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Gravitational closure of matter field equations

General theory & symmetrization

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Zusammenfassung

Schon bei der Herleitung der Relativitätstheorie nutzte Einstein Erkenntnisse über die Maxwellsche Elektrodynamik. Dies deutet auf die tiefe Verbindung, die zwischen Materie und Gravitation besteht. Diese führt sogar noch weiter als aus der Relativitätstheorie bekannt. In dieser Arbeit wird ein Verfahren vorgestellt, das es erlaubt, Gravitationstheorien aus gegebenen Materiemodellen herzuleiten. Diese konstruktive Methode kann für jede Materietheorie auf jeder tensoriellen Hintergrundstruktur angewendet werden, solange die Materiedynamik drei grundlegende physikalische Bedingungen erfüllt. Das zentrale Element dieser Herleitung ist die Lösung eines abzählbaren Systems von linearen partiellen Differentialgleichungen, den Abschlussgleichungen. Die Lösung dieser Abschlussgleichungen ist das Wirkungsfunktional der Gravitationstheorie. In der Praxis ist es jedoch sehr schwer, allgemeine Lösungen dieses Systems zu konstruieren. Man kann dies allerdings vereinfachen, indem man Raumzeit-Symmetrien bereits bei der Lösung der Abschlussgleichungen ausnutzt. Dies führt direkt zu den symmetrie-reduzierten Feldgleichungen. Ausgehend vom Standardmodell der Teilchenphysik erhält man so als Lösung der Abschlussgleichungen unter Benutzung kosmologischer Symmetrien direkt die Friedmann-Gleichungen. Die Einstein-Gleichungen müssen dafür nicht bekannt sein. Als zweites Fallbeispiel wird eine Verallgemeinerung Maxwellscher Elektrodynamik betrachtet und die zugehörigen verallgemeinerten Friedmann-Gleichungen werden hergeleitet. Diese Arbeit beschränkt sich in ihren Anwendungen auf kosmologische Symmetrien. Die Methoden lassen sich allerdings auch auf andere Raumzeitsymmetrien übertragen.

Abstract

Already Einstein used insights from Maxwell theory in order to develop the theory of relativity. This connection between matter and gravity theories is a lot deeper than it first seems. This thesis shows a constructive method to derive the gravitational dynamics from prescribed matter dynamics. It can be applied to any matter theory on any tensorial background if the matter dynamics satisfy three basic physical conditions. The heart of this mechanism are the gravitational closure equations, a countable set of partial differential equations, whose solution determines the gravitational action functional. Practically, it can be very difficult to find general solutions to the closure equations. A significant simplification can be achieved by exploiting spacetime symmetries during the solution of the closure equations. This allows a direct derivation of the symmetry-reduced field equations. This thesis demonstrates explicitly how the standard model of particle physics and cosmological symmetries give rise to the Friedmann equations without knowledge of the Einstein equations. Additionally, the cosmological dynamics of general linear electrodynamics are constructed. The result demonstrates how a refined theory of electromagnetism generalises the Friedmann equations. While this thesis considers cosmological symmetries, the methods presented here can be applied to any other spacetime symmetry.

Contents

Chapter 1

Introduction

It was Maxwell electrodynamics and the Maxwell equations which inspired Einstein to develop the theory of special and later, general relativity [1, 2]. This demonstrates the deep connection between matter and gravity theories. This connection is even deeper than it seems in this particular example of Maxwell-Einstein theory. Indeed, it is the central result of the constructive gravity program that gravitational dynamics can and hence have to be calculated from prescribed matter dynamics by requiring the common canonical evolution of both theories. This construction of gravitational dynamics is said to gravitationally close the matter theory. It can be applied to any matter theory on any geometric background if the matter dynamics satisfy three basic physical conditions.

Technically, it is the solution of a countable set of partial differential equations, the *gravitational closure equations*, which gives rise to the gravitational Lagrangian. The three coefficient functions of this system of differential equations arise from the matter field equations by straightforward algebraic calculations [3]. See Fig. 1.1 for an illustration.

$$S_{\text{matter}} \xrightarrow{\text{calculate}} \begin{array}{c}\text{closure}\\ \text{equations}\end{array} \xrightarrow{\text{solve}} S_{\text{gravity}} \xrightarrow{\text{vary}} \begin{array}{c}\text{field}\\ \text{equations}\end{array}$$

Figure 1.1: Structure of the constructive gravity program starting from a prescribed matter action. From there, the gravitational closure are set up and from their solution, the gravitational action is constructed.

This insight has two important consequences. First, it is clear that there can be no one-size-fits-all gravity theory which exists independent from all phenomenological and theoretical knowledge or new discoveries about the matter inhabiting the Universe. Secondly, it diminishes the role of gravity theories to a mere consistency condition as the information about the gravitational dynamics is already contained in the matter field dynamics. It just needs to be extracted from those by setting up and solving the gravitational closure equations.

In practice, this extraction – the solution of the gravitational closure equations – can still be a difficult task, often so difficult that one cannot find a general solution.

This thesis demonstrates that a significant simplification be achieved by exploiting spacetime symmetries already at the level of the gravitational closure equations. By doing so, one obtains the symmetry-reduced gravitational Lagrangian from which the symmetric field equations follow by variation. This symmetry reduction can be applied for any matter theory which satisfies the three basic conditions of the constructive gravity program. Besides, any spacetime symmetry can be chosen for which an insertion into the action is interchangeable with an insertion into the field equations. More precisely, the terms and conditions [4] of symmetric criticality [5] have to be met. For the example cosmological symmetries, the structure of the symmetry reduction is displayed in Fig. 1.2 where the thick arrows represent the new, direct way to the Friedmann equations or their refinement for other matter models.

The greatest simplification can be achieved by choosing the (spatially) maximal symmetry of cosmology – spatial homogeneity and isotropy, also called FLRW symmetries. If one chooses to test the closure mechanism with standard model matter, the exploitation of the FLRW symmetries directly yields

$$
\begin{tikzcd}
S_{\text{matter}} \arrow[r] & \text{closure equations} \arrow[r] \arrow[d, Rightarrow] & S_{\text{gravity}} \arrow[r] \arrow[d] & \text{field equations} \arrow[d] & \\
& \text{symm.-red. closure eqns} \arrow[r, Rightarrow] & S_{\text{cosmo}} \arrow[r] & \text{(refined) Friedmann equations} \arrow[r] & \text{cosmol. dynamics}
\end{tikzcd}
$$

Figure 1.2: Structure of the a cosmological symmetry reduction of the constructive gravity program. The thin arrows represent the paths usually taken towards cosmological field equations. For all of them, a general solution to the closure equation is required. The new, direct way is illustrated by thick arrows and requires a solution to the simplified, symmetry-reduced closure equations. Diagram inspired by Ref. [6].

the Friedmann equations as the solution to the closure equations [7]. The Einstein equations – the full gravitational field equations in this case – were not needed.

While there is an infinity of matter models beyond the standard model, it seems reasonable to study models with only slight generalizations. Restricting oneself to the electromagnetic sector, such a minimal generalization is general linear electrodynamics. This theory of electrodynamics is the most general one that still features linear field equations and that possesses an action. In contrast to Maxwell theory, the electromagnetic sector of the standard model, birefringence of light in vacuum is possible. Technically, these additional effects can be traced back to the refined spacetime geometry which is now a fourth rank tensor field and no longer a metric. The dynamics for this fourth rank geometry determine if, where and how strong the birefringence occurs. These gravitational dynamics need to be determined from the closure equations. A general solution for them seems out of reach as the closure equations are too involved. By exploiting cosmological symmetries in the closure equations, one derives the refined Friedmann equations behind general linear electrodynamics.

The thesis is structured as follows. It has two big parts. The first one, Chapter 2, deals with the general theory of the gravitational closure of matter field equations. The second part is constituted by the Chapters 3 – 6. It demonstrates applications to the two aforementioned matter models and the development of the symmetry reduction.

In Chapter 2, a detailed introduction to the constructive gravity program will be given. As already mentioned, the starting point is a prescribed matter action together with a tensorial background geometry G. The key quantity determining the causality of the matter field equations is the principal polynomial which is read off from the corresponding matter field equations. Section 2.1 deals with the imposition of the three matter conditions which need to be satisfied for the matter dynamics giving rise to complete spacetime kinematics. These cover the dispersion of both massless and massive (point) particles and notion of observer frames.

It is the requirement of common canonical evolution of initial data which turns out to be so strong that it determines the gravitational dynamics via the closure equations. Thus, the geometry on the data surfaces is made dynamical. Section 2.2 reviews the foliation of spacetime into three-dimensional data surfaces and defines the canonical geometry for any arbitrary tensorial geometry G. One then identifies the actual dynamical geomtric degrees of freedom by introducing generalized tensor field components which parametrize the canonical geometry. This removes practical problems of previous work [8, 9] if the separation of the lapse function and shift vector field in the spacetime foliation imposes non-linear constraints on the canonical geometry.

Section 2.3 demonstrates the setup of the canonical phase space by promoting the generalized tensor field components of the canonical geometry to configuration fields. By imposing two embedding properties on the theory, one arrives at the central object of the following derivation, the constraint algebra. From there, one generalizes the ideas of Kuchar et al. [10, 11] on how to exploit the constraint algebra in order to derive the corresponding gravitational dynamics. This is done by casting the functional differential equations of the constraint algebra into a set of linear partial differential equations for gravitational Lagrangian, the *gravitational closure equations*, in Section 2.4. It is then Section 2.5 of Chapter 2 which recovers the gravitational spacetime action from the solution of the gravitational closure equations. The techniques developed in this chapter will be put to use in the following.

The indispensable test case for the constructive gravity is the standard model of particle physics (or any subsector thereof) as the resulting gravitational action is already known to be the Einstein-Hilbert action. As already shown in previous work [8, 9, 12], the Einstein-Hilbert action is indeed the solution of the gravitational closure equations. These proofs were however made using mostly the ideas originally developed by Kuchar et al. [10, 11]. This thesis demonstrates that the Einstein-Hilbert action can actually be constructed without these techniques by just using the closure equations in their newly developed formulation using the actual geometric degrees of freedom. This is the central result of Chapter 3.

After showing a general solution of the closure equations, the actual goal of this thesis, the symmetry reduction, is developed in Chapter 4. Again, starting from the standard model, imposition of FLRW symmetries onto the gravitational closure equations gives rise directly to the Friedmann equations of cosmology.

These case studies provide valuable insights into the foundations of the constructive gravity program, the solution of the closure equations and the simplifications achieved by the symmetry reduction. This will be put to good use in the two following chapters. As already mentioned before, this thesis chooses to deal with general linear electrodynamics as a matter model of interest which is beyond the standard model. Instead of a metric, it is a fourth rank tensor field which now serves as the background geometry. As general linear electrodynamics satisfies the three matter conditions [13], the constructive gravity program can and hence has to be used to determine the dynamics of this refined geometry. Calculating the three coefficient functions entering the closure equations in Chapter 5, it becomes clear that these will be very involved. A general solution for them seems to be out of reach to the present date.

Here, the symmetry reduction developed in this thesis proves to be the only way to derive the refinement of Friedmann equations stemming from general linear electrodynamics. In Chapter 6, the symmetry reduction of the metric spacetime is generalized to the refined geometry of general linear electrodynamics. But even after applying the maximal cosmological symmetry, the resulting symmetry-reduced closure equations are more involved and a solution is more difficult to obtain. By developing additional solution techniques whose detailed character is deferred to Chapter 6, the refined Friedmann equations will be obtained.

The thesis closes with a brief summary in Chapter 7 and a discussion of topics which follow from this thesis.

Chapter 2

Gravitational closure of matter field equations

The first part of this thesis is dedicated to the foundations of the constructive gravity program. Its starting point is a prescribed matter action containing tensorial matter fields A and in general, a collection of tensor fields G serving as the background geometry. The matter action depends locally[1] on the matter fields and ultralocally[2] on the geometry. Thus, the matter action only determines the dynamics of the matter fields by variation. If the matter theory satisfies three basic physical conditions, the matter conditions, the dynamics of the background geometry can also be extracted from the matter action using the constructive gravity program. For theories satisfying the three matter conditions, Section 2.1 shows how these matter theories determine the spacetime kinematics. These kinematics also contain the identification of three-dimensional initial value hypersurfaces of the four-dimensional spacetime. Section 2.2 demonstrates how the spacetime geometry induces a geometry on these hypersurfaces. The canonical geometry is then made dynamical in order to find the gravitational dynamics in Section 2.3. Two embedding properties are required for the gravitational theory which guarantee spacetime diffeomorphism invariance. This input is everything one needs to derive the gravitational closure equations in Section 2.4. While the entire construction is based on the canonical dynamics, there is a straightforward way to cast the canonical description back to a spacetime formulation. The central result of Section 2.5 is to write down a spacetime action using directly the solution of the closure equations. These results are also important for the symmetry reductions developed in later chapters. This result as well as the new techniques to identify the actual degrees of freedom of the canonical geometry present the significant innovations and advantages of this work compared to previous ones, such as Ref. [8, 9].

The results presented in this chapter have already been published as
M. Düll, F. P. Schuller, N. Stritzelberger and F. Wolz,
Phys. Rev. D97 (2018), 084036,
whose chapters II, III and IV are presented here.

2.1 Spacetime kinematics

This section reviews the steps from the matter field equations to the full kinematics of a tensorial spacetime. The essential object is the principal polynomial density $\tilde{P}$ which is constructed from the matter field equations. Subsequently, three matter conditions are imposed which can be understood as necessary conditions in order for the matter theory being canonically quantizable. This physically reasonable assumption already allows to define massless momenta, observer wordlines and an observer-independent split of momenta into those of positive and negative energy. The kinematical setup of spacetime is then

[1]that is, up to finite derivative order
[2]that is, up to zeroth derivative order

completed by a choice of a suitable de-densitization in order to form the principal polynomial tensor field P which gives rise to definitions of massive particles and observer frames.

Massless dispersion relations

In order to keep notation compact, the following construction assumes that there is only one matter field A and one spacetime geometry tensor field G. As laid out in Ref. [3], the entire construction automatically generalises to finite sets of matter and geometry fields.

The starting point of the constructive gravity program is a suitable scalar matter action $S_{\text{matter}}[\Phi; G)$ for a tensorial matter field Φ on a smooth, four-dimensional manifold M. Such a matter action also contains a tensor field G of a priori arbitrary valence such that the Lagrangian $\mathcal{L}$ is a scalar density of weight one. By variation with respect to the matter fields Φ, the matter field equations

$$\frac{\delta S_{\text{matter}}}{\delta\Phi(x)} = 0\,,$$

are obtained which are tensor density equations of weight one. It proves useful to assume the matter field equations to be linear in the matter field A in order to keep the derivation compact. This means that any solution Φ to the field equations can be scaled to arbitrary small values $\epsilon\Phi$ by a small factor $\epsilon > 0$ so that the source tensor density $\frac{\delta S_{\text{matter}}}{\delta G(x)}$ later appearing on the right hand side of the gravitational field equations can be scaled down to correspondingly small values. In other words, the back-reaction of the matter fields onto the spacetime geometry can be made arbitrarily small, just as expected for test matter particles.

By varying the matter action and potentially making implicit information such as integrability conditions explicit [14], the matter field equations can be written as

$$Q_{\mathcal{A}\mathcal{B}}^{i_1\ldots i_F}(G(x))\left(\partial_{i_1}\ldots\partial_{i_F}A^{\mathcal{B}}\right)(x) + \text{terms of lower derivative order in } A = 0\,,$$

where the indices $\mathcal{A}, \mathcal{B} = 1\ldots R$ label a basis of some R-dimensional $GL(4)$-representation under which the components of the matter fields transform.

The left hand side of the equation is a tensor density of weight one by construction although only partial derivatives appear in this equation. The highest order coefficient $Q_{\mathcal{A}\mathcal{B}}^{i_1\ldots i_F}(G(x))$ is a tensor density of weight one. Corresponding correction terms under a coordinate change appear in lower derivative order terms which are thus not tensorial ones.

If possible gauge ambiguities are fixed by choosing a certain gauge, the principal polynomial density $\tilde{P}(k)$ can be read off as

$$\tilde{P}(k) = \tilde{\rho}\,\det_{\mathcal{A},\mathcal{B}}\left(Q_{\mathcal{A}\mathcal{B}}^{i_1\ldots i_F}(G(x))\,k_{i_1}(x)\,\ldots k_{i_F}(x)\right)$$

for some covector k. The density factor ρ can be choosen freely in order to make $\tilde{P}$ actually a density of weight one which is not guaranteed by the determinant itself.

Inspect the following two simple examples for matter theories, Klein-Gordon theory and Maxwell electrodynamics, which share the same principal polynomial density $\tilde{P}$. The construction differs slightly as Maxwell theory is a gauge theory.

Klein-Gordon theory The most simplest example of a matter field theory is Klein-Gordon theory operating with one scalar field on a metric background. From the Klein-Gordon action[3]

$$S_{\text{KG}}[\phi; g) = \int \mathrm{d}^4x\,\sqrt{-\det g}\left(\frac{1}{2}\,g^{ab}\,(\partial_a\phi)\,(\partial_b\phi) + m^2\phi^2\right),$$

[3] The notation $S[\phi; g)$ denotes that S depends locally on the field ϕ, that is on ϕ and its derivatives, while it depends on g ultra-locally, that is only on g itself, but not on its derivatives.

variation with respect to the scalar field ϕ yield the field equations

$$0 = \sqrt{-g}\, g^{ab}\, \partial^2_{ab}\phi + \partial_a(\sqrt{-g} g^{ab})\, \partial_b\phi - m^2\, \sqrt{-g}\, \phi\,.$$

The first term is already the one with the highest derivative order of the scalar field ϕ. The principal polynomial density can be read off directly,

$$\tilde{P}(k) = \sqrt{-g}\, g^{ab} k_a k_b$$

which is already by construction a density of weight one. In general, reading off the principal polynomial density will require more effort. Already if a gauge ambiguity is present, one will have to perform extra steps as the following example of Maxwell electrodynamics demonstrates.

Maxwell electrodynamics Another prominent example of a matter theory is free Maxwell electrodynamics with a metric serving as the spacetime geometry. Starting point is the Maxwell action

$$S_{\text{Maxwell}}[A; g) = -\frac{1}{4} \int \mathrm{d}^4 x\, \sqrt{-g}\, g^{ac} g^{bd} F_{ab} F_{cd}$$

with the field strength $F = \mathrm{d}A$. Besides adding the identity $\mathrm{d}F = 0$ to the field equations, variation of the Maxwell action yields the field equations

$$0 = \partial_a \left(\sqrt{-g}\, g^{ac} g^{bd} F_{cd} \right)\,. \tag{2.1}$$

Sorting this out and using the definition of the field strength, one would read off that the principal polynomial density $\tilde{P}$ vanishes. This is caused by ignoring the gauge ambiguity of the theory. In order to fix this, one chooses a certain gauge. Here, the choice of Ref. [12] is adopted,

$$0 = \partial_a \left(\sqrt{-g}\, g^{ab} A_b \right)\,.$$

The field equations (2.1) are now

$$0 = \sqrt{-g}\, g^{ab} g^{cd}\, \partial^2_{ab} A_c + \text{lower derivative order in } A\,.$$

As sketched in Ref. [12], the principal polynomial density $\tilde{P}$ is constructed as

$$\tilde{P}(k) = (\det g)\, \det_{c,d}(\sqrt{-g}\, g^{cd}\, g^{ab}\, k_a\, k_b) = \left(\sqrt{-g}\, g^{ab}\, k_a\, k_b \right)^4$$

The prescription for constructing the principal polynomial density requires to remove repeated factors [13, 15]. Thus, one ends up at the second rank principal polynomial density

$$\tilde{P}(k) = \sqrt{-g}\, g^{ab} k_a k_b$$

just as expected from the previous result of Klein-Gordon theory. This result transfers to the entire standard model of particle physics which has precisely this second degree principal polynomial density [14]. Thus, it is admissible to study any subsector of the standard model in order to determine the gravitational dynamics behind it as all subsectors share the exact same geometric and causal structure. The resulting gravitational dynamics are the ones known from general relativity as recovered by Chapter 3 of this thesis.

Reading off the principal polynomial density by choosing a gauge is a very specific method which can hardly be generalized to refined matter theories. A more general approach of finding the principal polynomial density respecting the gauge ambiguities involves the infinite frequency limit of modes of the prescribed test matter dynamics which is physically indistinguishable from massless modes. Using a WKB expansion and a generalization of an argument first given by Itin [16], one finds a general recipe for deriving $\tilde{P}$. As the topic of constructing principal polynomial densities from matter field equations is not relevant for the practical purposes of this thesis, please refer to Ref. [3] for more details on this.

The roots of the principal polynomial density determine the dispersion of massless particles which satisfy

$$\tilde{P}(k) = 0\,.$$

As this equation is only concerned with the roots of the principal polynomial density, its density character is not relevant. This will be different in the following. Thus, the principal polynomial density $\tilde{P}$ is cast into the principal tensor field P by choosing an everywhere non-vanishing density factor ρ of opposite weight. This density factor is constructed purely from the geometry G itself and not of any of its derivatives. The choice of ρ only affects the definition of massive point particles, but has no influence on any field-theoretic considerations or the massless point particles which are governed only by $\tilde{P}$.

The principal polynomial $P(k)$ is the central quantity determining the spactime kinematics in the following. The three matter conditions are actually conditions on the principal polynomial.

Matter conditions

The principal tensor field P constructed from the matter field equations has to satisfy three matter conditions which actually are two hyperbolicity and one energy condition. It is important to note that the three matter conditions are purely classical. They are necessary criteria for the matter theory *to be canonically quantizable*.

First matter condition: Predictivity

The first technical requirement is the hyperbolicity of the principal polynomial $P(x)$ at every point $x \in M$ which physically means that there is a well-posed initial value problem for the classical matter theory. For the polynomial P being hyperbolic, there has to be a covector $h \in T^*_xM$ such that $P(h)(x) \neq 0$ and the equation $P(q + \lambda h)(x) = 0$ has only real solutions $\lambda \in \mathbb{R}$ for any further covector $q \in T^*_xM$. If such a covector h exists, there will be an open and convex cone $C_x(P, h)$ containing all hyperbolic covectors lying in one connected set with h.

Practically important is the case of reducible polynomials, that is, the polynomial P can be written as a finite product

$$P(x) = P_1(x)\,P_2(x)\dots P_f(x)\,,$$

of lower order polynomials. The polynomial $P(x)$ is hyperbolic if and only if each of the lower order polynomials is hyperbolic. The corresponding hyperbolicity cone $C(x)$ is given by the intersection of the hyperbolicity cones of the lower order polynomials,

$$C(P, h) = C_1(P_1, h) \cap C_2(P_2, h) \cap \cdots \cap C_f(P_f, h)\,.$$

This shows that removing repeated factors in the principal polynomial does not result in a loss of information concerning the hyperbolicity cone. As Ref. [15] points out, removing repeated factors is a crucial and necessary technical requirements for the constructions to follow. From here on, it is thus assumed that all repeated factors have been removed from the principal tensor P.

For an illustration of different examples of hyperbolicity cones, see Fig. 2.1.

Second matter condition: Momentum-velocity duality

The second technical requirement is the hyperbolicity of the dual polynomial $P^\#$. It is defined as

$$P^\#(x) : T_xM \to \mathbb{R}$$
$$P^\#(x) := P^\#_1(x)\,P^\#_2(x)\dots P^\#_f(x)\,,$$

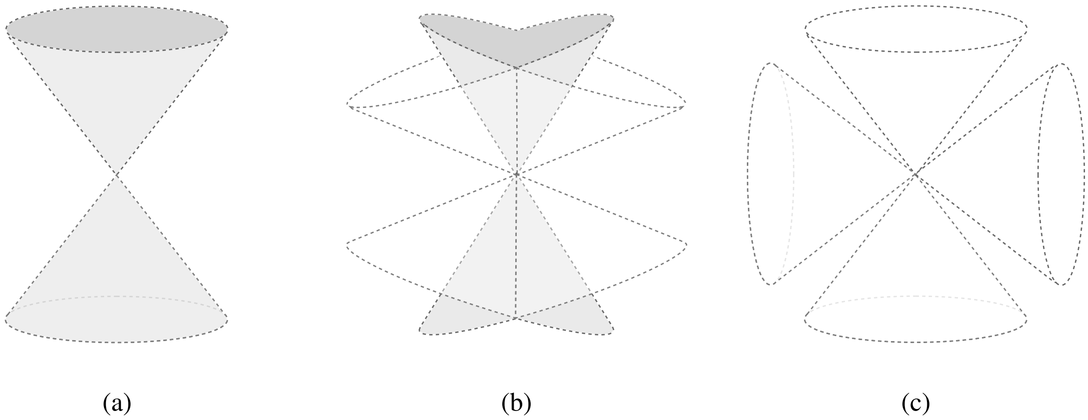

Figure 2.1: Three examples for hyperbolicity cones of polynomials; (a) for a hyperbolic polynomial of second degree, (b) for a hyperbolic, reducible one of fourth degree and (c) for a non-hyperbolic fourth degree polynomial for which there is no hyperbolicity cone. Figure designed by Florian Wolz, taken from Ref. [3].

with $P^\#_1(x), \ldots, P^\#_f(x)$ being polynomial maps $T_xM \to \mathbb{R}$ of minimal degree such that for all covectors k in the set

$$N_i^{\text{smooth}} := \left\{ k \in T_x^*M | P_i(x,k) = 0 \text{ and } \frac{\partial P_i}{\partial k}(x,k) \neq 0 \right\} ,$$

the gradients $\frac{\partial P_i}{\partial k} \in T_xM$ are the roots of the dual polynomials $P^\#_i$; that is,

$$P^\#_i \left(x, \frac{\partial P_i}{\partial k}(x,k) \right) = 0 \quad \forall k \in N_i^{\text{smooth}} .$$

This definition does not fully determine the dual polynomial, but leaves a real factor function open. The roots of the dual polynomial and with it, the hyperbolicity, are unaffected by this ambiguity. The hyperbolicity of the principal polynomial P guarantees the existence of its dual $P^\#$ while $P^\#$ is not necessarily hyperbolic [15]. Requiring $P^\#$ to be hyperbolic as well can be understood as a sufficient condition for P to be recoverable from its dual $P^\#$ as a double dual,

$$P(x) \sim P^{\#\#}(x) .$$

The dual principal polynomial can be interpreted using the characteristic curves $x : \mathbb{R} \to M$ of the initial matter field equation; these are stationary curves of the Hamiltonian action

$$S[k, x; \rho) := \int \mathrm{d}\lambda \left[k_a(\lambda)\, \dot{x}^a(\lambda) - \rho(\lambda)\, P(x(\lambda), k(\lambda)) \right] .$$

This Hamiltonian action can be rewritten in Lagrangian form as

$$S[x; \mu) := \int \mathrm{d}\lambda\, \mu(\lambda)\, P^\#(x(\lambda), \dot{x}(\lambda))$$

for any hyperbolic principal polynomial P [15]. Thus, hyperbolicity[4] of both P and $P^\#$ ensures the way back and forth from the Hamiltonian to the Lagrangian description of characteristic curves – physically speaking, the trajectories of massless particles. Correspondingly, there is – up to scale – a momentum associated to velocity of a massless particle and vice versa if the principal polynomial P is bi-hyperbolic.

[4]If both P and $P^\#$ are hyperbolic, it will be said that the prinicipal polynomial is *bi-hyperbolic* for the remainder of this thesis.

Third matter condition: Energy distinction

The first and second matter condition provided the motivation for the bi-hyperbolicity of the principal polynomial P. A missing piece in the construction is the physical meaning of the hyperbolicity cones of P and $P^{\#}$. This is subject to the third matter condition. As the starting point, recall that momenta of massless particles satisfy $P(x,k) = 0$. In order to divide the set of all massless momenta in those of either positive or negative energy in an observer-independent way, one needs to find the largest possible set of local observers that still agree on this split. More precisely, one wants to find the open set O_x in each tangent space T_xM such that for any particular non-trivial massless momentum k, one has either $k \in O_x^+$ or $k \in -O_x^+$; the dual cone

$$O_x^+ := \{q \in T_x^*M | U(q) > 0 \quad \forall U \in O_x\}$$

implements the observer-dependent positive energy condition when intersected with the set of all non-vanishing massless momenta. Formally, the cone N_x of massless momenta at every spacetime point is required to decompose into disjoint sets N_x^+ and N_x^-,

$$N_x \backslash \{0\} = N_x^+ \dot{\cup} N_x^- \,, \tag{2.2}$$

where the two subsets $N_x^\pm$ are defined as the intersections of N_x with $\pm O_x^+$,

$$N_x^\pm := N_x \cap (\pm O_x^+) \,.$$

It is thus question what the largest cone O_x is that can be chosen. If the condition (2.2) is satisfied, any one of the hyperbolicity cones of $P^{\#}$ will provide the largest cone O_x which reduces the choice to the finite set of hyperbolicity cones of $P^{\#}$ at each point x of the spacetime manifold. A smooth choice of the cone O_x through spacetime is then provided by a smooth vector field T that is everywhere hyperbolic with respect to the dual polynomial $P^{\#}$ such that there is a smooth distribution of future-directed observer cones $O_x = C_x(P^{\#}, T)$. For an illustration, see Fig. 2.2.

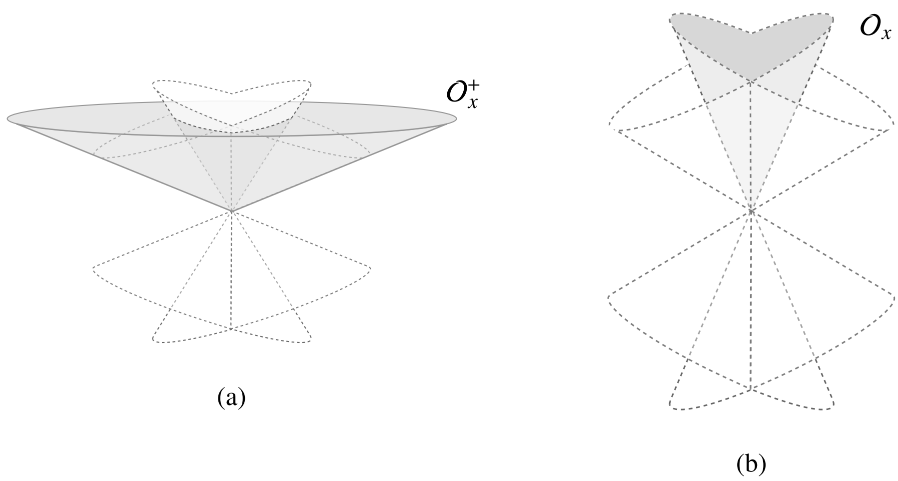

Figure 2.2: (a) Positive energy cone O_x^+ as the dual of the (b) observer cone O_x. The positive energy cone covers all momenta unanimously judged as positive by all observes. The observer cone contains all tangent vectors to observer wordlines at one point. Figure designed by Florian Wolz, taken from Ref. [3].

These three matter conditions up to now only exploit the roots of the prinicipal polynomial P and its dual $P^{\#}$ at each spacetime point. Also the hyperbolicity and observer cones are defined entirely in terms of the roots of P and $P^{\#}$ although all tangent vectors in the observer cone O_x are not roots of $P^{\#}$. The following section deals with the completion of the spacetime kinematics. First, the dispersion relation for massive particles is implemented. Secondly, the so called Legendre map is defined in order to define local observer frames.

Dispersion relation for massive particles

Recognize that for bi-hyperbolic principal polynomial, a smooth choice of hyperbolicity cones in cotangent space corresponds to the smooth choice of observer cones O_x on each tangent space. This cone is called cone C_x of positive energy massive particles satisfying $C_x \subseteq O_x^{\perp}$. Already in Ref. [17], it has been shown that hyperbolicity cones are open convex cones whose boundary is null with respect to the underlying polynomial while the interior has constant sign. Since up to now, only the roots of the polynomial have been used in the construction, the sign of the polynomial in the interior of the hyperbolicity can be chosen freely and in the course of this construction, it is set to

$$P(x, C_x) > 0 \quad \forall\, x \in M\,.$$

With this choice at hand, the mass $m > 0$ of a momentum $q \in C_x$ is defined by

$$P(x, q) = m^{\deg P}\,.$$

Note that in this definition, the density $\rho(x)$ that makes the functional density $\tilde{P}$ into a function plays a role for the first time in the construction of spacetime kinematics. For examples of a quadric and a quartic mass shell, see Fig. 2.3.

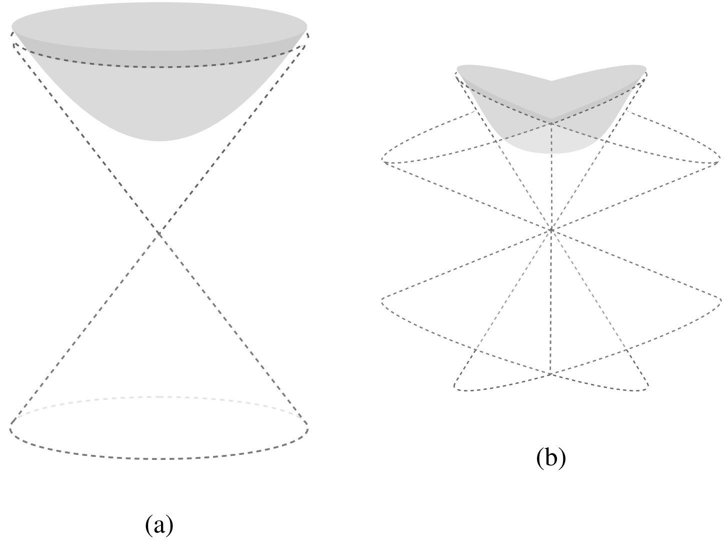

Figure 2.3: Two examples of positive energy mass shells; (a) Quadric mass shell of a prinicpal polynomial of second degree and (b) quartic mass shell of a fourth degree polynomial; both satisfy the three matter conditions. Figure designed by Florian Wolz, taken from Ref. [3].

In analogy to the dispersion relation for massless particles, the one for massive particles is written as a constraint in a Hamiltonian formulation as

$$S_{\text{massive}}[x, q; \mu) = \int \mathrm{d}\lambda \left[q_a(\lambda)\, \dot{x}^a(\lambda) - \mu(\lambda) \ln P(x(\lambda), m^{-1}\, q(\lambda)) \right].$$

The momentum q can be eliminated from this by virtue of the Legendre map

$$\begin{aligned} \ell_x &: C_x \to T_x M\,, \\ \ell_x(q) &:= \frac{1}{\deg P} \frac{\partial \ln P}{\partial q}(x, q) \end{aligned} \tag{2.3}$$

at each point x of the spacetime manifold M. The inverse Legendre map $\ell_x^{-1} : \ell_x(C_x) \to C_x$ exists due to the bi-hyperbolicity of P. Thus, the Lagrangian for the trajectory x of a massive particle of mass m can be written as

$$S_{\text{massive}}[x] = \int \mathrm{d}\lambda\, m \left[P^*(x(\lambda), \dot{x}(\lambda)) \right]^{\frac{1}{\deg P}} \tag{2.4}$$

with the (non-polynomial) map P^* defined as

$$P^*(x) : \ell_x(C_x) \to \mathbb{R}\,,$$
$$P^*(x, v) := P(x, \ell_x^{-1}(v))^{-1}\,.$$

The massive point particle action (2.4) is invariant under strictly monotonously increasing reparametrizations of the trajectory. This is the final piece of information that is needed for the construction of local observer frames; the parametrizations with

$$P^*(x(\lambda), \dot{x}(\lambda)) = 1$$

are distinguished as they correspond to trajectories with massive particle momenta

$$q(\lambda) = m\,\dot{x}(\lambda)$$

proportional to their velocity with the proportionality factor given by the particle's rest mass. Defining proper time with such parameters, the observer wordlines are defined by the two requirements

$$\dot{x}(\lambda) \in O_{x(\lambda)} \quad \text{and} \quad P^*(x(\lambda), \dot{x}(\lambda)) = 1\,.$$

One identifies the purely spatial directions $S(\lambda) \subset T_{x(\lambda)}M$ seen by an observer as

$$\ell_{x(\lambda)}\,(\dot{x}(\lambda))\;(S(\lambda)) = 0\,.$$

To sum up, the three matter conditions presented in this section have a physical motivation. They are necessary criteria for the matter theory to be canonically quantizable although all constructions made here are purely classical. This physically very reasonable assumption of possessing a quantum theory already proves to be sufficient to define local observer frames which are compatible with the original matter dynamics. This physical motivation tops off the mathematical character of the three matter conditions.

The dispersion relations for both massless and massive particles as well as the definition of local observer frames are the important ingredients for the full spacetime kinematics which were extracted purely from the prescribed matter field equations. In the following construction of the gravitational dynamics, the Legendre map ℓ_x will be the essential piece of information carried over from the matter field dynamics.

2.2 Canonical geometry

The spacetime kinematics constructed from the prescribed matter theory are now used to foliate the spacetime manifold into three-dimensional initial data surfaces. The spacetime geometry induces a geometry on these hypersurfaces. Normal and tangential deformation operators evolve data between the single leaves of the foliation. Calculating the commutation relations between them provides the hypersurface deformation algebra for any spacetime triplet (M, G, P) satisfying the three matter conditions imposed in the previous section.

In order to make the geometry on the leaves the dynamical object, the canonical geometry is introduced which mimics the induced geometry. The canonical geometry is potentially subject to non-linear constraints. In contrast to previous work such as Ref. [8], new techniques have been developed in order to capture such non-linear constraints automatically. One can then proceed to set up a canonical phase space and derive the gravitational dynamics.

Foliation of spacetime

Foliating spacetime into leaves of initial data hypersurfaces and subsequently inducing a canonical geometry is a standard technique in general relativity. It is straightforward to extend this to spacetimes (M, G, P) carrying canonically quantizable matter field dynamics. The only significant innovation is the projection of spacetime geometry tensor fields G of arbitrary valence to initial data surfaces. How this is done will be shown in the following when also the usual foliation techniques will be reviewed.

Let $X_t : \Sigma \hookrightarrow M$ be a one-parameter family of maps embedding a three-dimensional manifold Σ such that the four-dimensional manifold M is foliated into hypersurfaces $X_t(\Sigma)$ with everywhere hyperbolic co-normal $\epsilon^0(t,\sigma)$ for $\sigma \in \Sigma$. Choosing coordinates y^α on Σ, there is a one-parameter family of spacetime vectors

$$e_0(t,\sigma) := \ell_{X_t(\sigma)}\left(\epsilon^0(t,\sigma)\right) \quad , \quad e_\alpha(t,\sigma) := X_{t*}\left(\left(\frac{\partial}{\partial y^\alpha}\right)_\sigma\right)$$

for $t \in \mathbb{R}$ and $\sigma \in \Sigma$. Of course, the choice of a hyperbolic co-normal ϵ^0 is not unique but scales with an arbitrary factor. In order to make the choice unique, the co-normal is required to satisfy the normalization condition

$$P(X_t(\sigma), \epsilon^0(t,\sigma)) = 1\,.$$

Due to the construction of e_0 using the Legendre map ℓ, there is also the annihilation condition

$$P(X(t), \epsilon^\alpha(t,\sigma), \epsilon^0(t,\sigma), \dots, \epsilon^0(t,\sigma)) = 0$$

which together with the normalization condition imposes possibly nonlinear constraints on the geometry induced on the leaves of the foliation. While the induced geometry satisfies these constraints by construction, the constraints will have to be imposed by hand to the to be constructed canonical geometry.

The set of vectors $\{e_0(t,\sigma), \dots, e_3(t,\sigma)\}$ constitutes the orthogonal projection frame along each embedded hypersurface $X_t(\Sigma)$. Together with the unique dual frame $\left\{\epsilon^0(t,\sigma), \dots, \epsilon^3(t,\sigma)\right\}$, spacetime tensors of arbitrary valence can be projected to the hypersurface Σ.

In the following, such projections will be performed for the spacetime tangent vector field $\dot{X}_t$ constructed from the family of embedding maps, the spacetime geometry G and the principal polynomial P. The manifold Σ can be thought of as a „cinema screen“ on which the evolution of the four-dimensional spacetime is shown as a movie as the foliation parameter t evolves.

First, consider the spacetime tangent vector field $\dot{X}_t$. It is the tangent vector field to the congruence of spacetime curves corresponding to a point $\sigma \in \Sigma$ that does not move on Σ under evolution of t. Its projection to Σ gives rise to two one-parameter families of fields, the induced lapse and shift fields

$$\mathbf{n}(t) := \epsilon^0(t)\left(\dot{X}_t\right) \quad \text{and} \quad \mathbf{n}^\alpha(t) := \epsilon^\alpha(t)\left(\dot{X}_t\right)\,.$$

One of the most important intermediate steps towards setting up the gravitational closure equations for any admissible spacetime (M, G, P) is the projection of the spacetime geometry G to the hypersurface Σ and obtaining several one-parameter families of tensors on Σ. Practically, one obtains these components by inserting either the frame field $e_a(t,\sigma)$ into a slot of G that requires a vector or the dual field $\epsilon^a(t,\sigma)$ into those slots that require a covector. Consider the following example.

Projecting components of a metric Consider a metric tensor G^{ab} for which one completely ignores any symmetries and just has a $(2,0)$-tensor field. One identifies the components

$$\begin{aligned}
\mathbf{g}^{00} &:= G(\epsilon^0, \epsilon^0)\,,\\
\mathbf{g}^{0\alpha} &:= G(\epsilon^0, \epsilon^\alpha)\,,\\
\mathbf{g}^{\alpha 0} &:= G(\epsilon^\alpha, \epsilon^0)\,,\\
\mathbf{g}^{\alpha\beta} &:= G(\epsilon^\alpha, \epsilon^\beta)\,.
\end{aligned}$$

These are the 16 components of the induced geometry on Σ. Before setting up the canonical geometry, one needs to implement the constraints arising from the normalization and annihilation condition as well as the linear symmetry constraints as the spacetime metric G^{ab} is a symmetric $(2,0)$-tensor field. This will be performed in the next step.

After projecting the spacetime geometry, the last quantity to be projected to Σ is the principal tensor P. This results in $\deg P + 1$ many tensor fields

$$\mathbf{p}^{\alpha_1\ldots\alpha_i}(t,\sigma) := P\left(X(t,\sigma), \epsilon^{\alpha_1}(t,\sigma), \ldots, \epsilon^{\alpha_i}(t,\sigma), \epsilon^0(t,\sigma), \ldots, \epsilon^0(t,\sigma)\right)$$

for $i = 0, \ldots, \deg P$. Due to the total symmetry of P in its indices, one can use this much simpler index notation for the components $\mathbf{p}$. Normalization and annihilation condition translate to $\mathbf{p}(t,\sigma) = 1$ and $\mathbf{p}^\alpha(t,\sigma) = 0$. As an important remark, note that all projected components $\mathbf{g}$ and $\mathbf{p}$ are not only tensor fields on Σ, but also functionals of the embedding map X_t which will become technically important in the next step.

Hypersurface deformation algebra

In order to obtain the central quantity of this section, the hypersurface deformation algebra, consider the functional differential operators,

$$\mathbf{H}_t(n) := \int_\Sigma \mathrm{d}^3 z\, n(z)\, e_0^a(t,z)\, \frac{\delta}{\delta X_t^a(z)}\,,$$
$$\mathbf{D}_t(\vec{n}) := \int_\Sigma \mathrm{d}^3 z\, n^\alpha(z)\, e_\alpha^a(t,z)\, \frac{\delta}{\delta X_t^a(z)}$$

for arbitrary test functions n and $\vec{n}$ on Σ. The operators act on functionals of the embedding map $X_t : \Sigma \hookrightarrow M$. By identifying $n := \mathbf{n}$ and $\vec{n} := \vec{\mathbf{n}}$, the operators get their geometric interpretation as normal and tangential deformation operators. Note that the only piece of kinematical information is the Legendre map ℓ_x appearing in the definition of the frame vector e_0.

As the deformation operators $\mathbf{H}_t$ and $\mathbf{D}_t$ are vector fields over the infinite-dimensional manifold of embeddings $\mathrm{Emb}(\Sigma, M)$, the Lie brackets of the operators can be calculated. These constitute the *hypersurface deformation algebra*

$$[\mathbf{H}_t(n), \mathbf{H}_t(m)] = -\mathbf{D}_t\left((\deg P - 1)\, \mathbf{p}_t^{\alpha\beta}\, (m\partial_\beta n - n\partial_\beta m)\partial_\alpha\right)\,, \tag{2.5}$$
$$\left[\mathbf{D}_t(\vec{n}), \mathbf{H}_t(m)\right] = -\mathbf{H}_t\left(\mathcal{L}_{\vec{n}} m\right)\,, \tag{2.6}$$
$$\left[\mathbf{D}_t(\vec{n}), \mathbf{D}_t(\vec{m})\right] = -\mathbf{D}_t\left(\mathcal{L}_{\vec{n}} \vec{m}\right)\,. \tag{2.7}$$

Note that the only piece of kinematical information contained in the deformation algebra is the component $\mathbf{p}_t^{\alpha\beta}$ of the principal polynomial in the first algebra relation. This is already precisely the step in which the matter field equations directly inject information into the gravitational dynamics.

