $\alpha : [0,1] \to M$ be the reparametrized path that describe the the trajectory from x_p to x'_p such that $f(\alpha(t)) = (1-t)f(x_p) + tf(x'_p)$ for each $t \in [0,1]$. Let $\beta : [0,1] \to \mathbb{S}^n$ be the path describing the trajectory from y_N to y_S such that $g(\beta(t)) = 1-t$ for each $t \in [0,1]$. Then the mapping γ_p given by $\gamma_p(t) = (\alpha(t), \beta(1-t))$ for each $t \in [0,1]$ is a path from (x_p, y_S) to (x'_p, y_N) in $[f(x'_p)+1, f(x_p)] \subset W_p$ with the required properties.

Now applying the extended fusing lemma we get a function $F_p : W_p \to \mathbb{R}$ such that:

EFLp a) $K_{F_p} = K_{F_0} \cap W_p$.

EFLp b) $F_p = F_0$ in a neighborhood of ∂W_p.

EFLp c)

$$\begin{aligned} card(K_{F_p}/\sim_{F_p}) &= card(A_1^p \cup A_2^p \cup A_3^p/\sim_{F_p}) \\ &\leq card(A_1^p/\sim_{F_0}) + card(A_2^p/\sim_{F_0}) + card(A_3^p/\sim_{F_0}) - 1. \end{aligned}$$

With the triad functions F_p we construct a new function $F : M \times \mathbb{S}^n \to \mathbb{R}$ as follows:

$$F(x) = \begin{cases} F_p(x) & \text{if} \quad x \in W_p \\ F_0(x) & \text{if} \quad x \in cl\big(M \times \mathbb{S}^n \setminus \cup_{p=1}^{m(f)-1} W_p\big) := W \end{cases}$$

The function F is well defined because (" $\subseteq$ ") implies $W_p \cap W_{p'} = \emptyset$ for each $p \neq p' \in \{1, \ldots, m(f)-1\}$. The function F is smooth from (EFLb) and has the same critical set as F_0, $K_F = K_{F_0}$ from (EFLa). We have

$$(7.5) \qquad card(K_F/\sim_F) = \sum_{p=1}^{m(f)-1} card(K_{F_p}/\sim_{F_p}) + card(K_{F_0} \cap W/\sim_{F_0}).$$

Furthermore:

$$card(A_1^p/\sim_{F_0}) + card(A_2^p/\sim_{F_0}) + card(A_3^p/\sim_{F_0}) \leq$$

$$\leq card(A_1^p/\mathcal{R}_0 \cap \sim_{F_0}) + card(A_2^p/\mathcal{R}_0 \cap \sim_{F_0}) + card(A_3^p/\mathcal{R}_0 \cap \sim_{F_0})$$

because $\mathcal{R}_0 \cap \sim_{F_0} \subseteq \sim_{F_0}$. Since A_1^p, A_2^p and A_3^p are sets of equivalence classes of $\mathcal{R}_0$:

$$\begin{aligned} card(A_1^p/\mathcal{R}_0 \cap \sim_{F_0}) &+ card(A_2^p/\mathcal{R}_0 \cap \sim_{F_0}) + card(A_3^p/\mathcal{R}_0 \cap \sim_{F_0}) = \\ &= card(A_1^p \cup A_2^p \cup A_3^p/\mathcal{R}_0 \cap \sim_{F_0}) = card(K_{F_p}/\mathcal{R}_0 \cap \sim_{F_0}). \end{aligned}$$

From (EFLp c)) and the previous relations we get:

$$card(K_{F_p}/\sim_{F_p}) \leq card(A_1^p/\sim_{F_0}) + card(A_2^p/\sim_{F_0}) + card(A_3^p/\sim_{F_0}) - 1 \leq$$

$$\leq card(K_{F_p}/\mathcal{R}_0 \cap \sim_{F_0}) - 1.$$

On the other hand:

$$\begin{aligned} card(K_{F_0} \cap W/\sim_{F_0}) &= card(K_{F_0} \cap W/\sim_{F_0}) \leq \\ &\leq card(K_{F_0} \cap W/\mathcal{R}_0 \cap \sim_{F_0}). \end{aligned}$$

