INAUGURAL–DISSERTATION

zur

Erlangung der Doktorwürde

der

Naturwissenschaftlich–Mathematischen Gesamtfakultät

der

RUPRECHT–KARLS–UNIVERSITÄT

HEIDELBERG

vorgelegt von

Dipl.-Math. Jürgen Gutekunst

aus Tübingen

Tag der mündlichen Prüfung

13. Februar 2019

FEEDBACK CONTROL FOR AVERAGE OUTPUT SYSTEMS

Gutachter

PD DR. ANDREAS POTSCHKA

PROF. DR. DR. H. C. MULT. HANS GEORG BOCK

Zusammenfassung

In dieser Arbeit stellen wir neue Methoden zum Design ökonomischer nichtlinearer modellprädikativer Regler für Average Output Optimal Control Problems (AOCPs) vor. Bei AOCPs handelt es sich um Optimalsteuerungsprobleme mit unendlichen Zeithorizonten und Zielfunktionalen, welche die gemittelte Performance des Systems messen. Solche Probleme treten bei vielen kontinuierlich ablaufenden Prozessen auf, wie etwa beim Betrieb eines Kraftwerks. Aufgrund der unendlichen Zeithorizonte und der daraus resultierenden intrinsischen Nicht-Eindeutigkeit der Lösungen ist das Aufstellen geeigneter Nonlinear Model Predictive Control (NMPC) Schemata für AOCPs ein schwieriges Problem.

Oft basieren Untersuchungen zum Closed-Loop-Verhalten ökonomischer NMPC Schemata auf Dissipativitätsbedingungen an das dynamische System und das zugrundeliegenden Zielfunktionskriterium, welche schwierig zu überprüfen sind.

Die entwickelten Methoden basieren auf der Beobachtung, dass periodische Lösungen hervorragend zur Approximation von Lösungen von AOCPs geeignet sind. Diese Eigenschaft wird ausgenutzt, indem der Prädiktionshorizont in einen transienten und einen periodischen Teil aufgeteilt wird.

Zur Analyse des resultierenden Systemverhaltens werden neue Methoden entwickelt, die darauf beruhen, die Differenz der Lösungen von aufeinanderfolgenden NMPC Subproblemen zu analysieren. Es wird gezeigt, dass diese Differenz unter geeigneten Voraussetzungen mit fortschreitender Zeit gegen Null konvergiert. Dieser Ansatz beruht im Gegensatz zu vielen anderen ökonomischen NMPC Schemata nicht auf Dissipativitätsannahmen, sondern vielmehr auf Annahmen an die Steuerbarkeit des dynamischen Systems sowie der eindeutigen Lösbarkeit der auftretenden NMPC Subprobleme.

Als Resultat können wir zeigen, dass das resultierende System eine ökonomische Performance erzielt, die mit der optimalen periodischen Performance übereinstimmt.

Darüber hinaus erweitern wir den vorgestellten Ansatz in zwei Richtungen: Zuerst betrachten wir ein allgemeineres Szenario mit parameterabhängiger Dynamik und Parametern, die sich während des Betriebs ändern können. Die Parameteränderungen können zu einer Änderung des optimalen periodischen Verhaltens führen, insbesondere auch zur Änderung der optimalen Periodenlänge. Dieser Tatsache wird dadurch Rechnung getragen, dass die Periodenlänge als freie Optimierungsvariable in das NMPC Subproblem mitaufgenommen wird. Als zweites Szenario betrachten wir den Fall von Systemen mit zeitabhängigem, periodischem Zielfunktionskriterium und zeigen, dass die vorgestellten Methoden auch auf solche Systeme angewandt werden können.

Die vorgestellten Methoden sind im Rahmen des NMPC Toolkits `MLI` implementiert und werden an einer Reihe von anspruchsvollen Anwendungsproblemen getestet. Die Simulationsergebnisse bestätigen, dass die ökonomische Performance der resultierenden Closed-Loop Systeme tatsächlich mit der optimalen periodischen Performance übereinstimmt.

Abstract

In this work we propose new methods for the design of economic Nonlinear Model Predictive Control (NMPC) feedback schemes for Average Output Optimal Control Problems (AOCPs). AOCPs are Optimal Control Problems (OCPs) defined on infinite time horizons with averaging performance critera as objective functionals. Such problems arise frequently for continuously operating systems such as for example power plants. Due to the infinite time horizon and the resulting intrinsic non-uniqueness of solutions, the design of appropriate NMPC schemes for AOCPs is challenging.

Often, the analysis of the closed-loop behavior of economic NMPC schemes depends on dissipativity conditions on the dynamical system and the associated performance criterion, which sometimes can be hard to check.

The methods we develop are based on the observation that periodic solutions exhibit excellent approximation properties for AOCPs, which is exploited by splitting the time horizon and the objective functional of the NMPC subproblems into a transient and a periodic part.

For the analysis of the closed-loop behavior of the resulting controller we develop new methods that essentially work by showing that the (appropriately defined) difference of two subsequent NMPC subproblem solutions vanishes asymptotically. Complementary to many other economic NMPC schemes, this approach is not based on dissipativity assumptions on the dynamical system and the associated performance criterion but rather on assumptions on existence of periodic orbits, controllability of the dynamical system, and uniqueness of the NMPC subproblem solutions itself.

As a result, we can show that the economic performance of the closed-loop system is equal to the economic performance of the optimal periodic solutions.

Furthermore, the approach is extended in two directions. First, we consider the general setting of a parameter dependent dynamical system where the parameter can be subject to change during operation. This parameter change can lead to a change in the optimal periodic behavior, in particular also to a change of the optimal period, which we take into account by including the period as an optimization variable in the NMPC subproblem. Second, we show that the approach can also be applied to systems with time-dependent periodic performance criteria.

All the described methods are implemented within the MATLAB NMPC toolkit `MLI` and are applied to a number of demanding applications. The simulation results confirm that the generated closed-loop trajectories perform economically equally well as the optimal periodic trajectories.

Contents

Introduction

Nonlinear Model Predictive Control (NMPC) is an advanced feedback control method capable of controlling dynamic processes while at the same time satisfying process constraints. The method is based on repeatedly solving Optimal Control Problems (OCPs) on finite time horizons and updating the control inputs.

More recently, the interest in NMPC applications that directly consider an economic objective criterion has been rising. Contrary to tracking NMPC applications the setup of the underlying NMPC subproblems becomes more and more crucial for the closed-loop performance of the NMPC controller. The reason for this is that the economic objective criterion usually does not satisfy a dissipativity condition that can be used to prove stability via a LYAPUNOV-argument and therefore a simple finite horizon approximation of the infinite horizon OCP does not necessarily yield a stable NMPC controller.

In this work we consider systems with the objective to optimize the average economic output. Such systems are difficult to treat in the NMPC context because they usually do not have unique solutions on the infinite time horizon since the average performance only depends on the asymptotic behavior (and not on the current control inputs).

We focus on the question on how an economic NMPC subproblem has to be set up in order to produce an economically optimal and stable feedback while dealing with the aforementioned difficulties.

Contributions

The aim of this thesis is to develop an economic NMPC scheme for Average Output Optimal Control Problems (AOCPs) that requires as little a priori knowledge of the offline problem solution as possible while meeting requirements on recursive feasibility, economic performance and stability. The main contributions and results of this thesis can be described as follows.

Approximation of AOCP Solutions with Periodic Solutions

Periodic continuation allows to interpret periodic solutions of dynamical systems on finite horizons as solutions on the infinite horizon. We use this observation for the approximation of AOCP solutions with periodic solutions. According to a result of Grammel [47], a controllability assumption on the dynamical system is sufficient to guarantee arbitrarily good approximation properties of periodic solutions to the optimal average output objective value. In fact we show that under a certain compactness assumption on the set of feasible states and controls the solution of an infinite horizon AOCP can be approximated arbitrarily well with quasi-periodic solutions (independently of any controllability).

NMPC with Periodicity Constraint for Optimal Economic Performance

We extend and modify an idea of Limon et al. [77] to define an economic NMPC controller that is based on the good approximation properties of periodic solutions for AOCP. The prediction horizon of the NMPC subproblems is split into a transient and a periodic part, allowing the process to converge towards an optimal periodic orbit. The objective functional of the underlying NMPC subproblems depends purely (up to a small regularizing term) on the economic performance during the prediction horizon and the formulation includes a periodicity constraint. The necessary a priori knowledge for the setup of the NMPC subproblems is reduced drastically: only the period is required.

Stability Analysis of the Proposed NMPC Scheme

We analyze the closed-loop behavior of the proposed NMPC scheme and consider the questions of recursive feasibility, economic performance and stability.

Complementary to the usual approach in many stability proofs for economic NMPC schemes that rely on dissipativity properties of the performance criterion and the dynamical system and/or convex performance criteria, our approach is based on assumptions on controllability of the dynamical system, compactness of the set of feasible states and controls and uniqueness and continuous dependence of the NMPC subproblem solution with respect to the initial value.

NMPC with Variable Horizon Lengths

As an extension of the NMPC controller based on periodic approximations with fixed period, we propose a controller that takes the period as a free optimization variable into account. Such a controller can be advantageous in a scenario where the dynamical system is governed by a parameter-dependent right-hand side and the optimal period of the optimal periodic operation changes with the parameter.

Our investigations show that only minor modifications of the assumptions needed for the stability proof for the fixed-period controller are necessary to prove stability also for such a controller.

NMPC for Systems with Time-Periodic Performance Criterion

We also apply the feedback scheme based on periodic solutions to systems where periodicity is induced by a time-periodic objective criterion (e.g. a periodically varying electricity price). Stability of the resulting feedback trajectory at the optimal periodic trajectory is shown.

Numerical Case Studies

To showcase the performance of the proposed NMPC controllers we consider several numerical examples of increasing complexity.

In a first part we consider three examples with periodic performance criterion. The first example is a model of a hydrostorage power plant with one state, two controls, and linear dynamics. The second example is a doubletank system, which is also considered as a benchmark problem for economically oriented NMPC in Huang and Biegler [63]. It has two states, one control and nonlinear dynamics. As a third example we consider a four-tank system, which has four states, two controls, and nonlinear dynamics.

In a second part we consider the powerkite system, which is a flying kite connected via a long tether to a motor/generator at a ground platform. By controlling the kite in a smart way it is possible to generate electric energy. This system is described by a strongly nonlinear dynamical system with 9 state and 3 control variables and possesses a highly unstable behavior. For this system we consider the scenario of varying wind conditions and apply the controller using a fixed period as well as the controller using variable periods.

Thesis Outline

The thesis consists of three parts. Part I provides the necessary theoretical background on OCPs and NMPC, in Part II our contributions are presented and in Part III we describe the implementation as well as numerical case studies.

Part I starts with the introductory Chapter 1, where we present necessary essential mathematical background leading to the definition of OCPs. Furthermore a brief overview on solution methods for OCPs with a focus on the Direct Multiple Shooting method is given.

In Chapter 2 we introduce the principles behind NMPC as a feedback generating method. Existing approaches such as Tracking NMPC and Economic Nonlinear Model Predictive Control (E-NMPC) are reviewed including the analysis of the corresponding closed-loop behavior via the well known LYAPUNOV-stability arguments.

Part II begins with Chapter 3 where we introduce Average Output Optimal Control Problems (AOCPs) and discuss the difficulties that arise from the non-uniqueness of solutions in the context of feedback generation via NMPC. We observe good approximation properties of periodic solutions for AOCPs and, based on a compactness assumption on the set of feasible states and controls, we prove the existence of quasi-periodic solutions that can approximate the optimal average output arbitrarily well.

In Chapter 4 we briefly review existing economic NMPC schemes for AOCPs and develop a novel economic NMPC scheme that is based on the observation of the good approximation properties of periodic solutions for AOCPs. Based on a controllability assumption of the dynamical system and an assumption on uniqueness and continuous dependence of the NMPC subproblem solutions with respect to the initial value we develop a novel stability-theory for the proposed NMPC controller.

In Chapter 5 the NMPC scheme is extended in two directions. First, we consider a scenario where the system dynamics are subject to changing parameters and the optimal periodic operation (also the optimal period) can change with the parameter. This is taken into account by including the period as an additional optimization variable in the NMPC subproblem. Using time transformations this results in a new NMPC scheme for which we can prove a stability result using similar methods as in the previous chapter. As a second scenario we consider the case of systems with time-periodic performance criteria and show how the

NMPC scheme of Chapter 4 can also be applied to such systems by making only slight modifications.

Part III starts with Chapter 6 where we describe the implementation of the proposed NMPC schemes based on the software package `MLI` and give a brief overview over the numerical algorithm used therein.

Chapter 7 contains three numerical case studies where we apply the NMPC scheme for systems with time-periodic performance criterion presented in Chapter 5 to three examples from the energy sector.

In Chapter 8 we present a numerical case study in which we apply the controllers for time-independent performance criteria presented in Chapter 4 for fixed period and in Chapter 5 for variable period to the example of a flying energy producing powerkite. We compare the results of both controllers for two scenarios, one in which the wind speed stays constant and one in which the wind speed changes severely.

Part I

Nonlinear Model Predictive Control

Chapter 1

Optimal Control Problems

In this chapter, we give a brief introduction into the theory of Optimal Control Problems (OCPs) and introduce some fundamental definitions needed throughout this thesis. An OCP is the problem of finding a control for a dynamical system such that a predefined objective criterion is optimized while satisfying a set of constraints.

The dynamical system is described by a set of state and control variables (x and u) and the control variables make it possible to influence the behavior of the system. For a given control, the future behavior of the system is determined by a set of differential equations and the initial state of the system. In this thesis, we consider dynamical systems that are described by Ordinary Differential Equations (ODEs). An OCP usually includes an initial value constraint and additional constraints that are imposed throughout the time horizon, which could be for example fuel limitations or safety bounds.

The fields of application for OCPs are widespread and important applications can be found among others in biology [76], economics [28, 4], engineering (in particular chemical process engineering) [45, 75, 31] and aerospace [79].

1.1 Elements of Functional Analysis

We begin the chapter with some basic function space definitions that are necessary to define OCPs as infinite dimensional optimization problems on BANACH[1] spaces. We largely follow the presentation of Gerdts [46].

Definition 1.1 (Absolutely Continuous Functions and L_∞)
Let $\mathcal{T} := [t_0, t_\mathrm{f}] \subset \mathbb{R}$ be a compact interval with $t_0 < t_\mathrm{f}$.

- A function $f : \mathcal{T} \to \mathbb{R}$ is called absolutely continuous if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any finite sequence of pairwise disjoint subintervals $(a_i, b_i) \subset \mathcal{T}$ with $a_i, b_i \in \mathcal{T}$ it holds that

$$\sum_i |b_i - a_i| \le \delta_\varepsilon \quad \Rightarrow \quad \sum_i \left| f(b_i) - f(a_i) \right| \le \varepsilon. \tag{1.1}$$

  The set of all absolutely continuous functions on $\mathcal{T}$ is denoted by $\mathcal{AC}(\mathcal{T})$.

- The space $\mathcal{AC}^n(\mathcal{T})$ is defined as the product

$$\mathcal{AC}^n(\mathcal{T}) := \underbrace{\mathcal{AC}(\mathcal{T}) \times \cdots \times \mathcal{AC}(\mathcal{T})}_{n \text{ times}}. \tag{1.2}$$

[1] Stefan Banach 1892 - 1945

- The space $L_\infty(\mathcal{T})$ is defined as the set of all measurable functions $f : \mathcal{T} \to \mathbb{R}$ that are essentially bounded:

$$\operatorname*{ess\,sup}_{t\in\mathcal{T}} \left|f(t)\right| := \inf_{\substack{N\subset\mathcal{T}\\ N \text{ has measure zero}}} \sup_{t\in\mathcal{T}\setminus N} \left|f(t)\right| < \infty. \tag{1.3}$$

- The space $L^n_\infty(\mathcal{T})$ is defined as the product space

$$L^n_\infty(\mathcal{T}) := \underbrace{L_\infty(\mathcal{T}) \times \cdots \times L_\infty(\mathcal{T})}_{n \text{ times}}. \tag{1.4}$$

- The space $\mathcal{AC}^n_{\text{loc}}([t_0,\infty))$ is defined as the space of all functions $f : [t_0,\infty) \to \mathbb{R}^n$ such that on every compact subinterval $\tilde{\mathcal{T}} \subset [t_0,\infty)$ for the restriction it holds that

$$f|_{\tilde{\mathcal{T}}} \in \mathcal{AC}^n(\tilde{\mathcal{T}}). \tag{1.5}$$

- The space $L^n_{\infty,\text{loc}}([t_0,\infty))$ is defined as the space of all functions $f : [t_0,\infty) \to \mathbb{R}^n$ such that on every compact subinterval $\tilde{\mathcal{T}} \subset \mathbb{R}_{\geq 0}$ for the restriction it holds that

$$f|_{\tilde{\mathcal{T}}} \in L^n_\infty(\tilde{\mathcal{T}}). \tag{1.6}$$

△

It is well known that $\mathcal{AC}(\mathcal{T})$ and $L_\infty(\mathcal{T})$ endowed with the norm $\|f\|_\infty := \operatorname{ess\,sup}_{t\in\mathcal{T}} f(t)$ are BANACH spaces (see for example [103]). If it is clear from the context, we sometimes omit the interval $\mathcal{T}$ for notational convenience and write L^n_∞ for $L^n_\infty(\mathcal{T})$. The set of continuous functions from $\mathcal{T}$ or $\mathbb{R}$ to $\mathbb{R}$ is denoted by $\mathcal{C}(\mathcal{T})$ respectively $\mathcal{C}$.

In order to use the concepts of differentiability for mappings defined between BANACH spaces, we introduce the concept of the FRÉCHET[2] differentiability.

Definition 1.2 (FRÉCHET Differentiability)
Let $(V, \|\cdot\|_V), (W, \|\cdot\|_W)$ be BANACH spaces and $U \subset V$ an open subset. A function $f : U \to W$ is called FRÉCHET differentiable at $x \in U$, if there exists a continuous, linear operator $A : V \to W$ such that

$$\lim_{h\to 0} \frac{\left\|f(x+h) - f(x) - Ah\right\|_W}{\|h\|_V} = 0. \tag{1.7}$$

If such a linear operator A exists for $x \in U$, then it is unique, and we write $Df(x)$ and call it the FRÉCHET-derivative of f at x. △

[2] Maurice René Fréchet 1878 - 1973

1.2 Dynamical Systems

In this section, we define dynamical systems as a mathematical model to describe the evolution of continuous processes. We introduce important stability concepts, which are helpful to analyze the asymptotic behavior of such systems.

1.2.1 Initial Value Problems and the PICARD-LINDELÖF Theorem

A dynamical system can be seen as description of a process (a function of time) where future states of the process follow from the current state according to an evolution rule. In this thesis, we consider dynamical systems, where the states are represented by vectors in $\mathbb{R}^{n_x}$ and the evolution rule is defined by an Ordinary Differential Equation (ODE). With an explicit ODE, the time derivative of a process is a function of time and state and we write:

$$\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t)). \tag{1.8}$$

Due to this notation for ODEs, the function f is sometimes also called the right-hand side. If the right-hand side f is only a function of the state (there exists an $\tilde{f} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ such that $f(t,x) = \tilde{f}(x)$ for all possible arguments (t,x) in the domain of f), we call the ODE **“autonomous”**. With the function f and some time/state point we can define

Definition 1.3 (Initial Value Problem (IVP))
An Initial Value Problem (IVP) is a function $f : \Sigma \subset \mathbb{R} \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ (where Σ is an open subset) together with a pair $(t_0, x_0) \in \Sigma$ called the initial condition. For a given compact interval $\mathcal{T} \subset \mathbb{R}$ with $t_0 \in \mathcal{T}$, we call a function $\boldsymbol{x} \in \mathcal{AC}^n(\mathcal{T})$ a solution of the IVP if $\boldsymbol{x}(t_0) = x_0$ and $\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t))$ holds for almost all $t \in \mathcal{T}$. △

If the function f satisfies some regularity conditions, the PICARD-LINDELÖF[3] theorem provides local existence and uniqueness statements for the IVP.

Theorem 1.4 (Local PICARD-LINDELÖF Theorem)
Let $f : \Sigma \subset \mathbb{R} \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ be continuous and locally LIPSCHITZ continuous in the second variable on Σ:

$$\left\| f(t,x) - f(t,y) \right\| \leq M \left\| x - y \right\| \text{ for all } (t,x), (t,y) \in \Sigma.$$

Then there exists an open interval $I \subset \mathbb{R}$ containing t_0 and a unique, differentiable function $\boldsymbol{x} : I \to \mathbb{R}^{n_x}$ such that $\boldsymbol{x}(t_0) = y_0$ and $\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t))$ for all $t \in I$.

Proof See for example [110, Theorem 2.2]. □

This local version ensures existence of a solution for IVPs at least for a short time interval. If the domain Σ of f contains a set of the form $[a,b] \times \mathbb{R}^{n_x}$, the unique local solutions can be glued together and the existence of solution on the whole interval $[a,b]$ can be guaranteed:

[3] Émile Picard 1856 - 1941, Ernst Leonard Lindelöf 1870 - 1946

Theorem 1.5 (Global PICARD-LINEDLÖF THEOREM])
Let $\mathcal{T}$ be a compact interval and Σ an open subset of $\mathbb{R} \times \mathbb{R}^{n_x}$ such that $\mathcal{T} \times \mathbb{R}^{n_x} \subset \Sigma$. Let $f : \Sigma \to \mathbb{R}^{n_x}$ be continuous and uniformly LIPSCHITZ continuous in the second variable on Σ:

$$\left\| f(t,x) - f(t,y) \right\| \leq M \left\| x - y \right\| \text{ for all } (t,x), (t,y) \in \Sigma.$$

Then there exists a unique differentiable function $\boldsymbol{x} : \mathcal{T} \to \mathbb{R}^{n_x}$ such that $\boldsymbol{x}(t_0) = y_0$ and $\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t))$ for all $t \in \mathcal{T}$.

Proof See for example [2, Theorem 4.1.4]. □

In the following, we will assume that if the solution of an IVP exists, it will always be unique. The abbreviation $\Phi(\tau; x_0, t_0)$ stands for the value of the solution of the IVP $\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t))$ with initial value $\boldsymbol{x}(t_0) = x_0$ at time $\tau \in \mathbb{R}$. The mapping Φ can be interpreted as the time-dependent flow-mapping of the time-dependent vector-field $\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t))$. GRÖNWALL'S[4] Lemma implies the continuity of the flow-mapping for sufficiently smooth right-hand sides f (see for example Khalil [71, Theorem 3.5]).

1.2.2 Stability of Solutions

The question of how small changes of the initial value affect the (long-term) behavior of the solution of an IVP is analyzed in stability theory. In particular, stability concepts are of great importance for analyzing robustness properties of systems that are subject to perturbations.

The concept of class kappa functions as comparison functions, which were first introduced by Hahn [57, 56] and became a standard tool in stability theory for nonlinear systems with the work Sontag [106], are useful to formalize the definition of stability.

Definition 1.6 (Class $\mathcal{K}$ Functions)
We define the following subsets of $\mathcal{C}(\mathbb{R}_{\geq 0})$ and $\mathcal{C}(\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})$:

$$\begin{aligned}
\mathcal{K} &:= \{\alpha \in \mathcal{C}(\mathbb{R}_{\geq 0}) : \alpha \text{ is strictly increasing and } \alpha(0) = 0\}, \\
\mathcal{K}_\infty &:= \{\alpha \in \mathcal{K} \text{ and } \lim_{t\to\infty} \alpha(t) = \infty\}, \\
\mathcal{L} &:= \{\alpha \in \mathcal{C}(\mathbb{R}_{\geq 0}) : \alpha \text{ is strictly decreasing and } \lim_{t\to\infty} \alpha(t) = 0\}, \\
\mathcal{KL} &:= \{\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} : \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K} \text{ and } \beta(t, \cdot) \in \mathcal{L} \text{ for all } t \in \mathbb{R}_{\geq 0}\}.
\end{aligned}$$
△

Remark 1.1 Class $\mathcal{K}$ functions have the following properties concerning inversion and composition: Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{L}$. Then it holds:

- α_1 is invertible and for the inverse it holds $\alpha_1^{-1} \in \mathcal{K}_\infty$,
- $\alpha_1 \circ \alpha_2 \in \mathcal{K}_\infty$,
- $\alpha \circ \alpha_1 \in \mathcal{L}$,

[4] Thomas Hakon Grönwall 1877 - 1932

- $\alpha_1 \circ \alpha \in \mathcal{L}$.

Furthermore, for $\alpha_3 \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ it holds that $\alpha_3 \circ \beta \in \mathcal{KL}$. △

We introduce the following stability concepts for IVP*s* with autonomous ODEs.

Definition 1.7 (Stability of IVP Solutions)
A solution $\boldsymbol{x}_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_x}$ of the IVP $\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t))$ with initial value $\boldsymbol{x}(0) = x_0 \in \mathbb{R}^{n_x}$ is said to be

- **Lyapunov [5] stable**, if for any $\varepsilon \geq 0$ there exists $\delta \geq 0$ such that

$$\|x - \boldsymbol{x}_{\text{ref}}(t)\| \leq \varepsilon \quad \Rightarrow \quad \Phi(\tau; x, t) \text{ exists and } \|\Phi(\tau; x, t) - \boldsymbol{x}_{\text{ref}}(t)\| \leq \delta \text{ for all } \tau \geq t,$$

- **locally asymptotically stable**, if there exists a function $\beta \in \mathcal{KL}$ and an open set $U \subset \mathbb{R} \times \mathbb{R}^{n_x}$ containing $\{(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} : x = \boldsymbol{x}_{\text{ref}}(t)\}$ such that for any $(t, x) \in U$ it holds
  - $\Phi(\tau; x, t)$ exists for all $\tau \geq t$,
  - $\|\boldsymbol{x}_{\text{ref}}(\tau) - \Phi(\tau; x, t)\| \leq \beta(\|x - \boldsymbol{x}_{\text{ref}}(t)\|, \tau - t)$ for all $\tau \geq t$.
- **locally uniformly asymptotically stable**, if there exists $\delta \geq 0$ and a function $\beta \in \mathcal{KL}$ such that
  - $\|x - \boldsymbol{x}_{\text{ref}}(t)\| \leq \delta \quad \Rightarrow \quad \Phi(\tau; x, t)$ exists and is unique for all $\tau \geq t$,
  - $\|x - \boldsymbol{x}_{\text{ref}}(t)\| \leq \delta \quad \Rightarrow \quad \|\Phi(\tau; x, t) - \boldsymbol{x}_{\text{ref}}(t)\| \leq \beta(\|x - \boldsymbol{x}_{\text{ref}}(t)\|, \tau - t)$ for all $\tau \geq t$. △

To illustrate the concept of stability, we include some simple examples.

Example: Linear Systems

Let us consider the case of an IVP with a linear right-hand side, i.e. $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t)$ where $A \in \mathbb{R}^{n_x \times n_x}$ is a matrix. Let $x_0 \in \mathbb{R}^{n_x}$ be the initial value at time t_0. It is well known that the unique solution of this IVP is defined for all times and can be represented using the matrix exponential:

$$\Phi(\tau; x_0, t_0) = e^{A(\tau - t_0)} x_0. \tag{1.9}$$

It can be shown that for the induced matrix norm ($\|A\| := \sup\{\|Ax\|, x \in \mathbb{R}^{n_x} \text{ with } \|x\| = 1\}$) and the matrix exponential it holds:

$$\left\| e^{B} \right\| \leq e^{\lambda_{\max}(B)}, \text{ for all } B \in \mathbb{R}^{n_x \times n_x}, \tag{1.10}$$

where $\lambda_{\max}(B)$ denotes the largest real part of the eigenvalues of the matrix B. It follows for the solution

$$\|\Phi(\tau; x_0, t_0)\| \leq \left\| e^{A(\tau - t_0)} \right\| \|x_0\| \leq e^{\lambda_{\max}(A)(\tau - t_0)} \|x_0\|. \tag{1.11}$$

[5] Aleksandr Mikhailovich Lyapunov 1857 - 1918

This shows that if the largest real part of the eigenvalues of A is negative, the constant zero-solution ($\boldsymbol{x}_{\text{ref}}(\cdot) \equiv 0$) of the IVP $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t)$ with initial value $\boldsymbol{x}(0) = 0$ is locally uniformly asymptotically stable cf. Figure 1.1a.

Example: Steady-States of Affine Linear Systems

Let $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + b$ be an affine linear dynamical system with $A \in \mathbb{R}^{n_x \times n_x}$ and $b \in \mathbb{R}^{n_x}$. Any $x_0 \in \mathbb{R}^{n_x}$ with $Ax_0 + b = 0$ is called a steady-state for the system, because the right hand side vanishes and consequently for any $\tau \geq t_0$ it holds that $\Phi(\tau; x_0, t_0) = x_0$. By substituting $\tilde{\boldsymbol{x}}(t) := \boldsymbol{x}(t) - x_0$, the original ODE transforms to the equivalent linear system $\dot{\tilde{\boldsymbol{x}}}(t) = A\tilde{\boldsymbol{x}}(t)$ and the question is reduced to the question of stability of the zero solution for the linear system.

As we have seen in the previous example, 0 is a locally uniformly stable solution of a linear dynamical system if the eigenvalues of A have negative real part. Therefore, in this case the steady-state x_0 is locally uniformly asymptotically stable for the original system $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + b$.

Example: Lyapunov Stable but not Locally Asymptotically Stable

For $r_0 \geq 0$ and $\varphi_0 \in [0, 2\pi]$ consider the IVP

$$\begin{pmatrix} \dot{\boldsymbol{x}}_1(t) \\ \dot{\boldsymbol{x}}_2(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_2(t) \\ -\boldsymbol{x}_1(t) \end{pmatrix} \text{ with initial value } \boldsymbol{x}(t_0) = r_0 \begin{pmatrix} \sin(\varphi_0) \\ \cos(\varphi_0) \end{pmatrix} \in \mathbb{R}^2. \tag{1.12}$$

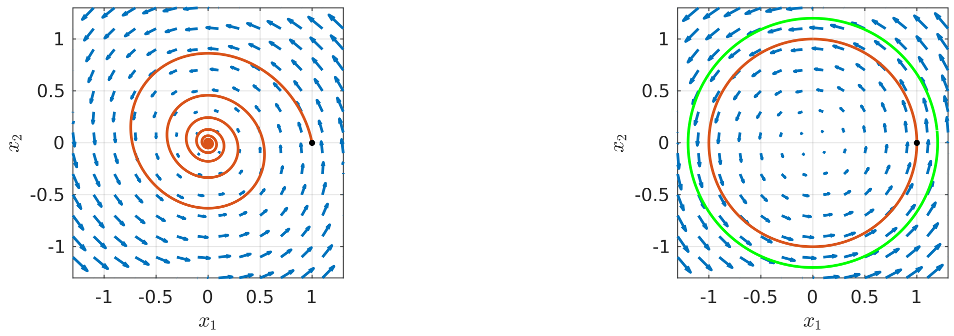
The solution of this system is then given by

$$\Phi(\tau; x_0, t_0) = r_0 \begin{pmatrix} \sin(\tau + \varphi_0) \\ \cos(\tau + \varphi_0) \end{pmatrix}. \tag{1.13}$$

The solutions move along circles of radius r_0 counterclockwise with constant angular velocity (Figure 1.1b). It follows that for any two solutions $\boldsymbol{x}_1, \boldsymbol{x}_2$, the difference $\|\boldsymbol{x}_1(\tau) - \boldsymbol{x}_2(\tau)\|$ will be independent of τ and therefore constant. This means that for all $\tau \geq t_0$ and for any $x, y \in \mathbb{R}^2$ it holds that

$$\left\| \Phi(\tau; x, t_0) - \Phi(\tau; y, t_0) \right\| = \left\| x - y \right\|, \tag{1.14}$$

and consequently any solution is Lyapunov stable. Equation (1.14) also implies that no solution is locally asymptotically stable.



a) $A = \begin{pmatrix} -0.2 & -1 \\ 1 & 0 \end{pmatrix}$.

b) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Figure 1.1: The above pictures show two linear systems that evolve according to $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t)$. For the system on the left, 0 is a locally uniformly asymptotically stable solution because the solution $\Phi(\tau; x_0, t_0)$ converges to zero for all $x_0 \in \mathbb{R}^2$. For the system on the right, any circle-parametrized solution is LYAPUNOV-stable but not asymptotically stable, because the solutions move on circular trajectories around the center with constant angular velocity and distance to the center.

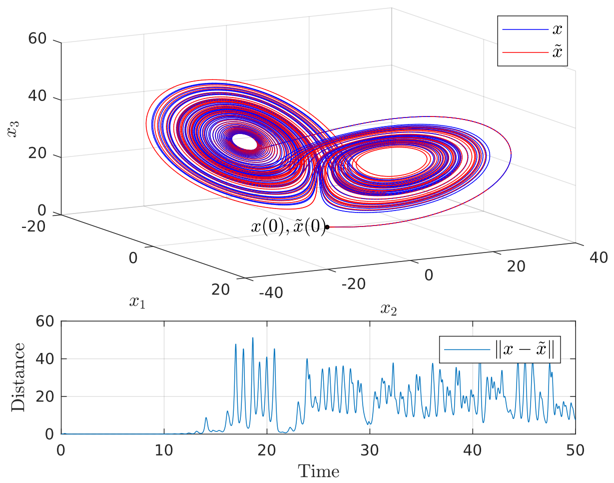


Figure 1.2: The above picture shows two sample solutions of the LORENZ system on the time interval $[0, 50]$. The blue trajectory corresponds to the solution $\boldsymbol{x}$ with initial value $\boldsymbol{x}(0) = (1,1,1)^T$ and the red trajectory corresponds to the solution $\tilde{\boldsymbol{x}}$ with a slightly perturbed initial value $\tilde{\boldsymbol{x}}(0) = (1.01, 1.01, 1.01)^T$. Below, the distance $\|\boldsymbol{x}(\tau) - \tilde{\boldsymbol{x}}(\tau)\|$ is plotted. It can be seen that although the two initial values are quite close, they soon have a large distance.

Example: The LORENZ System

The LORENZ[6] system is the prime example of an ODE showing chaotic solution behavior. The system is three dimensional and its evolution is described by the right-hand side

$$f\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}\sigma(x_2-x_1)\\x_1(\rho-x_3)-x_2\\x_1(x_2-\beta x_3)\end{pmatrix} \tag{1.15}$$

with the parameters $\sigma=10, \beta=\frac{8}{3}, \rho=28$. It can be shown that for many initial values, the solutions of this system behave chaotic, i.e. a small change of the initial value leads to a large difference in later states known as the Butterfly Effect and do not satisfy any of the above stability definitions. This effect is illustrated in Figure 1.2, where the evolution of two sample solutions with close initial values is depicted as well as the evolution of their distance.

Although being a deterministic system, making a prediction of the future behavior is very difficult because it would require perfect knowledge of the initial state. Such systems are the subject of chaos theory [109].

1.2.3 LYAPUNOV Functions

LYAPUNOV functions are of outstanding importance for stability theory. The method of analyzing stability properties of dynamical systems by means of a LYAPUNOV function was first introduced by LYAPUNOV in his Ph.D. thesis [81] and is also known as "LYAPUNOV'S second method". Roughly speaking, a LYAPUNOV function is a function that assigns a non-negative value to each possible state of the system with the additional properties that it is strictly decreasing along solutions of the dynamical system and that its minimum is attained at some reference (steady-)state of the system. Such functions can be used to show that every solution of the system is converging to the reference state.

If such a function exists, its values can be interpreted as an analogue to some kind of energy that is stored in the system. The energy is dissipating with time and the system is converging to the state with minimal energy, which is the reference state.

We introduce a variation of this method that can also be used to prove stability for time varying solutions of non-autonomous systems.

Let $\boldsymbol{x}_{\text{ref}}:\mathbb{R}_{\geq 0}\to\mathbb{R}^{n_x}$ be a solution of the IVP $\dot{\boldsymbol{x}}(t)=f(t,\boldsymbol{x}(t))$ with initial value $\boldsymbol{x}(0)=x_0$.

Definition 1.8 (LYAPUNOV-Like Functions)
Let $U\supset\{(t,x)\in\mathbb{R}_{\geq 0}\times\mathbb{R}^{n_x}: x=\boldsymbol{x}_{\text{ref}}(t)\}$ be an open set and $V:U\to\mathbb{R}_{\geq 0}\cup\{+\infty\}$ a continuous function. The function V is called a LYAPUNOV-like function for the solution $\boldsymbol{x}_{\text{ref}}$, if there exist functions $\alpha_1,\alpha_2\in\mathcal{K}_\infty$ and $\alpha_3\in\mathcal{K}$ such that

- for all $(t,x)\in U$ the following inequalities hold:

$$\alpha_1(\|x-\boldsymbol{x}_{\text{ref}}(t)\|)\leq V(t,x)\leq\alpha_2(\|x-\boldsymbol{x}_{\text{ref}}(t)\|), \tag{1.16}$$

[6] Edward Norton Lorenz 1917 - 2008

- for all $(t,x) \in U$ the solution $\Phi(\tau;x,t)$ exists for all times $\tau \geq t$, the solution stays inside U and satisfies

$$\frac{\partial}{\partial \tau} V(\tau, \Phi(\tau;x,t)) \leq -\alpha_3(\|\boldsymbol{x}_{\text{ref}}(\tau) - \Phi(\tau;x,t)\|). \tag{1.17}$$

△

The existence of a LYAPUNOV-like function for a reference solution now implies asymptotic stability for this solution.

Lemma 1.9 (Existence of LYAPUNOV-Like Function Implies Stability)
Let $V : U \to \mathbb{R}_{\geq 0}$ be a LYAPUNOV-like function for the solution $\boldsymbol{x}_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_x}$ of the IVP $\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t))$ with initial value $\boldsymbol{x}(0) = x_0$. Then $\boldsymbol{x}_{\text{ref}}$ is locally asymptotically stable.

Proof From (1.16) and (1.17), for any $(t,x) \in U$ and $\tau \geq t$ it follows

$$\frac{\partial}{\partial \tau} V(\tau, \Phi(\tau;x,t)) \leq -\alpha_3\left(\alpha_2^{-1} V(\tau, \Phi(\tau;x,t))\right). \tag{1.18}$$

This implies that $\tau \mapsto V(\tau, \Phi(\tau;x,t))$ is bounded from above by the solution of the IVP $\dot{\boldsymbol{y}}(\tau) = -\alpha_3(\alpha_2^{-1}(\boldsymbol{y}(\tau))$ with $\boldsymbol{y}(t) = V(t,x)$. Because $\alpha_3 \circ \alpha_2^{-1} \in \mathcal{K}$, according to Lemma A.2 there exists a $\mathcal{KL}$-function β that is an upper bound for those IVP solutions (and consequently also for $\tau \mapsto V(\tau, \Phi(\tau;x,t))$):

$$V(\tau, \Phi(\tau;x,t)) \leq \beta(V(t,x), \tau - t). \tag{1.19}$$

In combination with the inequalities (1.16) we get

$$\|\Phi(\tau;x,t) - \boldsymbol{x}_{\text{ref}}(\tau)\| \leq \alpha_1^{-1}\left(\beta(\alpha_2(\|x - \boldsymbol{x}_{\text{ref}}(t)\|), \tau - t)\right). \tag{1.20}$$

Since the function $(a,b) \mapsto \alpha_1^{-1}\left(\beta(\alpha_2(a), b)\right)$ is again a $\mathcal{KL}$ function (see Remark (1.1)) the proof is finished. □

In the context of feedback control, the following relaxed version of the previous Lemma is of interest. It can occur that the continuous condition (1.17) can't be checked but instead a discrete version for a sequence of times $(t_i = t_0 + i\Delta T)_{i \in \mathbb{N}}$ holds:

$$V(t_{i+1}, \Phi(t_{i+1};x,t_i)) \leq V(t_i, x) - \alpha(\|x - \boldsymbol{x}_{\text{ref}}(t_i)\|). \tag{1.21}$$

The discrete analog to Definition 1.8

Definition 1.10 (Discrete LYAPUNOV-Like Function)
Let $U \supset \{(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} : x = \boldsymbol{x}_{\text{ref}}(t)\}$ be an open set and $V : U \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ a continuous function. V is called a discrete LYAPUNOV-like function for the solution $\boldsymbol{x}_{\text{ref}}$, if for a sequence of times $(t_i = t_0 + i\Delta T)_{i \in \mathbb{N}}$ with $\Delta T > 0$ there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 \in \mathcal{K}$ such that

- for all $(t,x) \in U$ the following inequalities hold:

$$\alpha_1(\|x - \boldsymbol{x}_{\text{ref}}(t)\|) \leq V(t,x) \leq \alpha_2(\|x - \boldsymbol{x}_{\text{ref}}(t)\|), \tag{1.22}$$

- for all $i \in \mathbb{N}$ and all $(t_i, x) \in U$ the solution $\Phi(\tau; x, t_i)$ exists for all times $\tau \geq t_i$, the solution stays inside U and satisfies

$$V(t_{i+1}, \Phi(t_{i+1}; x, t_i)) \leq V(t_i, x) - \alpha_3(\|x - \boldsymbol{x}_{\mathrm{ref}}(t_i)\|), \tag{1.23}$$

- for $(t_i, x) \in U$ and $\tau \in [t_i, t_{i+1})$ it holds

$$V(\tau, \Phi(\tau; x, t_i)) \leq V(t_i, x). \tag{1.24}$$

△

With this definition, we can formulate a discrete version of Lemma 1.9:

Lemma 1.11 (Existence of Discrete LYAPUNOV-Like Function Implies Stability)
Let $V : U \to \mathbb{R}_{\geq 0}$ be a discrete LYAPUNOV-like function for the solution $\boldsymbol{x}_{\mathrm{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_x}$ of the IVP $\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t))$ with initial value $\boldsymbol{x}(0) = x_0$. Then there exists a $\mathcal{KL}$ function β such that for all $(t_i, x) \in U$ and $\tau \geq t_i$ it holds that

$$\|\boldsymbol{x}_{\mathrm{ref}}(\tau) - \Phi(\tau; x, t_i)\| \leq \beta(\|\boldsymbol{x}_{\mathrm{ref}}(t) - x\|, \tau - t_i). \tag{1.25}$$

Proof Because of (1.22) and (1.23), there exists a $\mathcal{K}_\infty$ function $\tilde{\alpha}$ such that for any $(t_i, x) \in U$ it holds

$$V(t_{i+1}, \Phi(t_{i+1}; x, t_i)) \leq V(t_i, x) - \tilde{\alpha}(V(t_i, x)). \tag{1.26}$$

Lemma A.3 guarantees the existence of a $\mathcal{KL}$ function $\tilde{\beta}$ with the property that

$$V(t_j, \Phi(t_j; x, t_i)) \leq \tilde{\beta}(V(t_i, x), t_j - t_k) \tag{1.27}$$

holds for all $i, j \in \mathbb{N}$ and $x \in \mathbb{R}^{n_x}$ such that $j \geq i$ and $(t_i, x) \in U$. Now let $\tau \geq t_i$ be arbitrary. Then, for the unique $k \in \mathbb{N}$ with $\tau \in [t_k, t_{k+1})$ we can calculate

$$\begin{aligned} V(\tau, \Phi(\tau; x, t_i)) \underbrace{\leq}_{(1.24)} & V(t_k, \Phi(t_k; x, t_i)) \leq \tilde{\beta}(V(t_i, x), t_k - t_i) \\ & \leq \tilde{\beta}(V(t_i, x), \tau - t_{i+1}) \text{ (because of } \tau - t_{i+1} \leq t_k - t_i \text{ and } \tilde{\beta} \in \mathcal{KL}) \\ & = \tilde{\beta}(V(t_i, x), \tau - t_i - \Delta T). \end{aligned}$$

For $(s, t) \in \mathbb{R}_{\geq 0} \times [\Delta T, \infty)$ we define the function $\hat{\beta}$ as follows:

$$\hat{\beta}(s, t) := \tilde{\beta}(s, t - \Delta T) + \frac{2s\Delta T}{t + \Delta T}. \tag{1.28}$$

By definition, the function $\hat{\beta}$ satisfies the inequality

$$V(\tau, \Phi(\tau; x, t_i)) \leq \hat{\beta}(V(t_i, x), \tau - t_i) \tag{1.29}$$

for all $\tau \geq t_i + \Delta T$. We can extend $\hat{\beta}$ continuously to $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ by setting

$$\hat{\beta}(s,t) := \hat{\beta}(s,\Delta T) + \frac{2s\Delta T}{t+\Delta T} \text{ for } (s,t) \in \mathbb{R}_{\geq 0} \times [0,\Delta T). \tag{1.30}$$

It can be checked that the resulting function $\hat{\beta}$ is a $\mathcal{KL}$ function. By definition of $\hat{\beta}$, which in particular implies $s \leq \hat{\beta}(s,t)$ for all $t \in [0,\Delta T)$, and the property (1.24) it follows

$$V(\tau, \Phi(\tau; x, t_i)) \leq V(t_i, x) \leq \hat{\beta}(V(t_i, x), \tau - t_i) \tag{1.31}$$

for all $\tau \in [t_i, t_{i+1})$. This shows that inequality (1.29) also holds for $\tau \in [t_i, t_{i+1})$. Together with the inequalities (1.22), $\hat{\beta}$ can now be used to construct a $\mathcal{KL}$ function β that satisfies (1.26) and the proof is finished. □

1.3 Discrete Time Systems

Discrete time systems can be seen as the discrete equivalent of dynamical systems. In such systems, the evolution is described by a transition map $F: \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ which maps the current state $x \in \mathbb{R}^{n_x}$ to the state at the next time instant, which is denoted by $x^+ := F(x)$. Since the "evolution rule" F is defined on $\mathbb{R}^{n_x}$, for any initial value $x_0 \in \mathbb{R}^{n_x}$ there automatically exists a unique sequence $(y_k)_{k\in\mathbb{N}}$ that solves the discrete initial value problem

$$y_0 = x_0 \text{ and } y_{k+1} = F(y_k) \text{ for all } k \in \mathbb{N}. \tag{1.32}$$

For a repeated application of the function F we also write $F^n := \underbrace{F \circ \cdots \circ F}_{n \text{ times}}$.

A dynamical system $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t))$ can be transformed into a discrete time system using a equidistant time grid $(t_i = i\Delta T)_{i\in\mathbb{N}}$.

The evolution rule F is then just defined by using the flow mapping Φ of the dynamical system (we suppose that the solution for the dynamical system exists for all initial values and all times, i.e. the flow map Φ is well defined):

$$F(x) = \Phi(\Delta T; x, 0). \tag{1.33}$$

It is clear that for any given initial value x_0 this evolution rule will generate a sequence $(x_k)_{k\in\mathbb{N}}$ that corresponds to the sequence $(\Phi(k\Delta T; x_0, 0))_{k\in\mathbb{N}}$.

Stability of Solutions

Similar to the stability theory of IVPs, the question of the asymptotic behavior of discrete time systems is of great interest. The stability notion of Definition 1.7 can be transferred to the discrete case:

Definition 1.12
A solution $(y_k)_{k\in\mathbb{N}}$ of the discrete time system $x^+ = F(x)$ with initial value $y_0 = x_0$ is called

- **LYAPUNOV-stable**, if for any $\varepsilon \geq 0$ there exists $\delta \geq 0$ such that

$$\left\|x - y_k\right\| \leq \varepsilon \Rightarrow \left\|F^j(x) - y_{k+j}\right\| \leq \delta \text{ for all } j \in \mathbb{N},$$

- **locally uniformly asymptotically stable**, if there exists a function $\beta \in \mathcal{KL}$ and $\delta > 0$ such that for any $k \in \mathbb{N}$ and any $x \in \mathbb{R}^{n_x}$ with $\left\|x - y_k\right\| < \delta$ it holds

$$\left\|y_{k+j} - F^j(x)\right\| \leq \beta\left(\left\|x - y_k\right\|, j\right) \text{ for all } j \in \mathbb{N}. \tag{1.34}$$

△

As in the continuous case, LYAPUNOV-like functions constitute an important tool in the analysis of the asymptotic behavior of discrete time systems.

Definition 1.13 (LYAPUNOV-Like Functions for Discrete Time Systems)
Let $(y_k)_{k\in\mathbb{N}}$ be a solution of the discrete time system $x^+ = F(x)$ and let $U \subset \mathbb{N} \times \mathbb{R}^{n_x}$ be a neighborhood of $\{(k, y_k), k \in \mathbb{N}\}$. A continuous function $V : U \to \mathbb{R}_{\geq 0}$ is called a LYAPUNOV-like function for the solution $(y_k)_{k\in\mathbb{N}}$ if there exist $\mathcal{K}_\infty$ functions α_1, α_2 and a $\mathcal{K}$ function α_3 such that

- U is invariant under F, i.e. $(k, x) \in U \Rightarrow (k+1, F(x)) \in U$,
- for all $(k, x) \in U$ it holds

$$\alpha_1\left(\left\|x - y_k\right\|\right) \leq V(k, x) \leq \alpha_2\left(\left\|x - y_k\right\|\right), \tag{1.35}$$

- $V(k+1, F(x)) \leq V(k, x) - \alpha_3(V(k, x))$ for all $(k, x) \in U$. △

The corresponding stability result is the following.

Lemma 1.14 (Stability of Discrete Time Systems)
If $V : U \to \mathbb{R}_{\geq 0}$ is a LYAPUNOV-like function for a solution solution $(y_k)_{k\in\mathbb{N}}$ of the discrete time system $x^+ = F(x)$, then $(y_k)_{k\in\mathbb{N}}$ is a uniformly asymptotic stable solution of the discrete time system.

Proof Application of Lemma A.3 shows the existence of a $\mathcal{KL}$-function β such that

$$V(k+j, F^j(x)) \leq \beta(V(k, x), j) \text{ for all } (k, x) \in U \text{ and } j \in \mathbb{N} \tag{1.36}$$

holds. The inequalities (1.35) finish the proof. □

1.4 Optimal Control Problems as Infinite Dimensional Optimization Problems

Often, the evolution of the dynamical process not only depends on the current state but also on a control input $u \in \mathbb{R}^{n_u}$ which can be used to influence the system. For a given control function $\boldsymbol{u} \in \mathrm{L}_\infty^{n_u}(\mathcal{T})$, the time-derivative of such a controlled system can then be expressed as $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t))$. Together with an initial value this gives rise to an IVP. The possibility to influence the system via an external input function $\boldsymbol{u}$ motivates the question of the "best" way to influence a system such that a prescribed objective is optimized.

We now introduce the formal definition of an OCP on the compact time interval $\mathcal{T} = [t_0, t_\mathrm{f}] \subset \mathbb{R}$ in general standard form.

Let

$$
\begin{aligned}
f &: \mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x} && (1.37a)\\
\ell &: \mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}, && (1.37b)\\
m &: \mathbb{R}^{n_x} \to \mathbb{R}, && (1.37c)\\
c &: \mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_c}, && (1.37d)\\
\Psi &: (\mathbb{R}^{n_x} \times \mathbb{R}^{n_u})^{m_\Psi} \to \mathbb{R}^{n_\Psi} && (1.37e)
\end{aligned}
$$

be sufficiently smooth functions.

Definition 1.15 (Standard Optimal Control Problem on a Finite Time Horizon)
An Optimal Control Problem (OCP) is an infinite dimensional optimization problem of the following form

$$
\begin{aligned}
\min_{\substack{\boldsymbol{x} \in \mathcal{AC}^{n_x}(\mathcal{T}),\\ \boldsymbol{u} \in \mathrm{L}_\infty^{n_u}(\mathcal{T})}} \quad & \varphi(\boldsymbol{x}, \boldsymbol{u}) := \int_{t_0}^{t_\mathrm{f}} \ell(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\mathrm{d}\tau + m(\boldsymbol{x}(t_\mathrm{f})) && (1.38a)\\
\text{s.t.} \quad & \boldsymbol{x}(t_0) = x_0, && (1.38b)\\
& \dot{\boldsymbol{x}}(\tau) = f(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \quad \tau \in \mathcal{T}, && (1.38c)\\
& 0 \le c(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \quad \tau \in \mathcal{T}, && (1.38d)\\
& 0 = \Psi(\boldsymbol{x}(t_0), \boldsymbol{u}(t_0), \ldots, \boldsymbol{x}(t_{m_\Psi}), \boldsymbol{u}(t_{m_\Psi})). && (1.38e)
\end{aligned}
$$

The optimal objective value of this OCP is denoted by $\varphi(x_0)$. The mapping $x \mapsto \varphi(x)$ is called the optimal value function. △

We call a pair $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}(\mathcal{T}) \times \mathrm{L}_\infty^{n_u}(\mathcal{T})$ admissible for problem (1.38), if it satisfies constraints (1.38b)-(1.38e). The performance criterion (1.38a) consists of an integral contribution (the LAGRANGE[7]-type objective) with integrand $\ell(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau))$ and of an end-point contribution $m(\boldsymbol{x}(t_\mathrm{f}))$, which is called MAYER[8]-type objective. An OCP with an objective function of this type is called a "BOLZA[9]-type" OCP.
The first constraint (1.38b) is the initial value constraint. Constraint (1.38c) is the ODE-constraint and has to be satisfied almost everywhere on $\mathcal{T}$. The expression $\dot{\boldsymbol{x}}(t)$ can be evaluated for almost all $t \in \mathcal{T}$ because the derivative of an absolutely continuous function exists almost everywhere. The path constraint (1.38d) is formulated as inequality constraint,

[7] Joseph-Louis Lagrange 1736 - 1813
[8] Christian Gustav Adolph Mayer 1839 - 1908
[9] Oskar Bolza 1887 - 1942

which in particular also includes equality constraints and has to be satisfied almost everywhere on $\mathcal{T}$. In most OCPs throughout this thesis, we consider path constraints that are time independent and decoupled in state and control, i.e. of the form

$$c(\tau, x, u) = \begin{pmatrix} c^x(x) \\ c^u(u) \end{pmatrix}. \tag{1.39}$$

This case also includes the so called simple bound constraints, which are box-constraints on state and control variables imposed on the whole time horizon. Constraint (1.38e) is a coupled interior point constraint which is evaluated at a finite grid of time points $t_0 < \ldots < t_{m_\Psi}$ in $\mathcal{T}$. Note that for notational convenience we often state the initial value constraint (1.38b) separately from Ψ. Another important type of a coupled interior point constraint that frequently occurs in this thesis is the periodicity constraint, which is of the form

$$\Psi_{\text{per}}(\boldsymbol{x}(t_i), \boldsymbol{x}(t_j)) = \boldsymbol{x}(t_i) - \boldsymbol{x}(t_j), \tag{1.40}$$

for some $t_i < t_j$ in $\mathcal{T}$.

We use the quite abstract general form of Definition (1.38) to define OCPs in order to highlight the underlying structure that all OCPs have in common. However, when we define a new OCP, most of the time we state the path constraints and the coupled constraints in their explicit form to facilitate the understanding.

OCPs as Optimization Problems on BANACH Spaces

In the following, we define general optimization problems on BANACH spaces and show how Problem 1.38 can be embedded in this framework.

Definition 1.16 (General Constrained BANACH Space Optimization Problem)
Let V, W, U be BANACH spaces, $J : V \to \mathbb{R}$ a functional, $C : V \to W$, $G : V \to U$ operators, $S \subset V$ a closed convex set and $K \subset U$ a closed convex cone with 0_U as vertex. The following problem is a general constrained BANACH space optimization problem

$$\begin{aligned} \min_{w \in S} \quad & J(w) && (1.40) \\ \text{s.t.} \quad & G(w) \in K, \\ & C(w) = 0_W. \end{aligned}$$

△

With the definitions

$$\begin{aligned} V &:= \mathcal{AC}^{n_x}(\mathcal{T}) \times \mathrm{L}_\infty^{n_u}(\mathcal{T}), \\ W &:= \mathrm{L}_\infty^{n_u}(\mathcal{T}) \times \mathrm{L}_\infty^{n_{\text{eq}}}(\mathcal{T}) \times \mathbb{R}^{n_\Psi}, \\ U &:= \mathrm{L}_\infty^{\text{ineq}}(\mathcal{T}), \\ K &:= \left\{ k \in \mathrm{L}_\infty^{Nn^x}(\mathcal{T}) : k(t) \geq 0_{\mathbb{R}^{n_{\text{ineq}}}} \text{ a.e. in } \mathcal{T} \right\}, \\ S &:= V, \end{aligned}$$

$$
\begin{aligned}
J(\boldsymbol{x},\boldsymbol{u}) &:= \int_{t_0}^{t_\mathrm{f}} \ell(\tau,\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\mathrm{d}\tau + m(\boldsymbol{x}(t_\mathrm{f})),\\
C(\boldsymbol{x},\boldsymbol{u}) &:= \begin{pmatrix} \boldsymbol{x}(t_0)-x_0 \\ \dot{\boldsymbol{x}}(\cdot)-f(\cdot,\boldsymbol{x}(\cdot),\boldsymbol{u}(\cdot)) \\ c_\mathrm{eq}(\cdot,\boldsymbol{x}(\cdot),\boldsymbol{u}(\cdot)), \\ \Psi(\boldsymbol{x}(t_0),\boldsymbol{u}(t_0),\ldots,\boldsymbol{x}(t_{m_\Psi}),\boldsymbol{u}(t_{m_\Psi})) \end{pmatrix},\\
G(\boldsymbol{x},\boldsymbol{u}) &:= c_\mathrm{ineq}(\cdot,\boldsymbol{x}(\cdot),\boldsymbol{u}(\cdot)),
\end{aligned}
$$

we can interpret the OCP (1.38) as a general constrained BANACH space optimization problem as in Definition 1.16.

Under certain smoothness assumptions on the functions f, c, ℓ, m, c, Ψ it can be shown that the BANACH space functionals J, C and G are FRÉCHET differentiable. For a more detailed discussion, we refer the reader to Gerdts [46, Chapter 2].

Existence of Solutions

In general, statements on the existence of solutions of the OCP (1.38) are difficult to make. However, in some special cases, it can be shown that the subset of functions that satisfy the constraints is a compact subset of $V = \mathcal{AC}^{n_x}(\mathcal{T}) \times \mathrm{L}_\infty^{n_u}(\mathcal{T})$. For example, Lee and Markus [73] prove existence for OCPs with control constraints and a right-hand side f where the control enters linearly and in Berkovitz [11] existence of optimal controls based on convexity assumptions is discussed.

1.4.1 Important Special Cases

In the following, we discuss some important frequently arising special types of OCPs and show how they can be transformed to an OCP of standard form as in Definition 1.15.

Autonomous OCPs

In the case the functions ℓ, f and c are time independent, i.e. of the form

$$\ell : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}, \tag{1.41}$$

$$f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}, \tag{1.42}$$

$$c : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_c}, \tag{1.43}$$

we call the OCP an autonomous OCP. By augmenting the time as auxiliary differential state $\boldsymbol{x}_\mathrm{aux}$ with $\dot{\boldsymbol{x}}_\mathrm{aux} := 1$ and initial condition $\boldsymbol{x}_\mathrm{aux}(t_0)(\cdot) = t_0$ every non-autonomous OCP can be transformed into an autonomous OCP.

OCPs with Pure MAYER-Type Objective or Pure LAGRANGE-Type

The LAGRANGE objective term $\int_{t_0}^{t_f} \ell(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\mathrm{d}\tau$ can be transformed into a MAYER-type objective contribution by augmenting the auxiliary state

$$\dot{\boldsymbol{x}}_\ell(\tau) := \ell(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) \;\text{ with }\; \boldsymbol{x}_\ell(t_0) := 0 \tag{1.44}$$

to the system. Conversely, if the MAYER term function m of the objective is differentiable, the equation

$$m(\boldsymbol{x}(t_f)) - m(\boldsymbol{x}(t_0)) = \int_{t_0}^{t_f} \frac{\mathrm{d}m}{\mathrm{d}x}(\boldsymbol{x}(\tau)) f(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\mathrm{d}\tau \tag{1.45}$$

can be used to transform a MAYER-type objective into a LAGRANGE-type objective. For some numerical solution approaches of the OCP (i.e. Direct Multiple Shooting or Direct Collocation methods) the MAYER-type form is preferred, as it allows to reduce the complexity of evaluating the objective-criterion, since the final value $\boldsymbol{x}(t_f)$ explicitly occurs as optimization variable.

OCPs with Variable Time Horizon

The OCP (1.38) has a fixed time horizon. With the help of the time transformation

$$t(\tau) := t_0 + (t_f - t_0)\tau \tag{1.46}$$

we can transform the general OCP of Definition 1.15 to an OCP on the normalized time horizon $[0,1]$. This substitution technique also allows the reduction of an OCP with free initial and free final time to an OCP on a fixed time horizon by including the initial time and final time as additional parameters that can also be subject to optimization.

1.4.2 Optimal Control Problems with Time-Delay Objective

In the course of this thesis, a special kind of OCP with time-delayed objective function will occur. This class of OCP has an objective function with a LAGRANGE term ℓ that depends not only on state and control variables at time t but also on state and control variables at the shifted time $t + T_s$. Let $\ell : \mathcal{T} \times (\mathbb{R}^{n_x} \times \mathbb{R}^{n_u})^2 \to \mathbb{R}$ be sufficiently smooth and let $T_s \in \mathbb{R}$ be such that $t_0 < t_f - T_s$.

Definition 1.17 (Optimal Control Problem with Time-Delay Objective)
An OCP with time-delay objective function is an infinite dimensional optimization problem of the following form

$$\min_{\substack{\boldsymbol{x} \in \mathcal{AC}^{n_x}(\mathcal{T}),\\ \boldsymbol{u} \in \mathrm{L}_\infty^{n_u}(\mathcal{T})}} \quad \varphi(\boldsymbol{x}, \boldsymbol{u}) := \int_{t_0}^{t_f - T_s} \ell(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau), \tau + T_s, \boldsymbol{x}(\tau + T_s), \boldsymbol{u}(\tau + T_s))\mathrm{d}\tau \tag{1.47a}$$

$$\text{s.t.} \quad \boldsymbol{x}(t_0) = x_0, \tag{1.47b}$$

$$\dot{\boldsymbol{x}}(\tau) = f(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \qquad \tau \in \mathcal{T}, \tag{1.47c}$$

$$0 \leq c(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \qquad \tau \in \mathcal{T}, \tag{1.47d}$$

$$0 = \Psi(\boldsymbol{x}(t_0), \boldsymbol{u}(t_0), \dots, \boldsymbol{x}(t_{m_\Psi}), \boldsymbol{u}(t_{m_\Psi})). \tag{1.47e}$$

△

Reduction to Non-Delayed OCP

We show how the time-delay OCP of Definition (1.47a) can be reduced to an OCP of standard form. The method we describe is a modification of a method described in Guinn [55]. For simplicity, we only consider a time-delayed OCP without the interior point constraints (1.47e). The extension to the case with coupled interior point constraint is straightforward. Let us suppose that the time delay T_s is an integral fraction of the length $T := t_\mathrm{f} - t_0$ of the time horizon, i.e. $T_s = \frac{K}{N} T$ with $K < N$ and $K, N \in \mathbb{N}^+$. Furthermore, let $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}(\mathcal{T}) \times \mathrm{L}_\infty^{n_u}(\mathcal{T})$. To reduce the problem, we introduce new state and control variables $y_i \in \mathbb{R}^{n_x}$ and $w_i \in \mathbb{R}^{n_u}$ for $i \in \{1, \dots, N\}$ that are defined on $\widetilde{\mathcal{T}} := \left[t_0, t_0 + \frac{1}{N} T\right]$ and set

$$\boldsymbol{y}_i(\tau) := \boldsymbol{x}\left(\tau - \frac{i-1}{N} T\right) \text{ and } \boldsymbol{w}_i(\tau) := \boldsymbol{u}\left(\tau - \frac{i-1}{N} T\right). \tag{1.48}$$

For $i \in \{1, \dots, N\}$ we define the functions $f_i : \widetilde{\mathcal{T}} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$

$$f_i(\tau, y_i, w_i) := f\left(\tau + \frac{i-1}{N} T, y_i, w_i\right), \tag{1.49}$$

for $i \in \{1, \dots, N-K\}$ the functions $\ell_i : \widetilde{\mathcal{T}} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$

$$\ell_i(\tau, y_i, w_i, y_{i+K}, w_{i+K}) := \ell\left(\tau + \frac{i-1}{N} T, y_i, w_i, \tau + \frac{K+i-1}{N} T, y_{i+K}, w_{i+K}\right) \tag{1.50}$$

and for $i \in \{1, \dots, N\}$ the functions $c_i : \widetilde{\mathcal{T}} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$

$$c_i(\tau, y_i, w_i) := c\left(\tau + \frac{i-1}{N} T, y_i, w_i\right). \tag{1.51}$$

Now we set

$$(\boldsymbol{y}, \boldsymbol{w}) := \left(\begin{pmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_N \end{pmatrix}, \begin{pmatrix} \boldsymbol{w}_1 \\ \vdots \\ \boldsymbol{w}_N \end{pmatrix}\right) \in \mathcal{AC}^{N n_x}(\widetilde{\mathcal{T}}) \times \mathrm{L}_\infty^{N n_u}(\widetilde{\mathcal{T}}). \tag{1.52}$$

It is clear that if $(\boldsymbol{x}, \boldsymbol{u})$ satisfies the ODE-constraint (1.47c), $(\boldsymbol{y}, \boldsymbol{w})$ satisfies the following ODE-constraint

$$\dot{\boldsymbol{y}}(\tau) = \tilde{f}(\tau, \boldsymbol{y}(\tau), \boldsymbol{w}(\tau)) := \begin{pmatrix} f_1(\tau, \boldsymbol{y}_1(\tau), \boldsymbol{w}_1(\tau)) \\ \vdots \\ f_N(\tau, \boldsymbol{y}_N(\tau), \boldsymbol{w}_N(\tau)) \end{pmatrix} \in \mathbb{R}^{N n_x}, \tau \in \widetilde{\mathcal{T}}. \tag{1.53}$$

Similarly, if $(\boldsymbol{x},\boldsymbol{u})$ satisfies the path constraint (1.47d), $(\boldsymbol{y},\boldsymbol{w})$ satisfies the following path constraint

$$\tilde{c}(\tau,\boldsymbol{y}(\tau),\boldsymbol{w}(\tau)) := \begin{pmatrix} c_1(\tau,\boldsymbol{y}_1(\tau),\boldsymbol{w}_1(\tau)) \\ \vdots \\ c_N(\tau,\boldsymbol{y}_N(\tau),\boldsymbol{w}_N(\tau)) \end{pmatrix} \leq 0_{Nn_c}, \tau \in \widetilde{\mathcal{T}}. \tag{1.54}$$

And finally, if $(\boldsymbol{x},\boldsymbol{u})$ satisfies the initial value constraint (1.47b), $(\boldsymbol{y},\boldsymbol{w})$ satisfies the following constraint

$$\boldsymbol{y}_1(t_0) = x_0. \tag{1.55}$$

From Definition 1.48 it follows that the following equations are satisfied for $(\boldsymbol{y},\boldsymbol{w})$:

$$\boldsymbol{y}_i\left(t_0 + \frac{1}{N}T\right) = \boldsymbol{y}_{i+1}(t_0) \text{ for } i \in \{1,\ldots,N-1\}. \tag{1.56}$$

Conversely, if $(\boldsymbol{y},\boldsymbol{w}) \in \mathcal{AC}^{Nn_x}(\widetilde{\mathcal{T}}) \times \mathrm{L}_\infty^{Nn_u}(\widetilde{\mathcal{T}})$ satisfies constraints (1.53) – (1.55) and the transversality constraints (1.56), we can construct a pair $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{AC}^{n_x}(\mathcal{T}) \times \mathrm{L}_\infty^{n_u}(\mathcal{T})$ that satisfies constraints (1.47c) – (1.47e). Altogether, if we merge the initial value constraint (1.55) and the constraints (1.56) into one coupled interior point constraint $\tilde{\Psi}$, we reduced the time-delayed OCP to the following OCP of standard form.

$$\min_{\substack{\boldsymbol{y}\in\mathcal{AC}^{Nn_x}(\widetilde{\mathcal{T}}),\\ \boldsymbol{w}\in\mathrm{L}_\infty^{Nn_x}(\widetilde{\mathcal{T}})}} \int_{\widetilde{\mathcal{T}}} \sum_{i=1}^{N-K} \ell_i(\tau,\boldsymbol{y}_i(\tau),\boldsymbol{w}_i(\tau),\boldsymbol{y}_{i+K}(\tau),\boldsymbol{w}_{i+K}(\tau))\mathrm{d}\tau \tag{1.57a}$$

$$\text{s.t.} \quad \dot{\boldsymbol{y}}(\tau) = \tilde{f}(\tau,\boldsymbol{y}(\tau),\boldsymbol{w}(\tau)), \qquad \tau \in \widetilde{\mathcal{T}}, \tag{1.57b}$$

$$0 \leq \tilde{c}(\tau,\boldsymbol{y}(\tau),\boldsymbol{w}(\tau)), \qquad \tau \in \widetilde{\mathcal{T}}, \tag{1.57c}$$

$$0 = \tilde{\Psi}\left(\boldsymbol{y}(t_0),\boldsymbol{w}(t_0),\boldsymbol{y}\left(t_0+\frac{1}{N}T\right),\boldsymbol{w}\left(t_0+\frac{1}{N}T\right)\right). \tag{1.57d}$$

1.5 Optimal Control Problems with Infinite Time Horizons

Up to this point we only considered OCPs with finite time horizons $\mathcal{T}$. It is also possible to define OCPs on infinite time horizons. However it is necessary to choose appropriate optimality criteria in this case. We begin this section with an instructive example and then introduce some optimality criteria for OCPs on infinite time horizons.

Example: Indefinite Objective

Consider the 2-dimensional dynamical system described by the linear ODE

$$\dot{\boldsymbol{x}}(t) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\boldsymbol{x}(t) + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\boldsymbol{u}(t) \tag{1.58}$$

with the state and control bounds

$$\begin{aligned} 0_2 \le \boldsymbol{x}(t) \le 2 \cdot \mathbb{1}_2 \;\text{ for } i = 1,2, t \in \mathbb{R}_{\ge 0}, \\ 0_2 \le \boldsymbol{u}(t) \le 2 \cdot \mathbb{1}_2 \;\text{ for } i = 1,2, t \in \mathbb{R}_{\ge 0}. \end{aligned} \tag{1.59}$$

Let $\ell(t,x,u) := -\|u\|_2^2$ be the performance criterion and $\boldsymbol{x}(0) = (0,0)^T$ the initial state of the system at time $t = 0$. We want to find an admissible pair $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{AC}^2_{\text{loc}} \times \mathrm{L}^2_{\infty,\text{loc}}$ such that $\int_0^\infty \ell(\boldsymbol{u})\mathrm{d}t$ is minimized. For a given function $\boldsymbol{u} \in \mathrm{L}^2_{\infty,\text{loc}}$ the solution at time $t > 0$ of the IVP defined by (1.58) with initial value $\boldsymbol{x}(0) = (0,0)^T$ can be expressed explicitly by the formula

$$\boldsymbol{x}(t) = \int_0^t \mathrm{e}^{-t+\tau} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \boldsymbol{u}(\tau)\mathrm{d}\tau. \tag{1.60}$$

Using this formula, we can verify that the following pair $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{AC}^2_{\text{loc}} \times \mathrm{L}^2_{\infty,\text{loc}}$ satisfies the bounds and solves the ODE for all times.

$$\boldsymbol{x}(t) := \begin{cases} \begin{pmatrix} 2(1-\mathrm{e}^{-t}) \\ 0 \end{pmatrix} & \text{for } t \in [0,\ln(2)) \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{for } t > \ln(2) \end{cases} \quad \text{and } \boldsymbol{u}(t) := \begin{cases} \begin{pmatrix} 2 \\ 0 \end{pmatrix} & \text{for } t \in [0,\ln(2)) \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{for } t > \ln(2) \end{cases} \tag{1.61}$$

Similarly, we can check that the pair $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{u}}) \in \mathcal{AC}^2_{\text{loc}} \times \mathrm{L}^2_{\infty,\text{loc}}$ defined by

$$\tilde{\boldsymbol{x}}(t) := \begin{cases} \begin{pmatrix} 0 \\ 2(1-\mathrm{e}^{-t}) \end{pmatrix} & \text{for } t \in [0,\ln(2)) \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{for } t > \ln(2) \end{cases} \quad \text{and } \tilde{\boldsymbol{u}}(t) := \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{for } t \in [0,\ln(2)) \\ \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} & \text{for } t > \ln(2) \end{cases} \tag{1.62}$$

solves the ODE (1.58) with the same initial value and also satisfies the bounds (1.59). In both cases, after an initial transient phase, the systems enter a constant steady-state and remain there for all times. In the first case, the steady-state is $(x_s, u_s) = \left((1,0)^T, (1,0)^T\right)$ and in the second case it is $(\tilde{x}_s, \tilde{u}_s) = \left((0,1)^T, (0,0.5)^T\right)$. The value of the performance criterion ℓ at the two steady-states can be calculated as

$$\ell(x_s, u_s) = -1 \text{ and } \ell(\tilde{x}_s, \tilde{u}_s) = -0.25 \tag{1.63}$$

and therefore, the performance integral is unbounded in both cases:

$$\int_0^\infty \ell(\boldsymbol{u}(t))\mathrm{d}t = \int_0^\infty \ell(\tilde{\boldsymbol{u}}(t))\mathrm{d}t = -\infty \tag{1.64}$$

However, one could argue that the first solution $(\boldsymbol{x},\boldsymbol{u})$ is preferable over the second one, because the performance of its steady-state beats the performance of the steady-state of the second solution. Therefore the first solution will perform better in an average sense.