From a mathematical point of view, it is important to note that the above Lie brackets fail to close with only structure constants, but instead require structure functions. This technical detail has some complicating implications, most prominently that one cannot simply represent the deformation algebra as a Lie algebra of functionals of some geometric phase space variables without making further requirements as shown later.

Canonical geometry

Up to now, the spacetime geometry was considered as given and the induced geometry on the leaves of the foliation as the derived and thus secondary quantity. Switching to the canonical point of view, the geometry on the leaves of the foliation is promoted to the primary quantity and the spacetime geometry is reconstructed from it. This change of perspective results in the fact that four generically non-linear constraints that the induced geometry satisfies by construction must now be reinstated for the canonical geometry.

Let $\mathbf{g}_t^{\mathcal{A}}$ be the induced geometry from the spacetime geometry G by virtue of a foliation $X_t : \Sigma \hookrightarrow M$ and let $\mathbf{n}_t$ and $\vec{\mathbf{n}}_t$ be the induced lapse and shift fields. Now, introduce $g^{\mathcal{A}}(t)$, $n(t)$ and $\vec{n}(t)$ as new, independent one-parameter families of tensor fields on Σ capturing the tensor structure of the induced geometry tensor fields. As already mentioned, the induced geometry satisfies constraints imposed by the normalization and annihilation condition. These constraints are in general non-linear and range beyond the mere valence of the tensor fields $g^{\mathcal{A}}$. The valence, however, is the only information carried over to the fields $g^{\mathcal{A}}$. Besides setting up these constraints, also all quantities built from the induced geometry have to be introduced as quantities built from $g^{\mathcal{A}}$. Among them are the components $p^{\alpha_1 \dots \alpha_i}$ of the principal tensor P which now have the same index structure $p^{\alpha_1 \dots \alpha_i}$ and are now functions of $g^{\mathcal{A}}$ in the same way as the $\mathbf{p}^{\alpha_1 \dots \alpha_i}$ were of $\mathbf{g}^{\mathcal{A}}$.

Reinstating the normalization condition $p(g)(t) = 1$ and annihilation condition $p^{\alpha}(g)(t) = 0$ results in four – generically non-linear – conditions on the canonical geometry $g^{\mathcal{A}}$ which remove four degrees of freedom from $g^{\mathcal{A}}$. These non-linear constraints can be covered by introducing a suitable parametrization of $g^{\mathcal{A}}$ in terms of configuration fields determining the actual geometric degrees of freedom in the next subsection. Similarly, any algebraic symmetries of the spacetime geometry G is automatically passed on to the induced geometry $\mathbf{g}^{\mathcal{A}}$ and has to be reinstated for the canonical geometry $g^{\mathcal{A}}$ as well. These constraints are however linear and homogeneous conditions for a suitable projector Π,

$$\left(\delta^{\mathcal{A}}_{\mathcal{B}} - \Pi^{\mathcal{A}}{}_{\mathcal{B}}\right) g^{\mathcal{B}} = 0 \,.$$

These additional constraints can usually be implemented without extra effort alongside the generically non-linear ones from the normalization and annihilation conditions.

Parametrization of the canonical geometry

The generically non-linear constraints on the canonical geometry stemming from the normalization and annihilation condition yield conditions that cannot be implemented by simply cutting away tensor field components among the $g^{\mathcal{A}}$ while keeping others. The situation is similar to the one of a particle in Euclidean space constrained to the submanifold of a circle. One cannot simply cut away a coordinate, but one has to respect a non-linear constraint in both coordinates. The conceptually and technically most suitable solution in classical mechanics is the introduction of generalized coordinates.

The same idea applies here. Introduce exactly as many configuration fields $\varphi^1, \dots, \varphi^F$ as needed in order to bijectively parametrize the field components $g^{\mathcal{A}}$ of the canonical geometry. These configuration fields satisfy normalization and annihilation condition and algebraic symmetries inherited from the spacetime geometry by construction. Technically, this requires to choose a suitable F-dimensional manifold Φ and smooth maps $\hat{g}^{\mathcal{A}} : \Phi \to \mathbb{R}$ such that any canonical geometry $g^{\mathcal{A}}$ generated by $\hat{g}^{\mathcal{A}}(\varphi^1, \dots, \varphi^F)$ satisfies

$$\begin{aligned} p(\hat{g}(\varphi(t,\sigma))) &= 1 \,, \\ p^{\alpha}(\hat{g}(\varphi(t,\sigma))) &= 0 \,, \\ \left(\delta^{\mathcal{A}}_{\mathcal{B}} - \Pi^{\mathcal{A}}{}_{\mathcal{B}}\right) \hat{g}^{\mathcal{B}}(\varphi(t,\sigma)) &= 0 \end{aligned}$$

for any $\sigma \in \Sigma$ and t in the range of the foliation parameter. If one single map $\hat{g}^{\mathcal{A}}$ does not suffice to cover the required range, the usual chart transition constructions can be invoked. Note that the total number F of configuration variables is the total number of all $g^{\mathcal{A}}$ minus one from the normalization condition, minus three from the annihilation condition and minus the dimension of the eigenspace $\mathrm{Eig}_1(\Pi)$ of the projector Π.

The following example demonstrates the just described procedure. Consider a metric spacetime $\left(M, G_{\mathrm{metric}}, G^{-1}_{\mathrm{metric}}\right)$. There are 16 components of the spacetime metric. There are $F = 16 - 1 - 3 - 6 = 6$ configuration field variables which can be written as a not further constrained metric tensor on the three-dimensional manifold Σ as all constraints are linear in this example.

The bijective parametrization requires that there are not only the maps $\hat{g}(\varphi)$, but also the inverse maps $\hat{\varphi}^{A}$. They take any collection $g^{\mathcal{A}}$ – even if symmetry conditions and constraints are not met – to a real number. The concetanation of both maps satisfies

$$\left(\hat{\varphi}^A \circ \hat{g}\right)(\varphi) = \varphi^A \quad \text{for} \quad A = 1 \ldots F\,.$$

The opposite composition $(\hat{g}^{\mathcal{A}} \circ \hat{\varphi})$ projects any set of $g^{\mathcal{A}}$ – even if the latter does not satisfy symmetry conditions or the constraints from normalization and annihilation conditions – to a set that does.

The parametrization maps $\hat{g}$ and $\hat{\varphi}$ need to be determined once during the setup of the theory. Additionally, also the two derivative maps

$$\frac{\partial \hat{\varphi}^A}{\partial g^{\mathcal{A}}}(\hat{g}(\varphi)) \quad \text{and} \quad \frac{\partial \hat{g}^{\mathcal{A}}}{\partial \varphi^A}(\varphi)$$

appear at various stages of the setup as intertwiners between the components of the canonical geometry – labelled by hypersurface indices collected in $\mathcal{A}$ – and the configuration fields, labelled by indices A. Using the just described features of the parametrization maps $\hat{g}$ and $\hat{\varphi}$, the following identities hold,

$$\begin{aligned}
\frac{\partial \hat{\varphi}^A}{\partial g^{\mathcal{A}}}(\hat{g}(\varphi)) \frac{\partial \hat{g}^{\mathcal{A}}}{\partial \varphi^B}(\varphi) &= \delta^A_B\,, \\
\frac{\partial \hat{g}^{\mathcal{A}}}{\partial \varphi^A}(\varphi) \frac{\partial \hat{\varphi}^A}{\partial g^{\mathcal{B}}}(\hat{g}(\varphi)) &= \mathcal{T}^{\mathcal{A}}{}_{\mathcal{B}}\,.
\end{aligned}$$

The left hand side of the second equation defines the right hand side $\mathcal{T}$ and one easily checks that it is a projector.

Introducing the canonical geometry and the configuration fields parametrizing the latter provides the preliminary stage for determining the associated canonical gravitational dynamics. They can be constructed by promoting the configuration fields to the configuration variables of a suitable geometric phase space. The setup of the geometric phase space and the subsequent way to the gravitational closure equations is presented in the following section.

2.3 Canonical gravitational dynamics

The central element of the canonical gravitational dynamics is the geometric phase space on which the dynamics are running. In the previous section, the configuration fields φ^A were introduced as the actual degrees of freedom of the canonical geometry. The configuration fields φ^A are now promoted to configuration variables of a phase space. The phase space structure is completed by adjoining canonically conjugate momentum fields π_A with respect to the field-theoretical Poisson bracket which is defined as

$$\{F, G\} := \int_\Sigma \mathrm{d}^3 z \left(\frac{\delta F}{\delta \varphi^A(z)} \frac{\delta G}{\delta \pi_A(z)} - \frac{\delta G}{\delta \varphi^A(z)} \frac{\delta F}{\delta \pi_A(z)} \right)$$

for any two functionals $F[\varphi, \pi]$ and $G[\varphi, \pi]$ of the canonical configuration variables φ^A and the canonical momenta π_A. Of course, there is an ambiguity in the choice of the associated momenta π_A and the Poisson bracket should be invariant under such ambiguities as well as under changes of coordinates on Σ. Check Ref. [3] for the associated proof.

Of course, the canonical gravitational dynamics using the configuration fields φ have to match the entire spacetime geometry G. In order to guarantee this, one has to define two phase space functionals whose action on the configuration variables mimic the action of the normal and tangential deformation operators defined in section 2.2 on the projected geometry. One then imposes two embedding properties on these functionals in order to give them the correct spacetime interpretation.

First embedding property: Phase space avatars of deformation operators

Define the two functionals

$$\mathcal{H}(n) := \int_\Sigma \mathrm{d}^3z\, n(z)\, \mathcal{H}[\varphi(z), \pi(z)]$$

$$\mathcal{D}(\vec{n}) := \int_\Sigma \mathrm{d}^3z\, n^\alpha(z)\, \mathcal{D}_\alpha[\varphi(z), \pi(z)]$$

in terms of local functionals $\mathcal{H}$ and $\mathcal{D}_\alpha$ of the geometric phase space variables. These local functionals evolve the canonical data between leaves of a given spacetime foliation in the same manner as the deformation operators $\mathbf{H}_t(n)$ and $\mathbf{D}_t(\vec{n})$ do when applied to the induced geometry on the leaves. As a technical expression, this requirement reads

$$\mathbf{H}_t(n)\, \mathbf{g}_t^{\mathcal{A}} \stackrel{!}{=} -\left\{\mathcal{H}(n), g^{\mathcal{A}}\right\}\,, \tag{2.8}$$

$$\mathbf{D}_t(\vec{n})\, \mathbf{g}_t^{\mathcal{A}} \stackrel{!}{=} -\left\{\mathcal{D}(\vec{n}), g^{\mathcal{A}}\right\}\,, \tag{2.9}$$

where the equal signs are to be understood as that the right hand side is the same function of the canonical geometry $g^{\mathcal{A}}$ as the left hand side is of the induced geometry $\mathbf{g}^{\mathcal{A}}$. As a direct consequence of this requirement, the Poisson algebra

$$\{\mathcal{H}(n), \mathcal{H}(m)\} = \mathcal{D}\Big((\deg P - 1)\, p^{\alpha\beta}\, (m\,\partial_\beta n - n\,\partial_\beta m)\,\partial_\alpha\Big)\,, \tag{2.10}$$

$$\{\mathcal{D}(\vec{n}), \mathcal{H}(m)\} = \mathcal{H}\left(\mathcal{L}_{\vec{n}} m\right)\,, \tag{2.11}$$

$$\{\mathcal{D}(\vec{n}), \mathcal{D}(\vec{m})\} = \mathcal{D}\left(\mathcal{L}_{\vec{n}} \vec{m}\right) \tag{2.12}$$

guarantees that there is no inconsistency with the hypersurface deformation algebra (2.5) – (2.7). The two requirements (2.8) and (2.9) and the three Poisson algebra relations (2.10) – (2.12) are the central quantities for the derivation of the gravitational closure equations which is shown in the following section. Before that, a second embedding property has to be established in order to determine the Hamiltonian describing the canonical gravitational dynamics in terms of the two functionals $\mathcal{H}$ and $\mathcal{D}$.

Second embedding property: Spacetime diffeomorphism invariance

The second embedding property requires the resulting gravitational theory to be invariant under spacetime diffeomorphisms, meaning that the evolution of initial data between two fixed Cauchy surfaces is independent of the intermediate foliation. Generalising the arguments given in Ref. [10] implies that the Hamiltonian is of totally constrained form

$$H[\varphi, \pi; n, \vec{n}) = \mathcal{H}(n) + \mathcal{D}(\vec{n})$$

The two functionals $\mathcal{H}$ and $\mathcal{D}$ are referred to as the superhamiltonian and supermomentum constraints. Due to the constraint algebra (2.10) – (2.12) closing, the Hamiltonian density $\mathcal{H}$ does not give rise to further constraints and does not contain any more terms than $\mathcal{H}$ and $\mathcal{D}$.

This means that the entire information about the canonical gravitational dynamics is already contained in the supermomentum $\mathcal{D}$ and the superhamiltonian $\mathcal{H}$. The constraint algebra and the two conditions (2.8) and (2.9) provide functional differential equations for the superhamiltonian and the supermomentum. The solution for $\mathcal{H}$ and $\mathcal{D}$ determines the Hamiltonian H which then generates the evolution of phase space curves $(\varphi^A(t), \pi_A(t))$ with respect to the foliation parameter t. Embedding this „geometry movie“ on the three-dimensional hypersurface Σ frame by frame by virtue of the one-parameter embedding map $X_t : \Sigma \hookrightarrow M$ and identifying lapse and shift, $n := \mathbf{n}$ and $\vec{n} := \vec{\mathbf{n}}$, results in the full spacetime geometry G. The crucial point of this construction is the determination of the superhamiltonian $\mathcal{H}$. While one can solve for the supermomentum $\mathcal{D}$ quite simply, a solution for the superhamiltonian $\mathcal{H}$ is most practically achieved by rewriting the functional differential equations for $\mathcal{H}$ in terms of partial differential equations – the gravitational closure equations. Their solution then provides the last missing piece of the canonical gravitational dynamics.

Determination of the supermomentum $\mathcal{D}(\vec{n})$

The supermomentum $\mathcal{D}$ can be determined using condition (2.9) and the third constraint algebra relation (2.12). Condition (2.9) can be written as

$$(\mathcal{L}_{\vec{n}}g)^{\mathcal{A}}(z) = \frac{\partial \hat{g}^{\mathcal{A}}}{\partial \varphi^A}(z)\,\frac{\delta \mathcal{D}(\vec{n})}{\delta \pi_A(z)}\,,$$

which is a functional differential equation for the supermomentum $\mathcal{D}$. Taking the derivative map to the left hand side by applying $\frac{\partial \hat{\varphi}^B}{\partial g^{\mathcal{A}}}(z)$ on both sides, one constructs the solution

$$\mathcal{D}(\vec{n}) = \int_\Sigma \mathrm{d}^3 z\, \pi_A(z)\,\frac{\partial \hat{\varphi}^A}{\partial g^{\mathcal{A}}}(\hat{g}(\varphi(z)))\,\left(\mathcal{L}_{\vec{n}}\hat{g}(\varphi)\right)^{\mathcal{A}}(z)\,, \tag{2.13}$$

which is also consistent with the third relation (2.12) of the constraint algebra.

Towards the superhamiltonian $\mathcal{H}(n)$

Following the same approach for the superhamiltonian $\mathcal{H}$ as for the supermomentum in the previous paragraph does not provide the full solution for the superhamiltonian. Expanding relation (2.8) yields the functional differential equation

$$\frac{\delta \mathcal{H}}{\delta \pi_B(z)} = n(z)\,k^B(z) + (\partial_\gamma n)(z)\,M^{B\gamma}(z) \tag{2.14}$$

with the coefficient

$$M^{A\gamma}(\varphi) := \frac{\partial \hat{\varphi}^A}{\partial g^{\mathcal{A}}}(\hat{g}(\varphi))\,e^a_0(t,\sigma)\,\frac{\partial \mathbf{g}^{\mathcal{A}}}{\partial(\partial_\gamma X^a)}(t,\sigma) \tag{2.15}$$

whose last factor can be calculated from the definition of the induced geometry $\mathbf{g}^{\mathcal{A}}$ using the identities

$$\begin{aligned} \frac{\partial e^m_\mu}{\partial(\partial_\gamma X^a)} &= \delta^m_a \delta^\gamma_\mu \quad , \quad \frac{\partial e^m_0}{\partial(\partial_\gamma X^a)} = -(\deg P - 1)\,e^m_\alpha\,\epsilon^0_a\,\mathbf{p}^{\alpha\gamma} \\ \frac{\partial \epsilon^0_m}{\partial(\partial_\gamma X^a)} &= -\epsilon^0_a \epsilon^\gamma_m \quad \text{and} \quad \frac{\partial \epsilon^\mu_m}{\partial(\partial_\gamma X^a)} = -\epsilon^\mu_a \epsilon^\gamma_m + (\deg P - 1)\,\epsilon^0_m \epsilon^0_a\,\mathbf{p}^{\mu\gamma}\,. \end{aligned} \tag{2.16}$$

The functional differential equation (2.14) can be solved to

$$\mathcal{H}(n) = \int_\Sigma \mathrm{d}^3 z\, n(z)\,\Big(\mathcal{H}_{\mathrm{local}}[\varphi;\pi) - \partial_\gamma(M^{A\gamma}\,\pi_A)\Big)(z) \tag{2.17}$$

where the undetermined functional $\mathcal{H}_{\mathrm{local}}[\varphi;\pi)$ depends on the momenta π_A, but not on their derivatives. It gives rise to the functionals

$$k^A[\varphi;\pi) := \frac{\partial \mathcal{H}_{\mathrm{local}}}{\partial \pi_A}[\varphi;\pi) \tag{2.18}$$

which also appears on the right hand side of Eq. (2.14). This definition already hints at the trick first introduced by Kuchar in Ref. [11] by performing a Legendre transformation of the unknown Hamiltonian functional $\mathcal{H}_{\mathrm{local}}$. One introduces generalized velocities k^A defined by (2.18) and rewrites

$$\mathcal{H}_{\mathrm{local}}[\varphi;\pi) = \pi_A\,k^A[\varphi;\pi) - \mathcal{L}[\varphi;k[\varphi;\pi))\,. \tag{2.19}$$

The introduction of the unknown Lagrangian density functional $\mathcal{L}$ allows to rewrite the first constraint algebra relation (2.10) – a functional differential equation quadratic in $\mathcal{H}$ – into a functional differential equation linear in the Lagrangian functional $\mathcal{L}$. This linear functional differential equation together with the second constraint algebra relation (2.11) which is already a linear equation can be cast into a set of partial differential equations, the gravitational closure equations.

While this construction is conceptually the same compared to previous work [8, 9], the calculations presented here significantly improve the understanding of the gravitational closure equations by introducing the configuration fields φ^A and the parametrization maps $\hat{g}(\varphi)$ of the canonical geometry. Besides, the number of coefficient functions is reduced and more precise construction algorithms are presented for them. Besides, the second constraint algebra relation (2.11) is now also expanded while the analog equations were introduced by less rigid arguments before.

2.4 Derivation of the gravitational closure equations

The results presented in this section were joint work with my colleague Florian Wolz.
They will also be presented in his PhD thesis,
Florian Wolz, *in preparation*, PhD thesis, Leibniz-Universität Hannover (2020).

In the previous section, it was shown that the Hamiltonian of the gravitational dynamics is of totally constrained form. While one of the two functionals, the supermomentum, is already determined, the second one, the superhamiltonian is only partially determined. It still contains a Lagrangian functional $\mathcal{L}$ which needs to be calculcated in order to construct the gravitational Hamiltonian H. The central result of this section is that the two remaining relations of the constraint algebra can be cast into a set of partial differential equations, the gravitational closure equations, whose solution determines the Lagrangian $\mathcal{L}$.

Before starting with the actual derivation, first introduce shorthand notation which allows to keep the notation more compact. First, consider a differentiable function Q of the configuration fields and their spatial derivatives on Σ. One defines the shorthand notation

$$Q_{:A}{}^{\alpha_1\ldots\alpha_N} := \frac{\partial Q}{\partial\partial^N_{\alpha_1\ldots\alpha_N}\varphi^A}\,.$$

The second definition introduces the third coefficient function $F^A{}_\mu{}^\gamma$ of the gravitational closure equations. It stems from the Lie derivative $\mathcal{L}_{\vec{n}}\hat{g}$ of the supermomentum (2.13). One expands it and introduces a coefficient $F^A{}_\mu{}^\gamma$ as

$$\frac{\partial\hat{\varphi}^A}{\partial g^{\mathcal{A}}}\,(\mathcal{L}_{\vec{n}}\hat{g}(\varphi))^{\mathcal{A}} =: n^\mu\,\varphi^A{}_{,\mu} - (\partial_\gamma n^\mu)\,F^A{}_\mu{}^\gamma\,. \tag{2.20}$$

The three coefficient functions $p^{\mu\nu}$, $M^{A\gamma}$ and $F^A{}_\mu{}^\gamma$ calculated from the matter action enter the gravitational closure equations. Therefore, they will be called *input coefficients*[5].

These two definitions provide the last piece of preliminary work. Now, the gravitational closure equations will be derived. The two constraint algebra relations (2.10) and (2.11) will be expanded separately one after another.

Evaluation of the first algebra relation

Starting point of the following construction is the first algebra relation (2.10)

$$\{\mathcal{H}(M),\mathcal{H}(N)\} = \mathcal{D}\Big((\deg P - 1)\,p^{\alpha\beta}\,(M\,\partial_\beta N - N\,\partial\delta_\beta N)\,\partial_\alpha\Big)\,.$$

In its original form, this relation provides a quadratic equation for the superhamiltonian $\mathcal{H}$. As mentioned before, identifying the non-local and local part of the superhamiltonian and using a Legendre transformation on the local part $\mathcal{H}_{\text{local}}$ turns this quadratic problem into a linear one for the Lagrangian $\mathcal{L}$.

[5] Restricting oneself to three independent input coefficients p, M and F provides the most striking difference of this thesis compared to the work of Ref. [18] in which similar results were presented.

First, the entire equation is localized by setting $M = \partial_y$ and $N = \partial_x$. The right hand side then is

$$\begin{aligned}
&\mathcal{D}\Big((\deg P - 1)\, p^{\alpha\beta}\, (\delta_y\, (\partial_\beta \delta_x - \delta_x\, \partial_\beta \delta_y)\, \partial_\alpha\Big) = \\
&\quad = (\deg P - 1)\, \Big[\pi_A(x)\, \varphi^A{}_{,\mu}(x)\, p^{\beta\mu}(x) + \partial_\gamma(\pi_A\, F^A{}_\mu{}^\gamma)(x)\, p^{\beta\mu}(x)\Big]\, (\partial_\beta \delta_x)(y) - x \leftrightarrow y\,,
\end{aligned}$$

where the identity

$$f(y)\, (\partial_\beta \delta_x)(y) - x \leftrightarrow y = f(x)\, (\partial_\beta \delta_x)(y) - x \leftrightarrow y \tag{2.21}$$

was used.

The left hand side of the first algebra relation is constituted by

$$\{\mathcal{H}(x), \mathcal{H}(y)\} = \int_\Sigma \mathrm{d}z \left(\frac{\delta \mathcal{H}(x)}{\delta \varphi^A(z)}\, \frac{\delta \mathcal{H}(y)}{\delta \pi_A(z)} - \frac{\delta \mathcal{H}(y)}{\delta \varphi^A(z)}\, \frac{\delta \mathcal{H}(x)}{\delta \pi_A(z)}\right)$$

where the two functional derivatives of the superhamiltonian are given by

$$\begin{aligned}
\frac{\delta \mathcal{H}(x)}{\delta \varphi^A(z)} &= -\frac{\delta \mathcal{L}(x)}{\delta \varphi^A(z)} - \partial_\gamma \left(\pi_B\, M^{B\gamma}{}_{:A}\right)(x)\, \delta_x(z) + \pi_B(x)\, M^{B\gamma}{}_{:A}(x)\, (\partial_\gamma \delta_x)(z) \quad \text{and} \\
\frac{\delta \mathcal{H}(y)}{\delta \pi_A(z)} &= k^A(y)\, \delta_y(z) - \left(\partial_\gamma M^{A\gamma}\right)(y)\, \delta_y(z) + M^{A\gamma}(y)\, (\partial_\gamma \delta_y)(z)\,.
\end{aligned}$$

Collecting all terms of the left hand side and using identity (2.21) as well as

$$f(x)\, g(y)\, (\partial_\mu \delta_x)(y) - x \;\leftrightarrow y = -f(y)\, g(y)\, (\partial_\mu \delta_y)(x) - x \leftrightarrow y\,, \tag{2.22}$$

and

$$\begin{aligned}
&\pi_B(x)\, M^{B\gamma}{}_{:A}(x)\, M^{A\mu}(y)\, (\partial^2_{\mu\gamma} \delta_x)(y) - x \leftrightarrow y = \\
&\quad = -\pi_B(x)\, M^{B\gamma}{}_{:A}(x)\, M^{A\mu}(x)\, (\partial^2_{\mu\gamma} \delta_x)(y) - 2\, \pi_B(x)\, M^{B(\gamma|}{}_{:A}(x)\, (\partial_\mu M^{A|\mu)})(x)\, (\partial_\gamma \delta_x)(y) - x \leftrightarrow y\,, \\
&-(\partial_\gamma \pi_B)(x)\, M^{B(\gamma|}{}_{:A}(x)\, M^{A|\mu)}(x)\, (\partial_\mu \delta_x)(y) - x \leftrightarrow y = \\
&\quad = -\pi_B(x)\, M^{B(\gamma|}{}_{:A}(x)\, M^{A|\mu)}(x)\, (\partial^2_{\mu\gamma} \delta_x)(y) + \partial_\mu (M^{B(\gamma|}{}_{:A}\, M^{A|\mu)})(x)\, (\partial_\gamma \delta_x)(y) - x \leftrightarrow y
\end{aligned}$$

and also collecting all terms of the equation on the left hand side, one obtains the functional differential equation for the Lagrangian $\mathcal{L}$,

$$\begin{aligned}
0 = &-k^B(y)\, \frac{\delta \mathcal{L}(x)}{\delta \varphi^B(y)} + (\partial_\gamma \delta_x)(y)\, k^B(y)\, M^{A\gamma}{}_{:B}(x)\, \frac{\partial \mathcal{L}}{\partial k^A}(x) + \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \varphi^B(\cdot)}\, M^{B\mu}\right)(y) \\
&+ \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial k^A}\right)(x)\, \Big[(\deg P - 1)\, p^{\rho\mu} F^A{}_\rho{}^\nu - M^{B[\mu|} M^{A|\nu]}{}_{:B}\Big](x)\, (\partial_\nu \delta_x)(y) \\
&+ \frac{\partial \mathcal{L}}{\partial k^A}(x) \Big[(\deg P - 1)\, p^{\rho\nu}(E^A{}_\rho + F^A{}_\rho{}^\gamma{}_{,\gamma}) + \partial_\mu (M^{B[\mu|} M^{A|\nu]}{}_{:B})\Big](x)\, (\partial_\nu \delta_x)(y) - (x \leftrightarrow y)\,.
\end{aligned}$$

Here, the associated momenta π_A were replaced by their definition $\pi_A = \frac{\partial \mathcal{L}}{\partial k^A}$ in the Lagrangian picture.

The next step of the derivation was first introduced by Kuchar in Ref. [11]. The Lagrangian $\mathcal{L}$ is expanded in powers of the generalized velocities k^A,

$$\mathcal{L}(x) = \sum_{N=0}^{\infty} C_{A_1 \ldots A_N}(x)\, k^{A_1}(x) \ldots k^{A_N}(x)\,. \tag{2.23}$$

The coefficients $C_{A_1 \ldots A_N}$ are accordingly called *expansion coefficients*. This expansion is inserted into the functional differential equation which then is

$$
\begin{aligned}
0 = &- \sum_{N=0}^{\infty} \frac{\delta C_{A_1\dots A_N}(x)}{\delta\varphi^A(y)}\, k^A(y)\, k^{A_1}(x)\dots k^{A_N}(x) \\
&+ \sum_{N=0}^{\infty} N\, C_{AA_2\dots A_N}(x)\, k^{A_2}(x)\dots k^{A_N}(x)\, M^{A\gamma}{}_{:B}\, k^B(y)\, (\partial_\gamma \delta_x)(y) \\
&+ \sum_{N=0}^{\infty} \partial_\mu \left(\frac{\delta C_{A_1\dots A_N}(x)}{\delta\varphi^A(\cdot)}\, M^{A\gamma}(\cdot) \right)(y)\, k^{A_1}(x)\dots k^{A_N}(x) \\
&+ \sum_{N=0}^{\infty} N\, \partial_\mu (C_{AA_2\dots A_N}\, k^{A_2}\dots k^{A_N})(x) \Big[(\deg P - 1)\, p^{\rho\mu}\, F^A{}_\rho{}^\nu - M^{B[\mu|} M^{A|\nu]}{}_{:B} \Big](x)\, (\partial_\nu \delta_x)(y) \\
&- \sum_{N=0}^{\infty} N\, C_{AA_2\dots A_N}(x)\, k^{A_2}(x)\dots k^{A_N}(x) \Big[(\deg P - 1)\, p^{\rho\nu}\, (\varphi^A{}_{,\rho} + F^A{}_\rho{}^\gamma{}_{,\gamma}) + \partial_\mu (M^{B[\mu|}\, M^{A|\nu]}{}_{:B}) \Big](x)\, (\partial_\nu \delta_x)(y) \\
&- x \leftrightarrow y\,.
\end{aligned}
\tag{2.24}
$$

This equation contains terms of different powers of the generalized velocities. The different powers can be considered separately. To read off the zeroth order equation, set all $k = 0$ which leaves the equation

$$
\begin{aligned}
0 = &\sum_{I=0}^{\infty} (-1)^I\, C_{:A}{}^{\alpha_1\dots\alpha_I}(x) \Big[(\partial^{I+1}_{\alpha_1\dots\alpha_I\mu} \delta_x)(y)\, M^{A\mu}(y) + (\partial^I_{\alpha_1\dots\alpha_I}\delta_x)(y)\, M^{A\mu}{}_{,\mu}(y) \Big] \\
&+ (\partial_\mu C_A)(x) \Big[(\deg P - 1)\, p^{\rho\mu}\, F^A{}_\rho{}^\nu - M^{B[\mu|} M^{A|\nu]}{}_{:B} \Big](x)\, (\partial_\nu \delta_x)(y) \\
&- C_A(x) \Big[(\deg P - 1)\, p^{\rho\nu}\, (\varphi^A{}_{,\rho} + F^A{}_\rho{}^\gamma{}_{,\gamma}) + \partial_\mu (M^{B[\mu|}\, M^{A|\nu]}{}_{:B}) \Big](x)\, (\partial_\nu \delta_x)(y) \\
&- x \leftrightarrow y\,,
\end{aligned}
$$

for which the definition of the functional derivative

$$
\frac{\delta C(y)}{\delta\varphi^A(x)} = \sum_{I=0}^{\infty} (-1)^I\, \frac{\partial C}{\partial \partial^I_{\alpha_1\dots\alpha_I}\varphi^A}(y)\, (\partial^I_{\alpha_1\dots\alpha_I}\delta_y)(x)\,, \tag{2.25}
$$

was used.

This differential equation now has to be integrated against a test function $f(x, y)$. After integration by parts and a subsequent integration of the δ-distribution, one obtains

$$
\begin{aligned}
0 = \int \mathrm{d}x \Big\{ &- \sum_{I=0}^{\infty} \sum_{J=0}^{I+1} \binom{I+1}{J}\, C_{:A}{}^{\alpha_1\dots\alpha_I}(x)\, (\partial^J_{2\,(\alpha_1\dots\alpha_J|} f)(x, x) (\partial^{I-J+1}_{|\alpha_{J+1}\dots\alpha_{I+1})} M^{A\alpha_{I+1}})(x) \\
&+ \sum_{I=0}^{\infty} \sum_{J=0}^{I} \binom{I}{J}\, C_{:A}{}^{\alpha_1\dots\alpha_I}(x)\, (\partial^J_{2\,(\alpha_1\dots\alpha_J|} f)(x, x) (\partial^{I-J+1}_{|\alpha_{J+1}\dots\alpha_I)\mu} M^{A\mu})(x) \\
\Big[&- (\partial_\mu C_A)(x) \Big[(\deg P - 1)\, p^{\rho\mu}\, F^A{}_\rho{}^\nu - M^{B[\mu|} M^{A|\nu]}{}_{:B} \Big](x) \\
&+ C_A(x) \Big[(\deg P - 1)\, p^{\rho\nu}\, (\varphi^A{}_{,\rho} + F^A{}_\rho{}^\gamma{}_{,\gamma}) + \partial_\mu (M^{B[\mu|}\, M^{A|\nu]}{}_{:B}) \Big](x) \Big] (\partial_{2\,\nu} f)(x, x) - \partial_2 \leftrightarrow \partial_1 \Big\}\,,
\end{aligned}
\tag{2.26}
$$

where the subscripts '1' and '2' denote that the partial derivative acts only on the first or second slot of the test function $f(\cdot, \cdot)$. Naively, one would state that for an arbitrary test function, the different derivative orders of the test function have to vanish.

However, both partial derivatives ∂_1 and ∂_2 appear in one equation and the entire equation is evaluated at the point (x, x). Thus, these two derivatives are not independent. As Ref. [9] already points out, the derivative $(\partial_\mu f)(x, x)$ can be written as

$$
(\partial_\mu f)(x, x) = (\partial_{1\,\mu} f)(x, x) + (\partial_{2\,\mu} f)(x, x)\,. \tag{2.27}
$$

This also generalizes to higher order derivatives as

$$(\partial^N_{2\,\alpha_1\ldots\alpha_N} f)(x,x) = \sum_{T=0}^{N} \binom{N}{T} (-1)^T \, (\partial^{N-T}_{(\alpha_1\ldots\alpha_{N-T}|} \, \partial^T_{1\,|\alpha_{N-T+1}\ldots\alpha_N)} f)(x,x)\,. \tag{2.28}$$

Adopting the simple example from Ref. [9], from the equation

$$0 = \int \mathrm{d}x \left[A(x)\, f(x,x) + B^\mu(x)\, (\partial_{1\,\mu} f)(x,x) + C^\mu(x)\, (\partial_{2\,\mu} f)(x,x) \right] ,$$

one must not naively read off that A, B^μ and C^μ vanish separately. Instead, using Eq. (2.27) and an integration by parts in order to free the term $\partial_\mu f$ of the derivative, one obtains

$$0 = \int \mathrm{d}x \left[(A(x) - (\partial_\mu C^\mu)(x))\, f(x,x) + (B^\mu(x) - C^\mu(x))\, (\partial_{1\,\mu} f)(x,x) \right]$$

and reads off that $A(x) - (\partial_\mu C^\mu)(x) = 0$ and $B^\mu - C^\mu = 0$ which is a significantly weaker statement compared to the naive one.

This insight proves to be immediately important. One applies Eq. (2.28) to the partial derivatives in Eq. (2.26) and subsequently frees the test function f of all derivatives ∂_μ by integration by parts. The equation now carries purely independent derivatives of the test function. As the entire equation vanishes and the test function is arbitrary, the coefficients in front of each derivative order have to vanish separately. One reads off the equations

$$0 = \sum_{K=0}^{\infty} \sum_{J=2}^{K+1} (-1)^J \binom{K}{J-1} \binom{J}{N} (J-1)\, \partial^{J+1}_{\gamma\alpha_1\ldots\alpha_J} \left(C_{:A}{}^{\beta_1\ldots\beta_{K-J}(\alpha_1\ldots\alpha_J|} \; M^{A|\gamma)}{}_{,\beta_1\ldots\beta_{K-J}} \right) , \tag{2.29}$$

$$\begin{aligned} 0 = {} & 2\,\partial_\mu \left(C_A\, M^{A[\mu|}{}_{:B}\, M^{B|\gamma]} \right) - 2\,(\deg P - 1)\, p^{\rho\gamma} [C_A\, \varphi^A{}_{,\rho} + \partial_\mu (C_A\, F^A{}_\rho{}^\mu\,)] \\ & + \sum_{K=0}^{\infty} C_{:A}{}^{\alpha_1\ldots\alpha_K} \; M^{A\gamma}{}_{,\alpha_1\ldots\alpha_K} + \sum_{K=0}^{\infty} \sum_{J=0}^{K} (-1)^J \binom{K}{J} (J+1)\, \partial^J_{\alpha_1\ldots\alpha_J} \left(C_{:A}{}^{\beta_1\ldots\beta_{K-J}(\alpha_1\ldots\alpha_J|} \; M^{A|\gamma)}{}_{,\beta_1\ldots\beta_{K-J}} \right) , \end{aligned} \tag{2.30}$$

$$0 = \sum_{K=N}^{\infty} \sum_{J=N+1}^{K+1} (-1)^J \binom{K}{J-1} \binom{J}{N} \partial^{J-N}_{\alpha_1\ldots\alpha_{J-N}} \left(C_{:A}{}^{\beta_J\ldots\beta_K(\alpha_1\ldots\alpha_{J-N}\mu_1\ldots\mu_{N-1}|} \; M^{A|\mu_N)}{}_{,\beta_J\ldots\beta_K} \right) \quad \text{for even } N \geq 2\,, \tag{2.31}$$

$$\begin{aligned} 0 = {} & 2 \sum_{K=N-1}^{\infty} \binom{K}{N-1} C_{:A}{}^{\beta_N\ldots\beta_K(\mu_1\ldots\mu_{N-1}|} \; M^{A|\mu_N)}{}_{,\beta_N\ldots\beta_K} \\ & - \sum_{K=N}^{\infty} \sum_{J=N+1}^{K+1} \binom{K}{J-1} \binom{J}{N} \partial^{J-N}_{\alpha_1\ldots\alpha_{J-N}} \left(C_{:A}{}^{\beta_J\ldots\beta_K(\alpha_1\ldots\alpha_{J-N}\mu_1\ldots\mu_{N-1}|} \; M^{A|\mu_N)}{}_{,\beta_J\ldots\beta_K} \right) \quad \text{for odd } N \geq 3\,. \end{aligned} \tag{2.32}$$

After reading off the equations, one uses the divergence ∂_γ(2.30) in order to eliminate terms and obtain Eq. (2.29) in the form presented here.

After evaluating the zeroth order equation (2.24), one proceeds in similar fashion for equations with larger powers of the generalized velocities k. To extract the N^{th} order contribution, apply the derivative

$$\left. \frac{\delta^N}{\delta k^{B_1}(x_1) \ldots \delta k^{B_N}(x_N)} \right|_{k=0}$$

to Eq. (2.24) which – after dropping a factor $(N-1)!$ – results in

$$
\begin{aligned}
0 = &-\sum_{J=1}^{N} \frac{\delta C_{B_1\ldots\widetilde{B_J}\ldots B_N}(x)}{\delta\varphi^{B_J}(y)}\, \delta_y(x_J)\, \delta_x(x_1)\ldots\widetilde{\delta_x(x_J)}\ldots\delta_x(x_N) \\
&+ N^2 C_{A(B_1\ldots B_{N-1}|}(x)\, M^{A\gamma}{}_{:|B_N)}(x)\, (\partial_\gamma\delta_x)(y)\, \delta_x(x_1)\ldots\delta_x(x_N) \\
&+ N\,\partial_\mu\left(\frac{\delta C_{B_1\ldots B_N}(x)}{\delta\varphi^A(\cdot)}\, M^{A\mu}(\cdot)\right)(y)\, \delta_x(x_1)\ldots\delta_x(x_N) \\
&+ N^2\, (\partial_\mu C_{AB_1\ldots B_N})(x)\, \Big[(\deg P-1)\, p^{\rho\mu} F^A{}_\rho{}^\nu - M^{B[\mu|}\, M^{A|\nu]}{}_{:B}\Big](x)\, (\partial_\nu\delta_x)(y)\, \delta_x(x_1)\ldots\delta_x(x_N) \\
&- N\,(N+1)\, C_{AB_1\ldots B_N}(x)\, \Big[(\deg P-1)\, p^{\rho\mu} F^A{}_\rho{}^\nu - M^{B[\mu|}\, M^{A|\nu]}{}_{:B}\Big](x)\, (\partial_\nu\delta_x)(y)\times \\
&\qquad \times \sum_{J=1}^{N} \delta_x(x_1)\ldots\widetilde{\delta_x(x_J)}\ldots\delta_x(x_N)\, (\partial_\mu\delta_x)(x_J) \\
&- N\,(N+1)\, C_{AB_1\ldots B_N}(x)\Big[(\deg P-1)\, p^{\rho\nu}(\varphi^A{}_{,\rho} + F^A{}_\rho{}^\gamma{}_{,\gamma}) + \partial_\mu(M^{B[\mu|}M^{A|\nu]}{}_{:B})\Big](x)\times \\
&\qquad \times\, (\partial_\nu\delta_x)(y)\, \delta_x(x_1)\ldots\delta_x(x_N) - x \leftrightarrow y\,.
\end{aligned}
$$

The notation $\sim$ instructs to omit this index or term. After spelling out the functional derivatives using their definition (2.25), the resulting equation has to be integrated against a test function $f(x, y, x_1, \ldots, x_N)$ with $N+2$ slots. Integration by parts and subsequent integration of the δ-distributions yields

$$
\begin{aligned}
0 = &\int \mathrm{d}x \Bigg\{ -\sum_{I=0}^{\infty}\sum_{J=1}^{N}\sum_{T=0}^{I} \binom{I}{T} C_{B_1\ldots\widetilde{B_J}\ldots B_N:B_J}{}^{\alpha_1\ldots\alpha_I}(x)\, (\partial^T_{2\,(\alpha_1\ldots\alpha_T|}\partial^{I-T}_{J+2\,|\alpha_{T+1}\ldots\alpha_I)} f)(x, x, x, \ldots, x) \\
&- N^2\, C_{A(B_1\ldots B_{N-1}|}(x)\, M^{A\gamma}{}_{:|B_N)}(x)\, (\partial_{2\,\gamma} f)(x, x, x, \ldots, x) \\
&- N\sum_{I=0}^{\infty}\sum_{T=0}^{\infty} \binom{I+1}{T} C_{B_1\ldots B_N:A}{}^{\alpha_1\ldots\alpha_I}(x) (\partial^T_{2\,(\alpha_1\ldots\alpha_T|} f)(x, x, x, \ldots, x)\, \Big(\partial^{I-T+1}_{|\alpha_{T+1}\ldots\alpha_{I+1})} M^{A\alpha_{I+1}}\Big)(x) \\
&+ N\sum_{I=0}^{\infty}\sum_{T=0}^{\infty} \binom{I}{T} C_{B_1\ldots B_N:A}{}^{\alpha_1\ldots\alpha_I}(x) (\partial^T_{2\,(\alpha_1\ldots\alpha_T|} f)(x, x, x, \ldots, x)\, \Big(\partial^{I-T+1}_{|\alpha_{T+1}\ldots\alpha_I)\mu} M^{A\mu}\Big)(x) \\
&- N^2\, \Big(\partial_\mu\, C_{AB_1\ldots B_N}\Big)(x)\, \Big[(\deg P-1)\, p^{\rho\mu} F^A{}_\rho{}^\nu - M^{B[\mu|}\, M^{A|\nu]}{}_{:B}\Big](x)\, (\partial_{2\,\nu} f)(x, x, x, \ldots, x) \\
&- N\,(N+1)\, C_{AB_1\ldots B_N}(x)\, \Big[(\deg P-1)\, p^{\rho\mu} F^A{}_\rho{}^\nu - M^{B[\mu|}\, M^{A|\nu]}{}_{:B}\Big](x)\, \sum_{J=1}^{N} (\partial_{2\,\nu}\, \partial_{J+2\,\mu} f)(x, x, x, \ldots, x) \\
&+ N\,(N+1)\, C_{AB_1\ldots B_N}(x)\, \Big[(\deg P-1)\, p^{\rho\nu}(\varphi^A{}_{,\rho} + F^A{}_\rho{}^\gamma{}_{,\gamma}) + \partial_\mu(M^{B[\mu|}M^{A|\nu]}{}_{:B})\Big](x)\, (\partial_{2\,\nu} f)(x, x, x, \ldots, x) \\
&- \partial_2 \leftrightarrow \partial_1 \Bigg\}
\end{aligned}
$$

As in the analysis of the zeroth order equation, not all partial derivatives of the test function f evaluated at the point $(x, x, x, \ldots, x)$ are independent and one has to be eliminated in favor of the other independent ones. This is relevant for the first and sixth line of the above equation. For the latter, use the analogon of Eq. (2.27)

$$
(\partial_\mu f)(x, \ldots, x) = (\partial_{1\,\mu} f)(x, \ldots, x) + (\partial_{2\,\mu} f)(x, \ldots, x) + \sum_{J=1}^{N} (\partial_{J+2\,\mu} f)(x, \ldots, x)
$$

in order to get rid of the sum over J. Subsequently, perform an integration by parts in order to free the test function of the derivative ∂_μ and cancel one of the emerging terms with the seventh line. Also note that there will appear a term with the second derivative $(\partial^2_{2\,\mu\nu} f)(x, x, x, \ldots, x)$ of the test function. Due to the symmetry of the derivative, the antisymmetric part of the summand drops out of the equation.