We know that $K_{F_0}/\mathcal{R}_0 \cap \sim_{F_0} \subseteq K_{F_0}/\sim_{F_0} \subseteq \{K_{F_p}\}_{p\in\overline{1,m(f)-1}}$ hence

$$\sum_{p=1}^{m(f)-1} card(K_{F_p}/\mathcal{R}_0 \cap \sim_{F_0}) + card(K_{F_0} \cap W/\mathcal{R}_0 \cap \sim_{F_0}) = \tag{7.6}$$

$$= card\big(\cup_{p=1}^{m(f)-1} K_p \cup (K_{F_0} \cap W)/\mathcal{R}_0 \cap \sim_{F_0} \big) = card(K_{F_0}/\mathcal{R}_0 \cap \sim_{F_0}). \tag{7.7}$$

From 7.5 using the inequalities obtained above we get:

$$card(K_F/\sim_F) \leq card(K_{F_0}/\mathcal{R}_0 \cap \sim_{F_0}) - (m(f) - 1).$$

Since $K_{F_0}/\mathcal{R}_0 \cap \sim_{F_0}$ is a sub-partition of $\{B'_j\}_{j\in J}$ we have

$$card(K_{F_0}/\mathcal{R}_0 \cap \sim_{F_0}) \leq \sum_{j\in J} card(B'_j) = crit(M) + m(f)$$

the case L1) is proved.

4. *The case L2).*

We consider now the case L2): there is some $p \in \overline{1, m(f)-1}$ such that $p < l(p)$. Then the (“ $\subseteq$ ”) condition implies that $f(x'_p) = f(x'_{p+1}) = \cdots = f(x'_{l(p)})$ and $W_p = W_{p+1} = \ldots = W_{l(p)}$. Note that $W_p = W_{p'}$ or $W_p \cap W_{p'} = \emptyset$ for each $p \neq p' \in \{1, \ldots, m(f)-1\}$. Let E_0 be the symmetrical relation defined at the beginning of the proof and let $\mathcal{R}_p = \mathcal{R}_0 \cap W_p$. For each $k \in \overline{p, l(p)-1}$ let $\mathcal{R}_{k+1}$ be the equivalence relation on K_{f_0} for $R \in \{=, <, >, E_{k+1}\}$ where E_{k+1} is recursively defined as:

$$E_{k+1} = E_k \cup \{\big((x_k, y_S), (x'_k, y_N)\big), \big((x'_k, y_N), (x_k, y_S)\big)\}$$

and $E_p = E_0$.

For each $k \in \overline{p, l(p)}$ we prove by induction the existence of a function $F_k : W_p \to \mathbb{R}$ such that:

EFLk a) $\vec{G}(F_k) = \vec{G}(F_0)$.

EFLk b) $F_k = F_0$ in a neighborhood of ∂W_p.

EFLk c) $(x_k, y_S) \sim_{F_k} (x'_k, y_N)$ and

$$card(K_{F_0} \cap W_p/\mathcal{R}_{k+1} \cap \sim_{F_k}) \leq card(K_{F_0} \cap W_p/\mathcal{R}_k \cap \sim_{F_{k-1}}) - 1.$$

For $k = p$ we are in the situation described in L1) and we apply the extended fusing lemma to the partition $\{A_1^p, A_2^p, A_3^p\}$. So we obtain the function F_p and the inequalities:

$$card(K_{F_0} \cap W_p / \sim_{F_p}) \leq card(A_1^p / \sim_{F_0}) + card(A_2^p / \sim_{F_0}) + card(A_3^p / \sim_{F_0}) - 1.$$

From $K_{F_0} \cap W_p / \mathcal{R}_p \subseteq \{A_1^p, A_2^p, A_3^p\}$ the previous inequalities get the form:

$$card(K_{F_0} \cap W_p / \mathcal{R}_{p+1} \cap \sim_{F_p}) \leq card(K_{F_0} \cap W_p / \mathcal{R}_p \cap \sim_{F_0}) - 1.$$

Since $F_0 = F_{p-1}$ on W_p, we obtain the inequality of (EFLp c).

