INAUGURAL – DISSERTATION

zur
Erlangung der Doktorwürde
der
Naturwissenschaftlich-Mathematischen Gesamtfakultät
der
Ruprecht–Karls–Universität
Heidelberg

vorgelegt von

Diplom–Mathematiker

Jan Timo Essig

aus Pforzheim

Tag der mündlichen Prüfung:

About a De Rham Complex Describing Intersection Space Cohomology in a Three Strata Case

Betreuer: Professor Dr. Markus Banagl

Abstract

The theory of intersection spaces assigns cell complexes to certain topological pseudomanifolds depending on a perversity function in the sense of intersection homology. The main property of the intersection spaces is Poincaré duality over complementary perversities for the reduced singular (co)homology groups with rational coefficients. Using differential forms, the resulting generalized cohomology theory for pseudomanifolds was extended to 2-strata pseudomanifolds with a geometrically flat link bundle in [Ban11]. In this thesis we use differential forms to generalize the intersection space cohomology theory to a class of 3-strata spaces with flatness assumptions for the links. The case of a zero-dimensional bottom stratum is treated as well as certain cases of positive-dimensional bottom strata. In both cases, we prove Poincaré duality over complementary perversities for the cohomology groups.

Zusammenfassung

Die Theorie der Schnitträume ordnet gewissen topologischen Pseudomannigfaltigkeiten Zellkomplexe zu, die von einer Perversitätsfunktion im Sinne der Schnitthomologie abhängen. Für zwei Schnitträume komplementärer Perversitäten existiert dann ein Poincaré-Dualitätsisomorphismus zwischen den reduzierten singulären (Ko)homologiegruppen mit rationalen Koeffizienten. Unter Benutzung von Differentialformen wurde diese verallgemeinerte Kohomologietheorie in [Ban11] auf Pseudomannigfaltigkeiten mit zwei Strata und einem geometrisch flachen Linkbündel erweitert. In dieser Dissertation erweitern wir die Schnittraum–Kohomologietheorie auf eine Klasse von Pseudomannigfaltigkeiten mit drei Strata und geometrisch flachen Linkbündeln. Dazu benutzen wir ebenfalls Differentialformen. Der Fall eines nulldimensionalen tiefsten Stratums wird genauso behandelt wie einige speziellere Fälle von positiv-dimensionalen. In beiden Fällen beweisen wir die Poincaré Dualität zwischen den Kohomologiegruppen komplementärer Perversitäten.

Contents

1 Introduction

The topic of this thesis is the description of the generalized intersection space (co)homology theory HI via differential forms on Thom-Mather-stratified pseudomanifolds with three strata. The result is a generalized cohomology theory with some kind of Poincaré duality. Since ordinary (co)homology theories such as singular (co)homology do not satisfy Poincaré duality for arbitrary pseudomanifolds, one has to generalize the concept of (co)homology if one wants the resulting theory to satisfy Poincaré duality on pseudomanifolds. Historically, the first approaches to Poincaré duality on singular spaces were Goresky-MacPherson's intersection homology, cf. [GM80, GM83], and Cheeger's L^2-cohomology with respect to some conical metrics on the regular part of the pseudomanifold, see [Che79, Che80, Che83]. Intersection homology was first defined using a suitable subcomplex of the simplicial chain complex on the pseudomanifold X depending on a perversity function and later axiomatized using sheaf theory. Moreover, there are descriptions using singular chains and differential forms. For Witt spaces Cheeger proved in loc. cit. that the L^2-cohomology with respect to the mentioned conical metric on the regular part of the pseudomanifold is isomorphic to the linear dual of middle perversity intersection homology with coefficients in $\mathbb{R}$.
The HI (co)homology theory was first defined in [Ban10] for pseudomanifolds with two strata and trivial link bundle using spatial homology truncation on the links of the singular stratum. The basic idea is the following: To get a generalized homology theory satisfying Poincaré duality for pseudomanifolds one truncates the links of the singular strata spatially to get a new space, which is actually a cell complex, such that Poincaré duality is satisfied for the singular (co)homology groups of that new space. For a pseudomanifold with only isolated singularities one, roughly speaking, cuts cone neighbourhoods of the singular points, then removes some high dimensional cells of the connected components of the boundary of the resulting manifold and glues back the cone on this new cell complexes. In other words, the fundamental principle is to replace links by their Moore approximations, which is Eckmann-Hilton dual to the concept of Postnikov approximations from homotopy theory.
This process results in the the intersection spaces $I^{\bar{p}}X$, which depend on a perversity function $\bar{p}$ and a choice of a subgroup of the cellular chain complex $C_k(L)$ of the link in the truncation degree (the homology of the intersection space is independent of that choice). It is a generalized geometric Poincaré complex, i.e. for closed oriented X, there is a Poincaré duality isomorphism $\widetilde{H}^i(I^{\bar{p}}X;\mathbb{Q}) \cong \widetilde{H}_{n-i}(I^{\bar{q}}X;\mathbb{Q})$ on reduced singular (co)homology with $\bar{p}$, $\bar{q}$ complementary perversities, n the dimension of X. See Section 3.1 for a more detailed explanation of intersection spaces.
In [Ban11], M. Banagl used differential forms to give a description of the intersection space cohomology with real coefficients. This approach allows

to apply the HI cohomology theory to a larger class of spaces: Let X be a Thom-Mather stratified pseudomanifold with a single singular stratum $\Sigma \subset X$ and a geometrically flat link bundle $p : E \to B$ (see Definition 4.1.3), with fiber a closed Riemannian manifold L. Then one defines the subcomplex of fiberwise cotruncated multiplicatively structured forms $ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B) \subset \Omega^\bullet(E)$ on E (see [Ban11, Sections 3-5]). Afterwards one defines the complex $\Omega I^\bullet_{\bar{p}}(M) \subset \Omega^\bullet(M)$ on the regular part of the pseudomanifold by taking forms with restriction to some open collarlike neighbourhood equaling the pullback of a form in $ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)$. For spaces with isolated singularities only, Banagl proved that integrating forms over some distinct smooth chains (that are part of a partially smooth cycle, see [Ban11, Section 9.1]) induces an isomorphism between the cohomology groups of $\Omega I^\bullet_{\bar{p}}(M)$ and the linear dual of the homology groups of the intersection space $I^{\bar{p}}X$ with real coefficient. That result was extended by the author to pseudomanifolds with one singular stratum of arbitrary dimension and trivial link bundle in [Ess12].
More recently M. Banagl and E. Hunsicker (see [BH15]) gave a Hodge theoretic description of $H^\bullet\big(\Omega I^\bullet_{\bar{p}}(M)\big)$ on pseudomanifolds with one singular stratum and trivial link bundle. They proved that the space of extended weighted harmonic forms with respect to a product type fibered scattering metric is isomorphic to $H^\bullet\big(\Omega I^\bullet_{\bar{p}}(M)\big)$. Moreover the authors gave a connection between HI and intersection homology IH and prove that for X a Witt space with one smooth singular stratum and trivial link bundle the signatures of IH and HI agree.
The theory of intersection space (co)homology (either via the actual intersection space or via the differential form approach) has applications in various different topics. Examples are K-theory ([Ban10, Chapter 2.8] and [Spi13]), deformation of singular varieties in algebraic geometry ([BM12]), perverse sheaves ([BBM14]), geometrically flat bundles and equivariant cohomology ([Ban13]) and string theory in theoretical physics ([Ban10, Chapter 3] and [BBM14]).

In this thesis we extend the definition of intersection space cohomology to pseudomanifolds with three strata, i.e. Thom-Mather stratified pseudomanifolds X with filtration

$$X = X_n \supset X_m \supset X_s = \Sigma \supset \emptyset\,.$$

First cases of intersection space homology of such spaces were investigated in [Ban12]. In Section 9 we recall the construction of intersection spaces in this setting. We tackle a much more general class of spaces here, starting with zero dimensional bottom stratum and treating a non-negative dimensional bottom stratum afterwards. As in [Ban12], we need additional assumptions on the spaces aside from the geometrical flatness condition for the link bundles from the two strata setting. Note that the case of a non-negative

dimensional bottom stratum is not a generalization of the first case.
We define a complex $\Omega I^{\bullet}_{\bar{p}}(M)$ depending on a perversity function $\bar{p}$ on the regular part M of the pseudomanifold (also called the blowup) as before. In the three strata case one obtains M by removing first a tubular neighbourhood of the bottom singular stratum resulting in a Thom-Mather stratified pseudomanifold-with-boundary X' and singular set $X_m \cap X'$. Afterwards one removes a tubular neighbourhood of the singular stratum in X'. The result is a specific type of a smooth manifold with corners, a so called $\langle 2 \rangle$-manifold. The boundary ∂M of M consists of two parts, $\partial M = E \cup_{\partial E = \partial W} W$, with E, W smooth compact manifolds with boundary. Each of the boundary parts has a collar in ∂M which fit together nicely, i.e. the restriction of the collar of each boundary part to $\partial E \times [0,1) = \partial W \times [0,1)$ is a collar of the boundary in W or E, respectively.
To obtain a generalization of the $\Omega I^{\bullet}_{\bar{p}}$-complex of the two strata setting we then define $\Omega I^{\bullet}_{\bar{p}}$ as the complex of forms on the $\langle 2 \rangle$-manifold M that satisfy the following pullback conditions on the collar neighbourhoods of E and W. Restricted to the collar of E the forms shall be pullbacks of fiberwisely cotruncated multiplicatively structured forms on E. The condition on the collar of W depends on the dimension of the bottom stratum s. If $s = 0$ the restriction of the forms to the collar of W shall equal the pullbacks of cotruncated forms on W. If $s > 0$ we demand that the restriction of the forms to the collar of W equals the pullback of (in respect to a different degree and bundle) fiberwisely cotruncated multiplicatively structured forms on W. The main statement proved in this thesis is the generalized Poincaré duality over complementary perversities for the cohomology of $\Omega I^{\bullet}_{\bar{p}}$:

Theorems 8.4.1 and 10.5.2: (Poincaré duality for HI)
Integration induces nondegenerate bilinear forms

$$\int : HI^r_{\bar{p}}(X) \times HI^{n-r}_{\bar{q}}(X) \to \mathbb{R}$$
$$\big([\omega],[\eta]\big) \mapsto \int_M \omega \wedge \eta,$$

with $HI^r_{\bar{p}}(X) := H^r\big(\Omega I^{\bullet}_{\bar{p}}(X)\big)$.

The strategy of the proof is as follows: The difficult part of the definition of $\Omega I^{\bullet}_{\bar{p}}(M)$ for pseudomanifolds with three strata is, that restriction of the contained forms to the intersection of the two collar neighbourhoods of the boundary parts E and W has to be both the pullback of an appropriate form on E as well as the pullback of an appropriate form on W, hence the pullback of some form on $\partial E = \partial W$. In order to deal with this we first prove a version of Poincaré-Lefschetz duality for an intermediate complex $\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)$. This is the complex of forms on M with restriction to the collar neighbourhood of E equaling the pullback of some fiberwisely truncated multiplicatively structured form and the restriction to the collar neighbourhood

of W the pullback of any form on W (this last requirement can be obmitted up to quasi-isomorphism). We prove the mentioned Poincaré-Lefschetz duality theorem for $\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)$ using a 5-Lemma argument that involves some distinguished triangles in the derived category over the reals and pulling back forms to the boundary part E (see Section 7).
Afterwards can use the complex $\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)$ to prove the above theorem. Again we use a 5-Lemma argument involving the long exact sequences of two distinguished triangles in the derived category over $\mathbb{R}$. These distinguished triangles contain the pullback to the boundary part W. Unfortunately in order to deal with the difficulties arising from the two independent conditions for the forms in $\Omega I^{\bullet}_{\bar{p}}(M)$ on the overlap of the collar neighbourhoods of the two boundary parts E and W we need an additional analytic assumption that allows us to cotruncate the complex $\Omega I^{\bullet}_{\bar{p}}(W)$ in one special degree, which is determined by the perversity. See Sections 6.4, 8.1 and 10.2.

In Section 9, we test the de Rham approach by applying it to the 3-strata pseudomanifolds of [Ban12]. We show that all the additional assumptions are satisfied in this setting and moreover that the cohomology of the complex of intersection differential forms $\Omega I^{\bullet}_{\bar{m}}(M)$ with respect to the lower middle perversity is isomorphic via integration of forms over cycles to the linear dual of the reduced homology groups of the intersection space with real coefficients.

2 Notation

All manifolds are assumed to be smooth, possibly with boundary or corners. X will always be a pseudomanifold with two or three strata. If we work with differential forms on the top stratum, the pseudomanifold is assumed to be Thom-Mather stratified.

We work with fiber bundles of different type, with closed Riemannian fibers and bases compact manifolds with boundary or with base spaces closed manifolds and fibers compact Riemannian manifolds with boundaries.
$p : E \to B$ always denotes a fiber bundle with closed Riemannian fiber L and base B, a compact manifold with boundary ∂B and $q : W \to \Sigma$ always denotes a fiber bundle with closed base manifold Σ and link F, a compact Riemannian manifold with boundary ∂F.

By a collar we denote one of the following: For a manifold M with boundary ∂M, a collar is an embedding $c_{\partial M} : \partial M \times [0,1) \hookrightarrow M$ with $c|_{\partial M \times \{0\}}(x,0) = x$ for each $x \in \partial M$. In some cases we also consider other half open intervals but $[0,1)$. For a smooth manifold with corners M and boundary $\partial M = \partial M_1 \cup ... \cup \partial M_p$ a collar of a boundary part ∂M_i is an embedding $c_i : \partial M_i \times [0,1) \hookrightarrow M$ again with $c_i|_{\partial M_i \times \{0\}} = \mathrm{id}_{\partial M_i}$. We mainly work with $\langle 2 \rangle$-manifolds, i.e. manifolds with corners and two boundary parts $\partial M_1 = E$

and $\partial M_2 = W$, $\partial M = E \cup_{\partial E = \partial W} W$. The inclusion of the boundary parts is denoted by $j_E : E \hookrightarrow M$ and $j_W : W \hookrightarrow M$ and the inclusion of the corner $\partial E = \partial W$ by $j_{\partial W} = j_{\partial E} : \partial W \hookrightarrow M$. The image of a collar, $\operatorname{im} c \subset M$ is called a collar neighbourhood.
For a real vector space V, we denote the linear dual $\operatorname{Hom}(V, \mathbb{R})$ by $V^\dagger$.
Additionally, let us give an overview about the complexes of differential forms we use (if they are definable): $\Omega^\bullet_{\mathcal{MS}}$ denotes the complex of multiplicatively structured forms on the total space of a geometrically flat fiber bundle. Its truncation and cotruncation are denoted by $ft_{<K}\Omega^\bullet_{\mathcal{MS}}$ and $ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}$, respectively.
The complex of partial intersection forms $\widetilde{\Omega I}^\bullet_{\bar{p}}$ is the complex of forms on a $\langle 2 \rangle$-manifold with restriction to a collar neighbourhood of one boundary part equal to the pullback of a fiberwisely cotruncated multiplicatively structured form on that boundary part.
Finally, $\Omega I^\bullet_{\bar{p}}$ is the complex of intersection forms on a $\langle 2 \rangle$-manifold, the subcomplex of $\widetilde{\Omega I}^\bullet_{\bar{p}}$ of forms with restriction to a collar neighbourhood of the other boundary part equaling the pullback of some either cotruncated form if the bottom stratum is zero dimensional or a fiberwisely cotruncated multiplicatively structured form on that boundary part otherwise.
Note that all of the above complexes are also used in a relative notion, relative to some collar neighbourhood.

Acknowledgements

This dissertation would not have been possible without the constant support, advice and guidance of Markus Banagl. His suggestions and questions, as well as the encouragement and friendly words he has always had for me, have been of great help. For interesting discussions, helpful questions and important remarks, I am also deeply grateful to Eike Beck, Bryce Chriestenson, Eugenie Hunsicker, Benedikt Jost, Kathrin Maurischat, Laurentiu Maxim, Felipe Müller, Sebastian Nill, Fabrizio Savarino, Franz Schlöder, Matthias Spiegel, Manuel Gonzales Villa, Dominik Wrazidlo and many others. For generous financial support, I thank the Evangelische Studienstiftung. Finally my greatest thanks go to my parents and, of course, to Lisa.

3 Revision of Intersection Spaces

The extension of intersection space cohomology to three strata pseudomanifolds was first treated in [Ban12] for oriented, compact, PL stratified pseudomanifolds X^n of even dimension $n = 2k$ with stratification

$$X = X^n \supset X_1 \cong S^1 \supset X_0 = pt$$

and an additional assumption satisfied for the link of the codim $(n-1)$-stratum (see [Ban12, section 5]). We will return to this class of pseudomanifolds in Section 9.
Let us recall the definition of a topological pseudomanifold:

Definition 3.0.1 *A* 0*-dim* topological stratified pseudomanifold *is a countable set of points together with the discrete topology.*
A n*-dim* topological stratified pseudomanifold *is a paracompact topological Hausdorff space* X *together with a filtration by closed subsets* $X = X_n \supset X_{n-1} = X_{n-2} \supset X_{n-3} \supset ... \supset X_0 \supset X_{-1} = \emptyset$ *such that*

1. *All non-empty* $X_i - X_{i-1}$ *are topological manifolds of dimension* i*, called the strata of* X*.*

2. $X - X_{n-2}$ *is (open and) dense in* X*.*

3. *Local triviality: For every* k *we have the following: Let* $x \in X_{n-k} - X_{n-k-1}$*. Then there exists an open neighbourhood* U_x *of* x *in* X *and a compact* $(k-1)$*-dim topological stratified pseudomanifold* $L = L_{k-1} \supset L_{k-3} \supset L_{k-4} \supset ... \supset L_0 \supset L_{-1} = \emptyset$ *and a homeomorphism*

$$\Phi : U_x \xrightarrow{\cong} \mathbb{R}^{n-k} \times \mathring{\text{cone}}(L^{k-1})$$

which is stratum preserving. L *is called the* link.

In order to have some concept of smoothness, which we need to define e.g. differential forms, it is not sufficient to have just topological pseudomanifolds. Instead we use Thom-Mather stratified spaces, a concept worked out by Mather in [Mat12]. The definition will be recalled in Section 5.1.

3.1 Intersection Spaces

After recalling the definition of a topological pseudomanifold we will recap the basics about intersection spaces of topological pseudomanifolds. The idea is to use the homotopy-theoretic method of spatial homology truncation on the links, to assign a cell complex $I^{\bar{p}}X$ to an n-dimensional pseudomanifold X, depending on a perversity function $\bar{p}$, called the $\bar{p}$-intersection space, such that for complementary perversities $\bar{p}$ and $\bar{q}$ there is a Poincaré duality isomorphism

$$\widetilde{H}^i(I^{\bar{p}}X;\mathbb{Q}) \cong \widetilde{H}_{n-1}(I^{\bar{q}}X;\mathbb{Q})$$

on reduced singular (co)homology with rational coefficients.
Spatial homology truncation is based on Moore approximation, a concept that is Eckmann-Hilton dual to Postnikov approximation, but not functorial. Hence, the full homology truncation machine needs a category with objects not mere cell complexes but CW complexes with some additional structure and cellular maps that preserve this additional structure as morphisms. See [Ban10, Chapter 1.1] for more details. To form the intersection space it is sufficient to use spatial homology truncation on the object level, which is much simpler:

Definition 3.1.1 *(Spatial homology truncation of CW-complexes)*
Let K be a CW-complex. A spatial homology k-truncation of K is a CW-complex $K_{<k}$ together with a cellular map $f : K_{<k} \to K$ with $f|_{K^{k-1}} = \mathrm{id}_{K^{k-1}}$, such that $f_* : H^r(K_{<k}) \xrightarrow{\cong} H^r(K)$ is an isomorphism for $r < k$ and $H^r(K_{<k}) = 0$ for $r \geq k$.

The 1–truncation of a path connected CW–complex K ca be chosen to be $K_{<1} = k_0$, where k_0 is a zero-cell of K. For K simply connected, a 2–truncation is also given by $K_{<2} = k_0$. Using [Ban10, Proposition 1.6] for $k \geq 3$ we get that for any simply connected CW-complex and any $k \in \mathbb{Z}$ that there is a spatial homology k-truncation of K. This result was generalized by D. Wrazidlo to arbitrary CW-complexes in [Wra13, Corollary 1.4]. Note that in general $K_{<k}$ is not a subcomplex of K, whereas this is trivially true for $k \leq 2$ by definition.
For a stratified topological pseudomanifold X^n with filtration $X = X_n \supset \Sigma^b \supset \emptyset$ and trivial link bundle $E = \Sigma \times L$ for the singular set, with L a connected manifold of dimension l, set $k := l - \bar{p}(l+1)$ and let $f : L_{<k} \to L$ be the spatial homology k-truncation of L. Let further M be the blowup of X, a compact manifold with boundary $\partial M = \Sigma \times L$, and let $g : \Sigma \times L_{<k} \to M$ be the composition

$$\Sigma \times L_{<k} \xrightarrow{\mathrm{id} \times f} \Sigma \times L = \partial M \hookrightarrow M.$$

Then the $\bar{p}$-intersection space of X is defined as the homotopy cofiber, i.e. the mapping cone, of g

$$I^{\bar{p}}X := \operatorname{cone} g = M \cup_g \operatorname{cone}(\Sigma \times L_{<k}).$$

In this setting Poincaré duality for reduced cohomology with rational coefficients was proved in the following form in [Ban10, Theorem 2.47]: For complementary perversities $\bar{p}$ and $\bar{q}$ there is an isomorphism

$$D : \widetilde{H}^{n-r}(I^{\bar{p}}X) \xrightarrow{\cong} \widetilde{H}_r(I^{\bar{q}}X)$$

which is compatible with Poincaré-Lefschetz duality on the blowup M of X, i.e. the diagram

$$\begin{tikzcd} \widetilde{H}^{n-r}(I^{\bar{p}}X) \arrow[r] \arrow[d, "\cong"', "D"] & H^{n-r}(M) \arrow[d, "\cong"', "-\cap[M,\partial M]"] \\ \widetilde{H}_r(I^{\bar{q}}X) \arrow[r] & H_r(M, \partial M) \end{tikzcd}$$

commutes. Moreover the duality isomorphism D is also compatible with a kind of Poincaré duality on $\Sigma \times L_{<k}$, i.e. the diagram

$$\begin{tikzcd} H^{n-r-1}(\Sigma \times L_{<k}) \arrow[r, "\delta^*"] \arrow[d, "\cong"', "-\cap[\partial M]"] & \widetilde{H}^{n-r}(I^{\bar{p}}X) \arrow[d, "\cong"', "D"] \\ H_r(\partial M, \Sigma \times L_{<l-1-k}) \arrow[r] & \widetilde{H}_r(I^{\bar{q}}X). \end{tikzcd}$$

commutes, too.
Note that while the Poincaré duality isomorphism for intersection spaces is compatible with taking the cap product with the fundamental class on the smooth parts of the pseudomanifold in the above sense, it is in the best case questionable whether it can be written as a cap product with some fundamental class. This question has been tackled by M. Klimczak in [Kli15]. He uses the intersection spaces to construct a new space which he calls a "Poincaré duality space" $\mathcal{DP}(X)$. He introduces a fundamental class for this space such that capping with that class is a Poincaré duality isomorphism on rational (co)homology (see [Kli15, Theorem 3.2]).
We would like to mention that although the construction of a spatial homology k-truncation of the link L involves the choice of a subgroup of the cellular cell group $C_k(L)$, the rational, reduced cohomology groups of the intersection space is independent of this choice. This was proven for isolated singularties in [Ban10, Theorem 2.18] but should be true for the above setting by an analogous argument.

Finally we want to mention that intersection spaces can be constructed in more general settings by generalizing the fiberwise truncation to non-trivial bundles, as done by F. Gaisendrees in [Gai12]. For some special cases of pseudomanifolds with more than two strata the intersection spaces have been constructed in [Ban12] using so called 3-diagrams of spaces.

4 Technical Preliminaries

4.1 Properties of Fiber Bundles

Remark 4.1.1 *It follows from the definition of smooth fiber bundles that the total space E of any fiber bundle $p : E \to B$ with closed fiber and base space a manifold with boundary is itself a manifold with boundary $\partial E = p^{-1}(\partial B)$.*

4.1.1 Flat Fiber Bundles

Definition 4.1.2 *(Flat fiber bundles)*
A fiber bundle $p : E \to B$ of smooth manifolds with fiber L is called flat if there is an atlas $\mathfrak{U} := \{U_\alpha\}_{\alpha\in I}$ of the bundle such that the corresponding transition functions are locally constant. That means that for the local trivialization maps $\phi_\alpha : p^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times L$, $\pi_1 \circ \phi_\alpha = p$, it holds that

$$\phi_\beta \circ \phi_\alpha^{-1} = \mathrm{id} \times g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times L \to (U_\alpha \cap U_\beta) \times L$$

with $g_{\alpha\beta} \in$ Diffeo (L) if $U_\alpha \cap U_\beta$ is connected.

To be able to define the complex of multiplicatively structured forms on a total space of a fiber bundle, we need a refinement of flat bundles, namely

Definition 4.1.3 *(Geometrically flat fiber bundles)*
A fiber bundle is called geometrically flat if it is flat and if there is a Riemannian metric on the fiber such that the structure group of the bundle is the isometriy group of the link with respect to that metric, i.e. the $g_{\alpha\beta}$ in the above definition are isometries of L.

Next we give a statement about (geometrically) flat fiber bundles over base manifolds with boundaries:

Lemma 4.1.4 *Let E^e, B^b be compact manifold with boundaries and $p : E \to B$ a (geometrically) flat fiber bundle with closed Riemannian fiber L. Then the restriction*

$$p| : \partial E \to \partial B$$

also is a (geometrically) flat fiber bundle.

Proof: We first show that $p^{-1}(\partial B) = \partial E$, i.e. that $p| : \partial E \to \partial B$ is actually a fiber bundle. If $x \in \partial B$ then there is a neighbourhood $x \in U \subset B$ with $U \cong \mathbb{R}_+ \times \mathbb{R}^{b-1}$ and so for each $y \in p^{-1}(\{x\})$ there is a neighbourhood $E \supset V_y \cong \mathbb{R}_+ \times \mathbb{R}^{e-1}$ of y in E. Hence all those $y \in p^{-1}(\{x\})$ belong to ∂E. The bundle is flat since any (finite) atlas $\mathcal{U} := \left\{U_\alpha\right\}_{\alpha\in I}$ of the bundle with locally constant transition functions induces a finite atlas

$$\mathcal{U}| := \left\{U_\alpha \cap \partial B\right\}_{\alpha\in I}$$

such that the bundle $q := p| : \partial E \to \partial B$ trivializes with respect to this atlas and the transition functions are locally constant since they are the restriction of the transition functions of the bundle $p : E \to B$:

$$(U_\alpha \cap U_\beta \cap \partial B) \times L \xrightarrow{\phi_\alpha|^{-1}} (p|)^{-1}(U_\alpha \cap U_\beta \cap \partial B) \xrightarrow{\phi_\beta|} (U_\alpha \cap U_\beta \cap \partial B) \times L.$$

For $x, y \in (U_\alpha \cap U_\beta \cap \partial B) \times L$ lying in the same connected component we have that

$$\phi_\beta| \circ \phi_\alpha|^{-1}(x) = \phi_\beta| \circ \phi_\alpha^{-1}|(x) = \big(\phi_\beta \circ \phi_\alpha^{-1}\big)|(x) = \big(\phi_\beta \circ \phi_\alpha^{-1}\big)|(y) = \phi_\beta| \circ \phi_\alpha|^{-1}(y),$$

where the third equality holds since $\phi_\beta \circ \phi_\alpha^{-1}$ is locally constant. $\square$

By an analogous proof one gets the same statement for bundles over closed manifolds with fiber a compact manifold with boundary:

Lemma 4.1.5 *Let W^w, F^f be compact manifolds with non-empty boundary, Σ^s a closed manifold and let $q : W \to \Sigma$ be a (geometrically) flat fiber bundle with fiber F. Then the restriction*

$$q| : \partial W \to \Sigma$$

also is a (geometrically) flat fiber bundle with fiber ∂F.
Note: As Riemannian metric on the boundary ∂F of F we take the restriction of the metric on F.

Proof: The first steps of the proof are the same as in the previous one: Of course $q|$ is surjective and smooth. Since the local trivializations $\phi_\alpha : q^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times F$ and their inverses are maps between manifolds with boundary one has diffeomorphisms $\phi_\alpha : q^{-1}(U_\alpha) \cap \partial W \xrightarrow{\cong} U_\alpha \times \partial F$ with inverses $\phi_\alpha^{-1}|_{U_\alpha \times \partial F}$. For the transition maps of $q|$ we get:

$$\phi_\alpha| \circ \phi_\beta^{-1}| = \mathrm{id} \times g_{\alpha\beta}| : (U_\alpha \cap U_\beta) \times \partial F \to (U_\alpha \cap U_\beta) \times \partial F.$$

In the geometrically flat setting $g_{\alpha\beta}$ is an isometry and since we take the restriction of the Riemannian metric on F for ∂F, the restriction $g_{\alpha\beta}$ is also an isometry.

$\square$

4.2 Collars on Bundles and Manifolds with Corners

4.2.1 Width of a collar

In order to prove Poincaré duality for the later defined complexes on manifolds with boundaries we need the following simple relation between good open covers and collars on manifolds with boundary. Before stating the result we recall the definition of good open covers:

Definition 4.2.1 *(Good open covers)*
Let B^b be a b-dimensional manifold (possibly with non-empty boundary ∂B). A cover $\{U_\alpha\}_{\alpha\in I}$ of B is called good if every nonempty finite intersection

$$U_{\alpha_0\ldots\alpha_p} = \bigcap_{i=0}^{p} U_{\alpha_i}$$

is diffeomorphic to $\mathbb{R}^b$ (or $\mathbb{R}^b_+$ if it has non-empty intersection with the boundary).

Note that manifolds without boundary always have good covers and finite good covers if they are compact (see [Bot82, Theorem 5.1]). The same is true for manifolds with boundary.

Lemma 4.2.2 *Let B be a manifold with non-empty boundary ∂B, let $c : \partial B \times [0,1) \hookrightarrow B$ be an open collar of ∂B in B and let $\mathcal{U} := \{U_\alpha\}_{\alpha\in I}$ be a finite good open cover of B such that the bundle trivializes with respect to each $U_\alpha \in \mathcal{U}$. Then there is an $\epsilon \in (0,1]$ such that for the subcollar*

$$c| : \partial B \times [0,\epsilon) \hookrightarrow B$$

it holds that

$$U_\alpha \cap B_- \neq \emptyset, \tag{1}$$

for every $\alpha \in I$, where $B_- := B - c\big(\partial B \times [0,\epsilon)\big)$. We call a collar satisfying the relation (1) small with respect to the given cover.

<u>Proof:</u> Let $C = c\big(\partial B \times [0,1)\big)$. If there are no $U_\alpha \in \mathcal{U}$ such that $U_\alpha \subset C$ we take $\alpha = 1$ and are done. So suppose $U_\alpha \subset C$. Since $U_\alpha \subset B$ is open, there must be an $N_\alpha \in \mathbb{N}$ such that $U_\alpha \not\subset c\big(B \times [0,1/n)\big)$ for all $n \geq N_\alpha$. (Otherwise U_α would be contained in $\partial B = c(\partial B \times \{0\})$.)
Choose such an N_α for each $\alpha \in I$ and set $\epsilon := (\max_{\alpha\in I} N_\alpha)^{-1} \in (0,1]$. This is well defined since the index set I is finite and the above relations are satisfied for that ϵ.

□

4.2.2 p-related Collars on Fiber Bundles

We start with a proposition on p-related collars on a fiber bundle $p : E \to B$ over a base manifold B with boundary ∂B.

Definition 4.2.3 *(p-related collars)*
Let $p : E \to B$ be a smooth fiber bundle with closed smooth fiber F and B a compact smooth manifold with boundary ∂B. Let

$$c_{\partial E} : \partial E \times [0,1) \to E$$

be a smooth collar on the manifold with boundary E and

$$c_{\partial B} : \partial B \times [0,1) \to B$$

a smooth collar on B. Then $c_{\partial E}$ and $c_{\partial B}$ are called p-related if and only if the diagram

$$\begin{tikzcd} \partial E \times [0,1) \arrow[r, "c_{\partial E}"] \arrow[d, "p| \times \mathrm{id}"] & E \arrow[d, "p"] \\ \partial B \times [0,1) \arrow[r, "c_{\partial B}"] & B \end{tikzcd}$$

commutes.

<u>Example:</u> Let $E = L \times B$ be a trivial link bundle. We then can take any collar $c_{\partial B} : \partial B \times [0,1) \hookrightarrow$ of ∂B in B and take $c_{\partial E} := \mathrm{id}_L \times c_{\partial B} : \partial E \times [0,1) \hookrightarrow E$. $c_{\partial E}$ is indeed a collar of $\partial E = \partial B \times L$, since we work with closed fibers L. Hence the diagram

$$\begin{tikzcd} \partial E \times [0,1) \arrow[r, "c_{\partial E}"] \arrow[d, "\pi_2 \times \mathrm{id}"] & E = L \times B \arrow[d, "\pi_2"] \\ \partial B \times [0,1) \arrow[r, "c_{\partial B}"] & B \end{tikzcd}$$

commutes and the collars are p-related.

Proposition 4.2.4 *For any smooth fiber bundle $p : E \to B$ with base space a compact smooth manifold with boundary $(B, \partial B)$ and closed smooth fiber L there is a pair of p-related collars*

$$c_{\partial E} : \partial E \times [0,1) \hookrightarrow E,$$
$$c_{\partial B} : \partial B \times [0,1) \hookrightarrow B.$$

Moreover, if a collar $c_{\partial B} : \partial B \times [0,1) \hookrightarrow B$ is given then a collar $c_{\partial E} : \partial E \times [0,1) \hookrightarrow E$ can be chosen such that $c_{\partial E}$ and $c_{\partial B}|$ are p-related for some subcollar of $c_{\partial B}$. (In detail, we take a subcollar $c_{\partial B}|_{\partial B \times [0,\alpha)}$ for some $\alpha \in (0,1]$ and reparametrize it to get a map $\partial B \times [0,1) \hookrightarrow B$.)

<u>Proof:</u> We start with the first part and therefore proceed as follows:

1. First we construct a vector field X on B which is nowhere tangent to ∂B. The flow of this vector field will then give the collar $c_{\partial B}$ on B. One can compare this approach to the proof of [Hir76, Theorem 6.2.1].

2. By locally lifting this vector field, we construct a vector field Y on E that is nowhere tangent to ∂E and p-related to X, i.e. for each $e \in E$ we have
$$p_* Y_e = X_{p(e)}.$$

3. By [AMR88, Prop 4.2.4], we then have the relation
$$p \circ \eta_t^Y = \eta_t^X \circ p.$$
for the flows η^X of X and η^Y of Y. That relation implies the statement of the proposition.

The first step is quite simple and standard: Take a finite good open cover $\{U_\alpha\}_{\alpha \in I}$ of B such that the bundle trivializes with respect to this cover. Then let $J \subset I$ denote the set of those $\alpha \in I$ with $U_\alpha \cap \partial B \neq \emptyset$. For each $\alpha \in J$ define a vector field X_α on U_α by taking the induced vector field of ∂_b on $\mathbb{R}^b_+$ by the coordinate map ϕ_α. Then take a partition of unity $\{\rho_\alpha\}_{\alpha \in I}$ subordinate to the cover $\{U_\alpha\}$ and define
$$X := \sum_{\alpha \in I} \rho_\alpha X_\alpha.$$

To obtain the vector field $Y \in \mathfrak{X}(E) = \Gamma(TE)$ we proceed as follows: Since there is a natural isomorphism between vector bundles
$$T(U_\alpha) \times T(L) \xrightarrow{\cong} T(U_\alpha \times L)$$
for all $\alpha \in I$, we can lift the vector field $\rho_\alpha X_\alpha \in \mathfrak{X}(U_\alpha)$ to $(\rho_\alpha X_\alpha, 0)$, a section of $T(U_\alpha) \times T(L) \cong T(U_\alpha \times L)$, which still has compact support in $U_\alpha \times L$.
Since $p : E \to B$ is a fiber bundle with fiber L and $\{U_\alpha\}_{\alpha \in I}$ a covering of the base B with respect to which the fiber trivializes, we have a diffeomorphism $\psi_\alpha : U_\alpha \times L \xrightarrow{\cong} p^{-1}(U_\alpha)$, for all $\alpha \in I$, such that the diagram

$$\begin{tikzcd}
p^{-1}(U_\alpha) \arrow[dr, "p|"'] & & U_\alpha \times L \arrow[ll, "\psi_\alpha"', "\cong"] \arrow[dl, "\pi_1"] \\
& U_\alpha &
\end{tikzcd}$$

commutes. Note that since the ψ_α are diffeomorphisms, there exist push-forward vector fields $\psi_{\alpha *}(\rho_\alpha X_\alpha, 0) \in \mathfrak{X}(p^{-1}(U_\alpha))$, which still have compact support (in $p^{-1}(U_\alpha)$). Since the family $\{p^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open cover of E, such that the sets in $\{p^{-1}(U_\alpha)\}_{\alpha \in J}$ cover an open neighbourhood of the boundary ∂E of E, we can set
$$Y := \sum_{\alpha \in J} \psi_{\alpha *}(\rho_\alpha X_\alpha, 0)$$
to get a vector field $Y \in \mathfrak{X}(E)$ that is nowhere tangent to ∂E. Let $x \in \partial E$, then $x \in p^{-1}(U_{\alpha_1 \ldots \alpha_r} = U_{\alpha_1} \cap \ldots \cap U_{\alpha_r})$ for some $\alpha_1, \ldots, \alpha_r \in J$. Then

Y_x is not tangent to ∂E if and only if $((\psi_{\alpha_1}^{-1})_* Y)_{\psi_{\alpha_1}^{-1}(x)}$ is not tangent to $\partial B \times U_{\alpha_1} \times L$.

$$\begin{aligned}((\psi_{\alpha_1}^{-1})_* Y)_{\psi_{\alpha_1}^{-1}(x)} &= \sum_{i=1}^{r} \rho_{\alpha_i}(p(x))\big([\mathrm{id} \times (\pi_2 \circ \psi_{\alpha_1}^{-1} \circ \psi_{\alpha_i})]_*(X_{\alpha_i}, 0)\big)_{\psi_{\alpha_1}^{-1}(x)} \\ &= \sum_{i=1}^{r} \rho_{\alpha_i}(p(x))(X_{\alpha_i})_{p(x)}.\end{aligned}$$

Now this is of course not tangent to the boundary since by definition of the $X_\alpha \in \mathfrak{X}(U_\alpha)$ we have (with again the ϕ_α the coordinate maps of the base): $(\phi_{\alpha_1}^{-1})_* X_\alpha = (\phi_{\alpha_1}^{-1} \circ \phi_\alpha)_* \partial_b = \sum_{i=1}^{b} a_i \partial_i$ with $a_b > 0$ since the transition maps are maps between manifolds with boundary.

Further, we have to show that X and Y are p-related, i.e. it holds that $p_* Y_e = X_{p(e)}$ for every $e \in E$. This is equivalent to the statement that for all smooth functions on an open subset of B it holds that

$$Y(f \circ p) = (Xf) \circ p.$$

(see e.g. [Lee13, Lemma 3.17]). For let $f : U \to \mathbb{R}$ be a smooth function on an open subset $U \subset B$ and let $x \in p^{-1}(U)$. Then

$$\begin{aligned}Y(f \circ p)(x) &= Y_x(f \circ p) \\ &= \sum_{\alpha \in \widetilde{J}} \psi_{\alpha *}(\rho_\alpha X_\alpha, 0)_{\psi_\alpha^{-1}(x)}(f \circ p) \quad \text{with } \widetilde{J} = \{\alpha \in J | x \in p^{-1}(U_\alpha)\} \\ &= \sum_{\alpha \in \widetilde{J}} \big(\rho_\alpha X_\alpha, 0\big)_{\psi_\alpha^{-1}(x)}(f \circ p \circ \psi_\alpha) \\ &= \sum_{\alpha \in \widetilde{J}} \rho_\alpha\big(p(x)\big)\big((X_\alpha)_{p(x) = \pi_1 \circ \psi_\alpha^{-1}(x)}, 0_{\pi_2 \circ \psi_\alpha^{-1}(x)}\big)(f \circ \pi_1) \\ &= \sum_{\alpha \in \widetilde{J}} \rho_\alpha\big(p(x)\big)(X_\alpha)_{p(x)}(f) = X_{p(x)}(f) = (Xf)\big(p(x)\big).\end{aligned}$$

As mentioned, for every t, this implies the relation

$$p \circ \eta_t^Y = \eta_t^X \circ p \tag{2}$$

for the flows η^X of the vector field $X \in \mathfrak{X}(B)$ and η^Y of $Y \in \mathfrak{X}(E)$. This relation implies the claim since there are open neighbourhoods $W_B \subset \partial B \times [0, \infty)$ of ∂B and $W_E \subset \partial E \times [0, \infty)$ of ∂E respectively, such that the flows η^X and η^Y are defined on these open subsets. But then there are constants $\epsilon_B, \epsilon_E > 0$ such that

$$\partial B \times [0, \epsilon_B) \subset W_B$$

and

$$\partial E \times [0, \epsilon_E) \subset W_E.$$

Let $\epsilon := \min(\epsilon_B, \epsilon_E)$ and let $f : [0,\infty) \to [0,\epsilon)$ be a diffeomorphism. Then we have collar embeddings

$$c_{\partial B} : \partial B \times [0,\infty) \xrightarrow{\mathrm{id}\times f} \partial B \times [0,\epsilon) \xrightarrow{\eta_X} B$$

and

$$c_{\partial E} : \partial E \times [0,\infty) \xrightarrow{\mathrm{id}\times f} \partial E \times [0,\epsilon) \xrightarrow{\eta_Y} E$$

such that

$$\begin{aligned} p \circ c_{\partial E}\ (x,t) &= (p \circ \eta^Y_{f(t)})(x) = \eta^X_{f(t)}\big(p(x)\big) \quad \big(\text{by eq. (2)}\big) \\ &= c_{\partial B} \circ (p \times \mathrm{id})(x,t) \end{aligned}$$

For the second part of the proof we proceed likewisely, but take a special vector field in step 1: The collar allows us to define a vector field $\widetilde{X} \in \mathfrak{X}(C_{\partial B})$ (with $C_{\partial B} = \mathrm{im}\, c_{\partial B}$) by taking the pushforward of ∂_t: $\widetilde{X} = c_* \partial_t$. Then for any $q \in \partial B$ and any $f \in C^\infty(C_{\partial B})$ it holds that

$$(\widetilde{X} f)\big(c(\tau,q)\big) = \big(\partial_t (f \circ c)\big)(\tau,q) = \frac{d}{dt}\Big|_{t=\tau} (f \circ c)(t,q).$$

This means that the flow of the vector field restricted to the boundary ∂B is the given collar $c_{\partial B}$. We then "lift" this vector field as before, not to a vector field on the whole total space E but rather to a vector field $Y \in \mathfrak{X}\big(p^{-1}(C_{\partial B})\big)$, where $p^{-1}(C_{\partial B})$ is an open neighbourhood of the boundary, by setting

$$Y = \sum_{\alpha \in J} \psi_{\alpha *}(\rho_\alpha \widetilde{X}|, 0).$$

As before this defines a nowhere vanishing vector field which is nowhere tangent to the boundary ∂E. The rest is a complete analogy to the first step. Note that it suffices to have the vector fields on open neighbourhoods of the boundary since we later only need the flow of the vector fields restricted to the boundary. □

4.2.3 Fiber-related Collars on Bundles

In analogy to the previous subsection we now establish that on a geometrically flat fiber bundle $q : W \to \Sigma$ with fiber F, a compact Riemannian manifold with boundary ∂F, there is a collar on the fiber F which gives rise to a collar on the total space W.

Definition 4.2.5 *(Fiber-related collars) Let $q : W \to \Sigma$ be a fiber bundle over a smooth closed base Σ with fiber a compact manifold F with boundary ∂F. Two collars $c_{\partial F} : \partial F \times [0,\infty) \to F$ and $c_{\partial W} : \partial W \times [0,\infty) \to W$ are* *called* fiber-related *if there is a finite good open cover* $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ *with respect to which the bundle trivializes such that for each* $\alpha \in I$ *the diagram*

$$\begin{tikzcd}
U_\alpha \times \partial F \times [0,\infty) \arrow[rr, "\phi_\alpha^{-1}|\times \mathrm{id}"] \arrow[d, "\mathrm{id} \times c_{\partial F}"] & & \partial W \times [0,\infty) \arrow[d, "c_{\partial W}"] \\
U_\alpha \times F \arrow[r, "\phi_\alpha^{-1}"] & q^{-1}(U_\alpha) \arrow[r, hook] & W
\end{tikzcd}$$

commutes.

Proposition 4.2.6 *For a geometrically flat fiber bundle* $q : W \to \Sigma$ *over a smooth closed base* Σ *with fiber a compact Riemannian manifold* F *with boundary* ∂F *there exist fiber-related collars*

$$c_F : \partial F \times [0,\infty) \to F,$$
$$c_W : \partial W \times [0,\infty) \to W.$$

Proof: The basic idea of the proof is the same as in Proposition 4.2.4, but the technical details are a bit more involved:

1. We use Gaussian coordinates to define a vector field X_F on F which points in the normal direction of the boundary ∂F, is invariant under push-forwards of isometries and the flow of which gives a collar $C_{\partial F}$ on F.