As already laid out in Ref. [9], the summand $J = N$ in the first line can be re-written as

$$\sum_{T=0}^{I}\binom{I}{T}(\partial^{T}_{2\,(\alpha_1\ldots\alpha_T|}\,\partial^{I-T}_{N+2\,|\alpha_{T+1}\ldots\alpha_I)}f)(x,\ldots,x) = \sum_{S=0}^{I}\sum_{t=1,3,\ldots N+1}\binom{I}{S}(-1)^{I-S}(\partial^{S}_{(\alpha_1\ldots\alpha_S|}\,\partial^{I-S}_{t\,|\alpha_{S+1}\ldots\alpha_I)}f)(x,\ldots,x)$$

in order to get rid of the derivative ∂_{N+2}. After integration by parts in order to free the test function f of all derivatives ∂_μ, one can proceed to read off the individual terms that have to vanish for each derivative order of the test function.

After re-ordering and re-arranging of sums, the following equations are read off,

$$0 = C_{AB_1\ldots B_N}\left[(\deg P - 1)\,p^{\rho\mu}F^{A}{}_{\rho}{}^{\nu} - M^{B[\mu|}\,M^{A|\nu]}{}_{:B}\right] \quad \text{for } N \geq 1\,, \tag{2.33}$$

$$\begin{aligned} 0 = {} & N\,(N+1)\,C_{AB_1\ldots B_N}\left(p^{\mu\nu}\,\varphi^{A}{}_{,\nu} - p^{\mu\nu}{}_{,\gamma}\,F^{A}{}_{\nu}{}^{\gamma}\right) - N^2\,C_{A(B_1\ldots B_{N-1}|}\,M^{A\mu}{}_{:|B_N)} - \sum_{J=1}^{N-1} C_{B_1\ldots\widetilde{B_J}\ldots B_N:B_J}{}^{\mu} \\ & - N\sum_{K=0}^{\infty} C_{B_1\ldots B_N:A}{}^{\alpha_1\ldots\alpha_K}\;M^{A\mu}{}_{,\alpha_1\ldots\alpha_K} - \sum_{K=0}^{\infty}(-1)^K\,(K+1)\left(\partial^K_{\alpha_1\ldots\alpha_K}\,C_{B_1\ldots B_{N-1}:B_N}{}^{\alpha_1\ldots\alpha_K\mu}\right) \quad \text{for } N \geq 1\,, \end{aligned} \tag{2.34}$$

$$\begin{aligned} 0 = {} & N\,(N+1)\,(\deg P - 1)\,C_{AB_1\ldots B_N}\,p^{\rho(\mu|}\,F^{A}{}_{\rho}{}^{|\nu)} + N\sum_{K=0}^{\infty}(K+1)\,C_{B_1\ldots B_N:A}{}^{\alpha_1\ldots\alpha_K(\mu|}\;M^{A|\nu)}{}_{,\alpha_1\ldots\alpha_K} \\ & + \sum_{J=1}^{N-1} C_{B_1\ldots\widetilde{B_J}\ldots B_N:B_J}{}^{\mu\nu} - \sum_{K=0}^{\infty}\binom{K+2}{2}(-1)^K\left(\partial^K_{\alpha_1\ldots\alpha_K}\,C_{B_1\ldots B_{N-1}:B_N}{}^{\alpha_1\ldots\alpha_K\mu\nu}\right) \quad \text{for } N \geq 1\,, \end{aligned} \tag{2.35}$$

$$\begin{aligned} 0 = {} & N\sum_{K=0}^{\infty}\binom{K+S-1}{S-1}C_{B_1\ldots B_N:A}{}^{\alpha_1\ldots\alpha_K(\mu_1\ldots\mu_{S-1}|}\;M^{A|\mu_S)}{}_{,\alpha_1\ldots\alpha_K} + \sum_{J=1}^{N-1} C_{B_1\ldots\widetilde{B_J}\ldots B_N:B_J}{}^{\mu_1\ldots\mu_S} \\ & - \sum_{K=0}^{\infty}(-1)^{K+S}\binom{K+S}{K}\left(\partial^K_{\alpha_1\ldots\alpha_K}C_{B_1\ldots B_{N-1}:B_N}{}^{\alpha_1\ldots\alpha_K\mu_1\ldots\mu_S}\right) \quad \text{for } N \geq 1, S \geq 3\,, \end{aligned} \tag{2.36}$$

$$C_{B_1\ldots\widetilde{B_J}\ldots B_N:B_J}{}^{\mu_1\ldots\mu_{S+T}} = \sum_{K=0}^{\infty}(-1)^{K+S+T}\binom{K+S+T}{S+T}\left(\partial^K_{\alpha_1\ldots\alpha_K}C_{B_1\ldots B_{N-1}:B_N}{}^{\alpha_1\ldots\alpha_K\mu_1\ldots\mu_{S+T}}\right) \tag{2.37}$$
$$\text{for } N \geq 2, T \geq 1, S \geq 1, J = 1\ldots N-1\,,$$

$$0 = \sum_{K=0}^{\infty}(-1)^{K+S+T}\binom{K+S+T}{S+T}(\partial_{\alpha_1\ldots\alpha_K}C_{B_1\ldots B_{N-1}:B_N}{}^{\alpha_1\ldots\alpha_K\mu_1\ldots\mu_{S+T}}) \tag{2.38}$$
$$\text{for } N \geq 3, T \geq 2, S \geq 1, J = 1\ldots N-1$$

These equation still contain information which can be made explicit and plugged into the other equations in order to simplify them. The first observation is that the last two equations imply

$$C_{B_1\ldots B_N:A}{}^{\alpha_1\ldots\alpha_K} = 0$$

for $N \geq 2$ and $K \geq 3$. This means that expansion coefficients $C_{B_1\ldots B_N}$ for $N \geq 2$ depend at most on second derivatives of the configuration fields φ^A. This collapse to second derivative order proves to be useful for all closure equations containing such expansion coefficients as the infinite sums break down to a finite amount of terms which simplifies these equations drastically. Note that there is no such a collapse to second derivative order for the first two expansion coefficients C and C_A which still depend on arbitrary many derivatives of the configuration fields. Only if a theory has vanishing input coefficient $M^{A\gamma}$, the first expansion coefficient C will also feature a collapse to second derivative order in the configuration fields.

One proceeds with the rewriting of the equations by analyzing Eq. (2.37) for $N = 2$ and $L \geq 2$. One obtains

$$C_{B_2:B_1}{}^{\mu_1\dots\mu_L} = \sum_{K=0}^{\infty}(-1)^{K+L}\binom{K+L}{L}\left(\partial^K_{\alpha_1\dots\alpha_K}C_{B_1:B_2}{}^{\alpha_1\dots\alpha_K\mu_1\dots\mu_L}\right). \tag{2.39}$$

Also, Eq. (2.37) provides the sequence of equations

$$0 = C_{B_1\dots\widetilde{B_J}\dots B_N:B_J}{}^{\mu\nu} - C_{B_1\dots B_{N-1}:B_N}{}^{\mu\nu} \quad \text{for } J = 1\dots N-1 \tag{2.40}$$

when evaluating the case $N \geq 3$ and $S + T = 2$. This type of exchange symmetry is inserted into the remaining equations.

Next, combining these results with Eq. (2.36), one finds for $S = 3$,

$$0 = C_{B_1\dots B_N:A}{}^{(\mu_1\mu_2|}\ M^{A|\mu_3)} \quad \text{for } N \geq 2\,. \tag{2.41}$$

The case $N = 2$ and $S \geq 4$ in Eq. (2.36) provides no new information while the case $N = 1$, $S \geq 3$ does. Setting $L := S - 1$, one reads off for $L \geq 2$

$$0 = \sum_{K=0}^{\infty}\left[\binom{K+L}{L}C_{B:A}{}^{\alpha_1\alpha_K(\mu_1\dots\mu_L|}\ M^{A|\mu_{L+1})}{}_{,\alpha_1\dots\alpha_K} + (-1)^{K+L}\binom{K+L+1}{L+1}\left(\partial^K_{\alpha_1\dots\alpha_K}C_{:B}{}^{\alpha_1\dots\alpha_K\mu_1\dots\mu_{L+1}}\right)\right]. \tag{2.42}$$

For Eq. (2.35), inserting the collapse to second derivative order and the exchange symmetry (2.40) reduces the equation to

$$\begin{aligned}0 &= N\,(N+1)\,(\deg P - 1)\,C_{AB_1\dots B_N}\,p^{\rho(\mu|}\,F^A{}_\rho{}^{|\nu)} + N\,C_{B_1\dots B_N:A}{}^{(\mu|}\ M^{A|\nu)} + 2\,N\,C_{B_1\dots B_N:A}{}^{\alpha(\mu|}\ M^{A|\nu)}{}_{,\alpha}\\ &\quad + (N-2)\,C_{B_1\dots B_{N-1}:B_N}{}^{\mu\nu} \quad \text{for } N \geq 2\,.\end{aligned} \tag{2.43}$$

For $N = 1$, one cannot use the collapse of the expansion coefficients to second derivative order and the equation still contains infinite sums,

$$\begin{aligned}0 &= 2\,(\deg P - 1)\,C_{AB}\,p^{\rho(\mu|}\,F^A{}_\rho{}^{|\nu)} - \sum_{K=0}^{\infty}(K+1)\,C_{B:A}{}^{\alpha_1\dots\alpha_K(\mu|}\,M^{A|\nu)}{}_{,\alpha_1\dots\alpha_K}\\ &\quad + \sum_{K=0}^{\infty}(-1)^K\binom{K+2}{K}\left(\partial^K_{\alpha_1\dots\alpha_K}\,C_{:B}{}^{\alpha_1\dots\alpha_K\mu\nu}\right)\end{aligned} \tag{2.44}$$

Last but not least, relation (2.34) can also be simplified using the collapse to second derivative order and the exchange symmetry (2.40). One finds

$$\begin{aligned}0 &= (N+2)\,(N+1)\,(\deg P - 1)\,C_{AB_1\dots B_{N+1}}\left(p^{\mu\gamma}\varphi^A{}_{,\gamma} - p^{\mu\nu}{}_{,\gamma}\,F^A{}_\nu{}^\gamma\right) - (N+1)^2\,C_{A(B_1\dots B_N|}\,M^{A\mu}{}_{:|B_{N+1})}\\ &\quad - (N+1)\,C_{B_1\dots B_N:A}\ M^{A\mu} - (N+1)\,C_{B_1\dots B_N:A}{}^{\alpha}\ M^{A\mu}{}_{,\alpha} - (N+1)\,C_{B_1\dots B_N:A}{}^{\alpha\beta}\ M^{A\mu}{}_{,\alpha\beta}\\ &\quad - \sum_{K=0}^{N+1}C_{B_1\dots\widetilde{B_K}\dots B_{N+1}:B_K}{}^{\mu} + 2\left(\partial_\gamma C_{B_1\dots B_N:B_{N+1}}{}^{\mu\gamma}\right) \quad \text{for } N \geq 2\end{aligned} \tag{2.45}$$

and the two additional equations

$$\begin{aligned}0 &= 6\,(\deg P - 1)\,C_{AB_1B_2}\left(p^{\mu\nu}\,\varphi^A{}_{,\nu} - p^{\mu\nu}{}_{,\gamma}\,F^A{}_\nu{}^\gamma\right) - 4\,C_{A(B_1|}\,M^{A\mu}{}_{:|B_2)} - 2\,C_{B_1B_2:A}\ M^{A\mu}\\ &\quad - 2\,C_{B_1B_2:A}{}^{\alpha}\ M^{A\mu}{}_{,\alpha} - 2\,C_{B_1B_2:A}{}^{\alpha\beta}\ M^{A\mu}{}_{,\alpha\beta} - C_{B_2:B_1}{}^{\mu} - \sum_{K=0}^{\infty}(-1)^K\,(K+1)\,\left(\partial_{\alpha_1\dots\alpha_K}C_{B_1:B_2}{}^{\mu\alpha_1\dots\alpha_K}\right)\end{aligned} \tag{2.46}$$

and

$$0 = 2\,(\deg P - 1)\,C_{AB}\,\left(p^{\mu\nu}\,\varphi^A{}_{,\nu} - p^{\mu\nu}{}_{,\gamma}\,F^A{}_\nu{}^\gamma\right) - C_A\,M^{A\mu}{}_{:B} - \sum_{K=0}^{\infty} C_{B:A}{}^{\alpha_1\ldots\alpha_K}\;M^{A\mu}{}_{,\alpha_1\ldots\alpha_K}$$
$$- \sum_{K=0}^{\infty}(-1)^K\,(K+1)\,\left(\partial^K_{\alpha_1\ldots\alpha_K} C_{:A}{}^{\alpha_1\ldots\alpha_K\mu}\right)\,. \tag{2.47}$$

By writing down these equations, one has extracted all information contained in the first constraint algebra relation (2.10) and cast it into a set of partial differential equations, Eqns. (2.29), (2.30), (2.31), (2.32), (2.33), (2.39), (2.40), (2.41), (2.42), (2.43), (2.44), (2.45), (2.46) and (2.47). All equations will be listed at the end of this chapter with the appropriate numbering that they also carry in Ref. [3]. First, one needs to evaluate the second constraint algebra relation as well. The resulting partial differential equations together with the ones obtained up to now form the gravitational closure equations.

Evaluation of the second algebra relation

The second algebra relation (2.11) also contains the superhamiltonian $\mathcal{H}$ – in contrast to the first one already linearly which makes it a lot simpler to cast this algebra relation into partial differential equations. These partial differential equations then complete the set from the first algebra relation to form the gravitational closure equations. With the necessary techniques already developed in the previous analysis of the first algebra relation (2.10), it is now a simpler task to expand the second one.

The starting point is the second algebra relation (2.11),

$$\left\{\mathcal{D}(\vec{N}), \mathcal{H}(M)\right\} = \mathcal{H}\left(\mathcal{L}_{\vec{N}}M\right) = \mathcal{H}(N^\mu\,\partial_\mu M)\,.$$

Note that the superhamiltonian $\mathcal{H}$ can be replaced by the local part $\mathcal{H}_{\text{local}}$ as the other part drops out of this equation. Its localized version is

$$\left\{\mathcal{D}_\mu(x), \mathcal{H}_{\text{local}}(y)\right\} = \mathcal{H}_{\text{local}}\left(\delta_x\,\partial_\mu\delta_y\right)\,.$$

The right hand side is straightforwardly calculated to be

$$\mathcal{H}_{\text{local}}\left(\delta_x\,\partial_\mu\delta_y\right) = \left[\frac{\partial\mathcal{L}}{\partial k^A}(x)\,k^A(x) - \mathcal{L}(x)\right](\partial_\mu\delta_y)(x)\,.$$

The Poisson bracket on the left hand side of the algebra relation requires the four functional derivatives,

$$\begin{aligned}
\frac{\delta D_\mu(x)}{\delta\varphi^A(z)} &= -\pi_A(x)(\partial_\mu\delta_x)(z) - \pi_B(x)F^B{}_\mu{}^\gamma{}_{:A}(x)\,(\partial_\gamma\delta_x)(z) + \partial_\gamma(\pi_B\,F^B{}_\mu{}^\gamma{}_{:A})(x)\,\delta_x(z)\,,\\
\frac{\delta D_\mu(x)}{\delta\pi_A(z)} &= \varphi^A{}_{,\mu}(x)\,\delta_x(z) - F^A{}_\mu{}^\gamma\,(x)\,(\partial_\gamma\delta_x)(z) + (\partial_\gamma F^A{}_\mu{}^\gamma)(x)\,\delta_x(z)\,,\\
\frac{\delta\mathcal{H}_{\text{local}}(y)}{\delta\varphi^A(z)} &= -\frac{\delta\mathcal{L}(y)}{\delta\varphi^A(z)}\,,\\
\frac{\delta\mathcal{H}_{\text{local}}(y)}{\delta\pi_A(z)} &= k^A(y)\,\delta_y(z)\,.
\end{aligned}$$

In order to obtain the first and second derivative in their respective form presented here, one uses the following identity for the δ-distribution,

$$g(z)(\partial_\gamma\delta_x)(z) = g(x)\,(\partial_\gamma\delta_x)(z) - (\partial_\gamma g)(x)\,\delta_x(z)\,.$$

These four derivatives appearing in the Poisson bracket are now put together and the right hand side of the equation is brought to the left. Using the identity

$$f(x)\,g(y)\,(\partial_\mu\delta_x)(y) = -f(y)\,g(y)\,(\partial_\mu\delta_y)(x) + (\partial_\mu f)(y)\,g(y)\,\delta_y(x)\,,$$

one obtains the functional differential equation

$$
\begin{aligned}
0 &= \mathcal{L}(y)\,(\partial_\mu \delta_y)(x) + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial k^A}\,k^A - \mathcal{L}\right)(y)\,\delta_y(x) - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial k^A}\right)(y)\,k^A(y)\,\delta_y(x) \\
&+ \frac{\partial \mathcal{L}}{\partial k^B}(y)\,F^B{}_\mu{}^\gamma{}_{:A}(y)\,k^A(y)\,(\partial_\gamma \delta_y)(x) + \left[\varphi^A{}_{,\mu}(x) + (\partial_\gamma F^A{}_\mu{}^\gamma)(x)\right]\,\frac{\delta \mathcal{L}(y)}{\delta \varphi^A(x)} \\
&+ \partial_\gamma\left(\frac{\delta \mathcal{L}(y)}{\delta \varphi^A(\cdot)}\right)(x)\,F^A{}_\mu{}^\gamma(x)\,.
\end{aligned}
$$

Following the steps of the previous analysis, one expands the the Lagrangian in terms of the expansion coefficients $C_{A_1\ldots A_N}$ as

$$
\mathcal{L}(x) = \sum_{N=0}^{\infty} C_{A_1\ldots A_N}(x)\,k^{A_1}(x)\ldots k^{A_N}(x)\,.
$$

The functional differential equation is now

$$
\begin{aligned}
0 &= \sum_{N=0}^{\infty} C_{A_1\ldots A_N}(y)\,k^{A_1}(y)\ldots k^{A_N}(y)\,(\partial_\mu \delta_y)(x) \\
&+ \sum_{N=0}^{\infty} (N-1)\,\partial_\mu\left(C_{A_1\ldots A_N}\,k^{A_1}\ldots k^{A_N}\right)(y)\,\delta_y(x) \\
&- \sum_{N=0}^{\infty} N\,\partial_\mu\left(C_{AA_2\ldots A_N}\,k^{A_2}\ldots k^{A_N}\right)(y)\,k^A(y)\,\delta_y(x) \\
&+ \sum_{N=0}^{\infty} N\,C_{BA_2\ldots A_N}(y)\,k^{A_2}(y)\ldots k^{A_N}(y)\,k^A(y)\,F^B{}_\mu{}^\gamma{}_{:A}(y)\,(\partial_\gamma \delta_y)(x) \\
&+ \sum_{N=0}^{\infty} \frac{\delta C_{A_1\ldots A_N}(y)}{\delta \varphi^A(x)}\,\left(\varphi^A{}_{,\mu}(x) + (\partial_\gamma F^A{}_\mu{}^\gamma)(x)\right)\,k^{A_1}(y)\ldots k^{A_N}(y) \\
&+ \sum_{N=0}^{\infty} \partial_\gamma\left(\frac{\delta C_{A_1\ldots A_N}(y)}{\delta \varphi^A(\cdot)}\right)(x)\,k^{A_1}(y)\ldots k^{A_N}(y)\,F^A{}_\mu{}^\gamma(x)\,.
\end{aligned}
\tag{2.48}
$$

One follows the same procedure as before and inspects the different powers of the velocities separately. One extracts the zeroth equation by setting all k^A to zero. Using the definition of the functional derivative (2.25), the equation becomes

$$
\begin{aligned}
0 = C(y)\,(\partial_\mu \delta_y)(x) - (\partial_\mu C)(y)\,\delta_x(y) + \sum_{I=0}^{\infty} C_{:A}{}^{\alpha_1\ldots\alpha_I}\,(y)\,(\partial^I_{\alpha_1\ldots\alpha_I}\delta_y)(x)\,\left(\varphi^A{}_{,\mu} + F^A{}_\mu{}^\gamma{}_{,\gamma}\right)(x) \\
+ \sum_{I=0}^{\infty} C_{:A}{}^{\alpha_1\ldots\alpha_I}\,(y)\,(\partial^{I+1}_{\alpha_1\ldots\alpha_I\gamma}\delta_y)(x)\,F^A{}_\mu{}^\gamma(x)\,.
\end{aligned}
$$

Now, integrate this equation against a test function $f(x,y)$. Of the partial derivatives ∂_1 and ∂_2, only ∂_1 appears in the resulting equation. Thus, use the fact that the different derivative orders of the test functions are independent and have to vanish separately to read off

$$
0 = -\partial_\mu C + \sum_{I=0}^{\infty} C_{:A}{}^{\alpha_1\ldots\alpha_I}\;\varphi^A{}_{,\alpha_1\ldots\alpha_I\mu}\,,
\tag{2.49}
$$

$$
0 = -C\,\delta^\gamma_\mu + \sum_{I=0}^{\infty}(I+1)\left[C_{:A}{}^{\alpha_1\ldots\alpha_I\gamma}\;\left(\varphi^A{}_{,\mu\alpha_1\ldots\alpha_I} + F^A{}_\mu{}^{\alpha_{I+1}}{}_{,\alpha_1\ldots\alpha_{I+1}}\right) - C_{:A}{}^{(\alpha_1\ldots\alpha_I|}\;F^A{}_\mu{}^{|\gamma)}{}_{,\alpha_1\ldots\alpha_I}\right]
\tag{2.50}
$$

$$
0 = \sum_{I=0}^{\infty}\binom{I+L}{L}\left[C_{:A}{}^{\beta_1\ldots\beta_L\alpha_1\ldots\alpha_I}\;\left(\varphi^A{}_{,\mu\alpha_1\ldots\alpha_I} + F^A{}_\mu{}^{\alpha_{I+1}}{}_{,\alpha_1\ldots\alpha_{I+1}}\right) - C_{:A}{}^{(\beta_1\ldots\beta_L\alpha_1\ldots\alpha_{I-1}|}\;F^A{}_\mu{}^{|\alpha_I)}{}_{,\alpha_1\ldots\alpha_I}\right]
\tag{2.51}
$$

for $L \geq 2$.

Note that Eq. (2.49) is simply the chain rule. As it provides no new information, this equation can be dropped from the set of partial differential equations. As the expansion coefficient C depends on arbitrary many derivatives of the configuration fields, Eq. (2.50) and the sequence (2.51) cannot be simplified.

As for the first algebra relation, in order to obtain the N^{th}-order equation, apply the derivative

$$\left.\frac{\delta^N}{\delta k^{B_1}(x_1)\dots\delta k^{B_N}(x_N)}\right|_{k=0}$$

to the functional differential equation (2.48). Dropping a factor $N!$ and terms that cancel each other, one obtains

$$\begin{aligned}
0 &= C_{B_1\dots B_N}(y)\,(\partial_\mu\delta_y)(x)\,\delta_y(x_1)\dots\delta_y(x_N)\\
&\quad-(\partial_\mu C_{B_1\dots B_N})(y)\,\delta_y(x)\,\delta_y(x_1)\dots\delta_y(x_N)\\
&\quad+N\,C_{A(B_1\dots B_{N-1}}(y)\,F^A{}_\mu{}^\gamma{}_{:|B_N)}(y)\,(\partial_\gamma\delta_y)(x)\,\delta_y(x_1)\dots\delta_y(x_N)\\
&\quad+\sum_{I=0}^\infty(-1)^I\,C_{B_1\dots B_N:A}{}^{\alpha_1\dots\alpha_I}\,(y)\,\left(\varphi^A{}_{,\mu}+F^A{}_\mu{}^\gamma{}_{,\gamma}\right)(x)\,\delta_y(x_1)\dots\delta_y(x_N)\,(\partial^I_{\alpha_1\dots\alpha_I}\delta_y)(x)\\
&\quad+\sum_{I=0}^\infty(-1)^I\,C_{B_1\dots B_N:A}{}^{\alpha_1\dots\alpha_I}\,(y)\,F^A{}_\mu{}^\gamma(x)\,\delta_y(x_1)\dots\delta_y(x_N)\,(\partial^{I+1}_{\alpha_1\dots\alpha_I\gamma}\delta_y)(x)\,.
\end{aligned}$$

This equation is now integrated against a test function $f(x,y,x_1,\dots,x_N)$. Again using the fact that different derivative orders of the test function are independent of each other, one reads off two equations and one sequence of relations for fixed N,

$$0=-\partial_\mu C_{B_1\dots B_N}+\sum_{I=0}^\infty C_{B_1\dots B_N:A}{}^{\alpha_1\dots\alpha_I}\;\varphi^A{}_{,\alpha_1\dots\alpha_I\mu}\,,\tag{2.52}$$

$$\begin{aligned}
0&=-C_{B_1\dots B_N}\,\delta^\gamma_\mu-N\,C_{A(B_1\dots B_{N-1}}\,F^A{}_\mu{}^\gamma{}_{:|B_N)}\\
&\quad+\sum_{I=0}^\infty(I+1)\left[C_{B_1\dots B_N:A}{}^{\alpha_1\dots\alpha_I\gamma}\;\left(\varphi^A{}_{,\mu\alpha_1\dots\alpha_I}+F^A{}_\mu{}^{\alpha_{I+1}}{}_{,\alpha_1\dots\alpha_{I+1}}\right)-C_{B_1\dots B_N:A}{}^{(\alpha_1\dots\alpha_I|}\;F^A{}_\mu{}^{|\gamma)}{}_{,\alpha_1\dots\alpha_I}\right],
\end{aligned}\tag{2.53}$$

$$\begin{aligned}
0&=\sum_{I=0}^\infty\binom{I+L}{L}\Big[C_{B_1\dots B_N:A}{}^{\beta_1\dots\beta_L\alpha_1\dots\alpha_I}\;\left(\varphi^A{}_{,\alpha_1\dots\alpha_I\mu}+F^A{}_\mu{}^{\alpha_{I+1}}{}_{,\alpha_1\dots\alpha_{I+1}}\right)\\
&\qquad-C_{B_1\dots B_N:A}{}^{(\beta_1\dots\beta_L\alpha_1\dots\alpha_{I-1}|}\;F^A{}_\mu{}^{|\alpha_I)}{}_{,\alpha_1\dots\alpha_I}\Big]\qquad\text{for } L\geq 2\,,
\end{aligned}\tag{2.54}$$

where Eq. (2.52) is again the chain rule and can be neglected. For $N=1$, there is no collapse to a finite derivative order and thus, the relations cannot be simplified. Eq. (2.53) and the sequence (2.54) thus read

$$\begin{aligned}
0&=-C_B\,\delta^\gamma_\mu-C_A\,F^A{}_\mu{}^\gamma{}_{:B}+\sum_{I=0}^\infty(I+1)\left[C_{B:A}{}^{\gamma\alpha_1\dots\alpha_I}\;\left(\varphi^A{}_{,\mu\alpha_1\dots\alpha_I}+F^A{}_\mu{}^{\alpha_{I+1}}{}_{,\alpha_1\dots\alpha_{I+1}}\right)\right]\\
&\quad-\sum_{I=0}^\infty(I+1)\,C_{B:A}{}^{(\alpha_1\dots\alpha_I|}\;F^A{}_\mu{}^{|\gamma)}{}_{,\alpha_1\dots\alpha_I}\,,
\end{aligned}\tag{2.55}$$

$$0=\sum_{I=0}^\infty\binom{I+N}{N}\left[C_{B:A}{}^{\beta_1\dots\beta_N\alpha_1\dots\alpha_I}\;\left(\varphi^A{}_{,\mu\alpha_1\dots\alpha_I}+F^A{}_\mu{}^{\alpha_{I+1}}{}_{,\alpha_1\dots\alpha_{I+1}}\right)-C_{B:A}{}^{(\beta_1\dots\beta_N\alpha_1\dots\alpha_{I-1}|}\;F^A{}_\mu{}^{|\alpha_I)}{}_{,\alpha_1\dots\alpha_I}\right].\tag{2.56}$$

Expansion coefficients $C_{B_1\dots B_N}$ with $N\geq 2$ in contrast only depend on the configuration fields up to second derivatives. This simplifies the equations and they break down to the three equations,

$$\begin{aligned}0 = &- C_{B_1 \dots B_N}\,\delta^\gamma_\mu - N\, C_{A(B_1\dots B_{N-1}|}\, F^A{}_\mu{}^\gamma{}_{:|B_N)} - C_{B_1\dots B_N:A}\; F^A{}_\mu{}^\gamma + C_{B_1\dots B_N:A}{}^\gamma\; \varphi^A{}_{,\mu} \\ &- C_{B_1\dots B_N:A}{}^\alpha\; F^A{}_\mu{}^\gamma{}_{,\alpha} - C_{B_1\dots B_N:A}{}^{\alpha_1\alpha_2}\; F^A{}_\mu{}^\gamma{}_{,\alpha_1\alpha_2} + 2\, C_{B_1\dots B_N:A}{}^{\alpha\gamma}\; \varphi^A{}_{,\alpha\mu}\,, \qquad (2.57)\\ 0 = &\; C_{B_1\dots B_N:A}{}^{\beta_1\beta_2}\; \varphi^A{}_{,\mu} - 2\, C_{B_1\dots B_N:A}{}^{\alpha(\beta_1|}\; F^A{}_\mu{}^{|\beta_2)}{}_{,\alpha} - C_{B_1\dots B_N:A}{}^{(\beta_1|}\; F^A{}_\mu{}^{|\beta_2)} \qquad (2.58)\\ 0 = &\; C_{B_1\dots B_N:A}{}^{(\beta_1\beta_2|}\; F^A{}_\mu{}^{|\beta_3)}\;. \qquad (2.59)\end{aligned}$$

The partial differential equations (2.50), (2.51), (2.55), (2.56), (2.57), (2.58) and (2.59) complete the set of partial differential equations which are from now on called the gravitational closure equations. The complete set of the closure equations is listed at the end of this chapter where they will also carry the appropriate numbering which they also have in Ref. [3].

The derivation of the gravitational closure equations has cast the physically almost infinitely hard problem of finding viable gravity theories into the mathematically challenging, but manageable task of solving a system of partial differential equations. As a mathematical side note, the gravitational closure equations as a set of partial differential equations might not have all information explicit that is needed in order to find their exact solution. For example, derivatives of the closure equation with respect to the configuration fiels φ^A could also provide equations that need to be added to the set of differential equations in order to solve them. In mathematical language, the described property of the closure equation is called *involutivity*. Roughly speaking, a system of partial differential equations is called involutive if all information contained in the system is already made explicit. As an example, partial differential equations can contain integrability conditions which have to be revealed by studying further derivatives of the actual differential equations.

If the system is not involutive, it can be made into an involutive one, e. g. by the Cartan-Kuranishi algorithm. For details about this mathematical topic, see Ref. [19]. For a more general discussion of involutivity and the Cartan-Kuranishi algorithm for the constructive gravity program, see Ref. [20]. Without discussing these topics mathematically, the work of the following chapters shows that one indeed has to consider derivatives of the closure equations – called *prolongations* – as well in order to extract the relevant information about the solution from the closure equations.

For a given matter theory, the task is now to solve the closure equations given with the specific input coefficients $F^A{}_\mu{}^\gamma$, $M^{A\gamma}$ and $p^{\mu\nu}$. The solution to the closure equations are the expansion coefficients $C_{A_1\dots A_N}$ of the gravitational Lagrangian $\mathcal{L}$. The last section of this chapter demonstrates the construction of a gravitational spacetime action from such a solution of the gravitational closure equations.

2.5 Gravitational spacetime action and field equations

Up to now, the entire gravitational dynamics were formulated in the canonical picture. The actual goal was to determine the Hamiltonian H constituted by the constraints, supermomentum and superhamiltonian. It is, however, possible to rewrite the canonical description of gravity into a spacetime action. This section shows that the main ingredient of this spacetime action is the Lagrangian $\mathcal{L}$ constructed from the solution of the closure equations. This will be particularly useful for the symmetry reductions studied later in this thesis.

The starting point of the entire construction is given by the Hamiltonian equations of motion,

$$\begin{aligned}\dot{\varphi}^A_t(x) &= \left\{\varphi^A(x), H(n,\vec{n})\right\}_t\\ \dot{\pi}_A(x) &= \left\{\pi_A(x), H(n,\vec{n})\right\}_t\end{aligned}$$

with the dot denoting a derivative with respect to the foliation parameter t and with n, $\vec{n}$ representing the lapse function and shift vector field, respectively. Including a matter Hamiltonian $H_{\text{matter}}[A;\varphi,n,n^\alpha]$, the gravitational field equations can be represented by geometric evolution equations

$$\frac{\delta H_{\text{matter}}}{\delta \varphi^A(x)} = -\left[\partial_t - n^\mu \partial_\mu - \partial_\mu n^\mu\right] \frac{\partial \mathcal{L}}{\partial k^A}(x) + \left[(\partial_\gamma n)\, M^{B\gamma}{}_{:A} - (\partial_\gamma n^\mu)\, F^B{}_\mu{}^\gamma{}_{:A}\right] \frac{\partial \mathcal{L}}{\partial k^B}(x) + \int \mathrm{d}^3 y\, n(y)\, \frac{\delta \mathcal{L}(y)}{\delta \varphi^A(x)}$$

and four constraint equations

$$\begin{aligned} \frac{\delta H_{\text{matter}}}{\delta n(x)} &= -\left[k^A - M^{A\gamma}{}_{,\gamma} - M^{A\gamma}\,\partial_\gamma\right] \frac{\partial \mathcal{L}}{\partial k^A}(x) + \mathcal{L}(x)\,, \\ \frac{\delta H_{\text{matter}}}{\delta n^\mu(x)} &= -\left[\varphi^A{}_{,\mu} + F^A{}_\mu{}^\gamma{}_{,\gamma} + F^A{}_\mu{}^\gamma\,\partial_\gamma\right] \frac{\partial \mathcal{L}}{\partial k^A}(x)\,, \end{aligned}$$

where the velocities k^A need to be replaced by

$$k^A(x) = \frac{1}{n(x)}\left[\dot{\varphi}^A - (\partial_\gamma n)\, M^{A\gamma} - n^\mu\, \varphi^A{}_{,\mu} + (\partial_\gamma n^\mu)\, F^A{}_\mu{}^\gamma\right](x)$$

after evaluation of all respective derivatives. This directly shows that the constraint equations are at most of first derivative order in the foliation parameter t while the evolution equations are of at most second derivative order in t. In particular, this guarantees that there are no Ostrogradsky ghosts. The Helmholtz action giving rise to these canonical field equations is given by

$$S[\varphi, \pi, n, n^\alpha] = \int \mathrm{d}t \left[-H_t[\varphi, \pi, n, n^\alpha] + \int_\Sigma \mathrm{d}^3 z\, (\pi_A\, \dot{\varphi}^A)(z)\right].$$

This form of the action can be made into a more practical one using directly the Lagrangian functional $\mathcal{L}$ from the solution of the gravitational closure equations. Following the steps laid out in Ref. [3], one writes the derivative of the configuration fields with respect to the foliation parameter t as

$$\dot{\varphi}^A = n\, k^A[\varphi; \pi) + (\partial_\gamma n)\, M^{A\gamma} + n^\mu\, \varphi^A{}_{,\mu} - (\partial_\gamma n^\mu)\, F^A{}_\mu{}^\gamma$$

using the first Hamilton equation of motion. Inserting this, the supermomentum (2.13) and the partially determined superhamiltonian (2.17) into the Helmholtz action, one observes that almost all terms drop out. Only the ones from the local superhamiltonian remain. One introduces capitalized spacetime quantities

$$\phi(t, z) = \varphi_t(z) \quad , \quad N(t, z) = n_t(z) \quad \text{and} \quad \vec{N}(t, z) = \vec{n}_t(z)$$

which are numerically identical to the canonical quantities φ, n and $\vec{n}$ but now interpreted as spacetime variables. The gravitational action can then be written as

$$S[\phi, N, \vec{N}] = \int \mathrm{d}t \int_\Sigma \mathrm{d}^3 z\, \mathscr{L}_{\text{geometry}}[\phi, N, \vec{N}](t, z)\,. \tag{2.60}$$

The Lagrangian $\mathscr{L}$ is directly constructed from the Lagrangian functional $\mathcal{L}$ of the gravitational closure equations by

$$\mathscr{L}_{\text{geometry}}[\phi, N, N^\alpha] = N\, \mathcal{L}\left[\phi; \frac{1}{N}\left(\partial_\gamma N)\, M^{A\gamma}(\phi)\right) + N^\mu\, \phi^A{}_{,\mu} - (\partial_\gamma N^\mu)\, F^A{}_\mu{}^\gamma\,(\phi)\right).$$

The field equations obtained from this spacetime picture are identical to the ones previously obtained in the canonical picture. For the purpose of this thesis – the derivation of the Friedmann equations and their refinement for general linear electrodynamics – it will be much simpler to construct the Lagrangian $\mathscr{L}_{\text{geometry}}$ and subsequently vary this Lagrangian with respect to the remaining symmetric degrees of freedom in order to obtain the Friedmann equations and their refinement. For more details see Chapters 4 and 6.

The construction of the gravitational spacetime action is the final piece of information about the desired gravitational dynamics and closes the review of the constructive gravity program. The last section of this chapter shows a complete list of all gravitational closure equations – the heart of the constructive gravity program – in the numbering of Ref. [3] which will be used throughout this thesis and other work such as in Ref. [7, 21].

After this theoretical work, the question arises whether the technology developed in this chapter is actually valid and reproduces the existing results – that is, general relativity being the gravity theory of the standard model of particle physics (or any subsector thereof). Before any kind of modified matter theory or a symmetry reduction of the constructive gravity program is studied, this test has to be passed. Previous works such as Ref. [9, 12] have shown that general relativity follows from Maxwell theory as a solution to the closure equations which at that time were called differently. However, their arguments and techniques relied heavily on arguments specific to the canonical geometry being a metric. By introducing the configuration fields, such arguments are no longer valid. It therefore remains to be seen how the Einstein-Hilbert action is constructed as a solution to the gravitational closure equations using purely the configuration fields and dropping arguments specific for a metric as the canonical geometry as given in Ref. [9, 12]. The next chapter will address precisely this topic and present the general solution of the gravitational closure equations starting from Maxwell theory as matter input.

2.6 The complete list of the gravitational closure equations

$$(C1) \quad 0 = -C\,\delta^{\gamma}_{\mu} + \sum_{I=0}^{\infty}(I+1)\left[C_{:A}{}^{\alpha_1\ldots\alpha_I\gamma}\;\left(\varphi^A{}_{,\mu\alpha_1\ldots\alpha_I} + F^A{}_{\mu}{}^{\alpha_{I+1}}{}_{,\alpha_1\ldots\alpha_{I+1}}\right) - C_{:A}{}^{(\alpha_1\ldots\alpha_I|}\;F^A{}_{\mu}{}^{|\gamma)}{}_{,\alpha_1\ldots\alpha_I}\right]$$

$$\begin{aligned}(C2) \quad 0 &= -C_B\,\delta^{\gamma}_{\mu} - C_A\,F^A{}_{\mu}{}^{\gamma}{}_{:B} + \sum_{I=0}^{\infty}(I+1)\left[C_{B:A}{}^{\gamma\alpha_1\ldots\alpha_I}\;\left(\varphi^A{}_{,\mu\alpha_1\ldots\alpha_I} + F^A{}_{\mu}{}^{\alpha_{I+1}}{}_{,\alpha_1\ldots\alpha_{I+1}}\right)\right] \\ &\quad - \sum_{I=0}^{\infty}(I+1)\,C_{B:A}{}^{(\alpha_1\ldots\alpha_I|}\;F^A{}_{\mu}{}^{|\gamma)}{}_{,\alpha_1\ldots\alpha_I}\end{aligned}$$

$$\begin{aligned}(C3) \quad 0 &= 2\,(\deg P - 1)\,C_{AB}\,p^{\rho(\mu|}\,F^A{}_{\rho}{}^{|\nu)} - \sum_{K=0}^{\infty}(K+1)\,C_{B:A}{}^{\alpha_1\ldots\alpha_K(\mu|}\,M^{A|\nu)}{}_{,\alpha_1\ldots\alpha_K} \\ &\quad + \sum_{K=0}^{\infty}(-1)^K\binom{K+2}{K}\left(\partial^K_{\alpha_1\ldots\alpha_K}\,C_{:B}{}^{\alpha_1\ldots\alpha_K\mu\nu}\right)\end{aligned}$$

$$\begin{aligned}(C4) \quad 0 &= 2\,(\deg P - 1)\,C_{AB}\left(p^{\mu\nu}\,\varphi^A{}_{,\nu} - p^{\mu\nu}{}_{,\gamma}\,F^A{}_{\nu}{}^{\gamma}\right) - C_A\,M^{A\mu}{}_{:B} - \sum_{K=0}^{\infty}C_{B:A}{}^{\alpha_1\ldots\alpha_K}\;M^{A\mu}{}_{,\alpha_1\ldots\alpha_K} \\ &\quad - \sum_{K=0}^{\infty}(-1)^K\,(K+1)\left(\partial^K_{\alpha_1\ldots\alpha_K}C_{:A}{}^{\alpha_1\ldots\alpha_K\mu}\right)\end{aligned}$$

$$\begin{aligned}(C5) \quad 0 &= 2\,\partial_{\mu}\left(C_A\,M^{A[\mu|}{}_{:B}\,M^{B|\gamma]}\right) - 2\,(\deg P - 1)\,p^{\rho\gamma}[C_A\,\varphi^A{}_{,\rho} + \partial_{\mu}(C_A\,F^A{}_{\rho}{}^{\mu})] \\ &\quad + \sum_{K=0}^{\infty}C_{:A}{}^{\alpha_1\ldots\alpha_K}\;M^{A\gamma}{}_{,\alpha_1\ldots\alpha_K} + \sum_{K=0}^{\infty}\sum_{J=0}^{K}(-1)^J\binom{K}{J}(J+1)\,\partial^J_{\alpha_1\ldots\alpha_J}\left(C_{:A}{}^{\beta_1\ldots\beta_{K-J}(\alpha_1\ldots\alpha_J|}\;M^{A|\gamma)}{}_{,\beta_1\ldots\beta_{K-J}}\right)\end{aligned}$$

$$\begin{aligned}(C6) \quad 0 &= 6\,(\deg P - 1)\,C_{AB_1B_2}\left(p^{\mu\nu}\,\varphi^A{}_{,\nu} - p^{\mu\nu}{}_{,\gamma}\,F^A{}_{\nu}{}^{\gamma}\right) - 4\,C_{A(B_1|}\,M^{A\mu}{}_{:|B_2)} - 2\,C_{B_1B_2:A}\;M^{A\mu} \\ &\quad - 2\,C_{B_1B_2:A}{}^{\alpha}\;M^{A\mu}{}_{,\alpha} - 2\,C_{B_1B_2:A}{}^{\alpha\beta}\;M^{A\mu}{}_{,\alpha\beta} - C_{B_2:B_1}{}^{\mu} - \sum_{K=0}^{\infty}(-1)^K\,(K+1)\left(\partial^K_{\alpha_1\ldots\alpha_K}C_{B_1:B_2}{}^{\mu\alpha_1\ldots\alpha_K}\right)\end{aligned}$$

$$(C7) \quad 0 = \sum_{K=0}^{\infty}\sum_{J=2}^{K+1}(-1)^J\binom{K}{J-1}\binom{J}{N}(J-1)\,\partial^{J+1}_{\gamma\alpha_1\ldots\alpha_J}\left(C_{:A}{}^{\beta_1\ldots\beta_{K-J}(\alpha_1\ldots\alpha_J|}\;M^{A|\gamma)}{}_{,\beta_1\ldots\beta_{K-J}}\right)$$

$$(C8_{N\geq2}) \quad 0 = \sum_{I=0}^{\infty}\binom{I+L}{L}\left[C_{:A}{}^{\beta_1\ldots\beta_L\alpha_1\ldots\alpha_I}\;\left(\varphi^A{}_{,\mu\alpha_1\ldots\alpha_I} + F^A{}_{\mu}{}^{\alpha_{I+1}}{}_{,\alpha_1\ldots\alpha_{I+1}}\right) - C_{:A}{}^{(\beta_1\ldots\beta_L\alpha_1\ldots\alpha_{I-1}|}\;F^A{}_{\mu}{}^{|\alpha_I)}{}_{,\alpha_1\ldots\alpha_I}\right]$$

$$(C9_{N\geq2}) \quad 0 = \sum_{I=0}^{\infty}\binom{I+N}{N}\left[C_{B:A}{}^{\beta_1\ldots\beta_N\alpha_1\ldots\alpha_I}\;\left(\varphi^A{}_{,\mu\alpha_1\ldots\alpha_I} + F^A{}_{\mu}{}^{\alpha_{I+1}}{}_{,\alpha_1\ldots\alpha_{I+1}}\right) - C_{B:A}{}^{(\beta_1\ldots\beta_N\alpha_1\ldots\alpha_{I-1}|}\;F^A{}_{\mu}{}^{|\alpha_I)}{}_{,\alpha_1\ldots\alpha_I}\right]$$

$$\begin{aligned}(C10_{N\geq2}) \quad 0 &= -\,C_{B_1\ldots B_N}\,\delta^{\gamma}_{\mu} - N\,C_{A(B_1\ldots B_{N-1}|}\,F^A{}_{\mu}{}^{\gamma}{}_{:|B_N)} - C_{B_1\ldots B_N:A}\;F^A{}_{\mu}{}^{\gamma} + C_{B_1\ldots B_N:A}{}^{\gamma}\;\varphi^A{}_{,\mu} \\ &\quad - C_{B_1\ldots B_N:A}{}^{\alpha}\;F^A{}_{\mu}{}^{\gamma}{}_{,\alpha} - C_{B_1\ldots B_N:A}{}^{\alpha_1\alpha_2}\;F^A{}_{\mu}{}^{\gamma}{}_{,\alpha_1\alpha_2} + 2\,C_{B_1\ldots B_N:A}{}^{\alpha\gamma}\;\varphi^A{}_{,\alpha\mu}\end{aligned}$$

$$(C11_{N\geq2}) \quad 0 = C_{B_1\ldots B_N:A}{}^{\beta_1\beta_2}\;\varphi^A{}_{,\mu} - 2\,C_{B_1\ldots B_N:A}{}^{\alpha(\beta_1|}\;F^A{}_{\mu}{}^{|\beta_2)}{}_{,\alpha} - C_{B_1\ldots B_N:A}{}^{(\beta_1|}\;F^A{}_{\mu}{}^{|\beta_2)}$$

$$(C12_{N\geq2}) \quad 0 = C_{B_1\ldots B_N:A}{}^{(\beta_1\beta_2|}\;F^A{}_{\mu}{}^{|\beta_3)}$$

$$(C13_{N\geq2}) \quad 0 = C_{B_1\ldots B_N:A}{}^{(\beta_1\beta_2|}\;M^{A|\beta_3)}$$

$$(C14_{N\geq2}) \quad 0 = C_{AB_1\ldots B_{N-1}}\left[(\deg P - 1)\,p^{\rho\mu}F^A{}_{\rho}{}^{\nu} - M^{B[\mu|}\,M^{A|\nu]}{}_{:B}\right]$$

$$(C15_{N\geq2}) \quad 0 = C_{B_1\ldots\widetilde{B_J}\ldots B_{N+1}:B_J}{}^{\mu\nu} - C_{B_1\ldots B_N:B_{N+1}}{}^{\mu\nu} \quad \text{for } J = 1\ldots N+1$$

$$\begin{aligned}(C16_{N\geq2}) \quad 0 &= N\,(N+1)\,(\deg P - 1)\,C_{AB_1\ldots B_N}\,p^{\rho(\mu|}\,F^A{}_{\rho}{}^{|\nu)} + N\,C_{B_1\ldots B_N:A}{}^{(\mu|}\;M^{A|\nu)} + 2\,N\,C_{B_1\ldots B_N:A}{}^{\alpha(\mu|}\;M^{A|\nu)}{}_{,\alpha} \\ &\quad + (N-2)\,C_{B_1\ldots B_{N-1}:B_N}{}^{\mu\nu}\end{aligned}$$

$$
\begin{aligned}
(C17_{N\geq 2})\quad 0 &= (N+2)\,(N+1)\,(\deg P - 1)\, C_{AB_1\ldots B_{N+1}}\,\left(p^{\mu\gamma}\varphi^{A}{}_{,\gamma} - p^{\mu\nu}{}_{,\gamma}\,F^{A}{}_{\nu}{}^{\gamma}\right)\\
&\quad - (N+1)^2\, C_{A(B_1\ldots B_N|}\, M^{A\mu}{}_{:|B_{N+1})} - (N+1)\, C_{B_1\ldots B_N:A}\;\; M^{A\mu}\\
&\quad - (N+1)\, C_{B_1\ldots B_N:A}{}^{\alpha}\;\; M^{A\mu}{}_{,\alpha} - (N+1)\, C_{B_1\ldots B_N:A}{}^{\alpha\beta}\;\; M^{A\mu}{}_{,\alpha\beta}\\
&\quad - \sum_{K=0}^{N+1} C_{B_1\ldots\widetilde{B_K}\ldots B_{N+1}:B_K}{}^{\mu} + 2\,\left(\partial_\gamma C_{B_1\ldots B_N:B_{N+1}}{}^{\mu\gamma}\right)
\end{aligned}
$$

$$
(C18_{N\geq 2})\quad 0 = C_{B_2:B_1}{}^{\mu_1\ldots\mu_L} - \sum_{K=0}^{\infty}(-1)^{K+L}\binom{K+L}{L}\left(\partial^{K}_{\alpha_1\ldots\alpha_K} C_{B_1:B_2}{}^{\alpha_1\ldots\alpha_K\mu_1\ldots\mu_L}\right)
$$

$$
(C19_{N\geq 2})\quad 0 = \sum_{K=0}^{\infty}\left[\binom{K+L}{L} C_{B:A}{}^{\alpha_1\alpha_K(\mu_1\ldots\mu_L|}\;\; M^{A|\mu_{L+1})}{}_{,\alpha_1\ldots\alpha_K} + (-1)^{K+L}\binom{K+L+1}{L+1}\left(\partial^{K}_{\alpha_1\ldots\alpha_K} C_{:B}{}^{\alpha_1\ldots\alpha_K\mu_1\ldots\mu_{L+1}}\right)\right]
$$

$$
(C20_{\text{even } N\geq 2})\quad 0 = \sum_{K=N}^{\infty}\sum_{J=N+1}^{K+1}(-1)^J\binom{K}{J-1}\binom{J}{N}\,\partial^{J-N}_{\alpha_1\ldots\alpha_{J-N}}\left(C_{:A}{}^{\beta_J\ldots\beta_K(\alpha_1\ldots\alpha_{J-N}\mu_1\ldots\mu_{N-1}|}\;\; M^{A|\mu_N)}{}_{,\beta_J\ldots\beta_K}\right)
$$

$$
\begin{aligned}
(C21_{\text{odd } N\geq 3})\quad 0 &= 2\sum_{K=N-1}^{\infty}\binom{K}{N-1} C_{:A}{}^{\beta_N\ldots\beta_K(\mu_1\ldots\mu_{N-1}|}\;\; M^{A|\mu_N)}{}_{,\beta_N\ldots\beta_K}\\
&\quad - \sum_{K=N}^{\infty}\sum_{J=N+1}^{K+1}\binom{K}{J-1}\binom{J}{N}\,\partial^{J-N}_{\alpha_1\ldots\alpha_{J-N}}\left(C_{:A}{}^{\beta_J\ldots\beta_K(\alpha_1\ldots\alpha_{J-N}\mu_1\ldots\mu_{N-1}|}\;\; M^{A|\mu_N)}{}_{,\beta_J\ldots\beta_K}\right)
\end{aligned}
$$

