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vorgelegt von: Hao Yu, M. Sc.
geb. in: Jiangsu, China

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Dual flows in hyperbolic space and de Sitter space

Gutachter: Prof. Dr. Claus Gerhardt

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DUAL FLOWS IN HYPERBOLIC SPACE AND DE SITTER SPACE

HAO YU

Abstract. We consider contracting flows in $(n+1)$-dimensional hyperbolic space and expanding flows in $(n+1)$-dimensional de Sitter space. When the flow hypersurfaces are strictly convex we relate the contracting hypersurfaces and the expanding hypersurfaces by the Gauß map. The contracting hypersurfaces shrink to a point x_0 in finite time while the expanding hypersurfaces converge to the maximal slice $\{\tau = 0\}$. After rescaling, by the same scale factor, the resclaed contracting hypersurfaces converge to a unit geodesic sphere, while the rescaled expanding hypersufaces converge to slice $\{\tau = -1\}$ exponential fast in $C^\infty(\mathbb{S}^n)$.

Zusammenfassung. Wir betrachten kontrahierende Flüsse im $(n+1)$-dimensionalen hyperbolischen Raum und expandierende Flüsse im $(n+1)$-dimensionalen de Sitter Raum. Wir verbinden die kontrahierenden Hyperflächen mit den expandierenden Hyperflächen durch die Gaußsche Abbildung, falls die Hyperflächen der Flüsse strikt konvex sind. Die kontrahierenden Hyperflächen schrumpfen zu einem Punkt x_0 in endlicher Zeit, während die expandierenden Hyperflächen zu dem maximalen Schnitt $\{\tau = 0\}$ konvergieren. Nach Reskalierung mit dem gleichen Faktor konvergieren die reskalierten kontrahierenden Hyperflächen nach einem geodätischen Einheitsspäre, während die reskalierten expandierenden Hyperflächen nach dem Schnitt $\{\tau = -1\}$ exponentiell schnell in $C^\infty(\mathbb{S}^n)$ konvergieren.

Contents

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1. Introduction

In a recent paper [7] a pair of dual flows was considered in $\mathbb{S}^{n+1}$. The one flow is the contracting flow

$$\dot{x} = -F\nu, \tag{1.1}$$

while the other is an expanding flow

$$\dot{x} = \tilde{F}^{-1}\nu, \tag{1.2}$$

where $F \in C^\infty(\Gamma_+)$ and $\tilde{F}$ is its inverse

$$\tilde{F}(\kappa_i) = \frac{1}{F(\kappa_i^{-1})}. \tag{1.3}$$

There is a Gauß map for the pair $(\mathbb{S}^{n+1}, \mathbb{S}^{n+1})$, which maps closed, strictly convex hypersurfaces M to their polar sets $\tilde{M}$, cf. [5, Chapter 9]. Gerhardt [7] proved, that the flow hypersurfaces of (1.1) and (1.2) are polar sets of each other, if the initial hypersurface have this property. Under the assumption that F is symmetric, monotone, positive, homogeneous of degree 1, F strictly concave (cf. 3.1) and $\tilde{F}$ concave, it is proved in [7] that the contracting flows contract to a round point and the expanding flows converge to an equator such that after appropriate rescaling, both flows converge to a geodesic sphere exponential fast.

The Gauß map exists also for the pair $(\mathbb{H}^{n+1}, N)$, where $\mathbb{H}^{n+1}$ is the $(n+1)$-dimensional hyperbolic space and N is the $(n+1)$-dimensional de Sitter space, cf. [5, Chapter 10]. We prove in this work similar results as in [7] by using this duality. Let $M(t)$ resp. $\tilde{M}(t)$ be solutions of the contracting flows

$$\dot{x} = -F\nu \tag{1.4}$$

in $\mathbb{H}^{n+1}$ resp. the dual flows

$$\dot{x} = -\tilde{F}^{-1}\nu \tag{1.5}$$

in N, where $\tilde{F}$ is the inverse of F defined by (1.3). We impose the following assumptions.

1.1. **Assumption.** Let $F \in C^\infty(\Gamma_+)$ be a symmetric, monotone, 1-homogeneous and concave curvature function satisfying the normalization

$$F(1, \dots, 1) = 1. \tag{1.6}$$

We assume further, either

(1) F is concave and $\tilde{F}$ is concave and the initial hypersurface M_0 is horoconvex (i.e. all principal curvatures $\kappa_i \geq 1$),

or

(2) $\tilde{F}$ is convex and M_0 is strictly convex.

We now state our main results

1.2. **Theorem.** *We consider curvature flows* (1.4) *and* (1.5) *under assumption* 1.1 *with initial smooth hypersurfaces* M_0 *and* $\tilde{M}_0$*, where* $\tilde{M}_0$ *is the polar hypersurface of* M_0*. Then the both flows exist on the maximal time interval* $[0, T^*)$ *with finite* T^**. The hypersurfaces* $\tilde{M}(t)$ *are the polar hypersurfaces of* $M(t)$ *and vice versa during the evolution. The contracting flow hypersurfaces in* $\mathbb{H}^{n+1}$ *shrink to a point* x_0 *while the expanding flow hypersurfaces in* N *converge to a totally geodesic hypersurface which is isometric to* $\mathbb{S}^n$*. We may assume the point* x_0 *is the Beltrami point by applying an isometry such that the hypersurfaces of the expanding flow are all contained in* N_- *and converge to the coordinate slice* $\{\tau = 0\}$*.*
Viewing $\mathbb{H}^{n+1}$ *and* N *as submanifolds of* $\mathbb{R}^{n+1,1}$ *and by introducing polar coordinates in the Euclidean part of* $\mathbb{R}^{n+1,1}$ *centered in* $(0, \dots, 0) \in \mathbb{R}^{n+1}$*, we can write flow hypersurfaces in* $\mathbb{H}^{n+1}$ *resp.* N *as graphs of functions* u *resp.* u^* *over* $\mathbb{S}^n$*. Let* $\Theta = \Theta(t, T^*)$ *be the solution of* (1.4) *with spherical initial hypersurface and exitence intervall* $[0, T^*)$*. Then the rescaled functions*

$$\tilde{u} = u\Theta^{-1} \tag{1.7}$$

and

$$w = u^*\Theta^{-1} \tag{1.8}$$

are uniformly bounded in $C^\infty(\mathbb{S}^n)$*. The rescaled principal curvatures* $\kappa_i\Theta$ *as well as* $\tilde{\kappa}_i\Theta^{-1}$ *are uniformly positiv, where* $\tilde{\kappa}_i$ *are the principal curvatures of* $\tilde{M}(t)$*.*
If the curvature function F *is further strictly concave or* $F = \frac{1}{n}H$*, then the rescaled functions* (1.7) *resp.* (1.8) *converge to the constant functions* 1 *resp.* -1 *in* $C^\infty(\mathbb{S}^n)$ *exponentially fast.*

Let us review some results concerning the contracting flows in $\mathbb{H}^{n+1}$. Under the assumption that the initial hypersurface is strictly convex and satisfies the condition $\kappa_i H > n$ for each i, Huisken [11] proved that the flow (1.4) with $F = H$ converges in finite time to a round sphere. Andrews [2] proved similar results for a general class of curvature function with argument $\kappa_i - 1$. Makowski [13] proved the contracting flow with a volume preserving term exists for all times and converges to a geodesic sphere exponentially fast.
The key ingredient treating the contracting flow is the pinching estimates. Under assumption 1.1 (1) it follows by a similar calculation as in [13], while Gerhardt [8] proved the pinching estimates under assumption 1.1 (2).
The elementary symmetric polynomials are defined by

$$H_k(\kappa_1, \dots, \kappa_n) = \sum_{1 \le i_1 < \dots < i_k \le n} \kappa_{i_1} \dots \kappa_{i_k}, \quad 1 \le k \le n. \tag{1.9}$$

Examples of curvature functions F satisfying assumption 1.1 (1) (up to normalization condition (1.6)) are

- the power means $\left(\frac{1}{n}\sum_i \kappa_i^r\right)^{1/r}$ for $|r| \le 1$,
- $\sigma_k = H_k^{1/k}$ for $1 \le k \le n$,

- the inverse $\tilde{\sigma}_k$ of σ_k for $1 \le k \le n$,
- $(H_k/H_l)^{1/(k-l)}$ for $0 \le l < k \le n$,
- $H_n^{\alpha_n} H_{n-1}^{\alpha_{n-1}-\alpha_n} \cdots H_2^{\alpha_2-\alpha_3} H_1^{\alpha_1-\alpha_2}$ for $\alpha_i \ge 0$ and $\sum_i \alpha_i = 1$.

For a proof see [3, Chapter 2]. Moreover, the curvature functions in the above list are all strictly concave with exception of the mean curvature (cf. Section 3)

Examples of convex curvature functions $\tilde{F}$, which is used in assumption 1.1 (2) (up to normalization condition (1.6)) are (cf. [5, Remark 2.2.13])

- the mean curvature H,
- the length of the second fundamental form $|A| = \left(\sum_i \kappa_i^2\right)^{1/2}$,
- the complete symmetric functions

  $\gamma_k(\kappa_1, \ldots, \kappa_n) = \left(\sum_{|\alpha|=k} \kappa_1^{\alpha_1} \kappa_2^{\alpha_2} \ldots \kappa_n^{\alpha_n}\right)^{1/k}$ for $1 \le k \le n$.

Note that for convex $\tilde{F}$ under assumption 1.1 (2), F is of class (K) and homogeneous of degree 1, hence strictly concave. (cf. [5, Definition 2.2.1, Lemma 2.2.12, 2.2.14], [7, Lemma 3.6])

2. Setting and general facts

We now review some general facts about hypersurfaces from [5, Chapter 1]. Let N be a $(n+1)$-dimensional dimensional semi-Riemannian manifold and M be a hypersurface in N. Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta}), (\bar{R}_{\alpha\beta\gamma\delta})$, etc., where greek indices range from 0 to n. Quantities in M will be denoted by $(g_{ij}), (h_{ij})$ etc., where latin indices range from 1 to n. Generic coordinate systems in N resp. M will be denoted by (x^α) resp. (ξ^i).

Covariant differentiation will usually be denoted by indices, only if ambiguities are possible, by a semicolon, e.g. $h_{ij;k}$.