2. This gives vector fields on $U_\alpha \times F$ and hence on $q^{-1}(U_\alpha)$. Since the bundle is geometrically flat, they coincide on the overlaps and hence give a global vector field X_W on W. This field does not vanish on ∂W and points inwards everywhere, thus its flow yields a collar on W.

3. Again by [AMR88, Prop 4.2.4] we then have on each of the coordinate domains $q^{-1}(U_\alpha)$ that the flows η^α on $U_\alpha \times F$ and $\eta^W|$ on $q^{-1}(U_\alpha)$ satisfy the relation

$$\phi_\alpha^{-1} \circ \eta_t^\alpha = \eta_t^W \circ \phi_\alpha^{-1}|. \tag{3}$$

   This relation implies the statement of the proposition.

We start with the construction of a normal Gaussian collar $C^G_{\partial F}$ of ∂F in F, i.e. a collar such that the collar coordinate $x^0 = d_{\partial F}$ is the distance to the boundary and such that the Riemannian metric of F on $C^G_{\partial F}$ can be expressed as $ds^2 = (dx^0)^2 + \sigma_{ij} dx^i\, dx^j$ with $\sigma_{ij} = \sigma_{ij}(x^0, x^i)$ a semi-Riemannian metric. That is, $C^G_{\partial F}$ is covered by a normal Gaussian coordinate system. Note that the x^i are local coordinates but $x^0 = d_{\partial F}$ is globally defined.

Using [Ger06, Theorem 12.5.13] one finds such a normal Gaussian collar as follows. The cited theorem says that any connected, oriented hypersurface of a closed Riemannian manifold has a tubular neighbourhood covered by a normal Gaussian coordinate system such that x^0 is the distance to the hypersurface, cf. [Ger06, Definiton 12.5.3]. To apply this result in our case, we take any collar of ∂F in F, form the double of the manifold F and extend the metric on F to ∂F using a partition of unity. We then apply the cited theorem and get the described tubular neighbourhood. The intersection of this tubular neighbourhood with the original manifold F is then the desired normal Gaussian collar.

We proceed with the first step by using this normal Gaussian collar to define a vector field X_F on F that points in the normal direction of ∂F by extending the vector field $\partial_0 \in \mathfrak{X}(C^B_{\partial F})$ to all of F with the help of a smooth cutoff function in normal ($= x^0$) direction. Since this vector field is in particular nowhere tangent to ∂F and does not vanish on a neighbourhood of ∂F its flow defines a collar of ∂F.

We use the normal vector field X_F, since for isometries $f : F \to F$ it holds that

$$f_* X_F = X_F.$$

To prove this, we use the fact that if two isometries between connected manifolds coincide in one point and the same is true for their differentials at that point, then they are equal. The proof of this simple result can be found for example in [Pet06, Proposition 22]. We apply it to the isometries $f|$ and $\mathrm{id} \times f|_{\partial F}$ on $C^G_{\partial F}$.

Let $x \in \partial F$. Since $f|$ is a map between manifolds with boundaries it holds that $f(0, x) = (0, f|_{\partial F}(x)) = (\mathrm{id} \times f|_{\partial F})(0, x)$. The fact that f maps ∂F to ∂F also implies that for the differential of f it holds that

$$f_*\partial_i = f_* j_* \partial_i = j_*(f|_{\partial F})_*\partial_i = (\mathrm{id} \times f|_{\partial F})_*\partial_i$$

for all $i \neq 0$. The same is true for the vector ∂_0, pointing in the normal direction, since for all $i \neq 0$ the fact that f is an isometry and therefore an diffeomorphism gives

$$\langle f_*\partial_0, \partial_i\rangle = \langle f_*\partial_0, f_*(f^{-1})_*\partial_i\rangle = \langle \partial_0, (f^{-1})_*\partial_i\rangle = 0$$

due to the fact that $(f^{-1})_*\partial_i \in T_{(0,x)}\partial F \subset T_{(0,x)}F$ as before. Hence $f_*\partial_0 = s\partial_0$ for some $s \in \mathbb{R}$. Since isometries preserve the length of vectors, $s = 1$ and therefore $f_*\partial_0 = \partial_0 = (\mathrm{id} \times f|_{\partial F})_*\partial_0$.

In summary, $f(0, x) = (\mathrm{id} \times f|_{\partial F})(0, x)$ and $(f_*)_{(0,x)} = \big((\mathrm{id} \times f|_{\partial F})_*\big)_{(0,x)}$, implying $f = \mathrm{id} \times f|_{\partial F}$ on $C^G_{\partial F}$. But this equation directly implies that $f_* X_F = X_F$.

In particular this result holds for the elements $g_{\alpha\beta}$ of the structure group of the bundle, which are isometries of F since the bundle is geometrically flat.

For the second step we define the vector field $(0, X_F)$ on $U_\alpha \times F$, via the isomorphism of vector bundles $T(U_\alpha) \times T(F) \xrightarrow{\cong} T(U_\alpha \times F)$. By the above relation $(g_{\alpha\beta})_* X_F = X_F$, we have $(\phi_\alpha^{-1})_*(0, X_F) = \phi_\beta^{-1}(0, X_F)$ for all $\alpha, \beta \in I$. Hence we get a vector field X_W on W by gluing the $\phi_\alpha^{-1}(0, X_F)$ together. It does not vanish on ∂W, points in the normal direction (and is therefore nowhere tangent to ∂W) and satisfies the relation $(\phi_\alpha)_* X_W = (0, X_F)$ by definition. Hence the equation (3) holds by [AMR88, Prop. 4.2.4]. This is the third step. The proof is finished by the same argument as in the proof of Proposition 4.2.4.

□

4.2.4 Collars on Manifolds with Corners

We are going to work with differential forms on a smooth manifold with corners M^n, the boundary of which can be subdivided as $\partial M = E \cup_{\partial E = \partial W} W$, satisfying certain conditions near the boundary parts E and W. In order to define "near E, W" precisely we have to investigate how the concept of a collar on a manifold with boundary generalizes to manifolds with corners of that type.

We first follow [Jän68] and [Lau00] to the definition of $\langle n \rangle$-manifolds:

Definition 4.2.7 *(Manifolds with Faces)*
Let M^n be an n-dimensional manifold with corners and for each $x \in M$ let $c(x)$ denote the number of zeroes of $\phi(x) \in \mathbb{R}_+^n = [0, \infty)^n$ for any coordinate chart $\phi : U \to \mathbb{R}_+^n$ with $x \in U$. A face is the closure of a connected component of the set $\{p \in M | c(p) = 1\}$. Then M is called a manifold with faces if each $x \in M$ is contained in $c(x)$ different faces.

Example: A 2-dimensional disc with one corner is a manifold with corners but not with faces, since the corner point does not lie in 2 faces but only in one.

Definition 4.2.8 *($\langle n \rangle$-manifolds) [See [Jän68, Def. 1]]*
A manifold with faces M together with a n-tuple of faces $(\partial_0 M, ..., \partial_{n-1} M)$ is called an $\langle n \rangle$-manifold if

1. *$\partial M = \bigcup_{i=0}^{n-1} \partial_i M$,*
2. *$\partial_i M \cap \partial_j M$ is a face of both $\partial_i M$ and $\partial_j M$ if $i \neq j$.*

Note that a $\langle 0 \rangle$-manifold is just a usual manifold (without boundary) and a $\langle 1 \rangle$-manifold is a manifold with boundary. Simple examples of $\langle n \rangle$-manifolds for arbitrary $n \in \mathbb{N}$ are $\mathbb{R}_+^n$ or the standard n-simplex. We will focus on $n = 2$.

So let M^n be an n-dimensional $\langle 2\rangle$-manifold with faces E, W (hence $\partial E = \partial W$). By [Lau00, Lemma 2.1.6] there are collars

$$\begin{aligned} c_{\partial E} &: \partial E \times [0,1) \hookrightarrow E, \\ c_{\partial W} &: \partial W \times [0,1) \hookrightarrow W, \\ c_E &: E \times [0,1) \hookrightarrow M \text{ and} \\ c_W &: W \times [0,1) \hookrightarrow M, \end{aligned}$$

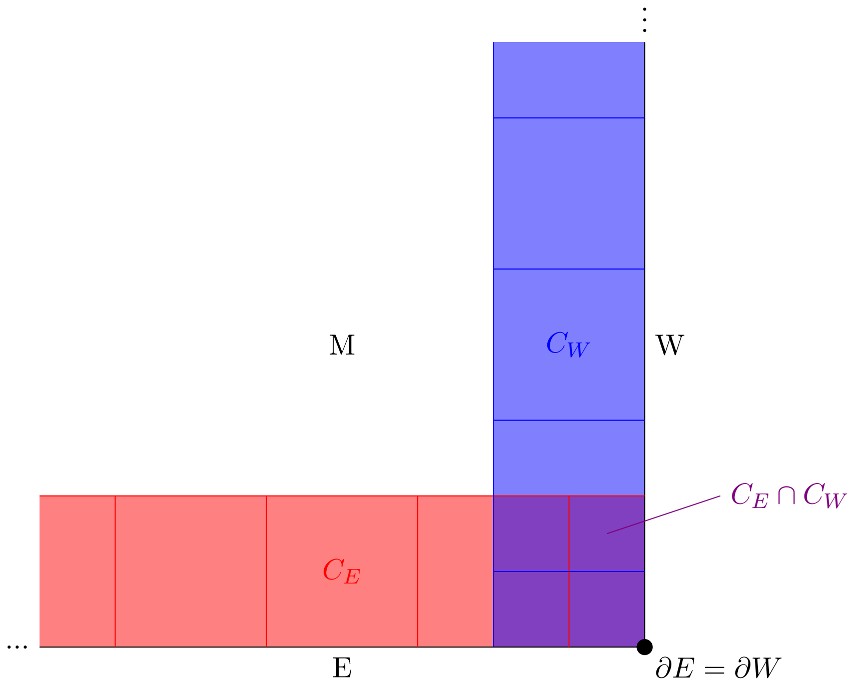
with $C_X := c_X(X \times [0,1))$ for $X = \partial E,\ \partial W,\ E,\ W$ such that

$$c_E|_{\partial E \times [0,1)} = c_{\partial W}$$

and

$$c_W|_{\partial W \times [0,1)} = c_{\partial E}.$$

This is illustrated by the following picture:



Proposition 4.2.9 *Let M^n be a $\langle 2\rangle$-manifold with boundary $\partial M = E \cup W$ as before. Then any two collars $c_{\partial E} : \partial E \times [0,1) \hookrightarrow E$ and $c_{\partial W} : \partial W \times [0,1) \hookrightarrow W$ extend to collars*

$$\begin{aligned} c_E &: E \times [0,1) \hookrightarrow M, \\ c_W &: W \times [0,1) \hookrightarrow M, \end{aligned}$$

i.e. $c_E|_{\partial W \times [0,1)} = c_{\partial W}$ *and* $c_W|_{\partial E \times [0,1)} = c_{\partial E}$.

Proof: The proof is simple: Interpret the collars as flows of vector fields on E, W which do not vanish on the boundaries and point inwards and extend them to vector fields on M (for example using an arbitrary collar on M) which do not vanish anywhere on W, E, respectively, and point into M. The flows of these vector fields are collars c_W and c_E with the desired properties. □

Corollary 4.2.10 *As before, let M be a $\langle 2\rangle$-manifold with boundary $\partial M = E \cup_{\partial E} W$. Assume furthermore that E is the total space of a geometrically flat fiber bundle $p : E \to B$ with closed fiber L and a compact base manifold with boundary B. Then there are collars c_E, c_W of E, W in M and $c_{\partial B}$ of ∂B in B such that $c_W|_{\partial E\times[0,1)}$ and $c_{\partial B}$ are p-related.*

Proof: Take a pair of p-related collars of ∂E in E and of ∂B in B and any collar of ∂W in W. Then use the previous Proposition to extend them to collars of E, W in M. □

Except of Section 10, in all of the following we use collars as in Corollary 4.2.10. In section 10 we use collars as in the following corollary:

Corollary 4.2.11 *Let M, E, W be as in the previous Corollary 4.2.10. In addition, suppose that W is the total space of a geometrically flat fiber bundle $q : W \to \Sigma$ with fiber a compact manifold F with boundary ∂F and a closed base manifold Σ. Then there are collars c_E, c_W of E, W in M, $c_{\partial B}$ of ∂B in B and $c_{\partial F}$ of ∂F in F such that $c_W|_{\partial W\times[0,1)}$ and $c_{\partial B}$ are p-related and $c_E|_{\partial E\times[0,1)}$ and $c_{\partial F}$ are q-fiber-related.*

Proof: The proof is the same as that of Corollary 4.2.10. □

5 Set–Up

5.1 Thom-Mather Stratifications

If one wants to work with differential forms there has to be some smooth structure. Hence we do not work with topological stratified pseudomanifold as defined for example in [Ban07, Definition 4.1.1] but use Thom-Mather smooth stratified spaces. We use the definition of B. Hughes and S. Weinberger, cf. [HW01, sect. 1.2]. (Another older reference would be e.g. [Mat12].)
Let X be a locally compact, seperable metric space and $\mathcal{S} = \{X_i\}$ a stratification, that is a locally finite decomposition of X into pairwise disjoint, locally closed subsets of X with every $X_i \in \mathcal{S}$ a topological manifold. We always assume that a stratification satisfies the *Frontier Condition*:

$\mathrm{cl}(X_i) \cap X_j \neq \emptyset$ if and only if $X_j \subset \mathrm{cl}(X_i)$. The elements of the stratification are called (open) strata and their closures $X^i = \mathrm{cl}(X_i)$ are called closed strata.

Definition 5.1.1 *(Thom-Mather stratified space)*
Let $k \in [0, \infty]$. A C^k-Thom-Mather stratified pseudomanifold is a triple $(X, \mathcal{S}, \mathbf{T})$ such that

1. *$\mathcal{S} = \{X_i\}$ is a stratification of X such that each stratum X_i is a C^k-manifold.*
2. *$\mathbf{T} = (T_i, \pi_i, \rho_i)$, called a tube system, consists of open neighbourhoods T_i of X_i (called tubular neighbourhoods), retractions $\pi_i : T_i \to X_i$ (called the local retraction of T_i) and maps $\rho_i : T_i \to [0, \infty)$ such that $X_i = \rho_i^{-1}(0)$.*
3. *For each pair $X_i, X_j \in \mathcal{S}$ the map*
$$(\pi_{ij},\ \rho_{ij}) : T_{ij} \to X_i \times [0, \infty),$$
where $T_{ij} = T_i \cap X_j$, $\pi_{ij} = \pi_i|_{T_{ij}}$ and $\rho_{ij} = \rho_i|_{T_{ij}}$, is a C^k-submersion.
4. *(Compability Conditions) For each triple $X_i, X_j, X_k \in \mathcal{S}$ and $x \in T_{jk} \cap T_{ik} \cap \pi_{jk}^{-1}(T_{ij})$ it holds that*
$$\pi_{ij} \circ \pi_{jk}(x) = \pi_{ik}(x),$$
$$\rho_{ij} \circ \pi_{jk}(x) = \rho_{ik}(x).$$

In this paper we will only work with C^∞-Thom-Mather stratified pseudomanifolds. In [Mat12] and [Mat73], Mather proved that every Whitney stratified space has a C^∞-Thom-Mather stratification. Since Whitney showed in [Whi65] that any complex or real analytic set admits a Whitney stratification, those are examples for the type of spaces we consider.
Note further that Mather also proved, using Thom's isotopy lemmas, that any stratum X_i in a Thom-Mather stratified space has a neighbourhood N such that the pair (N, X_i) is homeomorphic to the pair $(\mathrm{cyl}(f), X_i)$, with $\mathrm{cyl}(f)$ the mapping cylinder of some fiber bundle $p : E \to X_i$, which is called the link bundle of the stratum. We will later assume these bundles to satisfy some flatness conditions.
By a theorem of Goresky (see [Gor78]), each C^∞-Thom-Mather stratified pseudomanifold can be (smoothly) triangulated by a triangulation compatible with the filtration and hence is a PL-pseudomanifold.

5.2 The Two-Strata Case

In [Ban11] M. Banagl investigates oriented, compact smooth Thom-Mather stratified pseudomanifolds with filtration

$$X = X_n \supset X_b = \Sigma$$

with Σ^b a b-dimensional connected closed manifold with geometrically flat link bundle. That means there is an open neighbourhood N of Σ in X, such that the boundary of the compact manifold $M = X - N$ is the total space of a geometrically flat link bundle $p : \partial M \to \Sigma$ with fiber an oriented, closed smooth Riemannian manifold L^m of dimension $m = n - 1 - b$. There are two strata in this setting: $X_b = \Sigma$ and $X_n - X_b$.
Banagl defines a complex of differential forms $\Omega I^\bullet_{\bar{p}}$ on the nonsingular part M of X using cotruncation in the fiber direction for multiplicatively structured forms on the boundary ∂M.
The flat link bundle condition allows us to define a complex of multiplicatively structured differntial forms on the boundary. Let therefore $\mathcal{U} := \{U_\alpha\}_{\alpha \in I}$ be a good open cover of Σ such that the bundle trivializes with respect to this cover, i.e. for each $\alpha \in I$ there are diffeomorphisms $\psi_\alpha : U_\alpha \times L \to p^{-1}(U_\alpha)$ such that the following diagram commutes:

$$\begin{tikzcd}
p^{-1}(U_\alpha) \arrow[dr, "p|"'] & & U_\alpha \times L \arrow[ll, "\psi_\alpha"', "\cong"] \arrow[dl, "\pi_1"] \\
& U_\alpha &
\end{tikzcd}$$

We are then able to define the following subcomplex of the complex $\Omega^\bullet(\partial M)$ of differential forms on ∂M, using the projections $\pi_1 : U_\alpha \times L \to U_\alpha$ and $\pi_2 : U_\alpha \times L \to L$:

$$\Omega^\bullet_{\mathcal{MS}}(\Sigma) := \Big\{ \omega \in \Omega^\bullet(\partial M) \Big| \; \omega|_{U_\alpha} = \phi^*_\alpha \sum_j \pi_1^* \eta_j \wedge \pi_2^* \gamma_j \\ \text{with } \eta_j \in \Omega^\bullet(U_\alpha), \gamma_j \in \Omega^\bullet(L) \Big\}.$$

These forms can be truncated or cotruncated in the link direction (see [Ban11, section5]) and the mentioned complex $\Omega I^\bullet_{\bar{p}}$ is defined as containing the forms that look like the pullback of a fiberwise cotruncated multiplicative structured form near ∂M in a collar neighbourhood of the boundary.
The cohomology of that complex then satisfies generalized Poincaré-duality over complementary perversities and is isomorphic to the cohomology of the associated intersection space if the link bundle is trivial. For arbitrary flat link bundle we do not yet know how to construct the intersection space.
The de Rham result for the cohomology of $\Omega I^\bullet_{\bar{p}}$ and the cohomology of intersection spaces was proven by Banagl for pseudomanifolds with isolated singularities in Theorem 9.13 of loc. cit. and by the author in [Ess12].

5.3 The Three-Strata Case

The aim of the thesis is to generalize the above construction to certain classes of pseudomanifolds with three strata. Strictly speaking we consider smooth Thom-Mather stratified pseudomanifolds X of dimension n with filtration $X = X_n \supset X_m \supset X_b$ with $n-2 \geq m > b$ and additional conditions on the regular neighbourhoods of the singular strata. The strata here are X_b and $X_m - X_b$, which are the singular strata, and $X_n - X_b$. We begin with zero dimensional bottom stratum, i.e. $b = 0$ and $X_b = \{x_0, ..., x_d\}$, and treat a setting with positive dimensional bottom stratum afterwards (see Section 10).

5.3.1 Zero Dimensional Bottom Stratum

We first consider stratified pseudomanifolds X with filtration

$$X = X_n \supset X_b \supset X_0 = \{x_0, ..., x_d\},$$

where the bottom stratum is zero dimensional and the middle stratum has some flatness condition described below.
To define intersection space cohomology on these stratified pseudomanifolds we first remove a regular neighbourhood R_0 of X_0 homeomorphic to $\mathring{\text{cone}}(L_0)$, with L_0 a stratified pseudomanifold of dimension $n-1$. The result is a stratified pseudomanifold $X' = X - R_0$ with boundary and one singular stratum

$$B := X'_b = X_b - R_0 \cap X_b,$$

a $b-$-dimensional compact smooth manifold with boundary ∂B. We assume that this singular stratum has a geometrically flat link bundle in X', i.e. there is an open tubular neighbourhood T_b of B in X' such that

$$M := X' - R_b$$

is a smooth $\langle 2 \rangle$-manifold with boundary decomposed as

$$\partial M = E \cup_{\partial E = \partial W} W.$$

Flatness in this setting means that E is the total space of a flat link bundle

$$\begin{tikzcd} L \arrow[r] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

over the compact base manifold with boundary B with link a closed smooth Riemannian manifold L^m, such that the structure group of the bundle is the isometries of L. The manifold with boundary W is

$$W = L_0 - T_b \cap L_0,$$

with boundary $\partial W = \partial E$. The manifold W can be seen as the regular part of the pseudomanifold $L_0 = \partial X'$, the link of X_0.

Note that in order to prove Poincaré duality for $HI^{\bullet}_{\bar{p}}$ over complementary perversities, we have to impose an additional Witt-type condition on W.

5.3.2 A Positive Dimensional Bottom Stratum

Afterwards we will treat 3-strata stratified pseudomanifolds X with positive dimensional bottom stratum. That means X has a decomposition into closed strata

$$X = X_n \supset X_s \supset X_b,$$

with $b > 0$. The details of this setup are explained in subsection 10.1. In short we again get a $\langle 2 \rangle$-manifold M with boundary $\partial M = E \cup_{\partial E = \partial W} W$ as regular part of the pseudomanifold. However, here W is not the regular part of the link of the bottom singular stratum $X_b = \Sigma$ anymore. Instead there is a geometrically flat link bundle

$$\begin{tikzcd} F \arrow[r] & W \arrow[d, "q"] \\ & \Sigma, \end{tikzcd}$$

with fiber a compact smooth manifold F with boundary ∂F. In order to prove Poincaré duality we need additional assumptions on $\partial W = \partial E$ and the restrictions of the flat bundles p (which is as in the $b = 0$ case) and q to these boundaries.
Therefore this setting is not a generalization of the previous case.

5.3.3 Cotruncation Values

If we have a stratified pseudomanifold X with stratification $X = X_n \supset X_b \supset X_s$ and complementary perversities $\bar{p}$, $\bar{q}$, then, unless otherwisely stated, we set $\dim(L = \text{Link } X_b) := m := n - 1 - b$ and $\dim(F = \text{Link } X_s) := f := n - 1 - s$ and define the cutoff values

$$K := m - \bar{p}(m+1), \; K' := m - \bar{q}(m+1) \text{ and}$$
$$L := f - \bar{p}(f+1), \; L' := f - \bar{q}(f+1).$$

These cutoff values are the cotruncation degrees for complexes of multiplicatively structured differential forms near the respective strata.

6 Cotruncation on Manifolds with Boundary

In this section we establish the cotruncation of the cochain complex of smooth differential forms on manifolds with boundary. Recall that on a closed Riemannian manifold M the Hodge decomposition provides orthogonal splittings

$$\Omega^r(M) = \operatorname{im} d \oplus \ker d^* = \operatorname{im} d \oplus \operatorname{im} d^* \oplus \mathcal{H}^r(M) \tag{4}$$

with $\mathcal{H}^r(M) := \{\omega \mid \Delta\omega = (d\, d^* + d^*\, d)\omega = 0\}$ the harmonic r-forms on M. This allows us to define the cotruncated subcomplex of smooth differential forms:

$$\tau_{\geq L}\Omega^\bullet(M) := \ldots \to 0 \to \ker d^* \to \Omega^{L+1}(M) \to \Omega^{L+2}(M) \to \ldots \subset \Omega^\bullet(M).$$

By the Hodge decomposition this complex has the following properties:

$$H^r\big(\tau_{\geq L}\Omega^\bullet(M)\big) = \begin{cases} 0 & \text{for } r < L, \\ H^r(M) & \text{for } r \geq L \end{cases}$$

and

$$\tau_{\geq L}\Omega^\bullet(M) \oplus \tau_{<L}\Omega^\bullet = \Omega^\bullet(M),$$

where $\tau_{<L}\Omega^\bullet(M) := \ldots \to \Omega^{L-1}(M) \to \operatorname{im} d \to 0 \to \ldots \subset \Omega^\bullet(M)$.
For manifolds with boundary the Hodge decomposition (4) is not true in general, so we cannot define the subcomplex of cotruncated differential forms $\tau_{\geq L}\Omega^\bullet(M)$ in the same way as before. But there is a natural substitute for the Hodge decomposition: The so called Hodge–Morrey–Friedrichs decomposition. In principle, the difference to the Hodge decomposition for closed manifolds is that one has to impose boundary conditions for the differential forms. In particular, if the boundary is the empty set (M closed) these conditions vanish and the decomposition reduces to the well known Hodge decomposition on closed manifolds.

6.1 Tangential and Normal Components of Differential Forms

Let (M^n, g) be an oriented and compact smooth Riemannian manifold with boundary ∂M and Riemannian metric g. Let $\omega \in \Omega^r(M)$ be a smooth r-form on M. Then we call

$$\begin{aligned} \omega|_{\partial M} : \Gamma(TM|_{\partial M}) \times \ldots \times \Gamma(TM|_{\partial M}) &\to C^\infty(M) \\ (X_1, \ldots, X_r) &\mapsto \omega(X_1, \ldots, X_r) \end{aligned}$$

the boundary value of ω. It is a smooth section in the restricted r-form bundle $\Lambda^r(M)|_{\partial M}$, i.e we can write $\omega|_{\partial M} \in \Omega^r(M)|_{\partial M} := \Gamma(\Lambda^r(M)|_{\partial M})$. We emphasize that

$$\omega|_{\partial M} \neq j^*_{\partial M}\, \omega \in \Omega^r(\partial M),$$

the pullback of ω to ∂M under the inclusion $j_{\partial M} : \partial M \to M$.

Restriction to the boundary is both compatible with the wedge product and with the Hodge star operator corresponding to the metric g, i.e.

$$(\omega \wedge \eta)|_{\partial M} = \omega|_{\partial M} \wedge \eta|_{\partial M} \qquad \text{and} \qquad * (\omega|_{\partial M}) = (*\omega)|_{\partial M}.$$

The boundary value $\omega|_{\partial M}$ can then be decomposed into a tangential and normal part as follows: Using a collar of the boundary ∂M and the corresponding (normalized) normal field $\mathcal{N}$ one can decompose every vector field $X \in \Gamma(TM|_{\partial M})$ into its tangential and normal parts by setting

$$X^{\perp} := g(X, \mathcal{N})\mathcal{N}, \qquad X^{\parallel} := X - X^{\perp}.$$

Then we define the tangential and normal component of $\omega \in \Omega^r(M)$ as follows:

$$\mathbf{t}\ \omega(X_1, ..., X_r) := \omega(X_1^{\parallel}, ..., X_r^{\parallel}) \qquad \forall X_1, ..., X_r \in \Gamma(TM|_{\partial M}) \quad \text{and}$$
$$\mathbf{n}\ \omega := \omega|_{\partial M} - \mathbf{t}\omega.$$

for $r \geq 1$ and $\mathbf{t}\ \omega := \omega$ for $r = 0$.
Note that the tangential component $\mathbf{t}\ \omega$ is uniquely determined by the pullback $j_{\partial M}^*\ \omega$ of ω to ∂M under the inclusion $j_{\partial M} : \partial M \hookrightarrow M$.

Lemma 6.1.1 *(Dirichlet and Neumann boundary conditions)*
The spaces

$$\Omega_D^r(M) := \left\{\omega \in \Omega^r(M) \middle| \mathbf{t}\ \omega = 0\right\}$$

of smooth r-forms on the manifold with boundary M satisfying a Dirichlet boundary condition form a subcomplex $(\Omega_D^{\bullet}(M), d) \subset (\Omega^{\bullet}(M), d)$.
The spaces

$$\Omega_N^r(M) := \left\{\omega \in \Omega^r(M) \middle| \mathbf{n}\ \omega = 0\right\}$$

of smooth r-forms on the manifold with boundary M satisfying a Neumann boundary condition form a subcomplex $(\Omega_N^{\bullet}(M), d^*) \subset (\Omega^{\bullet}(M), d^*)$, where d^* denotes the codifferential with respect to g.

Proof: As mentioned above $\mathbf{t}\ \omega = 0 \quad \Leftrightarrow \quad j_{\partial M}^*\omega = 0$. But since pullbacks commute with the exterior differential, the first part of the lemma is established. The second part follows from [Sch95, Prop. 1.2.6], which states that the normal and tangential components of a differential form are Hodge adjoint to each other, i.e.

$$*(\mathbf{n}\ \omega) = \mathbf{t}\ (*\omega) \quad \text{and} \quad * (\mathbf{t}\ \omega) = \mathbf{n}\ (*\omega),$$

where $*(\mathbf{n}\ \omega)$ and $*(\mathbf{t}\ \omega)$ are understood by the action of the Hodge star on any extension of $\mathbf{n}\ \omega$, respectively $\mathbf{t}\ \omega$, followed by restriction to ∂M.

Hence for $\omega \in \Omega^r(M)$ with $\mathbf{n}\ \omega = 0$ we have $0 = *(\mathbf{n}\ \omega) = \mathbf{t}\ (*\omega)$ and therefore

$$\mathbf{n}\ d^*\omega = \mathbf{n}\ (\pm\ * d * \omega) = \pm * (\mathbf{t}\ d * \omega) = 0$$

by the previous result.

□

6.2 The Hodge–Morrey–Friedrichs Decomposition

As before, let (M^n, g) be a compact, oriented smooth Riemannian manifold with boundary ∂M.

Definition 6.2.1 *(Some subspaces of $\Omega^r(M)$)*
For $r \in \mathbb{Z}$ we define the spaces of coclosed Neumann forms $cC^r_N(M)$, exact Dirichlet forms $E^r_D(M)$, coexact Neumann forms $cE^r_N(M)$, exact harmonic forms $\mathcal{H}^r_{ex}(M)$ and Neumann harmonic forms $\mathcal{H}^r_N(M)$ as follows:

$$\begin{aligned} cC^r_N(M) &:= \big\{\omega \in \Omega^r_N(M) \big| d^*\omega = 0\big\} \\ E^r_D(M) &:= \big\{d\alpha \big| \alpha \in \Omega^{r-1}_D(M)\big\} \\ cE^r_N(M) &:= \big\{d^*\xi \big| \xi \in \Omega^{r+1}_N(M)\big\} \\ \mathcal{H}^r_{ex}(M) &:= \big\{d\eta \big| \eta \in \Omega^{r-1}(M), d^*d\eta = 0\big\} \\ \mathcal{H}^r_N(M) &:= \big\{\eta \in \Omega^r_N(M) \big| d^*\eta = 0,\ d\eta = 0\big\} \end{aligned}$$

Note that by 6.1.1 it holds that $E^r_D(M) \subset \Omega^r_D(M)$ and $cE^r_N(M) \subset \Omega^r_N(M)$ for all $r \in \mathbb{Z}$.
We then have the following decomposition of $\Omega^r(M)$ into orthogonal direct summands established by C. B. Morrey and K. O. Friedrichs:

Theorem 6.2.2 ***(The Hodge–Morrey–Friedrichs Decomposition)***
On a compact oriented smooth Riemannian manifold (M^n, g) with boundary ∂M we have, for each $r \in \mathbb{Z}$, the orthogonal direct sum decomposition

$$\Omega^r(M) = E^r_D(M) \oplus \mathcal{H}^r_{ex}(M) \oplus cE^r_N(M) \oplus \mathcal{H}^r_N(M).$$

Proof: By [Sch95, Corollary 2.4.9] the above orthogonal decomposition holds for L^2-forms and forms of arbitrary Sobolev class. But then a standard argument involving the Sobolev lemma and some regularity results gives the desired decomposition for smooth forms. For more details, see page 85 of [Sch95] and [Sch95, Section 2.2]).

□

Corollary 6.2.3 *Let $r \in \mathbb{Z}$. Then for $E^r(M) := \{d\omega | \omega \in \Omega^{r-1}(M)\}$ and $cC^r_N(M)$ there are orthogonal direct splittings*

$$E^r(M) = E^r_D(M) \oplus \mathcal{H}^r_{ex}(M)$$

and

$$cC^r_N(M) = cE^r_N(M) \oplus \mathcal{H}^r_N(M).$$

Proof: The main tool for proving this corollary is **Green's formula** ([Sch95, Prop. 2.1.2]). It implies that for two smooth forms $\omega \in \Omega^{r-1}(M)$, $\eta \in \Omega^r(M)$ we have

$$\ll d\omega, \eta \gg = \ll \omega, d^*\eta \gg + \int_{\partial M} \mathbf{t}\ \omega \wedge *\mathbf{n}\ \eta,$$

where $\ll \alpha, \beta \gg = \int_M \alpha \wedge *\beta$ denotes the L^2-metric on $\Omega^r(M)$. We first show that

$$E^r(M) = E^r_D(M) \oplus \mathcal{H}^r_{ex}(M): \tag{5}$$

For let $d\omega \in E^r(M)$, $d^*\alpha \in cE^r_N(M)$ and $\beta \in \mathcal{H}^r_N(M)$. Then by Green's formula

$$\ll d\omega, d^*\alpha \gg = \ll \omega, (d^*)^2\alpha \gg + \int_{\partial M} \mathbf{t}\ \omega \wedge *\mathbf{n}\ \alpha = 0,$$

since $(d^*)^2 = 0$ and $\mathbf{n}\ \alpha = 0$. On the other hand

$$\ll d\omega, \beta \gg = \ll \omega, d^*\beta \gg + \int_{\partial M} \mathbf{t}\ \omega \wedge *\mathbf{n}\ \beta = 0,$$

since $\beta \in \mathcal{H}^r_N(M)$. Therefore by the above Theorem 6.2.2 we have $\omega \in E^r_D(M) \oplus \mathcal{H}^r_{ex}(M)$, i.e. $E^r(M) \subset E^r_D(M) \oplus \mathcal{H}^r_{ex}(M)$. This implies (5), since the converse is trivially true.
The second step is to show that

$$cC^r_N(M) = cE^r_N(M) \oplus \mathcal{H}^r_N(M). \tag{6}$$

For let $\omega \in cC^r_N(M)$, $d\alpha \in E^r_D(M)$ and $d\beta \in \mathcal{H}^r_{ex}(M)$. Again by Green's formula we obtain

$$\ll \omega, d\alpha \gg = \ll d^*\omega, \alpha \gg + \int_{\partial M} \mathbf{t}\ \alpha \wedge *\mathbf{n}\ \omega = 0$$

and

$$\ll \omega, d\beta \gg = \ll d^*\omega, \beta \gg + \int_{\partial M} \mathbf{t}\ \beta \wedge *\mathbf{n}\ \omega = 0$$

(by definition of $cC^r_N(M)$). Therefore by Theorem 6.2.2 $\omega \in E^r_N(L) \oplus H^r_N(M)$, i.e. $cC^r_N(M) \subset E^r_N(L) \oplus H^r_N(M)$, and since the converse inclusion is trivially true, the corollary is established.

$\square$

6.3 Cotruncation on Manifolds with Boundary

Using the results of the previous subsection, in particular the Hodge–Morrey–Friedrichs decomposition we now can establish the cotruncated subcomplex of the complex of differential forms on a Riemannian manifold with boundary M.

Definition 6.3.1 *Let $k \in \mathbb{N}$. Then we define*

$$\tau_{\geq k}\Omega^\bullet(M) := ... \to 0 \to cC_N^k(M) \to \Omega^{k+1}(M) \to \Omega^{k+2}(M) \to ...$$

Lemma 6.3.2 *The subcomplex inclusion $i : \tau_{\geq k}\Omega^\bullet(M) \hookrightarrow \Omega^\bullet(M)$ induces an isomorphism*

$$i^* : H^r\big(\tau_{\geq k}\Omega^\bullet(M)\big) \xrightarrow{\cong} H^r(M) \quad \textit{for } r \geq k.$$

On the other hand

$$H^r\big(\tau_{\geq k}\Omega^\bullet(M)\big) = 0 \quad \textit{for } r < k.$$

Proof: For $r \geq k+2$ the statement is obvious since then $\tau_{\geq k}\Omega^r(M) = \Omega^r(M)$ and $\tau_{\geq k}\Omega^{r-1} = \Omega^{r-1}(M)$.

Let $r = k+1$.

$$H^{k+1}\big(\tau_{\geq k}\Omega^\bullet(M)\big) = \frac{\ker d^{k+1}}{d^k(cC_N^k(M))}$$

But Corollary 6.2.3 implies that

$$d^k\big(cC_N^k(M)\big) = d^k\big(cC_N^k(M) \oplus E^k(M)\big) = d^k\big(\Omega^k(M)\big) = \operatorname{im} d^k$$

and hence

$$H^k\big(\tau_{\geq k}\Omega^\bullet(M)\big) = \frac{\ker d^{k+1}}{\operatorname{im} d^k}.$$

Now let $r = k$.

$$H^k\big(\tau_{\geq k}\Omega^\bullet(M)\big) = \frac{\ker d^k \cap cC_N^k(M)}{d^{k-1}(0)} = \ker d^k \cap cC_N^k(M) = \mathcal{H}_N^r(M)$$

Let $\omega \in \Omega^k(M)$. By Theorem 6.2.2 and Corollary 6.2.3 there are forms $d\alpha \in E^k(M)$, $d^*\beta \in cE_N^k(M)$ and $\sigma \in \mathcal{H}_N^k(M)$ such that $\omega = d\alpha + d^*\beta + \sigma$. Now if ω is closed, $d\omega = 0$, then $dd^*\beta = d\omega - d^2\alpha - d\sigma = 0$ and by Green's formula we therefore have

$$\ll d^*\beta, d^*\beta \gg = \ll dd^*\beta, \beta \gg - \int_{\partial M} \mathbf{t}\ d^*\beta \wedge \mathbf{n}\ \beta = 0$$

by the definition of $\Omega_N^{k+1}(M)$. Therefore $d^*\beta = 0$ and we have

$$\omega \in \ker d^k \quad \Leftrightarrow \quad \omega = d\alpha + \sigma.$$

Hence, by the orthogonality of the Hodge–Morrey–Friedrichs decomposition we have

$$H^k\big(\tau_{\geq k}\Omega^\bullet(M)\big) = \ker d^k \cap cC_N^k(M) = \mathcal{H}_N^r(M) \cong \frac{\ker d^k}{\operatorname{im} d^{k-1}} = H^k(M).$$

The second statement is obvious since, by definition, $\tau_{\geq k}\Omega^r(M) = 0$ for $r < k$.

$\square$

6.4 Cotruncation of other Complexes of Differential Forms

To be able to prove Poincaré duality for HI on 3-strata pseudomanifolds we will need cotruncation of subcomplexes of the complex of differential forms. Even more we need this cotruncation to be consistent with the cotruncation $\tau_{\geq k}\Omega^\bullet(M)$. This means, if we have a subcomplex $S^\bullet \subset \Omega^\bullet(M)$, where M is a smooth compact manifold (with or without boundary), we want a cotruncated subcomplex $\tau_{\geq k}S^\bullet$ that satisfies

1. $H^r(\tau_{\geq k}S^\bullet) = \begin{cases} 0 & \text{if } r < k, \\ H^r(S^\bullet) & \text{if } r \geq k, \end{cases}$
2. $\tau_{\geq k}S^\bullet \subset \tau_{\geq k}\Omega^\bullet(M)$ is a subcomplex.

Definition 6.4.1 *(Geometrically cotruncatable subcomplexes of $\Omega^\bullet(M)$)*
A subcomplex $S^\bullet \subset \Omega^\bullet(M)$ is called <u>*geometrically cotruncatable in degree*</u> *$k \in \mathbb{N}$ if*

$$\operatorname{im} d_M^{k-1} \cap S^k = d_S^{k-1}(S^{k-1}).$$

Note that the inclusion "$\supset$" in the relation is satisfied for any subcomplex while the inclusion "$\subset$" does usually not hold.

Lemma 6.4.2 *(An equivalent condition for cotruncatability)*
Let $S^\bullet \subset \Omega^\bullet(M)$ be a subcomplex. Then $S^\bullet$ is geometrically cotruncatable in degree $k \in \mathbb{N}$ if and only if subcomplex inclusion induces and injection $H^k(S^\bullet) \hookrightarrow H^k(M)$.

<u>Proof:</u> This is just the definition of the geometric cotruncatability and cohomology. □

The next lemma shows why we call a complex satisfying the condition of Definition 6.4.1 geometrically cotruncatable: The intersection of $S^\bullet$ with the cotruncated complex $\tau_{\geq k}\Omega^\bullet(M)$ is a cotruncation of $S^\bullet$ in degree k:

Lemma 6.4.3 *If $S^\bullet \subset \Omega^\bullet(M)$ is geometrically cotruncatable in degree $k \in \mathbb{N}$, then there is an orthogonal direct sum decomposition*

$$S^k = d^{k-1}(S^{k-1}) \oplus S^k \cap cC_N^k(M)$$

and the complex $\tau_{\geq k}S^\bullet$ defined by

$$\tau_{\geq k}S^\bullet := \ldots \to 0 \to S^k \cap cC_N^k(M) \to S^{k+1} \to \ldots$$

is a suitable cotruncation in the above sense, i.e. $\tau_{\geq k}S^\bullet \subset \tau_{\geq k}\Omega^\bullet(M)$ is a subcomplex and

$$H^r(\tau_{\geq k}S^\bullet) = \begin{cases} 0 & \text{if } r < k, \\ H^r(S^\bullet) & \text{if } r \geq k. \end{cases}$$

(Note that if M is closed then $cC_N^r(M) = cC^r(M) = \{\omega \in \Omega^r(M) | d^*\omega = 0\}$ for all r.)

Proof: The first equation follows from the orthogonal direct sum composition

$$\Omega^k(M) = \operatorname{im} d^{k-1} \oplus cC_N^k(M)$$

for $\Omega^k(M)$ given by Theorem 6.2.2 and Corollary 6.2.3. This gives the direct sum decomposition

$$S^k = S^k \cap \Omega^k(M) = \underbrace{S^k \cap \operatorname{im} d^{k-1}}_{=d^{k-1}(S^{k-1})} \oplus S^k \cap cC_N^k(M).$$

It remains to prove that the complex $\tau_{\geq k}S^\bullet$ from above is a suitable cotruncation: It is obvious that, with the definition in the lemma, $\tau_{\geq k}S^\bullet$ is a subcomplex of $\tau_{\geq k}\Omega^\bullet(M)$. We have to prove the cohomology relations: Of course, if $r < k$, then $\ker d \cap \tau_{\geq k}S^r = 0$, implying $H^r(\tau_{\geq k}S^\bullet) = 0$. If $r > k+1$ then $\tau_{\geq k}S^r = S^r$ and $\tau_{\geq k}S^{r+1} = S^{r+1}$ and hence $H^r(\tau_{\geq k}S^\bullet) = H^r(S^\bullet)$. The only nontrivial degrees are $r = k$ and $r = k+1$: We have

$$\begin{aligned} S^k &= S^k \cap \Omega^k(M) = S^k \cap (cC_N^k(M) \oplus \operatorname{im} d^{k-1}) \\ &= (S^k \cap cC_N^k(M)) \oplus \underbrace{(S^k \cap \operatorname{im} d^{k-1})}_{=d^{k-1}(S^{k-1})}. \end{aligned}$$

Hence

$$H^k(S^\bullet) = \frac{\ker d^k \cap S^k}{d^{k-1}(S^{k-1})} \cong S^k \cap cC_N^k(M) \cap \ker d^k = H^k(\tau_{\geq k}S^\bullet)$$

and

$$H^{k+1}(S^\bullet) = \frac{\ker d^{k+1} \cap S^{k+1}}{d^k(S^k)} = \frac{\ker d^{k+1} \cap S^{k+1}}{d^k(cC_N^k(M) \cap S^k)} = H^{k+1}(\tau_{\geq k}S^\bullet).$$

Remark: Note that the subscript "N" in $cC_N^\bullet(M)$ stands for Neumann boundary conditions and can be dropped if the manifold M has empty boundary. □

Example 6.4.4 *Let $S^\bullet \subset \Omega^\bullet(M)$ be a subcomplex with $H^k(S^\bullet) = 0$, for some $k \in \mathbb{Z}$. Then $\operatorname{im} d^{k-1} \cap S^k \subset \ker d^k \cap S^k = d^{k-1}(S^{k-1}) \subset \operatorname{im} d^{k-1} \cap S^k$ and hence $S^\bullet$ is geometrically cotruncatable in degree k.*

Remark 6.4.5 *(Truncation of arbitrary subcomplexes $S^\bullet \subset \Omega^\bullet(M)$)*
Note that it is always possible to define truncation for arbitrary subcomplexes $S^\bullet \subset \Omega^\bullet(M)$ (without additional assumptions on $S^\bullet$) as a subcomplex of $\tau_{<k}\Omega^\bullet(M)$:

$$\tau_{<k}S^\bullet := \ldots \to S^{k-1} \to d^{k-1}(S^{k-1}) \to 0 \to \ldots$$

Indeed, $\tau_{<k}S^\bullet$ satisfies

$$H^r(\tau_{<k}S^\bullet) = \begin{cases} H^r(S^\bullet) & \text{if } r < k, \\ 0 & \text{else.} \end{cases}$$

If in addition $S^\bullet$ is geometrically cotruncatable in degree k, then there is an orthogonal direct sum decomposition

$$S^\bullet = \tau_{<k}S^\bullet \oplus \tau_{\geq k}S^\bullet$$

(induced be the direct sum decomposition $\Omega^\bullet(M) = \tau_{<k}\Omega^\bullet(M) \oplus \tau_{\geq k}\Omega^\bullet(M)$) and hence the composition

$$\tau_{<k}S^\bullet \hookrightarrow S^\bullet \xrightarrow{proj} \frac{S^\bullet}{\tau_{\geq k}S^\bullet}$$

is an isomorphism of differential complexes.