Chapter 3

From Maxwell electrodynamics to Einstein gravity

The indispensable test run for the constructive gravity program is to recover already known results. Starting from the standard model of particle physics (or any subsector), the Einstein-Hilbert action of general relativity should be recovered as a solution to the gravitational closure equations.

In contrast to previous work such as Ref. [8, 9, 12] which already showed similar calculations, the following chapter provides a conceptually new solution to the closure equations. The closure equations are now formulated as partial differential equations in which the derivatives are taken with respect to the newly established configuration fields and no longer with respect to the components of the canonical geometry. This prohibits to use variables such as the Riemann curvature tensor with respect to whom derivatives can be taken in the closure equations. Instead, one may only consider derivatives with respect to the configuration fields and their spatial derivatives. This accounts for a more involved, but conceptually more appealing solution of the closure equations.

Adopting the diagram 1.2 to the present case study, the chapter is structured as shown in Figure 3.1.

$$S_{\text{Maxwell}}[A;g] \xrightarrow[\text{Sec. 3.1}]{} \begin{matrix}\text{closure}\\ \text{equations}\end{matrix} \xrightarrow[\text{Sec. 3.2 – 3.4}]{} S_{\text{Einstein–Hilbert}}[g] \longrightarrow \begin{matrix}\text{Einstein}\\ \text{equations}\end{matrix}$$

Figure 3.1: Construction of the Einstein equations as a solution to the closure equations from Maxwell electrodynamics. With the Einstein-Hilbert action being the actual solution of the closure equations, it is then straightforward to calculate the Einstein equations by variation.

3.1 Matter action, principal polynomial and kinematical setup

The results shown in this section have already been published in
M. Düll, F. P. Schuller, N. Stritzelberger and F. Wolz,
Phys. Rev. D97 (2018), 084036,
whose section V.A is shown here.

The starting point of the gravitational closure formalism is always the prescribed matter action. While one could in principle gravitationally close the entire standard model of particle physics, this thesis restricts the matter action to the classical electromagnetic sector – Maxwell theory. This is perfectly valid from the conceptual point of view as all subsectors of the standard model feature the same geometric setup and account for the same principal polynomial P and therefore also the same gravity theory. By restricting to the electromagnetic sector, the construction of the principal polynomial becomes a rather quick calculation compared to the one for the entire standard model [14]. The Maxwell action of electrodynamics is given by

$$S_{\text{Maxwell}} = -\frac{1}{4} \int_M \mathrm{d}^4x \sqrt{|\det g_{..}|}\, F_{ab} F_{cd}\, g^{ac} g^{bd} \,,$$

where M is a four-dimensional smooth manifold equipped with a metric tensor field g and $F = \mathrm{d}A$ the field strength tensor for the one-form potential A. From this action, the field equations arise by variation with respect to the matter field A. Besides, due to the definition of the field strength $F = \mathrm{d}A$, also $\mathrm{d}F = 0$ is part of the field equations. As mentioned before, the matter field equations are the starting point for the constructive gravity program. From those, the principal polynomial has to be read off and subsequently, the kinematical setup of the spacetime geometry is constructed.

As already demonstrated in Section 2.1, it is most convenient to choose a gauge condition, insert it into the Maxwell equations which allows to read off the principal polynomial

$$P(k) = g^{ab}\, k_a k_b \,.$$

Thus, for Maxwell electrodynamics and actually for any other matter of the standard model of particle physics, the principal polynomial coincides with the metric tensor field serving as the geometry of spacetime. In different matter models such as general linear electrodynamics, the relation between principal polynomial and geometry is more involved in general.

Up to now, the metric still has arbitrary signature. Imposing the first of the three matter conditions – predictivity – which technically means requiring that the principal polynomial P is hyperbolic, fixes the signature to be Lorentzian and thus, the spacetime geometry has to be a Lorentzian metric. Then, also the other two matter conditions – the existence of a momentum-velocity duality and energy distinguishability – are already satisfied [13].

Once these three matter conditions are satisfied, one can identify the canonical geometry, i. e. the geometry of spatial hypersurfaces. It is

$$\begin{array}{llcll} g^{00} & := g(\epsilon^0, \epsilon^0) & , & g^{0\alpha} & := g(\epsilon^0, \epsilon^\alpha) \\ g^{\alpha 0} & := g(\epsilon^\alpha, \epsilon^0) & , & g^{\alpha\beta} & := g(\epsilon^\alpha, \epsilon^\beta) \,. \end{array}$$

The normalisation and annihilation conditions

$$p = g^{00} = 1 \quad , \quad p^\alpha = g^{\alpha 0} = 0$$

together with the symmetry $g^{ab} = g^{ba}$ of the metric remove ten of the originally 16 entries of the spacetime metric. The remaining six degrees of freedom are encoded in a symmetric tensor field $g^{\alpha\beta}$, the spatial metric which also appears in the ADM split of the metric. One suitable parametrization of this spatial metric in terms of the six geometric degrees of freedom φ^A is given by

$$\hat{g}^{\alpha\beta}(\varphi^A) := I^{\alpha\beta}{}_A\, \varphi^A \,.$$

Vice versa, the geometric degrees of freedom φ^A can be extracted from the spatial metric by the inverse parametrization map

$$\hat{\varphi}^A(g) := I^A{}_{\alpha\beta}\, g^{\alpha\beta} \,.$$

The intertwining matrices I have to satisfy the two conditions

$$I^A{}_{\alpha\beta} I^{\alpha\beta}{}_B \stackrel{!}{=} \delta^A_B \quad \text{and} \quad I^{\alpha\beta}{}_A I^A{}_{\mu\nu} \stackrel{!}{=} \delta^\alpha_{(\mu} \delta^\beta_{\nu)}$$

in order to yield an admissible parametrization. These conditions are satisfied by the matrices

$$I^A{}_{\alpha\beta} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}^{A}_{\alpha\beta} \quad \text{and}$$

$$I^{\alpha\beta}{}_A = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}^{\alpha\beta}_{A} . \tag{3.1}$$

One is now fully equipped to calculate the three input coefficients entering the gravitational closure equations.

3.2 Solution to the gravitational closure equations

In order to set up the gravitational closure equations, the three input coefficients entering the closure equations have to be calculated. With the parametrization chosen above, those three are quickly calculated. While the coefficient $M^{A\gamma}$ vanishes, the remaining two are

$$\begin{aligned} p^{\alpha\beta} &= \hat{g}^{\alpha\beta} , \\ F^A{}_\mu{}^\gamma &= 2\, I^A{}_{\mu\sigma}\, \hat{g}^{\gamma\sigma} . \end{aligned} \tag{3.2}$$

Already the triviality of $M^{A\gamma}$ simplifies the closure equations a lot as many terms throughout all equations drop out and equations $(C7)$, $(C13_N)$, $(C20_N)$ and $(C21_N)$ are completely trivial. Besides, one can address the collapse of the coefficient C from arbitrary derivative order to second derivative order.

Dependence of the coefficient C on derivatives of the geometric d. o. f.

Due to the trivial input coefficient $M^{A\gamma}$, closure equation $(C19_{N\geq 2})$ simplifies to

$$0 = \sum_{K=0}^{\infty} (-1)^{K+N} \binom{K+N+1}{K+1} \partial^K_{\alpha_1 \ldots \alpha_K} C_{:B}{}^{\alpha_1 \ldots \alpha_K \mu_1 \ldots \mu_{N+1}} . \tag{3.3}$$

As mentioned before, the expansion coefficient C of the gravitational Lagrangian depends on arbitrary many derivatives of the geometric degrees of freedom φ^A. Assuming that there is some integer F for which

$$C_{:A}{}^{\mu_1 \ldots \mu_N} = 0 \quad \forall N > F ,$$

closure equation $(C19_{F-1})$ provides

$$C_{:A}{}^{\mu_1 \ldots \mu_F} = 0 .$$

This argument can be extended to all further N down to $N = 2$ for which one finally reads off that

$$C_{:A}{}^{\mu_1 \mu_2 \mu_3} = 0 .$$

This means that the expansion coefficient C depends at most on the zeroth, first and second derivatives of the configuration fields φ^A. For the expansion coefficient C_A, a similar argument has not been found yet from the closure equations and this coefficient depends on arbitrary many derivatives of φ^A. For details, see Section 3.3.

The dependence of the expansion coefficient C on first and second derivatives of the geometric degrees of freedom φ^A are governed by the equations $(C1)$, $(C8_2)$, $(C8_3)$ and a prolongation of $(C4)$. In order to see the relevance of this prolongation, first consider closure equation $(C4)$,

$$0 = 2\,C_{AB}\left(\varphi^A{}_{,\nu} p^{\mu\nu} - p^{\mu\nu}{}_{,\gamma} F^A{}_\nu{}^\gamma\right) - C_{:B}{}^\mu \; + 2\,\partial_\alpha\left(C_{:B}{}^{\alpha\mu}\right)$$

for which the divergence term can be expanded and the resulting equation reads

$$0 = 2\,C_{AB}\left(\varphi^A{}_{,\nu} p^{\mu\nu} - p^{\mu\nu}{}_{,\gamma} F^A{}_\nu{}^\gamma\right) - C_{:B}{}^\mu \; + 2\,C_{:B}{}^{\alpha\mu}{}_{:D}\; \varphi^A{}_{,\alpha} + 2\,C_{:B}{}^{\alpha\mu}{}_{:D}{}^\lambda\, \varphi^D{}_{,\alpha\lambda} + 2\,C_{:B}{}^{\alpha\mu}{}_{:D}{}^{\lambda_1\lambda_2}\, \varphi^D{}_{,\alpha\lambda_1\lambda_2} \tag{3.4}$$

The last term in this equation contains a third derivative of the configuration fields. No other term in the equation carries such a derivative as the expansion coefficients C and C_{AB} do not depend on them and also none of the appearing input coefficients has a third derivative in it. Thus, from the prolongation $(3.4)_{:E}{}^{\omega_1\omega_2\omega_3}$, one concludes that

$$0 = C_{:B}{}^{\mu(\omega_1}{}_{:E}{}^{\omega_2\omega_3)}\;. \tag{3.5}$$

If not for the symmetrization brackets, one would read off that the coefficient C depends at most linearly on second derivatives of φ^A. For the equation as stated here, this conclusion must not be made as only the symmetrized version of the second derivative of C vanishes and not the full derivative. It turns out that due to closure equation $(C8_3)$ and an argument first carried out by Lovelock, the full derivative already vanishes.

Before coming to this conclusion, it is helpful to introduce the metric $g_{\alpha\beta}$ as the inverse to $g^{\alpha\beta}$. This tensor field can also be parametrized by a set of six „inverse“ configuration fields ψ_A as

$$\hat{g}_{\alpha\beta} = I^A{}_{\alpha\beta}\,\psi_A$$

with the same intertwining matrices I as before. As $g_{\cdot\cdot}$ is the inverse to $g^{\cdot\cdot}$, it is clear that the inverse d. o. f. depend on the original d. o. f. in a complicated way which is not explicitly needed for the calculations. Only the derivative

$$\frac{\partial\psi_A}{\partial\varphi^B} = -I^{\alpha\mu}{}_A I^{\beta\nu}{}_B\, g_{\alpha\beta}\, g_{\mu\nu}$$

is needed. This allows to rewrite closure equation $(C8_3)$ and relation (3.5) in terms of the inverse d. o. f. and conduct the desired argument. First, closure equation $(C8_3)$,

$$0 = C_{:A}{}^{(\beta_1\beta_2|}\, F^A{}_\mu{}^{|\beta_3)}\;, \tag{3.6}$$

is transformed to

$$0 = \frac{\partial C}{\partial\psi_{B,(\beta_1\beta_2|}}\, I^{|\beta_3)\omega}{}_B\,.$$

Expanding the symmetrization and applying the relation three times to the resulting terms, one concludes that (3.6) admits the exchange symmetry

$$\frac{\partial C}{\partial\psi_{B,\beta_1\beta_2}}\, I^{\alpha_1\alpha_2}{}_B = \frac{\partial C}{\partial\psi_{B,\alpha_1\alpha_2}}\, I^{\beta_1\beta_2}{}_B\,. \tag{3.7}$$

Also, relation (3.5) obtained as a prolongation from (3.4) has to be rewritten in terms of the inverse d. o. f. ψ_B and reads

$$0 = I^{\nu_1\nu_2}{}_A I^{\lambda_1\lambda_2}{}_B \frac{\partial^2 C}{\partial\psi_{A,\mu(\omega_1|}\partial\psi_{B,|\omega_2\omega_3)}}\,. \tag{3.8}$$

The last two relations (3.7) and (3.8) now allow to conduct the argument why the expansion coefficient C depends at most linearly on second derivatives of φ^A. The core of this argument stems from arguments given by Lovelock [22] while the argument itself is almost identical to the one given in previous work on the topic, most prominently in Ref. [9] and also mentioned in Ref. [8, 12].

The derivative

$$0 = I^{\nu_1\nu_2}{}_A\, I^{\lambda_1\lambda_2}{}_B \frac{\partial^2 C}{\partial\psi_{A,\mu(\omega_1|}\partial\psi_{B,|\omega_2\omega_3)}}$$

contains eight free Greek indices. In three dimensions, at least three of these eight indices have to be equal. Due to the Schwarz's theorem and the exchange symmetry (3.7), it is possible for any choice of the eight indices to arrange them in such a way that three equal indices fall into the position of the symmetrized ω_1, ω_2 and ω_3 in Eq. (3.8). It is directly clear that this derivative always vanishes as a symmetrization of three equal indices in Eq. (3.8) always vanishes Thus, any second derivative

$$\frac{\partial^2 C}{\partial\psi_{A,\alpha_1\alpha_2}\partial\psi_{B,\beta_1\beta_2}} = 0$$

vanishes and as a direct consequence, also second derivatives of C with respect to $\partial^2\varphi^A$ vanish. In other words, the expansion coefficient C depends at most linearly on second derivatives of the configuration fields.

Next, one investigates how the coefficient C depends on first and second derivatives of φ^A. Therefore, investigate closure equation $(C8_2)$,

$$0 = C_{:A}{}^{\beta_1\beta_2}\ \varphi^A{}_{,\mu} - 2\, C_{:A}{}^{\gamma(\beta_1|}\ F^A{}_\mu{}^{|\beta_2)}{}_{,\gamma} - C_{:A}{}^{(\beta_1|}\ F^A{}_\mu{}^{|\beta_2)}\ . \tag{3.9}$$

There are 18 different derivatives $C_{:A}{}^{\mu}$. Closure equation (3.9) features precisely 18 different relations for these derivatives. It turns out that one can indeed express every derivative $C_{:A}{}^{\mu}$ in terms of a combination of derivatives of the form $C_{:A}{}^{\mu\nu}$. Considering the prolongation $(3.9)_{:D}{}^{\lambda_1\lambda_2}$, the only terms that remain in this equation are

$$0 = C_{:A}{}^{\mu}{}_{:B}{}^{\nu\sigma}$$

as C depends at most linearly on second derivatives $\partial^2\varphi^A$. Thus, one concludes that the expansion coefficient C also contains no terms with both first and second derivatives of the configuration fields. These two insights will now be used in order to construct a solution for the expansion coefficient C_{AB}. This coefficient is coupled to C by closure equation $(C3)$.

Solution of the coefficient C_{AB}

The expansion coefficient C_{AB} will be the first one which is actually solved for. First important information about it can be extracted from closure equation $(C3)$ which relates it with derivatives of C by

$$0 = 2\, C_{AB}\, F^A{}_\rho{}^{(\mu}\ p^{\nu)\rho} - C_{:B}{}^{\mu\nu}\ . \tag{3.10}$$

Before actually analyzing this equations, consider the prolongation $(3.10)_{:D}{}^{\lambda_1\lambda_2}$ to conclude that C_{AB} does not depend on second derivatives of the configuration fields. This is because C depends at most linearly on them and the combination $F^A{}_\rho{}^{(\mu}\ p^{\nu)\rho}$ can be inverted from the equation. Additionally, the prolongation $(3.10)_{:D}{}^{\lambda}$ shows that the coefficient C_{AB} also does not depend on first derivatives of the configuration fields. Thus, as C_{AB} is a function of only the configuration fields themselves which simplifies closure equation $(C10_2)$ to

$$0 = C_{B_1B_2}\delta^\gamma_\mu + 4\, C_{A(B_1|}\, I^A{}_{\mu\sigma} I^{\gamma\sigma}{}_{|B_2)} + C_{B_1B_2:A}\ F^A{}_\mu{}^\gamma\ . \tag{3.11}$$

In order to solve this differential equation, it is useful to split the tensor density C_{AB} into a scalar density of weight one $\chi(\varphi)$ and a tensor $\tilde{C}_{AB}$. A convenient choice for the scalar density is

$$\chi(\varphi) = \frac{1}{\sqrt{-\det \hat{g}^{\cdot\cdot}}}\ .$$

Plugging this ansatz into Eq. (3.11), one obtains a differential equation for the tensor components $\tilde{C}_{AB}$,

$$0 = 2\,\tilde{C}_{A(B_1|}\, I^A{}_{\mu\sigma} I^{\gamma\sigma}{}_{|B_2)} + \tilde{C}_{B_1 B_2 : A}\, I^A{}_{\mu\sigma} g^{\gamma\sigma} \,.$$

One can try and solve this differential equation. Practically, it is, however, simpler to first perform a change of variables and use the inverse configuration fields ψ_B in this equation. It is then

$$0 = 2\,\tilde{C}_{A(B_1|}\, I^A{}_{\mu\sigma} I^{\gamma\sigma}{}_{|B_2)} - \frac{\partial \tilde{C}_{B_1 B_2}}{\partial \psi_A}\, I^{\omega\gamma}{}_A\, g_{\omega\mu} \,,$$

for which one quickly checks that the expression

$$\tilde{C}_{B_1 B_2} = I^{\lambda_1\lambda_2}{}_{(B_1} I^{\omega_1\omega_2}{}_{B_2)} \left(a_0\, g_{\lambda_1\omega_1} g_{\lambda_2\omega_2} - a_1\, g_{\lambda_1\lambda_2} g_{\omega_1\omega_2}\right)$$

is a solution. There are two independent constants a_0 and a_1 appearing in this expression. The full expansion coefficient C_{AB} can be plugged together as

$$C_{AB} = \frac{1}{\sqrt{-\det \hat{g}^{\cdot\cdot}}}\, I^{\lambda_1\lambda_2}{}_{(A} I^{\omega_1\omega_2}{}_{B)} \left(a_0\, \hat{g}_{\lambda_1\omega_1} \hat{g}_{\lambda_2\omega_2} - a_1\, \hat{g}_{\lambda_1\lambda_2} \hat{g}_{\omega_1\omega_2}\right) \,. \tag{3.12}$$

This solution will now be used in closure equation $(C3)$ which will then provide a differential equation for the dependence of C on second derivatives of the configuration fields.

Solution of the coefficient C

Recall that closure equation $(C3)$ contained two essential terms, the coefficient C_{AB} and the derivative $C_{:A}{}^{\mu\nu}$. As a solution for C_{AB} has been obtained, one can rearrange Eq. (3.10) as

$$C_{:B}{}^{\mu\nu} = \frac{4}{\sqrt{-\det \hat{g}^{\cdot\cdot}}} \left(a_0 I^{\mu\nu}{}_B - a_1 I^{\lambda_1\lambda_2}{}_B\, \hat{g}_{\lambda_1\lambda_2}\, g^{\mu\nu}\right) .$$

It is already a differential equation for the dependence of C on second derivatives of the configuration fields. Integration yields the partial solution of C as

$$C = \frac{4}{\sqrt{-\det \hat{g}^{\cdot\cdot}}} \left(a_0 \hat{g}^{\mu\nu}{}_{,\mu\nu} - a_1 \hat{g}_{\lambda_1\lambda_2} \hat{g}^{\lambda_1\lambda_2}{}_{,\mu\nu}\, \hat{g}^{\mu\nu}\right) + f(\partial\varphi, \varphi) \,, \tag{3.13}$$

where f is a scalar density of weight one depending on zeroth and first derivatives of the geometric degrees of freedom. This function will be determined by analysing the dependence of C on first derivatives of φ^A using closure equation (3.4). After rearranging, it is a differential equation for the dependence of the coefficient C on first derivatives $\partial\varphi^A$ of the configuration fields,

$$C_{:B}{}^{\mu} = f_{:B}{}^{\mu} = 2\,C_{AB} \left(\hat{g}^{\mu\nu}\varphi^A{}_{,\nu} - 2\, I^A{}_{\nu\sigma} I^{\mu\nu}{}_E \varphi^E{}_{,\gamma} \hat{g}^{\gamma\sigma}\right) + 2\,C_{:B}{}^{\alpha\mu}{}_{:D}\; \varphi^D{}_{,\alpha} \,.$$

Inserting the solution (3.12) of the coefficient C_{AB} and the partial solution (3.13) of the coefficient C, this equation can only be integrated if the two constants a_0 and a_1 appearing in (3.12) are equal, $a := a_0 \overset{!}{=} a_1$. One finds

$$\begin{aligned} C = &\frac{a}{\sqrt{-\det \hat{g}^{\cdot\cdot}}} \big(4\, \hat{g}^{\mu\nu}{}_{,\mu\nu} - 4\, \hat{g}_{\lambda_1\lambda_2} \hat{g}^{\lambda_1\lambda_2}{}_{,\mu\nu}\, \hat{g}^{\mu\nu} + 5\, \hat{g}_{\lambda_1\omega_1} \hat{g}_{\lambda_2\omega_2} \hat{g}^{\lambda_1\lambda_2}{}_{,\mu} \hat{g}^{\omega_1\omega_2}{}_{,\nu}\, \hat{g}^{\mu\nu} \\ &\quad + \hat{g}_{\lambda_1\lambda_2} \hat{g}_{\omega_1\omega_2} \hat{g}^{\lambda_1\lambda_2}{}_{,\mu} \hat{g}^{\omega_1\omega_2}{}_{,\nu}\, \hat{g}^{\mu\nu} - 2\, \hat{g}_{\lambda\nu}\, \hat{g}^{\lambda\gamma}{}_{,\mu}\, \hat{g}^{\mu\nu}{}_{,\gamma} - 4\, \hat{g}_{\omega_1\omega_2}\, \hat{g}^{\omega_1\omega_2}{}_{,\mu}\, \hat{g}^{\mu\gamma}{}_{,\gamma}\big) \\ &+ \frac{1}{\sqrt{-\det \hat{g}^{\cdot\cdot}}} \bar{f}(\varphi) \end{aligned} \tag{3.14}$$

with an undetermined function $\bar{f}$ depending only on the degrees of freedom φ. The function $\bar{f}$ has to be determined from the remaining closure equation $(C1)$. After some calculation and use of the partial solution (3.14), one finds that the function $\bar{f}$ is actually a constant, $\bar{f} := a \cdot b$. The solution of expansion coefficients C is thus

$$C = \frac{a}{\sqrt{-\det \hat{g}^{\cdot\cdot}}} (4\, \hat{g}^{\mu\nu}{}_{,\mu\nu} - 4\, \hat{g}_{\lambda_1\lambda_2} \hat{g}^{\lambda_1\lambda_2}{}_{,\mu\nu}\, \hat{g}^{\mu\nu} + 5\, \hat{g}_{\lambda_1\omega_1} \hat{g}_{\lambda_2\omega_2} \hat{g}^{\lambda_1\lambda_2}{}_{,\mu} \hat{g}^{\omega_1\omega_2}{}_{,\nu}\, \hat{g}^{\mu\nu}$$
$$+ \hat{g}_{\lambda_1\lambda_2} \hat{g}_{\omega_1\omega_2} \hat{g}^{\lambda_1\lambda_2}{}_{,\mu} \hat{g}^{\omega_1\omega_2}{}_{,\nu}\, \hat{g}^{\mu\nu} - 2\, \hat{g}_{\lambda\nu}\, \hat{g}^{\lambda\gamma}{}_{,\mu}\, \hat{g}^{\mu\nu}{}_{,\gamma} - 4\, \hat{g}_{\omega_1\omega_2}\, \hat{g}^{\omega_1\omega_2}{}_{,\mu}\, \hat{g}^{\mu\gamma}{}_{,\gamma} + b) \tag{3.15}$$

It is important to stress that the two constants a and b are integration constants and thus, a priori undetermined. One would have to perform two independent experiments in order to determine their value and make the theory predictive.

As one expects to know the result of this calculation, one compares the obtained expression to the standard theory of general relativity. One discovers that the expression (3.15) is indeed the three-dimensional Ricci scalar. Additionally, one can identify the two integration constants a and b as the gravitational and the cosmological constant. Thus, one can write Eq. (3.15) more compactly as

$$C = -\frac{1}{2\kappa} \frac{1}{\sqrt{-\det \hat{g}^{\cdot\cdot}}} (R - 2\Lambda) \ ,$$

where R is the three-dimensional Ricci scalar, Λ the cosmological constant and κ the gravitational constant.

After solving for two expansion coefficients, one needs to solve the others as well. It will be demonstrated in the following that the other expansion coefficients will not contribute any further information to the theory as they either vanish or drop from the equations of motion as boundary terms in the Lagrangian.

Triviality of higher numbered expansion coefficients

The starting point for the solution of expansion coefficients $C_{A_1 \dots A_N}$ for $N \geq 3$ is the sequence of closure equation $(C16_N)$,

$$0 = N \cdot (N+1)\, C_{AB_1 \dots B_N}\, F^A{}_\rho{}^{(\mu}\ p^{\nu)\rho} + (N-2)\, C_{B_1 \dots B_{N-1}:B_N}{}^{\mu\nu} \ .$$

For $N = 2$, this equation simplifies to

$$0 = 6\, C_{AB_1B_2} F^A{}_\rho{}^{(\mu}\ p^{\nu)\rho} \ .$$

Inserting the expressions (3.2) of the input coefficient $F^A{}_\mu{}^\gamma$ and inverting it from the equation, one concludes that the expansion coefficient $C_{B_1B_2B_3}$ vanishes. Since this coefficient vanishes, the expansion coefficient $C_{B_1 \dots B_5}$ vanishes as well. This is a direct consequence from closure equation $(C16_4)$. Extending this argument to all even-numbered instances $(C16_{2N})$ for $N \geq 2$, it is apparent that all odd-numbered expansion coefficients $C_{B_1 \dots B_{2N+1}}$ for $N \geq 1$ vanish.

The analogous argument holds for their even-numbered counterparts. As C_{AB} does not depend on second derivatives of the geometric degrees of freedom, evaluating $(C16_3)$ yields that $C_{B_1B_2B_3B_4}$ vanishes. Evaluating all odd-numbered instances $(C16_{2N+1})$ demonstrates the triviality of all even-numbered expansion coefficients $C_{B_1 \dots B_{2N}}$ for $N \geq 2$.

From the infinity of expansion coefficients that were there in the beginning of the evaluation, only three can actually contribute to the Lagrangian, namely C, C_A and C_{AB}. As the next section shows, only C and C_{AB} will actually be used for the construction of the field equations as the coefficient C_A turns out to be dynamically irrelevant.

3.3 Remarks about the second expansion coefficient

The last unknown expansion coefficient of the metric gravitational Lagrangian is C_A. The closure equations that determine it form an autonomous set of equations $(C2)$, $(C5)$, $(C6)$, $(C9_N)$ and $(C18_N)$ with

$N \geq 2$. Thus, the solution of the two other expansion coefficients cannot be used here and additionally, one cannot find a certain highest derivative order of φ^A up to which C_A depends on. Nevertheless, one can find solutions for the coefficient C_A. In order to do so, inspect closure equation ($C5$),

$$0 = -2\, p^{\rho\gamma} \left(C_A\, \varphi^A{}_{,\rho} + \partial_\mu \left(C_A\, F^A{}_\rho{}^\mu \right) \right) . \tag{3.16}$$

First, one applies the previously introduced split of the tensor density C_A into a tensor $\tilde{C}_A$ and a scalar density $\chi(\varphi) = (-\det \hat{g}^{\cdot\cdot})^{-\frac{1}{2}}$. One then introduces the Levi-Civita connection and the covariant derivative. Using shorthand notation $C_{\alpha\beta} := C_A I^A{}_{\alpha\beta}$, closure equation ($C5$) is written as

$$0 = g^{\mu\sigma}\, \nabla_\mu(\tilde{C}_{\nu\sigma}) .$$

In order to find solutions for the coefficient C_A, the question arises, which symmetric, second-rank, divergence-free tensor can be built from the metric (and thus, the configuration fields) and arbitrary high spatial derivatives of it? An answer to this question is involved and pointing to a more fundamental question whether it is actually admissible that C_A depends on arbitrarily high derivatives of φ^A [20]. Here, the arguments from Ref. [9] will be applied. Restricting to zeroth, first and second derivatives of the metric, Lovelock shows that the only symmetric, divergence-free, second-rank tensor in three and four dimensions are the metric and the Einstein tensor [22]. Thus, a solution for C_A from closure equation (3.16) is

$$C_A = \frac{1}{\sqrt{-\det \hat{g}^{\cdot\cdot}}}\, I^{\alpha\beta}{}_A \left(c_1\, \hat{g}_{\alpha\beta} + c_2\, \hat{G}_{\alpha\beta} \right) . \tag{3.17}$$

Calculating the Einstein tensor in terms of derivatives $\partial\varphi$ and $\partial^2\varphi$ and substituting the result into the remaining closure equations for C_A yields that the obtained solution (3.17) is indeed a solution to the gravitational closure equations. It can be written as the functional derivative of the potential

$$\Phi = \frac{1}{\sqrt{-\det \hat{g}^{\cdot\cdot}}} \left(-2c_1 + c_2\, R \right) ,$$

where R is the three-dimensional Ricci scalar. Following the arguments of Ref. [9], such a term drops out of the gravitational field equations and thus, one can set the two constants to $c_1 = c_2 = 0$ in the first place.

3.4 Recovering the Einstein-Hilbert action

The last sections have shown how the gravitational closure equations are solved for the case of a metric spacetime with Maxwell electrodynamics as the initial matter theory on it. The two expansion coefficients of the gravitational Lagrangian that contribute are C and C_{AB} and collecting the results from Eq. (3.12) and Eq. (3.15), the gravitational Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{2\kappa} \frac{1}{\sqrt{-\det \hat{g}^{\cdot\cdot}}} \left(R - 2\,\Lambda - \frac{1}{8} I^{\alpha_1\alpha_2}{}_A I^{\beta_1\beta_2}{}_B \left(\hat{g}_{\alpha_1\beta_1}\, \hat{g}_{\alpha_2\beta_2} - \hat{g}_{\alpha_1\alpha_2}\, \hat{g}_{\beta_1\beta_2} \right) k^A k^B \right) ,$$

where already, the long expression for C is replaced by the three-dimensional Ricci scalar. Following the standard rules [9, 12], one can reconstruct the full spacetime gravitational action,

$$S_{\text{grav,met}} = \frac{1}{2\kappa} \int \mathrm{d}^4x \frac{1}{\sqrt{-\det \hat{g}^{\cdot\cdot}}}\; (R - 2\Lambda)$$

which is just the Einstein-Hilbert action with a gravitational and a cosmological constant.

It is this derivation of the Einstein-Hilbert action starting from (free) Maxwell theory[1] which replaces the old thinking that the Einstein equations need to be postulated or be the result of postulated Lovelock

[1]One is not restricted to Maxwell theory. Indeed, it suffices to choose e. g. Klein-Gordon theory for a scalar field in order to derive the Einstein-Hilbert action. One can also choose the entire standard model of particle physics and derive the Einstein-Hilbert action as the solution to the gravitational closure equations as the standard model is built in such a way that it operates with a Lorentzian metric and prinicpal polynomial $P = g$ [14].

criteria. Instead, one only needs to postulate the matter action – which one would have postulated anyway as the matter sourcing the gravitational dynamics. From this matter theory, here Maxwell theory, the Einstein-Hilbert action is derived as the solution of the gravitational closure equations.

This result has two direct implications on this thesis. First, it provides the indispensable test run for the constructive gravity program as it is already known that general relativity is the gravity theory behind Maxwell theory. Second, it provides the starting point for the symmetry reduction of the constructive gravity program which is the goal of this work. While the goal of this symmetry reduction is to actually circumvent the need of the full Einstein equations, the general setup of the closure equations including the input coefficients is still required as the starting point for the symmetry reduction. The next chapter will demonstrate how to properly impose spatial homogeneity and isotropy as symmetries on the closure equations. The result will be the derivation of the Friedmann equations directly from Maxwell theory – circumventing the full Einstein equations.

Chapter 4

Friedmann equations without Einstein equations

The last chapter demonstrated the first exact solution to the gravitational closure equations. Starting from Maxwell electrodynamics, the closure equations were solved and rendered the Einstein-Hilbert action. By variation, the Einstein equations can be obtained.

It is standard wisdom to then insert a symmetry ansatz, e. g. spatial homogeneity and isotropy constituting the FLRW spacetime, into the Einstein equations in order to obtain the Friedmann equations. The solutions to the Friedmann equations subsequently determine the dynamics of cosmology.

In this chapter, the Friedmann equations are derived directly from Maxwell theory by imposing the symmetry condition already on the gravitational closure equations. This circumvents the need for the Einstein equations as the full gravitational field equations in order to derive the Friedmann equations of cosmology. While this might seem redundant for Maxwell-Einstein theory, a proper symmetry reduction of the constructive gravity program is needed for theories such as the aforementioned general linear electrodynamics for which the full gravitational field equations are not yet known.

The structure of this chapter is also sketched in Fig. 4.1. The general setup of the gravitational closure equations is inherited from the previous chapter. The FLRW symmetries are then imposed onto the closure equations and their solution will be constructed.

$$\begin{tikzcd}
S_{\text{Maxwell}}[A; g] \arrow[r, "\text{Sec. 3.1}"'] & \text{closure equations} \arrow[r, "\text{Sec. 3.2 -- 3.4}"'] \arrow[d, "\text{Sec. 4.1}"'] & S_{\text{Einstein}-\text{Hilbert}}[g] \arrow[r] \arrow[d] & \text{Einstein equations} \arrow[d] \\
 & \text{symm.-red. closure eqns} \arrow[r, "\text{Sec. 4.2}"'] & S_{\text{cosmo}}[a(t)] \arrow[r, "\text{Sec. 4.3}"'] & \text{Friedmann equations}
\end{tikzcd}$$

Figure 4.1: The path from the Maxwell action to the Friedmann equations which arise from a solution of the symmetry-reduced gravitational closure equations. This path (*right – down – right – right*) provides the first demonstration of a symmetry reduction of the constructive gravity program.

The results of this chapter are to be published as
M. Düll, N. L. Fischer, B. M. Schäfer, F. P. Schuller,
Symmetric gravitational closure, arXiv:2003.07109, 2020
whose results are elaborated here.

4.1 From the FLRW metric to the symmetric input coefficients

It was already mentioned in the last chapter that Maxwell electrodynamics satisfies the three matter conditions of Chapter 2. Thus, the constructive gravity program can and has to be used in order to determine the underlying gravitational dynamics. The setup of the gravitational closure equations remains the same as in the previous chapter and the six configuration fields φ^A are distributed over the spatial part

$$\hat{g}^{\alpha\beta}(\varphi) = \begin{pmatrix} \varphi^1 & \frac{\varphi^2}{\sqrt{2}} & \frac{\varphi^3}{\sqrt{2}} \\ \frac{\varphi^2}{\sqrt{2}} & \varphi^4 & \frac{\varphi^5}{\sqrt{2}} \\ \frac{\varphi^3}{\sqrt{2}} & \frac{\varphi^5}{\sqrt{2}} & \varphi^6 \end{pmatrix}^{\alpha\beta}$$

of the full spacetime metric; the full metric is completed by a lapse function N and shift vector field $\vec{N}$. All quantities also depend on the foliation time t which will be suppressed in the following to keep notation more compact.

A symmetry condition is imposed on a tensorial quantity T by requiring the Killing condition

$$\mathcal{L}_{K_i} T \stackrel{!}{=} 0$$

for all generators K_i of the symmetry also called the Killing vector fields. For FLRW symmetries, there are six Killing vector fields. As shown in Ref. [23, 24], these are in spherical coordinates r, θ, ϕ,

$$\begin{aligned}
K_1 &= \sin\phi\,\partial_\theta + \cot\theta\,\cos\phi\,\partial_\phi\,, \\
K_2 &= \cos\phi\,\partial_\theta + \cot\theta\,\sin\phi\,\partial_\phi\,, \\
K_3 &= -\partial_\phi\,, \\
K_4 &= \sqrt{1-kr^2}\left(\sin\theta\,\cos\phi\,\partial_r + \frac{\cos\theta\,\cos\phi}{r}\,\partial_\theta - \frac{\sin\phi}{r\,\sin\theta}\,\partial_\phi\right), \\
K_5 &= \sqrt{1-kr^2}\left(\sin\theta\,\sin\phi\,\partial_r + \frac{\cos\theta\,\sin\phi}{r}\,\partial_\theta + \frac{\cos\phi}{r\,\sin\theta}\,\partial_\phi\right), \\
K_6 &= \sqrt{1-kr^2}\left(\frac{\sin\theta}{r}\,\partial_\theta - \cos\theta\,\partial_r\right).
\end{aligned}$$

The range of these coordinates depends on the exact symmetry condition, namely whether the universe is spatially spherical ($k = 1$), flat ($k = 0$) or hyperbolic ($k = -1$). Evaluating the Killing condition

$$(\mathcal{L}_{K_i} g)^{ab} \stackrel{!}{=} 0$$

provides simple differential equations for the components of the metric. Solving them and performing an usual 3 + 1-decomposition reveals that the shift vector $\vec{n}$ vanishes, the lapse is a function of the foliation time only and the configuration fields φ^A simplify and constitute the spatial part of the Friedmann-Robertson-Walker metric. Thus, only three fields

$$\varphi^1 = -\frac{1-kr^2}{a^2} \quad , \quad \varphi^4 = -\frac{1}{a^2\,r^2} \quad , \quad \varphi^6 = -\frac{1}{a^2\,r^2\,\sin^2\theta}$$

are non-trivial and now given as functions of the scale factor $a(t)$ depending only on the foliation time and two of three spherical coordinates r, θ, ϕ.