Let $x : M \hookrightarrow N$ be a spacelike hypersurface (i.e. the induced metric is Riemannian) with a differentiable normal ν, which is always supposed to be normalized, and (h_{ij}) be the second fundamental form, and set $\sigma = \langle \nu, \nu \rangle$. We have the *Gauß formula*

$$x_{ij}^\alpha = -\sigma h_{ij} \nu^\alpha, \tag{2.1}$$

the *Weingarten equation*

$$\nu_i^\alpha = h_i^k x_k^\alpha, \tag{2.2}$$

the *Codazzi equation*

$$h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_j^\gamma x_k^\delta, \tag{2.3}$$

and the *Gauß equation*

$$R_{ijkl} = \sigma\{h_{ik}h_{jl} - h_{il}h_{jk}\} + \bar{R}_{\alpha\beta\gamma\delta} x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta. \tag{2.4}$$

Let us review some properties of $\mathbb{H}^{n+1}$ and N, cf. [5, Section 10.2]. We label the coordinates in the $(n+2)$-dimensional Minkowski space $\mathbb{R}^{n+1,1}$ as $x = (x^a), 0 \le a \le n+1$, where x^0 is the time function. Recall that the hyperbolic space $\mathbb{H}^{n+1}$ and de Sitter space N are the subspaces of $\mathbb{R}^{n+1,1}$ defined by

$$\mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+1,1} : \langle x, x\rangle = -1, x^0 > 0\}, \tag{2.5}$$

$$N = \{x \in \mathbb{R}^{n+1,1} : \langle x, x\rangle = 1\}. \tag{2.6}$$

Introduce polar coordinates in the Euclidean part of $\mathbb{R}^{n+1,1}$ centered in $(0, \dots, 0) \in \mathbb{R}^{n+1}$ such that the metric in $\mathbb{R}^{n+1,1}$ is expressed as

$$d\bar{s}^2 = -d{x^0}^2 + dr^2 + r^2\sigma_{ij}d\xi^i d\xi^j, \tag{2.7}$$

where σ_{ij} is the spherical metric.

By viewing $\mathbb{H}^{n+1}$ as

$$\mathbb{H}^{n+1} = \{(x^0, r, \xi^i) : r = \sqrt{|x^0|^2 - 1}, x^0 > 0, \xi \in \mathbb{S}^n\}, \tag{2.8}$$

and by setting

$$\varrho = \operatorname{arccosh} x^0, \tag{2.9}$$

$\mathbb{H}^{n+1}$ has coordinates (ϱ, ξ^i) and the metric

$$d\bar{s}^2_{\mathbb{H}^{n+1}} = d\varrho^2 + \sinh^2 \varrho\, \sigma_{ij}\, d\xi^i d\xi^j. \tag{2.10}$$

Similarly,

$$N = \{(x^0, r, \xi^i) : r = \sqrt{1 + |x^0|^2}, x^0 \in \mathbb{R}, \xi \in \mathbb{S}^n\}, \tag{2.11}$$

and by setting the eigentime

$$\tau = \operatorname{arcsinh} x^0, \tag{2.12}$$

N has coordinates (τ, ξ^i) and the metric

$$d\bar{s}^2_N = -d\tau^2 + \cosh^2 \tau \sigma_{ij} d\xi^i d\xi^j. \tag{2.13}$$

3. Strictly concave curvature functions

For $\xi, \kappa \in \mathbb{R}^n$, we write $\xi \sim \kappa$, if there is $\lambda \in \mathbb{R}$ such that $\xi = \lambda\kappa$.

3.1. **Definition.** Let $F \in C^2(\Gamma)$ be a symmetric, monotone, 1-homogeneous and concave curvature function. We call F strictly concave (in non-radial directions), if

$$F_{ij}\xi^i\xi^j < 0 \quad \forall \xi \not\sim \kappa \text{ and } \xi \neq 0, \tag{3.1}$$

or equivalently, if the multiplicity of the zero eigenvalue for $D^2F(\kappa)$ is one for all $\kappa \in \Gamma$.

Note since F is homogeneous of degree 1, $\kappa \in \Gamma$ is an eigenvector of $D^2F(\kappa)$ with zero eigenvalue. In [7, Chapter 3] it is proved that σ_k, $2 \leq k \leq n$ and the inverses $\tilde{\sigma}_k$ of σ_k, $1 \leq k \leq n$ are strictly concave. In [12, Chapter 2] it is proved that $Q_k = H_{k+1}/H_k, 1 \leq k \leq n-1$ are strictly concave in Γ_+. We consider the rest of the concave and inverse concave curvature functions listed on page 3.

3.2. **Lemma.** *The curvature functions*

$$F = (\tfrac{1}{n}\sum_i \kappa_i^r)^{1/r} \quad -1 \le r < 1 \tag{3.2}$$

are strictly concave in Γ_+.

Proof. Note that F converges locally uniformly to $\sigma_n = (\kappa_1 \cdots \kappa_n)^{1/n}$ as $r \to 0$ and σ_n is strictly concave. Furthermore, for $-1 \le r < 1$ and $r \ne 0$,

$$\frac{\partial F}{\partial \kappa^i} = n^{-1/r}\left(\sum_l \kappa_l^r\right)^{\frac{1}{r}-1} \kappa_i^{r-1}, \tag{3.3}$$

$$\frac{\partial^2 F}{\partial \kappa^i \partial \kappa^j} = n^{-1/r}(1-r)\left(\sum_l \kappa_l^r\right)^{\frac{1}{r}-2} \kappa_i^{r-2}(\kappa_i \kappa_j^{r-1} - \sum_l \kappa_l^r \delta_{ij}). \tag{3.4}$$

Consider η such that $F_{ij}\eta^j = 0$. Since $r \ne 1$,

$$\eta_i = \left(\sum_l \kappa_l^r\right)^{-1} \kappa_j^{r-1}\eta^j \kappa_i. \tag{3.5}$$

Knowing that F is concave for $|r| \le 1$ we conclude that F is strictly concave for $-1 \le r < 1$. □

3.3. **Lemma.** *Let f^α be concave in Γ_+ for all $1 \le \alpha \le k$ and strictly concave in Γ_+ for at least one index in $1 \le \alpha \le k$. Let φ be strictly monotone increasing and concave in Γ_+, then*

$$F(\kappa_1, \cdots, \kappa_n) = \varphi(f^1(\kappa_1, \cdots, \kappa_n), \cdots, f^k(\kappa_1, \cdots, \kappa_n)) \tag{3.6}$$

is strictly concave in Γ_+.

Proof. Let $0 \ne \xi \in \mathbb{R}^n$ and $\xi \not\sim \kappa$, then

$$F_{ij}\xi^i\xi^j = \varphi_\alpha f^\alpha_{ij}\xi^i\xi^j + \varphi_{\alpha\beta} f^\alpha_i f^\beta_j \xi^i\xi^j < 0, \tag{3.7}$$

since by assumption

$$\varphi_\alpha > 0, \quad \varphi_{\alpha\beta} \le 0, \quad f^\alpha_{ij}\xi^i\xi^j \le 0 \tag{3.8}$$

and

$$f^\alpha_{ij}\xi^i\xi^j < 0 \text{ for at least one } 1 \le \alpha \le k. \tag{3.9}$$

□

Note that the weighted geometric mean

$$\varphi(f^1, \cdots, \varphi^k) = (f^1)^{\alpha_1} \cdots (f^k)^{\alpha_k} \quad \text{with } \sum_i \alpha_i = 1 \tag{3.10}$$

is a strictly monotone increasing and concave function. Knowing that $H_{k+1}/H_k, 1 \le k \le n-1$ are strictly concave in Γ_+, we conclude that

(3.11)
$$(H_k/H_l)^{1/(k-l)} = (H_{l+1}/H_l)^{1/(k-l)} \cdots (H_k/H_{k-1})^{1/(k-l)} \quad 0 \le l < k \le n$$

and

(3.12)
$$H_n^{\alpha_n} H_{n-1}^{\alpha_{n-1}-\alpha_n} \cdots H_2^{\alpha_2-\alpha_3} H_1^{\alpha_1-\alpha_2} = \left(\frac{H_1}{H_0}\right)^{\alpha_1} \left(\frac{H_2}{H_1}\right)^{\alpha_2} \cdots \left(\frac{H_n}{H_{n-1}}\right)^{\alpha_n}$$

with $\alpha_i \ge 0$, $\sum_i \alpha_i = 1$ and $\alpha_1 \ne 1$ are strictly concave in Γ_+.

4. Polar sets and dual flows

We state some facts about Gauß maps for $(\mathbb{H}^{n+1}, N)$, cf. [5, Section 10.4].

4.1. **Theorem.** *Let $x : M_0 \to M \subset \mathbb{H}^{n+1}$ be a closed, connected, strictly convex hypersurface. Consider M as a codimension 2 immersed submanifold in $\mathbb{R}^{n+1,1}$ such that*

$$x_{ij} = g_{ij}x - h_{ij}\tilde{x}, \tag{4.1}$$

where $\tilde{x} \in T_x(\mathbb{R}^{n+1,1})$ is the representation of the exterior normal vector $\nu = (\nu^\alpha)$ of M in $T_x(\mathbb{H}^{n+1})$. Then the Gauß map

$$\tilde{x} : M_0 \to N \tag{4.2}$$

is the embedding of a closed, spacelike, achronal, strictly convex hypersurface $\tilde{M} \subset N$. Viewing $\tilde{M}$ as a codimension 2 submanifold in $\mathbb{R}^{n+1,1}$, its Gaussian formula is

$$\tilde{x}_{ij} = -\tilde{g}_{ij}\tilde{x} + \tilde{h}_{ij}x, \tag{4.3}$$

where $\tilde{g}_{ij}, \tilde{h}_{ij}$ are the metric and second fundamental form of $\tilde{M}$ and x is the embedding of M which also represents the future directed normal vector of $\tilde{M}$. The second fundamental form $\tilde{h}_{ij}$ is defined with respect to the future directed normal vector, where the time orientation of N is inherited from $\mathbb{R}^{n+1,1}$. Furthermore, there holds

$$\tilde{h}_{ij} = h_{ij}, \tag{4.4}$$

$$\tilde{\kappa}_i = \kappa_i^{-1}. \tag{4.5}$$

□

We prove in the following that the duality is also valid in case of curvature flows.

4.2. **Lemma.** *Let $\Phi \in C^\infty(\mathbb{R}_+)$ be strictly monotone, $\dot{\Phi} > 0$, and let $F \in C^\infty(\Gamma_+)$ be a symmetric, monotone, 1-homogeneous curvature function such that $F|_{\Gamma_+} > 0$ and such that the flows*

$$\dot{x} = -\Phi(F)\nu \tag{4.6}$$

in $\mathbb{H}^{n+1}$ resp.

$$\dot{\tilde{x}} = -\Phi(\tilde{F}^{-1})\tilde{\nu} \tag{4.7}$$

in N with initial strictly convex hypersurfaces M_0 resp. $\tilde{M}_0$ exist on maximal time intervals $[0, T^*)$ resp. $[0, \tilde{T}^*)$, where ν and $\tilde{\nu}$ are the exterior normal resp. past directed normal. The flow hypersurfaces are then strictly conxex. Let $M(t)$ resp. $\tilde{M}(t)$ be the corresponding flow hypersurfaces, then $T^* = \tilde{T}^*$ and $M(t) = \tilde{M}(t)$ for all $t \in [0, T^*)$.