For $k \leq l(p) - 1$ we assume the existence of F_k and we apply the extended fusing lemma to the partition $\{A_1^{k+1}, A_2^{k+1}, A_3^{k+1}\}$, where:

$$\begin{aligned} A_1^{k+1} &= \{(x, y) \mid (x, y)\mathcal{R}_{k+1}(x_{k+1}, y_S)\}, \\ A_2^{k+1} &= \{(x, y) \mid (x, y)\mathcal{R}_{k+1}(x'_{k+1}, y_N)\}, \\ A_3^{k+1} &= K_{F_0} \cap W_p \setminus (A_1^{k+1} \cup A_2^{k+1}). \end{aligned}$$

We must verify the conditions of the extended fusing lemma again. Since the proof resembles the corresponding proof in L1) we only show that $A_1^{k+1} \cap A_2^{k+1} = \emptyset$. Assume $A_1^{k+1} = A_2^{k+1}$. Therefore $(x'_{k+1}, y_N) \in A_1^{k+1}$ and implicitly $(x, y_S)\mathcal{R}_{k+1}(x', y_N)$. Then by the definition of $\mathcal{R}_{k+1}$ there is a sequence of vertices $z_1, \ldots, z_{k'} \in K_{F_0} \cap W_p$ such that

$$(x_{k+1}, y_S) = z_1 R z_2 R \ldots R z_{k'} = (x'_{k+1}, y_N)$$

where $R \in \{=, <, >, E_{k+1}\}$. Then there is a sequence $pr_1(z_1), \ldots, pr_2(z_{k'}) \in K_f$ with the properties:

(Ra) $pr_1(z_i) \in f^{-1}[f(x'_{k+1}), f(x_{k+1})]$ for each $1 \leq i \leq k'$.

(Rb) $x_{k+1} = pr_1(z_1) R_f pr_1(z_2) R_f \ldots R_f pr_2(z_k) = x'_{k+1}$ where $R_f \in \{=, <_f, >_f\}$.

(Rc) for each $2 \leq i \leq k'$

either $pr_1(z_i) pr_1(z_{i-1}) \in J(f)$ and $\mid f(pr_1(z_i)) - f(pr_1(z_{i-1})) \mid = f(x_{k+1}) - f(x'_{k+1})$

or $\mid f(pr_1(z_i)) - f(pr_1(z_{i-1})) \mid \leq f(x_{k+1}) - f(x'_{k+1}) - 1$.

From (Rb) it follows that there is a path P in $V(\vec{G}(f))$ from x_{k+1} to x'_{k+1} passing through the vertices $x_{k+1} = pr_1(z_1), \ldots, pr_1(z_{k'}) = x'_{k+1}$. Therefore $\overrightarrow{x_{k+1} x'_{k+1}} \in \vec{J}(f)$ is by the property (Rc) J-fragmentable in $G(\vec{f})$ (and by the property (Ra) even in $\vec{G}(f \mid_{[f(x'_{k+1}), f(x_{k+1})]}))$. This contradicts the hypothesis of this proposition, so $A_1^{k+1} \cap A_2^{k+1} = \emptyset$. After we have verified all the conditions of the extended fusing lemma we obtain a function $F_{k+1} : W_p \to \mathbb{R}$. The graph of F_{k+1} is identical to the graph of F_k because F_{k+1} equals F_k plus a constant in some neighborhood of each trajectory going to or from a critical point in W_p. Furthermore a reasoning like that we use for $k = p$ leads us to the inequality:

$$card(K_{F_0} \cap W_p/\mathcal{R}_{k+1} \cap \sim_{F_k}) \leq card(K_{F_0} \cap W_p/\mathcal{R}_k \cap \sim_{F_{k-1}}) - 1.$$