The key question is now: How do the gravitational closure equations have to be evaluated for the configuration fields confined to the symmetry? The closure equations are formulated for the full set of local functions $C_{A_1\ldots A_N}[\varphi^A]$. The goal is to rewrite them in terms of new ultra-local functions

$$C^{\text{cosmo}}_{A_1\ldots A_N}(a, r, \theta) = C_{A_1\ldots A_N}[\varphi^A(a, r, \theta)]$$

with configuration fields $\varphi^A(a, r, \theta) := \left((1-kr^2)\,a^{-2}, 0, 0, a^{-2}\,r^{-2}, 0, a^{-2}\,r^{-2}\,\sin^{-2}\theta\right)^A$. The expansion coefficients $C^{\text{cosmo}}_{A_1\ldots A_N}$ are ultralocal functions as the scale factor a only depends on the foliation time and is thus spatially constant. Here, it is important to stress that the expansion coefficients in the full gravitational closure equations must not be simply replaced by their just defined symmetric counterparts

$C^{\text{cosmo}}_{A_1\ldots A_N}$ since the derivatives in the gravitational closure equations are always taken with respect to the independent configuration fields φ^A and not with respect to the variables a, r, θ of $C^{\text{cosmo}}_{A_1\ldots A_N}$. The way to relate these different derivatives is to use the chain rule which then provides differential equations for the symmetry-reduced expansion coefficients $C^{\text{cosmo}}_{A_1\ldots A_N}$,

$$\frac{\partial C^{\text{cosmo}}_{A_1\ldots A_N}}{\partial a} = C_{A_1\ldots A_N:B}{}^{\mu_1\ldots\mu_R}\big|_{\varphi^A(a,r,\theta)} \frac{\partial \varphi^B{}_{,\mu_1\ldots\mu_R}}{\partial a}\,, \tag{4.1}$$

$$\frac{\partial C^{\text{cosmo}}_{A_1\ldots A_N}}{\partial r} = C_{A_1\ldots A_N:B}{}^{\mu_1\ldots\mu_R}\big|_{\varphi^A(a,r,\theta)} \frac{\partial \varphi^B{}_{,\mu_1\ldots\mu_R}}{\partial r}\,, \tag{4.2}$$

$$\frac{\partial C^{\text{cosmo}}_{A_1\ldots A_N}}{\partial \theta} = C_{A_1\ldots A_N:6}{}^{\mu_1\ldots\mu_R}\big|_{\varphi^A(a,r,\theta)} \frac{\partial \varphi^6{}_{,\mu_1\ldots\mu_R}}{\partial \theta}\,, \tag{4.3}$$

where $|_{\varphi^A(a,r,\theta}$ denotes the evaluation on the symmetric configuration. The strategy is now clear: first, evaluate the gravitational closure equations on symmetry and from these equations, determine all required derivatives $C_{A_1\ldots A_N:B}{}^{\mu_1\ldots\mu_R}\big|_{\varphi^A(a,r,\theta)}$; plug them into the three chain rule equations (4.1) – (4.3) and solve for the symmetric expansion coefficients $C^{\text{cosmo}}_{A_1\ldots A_N}$.

Before doing so, first impose the symmetry condition on the three input coefficients and write down their symmetric form. From the general setup, one inherits that the input coefficient $M^{A\gamma}$ vanishes. Also, the components $p^{\mu\nu}|_{\varphi^A(a,c)}$ are simply the diagonal components of the spatial FLRW metric. The general expression of the last input coefficient $F^A{}_\mu{}^\gamma$ was written down in Eq. (3.2). Imposition of FLRW symmetries yields

$$\begin{aligned} F^A{}_\mu{}^\gamma\,|_{\varphi^A(a,r,\theta)} = &-\left(2\,\delta^A_1\delta^r_\mu\delta^\gamma_r + \sqrt{2}\left(\delta^A_2\delta^\theta_\mu\delta^\gamma_r + \delta^A_3\delta^\varphi_\mu\delta^\gamma_r\right)\right) a^{-2}\,(1-kr^2)\\ &-\left(2\,\delta^A_4\delta^\theta_\mu\delta^\gamma_\theta + \sqrt{2}\left(\delta^A_2\delta^r_\mu\delta^\gamma_\theta + \delta^A_5\delta^\varphi_\mu\delta^\gamma_\theta\right)\right) a^{-2}\,r^{-2}\\ &-\left(2\,\delta^A_6\delta^\varphi_\mu\delta^\gamma_\varphi + \sqrt{2}\left(\delta^A_3\delta^r_\mu\delta^\gamma_\varphi + \delta^A_5\delta^\theta_\mu\delta^\gamma_\varphi\right)\right) a^{-2}\,r^{-2}\sin^{-2}\theta\,. \end{aligned}$$

Due to the vanishing shift vector field and the trivial coefficient $M^{A\gamma}$, the symmetric gravitational spacetime action (2.60) can be more compactly written as

$$S_{\text{cosmo}}(a,\dot a,N) = \int \mathrm{d}t \int_\Sigma \mathrm{d}^3 z \sum_{M=0}^{\infty} C^{\text{cosmo}}_{A_1\ldots A_M}(a,r,\theta)\,\dot\varphi^{A_1}(a,r,\theta)\ldots\dot\varphi^{A_M}(a,r,\theta)\,N^{1-M}\,. \tag{4.4}$$

As there are only three non-trivial configuration fields, there are also only three non-trivial derivatives

$$\dot\varphi^1 = \frac{2\,(1-kr^2)\,\dot a}{a^3} \quad,\quad \dot\varphi^4 = \frac{2\,\dot a}{a^3\,r^2} \quad \text{and} \quad \dot\varphi^6 = \frac{2\,\dot a}{a^3\,r^2\,\sin^2\theta}\,.$$

This has an important consequence for the solution of the closure equations. As there are only three non-trivial derivatives $\dot\varphi^A$, one only needs to determine those components of the expansion coefficients which couple exclusively to the non-trivial derivatives $\dot\varphi^1$, $\dot\varphi^4$ and $\dot\varphi^6$.

4.2 Evaluation of the gravitational closure equations

The analysis of the gravitational closure equations on symmetric configurations $\varphi^A(a,r,\theta)$ differs technically from the general metric solution presented in the last chapter. The general solution presented in the previous chapter featured general arguments and solution techniques. In contrast to that, the analysis of the closure equations in the symmetric case is based on considering any combination of the respective free indices of the equations. This gives rise to a componentwise solution. This is only manageable since the symmetry simplifies the configuration fields and components of the input coefficients.

While there are a lot of different ways to evaluate the gravitational closure equations, it has proved to be most practical to start with equations containing the first expansion coefficient C. Only after extracting as much information about this coefficient as possible, one should proceed with other equations and expansion coefficients.

Derivatives of the first expansion coefficient C

The first closure equation to be evaluated is the same as in the general case – it is the sequence $(C19_N)$ which provides a collapse of C to second derivative order as

$$C_{:B}{}^{\mu_1 \ldots \mu_R}\Big|_{\varphi^A(a,r,\theta)} = 0 \quad \text{for } R \geq 3\,.$$

This is directly put to use when considering the instance $(C8_3)|_{\varphi^A(a,r,\theta)}$ of the sequence $(C8_N)$. Due to the collapse of C to second derivative order in the configuration fields, it simplifies to

$$0 = C_{:A}{}^{(\beta_1\beta_2|}\Big|_{\varphi^A(a,r,\theta)} \; F^A{}_{\mu}{}^{|\beta_3)}\Big|_{\varphi^A(a,r,\theta)}\,.$$

The 30 individual relations relate the 36 independent derivatives of the type $C_{:A}{}^{\mu\nu}|_{\varphi^A(a,r,\theta)}$ with each other. The result is that 15 derivatives vanish and the others are expressed in terms of six independent ones. The 15 vanishing derivatives are

$$\begin{aligned}
0 &= C_{:1}{}^{rr}\Big|_{\varphi^A(a,r,\theta)} = C_{:1}{}^{r\theta}\Big|_{\varphi^A(a,r,\theta)} = C_{:1}{}^{r\varphi}\Big|_{\varphi^A(a,r,\theta)} = C_{:2}{}^{rr}\Big|_{\varphi^A(a,r,\theta)} = C_{:2}{}^{\theta\theta}\Big|_{\varphi^A(a,r,\theta)} \\
&= C_{:3}{}^{rr}\Big|_{\varphi^A(a,r,\theta)} = C_{:3}{}^{\varphi\varphi}\Big|_{\varphi^A(a,r,\theta)} = C_{:4}{}^{r\theta}\Big|_{\varphi^A(a,r,\theta)} = C_{:4}{}^{\theta\theta}\Big|_{\varphi^A(a,r,\theta)} = C_{:4}{}^{\theta\varphi}\Big|_{\varphi^A(a,r,\theta)} \\
&= C_{:5}{}^{\theta\theta}\Big|_{\varphi^A(a,r,\theta)} = C_{:5}{}^{\theta\varphi}\Big|_{\varphi^A(a,r,\theta)} = C_{:6}{}^{r\varphi}\Big|_{\varphi^A(a,r,\theta)} = C_{:6}{}^{\theta\varphi}\Big|_{\varphi^A(a,r,\theta)} = C_{:6}{}^{\varphi\varphi}\Big|_{\varphi^A(a,r,\theta)}\,.
\end{aligned}$$

W. l. o. g. the six independent derivatives are chosen to be

$$C_{:3}{}^{\theta\theta}\Big|_{\varphi^A(a,r,\theta)}\,,\; C_{:4}{}^{rr}\Big|_{\varphi^A(a,r,\theta)}\,,\; C_{:5}{}^{rr}\Big|_{\varphi^A(a,r,\theta)}\,,\; C_{:6}{}^{rr}\Big|_{\varphi^A(a,r,\theta)}\,,\; C_{:6}{}^{r\theta}\Big|_{\varphi^A(a,r,\theta)}\,,\; C_{:6}{}^{\theta\theta}\Big|_{\varphi^A(a,r,\theta)}\,. \tag{4.5}$$

Thus, the remaining relations can be written compactly as

$$\begin{aligned}
C_{:3}{}^{\theta\theta}\Big|_{\varphi^A(a,r,\theta)} &= -2\,\sin^2\theta\; C_{:2}{}^{\theta\varphi}\Big|_{\varphi^A(a,r,\theta)} = \frac{\sqrt{2}\,\sin^2\theta}{r^2\,(1-kr^2)}\; C_{:4}{}^{r\varphi}\Big|_{\varphi^A(a,r,\theta)} \\
&= \frac{-2}{r^2\,(1-kr^2)}\; C_{:5}{}^{r\theta}\Big|_{\varphi^A(a,r,\theta)}\,, \\
C_{:4}{}^{rr}\Big|_{\varphi^A(a,r,\theta)} &= \left((1-kr^2)\,r^2\right)^2\; C_{:1}{}^{\theta\theta}\Big|_{\varphi^A(a,r,\theta)} = -\sqrt{2}\,(1-kr^2)\,r^2\; C_{:2}{}^{r\theta}\Big|_{\varphi^A(a,r,\theta)}\,, \\
C_{:5}{}^{rr}\Big|_{\varphi^A(a,r,\theta)} &= \sqrt{2}\,(1-kr^2)^2\,r^4\,\sin^2\theta\; C_{:1}{}^{\theta\varphi}\Big|_{\varphi^A(a,r,\theta)} = -2\,(1-kr^2)\,r^2\sin^2\theta\; C_{:2}{}^{r\varphi}\Big|_{\varphi^A(a,r,\theta)} \\
&= -2\,(1-kr^2)\,r^2\; C_{:3}{}^{r\theta}\Big|_{\varphi^A(a,r,\theta)}\,, \\
C_{:6}{}^{rr}\Big|_{\varphi^A(a,r,\theta)} &= \left((1-kr^2)\,r^2\,\sin^2\theta\right)^2\; C_{:1}{}^{\varphi\varphi}\Big|_{\varphi^A(a,r,\theta)} = -\sqrt{2}\,(1-kr^2)\,r^2\,\sin^2\theta\; C_{:3}{}^{r\varphi}\Big|_{\varphi^A(a,r,\theta)}\,, \\
C_{:6}{}^{r\theta}\Big|_{\varphi^A(a,r,\theta)} &= \frac{(1-kr^2)\,r^2\,\sin^4\theta}{\sqrt{2}}\; C_{:2}{}^{\varphi\varphi}\Big|_{\varphi^A(a,r,\theta)} = -\sqrt{2}\,(1-kr^2)\,r^2\sin^2\theta\; C_{:3}{}^{\theta\varphi}\Big|_{\varphi^A(a,r,\theta)} \\
&= -\sqrt{2}\,\sin^2\theta\; C_{:5}{}^{r\varphi}\Big|_{\varphi^A(a,r,\theta)}\,, \\
C_{:6}{}^{\theta\theta}\Big|_{\varphi^A(a,r,\theta)} &= \sin^4\theta\; C_{:4}{}^{\varphi\varphi}\Big|_{\varphi^A(a,r,\theta)} = -\sqrt{2}\,\sin^2\theta\; C_{:5}{}^{\theta\varphi}\Big|_{\varphi^A(a,r,\theta)}\,.
\end{aligned}$$

These relations will be used at several points throughout the analysis of the closure equations. They are already of central importance in the next instance $(C8_2)|_{\varphi^A(a,r,\theta)}$ of the sequence $(C8_N)$ which breaks down to

$$\begin{aligned}
0 = {} & C_{:A}{}^{\beta_1\beta_2}\Big|_{\varphi^A(a,r,\theta)}\; \varphi^A{}_{,\mu} - 2\; C_{:A}{}^{\gamma(\beta_1|}\Big|_{\varphi^A(a,r,\theta)}\; F^A{}_{\mu}{}^{|\beta_2)}{}_{,\gamma}\Big|_{\varphi^A(a,r,\theta)} \\
& - C_{:A}{}^{(\beta_1|}\Big|_{\varphi^A(a,r,\theta)}\; F^A{}_{\mu}{}^{|\beta_2)}\Big|_{\varphi^A(a,r,\theta)}\,.
\end{aligned}$$

The 18 individual relations allow to express all derivatives of the type $C_{:A}{}^{\mu}|_{\varphi^A(a,r,\theta)}$ in terms of the six independent derivatives from Eq. (4.5). The detailed relations are quite involved and will simplify later on anyway. Therefore, they are omitted here.

In similar fashion, closure equation $(C1)|_{\varphi^A(a,r,\theta)}$ which reads

$$
\begin{aligned}
0 = &-C^{\text{cosmo}}\, \delta^\gamma_\mu - C_{:A}\, |_{\varphi^A(a,r,\theta)}\; F^A{}_\mu{}^\gamma\big|_{\varphi^A(a,r,\theta)} + C_{:A}{}^\gamma\,\big|_{\varphi^A(a,r,\theta)}\; \varphi^A{}_{,\mu} \\
&- C_{:A}{}^\alpha\,\big|_{\varphi^A(a,r,\theta)}\; F^A{}_\mu{}^\gamma{}_{,\alpha}\big|_{\varphi^A(a,r,\theta)} + 2\; C_{:A}{}^{\alpha\gamma}\,\big|_{\varphi^A(a,r,\theta)}\; \varphi^A{}_{,\mu\alpha} \\
&- C_{:A}{}^{\beta_1\beta_2}\,\big|_{\varphi^A(a,r,\theta)}\; F^A{}_\mu{}^\gamma{}_{,\beta_1\beta_2}\big|_{\varphi^A(a,r,\theta)}
\end{aligned}
$$

provides six relations which express the derivatives $C_{:A}\,|_{\varphi^A(a,r,\theta)}$ in terms of C^{cosmo} and the six independent derivatives from Eq. (4.5). As soon as the six derivatives have been determined, the relations obtained from closure equation $(C1)|_{\varphi^A(a,r,\theta)}$ can be used to solve the differential equations (4.1) – (4.3).

As a side effect, the analysis of $(C1)|_{\varphi^A(a,c)}$ reveals that the derivative $C_{:3}\,|_{\varphi^A(a,r,\theta)}$ vanishes. This provides a relation between the two derivatives $C_{:3}{}^{\theta\theta}\,\big|_{\varphi^A(a,r,\theta)}$ and $C_{:5}{}^{rr}\,|_{\varphi^A(a,r,\theta)}$. Thus, there are now only five independent derivatives of the expansion coefficient C, w. l. o. g.

$$
C_{:3}{}^{\theta\theta}\,\big|_{\varphi^A(a,r,\theta)}\,,\; C_{:4}{}^{rr}\,\big|_{\varphi^A(a,r,\theta)}\,,\; C_{:6}{}^{rr}\,\big|_{\varphi^A(a,r,\theta)}\,,\; C_{:6}{}^{r\theta}\,\big|_{\varphi^A(a,r,\theta)}\,,\; C_{:6}{}^{\theta\theta}\,\big|_{\varphi^A(a,r,\theta)}\,. \tag{4.6}
$$

The relations obtained from the analysis of closure equation $(C1)|_{\varphi^A(a,r,\theta)}$ are even more involved than the ones from $(C8_2)|_{\varphi^A(a,r,\theta)}$. They will also simplify during the following analysis of other closure equations. Therefore, they are also omitted here. In the end, their simpler form will be presented which will also be the form in which they contribute to the solution of the expansion coefficient C.

There are still more equations containing derivatives of the expansion coefficient C – not only first, but also second derivatives. Closure equation $(C4)$ contains a divergence term $\partial_\alpha C_{:B}{}^{\mu\alpha}$ which generates, besides others, a term

$$
C_{:B}{}^{\mu\alpha}{}_{:A}{}^{\lambda\omega}\; \varphi^A{}_{,\alpha\lambda\omega}\,.
$$

No other term appearing in this equation depends on second derivatives and thus, the derivative

$$
(C4)_{:D}{}^{\lambda_1\lambda_2\lambda_3}|_{\varphi^A(a,r,\theta)}
$$

of closure equation $(C4)$ provides the relation

$$
C_{:A}{}^{\mu(\lambda_1}{}_{:B}{}^{\lambda_2\lambda_3)}\,\big|_{\varphi^A(a,r,\theta)} = 0\,. \tag{4.7}
$$

This equation alone does not allow to read off that the expansion coefficient C does not depend on two or more second derivatives of the configuration fields as the symmetrization bracket is still there. If one then also takes derivative

$$
(C8_3)_{:D}{}^{\lambda_1\lambda_2}\big|_{\varphi^A(a,r,\theta)}
$$

into consideration, the evaluation of the 630 independent relations of Eq. (4.7) reveals that the equation is indeed valid without symmetrization,

$$
C_{:A}{}^{\mu_1\mu_2}{}_{:B}{}^{\lambda_1\lambda_2}\,\big|_{\varphi^A(a,r,\theta)} = 0\,,
$$

or in other words, the expansion coefficient C depends at most linearly on second derivatives of the configuration fields – after evaluation on symmetric configurations[1].

As a direct consequence of this, the derivative

$$
(C8_2)_{:D}{}^{\lambda_1\lambda_2}\big|_{\varphi^A(a,r,\theta)}
$$

of closure equation $(C8_2)$ reveals that with the symmetry condition imposed, the expansion coefficient C also contains no terms with both a first and second derivative of the configuration fields in it as

$$
C_{:A}{}^{\mu}{}_{:B}{}^{\lambda_1\lambda_2}\,\big|_{\varphi^A(a,r,\theta)} = 0\,.
$$

[1] In the case presented here, this information is also valid in the full case as the last chapter showed. Still, it is necessary to stress this here as for the perspective of the symmetry reduction, the information is only valid on the symmetric configuration. In Chapter 6, there will be instances where similar statements are actually only valid with the symmetry imposed.

This is an implication from the previous evaluation of $(C8_2)|_{\varphi^A(a,r,\theta)}$ which allowed to express all derivatives of the form $C_{:A}{}^{\mu}\,|_{\varphi^A(a,r,\theta)}$ in terms of the five independent ones from Eq. (4.6).

Now, one has extracted all relevant information on derivatives of the first expansion coefficient C. The only equation containing meaningful new information is closure equation $(C3)$ which connects the expansion coefficient C_{AB} with C. Before one can construct a solution for the latter, one first uses equation $(C3)$ in order to determine the components of C_{AB}.

Solution for expansion coefficient C^{cosmo}_{AB}

Just as in the general solution of the closure equations in the previous chapter, closure equation $(C3)$ provides important information by considering derivatives of it. Thus, first consider the full closure equation $(C3)$,

$$0 = 2\,C_{AB}\,F^{A}{}_{\rho}{}^{(\mu}\;p^{\nu)\rho} - C_{:B}{}^{\mu\nu}\ ,$$

and its derivatives,

$$(C3)_{:D}{}^{\lambda}\Big|_{\varphi^A(a,r,\theta)} \quad \text{and} \quad (C3)_{:D}{}^{\lambda\omega}\Big|_{\varphi^A(a,r,\theta)}\ .$$

As the second derivatives of the first expansion coefficient C drop out of the respective equation due to the results of the previous subsection, one can read off that the expansion coefficient C_{AB} depends only on the configuration fields φ, but not on any derivatives when symmetry is imposed. This will simplify the closure equations containing the coefficient C_{AB} as well as the three chain rule equations (4.1) – (4.3).

First, analyse closure equation $(C3)|_{\varphi^A(a,r,\theta)}$,

$$0 = 2\,C^{\text{cosmo}}_{AB}\;\;F^{A}{}_{\rho}{}^{(\mu|}\Big|_{\varphi^A(a,r,\theta)}\;p^{|\nu)\rho}\Big|_{\varphi^A(a,r,\theta)} - C_{:B}{}^{\mu\nu}\,\Big|_{\varphi^A(a,r,\theta)}\ .$$

The 36 individual relations together with those from closure equation $(C8_3)|_{\varphi^A(a,r,\theta)}$ reveal that only the components

$$C^{\text{cosmo}}_{14}\,,\ C^{\text{cosmo}}_{16}\,,\ C^{\text{cosmo}}_{26}\,,\ C^{\text{cosmo}}_{34} \quad \text{and} \quad C^{\text{cosmo}}_{46}$$

are independent. All other components either vanish or can be expressed in terms of these five. These five components can be determined further by closure equation $(C10_2)|_{\varphi^A(a,r,\theta)}$,

$$0 = C^{\text{cosmo}}_{B_1B_2}\,\delta^{\gamma}_{\mu} + 4\,C^{\text{cosmo}}_{A(B_1|}\,I^{A}{}_{\mu\sigma}\,I^{\gamma\sigma}{}_{|B_2)} + C_{B_1B_2:A}\,\Big|_{\varphi^A(a,r,\theta)}\;F^{A}{}_{\mu}{}^{\gamma}\Big|_{\varphi^A(a,r,\theta)}\ .$$

The only independent component left is C^{cosmo}_{14} while all other non-trivial components can be expressed by it,

$$C^{\text{cosmo}}_{14} = \frac{1}{\sin^2\theta}\,C^{\text{cosmo}}_{16} = \frac{1}{(1-kr^2)\,r^2\,\sin^2\theta}\,C^{\text{cosmo}}_{46}\ .$$

While the components C^{cosmo}_{22}, C^{cosmo}_{33} and C^{cosmo}_{55} are also non-trivial and can be expressed by C^{cosmo}_{14}, they do not contribute to the spacetime action (4.4) as the associated $\dot{\varphi}^A$ are trivial. Therefore, they will not be written down here.

Also from closure equation $(C10_2)|_{\varphi^A(a,r,\theta)}$, one reads off the three relations entering the chain rule equations (4.1) – (4.3) as

$$\begin{aligned}
C_{14:1}\,|_{\varphi^A(a,r,\theta)} &= \frac{3\,a^2}{2\,(1-kr^2)}\,C^{\text{cosmo}}_{14}\,,\\
C_{14:4}\,|_{\varphi^A(a,r,\theta)} &= \frac{3\,a^2\,r^2}{2}\,C^{\text{cosmo}}_{14} \quad \text{and}\\
C_{14:6}\,|_{\varphi^A(a,r,\theta)} &= \frac{a^2\,r^2\,sin^2\theta}{2}\,C^{\text{cosmo}}_{14}\,.
\end{aligned}$$

The chain rule equations (4.1) – (4.3) for the component C_{14}^{cosmo} thus read

$$\begin{aligned}
\frac{\partial C_{14}^{\text{cosmo}}}{\partial a} &= C_{14:1}\,|_{\varphi^A(a,r,\theta)}\,\frac{2\,(1-kr^2)}{a^3} + C_{14:4}\,|_{\varphi^A(a,r,\theta)}\,\frac{2}{a^3\,r^2} + C_{14:6}\,|_{\varphi^A(a,r,\theta)}\,\frac{2}{a^3\,r^2\sin^2\theta}\,,\\
\frac{\partial C_{14}^{\text{cosmo}}}{\partial r} &= C_{14:1}\,|_{\varphi^A(a,r,\theta)}\,\frac{2kr}{a^2} + C_{14:4}\,|_{\varphi^A(a,r,\theta)}\,\frac{2}{a^2\,r^3} + C_{14:6}\,|_{\varphi^A(a,r,\theta)}\,\frac{2}{a^2\,r^3\sin^2\theta}\,,\\
\frac{\partial C_{14}^{\text{cosmo}}}{\partial \theta} &= C_{14:6}\,|_{\varphi^A(a,r,\theta)}\,\frac{2\,\cos\theta}{a^2\,r^3\sin^3\theta}\,.
\end{aligned}$$

Solving them step by step, one constructs the solution

$$C_{14}^{\text{cosmo}} = K_0\,\frac{r^4\,\sin\theta\,a^7}{(1-kr^2)^{\frac{3}{2}}}$$

with a constant of integration K_0 which remains undetermined here. From this result, the two components

$$C_{16}^{\text{cosmo}} = K_0\,\frac{r^4\,\sin^3\theta\,a^7}{(1-kr^2)^{\frac{3}{2}}} \quad \text{and} \quad C_{46}^{\text{cosmo}} = K_0\,\frac{r^6\,\sin^3\theta\,a^7}{(1-kr^2)^{\frac{1}{2}}}$$

complete the set of components of the expansion coefficient C_{AB}^{cosmo}. This solution now enables to invert the analysis of closure equation $(C3)|_{\varphi^A(a,r,\theta)}$. By doing so, the derivatives $C_{:A}{}^{\mu\nu}\,|_{\varphi^A(a,r,\theta)}$ of the expansion coefficient C are determined in terms of the solution for C_{AB}^{cosmo}.

Solution of the expansion coefficient C^{cosmo}

The solution of the symmetric expansion coefficient C_{AB}^{cosmo} allows to construct the solution of the first expansion coefficient C^{cosmo}. In order to do so, first write down all derivatives from the chain rule equations (4.1) – (4.3) that need to be determined,

$$\begin{aligned}
&C_{:1}\,|_{\varphi^A(a,r,\theta)}\;,\; C_{:4}\,|_{\varphi^A(a,r,\theta)}\;,\; C_{:6}\,|_{\varphi^A(a,r,\theta)}\;,\\
&C_{:1}{}^{r}\,\Big|_{\varphi^A(a,r,\theta)}\;,\; C_{:4}{}^{r}\,\Big|_{\varphi^A(a,r,\theta)}\;,\; C_{:6}{}^{r}\,\Big|_{\varphi^A(a,r,\theta)}\;,\; C_{:6}{}^{\theta}\,\Big|_{\varphi^A(a,r,\theta)}\;,\\
&C_{:4}{}^{rr}\,\Big|_{\varphi^A(a,r,\theta)}\;,\; C_{:6}{}^{rr}\,\Big|_{\varphi^A(a,r,\theta)}\;,\; C_{:6}{}^{r\theta}\,\Big|_{\varphi^A(a,r,\theta)}\;,\; C_{:6}{}^{\theta\theta}\,\Big|_{\varphi^A(a,r,\theta)}\;,
\end{aligned}$$

where the derivative $C_{:1}{}^{rr}\,|_{\varphi^A(a,r,\theta)}$ is already dropped as the analysis of $(C8_3)|_{\varphi^A(a,r,\theta)}$ before revealed that this derivative vanishes.

Closure equation $(C3)|_{\varphi^A(a,r,\theta)}$ does not only provide information about the components of the expansion coefficient C_{AB}^{cosmo}, it also provides information about the derivatives $C_{:A}{}^{\mu\nu}$. As the components C_{26}^{cosmo} and C_{34}^{cosmo} vanish, the (up to now independent) derivatives

$$C_{:6}{}^{r\theta}\,\Big|_{\varphi^A(a,r,\theta)} \quad \text{and} \quad C_{:3}{}^{\theta\theta}\,\Big|_{\varphi^A(a,r,\theta)}$$

vanish as well. The other three derivatives are no longer independent and now expressed in terms of the component C_{14}^{cosmo} as

$$\begin{aligned}
C_{:4}{}^{rr}\,\Big|_{\varphi^A(a,r,\theta)} &= 4\left(\frac{1-kr^2}{a^2}\right)^2 C_{14}^{\text{cosmo}} = K_0\,r^4\,\sin\theta\,(1-kr^2)^{\frac{1}{2}}\,a^3\,,\\
C_{:6}{}^{rr}\,\Big|_{\varphi^A(a,r,\theta)} &= 4\,\frac{(1-kr^2)^2\,\sin^2\theta}{a^4}\,C_{14}^{\text{cosmo}} = K_0\,r^4\,\sin^3\theta\,(1-kr^2)^{\frac{1}{2}}\,a^3\,,\\
C_{:6}{}^{\theta\theta}\,\Big|_{\varphi^A(a,r,\theta)} &= 4\,\frac{(1-kr^2)\,\sin^2\theta}{a^4\,r^2}\,C_{14}^{\text{cosmo}} = K_0\,\frac{r^2\,\sin^3\theta\,a^3}{(1-kr^2)^{\frac{1}{2}}}\,.
\end{aligned}$$

Subsequently, use the relations from closure equation $(C8_2)|_{\varphi^A(a,r,\theta)}$ in order to determine the derivatives $C_{:A}{}^{\mu}\,|_{\varphi^A(a,r,\theta)}$,

$$
\begin{aligned}
C_{:1}{}^{r}\,\Big|_{\varphi^A(a,r,\theta)} &= -\frac{8\,(1-kr^2)}{a^4\,r^3}\,C^{\text{cosmo}}_{14}\,,\\
C_{:4}{}^{r}\,\Big|_{\varphi^A(a,r,\theta)} &= \left(-\frac{4\,kr\,(1-kr^2)}{a^4}+28\,\frac{(1-kr^2)^2}{a^4\,r}\right)C^{\text{cosmo}}_{14}\,,\\
C_{:6}{}^{r}\,\Big|_{\varphi^A(a,r,\theta)} &= \left(-\frac{4\,kr\,(1-kr^2)\sin^2\theta}{a^4}+28\,\frac{(1-kr^2)^2\,\sin^2\theta}{a^4\,r}\right)C^{\text{cosmo}}_{14}\,,\\
C_{:6}{}^{\theta}\,\Big|_{\varphi^A(a,r,\theta)} &= \frac{12\,(1-kr^2)\,\sin\theta\,\cos\theta}{a^4\,r^2}\,C^{\text{cosmo}}_{14}\,.
\end{aligned}
$$

Last but not least, the relations from closure equation $(C1)|_{\varphi^A(a,r,\theta)}$ express the derivatives

$$
C_{:1}\,|_{\varphi^A(a,r,\theta)}\ ,\ C_{:4}\,|_{\varphi^A(a,r,\theta)}\quad\text{and}\quad C_{:6}\,|_{\varphi^A(a,r,\theta)}
$$

in terms of the expansion coefficient C^{cosmo} itself and the component C^{cosmo}_{14} which is already known. One finds

$$
\begin{aligned}
C_{:1}\,|_{\varphi^A(a,r,\theta)} &= \frac{a^2}{2\,(1-kr^2)}\,C^{\text{cosmo}}-\frac{8\,(1-kr^2)}{a^4\,r^4}\,C^{\text{cosmo}}_{14}\,,\\
C_{:4}\,|_{\varphi^A(a,r,\theta)} &= \frac{a^2\,r^2}{2}\,C^{\text{cosmo}}+\frac{40\,(1-kr^2)^2}{a^4\,r^2}\,C^{\text{cosmo}}_{14}\,,\\
C_{:6}\,|_{\varphi^A(a,r,\theta)} &= \frac{a^2\,r^2\,\sin^2\theta}{2}\,C^{\text{cosmo}}+\frac{40\,(1-kr^2)^2\,\sin^2\theta}{a^4\,r^2}\,C^{\text{cosmo}}_{14}\,.
\end{aligned}
$$

Putting all of this together into the chain rule equations (4.1) – (4.3), one obtains three differential equations for the coefficient C^{cosmo} whose solution is

$$
C^{\text{cosmo}} = \frac{r^2\,\sin\theta}{(1-kr^2)^{\frac{1}{2}}}\,\left(K_1\,a^3-24\,K_0\,k\,a\right)
$$

with a second constant of integration K_1 which is also undetermined here.

As the two expansion coefficients C^{cosmo} and C^{cosmo}_{AB} have been determined, attention will be turned to the remaining expansion coefficients which will not contribute to the Friedmann equations.

Comments on C^{cosmo}_{A}

As already laid out in Chapter 2, the expansion coefficient C_A features no collapse to a certain derivative order in general. This is also true after a symmetry condition has been imposed. Indeed, it turns out that in the case of a symmetry reduction, the closure equations containing the coefficients C^{cosmo}_{A} and the derivatives $C_{:A}{}^{\mu_1\cdots\mu_R}\,|_{\varphi^A(a,r,\theta)}$ form an autonomous set of equations – separated from all other equations and expansion coefficients. Dealing with this autonomous set and the arbitrary amount of spatial derivatives amounts for an infinity of index combinations one would have to evaluate. Thus, one inherits arguments from the general solution of the closure equations presented in the previous chapter. These said that the terms containing the expansion coefficients C^{cosmo}_{A} are boundary terms in the action and can therefore be neglected in the first place.

A special case is given for the case of flat ($k=0$) cosmology. In this case, one can change coordinates from spherical coordinates (r,θ,ϕ) to Cartesian ones (x,y,z) in the beginning. Due to the spatial flatness, all spatial derivatives drop from the input coefficients and the gravitational closure equations after symmetry reduction. This vastly reduces the complexity of the closure equations and also of the chain rule equations as there is only one involving the scale factor a,

$$
\frac{\partial C^{\text{cosmo}}_{A}}{\partial a} = \left(C_{A:1}\,|_{\varphi^A(a)}+C_{A:4}\,|_{\varphi^A(a)}+C_{A:6}\,|_{\varphi^A(a)}\right)\frac{2}{a^3}\,.
$$

The only relevant closure equation is $(C2)|_{\varphi^A(a)}$ and its result that the three non-trivial components can be expressed by the component C_1^{cosmo}. Due to the spatial derivatives dropping out of the closure equation, the solution for the component C_1^{cosmo} is quickly found to be

$$C_1^{\text{cosmo}} = C_4^{\text{cosmo}} = C_6^{\text{cosmo}} = K_2\, a^5 \,,$$

which will also drop out of the (flat) Friedmann equations as a boundary term in the action.

Either way, the coefficient C_A will not contribute to the action. The same is true for the remaining higher order expansion coefficients and those can be shown to vanish directly from the symmetry-reduced closure equations.

Higher order expansion coefficients

The three lowest order expansion coefficients have been determined now. The task of determining the remaining higher order expansion coefficients is simpler compared to the analysis up to now. The remaining expansion coefficients

$$C^{\text{cosmo}}_{A_1 \ldots A_N} \quad \text{for } N \geq 3$$

all vanish and do not contribute to the symmetry-reduced spacetime action.

In order to see this, start by inspecting the instance $(C16_2)|_{\varphi^A(a,r,\theta)}$ of the sequence of equations. It is directly read off that

$$C^{\text{cosmo}}_{A_1 A_2 A_3} = 0\,.$$

Additionally, the derivatives

$$(C16_2)_{:D_1}{}^{\lambda_1\lambda_2} \ldots {}_{:D_R}{}^{\lambda_{2R-1}\lambda_{2R}}\Big|_{\varphi^A(a,r,\theta)} \quad \text{for } R \geq 1$$

allow to read off that all derivatives

$$C_{A_1 A_2 A_3 :D_1}{}^{\lambda_1\lambda_2} \ldots {}_{:D_R}{}^{\lambda_{2R-1}\lambda_{2R}}\Big|_{\varphi^A(a,r,\theta)} \quad \text{for } R \geq 1$$

vanish as well. For $R = 1$, one finds from closure equation $(C16_4)|_{\varphi^A(a,r,\theta)}$ that

$$C^{\text{cosmo}}_{A_1 \ldots A_5} = 0\,.$$

Subsequently using this argument for all even numbered instances $(C16_{2N})|_{\varphi^A(a,r,\theta)}$, one finds that all odd numbered expansion coefficients

$$C^{\text{cosmo}}_{A_1 \ldots A_{2N+1}} = 0 \quad \text{for } N \geq 1$$

vanish.

The same argument holds for the odd numbered equations $(C16_{2N+1})|_{\varphi^A(a,r,\theta)}$. Beginning with $N = 3$, one finds that

$$C^{\text{cosmo}}_{A_1 A_2 A_3 A_4} = 0$$

as the expansion coefficient C^{cosmo}_{AB} does not depend on second derivatives of the configuration fields. Iterating this argument in the same way as before provides that all even numbered expansion coefficients

$$C^{\text{cosmo}}_{A_1 \ldots A_{2N}} = 0 \quad \text{for } N \geq 2$$

vanish. Thus, none of these expansion coefficients contributes to the symmetric spacetime action (4.4). Only the two expansion coefficients C^{cosmo} and C^{cosmo}_{AB} contribute and form the cosmological spacetime action.

Cosmological spacetime action

The gravitational closure equations simplified after the imposition of FLRW symmetries such that a componentwise solution was obtained. No general arguments were required unless for the second expansion coefficient C_A^{cosmo} which will not contribute to the Friedmann equation anyway. Instead, only the two expansion coefficients

$$C^{\text{cosmo}} = \frac{r^2 \sin\theta}{(1-kr^2)^{\frac{1}{2}}} \left(K_1\, a^3 - 24\, K_0\, k\, a\right)$$

$$C_{14}^{\text{cosmo}} = K_0 \frac{r^4 \sin\theta\, a^7}{(1-kr^2)^{\frac{3}{2}}} \quad , \quad C_{16}^{\text{cosmo}} = K_0 \frac{r^4 \sin^3\theta\, a^7}{(1-kr^2)^{\frac{3}{2}}} \quad , \quad C_{46}^{\text{cosmo}} = K_0 \frac{r^6 \sin^3\theta\, a^7}{(1-kr^2)^{\frac{1}{2}}}$$

constitute the cosmological spacetime action whose general form is given by Eq. (4.4). Recall the three non-trivial derivatives of the configuration fields also entering the cosmological spacetime action,

$$\dot{\varphi}^1 = \frac{2\,(1-kr^2)\,\dot{a}}{a^3} \quad , \quad \dot{\varphi}^4 = \frac{2\,\dot{a}}{a^3\, r^2} \quad \text{and} \quad \dot{\varphi}^6 = \frac{2\,\dot{a}}{a^3\, r^2\, \sin^2\theta}\,.$$

Plugging these derivatives and the solution of the expansion coefficients into Eq. (4.4), one obtains the cosmological action as

$$S^{\text{cosmo}} = \int \mathrm{d}t \int \mathrm{d}^3 z\, \frac{r^2 \sin\theta}{(1-kr^2)^{\frac{1}{2}}} \left[N\left(K_1\, a^3 - 24\, K_0\, k\, a\right) + 24\, K_0\, \frac{a\dot{a}^2}{N}\right], \tag{4.8}$$

where N is the lapse function depending only the foliation time t. Variation of this action with respect to the lapse function and the scale factor provides the gravitational part of the Friedmann equations. In order to obtain the correct Friedmann equations, it is also necessary to impose the symmetry condition of spatial homogeneity and isotropy on the matter sourcing the cosmological dynamics.

4.3 Matter sources and Friedmann equations

The usual way to obtain the Einstein equations including the matter sources is to vary the sum

$$S_{\text{grav}} + S_{\text{matter}}$$

of the gravitational and the matter action with respect to the spacetime metric g which provides the field equations

$$\frac{\delta S_{\text{grav}}}{\delta g^{ab}} = -\frac{\delta S_{\text{matter}}}{\delta g^{ab}}\,.$$

The quantity on the right hand side sources the gravitational dynamics. This quantity is however the half of the *source tensor density*

$$\widetilde{S}_{ab} := -2\,\frac{\delta S_{\text{matter}}}{\delta g^{ab}}\,.$$

As Gotay and Marsden already pointed out in Ref. [25], one must not mistake the tensor density $\widetilde{S}_{ab}$ for the stress-energy-momentum tensor density $\widetilde{T}$ which is a $(1,1)$-tensor density – independent of the specific spacetime geometry. In the present case of a metric spacetime, the relation is however trivial as the intertwining role between the two tensors is played by the metric,

$$\widetilde{S}_{ab} = \widetilde{T}^m{}_a\, g_{mb}\,.$$

Up to now, the definitions of this section were only general. When applying cosmological symmetries, it is well known that these have to hold at large scales only since their imposition at all scales might not even produce non-trivial gravitational sources $\frac{\delta S_{\text{matter}}}{\delta g^{ab}}$. Usually, appropriate averaging over matter field configurations is used in order to obtain an effective perfect fluid energy-momentum density. This argument is difficult to translate to other matter models on other background geometries. Instead, it is the Killing condition

$$(\mathcal{L}_{K_i} S)_{ab} \stackrel{!}{=} 0 \quad \text{and} \quad (\mathcal{L}_{K_i} T)^a{}_b \stackrel{!}{=} 0$$

for the six cosmological Killing vector fields on the tensorial versions $S_{ab} := \widetilde{S}_{ab}\,(-\det g)^{-\frac{1}{2}}$ and $T^a{}_b := \widetilde{T}^a{}_b\,(-\det g)^{-\frac{1}{2}}$ which provides the way to construct gravitational sources which are compatible with the imposition of cosmological symmetries.

Throughout this thesis, the FLRW symmetries – spatial homogeneity and isotropy – are always first imposed on the stress-energy-momentum tensor T which leaves it in diagonal form with two undetermined functions of the foliation time t which in accordance to the usual perfect fluid interpretation are labeled as $\rho(t)$ and $p(t)$,

$$T^a{}_b := \operatorname{diag}\left(\rho, -p, -p, -p\right)^a{}_b\,.$$

Subsequently, one finds for the source tensor S

$$S_{ab} = \operatorname{diag}\left(\rho\,N^2, \frac{p\,a^2}{1-kr^2}, p\,a^2\,r^2, p\,a^2\,r^2\sin^2\theta\right)_{ab}\,. \tag{4.9}$$

The determination of the gravitational sources was the last step towards the derivation of the Friedmann equations.

Derivation of the Friedmann equations

The cosmological action (4.8) and the gravitational sources (4.9) are now determined. Variation with respect to the lapse function N provides the constraint equation. Dividing out the volume element

$$\frac{a^3\,r^2\sin\theta}{(1-kr^2)^2}$$

as well as choosing a parametrization such that $N = 1$ everywhere, straightforward calculation gives

$$\left(\frac{\dot a}{a}\right)^2 = \frac{\rho}{24\,K_0} - \frac{k}{a^2} + \frac{K_1}{24\,K_0}\,.$$

In order to determine the two unknown constants K_0 and K_1, one has to perform two independent experiments. As one already knows the result – the two Friedmann equations from standard cosmology – one can simply identify the two constants with the cosmological and Newton's constant as

$$K_0 = \frac{1}{64\,\pi\,G} \quad \text{and} \quad K_1 = \frac{\Lambda}{8\,\pi\,G}\,.$$

The second Friedmann equation arises by variation with respect to the scale factor a. Again choosing a parametrization with $N = 1$ everywhere, dividing out the volume element and a factor a^2 provides the acceleration equation. Inserting the constraint equation recovers the acceleration equation in its well-known form as

$$\frac{\ddot a}{a} = -\frac{4\,\pi\,G}{3}\,(\rho + 3\,p) + \frac{\Lambda}{3}\,.$$

While the plain result may sound boring at the beginning – it is already well-known that the Friedmann equations are the gravitational field equations for a spatially homogeneous and isotropic metric spacetime – it is the way towards the resulting Friedmann equations which provides an innovation. The Einstein equations never appeared at any stage and provocatively speaking, one could even abandon them for the mere purpose of cosmology. Rather, it is possible to start directly at the specific matter model – Maxwell electrodynamics in this example – and use the constructive gravity program together with the FLRW symmetries. Then, the Friedmann equations are directly obtained as a solution to the closure equations.

As a helpful side effect, the task of solving the gravitational closure equations in order to come up with gravitational field equations is also simplified by imposing a symmetry ansatz on them. Last but not least, this successful example of deriving the Friedmann equations without using Einstein equations provides confidence to the constructive gravity program and its symmetry reduction. The methods developed in this chapter will be of central importance in Chapter 6. There, from general linear electrodynamics, a refinement of Maxwell theory, the refined Friedmann equations will be constructed as a solution to the symmetry-reduced closure equations.

Before going into these quite involved calculations in Chapter 6, the next chapter reviews the basics of general linear electrodynamics and shows the general setup of the gravitational closure equations stemming from general linear electrodynamics.

Chapter 5

From general linear electrodynamics to the closure equations

The previous two chapters developed solutions of the closure equations starting from Maxwell electrodynamics. First, the general solution was presented which provided the Einstein-Hilbert action of general relativity. The previous chapter demonstrated that the Friedmann equations can be obtained directly as a solution to the closure equations on which FLRW symmetries imposed. This symmetry reduction provides a conceptually and technical new way for obtaining solutions to the gravitational closure equations.