Proof. The arguments are similar to those in [7, Section 4] with combination with the results from [5, Section 10.4]. Since there holds

$$\langle x, x\rangle = 1,\ \langle \dot{x}, x\rangle = 0,\ \langle x_j, x\rangle = 0,\ \langle \tilde{x}, x\rangle = 0, \tag{4.8}$$

(see [5, Lemma 10.4.1] for the last identity) we can consider the flow (4.6) as flow in $\mathbb{R}^{n+1,1}$

$$\dot{x} = -\Phi\tilde{x}, \tag{4.9}$$

and we have the decomposition

$$T_x(\mathbb{R}^{n+1,1}) = T_x(\mathbb{H}^{n+1}) \oplus \langle x\rangle. \tag{4.10}$$

Furthermore, we conclude from

$$\langle \dot{\tilde{x}}, x_j\rangle = \Phi_j,\ \langle \dot{\tilde{x}}, \tilde{x}\rangle = 0,\ \langle \dot{\tilde{x}}, x\rangle = \Phi, \tag{4.11}$$

from the Weingarten equation (see [5, Lemma 10.4.3, 10.4.4])

$$x_j = \tilde{h}^k_j \tilde{x}_k, \tag{4.12}$$

and from (4.10) that

$$\dot{\tilde{x}} = \Phi x + \Phi^m x_m = \Phi x + \Phi^m \tilde{h}^k_m \tilde{x}_k, \tag{4.13}$$

where

$$\Phi^m = g^{mj}\Phi_j, \tag{4.14}$$

and the second fundamental form $\tilde{h}_{ij}$ is defined with respect to the future directed normal vector $\tilde{\nu}$. The corresponding flow equation in N has the form

$$\dot{\tilde{x}} = \Phi\tilde{\nu} + \Phi^m \tilde{h}^k_m \tilde{x}_k. \tag{4.15}$$

Let $t_0 \in [0, T^*)$ and introduce polar coordinates in the Euclidean part of the Minkowski space as well as an eigentime coordinate system in N as in Section 2. For ϵ small and $t_0 < t < t_0 + \epsilon$, $\tilde{M}(t)$ can be written as graph over $\mathbb{S}^n$

$$\tilde{M}(t) = \operatorname{graph} \tilde{u}|_{\mathbb{S}^n}, \tag{4.16}$$

and we obtain the scalar flow equation

$$\frac{d\tilde{u}}{dt} = \Phi\tilde{v}^{-1} + \Phi^m \tilde{h}^k_m \tilde{u}_k, \tag{4.17}$$

where

$$\tilde{v}^2 = 1 - |D\tilde{u}|^2 = 1 - \frac{1}{\cosh^2 \tilde{u}}\sigma^{ij}\tilde{u}_i\tilde{u}_j. \tag{4.18}$$

Note that $\tilde{\nu}$ in (4.15) is the future directed normal

$$(\tilde{\nu}^\alpha) = \tilde{v}^{-1}(1, \check{\tilde{u}}^i), \tag{4.19}$$

where

$$\check{\tilde{u}}^i = \frac{1}{\cosh^2 \tilde{u}} \sigma^{ij} \tilde{u}_j. \tag{4.20}$$

Thus it holds in view of (4.15)

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} &= \frac{d\tilde{u}}{dt} - \tilde{u}_i \dot{\tilde{x}}^i \\ &= \Phi \tilde{v}^{-1} + \Phi^m h^k_m \tilde{u}_k - \Phi \tilde{v}^{-1} |D\tilde{u}|^2 - \Phi^m \tilde{h}^k_m \delta^i_k \tilde{u}_i \\ &= \Phi \tilde{v}. \end{aligned} \tag{4.21}$$

This is exactly the scalar curvature equation of the flow equation

$$\dot{\tilde{x}} = -\Phi \tilde{\nu}, \tag{4.22}$$

where $\tilde{\nu}$ in (4.22) is the future directed normal and

$$\Phi = \Phi(F) = \Phi(\tilde{F}^{-1}). \tag{4.23}$$

Now $\tilde{h}_{ij}$ in N is defined with respect to the future directed normal. By adapting the convention in [5, p.307] we switch the light cone in N and by defining $\tau = -\operatorname{arcsinh} x^0$ in (2.12) we still derive the flow (4.22) in N, where $\tilde{\nu}$ is now the past directed normal and the second fundamental form is defined with respect to this normal. The rest of the proof is identical to [7, Theorem 4.2]. □

From now we shall employ this duality by choosing

$$\Phi(r) = r. \tag{4.24}$$

Note that the expanding flows in $\mathbb{H}^{n+1}$ was already considered in [6] and its non-scale-invariant version in [14].

5. Pinching estimates

We consider the contracting flow

$$\begin{aligned} \dot{x} &= -F\nu, \\ x(0) &= M_0 \end{aligned} \tag{5.1}$$

in $\mathbb{H}^{n+1}$ with initial smooth and strictly convex hypersurfaces M_0, where ν is the exterior normal vector.

Under the assumptions of Theorem 1.2 the curvature flow (5.1) exists on a maximal time interval $[0, T^*), 0 < T^* \leq \infty$, cf. [5, Theorem 2.5.19, Lemma 2.6.1].

5.1. **Theorem.** *Let $M(t)$ be a solution of the flow (5.1) in $\mathbb{H}^{n+1}$. If the initial hypersurface M_0 in $\mathbb{H}^{n+1}$ satisfies*

$$\kappa_i > 1, \tag{5.2}$$

then this condition will also be satisfied by the flow hypersurfaces $M(t)$ during the evolution.

Proof. The tensor

$$S_{ij} = h_{ij} - g_{ij} \tag{5.3}$$

satisfies the equation

(5.4)
$$\begin{aligned}\dot{S}_{ij} - F^{kl} S_{ij;kl} &= F^{kl} h_{rk} h_l^r h_{ij} - 2F h_i^k h_{kj} \\ &\quad + K_N \{2F g_{ij} - F^{kl} g_{kl} h_{ij}\} + 2F h_{ij} + F^{kl,rs} h_{kl;i} h_{rs;j} \\ &\equiv N_{ij} + \tilde{N}_{ij},\end{aligned}$$

where $\tilde{N}_{ij} = F^{kl,rs} h_{kl;i} h_{rs;j}$. At every point where $h_{ij}\eta^j = \eta_i$ there holds

$$N_{ij}\eta^i\eta^j = \{F^{kl} h_{rk} h_l^r - 2F + F^{kl} g_{kl}\}|\eta|^2 \geq 0. \tag{5.5}$$

It was proved in [3, Theorem 3.3, Lemma 4.4] that

$$\tilde{N}_{ij}\eta^i\eta^j + \sup_{\Gamma} 2F^{kl}\{2\Gamma_l^r S_{ir;k}\eta^i - \Gamma_k^r \Gamma_l^s S_{rs}\} \geq 0, \tag{5.6}$$

where only the inverse concavity of F was used. Andrews' maximum principle in [3, Theorem 3.2] implies that $S_{ij} > 0$ during the evolution. □

In the next step we use a constant rank theorem to allow the condition $\kappa_i \geq 1$ in the proof of the succeeding Lemma 5.4.

5.2. **Lemma.** *Let $M(t)$ be a solution of the flow (5.1) in $\mathbb{H}^{n+1}$ and assume that the tensor S_{ij} satisfies $S_{ij} \geq 0$ on the hypersurfaces $M(t)$ for $t \in [0, T^*)$, then S_{ij} is of constant rank $l(t)$ for every $t \in (0, T^*)$.*

Proof. The proof is similar to those in [15, Theorem 3.2], where the main part is based on the computation in [4, Theorem 3.2]. For $\epsilon > 0$, let

$$W_{ij} = S_{ij} + \epsilon g_{ij}. \tag{5.7}$$

Let $l(t)$ be the minimal rank of $S_{ij}(t)$. For a fixed $t_0 \in (0, T^*)$, let $x_0 \in M(t_0)$ be the point such that $S_{ij}(t_0, \xi)$ attains its minimal rank at x_0. Set

$$\phi(t, \xi) = H_{l+1}(W_{ij}(t, \xi)) + \frac{H_{l+2}(W_{ij}(t, \xi))}{H_{l+1}(W_{ij}(t, \xi))}, \tag{5.8}$$

where H_l is the elementary symmetric polynomials of eigenvalues of W_{ij}, homogeneous of order l. A direct computation shows

$$\begin{aligned}F^{kl} W_{ij;kl} - \dot{W}_{ij} = &- F^{kl} h_{rk} h_l^r W_{ij} - F^{kl} g_{kl} W_{ij} + 2F h_i^k W_{kj} \\ &- F^{kl,rs} W_{kl;i} W_{rs;j} + 2F\epsilon g_{ij} \\ &- (1-\epsilon)\{F^{kl} h_{rk} h_l^r - 2F + F^{kl} g_{kl}\} g_{ij}.\end{aligned} \tag{5.9}$$

As in [4], we consider a neighborhood $(t_0 - \delta, t_0] \times \mathcal{O}$ around (t_0, ξ_0). We use the notation $h = O(f)$ if $|h(\xi)| \leq C f(\xi)$ for every $(t, \xi) \in (t_0 - \delta, t_0] \times \mathcal{O}$, where C is a constant, depending on the $C^{1,1}$ norm of the second fundamental form on $(t_0-\delta, t_0]\times\mathcal{O}$, but independent of ϵ. It was proved in [4, Corollary 2.2] that ϕ is in $C^{1,1}$. And as in [4], let $G=\{n-l+1, n-l+2,\dots,n\}$ and $B=\{1,\dots,n-l\}$. We choose the coordinates such that $h_{ij}=\kappa_i\delta_{ij}$ and $g_{ij}=\delta_{ij}$. In view of [4, (3.14)], in such coordinates ϕ^{ij} is up to $O(\phi)$ non-negative in $\mathcal{O}$ and we have

(5.10)

$$\begin{aligned}F^{kl}\phi_{;kl}-\dot\phi\le\phi^{ij}\{&-F^{kl}h_{rk}h^r_l W_{ij}-F^{kl}g_{kl}W_{ij}+2Fh^k_iW_{kj}\\&+2F\epsilon g_{ij}-F^{kl,rs}W_{kl;i}W_{rs;j}\}+F^{kl}\phi^{ij,rs}W_{ij;k}W_{rs;l}+O(\phi).\end{aligned}$$

We can choose $\mathcal{O}$ small enough, such that $\epsilon=O(\phi)$ as in [4, (3.8)]. It was proved in [4, (3.14)] that $\phi^{ii}=O(\phi)$ for $i\in G$ and since $W_{ii}\le\phi$ for $i\in B$, we infer that

$$F^{kl}\phi_{;kl}-\dot\phi\le-\phi^{ij}F^{kl,rs}W_{kl;i}W_{rs;j}+F^{kl}\phi^{ij,rs}W_{ij;k}W_{rs;l}+O(\phi). \tag{5.11}$$

Using the inverse concavity of F and proceed as in [4, Theorem 3.2], we conclude

$$F^{kl}\phi_{;kl}-\dot\phi\le C\{\phi+|D\phi|\}, \tag{5.12}$$

where C is a constant independent of ϵ and ϕ. Taking $\epsilon\to0$, the strong maximum principle for parabolic equations yields

$$H_{l(t_0)+1}(S_{ij}(t,\xi))\equiv0\quad\forall(t,\xi)\in(t_0-\delta,t_0]\times\mathcal{O}. \tag{5.13}$$

Since $M(t_0)$ is a closed hypersurface, $S_{ij}(t_0,\xi)$ is of constant rank $l(t_0)$ on $M(t_0)$. □