7 The Partial de Rham Intersection Complex

Instead of starting with the final complex of intersection space forms on M, we first define intermediate complexes $\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)$ and $\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M, C_W)$. They consist of forms whose restriction to C_E is the pullback of a fiberwisely truncated form on E and whose restriction to C_W is either the pullback of some form on W or zero for the relative group. We show that the corresponding cohomology groups $H^r\big(\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)\big)$ and $H^{n-r}\big(\widetilde{\Omega I}^{\bullet}_{\bar{q}}(M, C_W)\big)$ are Poincaré dual to each other, see Theorem 7.5.5.
Not till then we define the actual complex of intersection space forms on M, $\Omega I^{\bullet}_{\bar{p}}$, and show Poincaré duality for it.

Before we give the definitions of $\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)$ and $\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M, C_W)$ we recall the definitions of the complex of multiplicatively structured forms as well as the complex of fiberwisely truncated and cotruncated multiplicatively structured forms from [Ban11, Sections 3 and 6]:

Definition 7.0.1 *(Multiplicatively structured forms)*
Let $p : E \to B$ be a flat bundle with base B a compact manifold with boundary ∂B and fiber a Riemannian manifold L and let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be a good open cover of B such that the bundle trivializes with respect to that cover. Let further $U \subset B$ be open. We then define

$$\Omega^{\bullet}_{\mathcal{MS}}(U) := \Big\{\omega \in \Omega^{\bullet}(p^{-1}(U)) \Big| \forall\alpha \in I : \omega|_{p^{-1}(U_\alpha)} = \phi_\alpha^* \sum_{j_\alpha} \pi_1^*\eta_{j_\alpha} \wedge \pi_2^*\gamma_{j_\alpha} \\ \text{with } \eta_{j_\alpha} \in \Omega^{\bullet}(U \cap U_\alpha),\ \gamma_{j_\alpha} \in \Omega^{\bullet}(L)\Big\} \tag{7}$$

Here, the $\phi_\alpha : p^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times L$ deonte the local trivializations of the bundle.

To define the complexes of fiberwisely truncated and cotruncated multiplicatively structured forms we need the complexes of truncated and cotruncated forms of the closed (Riemannian) manifold L from [Ban11, Section 4].

Definition 7.0.2 *(Fiberwisely (co)truncated forms)*
Let $p : E \to B$ be a flat bundle with base B a compact manifold with boundary ∂B and fiber a closed manifold L and let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be a good open cover of B such that the bundle trivializes with respect to that cover as in the previous definition. Let further $U \subset B$ be open. We then define, for any integer K, the complex of (in degree K) fiberwisely truncated multiplicatively structured forms by

$$ft_{<K}\Omega^{\bullet}_{\mathcal{MS}}(U) := \Big\{\omega \in \Omega^{\bullet}_{\mathcal{MS}}(U) \Big| \forall\alpha \in I : \omega|_{p^{-1}(U_\alpha)} = \phi_\alpha^* \sum_{j_\alpha} \pi_1^*\eta_{j_\alpha} \wedge \pi_2^*\gamma_{j_\alpha} \\ \text{with } \gamma_{j_\alpha} \in \tau_{<K}\Omega^{\bullet}(L)\Big\}$$

If the fiber is a Riemannian manifold and the bundle is geometrically flat, we moreover define the complex of fiberwisely cotruncated multiplicatively structured forms by

$$\begin{aligned} ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(U) := \big\{\omega \in \Omega^{\bullet}_{\mathcal{MS}}(U) \big| \forall \alpha \in I : \omega|_{p^{-1}(U_\alpha)} = \; & \textit{as in (7)} \\ & \textit{with } \gamma_{j_\alpha} \in \tau_{\geq K}\Omega^{\bullet}(L)\big\}. \end{aligned}$$

All of these complexes $\Omega^{\bullet}_{\mathcal{MS}}(U), ft_{<K}\Omega^{\bullet}_{\mathcal{MS}}(U)$, and $ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(U)$, for $U \subset B$ open, are subcomplexes of the complex of forms $\Omega^{\bullet}\big(p^{-1}(U)\big)$.

Definition 7.0.3 *(The partial de Rham intersection complex)*

$$\begin{aligned} \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M) := \big\{\omega \in \Omega^{\bullet}(M) \big| & c_E^*\omega = \pi_E^*\eta \textit{ for some } \eta \in ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(B) \\ & c_W^*\omega = \pi_W^*\rho \textit{ for some } \rho \in \Omega^{\bullet}(W)\big\} \end{aligned}$$

Here $\pi_E,\ \pi_W$ denote the projections

$$\pi_E : E \times [0,1) \to E$$

and

$$\pi_W : W \times [0,1) \to W.$$

This is a subcomplex of the complex $\Omega^{\bullet}(M)$ of forms on M.

Definition 7.0.4 *(The relative partial de Rham intersection complex)*

$$\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M, C_W) := \big\{\omega \in \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M) \big| c_W^*\omega = 0\big\} \subset \Omega^{\bullet}(M, C_W).$$

In the rest of this section we prove Poincaré duality between $\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)$ and $\widetilde{\Omega I}^{\bullet}_{\bar{q}}(M, W)$.

Theorem 7.5.5 *(Poincaré Duality for partial Intersection Forms)*
For complementary perversities $\bar{p}$ and $\bar{q}$, integration induces a nondegenerate bilinear form

$$\begin{aligned} \widetilde{HI}^{r}_{\bar{p}}(M) \times \widetilde{HI}^{n-r}_{\bar{q}}(M, C_W) &\to \mathbb{R} \\ ([\omega], [\eta]) &\mapsto \int_M \omega \wedge \eta, \end{aligned}$$

where $\widetilde{HI}^{r}_{\bar{p}}(M) := H^r(\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M))$ and $\widetilde{HI}^{n-r}_{\bar{q}}(M, C_W) := H^{n-r}(\widetilde{\Omega I}^{\bullet}_{\bar{q}}(M, C_W))$.

7.1 Proof of Poincaré Duality for Partial Intersection Forms

The proof of the Theorem 7.5.5 is geared on the proof of Poincaré duality for intersection forms in the two strata case, see [Ban11, Sect. 8]. However, the additional stratum produces additional technical difficulties, as one might have expected.
We first want to deal with the fact that in $\widetilde{\Omega I}^{\bullet}_{\bar{p}}$ we do not just demand that the forms restricted to a collar neighbourhood of E come from a form in $ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(B)$ but also that they are constant in the collar direction in a collar neighbourhood of W.

Definition 7.1.1 *(Fiberwise cotruncated forms that are in $\Omega^{\bullet}_{\partial c}(E)$)*
We recall that the collar $c_{\partial E} : \partial E \times [0,1) \hookrightarrow E$ of ∂E in E is the restriction of the collar of W in M, $c_{\partial E} = c_W|_{\partial W \times [0,1)}$, and define

$$P^{\bullet}(B) := \{\omega \in \Omega^{\bullet}_{\mathcal{MS}}(B) |\ \exists\ \eta \in \Omega^{\bullet}_{\mathcal{MS}}(\partial B) :\ c^*_{\partial E}\omega = \pi^*_{\partial E}\eta\}.$$

Analogously, we define

$$P^{\bullet}_{\geq K}(B) := \{\omega \in ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(B) |\ \exists\ \eta \in ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(\partial B) :\ c^*_{\partial E}\omega = \pi^*_{\partial E}\eta\}$$

and

$$P^{\bullet}_{< K}(B) := \{\omega \in ft_{< K}\Omega^{\bullet}_{\mathcal{MS}}(B) |\ \exists\ \eta \in ft_{< K}\Omega^{\bullet}_{\mathcal{MS}}(\partial B) :\ c^*_{\partial E}\omega = \pi^*_{\partial E}\eta\}.$$

We want to show that those complexes are quasi-isomorphic to the analogous complexes without the condition at the end of the manifold. The final statement can be found in Lemma 7.1.4. Compare that result to [Ban11, Prop. 2.5]. As before, let $c_{\partial E}\big(\partial E \times [0,1)\big) = C_{\partial E} \cong \partial E \times [0,1)$ be the collar neighbourhood of ∂E in E together with the collar map

$$c_{\partial E} : \partial E \times [0,1) \hookrightarrow E.$$

Lemma 7.1.2 *Let $X^{\bullet}_{\mathcal{MS}} = c^*_{\partial E}(\Omega^{\bullet}_{\mathcal{MS}}(B))$, $\pi_{\partial E} : \partial E \times [0,1) \to \partial E$ denote the collar projection and $\sigma_{\partial E} : \partial E \hookrightarrow \partial E \times [0,1)$ the inclusion at $\frac{1}{2} \in [0,1)$. Then the map $\pi^*_{\partial E}$ restricts to a map $\Omega^{\bullet}_{\mathcal{MS}}(\partial B) \to X^{\bullet}_{\mathcal{MS}}$ and*

$$X^{\bullet}_{\mathcal{MS}} \underset{\pi^*_{\partial E}}{\overset{\sigma^*_{\partial E}}{\rightleftarrows}} \Omega^{\bullet}_{\mathcal{MS}}(\partial B)$$

are mutually homotopy inverse chain homotopy equivalences.

<u>Proof:</u> Since $\pi_{\partial E} \circ \sigma_{\partial E} = \mathrm{id}_{\partial E}$ and therefore $\sigma^*_{\partial E} \circ \pi^*_{\partial E} = \mathrm{id}^*_{\partial E}$, we only have to show that $\pi^*_{\partial E} \circ \sigma^*_{\partial E} \simeq \mathrm{id}_{C^*_{\partial E}}$. We consider the homotopy operator

$$\begin{aligned} K : \Omega^{\bullet}(\partial E \times [0,1)) &\to \Omega^{\bullet -1}(\partial E \times [0,1)),\\ f(x,t)\ \pi^*_{\partial E}\rho &\mapsto 0,\\ f(x,t)dt \wedge \pi^*_{\partial E}\rho &\mapsto g(x,t)\ \pi^*_{\partial E}\rho, \text{ with } g(x,t) = \int_{1/2}^{t} f(x,\tau)d\tau, \end{aligned}$$

and claim that it restricts to $K : X^{\bullet}_{\mathcal{MS}} \to X^{\bullet-1}_{\mathcal{MS}}$:
We first show that for a multiplicatively structured form ω we have that $K(\omega)$ is also multiplicatively structured. For $\omega = f(x,t)\pi^*_{\partial E}\rho$ this is obvious, since 0 is multiplicatively structured. So let us consider $K(f(x,t)dt\wedge\pi^*_{\partial E}\rho)$: If $\omega = f(x,t)dt\wedge\pi^*_{\partial E}\rho$ is multiplicatively structured, then so is $\pi^*_{\partial E}\rho$ and that means that for each coordinate chart U_α, where the flat bundle trivializes, we have that:

$$\rho|_{p|^{-1}(U_\alpha\cap\partial B)} = \phi^*_\alpha \sum_j \pi_1^*\eta_j \wedge \pi_2^*\gamma_j$$

with $\eta_j \in \Omega^\bullet(U_\alpha \cap \partial B)$ and $\gamma_j \in \Omega^\bullet(L)$. Denote by y the coordinates on L and by z the coordinates on ∂B. Since ω is multiplicatively structured, without loss of generality we can assume that $f(x,t)$ is independent of y: The compatible collars (c_E, c_B) allow us to write

$$f(x,t) = f(\phi_\alpha^{-1}(y,z),t) = g(y)\ h(z,t),$$

where the last equality is implied by the multiplicative structure of ω. But then also

$$\int_{1/2}^t f(x,\tau)d\tau = g(y)\int_{1/2}^t h(z,\tau)d\tau$$

and we can write

$$K(\omega)|_{p|^{-1}(U_\alpha\cap\partial B)} = \phi^*_\alpha\sum_j \pi_1^*(\int_{1/2}^t h(\cdot,\tau)d\tau\ \eta_j)\wedge\pi_2^*(g\cdot\gamma_j).$$

Therefore, $K(\omega)$ is multiplicatively structured.

We now show that $K(X^{\bullet}_{\mathcal{MS}}) \subset X^{\bullet-1}_{\mathcal{MS}}$:
Let $c_{\partial E-2} : \partial E \times [0,2) \hookrightarrow E$ be a slightly larger collar with $c_{\partial E-2}|_{\partial E\times[0,1)} = c_{\partial E}$ and define

$$\begin{aligned} K_2 : \Omega^\bullet(\partial E\times[0,2)) &\to \Omega^{\bullet-1}(\partial E\times[0,2)) \\ f(x,t)\pi^*_{\partial E}\rho &\mapsto 0 \\ f(x,t)dt\wedge\pi^*_{\partial E}\rho &\mapsto \pi^*_{\partial E}\rho \int_{1/2}^t f(x,\tau)d\tau. \end{aligned}$$

Let $j_2 : \partial E\times[0,1) \hookrightarrow \partial E\times[0,2)$ be the inclusion. Then we have for $\omega_1 = f(x,t)\pi^*_{\partial E}\rho \in \Omega^p(\partial E\times[0,2))$, $\omega_2 = g(x,t)dt\wedge\pi^*_{\partial E}\rho \in \Omega^p(\partial E\times[0,2))$:

$$\begin{aligned} K(j_2^*\omega_1) &= K((f\circ j_2)\wedge j_2^*\pi^*_{\partial E}\rho) = 0 = j_2^*(K_2\omega_1), \\ K(j_2^*\omega_2) &= K((g\circ j_2)dt\wedge j_2^*\pi^*_{\partial E}\rho) = \big(\int_{1/2}^t g(x,\tau)d\tau\big)\ j_2^*\pi^*_{\partial E}\rho = j_2^*(K_2\omega_2). \end{aligned}$$

If we restrict to multiplicatively structured forms, we thereby get the following commutative diagram:

$$\begin{tikzcd}
\Omega^\bullet_{\mathcal{MS}}(\partial E \times [0,2)) \arrow[r, "K_2"] \arrow[d, "j_2^*"] & \Omega^{\bullet-1}_{\mathcal{MS}}(\partial E \times [0,2)) \arrow[d, "j_2^*"] \\
\Omega^\bullet_{\mathcal{MS}}(\partial E \times [0,1)) \arrow[r, "K"] & \Omega^{\bullet-1}_{\mathcal{MS}}(\partial E \times [0,1))
\end{tikzcd}$$

So let $\omega \in X^r_{\mathcal{MS}}$. Then by definition of $X^\bullet_{\mathcal{MS}}$ there is a $\eta \in \Omega^r_{\mathcal{MS}}(B)$ such that $c^*_{\partial E}\eta = \omega$ and hence there is $\xi := c^*_{\partial E-2}\eta \in \Omega^r_{\mathcal{MS}}(\partial E \times [0,2))$ with $j_2^*\xi = \omega$. Now for a smooth bump function ψ on $[0,2)$ with $\psi|_{[0,1)} = 1$ and $\psi|_{(3/2,2)} = 0$ and for $\overline{\pi} : \partial E \times [0,2) \to [0,2)$ the projection,we have that

$$j_2^*\big(\overline{\pi}^*(\psi)\ K_2(\xi)\big) = j_2^*\big(K_2(\xi)\big)$$

and hence for $\overline{\eta} \in \Omega^{r-1}_{\mathcal{MS}}(B)$ the extension of $\overline{\pi}^*(\psi)\ K_2(\xi)$ by zero to the whole of E we have:

$$\begin{aligned} c^*_{\partial E}\overline{\eta} &= j_2^*\ c^*_{\partial E-2}\overline{\eta} = j_2^*\big(\overline{\pi}^*(\psi)K_2(\xi)\big) \\ &= j_2^*K_2(\xi) = Kj_2^*(\xi) = K\omega. \end{aligned}$$

Hence the homotopy operator K restricts to a map

$$K : X^\bullet_{\mathcal{MS}} \to X^{\bullet-1}_{\mathcal{MS}}.$$

By standard computations $dK + Kd = \mathrm{id} - \pi^*_{\partial E} \circ \sigma^*_{\partial E}$ and hence the lemma is established.

□

Lemma 7.1.3 *The statement of Lemma 7.1.2 is also true for the complexes of fiberwisely truncated and cotruncated forms*

$$ft_{<K}X^\bullet_{\mathcal{MS}} := c^*_{\partial E}(ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B)) \quad \textit{and} \quad ft_{\geq K}X^\bullet_{\mathcal{MS}} := c^*_{\partial E}(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)),$$

i.e. the maps

$$ft_{<K}X^\bullet_{\mathcal{MS}} \underset{\pi^*_{\partial E}}{\overset{\sigma^*_{\partial E}}{\rightleftarrows}} ft_{<K}\Omega^\bullet_{\mathcal{MS}}(\partial B),$$

$$ft_{\geq K}X^\bullet_{\mathcal{MS}} \underset{\pi^*_{\partial E}}{\overset{\sigma^*_{\partial E}}{\rightleftarrows}} ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(\partial B)$$

are mutually homotopy inverse chain homotopy equivalences.

<u>Proof:</u> The proof is literally the same as the proof of Lemma 7.1.2. This works out since there we do not change anything in the link direction. □

Lemma 7.1.4 *The subcomplex inclusions $i : P^\bullet(B) \hookrightarrow \Omega^\bullet_{\mathcal{MS}}(B)$, $i_{\geq K}$: $P^\bullet_{\geq K}(B) \hookrightarrow ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)$, $i_{<K}$: $P^\bullet_{<K}(B) \hookrightarrow ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B)$ are quasi-isomorphisms.*

Proof: We will give a proof for the non-truncated case that transfers literally to the truncated and cotruncated one.
Set
$$\Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B}) := \{\omega \in \Omega^\bullet_{\mathcal{MS}}(B) | c^*_{\partial E}\omega = 0\}.$$
Then there is a short exact sequence
$$0 \longrightarrow \Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B}) \longrightarrow \Omega^\bullet_{\mathcal{MS}}(B) \xrightarrow{c^*_{\partial E}} X^\bullet_{\mathcal{MS}} \longrightarrow 0.$$
Notice that the maps
$$\pi^*_{\partial E}\Omega^\bullet_{\mathcal{MS}}(\partial B) \underset{\pi^*_{\partial E}}{\overset{\sigma^*_{\partial E}}{\rightleftarrows}} \Omega^\bullet_{\mathcal{MS}}(\partial B)$$
are mutually inverse cochain complex isomorphisms. Further any form $\pi^*_{\partial E}\eta \in \pi^*_{\partial E}\Omega^\bullet_{\mathcal{MS}}(\partial B)$ can be extended to a form in $P^\bullet(B) \subset \Omega^\bullet_{\mathcal{MS}}(B)$ by taking the extension by zero of the form
$$\overline{\pi}^*(\psi)\ \pi^*_{-2}(\eta),$$
with $\psi \in C^\infty_c([0,2))$ the same bump function as in the proof of Lemma 7.1.2. This fact implies that

1. The inclusion $\iota : \pi^*_{\partial E}\Omega^\bullet_{\mathcal{MS}}(\partial B) \hookrightarrow X^\bullet_{\mathcal{MS}}$ is a homotopy equivalence, since for $\pi^*_{\partial E}\eta \in \pi^*_{\partial E}\Omega^\bullet_{\mathcal{MS}}(\partial B)$ holds:
$$\pi^*_{\partial E}\sigma^*_{\partial E}(\pi^*_{\partial E}\eta) = \pi^*_{\partial E}(\sigma^*_{\partial E}\pi^*_{\partial E}\eta) = \pi^*_{\partial E}\eta = \iota(\pi^*_{\partial E}\eta)$$
and hence $\iota = \pi^*_{\partial E} \circ \sigma^*_{\partial E}$, which is the composition of an isomorphism with a homotopy equivalence and therefore also a homotopy equivalence.

2. The restriction $c^*_{\partial E}| : P^\bullet(B) \to \pi^*_{\partial E}\Omega^\bullet_{\mathcal{MS}}(\partial B)$ is surjective (with kernel also $= \Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B})$) and hence there is a short exact sequence
$$0 \longrightarrow \Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B}) \longrightarrow P^\bullet(B) \xrightarrow{c^*_{\partial E}} \pi^*_{\partial E}(\Omega^\bullet_{\mathcal{MS}}(\partial B)) \longrightarrow 0.$$

The subcomplex inclusions $P^\bullet(B) \hookrightarrow \Omega^\bullet_{\mathcal{MS}}(B)$ and $\iota : \pi^*_{\partial E}\Omega^\bullet_{\mathcal{MS}}(\partial B) \hookrightarrow X^\bullet_{\mathcal{MS}}$ yield the following commutative diagram:
$$\begin{tikzcd}
0 \arrow[r] & \Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B}) \arrow[r] & \Omega^\bullet_{\mathcal{MS}}(B) \arrow[r, "c^*_{\partial E}"] & X^\bullet_{\mathcal{MS}} \arrow[r] & 0 \\
0 \arrow[r] & \Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B}) \arrow[r] \arrow[u, equal] & P^\bullet(B) \arrow[r, "c^*_{\partial E}"] \arrow[u, hook, "i"] & \pi^*_{\partial E}\Omega^\bullet_{\mathcal{MS}}(\partial B) \arrow[r] \arrow[u, hook, "\iota"] & 0
\end{tikzcd}$$

Now the fact that ι is a homotopy equivalence implies, together with the 5–Lemma, that $i : P^\bullet(B) \to \Omega^\bullet_{\mathcal{MS}}(B)$ is a quasi-isomorphism. □

7.2 Two Distinguished Triangles for $\widetilde{\Omega I}^\bullet_{\bar{p}}$

To prove Theorem 7.5.5 we need some additional lemmata. We want to use a five lemma argument and therefore need two distinguished triangles in $\mathcal{D}(\mathbb{R})$, the derived category over the reals.

Definition 7.2.1 *(Forms that are multiplicative near E)*

$$\Omega^r_{E\mathcal{MS}}(M) := \left\{\omega \in \Omega^r(M) \middle|\ \exists\, \eta \in \Omega^\bullet_{\mathcal{MS}}(B) :\ c_E^*\omega = \pi_E^*\eta\right\}.$$

Lemma 7.2.2 *In $\mathcal{D}(\mathbb{R})$, the derived category of complexes of real vector spaces, there is a distinguished triangle*

$$\begin{tikzcd}
\widetilde{\Omega I}^\bullet_{\bar{p}}(M) \arrow[rr] & & \Omega^\bullet_{E\mathcal{MS}}(M) \arrow[dl] \\
& ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B) \arrow[ul, "+1"] &
\end{tikzcd} \tag{8}$$

Proof: There is a short exact sequence

$$0 \to \widetilde{\Omega I}^\bullet_{\bar{p}}(M) \to \Omega^\bullet_{E\mathcal{MS}}(M) \to Q^\bullet_E(M) := \frac{\Omega^\bullet_{E\mathcal{MS}}(M)}{\widetilde{\Omega I}^\bullet_{\bar{p}}(M)} \to 0 \tag{9}$$

We have to show that there is a quasi-isomorphism $Q^\bullet_E(M) \to ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B)$. Let $\sigma_E : E \hookrightarrow E \times [0,1)$ be the inclusion at 0. Then the map $J_E := c_E \circ \sigma_E$ induces maps

$$\begin{aligned}
J_E^* &: \Omega^\bullet_{E\mathcal{MS}}(M) \overset{c_E^*}{\twoheadrightarrow} \Omega^\bullet_{E\mathcal{MS}}(E \times [0,1)) \xrightarrow[\cong]{\sigma_E^*} \Omega^\bullet_{\mathcal{MS}}(B) \\
\widetilde{J_E}^* &: \widetilde{\Omega I}^\bullet_{\bar{p}}(M) \overset{c_E^*}{\twoheadrightarrow} \widetilde{\Omega I}^\bullet(E \times [0,1)) \xrightarrow[\cong]{\sigma_E^*} P^\bullet_{\geq K}(B) \\
\overline{J_E}^* &: Q^\bullet_E(M) \overset{\overline{c_E}^*}{\twoheadrightarrow} Q^\bullet_E(E \times [0,1)) \xrightarrow[\cong]{\overline{\sigma_E}^*} Q^\bullet(B) := \frac{\Omega^\bullet_{\mathcal{MS}}(B)}{P^\bullet_{\geq K}(B)}.
\end{aligned}$$

The induced maps J_E^* and $\widetilde{J_E}^*$ are surjective by the standard argument of enlarging the collar and using a bump function (compare e.g. [Ban11, Prop. 6.3]). By a 3×3-lemma argument the map $\overline{J_E}^* : Q^\bullet_E(M) \to Q^\bullet(B)$ is an isomorphism (see [Ban11, p.44]). By Lemma 7.1.4, subcomplex inclusion induces a quasi-isomorphism

$$\bar{i} : Q^\bullet(B) \xrightarrow{qis} \frac{\Omega^\bullet_{\mathcal{MS}}(B)}{ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)}$$

and since we work with a flat bundle E over B, there is a quasi-isomorphism

$$\gamma_B : ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B) \to \frac{\Omega^\bullet_{\mathcal{MS}}(B)}{ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)} \tag{10}$$

by [Ban11, Lemma 6.6]. All in all we get a fraction of quasi-isomorphisms

$$\begin{tikzcd}[column sep=small] & \frac{\Omega^\bullet_{\mathcal{MS}}(B)}{ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)} & \\ Q_E(M) \arrow[ur, "\bar{i}\circ J_E^*", "qis"'] & & ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B) \arrow[ul, "\gamma_B"', "qis"] \end{tikzcd}$$

in the derived category $\mathcal{D}(\mathbb{R})$ which allows us to replace $Q^\bullet_E(M)$ in (9) to get the desired distinguished triangle in $\mathcal{D}(\mathbb{R})$.

$\square$

What we want to have is a second distinguished triangle in $\mathcal{D}(\mathbb{R})$ such that together with the distinguished triangle (8) both give a commutative diagramm on cohomology that enables us to prove the Poincaré Duality statement, Theorem 7.5.5.

Definition 7.2.3 *(Relative de Rham complexes)*
As before let $C_E = im\, c_E$, $C_W = im\, c_W$ and set $C := C_E \cup C_W$. We define

$$\Omega^\bullet_{rel}(M) := \{\omega \in \Omega^\bullet(M) | \omega|_C = 0\}$$
$$\widetilde{\Omega I}^\bullet_{\bar{p}}(M, C_W) := \{\omega \in \widetilde{\Omega I}^\bullet_{\bar{p}}(M) | w|_{C_W} = 0\}$$
$$ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B}) := \{\omega \in ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B) | c^*_{\partial E}\omega = 0\}.$$

Remark 7.2.4 *Note that since we work with p-related collars $C_{\partial E}$, $C_{\partial B}$ on E and B, we can rewrite $ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B})$: For each coordinate chart $U \subset B$ with respect to which the bundle trivializes we have*

$$\phi_U(C_{\partial E} \cap U) = (C_{\partial B} \cap U) \times L.$$

Hence we have for $\omega \in ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B})$ and each coordinate chart $U \subset B$:

$$0 = (c^*_{\partial E}\omega)|_{p^{-1}(U)} = \phi^*_U \sum_j \pi^*_1 c^*_{\partial B}\eta_j \wedge \pi^*_2\gamma_j,$$

implying $c_{\partial B}^* \eta_j = 0$ *for all* j. *To see this, we use that* U *is a coordinate chart. Therefore, we can write*

$$\sum_j \pi_1^* \eta_j \wedge \pi_2^* \gamma_j = \sum_I \sum_{j=1}^{k_I} f_j^I dx^I \wedge \gamma_j^I,$$

where we sum over all multi-indizes I. *We then can treat each multi-index* I *seperately. Assume that there is an* $j_0 \in \{1, ..., k_I\}$ *and an* $x \in C_{\partial B}$ *such that* $f_{j_0}^I(x) \neq 0$. *Contracting with* ∂_x^I *and evaluating at* x, *this gives:*

$$\gamma_{j_0}^I = -\sum_{j \neq j_0} \frac{f_j^I(x)}{f_{j_0}^I(x)} \gamma_j^I.$$

Therefore we can write

$$\sum_{j=1}^{k_I} f_j^I dx^I \wedge \gamma_j^I = \sum_{j \neq j_0} (f_j^I - \underbrace{\frac{f_j^I(x)}{f_{j_0}^I(x)}}_{=:c_j^I} f_{j_0}^I) dx^I \wedge \gamma_j^I.$$

If these new coefficient functions $f_j^I - c_j^I f_{j_0}^I$ *vanish on* $C_{\partial B}$ *we are done. Otherwise, repeat this process inductively to reduce the above sum to just one summand* $\widetilde{f}^I dx^I \wedge \gamma^I$, *for some* γ^I, *which still must equal the sum we started with and is thereby zero on* $C_{\partial B} \times L$. *Then either* $\gamma^I = 0$ *or* $\widetilde{f}^I|_{C_{\partial B}} = 0$. *The result of this discussion enables us to write*

$$\begin{aligned} ft_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B}) = \big\{ \omega \in & ft_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B}) | \omega|_{p^{-1}(U)} = \sum_j \pi_1^* \eta_j \wedge \pi_2^* \gamma_j \\ & \text{with } \eta_j \in \Omega^\bullet(U, U \cap C_{\partial B}),\ \gamma_j \in \tau_{\geq K} \Omega^\bullet(L) \big\}. \end{aligned}$$

Remark 7.2.5 *Later, in Section 11, we will see, that the cohomology groups of the above defined complexes do not depend of the choice of a pair of p-related collars.*

The next lemma provides the second distinguished triangle:

Lemma 7.2.6 *There is a second distinguished triangle*

$$\begin{tikzcd} \Omega^\bullet_{rel}(M) \arrow[rr, "incl"] & & \widetilde{\Omega I}^\bullet_{\bar{p}}(M, C_W) \arrow[dl, "\widetilde{J_E}^*"] \\ & ft_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B}) \arrow[ul, "+1"] & \end{tikzcd} \tag{11}$$

Proof: The map $\widetilde{J_E}^* : \widetilde{\Omega I}^\bullet_{\bar{p}}(M, C_W) \to ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B})$ is surjective by the same arguments we gave in previous proofs. The kernel of $\widetilde{J_E}^*$ are those forms $\omega \in \widetilde{\Omega I}^\bullet_{\bar{p}}(M, C_W)$ with

$$\omega|_{C_E} = 0 \Rightarrow \omega|_C = 0$$

and hence $\ker \widetilde{J_E}^* = \Omega^\bullet_{rel}(M)$ and we therefore have a commutative diagram

$$0 \to \Omega^\bullet_{rel}(M) \longrightarrow \widetilde{\Omega I}^\bullet_{\bar{p}}(M, C_W) \xrightarrow{\widetilde{J_E}^*} ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B}) \to 0$$

and in particular the distinguished triangle (11).

□

7.3 Poincaré Duality for Fiberwisely (Co)truncated Forms

The next step is to prove the following proposition:

Proposition 7.3.1 *For any $r \in \mathbb{Z}$, integration induces a nondegenerate bilinear form*

$$\int : H^{r-1}(ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B)) \times H^{n-r}(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B})) \to \mathbb{R},$$

$$([\omega], [\eta]) \mapsto \int_E \omega \wedge \eta.$$

For being able to prove the above Proposition 7.3.1, we need two Poincaré Lemmata and a Bootstrap Principle:

Lemma 7.3.2 *(Poincaré Lemma for fiberwisely truncated forms)*
Let $U \subset B$ be a chart intersecting the boundary, i.e. there is a diffeomorphism $\psi : U \xrightarrow{\cong} V := \mathbb{R}^n_+$, such that the bundle $p : E \to B$ trivializes over U, i.e. there is a diffeomorphism $\phi_U : p^{-1}(U) \xrightarrow{\cong} U \times L$ with $p = \pi_1 \circ \phi_U$. Let further denote $\pi_2 : U \times L \to L$ the second factor projection and $S_x : L \xrightarrow{at\ x} U \times L$ the inclusion at $x \in U - (\partial B \cap U)$. Then the induced maps

$$ft_{<K}\Omega^\bullet_{\mathcal{MS}}(U) \underset{(\pi_2 \circ \phi_U)^*}{\overset{S_x^*}{\rightleftarrows}} \tau_{<K}\Omega^\bullet(L)$$

are chain homotopy inverses of each other. In particular both are homotopy equivalences.

Proof: The standard approach to prove such Poincaré Lemmata works also in this special case:
We will use an induction on n, the dimension of B. The induction basis is

obvious, since for $n = 0$ both S_x and π_2 are the identity. For $n > 0$ we assume without loss of generality that $\psi(x) = e_1 = (1, 0) \in [0, \infty) \times \mathbb{R}^{n-1} = \mathbb{R}^n_+$ and factor S_x as

$$\begin{tikzcd}
 & & & U \times L \arrow[d, "\cong"', "\psi"] \\
L = \mathbb{R}^0 \times L \arrow[r, hook, "S", "at1"'] \arrow[rrru, bend left, "S_x"] & \mathbb{R}^1_+ \times L \arrow[r, hook, "S"] & ... \arrow[r, hook, "S"] & \mathbb{R}^n_+ \times L
\end{tikzcd}$$

and π_2 as

$$\begin{tikzcd}
U \times L \arrow[d, "\cong"', "\psi"] \arrow[rrr, bend left, two heads, "\pi_2"] & & & \\
\mathbb{R}^n_+ \times L \arrow[r, two heads, "Q"] & \mathbb{R}^{n-1}_+ \times L \arrow[r, two heads, "Q"] & ... \arrow[r, two heads, "Q"] & \mathbb{R}^0 \times L = L.
\end{tikzcd}$$

We now show that for all $1 \le k \le n$ the maps $S : \mathbb{R}^{k-1}_+ \times L \hookrightarrow \mathbb{R}^k_+ \times L$ and $Q : \mathbb{R}^k_+ \times L \to \mathbb{R}^{k-1}_+ \times L$ induce mutually homotopy inverse homotopy equivalences

$$ft_{<K}\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^k_+) \underset{Q^*}{\overset{S^*}{\rightleftarrows}} ft_{<K}\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^{k-1}_+).$$

By the above factorization the claim of the lemma is then established by the principle of induction.

<u>First Case:</u> $k = 1$

This case is distinct from the other ones because here we embed $L \overset{S}{\hookrightarrow} \mathbb{R}^1_+ \times L$ at $1 \in \operatorname{int}(\mathbb{R}^1_+) = (0, \infty)$. We will construct a homotopy operator $K : ft_{<K}\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^1_+) \to ft_{<K}\Omega^{\bullet-1}_{\mathcal{MS}}(\mathbb{R}^1_+)$ such that

$$dK + Kd = \mathrm{id} - Q^* \circ S^*.$$

Let $\omega \in ft_{<K}\Omega^r_{\mathcal{MS}}(\mathbb{R}^1_+)$. Then

$$\omega = \sum_{j_0} \pi_1^* f_{j_0} \pi_2^* \gamma_{j_0} + \sum_{j_1} \pi_1^*(f_{j_1} dt) \wedge \pi_2^* \gamma_{j_1},$$

with $f_{j_0}, f_{j_1} \in C^\infty(\mathbb{R}^1_+)$, $\gamma_{j_0} \in \tau_{<K}\Omega^r(L)$ and $\gamma_{j_1} \in \tau_{<K}\Omega^{r-1}(L)$. We then define $K : ft_{<K}\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^1_+) \to ft_{<K}\Omega^{\bullet-1}_{\mathcal{MS}}(\mathbb{R}^1_+)$ by

$$K(\omega) := \sum_{j_1} \pi_1^*\Big(\int_1^t f_{j_1}(\tau) d\tau\Big) \pi_2^* \gamma_{j_1}.$$

The form $K(\omega)$ is obviously multiplicative and fiberwisely truncated. By the standard computation (compare to the Second Case below) we get that $dK + Kd = \mathrm{id} - Q^* \circ S^*$.

Second Case $k > 1$:
The second case is slightly more complicated but works in principle in the same way with the difference that now $S : \mathbb{R}_+^{k-1} \xhookrightarrow{at\ 0} \mathbb{R} \times \mathbb{R}_+^{k-1} = \mathbb{R}_+^k$ (Remark: Note that the distinguished factor $\mathbb{R}$ is described by the first coordinate): Again we want to construct a homotopy operator $K : ft_{<K}\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}_+^k =: V) \to ft_{<K}\Omega^{\bullet-1}_{\mathcal{MS}}(V)$ such that

$$dK + Kd = \mathrm{id} - Q^* \circ S^*.$$

For $\omega \in ft_{<K}\Omega^r_{\mathcal{MS}}(V)$ we have

$$\omega = \sum_j \pi_1^* \eta_j \wedge \pi_2^* \gamma_j,$$

where $\eta_j \in \Omega^\bullet(V)$ and $\gamma_j \in \tau_{<K}\Omega^\bullet(L)$.
In $\Omega^\bullet(V)$ there are two types of forms:

1. $f(t,x)Q^*\alpha$, with $\alpha \in \Omega^\bullet(\mathbb{R}_+^{k-1})$,
2. $f(t,x)dt \wedge Q^*\alpha$, α as above.

Define $K_V : \Omega^\bullet(V) \to \Omega^{\bullet-1}(V)$ by

$$K_V(f(t,x)Q^*\alpha) = 0$$
$$K_V(f(t,,x)dt \wedge Q^*\alpha)(t,x) = \Big(\int_0^t f(x^0,x)dx^0\Big)Q^*\alpha.$$

K_V induces a map

$$K : ft_{<K}\Omega^\bullet_{\mathcal{MS}}(U) \to ft_{<K}\Omega^{\bullet-1}_{\mathcal{MS}}(U),$$
$$K(\pi_1^*(f(t,x)Q^*\alpha) \wedge \pi_2^*\gamma) = 0,$$
$$K(\pi_1^*(f(t,x)dt \wedge Q^*\alpha) \wedge \pi_2^*\gamma) = \pi_1^*\Big(\Big(\int_0^t f(x^0,x)dx^0\Big)Q^*\alpha\Big) \wedge \pi_2^*\gamma.$$

Let us compute $(dK + Kd)\omega$ for $\omega \in ft_{<K}\Omega^\bullet_{\mathcal{MS}}(U)$:
Define $\omega_a := \pi_1^*(f(t,x)Q^*\alpha) \wedge \pi_2^*\gamma$ and $\omega_b := \pi_1^*(f(t,x)dt \wedge Q^*\alpha) \wedge \pi_2^*\gamma$.

$$\begin{aligned}
(dK + Kd)\omega_a &= Kd\omega_a \\
&= K\Big[\pi_1^*\big\{(\partial_t f)dt \wedge Q^*\alpha \\
&\quad + \sum_{i=1}^{n-1}(\partial_i f)dx^i \wedge Q^*\alpha + f(t,x)Q_*d\alpha\big\} \wedge \pi_2^*\gamma \\
&\quad \pm \pi_1^*(f(t,x)Q^*\alpha) \wedge \pi_2^* d\gamma\Big] \\
&= \pi_1^*\big[f(t,x) - f(0,x)\big]Q^*\alpha \wedge \pi_2^*\gamma \\
&= (\mathrm{id} - Q^* \circ S^*)\omega_a.
\end{aligned}$$

$$
\begin{aligned}
dK\omega_b &= d\big(\pi_1^*(\int_0^t f(x^0,x)dx^0)Q^*\alpha \wedge \pi_2^*\gamma\big) \\
&= \pi_1^*\big(f(t,x)dt \wedge Q^*\alpha + \sum_{i=1}^{n-1}(\int_0^t \partial_i f(t,x)dt)dx^i \wedge Q^*\alpha \\
&\quad + (\int_0^t f(x^0,x)dx^0)Q^*d\alpha\big) \wedge \pi_2^*\gamma \\
&\quad + (-1)^{\deg(\alpha)}\pi_1^*(\int_0^t f(t,x)dt)Q^*\alpha \wedge \pi_2^*d\gamma \\
Kd\omega_b &= K\Big[\pi_1^*\big\{\sum_{i=1}^{n-1}\partial_i f\ dx^i \wedge dt \wedge Q^*\alpha - f dt \wedge Q^*d\alpha\big\} \wedge \pi_2^*\gamma \\
&\quad - (-1)^{\deg(\alpha)}\pi_1^*(f(t,x)dt \wedge Q*\alpha) \wedge \pi_2^*d\gamma\Big] \\
&= -\pi_1^*\big\{\sum_{i=1}^{n-1}(\int_0^t \partial_i f dx^0)dx^i \wedge Q^*\alpha + (\int_0^t f dx^0) \wedge Q^*d\alpha\big\} \wedge \pi_2^*\gamma \\
&\quad - (-1)^{\deg(\alpha)}\pi_1^*\big((\int_0^t f dx^0)Q^*\alpha\big) \wedge \pi_2^*d\gamma \\
&\Rightarrow\ (dK + Kd)\omega_b = \omega_b = (\mathrm{id} - Q^* \circ S^*)\omega_b
\end{aligned}
$$

□

Remark: For charts $\mathbb{R}^n \cong U \subset B$, with respect to which the bundle trivializes, the statement of the above lemma is also true by [Ban11, Lemma 5.1].

Definition 7.3.3 *For any open subset $U \subset B$ we define*

$$\Omega^\bullet_{\mathcal{MS}}(U, U \cap C_{\partial B}) := \big\{\omega \in \Omega^\bullet_{\mathcal{MS}}(U) \big| \omega|_{\underbrace{p^{-1}(U \cap C_{\partial B})}_{=p^{-1}(U)\cap C_{\partial E}}} = 0\big\},$$

$$\Omega^\bullet_{\mathcal{MS},c}(U, U \cap C_{\partial B}) := \big\{\omega \in \Omega^\bullet_{\mathcal{MS},c}(U) \big| \omega|_{\underbrace{p^{-1}(U \cap C_{\partial B})}_{=p^{-1}(U)\cap C_{\partial E}}} = 0\big\}$$

Analogously, we define the fiberwisely truncated and cotruncated subcomplexes.

In the following lemma we give the induction start for an inductive Mayer Vietoris argument, which makes use of the fact that the collar we work with is small with respect to the chosen good open cover $\mathcal{U}$ (compare to 4.2.2).

Lemma 7.3.4 *(Poincaré Lemma for relative forms with compact supports) Let $U \in \mathcal{U}$ be an open chart (with respect to which the bundle trivializes, i.e. there is a diffeomorphism $\phi_U : p^{-1}(U) \to U \times L$ with $p|_{p^{-1}(U)} = \pi_1 \circ \phi_U$). Then in particular there is a diffeomorphism $\psi : U \overset{\cong}{\to} V$ with $V = \mathbb{R}^n_+$ or $V = \mathbb{R}^n$ and, by Lemma 4.2.2, U is <u>not</u> completely contained in the collar neighbourhood $C_{\partial B} \supset \partial B$ of the boundary of B. Then there is a form $e \in \Omega^n_c(U, U \cap C_{\partial B}) = \{\omega \in \Omega^n_c(U) \mid \omega|_{U \cap C_{\partial B}} = 0\}$ such that the maps*

$$ft_{\geq K}\Omega^\bullet_{\mathcal{MS},c}(U, U \cap C_{\partial B}) \underset{e_*}{\overset{(\pi_2)_* \circ (\phi_U^{-1})^*}{\rightleftarrows}} \tau_{\geq K}\Omega^{\bullet - n}(L),$$

where

$$\pi_{2*}(\pi_1^*\eta \wedge \pi_2^*\gamma) = \begin{cases} \left(\int_U \eta\right) \gamma & \text{if } \eta \in \Omega^n_c(U, U \cap C_{\partial B}), \\ 0 & \text{else,} \end{cases}$$

and

$$e_*(\gamma) = \phi_U^*(e \wedge \pi_2^*\gamma), \tag{12}$$

are chain homotopy inverses of each other and in particular are both chain homotopy equivalences.

<u>Proof:</u> *First step:* (Definition of the form e)
Independent of U being diffeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$ we can assume that $\psi(U) = V \subset \mathbb{R}^n$ is arranged in such a way that for, say the x^0 component of elements $x \in V$ large enough, $x^0 > s$, one has $x \notin \psi(C_{\partial B} \cap U)$ (for $V = \mathbb{R}^n_+$, x^0 is also a component such that $\partial\mathbb{R}^n_+ = \{x^0 = 0\}$). We then take bump functions $\epsilon_i \in C^\infty_0(\mathbb{R})$ with $\int_R \epsilon_i = 1$ for $i \in \{0, ..., n-1\}$, such that in addition $\text{supp}(\epsilon_0) \subset (s, \infty)$. But then

$$e := \psi^* \left(\prod_{i=0}^{n-1} \epsilon_i\right) dx^0 \wedge ... \wedge dx^{n-1} \in \Omega^n_c(U, U \cap C_{\partial B}).$$

The map $e_* : \tau_{\geq K}\Omega^r(L) \to ft_{\geq K}\Omega^{r+n}_{\mathcal{MS},c}(U, U \cap C_{\partial B})$ is defined by relation (12) and by the definition of the form e it holds that $(\pi_2)_* \circ \phi_U^* \circ e_* = \text{id}$.