Optimality Criteria for Infinite Horizons

The above example illustrates the need of more general optimality concepts for infinite horizon OCPs. Several optimality criteria for infinite horizon OCPs can be found in the literature [24, 30] that circumvent this problem. We introduce the following functionals.

Definition 1.18 (Infinite Horizon Objective Functionals)
For $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}_{\text{loc}}^{n_x} \times \mathrm{L}_{\infty,\text{loc}}^{n_u}$, $T \in \mathbb{R}_{\geq 0}$ and $\lambda > 0$ we define the functionals

$$\varphi_T(\boldsymbol{x}, \boldsymbol{u}) := \int_0^T \ell(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\mathrm{d}\tau, \tag{1.65}$$

$$\varphi_\infty(\boldsymbol{x}, \boldsymbol{u}) := \lim_{T\to+\infty} \varphi_T(\boldsymbol{x}, \boldsymbol{u}), \tag{1.66}$$

$$\varphi_{\text{avg}}(\boldsymbol{x}, \boldsymbol{u}) := \limsup_{T\to+\infty} \frac{1}{T}\varphi_T(\boldsymbol{x}, \boldsymbol{u}), \tag{1.67}$$

$$J_\lambda(\boldsymbol{x}, \boldsymbol{u}) := \limsup_{T\to+\infty} \int_0^T \mathrm{e}^{-\lambda\tau}\ell(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\mathrm{d}\tau. \tag{1.68}$$

In case any of the limits does not exists, the value of the functional is defined to be $+\infty$. △

Definition 1.19 (Infinite Horizon OCP)
Let φ be one of the functionals $\varphi_\infty, \varphi_{\text{avg}}, J_\lambda$. Then we call the following problem an infinite horizon OCP.

$$\inf_{(\boldsymbol{x},\boldsymbol{u})\in\mathcal{AC}_{\text{loc}}^{n_x}\times\mathrm{L}_{\infty,\text{loc}}^{n_u}} \varphi(\boldsymbol{x}, \boldsymbol{u}) \tag{1.69a}$$

$$\text{s.t.} \quad \boldsymbol{x}(t_0) = x_0 \tag{1.69b}$$

$$\dot{\boldsymbol{x}}(\tau) = f(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \tau \in \mathbb{R}_{\geq 0}, \tag{1.69c}$$

$$0 \leq c(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \tau \in \mathbb{R}_{\geq 0}. \tag{1.69d}$$

△

As in problem (1.38), we call a pair $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}_{\text{loc}}^{n_x} \times \mathrm{L}_{\infty,\text{loc}}^{n_u}$ admissible on the time horizon $\mathbb{R}_{\geq 0}$, if it satisfies constraints (1.69b)-(1.69d). Note that in problem (1.69) we use "inf" instead of "min" in the finite horizon OCP (1.38), as the space $\mathcal{AC}_{\text{loc}}^{n_x} \times \mathrm{L}_{\infty,\text{loc}}^{n_u}$ is not closed.

Definition 1.20 (Optimality on Infinite Horizons)
If in problem (1.69) $\varphi = \varphi_\infty$, we call an admissible pair $(\boldsymbol{x}^*, \boldsymbol{u}^*)$ strongly optimal, if $\varphi_\infty(\boldsymbol{x}^*, \boldsymbol{u}^*) < \infty$ and if for any admissible pair $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}_{\text{loc}}^{n_x} \times \mathrm{L}_{\infty,\text{loc}}^{n_u}$

$$\varphi_\infty(\boldsymbol{x}^*, \boldsymbol{u}^*) \leq \varphi_\infty(\boldsymbol{x}, \boldsymbol{u}) \tag{1.70}$$

holds.
If in problem (1.69) $\varphi = \varphi_{\text{avg}}$, we call an admissible pair $(\boldsymbol{x}^*, \boldsymbol{u}^*)$ average optimal, if $\varphi_{\text{avg}}(\boldsymbol{x}^*, \boldsymbol{u}^*) < \infty$ and if for any admissible pair $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}_{\text{loc}}^{n_x} \times \mathrm{L}_{\infty,\text{loc}}^{n_u}$

$$\varphi_{\text{avg}}(\boldsymbol{x}^*, \boldsymbol{u}^*) \leq \varphi_{\text{avg}}(\boldsymbol{x}, \boldsymbol{u}) \tag{1.71}$$

holds. △

If we reconsider the example at the start of this Section, we can calculate

$$\varphi_\infty(\boldsymbol{x},\boldsymbol{u}) = -\infty, \qquad \varphi_\infty(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{u}}) = -\infty, \tag{1.72}$$

$$\varphi_{\text{avg}}(\boldsymbol{x},\boldsymbol{u}) = -1, \qquad \varphi_{\text{avg}}(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{u}}) = -0.25. \tag{1.73}$$

This shows that the functional φ_{avg} can distinguish the two admissible pairs $(\boldsymbol{x},\boldsymbol{u})$ and $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{u}})$.

Another benefit of the functionals φ_{avg} and J_λ is, that contrary to φ_∞, they are bounded on the set of admissible pairs for a greater class of problems.

Lemma 1.21 (Boundedness of Objective Functionals)
Let the set $A := \{(x,u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} : \exists t \in \mathbb{R} : 0 \le c(t,x,u)\}$ be compact and let $L : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ be a smooth function such that for all $(t,x,u) \in \mathbb{R} \times A$

$$|\ell(t,x,u)| \le L(x,u) \tag{1.74}$$

holds. Then φ_{avg} and J_λ are bounded on the set of admissible pairs for problem (1.69).

Proof Because of (1.74), for any admissible pair $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{AC}^{n_x}_{\text{loc}} \times \mathrm{L}^{n_u}_{\infty,\text{loc}}$ and any $T \in \mathbb{R}_{\ge 0}$ it holds

$$\left|\varphi_T(\boldsymbol{x},\boldsymbol{u})\right| = \frac{1}{T}\left|\int_0^T \ell(\tau,\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\mathrm{d}\tau\right| \tag{1.75}$$

$$\le \frac{1}{T}\int_0^T L(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\mathrm{d}\tau. \tag{1.76}$$

Because $0 \le c(\tau,\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))$ holds for almost all $\tau \in \mathbb{R}$, it follows that $(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)) \in A$ for almost all $\tau \in \mathbb{R}$. The smooth function L is bounded from above by a number $C < \infty$ which then implies

$$\varphi_{\text{avg}}(\boldsymbol{x},\boldsymbol{u}) \le C. \tag{1.77}$$

Similarly, for any $T > 0$ it holds

$$\left|\int_0^T \mathrm{e}^{-\lambda\tau}\ell(\tau,\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\mathrm{d}\tau\right| \le \int_0^T \left|\mathrm{e}^{-\lambda\tau}C\right|\mathrm{d}\tau = \frac{C}{\lambda}(1-\mathrm{e}^{-\lambda T}) < \frac{C}{\lambda}, \tag{1.78}$$

which proves $J_\lambda(\boldsymbol{x},\boldsymbol{u}) \le \frac{C}{\lambda}$. □

1.6 Solution Methods

In this section we give a small overview on methods that can be used to solve OCPs numerically.

1.6.1 Indirect Approaches

PONTRYAGIN's[10] Maximum Principle is the fundament of indirect methods. It states necessary optimality conditions that can be used to transform the infinite dimensional OCP (1.38) into a Multi Point Boundary Value Problem (MPBVP). An overview of numerical methods for solving the arising MPBVP can be found in Ascher et al. [6]. Two of the most important methods used for discretizing the MPBVPs are the collocation method [105] and the multiple shooting method [90, 22, 17, 34] which both transform the MPBVP into a finite dimensional system of nonlinear equations. In the indirect approach, first the optimality conditions for the infinite dimensional problem are set up first and then a discretization is applied. This is also the reason why the indirect approach is sometimes referred to as **First Optimize then Discretize** approach. The solution process requires an initial guess for all primal and dual variables of the MPBVP, which can be difficult to obtain, especially for the dual variables of the constraints. Furthermore, the derivation of the necessary optimality conditions is a difficult process that requires insight in the problem structure and cannot be automated. For these reasons, indirect approaches are not very well suited for the fast solution of OCPs.

1.6.2 Direct Approaches

Contrary to the indirect approaches, the first step in the direct approaches is the discretization of the infinite dimensional OCP. This directly transforms the infinite dimensional OCP into a finite dimensional NLP. The resulting NLP then can be solved by an appropriate numerical method, e.g. the Sequential Quadratic Programming (SQP) method [89] or an Interior Point method [116]. Because of the order of discretization and optimization, direct approaches are sometimes referred to as **First Discretize then Optimize** approaches. In the following, we give a brief introduction into the Direct Multiple Shooting method due to Bock and Plitt [20] for discretizing OCPs, since it is the basis for our solution algorithms.

1.6.3 The Direct Multiple Shooting Method

Multiple shooting methods were first used for solving Boundary Value Problems (BVPs) [22, 16, 90, 17], as they also occur for example in the indirect approaches. The Direct Multiple Shooting method for OCPs is first described in [20] and [91]. We will describe how the Direct Multiple Shooting method transforms the infinite dimensional OCP (1.38) into a finite dimensional NLP.

Time Discretization

As a first step, we partition the time interval $\mathcal{T} = [t_0, t_\mathrm{f}]$ into N intervals

$$t_0 < t_1 < \ldots < t_N = t_\mathrm{f}. \tag{1.79}$$

[10] Lev Semyonovich Pontryagin 1908 - 1988

Often, the intervals are chosen to be equidistant, but this is not necessary. We assume that the times $\{t_0, \dots, t_{m_\Psi}\}$ (from the coupled interior point constraint Ψ (1.38e)) are all elements of the time grid.

Control Discretization

The control functions on $\mathcal{T}$ are defined piecewise on the intervals $\mathcal{T}_i := [t_i, t_{i+1}]$. In principle, any finite dimensional parametrization

$$B_i : \mathcal{T}_i \times \mathbb{R}^{n_i^q} \to \mathbb{R}^{n_u} \tag{1.80}$$

can be used. Common choices of such parametrizations include

- piecewise constant:
  $n_i^q = n_u$ and $B_i(\tau, q_i) := q_i$ for all $i = 0, \dots, N-1$,
- piecewise linear:
  $n_i^q = 2n_u$ and $B_i(\tau, (q_{i,1}, q_{i,2})) := q_{i,1} + \frac{\tau - t_i}{t_{i+1} - t_i} q_{i,2}$ for all $i = 0, \dots, N-1$.

For simplicity we assume that for all $i \in \{0, \dots, N-1\}$ the dimension n_i^q is the same $(= n^q)$. Then the vector

$$q := (q_0, \dots, q_{N-1}, q_N) \in \mathbb{R}^{(N+1)n^q} \tag{1.81}$$

represents the control function $\tilde{\boldsymbol{u}}_q$ on $\mathcal{T}$

$$\tilde{\boldsymbol{u}}_q : \mathcal{T} \to \mathbb{R}^{n_u}, \tag{1.82}$$

$$t \mapsto \begin{cases} B_i(\tau, q_i) & \text{for } \tau \in [t_i, t_{i+1}), \\ B_{N-1}(t_N, q_{N-1}) & \text{for } \tau = t_N. \end{cases} \tag{1.83}$$

Note that the last component q_N is not necessary to define the function $\tilde{\boldsymbol{u}}_q$ because $\{t_N\}$ has measure zero. We include it q_N for notational convenience and fix it with the auxiliary constraint $0 = q_N - q_{N-1}$. If a continuous control function is desired, additional continuity conditions can be added (e.g. $q_{i,2} = q_{i+1,1}$ for all i in the case of piecewise linear interpolations). In this thesis, we always use a piecewise constant control parametrization.

State Parametrization

For each grid point t_i, we introduce a state variable $s_i \in \mathbb{R}^{n_x}$. The vector

$$s := (s_0, \dots, s_{N+1}) \in \mathbb{R}^{(N+1)n_x} \tag{1.84}$$

together with the vector q then represents the following function:

$$\tilde{\boldsymbol{x}}_{(s,q)} : \mathcal{T} \to \mathbb{R}^{n_x}, \tag{1.85}$$

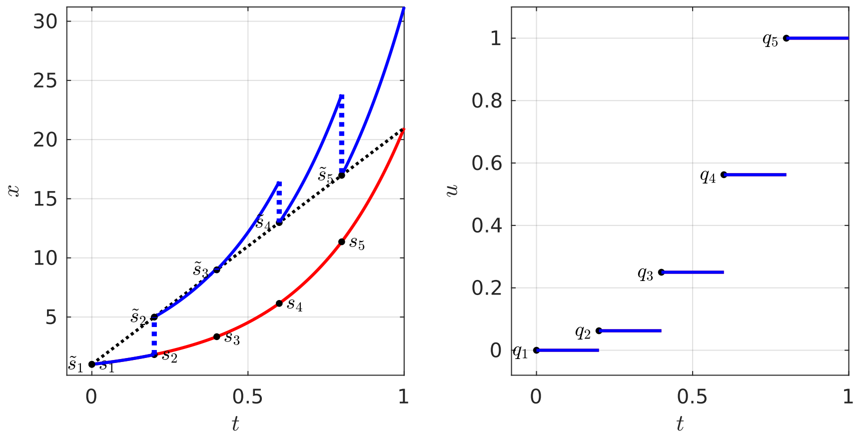


Figure 1.3: On the left, two state trajectories corresponding to the state/control variables $(\tilde{s}_i, q_i)$ (blue) and (s_i, q_i) (red) can be seen. The states $(\tilde{s}_i)$ are initialized along the dashed black line and the resulting trajectory has shooting-discontinuities (blue dashed lines). The states (s_i) correspond to a initialization that results in a continuous state trajectory. On the right, the piecewise constant control function corresponding to (q_i) is depicted.

$$\tau \mapsto \begin{cases} \boldsymbol{x}(\tau; s_i, B_i(\cdot, q_i), t_i) & \text{for } \tau \in [t_i, t_{i+1}), \\ s_{N+1} & \text{for } \tau = t_{N+1}. \end{cases} \tag{1.86}$$

Here, the expression $\boldsymbol{x}(\tau; s_i, B_i(\cdot, q_i), t_i)$ denotes the solution of the IVP

$$\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t), B_i(\cdot, q_i)), \boldsymbol{x}(t_i) = s_i, \tag{1.87}$$

at time τ. The function $\tilde{\boldsymbol{x}}_{(s,q)}$ is continuous on all subintervals $\mathcal{T}_i$, but it may contain jump discontinuities at the grid points. With the definition

$$g_i(s_i, q_i) := \boldsymbol{x}(t_{i+1}; s_i, B_i(\cdot, q_i), t_i), \tag{1.88}$$

the continuity of $\tilde{\boldsymbol{x}}_{(s,q)}$ at time t_{i+1} is equivalent to the equation (the matching condition)

$$g_i(s_i, q_i) - s_{i+1} = 0. \tag{1.89}$$

In other words, the pair (s, q) induces a solution $(\tilde{\boldsymbol{x}}_{(s,q)}, \tilde{\boldsymbol{u}}_q)$ of the IVP

$$\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t), \boldsymbol{u}(t)) \text{ with } \boldsymbol{x}(t_0) = s_0 \tag{1.90}$$

on the interval $\mathcal{T}$, if condition (1.89) is satisfied for $i \in \{0, \ldots, N-1\}$. An illustration of two pairs of state/control variables that induce a continuous trajectory respectively a trajectory with shooting discontinuities can be found in Figure 1.3.

Constraint Discretization

The path constraints (1.38d) are enforced at the grid-points $\{t_i, i = 0, \dots, N\}$ of the time discretization:

$$0 \le c(t_i, \tilde{\boldsymbol{x}}_{(s,q)}(t_i), \tilde{\boldsymbol{u}}_q(t_i)). \tag{1.91}$$

This directly translates to constraints at the coefficients (s, q):

$$0 \le c(t_i, s_i, q_i) \text{ for } i \in \{0, \dots, N\}. \tag{1.92}$$

Analogously, the coupled interior point constraint (1.38e) translates to

$$0 = \Psi(t_0, s_0, q_0, \dots, t_{m_\Psi}, s_{m_\Psi}, q_{m_\Psi}). \tag{1.93}$$

Note that even if constraint (1.92) is satisfied, constraint (1.38d) can be violated for $(\tilde{\boldsymbol{x}}_{(s,q)}, \tilde{\boldsymbol{u}}_q)$ between the shooting nodes. There are different ways to handle this problem, if strict satisfaction of the path constraint between the shooting nodes is important. First, refining the time grid can be considered. Theoretically, it is also possible to introduce a new auxiliary differential state $\dot{\boldsymbol{x}}_{\text{aux}}(t) := c(t, \boldsymbol{x}(t), \boldsymbol{u}(t))^-$ with initial condition $\boldsymbol{x}_{\text{aux}}(t_0) = 0$. Here v^- denotes the component-wise projection to the negative part for $v \in \mathbb{R}^n$. Then $\boldsymbol{x}_{\text{aux}}(t_f) = 0$ is equivalent to satisfaction of the path constraint $c(t, \boldsymbol{x}(t), \boldsymbol{u}(t))$ almost everywhere, and the path constraint could be eliminated from the OCP (1.38) by adding the final value constraint $\boldsymbol{x}_{\text{aux}}(t_f) = 0$. However, the projection onto the negative part is not differentiable at 0, which in practice would make it necessary to replace the projection with some smooth approximation. Another possibility is, to incorporate the path constraint for all time points $t \in \mathcal{T}$. This leads to a so-called semi-infinite program [59]. It is also possible to keep track of the local extrema of the path-constraint function c and incorporate additional constraints at these points as described in Potschka [93] and Potschka et al. [95].

Objective Function Discretization

Using definitions (1.85) and (1.82), we can calculate the objective value of the pair $(\tilde{\boldsymbol{x}}_{(s,q)}, \tilde{\boldsymbol{u}}_q)$ as follows:

$$\varphi(\tilde{\boldsymbol{x}}_{(s,q)}, \tilde{\boldsymbol{u}}_q) = \int_{t_0}^{t_f} \ell(\tau, \tilde{\boldsymbol{x}}_{(s,q)}(\tau), \tilde{\boldsymbol{u}}_q(\tau)) \mathrm{d}\tau + m(t_f, \tilde{\boldsymbol{x}}_{(s,q)}(t_f)) \tag{1.94}$$

$$= \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \ell(\tau, \boldsymbol{x}(\tau; s_i, B_i(\cdot, q_i), t_i), B_i(\tau, q_i)) \mathrm{d}\tau + m(t_f, s_N) \tag{1.95}$$

$$=: \sum_{i=0}^{N-1} L_i(s_i, q_i) + M(s_N). \tag{1.96}$$

The Multiple Shooting NLP

Using the described discretization, we can transform the infinite dimensional optimization problem (1.38) into the following finite dimensional optimization problem:

$$\min_{\substack{s\in\mathbb{R}^{(N+1)n_x},\\ q\in\mathbb{R}^{(N+1)n_u}}} \quad \phi(s,q) := \sum_{i=0}^{N-1} L_i(s_i,q_i) + M(s_N) \tag{1.97a}$$

$$\text{s.t.} \quad 0 = g_i(s_i,q_i) - s_{i+1}, \qquad i \in \{0,\dots,N-1\}, \tag{1.97b}$$

$$0 \le c(t_i,s_i,q_i), \qquad i \in \{0,\dots,N\}, \tag{1.97c}$$

$$0 = \Psi(s_0,q_0,\dots,s_{m_\Psi},q_{m_\Psi}). \tag{1.97d}$$

$$0 = q_N - q_{N-1}. \tag{1.97e}$$

A variety of numerical solution algorithms for NLPs exists cf. [12] or [89]. Suitable methods include Interior Point methods [116] and SQP methods [89]. The structure of problem (1.97) plays an important role in the efficient numerical treatment. For example in an SQP method, the fact that the objective function (1.97a) and the shooting constraints (1.97b) are decoupled, leads to a special block-diagonal structure of the Hessian approximations. By using condensing techniques as described in [91, 20], the continuity condition (1.97b) allows to reduce the number of optimization variables in the Quadratic Program (QP) subproblems considerably.

1.7 Summary

After introducing some fundamental function space definitions we started this chapter by defining the concepts of dynamical systems, IVPs and stability of solutions for IVPs. Based on these definitions we introduced OCPs with finite time horizons and BOLZA objective functionals as a class of infinite dimensional optimization problem on BANACH spaces. We showed how an OCP with delay objective functional can be transformed into a standard OCP with BOLZA objective functional. For the class of OCPs with infinite time horizons we motivated the need of special averaging objective functionals.

Finally we discussed how an infinite dimensional OCP can be transcribed into a finite dimensional NLP to make its solution numerically tractable. In particular we described the Direct Multiple Shooting method.

Chapter 2

Stability Theory of Nonlinear Model Predictive Control

In this chapter, we give a brief introduction of the main principles and the theory behind NMPC with a focus on stability theory for the resulting systems. Nonlinear Model Predictive Control (NMPC) is an advanced feedback control method for dynamical systems.

We define the moving horizon and the shrinking horizon NMPC schemes and show how the closed-loop behavior of a moving horizon NMPC scheme based on a tracking subproblem can be analyzed. Furthermore we define economic NMPC and present stability results for state-of-the-art economic NMPC schemes for dissipative systems.

2.1 Motivation and Historical Background

2.1.1 Feedback Control Methods

As we have seen in the previous chapter, solutions of dynamical systems can exhibit unstable behavior which leads to unpredictable development of the process. Open-loop control therefore is not suitable for controlling such processes and the use of feedback control methods that monitor the process and intervene correctively if necessary, is indispensable (see Figure 2.1).

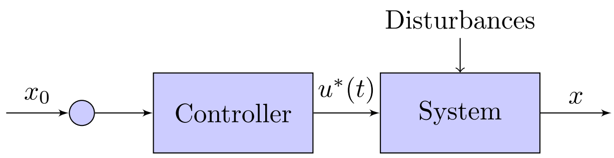
Simple examples for feedback controllers are Proportional-Integral-Derivative (PID) [1] or Fuzzy controllers [112], that can be used for keeping dynamical systems at desired setpoints by continuously monitoring the distance of the current process state to the desired setpoint and taking appropriate measures.

However, such controllers do not explicitly consider the dynamical model of the controlled process and thus have limited ability to anticipate the future behavior of the system accurately. Furthermore, many processes in practice are subject to constraints such as pressure, temperature or fuel consumption limits which cannot be taken into account by such controllers.

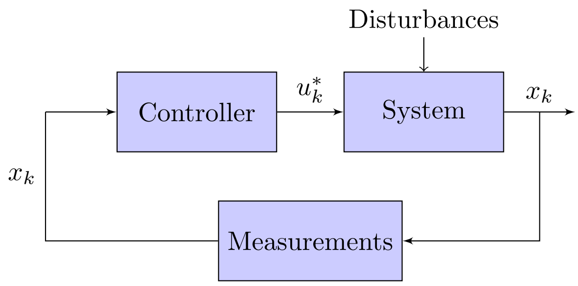
This motivates the need for a feedback control method that both has the ability to accurately predict the future behavior of the process based on a dynamical model and is capable to take into account operational constraints as well.

These requirements fit well in the framework of optimal control. In NMPC, the feedback is based on the repetitive process of solving an (appropriately chosen) OCP with the current system state as initial value and using the resulting control solution as feedback control.

The great flexibility offered by this approach furthermore allows to create controllers that go beyond simple setpoint-tracking tasks.



a) Open-loop control without feedback. The resulting system behavior will very likely deviate from the predicted open-loop control solution due to disturbances.



b) Closed-loop control with Feedback Loop. The current system state is measured and the control input is generated by the controller as a function of the current system state. Disturbances can be rejected by appropriate control reactions.

Figure 2.1: Open-loop control vs. closed-loop control.

2.1.2 Historical Development

In this section we want to give a short overview of the historical development of NMPC. Due to the great abundance of NMPC literature, this overview does not claim to be complete and rather has to be seen as a selection of important developments. The development of NMPC can be traced back to works of KALMAN[1] [69, 70] from the early 60's. He was working on Linear Quadratic Regulator (LQR) problems where a controlled process is governed by a discrete linear model and the objective is an infinite sum of squared deviations form the origin. It was shown that the optimal control solution of the LQR problem is a linear mapping from the initial value which can be represented by the gain matrix. This matrix can be calculated as the solution of a matrix RICATTI[2] equation.

This approach can be regarded as a precursor of NMPC and its strength was based on the fact that a general model was used to predict the systems future behavior. Despite its powerful stabilizing properties for linear systems, the theory of LQR had only little impact in the process industries due to its failure to handle constraints and nonlinear process behavior [102].

One of the first works where the central idea of NMPC can be found for discrete linear systems with control constraints is Propoi [97]. For nonlinear systems the idea was anticipated in the book of Lee and Markus [74] from 1967:

[1]Rudolf Emil Kálmán 1930 - 2016
[2]Jacopo Francesco Riccati 1676 - 1754

> "One technique for obtaining a feedback controller synthesis from knowledge of open-loop controllers is to measure the current control process state and then compute very rapidly for the open-loop control function. The first portion of this function is then used during a short time interval, after which a new measurement of the process state is made and a new open-loop control function is computed for this new measurement. The procedure is then repeated." Lee and Markus [74, Page 423]

This shows that the central ideas of NMPC were formulated relatively early considering that the seminal contributions of the theory of optimal control such as the maximum principle Pontryagin et al. [92] (1962) and the principle of dynamic programming Bellman [9] (1957) were formulated not long before.

In the beginning the formulation of the basic NMPC idea was rather a theoretical observation then a practical approach because neither powerful computers nor suitable software was available to "very rapidly" solve the arising OCPs. This changed with algorithmic progress for constrained linear and quadratic programs in the 1970s. Especially in the field of process control, where the system dynamics is often comparably slow, applications are reported in Richalet et al. [102], Cutler and Ramaker [33], Prett and Gillette [96].

While early versions of NMPC often did not automatically ensure stability, the systematic analysis of the stability properties of NMPC by means of LYAPUNOV-functions and terminal constraints started with the work of Chen and Shaw [26]. Within the 1990s, the stability analysis for NMPC schemes with terminal constraints and terminal costs was further developed in Nicolao et al. [88], Magni and Sepulchre [82], Chen and Allgöwer [27].

From the 2000's on, the interest in NMPC schemes that go beyond regulatory and tracking tasks and directly incorporate an economic performance criterion received increasing interest cf. Helbig et al. [58], Engell [38], Rawlings and Amrit [99], Grüne et al. [52], Faulwasser and Bonvin [39] and Grüne and Pannek [51, Chapter 8]. For an overview on recent developments in this direction we refer the reader to the overview articles of Ellis et al. [37] and Müller and Allgöwer [85].

2.2 Basic Principles of Nonlinear Model Predictive Control

In this section, we introduce the main ideas behind the Moving Horizon and Shrinking Horizon NMPC as an abstract sequence of solutions of parametrized OCPs.

Consider a dynamical system governed by the ODE $\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t), \boldsymbol{u}(t))$ with f sufficiently smooth and $\boldsymbol{u}(t)$ the control input. Let furthermore ℓ, m, c be sufficiently smooth functions as in (1.37).

2.2.1 The Moving Horizon Scheme

As the name already indicates, the Moving Horizon NMPC scheme is based on OCPs with time horizons moving forward in time. More specifically it is based on a family $P_{\text{move}}(t, x)$ of OCPs parametrized in initial time $t \in \mathbb{R}$ and initial state $x \in \mathbb{R}^{n_x}$ based on the dynamical system $\dot{\boldsymbol{x}}(\tau) = f(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau))$ with time horizons $\mathcal{T}^t := [t, t+T]$ of constant length T.

As an example of such a parametrized family of OCPs, we consider the following standard OCP $P_{\text{move}}(t,x)$ based on the functions ℓ, m, c and Ψ:

$$\min_{\substack{\boldsymbol{x}\in\mathcal{AC}^{n_x}(\mathcal{T}^t),\\ \boldsymbol{u}\in\mathrm{L}_\infty^{n_x}(\mathcal{T}^t)}} \quad \varphi(\boldsymbol{x},\boldsymbol{u}) := \int_{\mathcal{T}^t} \ell(\tau,\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\mathrm{d}\tau + m(\boldsymbol{x}(t+T)), \tag{2.1a}$$

$$\text{s.t.} \qquad \boldsymbol{x}(t) = x, \tag{2.1b}$$

$$\dot{\boldsymbol{x}}(\tau) = f(\tau,\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)), \quad \tau\in\mathcal{T}^t, \tag{2.1c}$$

$$0 \le c(\tau,\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)), \quad \tau\in\mathcal{T}^t \tag{2.1d}$$

$$0 \le \Psi(t+T,\boldsymbol{x}(t+T)). \tag{2.1e}$$

Let $\varphi^*(t,x)$ denote the optimal objective value for problem $P_{\text{move}}(t,x)$. We could as well consider any other OCP with time horizon length T containing the dynamical system constraint (2.1c) as well as the initial value constraint (2.1b).

The problem $P_{\text{move}}(t,x)$ is called the "NMPC Subproblem" for the Moving Horizon NMPC scheme:

Algorithm 1 Moving horizon NMPC scheme.

```
Choose a sampling time grid 𝕋 := {t_i ∈ ℝ : t_j < t_{j+1} for all j ∈ ℕ},
i ← 0,
while true do
    At time t_i determine the state of the system x_i ∈ ℝ^{n_x}
    Solve P_move(t_i, x_i), (x_i^⋆, u_i^⋆) (the "Open-Loop" prediction)
    For τ ∈ [t_i, t_{i+1}) apply the control u_i^⋆(τ) to the system
    i ← i + 1
end while
```

The resulting **"Closed-Loop"** state trajectory $\boldsymbol{x}_\mu(\cdot;x_0,t_0):[t_0,\infty)\to\mathbb{R}^{n_x}$ and control trajectory $\boldsymbol{u}_\mu(\cdot;x_0,t_0):[t_0,\infty)\to\mathbb{R}^{n_u}$ are defined as

$$\boldsymbol{x}_\mu(\tau;x_0,t_0) := \boldsymbol{x}_i^\star(\tau) \text{ for } \tau\in[t_i,t_{i+1}), \tag{2.2}$$

$$\boldsymbol{u}_\mu(\tau;x_0,t_0) := \boldsymbol{u}_i^\star(\tau) \text{ for } \tau\in[t_i,t_{i+1}). \tag{2.3}$$

As **"Cost-to-Go"** function for the NMPC scheme we denote the optimal objective function evaluated along the closed-loop trajectory:

$$t \mapsto \varphi^*(t,\boldsymbol{x}_\mu(t;x_0,t_0)). \tag{2.4}$$

This function plays an important role in the analysis of the closed-loop behavior.

Remark 2.1 (Measurement of Initial State) In practice, it is often not possible to determine the exact state of the system at a given time. The measurement process rather has to be interpreted as a function of the true state plus a measurement error term:

$$\eta_k = f^{\text{meas}}(x_k) + \mathrm{e}_k. \tag{2.5}$$

The problem of recovering the true state x_k of the process as accurately as possible from the measurement η_k at the current time (and possible also at past times) is called the "State Estimation" problem. The probably most commonly used method (cf. [67]) for state estimation is based on KALMAN filter methods (for linear models) (Kalman [69]) or extended KALMAN filters (nonlinear case) (Julier and Uhlmann [68]). Another method is based on "Moving Horizon Estimation", which can be viewed as the dual problem to the moving horizon NMPC subproblem. In this method, the estimation problem is interpreted and formulated as a parameter estimation problem with a time horizon that includes measurements of the past and ends at the current time.

For more detailed overview on state estimation methods we refer the reader to [98, Chapter 4]. In this work, we assume that full state information is always available. △

The characterizing property of any moving horizon NMPC scheme is, that it is based on NMPC subproblems of constant horizon length and the decisions are based on the predicted behavior of the system on a time horizon moving forward in time.

Under the assumption that the problems $P_{\text{move}}(t,x)$ always have a unique solution, the initial state x_0 and initial time t_0 are determining the closed-loop trajectory $\boldsymbol{x}_\mu(\cdot;x_0,t_0)$ for all times. To avoid notational clutter, we sometimes omit the initial time t_0 and the initial value x_0 and write $\boldsymbol{x}_\mu(\tau)$ for $\boldsymbol{x}_\mu(\tau;x_0,t_0)$ and $\boldsymbol{u}_\mu(\tau)$ for $\boldsymbol{u}_\mu(\tau;x_0,t_0)$ if it becomes clear from the context.

2.2.2 The Shrinking Horizon Scheme

Some processes are only defined on a fixed time horizon, i.e., they end at an a priori known end time T. A moving horizon scheme does not make sense in this case because at some time the end of the moving horizon will pass the time T. For such processes, the "Shrinking Horizon" scheme can be an alternative.

Contrary to the moving horizon scheme, the shrinking horizon NMPC scheme is based on a family of OCPs $P_{\text{shrink}}(t,x)$ with time horizons $\mathcal{T}^t_{\text{shrink}} = [t,T]$ ending at the pre-specified fixed time $T \in \mathbb{R}$.

An example for such a parametrized family of OCPs could be the following problem :

$$\begin{aligned}
\min_{\substack{\boldsymbol{x}\in\mathcal{AC}^{n_x}(\mathcal{T}^t_{\text{shrink}}),\\ \boldsymbol{u}\in L^{n_u}_\infty(\mathcal{T}^t_{\text{shrink}})}} \quad & \varphi(\boldsymbol{x},\boldsymbol{u}) := \int_t^T \ell(\tau,\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\mathrm{d}\tau + m(\boldsymbol{x}(T)) \qquad (2.5)\\
\text{s.t.} \quad & \boldsymbol{x}(t) = x,\\
& \dot{\boldsymbol{x}}(\tau) = f(\tau,\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)), \tau\in\mathcal{T}^t_{\text{shrink}},\\
& 0 \le c(\tau,\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)),\ \tau\in\mathcal{T}^t_{\text{shrink}},\\
& 0 \le \Psi(T,\boldsymbol{x}(T)).
\end{aligned}$$

Analogously to the moving horizon scheme we denote the solution of $P_{\text{shrink}}(t_i,x_i)$ by $(\boldsymbol{x}_i^\star,\boldsymbol{u}_i^\star)$. The shrinking horizon NMPC scheme is defined as the moving horizon NMPC scheme in Algorithm 1 with the only difference that the sampling time grid $\mathbb{T}$ is a partition of $[t_0, T]$ and that the problems $P_{\text{move}}(t_i, x_i)$ are replaced by the problems $P_{\text{shrink}}(t_i, x_i)$.

The closed-loop state and control trajectories for the shrinking horizon scheme are defined in the same way as for the moving horizon scheme:

$$\boldsymbol{x}_\mu(\tau; x_0, t_0) := \boldsymbol{x}_i^\star(\tau) \text{ for } \tau \in [t_i, t_{i+1}), \tag{2.6}$$

$$\boldsymbol{u}_\mu(\tau; x_0, t_0) := \boldsymbol{u}_i^\star(\tau) \text{ for } \tau \in [t_i, t_{i+1}). \tag{2.7}$$

An essential difference of moving and shrinking horizon scheme arises due to the **"Principle of Optimality"**

> "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." Bellman [9, R.E. Bellman 1957]

which for the solutions of the shrinking horizon problems $P_{\text{shrink}}(t_i, x_i)$ guarantees that

$$(\boldsymbol{x}_{i+1}^\star, \boldsymbol{u}_{i+1}^\star) = (\boldsymbol{x}_i^\star, \boldsymbol{u}_i^\star)|_{\mathcal{T}_{\text{shrink}}^{t_{i+1}}} \tag{2.8}$$

holds for all i, provided $x_{i+1} = \boldsymbol{x}_i^\star(t_{i+1})$ holds for all i (perturbation free, no plant-model mismatch). As a consequence, for a shrinking horizon NMPC scheme it holds that

$$(\boldsymbol{x}_\mu, \boldsymbol{u}_\mu) = (\boldsymbol{x}_0^\star, \boldsymbol{u}_0^\star). \tag{2.9}$$

This means that the closed-loop behavior of the system will be exactly as predicted by the first open-loop prediction. In particular, this implies recursive feasibility and stability of the shrinking horizon NMPC scheme.

This is a fundamental difference to the behavior of a moving horizon NMPC controller, where the closed-loop trajectory can be very different from the open-loop predictions.

2.3 Closed-Loop Behavior

As we have seen above, for the shrinking horizon NMPC scheme the closed-loop trajectory is identical to the solution of the first NMPC subproblem. This means that, in theory, already from the first sampling time on, the behavior of the system until the end time T is known.

This is fundamentally different for moving horizon NMPC schemes, where the principle of optimality cannot be applied and a number of questions concerning the properties of the resulting controlled system arise. These questions deal with important topics such as recursive feasibility, stability and economic performance of the closed-loop system.

Since the closed-loop behavior describes how the system will actually behave (contrary to "Open-Loop" predictions of the NMPC subproblems), it is of great importance to understand how the setup of the NMPC subproblem affects the closed-loop behavior of the system.

2.3.1 Recursive Feasibility

From the definition of the moving horizon NMPC Algorithm 1 it is not clear whether the closed-loop feedback trajectory $(\boldsymbol{x}_\mu, \boldsymbol{u}_\mu)$ exists at all, i.e. the NMPC method is well defined. Since NMPC is a feedback method for controlling processes, it is of great importance that the feedback trajectory exists. Any infeasibility of the NMPC subproblem leads to a failure of the feedback method and the controlled process will behave in an uncontrolled manner, which could be catastrophic.

The desired property of the NMPC algorithm that guarantees the existence of the closed-loop feedback trajectory is called **"Recursive Feasibility"**. Put in simple words, this property guarantees that if the NMPC subproblem at some sampling-time t_i is feasible, then the subsequent NMPC subproblem at the next sampling time t_{i+1} is also feasible.

Definition 2.1 (Recursive Feasibility)
An NMPC scheme with sampling times $\mathbb{T} = \{t_i, i \in \mathbb{N}\}$ based on the subproblems $P(t,x)$ is called recursively feasible, if for any $i \in \mathbb{N}$ and $x \in \mathbb{R}^{n_x}$ it holds

$$P(t_i, x) \text{ is feasible} \quad \Rightarrow \quad P(t_{i+1}, \boldsymbol{x}_i^\star(t_{i+1})) \text{ is feasible} . \tag{2.10}$$

△

Often, recursive feasibility is guaranteed by including appropriate terminal weights and / or terminal constraint of the form $0 \le \Psi(\boldsymbol{x}(t_i + T))$ to the formulation of the NMPC subproblem (cf. the survey [84]).

In general, recursive feasibility will depend on the formulation of the NMPC subproblems. In particular the functions ℓ, m, c, the sampling time grid $\mathbb{T}$ and the length of the time horizon T are important parameters that have to be chosen appropriately.

The recursive feasibility property can be considered as the minimal requirement for an NMPC controller, because then an NMPC controller is at least able to deliver a feedback that keeps the controlled process within the imposed operational bounds $(0 \le c\left(\tau, \boldsymbol{x}_\mu(\tau), \boldsymbol{u}_\mu(\tau)\right)$ for almost all $\tau \ge t_0)$ provided the first NMPC subproblem $P(x_0, t_0)$ is feasible.

2.3.2 Stability of NMPC Schemes

Provided the NMPC algorithm has the recursive feasibility property, the next important question is asking for the long-term behavior of the closed-loop feedback. Suppose the purpose of the NMPC method is stabilization of the process at a predefined reference trajectory $\boldsymbol{x}_{\text{ref}} : [t_0, \infty) \to \mathbb{R}^{n_x}$. Then, stability of the NMPC scheme at this reference can be defined in an analogue way as it is done for solutions of IVPs in Section 1.2.2.

Definition 2.2 (Stability of an NMPC Feedback at a Reference Solution)
Let $(\boldsymbol{x}_{\text{ref}}, \boldsymbol{u}_{\text{ref}}) : \mathbb{R}_{\ge 0} \to \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ be a solution of the IVP $\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t), \boldsymbol{u}(t))$ with initial value $\boldsymbol{x}(0) = x_0 \in \mathbb{R}^{n_x}$. We say an NMPC algorithm is

- **Lyapunov**-stabilizing the system at $(\boldsymbol{x}_{\text{ref}}, \boldsymbol{u}_{\text{ref}})$ if for any $\varepsilon \ge 0$ there exists $\delta > 0$ such that for all $i \in \mathbb{N}$ it holds:

$$\|x - \boldsymbol{x}_{\text{ref}}(t_i)\| \le \varepsilon \;\Rightarrow\; \left\|\boldsymbol{x}_\mu(\tau; x, t_i) - \boldsymbol{x}_{\text{ref}}(\tau)\right\| \le \delta \text{ for all } \tau \ge t_i,$$

- **locally asymptotically** stabilizing the system at $(\boldsymbol{x}_{\text{ref}}, \boldsymbol{u}_{\text{ref}})$, if for any $i \in \mathbb{N}$ there exists $\delta > 0$ and a function $\beta \in \mathcal{KL}$ such that

$$\|\boldsymbol{x}_{\text{ref}}(t_i) - x\| \leq \delta \Rightarrow \left\|\boldsymbol{x}_{\text{ref}}(\tau) - \boldsymbol{x}_{\mu}(\tau; x, t_i)\right\| \leq \beta(\|x - \boldsymbol{x}_{\text{ref}}(t_i)\|, \tau - t_i) \text{ for all } \tau \geq t_i, \tag{2.11}$$

- **locally uniformly asymptotically** stabilizing the system at $(\boldsymbol{x}_{\text{ref}}, \boldsymbol{u}_{\text{ref}})$, if there exists a $\delta > 0$ and a function $\beta \in \mathcal{KL}$ such that

$$\begin{aligned} &\|\boldsymbol{x}_{\text{ref}}(t_i) - x\| \leq \delta \Rightarrow \\ &\left\|\boldsymbol{x}_{\text{ref}}(\tau) - \boldsymbol{x}_{\mu}(\tau; x, t_i)\right\| \leq \beta(\|x - \boldsymbol{x}_{\text{ref}}(t_i)\|, \tau - t_i) \text{ for all } i \in \mathbb{N} \text{ and for all } \tau \geq t_i. \end{aligned} \tag{2.12}$$

△

The knowledge that an NMPC scheme is stabilizing a system at a trajectory $\boldsymbol{x}_{\text{ref}}$, is useful to understand the closed-loop behavior of the resulting system as it allows to transfer properties of $(\boldsymbol{x}_{\text{ref}}, \boldsymbol{u}_{\text{ref}})$ to $(\boldsymbol{x}_{\mu}, \boldsymbol{u}_{\mu})$.

2.3.3 Economic Performance

If some performance criterion $\ell_{\text{e}} : \mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ (which can be different from the function ℓ used in the NMPC subproblem) is associated to the dynamical system, it is an important question how the closed-loop trajectory $(\boldsymbol{x}_{\mu}, \boldsymbol{u}_{\mu})$ performs with respect to ℓ_{e}. This means that the development of

$$t \mapsto \int_{t_0}^{t} \ell_{\text{e}}(\tau, \boldsymbol{x}_{\mu}(\tau), \boldsymbol{u}_{\mu}(\tau)) \mathrm{d}\tau \tag{2.13}$$

has to be understood. This question is closely related to the stability properties of the NMPC scheme. If it is known that the controller stabilizes the system at some reference trajectory $(\boldsymbol{x}_{\text{ref}}, \boldsymbol{u}_{\text{ref}})$, the asymptotic behavior of $\int_{t_0}^{t} \ell_{\text{e}}(\tau, \boldsymbol{x}_{\mu}(\tau), \boldsymbol{u}_{\mu}(\tau)) \mathrm{d}\tau$ will be equal to that of the reference trajectory provided the initial value $\boldsymbol{x}_{\mu}(t_0) = x_0$ of the closed-loop system is close enough to the reference value $\boldsymbol{x}_{\text{ref}}(t_0)$.

It could also happen that none of the presented stability definitions (in Section 2.3.2) applies but the system nevertheless produces closed-loop trajectories that perform economically well.

2.4 Tracking/Stabilizing Nonlinear Model Predictive Control

Stabilization of a given dynamical system at a desired reference trajectory is one of the first and most important applications of NMPC [102, 44].

In the following, we introduce tracking NMPC as an example of a moving horizon NMPC scheme based on an NMPC subproblem, which is an OCP of tracking type, and analyze its closed-loop behavior with respect to recursive feasibility and stability.

2.4.1 Problem Formulation

The informal formulation of a tracking problem is quite straight-forward. Given a dynamical system $\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t), \boldsymbol{u}(t))$ together with some path constraints $0 \leq c(t, \boldsymbol{x}(t), \boldsymbol{u}(t))$ and a reference solution $(\boldsymbol{x}_{\text{ref}}, \boldsymbol{u}_{\text{ref}}) \in \mathcal{AC}^{n_x}_{\text{loc}} \times \mathrm{L}^{n_u}_{\infty,\text{loc}}$ of the system that satisfies the path constraint almost everywhere, a way of generating feedback is searched that reliably keeps the system close to the reference trajectory and steers the system back to the reference trajectory in case of perturbations.

2.4.2 The Tracking Subproblem

We set up the NMPC subproblem for the tracking controller by defining a tracking OCP. The tracking OCP minimizes the L^2-distance of the predicted trajectory to the reference trajectory subject to some additional constraints that guarantee recursive feasibility and stability of the resulting NMPC controller.

Tracking Objective Functional

For $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}(\mathcal{T}^t) \times \mathrm{L}^{n_x}_{\infty}(\mathcal{T}^t)$ (with $\mathcal{T}^t := [t, t+T]$) we define tracking LAGRANGE term

$$\ell_{\text{track}}(\tau, x, u) := \|x - \boldsymbol{x}_{\text{ref}}(\tau)\|^2 + \|u - \boldsymbol{u}_{\text{ref}}(\tau)\|^2 \tag{2.14}$$

and the associated tracking objective functional

$$\varphi_{t,\text{track}}(\boldsymbol{x}, \boldsymbol{u}) := \int_{\mathcal{T}^t} \ell_{\text{track}}(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\mathrm{d}\tau. \tag{2.15}$$

The functional $\varphi_{t,\text{track}}$ is bounded from below by 0 and it vanishes if and only if $(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) = (\boldsymbol{x}_{\text{ref}}(\tau), \boldsymbol{u}_{\text{ref}}(\tau))$ holds for almost all $\tau \in \mathcal{T}^t$.

Terminal Constraints

The terminal constraints in the tracking NMPC subproblem are often used to guarantee recursive feasibility and stability.

They can be of the following type:

- **Terminal Region Constraint**:

$$\boldsymbol{x}(t+T) \in X(t+T), \tag{2.16}$$

  where $X(t+T)$ would be a vicinity of $\boldsymbol{x}_{\text{ref}}(t+T)$, for example $X(t+T) = B_\delta(\boldsymbol{x}_{\text{ref}}(t+T)) = \{x \in \mathbb{R}^{n_x} : \|x - \boldsymbol{x}_{\text{ref}}(t+T)\| \leq \delta\}$ with δ not too big.

- **Terminal Equality Constraint**:

$$\boldsymbol{x}(t+T) = \boldsymbol{x}_{\text{ref}}(t+T). \tag{2.17}$$

Such a terminal constraint immediately implies recursive feasibility of the NMPC scheme, since it ensures that the admissible trajectories can be extended for arbitrarily long time intervals by $(\boldsymbol{x}_{\text{ref}}, \boldsymbol{u}_{\text{ref}})$.

In practice, the terminal constraint is usually represented by a function $\Psi_{\text{ter}} : \mathbb{R} \times \mathbb{R}^{n_x} \to \mathbb{R}$ and then is of the form $0 \le \Psi_{\text{ter}}(t+T, \boldsymbol{x}(t+T))$.

The Tracking NMPC Subproblem

The complete tracking subproblem $P_{\text{track}}(t,x)$ (here with terminal equality constraint (2.17)) then looks as follows:

$$\min_{\substack{\boldsymbol{x}\in\mathcal{AC}^{n_x}(\mathcal{T}^t),\\ \boldsymbol{u}\in\mathrm{L}_\infty^{n_x}(\mathcal{T}^t)}} \quad \varphi_{t,\text{track}}(\boldsymbol{x},\boldsymbol{u}) \tag{2.18a}$$

$$\text{s.t.} \quad \boldsymbol{x}(t) = x_i, \tag{2.18b}$$

$$\dot{\boldsymbol{x}}(\tau) = f(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \qquad \tau \in \mathcal{T}^t, \tag{2.18c}$$

$$0 \le c(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \qquad \tau \in \mathcal{T}^t, \tag{2.18d}$$

$$0 = \boldsymbol{x}(t+T) - \boldsymbol{x}_{\text{ref}}(t+T). \tag{2.18e}$$

2.4.3 Closed-Loop Behavior of the Tracking NMPC Controller

We discuss the properties of the moving horizon NMPC controller that is based on the tracking NMPC subproblem $P_{\text{track}}(t,x)$. We assume that close to the reference trajectory, the dynamical system $\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t), \boldsymbol{u}(t))$ has the following controllability property.

Assumption 2.1 (Controllability in Vicinity of Reference Trajectory)
There exists $\delta > 0$ and a $\mathcal{K}$-function η such that for any $(t,x) \in \mathbb{R} \times \mathbb{R}^{n_x}$ with $\|x - \boldsymbol{x}_{\text{ref}}(t)\| \le \delta$ there exists a pair $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{AC}^{n_x}(\mathcal{T}^t) \times \mathrm{L}_\infty^{n_x}(\mathcal{T}^t)$ such that $(\boldsymbol{x},\boldsymbol{u})$ is admissible for $P_{\text{track}}(t,x)$ and

$$\int_{\mathcal{T}^t} \left(\|\boldsymbol{x}(\tau) - \boldsymbol{x}_{\text{ref}}(\tau)\| + \|\boldsymbol{u}(\tau) - \boldsymbol{u}_{\text{ref}}(\tau)\|\right) \mathrm{d}\tau \le \eta(\|x - \boldsymbol{x}_{\text{ref}}(t)\|). \tag{2.19}$$

△

We include the following stability result for tracking NMPC because it nicely illustrates the role of the design of the tracking NMPC subproblem in particular the role of the terminal constraint (2.18e) for the stability properties of the resulting closed-loop system.

Lemma 2.3 (Recursive Feasibility and Stability of the Tracking NMPC Algorithm)
The moving horizon NMPC controller with NMPC subproblem $P_{\text{track}}(t_i, x_i)$ defines a recursively feasible, at $(\boldsymbol{x}_{\text{ref}}, \boldsymbol{u}_{\text{ref}})$ locally uniformly asymptotically stabilizing controller.

Proof Recursive Feasibility: Let $i \in \mathbb{N}$ and $x \in \mathbb{R}^{n_x}$ be such that $P_{\text{track}}(t_i, x)$ is feasible. Let $(\boldsymbol{x}_i, \boldsymbol{u}_i) \in \mathcal{AC}^{n_x}(\mathcal{T}^{t_i}) \times \mathrm{L}_\infty^{n_u}(\mathcal{T}^{t_i})$ be admissible for $P_{\text{track}}(t_i, x)$. We can now extend this solution to the interval $\mathcal{T}^{t_i} \cup \mathcal{T}^{t_{i+1}} = [t_i, t_{i+1}+T]$ by concatenating it with $(\boldsymbol{x}_{\text{ref}}, \boldsymbol{u}_{\text{ref}})$ the following way:

$$(\tilde{\boldsymbol{x}}_{i+1}, \tilde{\boldsymbol{u}}_{i+1})(\tau) := \begin{cases} (\boldsymbol{x}_i, \boldsymbol{u}_i)(\tau) & \text{if } \tau \in \mathcal{T}^{t_i}, \\ (\boldsymbol{x}_{\text{ref}}, \boldsymbol{u}_{\text{ref}})(\tau) & \text{if } \tau \in \mathcal{T}^{t_{i+1}} \setminus \mathcal{T}^{t_i}. \end{cases} \tag{2.20}$$

From this definition it clearly follows that that the restriction $(\boldsymbol{x}_{i+1}, \boldsymbol{u}_{i+1}) := (\tilde{\boldsymbol{x}}_{i+1}, \tilde{\boldsymbol{u}}_{i+1})|_{\mathcal{T}^{t_{i+1}}}$ is admissible for the OCP $P_{\text{track}}(t_{i+1}, \boldsymbol{x}_i(t_{i+1}))$.

Stability: Let $(\boldsymbol{x}_i^\star, \boldsymbol{u}_i^\star) \in \mathcal{AC}^{n_x}(\mathcal{T}^{t_i}) \times \mathrm{L}_\infty^{n_u}(\mathcal{T}^{t_i})$ be the solution of problem $P_{\text{track}}(t_i, \boldsymbol{x}_i(t_i))$ at sampling time t_i. Then, just as in the proof of recursive feasibility, by concatenating this solution with $(\boldsymbol{x}_{\text{ref}}, \boldsymbol{u}_{\text{ref}})$, we get an admissible pair $(\boldsymbol{x}_{i+1}, \boldsymbol{u}_{i+1})$ for problem $P_{\text{track}}(t_{i+1}, \boldsymbol{x}_i^\star(t_{i+1}))$. By evaluating $\varphi_{t_{i+1},\text{track}}(\boldsymbol{x}_{i+1}, \boldsymbol{u}_{i+1})$, we get an upper bound for $\varphi_{t_{i+1},\text{track}}(\boldsymbol{x}_i^\star(t_{i+1}))$:

$$\begin{aligned}
\varphi_{t_{i+1},\text{track}}(\boldsymbol{x}_{i+1}, \boldsymbol{u}_{i+1}) &= \int_{t_{i+1}}^{t_{i+1}+T} \|\boldsymbol{x}_{i+1}(\tau) - \boldsymbol{x}_{\text{ref}}(\tau)\|^2 + \|\boldsymbol{u}_{i+1}(\tau) - \boldsymbol{u}_{\text{ref}}(\tau)\|^2 \,\mathrm{d}\tau \\
&= \int_{t_{i+1}}^{t_i+T} \left\|\boldsymbol{x}_i^\star(\tau) - \boldsymbol{x}_{\text{ref}}(\tau)\right\|^2 + \left\|\boldsymbol{u}_i^\star(\tau) - \boldsymbol{u}_{\text{ref}}(\tau)\right\|^2 \,\mathrm{d}\tau \\
&\quad + \underbrace{\int_{t_i+T}^{t_i+T} \|\boldsymbol{x}_{\text{ref}}(\tau) - \boldsymbol{x}_{\text{ref}}(\tau)\|^2 + \|\boldsymbol{u}_{\text{ref}}(\tau) - \boldsymbol{u}_{\text{ref}}(\tau)\|^2 \,\mathrm{d}\tau}_{=0} \\
&= \varphi_{t_i,\text{track}}(x_i) - \int_{t_i}^{t_{i+1}} \left\|\boldsymbol{x}_i^\star(\tau) - \boldsymbol{x}_{\text{ref}}(\tau)\right\|^2 + \left\|\boldsymbol{u}_i^\star(\tau) - \boldsymbol{u}_{\text{ref}}(\tau)\right\|^2 \,\mathrm{d}\tau.
\end{aligned}$$

Now we give an upper bound for the term $\int_{t_i}^{t_{i+1}} \underbrace{\left\|\boldsymbol{x}_i^\star(\tau) - \boldsymbol{x}_{\text{ref}}(\tau)\right\|^2}_{=:\|\Delta(\tau)\|^2} \mathrm{d}\tau$. We calculate

$$\begin{aligned}
\left|\frac{\partial}{\partial\tau}\|\Delta(\tau)\|^2\right| &= \left|2\Delta(\tau)\dot{\Delta}(\tau)\right| \\
&= \left|2\Delta(\tau)(\dot{\boldsymbol{x}}_i^\star(\tau) - \dot{\boldsymbol{x}}_{\text{ref}}(\tau))\right| \\
&= \left|2\Delta(\tau)(f(\boldsymbol{x}_i^\star(\tau), \boldsymbol{u}_i^\star(\tau)) - f(\boldsymbol{x}_{\text{ref}}(\tau), \boldsymbol{u}_{\text{ref}}(\tau)))\right| \\
&\le 4LM\|\Delta(\tau)\|
\end{aligned} \tag{2.21}$$

Here L is the global LIPSCHITZ-constant of the right-hand side f and M the diameter of the set of the feasible states/controls. Because of $\frac{\partial}{\partial\tau}\|\Delta(\tau)\|^2 \ge -4LM\|\Delta(\tau)\|$, the solution of the IVP $\dot{\gamma}(t) = -4LM\sqrt{\gamma(t)}$ with $\gamma(t_i) = \|\Delta(t_i)\|^2$ is a lower bound for $\|\Delta(\tau)\|^2$. The solution γ can be expressed analytically and we get (with $c = 4LM$):

$$\|\Delta(\tau)\|^2 \ge \gamma(\tau) = \frac{1}{4}(c^2(\tau - t_i)^2 + 2c\delta(\tau - t_i) + \delta^2). \tag{2.22}$$

This shows that the difference $\|\Delta(\tau)\|^2$ doesn't decrease arbitrarily fast and in fact implies the existence of a $\mathcal{K}_\infty$-function α_1 such that

$$\int_{t_i}^{t_{i+1}} \left\|\boldsymbol{x}_i^\star(\tau) - \boldsymbol{x}_{\mathrm{ref}}(\tau)\right\|^2 \mathrm{d}\tau \geq \alpha_1\left(\left\|\boldsymbol{x}_i^\star(t_i) - \boldsymbol{x}_{\mathrm{ref}}(t_i)\right\|\right) \tag{2.23}$$

holds. This implies that for the closed-loop trajectory $\boldsymbol{x}_\mu$ and for all $i \in \mathbb{N}$ it holds

$$\varphi_{\mathrm{track},i+1}(\boldsymbol{x}_\mu(t_{i+1})) \leq \varphi_{\mathrm{track},i}(\boldsymbol{x}_\mu(t_i)) - \alpha_1\left(\left\|\boldsymbol{x}_\mu(t_i) - \boldsymbol{x}_{\mathrm{ref}}(t_i)\right\|\right). \tag{2.24}$$

The same calculation also shows

$$\varphi_{\mathrm{track},i}(x) \geq \alpha_1(\|x - \boldsymbol{x}_{\mathrm{ref}}(t_i)\|) \tag{2.25}$$

for all $i \in \mathbb{N}$ and $x \in B_\delta(\boldsymbol{x}_{\mathrm{ref}}(t_i))$. Furthermore, the controllability assumption implies the existence of a $\mathcal{K}$-function α_2 such that

$$\varphi_{\mathrm{track},i}(x) \leq \alpha_2(\|x - \boldsymbol{x}_{\mathrm{ref}}(t_i)\|) \tag{2.26}$$

holds for all $i \in \mathbb{N}$ and $x_i \in B_\delta(\boldsymbol{x}_{\mathrm{ref}}(t_i))$.

Now, similar to the proof of the stability Lemma (1.11) for IVPs, the properties (2.23), (2.24) and (2.26) are used to prove the existence of a $\mathcal{KL}$-function β such that

$$\left\|\boldsymbol{x}_\mu(\tau; x_i, t_i) - \boldsymbol{x}_{\mathrm{ref}}(\tau)\right\| \leq \beta(\|x_i - \boldsymbol{x}_{\mathrm{ref}}(t_i)\|, \tau - t_i) \tag{2.27}$$

holds for all $i \in \mathbb{N}$, $x \in B_\delta(\boldsymbol{x}_{\mathrm{ref}}(t_i))$ and $\tau \geq t_i$. □

The Lemma shows that the tracking NMPC controller, under reasonable assumptions (2.1), is capable of producing a locally asymptotically stabilizing feedback.

2.5 Economic Nonlinear Model Predictive Control

In tracking NMPC the reference trajectory is usually the result of some OCP itself (often with an economically motivated running-cost ℓ_{e}). The tracking NMPC controller then has a purely regulatory function and ensures that the process is stabilized at the desired reference trajectory. However, from an economical viewpoint, the tracking controller can perform suboptimal. In particular, perturbations or parameter changes can result in the fact that the reference trajectory itself is not economically optimal anymore.

The idea of E-NMPC is, to not only let the controller act as stabilizing instance but rather let it do the optimization simultaneously. In E-NMPC instead of a tracking functional, the original economic performance criterion ℓ_{e} is used in the NMPC subproblem. This allows for greater flexibility in the controller design, since the controller is not just steering the system to a predefined reference trajectory.

The convenient stability properties of tracking NMPC do not easily transfer to such economic NMPC schemes. It is not at all clear how good (in the sense of the economic performance criterion ℓ_{e}) the resulting closed-loop system of such a controller performs or if any asymptotic stability properties hold. The reason for this insufficiency is that, contrary to a tracking NMPC controller, the cost-to-go function not necessarily has the properties of a LYAPUNOV-function.

For the case of dissipative steady-states or dissipative periodic steady-states, there exist stability results for economic NMPC, which we will present in the following. A comprehensive overview for the dissipative steady-state case can be found in Ellis et al. [37], for the dissipative periodic case we also refer the reader to Zanon et al. [120] and Zanon et al. [123].

2.5.1 Economic Nonlinear Model Predictive Control for Systems with Optimal Dissipative Equilibria

Consider the autonomous system $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t))$ with LIPSCHITZ continuous economic performance criterion $\ell_\mathrm{e} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ and a path-constraint $0 \le c(x,u)$. The working assumption in this section is the following.

Assumption 2.2 (Existence of Strictly Dissipative Steady-State)
The system has a steady-state $(x_\mathrm{e}^*, u_\mathrm{e}^*) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ that is strictly dissipative, i.e. there exists a $\mathcal{K}_\infty$-function σ and a continuously differentiable function $\lambda : \mathbb{R}^{n_x} \to \mathbb{R}$ such that

- λ is bounded on $A_x := \{x \in \mathbb{R}^{n_x} : \exists u \in \mathbb{R}^{n_u} \text{ with } 0 \le c(x,u)\}$,
- The inequality

$$\sigma\left(\left\|x - x_\mathrm{e}^*\right\|\right) + \nabla\lambda(x)^T f(x,u) \le \ell_\mathrm{e}(x,u) - \ell_\mathrm{e}(x_\mathrm{e}^*, u_\mathrm{e}^*) \tag{2.28}$$

  holds for all $(x,u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ with $0 \le c(x,u)$. △

The existence of a strictly dissipative steady-state $(x_\mathrm{e}^*, u_\mathrm{e}^*)$ has strong consequences. For example, it implies that the best average output with respect to ℓ_e is equal to $\ell_\mathrm{e}(x_\mathrm{e}^*, u_\mathrm{e}^*)$.

To see this, let $t \in \mathbb{R}_{\ge 0}$ and $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}_\mathrm{loc}^{n_x} \times \mathrm{L}_{\infty,\mathrm{loc}}^{n_u}$ be any state/control trajectory that satisfies the ODE and path constraint $0 \le c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))$ for almost all $\tau \in \mathbb{R}_{\ge 0}$. Then, by using the strict dissipativity property (2.28), we can calculate

$$\int_0^t \ell_\mathrm{e}(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\mathrm{d}\tau \ge \int_0^t \left(\ell_\mathrm{e}(x_\mathrm{e}^*, \boldsymbol{u}_\mathrm{e}^*) + \sigma\left(\left\|\boldsymbol{x}(\tau) - x_\mathrm{e}^*\right\|\right) + \nabla\lambda(\boldsymbol{x}(\tau))^T f(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\right)\mathrm{d}\tau \tag{2.29}$$

$$= t\ell_\mathrm{e}(x_\mathrm{e}^*, u_\mathrm{e}^*) + \int_0^t \sigma\left(\left\|\boldsymbol{x}(\tau) - x_\mathrm{e}^*\right\|\right)\mathrm{d}\tau + \underbrace{\lambda(\boldsymbol{x}(t)) - \lambda(\boldsymbol{x}(0))}_{\text{bounded, because } A_x \text{ is bounded}} \tag{2.30}$$

It follows

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t \ell_\mathrm{e}(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\mathrm{d}\tau \ge \ell_\mathrm{e}(x_\mathrm{e}^*, u_\mathrm{e}^*). \tag{2.31}$$

Furthermore, we assume that the system is controllable in a vicinity of the steady-state $(x_\mathrm{e}^*, u_\mathrm{e}^*)$ in a sense similar to Assumption 2.1.

Assumption 2.3 (Controllability in Vicinity of x_e^*)
There exists a $\delta > 0$ and a $\mathcal{K}$-function η such that for any $x \in \mathbb{R}^{n_x}$ with $\|x - x_e^*\| \leq \delta$ there exists a pair $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}([0,T]) \times \mathrm{L}_\infty^{n_u}([0,T])$ that satisfies $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t))$ and $0 \leq c(\boldsymbol{x}(t), \boldsymbol{u}(t))$ almost everywhere and

$$\int_0^T \left(\|\boldsymbol{x}(\tau) - x_e^*\| + \|\boldsymbol{u}(\tau) - u_e^*\|\right) \mathrm{d}\tau \leq \eta\left(\|x - x_e^*\|\right). \tag{2.32}$$

△

For such systems, we now discuss the properties of the NMPC scheme based on the following NMPC subproblem $P_e(t_i, x_i)$ defined on the time horizons $\mathcal{T}^{t_i} = [t_i, t_i + T]$:

$$\min_{\substack{\boldsymbol{x} \in \mathcal{AC}^{n_x}(\mathcal{T}^{t_i}),\\ \boldsymbol{u} \in \mathrm{L}_\infty^{n_u}(\mathcal{T}^{t_i})}} \quad \varphi_e(\boldsymbol{x}, \boldsymbol{u}) := \int_{\mathcal{T}^{t_i}} \ell_e(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) \mathrm{d}\tau + M(\boldsymbol{x}(t_i + T)) \tag{2.33a}$$

$$\text{s.t.} \quad \boldsymbol{x}(t_i) = x_i, \tag{2.33b}$$

$$\dot{\boldsymbol{x}}(\tau) = f(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \tau \in \mathcal{T}^{t_i}, \tag{2.33c}$$

$$0 \leq c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \tau \in \mathcal{T}^{t_i}, \tag{2.33d}$$

$$\boldsymbol{x}(t_i + T) \in \mathbb{X}. \tag{2.33e}$$

Contrary to the tracking approach, here the objective functional directly measures the economic performance of the trajectory itself. The function $M : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ is a penalty term with the property that there exists a $\mathcal{K}_\infty$-function α such that

$$M(x) \leq \alpha\left(\|x_e^* - x\|\right) \tag{2.34}$$

holds for all $x \in \mathbb{R}^{n_x}$. The terminal region $\mathbb{X}$ is a closed set containing x_e^* and the purpose of the terminal constraint (2.33e) is, similar to constraint (2.18e) in the tracking NMPC subproblem, to guarantee recursive feasibility via an auxiliary control law defined on $\mathbb{X}$.

Assumption 2.4 (Auxiliary Control Law)
There exists a smooth map $\kappa : \mathbb{X} \subset \mathbb{R}^{n_x} \to \mathbb{R}^{n_u}$ such that the solution $\Phi_\kappa(\cdot; x_0, t_0)$ of the IVP $\dot{\boldsymbol{x}}(\tau) = f(\boldsymbol{x}(\tau), \kappa(\boldsymbol{x}(\tau)))$ with $\boldsymbol{x}(t_0) = x_0 \in A_x$ exists for all initial values $x_0 \in A_x$ and for all times $\tau \geq t_0$ and has the following properties:

- it stays in $\mathbb{X}$:

$$\Phi_\kappa(\tau; x_0, t_0) \in \mathbb{X} \text{ for all } \tau \geq t_0, \tag{2.35}$$

- it satisfies the path constraints

$$0 \leq c(\Phi_\kappa(\tau; x_0, t_0), \kappa(\Phi_\kappa(\tau; x_0, t_0))) \text{ for all } \tau \geq t_0, \tag{2.36}$$

- the terminal penalty M satisfies

$$\frac{\partial}{\partial \tau} M(\Phi_\kappa(\tau; x_0, t_0)) \leq -\ell_{\mathrm{e}}(\Phi_\kappa(\tau; x_0, t_0), \kappa(\Phi_\kappa(\tau; x_0, t_0))) \text{ for all } \tau \geq t_0. \tag{2.37}$$

△

This assumption implies that the cost-to-go function is decreasing along closed-loop trajectories.

Lemma 2.4 (Monotonically Decreasing Cost-to-Go)
Let Assumption 2.4 hold for the system $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t))$. Then the economic NMPC scheme based on the NMPC subproblem $P_{\mathrm{e}}(t, x)$ is recursively feasible and the sequence $(\varphi_{\mathrm{e}}(\boldsymbol{x}_i^\star, \boldsymbol{u}_i^\star))_{i\in\mathbb{N}}$ is monotonically decreasing.

Proof Recursive feasibility directly follows from Assumption 2.4. In particular (2.35) and (2.36) can be used to extend any admissible pair for problem $P_{\mathrm{e}}(t_i, x_i)$ to an arbitrary long time horizon. Property (2.37) can be used for the following calculation with $x_f := \boldsymbol{x}_i^\star(t_i + T)$:

$$M(\Phi_\kappa(t_{i+1} + T; x_f, t_i + T)) - M(x_f) \leq -\int_{\mathcal{T}^{t_i}} \ell_{\mathrm{e}}(\Phi_\kappa(\tau; x_f, t_i + T), \kappa(\Phi_\kappa(\tau; x_f, t_i + T)))\mathrm{d}\tau. \tag{2.38}$$

Therefore, with the optimality of $(\boldsymbol{x}_{i+1}^\star, \boldsymbol{u}_{i+1}^\star)$ it follows:

$$\begin{aligned}
\varphi_{\mathrm{e}}(\boldsymbol{x}_{i+1}^\star, \boldsymbol{u}_{i+1}^\star) \leq & \int_{t_{i+1}}^{t_i+T} \ell_{\mathrm{e}}(\boldsymbol{x}_i^\star(\tau), \boldsymbol{u}_i^\star(\tau))\mathrm{d}\tau \\
& + \int_{t_i+T}^{t_{i+1}+T} \ell_{\mathrm{e}}(\Phi_\kappa(\tau; x_f, t_i + T), \kappa(\Phi_\kappa(\tau; x_f, t_i + T)))\mathrm{d}\tau \\
& + M(\Phi_\kappa(t_{i+1} + T; x_f, t_i + T))
\end{aligned} \tag{2.39}$$

$$\leq \int_{t_{i+1}}^{t_i+T} \ell_{\mathrm{e}}(\boldsymbol{x}_i^\star(\tau), \boldsymbol{u}_i^\star(\tau))\mathrm{d}\tau + M(x_f) \tag{2.40}$$

$$= \varphi_{\mathrm{e}}(\boldsymbol{x}_i^\star, \boldsymbol{u}_i^\star) - \int_{t_i}^{t_{i+1}} \ell_{\mathrm{e}}(\boldsymbol{x}_i^\star(\tau), \boldsymbol{u}_i^\star(\tau))\mathrm{d}\tau \tag{2.41}$$

Since we may assume that $\ell_{\mathrm{e}} \geq 0$ holds for all admissible states/controls, this shows that the sequence $\left(\varphi_{\mathrm{e}}(\boldsymbol{x}_i^\star, \boldsymbol{u}_i^\star)\right)_{i\in\mathbb{N}}$ is monotonically decreasing. □

Contrary to the case of tracking NMPC, the decreasing cost-to-go function not necessarily implies stability at the steady-state $(x_{\mathrm{e}}^*, u_{\mathrm{e}}^*)$. The reason for this is that $(x_{\mathrm{e}}^*, u_{\mathrm{e}}^*)$ not necessarily minimizes ℓ_{e}.