Note that the proof of the Lemma 5.1 implies, if the initial hypersurface satisfies $\kappa_i\ge1$, then this condition remains true during the evolution. Furthermore, every closed hypersurface in $\mathbb{H}^{n+1}$ contains a point on which holds $\kappa_i>1$. Thus we conclude

5.3. **Corollary.** *Let $M(t)$ be a solution of the flow (5.1) in $\mathbb{H}^{n+1}$. If the initial hypersurface M_0 in $\mathbb{H}^{n+1}$ satisfies $\kappa_i\ge1$, then $\kappa_i>1$ for every $t\in(0,T^*)$.*

5.4. **Lemma.** *Let $M(t)$ be a solution of the flow (5.1) in $\mathbb{H}^{n+1}$ under assumption 1.1 (1), then there exists a uniform positive constant $\epsilon>0$ such that*

$$\kappa_1\ge\epsilon\kappa_n \tag{5.14}$$

during the evolution, where the principal curvatures are labeled as

$$\kappa_1\le\cdots\le\kappa_n. \tag{5.15}$$

Proof. The proof is similar to [13, Lemma 4.2]. By Replacing M_0 by $M(t_0)$ for a $t_0 \in (0, T^*)$ as initial hypersurface, we can assume that $\kappa_i > 1$ on M_0 Let F be a concave and inverse concave curvature function, then

$$T_{ij} = h_{ij} - g_{ij} - \epsilon(H - n)g_{ij} \tag{5.16}$$

satisfies the equation

$$\begin{aligned} \dot{T}_{ij} - F^{kl}T_{ij;kl} =& F^{kl}h_{rk}h^r_l\{h_{ij} - \epsilon H g_{ij}\} - 2Fh^k_i\{h_{kj} - \epsilon H g_{kj}\} \\ &+ 2K_N F g_{ij} - 2n\epsilon K_N F g_{ij} - K_N F^{kl} g_{kl}\{h_{ij} - \epsilon H g_{ij}\} \\ &- 2F(\epsilon n - 1)h_{ij} + F^{kl,rs}h_{kl;i}h_{rs;j} - \epsilon F^{kl,rs}h_{kl;p}h_{rs;q}g^{pq}g_{ij} \\ \equiv& N_{ij} + \tilde{N}_{ij}, \end{aligned} \tag{5.17}$$

where $\tilde{N}_{ij} = F^{kl,rs}h_{kl;i}h_{rs;j} - \epsilon F^{kl,rs}h_{kl;p}h_{rs;q}g^{pq}g_{ij}$.
At every point where $T_{ij}\eta^j = 0$ there holds

$$\begin{aligned} N_{ij}\eta^i\eta^j =& F^{kl}h_{rk}h^r_l(1 - \epsilon n)|\eta|^2 + 2Fh_{ij}(\epsilon n - 1)\eta^i\eta^j \\ &+ \{F^{kl}g_{kl} - 2F\}(1 - \epsilon n)|\eta|^2 - 2F(\epsilon n - 1)h_{ij}\eta^i\eta^j \\ =& (1 - \epsilon n)\sum_i F_i(\kappa_i^2 - 2\kappa_i + 1)|\eta|^2 \geq 0. \end{aligned} \tag{5.18}$$

It is proved in [1, Theorem 4.1] (see also the modification in [13, Theorem B.2]) that

$$\tilde{N}_{ij}\eta^i\eta^j + \sup_{\Gamma} 2F^{kl}\{2\Gamma^r_l T_{ir;k}\eta^i - \Gamma^r_k\Gamma^s_l T_{rs}\} \geq 0, \tag{5.19}$$

We can choose $\epsilon > 0$ sufficiently small, such that $T_{ij} \geq 0$ on M_0, then the Andrews' maximum principle [3, Theorem 3.2] implies $T_{ij} \geq 0$ and hence

$$\kappa_1 - 1 \geq \epsilon(H - n) \tag{5.20}$$

during the evolution. □

The following pinching results is due to Gerhardt. By using [8, Theorem 1.1] and the duality result Lemma 4.2 we obtain

5.5. **Theorem.** *Let $M(t)$ be a solution of the flow (5.1) in $\mathbb{H}^{n+1}$ under the assumption 1.1 (2), then there exists a uniform constant $\epsilon > 0$ such that*

$$\kappa_1 \geq \epsilon\kappa_n \tag{5.21}$$

during the evolution.

6. Contracting flows - convergence to a point

Fix a point $p_0 \in \mathbb{H}^{n+1}$, the hyperbolic metric in the geodesic polar coordinates centered at p_0 can be expressed as

$$\mathrm{d}\bar{s}^2 = dr^2 + \sinh^2 r \sigma_{ij} dx^i dx^j, \tag{6.1}$$

where σ_{ij} is the canonical metric of $\mathbb{S}^n$.
Geodesic spheres with center in p_0 are totally umbilic. The induced metric, second fundamental form and the principal curvatures of the coordinate slices $S_r = \{x^0 = r\}$ are given by

$$\bar{g}_{ij} = \sinh^2 r \sigma_{ij}, \; \bar{h}_{ij} = \tfrac{1}{2}\dot{\bar{g}}_{ij} = \coth r \bar{g}_{ij}, \; \bar{\kappa}_i = \coth r, \tag{6.2}$$

respectively. See [5, (1.5.12)].

6.1. **Lemma.** *Consider (5.1) with initial hypersurface $x(0) = S_{r_0}$, then the corresponding flow exists in a maximal time intervall $[0, T^*)$ with T^* finite and will shrink to a point. The flow hypersurfaces $M(t)$ are all geodesic spheres with the same center and their radii $\Theta = \Theta(t)$ solve the ODE*

$$\begin{aligned} \dot{\Theta} &= -\coth \Theta, \\ \Theta(0) &= r_0. \end{aligned} \tag{6.3}$$

Proof. We set

$$\begin{aligned} x^0(t,\xi) &= \Theta(t), \\ x^i(t,\xi) &= x^i(0,\xi). \end{aligned} \tag{6.4}$$

In view of [5, (1.5.7)] the exterior normal of a geodesic sphere is (1,0,...,0). Using that $F(\bar{h}^i_j) = \coth \Theta$, we see that x in (6.4) solves the flow equation (5.1). Now the solution of (6.3) is given by

$$\cosh \Theta = (\cosh r_0) e^{-t}. \tag{6.5}$$

Thus the spherical flow exists only for a finite time $[0, T^*)$. Note that (6.5) can be rewritten as

$$\Theta = \operatorname{arccosh} e^{(T^*-t)}. \tag{6.6}$$

□

Next we want to prove that the flow (5.1) shrinks to a point. Using the inverse of the Beltrami map, $\mathbb{H}^{n+1}$ is parametrizable over $B_1(0)$ yielding the metric (cf. [5, Section 10.2])

$$d\bar{s}^2 = \frac{1}{(1-r^2)^2} dr^2 + \frac{r^2}{1-r^2} \sigma_{ij} d\xi^i d\xi^j. \tag{6.7}$$

Define the variable ϱ by

$$\varrho = \operatorname{arctanh} r = \tfrac{1}{2}(\log(1+r) - \log(1-r)), \tag{6.8}$$

then

$$d\bar{s}^2 = d\varrho^2 + \sinh^2 \varrho \, \sigma_{ij} \, d\xi^i d\xi^j. \tag{6.9}$$

Let

$$d\tilde{s}^2 = dr^2 + r^2\sigma_{ij}d\xi^i d\xi^j \tag{6.10}$$

be the Euclidean metric over $B_1(0)$. Define

$$d\tau = \frac{1}{r\sqrt{1-r^2}}dr, \quad d\tilde{\tau} = r^{-2}dr, \tag{6.11}$$

we have further

$$\begin{aligned} d\bar{s}^2 &= \frac{r^2}{1-r^2}\{d\tau^2 + \sigma_{ij}d\xi^i d\xi^j\} \equiv e^{2\psi}\{d\tau^2 + \sigma_{ij}d\xi^i d\xi^j\}, \\ d\tilde{s}^2 &= r^2\{d\tilde{\tau}^2 + \sigma_{ij}d\xi^i d\xi^j\} \equiv e^{2\tilde{\psi}}\{d\tilde{\tau}^2 + \sigma_{ij}d\xi^i d\xi^j\}. \end{aligned} \tag{6.12}$$

An arbitary closed, connected, strictly embedded hypersurface $M \subset \mathbb{H}^{n+1}$ bounds a convex body and we can write M as a graph in geodesic polar coordinates.

$$M = \operatorname{graph} u = \{\tau = u(x) : x \in \mathbb{S}^n\}. \tag{6.13}$$

M can also be viewed as a graph $\tilde{M}$ in $B_1(0)$ with respect to the Euclidean metric

$$\tilde{M} = \operatorname{graph} \tilde{u} = \{\ \tilde{\tau} = \tilde{u}(x) : x \in \mathbb{S}^n\}. \tag{6.14}$$

Writing $\tilde{u} = \varphi(u)$, then there holds (see [5, (10.2.18)])

$$\dot{\varphi}^2 = 1 - r^2. \tag{6.15}$$

The same argument as in [7, Lemma 6.1] yields

6.2. **Lemma.** *Let $M(t)$ be a solution of (5.1) on a maximal time interval $[0, T^*)$ and represent $M(t)$, for a fixed $t \in [0, T^*)$, as a graph in polar coordinates with center in $x_0 \in \hat{M}(t)$*

$$M(t) = \operatorname{graph} u(t, \cdot), \tag{6.16}$$

then

$$\inf_{M(t)} u \le \Theta(t, T^*) \le \sup_{M(t)} u, \tag{6.17}$$

where the solution of the spherical flow $\Theta(t, T^*)$ is given by (6.6). □

6.3. **Lemma.** *Let $x_0 \in \hat{M}(t)$ be as above and represent $M(t)$ in Euclidean polar coordinates (6.10), then there exists a constant $c_0 = c_0(M_0) < 1$ such that the estimate*

$$r \le c_0 \tag{6.18}$$

holds for any $t \in [0, T^*)$.

Proof. The argument is similar to those in [7, Lemma 6.3, Remark 6.5]. Looking at the scalar flow equation for a short time interval, we conclude that the convex bodies $\hat{M}(t) \subset \mathbb{H}^{n+1}$ are decreasing with respect to t. Furthermore, $\hat{M}_0$ is strictly convex. Thus ϱ is uniformly bounded and the claim follows from the relation

$$r = \tanh \varrho = 1 - \frac{2}{e^{2\varrho} + 1}. \tag{6.19}$$

□

Denote h_{ij} resp. $\tilde{h}_{ij}$ the second fundamental forms and κ_i resp $\tilde{\kappa}_i$ the principal curvatures of M with respect to the ambient metric $\bar{g}_{\alpha\beta}$ resp. $\tilde{g}_{\alpha\beta}$.