Second step: (Construction of the homotopy operator)
As in the proof of [Ban11, Lemma 5.5] and in the proof of the previous Lemma 7.3.2, we prove by induction on n that $e_* \circ (\pi_2)_* \circ \phi_U^* \simeq \text{id}$. In detail, we proceed as follows:

1. First we show that the maps

$$e_{0*} : ft_{\geq K}\Omega^{\bullet-1}_{\mathcal{MS},c}(\mathbb{R}^{n-1}) \to ft_{\geq K}\Omega^\bullet_{\mathcal{MS},c}(U, U \cap C_{\partial B})$$

defined by

$$e_{0*}(\pi_1^*\eta \wedge \pi_2^*\gamma) := \phi_U^*\left(\pi_1^*\psi^*(e_0 \wedge \pi^*\eta) \wedge \pi_2^*\gamma\right)$$

with $\pi : V \to \mathbb{R}^{n-1}$ the projection, and

$$\pi_* : ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(U, U \cap C_{\partial B}) \to ft_{\geq K}\Omega^{\bullet-1}_{\mathcal{MS},c}(\mathbb{R}^{n-1})$$

(integration along the first fiber coordinate) defined by

$$\pi_*\big(\phi_U^*(\pi_1^*\psi^*(\underbrace{f(x,t)du^J}_{\text{no } dt \text{ contained}}) \wedge \pi_2^*\gamma)\big) = 0$$

$$\pi_*\big(\phi_U^*(\pi_1^*\psi^*(g(x,t)dt \wedge du^J) \wedge \pi_2^*\gamma)\big) = \pi_1^* \int_{\mathbb{R}} g(x,t)dt\ du^J \wedge \pi_2^*\gamma$$

satisfy the relation $e_{0*} \circ \pi^* \simeq \mathrm{id}$. (Actually the two maps are again mutually inverse homotopy equivalences.)
The homotopy operator

$$K : ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS},c}(U, U \cap C_{\partial B}) \to ft_{\geq K}\Omega^{\bullet-1}_{\mathcal{MS},c}(U, U \cap C_{\partial B})$$

satisfying $dK + Kd = e_{0*} \circ \pi_*$ is defined by

$$K\big(\phi_U^*(\pi_1^*\psi^*(f(x,t)du^J) \wedge \pi_2^*\gamma)\big) = 0$$
$$K\big(\phi_U^*(\pi_1^*\psi^*(g(t,x)dt \wedge du^J) \wedge \pi_2^*\gamma)\big)$$
$$= \phi_U^*(\pi_1^*\psi^*(\int_{-\infty}^t g(\tau,x)d\tau - \int_{(\infty}^t e_0)\ \int_{\mathbb{R}} g(\tau,x)d\tau)du^J \wedge \pi_2^*\gamma),$$

as usual. Note that by our definition of e_0, K respects the vanishing condition. A standard calculation shows that $Kd + dK = e_{0*} \circ \pi_* - \mathrm{id}$.

2. The second step is to put together the first step with the result of [Ban11, Lemma 5.5]: The following diagram commutes

$$\begin{tikzcd}
ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS},c}(U, U \cap C_{\partial B}) \arrow[d, shift left, "\pi_*"] \arrow[dd, bend left=60, "\pi_{2*}\circ(\phi_U^{-1})^*"] \\
ft_{\geq K}\Omega^{\bullet-1}_{\mathcal{MS},c}(\mathbb{R}^{n-1}) \arrow[u, shift left, "e_{0*}"] \arrow[d, shift left, "\widetilde{\pi}_*"] \\
\tau_{\geq K}\Omega^{\bullet-n}(L) \arrow[u, shift left, "\widetilde{e}_*"] \arrow[uu, bend left=60, "e_*"]
\end{tikzcd}$$

Note that $\widetilde{e}_*$ and $\widetilde{\pi}_*$ denote the mutually inverse homotopy equivalences of [Ban11, Lemma 5.5]. The commutativity of this diagram then implies the statement of the lemma: Since $e_* = \widetilde{e}_* \circ e_{0*}$ and $\pi_{2*} = \pi_* \circ \widetilde{\pi}_*$ are the composition of mutually inverse homotopy equivalences, they are also mutually inverse homotopy equivalences. $\square$

To use a Mayer–Vietoris type argument we need a bootstrap principle. The following lemma will provide one in our case:

Lemma 7.3.5 *(Bootstrap principle)*
Let $U, V \subset B$ be open sets and let $b := \dim B$, $m = \dim L$. Then if

$$\int : H^r\big(ft_{<K}\Omega^\bullet_{\mathcal{MS}}(Y)\big) \times H^{b+m-r}\big(ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS},c}(Y, Y\cap C_{dB})\big) \to \mathbb{R}$$

$$\big([\omega],[\eta]\big) \mapsto \int_{p^{-1}(Y)} \omega \wedge \eta$$

is nondegenerate for $Y = U, V, U\cap V$, so it is for $Y = U \cup V$.

Proof: We show that

1. There is a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & ft_{<K}\Omega^\bullet_{\mathcal{MS}}(U\cup V) \arrow[d] \\ & ft_{<K}\Omega^\bullet_{\mathcal{MS}}(U) \oplus ft_{<K}\Omega^\bullet_{\mathcal{MS}}(V) \arrow[d] \\ 0 & ft_{<K}\Omega^\bullet_{\mathcal{MS}}(U\cap V) \arrow[l] \end{tikzcd}$$

2. There is also a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS},c}(U\cap V, U\cap V\cap C_{\partial B}) \arrow[d] \\ & ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS},c}(U, U\cap C_{\partial B}) \oplus ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS},c}(V, V\cap C_{\partial B}) \arrow[d] \\ 0 & ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS},c}(U\cup V, (U\cup V)\cap C_{\partial B}) \arrow[l] \end{tikzcd}$$

The proof of the first statement just follows the argument of Banagl's proof in [Ban11, Lemma 5.10]. The fact that B is a compact manifold with boundary instead of a closed manifold does not give rise to any problems here. The proof of the second statement makes use of the following *Claim:* For $\omega \in ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS},c}(U, U\cap \partial B)$ and $f \in C^\infty(U)$ it holds that

$$p^*(f)\,\omega \in ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS},c}(U, U\cap C_{\partial B}).$$

Proof of the Claim: Since, by definition of a fiber bundle, for a coordinate chart U_α it holds that $\pi_1 \circ \phi_\alpha = p|_{U_\alpha}$, $p^*(f)\,\omega \in ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS},c}(U)$. Further

we have $\big(p^*(f)\ \omega\big)|_{C_{\partial E}} = 0$ since $\omega|_{C_{\partial E}} = 0$. Hence the claim is established and the argumentation of [Ban11, Lemma 5.10] is applicable.
The two short exact sequences induce long exact sequences on cohomology, which are dually paired by bilinear forms due to integration. By [Bot82, Lemma 5.6, p.45] the originated diagram commutes up to sign and hence the 5-Lemma implies the bootstrap principle.

□

Remark 7.3.6 *Note, that the compactness of B implies that*

$$ft_{\geq K}\Omega^\bullet_{\mathcal{MS},c}(B, C_{\partial B}) = ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B}).$$

The proof is literally the same as the proof of [Ban11, Lemma 5.11].

Together with the bootstrap principle of the above Lemma 7.3.5, we need an induction basis for being able to use the inductive Mayer-Vietoris argument.

Lemma 7.3.7 *(Local Poincaré Duality)*
For $U \in \mathcal{U}$ a coordinate chart, the bilinear form

$$\int : H^r\big(ft_{<K}\Omega^\bullet_{\mathcal{MS}}(U)\big) \times H^{b+m-r}\big(ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS},c}(U, U \cap C_{\partial B})\big) \to \mathbb{R},$$

where again $b = \dim B$, $m = \dim L$, is nondegenerate.

<u>Proof:</u> By the Lemmata 7.3.2 and 7.3.4, we have isomorphisms:

$$H^r\big(\tau_{<K}\Omega^\bullet(L)\big) \xrightarrow[\cong]{\phi_U^*\circ\pi_2^*} H^r\big(ft_{<K}\Omega^\bullet_{\mathcal{MS}}(U)\big)$$

$$H^{b+m-r}\big(ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS},c}(U, U \cap C_{\partial B})\big) \xrightarrow[\cong]{\pi_{2*}\circ(\phi_U^{-1})^*} H^{m-r}\big(\tau_{\geq K^*}\Omega^\bullet(L)\big).$$

Since the map

$$\int : H^r\big(\tau_{<K}\Omega^\bullet(L)\big) \to H^{m-r}\big(\tau_{\geq K^*}\Omega^\bullet(L)\big)^\dagger$$

$$[\omega] \mapsto \int_L \_ \wedge \omega$$

is also an isomorphism, proving the commutativity of the diagram

$$\begin{tikzcd}
H^r\big(ft_{<K}\Omega^\bullet_{\mathcal{MS}}(U)\big) \arrow[d, "\int"] & H^r\big(\tau_{<K}\Omega^\bullet(L)\big) \arrow[l, "\cong"', "(\phi\circ\pi_2)^*"] \arrow[d, "\int", "\cong"'] \\
H^{b+m-r}\big(ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS},c}(U, U \cap C_{\partial B})\big)^\dagger & H^{m-r}\big(\tau_{\geq K^*}\Omega^\bullet(L)\big)^\dagger \arrow[l, "\cong"', "(\pi_{2*}\circ\phi^{-1*})^\dagger"]
\end{tikzcd}$$

will prove the lemma.
Let $\gamma \in \tau_{<K}\Omega^r(L)$ and $\omega \in ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS},c}(U, U \cap C_{\partial B})$ closed. Then

$$\begin{aligned}\int_L \pi_{2*}\phi_U^{-1*}\omega \wedge \gamma &= \int_L \pi_{2*}\big(\sum_j \pi_1^*\eta_j \wedge \pi_2^*\gamma_j\big) \wedge \gamma \\ &= \int_L \pi_{2*}\big(\sum_j \pi_1^*\eta_j \wedge \pi_2^*(\gamma_j \wedge \gamma)\big) \\ &= \int_{U\times L} \big(\sum_j \pi_1^*\eta_j \wedge \pi_2^*\gamma_j\big) \wedge \pi_2^*\gamma \\ &= \int_{p^{-1}U} \phi_U^*\Big\{\big(\sum_j \pi_1^*\eta_j \wedge \pi_2^*\gamma_j\big) \wedge \pi_2^*\gamma\Big\} \\ &= \int_{p^{-1}U} \omega \wedge \phi_U^*\pi_2^*\gamma,\end{aligned}$$

where the fourth equality is due to the transformation law, see e.g. [Lee13, Proposition 10.20 (d)]. □

So now we have all the tools to establish the Poincaré Duality of Proposition 7.3.1:
Proof of Proposition 7.3.1: By Remark 7.3.6, the statement of the proposition is equivalent to the statement that integration induces a map

$$\int : H^r\big(ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B)\big) \times H^{b+m-r}\big(ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS},c}(B, C_{\partial B})\big) \to \mathbb{R}$$

that is nondegenerate for all r.
In fact, we prove that the bilinear map

$$\int : H^r\big(ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B)\big) \times H^{b+m-r}\big(ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS},c}(B, C_{\partial B})\big) \to \mathbb{R}$$

is nondegenerate for all r and all open subsets $U \subset B$ of the form

$$U = \bigcup_{i=1}^{s} U_{\alpha_0^i \dots \alpha_{p_i}^i}$$

with $s \leq |I|$ by an induction on s.
For $s = 1$ the statement was already proven in Lemma 7.3.7. Suppose the statement is true for all open subsets $\overline{U} \subset B$ of the form

$$\overline{U} = \bigcup_{i=1}^{s-1} U_{\alpha_0^i \dots \alpha_{p_i}^i}.$$

Let now

$$U = \bigcup_{i=1}^{s} U_{\alpha_0^i \dots \alpha_{p_i}^i}$$

and set $V = U_{\alpha_0^s \dots \alpha_{p_s}^s}$ and $\overline{U} = U - V$. Then the statement holds for V, $\overline{U}$, $\overline{U} \cap V$, since

$$\overline{U} = \bigcup_{i=1}^{s-1} U_{\alpha_0^i \dots \alpha_{p_i}^i},$$

$$\overline{U} \cap V = (\bigcup_{i=1}^{s-1} U_{\alpha_0^i \dots \alpha_{p_i}^i}) \cap U_{\alpha_0^s \dots \alpha_{p_s}^s} = \bigcup_{i=1}^{s-1} U_{\alpha_0^i \dots \alpha_{p_i}^i \alpha_0^s \dots \alpha_{p_s}^s}.$$

By the bootstrap principle of Lemma 7.3.5 the statement also holds for $U = \overline{U} \cup V$.
Therefore the statement also holds for $U = B$ since B is the finite union $B = \bigcup_{\alpha \in I} U_\alpha$.

□

7.4 Integration on $\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)$

We recall the definition of the subcomplex $\Omega^\bullet_{E\mathcal{MS}}(M) \subset \Omega^\bullet(M)$ (see Definition 7.2.1):

$$\Omega^\bullet_{E\mathcal{MS}}(M) := \left\{\omega \in \Omega^\bullet(M) \middle| c_E^*\omega = \pi_E^*\eta \text{ for some } \eta \in \Omega^\bullet_{\mathcal{MS}}(B)\right\}.$$

We now deal with integration on $\Omega^\bullet_{E\mathcal{MS}}(M)$ and afterwards on $\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)$:

Lemma 7.4.1 *For any $r \in \mathbb{Z}$, integration defines a bilinear form*

$$\int : \Omega^r_{E\mathcal{MS}}(M) \times \Omega^{n-r}_{E\mathcal{MS}}(M) \to \mathbb{R}.$$

<u>Proof:</u> Bilinearity is obvious and the finiteness of the integral is ensured by the compactness of M.

□

Corollary 7.4.2 *For any $r \in \mathbb{Z}$, integration defines bilinear forms*

$$\int : \widetilde{\Omega I}^{r}_{\bar{p}}(M) \times \widetilde{\Omega I}^{n-r}_{\bar{p}}(M, C_W) \to \mathbb{R}.$$

To be able to prove Poincaré duality for $\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)$ we need two technical lemmas:

Lemma 7.4.3 *For $\nu_0 \in ft_{\geq K}\Omega^{r-1}_{\mathcal{MS}}(B)$ and $\eta_0 \in ft_{\geq K^*}\Omega^{n-r}_{\mathcal{MS}}(B, C_{\partial B})$ we have*

$$\int_E \nu_0 \wedge \eta_0 = 0.$$

Proof: The proof is literally the same as the proof of [Ban11, Lemma 7.3]. □

Lemma 7.4.4 *For $\nu \in \widetilde{\Omega I}_{\bar{p}}^{r-1}(M),\ \eta \in \widetilde{\Omega I}_{\bar{q}}^{n-r}(M, C_W)$ we have*

$$\int_M d(\nu \wedge \eta) = 0.$$

Proof: The boundary of M is

$$\partial M = E \cup_{\partial E} W.$$

To prove the lemma we compute

$$\begin{aligned}
\int_M d(\nu \wedge \eta) &= \int_{\partial M} (\nu \wedge \mu)|_{\partial M} \quad \text{by Stokes' Theorem} \\
&= \int_E \nu_0 \wedge \eta_0 + \int_W c_W^*(\nu \wedge \eta) \\
&\quad \text{for some } \nu_0 \in ft_{\geq K}\Omega_{\mathcal{MS}}^{r-1}(B),\ \eta_0 \in ft_{\geq K^*}\Omega_{\mathcal{MS}}^{n-r}(B, C_{\partial B}) \\
&= 0 + \int_W c_W^*(\nu) \wedge c_W^*(\eta) \quad \text{by Lemma 7.4.3} \\
&= 0 \quad \text{since } \eta \in \widetilde{\Omega I}_{\bar{q}}^{n-r}(M, C_W).
\end{aligned}$$

□

7.5 Poincaré Duality for $\widetilde{\Omega I}_{\bar{p}}^{\bullet}(M)$

Proposition 7.5.1 *For any $r \in \mathbb{Z}$, integration on $\widetilde{\Omega I}_{\bar{p}}^{\bullet}(M)$ induces a bilinear form*

$$\int : \widetilde{HI}_{\bar{p}}^{r}(M) \times \widetilde{HI}_{\bar{q}}^{n-r}(M, C_W) \to \mathbb{R}$$
$$\big([\omega], [\eta]\big) \mapsto \int_M \omega \wedge \eta.$$

Proof: Let $\omega \in \widetilde{\Omega I}_{\bar{p}}^{r}(M)$ closed, $\widetilde{\omega} \in \widetilde{\Omega I}_{\bar{p}}^{r-1}(M)$, $\eta \in \widetilde{\Omega I}_{\bar{q}}(M, C_W)$ closed and $\widetilde{\eta} \in \widetilde{\Omega I}_{\bar{q}}^{n-r-1}(M, C_W)$.

$$\int_M (\omega + d\widetilde{\omega}) \wedge \eta = \int_M \omega \wedge \eta + \int_M d(\widetilde{\omega} \wedge \eta) = \int_M \omega \wedge \eta,$$

where the last step holds by the previous Lemma 7.4.4. By an analogous argument

$$\int_M \omega \wedge (\eta + d\widetilde{\eta}) = \int_M \omega \wedge \eta.$$

□

Lemma 7.5.2 *The subcomplex inclusion*

$$\Omega^\bullet_{E\mathcal{MS}}(M) \hookrightarrow \Omega^\bullet(M)$$

is a quasi-isomorphism.

Proof: By the usual arguments (taking a slightly larger collar) the pullbacks

$$c_E^* : \Omega^\bullet_{E\mathcal{MS}}(M) \to \pi_E^* \Omega^\bullet_{\mathcal{MS}}(B)$$

and

$$c_E^* : \Omega^\bullet(M) \to \pi_E^* \Omega^\bullet(E),$$

with $\pi_E : E \times [0,1) \to E$ the projection, are surjective with kernel $\Omega^\bullet(M, C_E)$. Hence there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega^\bullet(M, C_E) \arrow[r] \arrow[d, equal] & \Omega^\bullet_{E\mathcal{MS}}(M) \arrow[r, "c_E^*"] \arrow[d, hook] & \pi_E^* \Omega^\bullet_{\mathcal{MS}}(B) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \Omega^\bullet(M, C_E) \arrow[r] & \Omega^\bullet(M) \arrow[r, "c_E^*"] & \pi_E^* \Omega^\bullet(E) \arrow[r] & 0 \end{tikzcd}$$

As in [Ban11, Lemma 6.2] and the subsequent, the map

$$E \xhookrightarrow{\text{at } 0} E \times [0,1)$$

induces isomorphisms $\pi_E^* \Omega^\bullet_{\mathcal{MS}}(B) \xrightarrow{\cong} \Omega^\bullet_{\mathcal{MS}}(B)$ and $\pi_E^* \Omega^\bullet(E) \xrightarrow{\cong} \Omega^\bullet(E)$. The commutativity of the diagram

$$\begin{tikzcd} \pi_E^* \Omega^\bullet_{\mathcal{MS}}(B) \arrow[r, "\cong"] \arrow[d, hook] & \Omega^\bullet_{\mathcal{MS}}(B) \arrow[d, hook] \\ \pi_E^* \Omega^\bullet(E) \arrow[r, "\cong"] & \Omega^\bullet(E) \end{tikzcd}$$

together with an analogous statement as [Ban11, Theorem 3.13] for geometrically flat bundles over compact base manifolds with boundary, gives that the subcomplex inclusion $\pi_E^* \Omega^\bullet_{\mathcal{MS}}(B) \hookrightarrow \pi_E^* \Omega^\bullet(E)$ is a quasi-isomorphism. If we then apply the 5-Lemma to the commuting diagram on cohomology induced by the above diagram of short exact sequences we get the statement of the lemma. $\square$

Lemma 7.5.3 *For any manifold X (compact or not compact, possibly with corners) and any $a < b \in \mathbb{R}$ it holds that*

$$H_c^\bullet\big(X \times [a,b)\big) = 0.$$

Proof: There are two types of forms $\omega \in \Omega_c^r\big(X \times [a,b)\big)$:

1. $\omega = f(x,t)\ \pi_1^*\eta$,
2. $\omega = f(x,t)dt \wedge \pi_1^*\eta$.

Here f is a smooth function with compact support on $X \times [a,b)$ and $\eta \in \Omega_c^\bullet(X)$. If ω is closed and of the first form, then $\omega = 0$: $d\omega = 0$ implies $\partial_t f = 0$ and hence $f(x,t) = \lim_{\tau \to b} f(x,\tau) = 0$ for each $t \in [a,b)$, since f is smooth with compact support. So let ω be closed and of the first type. Then the form $\pi_1^*\eta \int_b^t f(x,\tau)d\tau$ has compact support since $f|_{X\times(b-\epsilon,b)} = 0$ for some $\epsilon > 0$ due to the fact that the support of f is compact in $X \times [a,b)$. Moreover

$$\begin{aligned} d\Big(\pi_1^*\eta \int_b^t f(x,\tau)d\tau\Big) &= f(x,t)dt \wedge \pi_1^*\eta + \int_b^t d(f(x,\tau)d\tau \wedge \pi_1^*\eta) \\ &= \omega + \int_b^t (\underbrace{d\omega}_{=0}) = \omega. \end{aligned}$$

and therefore there are no forms in $\Omega_c^\bullet\big(X \times [a,b)\big)$ that are closed but not boundaries. $\square$

Proposition 7.5.4 *Let* $N := M - \partial M$ *and*

$$\Omega_{rel}^\bullet(N) = \Omega^\bullet(N, N \cap C) = \{\omega \in \Omega^\bullet(N) | \omega|_{C\cap N} = 0\}.$$

Then the subcomplex inclusion

$$\Omega_{rel}^\bullet(N) \hookrightarrow \Omega_c^\bullet(N)$$

is a quasi-isomorphism.

Proof: Recall that for $X = E, W$ we denote

$$C_X := c_X\big((0,1) \times X\big) \subset N$$

(with c_X the collar and hence a diffeomorphism onto its image). We then set

$$\widetilde{C}_X := c_X\big((0,1+\epsilon) \times X\big) \subset N,$$

for a small $\epsilon > 0$, i.e. $\widetilde{C}_X$ is a slightly larger collar neighbourhood. This is always possible as one can see for example by viewing the collar as the flow of a vector field that is nowhere tangent to the boundary, as we did before. We first show that for the smooth open manifold

$$N_> := N - \big[c_E\big((0,1+\epsilon/2] \times E\big) \cup c_W\big((0,1+\epsilon/2] \times W\big)\big]$$

extension of forms by zero

$$\rho : \Omega_c^\bullet(N_>) \to \Omega_c^\bullet(N)$$

is a quasi-isomorphism: Let ψ_s^X, $s \in \mathbb{R}$, be a smooth one-parameter family of diffeomorphisms $\psi_s^X : N \to N$ such that $\psi_0^X = \mathrm{id}_N$, $\psi_s^X(C_X) \subset C_X$ for all $s \in [0,1]$ and $\psi_1^X(\widetilde{C}_X) = C_X$. By the same arguments as in the Lemmata 2.6, 2.7 and 2.8 of [Ban11, section 2.2], the maps $\psi_1^{X*} : \Omega_{rel}^\bullet(N) \to \Omega_{rel}^\bullet(N)$ are homotopic to the identity.
Now set $\psi_1 := \psi_1^W \circ \psi_1^E$. Then $\psi_1^* : \Omega_{rel}^\bullet(N) \to \Omega_{rel}^\bullet(N)$ is homotopic to the identity. Denote $\widetilde{C}_{\partial M} := \widetilde{C}_E \cup \widetilde{C}_W$ and

$$\Omega_{\widetilde{rel}}^\bullet(N) := \big\{\omega \in \Omega^\bullet(N) \big| \omega|_{\widetilde{C}_{\partial M}} = 0\big\}.$$

Then ψ_1^* factors as

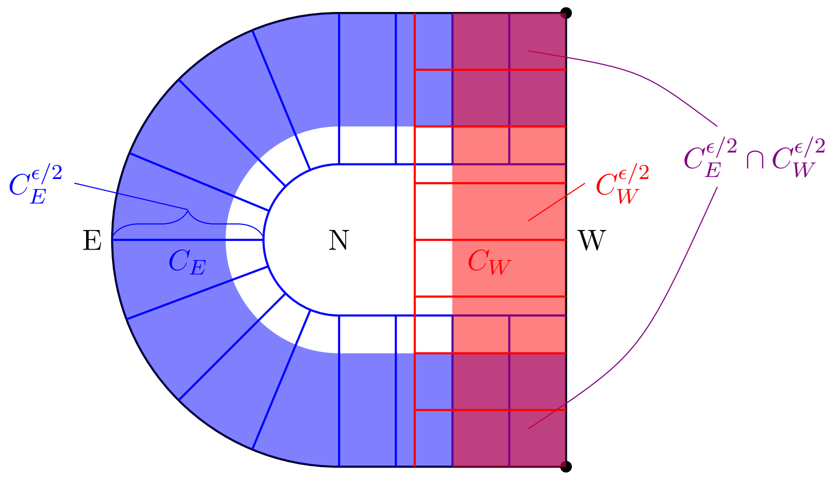
$$\psi_1^* : \Omega_{rel}^\bullet(N) \to \Omega_{\widetilde{rel}}^\bullet(N) \hookrightarrow \Omega_c^\bullet(N_>) \xrightarrow{\rho} \Omega_{rel}^\bullet(N)$$

with ρ the extension by zero (compare to the proof of [Ban11, Prop. 2.9]). Since, on the chain level, ψ_1^* is homotopic to the identity, the composition

$$H_{rel}^\bullet(N) \to H_c^\bullet(N_>) \xrightarrow{\rho^*} H_{rel}^\bullet(N)$$

is equal to the identity and hence ρ^* is surjective. To verify that it is also injective we prove the following claim:

<u>Claim:</u> Extension by zero defines an isomorphism $\gamma : H_c^\bullet(N_>) \to H_c^\bullet(N)$. For convenience we visualize the setting of this proof:



We use the definition $C_X^{\epsilon/2} := c_X\big((0, 1 + \frac{\epsilon}{2}] \times X\big)$ for $X = E, W$ and set

$$N_{W,>} := N - C_W^{\epsilon/2},$$

which is a smooth open manifold.

1. The first step of proving the above claim is to show that extension by zero defines an isomorphism

$$H_c^\bullet(N_{W,>}) \xrightarrow{\cong} H_c^\bullet(N).$$

Since $C_W^{\epsilon/2} \subset N$ is a closed subset, forms on $C_W^{\epsilon/2}$ are extendable to all of N. (By definition every form on a closed subset is locally extendable and hence also globally by using partitions of unity.) By the same argument, each form with compact support on $C_W^{\epsilon/2}$ can be extended to a form with compact support on N. The embedding $j : C_W^{\epsilon/2} \hookrightarrow N$ gives rise to a map

$$j^* : \Omega_c^\bullet(N) \twoheadrightarrow \Omega_c^\bullet(C_W^{\epsilon/2}),$$

since for each $K \subset N$ compact we have that $K \cap C_W^{\epsilon/2} \subset K$ is also compact as a closed subset of a compact set. By the previous argument, this map is surjective. Hence we get a short exact sequence

$$0 \to \ker j^* \to \Omega_c^\bullet(N) \to \Omega_c^\bullet(C_W^{\epsilon/2}) \to 0$$

with $\ker j^* = \{\omega \in \Omega_c^\bullet(N) | j^*\omega = 0\}$. Since $C_W^{\epsilon/2} \cong W \times (0, 1 + \frac{\epsilon}{2}]$ we have that its cohomology with compact support vanishes (compare to Lemma 7.5.3). Therefore, proving that the subcomplex inclusion

$$\Omega_c^\bullet(N_{W,>}) \hookrightarrow \ker j^*$$

is a quasi-isomorphism finishes the first part of the argument. We focus on the induced map

$$H_c^\bullet(N_{W,>}) \to H^\bullet(\ker j^*).$$

Injectivity: Let $\omega \in \Omega_c^\bullet(N_{W,>})$ be a closed form such that the cohomology class $[\omega] = 0 \in H^\bullet(\ker j^*)$ is zero, i.e. $\omega = d\widehat{\eta}$ for some $\widehat{\eta} \in \ker j^*$. To prove injectivity, we have to show that then the cohomology class of ω in $H_c^\bullet(N_{W,>})$ is also zero. We first split $\widehat{\eta}|_{C_W}$ into its tangential and normal component:

$$\widehat{\eta}|_{\widetilde{C}_W} = \widehat{\eta}_T(t) + dt \wedge \widehat{\eta}_N(t)$$

We then define a new form

$$\widetilde{\eta} := \widehat{\eta} - d\big(\xi \int_{1+\epsilon/2}^t \widehat{\eta}_N(\tau) d\tau\big)$$

with ξ a smooth cutoff function on N with $\xi|_{N-C_W} = 0$ and $\xi|_{C'_W} = 1$, where $C'_W := c_W\big((0, 1 + \frac{3}{4}\epsilon)\big)$.

$$j^*\widetilde{\eta} = j^*\widehat{\eta} - dj^*\big(\xi \int_{1+\epsilon/2}^t \widehat{\eta}_N(\tau) d\tau\big) = 0$$

since $j^*\widehat{\eta}_N = 0$. Hence $\widetilde{\eta} \in \ker j^*$, $d\widetilde{\eta} = \omega$ and $\widetilde{\eta}|_{C'_W} = \widetilde{\eta}_T(t) + dt \wedge 0$. Since $\omega \in \Omega^\bullet_c(N_{W,>})$ there is a $\delta > 0$ such that $\omega|_{C^\delta_W} = 0$, where $C^\delta_W := c_W\big((1+\epsilon/2, 1+\epsilon/2+\delta) \times W\big)$. Since $\omega = d\widetilde{\eta}$ and $C^\delta_W \subset C'_W$ this gives

$$0 = (d\widetilde{\eta})|_{C^\delta_W} = (d_W \widetilde{\eta}_T(t))|_{C^\delta_W} + dt \wedge \widetilde{\eta}'_T(t)|_{C^\delta_W}.$$

This gives $\widetilde{\eta}'_T(t)|_{C^\delta_W} = 0$, i.e. $\widetilde{\eta}_T|_{C^\delta_W}$ is independent of the coordinate in the collar direction. Equivalently, this means that the form is the pullback of its restriction to $c_W\big((1+\epsilon/2) \times W\big)$, which is zero. Hence $\widetilde{\eta}|_{C^\delta_W} = 0$ implying $\widetilde{\eta} \in \Omega^\bullet_c(N_{W,>})$. This finishes the proof of the injectivity.
<u>Surjectivity:</u> Let $\widehat{\omega} \in \ker j^*$ be a closed form. We want to show that there is a closed form $\omega \in \Omega^\bullet_c(N_>)$ and $\eta \in \ker j^*$ such that $\widehat{\omega} = \omega + d\eta$. As in the previous step we have $\widehat{\omega}|_{\widetilde{C}_W} = \widehat{\omega}_T(t) + dt \wedge \widehat{\omega}_N(t)$ and define

$$\omega := \widehat{\omega} - d\big(\xi \int_{1+\epsilon/2}^{t} \widehat{\omega}_N(\tau) d\tau\big).$$

Hence as before the normal part of $\omega|_{C_W}$ is zero and since ω is closed also $\omega'_T(t) = 0$, implying that $\omega \in \Omega^\bullet_c(N_>)$.

2. $N_> \hookrightarrow N_{W,>}$ is a smooth submanifold and $N_> = N_{W,>} - C^{\epsilon/2}_E$ with $C^{\epsilon/2}_E = N_{W,>} c_E\big((0, 1+\epsilon/2) \times E\big)$. By the same arguments as in the first step, we have that extension by zero induces an isomorphism $H^\bullet_c(N_>) \xrightarrow{\cong} H^\bullet_c(N_{W,>})$. (Recall that the collar c_W restricted to $\partial W \times [0,1)$ gives a collar of the boundary $\partial E \subset E$.) Composition of these maps gives the desired isomorphism

$$\gamma : H^\bullet_c(N_>) \xrightarrow{\cong} H^\bullet_c(N_{W,>}) \xrightarrow{\cong} H^\bullet_c(N).$$

Since γ factors as

$$\begin{tikzcd}
H^\bullet_c(N_>) \arrow[d, two heads, "\rho"] \arrow[dr, "\gamma"] \arrow[dr, phantom, "\cong", near end, swap] & \\
H^\bullet_{rel}(N) \arrow[r, "\alpha"'] & H^\bullet_c(N),
\end{tikzcd}$$

with $\alpha : H^\bullet_{rel}(N) \to H^\bullet_c(N)$ induced by the subcomplex inclusion, ρ is also injective and hence an isomorphism. Therefore, α is also an isomorphism.

$\square$

Finally we are able to prove Poincaré Duality for $\widetilde{\Omega I}^\bullet_{\bar{p}}(M)$:

Theorem 7.5.5 *(Poincaré duality for $\widetilde{HI}_{\bar{p}}(M)$)*
For any $r \in \mathbb{Z}$, the bilinear form

$$\int : \widetilde{HI}^r_{\bar{p}}(M) \times \widetilde{HI}^{n-r}_{\bar{q}}(M, C_W) \to \mathbb{R}$$

of Proposition 7.5.1 is nondegenerate.

Proof: *First Step*
For $\Omega^\bullet(M,C) := \{\omega \in \Omega^\bullet(M) \mid \omega|_C = 0\}$ and $H^r(M, \partial M) := H^r\big(\Omega^\bullet(M,C)\big)$ (by an analogue to the de Rham Theorem, this is isomorphic to the relative singular cohomology complex) integration induces an isomorphism

$$\int : H^r(M) \to H^{n-r}(M, \partial M)^\dagger$$

for all $r \in \mathbb{Z}$:
By the previous Lemma 7.5.2, the subcomplex inclusion $\Omega^\bullet_{E\mathcal{MS}}(M) \subset \Omega^\bullet(M)$ induces an isomorphism

$$H^r_{E\mathcal{MS}}(M) := H^r\big(\Omega^\bullet_{E\mathcal{MS}}(M)\big) \xrightarrow{\cong} H^r(M)$$

for any $r \in \mathbb{Z}$. The inclusion $i : N \hookrightarrow M$ is a homotopy equivalence and hence induces an isomorphism

$$i^* : H^r(M) \xrightarrow{\cong} H^r(N)$$

for all $r \in \mathbb{Z}$, as well as the isomorphism

$$i^* : H^r(M, \partial M) \xrightarrow{\cong} H^r_{rel}(N).$$

Since integration gives an isomorphism

$$\int : H^r(N) \xrightarrow{\cong} H^{n-r}_c(N)^\dagger$$

for all $r \in \mathbb{Z}$ and the diagram

$$\begin{tikzcd}
H^r(M) \arrow[r, "\int"] \arrow[dd, "\cong"', "i^*"] & H^{n-r}(M, \partial M)^\dagger \\
& H^{n-r}_{rel}(N)^\dagger \arrow[u, "i^{*\dagger}", "\cong"'] \\
H^r(N) \arrow[r, "\int", "\cong"'] & H^{n-r}_c(N)^\dagger \arrow[u, "\mathrm{incl}_*^\dagger", "\cong"']
\end{tikzcd}$$

commutes for any r, the first statement is established.

Second Step
By Proposition 7.3.1, integration gives an isomorphism

$$\int : H^r\big(ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B)\big) \xrightarrow{\cong} H^{n-r-1}\big(ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B})\big)^\dagger.$$

Third Step
The distinguished triangles of the two Lemmata 7.2.2 and 7.2.6 give the long exact sequences on cohomology

$$\begin{tikzcd}
... \arrow[r] & H^{r-1}\big(ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B)\big) \arrow[r] & \widetilde{HI}^r_{\bar{p}}(M) \arrow[d] \\
... & H^r\big(ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B)\big) \arrow[l] & H^r_{E\mathcal{MS}}(M) \arrow[l]
\end{tikzcd}$$

and

$$\begin{tikzcd}
... \arrow[r] & H^{n-r-1}\big(ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B})\big) \arrow[r] & H^{n-r}(M, \partial M) \arrow[d] \\
... & H^r\big(ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B})\big) \arrow[l] & \widetilde{HI}^r_{\bar{q}}(M, C_W) \arrow[l]
\end{tikzcd}$$

Fourth Step We prove that the diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H^{r-1}\big(ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B)\big) \arrow[r, "\int", "\cong"'] \arrow[d, "\delta"] & H^{n-r}\big(ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B})\big)^\dagger \arrow[d] \\
\widetilde{HI}^r_{\bar{p}}(M) \arrow[r, "\int"] \arrow[d] & \widetilde{HI}^{n-r}_{\bar{q}}(M, C_W)^\dagger \arrow[d] \\
H^r_{E\mathcal{MS}}(M) \arrow[r, "\int", "\cong"'] \arrow[d, "Q"] & H^{n-r}(M, \partial M)^\dagger \arrow[d, "D^\dagger"] \\
H^r\big(ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B)\big) \arrow[r, "\int", "\cong"'] \arrow[d] & H^{n-r-1}\big(ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS}}(B, C_{\partial B})\big)^\dagger \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

commutes (up to sign). Once we have done that, the statement of the theorem is implied by the 5-Lemma.
We first prove that the top square (TS) in the diagram commutes and therefore describe the connecting homomorphism

$$\delta : H^{r-1}\big(ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B)\big) \to \widetilde{HI}^r_{\bar{p}}(M) :$$

Let $\omega \in ft_{<K}\Omega^r_{\mathcal{MS}}(B)$ closed, i.e. $d\omega = 0$. Then $d\gamma_B\omega = 0$ holds as well, where $\gamma_B : ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B) \to Q^\bullet(B)$ is the quasi-isomorphism defined in equation (10). Since the map

$$\overline{J}^*_E : Q^\bullet_E(M) \to Q^\bullet(B)$$

defined in Lemma 7.2.2 is an isomorphism, there is a $\overline{\omega} \in Q^r_E(M) :\ \overline{J}^*_E\overline{\omega} = \gamma_B\omega$. Since $\overline{J}^*_E$ is an isomorphism and $0 = d\gamma_B\omega = d\overline{J}^*_E\overline{\omega} = \overline{J}^*_E d\overline{\omega}$, we have $d\overline{\omega} = 0$ in $Q^\bullet_E(M)$. Let $\xi \in \Omega^r_{E\mathcal{MS}}(M)$ be a representative of $\overline{\omega}$, i.e. $q(\xi) = \overline{\omega}$. Then $d\xi \in \widetilde{\Omega I}^\bullet_{\bar{p}}(M)$ since

$$q(d\xi) = dq(\xi) = d\overline{\omega} = 0.$$

Hence $(-d\xi, \xi) \in C^r(i)$, the mapping cone of the subcomplex inclusion $i : \widetilde{\Omega I}^\bullet_{\bar{p}}(M) \hookrightarrow \Omega^\bullet_{E\mathcal{MS}}(M)$ with $d(-d\xi, \xi) = 0$. Therefore by the definition of distinguished triangles and the induced long exact cohomology sequences,

$$\delta[\omega] = [-d\xi].$$

Since

$$q(\text{incl } \omega) =: \gamma_B(\omega) = \overline{J}^*_E q(\xi) = q(J^*_E\xi) = q(\sigma^*_E c^*_E \xi) \in Q^r(B)$$

it holds that

$$\alpha := \sigma^*_E c^*_E \xi - \omega \in ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B).$$

For closed forms $\omega \in ft_{<K}\Omega^{r-1}_{\mathcal{MS}}(B),\ \eta \in \widetilde{\Omega I}^{n-r}_{\bar{q}}(M, C_W)$ we hence get:

$$\begin{aligned}
\int_M \delta(\omega) \wedge \eta &= -\int_M d\xi \wedge \eta = -\int_M d(\xi \wedge \eta) \\
&= -\int_{M-C_E} d(\xi \wedge \eta) - \int_{C_E} d(\xi \wedge \eta) \\
&= -\int_{M-C_E} d(\xi \wedge \eta),
\end{aligned}$$

since

$$d(\xi \wedge \eta)|_{C_E} = \psi^*_E \pi^*_E d(\xi_0 \wedge \eta_0)$$

for some $\xi_0 \in \Omega^r_{MS}(B)$, $\eta_0 \in ft_{\geq K^*}\Omega^{n-r}_{\mathcal{MS}}(B, C_{\partial B})$ and hence

$$\int_{C_E} d(\xi \wedge \eta) = \int_E d(\xi_0 \wedge \eta_0) = 0$$

as an integral of an n-form over a $(n-1)$-dimensional manifold. Let $J_W : W - C_{\partial W} \hookrightarrow W \hookrightarrow M$. Then by Stokes' Theorem for manifolds with corners

$$\begin{aligned} -\int_{M-C_E} d(\xi \wedge \eta) &= -\int_E \sigma_E^* c_E^* \xi \wedge \widetilde{J_E}^* \eta + \int_{W-C_{\partial W}} J_W^*(\xi \wedge \eta) \\ &= -\int_E \omega \wedge \widetilde{J_E}^* \eta - \int_E \alpha \wedge \widetilde{J_E}^* \eta = -\int_E \omega \wedge \widetilde{J_E^*} \eta, \end{aligned}$$

where

$$\int_{W-C_{\partial W}} J_W^*(\xi \wedge \eta) = 0,$$

since $\eta|_{C_W} = 0$, and

$$\int_E \alpha \wedge \widetilde{J_E}^* \eta = 0$$

by Lemma 7.4.3. Thus (TS) commutes up to sign.

The commutativity of the middle square (MS) is obviously fullfilled since both the vertical maps are induced by the subcomplex inclusions $\widetilde{\Omega I}_{\bar{p}}^{\bullet}(M) \hookrightarrow \Omega_{E\mathcal{MS}}^{\bullet}(M)$ and $\Omega_{rel}^{\bullet}(M) \hookrightarrow \widetilde{\Omega I}_{\bar{q}}^{\bullet}(M, C_W)$.

To prove the commutativity of the bottom square (BS), we first investigate the connecting homomorphism

$$D : H^{n-r-1}\big(ft_{\geq K^*} \Omega_{\mathcal{MS}}^{\bullet}(B, C_{\partial B})\big) \to H_{rel}^{n-r}(M).$$

We look at the distinguished triangle (11). For $\eta \in ft_{\geq K^*} \Omega_{\mathcal{MS}}^{n-r-1}(B, C_{\partial B})$ closed, the surjectivity of $\widetilde{J_E}^*$ implies that there is a form $\overline{\eta} \in \widetilde{\Omega I}_{\bar{q}}^{n-r-1}(M, C_W)$ such that $\widetilde{J_E}^* \overline{\eta} = \eta$. Since $\widetilde{J_E}^*$ is a chain map, $d\overline{\eta} \in \ker \widetilde{J_E}^* = \Omega_{rel}^{n-r}(M)$. Let $\rho : \Omega_{rel}^{\bullet}(M) \hookrightarrow \widetilde{\Omega I}_{\bar{q}}^{\bullet}(M, C_W)$ denote the subcomplex inclusion and $C^{\bullet}(\rho)$ its algebraic mapping cone. Then the map

$$f : C^{\bullet}(\rho) \to ft_{\geq K^*} \Omega_{\mathcal{MS}}^{\bullet}(B, C_{\partial B}),\ (\tau, \sigma) \mapsto \widetilde{J_E}^*(\sigma)$$

is a quasi-isomorphism (by the standard argumentation). The cocycle

$$c := (-d\overline{\eta}, \overline{\eta}) \in C^{n-r-1}(\rho)$$

satisfies the equation $f(c) = \widetilde{J_E}^* \overline{\eta} = \eta$ and hence $D[\eta]$ can be described as

$$D[\eta] = [-d\overline{\eta}].$$

We next describe the map

$$Q : H^r\big(\Omega_{E\mathcal{MS}}^{\bullet}(M)\big) \to H^r\big(ft_{<K} \Omega_{\mathcal{MS}}^{\bullet}(B)\big),$$

induced by the corresponding map in the distinguished triangle (8).
Let $\omega \in \Omega^r_{E\mathcal{MS}}(M)$ be a closed form. Then $J_E^*\omega \in \Omega^r_{\mathcal{MS}}(B)$ represents the image of ω under

$$\Omega^\bullet_{E\mathcal{MS}}(M) \xrightarrow{q} Q^\bullet_E(M) \xrightarrow[\cong]{\overline{J_E}^*} Q^\bullet(B).$$

Since $\gamma_B : ft_{<K}\Omega^\bullet_{\mathcal{MS}}(B) \to Q^\bullet(B)$ is a quasi-isomorphism, there are forms $\overline{\omega} \in ft_{<K}\Omega^r_{\mathcal{MS}}(B)$, $d\overline{\omega} = 0$, and $\xi \in \Omega^{r-1}_{\mathcal{MS}}(B)$ such that

$$\gamma_B(\overline{\omega}) = \overline{J_E}^* q(\omega) + dq_B(\xi).$$

The above map Q is then described by

$$Q[\omega] = [\overline{\omega}].$$

Note that the form $\alpha := \overline{\omega} - J_E^*\omega - d\xi \in ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)$. We can now verify the commutativity of the (BS) by proving

$$\int_M \omega \wedge (-d\overline{\eta}) = \pm \int_E \overline{\omega} \wedge \eta, \tag{13}$$

with $[-d\overline{\eta}] = D[\eta]$ and $[\overline{\omega}] = Q[\omega]$.

$$\begin{aligned}
\int_M \omega \wedge (-d\overline{\eta}) &= -\int_M \omega \wedge d\overline{\eta} = \pm \int_{M-C_E} d(\omega \wedge \overline{\eta}) - \int_{C_E} \omega \wedge d\overline{\eta} \\
&= \pm \int_{M-C_E} d(\omega \wedge \overline{\eta}) \quad (\text{since } d\overline{\eta} \in \Omega^\bullet_{rel}(M)) \\
&= \pm \int_E J_E^*\omega \wedge \widetilde{J_E}^*\overline{\eta} \pm \int_{W-C_{\partial W}} J_W^*(\omega \wedge \overline{\eta}) \quad (\text{Stokes}) \\
&= \pm \int_E (\overline{\omega} - \alpha - d\xi) \wedge \widetilde{J_E}^*\overline{\eta} \quad (\text{above } + \ \eta|_{C_W} = 0) \\
&= \pm \int_E \overline{\omega} \wedge \eta,
\end{aligned}$$

since $\widetilde{J_E}^*\overline{\eta} = \eta$,

$$\int_E \alpha \wedge \eta = 0$$

by Lemma 7.4.3 and

$$\int_E d\xi \wedge \eta = \int_E d(\xi \wedge \eta) = \int_{\partial E} \xi \wedge \eta = 0$$

by Stokes' Theorem and since $\overline{\eta}|_{C_W} = 0$ and hence $\eta|_{\partial E} = 0$. Thus (BS) commutes and the theorem is proven. □

8 The De Rham Intersection Complex $\Omega I^\bullet_{\bar p}(M)$

8.1 Truncation and Cotruncation of $\Omega I^\bullet_{\bar p}(W)$

To be able to define the de Rham intersection complex $\Omega I^\bullet_{\bar p}(M)$, we need to note some obeservations about the boundary part $W \subset \partial M$.