It turns out that the dissipativity condition (2.28) is the crucial ingredient that ensures asymptotic stability of the economic NMPC controller [36, 100].

Lemma 2.5 (Asymptotic Stability of Economic NMPC)
Let Assumptions 2.2, 2.3 and 2.4 hold. Then the economic NMPC scheme defined by the NMPC subproblems $P_{\mathrm{e}}(t_i, x_i)$ (2.33) is locally asymptotically stable at $(x_{\mathrm{e}}^*, u_{\mathrm{e}}^*)$.

Proof As the proof works similarly to the stability proof for tracking NMPC we only give an outline. Without loss of generality we may assume that $\ell_{\mathrm{e}}(x_{\mathrm{e}}^*, u_{\mathrm{e}}^*) = 0$ and $\lambda(x_{\mathrm{e}}^*) = 0$. We consider the functional $\tilde{\varphi} : \mathcal{AC}^{n_x}(\mathcal{T}^t) \times \mathrm{L}_\infty^{n_u}(\mathcal{T}^t) \to \mathbb{R}$ defined by

$$\tilde{\varphi}(\boldsymbol{x}, \boldsymbol{u}) := \varphi(\boldsymbol{x}, \boldsymbol{u}) + \lambda(\boldsymbol{x}(t)). \tag{2.42}$$

Slightly abusing notation, we consider the function $x \mapsto \tilde{\varphi}(\boldsymbol{x}^*, \boldsymbol{u}^*)$, where $(\boldsymbol{x}^*, \boldsymbol{u}^*)$ is the solution of problem $P_{\mathrm{e}}(t, x)$ and show that this functions serves as a LYAPUNOV-like function for the NMPC scheme.

Let $(\boldsymbol{x}_k^\star, \boldsymbol{u}_k^\star)_{k\in\mathbb{N}}$ be the sequence of trajectories resulting from the economic NMPC scheme with initial value x_0 at time t_0. The dissipativity condition (2.28) now allows to show that $\tilde{\varphi}$ is decreasing along the sequence $(x_k)_{k\in\mathbb{N}} = (\boldsymbol{x}_k^\star(t_k))_{k\in\mathbb{N}}$:

$$\tilde{\varphi}(x_{k+1}) = \varphi(x_{k+1}) + \lambda(x_{k+1}) \tag{2.43}$$

$$\overset{(2.41)}{\leq} \varphi(x_k) - \int_{t_k}^{t_{k+1}} \ell_{\mathrm{e}}(\boldsymbol{x}_k^\star(\tau), \boldsymbol{u}_k^\star(\tau))\mathrm{d}\tau + \lambda(x_{k+1}) \tag{2.44}$$

$$= \tilde{\varphi}(x_k) - \lambda(x_k) - \int_{t_k}^{t_{k+1}} \ell_{\mathrm{e}}(\boldsymbol{x}_k^\star(\tau), \boldsymbol{u}_k^\star(\tau))\mathrm{d}\tau + \lambda(x_{k+1}) \tag{2.45}$$

$$= \tilde{\varphi}(x_k) - \int_{t_k}^{t_{k+1}} \big(\ell_{\mathrm{e}}(\boldsymbol{x}_k^\star(\tau), \boldsymbol{u}_k^\star(\tau)) - \nabla\lambda(\boldsymbol{x}_k^\star(\tau))^T f(\boldsymbol{x}_k^\star(\tau), \boldsymbol{u}_k^\star(\tau))\big)\mathrm{d}\tau \tag{2.46}$$

$$\leq \tilde{\varphi}(x_k) - \int_{t_k}^{t_{k+1}} \sigma\left(\left\|\boldsymbol{x}_k^\star(\tau) - x_{\mathrm{e}}^*\right\|\right)\mathrm{d}\tau. \tag{2.47}$$

This implies the existence of a $\mathcal{K}_\infty$-function α_3 such that

$$\tilde{\varphi}(x_{k+1}) \leq \tilde{\varphi}(x_k) - \alpha_3\left(\left\|x_k - x_{\mathrm{e}}^*\right\|\right) \tag{2.48}$$

holds for all $k \in \mathbb{N}$. The controllability Assumption 2.3 together with the LIPSCHITZ- continuity of ℓ_{e} and the continuity of λ imply the existence of a $\mathcal{K}_\infty$-function α_2 such that

$$\tilde{\varphi}(x) \leq \alpha_2\left(\left\|x - x_{\mathrm{e}}^*\right\|\right) \tag{2.49}$$

holds for all $x \in \mathbb{X}$. Furthermore, the dissipativity Assumption 2.2 implies

$$\tilde{\varphi}(x_k) \geq \int_{t_k}^{t_k+T} \sigma\left(\left\|\boldsymbol{x}_i^\star(\tau) - x_{\mathrm{e}}^*\right\|\right)\mathrm{d}\tau + \lambda(\boldsymbol{x}_i^\star(t_k + T)) \tag{2.50}$$

Since λ is non-negative, this implies the existence of a $\mathcal{K}_\infty$ function such that

$$\tilde{\varphi}(x) \geq \alpha_1\left(\left\|x - x_{\mathrm{e}}^*\right\|\right) \tag{2.51}$$

holds for all $x \in A_x$. Altogether, with Lemma 1.11 this implies the existence of a $\mathcal{KL}$-function β such that

$$\left\|\boldsymbol{x}_\mu(t_k; x, t_i) - x_{\mathrm{e}}^*\right\| \leq \beta\left(\left\|x - x_{\mathrm{e}}^*\right\|, t_k - t_i\right) \tag{2.52}$$

holds for all $k, i \in \mathbb{N}$ with $k \geq i$ and all $x \in \mathbb{X}$. □

As can be seen, strict dissipativity is a property of the model and the performance criterion that guarantees proper performance of an economic NMPC scheme. However, there may be cases where the dissipativity of the performance criterion is difficult to verify or not given at all. In these cases, it is possible to modify the stage-cost in a way that makes it strictly dissipative for a desired steady-state (x_s, u_s). This modification can be achieved by adding a tracking term $\alpha : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}_{\geq 0}$ that is positive definite at (x_s, u_s) to the stage-cost ℓ_e. The benefit of this method proposed by Rawlings et al. [100] is that it allows to apply the economic-NMPC scheme to a greater class of stage-costs than only the strictly dissipative ones.

Strong Duality

Whether a combination of system and performance criteria satisfies the strict dissipativity condition (2.2) for a steady state can be difficult to verify.

A property that is easier to verify and that implies strict dissipativity is called strong duality.

Definition 2.6 (Strong Duality of the Steady-State Problem)
If for the steady-state (x_s, u_s) there exists a $\mathcal{K}_\infty$-function σ and a multiplier $\tilde{\lambda} \in \mathbb{R}^{n_x}$ such that for the "rotated" stage cost $L(x, u) := \ell_\text{e}(x, u) + \tilde{\lambda}^T f(x, u) - \ell_\text{e}(x_s, u_s)$ it holds

$$L(x, u) \geq \alpha(\|x - x_s\|) \text{ for all } (x, u) \text{ with } 0 \leq c(x, u), \tag{2.53}$$

the steady-state (x_s, u_s) is called "Strongly dual". △

Strong duality clearly implies the strict dissipativity assumption (2.2), since the multiplier $\tilde{\lambda}$ can be interpreted as linear function $\lambda : \mathbb{R}^{n_x} \to \mathbb{R}, x \mapsto \tilde{\lambda}^T x$ which satisfies the strict-dissipativity property.

2.5.2 Variants of Economic Nonlinear Model Predictive Control

There exist a number of variants of the proposed economic NMPC controller discussed in the previous section. We briefly point out some of the important extensions. For a more detailed discussion, we refer the reader to the "Economic NMPC" chapter in [51].

Fixed Terminal-Constraint

If the terminal-region constraint (2.33e) is replaced by the constraint

$$\boldsymbol{x}(t_i + T) = x_e^*, \tag{2.54}$$

the resulting controller will automatically have the recursive feasibility property. This is because any admissible combination $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}(\mathcal{T}_i) \times \mathrm{L}_\infty^{n_u}(\mathcal{T}_i)$ for problem $P_\text{e}(t_i, x_i)$ then can be extended to an admissible pair for problem $P_\text{e}(t_{i+1}, \boldsymbol{x}(t_{i+1}))$ by applying the (steady-state)-control u_e^*. The terminal penalty term M can be omitted for such a controller, since the terminal constraint (2.54) will guarantee that the state x_e^* is reached at the end of the prediction horizon $t_i + T$ and the value of $M(\boldsymbol{x}(t_i + T))$ would be constant for all admissible trajectories. Consequently, also the auxiliary control law of Assumption 2.4 can be dropped. Assumption 2.2 also leads to asymptotic stability for this controller, see e.g. [100, Theorem 2].

Steady-State Terminal Constraint

All the presented economic-NMPC schemes have in common that the steady state (x_e^*, u_e^*) has to be known a priori, because the terminal penalty M has to be designed such that it is positive definite at x_e^* and the terminal region $\mathbb{X}$ in constraint (2.33e) or respectively the constraint (2.54) depend explicitly on x_e^*.

An alternative controller design that does not require a terminal penalty and a priori knowledge of the steady-state x_e^* replaces the terminal region constraint (2.33e) by the following steady-state constraint

$$0 = f(\boldsymbol{x}(t_i + T), \boldsymbol{u}(t_i + T)). \tag{2.55}$$

Similar to the case of the fixed terminal constraint, the auxiliary control-law Assumption 2.4 can then be dropped, because constraint (2.55) allows continuation of any admissible pair, as $(\boldsymbol{x}(t_i + T), \boldsymbol{u}(t_i + T)) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ is a steady state.

No Terminal Constraints

There also exist economic NMPC schemes that do not require terminal constraints. Under certain assumptions on the system, it is possible to prove near optimal performance of the closed-loop at least for a neighborhood of x_e^*. The "Turnpike-Property" [25] plays a central role in this context. Roughly speaking, the steady-state x_e^* satisfies the turnpike property, if the better (with respect to the economic performance) an admissible state trajectory is, the longer it stays in a vicinity of the steady state.

For a thorough discussion on the design of economic NMPC schemes based on the turnpike properties we refer the reader to [50, 39, 86].

2.5.3 Economic Nonlinear Model Predictive Control for Dissipative Periodic Systems

As we will also discuss in greater detail in the next chapter, steady-state operation not always constitutes the economically optimal way to operate a system. There may exist admissible periodic trajectories that outperform the best steady state. For such systems, the theory presented in the previous sections does not apply, because any system that satisfies Assumption 2.2 is economically optimally operated at the steady-state (x_e^*, u_e^*). Therefore, a generalization of the presented economic-NMPC scheme is necessary for such systems. In the following, we present the necessary generalizations that allow to set up an economic NMPC controller that extends the stability theory to the case of periodic systems. These generalizations are well known and for a discrete setting presented for example in Zanon et al. [123].

Similar to the case of dissipative steady-states (see Assumption 2.2), there exists a notion of dissipativity for periodic solutions. The following assumption is a generalization of Assumption 2.2.

Assumption 2.5 (Strict Periodic Dissipativity)
There exists a $T_\mathrm{p}(>0)$-periodic solution $(\boldsymbol{x}_p, \boldsymbol{u}_p) \in \mathcal{AC}^{n_x}([0, T_\mathrm{p}]) \times \mathrm{L}_\infty^{n_u}([0, T_\mathrm{p}])$ of $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t))$ that satisfies $0 \le c(\boldsymbol{x}_p(t), \boldsymbol{u}_p(t))$ for all $t \in [0, T_\mathrm{p}]$ (periodic in the sense that $\boldsymbol{x}_p(0) = \boldsymbol{x}_p(T_\mathrm{p})$), a continuously differentiable function $\lambda : \mathbb{R} \times \mathbb{R}^{n_x} \to \mathbb{R}_{\ge 0}$ and a $\mathcal{K}_\infty$ function σ such that

- λ is T_p-periodic in the first argument:

$$\lambda(t, x) = \lambda(t + T_\mathrm{p}, x) \text{ for all } (t, x) \in \mathbb{R} \times \mathbb{R}^{n_x}, \tag{2.56}$$

- λ is bounded on the set $\mathbb{R} \times A_x$ (with $A_x := \{x \in \mathbb{R}^{n_x} : \exists u \in \mathbb{R}^{n_u} \text{ with } 0 \le c(x, u)\}$)

- for all $(t, x, u) \in \mathbb{R} \times A_x \times \mathbb{R}^{n_u}$ with $0 \le c(x, u)$ it holds that

$$\sigma(\mathrm{dist}_\mathrm{p}(x)) + \nabla_t \lambda(t, x) + \nabla_x \lambda(t, x) f(x, u) \le \ell_\mathrm{e}(x, u) - \ell_\mathrm{e}(\boldsymbol{x}_p(t), \boldsymbol{u}_p(t)), \tag{2.57}$$

  where the distance of $(x, u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ to the trajectory $\boldsymbol{x}_p$ is denoted by

$$\mathrm{dist}_\mathrm{p}(x) := \min_{\tau \in [0, T_\mathrm{p}]} \left\| x - \boldsymbol{x}_p(\tau) \right\|. \tag{2.58}$$

△

Similar to the calculation for the steady-state case after Assumption 2.2, it can be shown that Assumption 2.5 implies that the average economic performance of any admissible solution for the system cannot be better than the economic performance of $(\boldsymbol{x}_p, \boldsymbol{u}_p)$.

To see this, let $k \in \mathbb{N}$ and $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}_\mathrm{loc}^{n_x} \times \mathrm{L}_{\infty,\mathrm{loc}}^{n_u}$ be any state/control trajectory that satisfies the ODE and path constraint $0 \le c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))$ for almost all $\tau \in \mathbb{R}_{\ge 0}$.

Then, by using the strict dissipativity property (2.57) we can calculate

$$\begin{aligned}
\int_0^{kT_\mathrm{p}} \ell_\mathrm{e}(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) \mathrm{d}\tau &\ge \int_0^{kT_\mathrm{p}} \big(\ell_\mathrm{e}(\boldsymbol{x}_p(\tau), \boldsymbol{u}_p(\tau)) + \sigma(\mathrm{dist}_\mathrm{p}(\boldsymbol{x}(\tau)))\big) \mathrm{d}\tau \\
&\quad + \int_0^{kT_\mathrm{p}} \big(\nabla_t \lambda(\tau, \boldsymbol{x}(\tau)) + \nabla_x \lambda(\tau, \boldsymbol{x}(\tau)) f(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\big) \mathrm{d}\tau \qquad (2.59) \\
&= k \int_0^{T_\mathrm{p}} \ell_\mathrm{e}(\boldsymbol{x}_p(\tau), \boldsymbol{u}_p(\tau)) \mathrm{d}\tau + \int_0^{kT_\mathrm{p}} \sigma(\mathrm{dist}_\mathrm{p}(\boldsymbol{x}(\tau))) \mathrm{d}\tau \\
&\quad + \lambda(kT_\mathrm{p}, \boldsymbol{x}(kT_\mathrm{p})) - \lambda(0, \boldsymbol{x}(0)) \qquad (2.60) \\
&= k \int_0^{T_\mathrm{p}} \ell_\mathrm{e}(\boldsymbol{x}_p(\tau), \boldsymbol{u}_p(\tau)) \mathrm{d}\tau + \underbrace{\int_0^{kT_\mathrm{p}} \sigma(\mathrm{dist}_\mathrm{p}(\boldsymbol{x}(\tau))) \mathrm{d}\tau}_{\ge 0} \\
&\quad + \underbrace{\lambda(0, \boldsymbol{x}(kT_\mathrm{p})) - \lambda(0, \boldsymbol{x}(0))}_{\text{bounded}} \qquad (2.61)
\end{aligned}$$

It follows

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t \ell_{\mathrm{e}}(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\mathrm{d}\tau \geq \frac{1}{T_{\mathrm{p}}}\int_0^{T_{\mathrm{p}}} \ell_{\mathrm{e}}(\boldsymbol{x}_p(\tau),\boldsymbol{u}_p(\tau))\mathrm{d}\tau. \tag{2.62}$$

This calculation shows that a system that satisfies Assumption 2.5 can be operated optimally at the "Periodic Steady-State" $(\boldsymbol{x}_p,\boldsymbol{u}_p) \in \mathcal{AC}^{n_x}([0,T_{\mathrm{p}}]) \times \mathrm{L}_\infty^{n_u}([0,T_{\mathrm{p}}]))$. Therefore, it makes sense to ask for an economic NMPC scheme that is asymptotically stable at $(\boldsymbol{x}_p,\boldsymbol{u}_p)$.

For an economic NMPC controller a certain degree of controllability is necessary and the following assumption can be seen as an extension of the controllability Assumption 2.3 in the steady-state case to the periodic case.

Assumption 2.6 (Controllability at Periodic Steady-State)
There exists a $\delta \geq 0$ and a $\mathcal{K}_\infty$-function η such that for any $x_0 \in \mathbb{R}^{n_x}$ with $\mathrm{dist}_{\mathrm{p}}(x) \leq \delta$ there exists a pair $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{AC}^{n_x}([0,T]) \times \mathrm{L}_\infty^{n_u}([0,T]))$ such that $\boldsymbol{x}(0) = x_0$, $0 \leq c(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))$ for all $\tau \in [0,T]$ and

$$\int_0^T \big(\|\boldsymbol{x}(\tau)-\boldsymbol{x}_p(\tau)\| + \|\boldsymbol{u}(\tau)-\boldsymbol{u}_p(\tau)\|\big)\,\mathrm{d}\tau \leq \eta\big(\mathrm{dist}_{\mathrm{p}}(x)\big). \tag{2.63}$$

△

The generalization of the auxiliary control law Assumption 2.4 to the periodic case reads as follows:

Assumption 2.7 (Periodic Auxiliary Control Law and Terminal Penalty)
There exists

- a family $(\mathbb{X}_t)_{t\in\mathbb{R}}$ of compact subsets of $\mathbb{R}^{n_x}$ such that $\boldsymbol{x}_p(t) \in \mathbb{X}_t$ and $\mathbb{X}_{t+T_{\mathrm{p}}} = \mathbb{X}_t$ for all $t \in \mathbb{R}$,
- a smooth map $\kappa : \mathbb{R}^{n_x} \to \mathbb{R}^{n_u}$,
- a terminal penalty function $M : \mathbb{R}\times\mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ with a $\mathcal{K}_\infty$-function η satisfying

$$M(\tau,x) \leq \eta(\mathrm{dist}_{\mathrm{p}}(x)) \text{ for all } (\tau,x) \in \mathbb{R}\times\mathbb{R}^{n_x}, \tag{2.64}$$

such that the solution $\Phi_\kappa(\cdot;x_0,t_0)$ of the IVP $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t),\kappa(\boldsymbol{x}(t)))$ with $\boldsymbol{x}(t_0) = x_0 \in \mathbb{X}$ exists for all initial values $x_0 \in \mathbb{X}_{t_0}$ and for all times $t \geq t_0$ and has the following properties:

- it remains inside the family $(\mathbb{X}_t)_{t\in\mathbb{R}}$

$$\Phi_\kappa(\tau;x_0,t_0) \in \mathbb{X}_\tau \text{ for all } \tau \geq t_0, \tag{2.65}$$

- it satisfies the path constraints

$$0 \leq c(\Phi_\kappa(\tau;x_0,t_0),\kappa(\Phi_\kappa(\tau;x_0,t_0))) \text{ for all } \tau \geq t_0, \tag{2.66}$$

- the terminal penalty M satisfies

$$\frac{\partial}{\partial\tau} M(\tau,\Phi_\kappa(\tau;x_0,t_0)) \leq -\ell_{\mathrm{e}}(\Phi_\kappa(\tau;x_0,t_0),\kappa(\Phi_\kappa(\tau;x_0,t_0))) \text{ for all } \tau \geq t_0 \tag{2.67}$$

along the solution. △

Now consider the economic NMPC scheme based on the NMPC subproblem $P_{\mathrm{e}}(t_i, x_i)$ defined on the time horizons $\mathcal{T}^{t_i} = [t_i, t_i + T]$:

$$\min_{\substack{\boldsymbol{x} \in \mathcal{AC}^{n_x}(\mathcal{T}^{t_i}),\\ \boldsymbol{u} \in \mathrm{L}_\infty^{n_u}(\mathcal{T}^{t_i})}} \quad \varphi(\boldsymbol{x}, \boldsymbol{u}) := \int_{\mathcal{T}^{t_i}} \ell_{\mathrm{e}}(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\mathrm{d}\tau + M(t_i + T, \boldsymbol{x}(t_i + T)) \tag{2.68a}$$

$$\text{s.t.} \quad \boldsymbol{x}(t_i) = x_i, \tag{2.68b}$$

$$\dot{\boldsymbol{x}}(\tau) = f(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \quad \tau \in \mathcal{T}^{t_i}, \tag{2.68c}$$

$$0 \le c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \quad \tau \in \mathcal{T}^{t_i}, \tag{2.68d}$$

$$\boldsymbol{x}(t_i + T) \in \mathbb{X}_{t_i+T}. \tag{2.68e}$$

Note that beside the adapted terminal penalty function M, the structure of this OCP is almost identical to the NMPC subproblem for the dissipative steady-state case defined in (2.33). The important difference is the time-dependent terminal-constraint (2.68e), which was constant in the steady-state case. The time-varying terminal constraint accounts for the fact that the economic optimal way to operate the system is periodic and not just a steady-state.

With the assumptions that are generalizing the case of the strict dissipative steady-state in the previous section, it is possible to extend the stability result for strictly dissipative steady-states (Lemma 2.5) to the case of strict periodic dissipative systems.

Lemma 2.7 (Asymptotic Stability of E-NMPC for Periodic Dissipative Systems)
If Assumptions 2.5, 2.6 and 2.7 hold, the set $\{(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \tau \in [0, T_{\mathrm{p}}]\}$ is asymptotically stabilized by the closed-loop feedback resulting from the economic NMPC scheme defined by the subproblems $P_{\mathrm{e}}(t_i, x_i)$, i.e. there exist a $\mathcal{KL}$-function β, such that

$$\left\| \boldsymbol{x}_\mu(t_j; x, t_k) \right\|_p \le \beta(\mathrm{dist}_{\mathrm{p}}(x), t_j - t_k) \text{ for all } j \ge k \in \mathbb{N} \text{ and } x \in \mathbb{X}_{t_k}. \tag{2.69}$$

Proof The proof is almost identical to the proof of Lemma 2.5. Again, a modified objective functional will serve as LYAPUNOV-like function. The "rotated" functional $\tilde{\varphi} : \mathcal{AC}^{n_x}([t, t + T]) \times \mathrm{L}_\infty^{n_u}([t, t + T]) \to \mathbb{R}$ this time is defined by

$$\tilde{\varphi}(\boldsymbol{x}, \boldsymbol{u}) := \varphi(\boldsymbol{x}, \boldsymbol{u}) + \lambda(t, \boldsymbol{x}(t)). \tag{2.70}$$

Again let $(\boldsymbol{x}_k^\star, \boldsymbol{u}_k^\star)_{k \in \mathbb{N}}$ be the sequence of trajectories resulting from the E-NMPC scheme with initial value x_0 at time t_0. A similar calculation as in the proof of Lemma 2.5 shows that the strict periodic dissipativity Assumption 2.5 and the auxiliary control law Assumption 2.7 imply the following inequality

$$\tilde{\varphi}(x_{k+1}) \le \tilde{\varphi}(x_k) - \int_{t_k}^{t_{k+1}} \sigma\left(\mathrm{dist}_{\mathrm{p}}(\boldsymbol{x}_k^\star(\tau))\right) \mathrm{d}\tau \tag{2.71}$$

for all $k \in \mathbb{N}$. Note that the distance function $\mathrm{dist_p}(\cdot)$ is LIPSCHITZ-continuous (with constant 1) and therefore (2.71) implies the existence of a $\mathcal{K}_\infty$-function α_3 such that

$$\tilde{\varphi}(x_{k+1}) \leq \tilde{\varphi}(x_k) - \alpha_3\left(\mathrm{dist_p}(x_k)\right) \tag{2.72}$$

holds for all $k \in \mathbb{N}$. The controllability Assumption 2.6 implies the existence of a $\mathcal{K}_\infty$-function α_2 such that

$$\tilde{\varphi}(x) \leq \alpha_2\left(\mathrm{dist_p}(x)\right) \tag{2.73}$$

holds for all $x \in \mathbb{R}^{n_x}$. On the other hand the periodic dissipativity property (2.57) implies the existence of a $\mathcal{K}_\infty$-function α_1 such that

$$\tilde{\varphi}(x) \geq \alpha_1\left(\mathrm{dist_p}(x)\right) \tag{2.74}$$

holds for all $x \in \mathbb{X}$. Altogether this implies the existence of a $\mathcal{KL}$-function β such that

$$\left\|\boldsymbol{x}_\mu(t_k; x, t_i)\right\|_p \leq \beta\left(\mathrm{dist_p}(x), t_k - t_i\right) \tag{2.75}$$

holds for all $k, i \in \mathbb{N}$ with $k \geq i$ and all $x \in \mathbb{X}_{t_i}$. □

Remark 2.2 The above stability Lemma only shows that the distance of the closed-loop trajectory to the set $X_p := \{x \in \mathbb{R}^{n_x} : \mathrm{dist_p}(x) = 0\}$ asymptotically decreases and not that the closed-loop feedback is stabilized at the trajectory $(\boldsymbol{x}_p, \boldsymbol{u}_p)$. This means that the distance $\left\|\boldsymbol{x}_\mu(t; x_0, t_0) - \boldsymbol{x}_p(t)\right\|$ does not necessarily decrease asymptotically. In a discrete setting, the situation is different because there the next state is completely determined by the current state and the current control input which can the be used for the following conclusion:

$$\begin{aligned} &(\boldsymbol{x}_\mu(t_i), \boldsymbol{u}_\mu(t_i)) \approx (\boldsymbol{x}_p(\tau), \boldsymbol{u}_p(\tau)) \\ \Rightarrow\ &(\boldsymbol{x}_\mu(t_{i+1}), \boldsymbol{u}_\mu(t_{i+1})) \approx (\boldsymbol{x}_p(\tau + t_{i+1} - t_i), \boldsymbol{u}_p(\tau + t_{i+1} - t_i)), \end{aligned} \tag{2.76}$$

Contrary to this, in the continuous setting, the control input at the time-instant t_i does not determine $\boldsymbol{x}_\mu(t_{i+1})$ uniquely (the control input on the whole interval $[t_i, t_{i+1}]$ is necessary), making the conclusion (2.76) in this case difficult. For a more detailed analysis of the discrete case we refer the reader to Zanon et al. [123]. △

A-Priori Knowledge of $(\boldsymbol{x}_p, \boldsymbol{u}_p)$

As we have seen, the periodic E-NMPC controller based on the subproblems $P_\mathrm{e}(t_i, x_i)$ requires a priori knowledge of the period T_p (as the terminal constraint sets have to be chosen periodic with period T_p), the terminal penalty function M and the periodic terminal regions $(\mathbb{X}_t)_{t\in\mathbb{R}}$. This means that a certain amount of knowledge of the periodic trajectory $(\boldsymbol{x}_p, \boldsymbol{u}_p)$ is necessary to set up such a controller.

Furthermore, it is clearly the case that any phase-shifted version of $(\boldsymbol{x}_p, \boldsymbol{u}_p)$ yields the same economic output as $(\boldsymbol{x}_p, \boldsymbol{u}_p)$, since the system $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t))$ is autonomous and ℓ_e does not depend on time. Therefore, when defining the terminal region sets $(\mathbb{X}_t)_{t\in\mathbb{R}}$, there is a somehow arbitrary choice involved since these sets have to be neighborhoods of $(\mathbb{X}_t)_{t\in\mathbb{R}}$.

2.6 Summary

After a brief overview on the historical development of NMPC, we gave a quick introduction to the central ideas behind NMPC. The repetitive process of measuring the current system state, solving an OCP with the obtained state as initial value and giving the obtained control-input to the plant was explained. Important questions concerning the closed-loop behavior of the resulting controller such as recursive feasibility and asymptotic stability properties were discussed. The stability properties of a tracking NMPC controller were analyzed and E-NMPC was introduced as an NMPC variant that attempts to combine the regulatory tasks of (tracking) NMPC with a simultaneous optimizing task. Based on dissipativity assumptions, stability results for economic NMPC schemes for steady-states and periodic steady-states were discussed.

What remains open is the question on whether it is possible to design economic NMPC schemes that do not explicitly rely on dissipativity assumptions and a-priori knowledge of economic optimal trajectories. In the remaining part of this thesis we develop a design approach for economic NMPC that, complementary to the schemes presented in this chapter, is based on good approximation properties of periodic solutions and assumptions on the NMPC subproblems rather than on dissipativity assumptions.

Part II

Nonlinear Model Predictive Control for Average Output Systems

Chapter 3

Average Output Optimal Control Problems

In this chapter we define and discuss one of the main subjects of this thesis, Average Output Optimal Control Problems (AOCPs). We introduce AOCPs as an extension of the class of finite horizon OCPs to infinite horizons, dealing with possibly indefinite objective integrals by using an averaging formulation. The averaging formulation usually will come at the cost of non-uniqueness of the solution of the AOCP.

We discuss the approximation properties of periodic solutions that can, under certain controllability assumptions, approximate the optimal average output arbitrarily well. Furthermore, we show that in case the set of admissible states is compact, a natural approximation property of quasi-periodic solutions arises that does not require any assumptions on controllability.

3.1 OCPs with Average Output Objective on Infinite Time Horizons

3.1.1 Problem Setup

We consider a controlled dynamical system $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t))$ that is subject to a path constraint of the form $0 \le c(\boldsymbol{x}(t), \boldsymbol{u}(t))$ for all times. Associated to this system is a performance criterion $\ell : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$.

We are interested in the question of how to optimally operate this system on the infinite time horizon $\mathcal{T} := \mathbb{R}_{\ge 0}$. To put this question in mathematical terms, it can be formulated as an infinite horizon OCP P_∞:

$$\inf_{\substack{\boldsymbol{x} \in \mathcal{AC}^{n_x}_{\mathrm{loc}}(\mathcal{T}),\\ \boldsymbol{u} \in \mathrm{L}^{n_u}_{\infty,\mathrm{loc}}(\mathcal{T})}} \int_0^\infty \ell(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\mathrm{d}\tau \tag{3.1a}$$

$$\text{s.t.} \quad \dot{\boldsymbol{x}}(\tau) = f(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \quad \tau \in \mathcal{T}, \tag{3.1b}$$

$$0 \le c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \quad \tau \in \mathcal{T}. \tag{3.1c}$$

As we have already seen in Section 1.5, even for simple dynamical systems, the objective functional of Problem P_∞ can be unbounded and thus this formulation in many cases is not very useful.

For this reason, we replace the objective functional by the average objective output functional of Definition 1.18

$$\varphi_{\mathrm{avg}}(\boldsymbol{x}, \boldsymbol{u}) := \limsup_{T \to +\infty} \frac{1}{T} \int_0^T \ell(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\mathrm{d}\tau, \tag{3.2}$$

and work with the following compactness Assumption.

Assumption 3.1 (Compactness of the Admissible States and Controls)
The set of admissible states and controls $\mathrm{A}^c := \{(x,u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} : 0 \leq c(x,u)\}$ is compact. △

Lemma 1.21 then guarantees that for the following modified version P_{avg} of Problem P_∞

$$\inf_{\substack{\boldsymbol{x}\in\mathcal{AC}^{n_x}_{\text{loc}}(\mathcal{T}),\\ \boldsymbol{u}\in\mathrm{L}^{n_u}_{\infty,\text{loc}}(\mathcal{T})}} \quad \varphi_{\text{avg}}(\boldsymbol{x},\boldsymbol{u}) \tag{3.3a}$$

$$\text{s.t.} \quad \dot{\boldsymbol{x}}(\tau) = f(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)), \quad \tau \in \mathcal{T}, \tag{3.3b}$$

$$0 \leq c(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)), \quad \tau \in \mathcal{T}. \tag{3.3c}$$

the objective functional is bounded on the set of admissible state/control trajectories. Therefore the existence of an admissible trajectory $(\boldsymbol{x},\boldsymbol{u})$ for problem P_{avg} is sufficient to guarantee the existence of the infimum which we denote by φ^*_{avg}.

If an initial value value constraint $\boldsymbol{x}(t_0) = x_0$ is added to the above Problem, we denote it by $P_{\text{avg}}(x_0)$ and the infimum of the objective functional by $\varphi^*_{\text{avg}}(x_0)$.

We call an infinite horizon OCP of the form P_{avg} or $P_{\text{avg}}(x_0)$ an AOCP.

3.1.2 Non-Uniqueness of Average Output Optimal Control Problem Solutions

Although the compactness Assumption 3.1 guarantees the well-definedness of the infimum $\varphi^*_{\text{avg}}(x_0)$, the existence of a unique solution for $P_{\text{avg}}(x_0)$ cannot be expected. First, there may be no admissible state/control trajectory pair with $\varphi^*_{\text{avg}}(\boldsymbol{x},\boldsymbol{u}) = \varphi^*_{\text{avg}}(x_0)$. Second, even in the case the infimum $\varphi^*_{\text{avg}}(x_0)$ is attained by an admissible pair $(\boldsymbol{x},\boldsymbol{u})$, the combination of the infinite time horizon and the averaging objective functional often imply that uniqueness of solutions of problem $P_{\text{avg}}(x_0)$ cannot be expected. This is due to the fact that the averaging functional φ_{avg} does not depend on the initial behavior of $(\boldsymbol{x},\boldsymbol{u})$, but only on its asymptotic behavior, as can be seen in the following calculation.

Invariance with Respect to Phase Shifts and Independence of Initial Behavior

Let $t_1 \in \mathbb{R}_{\geq 0}$ and $T > t_1$. For any $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{AC}^{n_x}_{\text{loc}}([0,\infty)) \times \mathrm{L}^{n_u}_{\infty,\text{loc}}([0,\infty))$ we have

$$\frac{1}{T}\int_0^T \ell(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\mathrm{d}\tau = \frac{1}{T}\left(\int_0^{t_1} \ell(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\mathrm{d}\tau + \int_{t_1}^T \ell(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\mathrm{d}\tau\right) \tag{3.4}$$

$$= \frac{1}{T}\int_0^{t_1} \ell(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\mathrm{d}\tau + \frac{T-t_1}{T}\frac{1}{T-t_1}\int_{t_1}^T \ell(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\mathrm{d}\tau. \tag{3.5}$$

Because $\lim\limits_{T\to\infty} \frac{T-t_1}{T} = 1$, this shows

$$\varphi_{\text{avg}}(\boldsymbol{x},\boldsymbol{u}) = \varphi_{\text{avg}}\left((\boldsymbol{x},\boldsymbol{u}) \circ [\tau \mapsto \tau + t_1]\right). \tag{3.6}$$

From this equation it follows that the average output does not depend on the behavior of $(\boldsymbol{x},\boldsymbol{u})$ on the interval $[0,t_1]$. Moreover, it shows that the average output is invariant with respect to phase shifts: If for $(\boldsymbol{x}_1, \boldsymbol{u}_1), (\boldsymbol{x}_2, \boldsymbol{u}_2) \in \mathcal{AC}^{n_x}_{\mathrm{loc}}([0,\infty)) \times \mathrm{L}^{n_u}_{\infty,\mathrm{loc}}([0,\infty))$ there exist $t_1 < t_2 \in \mathbb{R}_{\geq 0}$ such that

$$(\boldsymbol{x}_1, \boldsymbol{u}_1)(\tau + t_1) = (\boldsymbol{x}_2, \boldsymbol{u}_2)(\tau + t_2) \tag{3.7}$$

holds for all $\tau \in \mathbb{R}_{\geq 0}$, then it follows

$$\begin{aligned}\varphi_{\mathrm{avg}}(\boldsymbol{x}_1, \boldsymbol{u}_1) &= \varphi_{\mathrm{avg}}\left((\boldsymbol{x}_1, \boldsymbol{u}_1) \circ [\tau \mapsto \tau + t_1]\right) \\ &= \varphi_{\mathrm{avg}}\left((\boldsymbol{x}_2, \boldsymbol{u}_2) \circ [\tau \mapsto \tau + t_2]\right) = \varphi_{\mathrm{avg}}(\boldsymbol{x}_2, \boldsymbol{u}_2).\end{aligned}$$

Example: Controllability implies Non-Uniqueness

In the following, we want to illustrate how a certain degree of controllability of a dynamical system together with the observed invariance of the averaging objective functional with respect to phase shifts implies that the corresponding AOCP solutions are not unique.
Consider a system with the following controllability property.

Definition 3.1 ($T_{\max}$-Controllability (cf. Grammel [47]))
A system $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t))$ with path-constraint $0 \leq c(x,u)$ is said to be $T_{\max}$-controllable, if for any two $y_0, y_1 \in \mathrm{A}^c_x := \{x \in \mathbb{R}^{n_x} : \exists u \in \mathbb{R}^{n_u} : (x,u) \in \mathrm{A}^c\}$ there exists a pair $(\boldsymbol{x}_{y_0,y_1}, \boldsymbol{u}_{y_0,y_1}) \in \mathcal{AC}^{n_x}([0, T_{\max}]) \times \mathrm{L}^{n_u}_{\infty}([0, T_{\max}]))$ and a time $0 \leq t_{y_0,y_1} \leq T_{\max}$ such that the following holds

- $\dot{\boldsymbol{x}}_{y_0,y_1}(\tau) = f(\boldsymbol{x}_{y_0,y_1}(\tau), \boldsymbol{u}_{y_0,y_1}(\tau))$ for almost all $\tau \in [0, T_{\max}]$,
- $0 \leq c(\boldsymbol{x}_{y_0,y_1}(\tau), \boldsymbol{u}_{y_0,y_1}(\tau))$ for almost all $\tau \in [0, T_{\max}]$,
- $\boldsymbol{x}_{y_0,y_1}(0) = y_0$ and $\boldsymbol{x}(t_{y_0,y_1}) = y_1$. △

For a $T_{\max}$-controllable system, an admissible trajectory $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}_{\mathrm{loc}}([0,\infty)) \times \mathrm{L}^{n_u}_{\infty,\mathrm{loc}}([0,\infty))$ and an arbitrary element $y_1 \in \mathrm{A}^c_x$, we can define the pair $(\tilde{\boldsymbol{x}}_{y_1}, \tilde{\boldsymbol{u}}_{y_1}) \in \mathcal{AC}^{n_x}_{\mathrm{loc}}([0,\infty)) \times \mathrm{L}^{n_u}_{\infty,\mathrm{loc}}([0,\infty))$ as follows:

$$(\tilde{\boldsymbol{x}}_{y_1}, \tilde{\boldsymbol{u}}_{y_1})(\tau) := \begin{cases} (\boldsymbol{x}_{x_0,y_1}, \boldsymbol{u}_{x_0,y_1})(\tau), & \text{if } \tau \in [0, t_{x_0,y_1}), \\ (\boldsymbol{x}_{y_1,x_0}, \boldsymbol{u}_{y_1,x_0})(\tau - t_{x_0,y_1}), & \text{if } \tau \in [t_{x_0,y_1}, t_{x_0,y_1} + t_{y_1,x_0}), \\ (\boldsymbol{x}, \boldsymbol{u})(\tau - (t_{x_0,y_1} + t_{y_1,x_0})), & \text{if } \tau \in [t_{x_0,y_1} + t_{y_1,x_0}, \infty). \end{cases} \tag{3.8}$$

The average output of $(\tilde{\boldsymbol{x}}_{y_1}, \tilde{\boldsymbol{u}}_{y_1})$, independently of y_1, is equal to the average output of $(\boldsymbol{x}, \boldsymbol{u})$ because $(\boldsymbol{x}, \boldsymbol{u}) = (\tilde{\boldsymbol{x}}_{y_1}, \tilde{\boldsymbol{u}}_{y_1}) \circ [\tau \mapsto \tau + t_{x_0,y_1} + t_{y_1,x_0}]$. This shows that for $T_{\max}$-controllable systems, AOCPs do not have unique solutions.

Remark 3.1 (Non-Uniqueness and NMPC) The independence of the averaging functional φ_{avg} with respect to the initial behavior of the trajectories and the resulting non-uniqueness are the reasons why AOCPs in principle are not suited as NMPC subproblems. This becomes clear when one remembers the NMPC principles according to which only the (in this case arbitrary) initial part of the open-loop control solution (of the NMPC subproblem) is applied to the system. △

3.2 Periodic Approximations

As we have seen in the previous section, in general there exists no unique solutions for AOCPs. In this section we discuss how at least good approximations for AOCPs can be found, in particular using periodic solutions.

Motivation for Periodic Operation

A straightforward and simple approach for approximating a solution of the AOCP P_{avg}, is to limit the search to the set of feasible steady-states. This reduces the infinite dimensional problem to the following finite dimensional steady-state optimization problem

$$\inf_{(x,u)\in\mathbb{R}^{n_x}\times\mathbb{R}^{n_u}} \ell(x,u), \tag{3.9a}$$

$$\text{s.t.} \quad 0 = f(x,u), \tag{3.9b}$$

$$0 \le c(x,u). \tag{3.9c}$$

This simplification certainly is not satisfactory because the set of steady states could be empty and furthermore, the optimal way to operate a system with respect to average output not always consists in steady-state operation, as we already have seen in Section 2.5.3. There it was shown that strict periodic dissipativity implies optimality of periodic operation.

Steady state solutions can be interpreted as a special class of periodic solutions (the ones with period zero) and therefore it is not surprising that periodic solutions have greater approximation potential then steady-state solutions. The fact that periodic operation (with period greater then zero) of dynamical systems in some cases can outperform steady-state operation started to receive attention in the 1970s, in particular in the field of chemical process engineering [7, 54]. As early examples, periodic operation of consecutive-competitive reactions in a Continuous Stirred-Tank Reactor (CSTR) is considered in [101] and [72].

Criteria that help to figure out whether periodic operation can outperform steady-state operation are analyzed for discrete and continuous systems in [15] and [14].

Focusing on periodic solutions has the benefit that it reduces the time horizon and transforms the infinite horizon AOCP to a finite horizon OCP. In the following, we discuss a number of approximation properties of periodic solutions that justify this approach.

3.2.1 Low Dimensional Cases

For a two dimensional state-space, there is a remarkable existence result for average-optimal periodic trajectories that is, however, based on the concept of "relaxed controls". A relaxed control function $\boldsymbol{u}:\mathbb{R}_{\ge 0}\to\mathcal{P}(\mathbb{R}^{n_u})$ takes values in the set $\mathcal{P}(\mathbb{R}^{n_u})$ of probability measures on $\mathbb{R}^{n_u}$. For $x\in\mathbb{R}^{n_x}$ and $\mu\in\mathcal{P}(\mathbb{R}^{n_u})$ the right-hand side and the performance criterion then have to be interpreted as

$$f(x,\mu) := \int_{\mathbb{R}^{n_u}} f(x,u)\mu(\mathrm{d}u) \text{ resp. } \ell(x,\mu) := \int_{\mathbb{R}^{n_u}} \ell(x,u)\mu(\mathrm{d}u). \tag{3.10}$$

For the theory of relaxed controls we refer the reader to Young [119] and Warga [113].

Employing relaxed controls, Artstein and Bright [5] prove the following result.

Lemma 3.2 (Existence of Average Optimal Periodic Trajectories [5])
Let $n_x = 2$ and let compactness Assumption 3.1 hold. If the path constraint c is only state dependent, then there exists a $T_\mathrm{p} < \infty$, a T_p-periodic state trajectory $\boldsymbol{x}_p$ and a relaxed control function $\boldsymbol{u}_p$ such that $(\boldsymbol{x}_p, \boldsymbol{u}_p)$ satisfy the relaxed ODE constraint, the path constraint and

$$\frac{1}{T_\mathrm{p}} \int_0^{T_\mathrm{p}} \ell(\boldsymbol{x}_p(\tau), \boldsymbol{u}_p(\tau)) \mathrm{d}\tau = \varphi^*_{\mathrm{avg}}. \tag{3.11}$$

Proof See [5, Theorem A]. □

It has to be noted that the fact that the system is described by an ordinary differential equation is not essential for the existence of optimal periodic trajectories.

The result rather is based on a topological property of $\mathbb{R}^2$ that is exploited in a similar manner as it is done in the POINCARÉ[1]-BENDIXSON[2] theorem [10]. The POINCARÉ-BENDIXSON theorem states that a bounded solution of an ODE without stationary points in the ω-limit set converges to a trajectory that is a JORDAN[3]-curve [4].

3.2.2 Periodic Approximations via Controllability

For higher dimensional state-spaces a controllability assumptions on the system $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u})$ implies good approximation properties of periodic solutions. The following result of Grammel [47] is remarkable, because it not only states that periodic solutions exist that approximate the best average output arbitrarily well, but it also gives a bound on the required period to achieve a desired approximation quality.

Lemma 3.3 (Periodic Approximation Lemma)
Let compactness Assumption 3.1 hold and let the system have the $T_{\max}$-controllability property (3.1). Let $P \in \mathbb{R}_{\geq 0}$ be an upper bound for $|\ell(\cdot)|$ on the set of admissible states/controls A. Then for any $\varepsilon > 0$ there exists a time $0 < T_\varepsilon \leq \frac{6PT_{\max}}{\varepsilon}$ and a pair $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}([0, T_\varepsilon]) \times \mathrm{L}_\infty^{n_u}([0, T_\varepsilon])$ such that

- $(\boldsymbol{x}, \boldsymbol{u})$ satisfies the ODE constraint $\dot{\boldsymbol{x}}(\tau) = f(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))$ and the path constraint $0 \leq c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))$ for almost all $\tau \in [0, T_\varepsilon]$,
- $\boldsymbol{x}$ is periodic: $\boldsymbol{x}(0) = \boldsymbol{x}(T_\varepsilon)$,
- $\frac{1}{T_\varepsilon} \int_0^{T_\varepsilon} \ell(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) \mathrm{d}\tau \leq \varphi^*_{\mathrm{avg}} + \varepsilon$.

Proof The proof can be found in Grammel [47, Lemma 3.1]. □

[1] Henri Poincaré 1854 - 1912
[2] Ivar Otto Bendixson 1861 - 1935
[3] Marie Ennemond Camille Jordan 1838 - 1922
[4] A JORDAN-curve is the image of an injective continuous map of the circle in the two-dimensional plane.

The version in [47] is even stronger, as it states that for an arbitrary initial value $x_0 \in A_x$, there exists a T_ε periodic pair $(\boldsymbol{x}, \boldsymbol{u})$ starting at x_0 with all the properties stated in the lemma above. This indicates that the estimate for the required period $T_\varepsilon \leq \frac{6PT_{\max}}{\varepsilon}$ might be conservative, since it really shows that any point $x_0 \in A_x$ is contained in a T_ε-periodic ε suboptimal trajectory, which is a stronger statement then just the existence of any T_ε-periodic ε suboptimal trajectory, and there might very well be ε suboptimal periodic trajectories with shorter periods.

3.2.3 Quasi Periodic Approximations

As we will prove in the following, in a general setting the compactness Assumption 3.1 alone (without any controllability property) is already sufficient for the following approximation property of quasi-periodic solutions for AOCPs.

Lemma 3.4 (Quasi Periodic Approximations)
Let Assumption 3.1 hold and let the set of admissible trajectories for problem P_{avg} be non-empty. Then for any $\delta, \varepsilon > 0$, there exist a positive number $T_{\delta,\varepsilon} \in \mathbb{R}_{\geq 0}$ and a pair $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}([0, T_{\delta,\varepsilon}]) \times \mathrm{L}_\infty^{n_u}([0, T_{\delta,\varepsilon}])$ such that

- $(\boldsymbol{x}, \boldsymbol{u})$ satisfies the ODE constraint $\dot{\boldsymbol{x}}(\tau) = f(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))$ and the path constraint $0 \leq c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))$ for almost all $\tau \in [0, T_{\delta,\varepsilon}]$,
- $\boldsymbol{x}$ is "almost" periodic: $\left\| \boldsymbol{x}(0) - \boldsymbol{x}(T_{\delta,\varepsilon}) \right\| \leq \delta$,
- the average output of $(\boldsymbol{x}, \boldsymbol{u})$ is close to φ_{avg}^*:

$$\frac{1}{T_{\delta,\varepsilon}} \int_0^{T_{\delta,\varepsilon}} \ell(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) \mathrm{d}\tau \leq \varphi_{\text{avg}}^* + \varepsilon. \tag{3.12}$$

Proof Since the set of admissible trajectories for problem P_{avg} is not empty, we can choose a pair $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}_{\text{loc}}^{n_x}([0,\infty)) \times \mathrm{L}_{\infty,\text{loc}}^{n_u}([0,\infty))$ with $\varphi_{\text{avg}}(\boldsymbol{x}, \boldsymbol{u}) \leq \varphi_{\text{avg}}^* + \varepsilon$. Let $(t_i)_{i \in \mathbb{N}}$ be a monotonic sequence with $t_i \to \infty$. We consider the sequence of the average outputs in the intervals $[t_k, t_{k+1}]$,

$$d_k := \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \ell(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) \mathrm{d}\tau. \tag{3.13}$$

With this definition and any $n \in \mathbb{N}$ it holds that

$$\frac{1}{t_n} \int_0^{t_n} \ell(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) \mathrm{d}\tau = \frac{1}{t_n} \sum_{k=0}^{n-1} (t_{k+1} - t_k) d_k. \tag{3.14}$$

Taking the limit $n \to \infty$ on both sides of the equation yields the average output $\varphi_{\text{avg}}(\boldsymbol{x}, \boldsymbol{u})$ and shows that $d_k \leq \varphi_{\text{avg}}(\boldsymbol{x}, \boldsymbol{u})$ must hold for an infinite number of indices $k \in \mathbb{N}$. If $d_k \leq \varphi_{\text{avg}}(\boldsymbol{x}, \boldsymbol{u})$ would only hold for a finite number of indices, the limit of the right hand side of the equation would be greater than $\varphi_{\text{avg}}(\boldsymbol{x}, \boldsymbol{u})$. Because the system is autonomous, we may assume that $d_k \leq \varphi_{\text{avg}}(\boldsymbol{x}, \boldsymbol{u})$ holds for all $k \in \mathbb{N}$. Since the set A^c is compact, also the projection onto the first factor $\mathrm{A}_x^c = \pi_x(\mathrm{A}^c) \subset \mathbb{R}^{n_x}$ is compact and there exists $x^* \in \mathrm{A}_x^c$ and a subsequence $(\tilde{t}_i)_{i\in\mathbb{N}}$ of $(t_i)_{i\in\mathbb{N}}$ such that $\boldsymbol{x}(\tilde{t}_i) \to x^* \in \mathrm{A}_x^c$. Then there exists an index k_0 with

$$\left\| \boldsymbol{x}(\tilde{t}_{k_0}) - \boldsymbol{x}(\tilde{t}_{k_0+1}) \right\| \le \delta. \tag{3.15}$$

By construction, the pair $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}) \in \mathcal{AC}^{n_x}([0, \tilde{t}_{k_0+1} - \tilde{t}_{k_0}]) \times \mathrm{L}_\infty^{n_u}([0, \tilde{t}_{k_0+1} - \tilde{t}_{k_0}])$ defined by

$$(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})(\tau) := (\boldsymbol{x}, \boldsymbol{u})(\tau + \tilde{t}_{k_0}) \tag{3.16}$$

has all the desired properties and the proof is finished. □

3.2.4 The Associated Periodic Optimal Control Problem

The approximation properties of periodic solutions discussed in this section allow the approximate solution of an AOCP by transforming it to an OCP with finite time horizon using a periodicity constraint.

Approximation with Fixed Period

Suppose a suitable period T_p is known. Then the following OCP $\mathrm{Per}_{T_\mathrm{p}}^{\mathrm{fix}}$ with periodicity constraint can be defined on the time horizon $\mathcal{T} = [0, T_\mathrm{p}]$.

$$\min_{\substack{\boldsymbol{x}\in\mathcal{AC}^{n_x}(\mathcal{T}),\\ \boldsymbol{u}\in\mathrm{L}_\infty^{n_u}(\mathcal{T})}} \quad \phi_{T_\mathrm{p},\mathrm{per}}^{\mathrm{fix}}(\boldsymbol{x}, \boldsymbol{u}) := \frac{1}{T_\mathrm{p}} \int_0^{T_\mathrm{p}} \ell(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\mathrm{d}\tau \tag{3.17a}$$

$$\text{s.t.} \quad \dot{\boldsymbol{x}}(\tau) = f(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \quad \tau \in \mathcal{T}, \tag{3.17b}$$

$$0 \le c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \quad \tau \in \mathcal{T}, \tag{3.17c}$$

$$0 = \boldsymbol{x}(T_\mathrm{p}) - \boldsymbol{x}(0). \tag{3.17d}$$

It is clear that any admissible pair $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}(\mathcal{T}) \times \in \mathrm{L}_\infty^{n_u}(\mathcal{T})$ of problem $\mathrm{Per}_{T_\mathrm{p}}^{\mathrm{fix}}$ can be extended to the infinite time horizon $\mathbb{R}_{\ge 0}$ by periodic continuation and thus induces an admissible pair $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}) \in \mathcal{AC}_{\mathrm{loc}}^{n_x}(\mathbb{R}_{\ge 0}) \times \mathrm{L}_{\infty,\mathrm{loc}}^{n_u}(\mathbb{R}_{\ge 0}))$ for problem P_{avg} with the property $\phi_{T_\mathrm{p},\mathrm{per}}^{\mathrm{fix}}(\boldsymbol{x}, \boldsymbol{u}) = \varphi_{\mathrm{avg}}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})$.

Approximation with Free Periods

It is also possible to define a periodicity-constrained OCP without explicitly fixing the period. In this case the period T can be considered as an additional optimization variable (within some simple bounds $\underline{T} \le T \le \overline{T}$). Since the straight-forward formulation of the resulting problem would result in optimization over the spaces $\mathcal{AC}^{n_x}([0, T])$ and $\mathrm{L}_\infty^{n_u}([0, T])$, which then are varying with the optimization variable T, a time-transformation to the standard time horizon $[0, 1]$ is used to get a proper OCP.

With the substitution $\tau(t) = Tt$, the objective averaging integral transforms as follows:

$$\frac{1}{T}\int_0^T \ell(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\mathrm{d}\tau = \int_0^1 \ell(\boldsymbol{x}(Tt), \boldsymbol{u}(Tt))\mathrm{d}t. \tag{3.18}$$

Then, by replacing the original state and control variables by $\tilde{\boldsymbol{x}}(t) := \boldsymbol{x}(Tt)$ and $\tilde{\boldsymbol{u}}(t) := \boldsymbol{u}(Tt)$, the transformed OCP $\mathrm{Per}^{\mathrm{free}}$ looks as follows.

$$\min_{\substack{\tilde{\boldsymbol{x}}\in\mathcal{AC}^{n_x}([0,1]),\\ \tilde{\boldsymbol{u}}\in\mathrm{L}_\infty^{n_u}([0,1]),\\ T\in\mathbb{R}}} \quad \phi_{\mathrm{per}}^{\mathrm{free}}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}) := \int_0^1 \ell(\tilde{\boldsymbol{x}}(\tau), \tilde{\boldsymbol{u}}(\tau))\mathrm{d}\tau \tag{3.19a}$$

$$\text{s.t.} \quad \dot{\tilde{\boldsymbol{x}}}(\tau) = Tf(\tilde{\boldsymbol{x}}(\tau), \tilde{\boldsymbol{u}}(\tau)), \tau \in [0,1], \tag{3.19b}$$

$$0 \le c(\tilde{\boldsymbol{x}}(\tau), \tilde{\boldsymbol{u}}(\tau)), \quad \tau \in [0,1], \tag{3.19c}$$

$$0 = \tilde{\boldsymbol{x}}(1) - \tilde{\boldsymbol{x}}(0), \tag{3.19d}$$

$$\underline{T} \le T \le \overline{T}. \tag{3.19e}$$

It is clear that any admissible triple $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}, T) \in \mathcal{AC}^{n_x}([0,1]) \times \mathrm{L}_\infty^{n_u}([0,1]) \times \mathbb{R}$ for the above problem corresponds to an admissible pair $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}([0,T]) \times \mathrm{L}_\infty^{n_u}([0,T])$ for problem $\mathrm{Per}_T^{\mathrm{fix}}$ with $\phi_{T,\mathrm{per}}^{\mathrm{fix}}(\boldsymbol{x}, \boldsymbol{u}) = \phi_{\mathrm{per}}^{\mathrm{free}}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})$, simply by reversing the time-substitution and setting $(\boldsymbol{x}, \boldsymbol{u})(\tau) := (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})(\tau/T)$ for $\tau \in [0, T]$.

In combination with the approximation Lemma 3.3, this shows that under suitable controllability assumptions, problem $\mathrm{Per}^{\mathrm{free}}$ offers a way to calculate approximations of any desired quality to the infinite horizon AOCP by adjusting the bounds for T according to Lemma 3.3. The smaller the lower bound $\underline{T}$ and the greater the upper bound $\overline{T}$ is set, the better is the possible approximation.

3.3 Summary

In this chapter we introduced AOCPs as OCPs with averaging performance criterion on infinite time horizons. The averaging objective allows to consider problems that otherwise would have an indefinite objective integral. Together with a compactness assumption, it is possible to show that the infimum of the average outputs always exists provided that the set of admissible trajectories is not empty.

Due to the averaging nature of the objective functional, uniqueness cannot be expected for solutions of AOCPs, since the objective does not depend on the initial behavior but only on the asymptotic behavior. We illustrated this with a simple example for a system satisfying a certain controllability property.

This is also the reason why in principle AOCPs with initial value constraint are not a suitable base for an NMPC subproblem.

In the second section we focused on periodic approximations for AOCPs. In a simple, two-dimensional case, the topological properties of $\mathbb{R}^2$ allow to conclude that the optimal average-output can be realized with a periodic solution. In a more general setting we have shown that a compactness assumption on the set of admissible states/controls is sufficient to prove the existence of quasi-periodic solutions with arbitrary close average output compared to the optimal average output.

Finally we have shown how to set up corresponding periodic OCPs that exploit these approximation properties and allow to solve AOCPs approximately by reducing them to finite horizon OCPs with a periodicity constraint.

Chapter 4

Economic Nonlinear Model Predictive Control for Average Output Optimal Control Problems

In this chapter, we address the main topic of this thesis: feedback control for AOCPs. As we have seen in the previous chapter, the problem formulation for AOCPs is fairly straightforward. However, the infinitely long time horizon and the thereby induced uniqueness-issues for solutions pose great problems in the context of feedback generation, in particular because the average output functional φ_{avg} does not depend on the initial behavior of the trajectories (cf. Remark 3.1).

We begin this chapter by discussing some existing approaches for economic NMPC for AOCPs. Then we present our approach which is based on periodic solutions and extends an existing "Self Tracking" approach by Limon et al. [77] to a more general nonlinear setting.

By means of controllability assumptions on the dynamical system and regularity assumptions on the proposed NMPC subproblems we can prove a stability result for the resulting controller which shows that the closed-loop trajectories have an economic performance that is equal to the economic performance of the optimal periodic trajectory.

4.1 Problem Formulation and Existing Approaches

Problem Formulation

Let $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t))$ be a dynamical system with a path constraint $0 \leq c(x,u)$ and a performance criterion $\ell(x,u)$ such that the functions f, c, ℓ are sufficiently smooth and the set of admissible states/controls $\mathrm{A}^c := \{(x,u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} : 0 \leq c(x,u)\}$ is compact (Assumption 3.1). Our aim is to define an economic NMPC subproblem for this system such that the resulting moving horizon NMPC scheme is recursively feasible and has an average economic output as close as possible to the optimal average output φ^*_{avg} of the corresponding AOCP.

Furthermore, we try to keep the amount of a priori information needed to set up the NMPC subproblem as small as possible. Examples for such a priori information are pre-calculated trajectories, special terminal constraints and terminal penalty terms.

In the following, we give a brief overview of existing NMPC schemes for AOCPs and point out the strengths and limitations.

4.1.1 Tracking

Although it is not an economic NMPC scheme, we add tracking NMPC in the list of existing methods, because tracking NMPC schemes can locally be equivalent (producing the same feedback) to an economic NMPC cf. [121] and[122].

Tracking NMPC schemes are based on a pre-calculated reference orbit $(\boldsymbol{x}_{\text{ref}}, \boldsymbol{u}_{\text{ref}}) \in \mathcal{AC}^{n_x}_{\text{loc}}(\mathbb{R}_{\geq 0}) \times \mathrm{L}^{n_u}_{\infty,\text{loc}}(\mathbb{R}_{\geq 0})$ which for example could be a solution of the AOCP P_{avg}.

With this reference solution, a tracking controller can be set up as it is described in Section 2.4. The well understood stability properties of tracking NMPC imply that the resulting feedback trajectory will robustly converge to the reference trajectory.

However, such a controller requires the solution of the corresponding AOCP beforehand. Furthermore, because the tracking objective criterion of the NMPC subproblems is decoupled form the real economic performance criterion, possible changes of the optimal reference solution (e.g. resulting from parameter-changes of the dynamical system) would have to be taken into account by updating the tracking reference trajectory. An example of such a tracking NMPC scheme that is based on multiple pre-calculated solutions can be found in Ilzhöfer et al. [64].

The Phase Problem

Another issue with tracking controllers arises when the reference solution is not constant, i.e. not a steady-state. In this case, any time shifted version of the reference trajectory has an equally good average economic output. Consequently the tracking controller could be defined with any of the shifted reference trajectories, which implies that in the setup of the NMPC subproblem an additional choice of phase is necessary. A tracking NMPC scheme without a terminal region constraint as in (2.16) or a terminal equality constraint as in (2.17) can furthermore lead to suboptimal performance if the wrong phase of the reference solution is tracked.

4.1.2 Economic Nonlinear Model Predictive Control for Dissipative Systems

In case the system satisfies a strict dissipativity condition for a steady state (Assumption 2.2) or for a periodic solution (Assumption 2.5), it is possible to set up an NMPC controller with the original performance criterion ℓ as performance criterion in the NMPC subproblem.

We described these approaches in Sections 2.5.1 (steady state case) and 2.5.3 (periodic case). The objective functional for the NMPC subproblem at sampling-time t_i with horizon $\mathcal{T}^{t_i} = [t_i, t_i + T]$ for such schemes is of the form

$$\varphi(\boldsymbol{x}, \boldsymbol{u}) = \int_{\mathcal{T}^{t_i}} \ell(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\mathrm{d}\tau + M(t_i + T, \boldsymbol{x}(t_i + T)). \tag{4.1}$$

The MAYER-term function M in these approaches guarantees asymptotic stability of the resulting closed-loop trajectories. It has to satisfy a descent condition (2.37) (steady-state case) respectively (2.65) (periodic case) along trajectories resulting from an auxiliary control law.

Such schemes often also require terminal constraints such as (2.33e) or (2.68e) that ensure recursive feasibility of the NMPC scheme by means of an auxiliary control law.

In Müller and Grüne [86], an economic NMPC scheme without terminal constraint for a strictly periodic dissipative system with period T_{p} is presented. It is based on a multi-step controller, where the applied control solution only is updated after a full period. For this controller, using suitable controllability assumptions, the authors prove near average optimal performance and convergence to the optimal periodic orbit.

The existing economic NMPC schemes offer closed-loop stability and performance guarantees for dissipative systems. However, dissipativity conditions can be difficult to check, especially for the case of periodic dissipativity. Furthermore a certain amount of a priori information on the optimal orbit is necessary to set up the controllers with time-dependent terminal regions and terminal penalties.

4.1.3 Self-Tracking Economic Nonlinear Model Predictive Control

An approach that also serves as foundation for our method is the single-layer E-NMPC proposed by Limon et al. [77] for periodic (with period $T_\mathrm{p} \in \mathbb{N}$) time-varying discrete affine linear systems, i.e. systems of the form

$$x(k+1) = f(k, x(k), u(k)) := A(k)x(k) + B(k)u(k), \tag{4.2}$$

with $A(k) \in \mathbb{R}^{n_x \times n_n}$ and $B(k) \in \mathbb{R}^{n_x \times n_u}$ time periodic such that $A(k) = A(k+T_\mathrm{p})$ and $B(k) = B(k+T_\mathrm{p})$ for all $k \in \mathbb{N}$. The time-dependent performance criterion is given by a function $\ell : \mathbb{N} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ that satisfies $\ell(k, x, u) = \ell(k+T_\mathrm{p}, x, u)$ for all $(x, u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$. Furthermore ℓ is assumed to be non-negative and convex in (x, u) for all $k \in \mathbb{N}$. As set of feasible states and controls $Z(k) \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ at time $k \in \mathbb{N}$ they consider closed and convex polyhedrons that contain the origin and also satisfy the periodicity condition $Z(k) = Z(k+T_\mathrm{p})$ for all $k \in \mathbb{N}$.

As discrete NMPC subproblem with horizon length $N (\leq T_\mathrm{p})$ at sampling time k with initial value $x \in \mathbb{R}^{n_x}$ the following NLP is chosen (where $\|\cdot\|_\mathbb{X}$ and $\|\cdot\|_\mathbb{U}$ are some weighted norms on $\mathbb{R}^{n_x}$ and $\mathbb{R}^{n_u}$ and $Z_r(i)$ a closed convex polyhedron contained in the relative interior of $Z_r(i)$):

$$\min_{\substack{\bar{x}_0,\dots,\bar{x}_N \in \mathbb{R}^{n_x},\\ x_0^a,\dots,x_{T_\mathrm{p}}^a \in \mathbb{R}^{n_x},\\ \bar{u}_0,\dots,\bar{u}_{N-1} \in \mathbb{R}^{n_u},\\ u_0^a,\dots,u_{T_\mathrm{p}-1}^a \in \mathbb{R}^{n_u}}} \sum_{i=0}^{N-1} \left(\left\| \bar{x}_i - x_i^a \right\|_\mathbb{X}^2 + \left\| \bar{u}_i - u_i^a \right\|_\mathbb{U}^2 \right) + \sum_{i=0}^{T_\mathrm{p}-1} \ell(k+i, x_i^a, u_i^a) \tag{4.3a}$$

$$\text{s.t.} \quad \bar{x}_0 = x, \tag{4.3b}$$

$$\bar{x}_{i+1} = f(i+k, \bar{x}_i, \bar{u}_i), \quad i = 0, \dots, N-1, \tag{4.3c}$$

$$(\bar{x}_i, \bar{u}_i) \in Z(i+k), \quad i = 0, \dots, N-1, \tag{4.3d}$$

$$\bar{x}_N = x_0^a, \tag{4.3e}$$

$$x_{i+1}^a = f(i+k, x_i^a, u_i^a), \quad i = 0, \dots, T_\mathrm{p}-1, \tag{4.3f}$$

$$(x_i^a, u_i^a) \in Z_r(i+k), \quad i = 0, \dots, T_\mathrm{p}-1, \tag{4.3g}$$

$$x_0^a = x_{T_\mathrm{p}}^a. \tag{4.3h}$$

Here, beside the predicted trajectory represented by $(\bar{x}_0, \dots, \bar{x}_N)$ an artificial trajectory represented by $(x_0^a, \dots, x_{T_\mathrm{p}}^a)$ is considered and the objective functional is the combination of the difference of artificial trajectory and predicted trajectory (measured with the norms $\|\cdot\|_\mathbb{X}$ and

$\|\cdot\|_{\mathbb{U}}$) and the economic performance of the artificial trajectory. Both artificial and predicted trajectory are subject to the discrete evolution of the system ((4.3c) and (4.3f)) and the end-value of the predicted trajectory is set to be equal to the initial value of the artificial trajectory (4.3e). The path constraint sets $Z_r(\cdot)$ in (4.3g) are slightly restricted versions of the sets $Z(\cdot)$, which ensures that the path constraint (4.3d) is not active in case the first sum of the objective (4.3a) vanishes. Furthermore, a periodicity constraint is imposed on the artificial trajectory (4.3h).

Using the convexity of the performance criterion ℓ and a controllability condition of the linear system, the authors prove stability of the resulting closed-loop feedback trajectory at the optimal T_{p}-periodic trajectory via a LYAPUNOV-argument.

What makes this approach different from other economic NMPCapproaches is, that the objective is split up in an economic part and a self-tracking-part which vanishes if the optimal periodic orbit is reached. Furthermore no a priori calculation of the optimal periodic trajectory is necessary (e.g. for setting up a terminal constraint).

4.1.4 Limitations

All of the three above mentioned NMPC schemes have their limitations. First, each tracking approach requires an a priori calculated steady-state or a reference solution. In case of a parameter change in the process dynamics or the objective function, this reference needs to be updated accordingly.

The pure economic schemes rely on dissipativity properties of the system which can be hard to verify. Adding a regularizing tracking term to the cost-function as described e.g. in [100] can help to guarantee stability for non-dissipative systems. Such schemes also often require terminal constraint regions that have to be computed beforehand and have to be updated accordingly in case of a change in the system parameters.

The self-tracking approach by Limon et al. [77] does not require the offline solution of a steady-state or periodic OCP, however the self-tracking weights (represented by the weighted norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{U}}$ in (4.3a)) have to be chosen and the setting is discrete affine linear with a performance criterion that is positive and convex in the state and control variables.

4.2 A Controller Based On Periodic Solutions

In this section we propose a new NMPC controller for Average Output Systems working with periodic solutions. The controller is based on similar ideas as the one presented in [77] (cf. Section 4.1.3), as we also use a periodicity constraint and a objective functional which is the combination of a self-tracking term and an economic contribution.

However, in comparison to [77], our setting is quite different as we consider systems with nonlinear dynamics and the approach we present is not restricted to periodic and convex performance criteria. Furthermore, the objective functional of the NMPC subproblem we chose does not contain a state self-tracking term as in (4.3a) but rather a performance self-tracking term and our stability theory is based on assumptions on the existence of periodic orbits and uniqueness and continuous dependence of solutions of the NMPC subproblems rather than on the convexity of the performance criterion.

We proceed with the definition of the NMPC subproblem and give a detailed explanation of the objective functional and the imposed constraints. Then we outline the strategy we use for analyzing the closed-loop behavior of the resulting system.

4.2.1 The Subproblem $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t,x)$

For a uniformly spaced grid of sampling times $\mathbb{T} := \{t_i = i\Delta T, i \in \mathbb{N}\}$ we define an NMPC subproblem in the following way.

Objective Functional

The time horizon $\mathcal{T}^{t_i} = [t_i, t_i + T_\mathrm{t} + T_\mathrm{p}]$ for the NMPC subproblem at the sampling time t_i is split in a transient phase of length T_t, $\mathcal{T}^{t_i}_{\mathrm{trans}} = [t_i, t_i + T_\mathrm{t}]$ and a periodic phase of length T_p, $\mathcal{T}^{t_i}_{\mathrm{per}} = [t_i + T_\mathrm{t}, t_i + T_\mathrm{t} + T_\mathrm{p}]$. The objective functional is split into parts corresponding to transient and periodic phase.