6.4. **Lemma.** *The principal curvatures $\tilde{\kappa}_i$ of $M(t)$ are pinched, i.e., there exists a uniform constant c such that*

$$\tilde{\kappa}_n \le c\tilde{\kappa}_1, \tag{6.20}$$

where the $\tilde{\kappa}_i$ are labeled as

$$\tilde{\kappa}_1 \le \cdots \le \tilde{\kappa}_n. \tag{6.21}$$

Proof. The h_{ij} and $\tilde{h}_{ij}$ are related through the formula (see [5, (10.2.33)])

$$\tilde{h}_{ij}\tilde{v} = (1 - r^2)h_{ij}v, \tag{6.22}$$

where

$$\begin{aligned} v^2 &= 1 + \sigma^{ij}u_iu_j, \\ \tilde{v}^2 &= 1 + \dot{\varphi}^2\sigma^{ij}u_iu_j. \end{aligned} \tag{6.23}$$

Because of Lemma 6.3 there exists $0 < \delta < 1$ such that

$$r^2 \le 1 - \delta, \tag{6.24}$$

and thus

$$\delta v^2 \le \tilde{v}^2 \le v^2, \tag{6.25}$$

$$\delta h_{ij} \le \tilde{h}_{ij} \le \delta^{-1}h_{ij}. \tag{6.26}$$

Furthermore, there holds

$$\begin{aligned} g_{ij} &= \frac{r^2}{1 - r^2}\{u_iu_j + \sigma_{ij}\}, \\ \tilde{g}_{ij} &= r^2\{\dot{\varphi}^2u_iu_j + \sigma_{ij}\}. \end{aligned} \tag{6.27}$$

and we conclude

$$\delta^2 g_{ij} \le \tilde{g}_{ij} \le g_{ij}. \tag{6.28}$$

Now the claim follows from the maximum-minimum principle. □

For $\hat{M}(t) \subset \mathbb{H}^{n+1}$, the inradius $\rho_-(t)$ and circumradius $\rho_+(t)$ of $\hat{M}(t)$ are defined by

$$\begin{aligned} \rho_-(t) &= \sup\{r : B_r(y) \text{ is enclosed by } \hat{M}(t) \text{ for some } y \in \mathbb{H}^{n+1}\}, \\ \rho_+(t) &= \inf\{r : B_r(y) \text{ encloses } \hat{M}(t) \text{ for some } y \in \mathbb{H}^{n+1}\}. \end{aligned} \tag{6.29}$$

Now, choose $x_0 \in \hat{M}(t)$ to be the center of the inball of $\hat{M}(t) \subset \mathbb{H}^{n+1}$ and let x_0 be the center of the geodesic polar coordinates. Note that the center of the Euclidean inball is also x_0. Let $\rho_-(t)$ resp. $\rho_+(t)$ be the inradius resp. circumradius of $\hat{M}(t) \subset \mathbb{H}^{n+1}$, and let $\tilde{\rho}_-(t)$ resp. $\tilde{\rho}_+(t)$ be the inradius resp. circumradius of $\hat{M}(t) \subset \mathbb{R}^{n+1}$.

6.5. **Lemma.** *Let $B_{\rho_-(t)}(x_0) \subset \hat{M}(t)$ be a geodesic inball, then there exist positive constants c and δ, such that*

$$\hat{M}(t) \subset B_{4c\rho_-(t)}(x_0) \quad \forall t \in [T^* - \delta, T^*). \tag{6.30}$$

Proof. The pinching estimates in the Euclidean ambient space (6.20) and [1, Theorem 5.1, Theorem 5.4] imply

$$\tilde{\rho}_+(t) \le c\tilde{\rho}_-(t) \tag{6.31}$$

with a uniform constant c, hence $\hat{M}(t)$ is contained in the Euclidean ball $B_{\tilde{\rho}}(0)$,

$$\hat{M}(t) \subset B_{\tilde{\rho}}(0), \quad \tilde{\rho}(t) = 2c\tilde{\rho}_-(t). \tag{6.32}$$

Furthermore, we deduce from Lemma 6.2 that

$$\inf_{M(t)} \tilde{u} \le \tilde{\Theta} \le \sup_{M(t)} \tilde{u}, \tag{6.33}$$

where $M(t) = \text{graph } \tilde{u}$ is a representation of $M(t)$ in Euclidean polar coordinates. We conclude further

$$\tilde{\rho}(t) = 2c\tilde{\rho}_-(t) \le 2c\tilde{\Theta}. \tag{6.34}$$

Choose now $\delta > 0$ small such that

$$2c\tilde{\Theta}(t, T^*) \le 1 \quad \forall t \in [T^* - \delta, T^*). \tag{6.35}$$

Now it holds for

$$\rho(t) = \operatorname{arctanh} \tilde{\rho}(t) \tag{6.36}$$

$$\hat{M}(t) \subset B_{\rho(t)}(x_0) \subset \mathbb{H}^{n+1}. \tag{6.37}$$

Since

$$\tilde{\rho}(t) \le 1, \tag{6.38}$$

we conclude further

$$\tilde{\rho} \le \rho \le 2\tilde{\rho}, \quad \tilde{\rho}_- \le \rho_-. \tag{6.39}$$

Thus

$$\rho \le 2\tilde{\rho} = 4c\tilde{\rho}_- \le 4c\rho_- \tag{6.40}$$

and the claim follows. □

6.6. **Lemma.** *During the evolution the flow hypersurfaces $M(t)$ are smooth and uniformly convex satisfying a priori estimates in any compact subinterval $[0,T] \subset [0,T^*)$.*

Proof. Let $0 < T < T^*$ be fixed. From (6.31) and (6.33) we infer

$$c\tilde{\Theta}(T,T^*) \le \tilde{\rho}_-(T). \tag{6.41}$$

Since

$$\Theta(T,T^*) = \operatorname{arctanh}\tilde{\Theta}(T,T^*), \quad \rho_-(T) = \operatorname{arctanh}\tilde{\rho}_-(T), \tag{6.42}$$

and $\tilde{\rho}_-(T)$, $\tilde{\Theta}(T,T^*)$ are uniformly bounded from above by 1 we infer that

$$0 < \tfrac{c}{2}\Theta = \tfrac{c}{2}\operatorname{arctanh}\tilde{\Theta} \le c\tilde{\Theta} \le \operatorname{arctanh}(c\tilde{\Theta}) \le \rho_-(T). \tag{6.43}$$

Let $x_0 \in \hat{M}(T)$ be the center of an inball and introduce geodesic polar coordinates with center x_0. This coordinate system will cover the flow in $0 \le t \le T$. Writing the flow hypersurfaces as graphs $u(t,\cdot)$ of a function we have

$$0 < c^{-1} \le u \le c. \tag{6.44}$$

And since $M(t)$ are convex,

$$v^2 = 1 + \sinh^{-2} u\, \sigma^{ij} u_i u_j \tag{6.45}$$

is uniformly bounded. Under assumption 1.1 (1) we have $\kappa_i \ge 1$. And under assumption 1.1 (2) it is proved in [8, Lemma 4.4] that

$$\frac{1}{n}\tilde{\kappa}_n \le \tilde{F} \le c \tag{6.46}$$

in N or equivalently, $\kappa_i \ge c$ in $\mathbb{H}^{n+1}$. The proof of uniform boundedness of κ_i from above is similar to those in [7, Theorem 6.6]. Since F is concave, we may first apply the Krylov-Safonov and then the parabolic Schauder estimates to obtain the desired a priori estimates. □

In view of Lemma 6.1, 6.2, 6.5 and 6.6, the flow (5.1) shrinks in finite time to a point x_0.

7. The rescaled flow

In view of Lemma 6.2 and 6.5 we can choose $\delta > 0$ small and define

$$t_\delta = T^* - \delta, \tag{7.1}$$

such that

$$\hat{M}(t_\delta) \subset B_{8c\rho_-(t_\delta)}(x_0) \quad \forall x_0 \in \hat{M}(t_\delta), \tag{7.2}$$

and

$$8c\rho_-(t_\delta) \le 8c\Theta(t_\delta, T^*) < 1. \tag{7.3}$$

Fix now a $t_0 \in (t_\delta, T^*)$ and let $B_{\rho_-(t_0)}(x_0)$ be an inball of $\hat{M}(t_0)$. Choose x_0 to be the center of a geodesic polar coordinate system, then the hypersurfaces $M(t)$ can be written as graphs

$$M(t) = \operatorname{graph} u(t,\cdot) \quad \forall t_\delta \le t \le t_0, \tag{7.4}$$

such that

$$\rho_-(t_0) \le u(t_0) \le u(t) \le 1. \tag{7.5}$$

7.1. **Lemma.** *Let*

$$\chi = \frac{v}{\sinh u} \equiv v\eta(r), \tag{7.6}$$

if $\chi_i = 0$*, then* $u_i = 0$.

Proof. Note that

$$\eta(r) = \frac{1}{\sinh r} \tag{7.7}$$

solves the equation

$$\dot{\eta} = -\frac{\bar{H}}{n}\eta, \tag{7.8}$$

hence the proof is same as those in [7, Lemma 7.1]. □

Similar to [7, Lemma 7.2, Corollary 7.3] we obtain

7.2. **Lemma.** *There exists a uniform constant* $c > 0$ *such that*

$$\Theta(t, T^*)F \le c \quad \forall t \in [t_\delta, T^*), \tag{7.9}$$

and that the rescaled principal curvatures $\tilde{\kappa}_i = \kappa_i\Theta$ *satisfy*

$$\tilde{\kappa}_i \le c \quad \forall t \in [t_\delta, T^*). \tag{7.10}$$

□

7.3. **Lemma.** *Let* $t_1 \in [t_\delta, T^*)$ *be arbitrary and let* $t_2 > t_1$ *be such that*

$$\Theta(t_2, T^*) = \tfrac{1}{2}\Theta(t_1, T^*). \tag{7.11}$$

Let $x_0 \in \hat{M}(t_2)$ *be the center of an geodesic inball and introduce polar coordinates around* x_0 *and write the hypersurface* $M(t)$ *as graphs*

$$M(t) = \operatorname{graph} u(t,\cdot). \tag{7.12}$$

Define ϑ *by*

$$\vartheta(r) = \sinh r, \tag{7.13}$$

and

$$\varphi = \int_{r_2}^{u} \vartheta^{-1}, \tag{7.14}$$

where $r_2 = \Theta(t_2, T^*)$. Then $\varphi(t,\cdot)$ is uniformly bounded in $C^2(\mathbb{S}^n)$ for any $t_1 \le t \le t_2$ independent of t_1, t_2. Furthermore, let Γ_{ij}^k and $\tilde\Gamma_{ij}^k$ be the Christoffel symbols of the metrics g_{ij} and σ_{ij} respectively, then the tensor $\Gamma_{ij}^k - \tilde\Gamma_{ij}^k$ is also uniformly bounded independent of t_1, t_2.