Remark 8.1.1 *The $(n-1)$-dimensional compact manifold with boundary W is the top stratum of the singular stratified space $\partial X'$ mentioned in section 5.3. The boundary ∂W of W is the total space of the flat link bundle $q : \partial W = \partial E \to \partial B$, with $B = \Sigma$ the bottom stratum of the stratified pseudomanifold-with-boundary X'. Hence, following [Ban11], we can construct the chain complex of intersection forms $\Omega I^\bullet_{\bar p}(W)$ as a subcomplex of the complex of differential forms of the top stratum for the stratified pseudomanifold $\partial X'$ with two strata and regular part W.*

Additional assumption

To be able to cotruncate $\Omega I^\bullet_{\bar p}(W)$, we additionally demand the following condition for W: Let $L := n - 1 - \bar p(n)$ and demand for W that the subcomplex $\Omega I^\bullet_{\bar p}(W) \subset \Omega^\bullet(W)$ is geometrically cotruncatable in degree L, see Definition 6.4.1.

Remark 8.1.2 *Recall that, by Example 6.4.4, $\Omega I^\bullet_{\bar p}(W)$ is geometrically cotruncatable in degree L if $H^L\big(\Omega I^\bullet_{\bar p}(W)\big) = 0$.*

Definition 8.1.3 *(Truncation and Cotruncation of $\Omega I^\bullet_{\bar p}(W)$ in degree L) Define in analogy to subsection 6.4*

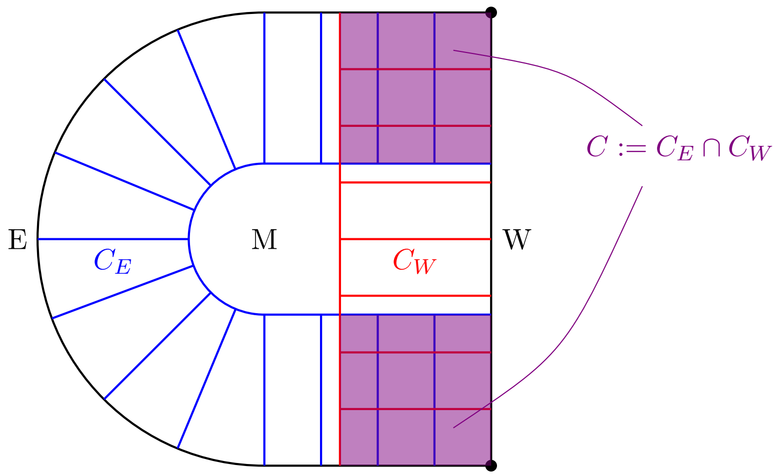
$$\tau_{<L}\Omega I^\bullet_{\bar p}(W) := \dots \to \Omega I^{L-1}_{\bar p}(W) \to \operatorname{im} d^{L-1}_{\Omega I^\bullet_{\bar p}} \to 0 \to \dots$$

and

$$\tau_{\geq L}\Omega I^\bullet_{\bar p}(W) := \dots \to 0 \to \Omega I^L_{\bar p}(W) \cap cC^L_N(W) \to \Omega I^{L+1}_{\bar p}(W) \to \dots$$

8.2 Definition of the De Rham Intersection Complex

In this section we finally give the definition of the de Rham intersection complex $\Omega I^\bullet_{\bar p}(M)$. Recall the setting of our considerations:



As before, for $X = E, W$, we denote by $c_X : X \times [0,1) \hookrightarrow M$ the collar of X in M and by $\pi_X : X \times [0,1) \to X$ the projection to the boundary factor of the collar.

Definition 8.2.1 *(The de Rham intersection complex $\Omega I^{\bullet}_{\bar{p}}(M)$)*

$$\Omega I^{\bullet}_{\bar{p}}(M) := \left\{ \omega \in \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M) \middle| \ \exists\ \eta \in \tau_{\geq L}\Omega^{\bullet}(W) : c_W^* \omega = \pi_W^* \eta \right\}.$$

Remark 8.2.2 *Let $C := C_E \cap C_W, C \cong \partial E \times [0,1)^2$, a collar neighbourhood of $\partial E = \partial W$ in M. Then for $\omega \in \widetilde{\Omega I}^{r}_{\bar{p}}(M)$ with $c_W^*\omega = \pi_W^* \eta_W$ for some $\eta_W \in \tau_{\geq L}\Omega^r(W)$ and $c_E^*\omega = \pi_E^*\eta_E$ for some $\eta_E \in ft_{\geq K}\Omega^r_{\mathcal{MS}}(B)$ we have that*

$$\omega|_C = (c_E^*\omega)|_C = (\pi_E^*\eta_E)|_C = \overline{\pi}^*(\eta_E|_{\partial E}), \tag{14}$$

with $\overline{\pi} : C \cong C_{\partial E} \times [0,1) \to C_{\partial E}$ the projection, as well as

$$\omega|_C = (c_W^*\omega)|_C = (\pi_W^*\eta_W)|_C = \widetilde{\pi}^*(\eta_W|_{\partial E}), \tag{15}$$

with $\widetilde{\pi} : C \cong C_{\partial W} \times [0,1) \to C_{\partial W}$ also the projection. Let now (x,t,s) denote the coordinates on $\partial E \times \underbrace{[0,1)}_{\text{directed to } E} \times \underbrace{[0,1)}_{\text{directed to } W} \cong C$. Then by (14), $\omega|_C$ is independent of s and by (15) it is independent of t. Hence there is a form $\eta \in \Omega^r(\partial E = \partial W)$ such that $\omega|_C = \pi^*\eta$ with $\pi : C \to \partial E$ the projection.
In particular we get

$$j_{\partial E}^*\eta_E = \pi_{\partial E}^*\eta$$

and hence $\eta \in ft_{\geq K}\Omega^r_{\mathcal{MS}}(\partial B)$. Since also

$$j_{\partial W}^*\eta_W = \pi_{\partial W}^*\eta$$

holds, we deduce that

$$\Omega I^{\bullet}_{\bar{p}}(M) = \left\{ \omega \in \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M) \middle| c_W^* \omega = \pi_W^* \eta \text{ for some } \eta \in \tau_{\geq L} \Omega I^{\bullet}_{\bar{p}}(W) \right\}$$

and

$$\omega \in \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M) \Rightarrow \sigma_W^* \circ c_W^* \omega \in \Omega I^{\bullet}_{\bar{p}}(W).$$

We now start to give the preparational material for the proof of Poincaré duality for $\Omega I^{\bullet}_{\bar{p}}(M)$:

Lemma 8.2.3 *There is a distinguished triangle*

$$\begin{tikzcd} \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M,W) \arrow[rr] & & \Omega I^{\bullet}_{\bar{p}}(M) \arrow[dl] \\ & \tau_{\geq L} \Omega I^{\bullet}_{\bar{p}}(W) \arrow[ul, "+1"] & \end{tikzcd}$$

in $\mathcal{D}(\mathbb{R})$.

Proof: The kernel of the surjective map $\sigma_W^* \circ c_W^* : \Omega I^{\bullet}_{\bar{p}}(M) \to \tau_{\geq L} \Omega I^{\bullet}_{\bar{p}}(W)$ is

$$\left\{ \omega \in \Omega I^{\bullet}_{\bar{p}}(M) \middle| c_W^* \omega = 0 \right\} = \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M,W).$$

Hence there is a short exact sequence

$$0 \longrightarrow \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M,W) \longrightarrow \Omega I^{\bullet}_{\bar{p}}(M) \xrightarrow{\sigma_W^* \circ c_W^*} \tau_{\geq L} \Omega I^{\bullet}_{\bar{p}}(W) \longrightarrow 0$$

and in particular a distinguished triangle of the desired form in $\mathcal{D}(\mathbb{R})$.

□

Lemma 8.2.4 *There is a short exact sequence*

$$0 \to \Omega I^{\bullet}_{\bar{p}}(M) \to \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M) \to \tau_{<L} \Omega I^{\bullet}_{\bar{p}}(W) \to 0.$$

In particular this induces another distinguished triangle

$$\begin{tikzcd} \Omega I^{\bullet}_{\bar{p}}(M) \arrow[rr] & & \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M) \arrow[dl] \\ & \tau_{<L} \Omega I^{\bullet}_{\bar{p}}(W) \arrow[ul, "+1"] & \end{tikzcd}$$

in $\mathcal{D}(\mathbb{R})$.

Proof: Since $\Omega I^{\bullet}_{\bar{p}}(M) \hookrightarrow \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)$ is a subcomplex, there is a short exact sequence

$$0 \to \Omega I^{\bullet}_{\bar{p}}(M) \to \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M) \to \frac{\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)}{\Omega I^{\bullet}_{\bar{p}}(M)} \to 0 \tag{16}$$

By Remark 8.2.2, for any $\omega \in \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)$ one has that $\sigma^*_W \circ c^*_W \omega \in \Omega I^{\bullet}_{\bar{p}}(W)$ and for $\omega \in \Omega I^{\bullet}_{\bar{p}}(M)$ one has $\sigma^*_W \circ c^*_W \omega \in \tau_{\geq L} \Omega I^{\bullet}_{\bar{p}}(W)$. By the standard arguments (enlarging the collar and using a cutoff function) the maps

$$\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M) \overset{\sigma^*_W \circ c^*_W}{\longrightarrow} \Omega I^{\bullet}_{\bar{p}}(W)$$

and

$$\Omega I^{\bullet}_{\bar{p}}(M) \overset{\sigma^*_W \circ c^*_W}{\longrightarrow} \tau_{\geq L} \Omega I^{\bullet}_{\bar{p}}(W)$$

are surjective and by the same argument as in [Ban11, sect.6,p.43] (using the 3×3-lemma) we get an isomorphism

$$\frac{\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)}{\Omega I^{\bullet}_{\bar{p}}(M)} \underset{\cong}{\xrightarrow{\sigma^*_W \circ c^*_W}} \frac{\Omega I^{\bullet}_{\bar{p}}(W)}{\tau_{\geq L} \Omega I^{\bullet}_{\bar{p}}(W)}.$$

By Remark 6.4.5, we have $\Omega I^{\bullet}_{\bar{p}}(W) = \tau_{<L} \Omega I^{\bullet}_{\bar{p}}(W) \oplus \tau_{\geq L} \Omega I^{\bullet}_{\bar{p}}(W)$ and hence the map

$$\tau_{<L} \Omega I^{\bullet}_{\bar{p}}(W) \underset{\cong}{\xrightarrow{\text{projoincl}}} \frac{\Omega I^{\bullet}_{\bar{p}}(W)}{\tau_{\geq L} \Omega I^{\bullet}_{\bar{p}}(W)}.$$

By composition we get an isomorphism

$$\frac{\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)}{\Omega I^{\bullet}_{\bar{p}}(M)} \overset{\cong}{\longrightarrow} \tau_{<L} \Omega I^{\bullet}_{\bar{p}}(W).$$

$\square$

Lemma 8.2.5 *Integration induces a nondegenerate bilinear form*

$$\int : HI^r_{\bar{p}}(W) \times HI^{n-1-r}_{\bar{q}}(W) \to \mathbb{R}.$$

Proof: Notice that $\Omega I^{\bullet}_{\bar{p}}(W) \cong \Omega I^{\bullet}_{\bar{p}}(W - \partial W)$ and consider [Ban11, Theorem 8.2].

$\square$

Lemma 8.2.6 *Also integration induces a nondegenerate bilinear form*

$$\int : H^r\big(\tau_{<L} \Omega I^{\bullet}_{\bar{p}}(W)\big) \times H^{n-1-r}\big(\tau_{\geq L^*} \Omega I^{\bullet}_{\bar{q}}(W)\big) \to \mathbb{R}.$$

Proof: For $r \geq L$ we have that $n-1-r < L^*$ and both complexes are zero and therefore also the cohomology groups. For $r < L$ we have that $n-r-1 \geq L^*$ and hence

$$H^r\big(\tau_{<L}\Omega I^{\bullet}_{\bar{p}}(W)\big) = HI^r_{\bar{p}}(W)$$

as well as

$$H^{n-1-r}\big(\tau_{\geq L^*}\Omega I^{\bullet}_{\bar{q}}(W)\big) \cong HI^{n-1-r}_{\bar{q}}(W).$$

Therefore we traced back the statement of the lemma to the result of the previous lemma.

□

8.3 Integration on $\Omega I^{\bullet}_{\bar{p}}(M)$

In analogy to [Ban11, Lemma 7.1,Cor. 7.2] we have

Lemma 8.3.1 *Integration defines bilinear forms*

$$\int : \Omega I^r_{\bar{p}}(M) \times \Omega I^{n-r}_{\bar{q}}(M) \to \mathbb{R}.$$

The following lemma is the extension of [Ban11, Lemma 7.4] to the 3-strata case:

Lemma 8.3.2 *Let $\omega \in \Omega I^{r-1}_{\bar{p}}(M),\ \eta \in \Omega I^{n-r}_{\bar{q}}(M)$, Then*

$$\int_M d(\omega \wedge \eta) = 0.$$

Proof: By Stokes' Theorem on manifolds with corners we get:

$$\int_M d(\omega \wedge \eta) = \int_W j_W^*(\omega \wedge \eta) + \int_E j_E^*(\omega \wedge \eta)$$

By definition of $\Omega I^{\bullet}_{\bar{p}}(M)$, we have

$$j_W^*(\omega \wedge \eta) = \omega_W \wedge \eta_W, \quad \omega_W \in Q^{r-1}_{\bar{p}}(W),\ \eta_W \in Q^{n-r}_{\bar{q}}(W)$$

and

$$j_E^*(\omega \wedge \eta) = \omega_E \wedge \eta_W, \quad \omega_E \in ft_{\geq K}\Omega^{r-1}_{\mathcal{MS}}(B),\ \eta_E \in ft_{\geq K^*}\Omega^{n-r}_{\mathcal{MS}}(B).$$

For $r-1 \geq L = n-1-\bar{p}(n-1) = 2+\bar{q}(n-1)$ we get $n-r \leq n-3-\bar{q}(n-1) < n-1-\bar{q}(n-1) = L^*$ and hence

$$Q^{n-r}_{\bar{q}}(W) = \Omega I^{n-r}_{\bar{q}}(W) \cap \tau_{\geq L^*}\Omega^{n-r}(W) = 0.$$

This implies that $\int_W \omega_W \wedge \omega_E = 0$ for $r-1 \geq L$.

For $r-1<L$ we have

$$Q_{\bar{p}}^{r-1}(W)=\Omega I_{\bar{p}}^{r-1}(W)\cap\tau_{\geq L}\Omega^{r-1}(W)=0$$

and therefore also $\int_W \omega_W\wedge\eta_W=0$ for $r-1<L$.
The relation

$$\int_E \omega_E\wedge\eta_E=0$$

holds by Lemma 7.4.3.

□

With the help of the previous lemma we get:

Proposition 8.3.3 *Integration induces bilinear forms*

$$\int : HI_{\bar{p}}^{r}(M)\times HI_{\bar{q}}^{n-r}(M)\to\mathbb{R},$$
$$\big([\omega],[\eta]\big)\mapsto\int_M\omega\wedge\eta.$$

Proof: Let $\omega\in\Omega I_{\bar{p}}^{r}(M)$ and $\eta\in\Omega I_{\bar{q}}^{n-r}(M)$ be closed forms and $\omega'\in\Omega I_{\bar{p}}^{r-1}(M)$, $\eta'\in\Omega I_{\bar{q}}^{n-r-1}(M)$ be any forms. We then have

$$\begin{aligned}\int_M(\omega+d\omega')\wedge\eta&=\int_M\omega\wedge\eta+\int_M d\omega'\wedge\eta\\&=\int_M\omega\wedge\eta+\int_M d(\omega'\wedge\eta)\quad\text{since } d\eta=0\\&=\int_M\omega\wedge\eta\quad\text{by Lemma 8.3.2.}\end{aligned}$$

and

$$\begin{aligned}\int_M\omega\wedge(\eta+d\eta')&=\int_M\omega\wedge\eta+\int_M\omega\wedge d\eta'\\&=\int_M\omega\wedge\eta+\int_M d(\omega\wedge\eta')\quad\text{since } d\omega=0\\&=\int_M\omega\wedge\eta\quad\text{by Lemma 8.3.2.}\end{aligned}$$

□

8.4 Poincaré Duality for the Intersection De Rham Complex

Finally we can state and prove the Poincaré duality theorem for $\Omega I_{\bar{p}}^{\bullet}(M)$:

Theorem 8.4.1 *(Poincaré duality for HI)*
Integration induces nondegenerate bilinear forms

$$\int : HI^r_{\bar{p}}(M) \times HI^{n-r}_{\bar{q}}(M) \to \mathbb{R},$$
$$\big([\omega], [\eta]\big) \mapsto \int_M \omega \wedge \eta.$$

Proof: The two distinguished triangles

$$\begin{tikzcd} \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M,W) \arrow[rr] & & \Omega I^{\bullet}_{\bar{p}}(M) \arrow[dl] \\ & \tau_{\geq L}\Omega I^{\bullet}_{\bar{p}}(W) \arrow[ul, "+1"] & \end{tikzcd}$$

and

$$\begin{tikzcd} \Omega I^{\bullet}_{\bar{p}}(M) \arrow[rr] & & \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M) \arrow[dl] \\ & \tau_{<L}\Omega I^{\bullet}_{\bar{p}}(W) \arrow[ul, "+1"] & \end{tikzcd}$$

of the Lemmata 8.2.3 and 8.2.4 induce long exact sequences on cohomology. We claim that these sequences fit into a commutative diagram of the following form:

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ H^{r-1}\big(\tau_{<L}\Omega I^{\bullet}_{\bar{p}}(W)\big) \arrow[r, "\int"] \arrow[d, "\delta"] & H^{n-r}\big(\tau_{\geq L^*}\Omega I^{\bullet}_{\bar{q}}(W)\big)^{\dagger} \arrow[d, "(\sigma^*_W \circ c^*_W)^{\dagger}"] \\ HI^r_{\bar{p}}(M) \arrow[r, "\int"] \arrow[d] & HI^{n-r}_{\bar{q}}(M)^{\dagger} \arrow[d] \\ \widetilde{HI}^r_{\bar{p}}(M) \arrow[r, "\int"] \arrow[d, "\Lambda"] & \widetilde{HI}^{n-r}_{\bar{q}}(M,W)^{\dagger} \arrow[d, "\Delta^{\dagger}"] \\ H^r\big(\tau_{<L}\Omega I^{\bullet}_{\bar{p}}(W)\big) \arrow[r, "\int"] \arrow[d] & H^{n-r-1}\big(\tau_{\geq L^*}\Omega I^{\bullet}_{\bar{q}}(W)\big)^{\dagger} \arrow[d] \\ \vdots & \vdots \end{tikzcd} \tag{17}$$

To show the commutativity of the diagram we will prove step by step that the individual squares in the diagram commute:
We start with the top square (TS):
We first describe the connecting homomorphism $\delta : H^{r-1}\big(\tau_{<L}\Omega I^\bullet_{\bar p}(W)\big) \to HI^r_{\bar p}(M)$. Let $\omega \in \tau_{<L}\Omega^{r-1}_{\bar p}(W)$ be a closed form. Then

$$d\lceil\omega\rceil = d\gamma_W\omega = \gamma_W(d\omega) = 0,$$

where $\lceil\omega\rceil \in \frac{\Omega I^\bullet_{\bar p}(W)}{Q^\bullet_{\bar p}(W)}$ denotes the equivalence class of ω in the quotient and

$$\gamma_W := \mathrm{proj}\circ\mathrm{incl} : \tau_{<L}\Omega I^\bullet_{\bar p}(W) \to \frac{\Omega I^\bullet_{\bar p}(W)}{Q^\bullet_{\bar p}(W)}$$

is a chain map. Let $i : \Omega I^\bullet_{\bar p}(M) \hookrightarrow \widetilde{\Omega I}^\bullet_{\bar p}(M)$ denote the subcomplex inclusion and $C^\bullet(i)$ its algebraic mapping cone, defined by

$$\begin{aligned} C^r(i) &:= \Omega I^{r+1}_{\bar p}(M) \oplus \widetilde{\Omega I}^r_{\bar p}(M), \\ d(\alpha,\beta) &= (-d\alpha, \alpha + d\beta). \end{aligned}$$

Since the map $J := c_W \circ \sigma_W$ induces an isomorphism

$$\overline{J}^* : \frac{\widetilde{\Omega I}^\bullet_{\bar p}(M)}{\Omega I^\bullet_{\bar p}(M)} \xrightarrow{\cong} \frac{\Omega I^\bullet_{\bar p}(W)}{Q^\bullet_{\bar p}(W)},$$

there is exactly one $\overline{\kappa} \in \frac{\widetilde{\Omega I}^\bullet_{\bar p}(M)}{\Omega I^\bullet_{\bar p}(M)}$ with representative $\kappa \in \widetilde{\Omega I}^\bullet_{\bar p}(M)$ (note that we will always denote equivalence classes of elements $\alpha \in \widetilde{\Omega I}^\bullet_{\bar p}(M)$ in $\frac{\widetilde{\Omega I}^\bullet_{\bar p}(M)}{\Omega I^\bullet_{\bar p}(M)}$ as $\overline{\alpha}$) such that

$$\overline{J}^*(\overline{\kappa}) = \lceil J^*\kappa\rceil = \lceil\omega\rceil.$$

We further have

$$\overline{d\kappa} = \mathrm{proj}(d\kappa) = d(\mathrm{proj}\ \kappa) = d\overline{\kappa} = 0$$

since $d\lceil\omega\rceil = 0$ and $\overline{J}^*$ is an isomorphism. Hence $d\kappa \in \Omega I^\bullet_{\bar p}(M)$ and $(-d\kappa,\kappa) \in C^\bullet(i)$ with

$$d(-d\kappa,\kappa) = (d^2\kappa, -d\kappa + d\kappa) = (0,0).$$

Finally, $\delta\big([\omega]\big)$ is described by

$$\delta\big([\omega]\big) = [-d\kappa] \in HI^r_{\bar p}(X).$$

To show that (TS) commutes we must show that for $\omega \in \tau_{<L}\Omega^{r-1}_{\bar p}(W)$ closed and $\eta \in \Omega I^{n-r}_{\bar q}(M)$ closed it holds that

$$\int_W \omega \wedge \sigma^*_W \circ c^*_W(\eta) = \pm\int_M -d\kappa \wedge \eta. \tag{18}$$

Since $d\eta = 0$, $-d\kappa \wedge \eta = -d(\kappa \wedge \eta)$ and hence by Stokes' Theorem for manifolds with corners

$$\int_M (-d\mathrm{w}) \wedge \eta = -\int_W \sigma_W^* \circ c_W^*(\kappa \wedge \eta) - \int_E \sigma_E^* \circ c_E^*(\kappa \wedge \eta).$$

Since $\kappa \in \widetilde{\Omega I}^{r-1}_{\bar{p}}(M)$ and $\eta \in \Omega I^{n-r}_{\bar{q}}(M)$, it holds that

$$\sigma_E^* \circ c_E^* \kappa \in ft_{\geq K} \Omega^r_{\mathcal{MS}}(B) \quad \text{and} \quad \sigma_E^* \circ c_E^* \eta \in ft_{\geq K^*} \Omega^{n-r}_{\mathcal{MS}}(B)$$

and hence 7.4.3 implies that

$$\int_E \sigma_E^* \circ c_E^*(\kappa \wedge \eta) = 0.$$

What remains is to calculate the integral $\int_W \sigma_W^* \circ c_W^*(\kappa \wedge \eta)$:
By definition we have

$$\lceil \sigma_W^* c_W^*(\kappa) \rceil = \lceil J_W^*(\kappa) \rceil = \overline{J}_W^* \mathrm{proj}(\kappa) = \overline{J}_W^* \overline{\kappa} = \lceil \omega \rceil$$

and hence there is a form $\alpha \in Q^{r-1}_{\bar{p}}(W)$ such that

$$\sigma_W^* c_W^*(\kappa) = \omega + \alpha.$$

That result gives

$$\int_W \sigma_W^* c_W^*(\kappa \wedge \eta) = \int_W \omega \wedge \sigma_W^* c_W^* \eta + \int_W \alpha \wedge \sigma_W^* c_W^* \eta = \int_W \omega \wedge \sigma_W^* c_W^* \eta$$

since $\alpha \in Q^{r-1}_{\bar{p}}(W)$ and $\sigma_W^* c_W^* \eta \in Q^{n-r}_{\bar{q}}(W)$ and hence

$$\int_W \alpha \wedge \sigma_W^* c_W^* \eta = 0,$$

by the same arguments as in the proof of Lemma 8.3.2. Summing up, we have shown that (TS) commutes.

Before proving the commutativity of the bottom square (BS) in (17), we describe the connecting homomorphism

$$\Delta : H^{n-r-1}\big(\tau_{\geq L^*} \Omega I^{\bullet}_{\bar{q}}(W)\big) \to \widetilde{HI}^{n-r}_{\bar{q}}(M, W)$$

and the map

$$\Lambda : \widetilde{HI}^{r}_{\bar{p}}(M) \to H^r\big(\tau_{<L} \Omega I^{\bullet}_{\bar{p}}(W)\big).$$

1. We first describe $\Delta[\eta]$ for $\eta \in \tau_{\geq L^*}\Omega I^{n-r-1}_{\bar{q}}(W)$ closed.
   Let
   $$\rho : \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M, W) \hookrightarrow \Omega I^{\bullet}_{\bar{q}}(M)$$
   denote the subcomplex inclusion and $C^{\bullet}(\rho)$ the corresponding mapping cone,
   $$C^r(\rho) = \widetilde{\Omega I}^{r+1}_{\bar{q}}(M, W) \oplus \Omega I^r_{\bar{q}}(M),$$
   $$d(\alpha, \beta) := (-d\alpha, \alpha + d\beta),$$
   and let $g : C^{\bullet}(\rho) \to \tau_{\geq L^*}\Omega I^{\bullet}_{\bar{q}}(W)$ be the quasi-isomorphism defined by
   $$g(\alpha, \beta) := \sigma^*_W c^*_W \beta.$$
   Let then $\eta \in \tau_{\geq L^*}\Omega I^{n-r-1}_{\bar{q}}(W)$ be a closed form. By Lemma 8.2.3, there is a form $\omega \in \Omega I^{n-r-1}_{\bar{q}}(M)$ such that
   $$\xi := \eta - \sigma^*_W c^*_W \omega \in d\big(\tau_{\geq L^*}\Omega I^{n-r-2}_{\bar{q}}(W)\big).$$
   Further we have
   $$\sigma^*_W c^*_W(d\omega) = d(\sigma^*_W c^*_W \omega) = d\eta + d\xi = 0,$$
   and therefore $d\omega \in \widetilde{\Omega I}^{n-r}_{\bar{q}}(M, W)$ and $c := (-d\omega, \omega) \in C^{n-r-1}(\rho)$ with $dc = (d^2\omega, -d\omega + d\omega) = (0, 0)$.
   Since $\big[g(c)\big] = [\sigma_W c^*_W \omega] = [\eta]$, we get
   $$\Delta[\eta] = [-d\omega] \in \widetilde{HI}^{n-r}_{\bar{q}}(M, W). \tag{19}$$

2. Secondly we give a description of the map Λ:
   Let $\theta \in \widetilde{\Omega I}^{r}_{\bar{p}}(M)$ be a closed form. Then $\mathrm{proj}(\sigma^*_W c^*_W \theta) = \lceil \sigma^*_W c^*_W \theta \rceil \in \frac{\Omega I^{\bullet}_{\bar{p}}(W)}{Q^{\bullet}_{\bar{p}}(W)}$ is also closed. By the arguments in the proof of Lemma 8.2.4, subcomplex inclusion followed by projection is a quasi-isomorphism $\tau_{<L}\Omega I^{\bullet}_{\bar{p}}(W) \to \frac{\Omega I^{\bullet}_{\bar{p}}(W)}{Q^{\bullet}_{\bar{p}}(W)}$ and thus there is a $\xi \in \tau_{<L}\Omega I^r_{\bar{p}}(W)$ closed such that
   $$\lceil \sigma^*_W c^*_W \theta - \xi \rceil = d\lceil \nu \rceil \tag{20}$$
   for some $\nu \in \Omega I^{r-1}_{\bar{p}}(W)$. We can then describe Λ by
   $$\Lambda[\theta] = [\xi] \in H^r\big(\tau_{<L}\Omega I^{\bullet}_{\bar{p}}(W)\big). \tag{21}$$

To prove the commutativity of the (BS) we have to show that for $\eta \in \tau_{\geq L^*}\Omega I^{n-r-1}_{\bar{q}}(W)$ and $\theta \in \widetilde{\Omega I}^{r}_{\bar{p}}(M)$ closed with $\Delta[\eta] = [-d\omega]$ and $\Lambda[\theta] = [\xi]$ as above it holds that
$$\int_M \theta \wedge (-d\omega) = \pm \int_W \xi \wedge \eta.$$

By Stokes' Theorem on manifolds with corners, we get

$$\int_M \theta\wedge(-d\omega) = \int_M d(\theta\wedge\omega) = \int_E \sigma_E^* c_E^*(\theta\wedge\omega) + \int_W \sigma_W^* c_W^*(\theta\wedge\omega).$$

By definition, there are $\theta_E \in ft_{\geq K}\Omega^r_{\mathcal{MS}}(B)$, $\omega_E \in ft_{\geq K^*}\Omega^{n-r-1}_{\mathcal{MS}}(B)$ with $\sigma_E^* c_E^*\theta = \theta_E$, $\sigma_E^* c_E^*\omega = \omega_E$. Hence by Lemma 7.4.3,

$$\int_E \sigma_E^* c_E^*(\theta\wedge\omega) = \int_E \theta_E\wedge\omega_E = 0.$$

This implies that

$$\int_M \theta\wedge d\omega = \int_W \sigma_W^* c_W^*(\theta\wedge\omega) = \int_W \sigma_W^* c_W^*\theta\wedge(\eta + d\alpha),$$

for some $\tau \in \tau_{\geq L^*}\Omega I^{n-r-2}_{\bar q}(W)$. Since $j_{\partial W}\theta \in ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(\partial B)$, $j^*_{\partial W}\tau \in ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS}}(\partial B)$, applying Stokes' Theorem and [Ban11, Lemma 7.3] afterwards we get

$$\int_W \sigma_W^* c_W^*\theta\wedge d\tau = \int_{\partial W} j^*_{\partial W}(\theta\wedge\tau) = 0.$$

So we arrive at the equation

$$\int_M \theta\wedge d\omega = \int_W \sigma_W^* c_W^*\theta\wedge\eta.$$

On the other hand, by (20) we get

$$\int_W \xi\wedge\eta = \int_W \sigma_W^* c_W^*(\theta)\wedge\eta + \int_W \alpha\wedge\eta - \int_W d\nu\wedge\eta,$$

where $\alpha := \xi + d\nu - \sigma_W^* c_W^*\theta \in Q^r_{\bar p}(W)$. As before we have: If $r \geq L$, then $n-r-1 < L^*$ and hence $\eta \in \tau_{\geq L^*}\Omega I^{n-r-1}_{\bar q}(W) = \{0\}$, implying $\eta = 0$. If $r < L$, then $\alpha \in \tau_{\geq L}\Omega^r(W) = \{0\}$, and hence $\alpha = 0$. In both cases we have

$$\int_W \alpha\wedge\eta = 0.$$

Since $\nu \in \Omega I^{r-1}_{\bar p}(W)$, there is a $\nu_0 \in ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(\partial B)$ such that $j^*_{\partial W}\nu = \pi^*_{\partial W}\nu_0$ and $\eta \in \tau_{\geq L^*}\Omega I^{n-r-1}_{\bar q}(W)$ implies that there exists a form $\eta_0 \in ft_{\geq K^*}\Omega^\bullet_{\mathcal{MS}}(\partial B)$ with $j^*_{\partial W}\eta = \pi^*_{\partial W}\eta_0$. Therefore (and since $d\eta = 0$), we get by Stokes' Theorem :

$$\int_W d\nu\wedge\eta = \int_W d(\nu\wedge\eta) = \int_{\partial W}(\nu\wedge\eta)|_{\partial W} = \int_{\partial W}\nu_0\wedge\eta_0 = 0$$

by [Ban11, Lemma 7.3]. Hence

$$\int_W \xi\wedge\eta = \int_W \sigma_W^* c_W^*(\theta)\wedge\eta = \int_N \theta\wedge(-d\omega),$$

which means that (BS) commutes (up to sign).

The middle square in (17) commutes, since the vertical maps are just inclusions and the horizontal maps both integration of wedge products of two forms.

The commutativity of the diagram (17) together with the fact that the map

$$\int : H^r\big(\tau_{<L}\Omega I^\bullet_{\bar{p}}(W)\big) \to H^{n-r-1}\big(\tau_{\geq L^*}\Omega I^\bullet_{\bar{q}}(W)\big)^\dagger$$

is an isomorphism for all $r \in \mathbb{Z}$ by Lemma 8.2.6 as well as the map

$$\int : \widetilde{HI}^r_{\bar{p}}(M) \to \widetilde{HI}^{n-r}_{\bar{q}}(M, W)^\dagger$$

by Proposition 7.5.5 then enables us to apply the 5-Lemma to conclude the statement of the theorem.

$\square$

9 A De Rham Theorem for Three Strata Spaces

To justify that our definition of the intersection form complex $\Omega I^{\bullet}_{\bar{p}}$ is plausible, we look at the pseudomanifolds with three strata for which Banagl defines an intersection space in [Ban12]. We will show that the additional assumption on $\Omega I^{\bullet}_{\bar{p}}(W)$ is satisfied in this setting, such that Theorem 8.4.1 holds, and that for $\bar{p} = \bar{m}$, the lower middle perversity, the cohomology groups $HI^{r}_{\bar{m}}(X)$ are isomorphic to the linear dual of the intersection space homology groups with real coefficients $H_r(IX)$. This de Rham type theorem will use an alternative description of IX (we use a space that is homotopy equivalent) and the de Rham results of [Ban11, Section 9] and [Ess12]. But let us first recall the class of spaces we work with and the results of [Ban12]: We work with pseudomanifolds X^n of even dimension $n = 2k$ and filtration

$$X = X_{2k} \supset \underbrace{X_1}_{\cong S^1} \supset X_0 = \{x_0\}$$

satisfying the strong Witt condition. To explain this condition let L be the link of X_1 in X. Then X satisfies the strong Witt condition if and only if the link L posseses a CW-structure such that the cellular boundary operator

$$\partial : C_{k-1}(L) \to C_{k-2}(L)$$

is injective (note that $k = n - 1 - \bar{m}(n-2)$ is the truncation value). This clearly implies $H_{k-1}(L) = 0$, the original Witt condition of [Sie83]. To be able to use differential forms we further demand that X is Thom-Mather stratified. For such spaces Banagl uses 3-diagrams to define intersection spaces as follows.

9.1 Intersection Spaces of such Pseudomanifolds

The bottom stratum $X_0 = \{x_0\}$ has a tubular neighbourhood $N_0 = \operatorname{cone} L_0$ with L_0 the link of $X_0 = \{x_0\}$. The space $X' := X - N_0$ is a pseudomanifold-with-boundary L_0 and singular stratum $X'_1 := X_1 \cap X' \cong \Delta^1$, a closed interval. The link of X'_1 is L, for which one chooses a CW-structure such that the strong Witt condition holds with respect to this structure. L_0 is a pseudomanifold with two isolated singularities $L_0 \cap X'_1 = \partial\Delta^1 = \Delta^0_0 \sqcup \Delta^0_1$ and link L.
In order to be able to perform spatial homology truncation we assume the links L and L_0 to be simply connected. Note that by the work of Wrazidlo, [Wra13], we would not have been forced to make this assumption to choose mere truncations on the object level. But we also want truncate the morphism f below so we need the simple connectivity assumption since one needs more assumptions to truncate a map otherwise (compare to [Wra13, Proposition 1.11]).

Regular neighbourhoods of the isolated singularities are PL-homeomorphic to cone L. We remove them from L_0 to get a manifold W with boundary $\partial W = L \sqcup L$. We fix a CW-structure on W. By the arguments of Banagl, a regular neighbourhood of the singular set $X_1' \subset X'$ is PL homeomorphic to $\Delta^1 \times \text{cone}(L)$. If we remove that regular neighbourhood of X' we get a compact n-manifold M with boundary ∂M, which is up to homotopy equivalence the homotopy colimit $|H(\Gamma)|$ of the following CW-3-diagram Γ:

$$W \xleftarrow{f} \ddot{L} \hookrightarrow \overline{L}$$

with $\ddot{L} = L \times \partial\Delta^1$, $\overline{L} = L \times \Delta^1$ and $f : \ddot{L} \to W$ a cellular approximation of the composition of the above PL homeomorphism $\ddot{L} \to \partial W$ with the inclusion $\partial W \hookrightarrow W$. So $\partial M \simeq |H(\Gamma)|$ and that will be the homotopy theoretic model of ∂M one uses to define the intersection space of X.
We will review the construction of the lower middle perversity intersection space $I^{\bar{m}}X$ here and do not recall the argument that by the strong Witt condition the intersection space for the upper middle perversity is equal to the lower middle perversity one: $I^{\bar{m}}X = I^{\bar{n}}X$. The (co)truncation values in this setting are

$$\begin{aligned} K &= 2k - 2 - \bar{m}(2k-1) = k \quad \text{and} \\ L &= 2k - 1 - \bar{m}(2k) = k. \end{aligned}$$

We note that W is also simply connected (see [Ban12, p.18] for an argument) and choose a completion of W to an object (W, Y_W) in $\mathbf{CW}_{n\supset\partial}$ and note that by the strong Witt condition $(L, Y_L = 0)$ is a completion of L to an object in $\mathbf{CW}_{n\supset\partial}$. Since for $f_i = f| : L = \Delta^0_i \times L \to W$ it holds that $(f_i)_*(0) \subset Y_W$ we can truncate these maps and hence also $f : L \times \Delta^0 \to W$ to a map $f_{<k} = (f_0)_{<k} \sqcup (f_1)_{<k} : \ddot{L}_{<k} \to W_{<k}$.
Let $\overline{L}_{<k} := \Delta^1 \times L_{<k}$ and $\Delta^{\bar{m}}$ be the 3-diagram

$$W_{<k} \xleftarrow{f_{<k}} \ddot{L}_{<k} \hookrightarrow \overline{L}_{<k}.$$

Then the following commutative diagram induces a cellular morphism $\epsilon : \Gamma^{\bar{m}} \to \Gamma$:

$$\begin{tikzcd} W_{<k} \arrow[d, "e_W"] & \ddot{L}_{<k} \arrow[l, "f_{<k}"'] \arrow[r, hook] \arrow[d, hook] & \overline{L}_{<k} \arrow[d, hook] \\ W & \ddot{L} \arrow[l, "f"'] \arrow[r, hook] & \overline{L} \end{tikzcd}$$

Here e_W is a cellular rel $(k-1)$-skeleton representative of $\text{emb}_k(W, Y_W)$ of [Ban10, Theorem 1.41]. Again, see [Ban12] for more details. The map ϵ then induces a cellular map $|H(\epsilon)| : |H(\Gamma)| \to |H(\Gamma^{\bar{m}})|$ and the lower middle perversity intersection space of X is defined as the homotopy cofiber, i.e. the mapping cone, of the composition

$$|H(\Gamma^{\bar{m}})| \xrightarrow{|H(\epsilon)|} |H(\Gamma)| \xrightarrow{\simeq} \partial M \hookrightarrow M.$$

As already stated, the strong Witt condition implies $I^{\bar{n}}X = I^{\bar{m}}X$ and hence we define $IX := I^{\bar{m}}X$ and can state the Poincaré duality theorem as proved in [Ban12, Section 6]:

Theorem 9.1.1 *(Poincaré Duality for Intersection Space Cohomology in Depth 2)*
Let X^n be an n-dimensional, compact, oriented PL pseudomanifold with $n = 2k$ that can be endowed with a PL filtration of the form $X = X_n \supset X_1 \supset X_0 = \{x_0\}$ with $X_1 \cong S^1$, such that the links of the two strata are simply connected and X satisfies the strong Witt condition. Then there exists a Poincaré duality isomorphism

$$D : \widetilde{H}^{n-r}(IX;\mathbb{Q}) \xrightarrow{\cong} \widetilde{H}_r(IX;\mathbb{Q})$$

that extends Poincaré-Lefschetz duality for the regular part $(M, \partial M)$ of X.

9.2 $\Omega I^{\bullet}_{\bar{p}}$– Differential Forms in this Setting

Since the pseudomanifolds treated in [Ban12] are the only 3-strata pseudomanifolds for which intersection spaces are defined yet such that they inherit Poincaré duality, we clearly want to be able to apply our differential forms approach. As mentioned at the beginning of this section, we therefore additionally assume that the pseudomanifolds are Thom-Mather stratified with strata $X_1 - X_0 \cong \Delta^1$ and $X_0 = \{x_0\}$ and that the regular neighbourhoods in the discussion above are tubular neighbourhoods contained in the control data. We additionally demand that the restriction of the PL homeomorphism $N_1' \xrightarrow{\cong} \Delta^1 \times \mathrm{cone}(L)$, where N_1' is the apropriate tubular neighbourhood of $X_1' \subset X'$, to $N_1' - X_1' \to \Delta^1 \times (0,1) \times L$ is a diffeomorphism (some authors call such maps controlled isomorphisms, see e.g. [Pfl01, p.127]).

We then are able to define the intersection form complex $\Omega I^{\bullet}_{\bar{m}}(M)$ for M defined as above. In this smooth setting it is a smooth $\langle 2\rangle$-manifold with boundary parts W an $E \cong \Delta^1 \times L$. To be able to apply the Poincaré duality Theorem 8.4.1 for $HI^{\bullet}_{\bar{m}}(X)$ in this setting, we need $\Omega I^{\bullet}_{\bar{m}}(M)$ to be geometrically cotruncatable in degree $k = n/2$. But this follows from the strong Witt condition for X:

Since L_0 is a pseudomanifold with two isolated singularities Δ^0_0, Δ^0_1. Hence by [Ban10, Theorem 2.12] there is a short exact sequence

$$0 \to H_k(W) \to \widetilde{H}_k(I^{\bar{m}}L_0) \to \operatorname{im}\partial_* \to 0$$

where $\partial^* : H_k(W, \partial W) \to H_{k-1}(\partial W)$ is the connecting homomorphism of the long exact homology sequence of the pair $(W, \partial W)$. The strong Witt condition in particular implies in particular that $H_{k-1}(L) = 0$. But $\partial W \cong L \sqcup L$ and hence $H_{k-1}(\partial W) = 0$, implying $\partial^* = 0$. That gives that the inclusion $W \hookrightarrow IL_0$ induces an isomorphism $H_k(W) \xrightarrow{\cong} \widetilde{H}(IL_0)$.
Moreover using the differential complex $C_\bullet(g_0)$, with $g_0 : \ddot{L}_{<k} \to \partial W \hookrightarrow W$, to describe the intersection space cohomology (making use of the fact $IL_0 = \text{cone}(g_0)$), together with the complex of partially smooth chains $S_\bullet^\propto(g_0)$ defined by $S_r^\propto(g_0) := H_{r-1}(\ddot{L}_{<k}) \oplus S_r^\infty(W)$ and with suitable boundary operator (see [Ban11, Section 9]) and the complex $U_\bullet(g_0)$ defined by $U_r(g_0) := S_{r-1}(L_0) \oplus S_r^\infty(W)$ with analogous boundary map we get the following commutative diagram:

$$\begin{tikzcd}
S_\bullet^\infty(W) \arrow[r, hook] \arrow[dd, hook] & S_\bullet^\propto(g_0) \arrow[d] \\
 & U_\bullet(g_0) \\
S_\bullet(W) \arrow[r, hook] & C_\bullet(g_0) \arrow[u]
\end{tikzcd}$$

This induces a commutative diagram on homology with the horizontal map on the bottom an isomorphism by the above argument and the vertical maps all isomorphisms by the Lemmata 9.1 and 9.2 of [Ban11] and for example [Lee13, Theorem 18.7]. Therefore the induced map on homology $H_k^\infty(W) \hookrightarrow H_k\big(S_\bullet^\propto(g_0)\big)$ is also an isomorphism.
Together with the de Rham Theorem [Ban11, Theorem 9.13] and the commutative diagram

$$\begin{tikzcd}
HI_{\bar{m}}^k(L_0) \arrow[r] \arrow[d, "\cong"', "\Psi_{\bar{m}}"] & H^k(W) \arrow[d, "\cong"', "\Psi_W"] \\
H_k\big(S_\bullet^\propto(g_0)\big) \arrow[r, "\text{inc}_*^\dagger", "\cong"'] & H_k\big(S_\bullet^\infty(W)\big)^\dagger
\end{tikzcd}$$

of [Ban11, Lemma 9.11] this implies that subcomplex inclusion induces an isomorphism $HI_{\bar{m}}^k(L_0) \xrightarrow{\cong} H^k(W)$, in particular this map is injective, implying that $\Omega I_{\bar{m}}^\bullet(W)$ is geometrically cotruncatable in degree k by Lemma 6.4.2.