Summing the inequalities (EFLk c) for $k \in \overline{p, l(p)}$ we get:

$$\begin{aligned} card(K_{F_0} \cap W_p/\mathcal{R}_{l(p)+1} \cap \sim_{F_{l(p)}}) &\leq card(K_{F_0} \cap W_p/\mathcal{R}_p \cap \sim_{F_0}) - (l(p) - p + 1) = \\ &= card(K_{F_0} \cap W_p/\mathcal{R}_0 \cap \sim_{F_0}) - (l(p) - p + 1). \end{aligned}$$

With the triad functions F_p we construct a new function $F : M \times \mathbb{S}^n \to \mathbb{R}$ as follows:

$$F(x) = \begin{cases} F_{l(p)}(x) & \text{if} \quad x \in W_p \\ F_0(x) & \text{if} \quad x \in cl\big(M \times \mathbb{S}^n \setminus \cup_{p=1}^{m(f)-1} W_{l(p)}\big) := W \end{cases}$$

As in the case L1) we have $card(K_F/ \sim_F) \leq crit(M) + 1.$♣

An application of the two last propositions is the following:

Theorem 9. *Let M be a closed manifold of dimension ≤ 7. Then*

$$crit(M \times \mathbb{S}^n) \leq crit(M) + 1.$$

Proof. For a closed manifold M of dimension n we know that $crit(M) \leq n + 1$. Therefore a nice minimal function f has less than $n + 1$ critical points. If f has at most 2 local minima then each J-graph of f satisfies the (“ $\subseteq$ ”) condition and by the proposition 7.7 the *crit* inequality holds. If f has at most 2 local maxima then we apply the above argument to $-f$.

For $n \leq 4$ the function f has either at most 2 minima or at most 2 maxima, therefore the *crit* inequality holds.

For $n = 5$ the function f has at most 6 critical points. If the f has both 3 local minima and 3 local maxima and no other critical points then f is a Morse function. Computing the homology of M using the Morse function f (see Milnor [20] or Schwarz [28] for details) we get $H_0(M; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. But M is connected so f has either at most 2 local minima or at most 2 local maxima. Then we have the same situation as for $n \leq 4$.

For $n = 6$ we analyze the case of f with 3 minima and 3 maxima. In this situation each edge of $\vec{G}(f)$ has the length at most 2, because f has at most 3 critical levels. We prove that the (“ $\subseteq$ ”) condition is fulfilled.

Let $\vec{J}$ be a J-graph of f. If for some $\overrightarrow{uu'} \in \vec{J}$ and $\overrightarrow{vv'} \in \vec{J}$ we have $f(u) - f(u') \geq 2$ and $f(v) - f(v') \geq 2$ then $f(u) - f(u') = 2$ and $f(v) - f(v') = 2$, because each edge has the length at most 2. Furthermore $f(u) \leq 2$, $f(v) \leq 2$ hence $f(u) = f(v) = 2$ and $f(u') = f(v') = 0$. For this reason the graph $\vec{J}$ satisfies the (“ $\subseteq$ ”) condition and by the proposition 7.7 the *crit* inequality is verified, too.

For $n = 7$ if $card\big(Min(f) \cup Max(f)\big) \leq 5$ or f has only 3 critical levels then we use the same arguments as in the case $n \leq 4$ resp. $n = 6$. It remain to analyze the case of f with 3 minima, 3 maxima and 4 critical levels. We prove that in this case f has an J-graph with all the edges of length at most 2. Let $V(\vec{G}(f)) = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Without loss of generality suppose $Min(f) = \{1, 2, 3\}$, $Max(f) = \{6, 7, 8\}$ and $3 < 4 < 5 < 6$. Let $\vec{F}$ be an F-graph of f and T_i the tree of $\vec{F}$ containing i, for each $i \in \overline{1, 3}$. It is not possible that all the trajectories from 7 go to 1 or to 2 or to 3, hence there is a trajectory that goes from 7 to 4 or to 5. Then 7 is an vertex of T_3. The same is valid for 8, hence 8 is an element of T_3. It is not possible that all the trajectories going to 1 are coming from 6 or 7 or 8 hence there is a trajectory going to 1 from 4 or from 5. Let e_1 be the edge in $\vec{G}(f)$ corresponding to this trajectory. The length of e_1 is at most 2. The same is valid for 2, hence there is an edge e_2 containing 2 of length at most 2. The last assertions which begin with ”it is not possible” are all similar hence we prove only the last of them. The local minima 1 and 2 induce an mountain-pass point. The mountain pass point is not a local maximum. Suppose the contrary let x_M be this local maximum. Then there is some $\epsilon > 0$ such that the neighborhood $U = f^{>f(x_M)-\epsilon}$ of x_M is not empty and contains only one critical point, x_M. The dimension of M is 7, hence $f^{<f(x_M)} \cap U = U \setminus \{x_M\}$ is path connected. This fact contradicts a result of Hofer [12], that characterizes the points of mountain-pass type. Therefore x_M is not a local maximum.