Proof. As in [7, Lemma 7.4], we conclude from Lemma 6.2 and Lemma 6.5 that there exists a uniform constant $c > 1$, independent of t_1, t_2, such that

$$c^{-1}\Theta(t_2, T^*) \le u(t,\xi) \le c\Theta(t_2, T^*) \quad \forall t \in [t_1, t_2]. \tag{7.15}$$

Note that

$$\varphi = \{\log\sinh(\tfrac{r}{2}) - \log\cosh(\tfrac{r}{2})\}\big|_{r_2}^{u}, \tag{7.16}$$

thus we derive the C^0-estimates

$$|\varphi| \le \log c. \tag{7.17}$$

As in the proof of [7, Lemma 7.5], an upper bound for the principal curvatures of the slices $\{x^0 = \text{const}\}$ intersecting $M(t)$ satisfies

$$\bar\kappa \le \frac{\sup\cosh u(0,\cdot)}{\sinh u_{\min}} \le \frac{c}{u_{\min}}, \tag{7.18}$$

and from [5, (2.7.83)] we infer that the uniformly boundedness of v.

$$v \le e^{\bar\kappa(u_{\max} - u_{\min})} \le e^{c(\frac{u_{\max}}{u_{\min}})-1}, \tag{7.19}$$

concluding further that

$$|D\varphi|^2 = v^2 - 1 \le c. \tag{7.20}$$

Define

$$\tilde g^{ij} = \sigma^{ij} - v^{-2}\varphi^i\varphi^j, \tag{7.21}$$

where

$$\varphi^i = \sigma^{ik}\varphi_k. \tag{7.22}$$

Due to the boundedness of v the metrics $\tilde g_{ij}$ and σ_{ij} are equivalent, thus we can raise the indices of φ_{ij} by $\tilde g_{ij}$ and by employing the relation [6, (3.26)]

$$h_j^i = v^{-1}\vartheta^{-1}\{-(\sigma^{ik} - v^{-2}\varphi^i\varphi^k)\varphi_{jk} + \dot\vartheta\delta_j^i\}, \tag{7.23}$$

we infer

$$\tilde g^{ik}\varphi_{jk} = -v\vartheta h_j^i + \dot\vartheta\delta_j^i, \tag{7.24}$$

concluding further from (7.10)

$$\|\varphi_{ij}\|^2 \le c(v^2\vartheta^2|A|^2 + n\dot\vartheta^2) \tag{7.25}$$

is bounded from above for all $t \in [t_1, t_2]$. We choose coordinates such that $\tilde\Gamma_{ij}^k$ in a fixed point vanishes. Denote the covariant derivative with respect to σ_{ij} by a colon. In such coordinates

$$\Gamma_{ij}^k = \tfrac12 g^{km}(g_{mi:j} + g_{mj:i} - g_{ij:m}). \tag{7.26}$$

From

$$g^{ij} = \vartheta^{-2}\tilde{g}^{ij} \tag{7.27}$$

we compute

$$g^{km}g_{mi:j} = \tilde{g}^{km}\{\varphi_{mj}\varphi_i + \varphi_{ij}\varphi_m + 2\cosh u\,\varphi_j(\varphi_m\varphi_i + \sigma_{mi})\}. \tag{7.28}$$

Using the estimates for φ proved before, we conclude that $\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k$ are uniformly bounded independent of t_1 and t_2. □

Define a new time parameter as

$$\tau = -\log\Theta, \tag{7.29}$$

then

$$\frac{dt}{d\tau} = \Theta\frac{\sinh\Theta}{\cosh\Theta}. \tag{7.30}$$

In the following we denote the differentiation with respect to t by a dot and differentiation with respect to τ by a prime.

7.4. **Lemma.** *The rescaled quantity $\tilde{F} = F\Theta$ satisfies the inequality*

$$\sup_{M(t_1)} \tilde{F} \le c \inf_{M(t_2)} \tilde{F} \tag{7.31}$$

with a uniform constant $c > 0$.

Proof. $\tilde{F}$ satisfies the equation

$$\tilde{F}' = \dot{F}\Theta^2\frac{\sinh\Theta}{\cosh\Theta} - \tilde{F}, \tag{7.32}$$

and from the evolution equation of F in [7, (2.8)] we conclude further

$$\tilde{F}' + \tilde{F} - \{F^{ij}F_{;ij} + F^{ij}h_{ik}h_j^k F + K_N F^{ij}g_{ij}F\}\Theta^2\frac{\sinh\Theta}{\cosh\Theta} = 0. \tag{7.33}$$

We consider the non-trivial term in (7.33)

$$-F^{ij}F_{;ij}\Theta^2\frac{\sinh\Theta}{\cosh\Theta}. \tag{7.34}$$

In view of (7.27), the pinching estimate and the boundedness of v, Θ^2F^{ij} and σ^{ij} are equivalent and hence uniformly positive definite. Furthermore,

$$F_{;ij} = F_{:ij} - \{\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k\}F_k. \tag{7.35}$$

Hence we conclude from Lemma 7.3 that $\tilde{F}$ satisfies a uniform parabolic equation of the form

$$\tilde{F}' - a^{ij}\tilde{F}_{:ij} + b^i\tilde{F}_i + c\tilde{F} = 0 \tag{7.36}$$

in the cylinder $[\tau_1, \tau_2] \times \mathbb{S}^n$, where $\tau_i = -\log\Theta(t_i, T^*)$, with uniformly bounded coefficients. The statement follows then from the parabolic Harnack inequality. □

7.5. **Corollary.** *The rescaled principal curvatures* $\tilde\kappa_i = \kappa\Theta$ *are uniformly bounded from below.*

Proof. Consider a point (t,ξ) in $M(t)$ such that

$$u(t,\xi) = \sup_{M(t)} u. \tag{7.37}$$

In view of [5, (1.5.10)], it holds in (t,ξ)

$$h_{ij} \ge \bar h_{ij}, \quad g_{ij} = \bar g_{ij}, \quad \kappa_i \ge \bar\kappa_i = \frac{\cosh u}{\sinh u}, \tag{7.38}$$

where we denote the quantity of the slices $\{x^0 = \text{const}\}$ with a bar. In view of (7.15)

$$\sup_{M(t)} \tilde F \ge F(\tilde\kappa_i(t,\xi)) \ge F\left(\frac{\cosh u(t,\xi)}{\sinh u(t,\xi)}\Theta(t,T^*)\right) \ge c > 0. \tag{7.39}$$

The statement follows from the pinching estimates and Lemma 7.4. □

Let $x_0 \in \mathbb{H}^{n+1}$ be the point the flow hypersurfaces are shrinking to and introduce geodesic polar coordinates around it. Write $M(t) = \operatorname{graph} u(t,\cdot)$ and let

$$\tilde u(\tau,\xi) = u(t,\xi)\Theta(t,T^*)^{-1}, \tag{7.40}$$

$$\tau_\delta = -\log\Theta(t_\delta,T^*), \quad Q(\tau_\delta,\infty) = [\tau_\delta,\infty)\times\mathbb{S}^n. \tag{7.41}$$

Using the same argument as in [7, Lemma 7.9, Lemma 7.10] we conclude that

7.6. **Lemma.** *The quantities* v *and* $|D\tilde u|$ *are uniformly bounded form above and* $\tilde u$ *is uniformly bounded from below and above in* $Q(\tau_\delta,\infty)$. □

Let

$$\varphi = -\int_u^{\Theta(0,T^*)} \vartheta^{-1}, \tag{7.42}$$

then

$$\varphi_i = \vartheta^{-1}u_i, \quad \varphi_{ij} = \vartheta^{-1}u_{ij} - \cosh u\,\vartheta^{-2}\,u_i\,u_j, \tag{7.43}$$

and

$$\vartheta^{-2}|D^2u|^2 + |D\tilde u|^4\cosh^2 u - 2\vartheta^{-1}|D^2u||D\tilde u|^2\cosh u \le |D^2\varphi|^2. \tag{7.44}$$

Since $|D^2\varphi|$ and $|D\tilde u|$ are bounded, we conclude that the C^2-norm of $\tilde u$ is uniformly bounded, where the covariant derivatives of $\tilde u$ and φ are taken with respect to σ_{ij}. From [5, Remark 1.5.1, Lemma 2.7.6] we conclude that

$$\frac{\sinh\Theta}{\cosh\Theta}Fv = \Phi(x,\tau,\tilde u,\tilde u e^{-\tau}, D\tilde u, D^2\tilde u), \tag{7.45}$$

where Φ is a smooth function with respect to its arguments, and

$$\begin{aligned} \Phi^{ij} &\equiv \frac{\partial \Phi}{\partial(-\tilde{u}_{ij})} = F^{ij}\Theta\frac{\sinh\Theta}{\cosh\Theta}, \\ \Phi^{ij,kl} &= F^{ij,kl}v^{-1}\Theta^2\frac{\sinh\Theta}{\cosh\Theta}. \end{aligned} \tag{7.46}$$

Hence by applying first the Krylov and Safonov, then the Schauder estimates, we deduce (cf. [5, Remark 2.6.2])

7.7. **Theorem.** *The rescaled function $\tilde{u}$ satisfies the uniformly parabolic equation*

$$\tilde{u}' = -\Phi + \tilde{u} \tag{7.47}$$

in $Q(\tau_\delta, \infty)$ and $\tilde{u}(\tau, \cdot)$ satisfies a priori estimates in $C^\infty(\mathbb{S}^n)$ independently of τ.

8. Convergence to a sphere

The aim of this section is to prove that $\tilde{u}$ converges exponentially fast to the constant function 1 if F is strictly concave or $F = \frac{1}{n}H$. Comparing the proof in [7, Section 8], we should handle a term stemming from the negative curvature of the ambient space $K_N < 0$.

8.1. **Lemma.** *There exists a positive constant C such that*

$$F^{kl}g_{kl}|A|^2 - FH \le C\sum_{i<j}(\kappa_i - \kappa_j)^2. \tag{8.1}$$

Proof. The proof is similar to [7, Lemma 8.2]. Let

$$\varphi = F^{kl}g_{kl}|A|^2 - FH. \tag{8.2}$$

Denote the partial derivatives of φ with respect to κ_i by φ_i, then

$$\varphi_j = \sum_{i=1}^n F_{ij}|A|^2 + \sum_{i=1}^n 2F_i\kappa_j - F_jH - F, \tag{8.3}$$

$$\begin{aligned} \varphi_{jk} = &\sum_{i=1}^n F_{ijk}|A|^2 + \sum_{i=1}^n 2F_{ij}\kappa_k + \sum_{i=1}^n 2F_{ik}\kappa_j \\ &+ 2\delta_{jk}\sum_{i=1}^n F_i - F_{jk}H - F_j - F_k. \end{aligned} \tag{8.4}$$

Therefore

$$\varphi(\kappa_n, \cdots, \kappa_n) = 0, \quad \varphi_j(\kappa_n, \cdots, \kappa_n) = 0 \quad \forall j = 1, \dots n. \tag{8.5}$$

by using the Euler's homogeneous relation and the normalization (1.6). Furthermore, φ_{jk} are uniformly bounded from above, since φ_{jk} are homogeneous of grad 0 and $\frac{\kappa_i}{|A|}$ are compactly contained in the defining cone. The statement follows by an argument using Taylor's expansion up to the second order similar to those in [7, Lemma 8.2]. ☐

We want to estimate the function

$$f_\sigma = F^{-\alpha}(|A|^2 - nF^2), \tag{8.6}$$

where

$$\alpha = 2 - \sigma, \tag{8.7}$$

and $0 < \sigma < 1$ small. For simplicity we drop the subscript σ of f_σ. In the following we always assume that F satisfies the assumption 1.1.

By Lemma 8.1 we have the following inequality corresponding to [7, Lemma 8.3].