Let $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{AC}^{n_x}(\mathcal{T}^t) \times \mathrm{L}^{n_u}_\infty(\mathcal{T}^t)$. The contribution $\varphi^{\mathrm{fix}}_{\mathrm{per}}(\boldsymbol{x},\boldsymbol{u})$ of the periodic part is defined to be the average economic performance of the periodic part of the time horizon:

$$\varphi^{\mathrm{fix}}_{\mathrm{per}}(\boldsymbol{x},\boldsymbol{u}) := \frac{1}{T_\mathrm{p}} \int_{\mathcal{T}^{t_i}_{\mathrm{per}}} \ell(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\mathrm{d}\tau. \tag{4.4}$$

The main contribution of the transient part is defined as the L^2-difference of the economic performance along the transient phase and the economic performance shifted forward by T_p and weighted with a discount-factor term with $\rho > 1$:

$$\varphi^{\mathrm{fix}}_{\mathrm{trans},\ell}(\boldsymbol{x},\boldsymbol{u}) := \int_{\mathcal{T}^{t_i}_{\mathrm{trans}}} \rho^{\tau - t_i} \left|\ell(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)) - \ell(\boldsymbol{x}(\tau+T_\mathrm{p}),\boldsymbol{u}(\tau+T_\mathrm{p}))\right|^2 \mathrm{d}\tau. \tag{4.5}$$

Additionally, a contribution over the transient time horizon consisting of the L^2-deviation (measured with some weighted norm $\|\cdot\|_{\mathbb{U}}$) on $\mathbb{R}^{n_u}$ of the control function and the control function shifted forward by T_p, also weighted with the discount-factor term is added:

$$\varphi^{\mathrm{fix}}_{\mathrm{trans},u}(\boldsymbol{u}) := \int_{\mathcal{T}^{t_i}_{\mathrm{trans}}} \rho^{\tau - t_i} \left\|\boldsymbol{u}(\tau) - \boldsymbol{u}(\tau+T_\mathrm{p})\right\|^2_{\mathbb{U}} \mathrm{d}\tau. \tag{4.6}$$

Both parts $\varphi^{\mathrm{fix}}_{\mathrm{trans},\ell}(\boldsymbol{x},\boldsymbol{u})$ and $\varphi^{\mathrm{fix}}_{\mathrm{trans},u}(\boldsymbol{u})$ are combined and added up to the complete transient objective functional:

$$\varphi^{\mathrm{fix}}_{\mathrm{trans}}(\boldsymbol{x},\boldsymbol{u}) := \varphi^{\mathrm{fix}}_{\mathrm{trans},\ell}(\boldsymbol{x},\boldsymbol{u}) + \varphi^{\mathrm{fix}}_{\mathrm{trans},u}(\boldsymbol{u}). \tag{4.7}$$

For notational convenience, we introduce the shifted-difference operator [1]

$$D_{T_\mathrm{p}} : [t, t+T_\mathrm{t}+T_\mathrm{p}]^{\mathbb{R}} \to [t, t+T_\mathrm{t}]^{\mathbb{R}} \tag{4.8}$$

[1] For two sets A, B we denote the set of mappings from A to B by A^B.

which is defined to map a function $\alpha \in [t, t + T_\mathrm{t} + T_\mathrm{p}]^\mathbb{R}$ to the difference of α and a α shifted forward by T_p:

$$D_{T_\mathrm{p}}(\alpha)(\tau) := \alpha(\tau) - \alpha(\tau + T_\mathrm{p}). \tag{4.9}$$

By making use of the shift operator D_{T_p}, the contribution of the transient objective can be expressed as

$$\varphi^{\mathrm{fix}}_{\mathrm{trans}}(\boldsymbol{x},\boldsymbol{u}) := \int_{\mathcal{T}^{t_i}_{\mathrm{trans}}} \rho^{\tau - t_i} \left| D_{T_\mathrm{p}}\left(\ell(\boldsymbol{x},\boldsymbol{u})\right)(\tau)\right|^2 \mathrm{d}\tau + \int_{\mathcal{T}^{t_i}_{\mathrm{trans}}} \rho^{\tau - t_i} \left\| D_{T_\mathrm{p}}(\boldsymbol{u})(\tau)\right\|^2_{\mathbb{U}} \mathrm{d}\tau. \tag{4.10}$$

Remark 4.1 (Purpose of the Control Deviation Term (4.5) **of the Objective Functional)** The purpose of the control self-tracking term

$$\varphi^{\mathrm{fix}}_{\mathrm{trans},u}(\boldsymbol{u}) = \int_{\mathcal{T}^{t_i}_{\mathrm{trans}}} \rho^{\tau - t_i} \left\| D_{T_\mathrm{p}}(\boldsymbol{u})(\tau)\right\|^2_{\mathbb{U}} \mathrm{d}\tau$$

becomes apparent in the proof of Lemma 4.2. In the case the transient objective vanishes, it guarantees that $\boldsymbol{u}(\tau) = \boldsymbol{u}(\tau + T_\mathrm{p})$ holds for almost all τ in the interval $[t_i, t_i + T_\mathrm{t}]$. This fact together with the slightly relaxed path constraints in the transient part of the horizon (cf. constraint 4.15d) and the continuity of the path constraint function c then allow the conclusion that at least for a small time-interval I around $t_i + T_\mathrm{t}$ (in particular in the transient phase) the relaxed path constraint $0 \le c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) + \varepsilon$ is not active.

It has to be noted that $\varphi^{\mathrm{fix}}_{\mathrm{trans},u}$ can be omitted in case the path constraint function c is independent of the control input, because in this situation already the continuity of the state trajectory is sufficient to conclude that the relaxed path constraint is not active for a small time interval at the end of the transient horizon. △

The objective functional is now the sum of transient and periodic contribution, where the transient part additionally is weighted with a factor $w_{\mathrm{trans}} \in \mathbb{R}_{>0}$:

$$\varphi^{\mathrm{fix}}(\boldsymbol{x},\boldsymbol{u}) = w_{\mathrm{trans}}\varphi^{\mathrm{fix}}_{\mathrm{trans}}(\boldsymbol{x},\boldsymbol{u}) + \varphi^{\mathrm{fix}}_{\mathrm{per}}(\boldsymbol{x},\boldsymbol{u}). \tag{4.11}$$

Constraints

As in every NMPC scheme, the initial value constraint and the ODE-constraint are included in the NMPC subproblem.

The path constraint is treated differently in the transient and the periodic part of the time horizon. In the transient part it is slightly relaxed (by using a small $\varepsilon > 0$) while in the periodic part it remains unchanged:

$$0 \le c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) + \varepsilon \mathbb{1}_{n_c} \qquad \text{for } \tau \in \mathcal{T}^{t_i}_{\mathrm{trans}}, \tag{4.12}$$

$$0 \le c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) \qquad \text{for } \tau \in \mathcal{T}^{t_i}_{\mathrm{per}}. \tag{4.13}$$

The relaxation in the transient part is necessary to avoid issues with controllability. This will become clear in the proof of Lemma 4.2 which plays an important part in our closed-loop analysis. The practical relevance is rather negligible since the factor ε can be chosen arbitrarily small.

In addition to these constraints, we include the periodicity constraint

$$\boldsymbol{x}(t_i + T_\mathrm{t}) = \boldsymbol{x}(t_i + T_\mathrm{t} + T_\mathrm{p}) \tag{4.14}$$

in the formulation.

This constraint will guarantee recursive feasibility of the NMPC scheme. Combined with the objective contribution of the periodic part, this constraint also allows the conclusion that an admissible pair $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}(\mathcal{T}^{t_i}) \times \mathrm{L}_\infty^{n_u}(\mathcal{T}^{t_i})$ can be periodically extended to a pair $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})$ on the infinite time horizon with average economic performance $\varphi_\mathrm{avg}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}) = \varphi^\mathrm{fix}_\mathrm{per}(\boldsymbol{x}, \boldsymbol{u})$.

The NMPC Subproblem $\mathrm{NP}^\mathrm{fix}_{T_\mathrm{p}}(t, x)$

Summarizing the above explanations, the NMPC subproblem we consider at time t for the initial value x can be stated as follows ($\mathrm{NP}^\mathrm{fix}_{T_\mathrm{p}}(t, x)$):

$$\min_{\substack{\boldsymbol{x} \in \mathcal{AC}^{n_x}(\mathcal{T}^t),\\ \boldsymbol{u} \in \mathrm{L}_\infty^{n_u}(\mathcal{T}^t)}} \quad \varphi^\mathrm{fix}(\boldsymbol{x}, \boldsymbol{u}) = w_\mathrm{trans} \varphi^\mathrm{fix}_\mathrm{trans}(\boldsymbol{x}, \boldsymbol{u}) + \varphi^\mathrm{fix}_\mathrm{per}(\boldsymbol{x}, \boldsymbol{u}) \tag{4.15a}$$

$$\text{s.t.} \quad \boldsymbol{x}(t) = x, \tag{4.15b}$$

$$\dot{\boldsymbol{x}}(\tau) = f(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \qquad \tau \in \mathcal{T}^t, \tag{4.15c}$$

$$0 \le c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) + \varepsilon \mathbb{1}_{n_c}, \qquad \tau \in \mathcal{T}^t_\mathrm{trans}, \tag{4.15d}$$

$$0 \le c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \qquad \tau \in \mathcal{T}^t_\mathrm{per}, \tag{4.15e}$$

$$0 = \boldsymbol{x}(t + T_\mathrm{t}) - \boldsymbol{x}(t + T_\mathrm{t} + T_\mathrm{p}). \tag{4.15f}$$

We denote the optimal objective value for problem $\mathrm{NP}^\mathrm{fix}_{T_\mathrm{p}}(t, x)$ by $\varphi^{\mathrm{fix},*}(x)$ (skipping t since it is independent of the initial time t) and define

$$\mathrm{Ad}^\mathrm{fix}_{T_\mathrm{p}} := \left\{ x \in \mathbb{R}^{n_x} : \mathrm{NP}^\mathrm{fix}_{T_\mathrm{p}}(t, x) \text{ is feasible} \right\}. \tag{4.16}$$

4.2.2 Interpretation of the Transient Weight

In the following, we will describe how the transient weight w_trans in the objective functional (4.15a) can be interpreted economically.

The periodicity constraint (4.15f) allows the extension of any admissible trajectory $(\boldsymbol{x}, \boldsymbol{u})$ of problem $\mathrm{NP}^\mathrm{fix}_{T_\mathrm{p}}(t, x)$ to an arbitrary prolonged horizon (for simplicity we assume $t = 0$). For example, for any $k \in \mathbb{N}^+$ we have the identity

$$\varphi^\mathrm{fix}_\mathrm{per}(\boldsymbol{x}, \boldsymbol{u}) = \int_{T_\mathrm{t}}^{T_\mathrm{t} + T_\mathrm{p}} \ell(\boldsymbol{x}, \boldsymbol{u}) \mathrm{d}\tau = \frac{1}{k} \int_{T_\mathrm{t}}^{T_\mathrm{t} + kT_\mathrm{p}} \ell(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}) \mathrm{d}\tau, \tag{4.17}$$

where $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})$ is the periodic extension of $(\boldsymbol{x}, \boldsymbol{u})$ from $[T_\mathrm{t}, T_\mathrm{t} + T_\mathrm{p}]$ to the (probably much longer) interval $[T_\mathrm{t}, T_\mathrm{t} + kT_\mathrm{p}]$. If the transient weight is chosen as $w_\mathrm{trans} = \frac{1}{k}$, it holds that

$$\varphi^\mathrm{fix}(\boldsymbol{x}, \boldsymbol{u}) = \frac{1}{k}\varphi^\mathrm{fix}_\mathrm{trans}(\boldsymbol{x}, \boldsymbol{u}) + \frac{1}{k}\int_{T_\mathrm{t}}^{T_\mathrm{t}+kT_\mathrm{p}} \ell(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})\mathrm{d}\tau. \tag{4.18}$$

The objective functional $\varphi^\mathrm{fix}(\boldsymbol{x}, \boldsymbol{u})$ of problem $\mathrm{NP}^\mathrm{fix}_{T_\mathrm{p}}(t, x)$ then can be replaced by

$$\widetilde{\varphi^\mathrm{fix}}(\boldsymbol{x}, \boldsymbol{u}) := \varphi^\mathrm{fix}_\mathrm{trans}(\boldsymbol{x}, \boldsymbol{u}) + \int_{T_\mathrm{t}}^{T_\mathrm{t}+kT_\mathrm{p}} \ell(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})\mathrm{d}\tau \tag{4.19}$$

without changing the solution of $\mathrm{NP}^\mathrm{fix}_{T_\mathrm{p}}(t, x)$ (only multiplication with the constant k happened). Clearly, for large k the economic contribution of the integral $\int_{T_\mathrm{t}}^{T_\mathrm{t}+kT_\mathrm{p}} \ell(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})\mathrm{d}\tau$ becomes the dominant term in $\widetilde{\varphi^\mathrm{fix}}(\boldsymbol{x}, \boldsymbol{u})$ because the interval $[T_\mathrm{t}, T_\mathrm{t} + kT_\mathrm{p}]$ gets longer while the transient contribution $\varphi^\mathrm{fix}_\mathrm{trans}(\boldsymbol{x}, \boldsymbol{u})$ remains the same.

This shows how the transient weight w_trans can be interpreted as a parameter that can be used to artificially prolong the periodic horizon of the problem.

4.3 Closed-Loop Behavior of the Controller Based on $\mathrm{NP}^\mathrm{fix}_{T_\mathrm{p}}(t, x)$

In this section we analyze the properties of the controller based on the NMPC subproblem $\mathrm{NP}^\mathrm{fix}_{T_\mathrm{p}}(t, x)$. Often when analyzing the closed-loop properties of an NMPC controller, the goal is to prove asymptotic stability of the closed-loop trajectory at some reference trajectory $(\boldsymbol{x}_\mathrm{ref}, \boldsymbol{u}_\mathrm{ref})$.

But for several reasons, we pursue a different strategy.

First we have to remember what the purpose of the NMPC controller is. The purpose of the controller first and foremost is to generate a feedback for the system that results in an economically optimal performance. We have already seen that the economic average performance of any pair $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}_\mathrm{loc}(\mathbb{R}) \times \mathrm{L}^{n_u}_{\infty,\mathrm{loc}}(\mathbb{R})$ is invariant with respect to time-shifts, therefore there exist a lot of trajectories with equally good economic performance. There is no reason why the controller should stabilize the system at a particular reference solution when there exist a whole family of equally well performing alternatives.

A second scenario could be a perturbation that happens at time t and suddenly shifts the system state from $\boldsymbol{x}_\mathrm{ref}(t)$ to $\boldsymbol{x}_\mathrm{ref}(\tilde{t})$ with $\tilde{t} \neq t$. In case the controller is stabilizing the system at $\boldsymbol{x}_\mathrm{ref}$, the system will converge back to $\boldsymbol{x}_\mathrm{ref}$ in the long run. However, by applying the control $\tau \mapsto \boldsymbol{u}_\mathrm{ref}(\tau + \tilde{t} - t)$ from time t on it would be possible to keep the system on the shifted reference trajectory $(\boldsymbol{x}_\mathrm{ref}, \boldsymbol{u}_\mathrm{ref}) \circ \big(\tau \mapsto (\tau + \tilde{t} - t)\big)$ from time t on which has an equally good economic performance as $(\boldsymbol{x}_\mathrm{ref}, \boldsymbol{u}_\mathrm{ref})$.

For this reasons it is not necessarily a shortcoming for a NMPC controller if it does not stabilize a system at a given reference solution and we focus our considerations on a different interpretation of stability.

4.3.1 Strategy Outline

Instead of attempting to prove stability of the resulting controller at some reference solution, we pursue a different strategy. After observing that the controller has the recursive feasibility property (in Section 4.3.3), which follows from the periodicity constraint (4.15f), we begin analyzing the development of the optimal objective value function along the closed-loop trajectory. Let $x_k \in \mathbb{R}^{n_x}$ denote the state of the closed-loop system at sampling time t_k, i.e. $x_k = \boldsymbol{x}_\mu(t_k; x_0, t_0)$.

Assumptions on controllability, regularity and uniqueness of the NMPC subproblems and the periodic OCP $\mathrm{Per}^{\mathrm{fix}}_{T_\mathrm{p}}$ (3.17) then are used to show that the transient objective part of the solution $(\boldsymbol{x}_{t,x}, \boldsymbol{u}_{t,x})$ of the subproblem $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t,x)$ reaches its optimal value (0) if and only if the periodic part reaches its optimal value (Lemma 4.2). This observation together with the discount factor ρ of the transient objective part (4.10) allows us to conclude that the transient $(\varphi^{\mathrm{fix}}_{\mathrm{trans}}(\boldsymbol{x}_{t_k,x_k}, \boldsymbol{u}_{t_k,x_k}))$ and periodic parts $(\varphi^{\mathrm{fix}}_{\mathrm{per}}(\boldsymbol{x}_{t_k,x_k}, \boldsymbol{u}_{t_k,x_k}))$ of the objective function asymptotically converge to their respective optimal values for $k \to \infty$ (Lemma 4.4). This shows in particular that the average economic performance of the periodic part of the predicted open-loop solutions $(\boldsymbol{x}_{t_k,x_k}, \boldsymbol{u}_{t_k,x_k})$ asymptotically converges to the optimal periodic average performance.

Then we have to transfer the knowledge we gained for the development of the predicted open-loop solutions to the development of the actual closed-loop trajectory $(\boldsymbol{x}_\mu, \boldsymbol{u}_\mu)$. To do so, we analyze the rate of change of solutions of subsequent NMPC subproblems. We compare the solutions of the subproblems at sampling time t_k and t_{k+1} for increasing $k \in \mathbb{N}$. This comparison can be done although the solutions are defined on different time horizons $\mathcal{T}^{t_k}$ and $\mathcal{T}^{t_{k+1}}$ simply by restricting both solutions to their intersection $\mathcal{T}^{t_k} \cap \mathcal{T}^{t_{k+1}} = [t_{k+1}, t_k + T_\mathrm{t} + T_\mathrm{p}]$.

If the rate of change between subsequent NMPC subproblems can be controlled, it is possible to transfer information from the solutions of the subproblems to information on the behavior of the closed-loop trajectory. This transfer can be accomplished by making use of a telescope argument. In Theorem 4.9 we summarize the results and prove that the average output of the closed-loop system will be equal to the average output of the best T_p-periodic solution.

To illustrate the explained procedure, we include the dependency graph that connects the assumptions and statements in Figure 4.1.

4.3.2 Assumptions on Subproblems and on Optimal Periodic Orbits

The strategy outlined above is based on a number of regularity and uniqueness assumptions on the NMPC subproblems $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t_k, x_k)$ and on the periodic OCP $\mathrm{Per}^{\mathrm{fix}}_{T_\mathrm{p}}$ we introduced in Section 3.2.4 of the previous chapter.

The first assumption ensures the existence of optimal T_p-periodic solutions of the periodic OCP (3.17).

Assumption 4.1 (Existence of Optimal Periodic Solutions)
Problem $\mathrm{Per}^{\mathrm{fix}}_{T_\mathrm{p}}$ (3.17) has a solution. By $\phi^{\mathrm{fix},*}_{T_\mathrm{p},\mathrm{per}}$ we denote the objective value of the solution. $\triangle$

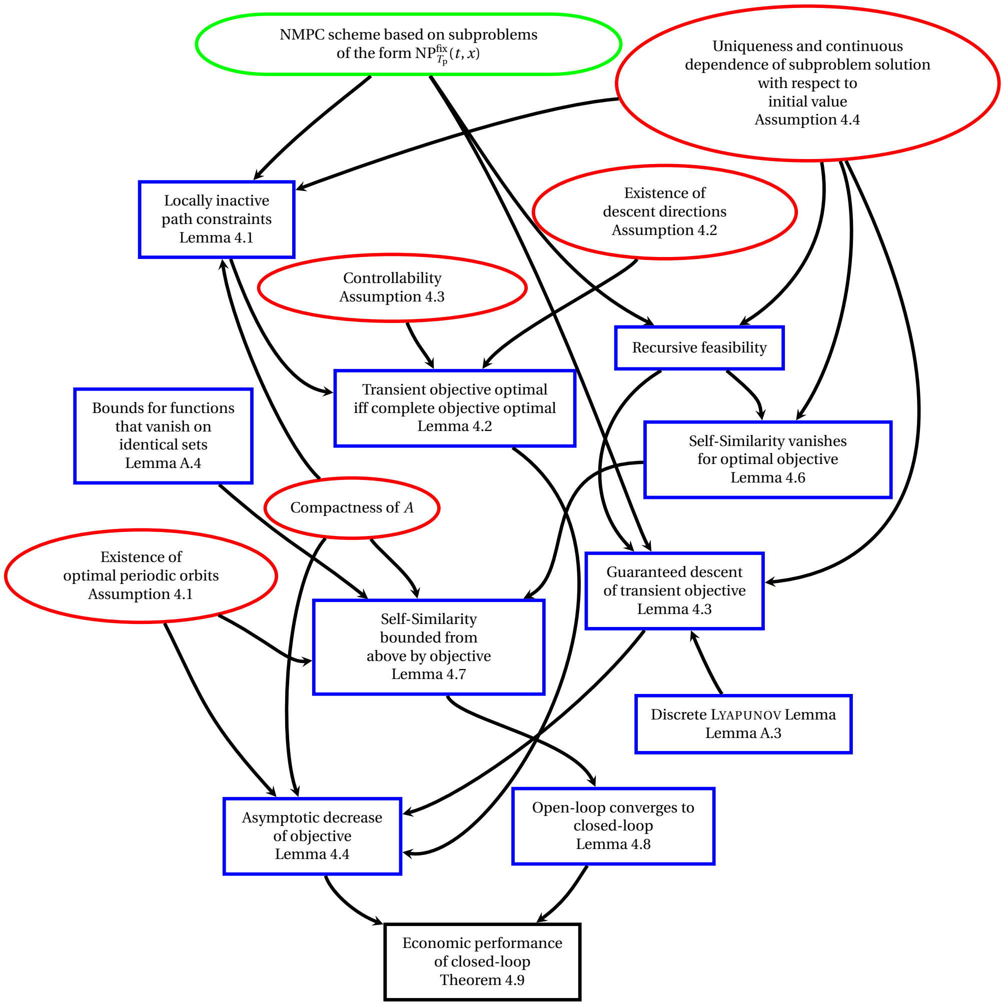


Figure 4.1: Dependency graph illustrating the strategy of the closed-loop analysis described in Section 4.3.1 for the NMPC scheme based on the subproblems $NP^{fix}_{T_p}$.

Remark 4.2 For the sake of a simplified notation, for the rest of this chapter we may assume without loss of generality that

$$\phi^{\text{fix},*}_{T_\text{p},\text{per}} = 0 \tag{4.20}$$

holds. This can be achieved by adding a suitable constant to the performance criterion ℓ. △

Additionally we need the existence of descent directions for suboptimal T_p-periodic solutions.

Assumption 4.2 (Existence of Descent Directions)
Let $(\boldsymbol{x}, \boldsymbol{u})$ be an admissible T_p-periodic pair for problem $\text{Per}^{\text{fix}}_{T_\text{p}}$ (3.17) with suboptimal objective $\phi^{\text{fix}}_{T_\text{p},\text{per}}(\boldsymbol{x}, \boldsymbol{u}) > \phi^{\text{fix},*}_{T_\text{p},\text{per}}$. Then there exists a neighborhood $U_t \subset \mathbb{R}$ of 0 and a FRÉCHET-differentiable homotopy $H = (H_x, H_u) : U_t \to \mathcal{AC}^{n_x}([0, T_\text{p}]) \times \text{L}^{n_u}_\infty([0, T_\text{p}])$ that satisfies the following conditions:

1. it starts at $(\boldsymbol{x}, \boldsymbol{u})$:

$$H(0) = (\boldsymbol{x}, \boldsymbol{u}), \tag{4.21}$$

2. for all $s \in U_t$, $H(s)$ is admissible for $\text{Per}^{\text{fix}}_{T_\text{p}}$,

3. the derivative of the objective functional evaluated along the homotopy is negative at $s = 0$:

$$\frac{\partial}{\partial s}\phi^{\text{fix}}_{T_\text{p},\text{per}}(H(s))|_{s=0} < 0. \tag{4.22}$$

△

Furthermore, we need a controllability assumption on the dynamical system and a regularity assumption on the NMPC subproblems.

Assumption 4.3 (Controllability)
For any interval $\mathcal{T} = [t_\text{s}, t_\text{f}] \subset \mathbb{R}$ and any pair $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}(\mathcal{T}) \times \text{L}^{n_u}_\infty(\mathcal{T})$ that satisfies $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t))$ for $t \in \mathcal{T}$ there exists an open neighborhood $U \subset \mathbb{R}^{n_x}$ of $\boldsymbol{x}(t_\text{s})$ and a FRÉCHET-differentiable mapping $C = (C_x, C_u) : U \to \mathcal{AC}^{n_x}(\mathcal{T}) \times \text{L}^{n_u}_\infty(\mathcal{T})$ that satisfies

- $C(\boldsymbol{x}(t_\text{s})) = (\boldsymbol{x}, \boldsymbol{u})$,
- for any $y \in U$, the pair $C(y) \in \mathcal{AC}^{n_x}(\mathcal{T}) \times \text{L}^{n_u}_\infty(\mathcal{T})$ satisfies the ODE,
- for any $y \in U$ it holds $C_x(y)(t_\text{s}) = \boldsymbol{x}(t_\text{s})$,
- for any $y \in U$ it holds $C_x(y)(t_\text{f}) = y$. △

Assumption 4.4 (Uniqueness and Continuous Dependence of Solutions)
The family of NMPC subproblems $\text{NP}^{\text{fix}}_{T_\text{p}}(t, x)$ has the following properties:

- For $x \in \mathrm{Ad}^{\mathrm{fix}}_{T_\mathrm{p}}$ problem $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t,x)$ has a unique solution denoted by $(\boldsymbol{x}_{t,x}, \boldsymbol{u}_{t,x})$ which depends on the initial time $t \in \mathbb{R}$ and the initial value $x \in \mathbb{R}^{n_x}$,
- the solution-mapping

$$\begin{aligned} \mathrm{Sol} : \mathbb{R} \times \mathrm{Ad}^{\mathrm{fix}}_{T_\mathrm{p}} &\to \mathcal{AC}^{n_x}(\mathcal{T}^t) \times \mathrm{L}^{n_u}_\infty(\mathcal{T}^t), \\ (t,x) &\mapsto (\boldsymbol{x}_{t,x}, \boldsymbol{u}_{t,x}) \end{aligned}$$

  is continuous. △

Since the problem $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t,x)$ is autonomous, i.e., neither the performance criterion nor the dynamics or the path constraint explicitly depend on time, it is clear that

$$(\boldsymbol{x}_{t,x}, \boldsymbol{u}_{t,x}) = (\boldsymbol{x}_{0,x}, \boldsymbol{u}_{0,x}) \circ [\tau \mapsto \tau - t] \tag{4.23}$$

holds and can always be used to transform problem $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t,x)$ to the problem $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(0,x)$ on the standard time horizon $\mathcal{T} := \mathcal{T}^0 = [0, T_\mathrm{t} + T_\mathrm{p}]$.

4.3.3 Recursive Feasibility

As we already pointed out in Section 2.3, recursive feasibility is the minimal requirement for any NMPC controller. The NMPC scheme based on the subproblems $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t_k, x_k)$ will have this property independently of the choice of the transient horizon length $T_\mathrm{t} \in \mathbb{R}_{\geq 0}$, the transient weighting factor $w_{\mathrm{trans}} \in \mathbb{R}_{\geq 0}$ and the discount factor $\rho > 1$. This is due to the periodicity constraint (4.15f) that is incorporated in the OCP-formulation.

To see this, we define the following extension operator

$$\mathrm{Ext}_k : \mathcal{AC}^{n_x}(\mathcal{T}^{t_k}) \times \mathrm{L}^{n_u}_\infty(\mathcal{T}^{t_k}) \to \mathcal{AC}^{n_x}(\mathcal{T}^{t_{k+1}}) \times \mathrm{L}^{n_u}_\infty(\mathcal{T}^{t_{k+1}})$$

by setting

$$\mathrm{Ext}_k(x,u)(t) := \begin{cases} (\boldsymbol{x}, \boldsymbol{u})(t) & \text{for } t \in [t_{k+1}, t_k + T_\mathrm{t} + T_\mathrm{p}] = \mathcal{T}^{t_k} \cap \mathcal{T}^{t_{k+1}}, \\ (\boldsymbol{x}, \boldsymbol{u})(t - T_\mathrm{p}) & \text{for } t \in [t_k + T_\mathrm{t} + T_\mathrm{p}, t_{k+1} + T_\mathrm{t} + T_\mathrm{p}] = \mathcal{T}^{t_{k+1}} \setminus \mathcal{T}^{t_k}, \end{cases} \tag{4.24}$$

and apply it to an admissible pair $(\boldsymbol{x}, \boldsymbol{u})$ for problem $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t_k, x_k)$.

We can check that $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}) := \mathrm{Ext}_k(x,u)$ is an admissible pair for problem $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t_{k+1}, \boldsymbol{x}(t_{k+1}))$:

- (per definition) it satisfies the initial value constraint (4.15b)

$$\tilde{\boldsymbol{x}}(t_{k+1}) = \boldsymbol{x}(t_{k+1}),$$

- it satisfies the ODE-constraint (4.15c) almost everywhere on $\mathcal{T}^{t_{k+1}}$, because $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})$ is just the periodic extension of $(\boldsymbol{x}, \boldsymbol{u})$ which satisfies it almost everywhere on $\mathcal{T}^{t_k}$,
- it satisfies the relaxed path constraint in the transient phase (4.15d) $\mathcal{T}^{t_{k+1}}_{\mathrm{trans}}$, because $(\boldsymbol{x}, \boldsymbol{u})$ satisfies it on $\mathcal{T}^{t_k}_{\mathrm{trans}}$ and (even the "unrelaxed" path-constraint) on $\mathcal{T}^{t_k}_{\mathrm{per}}$,

- it satisfies the path constraint in the periodic phase (4.15e) $\mathcal{T}_{\text{per}}^{t_{k+1}}$ because on $\mathcal{T}_{\text{per}}^{t_{k+1}} \cap \mathcal{T}_{\text{per}}^{t_k}$ it is identical to $(\boldsymbol{x}, \boldsymbol{u})$ and on $\mathcal{T}_{\text{per}}^{t_{k+1}} \setminus \mathcal{T}_{\text{per}}^{t_k}$ per definition it corresponds to $(\boldsymbol{x}, \boldsymbol{u})$ on $\mathcal{T}_{\text{per}}^{t_k} \setminus \mathcal{T}_{\text{per}}^{t_{k+1}}$,
- it satisfies the periodicity constraint (4.15f) because the extension operator is defined to periodically extend $(\boldsymbol{x}, \boldsymbol{u})$.

This shows that the NMPC controller based on the subproblems $\text{NP}_{T_\text{p}}^{\text{fix}}(t_k, x_k)$ has the recursive feasibility property.

4.3.4 Asymptotic Decrease of the Optimal Objective Value

The key observation we exploit in the process of showing that the objective value function is asymptotically decreasing along the closed-loop, is that the transient part of the objective functional vanishes if and only if the complete combined objective (transient and periodic part) attains its optimal value.

In accordance with our notation of the optimal objective value $\varphi^{\text{fix},*}(x)$ of problem $\text{NP}_{T_\text{p}}^{\text{fix}}(t, x)$, we define the functions $\varphi_{\text{per}}^{\text{fix},*}$, $\varphi_{\text{trans}}^{\text{fix},*}$ and $\varphi^{\text{fix},*}$ on the set $\text{Ad}_{T_\text{p}}^{\text{fix}}$ as follows:

$$\varphi_{\text{trans}}^{\text{fix},*}(x) := \varphi_{\text{trans}}^{\text{fix}}(\boldsymbol{x}_{t,x}, \boldsymbol{u}_{t,x}), \quad \varphi_{\text{per}}^{\text{fix},*}(x) := \varphi_{\text{per}}^{\text{fix}}(\boldsymbol{x}_{t,x}, \boldsymbol{u}_{t,x}), \quad \varphi^{\text{fix},*}(x) := \varphi^{\text{fix}}(\boldsymbol{x}_{t,x}, \boldsymbol{u}_{t,x}). \tag{4.25}$$

From the definition of the objective functionals it follows that for all $x \in \text{Ad}_{T_\text{p}}^{\text{fix}}$ it holds that

$$\varphi^{\text{fix},*}(x) = w_{\text{trans}} \varphi_{\text{trans}}^{\text{fix},*}(x) + \varphi_{\text{per}}^{\text{fix},*}(x). \tag{4.26}$$

The following Lemma is a technical result that shows that whenever the transient objective vanishes for an admissible pair of trajectories, there is a small time-interval at the end of the transient time horizon in which the path constraint (4.15d) is not active. We will need this Lemma in the proof of Lemma 4.2.

Lemma 4.1 (Locally Inactive Path Constraints)
Let $x \in \text{Ad}_{T_\text{p}}^{\text{fix}}$ and let $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}(\mathcal{T}) \times \text{L}_\infty^{n_u}(\mathcal{T})$ be admissible for problem $\text{NP}_{T_\text{p}}^{\text{fix}}(t, x)$ with $\varphi^{\text{fix}}(\boldsymbol{x}, \boldsymbol{u}) = 0$. Then, for any $\tilde{\varepsilon} > 0$ there exists a vicinity $U_{\tilde{\varepsilon}} \subset \mathbb{R}$ of $t + T_\text{t}$ such that for all $\tau \in U_{\tilde{\varepsilon}}$ it holds:

$$0_{n_c} < c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) + \tilde{\varepsilon} \mathbb{1}_{n_c}. \tag{4.27}$$

Proof Because of the path-constraint (4.15e) in the periodic part of the time horizon, $0 \leq c(\boldsymbol{x}(t + T_\text{t}), \boldsymbol{u}(t + T_\text{t}))$ holds. Since $\varphi_{\text{trans}}^{\text{fix}}(\boldsymbol{x}, \boldsymbol{u}) = 0$ it follows that

$$\varphi_{\text{trans},u}^{\text{fix}}(\boldsymbol{u}) = \int_t^{t+T_\text{t}} \rho^{\tau - t} \left\| D_{T_\text{p}}(\boldsymbol{u})(\tau) \right\|_{\mathbb{U}}^2 \text{d}\tau$$

vanishes and thus $\boldsymbol{u}(\tau) = \boldsymbol{u}(\tau + T_\mathrm{p})$ holds for almost all $\tau \in [t, t + T_\mathrm{t}]$. Combining this with the LIPSCHITZ-continuity of c (c is smooth and $c^{-1}([0,\infty)) = \mathrm{A}^c$ is compact and therefore in particular LIPSCHITZ-continuous with some constant $L_c > 0$) implies

$$\left\| c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) - c\big(\boldsymbol{x}(\tau + T_\mathrm{p}), \boldsymbol{u}(\tau + T_\mathrm{p})\big) \right\| \le L_c \left\| \boldsymbol{x}(\tau) - \boldsymbol{x}(\tau + T_\mathrm{p}) \right\| \tag{4.28}$$

for almost all $\tau \in [t, t + T_\mathrm{t}]$. Since all norms on finite dimensional real vector spaces are equivalent, this inequality (with an appropriately adjusted LIPSCHITZ constant) also holds for the maximum norm $\|\cdot\|_{\max}$.

The periodicity constraint (4.15f) and the continuity of $\boldsymbol{x}$ then imply the existence of a vicinity $U_{\tilde{\varepsilon}}$ of $t + T_\mathrm{t}$ such that

$$\left\| c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) - c\big(\boldsymbol{x}(\tau + T_\mathrm{p}), \boldsymbol{u}(\tau + T_\mathrm{p})\big) \right\|_{\max} < \tilde{\varepsilon} \tag{4.29}$$

holds for all $\tau \in U_{\tilde{\varepsilon}}$. Let now $\tau \in U_{\tilde{\varepsilon}}$. The path-constraint (4.15e) in the periodic part of the horizon ensures $0 \le c\big(\boldsymbol{x}(\tau + T_\mathrm{p}), \boldsymbol{u}(\tau + T_\mathrm{p})\big)$. In combination with the inequality (4.29) this implies that $0 < c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) + \tilde{\varepsilon}\mathbb{1}_{n_c}$. □

With a contradiction argument we can now show that the vanishing of the transient objective part is equivalent to the periodic part of the objective reaching its optimal value.

Lemma 4.2 (Transient Objective is Optimal iff Complete Objective is Optimal)
Let Assumptions 4.2 (existence of descent directions) and 4.3 (controllability) hold and let $x \in \mathrm{Ad}^{\mathrm{fix}}_{T_\mathrm{p}}$. Then $\varphi^{\mathrm{fix},*}_{\mathrm{trans}}(x) = 0$ if and only if $\varphi^{\mathrm{fix},*}(x) = \phi^{\mathrm{fix},*}_{T_\mathrm{p},\mathrm{per}}$.

Proof It is clear that if $\varphi^{\mathrm{fix},*}(x) = \phi^{\mathrm{fix},*}_{T_\mathrm{p},\mathrm{per}}$ holds, $\varphi^{\mathrm{fix},*}_{\mathrm{trans}}(x)$ has to vanish because it is by definition non-negative and

$$\varphi^{\mathrm{fix},*}(x) = w_{\mathrm{trans}}\varphi^{\mathrm{fix},*}_{\mathrm{trans}}(x) + \varphi^{\mathrm{fix},*}_{\mathrm{per}}(x) \ge w_{\mathrm{trans}}\varphi^{\mathrm{fix},*}_{\mathrm{trans}}(x) + \phi^{\mathrm{fix},*}_{T_\mathrm{p},\mathrm{per}} \tag{4.30}$$

holds with $w_{\mathrm{trans}} > 0$.

On the other hand, let us assume that $\varphi^{\mathrm{fix},*}_{\mathrm{trans}}(x) = 0$. We prove $\varphi^{\mathrm{fix},*}(x) = \phi^{\mathrm{fix},*}_{T_\mathrm{p},\mathrm{per}}$ by a contradiction argument. To this end we assume that $\varphi^{\mathrm{fix},*}(x) > \phi^{\mathrm{fix},*}_{T_\mathrm{p},\mathrm{per}}$. For notational convenience we use the abbreviation $(\boldsymbol{x}, \boldsymbol{u}) := (\boldsymbol{x}_{t,x}, \boldsymbol{u}_{t,x})$ for the rest of this proof and furthermore we may assume $t = 0$ since problem $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t, x)$ (4.15) is autonomous. Then, since $(\boldsymbol{x}, \boldsymbol{u})|_{\mathcal{T}_{\mathrm{per}}}$ can be interpreted as admissible pair for $\mathrm{Per}^{\mathrm{fix}}_{T_\mathrm{p}}$ (3.17), Assumption 4.2 guarantees the existence of a neighborhood $U_t \subset \mathbb{R}$ of 0 and a FRÉCHET-differentiable homotopy $H : U_t \to \mathcal{AC}^{n_x}(\mathcal{T}_{\mathrm{per}}) \times \mathrm{L}^{n_u}_{\infty}(\mathcal{T}_{\mathrm{per}})$ that satisfies the following conditions:

1. it starts at $(\boldsymbol{x}, \boldsymbol{u})|_{\mathcal{T}_{\mathrm{per}}}$:

$$H(0) = (\boldsymbol{x}, \boldsymbol{u})|_{\mathcal{T}_{\mathrm{per}}}, \tag{4.31}$$

2. for all $s \in U_t$, $H(s)$ is an admissible pair for $\mathrm{Per}^{\mathrm{fix}}_{T_\mathrm{p}}$,

3. the derivative of the objective functional evaluated along the homotopy is negative at $s = 0$

$$\frac{\partial}{\partial s}\phi^{\text{fix}}_{T_\text{p},\text{per}}(H(s))|_{s=0} < 0. \tag{4.32}$$

Lemma 4.1 guarantees the existence of a $\tilde{\delta} > 0$ such that $(\boldsymbol{x}, \boldsymbol{u})$ satisfies the inequality

$$0 \le c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) + \frac{\varepsilon}{2}\mathbb{1}_{n_c} \text{ for all } \tau \in [T_\text{t} - \tilde{\delta}, T_\text{t}]. \tag{4.33}$$

Furthermore, the controllability assumption 4.3 guarantees the existence of a neighborhood $U_x \subset \mathbb{R}^{n_x}$ of $\boldsymbol{x}(T_\text{t})$ and a FRÉCHET-differentiable mapping $C : U_x \to \mathcal{AC}^{n_x}([T_\text{t} - \tilde{\delta}, T_\text{t}]) \times \text{L}^{n_u}_\infty([T_\text{t} - \tilde{\delta}, T_\text{t}])$ satisfying

- $C(\boldsymbol{x}(T_\text{t})) = (\boldsymbol{x}, \boldsymbol{u})|_{[T_\text{t} - \tilde{\delta}, T_\text{t}]}$,
- for any $y \in U_x$, the pair $C(y) \in \mathcal{AC}^{n_x}([T_\text{t} - \tilde{\delta}, T_\text{t}]) \times \text{L}^{n_u}_\infty([T_\text{t} - \tilde{\delta}, T_\text{t}])$ is a solution of the dynamical system,
- for any $y \in U_x$ it holds $C(y)(T_\text{t} - \tilde{\delta}) = (\boldsymbol{x}, \boldsymbol{u})(T_\text{t} - \tilde{\delta})$.

Note that because of the continuity of C (remember that $\mathcal{AC}^{n_x}([T_\text{t} - \tilde{\delta}, T_\text{t}]) \times \text{L}^{n_u}_\infty([T_\text{t} - \tilde{\delta}, T_\text{t}])$ is endowed with the norm $\|\cdot\|_\infty$ cf. Definition 1.1), the neighborhood U_x can be chosen such that $C(y)(\tau)$ satisfies the relaxed path-constraint (4.15d) for all $\tau \in [T_\text{t} - \tilde{\delta}, T_\text{t}]$ and all $y \in U_x$.

The mappings H and C can be glued together (see Figure 4.2) to define the homotopy $\Psi = (\Psi_x, \Psi_u) : U_t \to \mathcal{AC}^{n_x}(\mathcal{T}) \times \text{L}^{n_u}_\infty(\mathcal{T})$ as follows:

$$\Psi(s)(\tau) := \begin{cases} (\boldsymbol{x}, \boldsymbol{u})(\tau) & \text{for } \tau \in [0, T_\text{t} - \tilde{\delta}], \\ C(H(s)(0))(\tau) & \text{for } \tau \in [T_\text{t} - \tilde{\delta}, T_\text{t}], \\ H(s)(\tau) & \text{for } \tau \in [T_\text{t}, T_\text{t} + T_\text{p}]. \end{cases} \tag{4.34}$$

The neighborhood U_t may have to be shrinked such that $H(s)(0) \in U_x$ for all $s \in U_t$. Then, per construction of Ψ, it can be seen that $\Psi(s) \in \mathcal{AC}^{n_x}(\mathcal{T}) \times \text{L}^{n_u}_\infty(\mathcal{T})$ satisfies the path constraints in the transient phase (4.15d) as well as in the periodic phase (4.15e) for all $s \in U_t$. It can also be verified that $\Psi(s)$ satisfies the initial value constraint (4.15b), the ODE-constraint (4.15c) and the periodicity constraint (4.15f) for all $s \in U_t$ and thus $\Psi(s) \in \mathcal{AC}^{n_x}(\mathcal{T}) \times \text{L}^{n_u}_\infty(\mathcal{T})$ is admissible for $\text{NP}^{\text{fix}}_{T_\text{p}}(t, x)$ for all $s \in U_t$.

Now we consider the derivative of the objective functional evaluated along this homotopy

$$\frac{\partial}{\partial s}\varphi^{\text{fix}}(\Psi(s)) = \frac{\partial}{\partial s}\left(w_\text{trans}\varphi^{\text{fix}}_\text{trans}(\Psi(s)) + \varphi^{\text{fix}}_\text{per}(\Psi(s))\right) \tag{4.35}$$

at $s = 0$. Since $\varphi^{\text{fix}}_\text{trans} : \mathcal{AC}^{n_x}(\mathcal{T}) \times \text{L}^{n_u}_\infty(\mathcal{T}) \to \mathbb{R}_{\ge 0}$ is differentiable, the composition $\varphi^{\text{fix}}_\text{trans} \circ \Psi : U_t \to \mathbb{R}_{\ge 0}$ is also a differentiable mapping. But because $\Psi(0) = (\boldsymbol{x}, \boldsymbol{u})$ we know $\varphi^{\text{fix}}_\text{trans}(\Psi(0)) =$

0. Therefore $\varphi^{\text{fix}}_{\text{trans}} \circ \Psi$ attains its global minimum at $s = 0$ and the derivative must vanish:

$$\frac{\partial}{\partial s}\varphi^{\text{fix}}_{\text{trans}}(\Psi(s))|_{s=0} = 0. \tag{4.36}$$

From the definition of $\Psi(s)(\tau)$ for $\tau \in [T_{\text{t}}, T_{\text{t}} + T_{\text{p}}]$ in (4.34) and the properties of the homotopy H, it follows that

$$\frac{\partial}{\partial s}\varphi^{\text{fix}}_{\text{per}}(\Psi(s))|_{s=0} < 0. \tag{4.37}$$

Now, (4.36) and (4.37) imply

$$\frac{\partial}{\partial s}\varphi^{\text{fix}}(\Psi(s))|_{s=0} < 0. \tag{4.38}$$

Therefore in a vicinity of $s = 0$ there exist an $\tilde{s}$ such that

$$\varphi^{\text{fix}}(\Psi(\tilde{s})) < \varphi^{\text{fix},*}(x). \tag{4.39}$$

But because $\Psi(\tilde{s})$ is an admissible pair for problem $\text{NP}^{\text{fix}}_{T_{\text{p}}}(t, x)$, this is a contradiction to the optimality of $(\boldsymbol{x}, \boldsymbol{u})$ and the proof is finished. □

We now proceed with the analysis of the development of the objective function $\varphi^{\text{fix},*}(\boldsymbol{x}_{\mu}(t))$ along closed-loop trajectories.

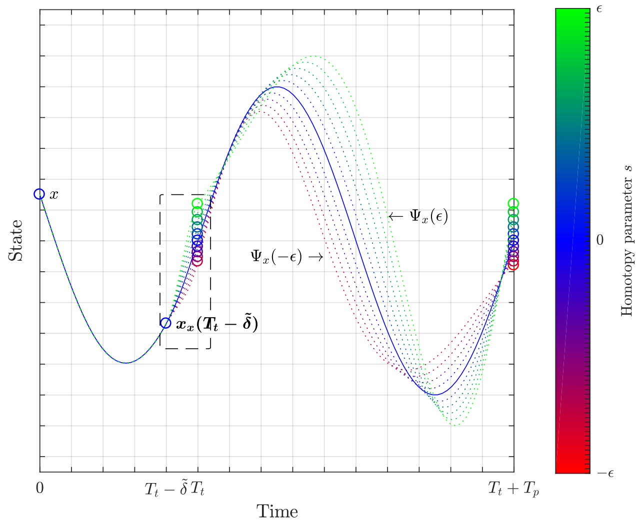
Lemma 4.3 (Guaranteed Objective Decrease)
Let $\varphi^{\text{fix},*}(x_k) = w_{\text{trans}}\varphi^{\text{fix},*}_{\text{trans}}(x_k) + \varphi^{\text{fix}}_{\text{per}}(x_k)$ be the objective value of the NMPC subproblem $\text{NP}^{\text{fix}}_{T_{\text{p}}}(t_k, x_k)$ at sampling time t_k. Then for the optimal objective value of the NMPC subproblem $\text{NP}^{\text{fix}}_{T_{\text{p}}}(t_{k+1}, x_{k+1})$ with initial value $x_{k+1} := \boldsymbol{x}_{t_k, x_k}(t_{k+1})$, the following inequality holds:

$$\varphi^{\text{fix},*}(x_{k+1}) \leq \rho^{-\Delta T}\, w_{\text{trans}}\varphi^{\text{fix},*}_{\text{trans}}(x_k) + \varphi^{\text{fix},*}_{\text{per}}(x_k). \tag{4.40}$$

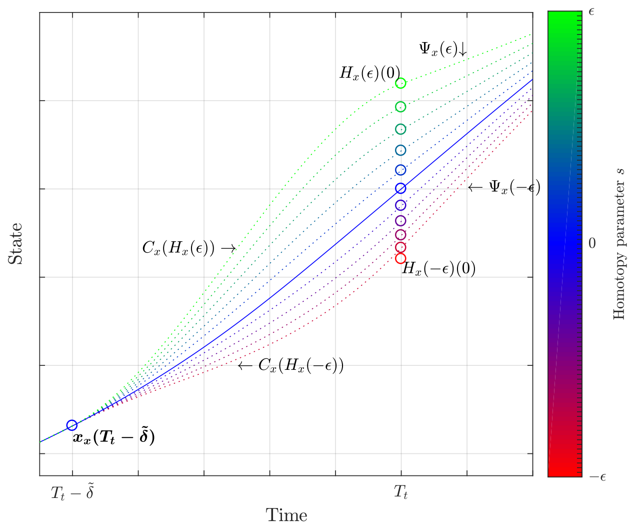
Proof Let $(\boldsymbol{x}, \boldsymbol{u}) := (\boldsymbol{x}_{t_k, x_k}, \boldsymbol{u}_{t_k, x_k})$. As it was shown in Section 4.3.3 (recursive feasibility), the pair $\text{Ext}_k(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}(\mathcal{T}^{t_{k+1}}) \times \text{L}^{n_u}_{\infty}(\mathcal{T}^{t_{k+1}})$ is admissible for problem $\text{NP}^{\text{fix}}_{T_{\text{p}}}(t_{k+1}, x_{k+1})$. Therefore, it can be used to bound $\varphi^{\text{fix},*}(x_{k+1})$ from above. To do so, we evaluate $\varphi^{\text{fix}}(\text{Ext}_k(\boldsymbol{x}, \boldsymbol{u}))$. Since the extension operator Ext_k preserves the economic performance in the periodic part, it holds that $\varphi^{\text{fix}}_{\text{per}}(\text{Ext}_k(\boldsymbol{x}, \boldsymbol{u})) = \varphi^{\text{fix},*}_{\text{per}}(x_k)$. Now we evaluate the transient part of the objective $\varphi^{\text{fix}}_{\text{trans}}(\text{Ext}_k(\boldsymbol{x}, \boldsymbol{u}))$.

From the definition of the Ext_k-operator it follows $\text{Ext}_k(\alpha)(\tau) = \text{Ext}_k(\alpha)(\tau + T_{\text{p}})$ for any $\tau \in [t_k + T_{\text{t}}, t_{k+1} + T_{\text{t}}] = \mathcal{T}^{t_{k+1}}_{\text{trans}} \setminus \mathcal{T}^{t_k}_{\text{trans}}$ and $\alpha : \mathcal{T}^{t_k} \to \mathbb{R}$. Therefore

$$D_{T_{\text{p}}}(\ell(\text{Ext}_k(\boldsymbol{x}, \boldsymbol{u})))(\tau) = 0 \text{ for } \tau \in \mathcal{T}^{t_{k+1}}_{\text{trans}} \setminus \mathcal{T}^{t_k}_{\text{trans}} \tag{4.41}$$



a) On the interval $[0, T_\mathrm{t} - \tilde{\delta}]$ all trajectories $\Psi_x(s)$ coincide with $\boldsymbol{x}_x$. On the periodic part $[T_\mathrm{t}, T_\mathrm{t} + T_\mathrm{p}]$, $\Psi_x(s)$ is defined by the homotopy $H_x(s)$ of Assumption 4.2 and on the part $[T_\mathrm{t} - \tilde{\delta}, T_\mathrm{t}]$ it is given by $s \mapsto C_x(H_x(s)(0))$, where C_x is the mapping of Assumption 4.3. Ψ is defined by gluing together both homotopies, see Figure (b).



b) Detailed picture of $(\Psi_x(s))_{s\in(-\varepsilon,\varepsilon)}$ on the interval $[T_\mathrm{t} - \tilde{\delta}, T_\mathrm{t}]$. For each $s \in (-\varepsilon, \varepsilon)$, the state $H_x(s)(0) \in \mathbb{R}^{n_x}$ is reached from the state $\boldsymbol{x}_x(T_\mathrm{t} - \tilde{\delta})$ in time $\tilde{\delta}$ via the state trajectory $C_x(H_x(s))$.

Figure 4.2: Illustration of the family of state trajectories $(\Psi_x(s))_{s\in(-\varepsilon,\varepsilon)}$. The color of the trajectories is associated to the homotopy parameter s according to the color bar on the right.

and

$$D_{T_\mathrm{p}}\left(\mathrm{Ext}_k(\boldsymbol{u})\right)(\tau) = 0 \text{ for } \tau \in \mathcal{T}_{\mathrm{trans}}^{t_{k+1}} \setminus \mathcal{T}_{\mathrm{trans}}^{t_k}. \tag{4.42}$$

Similarly, for any $\alpha : \mathcal{T}^{t_k} \to \mathbb{R}$ and $\tau \in [t_{k+1}, t_k + T_\mathrm{t}] = \mathcal{T}_{\mathrm{trans}}^{t_{k+1}} \cap \mathcal{T}_{\mathrm{trans}}^{t_k}$ it holds

$$\mathrm{Ext}_k(\alpha)(\tau) = \alpha(\tau) \text{ and } \mathrm{Ext}_k(\alpha)(\tau + T_\mathrm{p}) = \alpha(\tau + T_\mathrm{p}), \tag{4.43}$$

and therefore

$$D_{T_\mathrm{p}}\left(\ell(\mathrm{Ext}_k(\boldsymbol{x}, \boldsymbol{u}))\right)(\tau) = D_{T_\mathrm{p}}\left(\ell(\boldsymbol{x}, \boldsymbol{u})\right)(\tau) \text{ for } \tau \in \mathcal{T}_{\mathrm{trans}}^{t_{k+1}} \cap \mathcal{T}_{\mathrm{trans}}^{t_k} \tag{4.44}$$

and

$$D_{T_\mathrm{p}}(\mathrm{Ext}_k(\boldsymbol{u}))(\tau) = D_{T_\mathrm{p}}(\boldsymbol{u})(\tau) \text{ for } \tau \in \mathcal{T}_{\mathrm{trans}}^{t_{k+1}} \cap \mathcal{T}_{\mathrm{trans}}^{t_k}. \tag{4.45}$$

We evaluate the transient objective functional $\varphi_{\mathrm{trans}}^{\mathrm{fix}}$ at $\mathrm{Ext}_k(\boldsymbol{x}, \boldsymbol{u})$:

$$\begin{aligned} \varphi_{\mathrm{trans}}^{\mathrm{fix}}(\mathrm{Ext}_k(\boldsymbol{x}, \boldsymbol{u})) = &\int_{\mathcal{T}_{\mathrm{trans}}^{t_{k+1}} \cap \mathcal{T}_{\mathrm{trans}}^{t_k}} \rho^{\tau - t_{k+1}} \left(\left| D_{T_\mathrm{p}}\left(\ell(\mathrm{Ext}_k(\boldsymbol{x}, \boldsymbol{u}))\right)(\tau) \right|^2 + \left\| D_{T_\mathrm{p}}(\mathrm{Ext}_k(\boldsymbol{u}))(\tau) \right\|_{\mathbb{U}}^2 \right) \mathrm{d}\tau \\ &+ \int_{\mathcal{T}_{\mathrm{trans}}^{t_{k+1}} \setminus \mathcal{T}_{\mathrm{trans}}^{t_k}} \rho^{\tau - t_{k+1}} \left(\left| D_{T_\mathrm{p}}\left(\ell(\mathrm{Ext}_k(\boldsymbol{x}, \boldsymbol{u}))\right)(\tau) \right|^2 + \left\| D_{T_\mathrm{p}}(\mathrm{Ext}_k(\boldsymbol{u}))(\tau) \right\|_{\mathbb{U}}^2 \right) \mathrm{d}\tau \end{aligned} \tag{4.46}$$

The integrals over $\mathcal{T}_{\mathrm{trans}}^{t_{k+1}} \setminus \mathcal{T}_{\mathrm{trans}}^{t_k}$ vanish because of (4.41) and (4.42). For the integrals over $\mathcal{T}_{\mathrm{trans}}^{t_{k+1}} \cap \mathcal{T}_{\mathrm{trans}}^{t_k}$, we can use (4.44) and (4.45) and get

$$\int_{\mathcal{T}_{\mathrm{trans}}^{t_{k+1}} \cap \mathcal{T}_{\mathrm{trans}}^{t_k}} \rho^{\tau - t_{k+1}} \left(\left| D_{T_\mathrm{p}}\left(\ell(\boldsymbol{x}, \boldsymbol{u})\right)(\tau) \right|^2 + \left\| D_{T_\mathrm{p}}(u)(\tau) \right\|_{\mathbb{U}}^2 \right) \mathrm{d}\tau \tag{4.47}$$

$$\leq \int_{\mathcal{T}_{\mathrm{trans}}^{t_k}} \rho^{\tau - t_{k+1}} \left(\left| D_{T_\mathrm{p}}\left(\ell(\boldsymbol{x}, \boldsymbol{u})\right)(\tau) \right|^2 + \left\| D_{T_\mathrm{p}}(u)(\tau) \right\|_{\mathbb{U}}^2 \right) \mathrm{d}\tau \tag{4.48}$$

$$= \rho^{t_k - t_{k+1}} \int_{\mathcal{T}_{\mathrm{trans}}^{t_k}} \rho^{\tau - t_k} \left(\left| D_{T_\mathrm{p}}\left(\ell(\boldsymbol{x}, \boldsymbol{u})\right)(\tau) \right|^2 + \left\| D_{T_\mathrm{p}}(u)(\tau) \right\|_{\mathbb{U}}^2 \right) \mathrm{d}\tau \tag{4.49}$$

$$\leq \rho^{-\Delta T} \varphi_{\mathrm{trans}}^{\mathrm{fix}}(\boldsymbol{x}, \boldsymbol{u}), \tag{4.50}$$

which finishes the proof. □

Remark 4.3 (Necessity of the Discount Factor) In the proof of the previous Lemma it becomes apparent why the discount factor ρ is used in the definition of the transient objective part (4.10). The estimate (4.48) is conservative since it neglects the integrals

$$\int_{\mathcal{T}_{\mathrm{trans}}^{t_k} \setminus \mathcal{T}_{\mathrm{trans}}^{t_{k+1}}} \rho^{\tau - t_{k+1}} \left| D_{T_\mathrm{p}}\left(\ell(\boldsymbol{x}_{x_k}, \boldsymbol{u}_{x_k})\right)(\tau) \right|^2 \mathrm{d}\tau + \int_{\mathcal{T}_{\mathrm{trans}}^{t_k} \setminus \mathcal{T}_{\mathrm{trans}}^{t_{k+1}}} \rho^{\tau - t_{k+1}} \left\| D_{T_\mathrm{p}}(\boldsymbol{u})(\tau) \right\|_{\mathbb{U}}^2 \mathrm{d}\tau. \tag{4.51}$$

This means that the transient objective value at least gets smaller by the amount corresponding to this term. However, the two integrals could be zero even if $\varphi^{\text{fix}}_{\text{trans}}(\boldsymbol{x}_{x_k}, \boldsymbol{u}_{x_k})$ is non-zero itself. In this case, only the discount factor $\rho > 1$ guarantees the decrease of the objective functional. △

This Lemma shows that as long the transient part of the objective is non-zero, a decrease of the objective value at the next sampling time can be expected. In Lemma 4.2 we have seen that a zero transient objective implies also optimal periodic objective part. This combined with the previous Lemma allows us now to prove that the optimal objective value is asymptotically decreasing along closed-loop trajectories.

Lemma 4.4 (Asymptotic Decrease of the Objective Function Along Closed-Loop)
Let Assumption 4.1 (existence of optimal periodic orbits), 4.2 (existence of descent directions), 4.3 (controllability) and 4.4 (uniqueness and continuous dependence of subproblem solutions) hold. Then there exists a $\mathcal{KL}$ function β such that for any $x_0 \in \text{Ad}^{\text{fix}}_{T_\text{p}}$ and the resulting sequence of states $x_k = \boldsymbol{x}_\mu(t_k; t_0, x_0)$ of the closed-loop trajectory it holds that:

$$\varphi^{\text{fix},*}(x_k) \leq \beta(\varphi^{\text{fix},*}(x_0), t_k). \tag{4.52}$$

Proof From Lemma 4.3 it follows

$$\varphi^{\text{fix},*}(x_{i+1}) \leq \rho^{-\Delta T} w_{\text{trans}} \varphi^{\text{fix},*}_{\text{trans}}(x_i) + \varphi^{\text{fix},*}_{\text{per}}(x_i) = \varphi^{\text{fix},*}(x_i) - \underbrace{(1-\rho^{-\Delta T})}_{\in(0,1]} w_{\text{trans}} \varphi^{\text{fix},*}_{\text{trans}}(x_i). \tag{4.53}$$

Furthermore, Lemma 4.2 states that $\varphi^{\text{fix},*}_{\text{trans}}$ does not vanish as long as $\varphi^{\text{fix},*}$ is suboptimal. Assumption 4.1 (existence of optimal periodic orbits) implies the existence of $x \in \text{Ad}^{\text{fix}}_{T_\text{p}}$ with $\varphi^{\text{fix},*}(x) = 0$ and thus, since $\text{Ad}^{\text{fix}}_{T_\text{p}} \subset \text{A}^c_x$ is compact, Lemma A.4 guarantees the existence of a $\mathcal{K}_\infty$-function $\tilde{\alpha}$ such that

$$\tilde{\alpha}\left(\varphi^{\text{fix},*}(x)\right) \leq \varphi^{\text{fix},*}_{\text{trans}}(x) \tag{4.54}$$

holds for all $x \in \text{Ad}^{\text{fix}}_{T_\text{p}}$. Combined with (4.53), this implies

$$\varphi^{\text{fix},*}(x_{i+1}) \leq \varphi^{\text{fix},*}(x_i) - (1-\rho^{-\Delta T}) w_{\text{trans}} \tilde{\alpha}\left(\varphi^{\text{fix},*}(x_i)\right). \tag{4.55}$$

Since also $\alpha : s \mapsto (1-\rho^{-\Delta T}) w_{\text{trans}} \tilde{\alpha}(s)$ is a $\mathcal{K}_\infty$-function, Lemma A.3 can be applied and proves the existence of a $\mathcal{KL}$-function β with the desired properties. □

4.3.5 Open-Loop Solutions vs. Closed-Loop Behavior

In the previous section we have seen that under suitable conditions on the dynamical system and on the NMPC subproblems, the optimal objective value function $\varphi^{\text{fix},*}$ converges to zero $(= \phi^{\text{fix},*}_{T_\text{p},\text{per}})$ along closed-loop trajectories. This fact alone is not yet sufficient to conclude that the closed-loop trajectory has a good economic average performance, it just proves that the

NMPC subproblem solutions (the open-loop solutions) have a good economic performance. For example, it does not exclude the possibility of a scenario in which the closed-loop trajectory moves closely along an optimal periodic orbit but changes its direction from sampling time to sampling time.

However, in this section we will show that the uniqueness and continuity Assumption 4.4 on the NMPC subproblems implies that such a case cannot occur and that the asymptotic descent of $\varphi^{\text{fix},*}$ also implies that the average economic performance of the closed-loop trajectory converges to the average economic performance of the best periodic trajectory.

To this end we define a self-similarity function, which measures the difference of subsequent NMPC subproblem solutions. We use this function as a tool to show that this difference tends to zero along the closed-loop trajectory because it can be bounded by means of the optimal objective value function $\varphi^{\text{fix},*}$.

Definition 4.5 (Self-Similarity Function)
We call the function $S : \mathrm{Ad}^{\text{fix}}_{T_\text{p}} \to \mathbb{R}_{\geq 0}$ defined by

$$S(x) := \int_{\Delta T}^{T_\text{t}+T_\text{p}} \left(\left\| \boldsymbol{x}_x(\tau) - \boldsymbol{x}_{\Delta T,y}(\tau) \right\| + \left\| \boldsymbol{u}_x(\tau) - \boldsymbol{u}_{\Delta T,y}(\tau) \right\| \right) \mathrm{d}\tau, \tag{4.56}$$

with $y := \boldsymbol{x}_x(\Delta T)$ the self-similarity function (see Figure 4.3) △

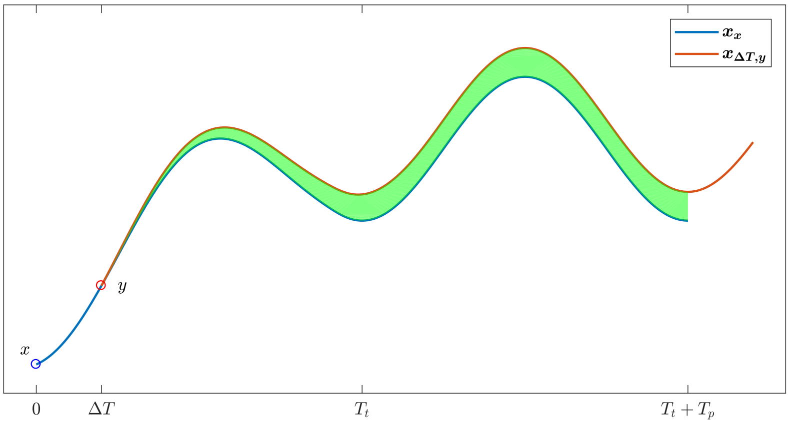


Figure 4.3: Illustration of the self- similarity function S. The green area between $\boldsymbol{x}_x$ and $\boldsymbol{x}_{\Delta T,y}$ with ($y = \boldsymbol{x}_x(\Delta T)$) corresponds to the integral $\int_{\Delta T}^{T_\text{t}+T_\text{p}} \left\| \boldsymbol{x}_x(\tau) - \boldsymbol{x}_{\Delta T,y}(\tau) \right\| \mathrm{d}\tau$ in the definition of $S(x)$. It vanishes if and only if $\boldsymbol{x}_x(\tau) = \boldsymbol{x}_{\Delta T,y}(\tau)$ holds for almost all $\tau \in [\Delta T, T_\text{t} + T_\text{p}]$.

From the continuity and uniqueness Assumption 4.4 it follows that the function S is continuous because it is the composition of continuous maps (in particular, the solution mapping $x \mapsto (\boldsymbol{x}_x, \boldsymbol{u}_x)$ and the evaluation mapping $\boldsymbol{x} \mapsto \boldsymbol{x}(\Delta T)$ are both continuous).

The function S has the following interesting property: It can easily be checked that if for any $x \in \mathrm{Ad}_{T_\mathrm{p}}^{\mathrm{fix}}$ it holds that $S(x) = 0$, then for $y = \boldsymbol{x}_{t_k,x}(t_{k+1})$ this implies

$$(\boldsymbol{x}_{t_k,x}, \boldsymbol{u}_{t_k,x})|_{\mathcal{T}^{t_k} \cap \mathcal{T}^{t_{k+1}}} = (\boldsymbol{x}_{t_{k+1},y}, \boldsymbol{u}_{t_{k+1},y})|_{\mathcal{T}^{t_k} \cap \mathcal{T}^{t_{k+1}}}. \tag{4.57}$$

The above equation is reminiscent of the Principle of Optimality for shrinking horizon NMPC (see Section 2.2.2), which states that the difference of subsequent NMPC subproblem solutions on the intersection of their time horizons vanishes.

The idea behind defining the function S is that if we can prove that S converges to 0 evaluated along the closed-loop trajectory $\boldsymbol{x}_\mu$, we know that the difference of subsequent NMPC subproblem solutions also converges to zero, which in turn implies that the difference of the NMPC subproblem solutions to the closed-loop trajectory itself also converges to zero.

The important observation that justifies this approach is the following:

Lemma 4.6 (Self-Similarity Vanishes if Objective is Optimal)
Let Assumption 4.4 (uniqueness and continuous dependence of subproblem solutions) hold. Then for any $x \in \mathrm{Ad}_{T_\mathrm{p}}^{\mathrm{fix}}$ with $\varphi^{\mathrm{fix},*}(x) = \phi^{\mathrm{fix},*}_{T_\mathrm{p},\mathrm{per}}$ it holds that $S(x) = 0$.

Proof Let $x \in \mathrm{Ad}_{T_\mathrm{p}}^{\mathrm{fix}}$ with $\varphi^{\mathrm{fix},*}(x) = 0$ and let $(\boldsymbol{x}, \boldsymbol{u}) := (\boldsymbol{x}_{t,x}, \boldsymbol{u}_{t,x})$. Without loss of generality we may again assume $t = 0$. As we have seen in the proof of Lemma 4.3, for the pair $\mathrm{Ext}(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}([\Delta T, \Delta T + T_\mathrm{t} + T_\mathrm{p}]) \times \mathrm{L}_\infty^{n_u}([\Delta T, \Delta T + T_\mathrm{t} + T_\mathrm{p}])$ (which is admissible for $\mathrm{NP}_{T_\mathrm{p}}^{\mathrm{fix}}(\Delta T, \boldsymbol{x}_x(\Delta T))$) it holds that:

$$\varphi_{\mathrm{per}}^{\mathrm{fix}}(\mathrm{Ext}(\boldsymbol{x}, \boldsymbol{u})) \leq \varphi^{\mathrm{fix}}(\boldsymbol{x}, \boldsymbol{u}) = 0. \tag{4.58}$$

But because $\varphi_{\mathrm{per}}^{\mathrm{fix}}$ is bounded from below by $\phi^{\mathrm{fix},*}_{T_\mathrm{p},\mathrm{per}} = 0$, it follows that $\varphi_{\mathrm{per}}^{\mathrm{fix}}(\mathrm{Ext}(\boldsymbol{x}, \boldsymbol{u})) = 0$. Because 0 is the optimal objective value, the uniqueness property in Assumption 4.4 implies that $\mathrm{Ext}(\boldsymbol{x}, \boldsymbol{u})$ is the solution of problem $\mathrm{NP}_{T_\mathrm{p}}^{\mathrm{fix}}(\Delta T, \boldsymbol{x}(\Delta T))$ (which is equivalent to $\mathrm{Ext}(\boldsymbol{x}, \boldsymbol{u}) \circ [\tau \mapsto \tau + \Delta T]$ being the solution of problem $\mathrm{NP}_{T_\mathrm{p}}^{\mathrm{fix}}(\boldsymbol{x}(\Delta T))$).

Therefore, according to the definition of S,

$$S(x) = \int_{\Delta T}^{T_\mathrm{t} + T_\mathrm{p}} \left(\|\boldsymbol{x}(\tau) - \mathrm{Ext}(\boldsymbol{x})(\tau)\| + \|\boldsymbol{u}(\tau) - \mathrm{Ext}(\boldsymbol{u})(\tau)\| \right) \mathrm{d}\tau. \tag{4.59}$$

But from the definition of the extension operator Ext (4.24) in Section 4.3.3, it follows that $\mathrm{Ext}(\boldsymbol{x})$ and $\mathrm{Ext}(\boldsymbol{u})$ coincide on $[\Delta T, T_\mathrm{t} + T_\mathrm{p}]$ with $\boldsymbol{x}$ and $\boldsymbol{u}$ respectively, which means the integral in (4.59) vanishes and $S(x) = 0$. □

Lemma 4.6 can be exploited to prove that the optimal objective function $\varphi^{\mathrm{fix},*}$ can be used as an upper bound for S in the following sense.

Lemma 4.7 (Upper Bound for Self-Similiarity)
Let Assumptions 4.1 (existence of optimal periodic solutions) and 4.4 (uniqueness and continuous dependence of subproblem solutions) hold. Then there exists a $\mathcal{K}$-function α such that

$$S(x) \leq \alpha(\varphi^{\mathrm{fix},*}(x)) \tag{4.60}$$

holds for all $x \in \mathrm{Ad}^{\mathrm{fix}}_{T_\mathrm{p}}$.