In particular, one can apply the symmetry reduction of the closure equations and find the symmetry-reduced solutions in case the general closure equations are too difficult to solve. This chapter will demonstrate that already for general linear electrodynamics, a slight generalization of Maxwell theory, a general solution to the resulting closure equations seems out of reach as they are too involved. Thus, the symmetry reduction developed in the previous chapter will be applied in order to construct the refinement of the Friedmann equations for the cosmology of general linear electrodynamics.

The most significant difference between Maxwell theory and general linear electrodynamics is the birefringence of light in vacuum. This effect is admissible due to a richer spacetime structure. The spacetime geometry is no longer a metric, but a fourth rank tensor field. The dynamics of this geometry have to be determined by the gravitational closure equations. The calculation of the three input coefficients in this chapter will reveal that the resulting closure equations are very involved and that a general solution for them seems out of reach. See also Fig. 5.1 for a diagrammatic scheme. The appropriate symmetry reduction presented in Chapter 6 will provide significant simplifications.

$$S_{\mathrm{GLED}}[A;G) \xrightarrow[\text{Cha. 5}]{} \begin{matrix}\text{closure}\\ \text{equations}\end{matrix} \quad \not\!\not\longrightarrow \quad S_{\mathrm{gravity}}[G] \quad \dashrightarrow \begin{matrix}\text{grav. field}\\ \text{equations}\end{matrix}$$

Figure 5.1: Setup of the gravitational closure formalism starting with general linear electrodynamics – a slight generalization of Maxwell theory. A general solution of the closure equations is out of reach due to the involved input coefficients.

The results of this chapter have already been published as
M. Düll, F. P. Schuller, N. Stritzelberger and F. Wolz,
Phys. Rev. D97 (2018), 084036,
whose section V.C is elaborated here.

Using an axiomatic approach to the phenomenology of electrodynamics [26, 27, 28], the requirement of e. g. charge conservation and magnetic flux conservation does not require to choose a certain spacetime geometry. Only if one wishes to have a linear constitutive law relating the field strength and the excitation of the electromagnetic field, one is required to introduce a tensorial spacetime geometry. For a detailed introduction to this axiomatic approach of electrodynamics also known as pre-metric electrodynamics, see Ref. [29].

Additionally requiring that the resulting theory of electrodynamics has to be formulated by an action, the action of this theory of electrodynamics is written as

$$S_{\mathrm{GLED}}[A;G) = -\frac{1}{8}\int\limits_{M} \mathrm{d}^4x\,\omega_G\,F_{ab}F_{cd}\,G^{abcd}\,, \tag{5.1}$$

which is the action of general linear electrodynamics[1] – the most general theory of linear electrodynamics with linear field equations on a four-dimensional manifold M. The field strength $F_{ab} := 2\,\partial_{[a}A_{b]}$ is defined as usual in terms of the electromagnetic one-form potential A. The scalar density ω_G is a nowhere vanishing function of the spacetime geometry G. While there are several choices for such a scalar density, this thesis sticks to the choice

$$\omega_G = \left(\frac{1}{24}\,\epsilon_{abcd}\,G^{abcd}\right)^{-1}$$

which requires that the fourth rank tensor field G satisfies $\epsilon_{abcd}\,G^{abcd} \neq 0$ everywhere. Due to the antisymmetry of the field strength tensor, the fourth rank tensor field G inherits the antisymmetry in the respective indices and satisfies

$$G^{abcd} = -G^{bacd} = G^{cdab}\,.$$

This reduces the amount of degrees of freedom of G to 21 compared to 256 of a general fourth rank tensor field. Still, 21 degrees of freedom are more than double as many as a spacetime metric features. These additional degrees of freedom account for effects like vacuum birefringence which are ruled out in Maxwell-Einstein theory.

Recall from Chapter 2 that it is the principal polynomial P which together with the manifold M and the geometry G forms the spacetime triple (M,G,P). For general linear electrodynamics, the principal polynomial was first calculated by Rubilar et al. [33, 34] as

$$P^{abcd} = -\frac{24}{\left(\epsilon_{ijkl}G^{ijkl}\right)^2}\,\epsilon_{mnpq}\,\epsilon_{rstu}\,G^{mnr(a}\,G^{b|ps|c}\,G^{d)qtu}\,. \tag{5.2}$$

Note that the principal polynomial P depends on the spacetime geometry non-polynomially. This provides technically particularly involved spacetime kinematics; at the same time, this is a first example of a matter theory for which principal polynomial and spacetime geometry differ significantly from each other.

In order to apply the constructive gravity program, general linear electrodynamics has to satisfy the three matter conditions of Section 2.1. Previous work has shown that this is the case if the fourth rank tensor G lies in one of seven admissible algebraic classes of which there are a total of 23. See Ref. [30] for details.

Following the steps from Chapter 2, the identification of the canonical geometry is the next step towards the gravitational closure equations. It was mentioned before that the geometry G inherits antisymmetric index pairs from the field strength tensor in the action (5.1). This antisymmetry is inherited to the induced geometry on the hypersurface Σ. With the appropriate volume form on Σ, these antisymmetric index pairs can be dualized. This leads to the identification of three different hypersurface

[1]For a detailied study of general linear electrodynamics, see e. g. Ref. [30] next to the already referenced works of Friedrich Hehl [29] and others. General linear electrodynamics can also be quantized [31]. Possible deviations from a metric spacetime within quantum (field) theory have been studied in Ref. [32].

fields,

$$
\begin{aligned}
\bar{\mathbf{g}}^{\alpha\beta} &:= -G\left(\epsilon^0, \epsilon^\alpha, \epsilon^0, \epsilon^\beta\right) , \\
\bar{\bar{\mathbf{g}}}_{\alpha\beta} &:= \frac{1}{4}\,\frac{1}{\det \bar{\mathbf{g}}^{\cdot\cdot}}\, \epsilon_{\alpha\mu\nu}\,\epsilon_{\beta\rho\sigma}\, G\left(\epsilon^\mu, \epsilon^\nu, \epsilon^\rho, \epsilon^\sigma\right) \;\; , \\
\bar{\bar{\bar{\mathbf{g}}}}_{\alpha\beta} &:= \left(\bar{\mathbf{g}}^{-1}\right)_{\alpha\mu} \left(\frac{1}{2}\,\frac{1}{\sqrt{\det \bar{\mathbf{g}}^{\cdot\cdot}}}\, \epsilon_{\beta\rho\sigma}\, G\left(\epsilon^0, \epsilon^\mu, \epsilon^\rho, \epsilon^\sigma\right) - \delta^\mu_\beta \right) .
\end{aligned}
$$

The two frame conditions $\mathbf{p} = 1$ and $\mathbf{p}^\alpha = 0$ transform to

$$
\bar{\mathbf{g}}^{\alpha\beta}\, \bar{\bar{\bar{\mathbf{g}}}}_{\alpha\beta} = 0 \quad \text{and} \quad \bar{\bar{\bar{g}}}_{[\alpha\beta]} = 0\,.
$$

This means that the induced geometry field $\bar{\bar{\bar{\mathbf{g}}}}$ is symmetric – which it is not by construction – and trace-free with respect to the field $\bar{\mathbf{g}}$. When constructing the canonical geometry $\bar{g}$, $\bar{\bar{g}}$ and $\bar{\bar{\bar{g}}}$, these conditions have to be imposed by hand as well as the two linear symmetry conditions

$$
\bar{g}^{[\alpha\beta]} = 0 \quad \text{and} \quad \bar{\bar{g}}_{[\alpha\beta]} = 0\,.
$$

These conditions reduce the 27 components of the three canonical tensor fields to 17 independent ones. Thus, there are 17 unconstrained configuration fields φ^A which are most practically denoted by

$$
\varphi^A := \left(\bar{\varphi}^1, \ldots, \bar{\varphi}^6, \bar{\bar{\varphi}}^1, \ldots, \bar{\bar{\varphi}}^6, \bar{\bar{\bar{\varphi}}}^1, \ldots, \bar{\bar{\bar{\varphi}}}^5\right) . \tag{5.3}
$$

Next in the construction are the parametrization maps. One chooses

$$
\hat{\bar{g}}^{\alpha\beta} = I^{\alpha\beta}{}_A\, \bar{\varphi}^A \quad , \quad \hat{\bar{\bar{g}}}_{\alpha\beta} = I^A{}_{\alpha\beta}\, \Delta_{AB}\, \bar{\bar{\varphi}}^B \quad \text{and} \quad \hat{\bar{\bar{\bar{g}}}}_{\alpha\beta} = I^A{}_{\alpha\beta} \left(\delta^B_A - \frac{n_A\, \bar{\varphi}^B}{n_C\, \bar{\varphi}^C}\right) \epsilon_{(m)B} \bar{\bar{\bar{\varphi}}}^m . \tag{5.4}
$$

Here and for the rest of this thesis, capital latin indices $A, B, C, \ldots$ range from 1 to 6 while the small latin indices m, n range from 1 to 5. The matrices $I^A{}_{\alpha\beta}$ and $I^{\alpha\beta}{}_A$ are the same as in the metric case presented in Eq. (3.1). The symbol Δ_{AB} denotes the components of the standard inner product on $\mathbb{R}^6$,

$$
\Delta_{AB} = \mathrm{diag}(1, 1, 1, 1, 1, 1)_{AB}\,.
$$

Additionally, there are constant orthonormal basis vectors $t, e^{(1)}, \ldots, e^{(5)}$ chosen in such a way that the 3×3 matrix

$$
I^A{}_{\alpha\beta}\, \Delta_{AB}\, t^B
$$

is positive definite. It is then the set of covectors

$$
n_A := \Delta_{AB}\, t^B \quad \text{and} \quad \epsilon_{(m)A} := \Delta_{AB}\, e^{(m)B}
$$

which provides the dual basis appearing in the parametrization map $\hat{\bar{\bar{\bar{g}}}}(\varphi)$. Next to the maps $\hat{g}(\varphi)$, there are also inverse parametrization maps $\hat{\varphi}^A(g)$ which allow to extract the 17 configuration fields from the canonical geometry by

$$
\hat{\bar{\varphi}}^A = I^A{}_{\alpha\beta}\, \bar{g}^{\alpha\beta} \quad , \quad \hat{\bar{\bar{\varphi}}}^A = \Delta^{AB}\, I^{\alpha\beta}{}_B\, \bar{\bar{g}}_{\alpha\beta} \quad \text{and} \quad \hat{\bar{\bar{\bar{\varphi}}}}^m = I^{\alpha\beta}{}_A\, e^{(m)A}\, \bar{\bar{\bar{g}}}_{\alpha\beta}\,.
$$

It is quickly checked that the parametrization maps give rise to symmetric tensor fields and also satisfy the frame condition

$$
\hat{\bar{g}}^{\alpha\beta}(\varphi)\, \hat{\bar{\bar{\bar{g}}}}_{\alpha\beta}(\varphi) = 0\,.
$$

As mentioned above, the parametrization maps require the introduction of an orthonormal $\mathbb{R}^6$-frame. While there are several choices for such a frame (and its dual), a particular one will be constructed in the following paragraph. After that, the three input coefficients will be calculated which enter the gravitational closure equations.

Construction of the orthonormal frame

Recall that the parametrization map of the field $\hat{\bar{\bar{g}}}_{\alpha\beta}$ requires the introduction of constant orthonormal basis vectors $t, e^{(1)}, \ldots, e^{(5)}$ with the requirement that the 3×3-matrix

$$I^A{}_{\alpha\beta}\, \Delta_{AB}\, t^B$$

is positive definite. The symbol Δ_{AB} denotes the constant components

$$\Delta_{AB} = \mathrm{diag}(1, 1, 1, 1, 1, 1)_{AB}$$

of the standard inner product of $\mathbb{R}^6$. A straightforward choice of such a vector t is

$$t^A = \left(\frac{1}{\sqrt{3}}, 0, 0, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right)^A .$$

It is normalized with respect to the inner product Δ and the matrix

$$I^A{}_{\alpha\beta}\, \Delta_{AB}\, t^B = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}_{\alpha\beta}$$

is clearly positive definite. In order to complete the set of basis vectors, consider the collection

$$\begin{aligned} f^{(1)A} &= (1, 0, 0, 0, 0, 0)^A , \\ f^{(2)A} &= (1, 1, 0, 0, 0, 0)^A , \\ f^{(3)A} &= (1, 1, 1, 1, 0, 0)^A , \\ f^{(4)A} &= (1, 1, 1, 1, 1, 0)^A \quad \text{and} \\ f^{(5)A} &= (1, 1, 1, 1, 1, 1)^A \end{aligned}$$

of vectors which are not yet normalized. The determinant of the 6×6 matrix whose columns are constituted by the set $\left\{t, f^{(1)}, \ldots, f^{(5)}\right\}$ is non-zero which guarantees that those six vectors are linearly independent. The application of the Gram-Schmidt orthonormalization algorithm provides the desired orthonormal basis vectors

$$\begin{aligned} t^A &= \left(\frac{1}{\sqrt{3}}, 0, 0, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right)^A , \\ e^{(1)A} &= \left(\sqrt{\frac{2}{3}}, 0, 0, -\frac{1}{\sqrt{6}}, 0, -\frac{1}{\sqrt{6}}\right)^A , \\ e^{(2)A} &= (0, 1, 0, 0, 0, 0)^A , \\ e^{(3)A} &= \left(0, 0, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}\right)^A , \\ e^{(4)A} &= (0, 0, 0, 0, 1, 0)^A \text{ and} \\ e^{(5)A} &= \left(0, 0, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right)^A . \end{aligned}$$

The dual basis $\{n, \epsilon_{(m)}\}$ is numerically equal to the components of the basis vectors as the dual basis is extracted from the basis by using the inner product Δ_{AB},

$$n_A = \Delta_{AB}\, t^B \quad \text{and} \quad \epsilon_{(m)B} = \Delta_{AB}\, e^{(m)B} .$$

This basis and its dual will be used in the remainder of this thesis. The general setup as well as the resulting gravitational action is independent of the choice of the frame. The calculations carried out for the symmetry reduction in the next chapter rely heavily on a componentwise evaluation of the closure equations which means that the particular choice of a frame has to be used.

Calculation of input coefficients

The construction of the parametrization maps and the orthonormal $\mathbb{R}^6$-basis appearing in it pave the way for the calculation of the three input coefficients. The first step is calculate the intertwining matrices $\frac{\partial \hat{g}}{\partial \varphi}(\varphi)$ and $\frac{\partial \hat{\varphi}}{\partial g}(\hat{g}(\varphi))$ which appear in the definition of the input coefficients $F^A{}_\mu{}^\gamma$ and $M^{A\gamma}$. These matrices are

$$\frac{\partial \hat{\overline{g}}^{\alpha\beta}}{\partial \overline{\varphi}^A} = I^{\alpha\beta}{}_A \quad , \quad \frac{\partial \hat{\overline{\overline{g}}}_{\alpha\beta}}{\partial \overline{\overline{\varphi}}^A} = \Delta_{AB}\, I^B{}_{\alpha\beta} \quad , \quad \frac{\partial \hat{\overline{\overline{\overline{g}}}}_{\alpha\beta}}{\partial \overline{\overline{\overline{\varphi}}}^m} = I^A{}_{\alpha\beta} \left(\delta^B_A - \frac{n_A\, \overline{\varphi}^B}{n_C\, \overline{\varphi}^C} \right) \epsilon_{(m)B} \,,$$

$$\frac{\partial \hat{\overline{\overline{\overline{g}}}}_{\alpha\beta}}{\partial \overline{\varphi}^A} = I^B{}_{\alpha\beta}\, n_B \left(\frac{n_A\, \overline{\varphi}^C\, \epsilon_{(m)\,C}\, \overline{\overline{\overline{\varphi}}}^m}{(n_F\, \overline{\varphi}^F)^2} - \frac{\epsilon_{(m)A}\, \overline{\overline{\overline{\varphi}}}^m}{n_F\, \overline{\varphi}^F} \right) ,$$

$$\frac{\partial \hat{\overline{\varphi}}^A}{\partial \overline{g}^{\alpha\beta}} = I^A{}_{\alpha\beta} \quad , \quad \frac{\partial \overline{\overline{\varphi}}^A}{\partial \overline{\overline{g}}_{\alpha\beta}} = \Delta^{AB}\, I^{\alpha\beta}{}_B \quad \text{and} \quad \frac{\partial \overline{\overline{\overline{\varphi}}}^m}{\partial \overline{\overline{\overline{g}}}_{\alpha\beta}} = I^{\alpha\beta}{}_A\, e^{(m)A} \,.$$

It is now straightforward to calculate the three input coefficients. First, the components $p^{\alpha\beta}$ of the principal polynomial (5.2) are

$$p^{\alpha\beta} = \frac{1}{6} \left(\hat{\overline{g}}^{\alpha\gamma}\, \hat{\overline{g}}^{\beta\delta}\, \hat{\overline{\overline{g}}}_{\gamma\delta} - \hat{\overline{g}}^{\alpha\beta}\, \hat{\overline{g}}^{\gamma\delta}\, \hat{\overline{\overline{g}}}_{\gamma\delta} - 2\, \hat{\overline{g}}^{\alpha\beta}\, \hat{\overline{g}}^{\gamma\nu}\, \hat{\overline{g}}^{\delta\mu}\, \hat{\overline{\overline{\overline{g}}}}_{\gamma\mu}\, \hat{\overline{\overline{\overline{g}}}}_{\delta\nu} + 3\, \hat{\overline{g}}^{\gamma\delta}\, \hat{\overline{g}}^{\alpha\mu}\, \hat{\overline{g}}^{\beta\nu}\, \hat{\overline{\overline{\overline{g}}}}_{\gamma\mu}\, \hat{\overline{\overline{\overline{g}}}}_{\delta\nu} \right) . \tag{5.5}$$

Second, as laid out in Chapter 2, the coefficient $F^A{}_\mu{}^\gamma$ follows from the Lie derivative

$$\frac{\partial \hat{\varphi}^A}{\partial g^{\mathcal{A}}}\, (\mathcal{L}_{\vec{n}} g)^{\mathcal{A}} =: n^\mu\, \varphi^A{}_{,\mu} - (\partial_\gamma n^\mu)\, F^A{}_\mu{}^\gamma \,.$$

One finds

$$\begin{aligned} \overline{F}^A{}_\mu{}^\gamma = 2\, I^A{}_{\mu\sigma}\, I^{\gamma\sigma}{}_B\, \overline{\varphi}^B \quad , \quad \overline{\overline{F}}^A{}_\mu{}^\gamma = -2\, \Delta^{AB}\, \Delta_{CD}\, I^{\gamma\sigma}{}_B\, I^C{}_{\mu\sigma}\, \overline{\overline{\varphi}}^D \,, \\ \overline{\overline{\overline{F}}}^m{}_\mu{}^\gamma = -2\, I^{\gamma\alpha}{}_A\, e^{(m)A}\, I^B{}_{\mu\alpha} \left(\delta^C_B - \frac{n_B\, \overline{\varphi}^C}{n_F\, \overline{\varphi}^F} \right) \epsilon_{(n)C}\, \overline{\overline{\overline{\varphi}}}^n . \end{aligned} \tag{5.6}$$

Last, the coefficient $M^{A\gamma}$ is determined by applying the rules (2.16) to the definition (2.15) of the input coefficient. In contrast to the metric spacetime of the previous chapter, the coefficient $M^{A\gamma}$ is non-trivial for the fourth rank tensor field G serving as the spacetime geometry. Just as for $F^A{}_\mu{}^\gamma$, the coefficient $M^{A\gamma}$ is split into three different sets of components, differing on the index range of the capital index A. The components are

$$\begin{aligned} \overline{M}^{A\gamma} &= 2\, \sqrt{\det \overline{g}^{\cdot\cdot}}\, I^A{}_{\alpha\beta}\, I^{\nu(\alpha|}{}_B\, \overline{\varphi}^B\, \epsilon^{|\beta)\mu\gamma}\, I^C{}_{\mu\nu} \left(\delta^D_C - \frac{n_C\, \overline{\varphi}^D}{n_F\, \overline{\varphi}^F} \right) \epsilon_{(m)D}\, \overline{\overline{\overline{\varphi}}}^m \,, \\ \overline{\overline{M}}^{A\gamma} &= \frac{6}{\sqrt{\det \overline{g}^{\cdot\cdot}}}\, \epsilon_{\alpha\mu\nu}\, \Delta^{AB}\, I^{\alpha\beta}{}_B\, I^{\lambda\nu}{}_C\, \overline{\varphi}^C\, p^{\mu\gamma}(g(\varphi))\, I^E{}_{\beta\lambda} \left(\delta^D_E - \frac{n_E\, \overline{\varphi}^D}{n_F\, \overline{\varphi}^F} \right) \epsilon_{(m)D}\, \overline{\overline{\overline{\varphi}}}^m \,, \\ \overline{\overline{\overline{M}}}^{m\gamma} &= -\sqrt{\det \overline{g}^{\cdot\cdot}}\, \epsilon^{\mu\nu\gamma} \left(\overline{g}^{-1} \right)_{\mu\alpha} I^{\alpha\beta}{}_A\, e^{(m)A} \left(I^{\kappa\lambda}{}_B\, \overline{\varphi}^B\, \frac{\partial \hat{\overline{\overline{\overline{g}}}}_{\beta\lambda}}{\partial \overline{\overline{\overline{\varphi}}}^n}\, \frac{\partial \hat{\overline{\overline{\overline{g}}}}_{\kappa\nu}}{\partial \overline{\overline{\overline{\varphi}}}^l}\, \overline{\overline{\overline{\varphi}}}^n\, \overline{\overline{\overline{\varphi}}}^l + I^B{}_{\beta\nu}\, \Delta_{BC}\, \overline{\overline{\varphi}}^C \right) . \end{aligned}$$

The two derivatives of the parametrization maps in the last components have not been written out in order to keep the notation compact.

The input coefficients for general linear electrodynamics are more involved when compared to those of the metric spacetime investigated in Chapter 3. In particular, there are 11 more configuration fields and the third input coefficient $M^{A\gamma}$ is non-trivial. As a direct consequence, the gravitational closure equations will be more involved. One should note that the closure equations for general linear electrodynamics as constructed here differ significantly from those in previous work [9]. The non-linear frame conditions are now treated automatically by introducing the configuration fields and the parametrization maps whereas the frame conditions had to be imposed by hand afterwards in previous work.

Due to the involved structure of the closure equations for general linear electrodynamics, a general solution has not yet been obtained. As the Chapter 4 demonstrated, such a general solution providing the full gravitational field equations is not necessary if one is interested in e. g. the cosmological dynamics. Instead of constructing the full gravitational field equations and imposing spatial homogeneity and isotropy on them, one rather imposes the symmetry condition on the gravitational closure equations and constructs the refined Friedmann equations as a solution to those. This circumvents the need for the full gravitational field equations.

This is precisely the goal of the last technical chapter of this thesis. As general linear electrodynamics provides a refined geometric structure, also the Friedmann equations of cosmology will be refined. In order to calculate them, the general setup of the closure equations developed in this chapter will be used. Imposing the FLRW symmetries on the closure equations simplifies them such that a solution can be constructed. Complementing this solution by suitably symmetry-reduced gravitational sources then yields the refined Friedmann equations.

Chapter 6

Refined Friedmann equations from general linear electrodynamics

The last chapter provided the setup of the gravitational closure equations starting from general linear electrodynamics. While general linear electrodynamics provides the possibly slightest generalization of Maxwell theory, the spacetime kinematics are far more diverse. This translates into more involved input coefficients p, F and M entering the gravitational closure equations. The solution techniques used for the closure equations from Maxwell theory in Chapter 3 do not suffice anymore for the more involved closure equations in this case. Thus, a general solution to the closure equations from general linear electrodynamics seems to be out of reach. Therefore, the symmetry reduction of the closure equations is the only convenient and promising way to obtain solutions to the closure equations. See Fig. 6.1 for illustration.

$$
\begin{tikzcd}
S_{\text{GLED}}[A;G) \arrow[r, "\text{Cha. 5}"'] & \text{closure equations} \arrow[r, "/\!/" description] \arrow[d, "\text{Sec. 6.1}"'] & S_{\text{gravity}}[G] \arrow[r, dashed] \arrow[d, dashed] & \text{grav. field equations} \arrow[d, dashed] \\
 & \text{symm.-red. closure eqns} \arrow[r, "\text{Sections}", "6.2-6.6"'] & S_{\text{cosmo}}[a(t), c(t)] \arrow[r, "\text{Sections}", "6.7, 6.8"'] & \text{refined Friedmann eqns}
\end{tikzcd}
$$

Figure 6.1: The construction of refined Friedmann equations behind general linear electrodynamics. As a general solution to the closure equations is out of reach, one applies the FLRW symmetries directly to the closure equations and thus obtains a cosmological spacetime action now featuring two scale factors instead of one. The refined Friedmann equations are derived by variation of this action.

It was one of the main goals of this thesis to develop a suitable symmetry reduction of the constructive gravity program. The direct derivation of the Friedmann equations from Maxwell theory in Chapter 4 demonstrated that and how the symmetry reduction works. The methods laid out in Maxwell-Friedmann theory will now be adopted – and extended – for the closure equations constructed from general linear electrodynamics. For the first time, the solution of the symmetry reduced closure equations will require to systematically study prolongations, that is, derivatives of the closure equations with respect to the configuration fields. The first step of the analysis is however the symmetry reduction of the spacetime geometry G and the three input coefficients. After that, the closure equations can be symmetry-reduced and evaluated.

6.1 The cosmological area metric & setup of the closure equations

The first step of the symmetry reduction is the imposition of spatial homogeneity and isotropy onto the spacetime geometry G. Technically, this is achieved by requiring the Killing condition

$$\mathcal{L}_{K_i} G \stackrel{!}{=} 0$$

to hold for the six Killing vector fields $K_1, \ldots, K_6$. They were already presented Chapter 4 in spherical coordinates. They contained a constant k which could take values of $+1$, 0 or -1 depending on the character of the spatial hypersurfaces (spherical, flat or hyperbolic ones). In order to achieve the largest simplification, the calculations in this chapter are restricted to flat spatial hypersurfaces ($k = 0$). This enables one to choose cartesian coordinates and write the Killing vector fields as

$$\begin{aligned} K_1 &= \partial_x \quad , \quad K_2 = \partial_y \quad , \quad K_3 = \partial_z \\ K_4 &= z\,\partial_y - y\,\partial_z \quad , \quad K_5 = x\,\partial_z - z\,\partial_x \quad , \quad K_6 = y\,\partial_x - x\,\partial_y \,. \end{aligned} \tag{6.1}$$

The Killing condition provides differential equations for the components G^{abcd} of the spacetime geometry. They can be solved straighforwardly and the spacetime geometry is written as

$$G^{abcd} = 2\,c^2(t)\,g^{a[c}g^{d]b} - c^3(t)\,\sqrt{-\det g^{\cdot\cdot}}\,\epsilon^{abcd} \tag{6.2}$$

with a Friedmann-Robertson-Walker metric

$$g^{ab} = \operatorname{diag}\left(\frac{1}{N^2}, -\frac{1}{a^2(t)}, -\frac{1}{a^2(t)}, -\frac{1}{a^2(t)}\right)^{ab} .$$

Note that the cosmological spacetime geometry G^{abcd} has three time-dependent degrees of freedom – the lapse function $N(t)$ and two scale factors $a(t)$ and $c(t)$. As for the metric FLRW spacetime, the shift vector field $\vec{N}$ vanishes. The principal polynomial P whose general form is given by Eq. (5.2) breaks down to

$$P^{abcd}(a, c, N) = g^{(ab}g^{cd)}$$

The roots of the principal polynomial describe the dispersion of light rays. As a direct consequence of the above result, light rays and redshift see only a Friedmann-Robertson-Walker metric with one scale factor $a(t)$ in this refined cosmology. Thus, there is no birefringence of light on cosmological scales. How the light dispersion precisely looks like, that is, how a solution for the scale factor $a(t)$ looks like, can only be determined by solutions to the refined Friedmann equations. These have to be constructed as a solution to the gravitational closure equations first.

The symmetric spacetime geometry (6.2) is now projected back to the spatial hypersurfaces. Following the definitions from the previous chapter and Ref. [3], the hypersurface fields are

$$\overline{g}^{\alpha\beta} = \frac{c^2}{a^2}\operatorname{diag}(1,1,1)^{\alpha\beta} \quad , \quad \overline{\overline{g}}_{\alpha\beta} = \frac{a^2}{c^4}\operatorname{diag}(1,1,1)_{\alpha\beta} \quad \text{and} \quad \overline{\overline{\overline{g}}}_{\alpha\beta} = 0 \,. \tag{6.3}$$

These hypersurface fields are mimiced by the canonical geometry and by the parametrization maps (5.4), the configuration fields. Using the labeling (5.3) from Chapter 5, there are six non-trivial configuration fields

$$\overline{\varphi}^1 = \overline{\varphi}^4 = \overline{\varphi}^6 = \frac{c^2}{a^2} \quad \text{and} \quad \overline{\overline{\varphi}}^1 = \overline{\overline{\varphi}}^4 = \overline{\overline{\varphi}}^6 = \frac{a^2}{c^4} \,.$$

These non-trivial configuration fields are now inserted into the general expressions for the three input coefficients $F^A{}_\mu{}^\gamma$, $M^{A\gamma}$ and $p^{\mu\nu}$. The input coefficient $F^A{}_\mu{}^\gamma$ whose general expression is given by Eq. (5.6) reduces to

$$
\begin{aligned}
F^{\overline{A}}{}_{\mu}{}^{\gamma}\Big|_{\varphi^A(a,c)} &= \frac{2\,c^2}{a^2}\left(\delta_1^{\overline{A}}\delta_\mu^x\delta_x^\gamma + \delta_4^{\overline{A}}\delta_\mu^y\delta_y^\gamma + \delta_6^{\overline{A}}\delta_\mu^z\delta_z^\gamma\right) \\
&\quad \frac{\sqrt{2}\,c^2}{a^2}\left(\delta_2^{\overline{A}}(\delta_\mu^x\delta_y^\gamma + \delta_\mu^y\delta_x^\gamma) + \delta_3^{\overline{A}}(\delta_\mu^x\delta_z^\gamma + \delta_\mu^z\delta_x^\gamma) + \delta_5^{\overline{A}}(\delta_\mu^y\delta_z^\gamma + \delta_\mu^z\delta_y^\gamma)\right) \\
F^{\overline{\overline{A}}}{}_{\mu}{}^{\gamma}\Big|_{\varphi^A(a,c)} &= -\frac{2\,a^2}{c^4}\left(\delta_1^{\overline{\overline{A}}}\delta_\mu^x\delta_x^\gamma + \delta_4^{\overline{\overline{A}}}\delta_\mu^y\delta_y^\gamma + \delta_6^{\overline{\overline{A}}}\delta_\mu^z\delta_z^\gamma\right) \\
&\quad -\frac{\sqrt{2}\,a^2}{c^4}\left(\delta_2^{\overline{\overline{A}}}(\delta_\mu^x\delta_y^\gamma + \delta_\mu^y\delta_x^\gamma) + \delta_3^{\overline{\overline{A}}}(\delta_\mu^x\delta_z^\gamma + \delta_\mu^z\delta_x^\gamma) + \delta_5^{\overline{\overline{A}}}(\delta_\mu^y\delta_z^\gamma + \delta_\mu^z\delta_y^\gamma)\right) \\
F^{\overline{\overline{\overline{m}}}}{}_{\mu}{}^{\gamma}\Big|_{\varphi^A(a,c)} &= 0\,.
\end{aligned}
$$

Secondly, the component $p^{\alpha\beta}$ of the principal polynomial P has three non-vanishing components

$$
p^{xx}\big|_{\varphi^A(a,c)} = p^{yy}\big|_{\varphi^A(a,c)} = p^{zz}\big|_{\varphi^A(a,c)} = -\frac{1}{3\,a^2(t)}\,.
$$

The third input coefficient $M^{A\gamma}$ also vanishes when symmetry is applied. It is conceptually and technically important that derivatives of the coefficient $M^{A\gamma}$ with respect to the configuration fields might be non-trivial although the coefficient itself vanishes after symmetry imposition. This is because one first has to take the derivative of the coefficient and may evaluate only afterwards. For the analysis of the closure equations, the derivatives which are required the most are

$$
\begin{aligned}
&M^{A\gamma}{}_{:\overline{\overline{\overline{m}}}}\Big|_{\varphi^A(a,c)} = \\
&= \delta^1_{\overline{\overline{\overline{m}}}}\sqrt{3}\left[\frac{c^5}{a^5}\left(-\delta_2^{\overline{A}}\delta_z^\gamma + \delta_3^{\overline{A}}\delta_y^\gamma\right) + \frac{1}{ac}\left(-\delta_2^{\overline{\overline{A}}}\delta_z^\gamma + \delta_3^{\overline{\overline{A}}}\delta_y^\gamma\right)\right] \\
&+ \delta^2_{\overline{\overline{\overline{m}}}}\left[\frac{c^5}{a^5}\left(\sqrt{3}\,\delta_3^{\overline{A}}\delta_x^\gamma + \delta_5^{\overline{A}}\delta_y^\gamma + \sqrt{2}\,\delta_z^\gamma\left(\delta_1^{\overline{A}} - \delta_4^{\overline{A}}\right)\right) + \frac{1}{ac}\left(\sqrt{3}\,\delta_3^{\overline{\overline{A}}}\delta_x^\gamma + \delta_5^{\overline{\overline{A}}}\delta_y^\gamma + \sqrt{2}\,\delta_z^\gamma\left(\delta_1^{\overline{\overline{A}}} - \delta_4^{\overline{\overline{A}}}\right)\right)\right] \\
&+ \delta^3_{\overline{\overline{\overline{m}}}}\left[\frac{c^5}{a^5}\left(\sqrt{\frac{2}{3}}\,\delta_x^\gamma\left(\delta_2^{\overline{A}} - \sqrt{2}\,\delta_5^{\overline{A}}\right) - \frac{2}{\sqrt{3}}\,\delta_y^\gamma\left(\delta_1^{\overline{A}} - \frac{1}{2}\delta_3^{\overline{A}} - \delta_6^{\overline{A}}\right) + \frac{1}{\sqrt{3}}\,\delta_z^\gamma\left(\delta_2^{\overline{A}} - \sqrt{2}\,\delta_5^{\overline{A}}\right)\right)\right. \\
&\quad\left. + \frac{1}{ac}\left(\sqrt{\frac{2}{3}}\,\delta_x^\gamma\left(\delta_2^{\overline{\overline{A}}} - \sqrt{2}\,\delta_5^{\overline{\overline{A}}}\right) - \frac{2}{\sqrt{3}}\,\delta_y^\gamma\left(\delta_1^{\overline{\overline{A}}} - \frac{1}{2}\delta_3^{\overline{\overline{A}}} - \delta_6^{\overline{\overline{A}}}\right) + \frac{1}{\sqrt{3}}\,\delta_z^\gamma\left(\delta_2^{\overline{\overline{A}}} - \sqrt{2}\,\delta_5^{\overline{\overline{A}}}\right)\right)\right] \\
&+ \delta^4_{\overline{\overline{\overline{m}}}}\left[\frac{c^5}{a^5}\left(\sqrt{2}\,\delta_x^\gamma\left(\delta_4^{\overline{A}} - \delta_6^{\overline{A}}\right) - \delta_2^{\overline{A}}\delta_y^\gamma + \delta_3^{\overline{A}}\delta_z^\gamma\right) + \frac{1}{ac}\left(\sqrt{2}\,\delta_x^\gamma\left(\delta_4^{\overline{A}} - \delta_6^{\overline{A}}\right) - \delta_2^{\overline{A}}\delta_y^\gamma + \delta_3^{\overline{A}}\delta_z^\gamma\right)\right] \\
&+ \delta^5_{\overline{\overline{\overline{m}}}}\left[\frac{c^5}{a^5}\left(\frac{1}{\sqrt{3}}\,\delta_x^\gamma\left(\delta_2^{\overline{A}} + 2\sqrt{2}\,\delta_5^{\overline{A}}\right) - \sqrt{\frac{2}{3}}\,\delta_y^\gamma\left(\delta_1^{\overline{A}} + \delta_3^{\overline{A}} - \delta_6^{\overline{A}}\right) - \frac{1}{\sqrt{3}}\,\delta_z^\gamma\left(\sqrt{2}\,\delta_2^{\overline{A}} + \delta_5^{\overline{A}}\right)\right)\right. \\
&\quad\left. + \frac{1}{ac}\left(\frac{1}{\sqrt{3}}\,\delta_x^\gamma\left(\delta_2^{\overline{\overline{A}}} + 2\sqrt{2}\,\delta_5^{\overline{\overline{A}}}\right) - \sqrt{\frac{2}{3}}\,\delta_y^\gamma\left(\delta_1^{\overline{\overline{A}}} + \delta_3^{\overline{\overline{A}}} - \delta_6^{\overline{\overline{A}}}\right) - \frac{1}{\sqrt{3}}\,\delta_z^\gamma\left(\sqrt{2}\,\delta_2^{\overline{\overline{A}}} + \delta_5^{\overline{\overline{A}}}\right)\right)\right].
\end{aligned}
$$

Further non-trivial derivatives are $M^{\overline{\overline{\overline{m}}}\gamma}{}_{:\overline{A}}\Big|_{\varphi^A(a,c)}$ and $M^{\overline{\overline{\overline{m}}}\gamma}{}_{:\overline{\overline{A}}}\Big|_{\varphi^A(a,c)}$ which however play only a minor role in the analysis of the symmetry-reduced closure equations. There is also a variety of second derivatives of $M^{A\gamma}$ with respect to the configuration fields φ^A. These are calculated using appropriate computer algebra system such as *Mathematica* if the need for these components arises.

After the symmetry reduction of the three input coeffients, the next step is to define the functions

$$
C^{\text{cosmo}}_{A_1\ldots A_N}(a,c) := C_{A_1\ldots A_N}[\varphi^A(a,c)]
$$

which serve as the symmetry-reduced expansion coefficients. The derivatives of these ultralocal functions $C^{\text{cosmo}}_{A_1\ldots A_N}$ have to be related with derivatives of the full expansion coefficients $C^{\text{cosmo}}_{A_1\ldots A_N}[\varphi^A]$ which are functionals of the configuration fields. As in the metric FLRW case presented in Chapter 4, this is done by employing the chain rule for every expansion coefficient ($N > 0$),

$$
\begin{aligned}
\frac{\partial C^{\text{cosmo}}_{A_1\ldots A_N}}{\partial a} &= \left(C_{A_1\ldots A_N:\bar{1}}\Big|_{\varphi^A(a,c)} + C_{A_1\ldots A_N:\bar{4}}\Big|_{\varphi^A(a,c)} + C_{A_1\ldots A_N:\bar{6}}\Big|_{\varphi^A(a,c)} \right) \frac{-2\,c^2}{a^3} \\
&\quad + \left(C_{A_1\ldots A_N:\bar{\bar{1}}}\Big|_{\varphi^A(a,c)} + C_{A_1\ldots A_N:\bar{\bar{4}}}\Big|_{\varphi^A(a,c)} + C_{A_1\ldots A_N:\bar{\bar{6}}}\Big|_{\varphi^A(a,c)} \right) \frac{2\,a}{c^4}\,,
\end{aligned} \tag{6.4}
$$

$$
\begin{aligned}
\frac{\partial C^{\text{cosmo}}_{A_1\ldots A_N}}{\partial c} &= \left(C_{A_1\ldots A_N:\bar{1}}\Big|_{\varphi^A(a,c)} + C_{A_1\ldots A_N:\bar{4}}\Big|_{\varphi^A(a,c)} + C_{A_1\ldots A_N:\bar{6}}\Big|_{\varphi^A(a,c)} \right) \frac{2\,c}{a^2} \\
&\quad + \left(C_{A_1\ldots A_N:\bar{\bar{1}}}\Big|_{\varphi^A(a,c)} + C_{A_1\ldots A_N:\bar{\bar{4}}}\Big|_{\varphi^A(a,c)} + C_{A_1\ldots A_N:\bar{\bar{6}}}\Big|_{\varphi^A(a,c)} \right) \frac{-4\,a^2}{c^5}\,.
\end{aligned} \tag{6.5}
$$

The derivatives appearing on the right hand side of these equations need to be determined from the gravitational closure equations evaluated on the symmetric configuration fields $\varphi^A(a, c)$. It will turn out that this determination is not as straightforward as the Maxwell-Friedmann calculation in Chapter 4 was. Instead, already the solution for the first expansion coefficient C^{cosmo} in the next section demonstrates that prolongations of the closure equations – that is, derivatives of the closure equations with respect to the configuration fields – can reveal more information. The additional information proves out to be useful in the solution of the chain rule equations (6.4) and (6.5).

The solution of the above differential equations provides the set of expansion coefficients $C^{\text{cosmo}}_{A_1\ldots A_N}$ constituting the cosmological spacetime action. The full gravitational spacetime action (2.60) is simplified to

$$
S_{\text{cosmo}} = \int \mathrm{d}t \sum_{M=0}^{\infty} C^{\text{cosmo}}_{A_1\ldots A_M}(a,c)\,\dot{\varphi}^{A_1}(a,c)\ldots\dot{\varphi}^{A_N}(a,c)\,N^{1-M}(t) \tag{6.6}
$$

with the lapse $N(t)$ being a function of the foliation time. The derivatives $\dot{\varphi}^A$ of the configuration fields with respect to the foliation time are given by

$$
\begin{aligned}
\dot{\bar{\varphi}}^1 = \dot{\bar{\varphi}}^4 = \dot{\bar{\varphi}}^6 &= \frac{2\,\dot{c}\,c}{a^2} - \frac{2\,\dot{a}\,c^2}{a^3} \\
\dot{\bar{\bar{\varphi}}}^1 = \dot{\bar{\bar{\varphi}}}^4 = \dot{\bar{\bar{\varphi}}}^6 &= \frac{2\,\dot{a}\,a}{c^4} - \frac{4\,\dot{c}\,a^2}{c^5}\,.
\end{aligned} \tag{6.7}
$$

As all other configuration fields vanish after imposing the symmetry condition, also their derivatives vanish. This also simplifies the analysis of the gravitational closure equations as only components of the expansion coefficients need to be computed which couple exclusively to the non-vanishing derivatives $\dot{\varphi}^A$. This becomes extremely important for the higher order expansion coefficients. First, however, the closure equations for the first expansion coefficient C will be studied.

6.2 Solution for the first expansion coefficient

The symmetric configuration fields depend only on the two scale factors, but not on any other coordinate or function thereof. In particular, all spatial derivatives of the symmetric configuration fields vanish and with them also spatial derivatives of the three input coefficients. This simplifies the gravitational closure equations when they are evaluated on symmetric configurations $\varphi^A(a, c)$. For the first expansion coefficient C^{cosmo}, this has two important consequences.

First, when inspecting the sequence of closure equations $(C19_{N\geq 2})|_{\varphi^A(a,c)}$. It breaks down to

$$
0 = C_{:B}{}^{\mu_1\ldots\mu_{N+1}}\Big|_{\varphi^A(a,c)} \qquad \text{for } N \geq 2\,, \tag{6.8}
$$

which provides a collapse of the coefficient C to second derivative order when FLRW symmetries are imposed.

Secondly, with the chain rule equations (6.4) and (6.5) containing only derivatives with respect to the configuration fields themselves, it is at first only closure equation $(C1)|_{\varphi^A(a,c)}$ which provides relations for these derivatives.

Analysis of closure equation $(C1)$

Closure equation $(C1)|_{\varphi^A(a,c)}$ provides an equation which contains only the symmetric expansion coefficient C^{cosmo} and derivatives of C with respect to the configuration fields. Relations obtained from this equation can be used directly in order to solve the two differential equations (6.4) and (6.5) for $N = 0$. As all spatial derivatives drop out of the equation, it is simply

$$0 = C^{\text{cosmo}} \delta^\gamma_\mu + C_{:A}\big|_{\varphi^A(a,c)} \; F^A{}_\mu{}^\gamma\big|_{\varphi^A(a,c)} \; .$$

Analyzing this for all combinations of the spatial indices γ and μ, one obtains the three relations

$$\begin{aligned}
C_{:\bar{\bar{1}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\; C_{:\bar{1}}\Big|_{\varphi^A(a,c)} + \frac{c^4}{2\,a^2}\, C^{\text{cosmo}} \, , \\
C_{:\bar{\bar{4}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\; C_{:\bar{4}}\Big|_{\varphi^A(a,c)} + \frac{c^4}{2\,a^2}\, C^{\text{cosmo}} \quad \text{and} \\
C_{:\bar{\bar{6}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\; C_{:\bar{6}}\Big|_{\varphi^A(a,c)} + \frac{c^4}{2\,a^2}\, C^{\text{cosmo}}
\end{aligned}$$

which can be plugged into Eq. (6.4) and Eq. (6.5). One finds the two differential equations

$$\frac{\partial C^{\text{cosmo}}}{\partial a} = \frac{3}{a}\, C^{\text{cosmo}} \tag{6.9}$$

$$\frac{\partial C^{\text{cosmo}}}{\partial c} = -\frac{2\,c}{a^2}\left(C_{:\bar{1}}\Big|_{\varphi^A(a,c)} + C_{:\bar{4}}\Big|_{\varphi^A(a,c)} + C_{:\bar{6}}\Big|_{\varphi^A(a,c)} \right) - \frac{6}{c}\, C^{\text{cosmo}} \tag{6.10}$$

of which the first one can be solved as

$$C^{\text{cosmo}} = f_0(c)\, a^3 \tag{6.11}$$

with one undetermined function f_0 of the second scale factor c. The second differential equation concerning the dependence of C^{cosmo} on this second scale factor cannot be solved yet. Instead, one needs to extract further relations for the three derivatives $C_{:\bar{1}}\big|_{\varphi^A(a,c)}$, $C_{:\bar{4}}\big|_{\varphi^A(a,c)}$ and $C_{:\bar{6}}\big|_{\varphi^A(a,c)}$ from the gravitational closure equations. As there is no further equation which contains the desired derivatives when evaluated on cosmological configurations $\varphi^A(a,c)$, one has to consider prolongations of the closure equations – that is, derivatives of the closure equations with respect to the configuration fields on which the symmetry is imposed afterwards. Investigating all closure equations, one sees that closure equation $(C5)$ is a candidate for providing relevant prolongations.