9.3 A De Rham Theorem in this Setting

To prove a de Rham Theorem in this setting we will use the de Rham theorems for HI of [Ban11] and [Ess12]. The approach will be similar to

the proof of the Poincaré Duality Theorem 8.4.1: We first show that integrating forms over cycles induces an isomorphism from $H^r\big(\widetilde{\Omega I}^{\bullet}_{\bar{m}}(M)\big)$ to $\big(H_r(I^{\bar{m}}X')\big)^{\dagger}$ and then use this result to prove the de Rham Theorem for HI.
Note that $I^{\bar{m}}X' = \operatorname{cone} g'$ with $g' : L_{<k} \times \Delta^1 = \overline{L}_{<k} \hookrightarrow L \times \Delta^1 = \overline{L} \hookrightarrow M$.

Remark 9.3.1 *(On Lee's smoothing operator on smooth $\langle n\rangle$-manifolds)*
The de Rham statements that follows will use the partial smoothing technique of [Ban11, Section 9] which makes use of Lee's smoothing operator $s : S_\bullet(M) \to S^\infty_\bullet(M)$ on smooth manifolds with boundary M. The question is whether there is also a smoothing operator if M is an $\langle n\rangle$-manifold. As one might expect, the answer is yes: It boils down to the fact, that Whitney's approximation Theorem (see [Lee13, Theorem 9.27] for a version for manifolds with boundary) is also true for $\langle n\rangle$-manifolds. The proof is the same as the proof of [Lee13, Theorem 9.27]:
By choosing a compatible system of collars as in [Lau00, Lemma 2.1.6] one gets proper smooth embeddings $R_0 : M \to M - \partial M_0$, where the set $\{\partial M_i\}_{0\le i\le n-1}$ denotes the boundary parts of M such that the following holds: For $\iota_0 : M - \partial M_0 \hookrightarrow M$ each of the two maps $R_0 \circ \iota_0 : M - \partial M_0 \to M - \partial M_0$ and $\iota_0 \circ R_0 : M \to M$ are smoothly homotopic to the identity (the proof is the same as the proof for manifolds with boundary, see [Lee13, Theorem 9.26]). Inductively one gets proper smooth embeddings $R_i : M - \partial M_0 - ... - \partial M_{i-1} \to M - \partial M_0 - ... - \partial M_i$ such that for $\iota_i : M - \partial M_0 - ... - \partial M_i \hookrightarrow M - \partial M_0 - ... - \partial M_{i-1}$ both compositions $R_i \circ \iota_i$ and $\iota_i \circ R_i$ are smoothly homotopic to the identity and hence the same result holds for $R := R_{n-1} \circ ... \circ R_0 : M \to M - \partial M$ and $\iota : M - \partial M \hookrightarrow M$. To see that int$(M) \hookrightarrow M$ is a homotopy equivalence you show by induction that removing the boundary parts ∂M_i one after another yields to homotopy equivalences.

Hence by the same arguments as in [Lee13, pp. 473-480], there is a smoothing operator on $\langle n\rangle$-manifolds. In particular it can be chosen such that for any $j \le k \le n$ and for any $i_1, ...i_k \in \{0, ..., n-1\}$ one has that

$$\begin{array}{ccc} S_\bullet(\partial M_{i_1} \cap ... \cap \partial M_{i_k}) & \longrightarrow & S_\bullet(\partial M_{i_1} \cap ... \cap \partial M_{i_{j-1}} \cap \partial M_{i_{j+1}} \cap ...\partial M_{i_k}) \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ S^\infty_\bullet(\partial M_{i_1} \cap ... \cap \partial M_{i_k}) & \longrightarrow & S^\infty_\bullet(\partial M_{i_1} \cap ... \cap \partial M_{i_{j-1}} \cap \partial M_{i_{j+1}} \cap ...\partial M_{i_k}) \end{array}$$

commutes, i.e. each simplex in a boundary part or the intersection of boundary parts is smoothened within the same. Finally this allows us to use partial smoothing as in [Ban11] here.

To prove the de Rham Theorem for HI we will use the following homotopy theoretic description of the intersection space of X. The idea is to write the intersection space (up to homotopy equivalence) as the mapping cone of a map $h : (IL_0)_{<k} \to IX'$, where $(IL_0)_{<k}$ is a spatial homology k-truncation of the middle perversity intersection space of the link L_0 (in the sense that the homology conditions hold) and IX' is the middle perversity intersection space of X'.

Proposition 9.3.2 *(Alternative description of the homotopy type of IX)* *Let $h : W_{<k} \cup_{f_{<k}} \mathrm{cone}(\ddot{L}_{<k}) \xrightarrow{e_W \cup_{f_k} \mathrm{id}} IL_0 \hookrightarrow IX'$, where IL_0 is the middle perversity intersection space of L_0 and IX' is the middle perversity intersection space of X'. Let $c \in IL_0$ denote the conepoint. Then collapsing $c \times [0,1] \subset \mathrm{cone}\big(W_k \cup \mathrm{cone}(\ddot{L}_{<k})\big) \subset \mathrm{cone}\, h$ in the mapping cone of h to a point yields a homotopy equivalence $\mathrm{cone}\, h \xrightarrow{\simeq} IX$.*

<u>Proof:</u> Since h is a cellular map between CW-complexes, the mapping cone of h is a CW-complex. $c \times [0,1]$ is a 1-cell in this CW-complex, so the pair $\big(\mathrm{cone}\, h, c \times [0,1]\big)$ has the homotopy extension property. Also, $c \times [0,1]$ is contractible, so by the homotopy extension property, the homtopy between $c \times [0,1]$ and $c \times \{0\}$ together with the identity on $\mathrm{cone}\, h$ can be extended to a homotopy equivalence $\mathrm{cone}\, h \xrightarrow{\simeq} \frac{\mathrm{cone}\, h}{c \times [0,1]} = IX$ which is the mapping cone of the map $|H(\Gamma^{\bar{m}})| \xrightarrow{|H(\epsilon)|} |H(\Gamma)| \xrightarrow{\simeq} \partial M \hookrightarrow M$, see Subsection 9.1. □
We will henceforth use $\mathrm{cone}\, h$ as model (up to homotopy equivalence) of IX. To describe the reduced homology of $\mathrm{cone}\, h$, we will use the algebraic mapping cone together with the following lemma. There and henceforth for a topological space X, $S_\bullet(X)$ denotes the singular chain complex of X. For a map $f : X \to Y$ between topological spaces, $C_\bullet(f)$ denotes the algebraic mapping cone of $f_* : S_\bullet(X) \to S_\bullet(Y)$.

Lemma 9.3.3 *Given the following diagram in the category* **Top**

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow f & & \downarrow g \\ X & \xrightarrow{j} & Y \end{array}$$

there is a continuous map $\phi : \mathrm{cone}(f) \to \mathrm{cone}(g)$ given by j on X and by $(x,t) \mapsto \big(i(x), t\big)$ on $\mathrm{cone}\, A$. Further there is a chain complex $C_\bullet(f,j,i,g)$ given by $C_r(f,j,i,g) := C_{r-1}(f) \oplus C_r(g) = S_{r-2}(A) \oplus S_{r-1}(X) \oplus S_{r-1}(B) \oplus S_r(Y)$ and the boundary formula

$$\begin{aligned} \partial(a,x,b,y) :&= \big(-\partial(a,x), \partial(v,x) - (i_*(a), j_*(x))\big) \\ &= \big(\partial a, -\partial x + f_*(a), -\partial b - i_*(a), \partial y - g_*(b) - j_*(x)\big). \end{aligned}$$

Then the chain map

$$\alpha : C_\bullet(f,j,i,g) \to C_\bullet(\phi), \quad (a,x,b,y) \mapsto (-c(f_* a) + x, -c(g_* b) + y),$$

with ca, cb the cones of the singular chains a, b, see e.g. [Dol80, pp. 34-35] for a definition, is a quasi-isomorphism.

Proof: We will first show, that α is indeed a chain map. We use the formula $\partial(ca) = a - c(\partial a)$ for the cones of singular chains:

$$\begin{aligned}\partial\alpha(a,x,b,y) &= \big(\partial c(f_*a) - \partial x, \partial y - \partial(c(g_*b)) - \phi_*(x) + \phi_*(c(f_*a))\big)\\ &= \big(f_*a - c(\partial f_*a) - \partial x, \partial y - g_*b + c(\partial(g_*b)) - j_*x + c(g_*i_*a)\big),\end{aligned}$$

while

$$\begin{aligned}\alpha\partial(a,x,b,y) &= \alpha(\partial a, -\partial x + f_*a, -\partial b - i_*a, \partial y - g_*b - j_*x)\\ &= \big(-c(\partial f_*a) - \partial x + f_*a, c(\partial g_*b) + c(g_*i_*a) + \partial y - g_*b - j_*x\big).\end{aligned}$$

So both are equal, showing that α is indeed a chain map. The commutativity of the following diagram together with the 5-Lemma then implies the statement:

$$\begin{tikzcd}
0 \arrow[r] & C_\bullet(g) \arrow[r, "incl"] \arrow[d, "qis"', "\xi"] & C_\bullet(f,j,i,g) \arrow[r, "pr"] \arrow[d, "\alpha"] & C_{\bullet-1}(f) \arrow[r] \arrow[d, "qis"', "\rho"] & 0\\
0 \arrow[r] & S_\bullet(\operatorname{cone} g) \arrow[r, "incl"] & C_\bullet(\phi) \arrow[r, "pr"] & S_{\bullet-1}(\operatorname{cone} f) \arrow[r] & 0
\end{tikzcd}$$

where $\xi(b,y) := -c(g_*b) + y,\ \rho(a,x) := -c(f_*a) + x.$ □

This is the final tool to be able to prove the de Rham Theorem for HI:
Let $i : \ddot{L}_{<k} \hookrightarrow \overline{L}_{<k}$, $j := j_W \circ e_W : W_{<k} \to W \hookrightarrow M$, $f_{<k} : \ddot{L}_{<k} \to W_{<k}$ and $g' : \overline{L}_{<k} \hookrightarrow L \hookrightarrow M$. These fit into a commutative diagram

$$\begin{tikzcd}
\ddot{L}_{<k} \arrow[r, "i"] \arrow[d, "f_{<k}"] & \overline{L}_{<k} \arrow[d, "g'"]\\
W_{<k} \arrow[r, "j"] & M
\end{tikzcd}$$

and hence by the previous lemma there is a quasi-isomorphism

$$\alpha : C_\bullet(f_{<k}, j, i, g') \to C_\bullet(h),$$

where $h = \phi$ of the lemma and h is the map of Proposition 9.3.2. This gives isomorphisms $H_r\big(C_\bullet(f_{<k}, j, i, g')\big) \cong H_r\big(C_\bullet(h)\big) \cong \widetilde{H}_r(\operatorname{cone} h) \cong \widetilde{H}_r(IX)$ for all $r \in \mathbb{Z}$.

Theorem 9.3.4 *(A de Rham Theorem in a 3-strata setting)*
There is an isomorphism $\Phi_{\bar{m}} : HI^r_{\bar{m}}(X) \to H_r\big(C_\bullet(f_{<k}, j, i, g')\big)^\dagger \cong \widetilde{H}_r(IX)^\dagger$.

Proof: We use the following short exact sequences:

$$0 \to C_\bullet(g') \to C_\bullet(f_{<k}, j, i, g') \to C_{\bullet-1}(f_{<k}) \to 0$$

and the sequence of Lemma 8.2.4

$$0 \to \Omega I^\bullet_{\bar{m}}(M) \to \widetilde{\Omega I}^\bullet_{\bar{m}}(M) \to \tau_{<k}\Omega I^\bullet_{\bar{p}}(W) \to 0.$$

To get the isomorphism $\Phi_{\bar{m}}$, we use the maps $g_0 : \ddot{L}_{<k} \to W$ and $g' : \overline{L}_{<k} \to M$. Choose chain maps $\ddot{q} : H_\bullet(\ddot{L}_{<k}) \to S_\bullet(\ddot{L}_{<k})$ and $\overline{q} : H_\bullet(\overline{L}_{<k}) \to S_\bullet(\overline{S}_{<k})$ as in [Ban11, Section 9.1], i.e. such that $\big[\ddot{q}(x)\big] = x$ for every $x \in H_\bullet(\ddot{L}_{<k})$ and the same for $\overline{q}$. Then the partial smooth chain complexes $S^{\propto}_\bullet(g_0)$ and $S^{\propto}_\bullet(g')$ are defined by setting

$$\begin{aligned} S^{\propto}_r(g_0) &:= H_{r-1}(\ddot{L}_{<k}) \oplus S^{\infty}_r(W) \\ \partial(x,w) &:= \big(0, \partial w - s g_{0*}\ddot{q}(x)\big) \end{aligned}$$

and

$$\begin{aligned} S^{\propto}_r(g') &:= H_{r-1}(\overline{L}_{<k}) \oplus S^{\infty}_r(M) \\ \partial(y,v) &:= \big(0, \partial v - s g'_*\overline{q}(y)\big), \end{aligned}$$

where $s : S_\bullet(\cdot) \to S^{\infty}_\bullet(\cdot)$ is Lee's smoothing operator, see [Lee13, Proof of Theorem 18.7]. The cohomology groups of these complexes are isomorphic to the cohomology groups of the ordinary mapping cones by the Lemmata 9.1 and 9.2 of [Ban11]. In detail, for a map $g : X \to M$ from a topological space X to a smooth manifold M, Banagl uses the complex $U_\bullet(g)$, defined by

$$\begin{aligned} U_r(g) &:= S_{r-1}(X) \oplus S^{\infty}_r(M), \\ \partial(x,w) &:= (-\partial x, \partial w - s g_* x). \end{aligned}$$

He proves that the chain maps $\mathrm{id} \oplus s : C_\bullet(g) \to U_\bullet(g)$ and $q \oplus \mathrm{id} : S^{\propto}_\bullet(g) \to U_\bullet(g)$ are quasi isomorphisms, where $S^{\propto}_\bullet(g)$ is defined in analogy to the complexes $S^{\propto}_\bullet(g_0), S^{\propto}_\bullet(g')$. We will apply this to the maps g_0 and g'. Before doing so, note that the de Rham Theorem for HI in the two strata setting implies that the following maps induced by integration of forms over cycles are isomorphisms:

$$\begin{aligned} \widetilde{\Psi}_{\bar{m}} : H^r\big(\widetilde{\Omega I}^\bullet_{\bar{m}}(M)\big) &\to H_r\big(S^{\propto}_\bullet(g')\big)^\dagger \\ \widetilde{\Psi}_{\bar{m}}\big([\omega]\big)\big[(x,v)\big] &:= \int_v \omega \end{aligned}$$

and

$$\begin{aligned} \Phi_W : HI^r_{\bar{m}}(L_0)\big) &\to H_r\big(S^{\propto}_\bullet(g_0)\big)^\dagger \\ \Phi_W\big([\omega]\big)\big[(x,v)\big] &:= \int_v \omega. \end{aligned}$$

As in [Ess12, Proposition 3.3.2], these are well defined maps. Φ_W is an isomorphism by [Ban11, Theorem 9.13] and $\Phi_{\bar{m}}$ is an isomorphism by [Ess12, Theorem 9.4.1], which is applicable analogously.
We use Φ_W to define the map $\Phi_W^{<k} : H_r\big(\tau_{<k}\Omega I^\bullet_{\bar{m}}(W)\big) \to H_r\big(C_\bullet(f_{<k})\big)$. To do so, we need to investigate the mapping cone $C_\bullet(f_{<k})$ first. For $r \geq k$, the homology groups $H_r(f_{<k})$ vanish by a Mayer–Vietoris argument and the strong Witt condition. For a detailed proof, we refer to [Ban12, Lemma 5.1]. For $r < k$, the chain map $\mathrm{id} \oplus e_{W*} : C_\bullet(f_{<k}) \to C_\bullet(g_0)$ is a quasi isomorphism by an argument using the 5-Lemma, the commutativity of the diagram

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet(W_{<k}) \arrow[r] \arrow[d, "e_W"] & C_\bullet(f_{<k}) \arrow[r] \arrow[d, "\mathrm{id}\oplus e_W"] & S_{\bullet-1}(\ddot{L}_{<k}) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & S_\bullet(W) \arrow[r] & C_\bullet(g_0) \arrow[r] & S_{\bullet-1}(\ddot{L}_{<k}) \arrow[r] & 0
\end{tikzcd}$$

and that $e_W : S_\bullet(W_{<k}) \to S_\bullet(W)$ induces an isomorphism on homology in degrees $r < k$. Note, that $\mathrm{id} \oplus e_{W*}$ is a chain map since $g_0 = e_W \circ f_{<k}$. We then define $\Phi_W^{<k}$ as follows: For $r \geq k$, we set $\Phi_W^{<k} = 0$. For $r < k$ we let $\Phi_W^{<k}$ be the composition

$$\begin{tikzcd}
H^r\big(\tau_{<k}\Omega I^\bullet_{\bar{m}}(W)\big) \arrow[r, "\cong", "\mathrm{incl}^*"'] & HI^r_{\bar{m}}(W) \arrow[r, "\cong", "\Phi_W"'] & H_r\big(S^\propto_\bullet(g_0)\big)^\dagger \arrow[d, "\cong"', "{\left[(\ddot{q}\oplus\mathrm{id})_*^{-1}\right]^\dagger}"] \\
H_r\big(C_\bullet(f_{<k})\big)^\dagger & H_r\big(C_\bullet(g_0)\big)^\dagger \arrow[l, "\cong", "(\mathrm{id}\oplus e_{W*})^\dagger"'] & H_r\big(U_\bullet(g_0)\big)^\dagger \arrow[l, "\cong", "(\mathrm{id}\oplus s)_*^\dagger"']
\end{tikzcd}$$

We further define $\widetilde{\Phi}_{\bar{m}} : H^\bullet\big(\widetilde{\Omega I}^\bullet_{\bar{m}}(M)\big) \to H_\bullet(g')^\dagger$ as the following composition:

$$H^\bullet\big(\widetilde{\Omega I}^\bullet_{\bar{m}}(M)\big) \xrightarrow{\widetilde{\Psi}_{\bar{m}}} H_\bullet\big(S^\propto_\bullet(g')\big)^\dagger \xrightarrow{\alpha^\dagger} H_\bullet\big(U_\bullet(g')\big)^\dagger \xrightarrow{(\mathrm{id}\oplus s)^\dagger} H_\bullet(g')^\dagger,$$

where $\alpha := (\bar{q} \oplus \mathrm{id})_*^{-1} : H_\bullet\big(U_\bullet(g')\big) \xrightarrow{\cong} H_\bullet\big(S^\propto_\bullet(g')\big)$.

We claim that the following diagram commutes for arbitrary $r \in \mathbb{Z}$:

$$\begin{tikzcd}
H^{r-1}\big(\widetilde{\Omega I}^{\bullet}_{\bar{m}}(M)\big) \arrow[r, "\cong", "\widetilde{\Phi}_{\bar{m}}"'] \arrow[d] & H_{r-1}\big(C_\bullet(g')\big)^\dagger \arrow[d] \\
H^{r-1}\big(\tau_{<k}\Omega I^{\bullet}_{\bar{m}}(W)\big) \arrow[r, "\cong", "\Phi_W^{<k}"'] \arrow[d] & H_{r-1}\big(C_\bullet(f_{<k})\big)^\dagger \arrow[d] \\
HI^r_{\bar{m}}(X) \arrow[d] & H_r\big(C_\bullet(f_{<k}, i, j, g')\big)^\dagger \arrow[d] \\
H^{r}\big(\widetilde{\Omega I}^{\bullet}_{\bar{m}}(M)\big) \arrow[r, "\cong", "\widetilde{\Phi}_{\bar{m}}"'] \arrow[d] & H_{r}\big(C_\bullet(g')\big)^\dagger \arrow[d] \\
H^{r}\big(\tau_{<k}\Omega I^{\bullet}_{\bar{m}}(W)\big) \arrow[r, "\cong", "\Phi_W^{<k}"'] & H_{r}\big(C_\bullet(f_{<k})\big)^\dagger
\end{tikzcd} \tag{22}$$

This is the case if and only if the square

$$\begin{tikzcd}
H^{r}\big(\widetilde{\Omega I}^{\bullet}_{\bar{m}}(M)\big) \arrow[r, "\widetilde{\Phi}_{\bar{m}}"] \arrow[d] & H_{r}\big(C_\bullet(g')\big)^\dagger \arrow[d, "i_*\oplus j_*"] \\
H^{r}\big(\tau_{<k}\Omega I^{\bullet}_{\bar{m}}(W)\big) \arrow[r, "\Phi_W^{<k}"] & H_{r}\big(C_\bullet(f_{<k})\big)^\dagger
\end{tikzcd} \tag{23}$$

commutes for arbitrary $r \in \mathbb{Z}$. This is trivially true for $r \geq k$ since the groups $H_r\big(C_\bullet(f_{<k})\big)$ are zero in these degrees. We will subdivide the proof for $r < k$ into several steps:

(i) First note that by definition $j = j_W \circ e_W$ and hence the triangle diagram

$$\begin{tikzcd}
C_\bullet(g_0) \arrow[dr, "i_*\oplus j_{W*}"'] & & C_\bullet(f_{<k}) \arrow[ll, "\mathrm{id}\oplus e_{W*}"'] \arrow[dl, "i_*\oplus j_*"] \\
& C_\bullet(g') &
\end{tikzcd}$$

commutes.

(ii) Remark 9.3.1 implies that the smoothing operator s commutes with the inclusion of the boundary part W, i.e. $s \circ j_{W*} = j_{W*} \circ s$. Hence the square

$$\begin{tikzcd}
C_\bullet(g') \arrow[r, "\mathrm{id}\oplus s"] & U_\bullet(g') \\
C_\bullet(g_0) \arrow[r, "\mathrm{id}\oplus s"] \arrow[u, "i_*\oplus j_{W*}"] & U_\bullet(g_0) \arrow[u, "i_*\oplus j_{W*}"']
\end{tikzcd}$$

commutes.

(iii) The main point of the proof is the commutativity of the following diagram for $r < k$:

$$\begin{tikzcd}
H^r\big(\widetilde{\Omega I}^{\bullet}_{\bar{m}}(M)\big) \arrow[r, "\widetilde{\Phi}_{\bar{m}}"] \arrow[d, "j_W^*"] & H_r\big(S^{\propto}_{\bullet}(g')\big)^{\dagger} & H_r\big(U_{\bullet}(g')\big)^{\dagger} \arrow[l, "(\overline{q}\oplus\mathrm{id})_*^{\dagger}"'] \arrow[d, "(i_*\oplus j_*)^{\dagger}"] \\
HI^r_{\bar{m}}(W) \arrow[r, "\Phi_W"] & H_r\big(S^{\propto}_{\bullet}(g_0)\big)^{\dagger} & H_r\big(U_{\bullet}(g_0)\big)^{\dagger} \arrow[l, "(\ddot{q}\oplus\mathrm{id})_*^{\dagger}"']
\end{tikzcd}$$

Let $\omega \in \widetilde{\Omega I}^{r}_{\bar{m}}(M)$ be a closed form and $(x, v) \in S^{\propto}_r(g_0)$ a cycle. We show that

$$\widetilde{\Phi}_{\bar{m}}\big([\omega]\big)\big((\overline{q}\oplus\mathrm{id})_*^{-1}[i_*\ddot{q}(x), j_{W*}v]\big) = \Phi_W\big(j_W^*[\omega]\big)\big([x,v]\big) = \int_v j_W^*\omega.$$

We must find a closed representative of $(\overline{q}\oplus\mathrm{id})_*^{-1}[i_*\ddot{q}(x), j_{W*}v]$ in $S^{\propto}_r(g')$. Since $\overline{q}(i_*x)$ and $i_*\ddot{q}(x)$ are closed with $\big[\overline{q}(i_*x)\big] = i_*(x) = i_*\big[\ddot{q}(x)\big] = \big[i_*\ddot{q}(x)\big]$, there is a form $a \in S_r(\overline{L}_{<k})$ with $\overline{q}(i_*x) - i_*\ddot{q}(x) = \partial a$. Consider the form $\big(i_*x, j_{W*}v + sg'_*(a)\big) \in S^{\propto}_r(g')$. It is closed since $(x, v) \in S^{\propto}_r(g_0)$ is closed:

$$\begin{aligned}
\partial\big(i_*x, j_{W*}v + sg'_*(a)\big) &= \big(0, \partial j_{W*}v - sg'_*(\overline{q}i_*x - a)\big) \\
&= \big(0, j_{W*}(\partial v - sg_{0*}\ddot{q}(x))\big) = 0,
\end{aligned}$$

where we used that $g' \circ i = j_W \circ g_0$ to get the second equality. It is the desired representative of $(\overline{q}\oplus\mathrm{id})_*^{-1}[i_*\ddot{q}(x), j_{W*}v]$:

$$\begin{aligned}
&(\overline{q}\oplus\mathrm{id})\big(i_*x, j_{W*}v + sg'_*(a)\big) = \big(\overline{q}i_*x, j_{W*}v + sg'_*(a)\big) \\
&= (i_*\ddot{q}(x), j_{W*}v) + (\partial a, sg'_*a) = (i_*\ddot{q}(x), j_{W*}v) - \partial(a, 0).
\end{aligned}$$

Again by Remark 9.3.1, $sg'_*(a) = j_{L*}s(a)$ and therefore we have

$$\begin{aligned}
&\widetilde{\Phi}_{\bar{m}}\big([\omega]\big)\big((\overline{q}\oplus\mathrm{id})_*^{-1}[i_*\ddot{q}(x), j_{W*}v]\big) \\
&\quad = \widetilde{\Phi}_{\bar{m}}\big([\omega]\big)\big([i_*x, j_{W*}v + j_{L*}sa]\big) = \int_v j_W^*\omega + \int_{sa} j_L^*\omega \\
&\quad = \int_v j_W^*\omega = \Phi_W\big(j_W^*[\omega]\big)\big([x,v]\big),
\end{aligned}$$

where we used the definition of $\widetilde{\Omega I}^{\bullet}_{\bar{m}}(M)$ in the last line: $j_L^*\omega \in ft_{\geq k}\Omega^r_{\mathcal{MS}}(I) = \{0\}$ for $r < k$.

Putting the three commutative diagrams of (i)–(iii) together, we get the commutativity of the square (23), since for $r < k$ the map $H^r\big(\widetilde{\Omega I}^{\bullet}_{\bar{m}}(M)\big) \to H^r\big(\tau_{<k}\Omega I^{\bullet}_{\bar{p}}(W)\big)$ is the pullback to the boundary part W, j_W^*. This implies

that diagram (22) commutes and hence [Ban10, Lemma 2.46] finishes the proof of the theorem.

□

Note that we would like to write the de Rham isomorphism $\Phi_{\bar{m}}$ as integration of forms over some smooth cycles. In analogue to the two strata setting, we would need a paritial smooth version of the mapping cone $C_\bullet(f_{<k}, i, j, g')$. But this is not defineable analogously. The reason is that the maps

$$\overline{q} : H_\bullet(\overline{L}_{<k}) \to S_\bullet(\overline{L}_{<k}) \quad \text{and} \quad \ddot{q} : H_\bullet(\ddot{L}_{<k}) \to S_\bullet(\ddot{L}_{<k})$$

do not commute with the inclusion map i_*: Embedding a cycle in $S_\bullet(L_{<k})$ into $\overline{L}_{<k}$ at zero or one gives different chains but the same homology class in $H_\bullet(\overline{L}_{<k})$.

10 A Positive Dimensional Bottom Stratum

In this section we deal with stratified pseudomanifolds with three strata and a bottom stratum of positive dimension.

10.1 Matching Flat Bundles

Let us start with a compact n-dimensional smooth $\langle 2\rangle$-manifold M^n with

$$\partial M = E \cup_{\partial E = \partial W} W.$$

For a definition see [Lau00] or Section 4.2.4.
We demand that E and W are the total spaces of geometrically flat fiber bundles

$$p : E \to B$$

and

$$q : W \to \Sigma.$$

The fiber L^m of p shall be a closed Riemannian manifold and the base B^b a compact manifold with boundary ∂B^b. In contrast, the fiber of q shall be a compact Riemannian manifold F^f with boundary $\partial F = Z \times L$ and the base Σ^s, Z^z of q a closed manifolds. In addition we demand that

$$\partial W = \Sigma \times \partial F = \Sigma \times Z \times L = \partial B \times L$$

and that the bundle maps restrict to projections

$$p|_{\partial E} = \pi_1 : \partial E = \partial B \times L \to \partial B$$

and

$$q|_{\partial W} = \widetilde{\pi}_1 : \partial W = \Sigma \times \partial F \to \Sigma.$$

Note, that we in particular want that the restriction of the local trivializations to the boundary equal the identity: There is (a good open) atlas $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ with respect to which the bundle trivializes, i.e. there are diffeomorphisms $\phi_\alpha : q^{-1}(U_\alpha) \to U_\alpha \times \partial F$ with $q|_{U_\alpha} = \pi_1 \circ \phi_\alpha$. What we demand is that the following diagram commutes:

$$\begin{tikzcd} U_\alpha \times \partial F = \partial W \cap q^{-1}(U_\alpha) \arrow[r, equal] \arrow[d, "j_{\partial W}"] & U_\alpha \times \partial F \arrow[d, "\mathrm{id} \times j_{\partial F}"] \\ q^{-1}(U_\alpha) \arrow[r, "\phi_\alpha", "\cong"'] & U_\alpha \times F \end{tikzcd}$$

for all $\alpha \in I$.

We construct a stratified pseudomanifold with three strata out of M by the following process:

1. First, add $E\times[0,1]$ to M via gluing the bottom $E\times\{0\}$ of the cylinder to the boundary part $E\subset\partial M$. Then by collapsing the fibers in $E\times\{1\}$ we get a pseudomanifold X' with boundary
$$W\cup_{\partial W}(\Sigma\times Z\times \mathrm{cone}(L))=\Sigma\times(F\cup_{\partial F} Z\times \mathrm{cone}\, L)$$
and singular stratum B.

2. We get X by coning off $F\cup Z\times\mathrm{cone}(L)$: Glue the bottom $W\times\{0\}$ of the cylinder $W\times[0,1]$ to $W=\partial X'$ and then collapse $F\cup Z\times\mathrm{cone}(L)$ in $W\times 1$ to a point. The result is a compact n-dimensional pseudomanifold
$$X=X_n\supset X_{\text{middle}}\supset X_{\text{bottom}}$$
with
$$\begin{aligned} X_{\text{bottom}}&=\Sigma,\\ X_{\text{middle}}-X_{\text{bottom}}&\simeq B,\\ X-X_{\text{middle}}&\cong N=M-\partial M.\end{aligned}$$

The question is: Given a perversity $\bar{p}$, how do we define the differential complex $\Omega I^\bullet_{\bar{p}}(M)$ such that its cohomology $HI^\bullet_{\bar{p}}(X):=H^\bullet(\Omega I^\bullet_{\bar{p}}(M))$ satisfies Poincaré duality over complementary perversities and is an invariant of X? The idea is the same as in the previous case, where the bottom stratum was an isolated singular point: On both boundary parts of M we have geometrically flat link bundles $E\to B$ and $W\to\Sigma$. So on both we can cotruncate the complex of multiplicatively structured forms in link direction. The intuitive idea is to let $\Omega I^\bullet_{\bar{p}}(M)$ be the complex of forms which are the pullback of fiberwisely cotruncated multiplicatively structured forms on E in a collarlike neighbourhood of $E\subset\partial M$ and the pullback of fiberwisely cotruncated multiplicatively structured forms on W in a collarlike neighbourhood of $W\subset\partial M$. Of course, on the intersection of both collarlike neighbourhoods the forms must satisfy both conditions. We use a pair of p–related collars for B and E and a pair of fiber–related collars for F and W. As in the previous sections, where $X_{\text{bottom}}=pt$, we prove Poincaré duality for a complex quasi isomorphic to $\Omega I^\bullet_{\bar{p}}(M)$ by using the method of iterated triangles:

Definition 10.1.1 *For $\langle 2\rangle$-manifolds M with boundary $\partial M=W\cup_{\partial W=\partial E}E$ as above we define*
$$\begin{aligned}\Omega I^r_{\bar{p}}(M):=\big\{\omega\in\Omega^r(M)\big|\ &\exists E\subset U_E\subset C_E,\ \exists W\subset U_W\subset C_W,\ \textit{both open}:\\ &\omega|_{U_E}=\pi_E^*\eta_E,\ \eta_E\in ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\\ &\omega|_{U_W}=\pi_W^*\eta_W,\ \eta_W\in ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(\Sigma)\big\},\end{aligned}$$
where $K=m-\bar{p}(m+1),\ m:=\dim L$ and $L=f-\bar{p}(f+1),\ f:=\dim F$.

As in the previous setting, where $\Sigma = pt$, we encounter one major problem proving Poinaré duality for the cohomology groups of $\Omega I^{\bullet}_{\bar{p}}(M)$ with our above method of iterated triangles: It is very difficult to prove Poincaré duality for the complex $ft_{\geq L}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma) \cap \Omega I^{\bullet}_{\bar{p}}(W)$, at least directly. As before we will replace this complex by a complex which is quasi isomorphic to it and then prove Poincaré duality for that complex.

Remark 10.1.2 *Note that this setting is not a generalization to what we have done before: In the previous sections we did* <u>*not*</u> *demand the bundle on the boundary part E to be trivial on the corner $\partial E = \partial W$ of M and did also not demand that $\partial W = \partial E = \partial B \times L$, which we do in this section.*

10.2 Fiberwisely $\Omega I^{\bullet}_{\bar{p}}$-Cotruncated Forms

Remark 10.2.1 *Since $\partial W = \Sigma \times \partial F = \Sigma \times Z \times L$, the notion of $\Omega I^{\bullet}_{\bar{p}}(W)$ is not unambiguous in this setting. So what complex is $\Omega I^{\bullet}_{\bar{p}}(W)$? The answer to the question becomes clear as soon as we ask ourselves why to consider forms in the intersection $ft_{\geq L}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma) \cap \Omega I^{\bullet}_{\bar{p}}(W)$: The restriction of forms $\omega \in \Omega I^{\bullet}_{\bar{p}}(M)$ to a collarlike neighbourhood $E \subset U \subset C_E$ of $E \subset M$ must be the pullback of some form in $ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(B)$:*

$$\omega|_U = \pi^*\eta, \text{ for some } \eta \in ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(B).$$

Pulling $\omega|_U$ back to W gives:

$$j_W^*\omega|_U = \pi^* j_W^*(\eta)$$

with $j_W^*\eta \in ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(\partial B = \Sigma \times Z)$. Hence we define and use henceforth

$$\Omega I^r_{\bar{p}}(W) := \big\{\omega \in \Omega^{\bullet}(W) \big| \exists \partial W \subset U \subset C_{\partial W} : \omega|_U = \pi^*\eta, \\ \eta \in ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times Z)\big\}$$

The candidate to replace $ft_{\geq L}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma) \cap \Omega I^{\bullet}_{\bar{p}}(W)$ in the above argument will be defined in 10.2.3. To do that, we first note that since $\partial F = Z \times L$, we can define

$$\Omega I^r_{\bar{p}}(F) := \big\{\omega \in \Omega^r(F) |\ \exists \partial F \subset U_{\partial F} \subset C_{\partial F} :\ \omega|_{U_{\partial F}} = \pi^*\eta, \\ \eta \in ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(\partial F)\big\},$$

where as before $K = m - \bar{p}(m+1)$. In analogue to the previous setting, where we used an additional assumption to cotruncate $\Omega I^{\bullet}_{\bar{p}}(W)$, we need the following assumption to be able to cotruncate $\Omega I^{\bullet}_{\bar{p}}(F)$:

Additional assumption: We assume that the complex $\Omega I^{\bullet}_{\bar{p}}(F)$ is geometrically cotruncateable in degree $L = f - \bar{p}(f)$, i.e.

$$\operatorname{im} d^{L-1} \cap \Omega I^{\bullet}_{\bar{p}}(F) = d^{L-1}(\Omega I^{L-1}_{\bar{p}}(F)).$$

Then we can cotruncate $\Omega I^\bullet_{\bar{p}}(F)$ in that degree.

Later, in the proofs of Proposition 10.2.7 and Lemma 10.2.10 we will need the following fact:

Lemma 10.2.2 *If $\Omega I^\bullet_{\bar{p}}(F)$ is cotruncateable in degree L, then so is*

$$ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Z) \subset \Omega^\bullet(Z \times L).$$

Proof: We must show that $\operatorname{im} d^{L-1} \cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Z) = d^{L-1}\big(ft_{\geq K}\Omega^{L-1}_{\mathcal{MS}}(Z)\big)$. So let $\eta = d\beta \in ft_{\geq K}\Omega^L_{\mathcal{MS}}(Z) \cap \operatorname{im} d^{L-1}$. We will extend β to a form in $\Omega^\bullet_{\partial C}(F)$ using the usual approach: Taking a smooth cutoff function ψ on $\mathbb{R}$ with $\psi(x) = 0$ for $x \leq -1$ and $\psi(x) = 1$ for $x \geq 1$, we use a collar neighbourhood $\widetilde{C}_{\partial F}$ (slightly bigger than $C_{\partial F}$) to get a smooth cutoff function (also denoted by) ψ on F with $\psi|_{C_{\partial F}} = 1$ and $\psi|_{F-\widetilde{C}_{\partial F}} = 0$. Then using the pullback $\pi^*\beta \in \Omega^\bullet(\widetilde{C}_{\partial F})$ extension by zero gives a form

$$\psi\pi^*\beta \in \Omega^\bullet_{\partial C}(F).$$

We look at the derivative of this form:

$$d(\psi\pi^*\beta) = (d\psi)\pi^*\beta + \psi\pi^* d\beta = (d\psi)\pi^*\beta + \psi\pi^*\eta.$$

Restricted to $C_{\partial F}$ this gives:

$$\begin{aligned}\big(d(\psi\pi^*\beta)\big)|_{C_{\partial F}} &= \underbrace{(d\psi)|_{C_{\partial F}}}_{=0}\pi^*\beta) + \underbrace{\psi|_{C_{dF}}}_{=1}\pi^*\eta \\ &= \pi^*\eta\end{aligned}$$

Hence $d(\psi\pi^*\beta) \in \Omega I^L_{\bar{p}}(F) \cap \operatorname{im} d = d^{L-1}\big(\Omega I^{L-1}_{\bar{p}}(F)\big)$, i.e. $d(\psi\pi^*\beta) = d\alpha$ with $\alpha \in \Omega I^\bullet_{\bar{p}}(F)$, so in particular $j^*_{\partial F}\alpha \in ft_{\geq K}\Omega^{L-1}_{\mathcal{MS}}(Z)$, and $dj^*_{\partial F}\alpha = j^*_{\partial F}(d(\psi\pi^*\beta)) = \eta$. $\square$

Let $\mathfrak{U} := \{U_\alpha\}_{\alpha\in I}$ be a finite good open cover of Σ with respect to which $q : W \to \Sigma$ trivializes, i.e. there are diffeomorphisms $\phi_\alpha : q^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times F$ with $q = \pi_1 \circ \phi_\alpha$. Let further $U \subset \Sigma$ be open. We then define

Definition 10.2.3 *(Fiberwisely $\Omega I^\bullet_{\bar{p}}$-cotruncated forms)*

$$\begin{aligned} f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^r_{\mathcal{MS}}(U) := \big\{\omega \in \Omega^\bullet_{\mathcal{MS}}(U) \big| \forall\alpha \in I :& \\ \omega|_{q^{-1}(U\cap U_\alpha)} &= \phi^*_\alpha \sum_{j_\alpha} \pi_1^*\eta^\alpha_{j_\alpha} \wedge \pi_2^*\gamma^\alpha_{j_\alpha}, \\ \textit{with } \gamma^\alpha_{j_\alpha} &\in \tau_{\geq L}\Omega I^\bullet_{\bar{p}}(F)\big\}\end{aligned}$$

Lemma 10.2.4 *$f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma) \subset \Omega I^{\bullet}_{\bar{p}}(W)$ is a subcomplex.*

Proof: Let $\omega \in f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)$ such that

$$\omega|_{q^{-1}(U_\alpha)} = \phi^*_\alpha \sum_{j_\alpha \in J_\alpha} \pi_1^*\eta^\alpha_{j_\alpha} \wedge \pi_2^*\gamma^\alpha_{j_\alpha} \in f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma).$$

Further let, for each α and each j_α, $\partial F \subset V^\alpha_{j_\alpha} \subset C_{\partial F}$ denote the collarlike neigbourhood where $\gamma^\alpha_{j_\alpha}$ is the pullback of some form $\sigma^\alpha_{j_\alpha} \in ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(\partial F)$. Let $\{\rho_\alpha\}_{\alpha \in I}$ be a partition of unity with respect to $\mathfrak{U}$.
Claim: $\overline{\omega} := \sum_\alpha \sum_{j_\alpha} \pi_1^*(\rho_\alpha \eta^\alpha_{j_\alpha}) \wedge \pi_2^*\sigma^\alpha_{j_\alpha} \in ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(\partial B) \subset \Omega^\bullet(\partial W)$.
Proof of the claim: Since $\partial F = Z \times L$, we can write $\sigma^\alpha_{j_\alpha} = \sum_{l_\alpha} \widetilde{\pi}_1\delta_{l_\alpha} \wedge \widetilde{\pi}_2\xi_{l_\alpha}$ with $\xi_{l_\alpha} \in \tau_{\geq K}\Omega^\bullet(L)$. Hence

$$\overline{\omega} = \sum_\alpha \sum_{j_\alpha} \sum_{l=l(\alpha,j_\alpha)} \widehat{\pi}_1(\pi_1(\rho_\alpha\eta_\alpha) \wedge \pi_2\delta_l) \wedge \widehat{\pi}_2\xi_l,$$

which shows that $\overline{\omega} \in ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(\partial B)$.
Since all of the index sets J_α are finite,

$$\partial F \subset V^\alpha := \bigcap_{j_\alpha \in J_\alpha} V^\alpha_{j_\alpha} \subset C_{\partial F}$$

are all open collarlike neighbourhoods of ∂F in F. They gives rise to an open collarlike neighbourhood of ∂W in W by taking the union

$$V := \bigcup_{\alpha \in I} \phi_\alpha^{-1}(U_\alpha \times V^\alpha) \subset C_{\partial W}.$$

For this collarlike neighbourhood $\partial W \subset V \subset C_{\partial W}$, Lemma 4.2.6 implies

$$\begin{aligned}
\omega|_V &= \sum_{\alpha \in I}(\rho_\alpha \circ q)\ \omega|_V = \sum_{\alpha \in I}(\phi^*_\alpha \sum_{j_\alpha} \pi_1^*(\rho_\alpha\eta_{j_\alpha}) \wedge \pi_2^*\pi^*\sigma^\alpha_{j_\alpha})|_V \\
&= \sum_{\alpha \in I} \phi^*_\alpha \pi^* \sum_{j_\alpha} \pi_1^*(\rho_\alpha\eta_{j_\alpha}) \wedge \pi_2^*\sigma^\alpha_{j_\alpha} = \pi^* \sum_{\alpha \in I} \phi^*_\alpha \sum_{j_\alpha} \pi_1^*(\rho_\alpha\eta_{j_\alpha}) \wedge \pi_2^*\sigma^\alpha_{j_\alpha} \\
&= \pi^*\overline{\omega}.
\end{aligned}$$

(The mentioned Lemma was used in the second line.) This implies $\omega \in \Omega I^{\bullet}_{\bar{p}}(W)$.

□

Remark 10.2.5 *Note that obviously $f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)$ is also a subcomplex of $ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)$.*

Before formulating the proposition which allows to replace $ft_{\geq L}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma) \cap \Omega I^{\bullet}_{\bar{p}}(W)$ by $f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)$, we prove the following technical lemma:

Lemma 10.2.6 *The pullback of any form $\omega \in \ker d_N^* \cap \Omega I_{\bar{p}}^\bullet(F)$ to the boundary ∂F is coclosed, i.e. $j_{\partial F}^* \omega \in \ker d_{\partial F}^*$.*

Proof: Since $\omega \in \Omega I_{\bar{p}}^\bullet(F)$, there is some collarlike boundary neighbourhood $\partial F \subset U \subset C_{\partial F}$ with $\omega|_U = \pi^*(j_{\partial F}^*\omega)$. Then

$$\begin{aligned} 0 = d_F * (\omega|_U) &= d_F\big(dx^0 \wedge \pi^*(* \, j_{\partial F}^*\omega)\big) \\ &= -dx^0 \wedge \pi^*(d_{\partial F} * j_{\partial F}^*\omega) \end{aligned}$$

and hence $d^*(j_{\partial F}^*\omega) = \pm * d * (j_{\partial F}^*\omega) = 0$. $\square$

Proposition 10.2.7 *For any $L \in \mathbb{Z}$, the subcomplex inclusion*

$$f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma) \hookrightarrow ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(\Sigma) \cap \Omega I_{\bar{p}}^\bullet(W)$$

is a quasi isomorphism.