Proof The previous Lemma shows that for the continuous functions S and $\varphi^{\mathrm{fix},*}$, which are both defined on the compact set $\mathrm{Ad}^{\mathrm{fix}}_{T_\mathrm{p}}$, it holds

$$\varphi^{\mathrm{fix},*}(x) = 0 \Rightarrow S(x) = 0. \tag{4.61}$$

Furthermore, because of Assumption 4.1, the function $\varphi^{\mathrm{fix},*}$ does vanish for some $x \in \mathrm{Ad}^{\mathrm{fix}}_{T_\mathrm{p}}$. Therefore, according to Lemma A.4, there exists a $\mathcal{K}$-function α with the desired property. □

The asymptotic decrease of the optimal objective value function $\varphi^{\mathrm{fix},*}$ and the fact that the self-similarity function S can be bounded by $\varphi^{\mathrm{fix},*}$ as stated in the above Lemma implies that also the function S decreases asymptotically along closed-loop trajectories. In the following Lemma we show how this in turn implies that the difference between the predicted NMPC subproblem solutions and the actual closed-loop trajectory also decreases asymptotically.

For the sampling time t_i and the state $x_i \in \mathrm{Ad}^{\mathrm{fix}}_{T_\mathrm{p}}$ we define

$$\mathrm{diff}(t_i, x_i) := \int_{\mathcal{T}^{t_i}} \left(\left\|\boldsymbol{x}_{t_i,x_i}(\tau) - \boldsymbol{x}_\mu(\tau)\right\| + \left\|\boldsymbol{u}_{t_i,x_i}(\tau) - \boldsymbol{u}_\mu(\tau)\right\|\right) \mathrm{d}\tau. \tag{4.62}$$

Lemma 4.8 (Telescope Argument for the Closed-Loop)
Let Assumptions 4.1 (existence of optimal periodic solutions), 4.2 (existence of descent directions), 4.3 (controllability) and 4.4 (uniqueness and continuous dependence of subproblem solutions) hold. Then there exists a $\mathcal{KL}$-function β such that for all $x_0 \in \mathrm{Ad}^{\mathrm{fix}}_{T_\mathrm{p}}$ and the resulting closed-loop sequence $(x_i)_{i\in\mathbb{N}}$ it holds that

$$\mathrm{diff}(t_i, x_i) \le \beta(\varphi^{\mathrm{fix},*}(x_0), t_i). \tag{4.63}$$

Proof According to Lemma 4.4, there exists a $\mathcal{KL}$-function β_1 such that for the objective $\varphi^{\mathrm{fix},*}$ and all $i \in \mathbb{N}^+$ it holds that

$$\varphi^{\mathrm{fix},*}(x_i) \le \beta_1(\varphi^{\mathrm{fix},*}(x_0), t_i). \tag{4.64}$$

Using the estimate (4.60) of Lemma 4.7, for all $i \in \mathbb{N}$ it holds

$$S(x_i) \le \alpha\left(\beta_1\left(\varphi^{\mathrm{fix},*}(x_0), t_i\right)\right). \tag{4.65}$$

Since α is a $\mathcal{K}$-function and β_1 is a $\mathcal{KL}$-function, $\tilde{\beta} := \alpha \circ \beta_1$ is also a $\mathcal{KL}$-function. Now we calculate the integral of (4.63) using the fact that $(\boldsymbol{x}_\mu, \boldsymbol{u}_\mu)$ is per definition identical to $(\boldsymbol{x}_{t_j,x_j}, \boldsymbol{u}_{t_j,x_j})$ on the interval $\mathcal{T}^{t_j} \setminus \mathcal{T}^{t_{j+1}} = [t_j, t_{j+1}]$. Let $n \in \mathbb{N}$ such that $n\Delta T \le T_\mathrm{t} + T_\mathrm{p} < (n+1)\Delta T$, such that for all sampling times t_i it holds that $\mathcal{T}^{t_i} \subset [t_i, t_{i+n}]$. Then we calculate

$$\mathrm{diff}(t_i, x_i) = \int_{\mathcal{T}^{t_i}} \left(\left\|\boldsymbol{x}_{t_i,x_i}(\tau) - \boldsymbol{x}_\mu(\tau)\right\| + \left\|\boldsymbol{u}_{t_i,x_i}(\tau) - \boldsymbol{u}_\mu(\tau)\right\|\right) \mathrm{d}\tau \tag{4.66}$$

$$= \sum_{j=i+1}^{i+n} \int_{t_j}^{t_{j+1}} \left(\left\| \boldsymbol{x}_{t_i,x_i}(\tau) - \boldsymbol{x}_{t_j,x_j}(\tau) \right\| + \left\| \boldsymbol{u}_{t_i,x_i}(\tau) - \boldsymbol{u}_{t_j,x_j}(\tau) \right\| \right) \mathrm{d}\tau. \tag{4.67}$$

With the abbreviations

$$d_{i,j}^k := \int_{t_k}^{t_{k+1}} \left(\left\| \boldsymbol{x}_{t_i,x_i}(\tau) - \boldsymbol{x}_{t_j,x_j}(\tau) \right\| + \left\| \boldsymbol{u}_{t_i,x_i}(\tau) - \boldsymbol{u}_{t_j,x_j}(\tau) \right\| \right) \mathrm{d}\tau, \tag{4.68}$$

the triangle-inequality and a telescope-sum we get the inequality

$$\operatorname{diff}(t_i, x_i) = \sum_{j=i+1}^{i+n} d_{i,j}^j \le \sum_{j=i+1}^{i+n} \sum_{l=i}^{j-1} d_{l,l+1}^j. \tag{4.69}$$

Rearranging the summation order gives

$$\operatorname{diff}(t_i, x_i) \le \sum_{j=i+1}^{i+n} \sum_{l=i}^{j-1} d_{l,l+1}^j = \sum_{l=i}^{i+n-1} \sum_{j=l+1}^{i+n} d_{l,l+1}^j. \tag{4.70}$$

Since $d_{l,l+1}^j \ge 0$ for all $l \le j \le l+n$ we get

$$\operatorname{diff}(t_i, x_i) \le \sum_{l=i}^{i+n-1} \sum_{j=l+1}^{i+n} d_{l,l+1}^j \le \sum_{l=i}^{i+n-1} \sum_{j=l+1}^{l+n} d_{l,l+1}^j. \tag{4.71}$$

Using the definition of $d_{l,l+1}^j$ and the estimate (4.65) we get

$$\begin{aligned} \operatorname{diff}(t_i, x_i) &\le \sum_{l=i}^{i+n-1} \sum_{j=l+1}^{l+n} \int_{t_j}^{t_{j+1}} \left(\left\| \boldsymbol{x}_{t_i,x_i}(\tau) - \boldsymbol{x}_{t_j,x_j}(\tau) \right\| + \left\| \boldsymbol{u}_{t_i,x_i}(\tau) - \boldsymbol{u}_{t_j,x_j}(\tau) \right\| \right) \mathrm{d}\tau && (4.72) \\ &= \sum_{l=i}^{i+n-1} S(x_l) \le \sum_{l=i}^{i+n-1} \tilde{\beta}(\varphi^{\text{fix},*}(x_0), t_l) && (4.73) \\ &\le n\tilde{\beta}(\varphi^{\text{fix},*}(x_0), t_i). && (4.74) \end{aligned}$$

Since $(s,t) \mapsto n\tilde{\beta}(s,t)$ is a $\mathcal{KL}$-function, the proof is finished. □

We summarize the results of the analysis of the closed-loop behavior in the following theorem.

Theorem 4.9 (Economic Closed-Loop Performance)
Let Assumptions 4.1 (existence of optimal periodic orbits), 4.2 (existence of descent directions), 4.3 (controllability) and 4.4 (uniqueness and continuous dependence of subproblem solutions) hold. Let $x_0 \in \mathrm{Ad}_{T_\mathrm{p}}^{\mathrm{fix}}$ be the state of the system at the initial time t_0. Then the resulting closed-loop trajctory $(\boldsymbol{x}_\mu, \boldsymbol{u}_\mu) \in \mathcal{AC}_{\mathrm{loc}}^{n_x} \times \mathrm{L}_{\infty,\mathrm{loc}}^{n_u}$ from the NMPC scheme based on the subproblems $\mathrm{NP}_{T_\mathrm{p}}^{\mathrm{fix}}$ (4.15) exists for all times and for the average economic performance it

holds

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} \ell(\boldsymbol{x}_\mu(\tau), \boldsymbol{u}_\mu(\tau))\mathrm{d}\tau = \phi^{\text{fix},*}_{T_\mathrm{p},\text{per}}. \tag{4.75}$$

Proof Via the estimate (4.63) of the previous Lemma we get the estimate

$$\int_{\mathcal{T}^{t_i}_{\text{per}}} \left(\left\|\boldsymbol{x}_{t_i,x_i}(\tau) - \boldsymbol{x}_\mu(\tau)\right\| + \left\|\boldsymbol{u}_{t_i,x_i}(\tau) - \boldsymbol{u}_\mu(\tau)\right\|\right)\mathrm{d}\tau \le \operatorname{diff}(t_i, x_i) \le \beta(\varphi^{\text{fix},*}(x_0), t_i). \tag{4.76}$$

This shows that for the economic performance of $(\boldsymbol{x}_\mu, \boldsymbol{u}_\mu)$ on the intervals $\mathcal{T}^{t_i}_{\text{per}}$ it holds

$$\lim_{i\to\infty} \left| \frac{1}{T_\mathrm{p}}\int_{\mathcal{T}^{t_i}_{\text{per}}} \ell(\boldsymbol{x}_\mu(\tau), \boldsymbol{u}_\mu(\tau))\mathrm{d}\tau - \underbrace{\frac{1}{T_\mathrm{p}}\int_{\mathcal{T}^{t_i}_{\text{per}}} \ell(\boldsymbol{x}_{t_i,x_i}(\tau), \boldsymbol{u}_{t_i,x_i}(\tau))\mathrm{d}\tau}_{=\varphi^{\text{fix},*}_{\text{per}}(x_i)} \right| = 0. \tag{4.77}$$

On the other hand, in Lemma 4.4 it was shown that $\lim_{i\to\infty}\varphi^{\text{fix},*}(x_i) = 0 (= \phi^{\text{fix},*}_{T_\mathrm{p},\text{per}})$ holds. In combination with (4.77) this implies the assertion. □

A Note on the Transient Part of the Objective Functional

The ideas presented in this chapter can be extended to a broader class of controllers by using different transient objective functionals $\varphi^{\text{fix}}_{\text{trans},\ell}$. The crucial property of the transient objective functional, which in particular is used in the proof of Lemma 4.3, is that the integrand of $\varphi^{\text{fix}}_{\text{trans},\ell}$ at time $\tau \in \mathcal{T}^t_{\text{trans}}$ needs to vanish if $(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) = (\boldsymbol{x}(\tau + T_\mathrm{p}), \boldsymbol{u}(\tau + T_\mathrm{p}))$ holds. This means that $\varphi^{\text{fix}}_{\text{trans},\ell}$ can be replaced by

$$\varphi^{\text{fix}}_{\text{trans},g}(\boldsymbol{x}, \boldsymbol{u}) := \int_{\mathcal{T}^t_{\text{trans}}} \rho^{\tau - t} \left| g(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) - g(\boldsymbol{x}(\tau + T_\mathrm{p}), \boldsymbol{u}(\tau + T_\mathrm{p}))\right|^2 \mathrm{d}\tau, \tag{4.78}$$

where $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ is an arbitrary function. As long as Assumption 4.4 (uniqueness and continuous dependence of subproblem solutions) holds, all the results presented in this chapter will carry over to the NMPC scheme where $\varphi^{\text{fix}}_{\text{trans},\ell}$ is replaced by $\varphi^{\text{fix}}_{\text{trans},g}$.

In the work of Limon et al. [77] the integrand of the transient contribution at a time τ directly measures the geometric distance between the states/controls in the transient part and the states/controls (evaluated at $\tau + kT_\mathrm{p}$ with k being an integer such that $\tau + kT_\mathrm{p}$ is in the periodic part of the horizon):

$$\left\|(\boldsymbol{x}, \boldsymbol{u})(\tau) - (\boldsymbol{x}, \boldsymbol{u})(\tau + kT_\mathrm{p})\right\|^2 \text{ such that } \tau + kT_\mathrm{p} \in [t_i + T_\mathrm{t}, t_i + T_\mathrm{t} + T_\mathrm{p}]. \tag{4.79}$$

In contrast to that, we take the "performance"-distance, which at a given time τ measures the difference of the performance criterion evaluated at time τ and at time $\tau + T_\mathrm{p}$, plus an additional control-distance term:

$$\left|D_{T_\mathrm{p}}(\ell(\boldsymbol{x},\boldsymbol{u}))(\tau)\right|^2 + \left\|D_{T_\mathrm{p}}(\boldsymbol{u})(\tau)\right\|_{\mathbb{U}}^2 = \left|\ell(\boldsymbol{x},\boldsymbol{u})(\tau) - \ell(\boldsymbol{x},\boldsymbol{u})(\tau + T_\mathrm{p})\right|^2 + \left\|\boldsymbol{u}(\tau) - \boldsymbol{u}(\tau + T_\mathrm{p})\right\|_{\mathbb{U}}^2. \tag{4.80}$$

There are two reasons behind this choice. First it gives the controller more freedom to possibly find economically better ways to steer the system to an economically optimal periodical orbit, since it does not penalize the geometric distance to the periodic part of the trajectory but rather the difference of the economic performance in the transient and periodic part.

Second, it reduces the need of weighting factors that balance the different scales of the transient objective contribution and the periodic objective contribution. The different scales stem from the fact that the periodic objective contribution measures an economic quantity (the performance of the system in the periodic part of the time horizon) and the transient part of the objective contribution (4.79) measures a geometric quantity, whereas in (4.80) it measures the difference of economic contributions. Still, in our approach a weighting factor for the control difference term $\left\|\boldsymbol{u}(\tau) - \boldsymbol{u}(\tau + T_\mathrm{p})\right\|_{\mathbb{U}}^2$ is needed, which is encoded in the norm $\|\cdot\|_{\mathbb{U}}$.

4.4 Summary

After a brief overview on existing NMPC approaches for AOCPs, we developed a novel NMPC approach, that is based on ideas similar to [77] and exploits the good approximation properties of periodic solutions to AOCPs.

The presented NMPC scheme is based on an NMPC subproblem with periodicity constraint and an objective functional that is split up in a transient and periodic part. We proceed with a thorough analysis of the closed-loop behavior of the resulting controller, which is based on controllability assumptions on the dynamic system and regularity assumptions on the NMPC subproblems. Independent of explicit dissipativity assumptions we prove asymptotic decrease of the cost-to-go function along the closed-loop.

Furthermore we define a self-similarity function, which measures the distance of subsequent NMPC subproblem solutions. We show that this function asymptotically decreases along the closed-loop trajectories and we use it to prove that the closed-loop trajectory resulting from the controller has asymptotically the same economic performance as the optimal periodic operation.

Chapter 5

Extensions to Free Period and Systems with Periodic Performance Criterion

In this chapter, we generalize the ideas behind the NMPC controller presented in the previous chapter in two directions.

First, we consider a scenario where the system dynamics are parameter dependent assuming that the current parameter is known to the controller at all times and can change abruptly. Our working assumption in this case will be that for different parameters different periodic solutions will be optimal. In particular we assume that also the period of the optimal periodic solution can change with the parameter. To take this into account in the NMPC context, we include the period into the NMPC subproblems as optimization variable and analyze the properties of the resulting controller.

In a second scenario, we consider systems with time-periodic performance criterion and apply a modified version of the NMPC controller presented in the previous chapter. In this scenario, we choose the period of the performance criterion also as length of the periodic part of the NMPC subproblem horizon. The properties of the resulting controller are analyzed.

Throughout this chapter, we assume that the feasible region defined by the path constraint

$$\mathrm{A}^c := \left\{(x,u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} : 0_{n_c} \leq c(x,u)\right\} \tag{5.1}$$

is compact.

5.1 Extension to Free Periods

5.1.1 Parameter Dependent Right-Hand Sides

In this section we consider a scenario where the right-hand side f additionally depends on a parameter $p \in \mathbb{R}^{n_p}$, i.e., is of the form $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times P \to \mathbb{R}^{n_x}$. The set P of possible parameters is assumed to be a compact subset of $\mathbb{R}^{n_p}$. For a given parameter $p \in P$ the dynamical system then is governed by the ODE

$$\dot{\boldsymbol{x}}(t) = f_p(\boldsymbol{x}(t), \boldsymbol{u}(t)) := f(\boldsymbol{x}(t), \boldsymbol{u}(t), p). \tag{5.2}$$

The parameter can change with time and is continuously monitored. We assume that the information of its exact value at the current time is always available to the controller. Our goal is to design a NMPC scheme that can use this information and automatically react appropriately in case a parameter change occurs.

The scenario of changing parameters has to be taken into account in the NMPC subproblems, because the controller relies on the accurate prediction quality of the dynamic model equations. For the NMPC scheme presented in the previous chapter this means that the ODE-constraint $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t))$ of the subproblem $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t_i, x_i)$ at sampling time t_i obviously has to be changed to $\dot{\boldsymbol{x}}(t) = f_{p_i}(\boldsymbol{x}(t), \boldsymbol{u}(t))$, where p_i is the value of the parameter at this sampling time.

However, it is not sufficient to only replace the ODE constraint. Also the periodicity constraint $\boldsymbol{x}(t_i + T_\mathrm{t}) = \boldsymbol{x}(t_i + T_\mathrm{t} + T_\mathrm{p})$ could cause problems. This is because a T_{p_0}-periodic solution $(\boldsymbol{x}, \boldsymbol{u})$ for $\dot{\boldsymbol{x}}(t) = f_{p_0}(\boldsymbol{x}(t), \boldsymbol{u}(t))$ does not need to be a solution, let alone a periodic solution with period T_{p_0} of $\dot{\boldsymbol{x}}(t) = f_{p_1}(\boldsymbol{x}(t), \boldsymbol{u}(t))$ for a different parameter $p_1 \neq p_0$. It may even be the case that for the new parameter p_1 there exist no T_{p_0}-periodic solutions of $\dot{\boldsymbol{x}}(t) = f_{p_1}(\boldsymbol{x}(t), \boldsymbol{u}(t))$ at all. In this case the NMPC controller that is based on T_{p_0} periodic solutions will generate a infeasible NMPC subproblem.

Even if there exist T_{p_0}-periodic solutions of $\dot{\boldsymbol{x}}(t) = f_{p_1}(\boldsymbol{x}(t), \boldsymbol{u}(t))$ it could happen that for a different period $T_{p_1} \neq T_{p_0}$ there exists periodic solutions with better average economic performance (see the powerkite application example in Chapter 8).

We formalize the scenario of parameter dependent optimal periodic orbits with the following assumption, which will be the basis for our considerations.

Assumption 5.1 (Existence of Optimal Periodic Solution for Different Parameters)
For any $p \in P$ there exists a solution of the problem $\mathrm{Per}^{\mathrm{free}}_p$

$$\min_{\substack{\boldsymbol{x}\in\mathcal{AC}^{n_x}([0,1]),\\ \boldsymbol{u}\in\mathrm{L}^{n_u}_\infty([0,1]),\\ T\in\mathbb{R}}} \quad \phi^{\mathrm{free}}_{p,\mathrm{per}}(\boldsymbol{x}, \boldsymbol{u}) := \int_0^1 \ell(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))\mathrm{d}\tau \tag{5.3a}$$

$$\text{s.t.} \quad \dot{\boldsymbol{x}}(t) = T f_p(\boldsymbol{x}(t), \boldsymbol{u}(t)), t \in [0,1], \tag{5.3b}$$

$$0 \le c(\boldsymbol{x}(t), \boldsymbol{u}(t)), \quad t \in [0,1], \tag{5.3c}$$

$$0 = \boldsymbol{x}(1) - \boldsymbol{x}(0), \tag{5.3d}$$

$$\underline{T} \le T \le \overline{T}. \tag{5.3e}$$

The optimal objective value is denoted by $\phi^{\mathrm{free},*}_{p,\mathrm{per}}$. △

Remark 5.1 As in problem $\mathrm{Per}^{\mathrm{free}}$ in Section 3.2.4, a time-transformation resulting from the substitution $t(\tau) = T\tau$ is used in the formulation of problem $\mathrm{Per}^{\mathrm{free}}_p$. Any admissible triple $(\boldsymbol{x}, \boldsymbol{u}, T) \in \mathcal{AC}^{n_x}([0,1]) \times \mathrm{L}^{n_u}_\infty([0,1]) \times \mathbb{R}$ for problem $\mathrm{Per}^{\mathrm{free}}_p$ corresponds to a T-periodic solution $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}) := (\boldsymbol{x}, \boldsymbol{u}) \circ [t \mapsto t/T] \in \mathcal{AC}^{n_x}([0,T]) \times \mathrm{L}^{n_u}_\infty([0,T])$ of the system $\dot{\boldsymbol{x}}(t) = f_p(\boldsymbol{x}(t), \boldsymbol{u}(t))$ with average economic performance $\frac{1}{T}\int_0^T \ell(\tilde{\boldsymbol{x}}(\tau), \tilde{\boldsymbol{u}}(\tau))\mathrm{d}\tau = \phi^{\mathrm{free}}_{p,\mathrm{per}}(\boldsymbol{x}, \boldsymbol{u})$. △

In the following, we discuss the necessary modifications of the NMPC subproblems that allow us to extend the NMPC scheme presented in the previous chapter to the described scenario.

5.1.2 NMPC Subproblem with Free Period

Inclusion of the Period T_p as Optimization Variable

The main idea for extending the controller based on the subproblems $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t_i, x_i)$ to the varying parameter case is to include the period T_p as an optimization variable (within reasonable bounds) in the NMPC subproblem rather than have it fixed to a given number.

To illustrate our approach of including the period as optimization variable we first state a “naive” version of the modified NMPC subproblem and then apply a time transformation to normalize the time horizons and get a proper OCP.

The “Naive” NMPC Subproblem

As in the NMPC subproblems for a fixed period $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t_i, x_i)$, we split the time horizon in a transient and a periodic part. In order to have time horizons of constant length after the reparametrization, we choose the transient horizon length T_t to be a constant multiple of the periodic horizon length:

$$T_\mathrm{t} := c_t T_\mathrm{p} \text{ with fixed } c_t \in \mathbb{R}_{>0}. \tag{5.4}$$

The variable T_p then will be the variable controlling the period.

For a pair $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}([t_i, T_\mathrm{t} + T_\mathrm{p}]) \times \mathrm{L}^{n_u}_\infty([t_i, t_i + T_\mathrm{t} + T_\mathrm{p}])$, the objective is split up in a transient part

$$\tilde{\varphi}^{\mathrm{free}}_{p,\mathrm{per}}(\boldsymbol{x}, \boldsymbol{u}, T_\mathrm{p}) := \frac{1}{T_\mathrm{p}} \left(\int_0^{c_t T_\mathrm{p}} \rho^\tau \left| D_{T_\mathrm{p}} (\ell(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau))) \right|^2 \mathrm{d}\tau + \int_0^{c_t T_\mathrm{p}} \rho^\tau \left\| D_{T_\mathrm{p}}(\boldsymbol{u})(\tau) \right\|^2_{\mathbb{U}} \mathrm{d}\tau \right), \tag{5.5}$$

and the periodic part

$$\tilde{\varphi}^{\mathrm{free}}_{p,\mathrm{trans}}(\boldsymbol{x}, \boldsymbol{u}, T_\mathrm{p}) := \frac{1}{T_\mathrm{p}} \int_{c_t T_\mathrm{p}}^{c_t T_\mathrm{p} + T_\mathrm{p}} \ell\left(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)\right) \mathrm{d}\tau. \tag{5.6}$$

Remark 5.2 The integral $\int_0^{c_t T_\mathrm{p}} \rho^\tau \left\| D_{T_\mathrm{p}}(\boldsymbol{u})(\tau) \right\|^2_{\mathbb{U}} \mathrm{d}\tau$ is included for the purpose also explained in Remark 4.1. It guarantees that in case the transient objective $\tilde{\varphi}^{\mathrm{free}}_{p,\mathrm{trans}}$ vanishes, then for almost all $\tau \in [0, c_t T_\mathrm{p}]$ the term $D_{T_\mathrm{p}}(\boldsymbol{u})(\tau) = \boldsymbol{u}(\tau) - \boldsymbol{u}(\tau + T_\mathrm{p})$ also vanishes. △

Since T_p will be subject to optimization, the factor $1/T_\mathrm{p}$ has to be included in the transient and the periodic part of the objective. In the periodic part, this factor makes sure that the contribution in (5.6) corresponds to the average performance during one period, and in the transient part it is necessary to avoid that the contribution in (5.5) gets smaller just by decreasing T_p which would skew the transient objective functional towards smaller periods.

The path-constraints are treated in the same way as in the NMPC subproblem $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t_i, x_i)$ and are slightly relaxed in the transient part.

We then arrive at the following problem $\widetilde{\mathrm{NP}}_p^{\mathrm{free}}(x_i)$ (with $\mathcal{T}^{t_i} := [t_i, t_i + T_\mathrm{t} + T_\mathrm{p}]$):

$$\min_{\substack{\boldsymbol{x}\in\mathcal{AC}^{n_x}(\mathcal{T}^{t_i}),\\ \boldsymbol{u}\in\mathrm{L}_\infty^{n_u}(\mathcal{T}^{t_i}),\\ T_\mathrm{p}\in\mathbb{R}}} \quad \tilde{\varphi}_p^{\mathrm{free}}(\boldsymbol{x},\boldsymbol{u},T_\mathrm{p}) = w_\mathrm{trans}\tilde{\varphi}_{p,\mathrm{trans}}^{\mathrm{free}}(\boldsymbol{x},\boldsymbol{u},T_\mathrm{p}) + \tilde{\varphi}_{p,\mathrm{per}}^{\mathrm{free}}(\boldsymbol{x},\boldsymbol{u},T_\mathrm{p}) \tag{5.7a}$$

$$\text{s.t.} \quad \boldsymbol{x}(t_i) = x_i, \tag{5.7b}$$

$$\dot{\boldsymbol{x}}(\tau) = f_p(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)), \qquad \tau\in[t_i, t_i+T_\mathrm{t}+T_\mathrm{p}], \tag{5.7c}$$

$$0 \le c(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)) + \varepsilon\mathbb{1}_{n_c}, \qquad \tau\in[t_i, t_i+T_\mathrm{t}], \tag{5.7d}$$

$$0 \le c(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)), \qquad \tau\in[t_i+T_\mathrm{t}, t_i+T_\mathrm{t}+T_\mathrm{p}], \tag{5.7e}$$

$$0 = \boldsymbol{x}(t_i+T_\mathrm{t}) - \boldsymbol{x}(t_i+T_\mathrm{t}+T_\mathrm{p}), \tag{5.7f}$$

$$\underline{T} \le T_\mathrm{p} \le \overline{T}. \tag{5.7g}$$

Problem $\widetilde{\mathrm{NP}}_p^{\mathrm{free}}(x_i)$ is not an OCP because the spaces $\mathcal{AC}^{n_x}([t_i, t_i+T_\mathrm{t}+T_\mathrm{p}])$ and $\mathrm{L}_\infty^{n_u}([t_i, t_i+T_\mathrm{t}+T_\mathrm{p}])$ are varying with the length of the periodic horizon, T_p, which also is an optimization variable of the problem.

To overcome this issue, we need to normalize the time horizons.

Normalization of the Time Horizon

To transform $\widetilde{\mathrm{NP}}_p^{\mathrm{free}}(x_i)$ to an OCP with fixed time horizon, we reparametrize it to the time horizon $[0, c_t+1] = [0, c_t T_\mathrm{p} + T_\mathrm{p}]/T_\mathrm{p}$ via the same time transformation $[\tau \mapsto T_\mathrm{p}\tau]$ which is also used in the formulation of $\mathrm{Per}_p^{\mathrm{free}}$ in Assumption 5.1.

The objective functionals $\tilde{\varphi}_{p,\mathrm{trans}}^{\mathrm{free}}$ (5.5) and $\tilde{\varphi}_{p,\mathrm{per}}^{\mathrm{free}}$ (5.6) transform to the functionals $\varphi_{p,\mathrm{trans}}^{\mathrm{free}}$ and $\varphi_{p,\mathrm{per}}^{\mathrm{free}}$ on $\mathcal{AC}^{n_x}([0, c_t+1]) \times \mathrm{L}_\infty^{n_u}([0, c_t+1]) \times \mathbb{R}_{\ge 0}$ which are defined as follows. For $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{AC}^{n_x}([0, c_t+1]) \times \in \mathrm{L}_\infty^{n_u}([0, c_t+1])$ the transient part is

$$\varphi_{p,\mathrm{trans}}^{\mathrm{free}}(\boldsymbol{x},\boldsymbol{u},T) := \int_0^{c_t} \rho^{\tau T}\left|D_1\left(\ell(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\right)\right|^2 \mathrm{d}\tau + \int_0^{c_t} \rho^{\tau T}\left\|D_1(\boldsymbol{u})(\tau)\right\|_{\mathbb{U}}^2 \mathrm{d}\tau, \tag{5.8}$$

and the periodic part is

$$\varphi_{p,\mathrm{per}}^{\mathrm{free}}(\boldsymbol{x},\boldsymbol{u}) := \int_{c_t}^{c_t+1} \ell\left(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)\right)\mathrm{d}\tau. \tag{5.9}$$

Note that the multiplication with the averaging factor $1/T$ that is present in the objective functionals $\tilde{\varphi}_{p,\mathrm{per}}^{\mathrm{free}}$ (5.5) and $\varphi_{p,\mathrm{trans}}^{\mathrm{free}}$ (5.6) only seemingly disappeared due the time-transformation. The values of the integrals still correspond to the averaged values of $\tilde{\varphi}_{p,\mathrm{trans}}^{\mathrm{free}}$ and $\tilde{\varphi}_{p,\mathrm{per}}^{\mathrm{free}}$.

This results in the problem $\mathrm{NP}_p^{\mathrm{free}}(x_i)$:

$$\min_{\substack{\boldsymbol{x}\in\mathcal{AC}^{n_x}([0,c_t+1]),\\ \boldsymbol{u}\in\mathrm{L}_\infty^{n_u}([0,c_t+1]),\\ T\in\mathbb{R}}} \varphi_p^{\mathrm{free}}(\boldsymbol{x},\boldsymbol{u},T) = w_{\mathrm{trans}}\varphi_{p,\mathrm{trans}}^{\mathrm{free}}(\boldsymbol{x},\boldsymbol{u},T) + \varphi_{p,\mathrm{per}}^{\mathrm{free}}(\boldsymbol{x},\boldsymbol{u}) \tag{5.10a}$$

$$\text{s.t.} \quad \boldsymbol{x}(0) = x_i, \tag{5.10b}$$

$$\dot{\boldsymbol{x}}(\tau) = T f_p(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)), \qquad \tau\in[0,c_t+1], \tag{5.10c}$$

$$0 \le c(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)) + \varepsilon\mathbb{1}_{n_c}, \quad \tau\in[0,c_t], \tag{5.10d}$$

$$0 \le c(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)), \qquad \tau\in[c_t,c_t+1], \tag{5.10e}$$

$$0 = \boldsymbol{x}(c_t) - \boldsymbol{x}(c_t+1), \tag{5.10f}$$

$$\underline{T} \le T \le \overline{T}. \tag{5.10g}$$

Remark 5.3 The solution at time $t \ge t_0$ of the IVP $\dot{\boldsymbol{x}}(t) = T f_p(\boldsymbol{x}(t),\boldsymbol{u}(t))$ with initial value $\boldsymbol{x}(t_0) = x_0$ is denoted by $\boldsymbol{x}(t; x_0, \boldsymbol{u}, p, T, t_0)$. We use the notation

$$\mathrm{Ad}_p^{\mathrm{free}} := \{x\in\mathbb{R}^{n_x} : \mathrm{NP}_p^{\mathrm{free}}(x) \text{ is feasible}\} \tag{5.11}$$

for the set of admissible initial values for this problem. △

5.1.3 Assumptions for the Free-Time Case

In order to generalize the results obtained for the NMPC controller of the previous chapter to the scenario of parameter dependent dynamics described in the beginning of this section, we need extended versions of Assumptions 4.2 (existence of descent directions), 4.3 (controllability) and 4.4 (uniqueness and continuous dependence of subproblem solutions) which we state in the following.

The modified version of Assumption 4.2 reads as follows:

Assumption 5.2 (Existence of Descent Directions)
Let $p\in P$ and $(\boldsymbol{x},\boldsymbol{u},T)\in\mathcal{AC}^{n_x}([0,1])\times\mathrm{L}_\infty^{n_u}([0,1])\times\mathbb{R}$ be an admissible triple for problem $\mathrm{Per}_p^{\mathrm{free}}$ with suboptimal objective $\phi_{p,\mathrm{per}}^{\mathrm{free}}(\boldsymbol{x},\boldsymbol{u},T) > \phi_{p,\mathrm{per}}^{\mathrm{free},*}$. Then there exists a neighborhood $U_t\subset\mathbb{R}$ of 0 and a Fréchet-differentiable homotopy $H: U_t \to \mathcal{AC}^{n_x}(\mathcal{T})\times\mathrm{L}_\infty^{n_u}(\mathcal{T})\times\mathbb{R}$ that satisfies the following conditions:

1. it starts at $(\boldsymbol{x},\boldsymbol{u},T)$:

$$H(0) = (\boldsymbol{x},\boldsymbol{u},T), \tag{5.12}$$

2. for all $s\in U_t$, $H(s)$ is an admissible pair for $\mathrm{Per}_p^{\mathrm{free}}$,

3. the derivative of the objective functional evaluated along the homotopy is negative at $s = 0$:

$$\frac{\partial}{\partial s}\phi^{\text{free}}_{p,\text{per}}(H(s))|_{s=0} < 0. \tag{5.13}$$

△

As the modified NMPC subproblem will have a transient horizon length proportional to the period T_{p} (see equation (5.4)), the controllability Assumption 4.3 has to be modified accordingly.

To understand the reason for this, we need to recapitulate the proof of Lemma 4.2, which plays a central role in the analysis of the closed-loop behavior. In this proof, a homotopy of admissible trajectories for the NMPC subproblem is constructed by first using the existence of descent directions assumptions to get a family of periodic trajectories of improving output and then gluing it together with a transient homotopy consisting of trajectories that start at the initial value and end at the start values of the trajectories of the periodic homotopy (see Figure 4.2b). The controllability assumption ensures that such the transient homotopy exists. When transferring this proof to the new situation with varying periods, it has to be taken into account that the length of the transient phase is varying with the period since both are proportional.
This means that a transient homotopy has to be constructed in such a way that not only the desired state is reached at the end of the transient phase, but also (since the length of the transient phase also changes) it has to be reached at the right time.

Since we cannot define homotopies in function spaces of varying horizon-length, we have to use time-reparametrizations to define the homotopy on function spaces with horizon-length normalized to the interval $[0,1]$.

The modified controllability assumption then reads as follows:

Assumption 5.3 (Time-Controllability)
For any $p \in P$, any $T \in \mathbb{R}_{>0}$ and any $(\boldsymbol{x},\boldsymbol{u}) \in \mathcal{AC}^{n_x}([0,1]) \times \mathrm{L}^{n_u}_\infty([0,1])$ that satisfy $\dot{\boldsymbol{x}}(t) = T f_p(\boldsymbol{x}(t),\boldsymbol{u}(t))$ on $[0,1]$ there exist neighborhoods $U_x \subset \mathbb{R}^{n_x}$ of $\boldsymbol{x}(1)$ and $U_t \subset \mathbb{R}_{>0}$ of T and a FRÉCHET-differentiable mapping $C : U_x \times U_t \to \mathcal{AC}^{n_x}([0,1]) \times \mathrm{L}^{n_u}_\infty([0,1])$ such that

- $C(\boldsymbol{x}(1), T) = (\boldsymbol{x},\boldsymbol{u})$,
- for any $(y,s) \in U_x \times U_t$, the pair $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{u}}) := C(y,s) \in \mathcal{AC}^{n_x}([0,1]) \times \mathrm{L}^{n_u}_\infty([0,1])$ satisfies $\dot{\tilde{\boldsymbol{x}}}(t) = s f_p(\tilde{\boldsymbol{x}}(t),\tilde{\boldsymbol{u}}(t))$ on $[0,1]$, $\tilde{\boldsymbol{x}}(0) = \boldsymbol{x}(0)$ and $\tilde{\boldsymbol{x}}(1) = y$. △

Remark 5.4 As we show in Lemma B.2, for affine linear systems the time-controllability assumption follows from the KALMAN rank condition. △

The last modified assumption we need is the extension of Assumption 4.4 to the NMPC subproblem $\mathrm{NP}^{\text{free}}_p(x)$.

Assumption 5.4 (Existence, Uniqueness and Continuous Dependence of Solutions)
The family of NMPC subproblems $\mathrm{NP}^{\text{free}}_p(x)$ has the following properties:

- The set of admissible initial values is independent of $p \in P$:

$$\mathrm{Ad}_{p_1}^{\mathrm{free}} = \mathrm{Ad}_{p_2}^{\mathrm{free}} \text{ for any } p_1, p_2 \in P. \tag{5.14}$$

  We denote this set by $\mathrm{Ad}^{\mathrm{free}}$.

- For any $x \in \mathrm{Ad}^{\mathrm{free}}$ and any $p \in P$, the solution of problem $\mathrm{NP}_p^{\mathrm{free}}(x)$ is unique. We denote it by $(\boldsymbol{x}_{p,x}, \boldsymbol{u}_{p,x}, T_{p,x})$.

- the solution-mapping

$$\begin{aligned}\mathrm{Sol}_{\mathrm{free}} : P \times \mathrm{Ad}^{\mathrm{free}} &\to \mathcal{AC}^{n_x}([0, c_t + 1]) \times \mathrm{L}_\infty^{n_u}([0, (c_t + 1)]) \times \mathbb{R}, \\ (p, x) &\mapsto (\boldsymbol{x}_{p,x}, \boldsymbol{u}_{p,x}, T_{p,x})\end{aligned}$$

  is continuous. △

5.1.4 NMPC Scheme Based on $\mathrm{NP}_p^{\mathrm{free}}(x)$

After we have defined the NMPC subproblem we solve at sampling time t_i with initial state x_i and parameter p, there are still some extra steps to consider compared to the NMPC schemes presented until now. The solution of problem $\mathrm{NP}_p^{\mathrm{free}}(x_i)$ is defined on a normalized time horizon and has to be transformed back to the physical time-scale before the resulting control can be applied until the next sampling time.

Back-Transformation of the Subproblem Solution to the Physical Time-Scale

The solution $(\boldsymbol{x}_{p,x_i}, \boldsymbol{u}_{p,x_i}, T_{p,x_i}) \in \mathcal{AC}^{n_x}([0, c_t+1]) \times \mathrm{L}_\infty^{n_u}([0, c_t+1]) \times \mathbb{R}$ of $\mathrm{NP}_p^{\mathrm{free}}(x_i)$ at sampling time t_i corresponds to a pair $(\tilde{\boldsymbol{x}}_{p,x_i}, \tilde{\boldsymbol{u}}_{p,x_i}) \in \mathcal{AC}^{n_x}([t_i, t_i + T_{p,x_i}(c_t+1)]) \times \mathrm{L}_\infty^{n_u}([t_i, t_i + T_{p,x_i}(c_t + 1)])$ in the physical time-scale, which is obtained by reversing the time-transformation as follows:

$$(\tilde{\boldsymbol{x}}_{p,x_i}, \tilde{\boldsymbol{u}}_{p,x_i})(\tau) := (\boldsymbol{x}_{p,x_i}, \boldsymbol{u}_{p,x_i})\big((\tau - t_i)/T_{p,x_i}\big). \tag{5.15}$$

The pair $(\tilde{\boldsymbol{x}}_{p,x_i}, \tilde{\boldsymbol{u}}_{p,x_i})$ then satisfies (5.7b)-(5.7f) for $T_{\mathrm{p}} = T_{p,x_i}$ and $T_{\mathrm{t}} = c_t T_{p,x_i}$.

Sampling Times

Complementary to the NMPC approaches presented in this thesis until this point, we do not use a uniformly spaced sampling time grid for the NMPC scheme based on the subproblems $\mathrm{NP}_p^{\mathrm{free}}(x_i)$.: Instead we (recursively) choose the sampling time t_{i+1} by adding a constant fraction $\Delta T \cdot T_{p,x_i}$ to t_i. The next sampling time then is defined as

$$t_{i+1} := t_i + \Delta T \cdot T_{p,x_i}, \tag{5.16}$$

and the state x_{i+1} then accordingly is

$$x_{i+1} := \tilde{\boldsymbol{x}}_{p,x_i}(t_{i+1}) = \boldsymbol{x}_{p,x_i}(\Delta T). \tag{5.17}$$

This sampling time choice clearly corresponds to a sampling time of constant length ($= \Delta T$) in the normalized time-scale of $\mathrm{NP}_p^{\mathrm{free}}(x_i)$. Note also that $\Delta T \cdot T_{p,x_i}$ cannot become arbitrarily small or big because T_{p,x_i} satisfies the constraint (5.10g).

Remark 5.5 The choice of the sampling times is motivated by the numerical implementation of the NMPC scheme. The problem $\mathrm{NP}_p^{\mathrm{free}}(x_i)$, which is defined on the normalized time horizon $[0, 1 + c_t]$, is solved using a Multiple Shooting Discretization approach. In our implementation the shooting intervals are of constant length and correspond to the sampling time intervals. If the expected initial state of the NLP arising from the next NMPC subproblem is given by the state at a shooting node of the current NMPC subproblem, then the NLP-solution corresponding to the current NMPC subproblem can be used to generate a good initial guess for the NLP-solution corresponding to the next NMPC subproblem. If the sampling times are chosen according to (5.16), this will be the case. $\triangle$

We summarize the proposed NMPC scheme for the parameter dependent scenario including the back-transformation step and the adaptive choice of the sampling times in the form of the following algorithm.

Algorithm 2 NMPC scheme for parameter dependent scenario with free period.

```
Choose an initial sampling time t_0 ∈ ℝ, i ← 0,
while true do
    At time t_i determine the state x_i ∈ ℝ^{n_x} and parameter p_i ∈ P of the system,
    Solve the problem NP^free_{p_i}(x_i), denote its solution by (x_{p_i,x_i}, u_{p_i,x_i}, T_{p_i,x_i}),
    Back-transform the solution (x_{p_i,x_i}, u_{p_i,x_i}, T_{p_i,x_i}) to the pair (x̃_{p_i,x_i}, ũ_{p_i,x_i}) on the physical time-scale [t_i, (c_t + 1) T_{p_i,x_i}] as described in (5.15),
    Set t_{i+1} := t_i + ΔT T_{p_i,x_i} and for τ ∈ [t_i, t_{i+1}) apply the control u_μ(τ) := ũ_{p_i,x_i}(τ) to the system,
    i ← i + 1
end while
```

As can be seen, in the proposed algorithm the parameter p_i is continuously monitored and used as an input for the NMPC subproblems. In the following theoretical investigations of the asymptotic behavior of the controller we assume that the parameter stays constant (which is also why we drop the index in the following). Nevertheless the results can be applied to all scenarios where the parameter eventually stays constant or stays constant for longer periods of time.

5.1.5 Asymptotic Decrease of Optimal Objective Function and Closed-Loop Behavior

Based on the assumptions presented in Section 5.1.3, in this section we analyze the properties of the closed-loop trajectory resulting from the NMPC scheme of Algorithm 2.

Strategy

The approach for the analysis of the closed-loop behavior of the NMPC scheme based on the subproblems $\mathrm{NP}_p^{\mathrm{free}}(x_i)$ is similar to the approach in the previous Chapter 4.

Again, the assumptions on controllability 5.3 and existence of descent-directions 5.2 can be used to prove that the transient part of the objective $\varphi_{p,\mathrm{trans}}^{\mathrm{free},*}$ vanishes if and only if the complete objective $\varphi_p^{\mathrm{free},*} = w_{\mathrm{trans}}\varphi_{p,\mathrm{trans}}^{\mathrm{free},*} + \varphi_{p,\mathrm{per}}^{\mathrm{free},*}$ vanishes. As in Lemma 4.4, this result is used to prove asymptotic decrease of the objective function $\varphi_p^{\mathrm{free},*}$ along the closed-loop trajectory. Then again by means of a continuous self-similarity function it can be shown that the average economic performance of the closed-loop trajectory is converging to the performance of the best T_p-periodic admissible solution.

Recursive Feasibility

The NMPC scheme based on the subproblems $\mathrm{NP}_p^{\mathrm{free}}(x_i)$ presented in Section 5.1.4 has the recursive feasibility property which again is a direct consequence of the periodicity constraint (5.10f). To show this we define the extension operator for the problem on the normalized time horizon:

$$\begin{aligned}\mathrm{Ext}_{\Delta T} : \mathcal{AC}^{n_x}([0, c_t + 1]) \times \mathrm{L}_\infty^{n_u}([0, c_t + 1]) \times \mathbb{R}_{\geq 0} &\to \mathcal{AC}^{n_x}([0, c_t + 1]) \times \mathrm{L}_\infty^{n_u}([0, c_t + 1]) \times \mathbb{R}_{\geq 0},\\ (\boldsymbol{x}, \boldsymbol{u}, T) &\mapsto (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}, T),\end{aligned} \tag{5.18}$$

where $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})$ is defined as follows

$$(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})(\tau) := \begin{cases} (\boldsymbol{x}, \boldsymbol{u})(\tau + \Delta T) & \text{for } \tau \in [0, c_t + 1 - \Delta T], \\ (\boldsymbol{x}, \boldsymbol{u})(\tau + \Delta T - T) & \text{for } \tau \in [c_t + 1 - \Delta T, c_t + 1]. \end{cases} \tag{5.19}$$

With this operator, as in Section 4.3.3, it can be checked easily that if $(\boldsymbol{x}, \boldsymbol{u}, T)$ is admissible for $\mathrm{NP}_p^{\mathrm{free}}(x_i)$, then $\mathrm{Ext}_{\Delta T}(\boldsymbol{x}, \boldsymbol{u}, T)$ is admissible for $\mathrm{NP}_p^{\mathrm{free}}(x_{i+1})$ with $x_{i+1} := \boldsymbol{x}(\Delta T)$, which means the NMPC scheme is recursive feasible.

Transient Objective and Combined Objective Vanish on the Same Sets

The central ingredient in the closed-loop analysis is again the observation that the assumptions on existence of descent directions and on controllability imply that the transient part of the objective vanishes if and only if the combined objective with transient and periodic objective vanishes.

Lemma 5.1 (Transient Objective is Optimal iff Complete Objective is Optimal)
Let Assumptions 5.2 (existence of descent directions) and 5.3 (time-controllability) hold, let $p \in P$ and $x \in \mathrm{Ad}^{\mathrm{free}}$. Then $\varphi_{p,\mathrm{trans}}^{\mathrm{free},*}(x) = 0$ if and only if $\varphi_p^{\mathrm{free},*}(x) = \phi_{p,\mathrm{per}}^{\mathrm{free},*}$.

Proof The proof of this Lemma is very much along the lines of the proof of Lemma 4.2, except for that the optimization variable T of problem $\mathrm{NP}_p^{\mathrm{free}}(x)$ also has to be taken into account. Without loss of generality we may assume in this proof that $\phi_{p,\text{per}}^{\text{free},*} = 0$ holds. First, it is clear that $\varphi_p^{\text{free},*}(x) = 0$ implies $\varphi_{p,\text{trans}}^{\text{free},*}(x) = 0$.

Now lets assume $\varphi_{p,\text{trans}}^{\text{free},*}(x) = 0$ and $\varphi_{p,\text{per}}^{\text{free},*}(x) > 0$, i.e. the performance on the periodic part is suboptimal. Let $(\boldsymbol{x}, \boldsymbol{u}, T) := (\boldsymbol{x}_{p,x}, \boldsymbol{u}_{p,x}, T_{p,x})$ be the solution of $\text{NP}_p^{\text{free}}(x)$. Then because of $\varphi_{p,\text{per}}^{\text{free},*}(\boldsymbol{x}, \boldsymbol{u}, T) > 0$, according to Assumption 5.2, there exists a neighborhood $W_t \subset \mathbb{R}$ around 0 and a FRÉCHET-differentiable homotopy $H = (H_x, H_u, H_T) : W_t \to \mathcal{AC}^{n_x}([c_t, c_t + 1]) \times \text{L}_\infty^{n_u}([c_t, c_t + 1]) \times \mathbb{R}$ that satisfies the following conditions:

1. it starts at $(\boldsymbol{x}, \boldsymbol{u}, T)$:

$$H(0) = \left((\boldsymbol{x}, \boldsymbol{u})|_{[c_t, c_t+1]}, T \right), \tag{5.20}$$

2. for all $s \in W_t$, $H(s)$ is admissible pair for $\text{Per}_p^{\text{free}}$,

3. the derivative of the periodic objective functional evaluated along the homotopy is negative at $s = 0$:

$$\frac{\partial}{\partial s} \varphi_{p,\text{per}}^{\text{free}}(H(s))|_{s=0} < 0. \tag{5.21}$$

Similarly as in Lemma 4.1, it can be shown that there exists a $\tilde{\delta} > 0$ such that $(\boldsymbol{x}, \boldsymbol{u})$ satisfies the inequality

$$0 \le c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) + \frac{\varepsilon}{2} \mathbb{1}_{n_c} \text{ for all } \tau \in [c_t - \tilde{\delta}, c_t]. \tag{5.22}$$

Assumption 5.3 then guarantees the existence of neighborhoods U_x of $\boldsymbol{x}(c_t)$ and U_t of T and a FRÉCHET-differentiable mapping $C : U_x \times U_t \to \mathcal{AC}^{n_x}([c_t - \tilde{\delta}, c_t]) \times \text{L}_\infty^{n_u}([c_t - \tilde{\delta}, c_t])$ such that

- $C(\boldsymbol{x}(c_t), T) = (\boldsymbol{x}, \boldsymbol{u})|_{[c_t - \tilde{\delta}, c_t]}$,
- for any $(y, s) \in U_x \times U_t$, the pair $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}) := C(y, s) \in \mathcal{AC}^{n_x}([c_t - \tilde{\delta}, c_t]) \times \text{L}_\infty^{n_u}([c_t - \tilde{\delta}, c_t])$ satisfies $\dot{\tilde{\boldsymbol{x}}} = s f_p(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})$ on $[c_t - \tilde{\delta}, c_t]$, $\tilde{\boldsymbol{x}}(c_t - \tilde{\delta}) = \boldsymbol{x}(c_t - \tilde{\delta})$ and $\tilde{\boldsymbol{x}}(c_t) = y$.

Because the mapping C is continuous, the neighborhoods U_x and U_t can be chosen such that $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}) = C(y, s)$ satisfies the path constraint $0 \le c(\tilde{\boldsymbol{x}}(\tau), \tilde{\boldsymbol{u}}(\tau)) + \varepsilon \mathbb{1}_{n_c}$ for all $\tau \in [c_t - \tilde{\delta}, c_t]$. The homotopy H and the mapping C are now used to construct a family of admissible triples for $\text{NP}_p^{\text{free}}(x)$. We define $\Psi = (\Psi_x, \Psi_u, \Psi_T) : W_t \to \mathcal{AC}^{n_x}([0, c_t + 1]) \times \text{L}_\infty^{n_u}([0, c_t + 1]) \times \mathbb{R}_{\ge 0}$ by setting

$$(\Psi_x(s), \Psi_u(s))(\tau) := \begin{cases} (\boldsymbol{x}, \boldsymbol{u}) \left(\tau \frac{H_T(s)}{T} \right) & \text{for } \tau \in [0, c_t - \tilde{\delta}], \\ C\left(H_x(s)(c_t)\right)(\tau) & \text{for } \tau \in [c_t - \tilde{\delta}, c_t], \\ H_x(s)(\tau) & \text{for } \tau \in [c_t, c_t + 1]. \end{cases} \tag{5.23}$$

and $\Psi_T(s) := H_T(s)$. For an illustration of the definition of Ψ see Figure 5.1. Now, at least for s sufficiently close to 0, the triple $\Psi(s)$ satisfies all the constraints (5.10b) -(5.10g) and with the same argumentation as in the proof of Lemma 4.2 we get

$$\frac{\partial}{\partial s}\varphi_p^{\text{free}}(\Psi(s))|_{s=0} < 0, \tag{5.24}$$

which is a contradiction to the optimality of $(\boldsymbol{x}, \boldsymbol{u}, T)$ for problem $\text{NP}_p^{\text{free}}(x)$. □

Guaranteed Descent of the Optimal Objective Value Function

The extension operator $\text{Ext}_{\Delta T}$ not only is used to prove recursive feasibility, it also is used to prove the following Lemma which is the extension of Lemma 4.3 and guarantees the descent of $\varphi_p^{\text{free},*}$ along closed-loop trajectories.

Lemma 5.2 (Guaranteed Objective Decrease)
Let $p \in P$ and let $(x_i)_{i\in\mathbb{N}}$ be the sequence of initial states resulting from the NMPC scheme, i.e. $x_{i+1} := \boldsymbol{x}_{p,x_i}(\Delta T)$. Then for any $i \in \mathbb{N}$ the following estimate holds:

$$\varphi_p^{\text{free},*}(x_{i+1}) \leq \rho^{-\Delta T}\, w_{\text{trans}} \varphi_{p,\text{trans}}^{\text{free},*}(x_i) + \varphi_{p,\text{per}}^{\text{free},*}(x_i). \tag{5.25}$$

Proof The proof is similar to the proof of Lemma 4.3. First it can be observed that the extension operator $\text{Ext}_{\Delta T}$ leaves the economic performance in the periodic part invariant: $\varphi_{p,\text{per}}^{\text{free}}\left(\text{Ext}_{\Delta T}(\boldsymbol{x}_{p,x_i}, \boldsymbol{u}_{p,x_i})\right) = \varphi_{p,\text{per}}^{\text{free}}(\boldsymbol{x}_{p,x_i}, \boldsymbol{u}_{p,x_i})$. Furthermore by using the definition of the extension operator it can be shown that

$$\varphi_{p,\text{trans}}^{\text{free}}\left(\text{Ext}_{\Delta T}(\boldsymbol{x}_{p,x_i}, \boldsymbol{u}_{p,x_i})\right) \leq \rho^{-\Delta T} \varphi_{p,\text{trans}}^{\text{free}}(\boldsymbol{x}_{p,x_i}, \boldsymbol{u}_{p,x_i}), \tag{5.26}$$

which finishes the proof. □

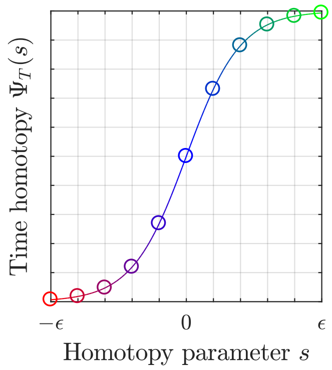
The two previous Lemmata 5.1 and 5.2 again allow us now to show asymptotic decrease of the objective functional $\varphi_p^{\text{free},*}$ along closed loop trajectories.

Lemma 5.3 (Asymptotic Decrease of the Objective Function Along the Closed-Loop)
Let Assumptions 5.1 (existence of optimal periodic trajectories), 5.2 (existence of descent directions), 5.3 (time-controllability) and 5.4 (uniqueness and continuous dependence of subproblem solutions) hold and let furthermore $p \in P$. Then there exists a $\mathcal{KL}$ function β such that for $x_0 \in \text{Ad}_p^{\text{free}}$ and the subsequent sequence of states $(x_i)_{i\in\mathbb{N}}$ of the closed-loop trajectory it holds

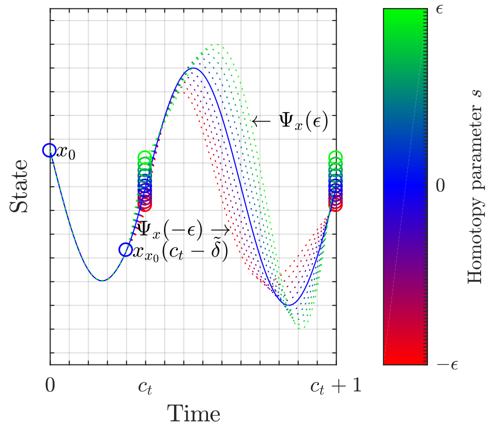
$$\varphi_p^{\text{free},*}(x_i) \leq \beta(\varphi_p^{\text{free},*}(x_0), i). \tag{5.27}$$

Proof Lemma 5.2 implies

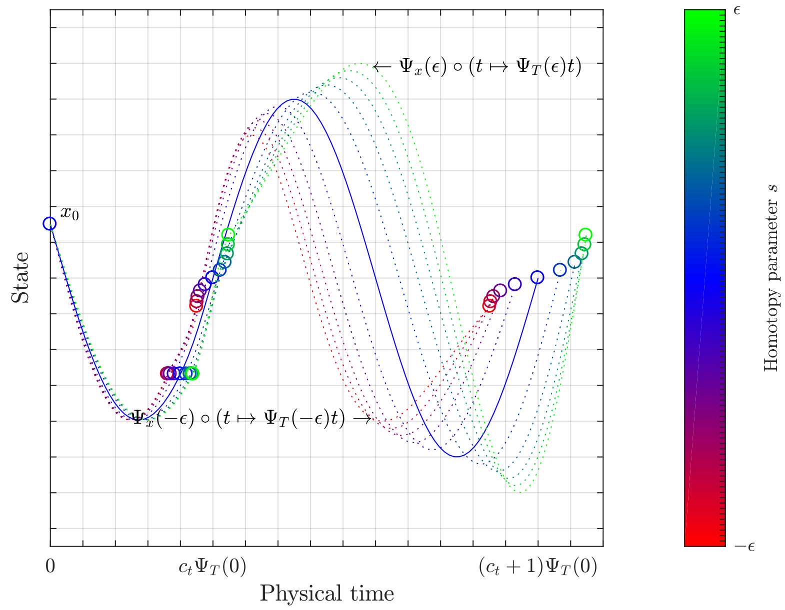
$$\varphi_p^{\text{free},*}(x_{i+1}) \leq \varphi_p^{\text{free},*}(x_i) - (1-\rho^{-\Delta T})\, w_{\text{trans}} \varphi_{p,\text{trans}}^{\text{free},*}(x_i) \tag{5.28}$$



a) The varying period variable $\Psi_T(s)$.



b) The family $(\Psi_x(s))_{s\in(-\varepsilon,\varepsilon)}$.



c) The family $(\Psi_x(s))_{s\in(-\varepsilon,\varepsilon)}$ after reversing the time-transformations.

Figure 5.1: Illustration of the families $(\Psi_T(s))_{s\in(-\varepsilon,\varepsilon)}$ and $(\Psi_x(s))_{s\in(-\varepsilon,\varepsilon)}$ in (5.1a) and (5.1b). All the state/control trajectories $(\Psi_x(s), \Psi_u(s))$ are defined on the normalized horizon $[0, c_t + 1]$ and solve the ODE $\dot{\boldsymbol{x}}(t) = \Psi_T(s) f(\boldsymbol{x}(t), \boldsymbol{u}(t))$. In (5.1c), the normalization to the time horizon $[0, c_t + 1]$ is reversed and each triple $(\Psi_x(s), \Psi_u(s), \Psi_T(s))$ corresponds to a pair $(\boldsymbol{y}_s, \boldsymbol{w}_s) \in \mathcal{AC}^{n_x}([0,(c_t+1)\Psi_T(s)]) \times \mathrm{L}_\infty^{n_u}([0,(c_t+1)\Psi_T(s)])$ defined on the physical time-scale by setting $(\boldsymbol{y}_s, \boldsymbol{w}_s)(\tau) := (\Psi_x(s)(\Psi_T(s)\tau), \Psi_u(s)(\Psi_T(s)\tau))$.

for all $i \in \mathbb{N}$. Since $\varphi_{p,\text{trans}}^{\text{free},*}$ and $\varphi_p^{\text{free},*}$ vanish on the same set (which is nonempty because of Assumption 5.1) and $Ad_{\text{free},p}$ is compact, Lemma A.4 implies the existence of a $\mathcal{K}_\infty$-function $\tilde{\alpha}$ such that $\tilde{\alpha}\left(\varphi_p^{\text{free},*}(y)\right) \leq \left(\varphi_{p,\text{trans}}^{\text{free},*}(y)\right)$ holds for all $y \in Ad_{\text{free},p}$. Therefore (5.28) implies

$$\varphi_p^{\text{free},*}(x_{i+1}) \leq \varphi_p^{\text{free},*}(x_i) - (1-\rho^{-\Delta T})\, w_{\text{trans}}\tilde{\alpha}\left(\varphi_p^{\text{free},*}(x_i)\right) \tag{5.29}$$

Since $\alpha : s \mapsto (1-\rho^{\Delta T})\, w_{\text{trans}}$ is also a $\mathcal{K}_\infty$ function, Lemma A.3 implies the existence of a $\mathcal{KL}$-function β such that

$$\varphi_p^{\text{free},*}(x_i) \leq \beta\left(\varphi_p^{\text{free},*}(x_0), i\right) \tag{5.30}$$

holds for all $i \in \mathbb{N}$ and the proof is finished. □

Closed-Loop Behavior

As for the controller with fixed period, we analyze the closed-loop behavior by comparing solutions of subsequent NMPC subproblems. As pointed out before, in our analysis we assume that the system parameter p remains constant, i.e. $p_i \equiv p$. Since the time horizons for the NMPC scheme based on $\text{NP}_p^{\text{free}}(x_i)$ are of varying length, this comparison is not as straight-forward as in Definition 4.5.

The solutions of the NMPC subproblems first have to be transformed back to the physical time-scale. After this back-transformation it can happen that two subsequent solutions are defined on time horizons of different length. However, exploiting the periodicity constraint (5.10f), both solutions can be extended to an arbitrary horizon length. We extend both solutions to the length $(c_t+1)\overline{T}$ and then can compare them on the overlapping parts of the horizon as in Definition 4.5.

The commutative diagram in Figure 5.2 illustrates how the extension step of a triple $(\boldsymbol{x}, \boldsymbol{u}, T) \in \mathcal{AC}^{n_x}([0, c_t+1]) \times \text{L}_\infty^{n_u}([0,(c_t+1)]) \times \mathbb{R}$ to the horizon $[0,(c_t+1)\overline{T}]$ can be formalized and interpreted as the composition of a periodic extension operator Per, a reparametrization operator Rep and a restriction operator Res.

The operator Per is defined as (where $\eta(s) := s - \lfloor s \rfloor$ denotes the fractional part of a number $s \in \mathbb{R}$)

$$(\boldsymbol{x}, \boldsymbol{u}, T) \mapsto \left(\left[\tau \mapsto \boldsymbol{x}\left(c_t + \eta(\tau - c_t)\right)\right], \left[\tau \mapsto \boldsymbol{u}\left(c_t + \eta(\tau - c_t)\right)\right], T\right) \tag{5.31}$$

For the solution $(\boldsymbol{x}_{p,y}, \boldsymbol{u}_{p,y}, T_{p,y})$ of problem $\text{NP}_p^{\text{free}}(y)$, the pair $\text{Ext}_{\max}(\boldsymbol{x}_{p,y}, \boldsymbol{u}_{p,y}, T_{p,y}) \in \mathcal{AC}^{n_x}([0,(c_t+1)\overline{T}]) \times \text{L}_\infty^{n_u}([0,(c_t+1)\overline{T}]$ can be seen as the predicted system behavior over the prolonged time horizon $[0,(c_t+1)\overline{T}]$.

With the help of the $\text{Ext}_{\max}$-operator, we can compare the behavior of the predicted solution $\text{Ext}_{\max}(\boldsymbol{x}_{p,x_i}, \boldsymbol{u}_{p,x_i}, T_{p,x_i})$ shifted to $\mathcal{T}^{t_i} := [t_i, t_i + (c_t+1)\overline{T}]$ with the behavior of the predicted solution $\text{Ext}_{\max}(\boldsymbol{x}_{p,x_{i+1}}, \boldsymbol{u}_{p,x_{i+1}}, T_{p,x_{i+1}})$ shifted to $\mathcal{T}^{t_{i+1}} := [t_i + \Delta T\, T_{p,x_i}, t_i + \Delta T\, T_{p,x_i} + (c_t+1)\overline{T}]$ by comparing both trajectories on the intersection $\mathcal{T}^{t_i} \cap \mathcal{T}^{t_{i+1}}$. This leads to an extension of the self-similarity function of Definition 4.5:

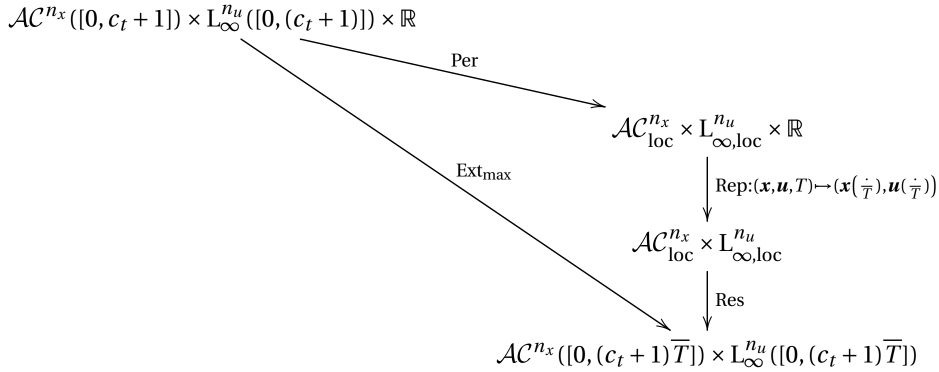


Figure 5.2: Definition of the operator $\mathrm{Ext}_{\max}$.

Definition 5.4 (Self-Similarity Function, Free Period Case)
The function $S_{\mathrm{free}} : \mathrm{Ad}_p^{\mathrm{free}} \to \mathbb{R}_{\geq 0}$ defined by

$$S_{\mathrm{free}}(x) := \int_{\Delta T T_{p,x}}^{(c_t+1)\overline{T}} \left(\left\| \mathrm{Ext}_{\max}(\boldsymbol{x}_{p,x}, \boldsymbol{u}_{p,x}, T_{p,x})(\tau + \Delta T T_{p,x}) - \mathrm{Ext}_{\max}(\boldsymbol{x}_{p,y}, \boldsymbol{u}_{p,y}, T_{p,y})(\tau) \right\| \right) \mathrm{d}\tau, \tag{5.32}$$

with $y := \boldsymbol{x}_{p,x}(\Delta T)$ is called the self-similarity function for problem $\mathrm{NP}_p^{\mathrm{free}}(x)$. △

Note that Assumption 5.4 implies that S_{free} is continuous , because it is the composition of continuous functions.

Similarly as in Lemma 4.6, Assumptions 5.1 (existence of optimal periodic solutions) and 5.4 (uniqueness and continuous dependence of subproblem solutions) can be used to show that the function S_{free} vanishes for any $x \in \mathrm{Ad}^{\mathrm{free}}$ with $\varphi_p^{\mathrm{free},*}(x) = 0$ and that there exists a $\mathcal{K}$-function α such that

$$S_{\mathrm{free}}(x) \leq \alpha\left(\varphi_p^{\mathrm{free},*}(x)\right) \tag{5.33}$$

holds for all $x \in \mathrm{Ad}^{\mathrm{free}}$.

This fact combined with the asymptotic decrease of the objective function $\varphi_p^{\mathrm{free},*}$ along the closed-loop can be used to show that the L^1-difference of the closed-loop trajectory to the extended predicted solutions

$$\mathrm{diff}(t_i, x_i) := \int_0^{(c_t+1)\overline{T}} \left\| \mathrm{Ext}_{\max}(\boldsymbol{x}_{p,x_i}, \boldsymbol{u}_{p,x_i}, T_{p,x_i})(\tau) - (\boldsymbol{x}_\mu, \boldsymbol{u}_\mu)(\tau + t_i) \right\| \mathrm{d}\tau \tag{5.34}$$

decreases asymptotically.

Via a telescope argument, we can calculate the L^1-distance of the closed-loop trajectory to the extended predicted solutions.

Lemma 5.5 (Asymptotic Improving Prediction Quality)
Let Assumptions 5.1 (existence of optimal periodic solutions), 5.2, 5.3 (existence of descent directions) and 5.4 (uniqueness and continuous dependence of subproblem solutions) hold. Then there exists a $\mathcal{KL}$-function β such that for any $x_0 \in Ad_{\text{free},p}$ and the resulting closed-loop sequence $(x_i)_{i\in\mathbb{N}}$ it holds

$$\text{diff}(t_i, x_i) \le \beta\left(\varphi_p^{\text{free},*}(x_0), i\right) \tag{5.35}$$

for all $i \in \mathbb{N}$.

Proof Similar as in Lemma 4.8, the proof is based on a telescope sum argument and the fact that per definition $(\boldsymbol{x}_\mu, \boldsymbol{u}_\mu)$ coincides with $\tau \mapsto \text{Ext}_{\max}(\boldsymbol{x}_{p,x_i}, \boldsymbol{u}_{p,x_i}, T_{p,x_i})(\tau - t_i)$ on the interval $[t_i, t_{i+1} := t_i + \Delta T T_{p,x_i}]$. With the telescope argument, $\text{diff}(t_i, x_i)$ can be bounded by a sum $\sum_{j=i}^{j=i+n} S_{\text{free}}(x_j)$. But $S_{\text{free}}(x_i)$ is asymptotically decreasing for increasing i, which is due to (5.33) and the fact that $\varphi_p^{\text{free},*}(x_i)$ is asymptotically decreasing (Lemma 5.3) for increasing i. □

The results of our analysis of the closed-loop behavior can be summarized in the following theorem.

Theorem 5.6 (Economic Closed-Loop Performance, Free Period Case)
Let Assumptions 5.1 (existence of optimal periodic orbits), 5.2 (existence of descent directions), 5.3 (time-controllability) and 5.4 (uniqueness and continuous dependence of subproblem solutions) hold. Let $x_0 \in \text{Ad}_{T_\text{p}}^{\text{free}}$ be the state of the system at the initial time t_0. Then the resulting closed-loop trajectory $(\boldsymbol{x}_\mu, \boldsymbol{u}_\mu) \in \mathcal{AC}_{\text{loc}}^{n_x} \times \text{L}_{\infty,\text{loc}}^{n_u}$ from the NMPC scheme described in Algorithm 2 based on the subproblems $\text{NP}_p^{\text{free}}$ (5.10) exists for all times and for the average economic performance it holds:

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \ell(\boldsymbol{x}_\mu(\tau), \boldsymbol{u}_\mu(\tau))\text{d}\tau = \phi_{p,\text{per}}^{\text{free},*}. \tag{5.36}$$

Proof The proof is analogue to the proof of Theorem 4.9. □

5.2 Systems with Time-Periodic Performance Criterion

The systems we considered in the previous chapter and in the previous section had a performance criterion ℓ which did not depend on time. In this section we want to consider systems with time-periodic performance criterion. We investigate the question whether it is possible to set up an NMPC controller similar to the one presented in Chapter 4 for such systems. As the length of the periodic part of the horizon we choose the a-priori given period of the performance criterion ℓ. Under assumptions similar as in the case of a time-independent performance criterion we can show that for such a controller the average economic performance of the closed-loop is equal to performance of the best periodic solution.

Problem Setting: Time-Periodic Performance Criterion

We consider a dynamical system that evolves according to $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t))$ with an associated path constraint $c : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_c}$ and an associated time-dependent periodic performance criterion ℓ with period $T_\text{p} \in \mathbb{R}_{>0}$, i.e. a function $\ell : \mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ such that $\ell(t, x, u) = \ell(t + T_\text{p}, x, u)$ holds for all $(t, x, u) \in \mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$. For such systems it is a natural question to ask for T_p-periodic solutions.