First prolongation of $(C5)$

One observes that although closure equation $(C5)|_{\varphi^A(a,c)}$ is identically satisfied, the derivative of the full closure equation $(C5)$ with respect to the five configuration fields $\varphi^{\bar{\bar{\bar{m}}}}$ and subsequent imposition of symmetry results in the equation

$$0 = C_{:\bar{A}}\Big|_{\varphi^A(a,c)} \; M^{\bar{A}\gamma}{}_{:\bar{\bar{\bar{m}}}}\Big|_{\varphi^A(a,c)} + C_{:\bar{\bar{A}}}\Big|_{\varphi^A(a,c)} \; M^{\bar{\bar{A}}\gamma}{}_{:\bar{\bar{\bar{m}}}}\Big|_{\varphi^A(a,c)} \; .$$

Evaluating this equation reveals that the three derivatives of C appearing in Eq. (6.10) are actually equal,

$$C_{:\bar{1}}\Big|_{\varphi^A(a,c)} = C_{:\bar{4}}\Big|_{\varphi^A(a,c)} = C_{:\bar{6}}\Big|_{\varphi^A(a,c)} \; .$$

This allows to rewrite Eq. (6.10) as

$$\frac{\partial C^{\text{cosmo}}}{\partial c} = -\frac{6\,c}{a^2}\; C_{:\bar{1}}\Big|_{\varphi^A(a,c)} - \frac{6}{c}\, C^{\text{cosmo}} \tag{6.12}$$

which still features the undetermined derivative $C_{:\bar{1}}\big|_{\varphi^A(a,c)}$. This means that the solution for the coefficient C^{cosmo} is still given by Eq. (6.11). The prolongation of closure equation $(C5)$ shows that derivatives of the closure equations indeed contain additional information about the expansion coefficients and their derivatives.

After evaluation of the first prolongation of $(C5)$, one checks whether there are any other closure equations whose prolongations might provide additional relations. This is not the case. It remains to be seen whether further prolongations of the two closure equations $(C1)$ and $(C5)$ provide additional information. This will be addressed next.

First prolongation of $(C1)$

The first prolongation of $(C1)$ is taken with respect to the configuration fields $\overline{\varphi}^B$ and $\overline{\overline{\varphi}}^B$. There are only three terms when the resulting equation is evaluated on symmetric configurations fields $\varphi^A(a,c)$. The respective equations are

$$0 = C_{:\overline{B}}\Big|_{\varphi^A(a,c)} \delta^\gamma_\mu + 2\ C_{:\overline{A}}\Big|_{\varphi^A(a,c)}\ I^{\overline{A}}{}_{\mu\sigma} I^{\gamma\sigma}{}_{\overline{B}} + C_{:\overline{B}:A}\ \Big|_{\varphi^A(a,c)}\ F^A{}_\mu{}^\gamma\Big|_{\varphi^A(a,c)} \quad \text{and} \tag{6.13}$$

$$0 = C_{:\overline{\overline{B}}}\Big|_{\varphi^A(a,c)} \delta^\gamma_\mu - 2\ C_{:\overline{\overline{A}}}\Big|_{\varphi^A(a,c)}\ I^{\overline{\overline{A}}\gamma\sigma} I_{\mu\sigma\overline{\overline{B}}} + C_{:\overline{\overline{B}}:A}\ \Big|_{\varphi^A(a,c)}\ F^A{}_\mu{}^\gamma\Big|_{\varphi^A(a,c)}\ . \tag{6.14}$$

Both equations have to be evaluated for all combinations of the free spatial indices μ and γ as well as of the capital index which results in 54 relations for each equation. The relations from Eq. (6.13) express all second derivatives of the form $C_{:\overline{A}:\overline{\overline{B}}}\Big|_{\varphi^A(a,c)}$ in terms of the second derivatives $C_{:\overline{A}:\overline{B}}\Big|_{\varphi^A(a,c)}$ and the first derivative $C_{:\overline{1}}\Big|_{\varphi^A(a,c)}$. Using the thus obtained relations in the analysis of Eq. (6.14), the second derivatives $C_{:\overline{\overline{A}}:\overline{\overline{B}}}\Big|_{\varphi^A(a,c)}$ are expressed in terms of their purely single-overlined counterparts $C_{:\overline{A}:\overline{B}}\Big|_{\varphi^A(a,c)}$, the first derivative $C_{:\overline{1}}\Big|_{\varphi^A(a,c)}$ and the coefficient C^{cosmo} itself.

Precisely, from Eq. (6.13), one finds the relations

$$\begin{aligned}
C_{:\overline{1}:\overline{\overline{1}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\ C_{:\overline{1}:\overline{1}}\Big|_{\varphi^A(a,c)} + \frac{3\,c^4}{2\,a^2}\ C_{:\overline{1}}\Big|_{\varphi^A(a,c)}\ , & C_{:\overline{4}:\overline{\overline{1}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\ C_{:\overline{1}:\overline{4}}\Big|_{\varphi^A(a,c)} + \frac{c^4}{2\,a^2}\ C_{:\overline{1}}\Big|_{\varphi^A(a,c)}\ ,\\
C_{:\overline{6}:\overline{\overline{1}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\ C_{:\overline{1}:\overline{6}}\Big|_{\varphi^A(a,c)} + \frac{c^4}{2\,a^2}\ C_{:\overline{1}}\Big|_{\varphi^A(a,c)}\ , & C_{:\overline{1}:\overline{\overline{4}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\ C_{:\overline{1}:\overline{4}}\Big|_{\varphi^A(a,c)} + \frac{c^4}{2\,a^2}\ C_{:\overline{1}}\Big|_{\varphi^A(a,c)}\ ,\\
C_{:\overline{4}:\overline{\overline{4}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\ C_{:\overline{4}:\overline{4}}\Big|_{\varphi^A(a,c)} + \frac{3\,c^4}{2\,a^2}\ C_{:\overline{1}}\Big|_{\varphi^A(a,c)}\ , & C_{:\overline{6}:\overline{\overline{4}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\ C_{:\overline{4}:\overline{6}}\Big|_{\varphi^A(a,c)} + \frac{c^4}{2\,a^2}\ C_{:\overline{1}}\Big|_{\varphi^A(a,c)}\ ,\\
C_{:\overline{1}:\overline{\overline{6}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\ C_{:\overline{1}:\overline{6}}\Big|_{\varphi^A(a,c)} + \frac{c^4}{2\,a^2}\ C_{:\overline{1}}\Big|_{\varphi^A(a,c)}\ , & C_{:\overline{4}:\overline{\overline{6}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\ C_{:\overline{4}:\overline{6}}\Big|_{\varphi^A(a,c)} + \frac{c^4}{2\,a^2}\ C_{:\overline{1}}\Big|_{\varphi^A(a,c)}\ ,\\
C_{:\overline{6}:\overline{\overline{6}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\ C_{:\overline{6}:\overline{6}}\Big|_{\varphi^A(a,c)} + \frac{3\,c^4}{2\,a^2}\ C_{:\overline{1}}\Big|_{\varphi^A(a,c)}\ .
\end{aligned}$$

In analogy, Eq. (6.14) provides

$$\begin{aligned}
C_{:\overline{\overline{1}}:\overline{\overline{1}}}\Big|_{\varphi^A(a,c)} &= \frac{c^{12}}{a^8}\ C_{:\overline{1}:\overline{1}}\Big|_{\varphi^A(a,c)} + \frac{c^{10}}{a^6}\ C_{:\overline{1}}\Big|_{\varphi^A(a,c)} - \frac{c^8}{4\,a^4}\ C^{\text{cosmo}}\ ,\\
C_{:\overline{\overline{1}}:\overline{\overline{4}}}\Big|_{\varphi^A(a,c)} &= \frac{c^{12}}{a^8}\ C_{:\overline{1}:\overline{4}}\Big|_{\varphi^A(a,c)} + \frac{c^{10}}{a^6}\ C_{:\overline{1}}\Big|_{\varphi^A(a,c)} + \frac{c^8}{4\,a^4}\ C^{\text{cosmo}}\ ,\\
C_{:\overline{\overline{1}}:\overline{\overline{6}}}\Big|_{\varphi^A(a,c)} &= \frac{c^{12}}{a^8}\ C_{:\overline{1}:\overline{6}}\Big|_{\varphi^A(a,c)} + \frac{c^{10}}{a^6}\ C_{:\overline{1}}\Big|_{\varphi^A(a,c)} + \frac{c^8}{4\,a^4}\ C^{\text{cosmo}}\ ,\\
C_{:\overline{\overline{4}}:\overline{\overline{4}}}\Big|_{\varphi^A(a,c)} &= \frac{c^{12}}{a^8}\ C_{:\overline{4}:\overline{4}}\Big|_{\varphi^A(a,c)} + \frac{c^{10}}{a^6}\ C_{:\overline{1}}\Big|_{\varphi^A(a,c)} - \frac{c^8}{4\,a^4}\ C^{\text{cosmo}}\ ,\\
C_{:\overline{\overline{4}}:\overline{\overline{6}}}\Big|_{\varphi^A(a,c)} &= \frac{c^{12}}{a^8}\ C_{:\overline{4}:\overline{6}}\Big|_{\varphi^A(a,c)} + \frac{c^{10}}{a^6}\ C_{:\overline{1}}\Big|_{\varphi^A(a,c)} + \frac{c^8}{4\,a^4}\ C^{\text{cosmo}}\ ,\\
C_{:\overline{\overline{6}}:\overline{\overline{6}}}\Big|_{\varphi^A(a,c)} &= \frac{c^{12}}{a^8}\ C_{:\overline{6}:\overline{6}}\Big|_{\varphi^A(a,c)} + \frac{c^{10}}{a^6}\ C_{:\overline{1}}\Big|_{\varphi^A(a,c)} - \frac{c^8}{4\,a^4}\ C^{\text{cosmo}}\ .
\end{aligned}$$

These relations alone provide information only about the second derivatives of the expansion coefficient C with respect to the configuration fields. Only together with the second prolongation of closure equation ($C5$), one might be able to read off relations which determine the unknown derivative $C_{:\overline{1}}\big|_{\varphi^A(a,c)}$.

Second prolongation of ($C5$)

Starting from the first prolongation, $(C5)_{:\overline{\overline{\overline{m}}}}$, the second prolongations $(C5)_{:\overline{\overline{\overline{m}}}:\overline{B}}$ and $(C5)_{:\overline{\overline{\overline{m}}}:\overline{\overline{B}}}$ have to be constructed. Only afterwards, symmetry is imposed. One obtains two relations

$$
\begin{aligned}
0 = {} & C_{:\overline{A}:\overline{B}}\big|_{\varphi^A(a,c)}\, M^{\overline{A}\gamma}{}_{:\overline{\overline{\overline{m}}}}\Big|_{\varphi^A(a,c)} + C_{:\overline{\overline{A}}:\overline{B}}\big|_{\varphi^A(a,c)}\, M^{\overline{\overline{A}}\gamma}{}_{:\overline{\overline{\overline{m}}}}\Big|_{\varphi^A(a,c)} \\
& + C_{:\overline{A}}\big|_{\varphi^A(a,c)}\, M^{\overline{A}\gamma}{}_{:\overline{\overline{\overline{m}}}:\overline{B}}\Big|_{\varphi^A(a,c)} + C_{:\overline{\overline{A}}}\big|_{\varphi^A(a,c)}\, M^{\overline{\overline{A}}\gamma}{}_{:\overline{\overline{\overline{m}}}:\overline{B}}\Big|_{\varphi^A(a,c)} \qquad \text{and}
\end{aligned}
\tag{6.15}
$$

$$
0 = C_{:\overline{A}:\overline{\overline{B}}}\big|_{\varphi^A(a,c)}\, M^{\overline{A}\gamma}{}_{:\overline{\overline{\overline{m}}}}\Big|_{\varphi^A(a,c)} + C_{:\overline{\overline{A}}:\overline{\overline{B}}}\big|_{\varphi^A(a,c)}\, M^{\overline{\overline{A}}\gamma}{}_{:\overline{\overline{\overline{m}}}}\Big|_{\varphi^A(a,c)} + C_{:\overline{\overline{A}}}\big|_{\varphi^A(a,c)}\, M^{\overline{\overline{A}}\gamma}{}_{:\overline{\overline{\overline{m}}}:\overline{\overline{B}}}\Big|_{\varphi^A(a,c)}\,. \tag{6.16}
$$

These two equations provide 90 relations each of which some are trivially satisfied. From a practical point of view, it is most helpful to first evaluate Eq. (6.16) as it contains only three instead of four terms. One finds

$$
C_{:\overline{1}:\overline{1}}\big|_{\varphi^A(a,c)} - C_{:\overline{1}:\overline{4}}\big|_{\varphi^A(a,c)} = -\frac{a^2}{2\,c^2}\, C_{:\overline{1}}\big|_{\varphi^A(a,c)} + \frac{a^4}{4\,c^4}\, C^{\mathrm{cosmo}}\,. \tag{6.17}
$$

The other non-trivial second derivatives $C_{:\overline{A}:\overline{B}}\big|_{\varphi^A(a,c)}$ can be expressed in terms of the two second derivatives $C_{:\overline{11}}\big|_{\varphi^A(a,c)}$ and $C_{:\overline{14}}\big|_{\varphi^A(a,c)}$ as

$$
\begin{gathered}
C_{:\overline{1}:\overline{1}}\big|_{\varphi^A(a,c)} = C_{:\overline{4}:\overline{4}}\big|_{\varphi^A(a,c)} = C_{:\overline{6}:\overline{6}}\big|_{\varphi^A(a,c)} \quad , \quad C_{:\overline{1}:\overline{4}}\big|_{\varphi^A(a,c)} = C_{:\overline{1}:\overline{6}}\big|_{\varphi^A(a,c)} = C_{:\overline{4}:\overline{6}}\big|_{\varphi^A(a,c)} \quad \text{and} \\
C_{:\overline{2}:\overline{2}}\big|_{\varphi^A(a,c)} = C_{:\overline{3}:\overline{3}}\big|_{\varphi^A(a,c)} = C_{:\overline{5}:\overline{5}}\big|_{\varphi^A(a,c)} = C_{:\overline{1}:\overline{1}}\big|_{\varphi^A(a,c)} - C_{:\overline{1}:\overline{4}}\big|_{\varphi^A(a,c)}\,.
\end{gathered}
$$

From the analysis of Eq. (6.16), one obtains the same relations and using the relations from the first prolongation of ($C1$), one can express all second derivatives of the form $C_{:\overline{\overline{A}}:\overline{B}}\Big|_{\varphi^A(a,c)}$ and $C_{:\overline{\overline{A}}:\overline{\overline{B}}}\Big|_{\varphi^A(a,c)}$ by the two independent second derivatives $C_{:\overline{1}:\overline{1}}\big|_{\varphi^A(a,c)}$ and $C_{:\overline{1}:\overline{4}}\big|_{\varphi^A(a,c)}$ as well as the undetermined lower order derivative $C_{:\overline{1}}\big|_{\varphi^A(a,c)}$ and the coefficient C^{cosmo}. Clearly, the first prolongation of ($C1$) and the second one of ($C5$) provided only information about the second derivatives of C, but not about the first derivative $C_{:\overline{1}}\big|_{\varphi^A(a,c)}$. This first derivatives needs to be determined in order to restrict the solution (6.11) for the expansion coefficient C^{cosmo}.

Summary

The only closure equations dealing with the expansion coefficient C^{cosmo} and its derivatives $C_{:\overline{A}}\big|_{\varphi^A(a,c)}$ are ($C1$) and the first prolongation of ($C5$). These equations determine the dependence of C^{cosmo} on the first scale factor $a(t)$, but not the dependence of C^{cosmo} on the second scale factor $c(t)$.

The first prolongation of ($C1$) and a second prolongation of ($C5$) provide relations for the second derivatives $C_{:A:B}\big|_{\varphi^A(a,c)}$. All second derivatives of C^{cosmo} are expressed in terms of two independent, but unknown derivatives $C_{:\overline{1}:\overline{1}}\big|_{\varphi^A(a,c)}$ and $C_{:\overline{1}:\overline{4}}\big|_{\varphi^A(a,c)}$. Besides leaving these two second derivatives undetermined, they also provide no new information on either $C_{:\overline{1}}\big|_{\varphi^A(a,c)}$ or C^{cosmo} itself.

Further prolongations of the two closure equations are not expected to reveal new information on the desired first derivative $C_{:\overline{1}}$. The N-th prolongation of ($C1$) with symmetry afterwards imposed is

$$
0 = C_{:B_1\cdots:B_N}\big|_{\varphi^A(a,c)}\,\delta^\gamma_\mu + N\, C_{:A:(B_1\cdots:B_{N-1}|}\big|_{\varphi^A(a,c)}\, F^A{}_\mu{}^\gamma{}_{:|B_N)}\Big|_{\varphi^A(a,c)} + C_{:A:B_1\cdots:B_N}\big|_{\varphi^A(a,c)}\, F^A{}_\mu{}^\gamma\Big|_{\varphi^A(a,c)}\,.
$$

It expresses $(N+1)$-th order derivatives in terms of the purely single-overlined ones $C_{:\overline{B}_1\cdots:\overline{B}_{N+1}}$ and N-th order derivatives $C_{:\overline{B}_1\cdots:\overline{B}_N}$. The $N+1$-th prolongation $(C5)_{:\overline{\overline{m}}:B_1\cdots:B_N}$ can be written as

$$0 = \sum_{K=0}^{N} \binom{N}{K} \, C_{:A:(B_1\cdots:B_K|}\Big|_{\varphi^A(a,c)} \, M^{A\gamma}{}_{:\overline{\overline{m}}|:B_{K+1}\cdots:B_N)}\Big|_{\varphi^A(a,c)} \, ,$$

where the capital indices range only over $\overline{B}$ and $\overline{\overline{B}}$. Due to the structure of the M-coefficient, these relations provide information only about the difference between two N^{th}-order derivatives, but not about the actual derivatives, as it was already seen in equation (6.17) for the case of $N = 2$. Thus, for every derivative order picked up by a prolongation of the closure equations $(C1)$ and $(C5)_{:\overline{\overline{m}}}$, there are new undetermined derivatives. This hampers the goal of determining the first derivative $C_{:\overline{1}}\big|_{\varphi^A(a,c)}$ required for the solution of the differential equation (6.12). Instead, there remains an undetermined function $f_0(c)$ in the solution of the first expansion coefficient C^{cosmo},

$$C^{\text{cosmo}} = f_0(c)\, a^3 \, .$$

6.3 Solution for the second expansion coefficient

The solution of the first expansion coefficient C^{cosmo} was the most simplest as there were only two closure equations involved after symmetry was imposed. Additionally, as the coefficient C^{cosmo} has no capital index, the amount of single relations is manageable. For the other expansion coefficients, each additional capital index accounts for more and more single relations which need to be evaluated. While this is not yet hindering for the solution of the second expansion coefficient C^{cosmo}_A, one will already have to evaluate significantly more relations than in the previous section.

For the second expansion coefficient C^{cosmo}_A, one will have to investigate three closure equations and their respective prolongations. Following the solution for the first expansion coefficient C^{cosmo}, one identifies closure equation $(C2)$ as the analog of $(C1)$. Closure equation $(C2)$ provides relations between the expansion coefficient C^{cosmo}_A and the derivatives $C_{A:B}|_{\varphi^A(a,c)}$.

Analysis of closure equation $(C2)$

The closure equations simplify as the spatial derivatives vanish after imposition of symmetry. This provides the symmetry-reduced equation $(C2)|_{\varphi^A(a,c)}$ as

$$0 = C^{\text{cosmo}}_B\, \delta^\gamma_\mu + C^{\text{cosmo}}_A \;\; F^A{}_\mu{}^\gamma{}_{:B}\Big|_{\varphi^A(a,c)} + C_{B:A} \;\; F^A{}_\mu{}^\gamma\Big|_{\varphi^A(a,c)} \tag{6.18}$$

Restricting the free capital index B to the range $\overline{B}$ results in 54 relations. One reads off that three components vanish,

$$C^{\text{cosmo}}_{\overline{2}} = C^{\text{cosmo}}_{\overline{3}} = C^{\text{cosmo}}_{\overline{5}} = 0 \, ,$$

while the other three components are equal,

$$C^{\text{cosmo}}_{\overline{1}} = C^{\text{cosmo}}_{\overline{4}} = C^{\text{cosmo}}_{\overline{6}} \, .$$

Thus, one only needs to determine the component $C^{\text{cosmo}}_{\overline{1}}$ and the other two non-trivial components are identical to it. For this component, closure equation $(C2)|_{\varphi^A(a,c)}$ implies the following relations,

$$\begin{aligned}
C_{\overline{1}:\overline{\overline{1}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\; C_{\overline{1}:\overline{1}}\Big|_{\varphi^A(a,c)} + \frac{3\,c^4}{2\,a^2}\, C^{\text{cosmo}}_{\overline{1}} \, , \\
C_{\overline{1}:\overline{\overline{4}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\; C_{\overline{1}:\overline{4}}\Big|_{\varphi^A(a,c)} + \frac{c^4}{2\,a^2}\, C^{\text{cosmo}}_{\overline{1}} \quad \text{and} \\
C_{\overline{1}:\overline{\overline{6}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\; C_{\overline{1}:\overline{6}}\Big|_{\varphi^A(a,c)} + \frac{c^4}{2\,a^2}\, C^{\text{cosmo}}_{\overline{1}} \, .
\end{aligned}$$

These are the relevant derivatives appearing in the chain rule equations (6.4) and (6.5) for the component $C^{\text{cosmo}}_{\bar{1}}$. Subsequent insertion into the chain rule equations provides the two differential equations

$$\frac{\partial C^{\text{cosmo}}_{\bar{1}}}{\partial a} = \frac{5}{a}\, C^{\text{cosmo}}_{\bar{1}} \quad \text{and} \quad \frac{\partial C^{\text{cosmo}}_{\bar{1}}}{\partial c} = -\frac{10}{c}\, C^{\text{cosmo}}_{\bar{1}} - \frac{2\,c}{a^2}\left(C_{\bar{1}:\bar{1}}\big|_{\varphi^A(a,c)} + C_{\bar{1}:\bar{\bar{4}}}\big|_{\varphi^A(a,c)} + C_{\bar{1}:\bar{\bar{6}}}\big|_{\varphi^A(a,c)} \right).$$

One can only solve the first equation and obtains a second undetermined function of the second scale factor c in the solution,

$$C^{\text{cosmo}}_{\bar{1}} = f_1(c)\, a^5\,.$$

In analogy, restricting the capital index B to the range $\bar{\bar{B}}$ in Eq. (6.18) again results in 54 relations now containing the components $C^{\text{cosmo}}_{\bar{\bar{B}}}$. In the same way as before, three components

$$C^{\text{cosmo}}_{\bar{\bar{2}}} = C^{\text{cosmo}}_{\bar{\bar{3}}} = C^{\text{cosmo}}_{\bar{\bar{5}}} = 0$$

vanish and the other three components are equal,

$$C^{\text{cosmo}}_{\bar{\bar{1}}} = C^{\text{cosmo}}_{\bar{\bar{4}}} = C^{\text{cosmo}}_{\bar{\bar{6}}}\,,$$

which means that only $C^{\text{cosmo}}_{\bar{\bar{1}}}$ needs to be determined. One extracts the three relations

$$\begin{aligned}
C_{\bar{\bar{1}}:\bar{\bar{1}}}\big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\; C_{\bar{\bar{1}}:\bar{1}}\big|_{\varphi^A(a,c)} - \frac{c^4}{2\,a^2}\, C^{\text{cosmo}}_{\bar{1}} \\
C_{\bar{\bar{1}}:\bar{\bar{4}}}\big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\; C_{\bar{\bar{1}}:\bar{4}}\big|_{\varphi^A(a,c)} + \frac{c^4}{2\,a^2}\, C^{\text{cosmo}}_{\bar{1}} \quad \text{and} \\
C_{\bar{\bar{1}}:\bar{\bar{6}}}\big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\; C_{\bar{\bar{1}}:\bar{6}}\big|_{\varphi^A(a,c)} + \frac{c^4}{2\,a^2}\, C^{\text{cosmo}}_{\bar{1}}\,.
\end{aligned}$$

Inserting these relations into the chain rule equations (6.4) and (6.5) provides the solution

$$C^{\text{cosmo}}_{\bar{\bar{1}}} = \hat{f}_1(c)\, a$$

with a third undetermined function $\hat{f}_1(\text{c})$. While it at first seems that the coefficient C^{cosmo}_A features two independent components $C^{\text{cosmo}}_{\bar{1}}$ and $C^{\text{cosmo}}_{\bar{\bar{1}}}$, it is a prolongation of other closure equations which will show that one component can actually be expressed by the other. The closure equations which have to inspected for that are $(C4)$ and again a prolongation of $(C5)$. For this case here, a different prolongation than in the previous section needs to be studied.

Analysis of closure equation $(C4)$

Closure equation $(C4)$ contains the expansion coefficient C_A together with a derivative of the input coefficient $M^{A\gamma}$. Of the remaining terms, all but one drop out of the equation since all spatial derivatives vanish after symmetry has been imposed. One is thus left with

$$0 = C^{\text{cosmo}}_A\; M^{A\mu}{}_{:B}\big|_{\varphi^A(a,c)} + C_{:B}{}^{\mu}\big|_{\varphi^A(a,c)}\,. \tag{6.19}$$

While this equation contains the desired expansion coefficient C^{cosmo}_A, it also features a derivative of the first expansion coefficient C. The equation is most reasonably analyzed by splitting the index range of the free capital index B.

Case 1: $B = \overline{\overline{\overline{m}}}$

If the range of the capital index B is restricted to the values of $\overline{\overline{\overline{m}}}$, the components of C^{cosmo}_{A} which will contribute to the cosmological spacetime action (6.6) appear in the equation,

$$0 = C^{\text{cosmo}}_{\overline{A}} \; M^{\overline{A}\mu}{}_{:\overline{\overline{\overline{m}}}}\Big|_{\varphi^A(a,c)} + C^{\text{cosmo}}_{\overline{\overline{A}}} \; M^{\overline{\overline{A}}\mu}{}_{:\overline{\overline{\overline{m}}}}\Big|_{\varphi^A(a,c)} + C_{:\overline{\overline{\overline{m}}}}{}^{\mu}\Big|_{\varphi^A(a,c)} \,.$$

One uses the expression of the derivative of $M^{A\gamma}$ from Section 6.1 as well as the results from closure equation $(C2)|_{\varphi^A(a,c)}$ which revealed that there are only two non-trivial, independent components, $C^{\text{cosmo}}_{\overline{1}}$ and $C^{\text{cosmo}}_{\overline{\overline{1}}}$ in order to see that all terms containing the coefficient C^{cosmo}_{A} drop out of this equation. Thus, one reads off

$$C_{:\overline{\overline{\overline{m}}}}{}^{\mu}\Big|_{\varphi^A(a,c)} = 0\,,$$

which provides no new information on neither the desired components of C^{cosmo}_{A} nor about the first expansion coefficient C^{cosmo} as this derivative was not needed for its solution.

Case 2: $B = \overline{B}$ **and** $B = \overline{\overline{B}}$

In case the capital index B of Eq. (6.19) ranges over $\overline{B}$ or $\overline{\overline{B}}$, the equation is

$$\begin{aligned} 0 &= C^{\text{cosmo}}_{\overline{\overline{\overline{m}}}} \; M^{\overline{\overline{\overline{m}}}\mu}{}_{:\overline{B}}\Big|_{\varphi^A(a,c)} + C_{:\overline{B}}{}^{\mu}\Big|_{\varphi^A(a,c)} \quad \text{and} \\ 0 &= C^{\text{cosmo}}_{\overline{\overline{\overline{m}}}} \; M^{\overline{\overline{\overline{m}}}\mu}{}_{:\overline{\overline{B}}}\Big|_{\varphi^A(a,c)} + C_{:\overline{\overline{B}}}{}^{\mu}\Big|_{\varphi^A(a,c)} \,, \end{aligned}$$

respectively. Studying closure equation $(C2)|_{\varphi^A(a,c)}$ for the free index B ranging in $\overline{\overline{\overline{m}}}$ reveals that the components $C^{\text{cosmo}}_{\overline{\overline{\overline{m}}}}$ actually vanish. These do not contribute to the cosmological spacetime action (6.6) anyway. The above two equations stemming from closure equation $(C4)|_{\varphi^A(a,c)}$ still simplify and reveal that

$$C_{:\overline{B}}{}^{\mu}\Big|_{\varphi^A(a,c)} = 0 \quad \text{and} \quad C_{:\overline{\overline{B}}}{}^{\mu}\Big|_{\varphi^A(a,c)} = 0\,.$$

This means that closure equation $(C4)$ provides no relations for the expansion coefficient C^{cosmo}_{A}, but rather determines the derivative

$$C_{:B}{}^{\mu}\Big|_{\varphi^A(a,c)} = 0$$

of the first expansion coefficient C. This result will become important in the analysis of a prolongation of closure equation $(C5)$ which will then provide new information about the expansion coefficient C^{cosmo}_{A}.

First prolongation of $(C5)$

It is again closure equation $(C5)$ which plays a crucial role in extracting further information about the expansion coefficient C^{cosmo}_{A}. In contrast to the analysis of coefficient C^{cosmo}, it is this time the prolongation $(C5)_{:D}{}^{\kappa}|_{\varphi^A(a,c)}$ which will provide a relation between the two components $C^{\text{cosmo}}_{\overline{1}}$ and $C^{\text{cosmo}}_{\overline{\overline{1}}}$. The prolongation with imposed symmetry is

$$\begin{aligned} 0 = {} & 2\, C^{\text{cosmo}}_{A} \; M^{A[\kappa|}{}_{:B}\Big|_{\varphi^A(a,c)} \, M^{B|\gamma]}{}_{:D}\Big|_{\varphi^A(a,c)} + 6\, C^{\text{cosmo}}_{D} \; p^{\kappa\gamma}\Big|_{\varphi^A(a,c)} - 6\, C^{\text{cosmo}}_{A} \; F^{A}{}_{\rho}{}^{\kappa}{}_{:D}\Big|_{\varphi^A(a,c)} \, p^{\rho\gamma}\Big|_{\varphi^A(a,c)} \\ & - 6\; C_{A:D}|_{\varphi^A(a,c)} \, F^{A}{}_{\rho}{}^{\kappa}\Big|_{\varphi^A(a,c)} \, p^{\rho\gamma}\Big|_{\varphi^A(a,c)} + 2\; C_{:A}{}^{[\kappa|}\Big|_{\varphi^A(a,c)} \, M^{A|\gamma]}{}_{:D}\Big|_{\varphi^A(a,c)} \,, \end{aligned}$$

from which the first and the last term drop out using the previously studied closure equation $(C4)|_{\varphi^A(a,c)}$. Restricting the range of the free capital index D to $\overline{D}$ and $\overline{\overline{D}}$, the equation simplifies to the two respective equations,

$$
\begin{aligned}
0 &= C^{\text{cosmo}}_{\overline{D}}\, p^{\kappa\gamma}\big|_{\varphi^A(a,c)} - C^{\text{cosmo}}_{\overline{A}}\, F^{\overline{A}}{}_{\rho}{}^{\kappa}{}_{:\overline{D}}\Big|_{\varphi^A(a,c)}\, p^{\rho\gamma}\big|_{\varphi^A(a,c)} - C_{A:\overline{D}}\big|_{\varphi^A(a,c)}\, F^{A}{}_{\rho}{}^{\kappa}\big|_{\varphi^A(a,c)}\, p^{\rho\gamma}\big|_{\varphi^A(a,c)} \quad \text{and} \\
0 &= C^{\text{cosmo}}_{\overline{\overline{D}}}\, p^{\kappa\gamma}\big|_{\varphi^A(a,c)} - C^{\text{cosmo}}_{\overline{A}}\, F^{\overline{\overline{A}}}{}_{\rho}{}^{\kappa}{}_{:\overline{\overline{D}}}\Big|_{\varphi^A(a,c)}\, p^{\rho\gamma}\big|_{\varphi^A(a,c)} - C_{A:\overline{\overline{D}}}\big|_{\varphi^A(a,c)}\, F^{A}{}_{\rho}{}^{\kappa}\big|_{\varphi^A(a,c)}\, p^{\rho\gamma}\big|_{\varphi^A(a,c)}\,.
\end{aligned}
$$

These equations each account for 54 relations. From the first equation, one obtains the three relations

$$
\begin{aligned}
C_{\overline{\overline{1}}:\overline{1}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\; C_{\overline{1}:\overline{1}}\Big|_{\varphi^A(a,c)} + \frac{c^4}{2\,a^2}\, C^{\text{cosmo}}_{\overline{1}}\,, \\
C_{\overline{\overline{1}}:\overline{4}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\; C_{\overline{1}:\overline{4}}\Big|_{\varphi^A(a,c)} - \frac{c^4}{2\,a^2}\, C^{\text{cosmo}}_{\overline{1}} \quad \text{and} \\
C_{\overline{\overline{1}}:\overline{6}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\; C_{\overline{1}:\overline{6}}\Big|_{\varphi^A(a,c)} - \frac{c^4}{2\,a^2}\, C^{\text{cosmo}}_{1}\,,
\end{aligned}
$$

while the second equation provides

$$
\begin{aligned}
C_{\overline{\overline{1}}:\overline{\overline{1}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\; C_{\overline{1}:\overline{\overline{1}}}\Big|_{\varphi^A(a,c)} - \frac{3\,c^4}{2\,a^2} C^{\text{cosmo}}_{\overline{\overline{1}}}\,, \\
C_{\overline{\overline{1}}:\overline{\overline{4}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\; C_{\overline{1}:\overline{\overline{4}}}\Big|_{\varphi^A(a,c)} - \frac{c^4}{2\,a^2}\, C^{\text{cosmo}}_{\overline{\overline{1}}} \quad \text{and} \\
C_{\overline{\overline{1}}:\overline{\overline{6}}}\Big|_{\varphi^A(a,c)} &= \frac{c^6}{a^4}\; C_{\overline{1}:\overline{\overline{6}}}\Big|_{\varphi^A(a,c)} - \frac{c^4}{2\,a^2}\, C^{\text{cosmo}}_{\overline{\overline{1}}}\,.
\end{aligned}
$$

Recall that closure equation $(C2)|_{\varphi^A(a,c)}$ provided the following relations,

$$
C_{\overline{\overline{1}}:\overline{\overline{1}}}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\; C_{\overline{\overline{1}}:\overline{1}}\Big|_{\varphi^A(a,c)} - \frac{c^4}{2\,a^2}\, C^{\text{cosmo}}_{\overline{\overline{1}}} \quad \text{and} \quad C_{\overline{1}:\overline{\overline{1}}}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\; C_{\overline{1}:\overline{1}}\Big|_{\varphi^A(a,c)} + \frac{3\,c^4}{2\,a^2}\, C^{\text{cosmo}}_{\overline{1}}\,.
$$

Combining the relations of $(C5)_{:D}{}^{\kappa}|_{\varphi^A(a,c)}$ and $(C2)|_{\varphi^A(a,c)}$, one concludes

$$
C^{\text{cosmo}}_{\overline{\overline{1}}} = \frac{c^6}{a^4}\, C^{\text{cosmo}}_{\overline{1}}\,.
$$

Thus, only component C^{cosmo}_{1} is independent and needs to be determined. All other components can either be expressed by it or vanish anyway. In particular, this also means the undetermined function $\hat{f}_1(c)$ which appeared in the solution of $C^{\text{cosmo}}_{\overline{\overline{1}}}$ can actually be expressed as $\hat{f}_1(c) = f_1(c)\, c^6$. The undetermined function $f_1(c)$ of the component $C^{\text{cosmo}}_{\overline{1}}$ remains in the solution. One can investigate whether additional prolongations will provide any relations which determine it.

First prolongation of $(C2)$

Symmetry-reduced closure equation $(C2)|_{\varphi^A(a,c)}$ provided the relations for the solution of the component $C^{\text{cosmo}}_{\overline{1}}$, the only independent component of expansion coefficient C^{cosmo}_{A}. Just as in the previous analysis, it might be worth to study the prolongations $(C2)_{:\overline{D}}|_{\varphi^A(a,c)}$ and $(C2)_{:\overline{\overline{D}}}|_{\varphi^A(a,c)}$. These might provide additional information about the first derivatives $C_{\overline{1}:A}\big|_{\varphi^A(a,c)}$. If all of the derivatives can be expressed, the free function f_1 can be determined.

For the respective index ranges of the free capital index B of $(C2)$, these equations are

$$
\begin{aligned}
0 =\; & C_{\bar{B}:\bar{D}:A}\big|_{\varphi^A(a,c)}\, F^{A}{}_{\mu}{}^{\gamma}\big|_{\varphi^A(a,c)} + C_{\bar{A}:\bar{D}}\big|_{\varphi^A(a,c)}\, F^{\bar{A}}{}_{\mu}{}^{\gamma}{}_{:\bar{B}}\big|_{\varphi^A(a,c)} + C_{\bar{B}:\bar{A}}\big|_{\varphi^A(a,c)}\, F^{\bar{A}}{}_{\mu}{}^{\gamma}{}_{:\bar{D}}\big|_{\varphi^A(a,c)} \\
& + C_{\bar{B}:\bar{D}}\big|_{\varphi^A(a,c)}\, \delta^{\gamma}_{\mu}\,, \\
0 =\; & C_{\bar{B}:\bar{\bar{D}}:A}\big|_{\varphi^A(a,c)}\, F^{A}{}_{\mu}{}^{\gamma}\big|_{\varphi^A(a,c)} + C_{\bar{A}:\bar{\bar{D}}}\big|_{\varphi^A(a,c)}\, F^{\bar{A}}{}_{\mu}{}^{\gamma}{}_{:\bar{B}}\big|_{\varphi^A(a,c)} + C_{\bar{B}:\bar{\bar{A}}}\big|_{\varphi^A(a,c)}\, F^{\bar{\bar{A}}}{}_{\mu}{}^{\gamma}{}_{:\bar{\bar{D}}}\big|_{\varphi^A(a,c)} \\
& + C_{\bar{B}:\bar{\bar{D}}}\big|_{\varphi^A(a,c)}\, \delta^{\gamma}_{\mu}\,, \\
0 =\; & C_{\bar{\bar{B}}:\bar{D}:A}\big|_{\varphi^A(a,c)}\, F^{A}{}_{\mu}{}^{\gamma}\big|_{\varphi^A(a,c)} + C_{\bar{\bar{A}}:\bar{D}}\big|_{\varphi^A(a,c)}\, F^{\bar{\bar{A}}}{}_{\mu}{}^{\gamma}{}_{:\bar{\bar{B}}}\big|_{\varphi^A(a,c)} + C_{\bar{\bar{B}}:\bar{A}}\big|_{\varphi^A(a,c)}\, F^{\bar{A}}{}_{\mu}{}^{\gamma}{}_{:\bar{D}}\big|_{\varphi^A(a,c)} \\
& + C_{\bar{\bar{B}}:\bar{D}}\big|_{\varphi^A(a,c)}\, \delta^{\gamma}_{\mu} \quad \text{and} \\
0 =\; & C_{\bar{\bar{B}}:\bar{\bar{D}}:A}\big|_{\varphi^A(a,c)}\, F^{A}{}_{\mu}{}^{\gamma}\big|_{\varphi^A(a,c)} + C_{\bar{\bar{A}}:\bar{\bar{D}}}\big|_{\varphi^A(a,c)}\, F^{\bar{\bar{A}}}{}_{\mu}{}^{\gamma}{}_{:\bar{\bar{B}}}\big|_{\varphi^A(a,c)} + C_{\bar{\bar{B}}:\bar{\bar{A}}}\big|_{\varphi^A(a,c)}\, F^{\bar{\bar{A}}}{}_{\mu}{}^{\gamma}{}_{:\bar{\bar{D}}}\big|_{\varphi^A(a,c)} \\
& + C_{\bar{\bar{B}}:\bar{\bar{D}}}\big|_{\varphi^A(a,c)}\, \delta^{\gamma}_{\mu}\,.
\end{aligned}
$$

Each equation provides 324 single relations. Evaluating these relations for the first equation, one concludes that there are only two independent derivatives of the form $C_{\bar{1}:A}\big|_{\varphi^A(a,c)}$, namely

$$C_{\bar{1}:\bar{1}}\big|_{\varphi^A(a,c)} \quad \text{and} \quad C_{\bar{1}:\bar{4}}\big|_{\varphi^A(a,c)}\,.$$

All other non-trivial derivatives can be expressed in terms of those two,

$$
\begin{aligned}
& C_{\bar{1}:\bar{4}}\big|_{\varphi^A(a,c)} = C_{\bar{1}:\bar{6}}\big|_{\varphi^A(a,c)} = C_{\bar{4}:\bar{1}}\big|_{\varphi^A(a,c)} = C_{\bar{4}:\bar{6}}\big|_{\varphi^A(a,c)} = C_{\bar{6}:\bar{1}}\big|_{\varphi^A(a,c)} = C_{\bar{6}:\bar{4}}\big|_{\varphi^A(a,c)}\,, \\
& C_{\bar{1}:\bar{1}}\big|_{\varphi^A(a,c)} = C_{\bar{4}:\bar{4}}\big|_{\varphi^A(a,c)} = C_{\bar{6}:\bar{6}}\big|_{\varphi^A(a,c)} \quad \text{and} \\
& C_{\bar{2}:\bar{2}}\big|_{\varphi^A(a,c)} = C_{\bar{3}:\bar{3}}\big|_{\varphi^A(a,c)} = C_{\bar{5}:\bar{5}}\big|_{\varphi^A(a,c)} = C_{\bar{1}:\bar{1}}\big|_{\varphi^A(a,c)} - C_{\bar{1}:\bar{4}}\big|_{\varphi^A(a,c)}\,.
\end{aligned}
$$

The remaining three equations merely provide relations which are consistent with the ones from the first equation. In particular, they provide no further information about the two independent derivatives $C_{\bar{1}:\bar{1}}\big|_{\varphi^A(a,c)}$ and $C_{\bar{1}:\bar{4}}\big|_{\varphi^A(a,c)}$. These however need to be determined in order to solve the second chain rule equation for $C^{\text{cosmo}}_{\bar{1}}$. Thus, also after evaluating the first prolongation of $(C2)$, one remains with the solution

$$C^{\text{cosmo}}_{\bar{1}} = f_1(c)\, a^5\,.$$

As already shown in this section, there are further closure equations containing the expansion coefficient C_A. One therefore studies whether prolongations of these equations might reveal relations for the two undetermined derivatives $C_{\bar{1}:\bar{1}}\big|_{\varphi^A(a,c)}$ and $C_{\bar{1}:\bar{4}}\big|_{\varphi^A(a,c)}$.

Further closure equations for C_A

The important criterion for a closure equation to be taken into account here is that it needs to contain either the expansion coefficient C^{cosmo}_{A} itself or the derivative of C_A with respect to the configuration fields φ^B. Clearly, this is the case for $(C2)$, $(C4)$ and the aforementioned prolongation of $(C5)$. While an extra piece of information could be extracted from the first prolongation of $(C2)$, it is not expected to gain more information from the second prolongation as it will only provide more relations for the second derivatives $C_{A:B_1:B_2}\big|_{\varphi^A(a,c)}$.

One might as well argue that further prolongations of closure equation $(C4)$ might reveal new information about the desired derivatives of C_A. The first prolongation $(C4)_{:D}|_{\varphi^A(a,c)}$ provides

$$
\begin{aligned}
0 = & - C_{A:D}|_{\varphi^A(a,c)}\, M^{A\mu}{}_{:B}\big|_{\varphi^A(a,c)} - C^{\text{cosmo}}_{A}\, M^{A\mu}{}_{:B:D}\big|_{\varphi^A(a,c)} \\
& - C_{B:A}|_{\varphi^A(a,c)}\, M^{A\mu}{}_{:D}\big|_{\varphi^A(a,c)} - C_{:B}{}^{\mu}{}_{:D}\big|_{\varphi^A(a,c)}\,.
\end{aligned}
$$

This equation only determines the second derivative of the expansion coefficient C which appears as the last summand. Further prolongations – also the first ones with respect to $\varphi^D{}_{,\nu}$ or $\varphi^D{}_{,\nu_1\nu_2}$ – either contain the not yet determined expansion coefficient C^{cosmo}_{AB} or derivatives of C_A which are not relevant for the actual goal of determining the component $C^{\text{cosmo}}_{\bar{1}}$. An expression for the two undetermined derivatives $C_{\bar{1}:\bar{1}}\big|_{\varphi^A(a,c)}$ and $C_{\bar{1}:\bar{4}}\big|_{\varphi^A(a,c)}$ is not obtained.