We continue the main line of the proof: with T_3 and the two edges e_1 and e_2 we get a tree $T = F \cup e_1 \cup e_2$ that contains $\vec{F}$. If e_1 is T-fragmentable in G(f) then there is an edge e_1' of length 1 going to 1. We get a new tree $T' = F \cup e_1' \cup e_2$ and e_1' is not T'-fragmentable. If e_2 is T'-fragmentable then there is a edge e_2' of length 1 going to 2. The tree $T'' = F \cup e_1' \cup e_2'$ is an J-graph of f. Concluding, in each of the above situation if the J-graph has a edge $\overrightarrow{xx'}$ of length 2 then $h(x') = 0$. Therefore the J-graph satisfies the condition (“ $\subseteq$ ”) and then the *crit* inequality is verified. ♣

8 Appendix: notions and results from the graph theory

The purpose of this appendix is to familiarize the reader with the basic concepts of graph theory. The majority of these concepts we take from the book of Bollobás [1]. Inevitably the appendix contains a large number of definitions. We should add at this stage that the terminology of graph theory is far from being standard, though that used in this appendix is well accepted.

l. Definitions

A *graph* G is an ordered pair of disjoint sets (V, E) such that E is a subset of the set of unordered pairs of V. Unless it is explicitly stated otherwise, we consider only finite graphs, that is V and E are always finite. The set V is the set of *vertices* and E is the set of *edges*. If G is a graph then $V = V(G)$ is the vertex set of G and $E = E(G)$ is the edge set. An edge $\{x, y\}$ is said to *join* the vertices x and y and is denoted by xy. Thus xy and yx mean exactly the same edge, the vertices x and y are the *end-vertices* of this edge. If $xy \in E(G)$ then x and y are *adjacent* or *neighboring* vertices of G and the vertices x and y are *incident* with the edge xy. Two edges are *adjacent* if they have exactly one common end-vertex. As the terminology suggests, we do not usually think of a graph as an ordered pair, but as a collection of vertices some of which are joined by edges. It is then a natural step to draw a picture of the graph. In fact, sometimes the easiest way to describe a graph is to draw it, the graph $G = (\{1, 2, 3, 4, 5, 6\}, \{12, 14, 16, 25, 34, 36, 45, 56\})$ is immediately comprehended if we draw it.

We say that $G' = (V', E')$ is a *sub-graph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. In this case we write $G' \subseteq G$. If G' contains all edges of G that join two vertices in V' then G' is said to be the sub-graph *induced* or *spanned* by V' and is denoted by $G[V']$. A sub-graph G' of G is an *induced* sub-graph if $G' = G[V(G')]$.

We shall often construct new graphs from old ones by deleting or adding some vertices and edges. If $W \subseteq V(G)$ then $G - W = G[V \setminus W]$ is the sub-graph of G obtained by detecting the vertices in W and all edges incident with them. Similarly if $E' \subseteq E(G)$ then $G - E' = (V(G), E(G) \setminus E')$. If $W = \{w\}$ and $E' = \{xy\}$ then this notation is simplified to $G - w$ and $G - xy$. Similarly, if x and y are non-adjacent vertices of G then $G + xy$ is obtained from G by joining x to y. If x is a vertex of a graph G then instead of $x \in V(G)$ we usually write $x \in G$. The *order* of G is the number of vertices, it is denoted by $|G|$. For the number of elements (cardinality) of a set is used the following notation: $card(X)$ denotes the number of elements of the set X. Thus $|G| = card(V(G))$. The *size* of G is the number of edges; it is denoted by $e(G)$.