The arising follow-up question in the NMPC context is, whether the controller presented in Chapter 4 is also applicable to such systems. This could be done by simply taking the period T_p of the performance criterion ℓ as the period for the periodic part of the time horizon of the NMPC subproblem.

The NMPC Subproblem

In order to apply the NMPC controller presented in Section 4.2, the objective functional has to be adapted to take into account the explicit time-dependency of the performance criterion ℓ. For $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}(\mathcal{T}^t) \times \mathrm{L}_\infty^{n_u}(\mathcal{T}^t)$, the transient and the periodic parts of the objective contribution for the NMPC subproblem at sampling time t are then of the form

$$\begin{aligned}\varphi_\text{trans}^{\text{fix},\ell}(t, \boldsymbol{x}, \boldsymbol{u}) := &\int_{\mathcal{T}_\text{trans}^t} \rho^{\tau - t} \left| \ell(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) - \ell(\tau + T_\text{p}, \boldsymbol{x}(\tau + T_\text{p}), \boldsymbol{u}(\tau + T_\text{p})) \right|^2 \mathrm{d}\tau \\ &+ \int_{\mathcal{T}_\text{trans}^t} \rho^{\tau - t} \left\| D_{T_\text{p}}(\boldsymbol{u})(\tau) \right\|^2 \mathrm{d}\tau.\end{aligned} \tag{5.37}$$

and

$$\varphi_\text{per}^{\text{fix},\ell}(t, \boldsymbol{x}, \boldsymbol{u}) := \int_{\mathcal{T}_\text{per}^t} \ell(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) \mathrm{d}\tau. \tag{5.38}$$

We explicitly include the initial time t in the arguments of $\varphi_\text{trans}^{\text{fix},\ell}$ and $\varphi_\text{per}^{\text{fix},\ell}$ to emphasize the time-dependency.

The full NMPC subproblem $\mathrm{NP}_{T_\text{p}}^{\text{fix},\ell}(t, x)$ we consider reads as follows:

$$\min_{\substack{\boldsymbol{x} \in \mathcal{AC}^{n_x}(\mathcal{T}^t), \\ \boldsymbol{u} \in \mathrm{L}_\infty^{n_u}(\mathcal{T}^t)}} \quad \varphi^{\text{fix},\ell}(t_i, \boldsymbol{x}, \boldsymbol{u}) = w_\text{trans} \varphi_\text{trans}^{\text{fix},\ell}(t, \boldsymbol{x}, \boldsymbol{u}) + \varphi_\text{per}^{\text{fix},\ell}(t, \boldsymbol{x}, \boldsymbol{u}) \tag{5.39a}$$

$$\text{s.t.} \quad \boldsymbol{x}(t) = x, \tag{5.39b}$$

$$\dot{\boldsymbol{x}}(\tau) = f(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \qquad \tau \in \mathcal{T}^t, \tag{5.39c}$$

$$0 \le c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) + \varepsilon \mathbb{1}_{n_c}, \qquad \tau \in \mathcal{T}_\text{trans}^t, \tag{5.39d}$$

$$0 \le c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)), \qquad \tau \in \mathcal{T}_\text{per}^t, \tag{5.39e}$$

$$0 = \boldsymbol{x}(t + T_\text{t}) - \boldsymbol{x}(t + T_\text{t} + T_\text{p}). \tag{5.39f}$$

By $\varphi^{\text{fix},\ell,*}(t,x)$ we denote the value $\varphi^{\text{fix},\ell}(t,\boldsymbol{x}_{t,x},\boldsymbol{u}_{t,x})$ at the solution $(\boldsymbol{x}_{t,x},\boldsymbol{u}_{t,x})$ of problem $\text{NP}^{\text{fix},\ell}_{T_\text{p}}(t,x)$. We define the set of admissible initial times and states for this problem

$$\text{Ad}^{\text{fix},\ell}_{T_\text{p}} := \{(t,x)\in\mathbb{R}\times\mathbb{R}^{n_x} : \text{NP}^{\text{fix},\ell}_{T_\text{p}}(t,x) \text{ is feasible }\}\subset\mathbb{R}\times\text{A}^c_x. \tag{5.40}$$

5.2.1 Assumptions for the Time-Dependent Performance Criterion Case

The analysis of the closed-loop behavior of the resulting NMPC controller is based on assumptions that are very similar to the ones used in the previous Chapter 4. The only difference stems from the fact that for the objective functionals now it has to be taken into account that the performance criterion ℓ is time-dependent.

We state these assumptions again in the following.

Assumption 5.5 (Existence of Optimal Periodic Solutions)
There exists a solution for the OCP $\text{Per}^{\text{fix},\ell}_{T_\text{p}}$

$$\min_{\substack{\boldsymbol{x}\in\mathcal{AC}^{n_x}([0,T_\text{p}]),\\ \boldsymbol{u}\in\text{L}^{n_u}_\infty([0,T_\text{p}])}} \phi^{\ell}_{T_\text{p},\text{per}}(\boldsymbol{x},\boldsymbol{u}) := \frac{1}{T_\text{p}}\int_0^{T_\text{p}} \ell(\tau,\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\mathrm{d}\tau \tag{5.41a}$$

$$\text{s.t.} \quad \dot{\boldsymbol{x}}(\tau) = f(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)), \quad \tau\in[0,T_\text{p}], \tag{5.41b}$$

$$0 \le c(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau)), \quad \tau\in[0,T_\text{p}], \tag{5.41c}$$

$$0 = \boldsymbol{x}(T_\text{p}) - \boldsymbol{x}(0). \tag{5.41d}$$

The optimal objective value is denoted by $\phi^{\ell,*}_{T_\text{p},\text{per}}$. △

This assumption ensures that the set of admissible initial times/states $\text{Ad}^{\text{fix},\ell}_{T_\text{p}}$ is nonempty.

Assumption 5.6 (Existence of Descent Directions)
Let $(\boldsymbol{x},\boldsymbol{u})$ be an admissible T_p-periodic pair for problem $\text{Per}^{\text{fix},\ell}_{T_\text{p}}$ with suboptimal objective $\phi^{\ell}_{T_\text{p},\text{per}}(\boldsymbol{x},\boldsymbol{u}) > \phi^{\ell,*}_{T_\text{p},\text{per}}$. Then there exists a neighborhood $U_t\subset\mathbb{R}$ of 0 and a FRÉCHET-differentiable homotopy $H=(H_x,H_u): U_t\to\mathcal{AC}^{n_x}([0,T_\text{p}])\times\text{L}^{n_u}_\infty([0,T_\text{p}])$ that satisfies the following conditions:

1. it starts at $(\boldsymbol{x},\boldsymbol{u})$:

$$H(0) = (\boldsymbol{x},\boldsymbol{u}), \tag{5.42}$$

2. for all $s\in U_t$, $H(s)$ is admissible for $\text{Per}^{\text{fix},\ell}_{T_\text{p}}$,

3. the derivative of the objective functional evaluated along the homotopy is negative at $s=0$:

$$\frac{\partial}{\partial s}\phi^{\ell}_{T_\text{p},\text{per}}(H(s))|_{s=0} < 0. \tag{5.43}$$

△

Assumption 5.7 (Uniqueness and Continuous Dependence of Solutions)
The family of NMPC subproblems $\mathrm{NP}^{\mathrm{fix},\ell}_{T_\mathrm{p}}(t,x)$ has the following properties:

- For $(t,x) \in \mathrm{Ad}^{\mathrm{fix},\ell}_{T_\mathrm{p}}$ problem $\mathrm{NP}^{\mathrm{fix},\ell}_{T_\mathrm{p}}(t,x)$ has a unique solution which we denote by $(\boldsymbol{x}_{t,x}, \boldsymbol{u}_{t,x})$ depending on the initial time $t \in \mathbb{R}$ and the initial value $x \in \mathbb{R}^{n_x}$,
- the solution-mapping

$$\begin{aligned} \mathrm{Sol} : \mathrm{Ad}^{\mathrm{fix},\ell}_{T_\mathrm{p}} &\to \mathcal{AC}^{n_x}(\mathcal{T}^t) \times \mathrm{L}^{n_u}_\infty(\mathcal{T}^t), \\ (t,x) &\mapsto (\boldsymbol{x}_{t,x}, \boldsymbol{u}_{t,x}) \end{aligned}$$

  is continuous. △

5.2.2 Closed-Loop Behavior of the NMPC Scheme Based on $\mathrm{NP}^{\mathrm{fix},\ell}_{T_\mathrm{p}}(t_i, x_i)$

As the strategy for the analysis of the closed-loop behavior is again similar to the strategy we used in chapter 4, we will not give detailed proofs for every statement but rather only point out where the arguments have to be adjusted to take into account the time-periodic nature of the performance criterion.

As before, we consider the closed-loop trajectory resulting from solving $\mathrm{NP}^{\mathrm{fix},\ell}_{T_\mathrm{p}}(t_i, x_i)$ for a equidistant grid of sampling times $t_i = i\Delta T$ with $x_i \in \mathbb{R}^{n_x}$ being the state of the system at time t_i.

Recursive Feasibility

The first observation is that the NMPC scheme has the recursive-feasibility property. As we already noticed in Section 4.3.3, recursive feasibility follows from the set of constraints (5.39b)- (5.39f). This property is independent of the structure of the objective functional because the periodicity constraint always allows to construct an admissible pair of trajectories for problem $\mathrm{NP}^{\mathrm{fix},\ell}_{T_\mathrm{p}}(t_{i+1}, x_{i+1})$ simply by periodically extending any admissible pair of trajectories for $\mathrm{NP}^{\mathrm{fix},\ell}_{T_\mathrm{p}}(t_i, x_i)$.

Asymptotic Objective Decrease

In order to show that the optimal objective value $\varphi^{\mathrm{fix},\ell,*}(t_k, x_k)$ decreases asymptotically to $\phi^{\ell,*}_{T_\mathrm{p},\mathrm{per}}$ we begin with the following Lemma which is the analogue to Lemma 4.2.

Lemma 5.7 (Transient Objective is Optimal iff Complete Objective is Optimal)
Let Assumptions 5.6 (existence of descent directions) and 4.3 (controllability) hold and let $(t,x) \in \mathrm{Ad}^{\mathrm{fix},\ell}_{T_\mathrm{p}}$. Then $\varphi^{\mathrm{fix},\ell,*}_{\mathrm{trans}}(t,x) = 0$ if and only if $\varphi^{\mathrm{fix},\ell,*}(t,x) = \phi^{\ell,*}_{T_\mathrm{p},\mathrm{per}}$.

Proof The proof is again analogue to the proof of Lemma 4.2. Therefore we only sketch the central contradiction argument. Let $(\boldsymbol{x}, \boldsymbol{u})$ be admissible for $\mathrm{NP}^{\mathrm{fix},\ell}_{T_\mathrm{p}}(t,x)$ with $\varphi^{\mathrm{fix},\ell}_{\mathrm{trans}}(t, \boldsymbol{x}, \boldsymbol{u}) = 0$ and $\varphi^{\mathrm{fix},\ell}_{\mathrm{per}}(t, \boldsymbol{x}, \boldsymbol{u}) > \phi^{\ell,*}_{T_\mathrm{p},\mathrm{per}}$. Then Assumption 5.6 (existence of descent directions) implies the existence of a homotopy $H : (-\varepsilon, \varepsilon) \to \mathcal{AC}^{n_x}(\mathcal{T}^t_{\text{per}}) \times \mathrm{L}^{n_u}_\infty(\mathcal{T}^t_{\text{per}})$ of periodic trajectories with improving economic output and $H(0) = (\boldsymbol{x}, \boldsymbol{u})|_{\mathcal{T}^t_{\text{per}}}$. Assumption 4.3 (controllability) allows to extend the homotopy $H : (-\varepsilon, \varepsilon) \to \mathcal{AC}^{n_x}(\mathcal{T}^t_{\text{per}}) \times \mathrm{L}^{n_u}_\infty(\mathcal{T}^t_{\text{per}})$ to a homotopy $\Psi : (-\varepsilon, \varepsilon) \to \mathcal{AC}^{n_x}(\mathcal{T}^t) \times \mathrm{L}^{n_u}_\infty(\mathcal{T}^t)$ consisting of of admissible pairs of trajectories for $\mathrm{NP}^{\text{fix},\ell}_{T_\text{p}}(t,x)$. But by construction, it has the property

$$\frac{\partial}{\partial s}\varphi^{\text{fix},\ell}_{\text{per}}(t, H(s))|_{s=0} < 0, \tag{5.44}$$

which proves that $(\boldsymbol{x}, \boldsymbol{u})$ is not optimal for $\mathrm{NP}^{\text{fix},\ell}_{T_\text{p}}(t,x)$. □

The following Lemma is the analogue version of Lemma 4.3.

Lemma 5.8 (Objective Decrease)
Let $(t_i, x_i) \in \mathrm{Ad}^{\text{fix},\ell}_{T_\text{p}}$. Then for $(t_{i+1}, x_{i+1}) := (t_i + \Delta T, \boldsymbol{x}_{t_i,x_i}(t_i + \Delta T))$ it holds

$$\varphi^{\text{fix},\ell,*}(t_{i+1}, x_{i+1}) \leq \rho^{-\Delta T} w_{\text{trans}} \varphi^{\text{fix},\ell,*}_{\text{trans}}(t_i, x_i) + \varphi^{\text{fix},\ell,*}_{\text{per}}(t_i, x_i). \tag{5.45}$$

Proof The proof is analogue to the proof of Lemma 4.3. □

This result will be exploited to show that the sequence $(\varphi^{\text{fix},\ell,*}(t_i, x_i))_{i\in\mathbb{N}}$ converges to the optimal T_p-periodic output $\phi^{\ell,*}_{T_\text{p},\text{per}}$. To do so, the values of $\varphi^{\text{fix},\ell,*}_{\text{trans}}(t_i, x_i)$ need to be bounded in terms of the values of $\varphi^{\text{fix},\ell,*}_{\text{trans}}(t_i, x_i)$.

For this purpose, we combine the two previous Lemmas 5.7 and 5.8 to prove the following.

Lemma 5.9 (Asymptotic Decrease of the Objective Function Along the Closed-Loop)
Let Assumptions 5.5 (existence of optimal periodic solutions), 5.6 (existence of descent directions), 4.3 (controllability) and 5.7 (uniqueness and continuous dependence of subproblem solutions) hold. Then there exists a $\mathcal{KL}$ function β such that for $(t_0, x_0) \in \mathrm{Ad}^{\text{fix},\ell}_{T_\text{p}}$ the following estimate holds:

$$\varphi^{\text{fix},\ell,*}(t_i, x_i) \leq \beta(\varphi^{\text{fix},\ell,*}(t_0, x_0), t_i). \tag{5.46}$$

Proof The proof of this Lemma is analogue to the proof of Lemma 4.4. Only one argument has to be extended to cope with the fact that the functions $\varphi^{\text{fix},\ell,*}$, $\varphi^{\text{fix},\ell,*}_{\text{trans}}$ and $\varphi^{\text{fix},\ell,*}_{\text{per}}$ this time additionally depend on the initial time of the OCP. First, Lemma 5.8 implies that the inequality

$$\varphi^{\text{fix},\ell,*}(t_{i+1}, x_{i+1}) \leq \varphi^{\text{fix},\ell,*}(t_i, x_i) - \underbrace{(1 - \rho^{-\Delta T})}_{\in(0,1]} \varphi^{\text{fix},\ell,*}_{\text{trans}}(t_i, x_i), \tag{5.47}$$

holds for all $i \in \mathbb{N}$. The time-periodicity of ℓ and the definition of the OCP $\mathrm{NP}^{\text{fix},\ell}_{T_\text{p}}(t,x)$ furthermore imply that the problems $\mathrm{NP}^{\text{fix},\ell}_{T_\text{p}}(t,x)$ and $\mathrm{NP}^{\text{fix},\ell}_{T_\text{p}}(t + T_\text{p}, x)$ are equivalent and therefore the functions $\varphi^{\text{fix},\ell,*}$, $\varphi^{\text{fix},\ell,*}_{\text{trans}}$ and $\varphi^{\text{fix},\ell,*}_{\text{per}}$ can be interpreted as functions defined on $[0, T_\text{p}] \times \mathbb{R}^{n_x} \cap \text{Ad}^{\text{fix},\ell}_{T_\text{p}}$. This intersection is compact because $\text{Ad}^{\text{fix},\ell}_{T_\text{p}}$ is a subset of $\mathbb{R} \times \text{A}^c_x$ where A^c_x is compact and $[0, T_\text{p}]$ is also compact. Furthermore,the term $\varphi^{\text{fix},\ell,*}_{\text{trans}}(t,x)$ vanishes if and only if $\varphi^{\text{fix},\ell,*}(t,x) = \phi^{\ell,*}_{T_\text{p},\text{per}}$ (Lemma 5.7) an application of Lemma A.4 guarantees the existence of a $\mathcal{K}_\infty$-function $\tilde{\alpha}$ such that

$$\tilde{\alpha}(\varphi^{\text{fix},\ell,*}(t,x)) \le \varphi^{\text{fix},\ell,*}_{\text{trans}}(t,x) \tag{5.48}$$

holds for all $(t,x) \in \mathbb{R} \times \mathbb{R}^{n_x}$. In combination with (5.47) this implies the existence of a $\mathcal{K}_\infty$-function α such that

$$\varphi^{\text{fix},\ell,*}(t_{i+1}, x_{i+1}) \le \alpha(\varphi^{\text{fix},\ell,*}(t_i, x_i)) \tag{5.49}$$

holds for all $i \in \mathbb{N}$. Applying Lemma A.3 then finishes the proof. □

By means of the following version of the self-similarity function, we can transfer the result from the previous Lemma to a result on the economic performance of the closed-loop.

Definition 5.10 (Self-Similarity Function)
We call the function $S^\ell_{\text{fix}} : \text{Ad}^{\text{fix},\ell}_{T_\text{p}} \to \mathbb{R}_{\ge 0}$ defined by

$$S^\ell_{\text{fix}}(t,x) := \int_{t+\Delta T}^{t+T_\text{t}+T_\text{p}} \left(\left\| \boldsymbol{x}_{t,x}(\tau) - \boldsymbol{x}_{t+\Delta T,y}(\tau) \right\| + \left\| \boldsymbol{u}_x(\tau) - \boldsymbol{u}_{t+\Delta T,y}(\tau) \right\| \right) \mathrm{d}\tau, \tag{5.50}$$

with $y := \boldsymbol{x}_{t,x}(t + \Delta T)$ the self-similarity function for problem $\text{NP}^{\text{fix},\ell}_{T_\text{p}}(t,x)$. △

Similar as in Lemma 4.7 it can be shown that in the situation of the previous lemma the self similarity function can be bounded from above as follows

Lemma 5.11 (Upper Bound for Self-Similarity Function)
Let Assumptions 5.5 (existence of optimal periodic solutions) and 5.7 (uniqueness and continuous dependence of subproblem solutions) hold. Then there exists a $\mathcal{K}$-function α such that for all $(t,x) \in \text{Ad}^{\text{fix},\ell}_{T_\text{p}}$ it holds that

$$S^\ell_{\text{fix}}(t,x) \le \alpha(\varphi^{\text{fix},\ell,*}(t,x)). \tag{5.51}$$

Proof Analogue to the proof of Lemma 4.7. □

With this function it is possible to bound the difference of the predicted open loop solutions and the closed-loop trajectory

$$\text{diff}(t_i, x_i) := \int_{\mathcal{T}^{t_i}} \left(\left\| \boldsymbol{x}_{t_i,x_i}(\tau) - \boldsymbol{x}_\mu(\tau) \right\| + \left\| \boldsymbol{u}_{t_i,x_i}(\tau) - \boldsymbol{u}_\mu(\tau) \right\| \right) \mathrm{d}\tau. \tag{5.52}$$

via a telescope argument. The result is the analogue to Lemma 4.8:

Lemma 5.12 (Telescope Argument for the Closed-Loop)
Let Assumptions 5.5 (existence of optimal periodic solutions), 5.6 (existence of descent directions), 4.3 (controllability) and 5.7 (uniqueness and continuous dependence of subproblem solutions) hold. Then there exists a $\mathcal{KL}$-function β such that for all $(t_0, x_0) \in \mathrm{Ad}_{T_\mathrm{p}}^{\mathrm{fix},\ell}$ and the resulting closed-loop sequence $(x_i)_{i\in\mathbb{N}}$ it holds that

$$\mathrm{diff}(t_i, x_i) \leq \beta(\varphi^{\mathrm{fix},\ell,*}(t_0, x_0), i). \tag{5.53}$$

Proof Analogue to the proof of Lemma 4.8. □

This lemma together with the asymptotic decrease of $\varphi^{\mathrm{fix},\ell,*}$ along the closed-loop (Lemma 5.9) implies that the average economic performance of the closed-loop is equal to the average economic performance of the optimal periodic operation $\phi_{T_\mathrm{p},\mathrm{per}}^{\ell,*}$ (cf. the calculation (4.77)).

Again we can summarize the results of our analysis of the closed-loop behavior in the following theorem.

Theorem 5.13 (Economic Closed-Loop Performance, Time-Dependent Case)
Let Assumptions 5.5 (existence of optimal periodic orbits), 5.6 (existence of descent directions), 4.3 (controllability) and 5.7 (uniqueness and continuous dependence of subproblem solutions) hold. Let $(t_0, x_0) \in \mathrm{Ad}_{T_\mathrm{p}}^{\mathrm{fix},\ell}$ be the pair of initial time and initial state of the system. Then the resulting closed-loop trajectory $(\boldsymbol{x}_\mu, \boldsymbol{u}_\mu) \in \mathcal{AC}_{\mathrm{loc}}^{n_x} \times \mathrm{L}_{\infty,\mathrm{loc}}^{n_u}$ from the NMPC scheme based on the subproblems $\mathrm{NP}_{T_p}^{\mathrm{fix},\ell}(t, x)$ (5.39) exists for all times and for the average economic performance it holds

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \ell(\boldsymbol{x}_\mu(\tau), \boldsymbol{u}_\mu(\tau))\mathrm{d}\tau = \phi_{T_\mathrm{p},\mathrm{per}}^{\ell,*}. \tag{5.54}$$

Proof The proof is analogue to the proof of Theorem 4.9. □

5.3 Summary

In this chapter we proposed the extension of the economic NMPC scheme we presented in the previous chapter in two directions.

First, we considered the scenario of a parameter dependent dynamical system where the optimal periodic orbit (including its period) is changing with the parameter. To take this into account we included the period as optimization variable in the NMPC subproblem. Under similar assumptions on controllability, existence of optimal periodic orbits and uniqueness of NMPC subproblem solutions as in the previous chapter, we proved that the resulting NMPC controller has an economic performance equal to the optimal periodic trajectory.

Second, we described how the controller can be extended to a scenario where the performance criterion ℓ is time dependent and time-periodic with an a priori given period. Similar results on the closed-loop performance for this controller were obtained.

Part III

Applications and Numerical Results

Chapter 6

Numerical Implementation

In this chapter we give a detailed description of how the proposed NMPC controllers are implemented within the MATLAB NMPC toolkit `MLI` (see Wirsching [114]). First we explain the transcription of the infinite dimensional NMPC subproblems to finite dimensional NLPs. Then a quick introduction on the interior point method used for solving the NLPs is given followed an overview of the numerical methods that are used to evaluate the arising nonlinear functions and their derivatives.

6.1 Discretization of the Nonlinear Model Predictive Control Subproblems

For the numerical solution of the infinite dimensional OCPs that occur in the proposed NMPC schemes as NMPC subproblems, the problems have to be transformed to a form that can be handled by a computer. This transformation is done by using the Direct Multiple Shooting method (Bock [17]) which belongs to the class of "First Discretize then Optimize" approaches (see Section 1.6.2).

We proceed with a detailed description of the multiple shooting NLPs that result from the NMPC subproblems. Since the objective functionals of the NMPC subproblems contain a non-standard time-delay contribution, we attribute a special focus on the discretization of the objective functionals.

6.1.1 The Fixed Period Case

In the following, we describe the multiple shooting discretization for problem $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t_i, x_i)$, which is the NMPC subproblem for the NMPC scheme presented in Section 4.2.

Time Discretization

The first step in setting up the discretized version of $\mathrm{NP}^{\mathrm{fix}}_{T_\mathrm{p}}(t_i, x_i)$ is to choose an appropriate partition of the time horizon. Since the dynamical system we consider is autonomous, we may assume (for notational convenience) that the time horizon is $\mathcal{T} = \mathcal{T}_{\mathrm{trans}} \cup \mathcal{T}_{\mathrm{per}} = [0, T_\mathrm{t}] \cup [T_\mathrm{t}, T_\mathrm{t} + T_\mathrm{p}]$ (instead of $[t_i, t_i + T_\mathrm{t} + T_\mathrm{p}]$). We choose an equidistant shooting time grid with interval lengths ΔT. The interval length ΔT is chosen equal to the sampling time of the NMPC scheme, which allows for efficient initialization of subsequent NMPC subproblems by shifting the NLP variables. Since the OCP contains the coupled periodicity constraint (4.15f), it is important that the times T_t and $T_\mathrm{t} + T_\mathrm{p}$ are elements of the shooting time grid.

As a consequence, the length T_t of the transient time horizon and the length T_p of the periodic time horizon are chosen to be multiples of the uniform shooting interval length ΔT.

In practice, we achieve this by fixing a number $N_p \in \mathbb{N}$ of shooting intervals for the periodic part of the horizon, which then fixes the shooting interval length to $\Delta T = T_\mathrm{p}/N_p$. Then the length T_t of the transient time horizon is defined as a multiple $N_T \Delta T$ of the interval length ΔT. This results in the shooting interval grid

$$\mathbb{T} := \left\{t_j = j\Delta T, j \in \left\{0, 1, \dots, N_t + N_p + 1\right\}\right\}. \tag{6.1}$$

Control Parametrization and State Discretization

The control functions are chosen to be constant on the shooting intervals. Therefore, we identify the $(N_t + N_p)$-tuple $q := (q_0, \dots, q_{N_t+N_p-1}) \in \mathbb{R}^{n_u \times (N_t+N_p)}$ with the control function $u_q \in \mathrm{L}_\infty^{n_u}(\mathcal{T})$

$$\boldsymbol{u}_q(\tau) := q_j \text{ for } \tau \in [t + (j-1)\Delta T, j\Delta T). \tag{6.2}$$

Corresponding to this control parametrization, the state trajectories are parametrized by the values at the shooting time-nodes t_j which are denoted by $s_j \in \mathbb{R}^{n_x}$. Then, as described in Section 1.6.3 the $(N_t + N_p + 1)$-tuple of $\mathbb{R}^{n_x}$ vectors

$$s := (s_0, s_1, \dots, s_{N_t+N_p}) \in \mathbb{R}^{n_x \times (N_t+N_p+1)} \tag{6.3}$$

together with the control vector $q = (q_0, \dots, q_{N_t+N_p-1}) \in \mathbb{R}^{n_u \times (N_t+N_p)}$ can be interpreted as the (possibly discontinuous) state trajectory $\boldsymbol{x}_{(s,q)} : \mathcal{T} \to \mathbb{R}^{n_x}$ which is defined as

$$\boldsymbol{x}_{(s,q)}(\tau) := \begin{cases} \boldsymbol{x}(\tau; s_j, q_j, t_j) \text{ for } \tau \in [t + (j-1)\Delta T, t + j\Delta T), \\ s_{N_t+N_p} \text{ for } \tau = t + T_\mathrm{t} + T_\mathrm{p}. \end{cases} \tag{6.4}$$

The pair $(s, q) \in \mathbb{R}^{n_x \times (N_t+N_p+1)} \times \mathbb{R}^{n_u (\times N_t+N_p)}$ thus can be identified with the pair $(\boldsymbol{x}_{(s,q)}, \boldsymbol{u}_q)$ of state and control functions. The resulting NLP has $N_x := n_x(N_t + N_p + 1)$ optimization variables corresponding to the state variables s and $N_u := n_u(N_t+N_p)$ optimization variables corresponding to the control parametrization variables q.

Constraint Discretization

The initial value constraint (4.15b) and the ODE-constraint (4.15c) are transformed into the equality constraint at $s_0 - x_0 = 0$ and the matching constraints. Because the dynamical system is assumed to be autonomous, all shooting continuity conditions are of the form

$$\boldsymbol{x}(\Delta T; s_j, q_j, 0) - s_{j+1} = 0 \text{ for } j = 0, \dots, N_t + N_p - 1. \tag{6.5}$$

With the notation $M(s_j, q_j) := \boldsymbol{x}(\Delta T; s_j, q_j, 0)$ the shooting continuity constraints transform to

$$M(s_j, q_j) - s_{j+1} = 0 \text{ for } j = 0, \ldots, N_t + N_p - 1. \tag{6.6}$$

The path constraints in the transient part (4.15d) and in the periodic part (4.15e) of the time horizon are imposed on the state/control values at the shooting nodes. The periodicity constraint (4.15f) simply transforms to

$$s_{N_t} - s_{N_t+N_p} = 0. \tag{6.7}$$

Objective Function Discretization

The calculation of the objective value corresponding to the state/control variable (s, q) is split into the calculation of the objective contribution corresponding to the transient part of the horizon and the contribution corresponding to the periodic part of the horizon.

For the periodic part, the objective contribution translates to the sum

$$\mathrm{V}_{\mathrm{per}}^{\mathrm{fix}}(s, q) := \varphi_{\mathrm{per}}^{\mathrm{fix}}(\boldsymbol{x}_{(s,q)}, \boldsymbol{u}_q) = \sum_{j=N_t}^{N_t+N_p} \int_{j\Delta T}^{(j+1)\Delta T} \ell(\tau; \boldsymbol{x}(\tau; s_j, q_j, t_j), q_j) \mathrm{d}\tau. \tag{6.8}$$

With the notation $L(s_j, q_j) := \int_0^{\Delta T} \ell(\tau; \boldsymbol{x}(\tau; s_j, q_j, 0), q_j) \mathrm{d}\tau$ every integral can be interpreted as function of (s_j, q_j) and the objective transforms to the sum

$$\mathrm{V}_{\mathrm{per}}^{\mathrm{fix}}(s, q) = \sum_{j=N_t}^{N_t+N_p} L(s_j, q_j). \tag{6.9}$$

According to (4.10), the transient contribution is given by

$$\begin{aligned}\mathrm{V}_{\mathrm{trans}}^{\mathrm{fix}}(s, q) &:= \varphi_{\mathrm{trans}}^{\mathrm{fix}}(\boldsymbol{x}_{(s,q)}, \boldsymbol{u}_q) \\ &= \int_{\mathcal{T}_{\mathrm{trans}}} \rho^\tau \left| D_{T_{\mathrm{p}}} \left(\ell(\boldsymbol{x}_{(s,q)}, \boldsymbol{u}_q) \right)(\tau) \right|^2 \mathrm{d}\tau + \int_{\mathcal{T}_{\mathrm{trans}}} \rho^\tau \left\| D_{T_{\mathrm{p}}}(\boldsymbol{u}_q)(\tau) \right\|_{\mathbb{U}}^2 \mathrm{d}\tau.\end{aligned} \tag{6.10}$$

Both integral terms can be split up into the sum of integrals over the shooting intervals of the transient horizon. By using the functions $LT : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ and $UT : \mathbb{R}^{n_u} \times \mathbb{R}^{n_u} \to \mathbb{R}$ for $j = 0, \ldots, N_t - 1$, which are defined as

$$LT(s, q, \tilde{s}, \tilde{q}) := \int_0^{\Delta T} \rho^\tau \left| \ell(\boldsymbol{x}(\tau; s, q, 0), q) - \ell(\boldsymbol{x}(\tau; \tilde{s}, \tilde{q}, 0), \tilde{q}) \right|^2 \mathrm{d}\tau, \tag{6.11}$$

$$UT(q, \tilde{q}) := \int_0^{\Delta T} \rho^\tau \left\| q - \tilde{q} \right\|_{\mathbb{U}}^2 \mathrm{d}\tau, \tag{6.12}$$

we can express $\mathrm{V}^{\mathrm{fix}}_{\mathrm{trans}}(s,q)$ as follows:

$$\mathrm{V}^{\mathrm{fix}}_{\mathrm{trans}}(s,q) = \sum_{j=0}^{N_{t-1}} \rho^{j\Delta T} LT(s_j, q_j, s_{j+N_p}, q_{j+N_p}) + \sum_{j=0}^{N_{t-1}} \rho^{j\Delta T} UT(q_j, q_{j+N_p}). \tag{6.13}$$

Summing up the transient and periodic contributions of the objective function, we can now define the NLP-objective function $\mathrm{V}^{\mathrm{fix}} : \mathbb{R}^{N_x} \times \mathbb{R}^{N_u} \to \mathbb{R}$ for the NMPC subproblem at sampling time t_i as follows:

$$\mathrm{V}^{\mathrm{fix}}(s,q) := w_{\mathrm{trans}} \mathrm{V}^{\mathrm{fix}}_{\mathrm{trans}}(s,q) + \mathrm{V}^{\mathrm{fix}}_{\mathrm{per}}(s,q) \tag{6.14}$$

$$= w_{\mathrm{trans}} \left(\sum_{j=0}^{N_{t-1}} \rho^{j\Delta T} LT(s_j, q_j, s_{j+N_p}, q_{j+N_p}) + \sum_{j=0}^{N_{t-1}} \rho^{j\Delta T} UT(q_j, q_{j+N_p}) \right) + \sum_{j=N_t}^{N_t+N_p} L(s_j, q_j). \tag{6.15}$$

The Complete NLP

With the above described multiple shooting discretization the infinite dimensional OCP $\mathrm{NP}^{\mathrm{fix}}_{T_{\mathrm{p}}}(t_i, x_i)$ is transcribed into the following finite dimensional $\mathrm{NLP}^{\mathrm{fix}}_{T_{\mathrm{p}}}(t_i, x_i)$.

$$\min_{(s,q)\in\mathbb{R}^{N_x}\times\mathbb{R}^{N_u}} \quad \mathrm{V}^{\mathrm{fix}}(s,q) \tag{6.16a}$$

$$\text{s.t.} \quad 0 = s_0 - x_i, \tag{6.16b}$$

$$0 = M(s_j, q_j) - s_{j+1}, \quad \text{for } j = 0, \dots, N_t + N_p - 1, \tag{6.16c}$$

$$0 \le c(s_j, q_j) + \varepsilon \mathbb{1}_{n_c}, \quad \text{for } j = 0, \dots, N_t, \tag{6.16d}$$

$$0 \le c(s_j, q_j), \quad \text{for } j = N_t + 1, \dots, N_t + N_p, \tag{6.16e}$$

$$0 = s_{N_t} - s_{N_t+N_p}. \tag{6.16f}$$

6.1.2 The Free Period Case

The discretization of the NMPC subproblems of the NMPC feedback controller with free periods presented in Section 5.1, is in principle similar to the case of the controller with fixed period. The main difference is that, since the period $T \in \mathbb{R}$ is an optimization variable, it also will be included in the resulting NLP.

As in the fixed period case, we give a detailed description of how the transcription of the (infinite dimensional) NMPC subproblem $\mathrm{NP}^{\mathrm{free}}_{p}(x_i)$ to a finite dimensional NLP is accomplished.

Time Discretization

The time horizon of the problem $\mathrm{NP}^{\mathrm{free}}_{p}(p, x_i)$ is the (normalized) interval $\mathcal{T} = [0, c_t + 1]$. Again we use a equidistant shooting time grid. Since we want the time c_t to be part of the shooting-grid (which makes it easy to evaluate the periodicity constraint), we need to choose the relative transient horizon length c_t appropriately. To do so, we first fix a number $N_p \in \mathbb{N}$ of shooting intervals for the periodic part of the horizon and then choose the transient horizon length c_t to be a multiple of the shooting interval length $1/N_p$: $c_t = N_t/N_p$. This results in the following time grid (with $t_j := j/N_p$):

$$0 = t_0 < t_1 < \ldots < \underbrace{t_{N_t}}_{=c_t} < \ldots < \underbrace{t_{N_t+N_p}}_{=c_t+1}. \tag{6.17}$$

Control Parametrization and State Discretization

Similar to the case of the fixed period controller, we use a control parametrization that identifies the $(N_t + N_p)$-tuple $q := (q_0, \ldots, q_{N_t+N_p-1}) \in \mathbb{R}^{n_u \times (N_t+N_p)}$ with the control function $u_q \in \mathrm{L}_\infty^{n_u}(\mathcal{T})$

$$\boldsymbol{u}_q(\tau) := q_j \text{ for } \tau \in [t_j, t_{j+1}). \tag{6.18}$$

Corresponding to this control parametrization, the state trajectories are parametrized by the values at the shooting nodes t_j which are denoted by s_j. Then the $(N_t + N_p + 1)$-tuple of $\mathbb{R}^{n_x}$ vectors $s = (s_0, \ldots, s_{N_t+N_p}) \in \mathbb{R}^{n_x \times (N_t+N_p+1)}$ together with the control function $\boldsymbol{u}_q$ and a period parameter $T \in \mathbb{R}_{\geq 0}$ can be interpreted as (possibly discontinuous) state trajectory $\boldsymbol{x}_{(s,q,T)} : \mathcal{T} \to \mathbb{R}^{n_x}$ that is defined as

$$\boldsymbol{x}_{(s,q,T)}(\tau) := \begin{cases} \boldsymbol{x}(\tau; s_j, q_j, p, T, t_j) \text{ for } \tau \in [t_j, t_{j+1}), \\ s_{N_t+N_p} \text{ for } \tau = c_t + 1. \end{cases} \tag{6.19}$$

Here, the expression $\boldsymbol{x}(\tau; s_j, q_j, p, T, t_j)$ stands for the solution of the IVP $\dot{x} = TF(x, q_j, p)$ at time τ with initial value s_j at time t_j, see Remark 5.3.

The triple $(s, q, T) \in \mathbb{R}^{n_x \times (N_t+N_p+1)} \times \mathbb{R}^{n_u \times (N_t+N_p)} \times \mathbb{R}$ is then identified with the triple $(\boldsymbol{x}_{(s,q,T)}, \boldsymbol{u}_q, T) \in \mathcal{AC}^{n_x}(\mathcal{T}) \times \mathrm{L}_\infty^{n_u}(\mathcal{T}) \times \mathbb{R}$ and the NLP will have $N_x := n_x(N_t + N_p + 1)$ optimization variables corresponding to the state variables s_j,$N_u := n_u(N_t + N_p)$ optimization variables corresponding to the control parametrization variables q_j and one optimization variable corresponding to the period variable T.

Constraint Discretization

The only difference to the fixed period case in the constraint discretization is that the matching conditions will also depend on the variable T that corresponds to the period variable.

For the state trajectory $\boldsymbol{x}_{(s,q,T)}$ to be continuous at the shooting node t_j, the shooting continuity conditions

$$\boldsymbol{x}(t_{j+1}; s_j, q_j, p, T, t_j) = s_{j+1} \tag{6.20}$$

has to be satisfied for $j = 0, \dots, N_t + N_p - 1$. Since the system is assumed to be autonomous and $t_{j+1} - t_j = 1/N_p$, condition (6.20) is equivalent to

$$\boldsymbol{x}(1/N_p; s_j, q_j, p, T, 0) = s_{j+1}. \tag{6.21}$$

Beside on s_j, s_{j+1} and q_j, this condition additionally depends on the optimization variable T. By introducing the function $M : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R} \to \mathbb{R}^{n_x}$ defined as
$M(s_j, q_j, T) := \boldsymbol{x}(1/N_p; s_j, q_j, p, T, 0)$, the shooting-continuity conditions then translate to the following NLP-constraints:

$$M(s_j, q_j, T) - s_{j+1} = 0 \text{ for } j = 0, \dots, N_t + N_p - 1. \tag{6.22}$$

Objective Function Discretization

For calculating the objective value of the triple $(\boldsymbol{x}_{(s,q,T)}, \boldsymbol{u}_q, T) \in \mathcal{AC}^{n_x}(\mathcal{T}) \times \mathrm{L}_\infty^{n_u}(\mathcal{T}) \times \mathbb{R}$ that is associated with the NLP-variable (s, q, T), the different objective contributions are first split in transient and periodic parts and then further into parts corresponding to the shooting intervals.

For the periodic part, according to (5.9), we get

$$\mathrm{V}_{p,\mathrm{per}}^{\mathrm{free}}(s, q, T) := \varphi_{p,\mathrm{per}}^{\mathrm{free}}(\boldsymbol{x}_{(s,q,T)}, \boldsymbol{u}_q) = \int_{c_t}^{c_t+1} \ell(\boldsymbol{x}_{(s,q,T)}(\tau), \boldsymbol{u}_q(\tau))\mathrm{d}\tau \tag{6.23}$$

$$= \sum_{j=N_t}^{N_t+N_p-1} \int_{t_j}^{t_{j+1}} \ell(\boldsymbol{x}(\tau; s_j, q_j, p, T, t_j), q_j)\mathrm{d}\tau. \tag{6.24}$$

With the notation $L(s_j, q_j, T) := \int_0^{1/N_p} \ell(\boldsymbol{x}(\tau; s_j, q_j, p, T, 0), q_j)\mathrm{d}\tau$, this simplifies to the sum

$$\mathrm{V}_{p,\mathrm{per}}^{\mathrm{free}}(s, q, T) = \sum_{j=N_t}^{N_t+N_p-1} L(s_j, q_j, T). \tag{6.25}$$

The transient part of the contribution is treated similarly as in the fixed period case. According to (5.8) the contribution is

$$\mathrm{V}_{p,\mathrm{trans}}^{\mathrm{free}}(s, q, T) := \varphi_{p,\mathrm{trans}}^{\mathrm{free}}(\boldsymbol{x}_{(s,q,T)}, \boldsymbol{u}_q, T) \tag{6.26}$$

$$= \int_0^{c_t} \rho^{\tau T} \left| D_1 \left(\ell\left(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)\right)\right)\right|^2 \mathrm{d}\tau + \int_0^{c_t} \rho^{\tau T} \left\| D_1(\boldsymbol{u})(\tau) \right\|_{\mathbb{U}}^2 \mathrm{d}\tau. \tag{6.27}$$

Both integrals can be split into sums over the shooting intervals of the transient horizon. By using the functions $LT : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R} \to \mathbb{R}$ and $UT : \mathbb{R}^{n_u} \times \mathbb{R}^{n_u} \times \mathbb{R} \to \mathbb{R}$, which are defined as

$$LT(s, q, \tilde{s}, \tilde{q}, T) := \int_0^{1/N_p} \rho^{\tau T} \left| \ell(\boldsymbol{x}(\tau; s, q, p, T, 0), q) - \ell(\boldsymbol{x}(\tau; \tilde{s}, \tilde{q}, p, T, 0), \tilde{q}) \right|^2 \mathrm{d}\tau, \tag{6.28}$$

$$UT(q, \tilde{q}, T) := \int_0^{1/N_p} \rho^{\tau T} \left\| q - \tilde{q} \right\|_{\mathbb{U}}^2 \mathrm{d}\tau, \tag{6.29}$$

we can express $\mathrm{V}_{p,\mathrm{trans}}^{\mathrm{free}}(s, q, T)$ as follows:

$$\mathrm{V}_{p,\mathrm{trans}}^{\mathrm{free}}(s, q, T) = \sum_{j=0}^{N_t - 1} \rho^{j/N_p} LT(s_j, q_j, s_{j+N_p}, q_{j+N_p}, T) + \sum_{j=0}^{N_t - 1} \rho^{j/N_p} UT(q_j, q_{j+N_p}, T). \tag{6.30}$$

Summing up the objective contributions of the transient and the periodic parts, the NLP-objective function $V_{\mathrm{free}} : \mathbb{R}^{N_x} \times \mathbb{R}^{N_u} \times \mathbb{R} \to \mathbb{R}$ can be written as

$$\begin{aligned} \mathrm{V}_p^{\mathrm{free}}(s, q, T) := & w_{\mathrm{trans}} \mathrm{V}_{p,\mathrm{trans}}^{\mathrm{free}}(s, q, T) + \mathrm{V}_{p,\mathrm{per}}^{\mathrm{free}}(s, q, T) \\ = & w_{\mathrm{trans}} \left(\sum_{j=0}^{N_t - 1} \rho^{j/N_p} LT(s_j, q_j, s_{j+N_p}, q_{j+N_p}, T) + \sum_{j=0}^{N_t - 1} \rho^{j/N_p} UT(q_j, q_{j+N_p}, T) \right) \\ & + \sum_{j=N_t}^{N_t + N_p - 1} L(s_j, q_j, T). \end{aligned} \tag{6.31}$$

The Complete NLP

With the above described Multiple Shooting discretization the infinite dimensional OCP $\mathrm{NP}_p^{\mathrm{free}}(x_i)$ is transcribed into the finite dimensional $\mathrm{NLP}_p^{\mathrm{free}}(x_i)$.

$$\min_{(s,q,T) \in \mathbb{R}^{N_x} \times \mathbb{R}^{N_u} \times \mathbb{R}} \quad \mathrm{V}_p^{\mathrm{free}}(s, q, T) \tag{6.32a}$$

$$\text{s.t.} \quad 0 = s_0 - x_i, \tag{6.32b}$$

$$0 = M(s_j, q_j, T) - s_{j+1}, \quad \text{for } j = 0, \ldots, N_t + N_p - 1, \tag{6.32c}$$

$$0 \le c(s_j, q_j) + \varepsilon \mathbb{1}_{n_c}, \quad \text{for } j = 0, \ldots, N_t, \tag{6.32d}$$

$$0 \le c(s_j, q_j), \quad \text{for } j = N_t + 1, \ldots, N_t + N_p, \tag{6.32e}$$

$$0 = s_{N_t} - s_{N_t + N_p}, \tag{6.32f}$$

$$\underline{T} \le T \le \overline{T}. \tag{6.32g}$$

6.1.3 The Time-Periodic Objective Criterion Case

The discretization of the NMPC subproblems arising in the NMPC scheme presented in Section 5.2 is done in the same way as we already described in Section 6.1.1. The only difference is that one has to take into account that the performance criterion ℓ explicitly depends on time. This results in the fact that the discretized objective functional also depends on the initial time of the time horizon of the NMPC subproblem. Since the generalization to this case is straight-forward, we refrain here from its explicit description.

6.1.4 Evaluating the NLP Functions

We discuss how the functions associated with the $\mathrm{NLP}^{\mathrm{fix}}_{T_\mathrm{p}}(t_i, x_i)$ and $\mathrm{NLP}^{\mathrm{free}}_{p}(x_i)$ are evaluated. To omit redundant descriptions, we only describe the evaluation process for the problem $\mathrm{NLP}^{\mathrm{free}}_{p}(x_i)$, as the case of the fixed period problem $\mathrm{NLP}^{\mathrm{fix}}_{T_\mathrm{p}}(t_i, x_i)$ is essentially the same just without the additional period variable T.

Constraint Evaluation

The evaluation of the initial value constraint, the discretized path constraints and the periodicity constraint is straight-forward.

For evaluating the shooting-constraint functions $M(s_j, q_j, T) = \boldsymbol{x}(1/N_p; s_j, q_j, p, T, 0)$ of $\mathrm{NLP}^{\mathrm{free}}_{p}(x_i)$, it is necessary to solve the IVP $\dot{\boldsymbol{x}}(\tau) = T f_p(\boldsymbol{x}(\tau), q_j)$ with $\boldsymbol{x}(0) = s_j$ on $[0, 1/N_p]$. As we describe below, the evaluation of the shooting constraints is done simultaneously with the evaluation of the terms $L_{\mathrm{per}}(s_j, q_j, T)$ occurring in the objective function.

Objective Evaluation

For the evaluation of the objective function $\mathrm{V}^{\mathrm{free}}_{p}(s, q, T)$, the parts corresponding to the transient and the periodic part of the time horizon have to be considered separately. We begin with the contribution corresponding to the periodic part of the horizon. To evaluate this part, terms of the form $L_{\mathrm{per}}(s_j, q_j, T) := \int_0^{1/N_p} \ell(\boldsymbol{x}(\tau; s_j, q_j, p, T, 0), q_j) \mathrm{d}\tau$ have to be computed. This is done by augmenting the auxiliary differential state x_{aux} to the original ODE and defining $\dot{x}_{\mathrm{aux}} := \ell(x_{\mathrm{aux}}, q_j)$. The solution of the $(n_x + 1)$-dimensional-dimensional IVP

$$\begin{pmatrix} \dot{\boldsymbol{x}}(\tau) \\ \dot{\boldsymbol{x}}_{\mathrm{aux}}(\tau) \end{pmatrix} = \begin{pmatrix} T \cdot f_p(\boldsymbol{x}(\tau), q_j) \\ \ell(\boldsymbol{x}(\tau), q_j) \end{pmatrix}, \begin{pmatrix} \boldsymbol{x}(0) \\ \boldsymbol{x}_{\mathrm{aux}}(0) \end{pmatrix} = \begin{pmatrix} s_j \\ 0 \end{pmatrix} \tag{6.33}$$

at time $1/N_p$ then is used to compute $L_{\mathrm{per}}(s_j, q_j, T)$. Note that from the solution of IVP (6.33) also the term $\boldsymbol{x}(1/N_p; s_j, q_j, p, T, 0)$ can be extracted. Since the computation of this term anyway is necessary for evaluating the shooting constraints, it is possible to combine the evaluation of the shooting constraints with the evaluation of the objective function by solving the IVP (6.33).

For the evaluation of the transient part of the objective, it is necessary to compute the integrals

$$LT_j(s_j, q_j, s_{j+N_p}, q_{j+N_p}, T) := \int_{j/N_p}^{(j+1)/N_p} \rho^{\tau T} \left| D_1 \left(\ell(\boldsymbol{x}_{(s,q,T)}, \boldsymbol{u}_q) \right)(\tau) \right|^2 \mathrm{d}\tau. \tag{6.34}$$

The integrand in this expression depends on $\boldsymbol{x}_{(s,q,T)}(\tau)$ and on $\boldsymbol{x}_{(s,q,T)}(\tau+1)$ because the term $D_1 \left(\ell(\boldsymbol{x}_{(s,q,T)}, \boldsymbol{u}_q) \right)(\tau)$ is given by

$$D_1 \left(\ell(\boldsymbol{x}_{(s,q,T)}, \boldsymbol{u}_q) \right)(\tau) = \ell(\boldsymbol{x}_{(s,q,T)}(\tau), \boldsymbol{u}_q(\tau)) - \ell(\boldsymbol{x}_{(s,q,T)}(\tau + 1), \boldsymbol{u}_q(\tau + 1)). \tag{6.35}$$

Evaluating this difference for $\tau \in [j/N_p, (j+1)/N_p]$, for the first summand it holds

$$\ell(\boldsymbol{x}_{(s,q,T)}(\tau), \boldsymbol{u}_q(\tau)) = \ell(\boldsymbol{x}(\tau; s_j, q_j, p, T, j/N_p), q_j) \tag{6.36}$$

$$= \ell(\boldsymbol{x}(\tau - j/N_p; s_j, q_j, p, T, 0), q_j) \tag{6.37}$$

and for the second summand it holds

$$\ell(\boldsymbol{x}_{(s,q,T)}(\tau+1), \boldsymbol{u}_q(\tau+1)) = \ell(\boldsymbol{x}(\tau+1; s_{j+N_p}, q_{j+N_p}, p, T, (j+N_p)/N_p), q_{j+N_p}) \tag{6.38}$$

$$= \ell(\boldsymbol{x}(\tau; s_{j+N_p}, q_{j+N_p}, p, T, j/N_p), q_{j+N_p}) \tag{6.39}$$

$$= \ell(\boldsymbol{x}(\tau - j/N_p; s_{j+N_p}, q_{j+N_p}, p, T, 0), q_{j+N_p}) \tag{6.40}$$

Since the integrand of $LT_j(s_j, q_j, s_{j+N_p}, q_{j+N_p}, T)$ contains the square of the absolute value of $D_1\big(\ell(\boldsymbol{x}_{(s,q,T)}, \boldsymbol{u}_q)\big)(\tau)$, the term $LT_j(s_j, q_j, s_{j+N_p}, q_{j+N_p}, T)$ depends on both (s_j, q_j) and (s_{j+N_p}, q_{j+N_p}) non-linearly.
For the computation of $LT_j(s_j, q_j, s_{j+N_p}, q_{j+N_p}, T)$ we have to solve the auxiliary (n_x+n_x+1)-dimensional IVP

$$\begin{pmatrix} \dot{\boldsymbol{x}}_1(\tau) \\ \dot{\boldsymbol{x}}_2(\tau) \\ \dot{\boldsymbol{x}}_{\text{aux}}(\tau) \end{pmatrix} = \begin{pmatrix} T \cdot f_p(\boldsymbol{x}_1(\tau), q_j) \\ T \cdot f_p(\boldsymbol{x}_2(\tau), q_{j+N_p}) \\ \rho^{(\tau + j/N_p)T} \left| \ell(\boldsymbol{x}_1(\tau), q_j) - \ell(\boldsymbol{x}_2(\tau), q_{j+N_p}) \right|^2 \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{x}_1(0) \\ \boldsymbol{x}_2(0) \\ \boldsymbol{x}_{\text{aux}}(0) \end{pmatrix} = \begin{pmatrix} s_j \\ s_{j+N_p} \\ 0 \end{pmatrix}, \tag{6.41}$$

on the interval $[0, 1/N_p]$. Then, according to (6.37) and (6.40) it holds

$$LT_j(s_j, q_j, s_{j+N_p}, q_{j+N_p}, T) = \boldsymbol{x}_{\text{aux}}(1/N_p). \tag{6.42}$$

For a brief description of the numerical methods used for the integration of the IVPs see Section 6.3.1.

6.2 Interior Point Methods

For our prototypical implementation of the proposed NMPC controller we solve the resulting NLPs with the software package IPOPT (Wächter and Biegler [117]) which is based on an Interior Point method ([89, Chapter 19]).

In the following, we briefly describe the main ideas behind the NLP solver IPOPT, for a more detailled description we refer the reader to [117]. The method is based on a primal-dual barrier approach. In a first step, the NLP is transformed into a standard form

$$\min_{x \in \mathbb{R}^n} \quad \varphi(x) \tag{6.43a}$$

$$\text{s.t.} \quad c(x) = 0, \tag{6.43b}$$

$$x \geq 0. \tag{6.43c}$$

This can be accomplished by introducing slack variables, if necessary. Then, for $\mu \in \mathbb{R}_{\geq 0}$ the equality constrained barrier problem

$$\min_{x \in \mathbb{R}^n} \quad \varphi_\mu(x) := \varphi(x) + \mu \sum_{i=1}^{n} \ln(x^{(i)}) \tag{6.44a}$$

$$\text{s.t.} \quad c(x) = 0 \tag{6.44b}$$

is solved approximately. By considering the barrier problems for a sequence of barrier parameters μ that converges to zero, the solution of the original problem is approximated with increasing accuracy. The barrier problems itself are solved by a damped NEWTON method applied to the primal-dual equations of the barrier problem which can be written in the form (with $\operatorname{diag}(x) \in \mathbb{R}^{n\times n}$ being the diagonal matrix with the entries of x on the diagonal)

$$\nabla_x \varphi(x) + \nabla_x c(x)\lambda - z = 0 \tag{6.45}$$

$$c(x) = 0 \tag{6.46}$$

$$\operatorname{diag}(x)\operatorname{diag}(z)\mathbb{1}_n - \mu\mathbb{1}_n = 0. \tag{6.47}$$

For each NEWTON-iterate (x_k, λ_k, z_k), a search direction $(d_k^x, d_k^\lambda, d_k^z)$ has to be calculated as the solution of the (at (x_k, λ_k, z_k)) linearized primal-dual equations of the barrier problem. This corresponds to the solution of the linear system

$$\begin{pmatrix} \nabla_{xx}^2\left(\varphi(x_k) - c(x_k)^T\lambda_k\right) & \nabla_x c(x_k) & -\mathrm{I}_n \\ \nabla_x c(x_k)^T & 0 & 0 \\ \operatorname{diag}(z_k) & 0 & \operatorname{diag}(x_k) \end{pmatrix} \begin{pmatrix} d_k^x \\ d_k^\lambda \\ d_k^z \end{pmatrix} = -\begin{pmatrix} \nabla_x\varphi(x_k) + \nabla_x c(x_k)\lambda_k - z_k \\ c(x_k) \\ \operatorname{diag}(x_k)\operatorname{diag}(y_k)\mathbb{1}_n - \mu\mathbb{1}_n \end{pmatrix}. \tag{6.48}$$

When the search direction is computed, a step size $0 < \alpha_k < 1$ is computed using a backtracking line-search procedure described in [118] and the next NEWTON-iterate then is defined as $(x_{k+1}, \lambda_{k+1}, z_{k+1}) := (x_k, \lambda_k, z_k) + \alpha_k(d_k^x, d_k^\lambda, d_k^z)$. The line-search method employed in IPOPT ensures global convergence of the procedure. When a certain optimality error of the NEWTON-iterates is below a threshold, a new barrier problem with a smaller μ is considered (FIACCO-MCCORMICK-approach cf. Fiacco [42]).

6.3 Numerical Integration and Derivative Generation

The process of solving the NLPs arising from the NMPC subproblems requires the evaluation of the objective and constraint functions as well as first- and second-order derivative information. This information is required for the computation of the search directions for the NEWTON-step according to (6.48).

As we have seen in section 6.1.4, the NLP objective function as well as the NLP constraints are computed by solving various IVPs. Therefore numerical methods for solving IVPs as well as, since our NLP-methods are derivative based, numerical methods for calculating derivatives of IVP solutions (also called sensitivities) are required.

In particular, for a given IVP,

$$\dot{\boldsymbol{x}}(\tau) = f(\tau, \boldsymbol{x}(\tau), q, p), \quad \boldsymbol{x}(t_s) = x_0, \quad \tau \in [t_s, t_e], \tag{6.49}$$

the following data has to be supplied by numerical methods:

- the final state $\boldsymbol{x}(t_e; x_0, q, p, t_s) \in \mathbb{R}^{n_x}$,
- the derivative of the final state with respect to initial value, controls and parameters

$$\frac{\partial \boldsymbol{x}(t_e; x_0, q, p, t_s)}{\partial(x_0, q, p)} \in \mathbb{R}^{n_x \times (n_x + n_q + n_p)}, \tag{6.50}$$

- second-order derivatives (Hessians) of the form

$$\frac{\partial^2 \left(\boldsymbol{x}(t_e; x_0, q, p, t_s)^T \lambda\right)}{\partial(x_0, q, p)^2}, \tag{6.51}$$

  where $\lambda \in \mathbb{R}^{n_x}$ corresponds to a multiplier arising in the NLP.

6.3.1 Numerical Integration

The evaluation of the final values $\boldsymbol{x}(t_e; x_0, q, p, t_s)$, we use the integrator package `SolvIND` with a RUNGE-KUTTA-FEHLBERG method [40, 41].

This method belongs to the class of one-step methods which divides the time horizon $[t_s, t_e]$ into a discretization grid $t_0 := t_s, < t_1 < \ldots, < t_N := t_e$ and approximates the solution of the IVP at these grid points according to the iteration formula

$$\eta_{k+1} := \eta_k + h_k \Phi(t_k, h_k, \eta_k), \quad h_k := t_{k+1} - t_k. \tag{6.52}$$

The generating function $\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ for a RUNGE-KUTTA-method with $s \in \mathbb{N}$ stages is defined as

$$\Phi(t, h, \eta) := \sum_{i=1}^{s} c_i k_i, \quad k_i := f\left(t + \alpha_i h, \eta + h \sum_{j=1}^{s} B_{ij} k_j, q, p\right) \tag{6.53}$$

with suitable chosen coefficients $\alpha, c \in \mathbb{R}^s$ and $B \in \mathbb{R}^{s \times s}$. The method we employ is of explicit type, since it uses a matrix B whose entries B_{ij} vanish for $j \geq i$. This means that the definition of k_i in (6.53) is well posed since the right-hand side of the equation only depends on k_j with $j < i$, i.e., k_i can be calculated explicitly using this formula.

With appropriately chosen coefficients c, α, B and number of stages s, the method is consistent and stable, see e.g. [108].

Since the choice of the discretization grid is of great importance for the accuracy of the obtained approximations η_k, the integrator package `SolvIND` employs an adaptive discretization scheme based on estimates of the local truncation error.

6.3.2 Derivative Generation

There are several ways how derivatives of a function f can be calculated numerically. First, there is the possibility to calculate the derivative of F symbolically "by hand". This has the advantage of high accuracy, but it requires differentiation by the user, which is error-prone especially if the function F is complicated.

Another possibility is differentiation by the method of finite differences, where the derivative in the direction v is approximated by the quotient $F'(x) \cdot v \approx \frac{F(x+hv)-F(x)}{h}$ with a sufficiently small number $h \in \mathbb{R}_{>0}$. This method is very easy to implement but it suffers from amplification of rounding errors if h is chosen too small.

For efficient numerical derivative generation, especially in the context of derivatives of IVP-solutions, the integrator package `SolvIND` uses a different method based on the principle of Internal Numerical Differentiation (IND) (Bock [18]) in combination with techniques of Automatic Differentiation (AD) (Speelpenning [107], Griewank [49]) and TAYLOR[1]-coefficient propagation (Bischof et al. [13]). Because of their importance in the efficient computation of derivatives, we briefly review the basic ideas behind these concepts.

AD and Taylor Coefficient Propagation

AD is a technique for evaluating derivatives for a large class of functions $F : \mathbb{R}^{n_x} \to \mathbb{R}^{n_m}$ that can be expressed as a sequence of elementary operations such as addition, multiplication, subtraction, division, sin, exp, etc. The computational evaluation of such functions is done by constructing an evaluation graph with elementary operations at the nodes and intermediate results at the edges.

The idea in AD is that the computational graph can also be used to compute the derivatives by repeatedly making use of the chain rule. We illustrate this with the example of a function F that is just the composition of various other functions:

$$F = F_n \circ F_{n-1} \circ \ldots \circ F_1. \tag{6.54}$$

Forward Mode

The forward mode of AD can be used to compute directional derivatives of the form $\frac{\partial F}{\partial x}(x) \cdot v \in \mathbb{R}^{n_m}$ for a direction $v \in \mathbb{R}^{n_x}$ without the need to compute the whole JACOBIAN [2]. By introducing the intermediate results w_k defined recursively as $w_{k+1} := F_k(w_k)$ and $w_1 = x$ and the intermediate derivatives $\overline{w}_k := \frac{\partial F_k}{\partial w_k}(w_k)$, the chain rule can be used to calculate

$$\frac{\partial F}{\partial x}(x) \cdot v = \overline{w}_n \cdot \overline{w}_{n-1} \ldots \overline{w}_1 \cdot v. \tag{6.55}$$

This expression can be evaluated simultaneously to traversing the evaluation graph of $F(x) = F_n(F_{n-1}(\ldots(F_1(x))))$ in the forward direction (hence the name forward mode), i.e., in the di-

[1] Brook Taylor 1685 - 1731
[2] Carl Gustav Jacob Jacobi 1804 - 1851

rection of increasing indices. The complete JACOBIAN can be calculated by applying the forward mode to a set of basis vectors of $\mathbb{R}^{n_x}$.

Reverse Mode

The reverse mode of AD is also based on the chain rule, but it is based on traversing the evaluation graph in the direction of decreasing indices, i.e., in the opposite direction of the evaluation order (hence the name reverse mode). It was first proposed in the master-thesis Linnainmaa [78] and can be used to calculate adjoint derivatives of the form $\frac{\partial(a^T \cdot F)}{\partial x}(x) \in \mathbb{R}^{n_m}$ for adjoint directions $a \in \mathbb{R}^{n_x}$. Applying the chain rule

$$\frac{\partial(a^T \cdot F)}{\partial x}(x) = a^T \frac{\partial F_n}{\partial w_n}(w_n) \circ \ldots \circ \frac{\partial F_1}{\partial w_1}(w_1) \tag{6.56}$$

and traversing from the outside to the inside gives rise to the following rule in which the adjoint direction a is propagated backwards according to $a_{k-1}^T := a_k^T \frac{\partial F_k}{\partial w_k}(w_k)$ for $k = n, \ldots, 2$ until the final value $a_1^T = \frac{\partial(a^T \cdot F)}{\partial x}(x)$ is reached. Since the intermediate results w_k are necessary for performing the backward propagation step, it is necessary to calculate these results beforehand and save them on tape or use a checkpointing strategy (see Griewank [48]) to reduce the necessary amount of memory.

Despite this additional memory requirements, the reverse mode can be more efficient compared to the forward mode of AD especially if $n_m \ll n_x$. For example, the calculation of the full JACOBIAN $J = \frac{\partial F}{\partial x}(x)$ with the forward mode requires one forward sweep for each column $J \cdot e_i$ of the JACOBIAN, i.e. n_x forward sweeps. Compared to this, in the reverse mode the JACOBIAN is built row by row, requiring one backward sweep per row, i.e. n_m backward sweeps.

The computational cost of one (forward or reverse) sweep is proportional to the computational cost of evaluating the function F itself.

TAYLOR-Coefficient Propagation

The fact that most elementary functions are locally analytic implies that they can be represented by local TAYLOR-series. The TAYLOR-coefficients can be used to calculate derivatives (also of higher order) since the derivatives can be extracted from the coefficients of the TAYLOR-polynomials. As an example, the directional derivative $\frac{\partial F}{\partial x}(x) \cdot v$ can simply be extracted as the degree one coefficient of the TAYLOR-expansion of $t \mapsto F(x + t \cdot v)$. The techniques of forward and reverse mode AD can be transferred and applied to truncated TAYLOR-expansions instead of directional or adjoint derivatives to calculate derivatives or arbitrary order efficiently. Instead of directions, TAYLOR-coefficients are propagated along the evaluation graph.

For a detailed overview on evaluating derivatives using AD-techniques in combination with TAYLOR-coefficients we refer the reader to Griewank [49] and Bischof et al. [13].

External Numerical Differentiation and Internal Numerical Differentiation

In order to compute sensitivities of IVP solutions, it is again possible to consider the integration procedure described in Section 6.3.1 as black-box and apply a finite-difference approximation. This method is referred to as External Numerical Differentiation (END) and suffers from rounding errors and high computational costs for applying the integration routine multiple times with perturbed initial values and parameters.
Therefore it is desirable to interpret the integration procedure as a sequence of elementary operations and apply techniques of AD. However, the adaptive discretization schemes that ensure a certain accuracy of the integration procedure are based on conditional statements which are not differentiable at some points and thus make it impossible to apply AD.

The remedy here is to first apply the integration procedure with the adaptive discretization scheme and then, for applying AD, freeze the adaptive discretization scheme and treat it as a fixed grid. This results in an integration scheme that can indeed be interpreted as a sequence of smooth elementary operations and thus AD can be applied. This principle is referred to as the principle of Internal Numerical Differentiation (IND) and was introduced in Bock [18, 19] and is used in the integrator package `SolvIND`.

6.4 Summary

In this chapter we gave an overview of the numerical implementation of the proposed NMPC schemes. We gave a detailed description of the transcription of the infinite dimensional OCPs to finite dimensional NLPs via the direct multiple shooting method and a brief description of the NLP solution algorithm which is based on an interior point method. Furthermore we described the numerical methods behind the evaluation and derivative generation for the NLP functions.

Chapter 7

Examples with Time-Periodic Objective

In this chapter we apply the controller presented in Section 5.2 to several examples with time-periodic performance criteria.

The first example is a toy-problem modelling the control of a hydrostorage power plant. The second example is a system of two connected water tanks with water flowing from tank 1 to tank 2 and a controllable inflow to tank 1. The last example is a system of 4 connected water tanks.

7.1 Hydrostorage Power Plant

The hydrostorage power plant example is a toy example modelling a system that can be used to convert potential energy of water stored in a elevated reservoir into electricity by using a generator. The upper water reservoir can be filled using a pump. The objective is to maximize the economic profit under the assumption that the electricity price is periodically varying with a period of $T_{\text{p}} = 24\text{h}$.

7.1.1 Model Equations and Performance Criterion

The system is described using an ODE with the water level $x_1 \in \mathbb{R}$ of the upper reservoir as state variable and two control variables $u_{\text{in}}, u_{\text{out}} \in \mathbb{R}$ representing the flow rates of the inflow/outflow pump at a given time instant. The higher the water level in the upper reservoir, the more power is needed to pump more water into the reservoir and the more power can be generated by letting water flow out through the generator.
If $A \in \mathbb{R}$ denotes the cross-surface of the upper water reservoir, the system is described by the following ODE

$$\dot{\boldsymbol{x}}_1(t) = f\left(\boldsymbol{x}_1(t), \begin{pmatrix} \boldsymbol{u}_{\text{in}}(t) \\ \boldsymbol{u}_{\text{out}}(t) \end{pmatrix}\right) := \frac{\boldsymbol{u}_{\text{in}}(t) - \boldsymbol{u}_{\text{out}}(t)}{\rho_{\text{water}} A}. \tag{7.1}$$

The (dimensionless) electricity price is assumed to vary periodically according to the formula:

$$p(t) := 0.1 \cdot (3.6 \cdot 10^6)^{-1} \cdot (\sin(t\pi / T_{\text{p}})^2 + 1) \quad [\text{J}^{-1}]. \tag{7.2}$$

This corresponds to a electricity price between 0.1 and 0.2 per [kWh]. The performance criterion is the electric power of the pump minus the electric power at the generator times the

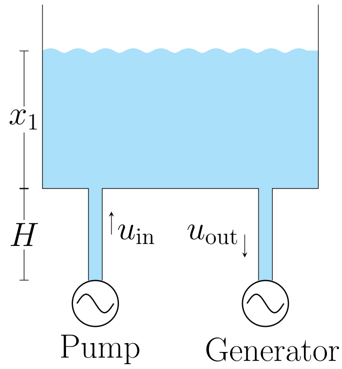


Figure 7.1: Illustration of the hydrostorage power plant.

(dimensionless) electricity price

$$\ell\left(t, x_1, \begin{pmatrix} u_{\text{in}} \\ u_{\text{out}} \end{pmatrix}\right) := \underbrace{p(t)}_{[1/\text{J}]} \underbrace{(u_{\text{in}} - u_{\text{out}}) g (x_1 + H)}_{[\text{J/h}]}. \tag{7.3}$$

It measures the economic performance of the system at at given time/state (the sign of ℓ is negative when the system generates electricity).

The objective of the system is to maximize the economic revenue while satisfying the operational constraints of the system. The constraints are given in form of simple bound constraints on the water level in the upper reservoir and the in-/outflow rates. The parameters describing the model with the operational constraints can be found in Table 7.1. For simplicity we assume the water density to be 1kg/l.

State	Description	Lower bound	Upper bound
x_1	Water level reservoir	0.1m	3m
Control	Description	Lower bound	Upper bound
u_{in}	Flow rate inflow-pump	0l/h	$20 \cdot 10^6$l/h
u_{out}	Flow rate generator	0l/h	$40 \cdot 10^6$l/h
Parameter	Description	Value	
g	Gravitational acceleration	9.81m/s^2	
A	Cross section of reservoir	100,000m^2	
H	Height of reservoir above pump/generator	250m	
ρ_{water}	Water density	1000kg/m^3	

Table 7.1: State and control bounds and parameters of the hydrostorage system.

7.1.2 Controller Setup

We consider three different scenarios for the hydrostorage system. The scenarios are distinguished by different initial states. In the first scenario we consider the initial state where the water level in the reservoir is close to the lower bounds, in the second scenario we consider the initial state where the water level in the reservoir is close to the upper bounds and in the third scenario we consider the initial state that corresponds to the optimal periodic operation. To each of the scenarios we apply the NMPC controller presented in Section 5.2 based on the NMPC subproblems of the form $\mathrm{NLP}_{24}^{\mathrm{fix},\ell}(t_i, x_i)$ with sampling time intervals corresponding to 1h. We test various controller configurations differing in the length of the transient horizon T_{t}. As transient weight we use $w_{\mathrm{trans}} = 1$. The weighted norm for the control self-tracking term is defined as

$$\|u\|_{\mathbb{U}} := 10^{-7}(|u_{\mathrm{in}}| + |u_{\mathrm{out}}|). \tag{7.4}$$

All controllers use the discount factor $\rho = 1.01$ and the transient part of the horizon is relaxed to $0 \le c(x,u) + \varepsilon$ with $\varepsilon = 10^{-6}$ (see constraint (6.16d) in $\mathrm{NLP}_{24}^{\mathrm{fix},\ell}(t_i, x_i)$.