Proof: We will use a Mayer-Vietoris argument:
We define

$$ft_{\geq L}\Omega_{\mathcal{MS},rel}(\Sigma) := \big\{\omega \in ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(\Sigma)\big|\exists \partial W \subset U \subset C_{\partial W} : \omega|_U = 0\big\}$$

and note that $\omega \in ft_{\geq L}\Omega_{\mathcal{MS},rel}(\Sigma)$ if and only if

$$\omega|_{q^{-1}(U_\alpha)} = \phi_\alpha^* \sum_{j_\alpha} \pi_1^*\eta_{j_\alpha}^\alpha \wedge \pi_2^*\gamma_{j_\alpha}^\alpha$$

with $\gamma_{j_\alpha}^\alpha|_{U_{j_\alpha}} = 0$ for some collarlike neighbourhood $\partial F \subset U_{j_\alpha} \subset C_{\partial F}$. This is the case since we work with fiber-related collars on F and W. Hence we have subcomplex inclusions

$$ft_{\geq L}\Omega_{\mathcal{MS},rel}(\Sigma) \hookrightarrow ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(\Sigma) \cap \Omega I_{\bar{p}}^\bullet(W)$$

and

$$ft_{\geq L}\Omega_{\mathcal{MS},rel}(\Sigma) \hookrightarrow f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma).$$

Restriction of forms to the boundary ∂W gives the following surjections:

$$j_{\partial W}^* : ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(\Sigma) \cap \Omega I_{\bar{p}}^\bullet(W) \twoheadrightarrow ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(\Sigma) \cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(\Sigma \times Z) \quad (24)$$

(which is obvious) as well as

$$j_{\partial W}^* : f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma) \twoheadrightarrow \underbrace{\Big\{\sum_j \pi_1^*\eta_j \wedge \pi_2^*\sigma_j \big| \sigma_j \in \tau_{\geq L}\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Z)\big)\Big\}}_{=:ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(\Sigma)}. \quad (25)$$

(The well-definedness of this complex follows from Lemma 10.2.2.) We will prove that $j^*_{\partial W}$ really maps $f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma)$ to $ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(\Sigma)$ and that the map is surjective:
First we show that for $\omega \in f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma)$ it holds that

$$j^*_{\partial W}\omega \in ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(\Sigma): \tag{26}$$

Let ρ_α be a partition of unity of Σ with respect to the cover $\mathfrak{U}$. Then

$$\omega = \sum_{\alpha\in I}(\rho_\alpha \circ q)\ \omega = \sum_{\alpha\in I}\phi^*_\alpha \sum_{j_\alpha}\pi^*_1(\rho_\alpha\ \eta^\alpha_{j_\alpha}) \wedge \pi^*_2\gamma^\alpha_{j_\alpha}$$

and hence the relation $\phi_\alpha \circ j_{\partial W} = \mathrm{id} \times j_{\partial F}$ gives

$$j^*_{\partial W}\omega = \sum_\alpha \sum_{j_\alpha}\pi^*_1(\rho_\alpha\ \eta^\alpha_{j_\alpha}) \wedge \pi^*_2 j^*_{\partial F}\gamma^\alpha_{j_\alpha}.$$

Therefore, equation (26) holds for all such ω if and only if for all $\gamma \in \tau_{\geq L}\Omega I^r_{\bar{p}}(F)$ it holds that $j^*_{\partial F}\gamma \in \tau_{\geq L}\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Z)\big)$. The only degree that is non-trivial is $r = L$, since for $r < L$ both complexes are zero and for $r > L$ the cotruncated complex is equal to the initial one. So let $\gamma \in \tau_{\geq L}\Omega I^L_{\bar{p}}(F)$, then $\gamma \in \ker d^*_N$. By definition of $\Omega I^\bullet_{\bar{p}}(F)$, we have $j^*_{\partial F}\gamma \in ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Z)$. We must show that $j^*_{\partial F}\gamma \in \ker d^*$:
Since $\gamma \in \Omega I^\bullet_{\bar{p}}(F)$, it is the pullback of a form on ∂F for some collarlike boundary neighbourhood $\partial F \subset U \subset C_{\partial F}$: $\gamma|_U = \pi^*\eta$. Then we have:

$$\begin{aligned} 0 &= d_F\ *(\underbrace{\gamma|_U}_{=\pi^*\eta}) = d_F(dx^0 \wedge \pi^*(*\eta)) \\ &= -dx^0 \wedge \pi^*(d*\eta) \end{aligned}$$

and hence $d^*\eta = 0$, i.e. $j^*_{\partial F}\gamma \in \ker d^*$.

Secondly, we show the surjectivity:
Let $\omega := \sum_j \pi^*_1\eta_j \wedge \pi^*_2\gamma_j \in ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(Z)$, $\deg\gamma_j = r_j$ and let $\psi : [0,\infty) \to \mathbb{R}$ be a smooth cutoff function with $\psi|_{[0,1)} = 1$ and $\psi|_{[2,\infty]} = 0$. This defines smooth cutoff functions on F by setting $\psi_F = \psi \circ \mathrm{proj}_2 \circ c_{\partial F}$. Extension of the forms $\psi_F\pi^*\gamma_j$ defines forms in $\Omega I^{r_j}_{\bar{p}}(F) = \tau_{\geq L}\Omega I^{r_j}_{\bar{p}}(F)$ for $r_j > L$ (and zero for $r_j < L$). Hence in this degrees we have a preimage of γ_j in $\tau_{\geq L}\Omega I^\bullet_{\bar{p}}(F)$.
It remains to construct a preimage for $r_j = L$. In this degree we have $\gamma_j \in \ker d^*_N$. By the additional assumption on $\Omega I^\bullet_{\bar{p}}(F)$ we have a direct sum decomposition

$$\Omega I^L_{\bar{p}}(F) = \ker d^*_N \cap \Omega I^L_{\bar{p}}(F) \oplus d\big(\Omega I^{L-1}_{\bar{p}}(F)\big).$$

Hence, we can write $\psi\ \pi^*\gamma_j = \alpha_j + d\beta_j$ for some $\alpha_j \in \ker d^*_N \cap \Omega I^\bullet_{\bar{p}}(F)$, $\beta_j \in d\big(\Omega I^{L-1}_{\bar{p}}(F)\big)$. We consider the form

$$\pi^*_1\eta_j \wedge \pi^*_2\alpha_j.$$

Since $dj^*_{\partial F}(\beta_j) = j^*_{\partial F}(d\beta_j) = j^*_{\partial F}(\psi\pi^*\gamma_j - \alpha_j) = \gamma_j - j^*_{\partial F}\alpha_j \in \ker d^* \cap \operatorname{im} d = \{0\}$ (by Lemma 10.2.6), we have $j^*_{\partial F}\alpha_j = \gamma_j$ and hence Proposition 4.2.6 implies that $\pi_1^*\eta_j \wedge \pi_2^*\alpha_j \in f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma)$ is the desired preimage:

$$j^*_{dW}\left(\pi_1^*\eta_j \wedge \pi_2^*\alpha_j\right) = \pi_1^*\eta_j \wedge \pi_2^* j^*_{\partial F}\alpha_j = \pi_1^*\eta_j \wedge \pi_2^*\gamma_j.$$

We define

$$\widetilde{\omega} := \sum_{\alpha\in I} \phi^*_\alpha \sum_j \pi_1^*(\rho_\alpha\ \eta_j) \wedge \pi_2^*(\sigma_j)$$

with

$$\sigma_j = \begin{cases} \psi_F\ \pi^*\gamma_j & \text{if } \deg\gamma_j \neq L \\ \alpha_j & \text{if } \deg\gamma_j = L. \end{cases}$$

Since we work with fiber-related collars on F and W, this form is well defined. It extends ω to all of W:

$$j^*_{\partial W}\widetilde{\omega} = \underbrace{\sum_{\alpha\in I} q^*(\rho_\alpha)}_{=1} \sum_j \pi_1^*\eta_j \wedge \pi_2^* \underbrace{j^*_{\partial F}\sigma_j}_{=\gamma_j} = \overline{\omega}.$$

Obviously, the kernel of $j^*_{\partial W}$ in both (24) and (25) is $ft_{\geq L}\Omega_{\mathcal{MS},rel}(W)$. We get the following commutative diagram:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
ft_{\geq L}\Omega^\bullet_{\mathcal{MS},rel}(W) & = & ft_{\geq L}\Omega^\bullet_{\mathcal{MS},rel}(W) \\
\downarrow & & \downarrow \\
f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma) & \hookrightarrow & ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(\Sigma) \cap \Omega I^\bullet_{\bar{p}}(W) \\
\downarrow{\scriptstyle j^*_{\partial W}} & & \downarrow{\scriptstyle j^*_{\partial W}} \\
ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(\Sigma) & \hookrightarrow & ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(\Sigma) \cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(\Sigma\times Z) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

In the following Lemmas 10.2.9, 10.2.10, 10.2.11 and 10.2.12 we will prove that the latter subcomplex inclusion

$$ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(\Sigma) \hookrightarrow ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(\Sigma) \cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(\Sigma\times Z) \tag{27}$$

is a quasi isomorphism. That will, together with the 5-Lemma, finish the proof. $\square$

Before finishing the proof of the above proposition, we will state the following corollary:

Corollary 10.2.8 *The subcomplex inclusion*

$$f(\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma) \hookrightarrow \Omega I^\bullet_{\bar{p}}(W)$$

is a quasi isomorphism.

Proof: Take $L < 0$ in the above Proposition 10.2.7. □

We will use a Majer-Vietoris argument to prove that the subcomplex inclusion in (27) is a quasi isomorphism. We need an induction start, i.e. a Poincaré Lemma, and a Bootstrap Lemma. For an open subset $U \subset \Sigma$ we define $ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(U)$ in the obvious way: The η_j's in the sums are required to lie inside $\Omega^\bullet(U)$. Let $x_0 \in U$ and let $S_0 : \{x_0\} \times Z \times L \hookrightarrow U \times Z \times L$ denote the inclusion at x_0.

Lemma 10.2.9 *(Poincaré Lemma for $ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}$)*
If $U \subset \Sigma$ is a coordinate chart, then

$$S_0^* : ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(U) \to \tau_{\geq L}\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Z)\big)$$

is a homotopy equivalence, with homotopy inverse the induced map of the projection $\pi_2 : U \times F \to F$.

Proof: It is obvious that S_0^* maps $ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(U)$ to $\tau_{\geq L}\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Z)\big)$. Using the homotopy operator $K_{\mathcal{MS}}$ of [Ban11, Prop. 3.9], which is also applicable in this setting, since it does not change anything in the $Z \times L = F$ direction, one gets $\mathrm{id} - \pi_2^* S_0^* = K_{\mathcal{MS}}d + dK_{\mathcal{MS}}$. Since $S_0^*\pi_2^* = \mathrm{id}$, the statement of the lemma is true. □

Lemma 10.2.10 *(Poincaré Lemma for $ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(U) \cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(U \times Z)$)*
If $U \subset \Sigma$ is a coordinate chart, then

$$S_0^* : ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(U) \cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(U \times Z) \to \tau_{\geq L}\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Z)\big)$$

is also a homotopy equivalence with homotopy inverse induced by the projection $\pi_2 : U \times F \to F$.

Proof: As before, the main point of the proof is the fact that

$$S_0^*\big(ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(U) \cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(U \times Z)\big) = \tau_{\geq L}\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Z)\big).$$

First, note that

$$S_0^*\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(U \times Z)\big) = ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Z) \subset \Omega^\bullet_{\mathcal{MS}}(Z \times L),$$

since the inclusion $\widetilde{S}_0 : \{x_0\} \times Z \to U \times Z$ induces a surjection

$$\widetilde{S}_0^* : \Omega^\bullet(U \times Z) \twoheadrightarrow \Omega^\bullet(Z).$$

We further have

$$S_0^*\big(ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(U)\big) = \tau_{\geq L}\Omega^\bullet(F)$$

and hence by Lemma 10.2.2

$$S_0^*\big(ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(U) \cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(U\times Z)\big) \subset \underbrace{ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Z) \cap \tau_{\geq L}\Omega^\bullet(F)}_{=\tau_{\geq L}\left(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Z)\right)}.$$

The equality follows, since for each $\eta \in \tau_{\geq L}\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Z)\big)$ we have $\pi_2^*\eta \in ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(U) \cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(U\times Z)$ and $S_0^*\pi_2^*\eta = \eta$.
That S_0^* is a homotopy equivalence follows as before from using the homotopy operator $K_{\mathcal{MS}}$ to show that $\pi_2^* S_0^* \simeq \mathrm{id}$. □

To finish the proof of 10.2.7 we make use of the following bootstrap statement:

Lemma 10.2.11 *(Bootstrap Lemma)*
Let $U, V \subset \Sigma$ be open subsets such that

$$ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(Y) \hookrightarrow ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(Y) \cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Y\times Z) \qquad (28)$$

is a quasi isomorphism for $Y \in \{U,\ V,\ U\cap V\}$. Then the subcomplex inclusion in (28) is also a quasi isomorphism for $Y = U \cup V$.

Proof: We will show that there are short exact sequences

$$0 \to X^\bullet(U\cup V) \to X^\bullet(U) \oplus X^\bullet(V) \to X^\bullet(U\cap V) \to 0 \qquad (29)$$

for both $X^\bullet = ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}},\ ft_{\geq L}\Omega^\bullet_{\mathcal{MS}} \cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(-\times Z)$. The argument is the standard one: See for example the proof of [Ban11, Lemma 5.10], where the argument is given for fiberwisely cotruncated multiplicatively structured forms. In principle, one uses a partition of unity $\{\rho_U, \rho_V\}$ of $U\cup V$ with respect to the open cover $\{U, V\}$. Then for $\omega \in X^\bullet(U\cap V)$ one has $\pi^*\rho_V\omega \in X^\bullet(U)$ and $\pi^*\rho_U\omega \in X^\bullet(V)$ (recall that $\pi : \Sigma\times Z\times L \to \Sigma$), for either $X^\bullet = ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}},\ ft_{\geq L}\Omega^\bullet_{\mathcal{MS}} \cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(-\times Z)$:
For $\pi_1^*\eta \wedge \pi_2^*\gamma \in ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(U\cap V)$ or $ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(U\cap V)$ one gets

$$(\pi^*\rho_V)(\pi_1^*\eta \wedge \pi_2^*\gamma) = \pi_1^*(\rho_V\eta) \wedge \pi_2^*\gamma \in ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(U) \text{ or } ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(U).$$

while for $\widetilde{\pi}_1^*\eta \wedge \widetilde{\pi}_2^*\gamma \in ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}((U\cap V)\times Z)$

$$(\pi^*\rho_V)(\widetilde{\pi}_1^*\eta \wedge \widetilde{\pi}_2^*\gamma) = \widetilde{\pi}_1^*(\widetilde{\pi}^*\rho_V\eta) \wedge \widetilde{\pi}_2^*\gamma \in ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(U\times Z),$$

where $\widetilde{\pi} : \Sigma\times Z \to \Sigma$. The analogous statements for U and V interchanged are true by the same arguments and hence surjectivity at the latter group of (29) follows, since $\omega = -(-\pi^*\rho_V\omega) + \pi^*\rho_U\omega$.

The exactness of (29) at the middle group follows since for two forms $\alpha \in X^\bullet(U)$, $\beta \in X^\bullet(V)$ with $\alpha|_{U\cap V} = \beta|_{U\cap V}$ there is a unique form $\kappa \in \Omega^\bullet((U\cup V)\times Z \times L)$ with $\kappa|_U = \alpha$ and $\kappa|_V = \beta$. But then

$$\kappa = (\pi^*\rho_U + \pi^*\rho_V)\kappa = \pi^*\rho_U\alpha + \pi^*\rho_V \in X^\bullet(U\cup V).$$

The two short exact sequences of (29) give rise to a commutative cohomology diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H^{r-1}\big(R^\bullet(U\cap V)\big) \arrow[r, "\cong"] \arrow[d] & H^{r-1}\big(Z^\bullet(U\cap V)\big) \arrow[d] \\
H^r\big(R^\bullet(U\cup V)\big) \arrow[r] \arrow[d] & H^r\big(Z^\bullet(U\cup V)\big) \arrow[d] \\
H^r\big(R^\bullet(U)\big)\oplus H^r\big(R^\bullet(V)\big) \arrow[r, "\cong"] \arrow[d] & H^r\big(Z^\bullet(U)\big)\oplus H^r\big(Z^\bullet(V)\big) \arrow[d] \\
H^r\big(R^\bullet(U\cap V)\big) \arrow[r, "\cong"] \arrow[d] & H^r\big(Z^\bullet(U\cap V)\big) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

with

$$R^\bullet(Y) := ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(Y),$$

and

$$Z^\bullet(Y) := ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(Y) \cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Y\times Z).$$

The statement of the lemma is then implicated by the 5-Lemma. □

Lemma 10.2.12 *The map*

$$ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(\Sigma) \hookrightarrow ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(\Sigma)\cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(\Sigma\times Z)$$

is a quasi isomorphism.

Proof: Let $\{U_\alpha\}_{\alpha\in I}$ be a finite good open cover of Σ. We will show by induction that for all open subsets $U\subset B$ of the form

$$U = \bigcup_{i=1}^{s} U_{\alpha_0^i\dots\alpha_{p_i}^i}$$

the map

$$H^r\big(ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(U)\big) \to H^r\big(ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(U) \cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(U \times Z)\big), \quad (30)$$

induced by subcomlex inclusion, is an isomorphism. For $s = 1$ we have $U = \bigcap_{j=0}^{p} U_{\alpha_j} \cong \mathbb{R}$, since the cover is good. Therefore by the Lemmata 10.2.9, 10.2.10 we have a commutative diagram

$$\begin{tikzcd}
ftft_{\geq K,L}\Omega^\bullet_{2\mathcal{MS}}(U) \arrow[d, hook] \arrow[dr, "\simeq", "S_0^*"'] & \\
& \tau_{\geq L}\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(Z)\big)(Z) \\
ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(U) \cap ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(U \times Z) \arrow[ur, "\simeq", "S_0^*"'] &
\end{tikzcd}$$

and the induced diagram on cohomology gives the statement for $s = 1$.
Now suppose that the map in (30) is an isomorphism for all $U = \bigcup_{i=1}^{s-1} U_{\alpha_0^i \dots \alpha_{p_i}^i}$.
Let $V = U_{\alpha_0^s \dots \alpha_{p_s}^s} \subset \Sigma$. Then by the induction hypotheses the map on cohomology is an isomorphism for U and $U \cap V$, since

$$U \cap V = \Big(\bigcup_{i=1}^{s-1} U_{\alpha_0^i \dots \alpha_{p_i}^i}\Big) \cap U_{\alpha_0^s \dots \alpha_{p_s}^s} = \bigcup_{i=1}^{s-1} U_{\alpha_0^i \dots \alpha_{p_i}^i\ \alpha_0^s \dots \alpha_{p_s}^s}.$$

Since the same is true for V by the induction basis, Lemma 10.2.11 implies that the cohomology map is an isomorphism for

$$U \cup V = \bigcup_{i=1}^{s} U_{\alpha_0^i \dots \alpha_{p_i}^i}.$$

The statement of the lemma is then implied by the above, since $\Sigma = \bigcup_{\alpha \in I} U_\alpha$ and $|I| < \infty$. □

10.3 Poincaré Duality for $f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma)$

To prove Poincaré duality for $HI^\bullet_{\bar{p}}(X)$, we use the method of triangles as before. To apply the argument, we first have to prove that the $\Omega I^\bullet_{\bar{p}}$-fiberwisely truncated and cotruncated forms satisfy Poincaré duality (over complementary perversities).
Let $U \subset \Sigma$ be an open subset. We then define

Definition 10.3.1 *(Fiberwise $\Omega I^\bullet_{\bar{p}}$ forms)*

$$\begin{aligned} f(\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(U) := \big\{\omega \in \Omega^\bullet_{\mathcal{MS}}(U) \big| \forall \alpha \in I :& \\ \omega|_{q^{-1}(U \cap U_\alpha)} &= \phi_\alpha^* \sum_{j_\alpha} \pi_1^* \eta^\alpha_{j_\alpha} \wedge \pi_2^* \gamma^\alpha_{j_\alpha}, \\ \textit{with } \gamma^\alpha_{j_\alpha} &\in \Omega I^\bullet_{\bar{p}}(F)\big\}. \end{aligned}$$

This defines a subcomplex $f(\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(U) \subset \Omega^\bullet_{\mathcal{MS}}(U)$ as well as, for $U = \Sigma$, $f(\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma) \subset \Omega I^\bullet_{\bar{p}}(W)$. Further we define

Definition 10.3.2 *(Fiberwise $\Omega I^\bullet_{\bar{p}}$ forms with compact support)*

$$f(\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS},c}(U) := \left\{\omega \in f(\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(U) : \omega \in \Omega^\bullet_{\mathcal{MS},c}(U)\right\}.$$

This defines a subcomplex $f(\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS},c}(U) \subset \Omega^\bullet_{\mathcal{MS},c}(U)$.

Definition 10.3.3 *(Fiberwise $\Omega I^\bullet_{\bar{p}}$-truncated forms)*

$$\begin{aligned} f(\tau_{<L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(U) := \big\{\omega \in \Omega^\bullet_{\mathcal{MS}}(U) \big| \forall \alpha \in I :& \\ \omega|_{q^{-1}(U\cap U_\alpha)} &= \phi^*_\alpha \sum_{j_\alpha} \pi_1^* \eta^\alpha_{j_\alpha} \wedge \pi_2^* \gamma^\alpha_{j_\alpha}, \\ &\textit{with } \gamma^\alpha_{j_\alpha} \in \tau_{<L}\Omega I^\bullet_{\bar{p}}(F)\big\}. \end{aligned}$$

Again, there are subcomplex inclusions $f(\tau_{<L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma) \subset \Omega I^\bullet_{\bar{p}}(W)$ and $f(\tau_{<L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma) \subset ft_{<K}\Omega^\bullet_{\mathcal{MS}}(\Sigma)$. These are proved as in Lemma 10.2.4. In the same manner, we define fiberwise $\Omega I^\bullet_{\bar{p}}$-truncated and cotruncated forms with compact support.

Lemma 10.3.4 *(Poincaré Lemma for $f(\tau_{<L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}$ and $f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}$)*
Let $U \subset \Sigma$ be a coordinate chart and let $\pi_2 : U \times F \to F$ denote the projection map. Then π_2 induces homotopy equivalences

$$\pi_2^* : \tau_{<L}\Omega I^\bullet_{\bar{p}}(F) \xrightarrow{\simeq} f(\tau_{<L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(U)$$

and

$$\pi_{2*} : f(\tau_{\geq L^*}\Omega I_{\bar{q}})\Omega^\bullet_{\mathcal{MS},c}(U) \xrightarrow{\simeq} \left(\tau_{\geq L}\Omega I^\bullet_{\bar{p}}(F)\right)^{\bullet+s}$$

with $s = \dim \Sigma$.

Proof: As in [Ban11, Lemmata 5.2 and 5.5]. □

Lemma 10.3.5 *(Poincaré Duality for (co)truncated $\Omega I^\bullet_{\bar{p}}$ forms)*
Let $f := \dim F$ and $\bar{p}, \bar{q}$ be complementary perversities and $L := f - \bar{p}(f+1)$, $L^* := f - \bar{q}(f+1)$. Then integration induces a nondegenerate bilinear form

$$\int : H^r(\tau_{<L}\Omega I^\bullet_{\bar{p}}(F)) \times H^{f-r}(\tau_{\geq L^*}\Omega I^\bullet_{\bar{q}}(F)) \to \mathbb{R}.$$

Proof: For $r \geq L$ you get $f - r \leq f - L = \bar{p}(f+1) = f - 1 - \bar{q}(f+1) < L^*$ hence both cohomologies groups are zero. For $r < L$ we have $f - r \geq L^*$ so the statement reduces to the nondegeneracy of

$$\int : HI^r_{\bar{p}}(F) \times HI^{n-r}_{\bar{q}}(F) \to \mathbb{R}$$

which is fulfilled by [Ban11, Theorem 8.2]. □

Corollary 10.3.6 *(Local P.D. for fiberwisely $\Omega I^{\bullet}_{\bar{p}}$-(co)truncated forms)*
Let $U \subset \Sigma$ be a coordinate chart. Then integration induces a nondegenerate bilinear form

$$\int : H^r(f(\tau_{<L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(U)) \times H^{n-1-r}(f(\tau_{\geq L^*}\Omega I_{\bar{q}})\Omega^{\bullet}_{\mathcal{MS},c}(U)) \to \mathbb{R}.$$

<u>Proof:</u> This is true by the two preceding lemmata: By [Ban11, Lemma 5.4] the following diagram commutes:

$$\begin{tikzcd}
H^r(\tau_{<L}\Omega I^{\bullet}_{\bar{p}}(F)) \arrow[r, "\cong", "\pi_2^*"'] \arrow[d, "\cong"', "\int"] & H^r(f(\tau_{<L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(U)) \arrow[d, "\int"] \\
H^{f-r}(\tau_{\geq L}\Omega I^{\bullet}_{\bar{q}}(F))^{\dagger} \arrow[r, "\cong", "\pi_{2*}^{\dagger}"'] & H^{n-1-r}(f(\tau_{\geq L^*}\Omega I_{\bar{q}})\Omega^{\bullet}_{\mathcal{MS},c}(U))^{\dagger}
\end{tikzcd}$$

(Recall that $n-1 = s+f$). ☐

By the same arguments as in [Ban11, Lemma 5.10] we have a bootstrap principle:

Lemma 10.3.7 *(Bootstrap Lemma)*
Let $U, V \subset \Sigma$ be open subsets such that

$$\int : H^r(f(\tau_{<L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(Y)) \times H^{n-1-r}(f(\tau_{\geq L^*}\Omega I_{\bar{q}})\Omega^{\bullet}_{\mathcal{MS},c}(Y)) \to \mathbb{R}.$$

is nondegenerate for $Y \in \{U, V, U \cap V\}$, then so it is for $Y = U \cup V$.

Using the Bootstrap Lemma 10.3.7 and Corollary 10.3.6, we arrive at the following proposition:

Proposition 10.3.8 *(P.D. for fiberwisely $\Omega I^{\bullet}_{\bar{p}}$-(co)truncated forms)*
For complementary perversities, integration induces nondegenerate bilinear forms

$$\int : H^r(f(\tau_{<L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)) \times H^{n-1-r}(f(\tau_{\geq L^*}\Omega I_{\bar{q}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)) \to \mathbb{R}.$$

<u>Proof:</u> Since Σ is compact, it holds that

$$f(\tau_{\geq L^*}\Omega I_{\bar{q}})\Omega^{\bullet}_{\mathcal{MS},c}(\Sigma) = f(\tau_{\geq L^*}\Omega I_{\bar{q}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma).$$

Now let $\{U_\alpha\}_{\alpha \in I}$ be a finite good open cover of Σ. We show that

$$\int : H^r(f(\tau_{<L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(U)) \times H^{n-1-r}(f(\tau_{\geq L^*}\Omega I_{\bar{q}})\Omega^{\bullet}_{\mathcal{MS},c}(U)) \to \mathbb{R}.$$

is nondegenerate for all $U \subset \Sigma$ of the form

$$U = \bigcup_{i=1}^{t} U_{\alpha_0^i \dots \alpha_{p_i}^i}$$

for arbitrary t by induction. Since the cover is good, $U_{\alpha_0 \dots \alpha_p} \cong \mathbb{R}^s$, and hence the statement for $t = 1$ holds by Corollary 10.3.6. Suppose the statement is true for all

$$U = \bigcup_{i=1}^{t-1} U_{\alpha_0^i \dots \alpha_{p_i}^i}.$$

Let $V := U_{\alpha_0^t \dots \alpha_{p_t}^t}$, then

$$U \cap V = \bigcup_{i=1}^{t-1} U_{\alpha_0^i \dots \alpha_{p_i}^i \alpha_0^t \dots \alpha_{p_t}^t}.$$

Hence by induction hypothesis, the statement holds for U and $U \cap V$. Since the statement is also true for V by the induction base, the Bootstrap Lemma 10.3.7 gives the nondegeneracy for $U \cup V$.
Since $\Sigma = \bigcup_{\alpha \in I} U_\alpha$, we have the nondegeneracy for $U = \Sigma$. $\square$

10.4 Distinguished Triangles for $\Omega I^\bullet_{\bar{p}}(M)$

Inspired by the previous setting, with an isolated bottom stratum, we want distinguished triangles for $\Omega I^\bullet_{\bar{p}}(M)$ analogous to the distinguished triangles of the Lemmata 8.2.3 and 8.2.4. The first one is rather obvious:

Lemma 10.4.1 *Subcomplex inclusion and pullback to the boundary part $W \subset \partial M$ induce the following distinguished triangle in $\mathcal{D}(\mathbb{R})$:*

$$\begin{tikzcd}
\widetilde{\Omega I^\bullet_{\bar{p}}}(M, W) \arrow[rr] & & \Omega I^\bullet_{\bar{p}}(M) \arrow[dl] \\
& f(\tau_{\geq L} \Omega I_{\bar{p}}) \Omega^\bullet_{\mathcal{MS}}(\Sigma) \arrow[ul, "+1"] &
\end{tikzcd} \tag{31}$$

with $\widetilde{\Omega I^\bullet_{\bar{p}}}(M, W)$ defined as in Definition 7.0.4.

<u>Proof:</u> Restriction to the boundary part $W \subset \partial M$ defines a surjective map

$$\Omega I^\bullet_{\bar{p}}(M) \twoheadrightarrow ft_{\geq L} \Omega^\bullet_{\mathcal{MS}}(\Sigma) \cap ft_{\geq K} \Omega^\bullet_{\mathcal{MS}}(\Sigma \times Z)$$

with kernel $\widetilde{\Omega I_{\bar{p}}}^\bullet(M, W)$. But since the complex $ft_{\geq L} \Omega^\bullet_{\mathcal{MS}}(\Sigma) \cap ft_{\geq K} \Omega^\bullet_{\mathcal{MS}}(\Sigma \times Z)$ ist quasi isomorphic to $f(\tau_{\geq L} \Omega I_{\bar{p}}) \Omega^\bullet_{\mathcal{MS}}(\Sigma)$ by Proposition 10.2.7, we can replace it in the distinguished triangle in $\mathcal{D}(\mathbb{R})$ that is induced by the resulting short exact sequence. $\square$

To state and prove the second distinguished triangle we need the following map, which is analogous to the one used in [Ban11, p.40] and the previous chapters, see e.g. equation (10):

$$\gamma_U : f(\tau_{<L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(U) \hookrightarrow f(\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(U) \twoheadrightarrow \underbrace{\frac{f(\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(U)}{f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(U)}}_{:=QI^\bullet(U)}$$

Lemma 10.4.2 *(Bootstrap for γ_U)*
Let $U, V \subset \Sigma$ be open sets. Then if γ_U, γ_V and $\gamma_{U\cap V}$ are quasi isomorphisms, so is $\gamma_{U\cup V}$.

Proof: The proof is an analogy to the proof of [Ban11, Lemma6.5]: One checks that there are exact Mayer–Vietoris sequences for $f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}$ and $f(\tau_{<L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}$. Hence by the 3×3-Lemma there is also a short exact Mayer–Vietoris sequence for $QI^\bullet$. The 5-Lemma then concludes the proof. □

Lemma 10.4.3 *(γ_Σ is a quasi isomorphism)*
For $U = \Sigma$ the map $\gamma_\Sigma : f(\tau_{<L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma) \to QI^\bullet(\Sigma)$ is a quasi isomorphism.

Proof: Again, we use a good open cover $\{U_\alpha\}_{\alpha\in I}$ of Σ and the above bootstrap principle. Before we apply the bootstrap argument we will show that for any $U_{\alpha_0\ldots\alpha_p} \cong \mathbb{R}$, $\gamma_{U_{\alpha_0\cdots\alpha_p}}$ is a quasi isomorphism:
Since $\Omega I^\bullet_{\bar{p}}(F)$ is geometrically cotruncateable in degree L, we have $\Omega I^\bullet_{\bar{p}}(F) = \tau_{<L}\Omega I^\bullet_{\bar{p}}(F) \oplus \tau_{\geq L}\Omega I^\bullet_{\bar{p}}(F)$, and hence the map

$$\gamma : \tau_{<L}\Omega I^\bullet_{\bar{p}}(F) \hookrightarrow \Omega I^\bullet_{\bar{p}}(F) \twoheadrightarrow \frac{\Omega I^\bullet_{\bar{p}}(F)}{\tau_{\geq L}\Omega I^\bullet_{\bar{p}}(F)}$$

is an isomorphism. Making use of Lemma 10.3.4 and the commutativity of the following diagram:

$$\begin{tikzcd}
\tau_{<L}\Omega I^\bullet_{\bar{p}}(F) \arrow[r, "\simeq", "\pi_2^*"'] \arrow[d, "\cong"', "\gamma"] & f(\tau_{<L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(U_{\alpha_0\ldots\alpha_p})) \arrow[d, "\gamma_{U_{\alpha_0\cdots\alpha_p}}"] \\
\frac{\Omega I^\bullet_{\bar{p}}(F)}{\tau_{\geq L}\Omega I^\bullet_{\bar{q}}(F)} \arrow[r, "qis", "{\pi_2}^*"'] & QI^\bullet(U_{\alpha_0\ldots\alpha_p})
\end{tikzcd}$$

we get that $\gamma_{U_{\alpha_0\cdots\alpha_p}}$ is a quasi isomorphism. The rest of the argument is as in previous proofs: We make use of the Bootstrap Lemma to show by induction over s that for all U of the form

$$U = \bigcup_{i=1}^{s} U_{\alpha_0^i\ldots\alpha_{p_i}^i}$$

it holds that γ_U is a quasi isomorphism. Then the statement of the Lemma is implied by the fact that $\Sigma = \bigcup_{\alpha \in I} U_\alpha$. $\square$

We next construct the other distinguished triangle for $\Omega I^\bullet_{\bar{p}}(M)$, which we need to prove Poincaré Duality for $HI(X)$:

Lemma 10.4.4 *Subcomplex inclusion and pullback to the boundary part $W \subset \partial M$ induce a distinguished triangle*

$$\begin{tikzcd} \Omega I^\bullet_{\bar{p}}(M) \arrow[rr] & & \widetilde{\Omega I}^\bullet_{\bar{p}}(M) \arrow[dl] \\ & f(\tau_{<L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma) \arrow[ul, "+1"] & \end{tikzcd} \tag{32}$$

in $\mathcal{D}(\mathbb{R})$, where $\widetilde{\Omega I}^\bullet_{\bar{p}}(M)$ is defined as in Definition 7.0.3.

<u>Proof:</u> Let $\mathrm{Cap}^\bullet(W) := ft_{\geq L}\Omega^\bullet_{\mathcal{MS}}(\Sigma) \cap \Omega I^\bullet_{\bar{p}}(W)$. Then pullback to the boundary part $W \subset \partial M$ induces the following comutative diagram of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \Omega I^\bullet_{\bar{p}}(M) \arrow[r] \arrow[d, two heads, "j^*_W"] & \widetilde{\Omega I}^\bullet_{\bar{p}}(M) \arrow[r] \arrow[d, two heads, "j^*_W"] & \frac{\widetilde{\Omega I}^\bullet_{\bar{p}}(M)}{\Omega I^\bullet_{\bar{p}}(M)} \arrow[r] \arrow[d, dotted, two heads, "\bar{j}^*_W"] & 0 \\ 0 \arrow[r] & \mathrm{Cap}^\bullet(W) \arrow[r] & \Omega I^\bullet_{\bar{p}}(W) \arrow[r] & \frac{\Omega I^\bullet_{\bar{p}}(W)}{\mathrm{Cap}^\bullet(W)} \arrow[r] & 0 \end{tikzcd}$$

Here the dotted arrow in the last column denotes that this map is induced by the two pullbacks at the left and in the middle. The kernel of both maps

$$j^*_W : \Omega I^\bullet_{\bar{p}}(M) \twoheadrightarrow \mathrm{Cap}^\bullet(W)$$

and

$$j^*_W : \widetilde{\Omega I}^\bullet_{\bar{p}}(M) \twoheadrightarrow \Omega I^\bullet_{\bar{p}}(W)$$

is $\Omega^\bullet(M, W)$ and hence the $\ker \bar{j}^*_W = \{0\}$. Hence by the 3×3-Lemma,

$$\bar{j}^*_W : \frac{\widetilde{\Omega I}^\bullet_{\bar{p}}(M)}{\Omega I^\bullet_{\bar{p}}(M)} \xrightarrow{\cong} \frac{\Omega I^\bullet_{\bar{p}}(W)}{\mathrm{Cap}^\bullet(W)}$$

is an isomorphism. We therefore get a distinguished triangle

$$\begin{tikzcd} \Omega I^\bullet_{\bar{p}}(M) \arrow[rr] & & \widetilde{\Omega I}^\bullet_{\bar{p}}(M) \arrow[dl] \\ & \frac{\Omega I^\bullet_{\bar{p}}(W)}{\mathrm{Cap}^\bullet(W)} \arrow[ul, "+1"] & \end{tikzcd} \tag{33}$$

in $\mathcal{D}(\mathbb{R})$. Now by Proposition 10.2.7 and Corollary 10.2.8 we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma) \arrow[r] \arrow[d, hook, "qis"] & f(\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma) \arrow[r] \arrow[d, hook, "qis"] & QI^{\bullet}(\Sigma) \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & \mathrm{Cap}^{\bullet}(W) \arrow[r] & \Omega I^{\bullet}_{\bar{p}}(W) \arrow[r] & \frac{\Omega I^{\bullet}_{\bar{p}}(W)}{\mathrm{Cap}^{\bullet}(W)} \arrow[r] & 0
\end{tikzcd}$$

(Recall that $QI^{\bullet}(\Sigma) = \frac{f(\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)}{f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)}$). By the 5-Lemma, the map in the last column is also a quasi isomorphism and hence together with the result of Lemma 10.4.3 we can replace $\frac{\Omega I^{\bullet}_{\bar{p}}(W)}{\mathrm{Cap}^{\bullet}(W)}$ in the distinguished triangle (33) by $f(\tau_{<L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)$ to get the desired distinguished triangle (32). □

10.5 Integration and Poincaré Duality for $HI^{\bullet}_{\bar{p}}(X)$

By the same arguments as in subsection 8.3, we get for positive dimensional bottom stratum:

Proposition 10.5.1 *Integration induces bilinear forms*

$$\int : HI^{r}_{\bar{p}}(X) \times HI^{n-r}_{\bar{q}}(X) \to \mathbb{R}$$
$$\big([\omega],\ [\eta]\big) \mapsto \int_M \omega \wedge \eta.$$

Proof: As in subsection 8.3. □

So finally we can prove Poincaré Duality for $HI(X)$ over complementary perversities:

Theorem 10.5.2 *(Poincaré Duality for HI in the positive dimensional bottom stratum case)*
The above bilinear form

$$\int : HI^{r}_{\bar{p}}(X) \times HI^{n-r}_{\bar{q}}(X) \to \mathbb{R}$$
$$\big([\omega],\ [\eta]\big) \mapsto \int_M \omega \wedge \eta,$$

which is induced by integration, is nondegenerate.

Proof: We make use of the long exact cohomology sequences induced by the distinguished triangles (31) and (32). We show that the following diagram commutes (up to sign):

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H^{r-1}\big(f(\tau_{<L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)\big) \arrow[r, "\cong", "\int_W"'] \arrow[d, "\delta"] & H^{n-r}\big(f(\tau_{\geq L^*}\Omega I_{\bar{q}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)\big)^{\dagger} \arrow[d, "(j_W^*)^{\dagger}"] \\
HI^{r}_{\bar{p}}(X) \arrow[r, "\int_M"'] \arrow[d] & HI^{n-r}_{\bar{q}}(X)^{\dagger} \arrow[d] \\
\widetilde{HI}^{r}_{\bar{p}}(M) \arrow[r, "\cong", "\int_M"'] \arrow[d, "\Lambda"] & \widetilde{HI}^{n-r}_{\bar{q}}(M,W)^{\dagger} \arrow[d, "\Delta^{\dagger}"] \\
H^{r}\big(f(\tau_{<L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)\big) \arrow[r, "\cong", "\int_M"'] \arrow[d] & H^{n-r-1}\big(f(\tau_{\geq L^*}\Omega I_{\bar{q}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)\big)^{\dagger} \arrow[d] \\
\vdots & \vdots
\end{tikzcd}
\tag{34}
$$

Note that the horizontal maps

$$\int : H^{r}\big(f(\tau_{<L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)\big) \xrightarrow{\cong} H^{n-r-1}\big(f(\tau_{\geq L^*}\Omega I_{\bar{q}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)\big)^{\dagger}$$

and

$$\int : \widetilde{HI}^{r}_{\bar{p}}(M) \xrightarrow{\cong} \widetilde{HI}^{n-r}_{\bar{q}}(M,W)^{\dagger}$$

are isomorphisms by the Propositions 10.3.8 and 7.5.5. We will be very brief in the arguments that imply the commutativity of the above diagram, since they equal the arguments used in Theorem 8.4.1.

We first show the commutativity of the top square (TS) in the above diagram: Let $\omega \in f(\tau_{<L}\Omega I^{\bullet}_{\bar{p}})\Omega_{\mathcal{MS}}(\Sigma)$ a closed form. We want to describe $\delta[\omega] \in HI^{r}_{\bar{p}}(X)$. Since $\gamma_{\Sigma}\omega \in \frac{\Omega I^{\bullet}_{\bar{p}}(W)}{\mathrm{Cap}^{\bullet}(W)}$ is still a closed form, there exists a unique closed element $\overline{\kappa} \in \frac{\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)}{\Omega I^{\bullet}_{\bar{p}}(M)}$ with representative $\kappa \in \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)$ such that $\overline{j}^{*}_{W}\overline{\kappa} = \gamma_{\Sigma}\omega$. (This uses the fact that $\overline{j}^{*}_{W}$ is an isomorphism). Then

$$\delta[\omega] = [-d\kappa].$$

Now let $\eta \in \Omega I^{n-r}_{\bar{q}}(M)$ be a closed form. To show the commutativity of (TS) we must argue that

$$\int_M (d\kappa) \wedge \eta = \pm \int_W \kappa \wedge (c_W^* \eta).$$

Since $d\eta = 0$, Stokes Theorem for manifolds with corners (see e.g. [Lee13, Theorem 10.32]) gives

$$\begin{aligned}\int_M (d\kappa)\wedge\eta = \int_M d(\kappa\wedge\eta) &= \int_E c_E^*\kappa\wedge c_E^*\eta + \int_W c_W^*\kappa\wedge c_W^*\eta\\ &= \int_W c_W^*\kappa\wedge c_W^*\eta \quad \text{by Lemma 7.4.3.}\end{aligned}$$

But by definition of κ, there is an $\alpha \in \mathrm{Cap}^\bullet(W)$ such that $c_W^*\kappa = \omega + \alpha$. But again, by an analogous argument as in Lemma 7.4.3 (definition of cotruncation: $r \geq L \Rightarrow \dim(F) - r < L^*$ and vice versa), it holds that

$$\int_W \alpha\wedge c_W^*\eta = 0, \tag{35}$$

since $c_W^*\eta \in ft_{\geq L^*}\Omega^\bullet_{\mathcal{MS}}(\Sigma)$. Therefore

$$\int_M (d\kappa)\wedge\eta = \int_W \omega\wedge c_W^*\eta,$$

i.e. the top square commutes.

The middle square (MS) commutes obviously since both vertical maps involved are induced by subcomplex inclusions. So it is only left to show that the bottom square (BS) commutes:

Note, that for a closed form $\eta \in f(\tau_{\geq L}\Omega I_{\bar{q}})\Omega^{n-r-1}_{\mathcal{MS}}(\Sigma)$ it holds that $\Delta[\eta] = [-d\omega]$ for some $\omega \in \Omega I^{n-r-1}_{\bar{q}}(M)$ with $c_W^*\omega - \eta \in d(f(\tau_{\geq L^*}\Omega I_{\bar{q}})\Omega^\bullet_{\mathcal{MS}}(\Sigma))$. Let us describe the map

$$\Lambda : \widetilde{HI}^r_{\bar{p}}(M) \to H^r\big(f(\tau_{<L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma)\big):$$

For this purpose let $\theta \in \widetilde{\Omega I}^r_{\bar{p}}(M)$ be a closed form. Then of course $\lceil c_W^*\theta\rceil \in \frac{\Omega I^\bullet_{\bar{p}}(W)}{\mathrm{Cap}^\bullet(W)}$ is also closed and hence there is a closed form $\xi \in f(\tau_{<L}\Omega I_{\bar{p}})\Omega^r_{\mathcal{MS}}(W)$ such that

$$\lceil c_W^*\theta - \xi\rceil = d\lceil\nu\rceil \in \frac{\Omega I^\bullet_{\bar{p}}(W)}{\mathrm{Cap}^\bullet(W)}, \tag{36}$$

for some $\nu \in \Omega I^\bullet_{\bar{p}}(W)$. Then $\Lambda[\theta] = [\xi] \in H^r(f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma))$. To prove that (BS) commutes we show that for $\eta \in f(\tau_{\geq L}\Omega I_{\bar{q}})\Omega^{n-r-1}_{\mathcal{MS}}$ and $\theta \in \widetilde{\Omega I}^r_{\bar{p}}(M)$ closed, with $\Delta[\eta] = [-d\omega]$, $\Lambda[\theta] = [\xi]$ one has

$$\int_M \theta\wedge d\omega = \pm\int_W \xi\wedge\eta.$$

By Stokes Theorem for manifolds with corners we get

$$\begin{aligned}\int_M \theta\wedge d\omega = \int_M d(\theta\wedge\omega) &= \int_W c_W^*(\theta\wedge\omega) + \int_E c_E^*(\theta\wedge\omega)\\ &= \int_W c_W^*(\theta\wedge\omega) \quad \text{by Lemma 7.4.3}\\ &= \int_W c_W^*\theta\wedge(\eta + d\tau),\end{aligned}$$

for some $\tau \in f(\tau_{\geq L}\Omega I_{\bar{q}})\Omega^{n-r-2}_{\mathcal{MS}}(\Sigma)$. Since by Stokes Theorem and [Ban11, Lemma 7.3] it holds that $\int_W c_W^*\theta \wedge d\tau = 0$ we arrive at the equation

$$\int_M \theta \wedge d\omega = \int_W c_W^* \theta \wedge \eta.$$

By (36) there is a $\nu \in \Omega I^\bullet_{\bar{p}}(W)$ such that $\alpha := \xi + d\nu - c_W^*\theta \in \mathrm{Cap}^\bullet(W)$ and thus we get

$$\int_W \xi \wedge \eta = \int_W c_W^*\theta \wedge \eta + \int_W \alpha \wedge \eta - \int_W d\nu \wedge \eta = \int_W c_W^*\theta \wedge \eta,$$

since $\int_W \alpha \wedge \eta = 0$ as in equation (35) and $\int_W d\nu \wedge \eta = 0$ by Stokes Theorem and [Ban11, Lemma 7.3] as before. Thus (BS) also commutes (up to sign).