8.2. **Lemma.** *Let F be strictly concave, then there exist uniform constants $\epsilon > 0$ and $C > 0$, such that*

$$\begin{aligned} -F^{ij}f_{ij}+2\epsilon^2F^{ij}h_{ki}h_j^kf \le \alpha F^{-1}F^{ij}F_{;ij}f + 2(\alpha-1)F^{-1}F^{ij}F_if_j \\ - 2\{h^{ij} - FnF^{ij}\}F^{-\alpha}F_{;ij} - 2\epsilon^2|DA|^2F^{-\alpha} + 2Cf. \end{aligned} \tag{8.8}$$

Corresponding to [7, Lemma 8.5] we have

8.3. **Lemma.** *Let F be strictly concave, then there exist positive constants C and c such that for any $p \ge 2$, any $\delta > 0$ and any $0 \le t < T^*$*

$$\begin{aligned} \epsilon^2\int_M F^{ij}h_{ki}h_j^kf^p \le \{\delta^{-1}c(p-1)+c\}\int_M F^{ij}f_if_jf^{p-2} \\ + \{\delta c(p-1)+c\}\int_M |DA|^2F^{-\alpha}f^{p-1} + 2C\int_M f^p. \end{aligned} \tag{8.9}$$

Parallel to [7, Lemma 8.6] we have

8.4. **Lemma.** *Let F be strictly concave, then there exist $C_1 > 0$ and $\sigma_0 > 0$ such that for all*

$$p \ge 4c\epsilon^{-2}, \quad \sigma \le \min(\tfrac14 c^{-1}\epsilon^3p^{-1/2}, \sigma_0), \tag{8.10}$$

the estimate

$$\|f\|_{p,M} \le C_1 \quad \forall t \in [0, T^*) \tag{8.11}$$

holds, where $C_1 = C_1(M_0, p)$ and $\sigma_0 = \sigma_0(F, M_0)$.

Proof. Multiply [7, (8.30)] with pf^{p-1} and integrate by parts, and note that

$$d\mu_t = \mu_t\, dx \;\text{ on }\; M_t, \tag{8.12}$$

where

$$\frac{d}{dt}\mu_t = \frac{d}{dt}\sqrt{g} = \tfrac12\mu_tg^{ij}\dot g_{ij} = -FH\mu_t, \tag{8.13}$$

thus

$$\frac{d}{dt}\int_M f^p = p\int_M f^{p-1}f' - \int_M HFf^p, \tag{8.14}$$

and

$$\begin{aligned}\frac{d}{dt}\int_M f^p + \tfrac{1}{2}p(p-1)\int_M F^{ij}f_if_jf^{p-2} + \epsilon^2 p\int_M |DA|^2F^{-\alpha}f^{p-1}\\ \le \sigma p\int_M F^{ij}h_{ki}h_j^k f^p + 4Cp\int_M f^p.\end{aligned} \tag{8.15}$$

By choosing

$$c_0 = \tfrac{1}{4}c, \quad \sigma \le \min(\epsilon^3 p^{-1/2}{c_0}^{-1}, \sigma_0), \quad \delta = \epsilon p^{-1/2}, \tag{8.16}$$

and by using (8.9), the right-hand side of inequality (8.15) can be estimated from above by

(8.17)

$$\begin{aligned}&\epsilon p^{1/2}c_0^{-1}\{\epsilon^2\int_M F^{ij}h_{ki}h_j^k f^p\} + 4Cp\int_M f^p\\ &\le \epsilon p^{1/2}c_0^{-1}\{\delta^{-1}c(p-1)+c\}\int_M F^{ij}f_if_jf^{p-2}\\ &\quad + \epsilon p^{1/2}c_0^{-1}\{\delta c(p-1)+c\}\int_M |DA|^2F^{-\alpha}f^{p-1} + \{2C\epsilon p^{1/2}c_0^{-1}+4Cp\}\int_M f^p\\ &= c_0^{-1}\{p(p-1)c + \epsilon p^{1/2}c\}\int_M F^{ij}f_if_jf^{p-2}\\ &\quad + c_0^{-1}\{\epsilon^2(p-1)c+\epsilon p^{1/2}c\}\int_M |DA|^2F^{-\alpha}f^{p-1} + \{2C\epsilon p^{1/2}c_0^{-1}+4Cp\}\int_M f^p\\ &\le \tfrac{1}{2}p(p-1)\int_M F^{ij}f_if_jf^{p-2} + \tfrac{1}{2}\epsilon^2(p-1)\int_M |DA|^2F^{-\alpha}f^{p-1} + 5Cp\int_M f^p.\end{aligned}$$

From (8.15), (8.17) we conclude that

$$\frac{d}{dt}\int_M f^p \le 5Cp\int_M f^p, \tag{8.18}$$

and the Gronwall's lemma leads to

$$\int_M f^p \le \int_M f^p\big|_{t=0}\cdot \exp(5CpT^*), \tag{8.19}$$

$$\|f\|_p = \left(\int_M f^p\right)^{\frac{1}{p}} \le e^{5CT^*}(|M_0|+1)\sup_{0\le\sigma\le 1/2}\ \sup_{M_0} f_\sigma. \tag{8.20}$$

□

To proceed further, we use the Stampacchia iteration scheme as in the Huisken's paper [10, Theorem 5.1], as well as [11, Theorem 5.1]. Note that $\mathbb{H}^{n+1}$ is simply connected and has constant sectional curvature $K_N = -1$, thus the Sobolev inequality in [9, Theorem 2.1] has the form

8.5. **Lemma.** *Let v be a nonnegative Lipschitz function on M, then there exists a constant $c = c(n) > 0$, such that*

$$(\int_M |v|^{\frac{n}{n-1}})^{\frac{n-1}{n}} \le c\{\int_M |Dv| + \int_M H|v|\}. \tag{8.21}$$

Corresponding to [7, Theorem 8.7], we have

8.6. **Theorem.** *Let F be strictly concave or $F = \frac{1}{n}H$, then there exist constants $\delta > 0$ and $c_0 > 0$, such that*

$$|A|^2 - nF^2 \le c_0 F^{2-\delta}. \tag{8.22}$$

Proof. As in the proof of [10, Theorem 5.1] let $f_{\sigma,k} = \max(f_\sigma - k, 0)$ for all $k \ge k_0 = \sup_{M_0} f_\sigma$ and denote by $A(k)$ the set where $f_\sigma > k$. We obtain with $v = f_{\sigma,k}^{p/2}$ for $p \ge 4c\epsilon^{-2}$,

$$\begin{aligned} \frac{d}{dt}\int_{A(k)} v^2 + \int_{A(k)} |Dv|^2 &\le \sigma p \int_{A(k)} H^2 f_\sigma^p + 5Cp \int_{A(k)} f_\sigma^p \\ &\le C(p) \int_{A(k)} H^2 f_\sigma^p. \end{aligned} \tag{8.23}$$

By applying Lemma 8.5 we can bound f_σ for σ small as in the proof of [10, Theorem 5.1]. The case $F = \frac{1}{n}H$ is proved in [11, Lemma 5.1]. □

8.7. **Lemma.** *Let F be strictly concave or $F = \frac{1}{n}H$ and $\tilde{M}(\tau)$ be the rescaled hypersurfaces, then there are constants $c, \delta > 0$ such that*

$$\int_{\tilde{M}} |D\tilde{A}|^2 \le ce^{-\delta\tau} \quad \forall \tau_0 \le \tau < \infty, \tag{8.24}$$

where

$$\tau_0 = -\log\Theta(0, T^*), \quad |D\tilde{A}|^2 = \Theta^2 g^{ij} h^k_{l;i}\Theta h^l_{k;j}\Theta. \tag{8.25}$$

Proof. Choose

$$f = F^{-2}\{|A|^2 - nF^2\}. \tag{8.26}$$

From Theorem 8.6 we infer

$$f \le c_0 F^{-\delta} \le c\Theta^\delta = ce^{-\delta\tau} \quad \forall \tau \ge \tau_0, \tag{8.27}$$

and from Theorem 7.7 we obtain

$$|D^m A| \le c|A| \quad \forall m \ge 1. \tag{8.28}$$

Integrating inequality (8.8) over M, using integraion by parts and using relation (8.28), we infer

$$2\epsilon^2 \int_M |DA|^2 F^{-2} \le c\int_M f. \tag{8.29}$$

Hence (8.24) follows by rescaling (8.29). □

Using the same proof of [7, Lemma 8.10] we have

8.8. **Lemma.** *There are positive constants c and δ such that for all $\tau \geq \tau_0$*

$$\tilde{F}_{\max} - \tilde{F}_{\min} \leq ce^{-\delta\tau}, \tag{8.30}$$

and

$$\|D\tilde{F}\| \leq ce^{-\delta\tau}. \tag{8.31}$$

□

8.9. **Lemma.** *There are positive constants c and δ such that for all $\tau \geq \tau_0$*

$$|D\tilde{u}| \leq ce^{-\delta\tau}, \tag{8.32}$$

where

$$|D\tilde{u}|^2 = \sigma^{ij}\tilde{u}_i\tilde{u}_j. \tag{8.33}$$

Proof. As in the proof of [7, Lemma 8.12], we let

$$\varphi = \log\tilde{u}, \quad w = \tfrac{1}{2}|D\varphi|^2, \tag{8.34}$$

then

$$\varphi' = -e^{-\varphi}\tilde{F}\Theta^{-1}\frac{\sinh\Theta}{\cosh\Theta}v + 1. \tag{8.35}$$

Differentiate now (8.35) with respect to $\varphi^k D_k$ we obtain

$$\begin{aligned} w' =& 2e^{-\varphi}w\tilde{F}\Theta^{-1}\frac{\sinh\Theta}{\cosh\Theta}v - e^{-\varphi}\tilde{F}\Theta^{-1}\frac{\sinh\Theta}{\cosh\Theta}v^{-1}\sinh^{-2}u\,u^2 w_k\varphi^k \\ &+ R_1 + R_2, \end{aligned} \tag{8.36}$$

where

$$\begin{aligned} R_1 &= -e^{-\varphi}\frac{\sinh\Theta}{\cosh\Theta}vF_k\varphi^k, \\ R_2 &= e^{-\varphi}\tilde{F}\frac{\sinh\Theta}{\Theta\cosh\Theta}v^{-1}|D\varphi|^4\sinh^{-3}u\{u^3\cosh u - u^2\sinh u\} \geq 0. \end{aligned} \tag{8.37}$$

In view of (8.31) R_1 decays exponentially. Thus the function

$$w_{\max} = \sup_{\tilde{M}(\tau)} w \tag{8.38}$$

is Lipschitz and satisfies

$$w'_{\max} \geq 2e^{-\varphi}w\tilde{F}\Theta^{-1}\frac{\sinh\Theta}{\cosh\Theta}v - ce^{-\delta\tau} \tag{8.39}$$

for almost every $\tau \geq \tau_0$. Using the same argument as in [7, Lemma 8.12] we conclude that

$$w_{\max}(\tau) \leq \tfrac{c}{\delta}e^{-\delta\tau} \quad \forall\tau \geq \tau_0. \tag{8.40}$$

□

The same arguments of [7, Corollary 8.13] and the interpolation inequalities for the C^m-norms (cf. [6, Corollary 6.2]) yield

8.10. **Theorem.** *Let F be strictly concave or $F = \frac{1}{n}H$, then the rescaled function $\tilde{u}$ converges in $C^\infty(\mathbb{S}^n)$ to the constant function 1 exponentially fast.*

□

8.11. **Lemma.** *Let F be strictly concave or $F = \frac{1}{n}H$, then there exist positive constants c and δ such that*

$$|\tilde{F}(\tau,\cdot) - 1| \le ce^{-\delta\tau} \quad \forall \tau \ge \tau_0. \tag{8.41}$$

Proof. Observe that for τ_1 sufficiently large we have

$$\left|\frac{\sinh\Theta}{\cosh\Theta} - \Theta\right| \le c\Theta^2 \quad \forall \tau \ge \tau_1. \tag{8.42}$$

The rest of the proof is identical to [7, Lemma 8.16]. □

9. Inverse curvature flows

Let $M(t)$ be the flow hypersurfaces of the direct flow in $\mathbb{H}^{n+1}$ and write $M(t)$ as graphs $M(t) = \operatorname{graph} u(t,\cdot)$. with respect to the geodesic polar coordinates centered in the point where the direct flow shrinks to. By applying an isometry we may assume that the point x_0 is the Beltrami point. The polar hypersurfaces $M(t)^*$ are the flow hypersurfaces of the corresponding inverse curvature flow in the de Sitter space. Write $M(t)^* = \operatorname{graph} u^*(t,\cdot)$ over $\mathbb{S}^n$.