Two graphs are *isomorphic* if there is a correspondence between their vertex sets that preserves adjacency. Thus $G = (V, E)$ is isomorphic to $G' = (V', E')$ if there is a bijection $\phi : V \to V'$ such that $xy \in E$ iff $\phi(x)\phi(y) \in E'$. Clearly isomorphic graphs have the same order and size. Usually we do not distinguish between isomorphic graphs, unless we consider graphs with a distinguished or labeled set of vertices (for example, sub-graphs of a given graph). In accordance with this convention, if G and H are isomorphic graphs then we write either $G \cong H$ or simply $G = H$.

The set of vertices adjacent to a vertex $x \in G$ is denoted by $\Gamma(x)$. The *degree* of x is $d(x) = |\Gamma(x)|$. If we want to emphasize that the underlying graph is G then we write $\Gamma_G(x)$ and $d_G(x)$.

A *path* is a graph P of the form

$$V(P) = \{x_o, x_1, \dots, x_l\}, \qquad E(P) = \{x_0x_1, x_1x_2, \dots, x_{l-1}x_l\}.$$

This path P is usually denoted by $x_0x_1 \dots x_l$. The vertices x_0 and x_l are the end-vertices of P and $l = e(P)$ is the *length* of P. We say that P is a path *from* x_0 to x_l or an x_0-x_l path. Of course, P is also a path from x_l to x_0 or an x_l-x_0 path. Sometimes we wish to emphasize that P is considered to go from x_0 to x_l and then we call x_0 the *initial* and x_l the *terminal* vertex of P. A path with initial vertex x is an *x-path.* The term *independent* will be used in connection with vertices, edges and paths of a graph. A set of vertices (edges) is *independent* if no elements of it are adjacent; a set of paths is *independent* if for any two paths each vertex belonging to both paths is an end-vertex of both. Thus $P_1, P_2, \dots, P_k$ are independent x-y paths iff $V(P_i) \cap V(P_j) = \{x, y\}$ whenever $i \neq j$. Also, $W \subseteq V(G)$ consists of independent vertices if $G[W]$ is an *empty* graph, i.e. a graph having no edges.

Most paths we consider are sub-graphs of a given graph G. A *walk* W in G is an alternating sequence of vertices and edges, say $x_0, \alpha_1, x_2, \alpha_2, \dots, \alpha_l, x_l$ where $\alpha_i = x_{i-1}x_i$, $1 \leq i \leq l$. In accordance with the terminology above, W is an x_0-x_l *walk* and is denoted by $x_0x_1 \dots x_l$; the *length* of W is l. This walk W is called a *trail* if all its edges are distinct. Note that a path is a walk with distinct vertices. A trail whose end-vertices coincide (a *closed* trail) is called a *circuit*. If a walk $W = x_0x_1 \dots x_l$ is such that $l \geq 3$, $x_0 = x_l$, and the vertices x_i, $0 < i < l$, are distinct from each other and x_0 then W is said to be a *cycle.* For simplicity this cycle is denoted by $x_1x_2 \dots x_l$. Note that the notation differs from that of a path since x_1x_l is also an edge of this cycle. Furthermore, $x_1x_2 \dots x_l$, $x_lx_{l-1} \dots x_1$, $x_2x_3 \dots x_lx_1$, $x_ix_{i-1} \dots x_1x_lx_{l-1} \dots x_{i+1}$, all denote the same cycle.

Given vertices x, y, their *distance* $d(x, y)$ is the minimal length of an x-y path. If there is no x-y path then $d(x, y) = \infty$.