7.1.3 Results of NMPC Simulations

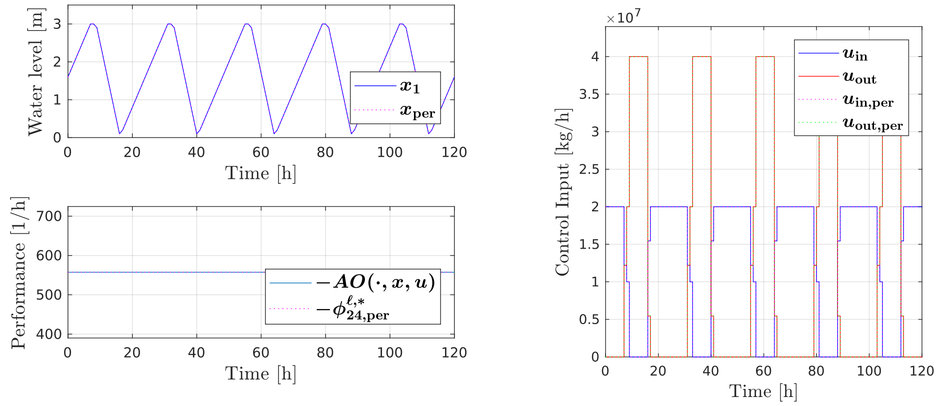
All NMPC simulations are run for 120h. To evaluate the economic performance for a generated closed-loop trajectory $(\boldsymbol{x}_\mu, \boldsymbol{u}_\mu)$, we consider the moving average output

$$AO(t, \boldsymbol{x}_\mu, \boldsymbol{u}_\mu) := \frac{1}{T_{\mathrm{p}}} \int_t^{t+T_{\mathrm{p}}} \ell(\tau, \boldsymbol{x}_\mu(\tau), \boldsymbol{u}_\mu(\tau))\mathrm{d}\tau \tag{7.5}$$

and compare the behavior of $t \mapsto AO(t, \boldsymbol{x}_\mu, \boldsymbol{u}_\mu)$ with the value of the optimal average output $\phi_{24,\mathrm{per}}^{\ell,*} \approx -5.57 \cdot 10^2 [\mathrm{h}^{-1}]$ (which we calculate in Section 7.1.4).

Initial Value on Optimal Periodic Trajectory

The initial value at time $t_0 = 0$ corresponding to the optimal periodic solution depicted in Figure 7.5 is $x_0 = 1.6$m. We apply NMPC controllers with the given data and transient horizon lengths of 12h and 24 h. The results for the controller with $T_{\mathrm{t}} = 12$h are shown in Figure 7.2. The closed-loop state trajectory stays on the optimal periodic solution and the economic average output corresponds to the optimal periodic average output. The controller with $T_{\mathrm{t}} = 24$h performed equally good and also shows no deviation from the optimal periodic deviation.



a) Closed-loop state trajectory and moving average output for $T_{\mathrm{t}} = 12\mathrm{h}$.

b) Closed-loop control profile for $T_{\mathrm{t}} = 12\mathrm{h}$.

Figure 7.2: NMPC simulations with initial value on the optimal periodic trajectory and transient horizon length $T_{\mathrm{t}} = 12\mathrm{h}$. No deviation to the optimal periodic behavior can be observed and the economic average output is equal to $\phi^{\ell,*}_{24,\mathrm{per}}$.

Initial Value close to Lower Bounds

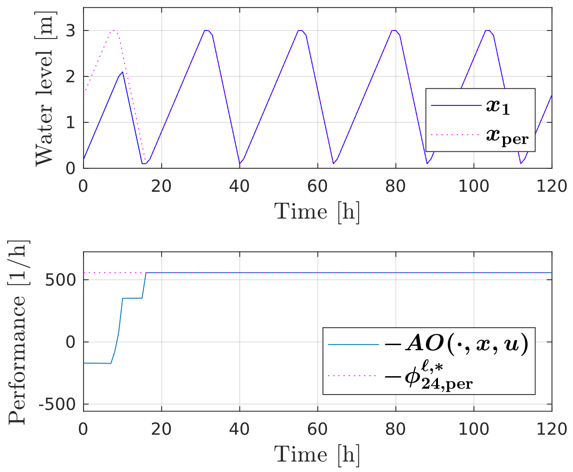
In this scenario we choose an initial value at time $t_0 = 0$ close to the lower bounds of the system $x_0 = 0.2\mathrm{m}$. We apply NMPC controllers with transient horizon lengths of 12h and 24 h. The results are shown in Figure 7.3.

Both controllers, the one with transient horizon length 12h and the one with horizon length 24h gradually fill up the water reservoir by pumping in water at times when the electricity price p is low. Both controllers reach the water level that corresponds to the optimal periodic trajectory after less then 20 h. The controllers sacrifice economic performance in the beginning to reach the optimal periodic operation.

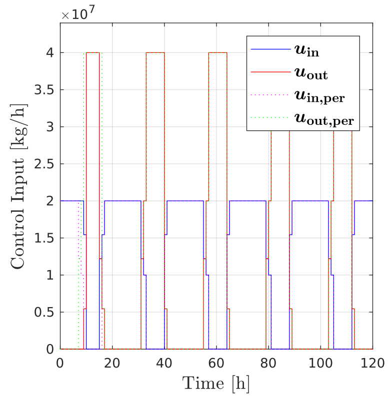
Initial Value Close to Upper Bounds

In this scenario we choose an initial value at time $t_0 = 0$ close to the upper bounds of the system $x_0 = 2.9\mathrm{m}$. We apply NMPC controllers with transient horizon lengths of 12h and 24h. The results are shown in Figure 7.4.

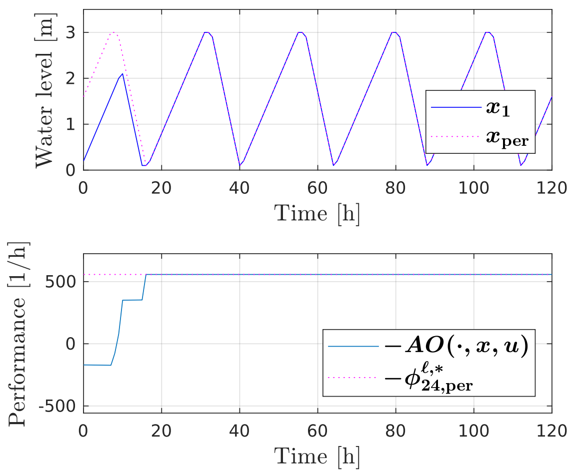
Both controllers, the one with transient horizon length 12h and the one with horizon length 24h reach the water level that corresponds to the optimal periodic trajectory in less then 10h. The controllers exploit the high water level in the beginning (no water needs to be pumped in the reservoir, which saves electricity).



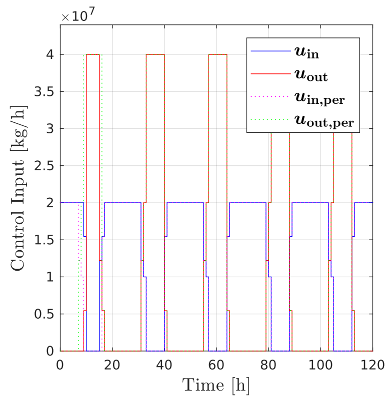
a) Closed-loop state trajectory and moving average output for $T_\mathrm{t} = 12\mathrm{h}$.



b) Closed-loop control profile for $T_\mathrm{t} = 12\mathrm{h}$.

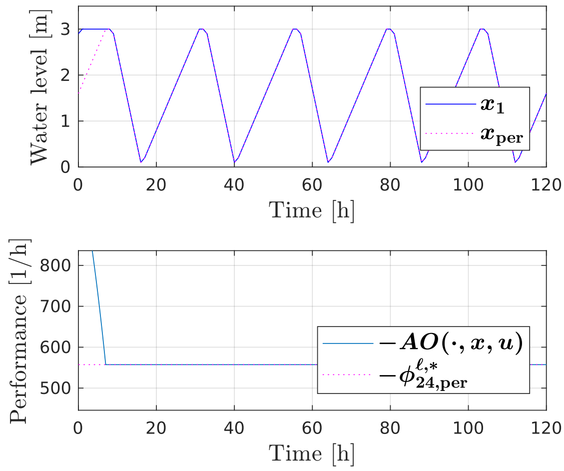


c) Closed-loop state trajectory and moving average output for $T_\mathrm{t} = 24\mathrm{h}$.

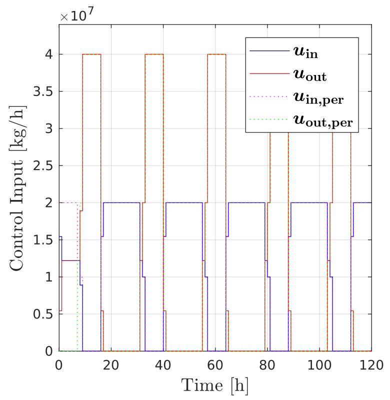


d) Closed-loop control profile for $T_\mathrm{t} = 24\mathrm{h}$.

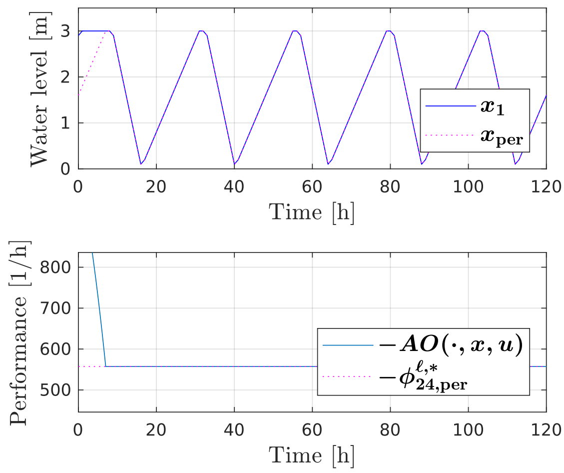
Figure 7.3: NMPC simulations with initial value close to lower bound and transient horizon length $T_\mathrm{t} = 12\mathrm{h}$ (upper plots) respectively $T_\mathrm{t} = 24\mathrm{h}$ (lower plots). Both controllers steer the system back to the optimal periodic behavior after an initial phase of filling up the water reservoir. After 20h, no deviation to the optimal periodic trajectory can be observed anymore.



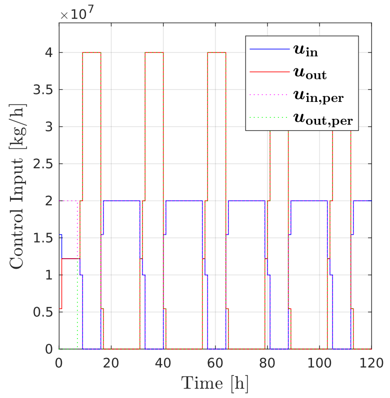
a) Closed-loop state trajectory and moving average output for $T_{\text{t}} = 12\text{h}$.



b) Closed-loop control profile for $T_{\text{t}} = 12\text{h}$.



c) Closed-loop state trajectory and moving average output for $T_{\text{t}} = 24\text{h}$.



d) Closed-loop control profile for $T_{\text{t}} = 24\text{h}$.

Figure 7.4: NMPC simulations with initial value close to upper bound and transient horizon length $T_{\text{t}} = 12\text{h}$ (upper plots) respectively $T_{\text{t}} = 24\text{h}$ (lower plots). Both controllers steer the system back to the optimal periodic behavior after an initial phase of letting water flow out from the water reservoir. This results in an economic performance better then $\phi^{\ell,*}_{24,\text{per}}$ in the beginning. After around 10h, no deviation to the optimal periodic trajectory can be observed anymore.

7.1.4 Optimal Periodic Operation

Although it is not necessary for the NMPC controller, for comparison reasons we determine the optimal T_p-periodic trajectory. We solve the periodic OCP $\mathrm{Per}_{24}^{\mathrm{fix},\ell}$ (5.41) using a multiple shooting discretization with uniform shooting interval length of 1 time-unit, i.e. 24 shooting intervals.
The solution $(\boldsymbol{x}_{\mathrm{per}}, \boldsymbol{u}_{\mathrm{per}})$ of the discretized OCP $\mathrm{Per}_{24}^{\mathrm{fix},\ell}$ with state trajectory and the corresponding control profile is depicted in Figure 7.5 and has an average output of

$$\phi_{24,\mathrm{per}}^{\ell,*} \approx -5.57 \cdot 10^2 [\mathrm{h}^{-1}]. \tag{7.6}$$

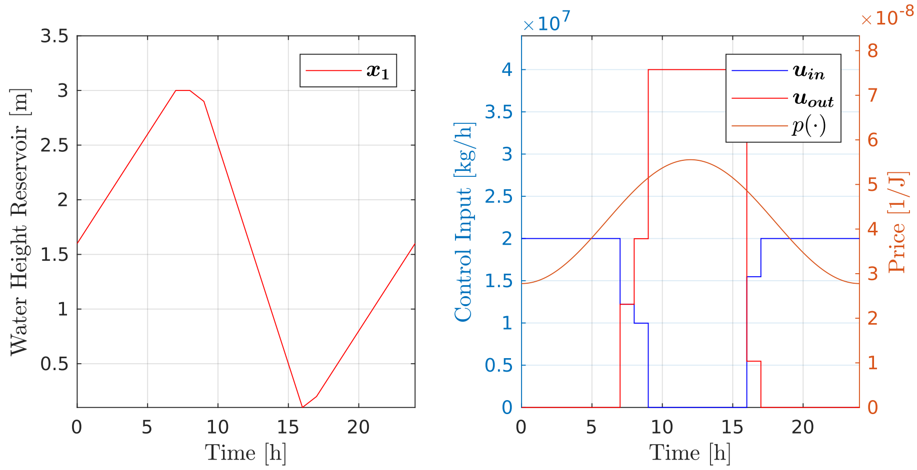


Figure 7.5: Optimal periodic operation of the hydrostorage system. On the left the optimal periodic water level development, on the right the optimal periodic control input. The water is pumped up when the electricity price is low and it is flowing out through the generator when the electricity price is high.

7.2 Double-Tank System

The double-tank example is a system of a two connected water tanks with water flowing from the first tank into the second. Apart from the connection of the tanks, the water flows out of the second tank and there is a controllable inflow u_{in} at the first tank. This example is a slightly modified version of the double-tank example in [63] where it is used as a benchmark problem for an economically oriented NMPC scheme with periodic constraints.

7.2.1 Model Equations and Performance Criterion

The states of the system are the water levels x_1, x_2 in each tank. The outflow-rate of each tank is proportional to the square-root of the water level and the outflow rate of the second tank is subject to a lower bound.

The system can be described by the ODE

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = f\left(\begin{pmatrix} \boldsymbol{x}_1(t) \\ \boldsymbol{x}_2(t) \end{pmatrix}, \boldsymbol{u}_{\text{in}}\right) := \begin{pmatrix} 0.16\boldsymbol{u}_{\text{in}}(t) - 0.4\sqrt{\boldsymbol{x}_1(t)} \\ 0.4(\sqrt{\boldsymbol{x}_1(t)} - \sqrt{\boldsymbol{x}_2(t)}) \end{pmatrix}. \tag{7.7}$$

Note that the right-hand side f is not differentiable for water levels 0, however, on the compact set of feasible states and controls defined by the simple bounds (see Table 7.2), the right-hand side is differentiable and LIPSCHITZ continuous. For this reason the simple bound are slightly moved away from 0 The performance criterion ℓ of the system is the inflow-rate u_{in} multiplied with a time-dependent price p:

$$\ell(t, u_{\text{in}}) := p(t)u_{\text{in}}. \tag{7.8}$$

The price p is sinusoidally varying with a period of $T_{\text{p}} = 10$:

$$p(t) := \sin(t\pi/T_{\text{p}})^2 + 0.1. \tag{7.9}$$

The task is to operate the system such that the average output with respect to the performance criterion ℓ is minimized. As the example is a toy-example, all states, controls and other parameters are dimensionless. The system is subject to constraints in the form of simple bounds on the water levels in the tanks and the inflow control that have to be satisfied during operation, see Table 7.2.

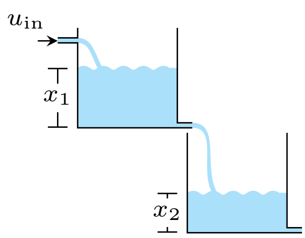


Figure 7.6: The setup of the two water tanks of the double-tank system.

State	Description	Lower Bound	Upper Bound
x_1	Water level tank 1	0.001	3
x_2	Water level tank 2	0.16	3
Control	Description	Lower Bound	Upper Bound
u_{in}	Water Inflow Tank 1	0	5

Table 7.2: State and control bounds of the double-tank system.

7.2.2 Controller Setup

We consider three different scenarios for the double-tank system. The scenarios are distinguished by different initial states. In the first scenario we consider the initial state where the water level in both water tanks is close to the lower bounds, in the second scenario we consider the initial state where the water level in both water tanks is close to the upper bounds and in the third scenario we consider the initial state that corresponds to the optimal periodic operation. To each of the scenarios we apply the NMPC controller presented in Section 5.2 based on the NMPC subproblems of the form $\mathrm{NLP}_{10}^{\mathrm{fix},\ell}(t_i, x_i)$ with sampling time intervals corresponding to 1 time step. We test various controller configurations differing in the length of the transient horizon T_t. As transient weight we use $w_{\mathrm{trans}} = 1$. The weighted norm for the control self-tracking term is defined as

$$\|u\|_{\mathbb{U}} := 0.1 \cdot |u_{\mathrm{in}}|. \tag{7.10}$$

All controllers use the discount factor $\rho = 1.01$. The path constraint in the transient part of the horizon is relaxed to $0 \le c(x,u) + \varepsilon \mathbb{1}_{n_c}$ with $\varepsilon = 10^{-6}$ (constraint (6.16d) in $\mathrm{NLP}_{10}^{\mathrm{fix},\ell}(t_i, x_i)$).

7.2.3 Results of NMPC Simulations

All NMPC simulations are run for 100 time units. To evaluate the economic performance for a generated closed-loop trajectory $(\boldsymbol{x}_\mu, \boldsymbol{u}_\mu)$, we consider the moving average output

$$AO(t, \boldsymbol{x}_\mu, \boldsymbol{u}_\mu) := \frac{1}{T_\mathrm{p}} \int_t^{t+T_\mathrm{p}} \ell(\tau, \boldsymbol{x}_\mu(\tau), \boldsymbol{u}_\mu(\tau)) \mathrm{d}\tau \tag{7.11}$$

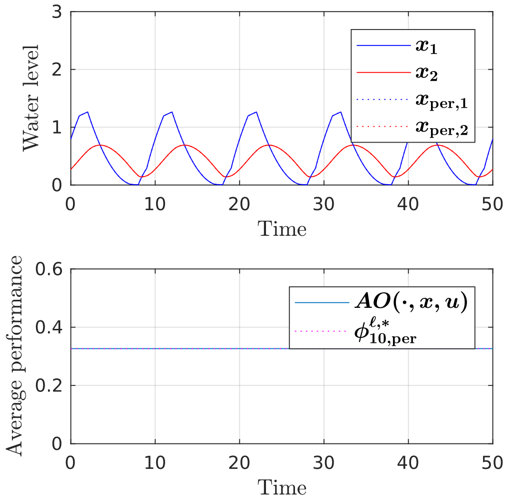
and compare the behavior of $t \mapsto AO(t, \boldsymbol{x}_\mu, \boldsymbol{u}_\mu)$ with the value of the optimal average output $\phi_{10,\mathrm{per}}^{\ell,*} \approx 0.3264$ (which we calculate in Section 7.2.4).

Initial Value on Optimal Periodic Trajectory

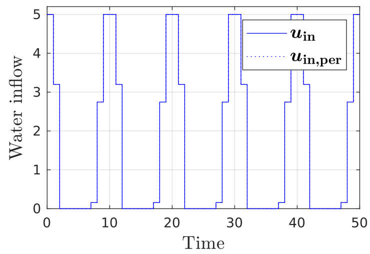
The initial value at time $t_0 = 0$ corresponding to the optimal periodic solution depicted in Figure 7.10 is $x_0 = (0.7954, 0.2728)^T$. We apply NMPC controllers with the given data and transient horizon lengths of 10 and 20 time units. The results for are shown in Figure 7.7.
In both NMPC simulations, the controller keeps the water levels in the two tanks on the optimal periodic trajectory and the moving average output corresponds to the optimal periodic performance.

Initial Value Close to Lower Bounds

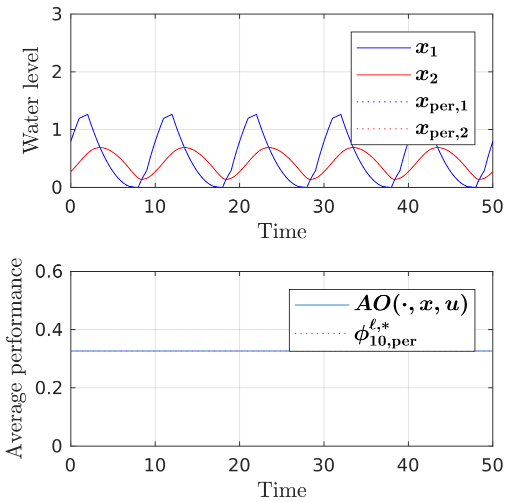
In this scenario we choose an initial value at time $t_0 = 0$ close to the lower bounds of the system $x_0 = (0.1, 0.26)^T$. We apply NMPC controllers with transient horizon lengths of 10 and 20 time units. The results are shown in Figure 7.8.
In both NMPC simulations, after around 10 time units the water levels of both tanks reach the water levels corresponding to the optimal periodic trajectories. In the beginning, the controllers pump a little more water into tank 1 then the reference control $\boldsymbol{u}_{\mathrm{in,per}}$, to reach the optimal periodic operation. This results in an initially slightly worse average output, before the optimal periodic average output is reached in around 10 time units.



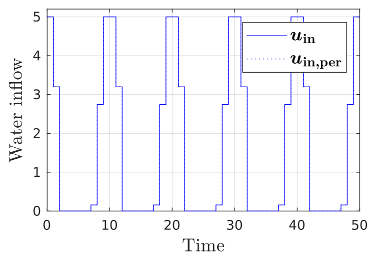
a) Closed-loop state trajectory and moving average output for $T_t = 10$.



b) Closed-loop control profile for $T_t = 10$.

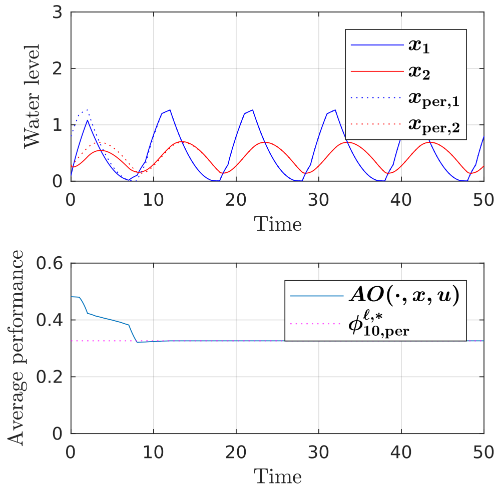


c) Closed-loop state trajectory and moving average output for $T_t = 20$.

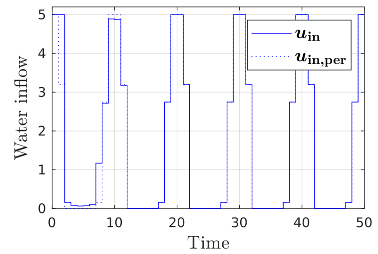


d) Closed-loop control profile for $T_t = 20$.

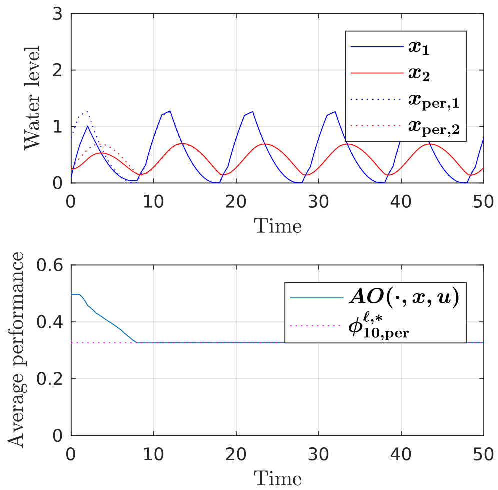
Figure 7.7: NMPC simulations for the double-tank example with initial value on optimal periodic trajectory. Plots (a) and (b) show the results of the controller with $T_t = 10$ and plots (c) and (d) show the results of the controller with $T_t = 20$. No deviation from the optimal periodic behavior (the dotted lines) can be observed, which shows that the controller keeps the process within the optimal periodic operation regime.



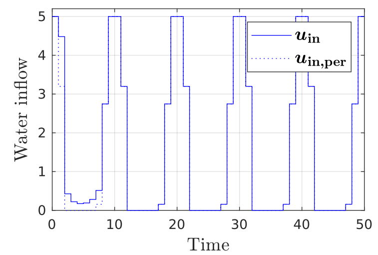
a) Closed-loop state trajectory and moving average output for $T_t = 10$.



b) Closed-loop control profile for $T_t = 10$.



c) Closed-loop state trajectory and moving average output for $T_t = 20$.



d) Closed-loop control profile for $T_t = 20$.

Figure 7.8: NMPC simulations for the double-tank example with initial value close to lower bounds. Plots (a) and (b) show the results of the controller with $T_t = 10$ and plots (c) and (d) show the results of the controller with $T_t = 20$. In both simulations, the optimal periodic trajectory is reached after around 10 time units. To reach the optimal periodic operation regime, in the beginning more water is pumped into the upper tank to which can be seen at the control profiles in (b) and (d).

Initial Value Close to Upper Bounds

In this scenario we choose an initial value at time $t_0 = 0$ close to the lower bounds of the system $x_0 = (2.9, 2.9)^T$. We again apply NMPC controllers with transient horizon lengths of 10 and 20 time units. The results are shown in Figure 7.9.
In both NMPC simulations, after around 20 time units the water levels of both tanks reach the water levels corresponding to the optimal periodic trajectories. It can also be observed that, compared to the closed-loop state trajectories, the moving average performance reaches the level of the optimal periodic performance significantly faster (in around 10 time units). This highlights the fact that the transient objective of the NMPC scheme is dominated by the contribution $\varphi_{\text{trans}}^{\text{fix},\ell}(t, \boldsymbol{x}, \boldsymbol{u})$ (5.37) which measures the deviation from periodicity of the economic output and is independent of the state-behavior.

7.2.4 Optimal Periodic Operation

For comparison reasons we determine the optimal periodic operation of the double-tank system. To do so, we solve the periodic OCP $\text{Per}_{10}^{\text{fix},\ell}$ (5.41) using a multiple shooting discretization with uniform shooting interval length of 1 time-unit, i.e. 10 shooting intervals.
The solution $(\boldsymbol{x}_{\text{per}}, \boldsymbol{u}_{\text{per}})$ of the discretized OCP $\text{Per}_{10}^{\text{fix},\ell}$ with state trajectory and the corresponding control profile is depicted in Figure 7.10 and has an average output of

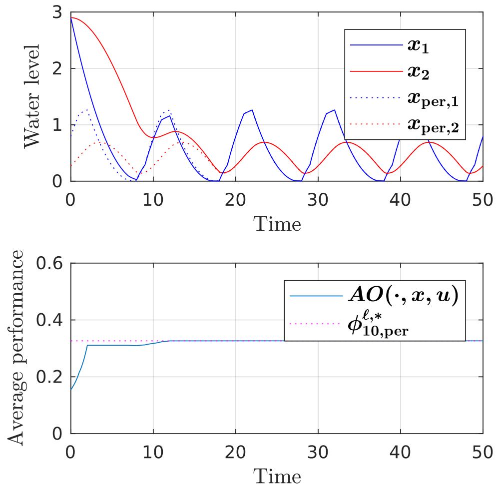
$$\phi_{10,\text{per}}^{\ell,*} \approx 0.3264. \tag{7.12}$$

7.3 Four-Tank System

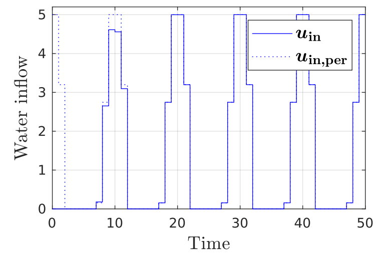
The four-tank system is a laboratory plant consisting of four water tanks that are interconnected with pumps and valves and an additional water reservoir. It was introduced as a control-benchmark model in Johansson [65]. It serves as a tracking Model Predictive Control (MPC) benchmark problem in [3] and, in the version we consider, as an E-NMPC-benchmark problem in [77].

7.3.1 Model Equations and Performance Criterion

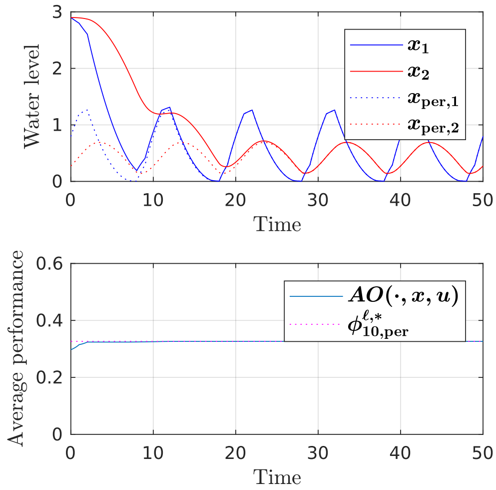
The system consists of four upper water tanks with $x = (x_1, x_2, x_3, x_4)$ denoting the water levels in each tank and one lower water tank. The lower water tank is connected with the two upper tanks via pumps which act as controls $u = (u_a, u_b)$ in the system (see Figure 7.11).



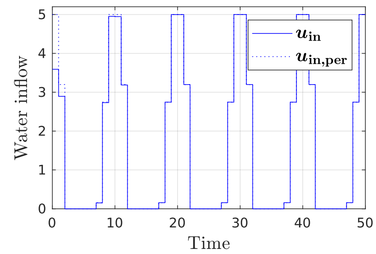
a) Closed-loop state trajectory and moving average output for $T_t = 10$.



b) Closed-loop control profile for $T_t = 10$.



c) Closed-loop state trajectory and moving average output for $T_t = 20$.



d) Closed-loop control profile for $T_t = 20$.

Figure 7.9: NMPC simulations for the double-tank example with initial value close to upper bounds. Plots (a) and (b) show the results of the controller with $T_t = 10$ and plots (c) and (d) show the results of the controller with $T_t = 20$. In both simulations, the optimal periodic operation regime is reached after around 20 times units. Compared to the closed-loop state trajectories, the average output of the closed-loop system in both simulations is already on the level of the optimal periodic performance after 10 time units.

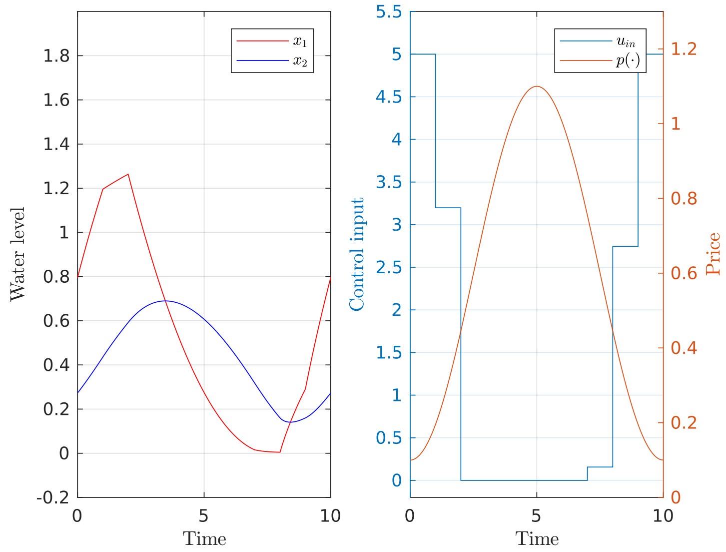


Figure 7.10: Optimal periodic operation of the double-tank system. As can be seen at the optimal periodic control $\boldsymbol{u}_{\text{in}}$ on the right, water is pumped into the upper water tank during the period when the electricity price is relatively low.

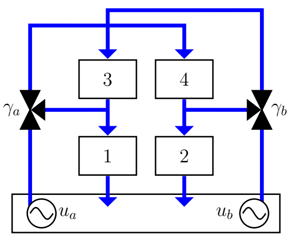


Figure 7.11: Illustration of the four-tank system.

Parameter	Value	Description
$H^{\max}$	1.2m	Maximum water level in each tank
$H^{\min}$	0.2m	Minimum water level in each tank
$Q^{\max}$	$2.5\text{m}^3/\text{h}$	Maximal flow through each pump
$Q^{\min}$	$0\text{m}^3/\text{h}$	Minimal flow through each pump
a_1	$1.341\cdot 10^{-4}\text{m}^2$	Discharge constant of tank 1
a_2	$1.533\cdot 10^{-4}\text{m}^2$	Discharge constant of tank 2
a_3	$9.322\cdot 10^{-5}\text{m}^2$	Discharge constant of tank 3
a_4	$9.061\cdot 10^{-5}\text{m}^2$	Discharge constant of tank 4
A	0.03m^2	Cross section of all tanks
γ_a	0.3	Parameter of three way valve a
γ_b	0.4	Parameter of three way valve b

Table 7.3: Parameters of the four-tank system.

The water outflow of each of the upper tanks is proportional to the square-root of its water-level. Pump A feeds tank 1 and 4 with γ_a being the relative part of the pumped water flowing into tank 1 and $1-\gamma_a$ the relative part flowing into tank 4. Pump B feeds tank 2 and 3 with γ_b being the relative part of the pumped water flowing into tank 2 and $1-\gamma_b$ the relative part flowing into tank 3.

The dynamic equations describing the system can be derived using BERNOULLI'S[1] law and mass balance:

$$\dot{x}_1 = -\frac{a_1}{A}\sqrt{2gx_1} + \frac{a_3}{A}\sqrt{2gx_3} + \frac{\gamma_a}{3600A}u_a \tag{7.13a}$$

$$\dot{x}_2 = -\frac{a_2}{A}\sqrt{2gx_2} + \frac{a_4}{A}\sqrt{2gx_4} + \frac{\gamma_b}{3600A}u_b \tag{7.13b}$$

$$\dot{x}_3 = -\frac{a_3}{A}\sqrt{2gx_3} + \frac{1-\gamma_b}{3600A}u_b \tag{7.13c}$$

$$\dot{x}_4 = -\frac{a_4}{A}\sqrt{2gx_4} + \frac{1-\gamma_a}{3600A}u_a \tag{7.13d}$$

Note that right-hand side defined by the above equations is not differentiable for water levels 0. However, on the compact set of feasible states and controls defined by the simple bounds (see Table 7.3), the right-hand side is differentiable and LIPSCHITZ continuous. The time-periodic economic performance criterion ℓ penalizes the use of the pumps and has a term that is inversely proportional to the combined water level in the lower two tanks. It is defined as follows

$$\ell(t,x,u) = \left(u_a^2 + p(t)u_b^2\right) + 15\frac{2H^{\min}}{A(x_1+x_2)} \tag{7.14}$$

where the function p is periodic with period $T_\text{p} = 150s$:

$$p(t) = 0.15\sin\left(\frac{2\pi t}{150}\right) + 1. \tag{7.15}$$

[1] Daniel Bernoulli 1700 - 1782

The task is to operate the system such that the average output with respect to the performance criterion ℓ is minimized.

7.3.2 Controller Setup

We consider three different scenarios for the four-tank system. The scenarios are distinguished by different initial states. In the first scenario we consider the initial state where the water levels in the four-tanks is close to the lower bounds, in the second scenario we consider the initial state where the water level in the reservoir is close to the upper bounds and in the third scenario we consider the initial state that was used in [77]. To each of the scenarios we apply the NMPC controller presented in Section 5.2 based on the NMPC subproblems of the form $\mathrm{NLP}_{150}^{\mathrm{fix},\ell}(t_i, x_i)$ with sampling time intervals of $5s$. We test various controller configurations differing in the length of the transient horizon T_t. As transient weight we use $w_{\mathrm{trans}} = 1$. The weighted norm for the control self-tracking term is defined as

$$\|u\|_{\mathbb{U}} := 0.1 \cdot |u_a| + 0.1 \cdot |u_b|. \tag{7.16}$$

The length of the transient horizon is chosen as $T_\mathrm{t} = 150\mathrm{s}$. All controllers use the discount factor $\rho = 1.01$ and the path constraint in the transient part of the horizon is relaxed to $0 \le c(x,u) + \varepsilon \mathbb{1}_{n_c}$ with $\varepsilon = 10^{-6}$ (see constraint (6.16d) in $\mathrm{NLP}_{150}^{\mathrm{fix},\ell}(t_i, x_i)$).

7.3.3 Results of NMPC Simulations

Each simulation is run for 150 sampling times, which equals 750s. To evaluate the economic performance for a generated closed-loop trajectory $(\boldsymbol{x}_\mu, \boldsymbol{u}_\mu)$, we consider the moving average output

$$AO(t, \boldsymbol{x}_\mu, \boldsymbol{u}_\mu) := \frac{1}{T_\mathrm{p}} \int_t^{t+T_\mathrm{p}} \ell(\tau, \boldsymbol{x}_\mu(\tau), \boldsymbol{u}_\mu(\tau)) \mathrm{d}\tau \tag{7.17}$$

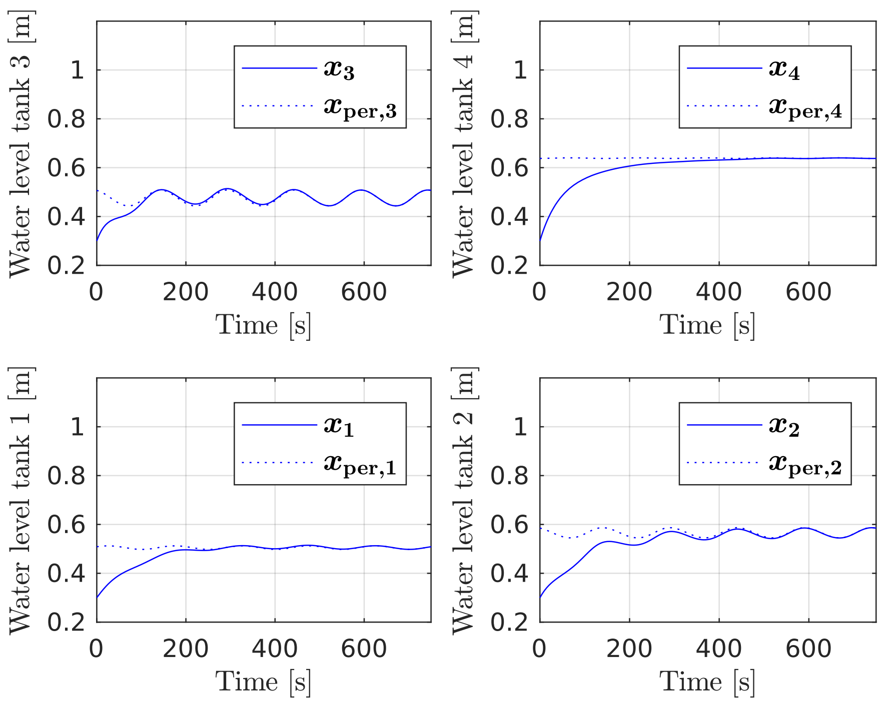
and compare the behavior of $t \mapsto AO(t, \boldsymbol{x}_\mu, \boldsymbol{u}_\mu)$ with the optimal periodic performance $\phi_{150,\mathrm{per}}^{\ell,*} \approx 11.21$ (for the calculation of $\phi_{150,\mathrm{per}}^{\ell,*}$ see Section 7.3.4).

Initial Value close to Lower Bounds

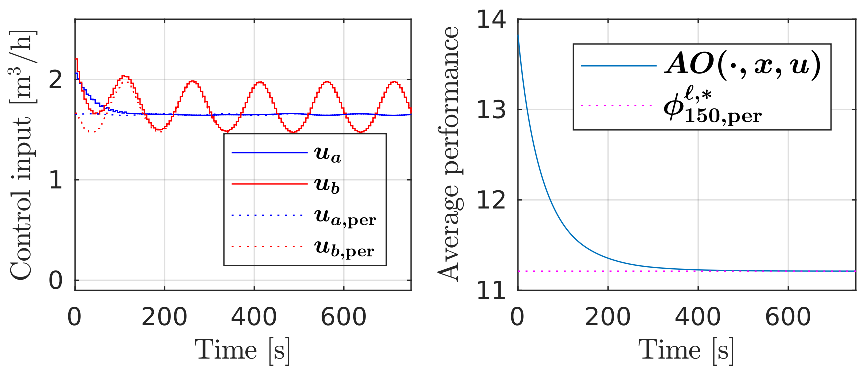
In this scenario we choose an initial value at time $t_0 = 0$ close to the lower bounds of the system $x_0 = (0.3, 0.3, 0.3, 0.3)^T$. The results are shown in Figure 7.12.
After an initial phase of pumping more water into the tanks compared to the periodic reference controls $\boldsymbol{u}_{a,\mathrm{per}}$ and $\boldsymbol{u}_{b,\mathrm{per}}$, the optimal periodic trajectories are reached at around 500s. The average performance converges smoothly to the optimal periodic performance after being worse initially due to the cost of heavier pump usage.

Initial Value Close to Upper Bounds

In this scenario we choose an initial value at time $t_0 = 0$ close to the upper bounds of the system $x_0 = (1.1, 1.1, 1.1, 1.1)^T$. The results are shown in Figure 7.13.



a) Closed-loop state trajectories.



b) Closed-loop control input and average performance.

Figure 7.12: NMPC simulations with $T_\mathrm{t} = 150\mathrm{s}$ for the four-tank system with initial values close to lower bounds. After an initial correction phase where the controller pumps more water into the tanks to reach the optimal periodic behavior, the optimal periodic regime is maintained.

After an initial phase heavy pump usage (first pump b then pump a are at full power see Figure 7.13b), the optimal periodic trajectories are reached at around 500s. While at $t = 100$s the closed-loop states still are far off the optimal periodic trajectory, at the same time the average performance already reaches the level of the optimal periodic performance. Again, as in the double-tank example (see Figure 7.9), this highlights the fact that the transient objective of the NMPC scheme is dominated by the contribution $\varphi_{\text{trans}}^{\text{fix},\ell}(t, \boldsymbol{x}, \boldsymbol{u})$ (5.37) which measures the deviation from periodicity of the economic output and is independent of the state-behavior.

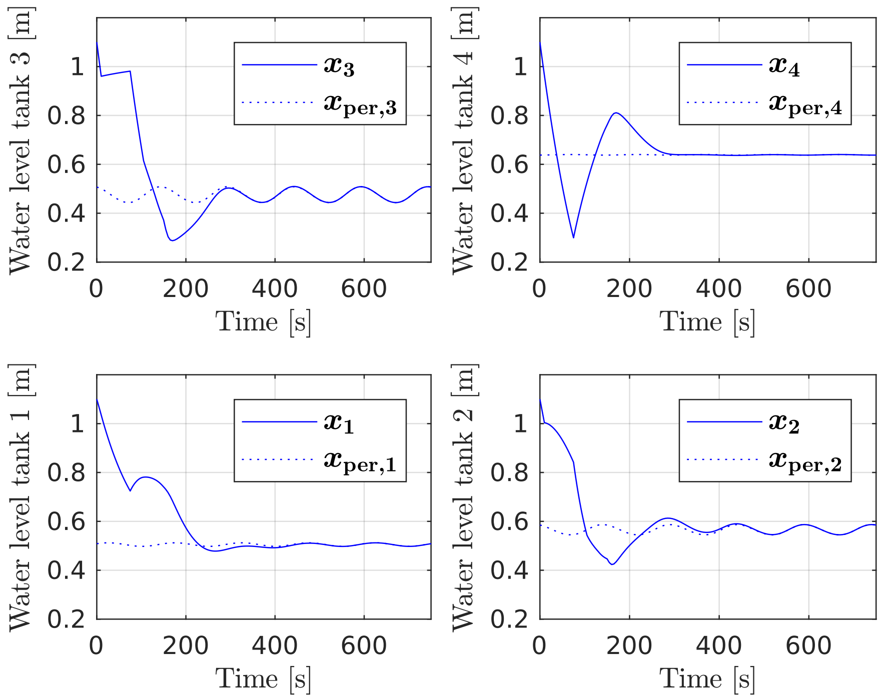
Initial Value as in [77]

In this scenario we choose the initial value at time $t_0 = 0$ that was also considered in the NMPC simulations in [77]: $x_0 = (0.4594, 0.9534, 0.4587, 0.9521)^T$. The results are shown in Figure 7.3.3.
It can be observed that after around 300s the optimal the optimal periodic operation is reached. In the beginning, the controller applies strong control actions that deviate drastically from the periodic reference controls $\boldsymbol{u}_{a,\text{per}}$ and $\boldsymbol{u}_{b,\text{per}}$. As can be seen at the average performance plot, after producing initially an economic output that is better than the optimal periodic output in the first 100s, the average output of the system reaches the level of the optimal periodic output after around 300s.

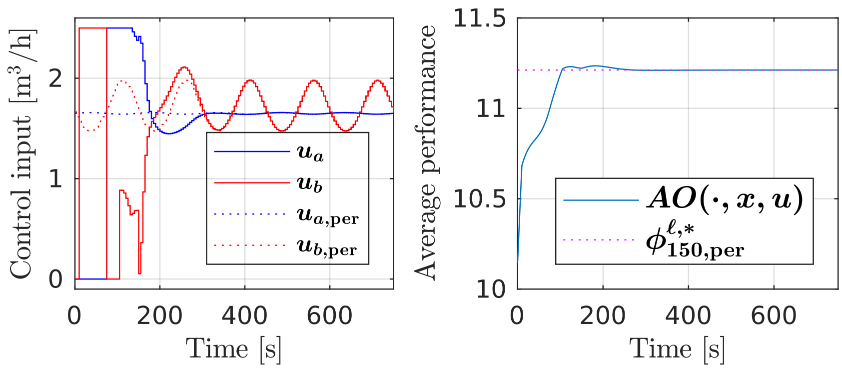
7.3.4 Optimal Periodic Operation

For comparison reasons we determine the optimal periodic operation of the four-tank system. To do so, we solve the periodic OCP $\text{Per}_{150}^{\text{fix},\ell}$ (5.41) using a multiple shooting discretization with uniform shooting interval length of $5s$, i.e. 30 shooting intervals.
The solution $(\boldsymbol{x}_{\text{per}}, \boldsymbol{u}_{\text{per}})$ of the discretized OCP $\text{Per}_{150}^{\text{fix},\ell}$ with state trajectory and the corresponding control profile is depicted in Figure 7.15 and has an average output of

$$\phi_{150,\text{per}}^{\ell,*} \approx 11.21. \tag{7.18}$$

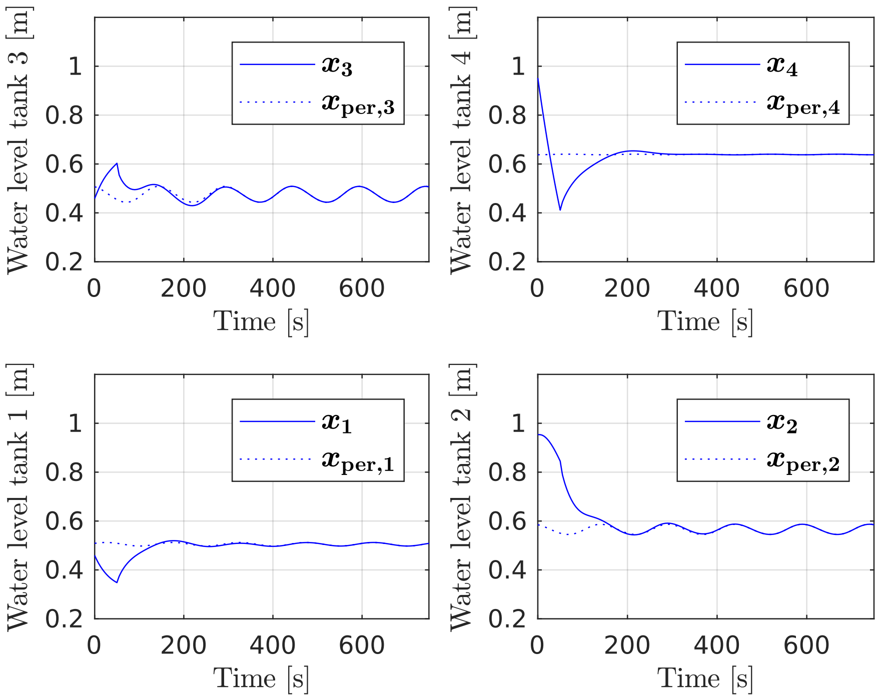


a) Closed-loop state trajectories.

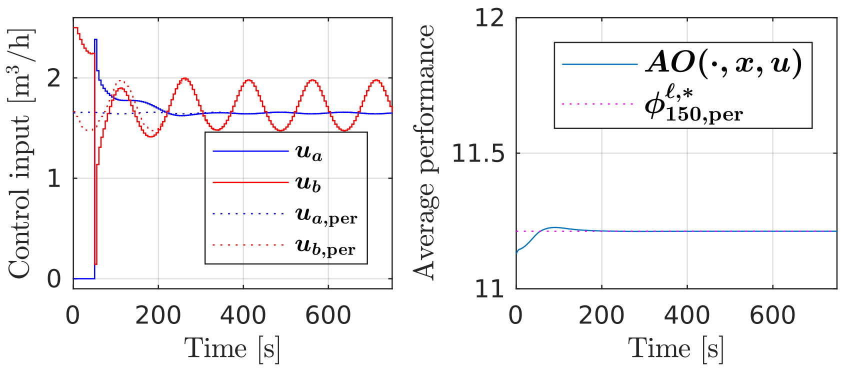


b) Closed-loop control input and average performance.

Figure 7.13: NMPC simulations with $T_t = 150$s for the four-tank with initial values close to the upper bound. In (b) it can be seen that in an initial phase the controller deviates drastically from the optimal periodic controls ($\boldsymbol{u}_{a,\text{per}}, \boldsymbol{u}_{b,\text{per}}$) to reach the optimal periodic trajectory. At the time around 100s, while the states are still far off the optimal periodic trajectory, the average performance gets already quite close to the optimal periodic performance.



a) Closed-loop state trajectories.



b) Closed-loop control input and average performance.

Figure 7.14: NMPC simulations with $T_t = 150$s for the four-tank system with initial value as in [77]. After an initial correction phase where the controller deviates strongly from the optimal periodic controls, the optimal periodic trajectories are reached after around 200s.

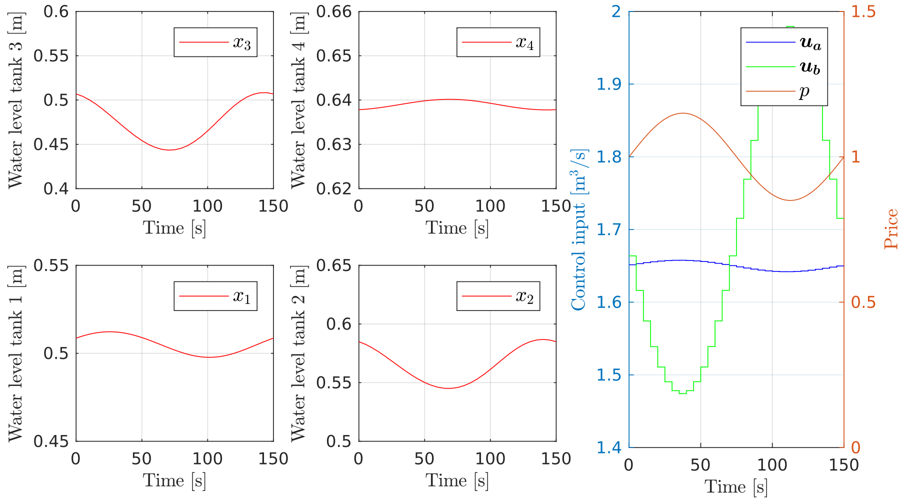


Figure 7.15: Optimal periodic operation of the four-tank system. The control profiles on the right plot show that the pump b, whose use is penalized by the time-periodic price p (see (7.15)), is mainly used in the second half of the period because in this phase the price is lower.

Chapter 8

The Powerkite Example

Converting wind-energy into electricity by using flying kites is an idea motivated first in LOYD [80]. The example we consider in this chapter is a single kite connected via a tether to an electricity generator at a ground platform. By pulling on the tether, the kite drives the generator and kinetic energy is transformed into electricity. The generator can also be used as a motor to pull the kite back.

Such a system has a number of advantages over traditional wind-turbines. Since the amount of energy that can be extracted out of the wind varies with the cube of the wind speed, a high flying kite can exploit the stronger winds speeds in higher altitudes. Because the relative wind speed on a wind-turbine blade is the highest at its fast moving wing-tips, the tips are the place where most of the energy is harvested of the wind (see Canale et al. [23, Figure 5]). A kite-wing can be flown in a way that it simulates the tips of a wind turbine but with a much higher surface. Additionally, such a system does not require a huge mast that needs to be strong enough to withstand the wind-forces generated by the blades and to support a heavy generator, nor does it require the same logistical effort for transportation of the necessary components which becomes more and more of a limiting factor for conventional wind-turbines (cf. Cotrell et al. [32], Jose et al. [66]).

However, it turns out (see Diehl [35], Hou and Diehl [60]) that the energy-optimal orbits of such system are not open-loop stable, which means that small disturbances can lead to entirely different trajectories. These instability problems need to be addressed by suitable feedback controllers.

In the following, we describe the dynamical model of the system and apply the NMPC controller based on the subproblems with fixed period $\mathrm{NP}^{\mathrm{fix}}_{T_{\mathrm{p}}}(t_i, x_i)$ and the NMPC controller based on the subproblems with free period $\mathrm{NP}^{\mathrm{free}}_{p}(t_i, x_i)$ to the system for different wind-speed scenarios and compare the results. We compare the results with the optimal periodic performance.

8.1 Model Description and Performance Criterion

We consider the system of a power generating flying kite as described in [61] and [64]. It consists of a flying wing connected to a generator/motor on a ground platform via a tether (see Figure 8.1). The wind blows in the e_x direction, where e_x is a unit vector. The unit vector e_z points to the sky and together with $e_y := e_z \times e_x$ the three vectors e_x, e_y, e_z form an orthonormal right-handed basis of the Euclidean space. For the purpose of describing the position of the kite wing relative to the ground platform, we use the convenient spherical coordinates $q = (r, \Phi, \theta)$. The position of the kite then can be written as $P = re_r$. A local

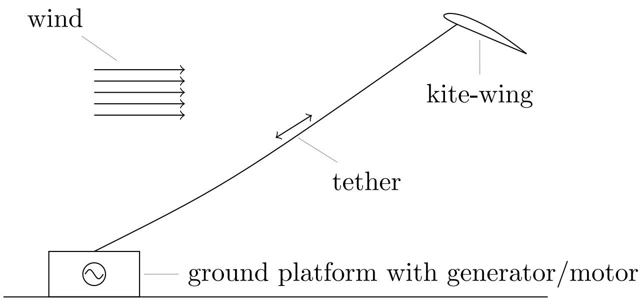


Figure 8.1: Illustration of the powerkite system.

orthonormal basis in spherical coordinates can be defined as follows

$$[e_r, e_\Phi, e_\theta] := \begin{pmatrix} \sin(\theta)\cos(\Phi) & \sin(\theta)\sin(\Phi) & \cos(\theta) \\ -\sin(\Phi) & \cos(\Phi) & 0 \\ -\cos(\theta)\cos(\Phi) & -\cos(\theta)\sin(\Phi) & \sin(\theta) \end{pmatrix}^T [e_x, e_y, e_z] \tag{8.1}$$

The wind at the ground level h_0 is assumed to blow in e_x direction with velocity w_0. We assume that the wind velocity w is height-dependent according the wind shear model [83, Chapter 2]

$$w(h) = \frac{\ln(h) - \ln(h_r)}{\ln(h_0) - \ln(h_r)} w_0, \tag{8.2}$$

where h_r is the roughness length. The effective wind vector (which is the difference of the wind vector and the kite velocity vector) can be written as

$$w_e = w e_x - \dot{P} = w e_x - \dot{r} e_r - r\sin(\theta)\dot{\Phi} e_\Phi + r\dot{\theta} e_\theta \tag{8.3}$$

By a suitable yaw control the transversal axis e_t of the kite wing (the unit vector pointing from the left to the right wingtip) is always perpendicular to the effective wind vector w_e. The roll angle Ψ (whose time-derivative $\dot{\Psi}$ acts as control variable) is the angle between the unit vector e_t going from the left to the right wingtip and the unit vector e_r:

$$\sin(\Psi) = e_t \cdot e_r. \tag{8.4}$$

The Equations of Motion

The Lagrangian formalism can be used (see e.g. [61]) to derive the equations of motion for spherical coordinates $q = (r, \theta, \Phi)^T$:

$$\ddot{q} = S^{-1} F / m + a_{\text{pseudo}}, \tag{8.5}$$

where S is a diagonal scaling matrix, a_{pseudo} a pseudo-acceleration, m the effective inertial mass:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r\sin(\theta) & 0 \\ 0 & 0 & -r \end{pmatrix}, \tag{8.6}$$

$$a_{\text{pseudo}} = \begin{pmatrix} -\ddot{r} + r(\dot{\theta}^2 + \sin(\theta)^2\dot{\Phi}^2) - \frac{\dot{m}}{m}\dot{r}, \\ -2(\cot(\theta)\dot{\Phi}\dot{\theta} + \frac{\dot{r}}{r}\dot{\Phi}) - \frac{\dot{m}}{m}\dot{\Phi} \\ \cos(\theta)\sin(\theta)\dot{\Phi}^2 - 2\frac{\dot{r}}{r}\dot{\theta} - \frac{\dot{m}}{m}\dot{\theta} \end{pmatrix}, \tag{8.7}$$

$$m = m_k + \underbrace{\rho_c \pi \left(\frac{d_c}{2}\right)^2 r}_{\text{Cable Mass}} /3. \tag{8.8}$$

The total force F acting on the kite is the sum of aerodynamical lift and drag forces

$$F_{\text{aer}} = F_{\text{lift}} + F_{\text{drag}} = \frac{1}{2}\rho A C_L \|w_e\|^2 e_n + \frac{1}{2}\rho A(c_{D,0} + K C_L^2)\|w_e\| w_e \tag{8.9}$$

with $e_n := \frac{w_e}{\|w_e\|} \times e_t$ being the direction of the aerodynamical lift, the gravitational force

$$F_g = (V\rho - m)g \begin{pmatrix} \cos(\theta) \\ 0 \\ \sin(\theta) \end{pmatrix}, \tag{8.10}$$

and the friction of the cable

$$F_{\text{f}} = \frac{c_{D,C}\rho d_c r}{8} \|w_e\| w_e. \tag{8.11}$$

To prevent the cable from coiling up, the winding number η is augmented to the states and its time derivative is given by

$$\dot{\eta} = \frac{\dot{\Phi}\ddot{\theta} - \dot{\theta}\ddot{\Phi}}{2\pi(\dot{\Phi}^2 + \dot{\theta}^2)}. \tag{8.12}$$

Altogether the system has 9 differential state variables

$$x := (r, \Phi, \theta, \dot{r}, \dot{\Phi}, \dot{\theta}, \Psi, C_L, \eta)^T \tag{8.13}$$

and 3 control variables

$$u := (\ddot{r}, \dot{\Psi}, \dot{C}_L)^T. \tag{8.14}$$

Summarizing the equations of motion and the differential equation for the winding number, the dynamics of the system can be described by a differential equation

$$\dot{x} = f_{w_0}(x, u), \tag{8.15}$$

with right-hand side $f_{w_0} : \mathbb{R}^9 \times \mathbb{R}^3 \to \mathbb{R}^9$. Here the index w_0 indicates the dependency of the equations with respect to the wind speed w_0. The parameters used in the description of the model-equations can be found in table 8.1. Furthermore, the system is subject to simple bounds on state and control variables which are listed in Table 8.2.

Parameter	Value	Description
m_k	850kg	mass of the kite
A	500m^2	effective wing area
V	720m^3	volume
$c_{D,0}$	0.04	aerodynamic drag coefficient
K	0.04	induced drag constant
g	9.81m/s^2	gravitational constant
ρ	1.23kg/m^3	density of the air
ρ_c	1450kg/m^3	density of the cable
$c_{D,C}$	1	frictional constant
d_c	0.05614m	diameter of the cable
w_0	10m/s	wind velocity at ground level
h_0	100m	ground level
h_r	0.1m	roughness length

Table 8.1: Parameters of the powerkite model.

State	Description	Lower bound	Upper bound
r	Radial coordinate	1250m	1550m
Φ	Azimuthal angle	−0.34rad	0.34rad
Θ	Polar angle	0.85rad	1.45rad
$\dot{r}$	Rope velocity	−40m/s	15m/s
$\dot{\Phi}$	Azimuthal angle change	−0.2rad/s	0.2rad/s
$\dot{\Theta}$	Polar angle change	−0.2rad/s	0.2rad/s
Ψ	Roll angle	−0.29rad	0.29rad
C_L	Lift coefficient	0.3	1.5
η	Winding number	−1	1
Control	Description	Lower bound	Upper bound
$\ddot{r}$	Rope acceleration	−50m/s^2	50m/s^2
$\dot{\Psi}$	Roll angle change	−0.065rad/s	0.065rad/s
$\dot{C}_L$	Lift coefficient change	−3.5/s	3.5/s

Table 8.2: State and control bounds of the powerkite model.

For a more detailed description and derivation of the right-hand side we refer the reader to [61]. An implementation of the right-hand side for this dynamical system can also be found as part of the Automatic Control and Dynamic Optimization environment `ACADO` [62].

Performance Criterion

The performance criterion ℓ we consider is the power output generated at the generator of the ground platform. Note that the generator may also act as a motor to pull the kite back. The objective is to generate as much average energy-output as possible. Therefore, since we formulate OCPs as minimization problems we use the convention that ℓ is negative when power is generated and positive when power is used.

The power output at the generator depends on the effective inertial mass of the kite and cable system and of the radial acceleration and velocity (power equals force times velocity):

$$\ell(x,u) = -m\ddot{r}\dot{r} \cdot S_\ell. \tag{8.16}$$

We use the scaling factor $S_\ell := 10^{-6}$ so that ℓ measures the power in [MW].

8.2 Wind Speed Scenarios

In our NMPC simulations we consider two different wind speed development scenarios, which we describe in the following.

8.2.1 Constant Wind Speed

As a first simulation scenario we consider the case where the wind speed is fluctuating around 10m/s with a normally distributed noise with standard deviation 0.1m/s changing every 5s. The wind speed development is depicted in Figure 8.2a.

8.2.2 Changing Wind Speed

As a second case we consider a simulation scenario where in the first 120s the wind speed is increasing form 10m/s to 16m/s, staying at 16m/s for 240s, decreasing again to 10m/s for 120s and then staying at 10m/s until the end of the simulation. As in the constant wind speed scenario, a noise which is normally distributed with standard deviation 0.1m/s changing every 5s is added. The wind speed development is depicted in Figure 8.4a.

8.3 Controller Setup

We carry out NMPC simulations with the controller with fixed period as well as with the controller for free period. For the free period controller we define the lower and upper bounds on the period as in problem $\mathrm{Per}_{w_0}^{\mathrm{free}}$, namely $\underline{T}_p = 6$s and $\overline{T}_p = 30$s. To showcase the different behavior of fixed vs. free period NMPC we choose a period $T_\mathrm{p} = 11.28$s for the fixed period NMPC controller, which corresponds to the optimal period for wind speed 10m/s.

All scenarios are tested with two different transient horizon lengths $T_\mathrm{t} = T_\mathrm{p}$ and $T_\mathrm{t} = \frac{1}{2}T_\mathrm{p}$ for the fixed period controller respectively $c_t = 1$ and $c_t = \frac{1}{2}$ for the free period controller. The shooting intervals are chosen such that one period is divided in $N_p = 20$ shooting intervals, which translates to a discretization of the complete prediction horizon into altogether 30 shooting intervals ($T_\mathrm{t} = \frac{1}{2}T_\mathrm{p}$ or $c_t = \frac{1}{2}$) respectively 40 shooting intervals ($T_\mathrm{t} = T_\mathrm{p}$ or $c_t = 1$).

The discount factor is set to $\rho = 1.01$. As transient weight we use $w_{\text{trans}} = 1$ and the weighted norm for the control self-tracking term is defined as

$$\|u\|_{\mathbb{U}} := 0.1 \cdot |\ddot{r}| + 10 \cdot |\dot{\Psi}| + |\dot{C}_L|. \tag{8.17}$$

The path constraint in the transient part of the horizon is relaxed to $0 \le c(x,u) + \varepsilon \mathbb{1}_{n_c}$ with $\varepsilon = 10^{-6}$ (see constraint (6.16d) in $\text{NLP}^{\text{fix}}_{T_\text{p},w}(t_i, x_i)$ or (6.32d) in $\text{NLP}^{\text{free}}_{w}(x_i)$). For the fixed period controller, the sampling intervals coincide with the shooting intervals (they are of the uniform length $T_\text{p}/N_p = 11.28\text{s}/20 = 0.564\text{s}$) and for the free period controller the sampling times are adaptively chosen as described in Section 5.1.4 and 6.1.2, i.e., between $\underline{T}_p/N_p = 0.3\text{s}$ and $\overline{T}_p/N_p = 1.5\text{s}$.

8.4 Results of NMPC Simulations

We discuss the obtained simulation results for the different controllers and wind speed scenarios. Each of the NMPC simulations is run for a duration of 1500 samples. As initial value x_0 for all simulations we choose a value on an optimal periodic orbit for wind speed 10m/s with period $T_\text{p} = 11.28\text{s}$. To facilitate the comparison of the fixed and the free period controller, we always present the results of both controllers for the same wind speed scenario together.

Notes on the Comparison of Results

In the following, we briefly explain how we compare the results for the two different controllers.

As a measure of the current average performance, we consider the moving average performance as we did in the previous chapter. However, since the sampling intervals and the employed periods are not pre-determined to a fixed value in the case of the free period controller, the definition of the moving average of the produced energy over one period is not as straightforward as in the case of the fixed period controller cf. (7.5). It does not make sense to consider a moving average energy output of the form $t \mapsto 1/T \int_t^{t+T} \ell(\boldsymbol{x}_\mu(\tau), \boldsymbol{u}_\mu(\tau))\mathrm{d}\tau$ with fixed T in this case.

Instead, for each sampling time t_i, we consider the average performance over the next $N_p = 20$ consecutive sampling time intervals:

$$AO(t_i, \boldsymbol{x}_\mu, \boldsymbol{u}_\mu) := \frac{1}{t_{i+N_p} - t_i} \int_{t_i}^{t_{i+N_p}} \ell(\boldsymbol{x}_\mu(\tau), \boldsymbol{u}_\mu(\tau))\mathrm{d}\tau. \tag{8.18}$$

This definition of the moving average output is in accordance with the definition (7.5) we used for closed-loop trajectories resulting from the fixed period controller as the two definitions coincide for the case of a fixed period.

The development of AO is included in Figures 8.2a, 8.3a, 8.4a and 8.5a as a measure of the average energy output at a given sampling time.

In Tables 8.3, 8.4 and 8.5 we compare the performance results for the different NMPC schemes. Each table considers a fixed time interval and compares the results of the fixed period controller with the results of the free period controller with transient horizon length of half a period respectively transient horizon length of a full period.
The considered time intervals are always rounded to the nearest elements of the sampling time grid of the underlying NMPC scheme. This is the reason why the considered length of the intervals differs slightly for the fixed period and the free period controller. We state the actual considered interval length in column 4 and the total produced energy during this time in column 3. The resulting average performance is stated in column 5. Column 6 contains the weighted mean value of the moving average output AO, which for an interval $[t_{n_s}, t_{n_e}]$ of sampling times is defined as

$$\text{Mean}(AO) = \frac{\sum_{i=n_s}^{n_e-1} AO(t_i, \boldsymbol{x}_\mu, \boldsymbol{u}_\mu)(t_{i+1} - t_i)}{t_{n_e} - t_{n_s}}. \tag{8.19}$$

We use this value because the simple average performance on an interval of fixed length is not necessarily a fair comparison measure since the fixed interval not always includes an integer number of periods.

For comparison purposes we use the optimal periodic average performance for different wind speeds with fixed and free period, which we calculate at the end of this chapter in Section 8.5.

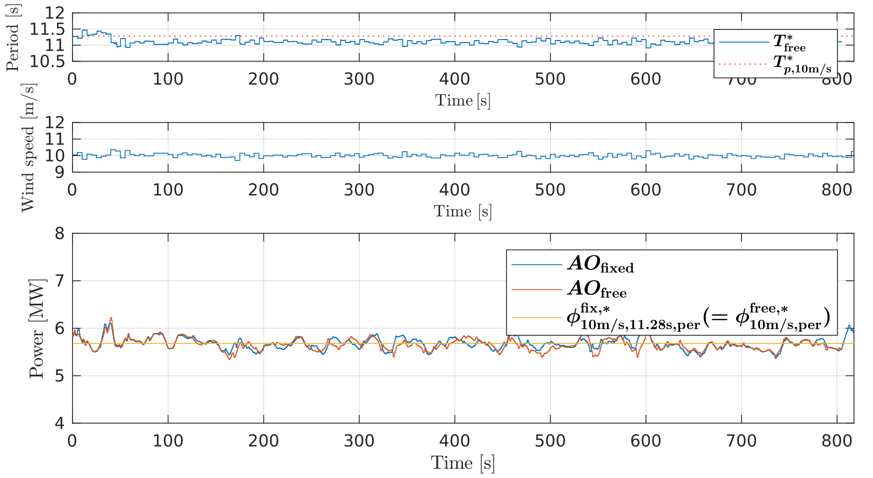
8.4.1 Results for the Constant Wind Speed Scenario

In the constant wind speed scenario, the moving average output of both the fixed period and free period controllers oscillates around the optimal values $\phi^{\text{fix},*}_{10\text{m/s},11.28\text{s},\text{per}}$, see Figure 8.2a (transient horizon length $T_\text{t} = 0.5T_\text{p}$ and $c_t = 0.5$) and Figure 8.3a (transient horizon length $T_\text{t} = T_\text{p}$ and $c_t = 1$). The employed period of the free period controller oscillates slightly below the optimal period for wind speed 10m/s. Table 8.3 shows that both controllers economically perform equally well up to a marginal difference (around 0.2% difference of the total energy produced). This can be explained due to the fact that free period controller most of the time uses a period that is slightly below the optimal period which seems to be an effect of the continuously oscillating wind speeds. The table also shows that the controllers with longer time horizon only very slightly improve the economic performance.