Using diagram (34) and applying the 5-Lemma to it then finishes the proof of the theorem. □

Remark 10.5.3 *Note, that we have also shown that the complex defined by*

$$\widehat{\Omega I}^r_{\bar{p}}(M) := \left\{\omega \in \Omega I^\bullet_{\bar{p}}(M) \;\middle|\; c_W^*\omega \in f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma)\right\} \subset \Omega I^\bullet_{\bar{p}}(M)$$

is quasi isomorphic via the subcomplex inclusion. This fact simply follows by applying the 5-Lemma to the cohomology diagram induced by the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{rel(W)} \arrow[r] \arrow[d, equal] & \widehat{\Omega I}^\bullet_{\bar{p}}(M) \arrow[r] \arrow[d, hook] & f(\tau_{\geq L}\Omega I_{\bar{p}})\Omega^\bullet_{\mathcal{MS}}(\Sigma) \arrow[r] \arrow[d, hook, "qis"] & 0 \\
0 \arrow[r] & \Omega_{rel(W)} \arrow[r] & \Omega I^\bullet_{\bar{p}}(M) \arrow[r] & Cap^\bullet(W) \arrow[r] & 0.
\end{tikzcd}$$

11 Independence of Choices

In this last section we want to discuss whether or not and to what extent the choices we made to construct the complexes of differential forms we used influence our results. There are two types of choices we made

1. Choices on collars of the boundaries and boundary parts ∂B of B, ∂E of E, ∂F of F, ∂W of W and E, W of M.
2. Choices of good open covers of the base manifolds B of the bundle $p : E \to B$ with fiber L and Σ of $q : W \to \Sigma$ with fiber F with respect to which the bundles trivialize.

We want to show that for each perversity $\bar{p}$ the cohomology groups $HI^r_{\bar{p}}(X)$ do not depend on the choices we made for all $r \in \mathbb{Z}$.

11.1 The Choice of the Collars

11.1.1 The Two–Strata Case

Let us first have a look onto the HI-cohomology of a pseudomanifold X with two strata $X = X_n \supset X_b = B$ such that the link bundle $p : E \to B$ is geometrically flat with link L. Then one defines the complex $\Omega I^\bullet_{\bar{p}}$ on the regular part M of X (defined by deleting a distinguished neighbourhood of the singular stratum) by choosing a collar $c_{\partial M} : \partial M \times [0, 1) \hookrightarrow M$ and setting

$$\Omega I^\bullet_{\bar{p}}(M) := \left\{\omega \in \Omega^\bullet(M) \middle| c^*_{\partial M}\omega = \pi^*\eta, \text{ with } \eta \in ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\right\}$$

with $\pi : \partial M \times [0, 1) \to \partial M$ the first factor projection. Note that in the original paper [Ban11] the author defines this complex as a subcomplex of $N = \mathrm{int}(M)$ but those two complexes are isomorphic.
That means that $\Omega I^\bullet_{\bar{p}}(M)$ is defined as the subcomplex of $\Omega^\bullet(M)$ consisting of forms satisfying a certain relation on a collarlike neighbourhood of the boundary. More precisely $\Omega I^\bullet_{\bar{p}}(M)$ consists of the forms with restriction to such a neighbourhood equaling the pullback of a form in some subcomplex of $\Omega^\bullet(\partial M)$. By a standard argument using integration in the collar direction this gives cohomology groups that are independent of the choice of a collar:

Proposition 11.1.1 *Let M be a compact manifold with boundary ∂M and let $S^\bullet \subset \Omega^\bullet(\partial M)$ be a subcomplex. Then any two collars $c, \widetilde{c}$ of ∂M in M define quasi-isomorphic complexes*

$$Q^\bullet(M) := \left\{\omega \in \Omega^\bullet(M) \middle| c^*\omega = \pi^*\eta,\ \eta \in S^\bullet\right\}$$

and

$$\widetilde{Q}^\bullet(M) := \left\{\omega \in \Omega^\bullet(M) \middle| \widetilde{c}^*\omega = \pi^*\eta,\ \eta \in S^\bullet\right\}.$$

Proof: This proof is given for the above complex $\Omega I^\bullet_{\bar{p}}(M)$ in a slightly different version in [BH15, Lemma 6.7].
We let $j : \partial M \hookrightarrow M$ denote the inclusion of the boundary in M and set $P^\bullet(M) := \{\omega \in \Omega^\bullet(M) | j^*\omega \in S^\bullet\}$. Since c and $\widetilde{c}$ both are collars of ∂M in M we have subcomplex inclusions $Q^\bullet(M) \subset P^\bullet(M)$ and $\widetilde{Q}^\bullet(M) \subset P^\bullet(M)$. We show that these subcomplex inclusions are quasi-isomorphisms:
Let $\omega \in P^\bullet(M)$ be a closed form. First note that we can enlarge the collar a little bit to get an embedding $c_l : \partial M \times [0,2) \hookrightarrow M$. Denoting $C_l = c_l\big(\partial M \times [0,2)\big)$ we have $\omega|_{C_l} = \omega_T(t) + dt \wedge \omega_N(t)$. Using a smooth cutoff function $\xi : [0,2) \to \mathbb{R}$ with $\xi|_{[0,1)} = 1$ and $\xi|_{[3/2,1)} = 0$ we define a form

$$\widehat{\omega} := \omega - d\big(\pi_2^*\xi \int_0^t \omega_N(\tau)d\tau\big)$$

by using extension by zero of the second summand and using the slightly sloppy notation $\pi_2 = \pi_2 \circ c_l^{-1} : C_l \to \partial M \times [0,2) \to [0,2)$. Then $0 = d\omega = d\widehat{\omega}$ and hence $0 = d\widehat{\omega}|_C = dt \wedge \widehat{\omega}'(t) + d_{\partial M}\widehat{\omega}(t)$. Moreover

$$\widehat{\omega}|_{C = c_l(\partial M \times [0,1))} = \omega|_C - dt \wedge \omega_N(t) - \int_0^t d_{\partial M}\omega_N(\tau)d\tau$$

and therefore $dt \lrcorner\, \widehat{\omega} = 0$. This implies that $0 = dt \lrcorner\, d\widehat{\omega}|_C = \widehat{\omega}'$ or, equivalently, $\widehat{\omega}|_C = \pi_1^*(j^*\widehat{\omega}) = \pi_1^*(j^*\omega)$. Hence $\widehat{\omega} \in Q(M)$ is a representative of the cohomology class $[\omega] \in H^\bullet(P^\bullet)$ and we have shown that subcomplex inclusion $Q^\bullet \subset P^\bullet$ induces a surjective map on cohomology.
Now let $\omega \in Q^\bullet$ be a closed form with $\omega = d\eta$ for some form $\eta \in P^\bullet$. We must show that then also $\omega = d\widehat{\eta}$ with $\eta \in Q^\bullet$. The arguments follow the above: $\eta|_{C_l} = \eta_T(t) + dt \wedge \eta_N(t)$. Set

$$\widehat{\eta} := \eta - d\big(\pi_2^* xi \int_0^t \eta_N(\tau)d\tau\big)$$

Then $d\widehat{\eta} = d\eta = \omega$ and hence $(\widehat{\eta}|_C)' = 0$ giving $\widehat{\eta}|_C = \pi_1^*(j^*\widehat{\eta}) = \pi_1^*(j^*\eta)$.
□

Corollary 11.1.2 *For a pseudomanifold with two strata, the subcomplex inclusion $\Omega I^\bullet_{\bar{p}}(M) \hookrightarrow \mathcal{AI}^\bullet_{\bar{p}}(M) := \{\omega \in \Omega^\bullet(M) | j^*\omega \in ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\}$ is a quasi-isomorphism. Hence $HI^\bullet_{\bar{p}}(M)$ is independent of the choice of a collar.*

Proof: Apply the previous Proposition.
□

11.1.2 Pseudomanifolds with Three Strata

Note that we restrict ourselves to the setting of Section 10. The setting with isolated bottom stratum can be treated by analogous proofs using the

analogous statements in that setting . To show that for pseudomanifolds X with three strata as in Section 10 it also holds that $HI^{\bullet}_{\bar{p}}(X)$ is independent of the choices of collars of E and W in M and of ∂B and ∂F in B and F we will make use of the distinguished triangle of the Lemma 10.4.4.
In detail we will prove the following: Let $j_E : E \hookrightarrow M$ and $j_W : W \hookrightarrow M$ denote the embedding of the boundary parts. We will show that the subcomplex inclusion $\Omega I^{\bullet}_{\bar{p}}(M) \hookrightarrow \mathcal{AI}^{\bullet}_{\bar{p}}(M)$ with

$$\mathcal{AI}^{\bullet}_{\bar{p}}(M) := \left\{\omega \in \Omega^{\bullet}(M) \middle| j_E^*\omega \in ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(B),\ j_W^*\omega \in ft_{\geq K^*}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)\right\}$$

is a quasi-isomorphism. This shows that the cohomology groups $HI^r_{\bar{p}}(X)$, $r \in Z$, do not depend on the choice of the collars.

Proposition 11.1.3 *Let* $\widetilde{\mathcal{AI}}^{\bullet}_{\bar{p}}(M) := \{\omega \in \Omega^{\bullet}(M) | j_E^*\omega \in ft_{\geq K}\Omega^{\bullet}_{\mathcal{MS}}(B)\}$*. Then the subcomplex inclusion* $\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M) \hookrightarrow \widetilde{\mathcal{AI}}^{\bullet}_{\bar{p}}(M)$ *is a quasi-isomporphism.*

<u>Proof:</u> We will use the distinguished triangle of Lemma 7.2.2. Note that the standard argument of Lemma 11.1.1 is also applicable to collars of boundary parts of manifolds with corners. This implies that the subcomplex inclusion $\Omega^{\bullet}_{E\mathcal{MS}}(M) \hookrightarrow \mathcal{A}^{\bullet}_{E\mathcal{MS}}(M) = \{\omega \in \Omega^{\bullet}(M) | j_E^*\omega \in \Omega^{\bullet}_{\mathcal{MS}}(B)\}$ is a quasi-isomorphism. Taking also the result of Lemma 7.1.4 into account we get the following commutative diagram of distinguished triangles:

$$\begin{tikzcd}
\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M) \arrow[r] \arrow[d, hook] & \Omega^{\bullet}_{E\mathcal{MS}}(M) \arrow[r] \arrow[d, hook] & ft_{<K}\Omega^{\bullet}_{\mathcal{MS}}(B) \arrow[r, "+1"] \arrow[d, hook] & {} \\
\widetilde{\mathcal{AI}}^{\bullet}_{\bar{p}}(M) \arrow[r] & \mathcal{A}^{\bullet}_{E\mathcal{MS}}(M) \arrow[r] & ft_{<K}\Omega^{\bullet}_{\mathcal{MS}}(B) \arrow[r, "+1"] & {}
\end{tikzcd}$$

Note that the bottom line is a distinguished triangle by the same argument as in the proof of Lemma 7.1.4. Looking at the induced sequence on cohomology and applying the 5-Lemma finishes the proof.

□

Lemma 11.1.4 *Let* $f(\tau_{<L}\mathcal{AI}^{\bullet}_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma) := \{\omega \in \Omega^{\bullet}_{\mathcal{MS}}(\Sigma) | \forall \alpha \in I :\ \omega_{q^{-1}(U_\alpha)} = \sum_j \pi_1^*\eta_j \wedge \pi_2^*\gamma_j,\ \eta_j \in \Omega^{\bullet}(\Sigma),\ \gamma_j \in \tau_{<L}\mathcal{AI}^{\bullet}_{\bar{p}}(F)\}$*. Then the subcomplex inclusion* $f(\tau_{<L}\mathcal{AI}^{\bullet}_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma) \hookrightarrow f(\tau_{<L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)$ *is a quasi-isomorphism and in particular the cohomology groups of the latter complex are independent of the choice of a collar.*

<u>Proof:</u> We will not give all the details since they are similar to previous proofs. By the result of Corollary 11.1.2 the subcomplex inclusion $\Omega I^{\bullet}_{\bar{p}}(F) \hookrightarrow \mathcal{AI}^{\bullet}_{\bar{p}}(F)$ is a quasi-isomorphism and hence also $\tau_{<L}\Omega I^{\bullet}_{\bar{p}}(F) \hookrightarrow \tau_{<L}\mathcal{AI}^{\bullet}_{\bar{p}}(F)$.

The rest is based on the usual Mayer-Vietoris argument which needs a bootstrap principle and as induction start the local Poincaré statements for coordinate charts U:

$$f(\tau_{<L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(U) \xrightarrow{\simeq} \tau_{<L}\Omega I^{\bullet}_{\bar{p}}(F)$$

as well as

$$f(\tau_{<L}\mathcal{AI}^{\bullet}_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(U) \xrightarrow{\simeq} \tau_{<L}\mathcal{AI}^{\bullet}_{\bar{p}}(F).$$

□

Finally we will use the preciding lemma and proposition to prove that $HI^{\bullet}_{\bar{p}}(X)$ is independent of the choice of collars:

Theorem 11.1.5 *(Independence of collars)*
The subcomplex inclusion $\Omega I^{\bullet}_{\bar{p}}(M) \hookrightarrow \mathcal{AI}^{\bullet}_{\bar{p}}(M)$ is a quasi-isomorphism. In particular, the cohomology groups $HI^{r}_{\bar{p}}(X)$, $r \in \mathbb{Z}$, are independent of the choice of collars.

Proof: We first argue that there is a distinguished triangle in $\mathcal{D}(\mathbb{R})$ of the following form:

$$\begin{tikzcd} \mathcal{AI}^{\bullet}_{\bar{p}}(M) \arrow[rr] & & \widetilde{\mathcal{AI}}^{\bullet}_{\bar{p}}(M) \arrow[dl] \\ & f(\tau_{<L}\mathcal{AI}^{\bullet}_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma) \arrow[ul, "+1"] & \end{tikzcd}$$

The proof is an analogon to the proof of Lemma 10.4.4. One starts with the short exact sequence

$$0 \to \mathcal{AI}^{\bullet}_{\bar{p}}(M) \to \widetilde{\mathcal{AI}}^{\bullet}_{\bar{p}}(M) \to \frac{\widetilde{\mathcal{AI}}^{\bullet}_{\bar{p}}(M)}{\mathcal{AI}^{\bullet}_{\bar{p}}(M)} \to 0$$

As in Lemma 10.4.4 pullback to $W \subset M$ induces an isomorphism

$$\frac{\widetilde{\mathcal{AI}}^{\bullet}_{\bar{p}}(M)}{\mathcal{AI}^{\bullet}_{\bar{p}}(M)} \xrightarrow[j^*_W]{\cong} \frac{\Omega I^{\bullet}_{\bar{p}}(W)}{\mathrm{Cap}(W)}.$$

Using the quasi-isomorphism $f(\tau_{<L}\Omega I_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma) \to \frac{\Omega I^{\bullet}_{\bar{p}}(W)}{\mathrm{Cap}(W)}$ of the mentioned lemma and the result of the previous Lemma 11.1.4 then gives the above distinguished triangle in $\mathcal{D}(\mathbb{R})$.
For the second step of the proof we note that the above distinguished triangle and the distinguished triangle (32) of Lemma 10.4.4 together give rise to the

following commutative diagram on cohomology

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H^{r-1}\big(f(\tau_{<L}\Omega I^{\bullet}_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)\big) \arrow[r, "\cong"] \arrow[d] & H^{r-1}\big(f(\tau_{<L}\mathcal{AI}^{\bullet}_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)\big) \arrow[d] \\
HI^{r}_{\bar{p}}(M) \arrow[r] \arrow[d] & H^{r}\big(\mathcal{AI}^{\bullet}_{\bar{p}}(M)\big) \arrow[d] \\
H^{r}\big(\widetilde{\Omega I}^{\bullet}_{\bar{p}}(M)\big) \arrow[r, "\cong"] \arrow[d] & H^{r}\big(\widetilde{\mathcal{AI}}^{\bullet}_{\bar{p}}(M)\big) \arrow[d] \\
H^{r}\big(f(\tau_{<L}\Omega I^{\bullet}_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)\big) \arrow[r, "\cong"] \arrow[d] & H^{r}\big(f(\tau_{<L}\mathcal{AI}^{\bullet}_{\bar{p}})\Omega^{\bullet}_{\mathcal{MS}}(\Sigma)\big) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

where the horizontal maps are all induced by subcomplex inclusion. The statement of the theorem is then implied by the 5-Lemma.

□

11.2 The Choice of an Atlas of the Flat Bundle

To define complexes of multiplicatively structured forms on the total space of a (geometrically) flat bundle one must choose an atlas of the bundle, i.e. a cover of the base with respect to which the bundle trivializes. In practise one often chooses a good open cover to make Mayer-Vietoris type arguments easier. A priori the cohomology of these complexes depends not only on the bundle but also on this atlas. As shown in [Ban11, Theorem 3.13], subcomplex inclusion of $\Omega^{\bullet}_{\mathcal{MS}}(B)$ into $\Omega^{\bullet}(E)$, the differential form complex on the total space, induces an isomorphism on cohomology. Hence these cohomology groups are independent of the atlas. A correspondent independence result for the cohomology groups of the complex of fiberwisely cotruncated multiplicatively structured forms is not that easy to show, however. To do so we use the results of [Ban13].

11.2.1 The Two–Strata Case

Again we first look onto the HI-cohomology of pseudomanifolds X with filtration $X = X_n \supset X_b = B$ and geometrically flat link bundle $p : E \to B$

for the singular set. The main tools for showing that the cohomology groups $HI^r(X)$, $r \in \mathbb{Z}$, are independent of the choice of a good atlas of the flat bundle, i.e. a good open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ such that the bundle trivializes with respect to that cover, are [Ban13, Lemma 4.1 and Theorem 5.1].

Lemma 11.2.1 *($H^\bullet\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\big)$ is independent of the atlas)*
The cohomology groups $H^r\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\big)$, $r \in \mathbb{Z}$, are independent of the choice of a good atlas for the bundle $p : E \to B$.

Proof: Let $\mathfrak{U} := \{U_\alpha\}_{\alpha \in I}$ denote a good open atlas for the bundle $p : E \to B$. We follow the notation of [Ban13] and let $K_{\geq K}$ denote the cotruncated double complex defined by

$$K^{p,q}_{\geq K} := C^p\big(\mathfrak{U}; (ft_{\geq K}\Omega^\bullet_{\mathcal{MS}})^q\big) = \prod_{\alpha_0,\dots,\alpha_p} ft_{\geq K}\Omega^q_{\mathcal{MS}}(U_{\alpha_0\dots\alpha_p}).$$

By [Bot82, Theorem 14.14] this double complex defines a spectral sequence $E(K_{\geq K}) = (E_{\geq K,r}, d_{\geq K,r})$ which converges to the total cohomology of the double complex which is by the generalized Mayer Vietoris principle isomorphic to the de Rham cohomology of $ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)$ (compare to [Bot82, Prop. 8.8]) and with second page

$$E_2^{p,q} = H^p\big(\mathfrak{U}, \mathbf{H}^q_{\geq K}(L)\big)$$

with $\mathbf{H}^q_{\geq K}(F)$ the Čech presheaf defined by

$$\mathbf{H}^q_{\geq K}(F)(U) = H^q(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(U)) \quad \forall U \in Ob\ C(\mathfrak{U}),\ U \neq \emptyset.$$

The arguments in the proof of [Ban13, Theorem 5.1, p.15] give that $d_{\geq K,2} = 0$, i.e. the spectral sequence collapses on the second page, $E_2^{p,q} = E_\infty^{p,q}$. So proving that the second page of the spectral sequence is independent of the atlas will show that the cohomology of $ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)$ is independent of the atlas. We will distinguish the cases $q < K$ and $q \geq K$.
$q < K$: By [Ban13, Lemma 4.1] we have $\mathbf{H}^q_{\geq K}(F) = 0$ the trivial Čech presheaf in this degrees and hence for all $p \in \mathbb{Z}$ we have $E_2^{p,q} = 0$ which is independent of $\mathfrak{U}$.
$q \geq K$: Again by [Ban13, Lemma 4.1] and the arguments preceding this lemma for these q there is an isomorphism of Čech presheaves

$$\mathbf{H}^q_{\geq K}(F) \xrightarrow{\cong} \mathbf{H}^q(F).$$

($\mathbf{H}^q(F)$ the Čech presheaf defined by $U \mapsto H^q(p^{-1}(U))$.) This isomorphism induces isomorphisms $H^p(\mathfrak{U}, \mathbf{H}^q_{\geq K}(F)) \xrightarrow{\cong} H^p(\mathfrak{U}, \mathbf{H}^q(F)) = E_2^{p,q}$ for all $p \in \mathbb{Z}$, with E_2 the second page of the cohomology spectral sequence of the fiber bundle (see [Bot82, Theorem 14.18]). Since B is a manifold, [Spa82, Corollary 3.2] gives an isomorphism $H^p(\mathfrak{U}, \mathbf{H}^q(F)) \xrightarrow{\cong} H^p(B, \mathbf{H}^q(F))$ to the singular cohomology with local coefficients $\mathbf{H}^q(F)$ which ist independent of the atlas.

□

Proposition 11.2.2 *(Independence of HI of the atlas)*
Let $\bar{p}$ be a perversity. Then the cohomology groups $HI^r_{\bar{p}}(X)$, $r \in \mathbb{Z}$, are independent of the choice of a good open atlas of the bundle $p : E = \partial M \to B$.

Proof: Let $\partial M \subset C \cong \partial M \times [0,1)$ be a collar neighbourhood of the boundary and recall the definition $\Omega^\bullet_{rel}(M) = \{\omega \in \Omega^\bullet(M) | \omega|_C = 0\}$. Then there is a short exact sequence

$$0 \longrightarrow \Omega^\bullet_{rel}(M) \xrightarrow{\iota} \Omega I^\bullet_{\bar{p}}(M) \xrightarrow{j^*} ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B) \longrightarrow 0$$

with ι the subcomplex inclusion and $j : \partial M \hookrightarrow M$ the inclusion map of the boundary. This gives the following long exact sequence on cohomology:

$$\begin{array}{ccccccc} ... & \longrightarrow & H^{r-1}\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\big) & \xrightarrow{\partial^*} & H^r\big(\Omega^\bullet_{rel}(M)\big) & \xrightarrow{\iota^*} & HI^r_{\bar{p}}(X) \\ & & & & & \swarrow{\scriptstyle j^*} & \\ ... & \xleftarrow{\iota^*} & H^{r+1}\big(\Omega^\bullet_{rel}(M)\big) & \xleftarrow{\partial^*} & H^r\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\big) & & \end{array}$$

By linear algebra we get that

$$\begin{aligned} HI^r(X) &\cong \ker j^* \oplus \operatorname{im} j^* = \operatorname{im} \iota^* \oplus \ker \partial^* \\ &\cong \frac{H^r\big(\Omega^\bullet_{rel}(M)\big)}{\ker \iota^*} \oplus \ker \partial^* = \frac{H^r\big(\Omega^\bullet_{rel}(M)\big)}{\operatorname{im} \partial^*} \oplus \ker \partial^*. \end{aligned}$$

Since for all $\alpha \in \mathbb{Z}$ the cohomology groups $H^\alpha\big(\Omega^\bullet_{rel}(M)\big)$ and by Lemma 11.2.1 also $H^\alpha\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\big)$ are independent of the choice of an atlas, all that is left to prove is that the map $\partial^* : H^r\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\big) \to H^{r+1}\big(\Omega^\bullet_{rel}(M)\big)$ is independent of the atlas as well. By the results of [Ban13, Lemma 4.1 and Theorem 5.1] we have

$$\begin{aligned} H^r\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\big) &\cong H^r_D\big(\mathfrak{U}; \mathbf{H}^q_{\geq K}(F)\big) = \bigoplus_{p+q=r} E^{p,q}_{\infty,\geq K} = \bigoplus_{p+q=r} E^{p,q}_{2,\geq K} \\ &= \bigoplus_{q=K}^{r} E_2^{r-q,q} = \bigoplus_{q=K}^{r} E_\infty^{r-q,q} \subset H^r(E). \end{aligned}$$

As mentioned, the groups $E_2^{r-q,q} = H^{r-q}(\mathfrak{U}; \mathbf{H}^q(F))$ are independent of the atlas. Let us recall the definition of the map ∂^*:

For $\eta \in ft_{\geq K}\Omega^r_{MS}(B)$ closed there is a preimage $\omega \in \Omega I^\bullet_{\bar{p}}(M)$ with $\eta = j^*\omega$. Then $\partial^*[\eta] = [d\omega] \in H^{r+1}\big(\Omega^\bullet_{rel}(M)\big)$. Now if there are two different good open atlases $\mathfrak{U}$ and $\mathfrak{U}'$of the bundle $p : E \to B$ with corresponding complexes $ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)$ and $(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B))'$ of fiberwise cotruncated multiplicatively structured forms, then for two cohomology classes $[\eta] \in H^r\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\big)$ and $[\theta]' \in H^r\big((ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B))'\big)$ that correspond to the same cohomology class in $\bigoplus_{q=K}^r H^{r-q}(\mathfrak{U}; \mathbf{H}^q(F))$ we have that $\eta - \theta = d\alpha$ for some form $\alpha \in \Omega^{r-1}(E)$. Let $\sigma \in \Omega^{r-1}(M)$ be a form with $j^*\sigma = \alpha$.
Then take an explicit preimage $\omega \in \Omega I^\bullet_{\bar{p}}(M)$ be letting $\psi \in C^\infty(M)$ denote a smooth cutoff function in the collar direction of a larger collar $\partial M \times [0, 2) \cong \widetilde{C} \supset C$ with $\psi(t) = 1$ for $t \in [0, 1)$ and $\psi(t) = 0$ for $t > 3/2$ and defining $\omega = \psi\pi^*\eta$ extended by zero to all of M. Analogously take $\xi = \psi\pi^*\theta$, a preimage of θ. Then

$$d(\psi' dt \wedge \pi^*\alpha) = -\psi' dt \wedge \pi^*(\eta - \theta) = d\xi - d\omega.$$

Since $\psi' dt \wedge \pi^*\alpha \in \Omega^r_{rel}(M)$ this implies that

$$\partial^*[\eta] = [d\omega] = [d\xi] = \partial'^*[\theta]' \in H^{r+1}\big(\Omega^\bullet_{rel}(M)\big).$$

Alltogether we have shown that $HI^r_{\bar{p}}(X)$ is independent of the atlas for all $r \in \mathbb{Z}$.

□

11.2.2 Pseudomanifolds with Three Strata

Finally we will show that also for pseudomanifolds with three strata, zero dimensional bottom stratum and the additional condition of Section 8.1, the cohomology of $\Omega I^\bullet_{\bar{p}}(M)$ is independent of the choice of a good open atlas for the flat bundle $p : E \to B$. The proof is inspired by the proof of the indepndency of choices of the homology of the intersection spaces in [Ban10, Theorem 2.18].
Note first that Lemma 11.2.1 is still true for a compact base B with boundary ∂B, ergo applicable in the three strata case. The arguments about Čech cohomology can be transferred literally and the result of [Spa82] does not require B to be a manifold (without boundary) and is also applicable to manifolds with boundary:

Lemma 11.2.3 *($H^\bullet\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\big)$ is independent of the atlas)*
The cohomology groups $H^r\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\big)$, $r \in \mathbb{Z}$, are independent of the choice of an good open atlas for the bundle $p : E \to B$ also if B is a compact manifold with boundary.

Proof: Literally the same as the proof of Lemma 11.2.1. □

We are then able to prove that the cohomology groups of $\widetilde{\Omega I}^\bullet_{\bar{p}}(M)$ do not depend on the choice of an atlas for the bundle $p : E \to B$:

Lemma 11.2.4 *($\widetilde{HI}^{\bullet}_{\bar{p}}(M)$ is independent of the atlas of $p : E \to B$)*
The cohomology groups $\widetilde{HI}^{r}_{\bar{p}}(M)$, $r \in \mathbb{Z}$, are independent of the choice of an good open atlas for the bundle $p : E \to B$. Moreover, for $r \in \mathbb{Z}$ the image of the map $i^* : \widetilde{HI}^{r}_{\bar{p}}(M) \to H^r(M)$, induced by subcomplex inclusion, is independent of the choice of the atlas for the bundle.

Proof: The proof of the first part of the statement is the same as the proof of Proposition 11.2.2 and will hence not be repeated. To prove the second part we will need the distinguished triangles

$$\Omega^\bullet(M, C_E) \to \widetilde{\Omega I}^{\bullet}_{\bar{p}}(M) \xrightarrow{J_E^*} ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B) \xrightarrow{+1}$$

of Lemma 7.2.6 and

$$\Omega^\bullet(M, C_E) \to \Omega^\bullet(M) \xrightarrow{J_E^*} \Omega^\bullet(E) \xrightarrow{+1} .$$

Those induce the following commutative diagram on cohomology

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & H^r(M, C_E) & \xrightarrow{f} & \widetilde{HI}^{r}_{\bar{p}}(M) & \xrightarrow{h} & H^r\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\big) & \xrightarrow{\partial} & \ldots \\
 & & \| & & \downarrow{\scriptstyle i^*} & & \downarrow & & \\
\ldots & \longrightarrow & H^r(M, C_E) & \xrightarrow{g} & H^r(M) & \xrightarrow{J_E^*} & H^r(E) & \xrightarrow{\partial} & \ldots
\end{array}$$

Note that the map $H^r\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\big) \hookrightarrow H^r(E)$ is injective by the Poincaré-Lefschetz duality statement of Proposition 7.3.1 or by the argument in the proof of Lemma 11.2.1 and further that we have renamed the map $h := J_E^* : \widetilde{HI}^{r}_{\bar{p}}(M) \to H^r\big(ft_{\geq K}\Omega^\bullet_{\mathcal{MS}}(B)\big)$ to distinguish it from the map J_E^* defined on the ordinary cohomology groups. By linear algebra, $\widetilde{HI}^{r}_{\bar{p}}(M) = \ker h \oplus U = \operatorname{im} f \oplus U$ with $U \cong \operatorname{im} h$.
By the above diagram we have

$$i^*\big(\widetilde{HI}^{r}_{\bar{p}}(M)\big) \cong \operatorname{im} g \oplus \operatorname{im} h.$$

To prove this we first show that $i^*(\operatorname{im} f) = \operatorname{im} g$: The "$\subset$" part of that proof is obvious since the diagram commutes. Let $g(y) \in H^r(M)$. Then $g(y) = i^* f(y) \in i^*(\operatorname{im} f)$.

For $x \in \ker i^* \cap U$ we have $0 = J_E^* i^*(x) = h(x)$. The injectivity of $h|_U$ then implies $x = 0$ and therefore $\ker i^* \cap U = \{0\}$. Next we use that $H^r(M) = \ker J_E^* \oplus V$ with $V \cong \operatorname{im} J_E^*$. Let $i^*(x) \in \ker J_E^*$, then $0 = J_E^* i^*(x) = h(x)$ and hence $x \in \ker h$, implying $i^*(U) \subset V$. This means that $J_E^*|_{\operatorname{im} i^*} : \operatorname{im} i^* \to \operatorname{im} h$ is an isomorphism. Last note that in particular the above gives $i^*(U) \cap i^*(\operatorname{im} f) = \{0\}$. Hence we have proven that $\operatorname{im} i^* \cong \operatorname{im} g \oplus \operatorname{im} h$. Both of these vector spaces are independent of the choice of the atlas, the latter by the proof of the first statement and since $\operatorname{im} h = \ker \partial$.

□

At last we are able to prove the independence of the cohomology groups $HI^r_{\bar{p}}(X)$, $r \in \mathbb{Z}$, of the flat atlas of $p : E \to B$.

Theorem 11.2.5 *(Independence of $HI^\bullet_{\bar{p}}(X)$ of the atlas of $p : E \to B$)*
The cohomology groups $HI^r_{\bar{p}}(X)$, $r \in \mathbb{Z}$, are independent of the choice of the atlas of the bundle $p : E \to B$.

Proof: We distinguish the cases $r > L$, $r < L$ and $r = L$.
For $r < L$ we use the long exact cohomology sequence induced by the distinguished triangle of Lemma 8.2.3:

$$... \to \underbrace{H^{r-1}\big(\tau_{\geq L}\Omega I^\bullet_{\bar{p}}(W)\big)}_{=0} \to \widetilde{HI}^r_{\bar{p}}(M, C_W) \to HI^r_{\bar{p}}(X) \to \underbrace{H^r\big(\tau_{\geq L}\Omega I^\bullet_{\bar{p}}(W)\big)}_{=0} \to ...$$

since both r and $r-1$ are smaller than L. This gives

$$HI^r_{\bar{p}}(X) \cong \widetilde{HI}^r_{\bar{p}}(M, C_W) \cong \widetilde{HI}^{n-r}_{\bar{q}}(M),$$

where the last isomorphism is Poincaré-Lefschetz duality for $\widetilde{HI}$ over complementary perversities, see Theorem 7.5.5. Therefore $HI^r_{\bar{p}}(X)$ is independent of the atlas in that degree by Lemma 11.2.4. For $r > L$ we use the long exact cohomology sequence induced by the distinguished triangle of Lemma 8.2.4:

$$... \to \underbrace{H^{r-1}\big(\tau_{<L}\Omega I^\bullet_{\bar{p}}(W)\big)}_{=0} \to \widetilde{HI}^r_{\bar{p}}(M) \to HI^r_{\bar{p}}(X) \to \underbrace{H^r\big(\tau_{<L}\Omega I^\bullet_{\bar{p}}(W)\big)}_{=0} \to ...$$

since here both r and $r-1$ are bigger or equal than L. Hence $HI^r_{\bar{p}}(X) \cong \widetilde{HI}^r_{\bar{p}}(M)$ and hence independent of the atlas of the bundle p.
The case $r = L$ is the only difficult one. We will also use the above long exact cohomology sequences. However for $r = L$ the result we can deduce is not as before. Since

$$H^{k-1}\big(\tau_{<L}\Omega I^\bullet_{\bar{p}}(W)\big) \cong HI^{L-1}_{\bar{p}}(W)$$

and

$$H^L\big(\tau_{\geq L}\Omega I^\bullet_{\bar{p}}(W)\big) \cong HI^L_{\bar{p}}(W),$$

all we can say is that the map $\widetilde{HI}^L_{\bar{p}}(M, C_W) \hookrightarrow HI^L_{\bar{p}}(M)$ is injective and that the map $HI^L_{\bar{p}}(M) \twoheadrightarrow \widetilde{HI}^L_{\bar{p}}(M)$ is surjective. Taking into account the long exact cohomology sequence induced by the distinguished triangle

$$\widetilde{\Omega I}^\bullet_{\bar{p}}(M, C_W) \to \widetilde{\Omega I}^\bullet_{\bar{p}}(M) \xrightarrow{J^*_W} \Omega I^\bullet_{\bar{p}}(W) \xrightarrow{+1},$$

these maps induce in the following commutative diagram fitting into the long exact sequence of this distinguished triangle:

$$\begin{tikzcd}
\ldots \arrow[r] & \widetilde{HI}^L_{\bar p}(M, C_W) \arrow[rr] \arrow[dr, hook, "\iota"] & & \widetilde{HI}^L_{\bar p}(M) \arrow[r, "J_W^*"] & HI^L_{\bar p}(W) \arrow[r] & \ldots \\
& & HI^L_{\bar p}(X) \arrow[ur, "s"] & & &
\end{tikzcd}$$

This gives a short exact sequence

$$0 \to \widetilde{HI}^L_{\bar p}(M, C_W) \xrightarrow{\iota} HI^L_{\bar p}(X) \xrightarrow{J_W^* \circ s} \operatorname{im} J_W^* \to 0$$

and hence (since this are all real vector spaces) an isomorphism

$$HI^L_{\bar p}(X) \cong \widetilde{HI}^L_{\bar p}(M, C_W) \oplus \operatorname{im} J_W^*.$$

Since by Lemma 11.2.4 and Poincaré-Lefschetz duality for $\widetilde{HI}$ the vector space $\widetilde{HI}^L_{\bar p}(M, C_W)$ is independent of the atlas, proving that $\operatorname{im} J_W^*$ is independent of the atlas of the bundle $p : E \to B$ will finish the proof of the theorem. We use the following commutative diagram:

$$\begin{tikzcd}
H^L(M) \arrow[r, "J_W^*"] & H^L(W) \\
\widetilde{HI}^L_{\bar p}(M) \arrow[u, "i^*"] \arrow[r, "J_W^*"] & HI^L_{\bar p}(W) \arrow[u, hook, "i_W^*"']
\end{tikzcd}$$

Note that the map $i_W^* : HI^L_{\bar p}(W) \hookrightarrow H^L(W)$, which is induced by subcomplex inclusion, is injective since $\Omega I^\bullet_{\bar p}(W)$ is assumed to be geometrically cotruncateable in degree L. This implies that $J_W^*(\ker i^*) = 0$. Hence $J_W^*\big(\widetilde{HI}^L_{\bar p}(M)\big) \cong i_W^* J_W^*\big(\widetilde{HI}^L_{\bar p}(M)\big) = J_W^*(\operatorname{im} i^*)$. But since $\operatorname{im} i^*$ is independent of the choice of an atlas by the previous lemma, $J_W^*(\operatorname{im} i^*)$ is also independent of the choice of an atlas of $p : E \to B$.

$\square$

References

[AMR88] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, tensor analysis, and applications*, volume 75 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1988.

[Ban07] Markus Banagl. *Topological Invariants of Stratified Space*. Springer Monographs in Mathematics. Springer Verlag, 2007.

[Ban10] Markus Banagl. *Intersection Spaces, Spatial Homology Truncation, and String Theory*, volume 1997 of *Lecture Notes in Mathematics*. Springer, 2010.

[Ban11] Markus Banagl. Foliated stratified spaces and a de rham complex describing intersection space cohomology, 2011. to appear in Journal of Differential Geometry.

[Ban12] Markus Banagl. First cases of intersection spaces in stratification depth 2. *J. Singul.*, 5:57–84, 2012.

[Ban13] Markus Banagl. Isometric group actions and the cohomology of flat fiber bundles. *Groups Geom. Dyn.*, 7(2):293–321, 2013.

[BBM14] Markus Banagl, Nero Budur, and Laurenţu Maxim. Intersection spaces, perverse sheaves and type IIB string theory. *Adv. Theor. Math. Phys.*, 18(2):363–399, 2014.

[BH15] Markus Banagl and Eugenie Hunsicker. Hodge theory for intersection space cohomology, 2015. submitted preprint.

[BM12] Markus Banagl and Laurentiu Maxim. Deformation of singularities and the homology of intersection spaces. *J. Topol. Anal.*, 4(4):413–448, 2012.

[Bot82] Raoul Bott. *Differential forms in algebraic topology*. Graduate Texts in Mathematics. Springer Verlag, 1982.

[Che79] J. Cheeger. On the spectral geometry of spaces with cone-like singularities. *Proc. Natl. Acad. Sci. USA*, 76:2103–2106, 1979.

[Che80] J. Cheeger. On the hodge theory of riemannian pseudomanifolds. *Proc. Sympos. Pure Math.*, 36:91–146, 1980.

[Che83] J. Cheeger. Spectral geometry of singular riemannian spaces. *Journal for Differential Geometry*, 18:575–657, 1983.

[Dol80] Albrecht Dold. *Lectures on Algebraic Topology*, volume 200 of *Grundlehren der mathematischen Wissenschaften*. Springer Verlag, 1980.

[Ess12] Timo Essig. About a de rham complex discribing intersection space cohomology in a non-isolated singularity case. Master's thesis, University of Heidelberg, 2012.

[Gai12] Florian Justus Gaisendrees. *"Fiberwise homology truncation"*. PhD thesis, Heidelberg, Univ., Diss., 2012, 2012. Zsfassung in engl. und dt. Sprache.

[Ger06] C. Gerhardt. *Analysis II*. Analysis. International Press, 2006.

[GM80] M. Goresky and R.D. MacPherson. Intersection homology theory. *Topology*, 19(2):135–162, 1980.

[GM83] M. Goresky and R.D. MacPherson. Intersection homology ii. *Invent. Math.*, 72(1):77–129, 1983.

[Gor78] R. Mark Goresky. Triangulation of stratified objects. *Proc. Amer. Math. Soc.*, 72(1):193–200, 1978.

[HHM04] T. Hausel, E. Hunsicker, and R. Mazzeo. Hodge cohomology of gravitational instantons. *Duke Mathematical Journal*, 122(3):485–548, 2004.

[Hir76] Morris W. Hirsch. *Differential Topology*, volume 33 of *Graduate Texts in Mathematics*. Springer, 1976.

[HW01] Bruce Hughes and Shmuel Weinberger. Surgery and stratified spaces. In *Surveys on surgery theory, Vol. 2*, volume 149 of *Ann. of Math. Stud.*, pages 319–352. Princeton Univ. Press, Princeton, NJ, 2001.

[Jän68] Klaus Jänich. On the classification of $O(n)$-manifolds. *Math. Ann.*, 176:53–76, 1968.

[Kli15] Mathieu Klimczak. Poincaré duality for spaces with isolated singularities, 2015.

[Lan75] S. Mac Lane. *Homology*, volume 114 of *Die Grundlagen der mathematischen Wissenschaften*. Springer, 3rd corrected printing edition, 1975.

[Lan05] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer, 3rd corrected printing edition, 2005.

[Lau00] Gerd Laures. On cobordism of manifolds with corners. *Trans. Amer. Math. Soc.*, 352(12):5667–5688 (electronic), 2000.

[Lee13] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[Mat73] John N. Mather. Stratifications and mappings. In *Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, pages 195–232. Academic Press, New York, 1973.

[Mat12] John Mather. Notes on topological stability. *Bull. Amer. Math. Soc. (N.S.)*, 49(4):475–506, 2012.

[Mil65] J. Milnor. *Lectures on the h-cobordism theorem*, volume 1 of *Mathematical Notes*. Princeton University Press, 1965.

[Mor99] D. Morrison. Through the looking glass, mirror symmetry iii. *Astérisque*, 100:1–171, 1999.

[Pet06] P. Petersen. *Riemannian Geometry*. Graduate Texts in Mathematics. Springer New York, 2006.

[Pfl01] Markus J. Pflaum. *Analytic and geometric study of stratified spaces*. Lecture notes in mathematics ; 1768. Springer, Berlin ; Heidelberg ; New York [u.a.], 2001.

[Poi95] H. Poincaré. Analysis situs. *Journal de l'École Polytechnique*, 1:193–288, 11895. Reprinted in Oeuvres der H. Poincaré VI, 193-288.

[Sch95] Günter Schwarz. *Hodge Decomposition - A Method for Solving Boundary Value Problems*, volume 1607 of *Lecture Notes in Mathematics*. Springer, 1995.

[Sie83] P. H. Siegel. Witt spaces: A geometric cycle theory for KO-homology at odd primes. *Amer. J. Math.*, 105(5):1067–1105, 1983.

[Spa82] E. Spanier. Cohomology isomorphisms. In *Symposium on Algebraic Topology in honor of José Adem (Oaxtepec, 1981)*, volume 12 of *Contemp. Math.*, pages 315–329. Amer. Math. Soc., Providence, R.I., 1982.

[Spi13] Matthias Jonas Spiegel. *K-Theory of intersection spaces*. PhD thesis, Heidelberg, Univ., Diss., 2013, 2013. Mit dt. Zsfassung.

[SZ88] R. Stöcker and H. Zieschang. *Algebraische Topologie*. B.G. Teubner Verlag, 1988.

[tD08] Tammo tom Dieck. *Algebraic Topology*. EMS Textbooks in Mathematics. European Mathematical Society, 2008. Corrected reprint of the 1971 edition.

[War83] Frank W. Warner. *Foundations of Differenriable Manifolds and Lie Groups*, volume 94 of *Graduate texts in Mathematics*. Springer Verlag, 1983.

[Whi65] Hassler Whitney. Tangents to an analytic variety. *Ann. of Math. (2)*, 81:496–549, 1965.

[Wra13] Dominik Wrazidlo. Induced maps between intersection spaces. Master's thesis, University of Heidelberg, 2013.