A graph is *connected* if for every pair $\{x, y\}$ of distinct vertices there is a path from x to y. Note that a connected graph of order at least 2 cannot contain an isolated vertex. A maximal connected sub-graph is a *component* of the graph. A *cut-vertex* is a vertex whose deletion increases the number of components. Similarly an edge is a *bridge* if its deletion increases the number of components. Thus an edge of a connected graph is a bridge if its deletion disconnects the graph. A graph without any cycles is a *forest* or an *acyclic* graph; a *tree* is a connected forest. The relation of a tree to a forest sounds logically from the semantics point of view if we note that a forest is a disjoint union of trees; in other words, a forest is a graph whose every component is a tree.

We shall write $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$ and kG for the union of k disjoint copies of G. We obtain the *join* $G + H$ from $G \cup H$ by adding all edges between G and H. There are several notions closely related to that of a graph. By definition a graph does not contain a *loop*, an "edge" joining a vertex to itself; neither does it contain *multiple edges*, that is several "edges" joining the same two vertices. In a *multigraph* both multiple edges and multiple loops are allowed; a loop is a special edge. In our thesis we never use multigraphs because the G-graphs that we get from pseudo-gradient vector fields are always graphs.

If the edges are *ordered* pairs of vertices then we get the notions of a *directed graph.* An ordered pair (a, b) is said to be an edge *directed from a to b* or an edge *beginning at a and ending at b*, and is denoted by $\overrightarrow{ab}$. The notions defined for graphs are easily carried over to directed graphs, mutatis mutandis.

An *oriented graph* is a directed graph obtained by orienting the edges, that is by giving the edge ab a direction $\overrightarrow{ab}$ or $\overrightarrow{ba}$. Thus an oriented graph is a directed graph in which at most one of $\overrightarrow{ab}$ and $\overrightarrow{ba}$ occurs. A *source* vertex a is a vertex incident with no edge ending at a and a *sink* vertex b is a vertex incident with no edge beginning at b. If a vertex a is incident with at least two distinct edges beginning at a then it is called an *MP-vertex.* An *interval* graph is an oriented connected graph with exactly one sink point and exactly one source point. Thus an interval graph is a tree.

2. Results

With the concepts defined so far we can enumerate some results about graphs. Though these results are hardly more than simple observations.

Proposition 8.1. *A graph is a forest iff for every pair $\{x, y\}$ of distinct vertices it contains at most one x-y path.*

Proposition 8.2. *The followings assertions are equivalent for a graph G:*

a. *G is a tree.*

b. *G is a minimal connected graph, that is G is connected and if $xy \in E(G)$ then $G - xy$ is disconnected. In other words, G is connected and every edge is a bridge.*

Corollary 8.3. *Every connected graph contains a spanning tree, that is a tree containing every vertex of a graph.*

We need the following result to count the critical points that fuse in the product manifold:

Proposition 8.4. *Let $\vec{T}$ be an oriented tree with m sink vertices. Then*

$$card\{x' \in V(\vec{T}) \mid \overrightarrow{xx'} \in \vec{T} \text{ and } x \text{ is a MP-vertex of } \vec{T}\}-$$

$$-card\{x \in V(\vec{T}) \mid x \text{ is a MP-vertex of } \vec{T}\} \geq m-1.$$

The previous proposition is very natural if we split it in two lemmas and we think in terms of homotopy theory. For this reason we regard $\vec{T}$ as a 1-dimensional simplicial space. If S is the set of sink vertices of $\vec{T}$ then we denote by T/S the factor space obtained from T by identifying all the sink vertices to one point. The 1-dimensional simplicial space T/S is the subject (and implicitly the oriented factor graph $\vec{T}/S$) of the followings lemmas:

Lemma 8.5. *The fundamental group of T/S is*

$$\pi_1(T/S) = \bigoplus_{card(S)-1} \mathbb{Z}.$$

Lemma 8.6. *If there is a nonnegative integer k such that $\pi_1(T/S) = \oplus_k \mathbb{Z}$ then*

$$card\{x' \in V(\vec{T}) \mid \overrightarrow{xx'} \in \vec{T} \text{ and } x \text{ is a MP-vertex of } \vec{T}\}-$$

$$-card\{x \in V(\vec{T}) \mid x \text{ is a MP-vertex of } \vec{T}\} \geq k-1.$$

The two lemmas lead straight to proposition 8.4.

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