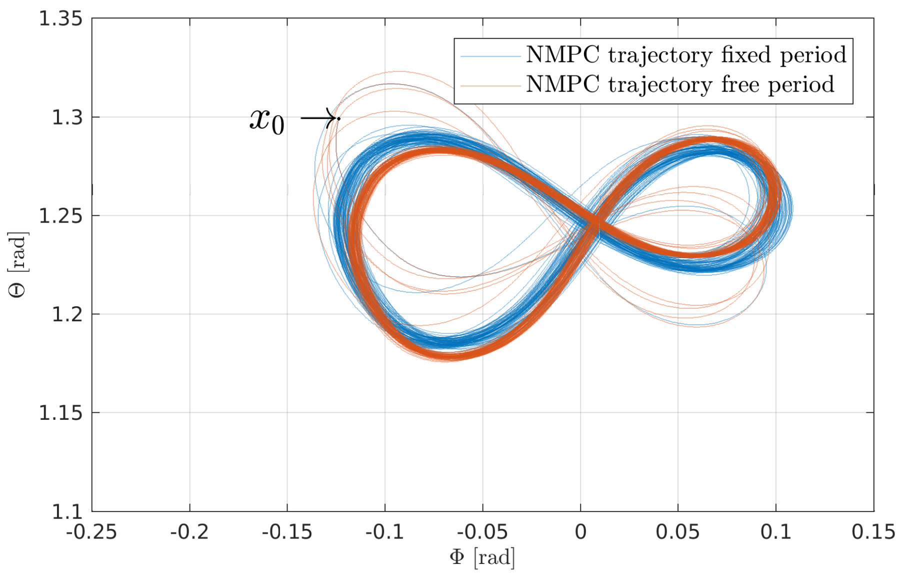
Summarizing the results for the constant wind speed scenario, we reach the conclusion that both the fixed period and free period controller perform equally well and very close to the economically optimal periodic performance if the fixed period is appropriately chosen.

8.4.2 Results for the Changing Wind Speed Scenario

In the changing wind speed scenario, the moving average output of the fixed period controller oscillates around the level of $\phi^{\text{fix},*}_{11.28\text{s},16\text{m/s},\text{per}}$ during the strong wind phase and around the level of $\phi^{\text{fix},*}_{11.28\text{s},10\text{m/s},\text{per}}$ during the phase with weaker wind, see Figure 8.4a (transient horizon length $T_\text{t} = 0.5T_\text{p}$ and $c_t = 0.5$) and Figure 8.5a (transient horizon length $T_\text{t} = T_\text{p}$ and

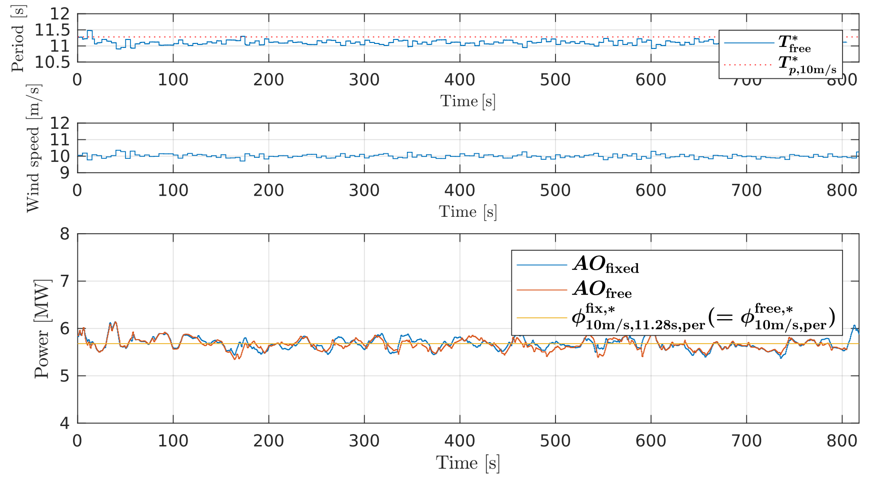


a) Employed period of free period controller, wind speed scenario and resulting average performance for both controllers. The employed period for the free period controller oscillates slightly below the optimal period length. The moving average output is oscillating around the optimal periodic average output for wind speed 10m/s. Both the free and fixed period controller perform roughly equally well.

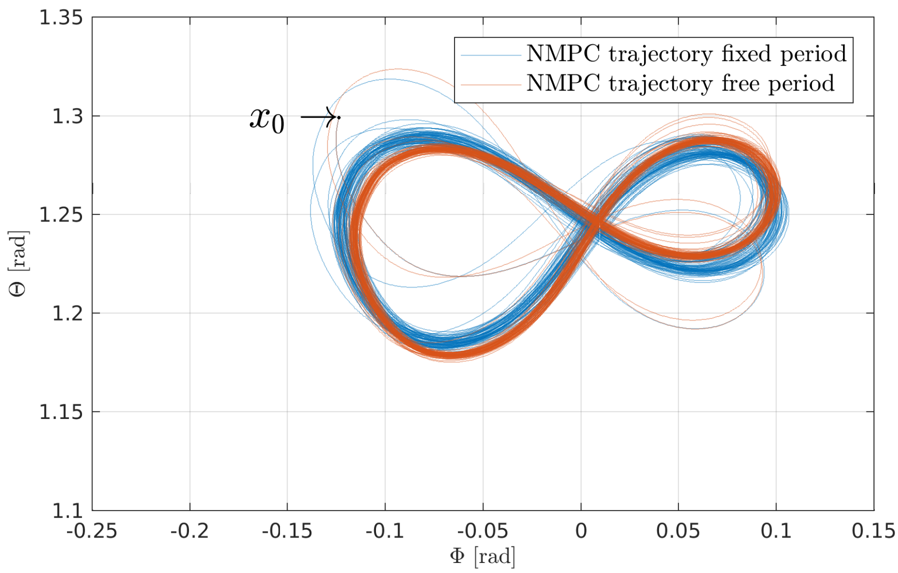


b) Phase portrait for powerkite NMPC simulation. The closed-loop trajectories from both controllers converge towards the form of a horizontal figure 8.

Figure 8.2: NMPC simulation results for constant wind speed scenario with transient horizon length $T_{\text{t}} = \frac{1}{2} T_{\text{p}}$ respectively $c_t = \frac{1}{2}$. Both the fixed and free period controller perform very similar.



a) Employed period of free period controller, wind speed scenario and resulting average performance for both controllers. Very similar behavior as in the case of the shorter transient horizon can be observed (cf. Figure 8.2a).



b) Phase portrait for powerkite NMPC simulation. The closed-loop trajectories from both controllers converge towards the form of a horizontal figure 8.

Figure 8.3: NMPC simulation results for constant wind speed scenario with transient horizon length $T_\mathrm{t} = T_\mathrm{p}$ respectively $c_t = 1$. The results are very similar to the results of the controller with shorter transient horizon (Figure 8.2).

transient horizon length	controller	total energy [MJ]	considered time [s]	average performance [MW]	Mean(AO) [MW]
$T_\mathrm{t} = 0.5T_\mathrm{p}$	fixed	4535.68	800.17	5.67	5.68
$c_t = 0.5$	free	4526.29	800.47	5.65	5.66
$T_\mathrm{t} = T_\mathrm{p}$	fixed	4539.61	800.17	5.67	5.68
$c_t = 1$	free	4528.43	800.35	5.66	5.67

Table 8.3: NMPC simulation results for the constant wind speed scenario for the time interval $[0s, 800s]$. Both controllers show very similar performance.

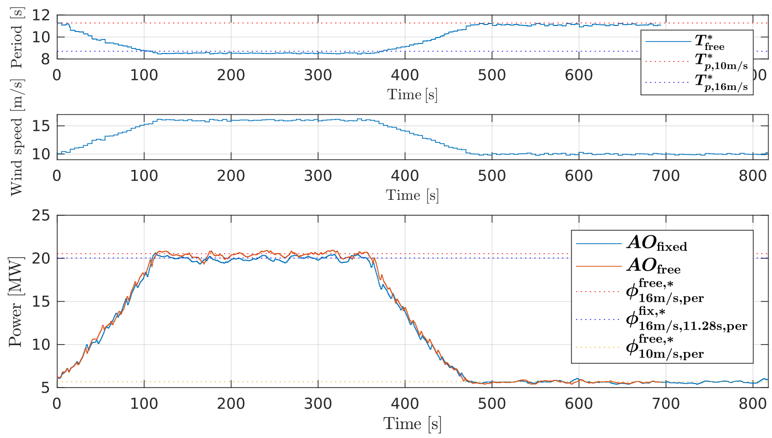
transient horizon length	controller	total energy [MJ]	considered time [s]	average performance [MW]	Mean(AO) [MW]
$T_\mathrm{t} = 0.5T_\mathrm{p}$	fixed	4818.22	239.82	20.09	19.99
$c_t = 0.5$	free	4916.47	239.99	20.49	20.46
$T_\mathrm{t} = T_\mathrm{p}$	fixed	4832.13	239.82	20.15	20.05
$c_t = 1$	free	4917.94	240.06	20.49	20.46

Table 8.4: NMPC simulation results for the varying wind speed scenario for the time interval $[120s, 360s]$ with strong wind ($w_0 = 16\mathrm{m/s}$). The free period controller outperforms the fixed period controller because it always employs the optimal period length.

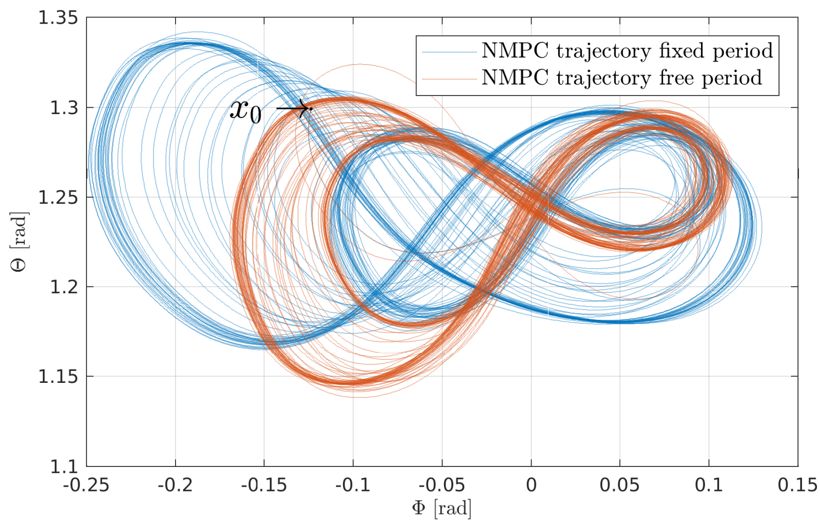
$c_t = 1$). Similarly, the moving average output of the free period controller oscillates around the level of $\phi^{\mathrm{free},*}_{16\mathrm{m/s,per}}$ during the strong wind phase and around the level of $\phi^{\mathrm{free},*}_{10\mathrm{m/s,per}}$ during the phase with weaker wind. The employed period of the free period controller is constantly adapting to the optimal period corresponding to the current wind speed.

Table 8.4 shows a detailed comparison of the economic performance during the phase of strong wind. The free period controller performs significantly better during this phase (around 1.7% for transient horizon length $T_\mathrm{t} = T_\mathrm{p}$ and $c_t = 1$ and around 2% for transient horizon length $T_\mathrm{t} = 0.5T_\mathrm{p}$ and $c_t = 0.5$). This effect can also be observed over the whole considered time interval, see Table 8.5. The tables also show that the controllers with longer time horizon only very slightly improve the economic performance.

Summarizing the results from the changing wind speed scenario, we reach the conclusion that the free period controller can outperform the fixed period controller because it always employs the optimal period which corresponds to the current operating conditions (in this case the changing wind speed parameter) of the system.

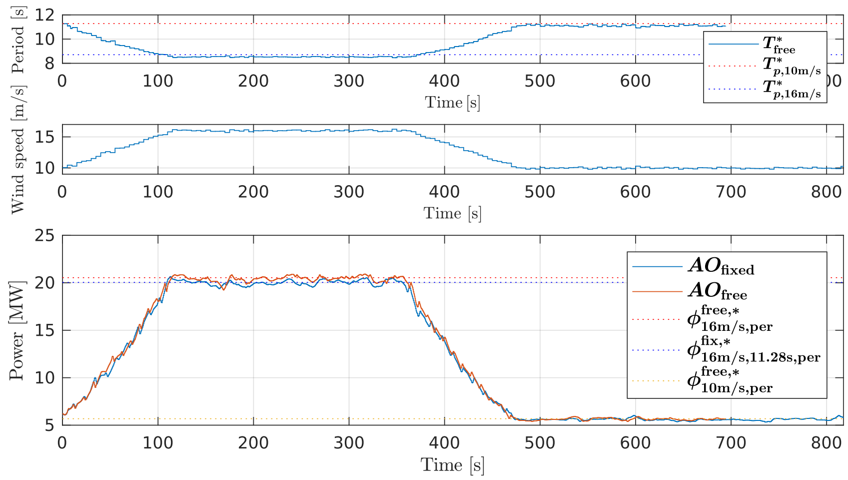


a) Employed period of free period controller, wind speed scenario and resulting average performance for both controllers. The period is adapted according to the current wind speed and converges towards the optimal period for the phase of strong wind. The moving average output for both controllers oscillates around the values of the optimal periodic operation. During the strong wind phase, the free period controller significantly outperforms the fixed period controller because it can adapt the period accordingly while the fixed period controller is using a suboptimal period.

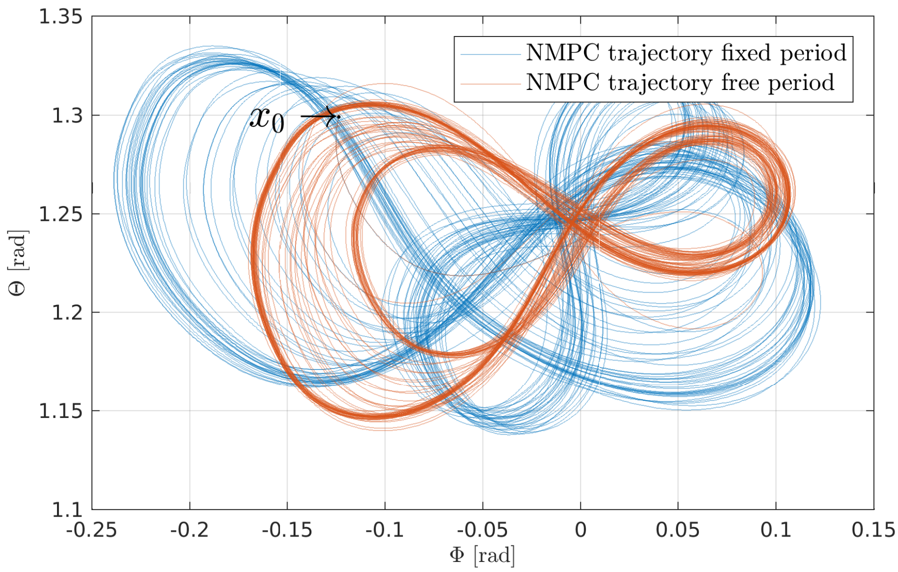


b) Phase portrait for powerkite NMPC simulation for both controllers. In both closed-loop trajectories, two different horizontal figure 8 can be observed. These correspond to the time intervals when the wind speed remains for a while at 16m/s respectively 10m/s. While the figure 8 are quite similar during the phase of 10m/s wind speed, they differ noticeably during the strong wind phase.

Figure 8.4: NMPC simulation results for changing wind speed scenario with transient horizon length $T_\mathrm{t} = \frac{1}{2} T_\mathrm{p}$ respectively $c_t = \frac{1}{2}$.



a) Employed period of free period controller, wind speed scenario and resulting average performance for both controllers. Compared to the results for shorter transient horizon in Figure 8.4a no substantial differences can be observed.



b) Phase portrait for powerkite NMPC simulation for both controllers. In both closed-loop trajectories, two different horizontal figure 8 can be observed. These correspond to the time intervals when the wind speed remains for a while at 16m/s respectively 10m/s.

Figure 8.5: NMPC simulation results for changing wind speed scenario with transient horizon length $T_t = T_p$ respectively $c_t = 1$. Compared to the results for transient horizon length $T_t = 0.5T_p$ respectively $c_t = 0.5$ (cf. Figure 8.4), similar performance can be observed.

transient horizon length	controller	total energy [MJ]	considered time [s]	average performance [MW]	Mean(AO) [MW]
$T_t = 0.5T_p$	fixed	8621.97	650.07	13.26	13.28
$c_t = 0.5$	free	8788.61	650.26	13.52	13.54
$T_t = T_p$	fixed	8637.58	650.07	13.29	13.30
$c_t = 1$	free	8789.18	650.48	13.51	13.53

Table 8.5: NMPC simulation results for the varying wind speed scenario for the time interval $[0s, 650s]$. The free period controller outperforms the fixed period controller because it always employs the optimal period length.

8.5 Optimal Periodic Operation

To get reference values for the optimal average performance, we calculate the optimal periodic orbits for the described system for different wind speeds w_0 reaching from 10m/s to 16m/s.

Free Period

To determine the optimal periodic operation for a given wind speed w_0, we consider the periodic OCP $\text{Per}_{w_0}^{\text{free}}$ (5.3) with the period bounded between $\underline{T}_p = 6\text{s}$ and $\overline{T}_p = 30\text{s}$.

Problem $\text{Per}_{w_0}^{\text{free}}$ is discretized using a Direct Multiple Shooting discretization with a uniform shooting grid consisting of 20 shooting intervals and piecewise constant controls. To get a unique solution of this problem we have to include an initial value constraint for the winding number $0 = \eta(0)$ (note that the objective value of the solution is not affected by this additional constraint). The results are summarized in Table 8.6. We note that in all of the solutions we obtained the bound $\underline{T} \le T \le \overline{T}$ (5.3e) is inactive. We denote the optimal period for wind speed w by $T^*_{p,w}$.

Fixed Period

As we did in the free period case, we determine the optimal periodic operation for the different wind speeds w_0 between 10m/s and 16m/s with the period fixed to $T_p = 11.28\text{s}$, which corresponds to the optimal period for the wind speed $w_0 = 10\text{m/s}$.

This is done by solving the periodic OCPs $\text{Per}^{\text{fix}}_{T_p,w_0}$ which is equivalent to $\text{Per}_{w_0}^{\text{free}}$, where the inequality constraint on the period (5.3e) is replaced by the appropriate equality constraint.

The results are summarized together with the results for the free period case in Table 8.6.

	Fixed Period		Free Period		
w_0 [m/s]	Average Output [MW]	Period [s]	Average Output [MW]	Period [s]	Improvement [%]
10	5.68	11.28	5.68	11.28	0.00
11	7.35	11.28	7.37	10.55	0.33
12	9.27	11.28	9.35	10.14	0.83
13	11.47	11.28	11.63	9.77	1.33
14	13.99	11.28	14.24	9.43	1.78
15	16.83	11.28	17.20	8.98	2.18
16	20.04	11.28	20.54	8.71	2.51

Table 8.6: Optimal periodic average energy output for different wind speeds. Column 2 and 3 show the results of $\mathrm{Per}^{\mathrm{fix}}_{T_\mathrm{p},w_0}$ where the period is fixed to $T_\mathrm{p} = 11.28\mathrm{s}$ (which is the optimal period for a wind speed of 10m/s). Column 4 and 5 show the results corresponding to the problems $\mathrm{Per}^{\mathrm{free}}_{w_0}$, where the period is included as an optimization variable. In general the average energy output increases strongly with the wind speed. Additionally, by comparing the optimal average outputs with fixed and with free period (columns 2 and 4) it can be observed that there is further potential for improvement by optimizing the period. The optimal period decreases for increasing wind speed. The relative improvement in percent can be seen the last column.

Conclusion and Outlook

In this thesis we have developed a novel approach for the design of economic NMPC schemes for AOCPs. The approach is based on the observation that periodic solutions exhibit excellent approximation properties to infinite horizon AOCPs. Compared to classical tracking NMPC or E-NMPC with terminal constraints, our approach only requires a minimal amount of a priori knowledge of the optimal economic behavior of the system, namely a period.

We have developed a stability theory for our method, which, complementary to the usual approaches in stability analysis for E-NMPC, does not rely on dissipativity conditions of the system (which are often hard to verify) but rather on assumptions on controllability and on the continuous dependence of the NMPC subproblem solutions with respect to the initial value. As a result of our closed-loop analysis we have shown that the proposed NMPC scheme achieves an economic performance which is on par with the optimal periodic economic performance.

Furthermore, we have developed two extensions of the presented NMPC scheme. The first one treats the period as an additional optimization variable and is suited for systems with changing parameters. The second is an adaption for systems with periodic performance criteria.

All the presented NMPC schemes are implemented within the `Matlab` NMPC framework `MLI` and tested in a number of challenging application examples. The theoretical properties of the closed-loop analysis have been confirmed in our NMPC simulations and the average economic performance of the controllers reached the same level as the average performance of the optimal periodic trajectories.

Real-Time Feasible Efficient Numerical Implementation

As part of the case studies in this work, we have solved the NMPC subproblems using an interior-point method. We have chosen this approach because of its robustness. However, for an efficient, real-time feasible implementation other solution algorithms have to be considered. For example, an SQP method can be used for the iterative solution of the occurring NLPs. For the transformation of the high-dimensional but due to the Direct Multiple Shooting discretization sparse QPs into smaller but dense QPs, structure-exploiting condensing techniques can be used. As the objective function and the periodicity constraint are both coupled, the standard condensing techniques have to be enhanced to take this additional structure into account. This can be done by first transforming the OCP into a standard (non-coupled) OCP as described in Section 1.4.2 and then apply standard techniques.

Furthermore, additional speedup of the calculations can be achieved by using Multi-Level Iteration (MLI) schemes as presented in [114] and [43]. Based on contraction-criteria for the SQP-iterations this method can yield considerable computational savings by adaptively updating the arising matrices and vectors.

Application to Differential Algebraic Equations (DAEs), Partial Differential Equations (PDEs) and Switched Systems

As we have pointed out in Chapter 3 the good approximation properties of periodic solutions do only partially rely on the fact that the system is described by a system of ODEs. Extensions of the theory in the direction of systems described by DAEs, PDEs or even switched systems (systems with discontinuities in the right hand side) with state-discontinuities could be considered [21].

Examples for systems that can be modeled using switches and that exhibit intrinsic periodic behavior include Simulated Moving Bed [111, 94] and Multicolumn Countercurrent Solvent Gradient Purification [8] processes.

NMPC with Adaptive Horizon Lengths

The NMPC controller we have proposed in Section 5.1 works with a time horizon of varying length since the period is included as a free optimization variable in the OCP. In our numerical case-studies we observed that, especially if the controlled process is close to the optimal periodic orbit, a controller with relatively short transient horizon can perform equally well as a controller with a longer transient horizon.

As a shorter time horizon reduces the computational complexity of the NMPC subproblems (and thus the required time for their numerical solution), a controller that reduces the length of the transient horizon in case the process is close to the optimal periodic promises further computational speedup.

Appendix A

Comparison Functions and $\mathcal{KL}$-Bound for Lyapunov-Like Functions

This appendix collects some elementary results concerning comparison functions which are important in the stability considerations in this thesis. The following elementary result allows that in many cases $\mathcal{K}$ functions can additionally be assumed to be smooth.

Lemma A.1 ($\mathcal{K}$ and $\mathcal{K}_\infty$ Functions Have Smooth Lower Bounds)
Every $\mathcal{K}$-function is bounded from below by a smooth $\mathcal{K}$ function and every $\mathcal{K}_\infty$-function is bounded from below by a smooth $\mathcal{K}_\infty$ function.

Proof The $\mathcal{K}_\infty$ part is part of Clarke et al. [29, Lemma 2.5]. However, since there only a sketch of the proof is provided, we include a different direct proof using convolutions. Let $\alpha \in \mathcal{K}$. Let $\gamma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a smooth function with support in $[0,1]$, positive on $(0,1)$ and strictly bounded from above by $+1$. An appropriately chosen bump-function can be used for this purpose. The convolution $\alpha * \gamma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is defined as

$$\alpha * \gamma(t) := \int_{\mathbb{R}} \alpha(\tau)\gamma(t-\tau)\mathrm{d}\tau. \tag{A.1}$$

Note that here we interpret α as a function on $\mathbb{R}$ by setting $\alpha(s) = 0$ for all negative s. The function $\alpha * \gamma$ is smooth, because the convolution of a continuous function with a smooth function is automatically also smooth. It is a lower bound for α because

$$\alpha * \gamma(t) = \int_{\mathbb{R}} \alpha(\tau)\gamma(t-\tau)\mathrm{d}\tau = \int_{t-1}^{t} \underbrace{\alpha(\tau)}_{<\alpha(t)} \underbrace{\gamma(t-\tau)}_{\leq 1} \mathrm{d}\tau \leq \alpha(t). \tag{A.2}$$

It is strictly increasing on $\mathbb{R}_{\geq 0}$, as the following calculation for $0 \leq t_1 < t_2$ shows.

$$\begin{aligned} \alpha * \gamma(t_1) = \gamma * \alpha(t_1) &= \int_{\mathbb{R}} \gamma(\tau)\alpha(t_1-\tau)\mathrm{d}\tau = \int_0^1 \gamma(\tau)\alpha(t_1-\tau)\mathrm{d}\tau \\ &< \int_0^1 \gamma(\tau)\alpha(t_2-\tau)\mathrm{d}\tau = \int_{\mathbb{R}} \gamma(\tau)\alpha(t_2-\tau)\mathrm{d}\tau = \gamma * \alpha(t_2) = \alpha * \gamma(t_2) \end{aligned} \tag{A.3}$$

This shows that $\alpha * \gamma$ is a $\mathcal{K}$ function. Furthermore, if α is a $\mathcal{K}_\infty$ function, the property $\int_0^1 \gamma(\tau)\mathrm{d}\tau > 0$ implies that $\lim_{t\to\infty} \alpha * \gamma(t) = \infty$ holds and therefore $\alpha * \gamma$ is also a $\mathcal{K}_\infty$ function. □

Lemma A.2 ($\mathcal{KL}$ Bound for LYAPUNOV-Like Functions with Continuous Descent Property)
Let $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a $\mathcal{K}$ function. Then there exists a $\mathcal{KL}$-function β such that for any $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}_{\geq 0}$, for any solution $\Phi(t; t_0, x_0)$ of the IVP

$$\dot{\boldsymbol{x}}(t) = -\alpha(\boldsymbol{x}(t)) \text{ with } \boldsymbol{x}(t_0) = x_0. \tag{A.4}$$

it holds

$$\Phi(t; t_0, x_0) \leq \beta(x_0, t - t_0). \tag{A.5}$$

Proof With Lemma A.1 let $\tilde{\alpha} \in \mathcal{K}$ be a smooth lower bound of α. It can be shown that the solution of the IVP

$$\dot{\boldsymbol{x}}(t) = -\tilde{\alpha}(\boldsymbol{x}(t)) \text{ with } \boldsymbol{x}(t_0) = x_0 \tag{A.6}$$

is unique, exists for all times (PICARD-LINDELÖF) and is an upper bound for any solution of the IVP (A.4). Then, according to Khalil [71, Lemma 4.4] there exists a $\mathcal{KL}$ function β with the desired property (A.5) for the solutions of the IVP (A.6). The Lemma follows since solutions of (A.6) are upper bounds for solutions of (A.4) with the same initial value. □

A similar statement holds in a discrete setting.

Lemma A.3 ($\mathcal{KL}$ Bound for Discrete LYAPUNOV-Like Functions)
Let α be a $\mathcal{K}_\infty$ function and let $\varphi_\alpha(t) := t - \alpha(t)$ for $t \in \mathbb{R}_{\geq 0}$. Then there exists a $\mathcal{KL}$ function β such that

$$\varphi_\alpha^k(t) \leq \beta(t, k) \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ and } k \in \mathbb{N} \tag{A.7}$$

holds (here $\varphi_\alpha^k(t)$ is an abbreviation for $\underbrace{\varphi_\alpha \circ \cdots \circ \varphi_\alpha}_{k \text{ times}}(t)$).

Proof Since $\alpha \in \mathcal{K}_\infty$, there is a smooth (in particular continuously differentiable) $\mathcal{K}_\infty$ function $\tilde{\alpha}$ that is a lower bound for α (see Lemma (A.1)). Now consider the function

$$\gamma : t \mapsto \int_0^t \frac{\dot{\tilde{\alpha}}(\tau)}{2\dot{\tilde{\alpha}}(\tau + 1)} \mathrm{d}\tau. \tag{A.8}$$

From its definition it is clear that $0 \leq \dot{\gamma}(t) \leq 0.5$ holds for all $t \in \mathbb{R}_{\geq 0}$. This implies that γ is LIPSCHITZ continuous with constant ≤ 0.5. From the definition it also follows that $\dot{\gamma}(t) \leq \dot{\tilde{\alpha}}(t)$ holds for all $t \in \mathbb{R}_{\geq 0}$ and therefore, since $\gamma(0) = \tilde{\alpha}(0) = 0$, the function γ is a lower bound for $\tilde{\alpha}$ (and for α). Furthermore γ is strictly increasing because $\dot{\gamma}(t) > 0$ for $t > 0$. This shows that γ is a $\mathcal{K}$ function. Now define the function φ_γ as follows

$$\varphi_\gamma(t) := t - \gamma(t) \text{ for } t \in \mathbb{R}_{\geq 0}. \tag{A.9}$$

The LIPSCHITZ continuity of γ with constant ≤ 0.5 implies that φ_γ is strictly increasing. Furthermore, since γ is a lower bound for α, φ_γ is an upper bound for φ_α. It follows that

$\varphi_\gamma^k(t) \geq \varphi_\alpha^k(t)$ holds for all $k \in \mathbb{N}$ and $t \in \mathbb{R}_{\geq 0}$. Now define $\tilde{\beta} : (t,k) \mapsto \varphi_\gamma^k(t)$. For any $k \in \mathbb{N}$ it holds that $\tilde{\beta}(0,k) = 0$ and $\tilde{\beta}(\cdot,k)$ is a strictly increasing function. Furthermore, for any $t \in \mathbb{R}_{\geq 0}$ the sequence $\left(\tilde{\beta}(t,k)\right)_{k\in\mathbb{N}}$ is strictly decreasing and converges to zero for $k \to \infty$ (since the sequence is strictly decreasing it converges to some limit $t^* \in \mathbb{R}$, and the continuity of φ_γ implies $t^* = \varphi_\gamma(t^*)$, which means $t^* = 0$). Now $\tilde{\beta}$ has all the properties of a $\mathcal{KL}$ function, except that it is only defined on $\mathbb{R}_{\geq 0} \times \mathbb{N}$ instead of $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. But for each $t \in \mathbb{R}_{\geq 0}$, the sequence $(\tilde{\beta}(t,k))_{k\in\mathbb{N}}$ can be extended to an $\mathcal{L}$ function $\beta(t,\cdot)$ by linear interpolation. The resulting function β is a $\mathcal{KL}$ function with all the desired properties. □

Lemma A.4 (Bounds for Functions that Vanish on the Same Sets)
Let $K \subset \mathbb{R}^{n_x}$ be compact and let $f, g : K \to \mathbb{R}_{\geq 0}$ be two continuous functions with the property

$$f(x) = 0 \Rightarrow g(x) = 0 \tag{A.10}$$

and let the set $f^{-1}(0)$ be nonempty. Then there exist $\mathcal{K}_\infty$-functions α_1 and α_2 such that

$$g(x) \leq \alpha_1(f(x)) \tag{A.11}$$

and

$$\alpha_2(g(x)) \leq f(x) \tag{A.12}$$

holds for all $x \in K$.

Proof Consider the function $d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$d(s) := \max_{x\in f^{-1}([0,s])} g(x). \tag{A.13}$$

For any $s \geq 0$ the set $f^{-1}([0,s])$ is a nonempty and compact set since g is continuous and $f^{-1}([0,s])$ is a closed subset of the compact set K. Therefore the function d is well defined. By definition, d is non-decreasing and for any $x \in K$ it holds that

$$g(x) \leq d(f(x)). \tag{A.14}$$

We now prove $\lim_{s\to 0} d(s) = 0$. Let $\varepsilon > 0$ and consider the set $A_\varepsilon := g^{-1}([0,\varepsilon))$. Since g is continuous, A_ε is open and its complement $A_\varepsilon^c \subset K$ is closed and thus compact. The function f therefore has a lower bound δ_ε on A_ε^c which can be chosen greater than 0 because f does not vanish on A_ε^c. This implies

$$f(x) \leq \delta_\varepsilon \Rightarrow g(x) \leq \varepsilon \tag{A.15}$$

and thus $d(\delta_\varepsilon) \leq \varepsilon$.

By setting

$$\alpha\left(1/2^k\right) := d\left(1/2^{k-1}\right) \cdot \left(1 + 1/2^k\right) \tag{A.16}$$

for $k \in \mathbb{Z}$ we can define a piecewise affine linear function $\alpha : (0, \infty) \to \mathbb{R}_{\geq 0}$ which per construction is strictly increasing, continuous and an upper bound of d. Furthermore, it can be checked that $\lim_{s \to 0} \alpha(s) = 0$ which shows that α is a $\mathcal{K}$ function. Since any $\mathcal{K}$ function can be bounded from above by a $\mathcal{K}_\infty$ function, this proves the existence of α_1.

Because $\mathcal{K}_\infty$ is closed under inversion and $\alpha_2 := \alpha_1^{-1}$ satisfies (A.12), the proof is finished. □

Appendix B

Controllability

In this Appendix we discuss controllability properties for affine linear dynamical systems. The explicit representation of control functions by means of the controllability Gramian allows to prove that the KALMAN rank condition implies both the controllability Assumption 4.3 and the time-controllability Assumption 5.3.

B.1 Controllability for Linear Systems

Consider the affine-linear controlled system

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

with $A \in \mathbb{R}^{n_x \times n_x}$ and $B \in \mathbb{R}^{n_x \times n_u}$. For the initial value $\boldsymbol{x}(0) = x_0$ and a control function $\boldsymbol{u} \in \mathrm{L}^{n_u}_{\infty,\mathrm{loc}}(\mathbb{R}_{\geq 0})$, the solution of the IVP is given by

$$\boldsymbol{x}(t) = \exp(At)x_0 + \int_0^t \exp\left(A(t-\tau)\right) B\boldsymbol{u}(\tau)\mathrm{d}\tau. \tag{B.1}$$

If the controllability Gramian

$$G(t) := \int_0^t \exp\left(A(t-\tau)\right) BB^T \exp\left(A^T(t-\tau)\right) \mathrm{d}\tau \in \mathbb{R}^{n_x \times n_x} \tag{B.2}$$

can be inverted for a given $t \in \mathbb{R}_{\geq 0}$, then the state $y \in \mathbb{R}^{n_x}$ can be reached from the origin $0 \in \mathbb{R}^{n_x}$ in time t by applying the control function

$$\boldsymbol{u}_{y,t}(\tau) := B^T \exp\left(A^T(t-\tau)\right) G(t)^{-1} y. \tag{B.3}$$

Furthermore, it can be shown that KALMAN Rank Condition

$$\operatorname{rank}\underbrace{\left(\left[B|AB|\cdots|A^{n_x-1}B\right]\right)}_{C(A,B)\in\mathbb{R}^{n_x\times n_u n_x}} = n_x \tag{B.4}$$

implies the invertibility of the matrix $G(t)$ for any $t \in \mathbb{R}_{\geq 0}$ (see e.g. Rugh [104, Theorem 9.5]).

The explicit expression of the required control to reach desired state leads to the following Lemma.

Lemma B.1 (KALMAN Rank Condition Implies Controllability Assumption 4.3)
If the affine-linear system $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$ satisfies the KALMAN rank condition, then the controllability Assumption 4.3 holds.

Proof Let $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}) \in \mathcal{AC}^{n_x}([0,T]) \times \mathrm{L}_\infty^{n_u}([0,T])$ be a solution of the dynamical system with initial value $\tilde{\boldsymbol{x}}(0) = x_0$ and final value $\tilde{\boldsymbol{x}}(T) = x_T$. Then for any $\Delta x \in \mathbb{R}^{n_x}$, the control function $\boldsymbol{u}_{\Delta x,T} \in \mathrm{L}_\infty^{n_u}([0,T])$ according to (B.3) is well defined because $G(T)$ is invertible due to the KALMAN rank condition. With the previous considerations, it is clear that the dynamical system with the control function $\tilde{\boldsymbol{u}} + \boldsymbol{u}_{\Delta x,T}$ reaches the state $x_T + \Delta x$ from the initial state x_0 in time T. This induces the mapping $C_u : \mathbb{R}^{n_x} \to \mathrm{L}_\infty^{n_u}([0,T])$ with $C_u(y) := \tilde{\boldsymbol{u}} + \boldsymbol{u}_{y-x_T}$. With $C_x : \mathbb{R}^{n_x} \to \mathcal{AC}^{n_x}([0,T])$ defined as $C_x(y)(\tau) := \boldsymbol{x}(\tau; x_0, C_u(y))$ it is straightforward to check that C_u induces a FRÉCHET-differentiable mapping $C := (C_x, C_u) : \mathbb{R}^{n_x} \to \mathrm{L}_\infty^{n_u}([0,T]) \times \mathcal{AC}^{n_x}([0,T])$ satisfying all the required properties imposed in Assumption 4.3. □

B.2 Time-Controllability for Linear Systems

For affine linear systems, the KALMAN rank condition (B.4) not only implies the controllability Assumption 4.3, but also the stronger time-controllability Assumption 5.3.

Lemma B.2 (KALMAN Rank Condition Implies Time-Controllability Assumption 5.3)
If the dynamical system $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$ satisfies the KALMAN Rank Condition, then the time-controllability Assumption 5.3 holds.

Proof For $x_0 \in \mathbb{R}^{n_x}$ and $T_0 \in \mathbb{R}_{>0}$ let $\Phi(t; x_0, \boldsymbol{u}, T_0)$ denote the solution at time t of the IVP $\dot{\boldsymbol{x}}(\tau) = T_0\,(A\boldsymbol{x}(\tau) + B\boldsymbol{u}(\tau))$ with initial value $\boldsymbol{x}(0) = x_0$. As we have seen in Lemma B.1, the KALMAN rank condition implies that the state $y \in \mathbb{R}^{n_x}$ can be reached from the origin in time $T_0 \in \mathbb{R}_{>0}$ by applying the control function (see equation (B.3))

$$\boldsymbol{u}_{y,T_0}(\tau) = B^T \exp\left(A^T (T_0 - \tau)\right) G(T_0)^{-1} y \tag{B.5}$$

to the original dynamical system. This implies that with the control input $\tilde{\boldsymbol{u}}_{y,T_0} := \boldsymbol{u}_{y,T_0} \circ (\tau \mapsto T_0\tau)$ the dynamical system $\dot{\boldsymbol{x}}(\tau) = T_0 \cdot (A\boldsymbol{x}(\tau) + B\boldsymbol{u}(\tau))$ reaches the state y from the origin in time 1. The induced mapping $S : \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0} \to L_\infty^{n_u}([0,1])$ defined by $S(y,t) := \tilde{\boldsymbol{u}}_{y,t}$ satisfies

$$\Phi(1; 0, S(y,T), T) = y \tag{B.6}$$

for all $(y,T) \in \mathbb{R}^{n_x} \times \mathbb{R}_{>0}$ and is FRÉCHET-differentiable, which follows from the explicit representation of $\boldsymbol{u}_{y,T}$ in (B.5). Let now $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{AC}^{n_x}([0,1]) \times L_\infty^{n_u}([0,1])$ be a solution of the IVP

$$\dot{\boldsymbol{x}}(\tau) = T_1\,(A\boldsymbol{x}(\tau) + B\boldsymbol{u}(\tau)), \quad \boldsymbol{x}(0) = x_0. \tag{B.7}$$

By setting $C(y,T) := \boldsymbol{u} + S(y - \Phi(1; x_0, \boldsymbol{u}, T), T)$ we can define a mapping C that satisfies

- $C(\boldsymbol{x}(1), T_1) = \boldsymbol{u}$,
- $\Phi(1; x_0, C(y,T), T) = y$ for all $(y,T) \in \mathbb{R}^{n_x} \times \mathbb{R}_{>0}$,

which proves that Assumption 5.3 holds for the affine-linear system $\dot{\boldsymbol{x}} = (A\boldsymbol{x} + B\boldsymbol{u})$. □

Danksagung

An dieser Stelle gebührt mein Dank allen, die mich während der Erstellung dieser Arbeit in Heidelberg unterstüzt haben. Herausheben möchte ich an dieser Stelle meine Mentoren und Lehrer der Fakultät für Mathematik und Informatik, Hans Georg Bock, Johannes Schlöder und Andreas Potschka für deren entgegengebrachtes Vertrauen und die zahlreichen Denkanstöße, die wesentlich zum Gelingen dieser Arbeit beigetragen haben.

Ich möchte ich mich bei allen Kollegen der Arbeitsgruppen Simulation und Optimierung, Model-Based Optimizing Control für die angeneheme und kooperative Arbeitsatmosphäre der vergangenen Jahre bedanken. Für die Diskussionen im Kontext meiner Arbeit danke ich besonders Alexander Buchner, Johannes Herold, Christian Kirches, Manuel Kudruss, Huu Chuong La, Conrad Leidereiter, Andreas Meyer, Andreas Schmidt und María Eléna Suaréz-Garcés.

Der Graduiertenschule HGS MathComp, die mir unter anderem die Teilnahme an Konferenzen in Hanoi und Pittsburgh ermöglicht hat, gebührt mein besonderer Dank. Für die Unterstützung in organisatorischen Angelegenheiten geht mein Dank an die Verwaltung der Arbeitsgruppe Simulation und Optimierung, Abir Al-Laham, Margret Rothfuß, Anastasia Valter, Anja Vogel und Jeannette Walsch sowie der Verwaltung der HGS, Ria Hillenbrand-Lynott und Sarah Steinbach. Für die Unterstützung in Promotionsangelegenheiten geht mein Dank an Dorothea Heukäufer vom Dekanat der Fakultät für Mathematik und Informatik.

Abschliessend möchte ich mich herzlich bei meiner Familie für die kontinuierliche Unterstützung und den Rückhalt während meiner Zeit in Heidelberg bedanken.

Bibliography

[1] K. J. Åström and T. Hägglund. *PID controllers.* International Society for Measurement and Control, Research Triangle Park, N.C, 1995. ISBN 1556175167.

[2] S. Ahmad and A. Ambrosetti. *A Textbook on Ordinary Differential Equations.* Springer International Publishing, 2015. DOI: 10.1007/978-3-319-16408-3.

[3] I. Alvarado. *Model Predictive Control for Tracking Constrained Linear Systems.* PhD thesis, Ph. D. dissertation, Univ. de Sevilla, 2007.

[4] S. Aniţa, V. Arnăutu, and V. Capasso. *An Introduction to Optimal Control Problems in Life Sciences and Economics.* Birkhäuser Boston, 2011. DOI: 10.1007/978-0-8176-8098-5.

[5] Z. Artstein and I. Bright. Periodic optimization suffices for infinite horizon planar optimal control. *SIAM Journal on Control and Optimization*, 48(8):4963–4986, Jan. 2010. DOI: 10.1137/090772617.

[6] U. M. Ascher, R. M. M. Mattheij, and R. D. Russell. *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations.* Society for Industrial and Applied Mathematics, Jan. 1995. DOI: 10.1137/1.9781611971231.

[7] J. E. Bailey and F. J. M. Horn. Comparison between two sufficient conditions for improvement of an optimal steady-state process by periodic operation. *Journal of Optimization Theory and Applications*, 7(5):378–384, May 1971. DOI: 10.1007/bf00934000.

[8] M. Behrens, H. G. Bock, S. Engell, P. Khobkhun, and A. Potschka. Real-time PDE constrained optimal control of a periodic multicomponent separation process. In *International Series of Numerical Mathematics*, pages 521–537. Springer International Publishing, 2014. DOI: 10.1007/978-3-319-05083-6_33.

[9] R. E. Bellman. *Dynamic Programming.* University Press, Princeton, N.J., 6th edition, 1957.

[10] I. Bendixson. Sur les courbes définies par des équations différentielles. *Acta Mathematica*, 24(0):1–88, 1901. DOI: 10.1007/bf02403068.

[11] L. D. Berkovitz. *Optimal Control Theory.* Springer New York, 1974. ISBN 978-1-4419-2804-7. DOI: 10.1007/978-1-4757-6097-2.

[12] D. Bertsekas. *Nonlinear programming.* Athena Scientific, Belmont, Mass, 2016. ISBN 1886529051.

[13] C. Bischof, G. Corliss, and A. Griewank. Structured second-and higher-order derivatives through univariate taylor series. *Optimization Methods and Software*, 2(3-4):211–232, Jan. 1993. DOI: 10.1080/10556789308805543.

[14] S. Bittanti, G. Fronza, and G. Guardabassi. Periodic control: A frequency domain approach. *IEEE Transactions on Automatic Control*, 18(1):33–38, Feb. 1973. DOI: 10.1109/tac.1973.1100225.

[15] S. Bittanti, G. Fronza, and G. Guardabassi. Optimal steady-state versus periodic operation in discrete systems. *Journal of Optimization Theory and Applications*, 18(4): 521–536, Apr. 1976. DOI: 10.1007/bf00932660.

[16] H. G. Bock. Zur numerischen Behandlung zustandsbeschränkter Steuerungsprobleme mit Mehrzielmethode und Homotopieverfahren. *Zeitschrift für Angewandte Mathematik und Mechanik*, 57(4):T266–T268, 1977.

[17] H. G. Bock. Numerical Solution of Nonlinear Multipoint Boundary Value Problems with Applications to Optimal Control. *(Gesellschaft fuer angewandte Mathematik und Mechanik, Wissenschaftliche Jahrestagung, Lyngby, Denmark, May 31-June 3, 1977.) Zeitschrift fuer angewandte Mathematik und Mechanik,*, 58, 1978.

[18] H. G. Bock. Numerical treatment of inverse problems in chemical reaction kinetics. In *Springer Series in Chemical Physics*, pages 102–125. Springer Berlin Heidelberg, 1981. DOI: 10.1007/978-3-642-68220-9_8.

[19] H. G. Bock. Recent advances in parameteridentification techniques for o.d.e. In *Numerical Treatment of Inverse Problems in Differential and Integral Equations*, pages 95–121. Birkhäuser Boston, 1983. DOI: 10.1007/978-1-4684-7324-7_7.

[20] H. G. Bock and K. J. Plitt. A multiple shooting algorithm for direct solution of optimal control problems. *IFAC Proceedings Volumes*, 17(2):1603–1608, July 1984. DOI: 10.1016/s1474-6670(17)61205-9.

[21] H. G. Bock, C. Kirches, A. Meyer, and A. Potschka. Numerical solution of optimal control problems with explicit and implicit switches. *Optimization Methods and Software*, 33(3):450–474, Mar. 2018. DOI: 10.1080/10556788.2018.1449843.

[22] R. Bulirsch. Die Mehrzielmethode zur numerischen Lösung von nichtlinearen Randwertproblemen und Aufgaben der optimalen Steuerung. *Report der Carl-Cranz-Gesellschaft*, 251, 1971.

[23] M. Canale, L. Fagiano, and M. Milanese. Power kites for wind energy generation [applications of control]. *IEEE Control Systems*, 27(6):25–38, Dec. 2007. DOI: 10.1109/mcs.2007.909465.

[24] D. A. Carlson. Uniformly overtaking and weakly overtaking optimal solutions in infinite-horizon optimal control: When optimal solutions are agreeable. *Journal of Optimization Theory and Applications*, 64(1):55–69, Jan. 1990. DOI: 10.1007/bf00940022.

[25] D. A. Carlson, A. B. Haurie, and A. Leizarowitz. *Infinite Horizon Optimal Control*. Springer Berlin Heidelberg, 1991. DOI: 10.1007/978-3-642-76755-5.

[26] C. Chen and L. Shaw. On receding horizon feedback control. *Automatica*, 18(3):349–352, May 1982. DOI: 10.1016/0005-1098(82)90096-6.

[27] H. Chen and F. Allgöwer. A computationally attractive nonlinear predictive control scheme with guaranteed stability for stable systems. *Journal of Process Control*, 8(5-6): 475–485, Oct. 1998. DOI: 10.1016/s0959-1524(98)00021-3.

[28] P. Chen. *Optimal Control Models in Finance*. Kluwer Academic Publishers, 2005. DOI: 10.1007/b101888.

[29] F. Clarke, Y. Ledyaev, and R. Stern. Asymptotic stability and smooth lyapunov functions. *Journal of Differential Equations*, 149(1):69–114, Oct. 1998. DOI: 10.1006/jdeq.1998.3476.

[30] F. Colonius and W. Kliemann. Infinite time optimal control and periodicity. *Applied Mathematics & Optimization*, 20(1):113–130, July 1989. DOI: 10.1007/bf01447651.

[31] J.-P. Corriou. *Process Control*. Springer International Publishing, 2018. DOI: 10.1007/978-3-319-61143-3.

[32] J. Cotrell, T. Stehly, J. Johnson, J. O. Roberts, Z. Parker, G. Scott, and D. Heimiller. Analysis of transportation and logistics challenges affecting the deployment of larger wind turbines: Summary of results. Technical report, Jan. 2014.

[33] C. R. Cutler and B. L. Ramaker. Dynamic Matrix Control-A Computer Control Algorithm. In *Joint Automatic Control Conference, 13-15 Aug.*, volume I, pages WP5–B/6 pp., 1980.

[34] P. Deuflhard. A modified newton method for the solution of ill-conditioned systems of nonlinear equations with application to multiple shooting. *Numerische Mathematik*, 22(4):289–315, Aug. 1974. DOI: 10.1007/bf01406969.

[35] M. Diehl. *Real-Time Optimization for Large Scale Nonlinear Processes*. PhD Thesis, Heidelberg University, 2001.

[36] M. Diehl, R. Amrit, and J. B. Rawlings. A lyapunov function for economic optimizing model predictive control. *IEEE Transactions on Automatic Control*, 56(3):703–707, Mar. 2011. DOI: 10.1109/tac.2010.2101291.

[37] M. Ellis, H. Durand, and P. D. Christofides. A tutorial review of economic model predictive control methods. *Journal of Process Control*, 24(8):1156–1178, Aug. 2014. DOI: 10.1016/j.jprocont.2014.03.010.

[38] S. Engell. Feedback control for optimal process operation. *Journal of Process Control*, 17(3):203–219, Mar. 2007. DOI: 10.1016/j.jprocont.2006.10.011.

[39] T. Faulwasser and D. Bonvin. On the design of economic NMPC based on approximate turnpike properties. In *2015 54th IEEE Conference on Decision and Control (CDC)*. IEEE, Dec. 2015. DOI: 10.1109/cdc.2015.7402995.

[40] E. Fehlberg. Klassische runge-kutta-formeln fünfter und siebenter ordnung mit schrittweiten-kontrolle. *Computing*, 4(2):93–106, June 1969. DOI: 10.1007/bf02234758.

[41] E. Fehlberg. Klassische runge-kutta-formeln vierter und niedrigerer ordnung mit schrittweiten-kontrolle und ihre anwendung auf wärmeleitungsprobleme. *Computing*, 6(1-2):61–71, Mar. 1970. DOI: 10.1007/bf02241732.

[42] A. Fiacco. *Nonlinear programming; sequential unconstrained minimization techniques.* Wiley, New York, 1968. ISBN 9780471258100.

[43] J. V. Frasch, L. Wirsching, S. Sager, and H. G. Bock. Mixed–-level iteration schemes for nonlinear model predictive control. *IFAC Proceedings Volumes*, 45(17):138–144, 2012. DOI: 10.3182/20120823-5-nl-3013.00085.

[44] C. E. García, D. M. Prett, and M. Morari. Model predictive control: Theory and practice—a survey. *Automatica*, 25(3):335–348, May 1989. DOI: 10.1016/0005-1098(89)90002-2.

[45] H. P. Geering. *Optimal Control with Engineering Applications.* Springer Berlin Heidelberg, 2007. DOI: 10.1007/978-3-540-69438-0.

[46] M. Gerdts. Optimal Control of Ordinary Differential Equations and Differential-Algebraic Equations. *Habilitation, University of Bayreuth*, 2006.

[47] G. Grammel. Estimate of periodic suboptimal controls. *Journal of Optimization Theory and Applications*, 99(3):681–689, Dec. 1998. DOI: 10.1023/a:1021707201815.

[48] A. Griewank. Achieving logarithmic growth of temporal and spatial complexity in reverse automatic differentiation. *Optimization Methods and Software*, 1(1):35–54, 1992. DOI: 10.1080/10556789208805505.

[49] A. Griewank. *Evaluating derivatives : principles and techniques of algorithmic differentiation.* Society for Industrial and Applied Mathematics, Philadelphia, PA, 2008. ISBN 9780898716597.

[50] L. Grüne. Economic receding horizon control without terminal constraints. *Automatica*, 49(3):725–734, Mar. 2013. DOI: 10.1016/j.automatica.2012.12.003.

[51] L. Grüne and J. Pannek. *Nonlinear Model Predictive Control.* Springer International Publishing, 2017. DOI: 10.1007/978-3-319-46024-6.

[52] L. Grüne, J. Pannek, M. Seehafer, and K. Worthmann. Analysis of unconstrained nonlinear MPC schemes with time varying control horizon. *SIAM Journal on Control and Optimization*, 48(8):4938–4962, Jan. 2010. DOI: 10.1137/090758696.

[53] L. Grüne, S. Sager, F. Allgöwer, H. G. Bock, and M. Diehl. Predictive planning and systematic action—on the control of technical processes. In *Production Factor Mathematics*, pages 9–37. Springer Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-11248-5_2.

[54] G. Guardabassi, A. Locatelli, and S. Rinaldi. Status of periodic optimization of dynamical systems. *Journal of Optimization Theory and Applications*, 14(1):1–20, July 1974. DOI: 10.1007/bf00933171.

[55] T. Guinn. Reduction of delayed optimal control problems to nondelayed problems. *Journal of Optimization Theory and Applications*, 18(3):371–377, Mar. 1976. DOI: 10.1007/bf00933818.

[56] W. Hahn. *Theorie Und Anwendung Der Direkten Methode von Ljapunov*, volume 22. Springer, 1959.

[57] W. Hahn. *Stability of Motion*, volume 138. Springer, 1967.

[58] A. Helbig, O. Abel, and W. Marquardt. Structural concepts for optimization based control of transient processes. In *Nonlinear Model Predictive Control*, pages 295–311. Birkhäuser Basel, 2000. DOI: 10.1007/978-3-0348-8407-5_16.

[59] R. Hettich and K. O. Kortanek. Semi-infinite programming: Theory, methods, and applications. *SIAM Review*, 35(3):380–429, Sept. 1993. DOI: 10.1137/1035089.

[60] B. Hou and M. Diehl. Robustness and stability optimization of power generating kite systems in a periodic pumping mode. In *2010 IEEE International Conference on Control Applications*. IEEE, Sept. 2010. DOI: 10.1109/cca.2010.5611288.

[61] B. Houska and M. Diehl. Optimal control for power generating kites. In *2007 European Control Conference (ECC)*. IEEE, July 2007. DOI: 10.23919/ecc.2007.7068861.

[62] B. Houska, H. J. Ferreau, and M. Diehl. ACADO toolkit-an open-source framework for automatic control and dynamic optimization. *Optimal Control Applications and Methods*, 32(3):298–312, May 2010. DOI: 10.1002/oca.939.

[63] R. Huang and L. Biegler. Stability of economically-oriented NMPC with periodic constraint. *IFAC Proceedings Volumes*, 44(1):10499–10504, Jan. 2011. DOI: 10.3182/20110828-6-it-1002.00137.

[64] A. Ilzhöfer, B. Houska, and M. Diehl. Nonlinear MPC of kites under varying wind conditions for a new class of large-scale wind power generators. *International Journal of Robust and Nonlinear Control*, 17(17):1590–1599, 2007. DOI: 10.1002/rnc.1210.

[65] K. Johansson. The quadruple-tank process: a multivariable laboratory process with an adjustable zero. *IEEE Transactions on Control Systems Technology*, 8(3):456–465, May 2000. DOI: 10.1109/87.845876.

[66] Z. Jose, D. Michael, G. Patrick, S. Ananthan, E. Lantz, J. Cotrell, F. Beck, and R. Tusing. Enabling wind power nationwide. Technical report, May 2015.

[67] S. Julier and J. Uhlmann. Unscented filtering and nonlinear estimation. *Proceedings of the IEEE*, 92(3):401–422, Mar. 2004. DOI: 10.1109/jproc.2003.823141.

[68] S. J. Julier and J. K. Uhlmann. New extension of the kalman filter to nonlinear systems. In I. Kadar, editor, *Signal Processing, Sensor Fusion, and Target Recognition VI*. SPIE, July 1997. DOI: 10.1117/12.280797.

[69] R. E. Kalman. A new approach to linear filtering and prediction problems. *Journal of Basic Engineering*, 82(1):35, 1960. DOI: 10.1115/1.3662552.

[70] R. E. Kalman and others. Contributions to the theory of optimal control. *Bol. Soc. Mat. Mexicana*, 5(2):102–119, 1960.

[71] H. K. Khalil. *Nonlinear Systems*, volume 3. 2002. ISBN 0-13-067389-7 978-0-13-067389-3.

[72] C. K. Lee and J. E. Bailey. Modification of consecutive-competitive reaction selectivity by periodic operation. *Industrial & Engineering Chemistry Process Design and Development*, 19(1):160–166, Jan. 1980. DOI: 10.1021/i260073a028.

[73] E. B. Lee and L. Markus. Optimal control for nonlinear processes. *Archive for Rational Mechanics and Analysis*, 8(1):36–58, Jan. 1961. DOI: 10.1007/bf00277429.

[74] E. B. Lee and L. Markus. *Foundations of Optimal Control Theory*. Wiley, New York, 1967.

[75] W. Lee and V. W. Weekman. Advanced control practice in the chemical process industry: A view from industry. *AIChE Journal*, 22(1):27–38, Jan. 1976. DOI: 10.1002/aic.690220103.

[76] S. Lenhart and J. T. Workman. *Optimal Control Applied to Biological Models*. Crc Press, 2007. ISBN 9781584886402.

[77] D. Limon, M. Pereira, D. M. de la Peña, T. Alamo, and J. Grosso. Single-layer economic model predictive control for periodic operation. *Journal of Process Control*, 24 (8):1207–1224, Aug. 2014. DOI: 10.1016/j.jprocont.2014.03.013.

[78] S. Linnainmaa. The representation of the cumulative rounding error of an algorithm as a Taylor expansion of the local rounding errors. *Master's Thesis (in Finnish), Univ. Helsinki*, pages 6–7, 1970.

[79] J. M. Longuski, J. J. Guzmán, and J. E. Prussing. *Optimal Control with Aerospace Applications*. Springer New York, 2014. DOI: 10.1007/978-1-4614-8945-0.

[80] M. LOYD. Crosswind kite power (for large-scale wind power production). *Journal of Energy*, 4(3):106–111, May 1980. DOI: 10.2514/3.48021.

[81] A. M. Lyapunov. A General Task about the Stability of Motion. *PhD Thesis*, 1892.

[82] L. Magni and R. Sepulchre. Stability margins of nonlinear receding-horizon control via inverse optimality. *Systems & Control Letters*, 32(4):241–245, Dec. 1997. DOI: 10.1016/s0167-6911(97)00079-0.

[83] J. F. Manwell. *Wind energy explained : theory, design and application*. Wiley, Chichester, U.K, 2009. ISBN 9780470015001.

[84] D. Mayne, J. Rawlings, C. Rao, and P. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, June 2000. DOI: 10.1016/s0005-1098(99)00214-9.

[85] M. A. Müller and F. Allgöwer. Economic and distributed model predictive control: Recent developments in optimization-based control. *SICE Journal of Control, Measurement, and System Integration*, 10(2):39–52, 2017. DOI: 10.9746/jcmsi.10.39.

[86] M. A. Müller and L. Grüne. Economic model predictive control without terminal constraints for optimal periodic behavior. *Automatica*, 70:128–139, Aug. 2016. DOI: 10.1016/j.automatica.2016.03.024.

[87] M. A. Müller, L. Grüne, and F. Allgöwer. On the role of dissipativity in economic model predictive control. *IFAC-PapersOnLine*, 48(23):110–116, 2015. DOI: 10.1016/j.ifacol.2015.11.269.

[88] G. D. Nicolao, L. Magni, and R. Scattolini. Robust predictive control of systems with uncertain impulse response. *Automatica*, 32(10):1475–1479, Oct. 1996. DOI: 10.1016/0005-1098(96)00082-9.

[89] J. Nocedal and S. Wright. *Numerical Optimization*. Springer Series in Operations Research and Financial Engineering. Springer Berlin Heidelberg, 2006. DOI: 10.1007/978-3-540-35447-5.

[90] M. Osborne. On shooting methods for boundary value problems. *Journal of Mathematical Analysis and Applications*, 27(2):417–433, Aug. 1969. DOI: 10.1016/0022-247x(69)90059-6.

[91] K. J. Plitt. *Ein Superlinear Konvergentes Mehrzielverfahren Zur Direkten Berechnung Beschränkter Optimaler Steuerungen*. Master's Thesis, Rheinische Friedrich–Wilhelms–Universität Bonn, 1981.

[92] L. S. Pontryagin, V. G. Boltyanski, R. V. Gamkrelidze, and E. F. Miscenko. *The Mathematical Theory of Optimal Processes*. Wiley, Chichester, 1962.

[93] A. Potschka. *Handling Path Constraints in a Direct Multiple Shooting Method for Optimal Control Problems*. Diploma Thesis, Universität Heidelberg, 2006.

[94] A. Potschka. Optimal control of a simulated moving bed process. In *A Direct Method for Parabolic PDE Constrained Optimization Problems*, pages 187–197. Springer Fachmedien Wiesbaden, Nov. 2013. DOI: 10.1007/978-3-658-04476-3_15.

[95] A. Potschka, H. G. Bock, and J. P. Schlöder. A minima tracking variant of semi-infinite programming for the treatment of path constraints within direct solution of optimal control problems. *Optimization Methods and Software*, 24(2):237–252, Apr. 2009. DOI: 10.1080/10556780902753098.

[96] D. M. Prett and R. Gillette. Optimization and constrained multivariable control of a catalytic cracking unit. In *Joint Automatic Control Conference*, number 17, page 73, 1980.

[97] A. I. Propoi. Use of Linear Programming Methods for Synthesizing Sampled-Data Automatic Systems. *Automn. Remote Control*, 24(7):837–844, 1963.

[98] J. Rawlings. *Model predictive control : theory, computation, and design*. Nob Hill Publishing, Madison, Wisconsin, 2017. ISBN 0975937731.

[99] J. B. Rawlings and R. Amrit. Optimizing process economic performance using model predictive control. In *Nonlinear Model Predictive Control*, pages 119–138. Springer Berlin Heidelberg, 2009. DOI: 10.1007/978-3-642-01094-1_10.

[100] J. B. Rawlings, D. Angeli, and C. N. Bates. Fundamentals of economic model predictive control. In *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*. IEEE, Dec. 2012. DOI: 10.1109/cdc.2012.6425822.

[101] A. Renken. The use of periodic operation to improve the performance of continuous stirred tank reactors. *Chemical Engineering Science*, 27(11):1925–1932, 1972. DOI: 10.1016/0009-2509(72)87051-9.

[102] J. Richalet, A. Rault, J. Testud, and J. Papon. Model predictive heuristic control. *Automatica*, 14(5):413–428, Sept. 1978. DOI: 10.1016/0005-1098(78)90001-8.

[103] H. L. Royden and P. Fitzpatrick. *Real Analysis*, volume 198. Macmillan New York, 1988.

[104] W. Rugh. *Linear system theory*. Prentice Hall, Upper Saddle River, N.J, 1996. ISBN 0134412052.

[105] R. D. Russell and L. F. Shampine. A collocation method for boundary value problems. *Numerische Mathematik*, 19(1):1–28, Feb. 1972. DOI: 10.1007/bf01395926.

[106] E. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34(4):435–443, Apr. 1989. DOI: 10.1109/9.28018.

[107] B. Speelpenning. Compiling fast partial derivatives of functions given by algorithms. Technical report, Illinois Univ., Urbana (USA). Dept. of Computer Science, 1980.

[108] J. Stoer and R. Bulirsch. *Introduction to Numerical Analysis*, volume 12. Springer Science & Business Media, 2013.

[109] S. Strogatz. *Nonlinear dynamics and chaos : with applications to physics, biology, chemistry, and engineering*. Westview Press, a member of the Perseus Books Group, Boulder, CO, 2015. ISBN 0813349109.

[110] G. Teschl. *Ordinary Differential Equations and Dynamical Systems*, volume 140. American Mathematical Society Providence, 2012.

[111] A. Toumi, S. Engell, M. Diehl, H. G. Bock, and J. Schlöder. Efficient optimization of simulated moving bed processes. *Chemical Engineering and Processing: Process Intensification*, 46(11):1067–1084, Nov. 2007. DOI: 10.1016/j.cep.2006.06.026.

[112] L. Tsoukalas. *Fuzzy and neural approaches in engineering*. Wiley, New York, 1997. ISBN 0471160032.

[113] J. Warga. *Optimal Control of Differential and Functional Equations*. Elsevier, 1972. ISBN 978-0-12-735150-6. DOI: 10.1016/c2013-0-11669-8.

[114] L. Wirsching. *Multi-Level Iteration Schemes with Adaptive Level Choice for Nonlinear Model Predictive Control*. PhD thesis, 2018.

[115] K. Worthmann. *Stability Analysis of Unconstrained Receding Horizon Control Schemes*. PhD thesis, 2011.

[116] S. J. Wright. *Primal-Dual Interior-Point Methods*, volume 54. Siam, 1997.

[117] A. Wächter and L. T. Biegler. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Mathematical Programming*, 106(1):25–57, Apr. 2005. DOI: 10.1007/s10107-004-0559-y.

[118] A. Wächter and L. T. Biegler. Line search filter methods for nonlinear programming: Motivation and global convergence. *SIAM Journal on Optimization*, 16(1):1–31, Jan. 2005. DOI: 10.1137/s1052623403426556.

[119] L. C. Young. *Lectures on the calculus of variations and optimal control theory*. Chelsea Pub. Co, New York, 1980. ISBN 978-0-8218-2690-4.

[120] M. Zanon, S. Gros, and M. Diehl. A lyapunov function for periodic economic optimizing model predictive control. In *52nd IEEE Conference on Decision and Control*. IEEE, Dec. 2013. DOI: 10.1109/cdc.2013.6760691.

[121] M. Zanon, S. Gros, and M. Diehl. A tracking MPC formulation that is locally equivalent to economic MPC. *Journal of Process Control*, 45:30–42, Sept. 2016. DOI: 10.1016/j.jprocont.2016.06.006.

[122] M. Zanon, S. Gros, and M. Diehl. A periodic tracking MPC that is locally equivalent to periodic economic MPC. *IFAC-PapersOnLine*, 50(1):10711–10716, July 2017. DOI: 10.1016/j.ifacol.2017.08.2224.

[123] M. Zanon, L. Grüne, and M. Diehl. Periodic optimal control, dissipativity and MPC. *IEEE Transactions on Automatic Control*, 62(6):2943–2949, June 2017. DOI: 10.1109/tac.2016.2601881.

Nomenclature

List of Symbols

$\triangle$	End of a definition, remark or assumption
$\square$	End of a proof
I_n	Identity matrix in $\mathbb{R}^{n\times n}$
$\setminus$	Set difference
$\mathrm{diag}(v)$	Diagonal matrix with the entries of the vector v on the diagonal

Function Spaces

A^B	Set of functions $f : A \to B$
$\mathcal{C}(\mathcal{T})$	Space of continuous functions $f : \mathcal{T} \to \mathbb{R}$
$\mathcal{AC}^n(\mathcal{T})$	Space of absolutely continuous functions $f : \mathcal{T} \to \mathbb{R}^n$
$\mathcal{AC}^n([t_0,\infty))$	Space of locally absolutely continuous functions $f : [t_0,\infty) \to \mathbb{R}^n$
$\mathrm{L}^n_\infty(\mathcal{T})$	Space of measurable functions $f : \mathcal{T} \to \mathbb{R}^n$ equipped with $\|\cdot\|_\infty$
$\mathrm{L}^n_{\infty,\mathrm{loc}}([t_0,\infty))$	Space of locally measurable functions $f : [t_0,\infty) \to \mathbb{R}^n$ equipped with $\|\cdot\|_\infty$
$\mathcal{K},\mathcal{K}_\infty,\mathcal{L},\mathcal{KL}$	Comparison functions cf. Definition 1.6

Black Board Symbols

$\mathbb{Z},\mathbb{N},\mathbb{N}^+$	Set of the integers, natural numbers including (excluding) zero
$\mathbb{R},\mathbb{R}_{\geq 0},\mathbb{R}_{>0}$	Set of the real numbers (non-negative, positive) numbers
$\mathbb{R}^n$	Space of n-dimensional real-valued vectors
$\mathbb{R}^{n\times m}$	Space of $n \times m$ matrices with real entries
$\mathbb{1}_n$	Vector with entries 1 in $\mathbb{R}^n$
0_n	Zero Element of $\mathbb{R}^n$

Optimal Control Problems

$\mathrm{NP}_T^{\mathrm{fix}}$	NMPC subproblem with fixed period T
$\mathrm{NP}_T^{\mathrm{fix},\ell}$	NMPC subproblem with fixed period T for periodic performance criterion ℓ
$\mathrm{NP}_p^{\mathrm{free}}$	NMPC subproblem with free period for parameter p
$\mathrm{Per}_T^{\mathrm{fix}}$	Periodic OCP with period T
$\mathrm{Per}_T^{\mathrm{fix},\ell}$	Periodic OCP with periodic performance criterion ℓ and period T
$\mathrm{Per}_p^{\mathrm{free}}$	Periodic OCP with free period for parameter p

NMPC Objective Functionals

$\varphi^{\text{fix}}_{\text{per}}$	Periodic objective functional of problem $\text{NP}^{\text{fix}}_{T}$
$\varphi^{\text{fix}}_{\text{trans}}$	Transient objective functional of problem $\text{NP}^{\text{fix}}_{T}$
φ^{fix}	Objective functional of problem $\text{NP}^{\text{fix}}_{T}$
$\varphi^{\text{fix},\ell}_{\text{per}}$	Periodic objective functional of problem $\text{NP}^{\text{fix},\ell}_{T}$
$\varphi^{\text{fix},\ell}_{\text{trans}}$	Transient objective functional of problem $\text{NP}^{\text{fix},\ell}_{T}$
$\varphi^{\text{fix},\ell}$	Objective functional of problem $\text{NP}^{\text{fix},\ell}_{T}$
$\varphi^{\text{free}}_{p,\text{per}}$	Periodic objective functional of problem $\text{NP}^{\text{free}}_{p}$
$\varphi^{\text{free}}_{p,\text{trans}}$	Transient objective functional of problem $\text{NP}^{\text{free}}_{p}$
$\varphi^{\text{free}}_{p}$	Objective functional of problem $\text{NP}^{\text{free}}_{p}$

Objective Functionals for periodic OCPs

$\phi^{\text{fix}}_{T,\text{per}}$	Objective Functional of problem $\text{Per}^{\text{fix}}_{T}$
$\phi^{\ell}_{T,\text{per}}$	Objective Functional of problem $\text{Per}^{\text{fix},\ell}_{T}$
$\phi^{\text{free}}_{p,\text{per}}$	Objective Functional of problem $\text{Per}^{\text{free}}_{p}$

Roman Symbols

$\boldsymbol{x}(\cdot)$	State trajectory
$\boldsymbol{u}(\cdot)$	Control trajectory

List of Acronyms

AD	Automatic Differentiation
AOCP	Average Output Optimal Control Problem
BVP	Boundary Value Problem
CSTR	Continuous Stirred-Tank Reactor
DAE	Differential Algebraic Equation
E-NMPC	Economic Nonlinear Model Predictive Control
END	External Numerical Differentiation
IND	Internal Numerical Differentiation
IVP	Initial Value Problem
LQR	Linear Quadratic Regulator
MLI	Multi-Level Iteration
MPBVP	Multi Point Boundary Value Problem
MPC	Model Predictive Control
NLP	Nonlinear Program
NMPC	Nonlinear Model Predictive Control
OCP	Optimal Control Problem
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
PID	Proportional-Integral-Derivative
QP	Quadratic Program
SQP	Sequential Quadratic Programming