9.1. **Lemma.** *The functions u, u^* satisfy the relations*

$$u_{\max} = -u^*_{\min} \quad \forall t \in [t_\delta, T^*), \tag{9.1}$$

$$u_{\min} = -u^*_{\max} \quad \forall t \in [t_\delta, T^*). \tag{9.2}$$

Proof. We use the relation [5, (10.4.65)]

$$\tilde{x}^0 = \frac{r}{\sqrt{1-r^2}}, \tag{9.3}$$

and note that by comparing [5, (10.2.5)] and the metric in the eigentime coordinate system in N (2.13) we infer that

$$\cosh^2 u^* = 1 + |\tilde{x}^0|^2. \tag{9.4}$$

From (6.8) we infer that

$$r = \tanh u. \tag{9.5}$$

Since we have switched the light cone such that the uniformly convex slices are contained in $\{\tau < 0\}$, we deduce that

$$u^* = -\operatorname{arcsinh}(\tilde{v}\sinh u) = -\operatorname{arcsinh}\tilde{\chi}. \tag{9.6}$$

In a point where u^* attains its minimum, there holds $v = 1$ in view of Lemma 7.1. Thus $u = -u^*$ and u attains its maximum in such a point. This proves (9.1). The proof of (9.2) is similar. □

9.2. **Corollary.** *There exists a positive constant c such that*

$$-c \le w \equiv u^*\Theta^{-1} \le -c^{-1} \quad \forall t \in [t_\delta, T^*). \tag{9.7}$$

□

Define $\vartheta(u) = \cosh(u)$ and $\bar{g}_{ij} = \vartheta^2\sigma_{ij}$. We prove in the following that w is uniformly bounded in $C^\infty(\mathbb{S}^n)$. For simplicity, we write in the following u instead u^* for the graphs of the flow hypersurfaces in the de Sitter space. The proof of C^1-estimates of w is similar to [5, Theorem 2.7.11].

9.3. **Lemma.** *There exists a positive constant c such that*

$$|Dw|^2 \equiv \sigma^{ij}w_iw_j \le c \quad \forall t \in [t_\delta, T^*). \tag{9.8}$$

Proof. Since

$$\|Du\|^2 \equiv g^{ij}u_iu_j = v^{-2}\bar{g}^{ij}u_iu_j \equiv v^{-2}|Du|^2, \tag{9.9}$$

we first estimate $\|Du\|\Theta^{-1}$. Let λ be a real parameter to be specified later and define

$$G = \tfrac{1}{2}\log(\|Du\|^2\Theta^{-2}) + \lambda u\Theta^{-1}. \tag{9.10}$$

There is $x_0 \in \mathbb{S}^n$ such that

$$G(x_0) = \sup_{\mathbb{S}^n} G, \tag{9.11}$$

and thus in x_0

$$0 = G_i = \|Du\|^{-2}u_{ij}u^j + \lambda u_i\Theta^{-1}, \tag{9.12}$$

where the covariant derivatives are taken with respect to g_{ij} and

$$u^i = g^{ij}u_j = v^{-2}\bar{g}^{ij}u_j. \tag{9.13}$$

Since

$$h_{ij}v^{-1} = -u_{ij} - \dot{\vartheta}\vartheta\sigma_{ij}, \tag{9.14}$$

we infer that

$$\begin{aligned} \lambda\|Du\|^{-4}\Theta^{-4} &= -u_{ij}u^iu^j\Theta^{-3} \\ &= v^{-1}h_{ij}u^iu^j\Theta^{-3} + \dot{\vartheta}\vartheta\sigma_{ij}u^iu^j\Theta^{-3}. \end{aligned} \tag{9.15}$$

By considering the dual flow in the hyperbolic space, we conclude that $h_{ij} > 0$. Furthermore,

$$\dot{\vartheta}\vartheta\sigma_{ij}u^iu^j\Theta^{-3} = (\dot{\vartheta}\Theta^{-1})\vartheta^{-1}v^{-2}\|Du\|^2\Theta^{-2}. \tag{9.16}$$

By applying [5, Theorem 2.7.11] directly, we conclude that v^{-2} is uniformly bounded. Note $\dot{\vartheta}\Theta^{-1} \le c$. Let c_0 be an upper bound for $(\dot{\vartheta}\Theta^{-1})\vartheta^{-1}v^{-2}$ and by choosing $\lambda < -c_0$ we conclude that $\|Du\|\Theta^{-1}$ can not be too large in x_0. Thus $\|Du\|\Theta^{-1}$ is uniformly bounded from above. We conclude that

$$\sigma^{ij}w_iw_j = \|Du\|^2\Theta^{-2}\theta^2v^2 \tag{9.17}$$

is uniformly bounded. □

9.4. **Lemma.** *There exists a positive constant c such that for all $m \ge 2$*

$$|D^m w|^2 \le c \quad \forall t \in [t_\delta, T^*). \tag{9.18}$$

Proof. Let $(\hat{h}^{ij}) = (h_{ij})^{-1}$ be the inverse of the second fundamental form in $\mathbb{H}^{n+1}$ and $\tilde{h}_{ij}$ the second fundamental form in N. We consider the mixed tensor

$$\hat{h}_i^j = g_{ik}\hat{h}^{kj}, \quad \tilde{h}_i^j = \tilde{g}^{kj}\tilde{h}_{ki}, \tag{9.19}$$

where g_{ij} and $\tilde{g}_{ij} = h_i^k h_{kj}$ are the metrics of hypersurfaces in $\mathbb{H}^{n+1}$ resp. N. From the relation

$$\tilde{\kappa}_i = \kappa_i^{-1}, \tag{9.20}$$

we infer that

$$\tilde{h}_i^j = \hat{h}_i^j. \tag{9.21}$$

From Theorem 7.7 we infer that $h_i^j\Theta$ are uniformly bounded in $C^\infty(\mathbb{S}^n)$ and due to Lemma 7.2 and Corollary 7.5 there are constants $c_1, c_2 > 0$ such that

$$0 < c_1\delta_i^j \le h_i^j\Theta \le c_2\delta_i^j, \tag{9.22}$$

and thus $\tilde{h}_i^j\Theta^{-1} = \hat{h}_i^j\Theta^{-1}$, as the inverse of $h_i^j\Theta$, are uniformly bounded in $C^\infty(\mathbb{S}^n)$. We switch now our notation by considering the quantities in N without writing a tilde. Denote the covariant derivatives with respect to $\bar{g}_{ij}$ resp. σ_{ij} by a semiconlon resp. a colon. In view of [5, Remark 1.6.1, Lemma 2.7.6] we have

(9.23)

$$\begin{aligned} v^{-1}h_{ij} &= -v^{-2}u_{;ij} - \dot{\vartheta}\vartheta\sigma_{ij} \\ &= -v^{-2}\{u_{:ij} - \tfrac{1}{2}\bar{g}^{km}\left((\vartheta^2)_j\sigma_{mi} + (\vartheta^2)_i\sigma_{mj} - (\vartheta^2)_m\sigma_{ij}\right)u_k\} - \dot{\vartheta}\vartheta\sigma_{ij}. \end{aligned}$$

Therefore,

$$u_{:ij} = -vh_{ij} + 2\vartheta^{-1}\dot{\vartheta}u_iu_j - \vartheta\dot{\vartheta}\sigma_{ij}. \tag{9.24}$$

By considering the dual flow in hyperbolic space, we infer that

$$|A|\Theta^{-1} \le c, \tag{9.25}$$

and note that

$$\bar{g}^{ij} \le \bar{g}^{ij} + v^{-2}\check{u}^i\check{u}^j = g^{ij}, \tag{9.26}$$

where

$$\check{u}^i = \bar{g}^{ij}u_j, \tag{9.27}$$

we conclude that

$$\sigma^{ik}\sigma^{jl}h_{ij}h_{kl} \le c|A|^2. \tag{9.28}$$

In view of $\dot{\vartheta}\Theta^{-1} \le c$ we conclude that $|D^2w|^2$ is uniformly bounded.
Contract (9.24) with g^{ij} we conclude further

$$-g^{ij}w_{:ij} - \vartheta^{-3}\dot{\vartheta}\Theta v^{-2}|Dw|^2 + vH\Theta^{-1} + n\vartheta^{-1}\dot{\vartheta}\Theta^{-1} = 0. \tag{9.29}$$

Since v is uniformly bounded, (9.29) is a uniformly elliptic equation in w with bounded coefficients. A bootstrapping procedure with Schauder theory yields for all $m \in \mathbb{N}$

$$|w|_{m,\mathbb{S}^n} \le c_m \quad \forall t \in [0, T^*). \tag{9.30}$$

□

From Lemma 8.10 and preceding results in Section 9 we conclude

9.5. **Theorem.** *Let the geodesic polar coordinates (τ, ξ^i) of N be specified in Section 2. Represent the inverse curvature flow (1.5) in N as graphs over $\mathbb{S}^n$, $M(t)^* = \operatorname{graph} u^*(t, \cdot)$, where the curvature function $\tilde{F}$ satisfies the assumption 1.1. Then u^* converges to the constant function 0 in $C^\infty(\mathbb{S}^n)$. The rescaled function $w = u^*\Theta^{-1}$ are uniformly bounded in $C^\infty(\mathbb{S}^n)$. When the curvature function F of the corresponding contracting flow is strictly concave or $F = \frac{1}{n}H$, then $w(\tau, \cdot)$ converges in $C^\infty(\mathbb{S}^n)$ to the constant function -1 exponentially fast.* □

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Ruprecht-Karls-Universität, Institut für Angewandte Mathematik, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany

E-mail address: h.yu@stud.uni-heidelberg.de

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