The actually most promising prolongation of a closure equation is the second prolongation of closure equation $(C5)$ in the form

$$(C5)_{:D}{}^{\lambda}{}_{:E}|_{\varphi^A(a,c)}|_{(\lambda\gamma)}$$

where only the symmetric part of the equation – with respect to the spatial indices – is taken into account; the antisymmetric part contains quite involved derivatives of the input coefficient $M^{A\gamma}$ and an undetermined derivative of the first expansion coefficient C. One obtains 824 single relations even if one restricts the index range of the two free capital indices to $\overline{D}$, $\overline{\overline{D}}$, $\overline{E}$ and $\overline{\overline{E}}$. Evaluating the 824 relations by hand is almost impossible and as one cannot expect success, this task should be delegated to a suitable and reliable computer algebra program. Implementing such a program exceeded the scope of this thesis and remains a task for future work and research on this topic.

Further closure equations deal only with derivatives of the expansion coefficient C_A that are not relevant for the solution of $C^{\text{cosmo}}_{\bar{1}}$. Therefore, one is left with the solution

$$\begin{aligned}
C^{\text{cosmo}}_{\bar{1}} &= C^{\text{cosmo}}_{\bar{4}} = C^{\text{cosmo}}_{\bar{6}} = f_1(c)\, a^5 \quad \text{and} \\
C^{\text{cosmo}}_{\bar{\bar{1}}} &= C^{\text{cosmo}}_{\bar{\bar{4}}} = C^{\text{cosmo}}_{\bar{\bar{6}}} = f_1(c)\, c^6\, a\,,
\end{aligned}$$

which leaves a second undetermined function f_1 in the cosmological spacetime action (6.6) after previously obtaining the first function f_0 in the solution of the first expansion coefficient C^{cosmo}.

Inspecting the contribution of the expansion coefficient C^{cosmo}_A to the cosmological spacetime action (6.6), one finds the expression

$$C^{\text{cosmo}}_A\, \dot{\varphi}^A = -6\, f_1(c)\, c\, \dot{c}\, a^3\,,$$

which allows to define a new function $F_1(c)$ as

$$F_1(c) := -6\, f_1(c)\, c\,.$$

Thus, the contribution to the cosmological spacetime action can be written more compactly as

$$C^{\text{cosmo}}_A\, \dot{\varphi}^A = F_1(c)\, \dot{c}\, a^3\,.$$

This result improves previous results, e. g. from Ref. [24] as it shows that the coefficient C^{cosmo}_A contributes only one undetermined function to the cosmological spacetime action. This result could only be achieved by taking prolongations of the closure equations into account.

6.4 Solution for the third expansion coefficient

The next coefficient to be determined is the third expansion coefficient C^{cosmo}_{AB}. There are even more closure equations involved in its determination. Next to closure equation $(C10)_2$ which is an equation only for C^{cosmo}_{AB} and its derivatives, it is particularly closure equation $(C3)|_{\varphi^A(a,c)}$,

$$0 = 2\, C^{\text{cosmo}}_{AB}\ F^A{}_{\rho}{}^{(\mu}\big|_{\varphi^A(a,c)}\, p^{\nu)\rho}\big|_{\varphi^A(a,c)} - C_{:B}{}^{\mu\nu}\big|_{\varphi^A(a,c)}$$

which relates the expansion coefficient C^{cosmo}_{AB} with a not yet determined derivative of the first expansion coefficient C. This derivative is part of other closure equations, most prominently in $(C8_3)|_{\varphi^A(a,c)}$. Additionally, a prolongation of closure equation $(C21_3)$ will have to be used in order to gain more information about this derivative.

Closure equation $(C8)_3$

Due to the collapse (6.8) of C to second derivative order, closure equation $(C8_3)|_{\varphi^A(a,c)}$ simplifies to

$$0 = C_{:A}{}^{(\beta_1\beta_2|}\Big|_{\varphi^A(a,c)} \, F^A{}_\mu{}^{|\beta_3)}\Big|_{\varphi^A(a,c)} \, ,$$

which provides relations for the derivatives $C_{:A}{}^{\mu\nu}|_{\varphi^A(a,c)}$ of the first expansion coefficient C. This equation amounts for 30 independent relations which leaves the system of 72 independent derivatives $C_{:\bar{B}}{}^{\mu\nu}\Big|_{\varphi^A(a,c)}$ and $C_{:\bar{\bar{B}}}{}^{\mu\nu}\Big|_{\varphi^A(a,c)}$ underdetermined. Yet, one finds 15 relations relating the exact single- and double-overlined counterparts of derivatives with each other,

$$\begin{aligned}
&C_{:\bar{\bar{1}}}{}^{xx}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\, C_{:\bar{1}}{}^{xx}\Big|_{\varphi^A(a,c)} \, , \; C_{:\bar{\bar{1}}}{}^{xy}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\, C_{:\bar{1}}{}^{xy}\Big|_{\varphi^A(a,c)} \, , \; C_{:\bar{\bar{1}}}{}^{xz}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\, C_{:\bar{1}}{}^{xz}\Big|_{\varphi^A(a,c)} \\
&C_{:\bar{\bar{2}}}{}^{xx}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\, C_{:\bar{2}}{}^{xx}\Big|_{\varphi^A(a,c)} \, , \; C_{:\bar{\bar{2}}}{}^{yy}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\, C_{:\bar{2}}{}^{yy}\Big|_{\varphi^A(a,c)} \\
&C_{:\bar{\bar{3}}}{}^{xx}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\, C_{:\bar{3}}{}^{xx}\Big|_{\varphi^A(a,c)} \, , \; C_{:\bar{\bar{3}}}{}^{zz}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\, C_{:\bar{3}}{}^{zz}\Big|_{\varphi^A(a,c)} \\
&C_{:\bar{\bar{4}}}{}^{xy}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\, C_{:\bar{4}}{}^{xy}\Big|_{\varphi^A(a,c)} \, , \; C_{:\bar{\bar{4}}}{}^{yy}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\, C_{:\bar{4}}{}^{yy}\Big|_{\varphi^A(a,c)} \, , \; C_{:\bar{\bar{4}}}{}^{yz}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\, C_{:\bar{4}}{}^{yz}\Big|_{\varphi^A(a,c)} \\
&C_{:\bar{\bar{5}}}{}^{yy}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\, C_{:\bar{5}}{}^{yy}\Big|_{\varphi^A(a,c)} \, , \; C_{:\bar{\bar{5}}}{}^{zz}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\, C_{:\bar{5}}{}^{zz}\Big|_{\varphi^A(a,c)} \\
&C_{:\bar{\bar{6}}}{}^{xz}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\, C_{:\bar{6}}{}^{xz}\Big|_{\varphi^A(a,c)} \, , \; C_{:\bar{\bar{6}}}{}^{yz}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\, C_{:\bar{6}}{}^{yz}\Big|_{\varphi^A(a,c)} \, , \; C_{:\bar{\bar{6}}}{}^{zz}\Big|_{\varphi^A(a,c)} = \frac{c^6}{a^4}\, C_{:\bar{6}}{}^{zz}\Big|_{\varphi^A(a,c)} \, .
\end{aligned}$$

Besides, there are 12 relations with four terms; these relate two pairs of single- and double-overlined derivatives with each other,

$$\begin{aligned}
0 &= \frac{c^6}{a^4}\, C_{:\bar{4}}{}^{xx}\Big|_{\varphi^A(a,c)} - C_{:\bar{\bar{4}}}{}^{xx}\Big|_{\varphi^A(a,c)} + \frac{\sqrt{2}\,c^6}{a^4}\, C_{:\bar{2}}{}^{xy}\Big|_{\varphi^A(a,c)} - \sqrt{2}\, C_{:\bar{\bar{2}}}{}^{xy}\Big|_{\varphi^A(a,c)} \, , \\
0 &= \frac{c^6}{a^4}\, C_{:\bar{1}}{}^{yy}\Big|_{\varphi^A(a,c)} - C_{:\bar{\bar{1}}}{}^{yy}\Big|_{\varphi^A(a,c)} + \frac{\sqrt{2}\,c^6}{a^4}\, C_{:\bar{2}}{}^{xy}\Big|_{\varphi^A(a,c)} - \sqrt{2}\, C_{:\bar{\bar{2}}}{}^{xy}\Big|_{\varphi^A(a,c)} \, , \\
0 &= \frac{c^6}{a^4}\, C_{:\bar{6}}{}^{xx}\Big|_{\varphi^A(a,c)} - C_{:\bar{\bar{6}}}{}^{xx}\Big|_{\varphi^A(a,c)} + \frac{\sqrt{2}\,c^6}{a^4}\, C_{:\bar{3}}{}^{xz}\Big|_{\varphi^A(a,c)} - \sqrt{2}\, C_{:\bar{\bar{3}}}{}^{xz}\Big|_{\varphi^A(a,c)} \, , \\
0 &= \frac{c^6}{a^4}\, C_{:\bar{1}}{}^{zz}\Big|_{\varphi^A(a,c)} - C_{:\bar{\bar{1}}}{}^{zz}\Big|_{\varphi^A(a,c)} + \frac{\sqrt{2}\,c^6}{a^4}\, C_{:\bar{3}}{}^{xz}\Big|_{\varphi^A(a,c)} - \sqrt{2}\, C_{:\bar{\bar{3}}}{}^{xz}\Big|_{\varphi^A(a,c)} \, , \\
0 &= \frac{c^6}{a^4}\, C_{:\bar{5}}{}^{xx}\Big|_{\varphi^A(a,c)} - C_{:\bar{\bar{5}}}{}^{xx}\Big|_{\varphi^A(a,c)} + \frac{2\,c^6}{a^4}\, C_{:\bar{3}}{}^{xy}\Big|_{\varphi^A(a,c)} - 2\, C_{:\bar{\bar{3}}}{}^{xy}\Big|_{\varphi^A(a,c)} \, , \\
0 &= \frac{c^6}{a^4}\, C_{:\bar{5}}{}^{xx}\Big|_{\varphi^A(a,c)} - C_{:\bar{\bar{5}}}{}^{xx}\Big|_{\varphi^A(a,c)} + \frac{2\,c^6}{a^4}\, C_{:\bar{2}}{}^{xz}\Big|_{\varphi^A(a,c)} - 2\, C_{:\bar{\bar{2}}}{}^{xz}\Big|_{\varphi^A(a,c)} \, , \\
0 &= \frac{c^6}{a^4}\, C_{:\bar{3}}{}^{yy}\Big|_{\varphi^A(a,c)} - C_{:\bar{\bar{3}}}{}^{yy}\Big|_{\varphi^A(a,c)} + \frac{2\,c^6}{a^4}\, C_{:\bar{5}}{}^{xy}\Big|_{\varphi^A(a,c)} - 2\, C_{:\bar{\bar{5}}}{}^{xy}\Big|_{\varphi^A(a,c)} \, , \\
0 &= \frac{c^6}{a^4}\, C_{:\bar{3}}{}^{yy}\Big|_{\varphi^A(a,c)} - C_{:\bar{\bar{3}}}{}^{yy}\Big|_{\varphi^A(a,c)} + \frac{2\,c^6}{a^4}\, C_{:\bar{2}}{}^{yz}\Big|_{\varphi^A(a,c)} - 2\, C_{:\bar{\bar{2}}}{}^{yz}\Big|_{\varphi^A(a,c)} \, , \\
0 &= \frac{c^6}{a^4}\, C_{:\bar{2}}{}^{zz}\Big|_{\varphi^A(a,c)} - C_{:\bar{\bar{2}}}{}^{zz}\Big|_{\varphi^A(a,c)} + \frac{2\,c^6}{a^4}\, C_{:\bar{5}}{}^{xz}\Big|_{\varphi^A(a,c)} - 2\, C_{:\bar{\bar{5}}}{}^{xz}\Big|_{\varphi^A(a,c)} \, , \\
0 &= \frac{c^6}{a^4}\, C_{:\bar{2}}{}^{zz}\Big|_{\varphi^A(a,c)} - C_{:\bar{\bar{2}}}{}^{zz}\Big|_{\varphi^A(a,c)} + \frac{2\,c^6}{a^4}\, C_{:\bar{3}}{}^{yz}\Big|_{\varphi^A(a,c)} - 2\, C_{:\bar{\bar{3}}}{}^{yz}\Big|_{\varphi^A(a,c)} \, , \\
0 &= \frac{c^6}{a^4}\, C_{:\bar{6}}{}^{yy}\Big|_{\varphi^A(a,c)} - C_{:\bar{\bar{6}}}{}^{yy}\Big|_{\varphi^A(a,c)} + \frac{\sqrt{2}\,c^6}{a^4}\, C_{:\bar{5}}{}^{yz}\Big|_{\varphi^A(a,c)} - \sqrt{2}\, C_{:\bar{\bar{5}}}{}^{yz}\Big|_{\varphi^A(a,c)} \, , \\
0 &= \frac{c^6}{a^4}\, C_{:\bar{4}}{}^{zz}\Big|_{\varphi^A(a,c)} - C_{:\bar{\bar{4}}}{}^{zz}\Big|_{\varphi^A(a,c)} + \frac{\sqrt{2}\,c^6}{a^4}\, C_{:\bar{5}}{}^{yz}\Big|_{\varphi^A(a,c)} - \sqrt{2}\, C_{:\bar{\bar{5}}}{}^{yz}\Big|_{\varphi^A(a,c)} \, .
\end{aligned}$$

Last, there are also three relations that each contain six different derivatives – three pairs of single- and double-overlined counterparts – as

$$
\begin{aligned}
0 &= \frac{c^6}{a^4}\left(C_{:\bar{3}}{}^{xy}\Big|_{\varphi^A(a,c)} + C_{:\bar{2}}{}^{xz}\Big|_{\varphi^A(a,c)} + \sqrt{2}\ C_{:\bar{1}}{}^{yz}\Big|_{\varphi^A(a,c)} \right) - \left(C_{:\bar{\bar{3}}}{}^{xy}\Big|_{\varphi^A(a,c)} + C_{:\bar{\bar{2}}}{}^{xz}\Big|_{\varphi^A(a,c)} + \sqrt{2}\ C_{:\bar{\bar{1}}}{}^{yz}\Big|_{\varphi^A(a,c)} \right), \\
0 &= \frac{c^6}{a^4}\left(C_{:\bar{2}}{}^{yz}\Big|_{\varphi^A(a,c)} + C_{:\bar{5}}{}^{xy}\Big|_{\varphi^A(a,c)} + \sqrt{2}\ C_{:\bar{4}}{}^{xz}\Big|_{\varphi^A(a,c)} \right) - \left(C_{:\bar{\bar{2}}}{}^{yz}\Big|_{\varphi^A(a,c)} + C_{:\bar{\bar{5}}}{}^{xy}\Big|_{\varphi^A(a,c)} + \sqrt{2}\ C_{:\bar{\bar{4}}}{}^{xz}\Big|_{\varphi^A(a,c)} \right), \\
0 &= \frac{c^6}{a^4}\left(C_{:\bar{3}}{}^{yz}\Big|_{\varphi^A(a,c)} + C_{:\bar{5}}{}^{xz}\Big|_{\varphi^A(a,c)} + \sqrt{2}\ C_{:\bar{6}}{}^{xy}\Big|_{\varphi^A(a,c)} \right) - \left(C_{:\bar{\bar{\bar{3}}}}{}^{yz}\Big|_{\varphi^A(a,c)} + C_{:\bar{\bar{\bar{5}}}}{}^{xz}\Big|_{\varphi^A(a,c)} + \sqrt{2}\ C_{:\bar{\bar{\bar{6}}}}{}^{xy}\Big|_{\varphi^A(a,c)} \right).
\end{aligned}
$$

It was clear from the beginning that the 30 relations of $(C8_3)|_{\varphi^A(a,c)}$ cannot provide the same strong results as its equivalent in the metric case in Chapter 4 could. This is simply because now there are 72 single derivatives in the equations compared to 36 in the metric FLRW case. This lack of information about the single derivatives is compensated by other closure equations which were trivially satisfied in the metric FLRW case. Here, it turns out that the prolongation $(C21_3)_{:\bar{\bar{\bar{m}}}}|_{\varphi^A(a,c)}$ which contains the precise same derivatives as $(C8_3)|_{\varphi^A(a,c)}$ will provide the additional information which is needed for a solution of the symmetry-reduced closure equations.

First prolongation of $(C21_3)$

The instance $(C21_3)$ of the sequence $(C21_{N\geq 3,\mathrm{odd}})$ of closure equations,

$$
\begin{aligned}
0 = 2 &\sum_{K=2}^{\infty} \binom{K}{2} C_{:A}{}^{\beta_3\ldots\beta_K(\mu_1\mu_2|}\ M^{A|\mu_3)}{}_{,\beta_3\ldots\beta_K} \\
&- \sum_{K=3}^{\infty}\sum_{J=4}^{K+1} (-1)^J \binom{K}{J-1}\binom{J}{3}\, \partial^{J-3}_{\alpha_1\ldots\alpha_{J-3}} \left(C_{:A}{}^{\beta_J\ldots\beta_K(\alpha_1\ldots\alpha_{J-3}\mu_1\mu_2|}\ M^{A|\mu_3)}{}_{,\beta_J\ldots\beta_K} \right),
\end{aligned}
$$

is satisfied when FLRW symmetries are imposed. The first term features a sum over the index K starting from the value 2. For this starting value, the summand reads

$$
C_{:A}{}^{(\mu_1\mu_2|}\ M^{A|\mu_3)}
$$

which vanishes when evaluated on $\varphi^A(a,c)$ due to the vanishing M coefficient. All further summands contain at least one spatial derivative which makes these terms drop from the equation when symmetry is imposed. In analogy, the second term always contains at least one spatial derivative and therefore also vanishes on symmetry. If one now considers the prolongation $(C21_3)_{:\bar{\bar{\bar{m}}}}|_{\varphi^A(a,c)}$ with respect to the configuration fields $\varphi^{\bar{\bar{\bar{m}}}}$, the second summand still vanishes completely due to the spatial derivatives. The first summand, however, will contribute one non-trivial term,

$$
0 = C_{:A}{}^{(\mu_1\mu_2|}\Big|_{\varphi^A(a,c)}\ M^{A|\mu_3)}{}_{:\bar{\bar{\bar{m}}}}\Big|_{\varphi^A(a,c)}\ .
$$

It provides 50 additional relations complementary to the ones obtained from $(C8_3)|_{\varphi^A(a,c)}$. One finds 20 derivatives to vanish,

$$
\begin{aligned}
0 &= C_{:\bar{1}}{}^{xy}\Big|_{\varphi^A(a,c)} = C_{:\bar{\bar{1}}}{}^{xy}\Big|_{\varphi^A(a,c)} = C_{:\bar{1}}{}^{xz}\Big|_{\varphi^A(a,c)} = C_{:\bar{\bar{1}}}{}^{xz}\Big|_{\varphi^A(a,c)} = C_{:\bar{2}}{}^{xx}\Big|_{\varphi^A(a,c)} = C_{:\bar{\bar{2}}}{}^{xx}\Big|_{\varphi^A(a,c)} \\
&= C_{:\bar{2}}{}^{yy}\Big|_{\varphi^A(a,c)} = C_{:\bar{\bar{2}}}{}^{yy}\Big|_{\varphi^A(a,c)} = C_{:\bar{3}}{}^{xx}\Big|_{\varphi^A(a,c)} = C_{:\bar{\bar{3}}}{}^{xx}\Big|_{\varphi^A(a,c)} = C_{:\bar{3}}{}^{zz}\Big|_{\varphi^A(a,c)} = C_{:\bar{\bar{3}}}{}^{zz}\Big|_{\varphi^A(a,c)} \\
&= C_{:\bar{4}}{}^{xy}\Big|_{\varphi^A(a,c)} = C_{:\bar{\bar{4}}}{}^{xy}\Big|_{\varphi^A(a,c)} = C_{:\bar{5}}{}^{yy}\Big|_{\varphi^A(a,c)} = C_{:\bar{\bar{5}}}{}^{yy}\Big|_{\varphi^A(a,c)} = C_{:\bar{5}}{}^{zz}\Big|_{\varphi^A(a,c)} = C_{:\bar{\bar{5}}}{}^{zz}\Big|_{\varphi^A(a,c)} \\
&= C_{:\bar{6}}{}^{xz}\Big|_{\varphi^A(a,c)} = C_{:\bar{\bar{6}}}{}^{xz}\Big|_{\varphi^A(a,c)}\ .
\end{aligned}
$$

Additionally, 13 double-overlined derivatives $C_{:\bar{\bar{B}}}^{\mu\nu}\big|_{\varphi^A(a,c)}$ can be expressed by their single-overlined counterparts,

$$
\begin{aligned}
&C_{:\bar{\bar{2}}}^{xy}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\ C_{:\bar{2}}^{xy}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{\bar{2}}}^{xz}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\ C_{:\bar{2}}^{xz}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{\bar{2}}}^{yz}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\ C_{:\bar{2}}^{yz}\big|_{\varphi^A(a,c)}\ ,\\
&C_{:\bar{\bar{2}}}^{zz}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\ C_{:\bar{2}}^{zz}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{\bar{3}}}^{xy}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\ C_{:\bar{3}}^{xy}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{\bar{3}}}^{xz}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\ C_{:\bar{3}}^{xz}\big|_{\varphi^A(a,c)}\ ,\\
&C_{:\bar{\bar{3}}}^{yy}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\ C_{:\bar{3}}^{yy}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{\bar{3}}}^{yz}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\ C_{:\bar{3}}^{yz}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{\bar{4}}}^{xz}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\ C_{:\bar{4}}^{xz}\big|_{\varphi^A(a,c)}\ ,\\
&C_{:\bar{\bar{5}}}^{xx}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\ C_{:\bar{5}}^{xx}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{\bar{5}}}^{xy}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\ C_{:\bar{5}}^{xy}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{\bar{5}}}^{xz}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\ C_{:\bar{5}}^{xz}\big|_{\varphi^A(a,c)}\ ,\\
&C_{:\bar{\bar{5}}}^{yz}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\ C_{:\bar{5}}^{yz}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{\bar{6}}}^{xy}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\ C_{:\bar{6}}^{xy}\big|_{\varphi^A(a,c)}
\end{aligned}
$$

Seven more double-overlined derivatives of the form $C_{:\bar{\bar{B}}}^{\mu\nu}\big|_{\varphi^A(a,c)}$ can be written as a combination of single-overlined derivatives,

$$
\begin{aligned}
&C_{:\bar{\bar{1}}}^{yy}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\left(C_{:\bar{1}}^{yy}\big|_{\varphi^A(a,c)} - 2\ C_{:\bar{4}}^{yy}\big|_{\varphi^A(a,c)}\right),\ C_{:\bar{\bar{1}}}^{yz}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\left(C_{:\bar{1}}^{yz}\big|_{\varphi^A(a,c)} - 2\ C_{:\bar{6}}^{yz}\big|_{\varphi^A(a,c)}\right),\\
&C_{:\bar{\bar{1}}}^{zz}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\left(C_{:\bar{1}}^{zz}\big|_{\varphi^A(a,c)} - 2\ C_{:\bar{6}}^{zz}\big|_{\varphi^A(a,c)}\right),\ C_{:\bar{\bar{4}}}^{xx}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\left(C_{:\bar{4}}^{xx}\big|_{\varphi^A(a,c)} - 2\ C_{:\bar{1}}^{xx}\big|_{\varphi^A(a,c)}\right),\\
&C_{:\bar{\bar{4}}}^{zz}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\left(C_{:\bar{4}}^{zz}\big|_{\varphi^A(a,c)} - 2\ C_{:\bar{6}}^{zz}\big|_{\varphi^A(a,c)}\right),\ C_{:\bar{\bar{6}}}^{xx}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\left(C_{:\bar{6}}^{xx}\big|_{\varphi^A(a,c)} - 2\ C_{:\bar{1}}^{xx}\big|_{\varphi^A(a,c)}\right),\\
&C_{:\bar{\bar{6}}}^{yy}\big|_{\varphi^A(a,c)} = -\frac{c^6}{a^4}\left(C_{:\bar{6}}^{yy}\big|_{\varphi^A(a,c)} - 2\ C_{:\bar{4}}^{yy}\big|_{\varphi^A(a,c)}\right).
\end{aligned}
$$

Thus, the combination of the two equations $(C8_3)|_{\varphi^A(a,c)}$ and $(C21_3)_{:\bar{\bar{\bar{m}}}}|_{\varphi^A(a,c)}$ allows to express all double-overlined derivatives $C_{:\bar{\bar{B}}}^{\mu\nu}\big|_{\varphi^A(a,c)}$ in terms of derivatives with respect to single-overlined derivatives. Additionally, nine single-overlined derivatives $C_{:\bar{A}}^{\mu\nu}\big|_{\varphi^A(a,c)}$ can be expressed in terms of four others,

$$
\begin{aligned}
&C_{:\bar{3}}^{xy}\big|_{\varphi^A(a,c)} = -\frac{1}{2}\ C_{:\bar{5}}^{xx}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{2}}^{xz}\big|_{\varphi^A(a,c)} = -\frac{1}{2}\ C_{:\bar{5}}^{xx}\big|_{\varphi^A(a,c)}\ ,\\
&C_{:\bar{5}}^{xy}\big|_{\varphi^A(a,c)} = -\frac{1}{2}\ C_{:\bar{3}}^{yy}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{2}}^{yz}\big|_{\varphi^A(a,c)} = -\frac{1}{2}\ C_{:\bar{3}}^{yy}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{4}}^{xz}\big|_{\varphi^A(a,c)} = \frac{1}{\sqrt{2}}\ C_{:\bar{3}}^{yy}\big|_{\varphi^A(a,c)}\ ,\\
&C_{:\bar{2}}^{zz}\big|_{\varphi^A(a,c)} = \sqrt{2}\ C_{:\bar{6}}^{xy}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{3}}^{yz}\big|_{\varphi^A(a,c)} = -\frac{1}{\sqrt{2}}\ C_{:\bar{6}}^{xy}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{5}}^{xz}\big|_{\varphi^A(a,c)} = -\frac{1}{\sqrt{2}}\ C_{:\bar{6}}^{xy}\big|_{\varphi^A(a,c)}\ ,\\
&C_{:\bar{4}}^{yz}\big|_{\varphi^A(a,c)} = C_{:\bar{6}}^{yz}\big|_{\varphi^A(a,c)}\ .
\end{aligned}
$$

There are seven more relations which express additional seven single-overlined derivatives as a combination of other single-overlined derivatives,

$$
\begin{aligned}
&C_{:\bar{1}}^{yy}\big|_{\varphi^A(a,c)} = C_{:\bar{4}}^{xx}\big|_{\varphi^A(a,c)} - C_{:\bar{1}}^{xx}\big|_{\varphi^A(a,c)} + C_{:\bar{4}}^{yy}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{1}}^{yz}\big|_{\varphi^A(a,c)} = \frac{1}{\sqrt{2}}\ C_{:\bar{5}}^{xx}\big|_{\varphi^A(a,c)} + C_{:\bar{6}}^{yz}\big|_{\varphi^A(a,c)}\ ,\\
&C_{:\bar{1}}^{zz}\big|_{\varphi^A(a,c)} = C_{:\bar{6}}^{xx}\big|_{\varphi^A(a,c)} - C_{:\bar{1}}^{xx}\big|_{\varphi^A(a,c)} + C_{:\bar{6}}^{zz}\big|_{\varphi^A(a,c)}\ ,\ C_{:\bar{2}}^{xy}\big|_{\varphi^A(a,c)} = \frac{1}{\sqrt{2}}\left(-\ C_{:\bar{4}}^{xx}\big|_{\varphi^A(a,c)} + C_{:\bar{1}}^{xx}\big|_{\varphi^A(a,c)}\right),\\
&C_{:\bar{3}}^{xz}\big|_{\varphi^A(a,c)} = \frac{1}{\sqrt{2}}\left(-\ C_{:\bar{6}}^{xx}\big|_{\varphi^A(a,c)} + C_{:\bar{1}}^{xx}\big|_{\varphi^A(a,c)}\right),\ C_{:\bar{4}}^{zz}\big|_{\varphi^A(a,c)} = C_{:\bar{6}}^{yy}\big|_{\varphi^A(a,c)} - C_{:\bar{4}}^{yy}\big|_{\varphi^A(a,c)} + C_{:\bar{6}}^{zz}\big|_{\varphi^A(a,c)}\ ,\\
&C_{:\bar{5}}^{yz}\big|_{\varphi^A(a,c)} = \frac{1}{\sqrt{2}}\left(-\ C_{:\bar{6}}^{yy}\big|_{\varphi^A(a,c)} + C_{:\bar{4}}^{yy}\big|_{\varphi^A(a,c)}\right).
\end{aligned}
$$

As a summary of the previous analysis, closure equation $(C8_3)|_{\varphi^A(a,c)}$ and the first prolongation $(C21_3)_{\overset{\equiv}{:m}}|_{\varphi^A(a,c)}$ express all 72 derivatives of the form

$$C_{:\overline{A}}{}^{\mu\nu}\Big|_{\varphi^A(a,c)} \quad \text{and} \quad C_{:\overline{\overline{A}}}{}^{\mu\nu}\Big|_{\varphi^A(a,c)}$$

in terms of only ten independent ones. Additionally, 20 of the 72 derivatives vanish completely. This knowledge and the obtained relations have now to be transferred to closure equation $(C3)|_{\varphi^A(a,c)}$ which connects the previous analysis to the actually desired third expansion coefficient C^{cosmo}_{AB}.

Analysis of closure equation $(C3)|_{\varphi^A(a,c)}$

Closure equation $(C3)|_{\varphi^A(a,c)}$ provides a connection between the third expansion coefficient C^{cosmo}_{AB} and the derivative $C_{:A}{}^{\mu\nu}|_{\varphi^A(a,c)}$. As the previous study revealed, only ten of these derivatives are independent. This will also reduce the amount of independent components of C^{cosmo}_{AB}.

Closure equation $(C3)|_{\varphi^A(a,c)}$ is

$$0 = 6\, C^{\text{cosmo}}_{AB}\ \ F^A{}_\rho{}^{(\mu|}\Big|_{\varphi^A(a,c)}\ p^{|\nu)\rho}\Big|_{\varphi^A(a,c)} - C_{:B}{}^{\mu\nu}\Big|_{\varphi^A(a,c)}$$

and will feature 72 single relations if one restricts the index range of B to $\overline{B}$ and $\overline{\overline{B}}$. As there are already 153 components of the expansion coefficient C^{cosmo}_{AB}, one should investigate how many of them actually contribute to the spacetime action. These components couple exclusively to the non-trivial derivatives $\dot{\varphi}^A$ of the configuration fields and thus have to be calculated. The other components are not needed for the construction of the spacetime action, but still might provide useful information, especially if prolongations of the closure equations are studied. The desired 21 components are

$$C^{\text{cosmo}}_{\overset{-}{\overline{11}}},\ C^{\text{cosmo}}_{\overset{-}{\overline{14}}},\ C^{\text{cosmo}}_{\overset{-}{\overline{16}}},\ C^{\text{cosmo}}_{\overset{-}{\overline{41}}},\ C^{\text{cosmo}}_{\overset{-}{\overline{44}}},\ C^{\text{cosmo}}_{\overset{-}{\overline{46}}},\ C^{\text{cosmo}}_{\overset{-}{\overline{61}}},\ C^{\text{cosmo}}_{\overset{-}{\overline{64}}},\ C^{\text{cosmo}}_{\overset{-}{\overline{66}}},$$
$$C^{\text{cosmo}}_{\overline{11}},\ C^{\text{cosmo}}_{\overline{14}},\ C^{\text{cosmo}}_{\overline{16}},\ C^{\text{cosmo}}_{\overline{44}},\ C^{\text{cosmo}}_{\overline{46}},\ C^{\text{cosmo}}_{\overline{66}},\ C^{\text{cosmo}}_{\overline{\overline{11}}},\ C^{\text{cosmo}}_{\overline{\overline{14}}},\ C^{\text{cosmo}}_{\overline{\overline{16}}},\ C^{\text{cosmo}}_{\overline{\overline{44}}},\ C^{\text{cosmo}}_{\overline{\overline{46}}} \text{ and } C^{\text{cosmo}}_{\overline{\overline{66}}}.$$

Other components might still be necessary and might contribute at later stages in the solution of either C^{cosmo}_{AB} or other expansion coefficients. The inevitable task is however to determine at least all components entering the cosmological spacetime action.

Careful evaluation of all 72 relations of $(C3)|_{\varphi^A(a,c)}$ shows that all components $C^{\text{cosmo}}_{\overset{-}{\overline{AB}}}$ and $C^{\text{cosmo}}_{\overline{\overline{AB}}}$ can be expressed in terms of the single-overlined components $C^{\text{cosmo}}_{\overline{AB}}$ and the ten independent derivatives $C_{:\overline{A}}{}^{\mu\nu}\Big|_{\varphi^A(a,c)}$. Precisely, for the components entering the cosmological action, one finds

$$\begin{aligned}
C^{\text{cosmo}}_{\overset{-}{\overline{11}}} &= \frac{c^6}{a^4}\, C^{\text{cosmo}}_{\overline{11}} + \frac{c^4}{4}\ C_{:\bar{1}}{}^{xx}\Big|_{\varphi^A(a,c)}\,,\ C^{\text{cosmo}}_{\overset{-}{\overline{44}}} = \frac{c^6}{a^4}\, C^{\text{cosmo}}_{\overline{44}} + \frac{c^4}{4}\ C_{:\bar{4}}{}^{yy}\Big|_{\varphi^A(a,c)}\,,\\
C^{\text{cosmo}}_{\overset{-}{\overline{14}}} &= \frac{c^6}{a^4}\, C^{\text{cosmo}}_{\overline{14}} + \frac{c^4}{4}\left(C_{:\bar{4}}{}^{xx}\Big|_{\varphi^A(a,c)} - C_{:\bar{1}}{}^{xx}\Big|_{\varphi^A(a,c)} + C_{:\bar{4}}{}^{yy}\Big|_{\varphi^A(a,c)}\right),\\
C^{\text{cosmo}}_{\overset{-}{\overline{16}}} &= \frac{c^6}{a^4}\, C^{\text{cosmo}}_{\overline{16}} + \frac{c^4}{4}\left(C_{:\bar{6}}{}^{xx}\Big|_{\varphi^A(a,c)} - \ C_{:\bar{1}}{}^{xx}\Big|_{\varphi^A(a,c)} + C_{:\bar{6}}{}^{zz}\Big|_{\varphi^A(a,c)}\right),\\
C^{\text{cosmo}}_{\overset{-}{\overline{41}}} &= \frac{c^6}{a^4}\, C^{\text{cosmo}}_{\overline{14}} + \frac{c^4}{4}\ C_{:\bar{4}}{}^{xx}\Big|_{\varphi^A(a,c)}\,,\ C^{\text{cosmo}}_{\overset{-}{\overline{61}}} = \frac{c^6}{a^4}\, C^{\text{cosmo}}_{\overline{16}} + \frac{c^4}{4}\ C_{:\bar{6}}{}^{xx}\Big|_{\varphi^A(a,c)}\,,\\
C^{\text{cosmo}}_{\overset{-}{\overline{46}}} &= \frac{c^6}{a^4}\, C^{\text{cosmo}}_{\overline{46}} + \frac{c^4}{4}\left(C_{:\bar{6}}{}^{yy}\Big|_{\varphi^A(a,c)} - C_{:\bar{4}}{}^{yy}\Big|_{\varphi^A(a,c)} + C_{:\bar{6}}{}^{zz}\Big|_{\varphi^A(a,c)}\right),\\
C^{\text{cosmo}}_{\overset{-}{\overline{64}}} &= \frac{c^6}{a^4}\, C^{\text{cosmo}}_{\overline{46}} + \frac{c^4}{4}\ C_{:\bar{6}}{}^{yy}\Big|_{\varphi^A(a,c)}\,,\ C^{\text{cosmo}}_{\overset{-}{\overline{66}}} = \frac{c^6}{a^4}\, C^{\text{cosmo}}_{\overline{66}} + \frac{c^4}{4}\ C_{:\bar{6}}{}^{zz}\Big|_{\varphi^A(a,c)}\,.
\end{aligned}$$

The purely double-overlined components $C^{\text{cosmo}}_{\overline{\overline{AB}}}$ can be expressed as

$$
\begin{aligned}
C^{\text{cosmo}}_{\overline{\overline{14}}} &= \frac{c^{12}}{a^8}\,C^{\text{cosmo}}_{\overline{14}} + \frac{c^{10}}{4\,a^4}\left(C_{:\overline{1}}{}^{xx}\Big|_{\varphi^A(a,c)} + C_{:\overline{4}}{}^{yy}\Big|_{\varphi^A(a,c)}\right), & C^{\text{cosmo}}_{\overline{\overline{11}}} &= \frac{c^{12}}{a^8}\,C^{\text{cosmo}}_{\overline{11}} + \frac{c^{10}}{2\,a^4}\;C_{:\overline{1}}{}^{xx}\Big|_{\varphi^A(a,c)}\,,\\
C^{\text{cosmo}}_{\overline{\overline{16}}} &= \frac{c^{12}}{a^8}\,C^{\text{cosmo}}_{\overline{16}} + \frac{c^{10}}{4\,a^4}\left(C_{:\overline{1}}{}^{xx}\Big|_{\varphi^A(a,c)} + C_{:\overline{6}}{}^{zz}\Big|_{\varphi^A(a,c)}\right), & C^{\text{cosmo}}_{\overline{\overline{44}}} &= \frac{c^{12}}{a^8}\,C^{\text{cosmo}}_{\overline{44}} + \frac{c^{10}}{2\,a^4}\;C_{:\overline{4}}{}^{yy}\Big|_{\varphi^A(a,c)}\,,\\
C^{\text{cosmo}}_{\overline{\overline{46}}} &= \frac{c^{12}}{a^8}\,C^{\text{cosmo}}_{\overline{46}} + \frac{c^{10}}{4\,a^4}\left(C_{:\overline{4}}{}^{yy}\Big|_{\varphi^A(a,c)} + C_{:\overline{6}}{}^{zz}\Big|_{\varphi^A(a,c)}\right), & C^{\text{cosmo}}_{\overline{\overline{66}}} &= \frac{c^{12}}{a^8}\,C^{\text{cosmo}}_{\overline{66}} + \frac{c^{10}}{2\,a^4}\;C_{:\overline{6}}{}^{zz}\Big|_{\varphi^A(a,c)}\,.
\end{aligned}
$$

The only independent components of C^{cosmo}_{AB} are the purely single-overlined ones. All other components can be expressed by those and the appropriate derivatives of the first expansion coefficient C.

This result becomes important when closure equation $(C10_2)|_{\varphi^A(a,c)}$ is studied. This closure equation is an equation purely for the expansion coefficient C_{AB} and its derivatives. It will provide the relations needed in order to solve the differential equations (6.4) and (6.5) for the desired components of C^{cosmo}_{AB}. Usage of the results from closure equation $(C3)|_{\varphi^A(a,c)}$ already reduces the amount of independent components of C^{cosmo}_{AB} which will be even further restricted by $(C10_2)|_{\varphi^A(a,c)}$.

Evaluation of closure equation $(C10_2)|_{\varphi^A(a,c)}$

Closure equation $(C10_2)|_{\varphi^A(a,c)}$ is the $N = 2$-analog of closure equations $(C1)|_{\varphi^A(a,c)}$ and $(C2)|_{\varphi^A(a,c)}$ for the first two expansion coefficients. All spatial derivatives drop out of the equation which remains as

$$
0 = C^{\text{cosmo}}_{B_1B_2}\delta^\gamma_\mu + 2\,C^{\text{cosmo}}_{A(B_1|}\;F^A{}_\mu{}^\gamma{}_{:|B_2)}\Big|_{\varphi^A(a,c)} + C_{B_1B_2:A}\Big|_{\varphi^A(a,c)}\;F^A{}_\mu{}^\gamma\Big|_{\varphi^A(a,c)}\,.
$$

As the analysis of $(C3)|_{\varphi^A(a,c)}$ revealed, only the components $C^{\text{cosmo}}_{\overline{AB}}$ are independent and need to be determined. All other components with double-overlined capital indices follow from the single-overlined ones and a derivative of the first expansion coefficient C. Thus, it suffices for now to restrict the index range of the free capital indices B_1 and B_2 to the single-overlined ones, $\overline{B}_1$ and $\overline{B}_2$. This results in 189 single relations. Their evaluation reveals that there are only two independent, non-vanishing components, $C^{\text{cosmo}}_{\overline{11}}$ and $C^{\text{cosmo}}_{\overline{14}}$. The other contributing components can be expressed by them as

$$
C^{\text{cosmo}}_{\overline{11}} = C^{\text{cosmo}}_{\overline{44}} = C^{\text{cosmo}}_{\overline{66}} \quad\text{and}\quad C^{\text{cosmo}}_{\overline{14}} = C^{\text{cosmo}}_{\overline{16}} = C^{\text{cosmo}}_{\overline{46}}\,.
$$

Additionally, there are three other non-trivial components which do not contribute to the cosmological spacetime action. They can be expressed by the two independent components $C^{\text{cosmo}}_{\overline{11}}$ and $C^{\text{cosmo}}_{\overline{14}}$ as

$$
C^{\text{cosmo}}_{\overline{22}} = C^{\text{cosmo}}_{\overline{33}} = C^{\text{cosmo}}_{\overline{55}} = C^{\text{cosmo}}_{\overline{11}} - C^{\text{cosmo}}_{\overline{14}}\,.
$$

All other components of $C^{\text{cosmo}}_{\overline{AB}}$ vanish. Using the results from closure equation $(C3)|_{\varphi^A(a,c)}$, one concludes that several derivatives $C_{:\overline{A}}{}^{\mu\nu}\big|_{\varphi^A(a,c)}$ vanish,

$$
\begin{aligned}
0 &= C_{:\overline{1}}{}^{yz}\Big|_{\varphi^A(a,c)} = C_{:\overline{2}}{}^{xz}\Big|_{\varphi^A(a,c)} = C_{:\overline{2}}{}^{yz}\Big|_{\varphi^A(a,c)} = C_{:\overline{2}}{}^{zz}\Big|_{\varphi^A(a,c)} = C_{:\overline{3}}{}^{xy}\Big|_{\varphi^A(a,c)} = C_{:\overline{3}}{}^{yy}\Big|_{\varphi^A(a,c)} = C_{:\overline{3}}{}^{yz}\Big|_{\varphi^A(a,c)}\\
&= C_{:\overline{4}}{}^{xz}\Big|_{\varphi^A(a,c)} = C_{:\overline{4}}{}^{yz}\Big|_{\varphi^A(a,c)} = C_{:\overline{5}}{}^{xx}\Big|_{\varphi^A(a,c)} = C_{:\overline{5}}{}^{xy}\Big|_{\varphi^A(a,c)} = C_{:\overline{5}}{}^{xz}\Big|_{\varphi^A(a,c)} = C_{:\overline{6}}{}^{xy}\Big|_{\varphi^A(a,c)} = C_{:\overline{6}}{}^{yz}\Big|_{\varphi^A(a,c)}
\end{aligned}
$$

as well as their respective double-overlined counterparts.

The 189 relations evaluated so far already provided all information needed in order to solve the differential equations (6.4) and (6.5) for $N = 2$. Evaluating the 513 remaining relations for the index ranges $\overline{B}_1$, $\overline{\overline{B}}_2$ and $\overline{\overline{B}}_1$ and $\overline{\overline{B}}_2$ provides further information about the derivatives $C_{:\overline{A}}{}^{\mu\nu}\big|_{\varphi^A(a,c)}$. They leave only two of them independent which allows to express all other non-trivial ones by them as

$$
\begin{aligned}
&C_{:\overline{1}}{}^{xx}\Big|_{\varphi^A(a,c)} = C_{:\overline{4}}{}^{yy}\Big|_{\varphi^A(a,c)} = C_{:\overline{6}}{}^{zz}\Big|_{\varphi^A(a,c)}\\
&C_{:\overline{1}}{}^{yy}\Big|_{\varphi^A(a,c)} = C_{:\overline{1}}{}^{zz}\Big|_{\varphi^A(a,c)} = C_{:\overline{4}}{}^{xx}\Big|_{\varphi^A(a,c)} = C_{:\overline{4}}{}^{zz}\Big|_{\varphi^A(a,c)} = C_{:\overline{6}}{}^{xx}\Big|_{\varphi^A(a,c)} = C_{:\overline{6}}{}^{yy}\Big|_{\varphi^A(a,c)}\,.
\end{aligned}
$$

There are three additional non-trivial derivatives which can be expressed by the difference of the two independent derivatives as

$$C_{:\bar{2}}{}^{xy}\Big|_{\varphi^A(a,c)} = C_{:\bar{3}}{}^{xz}\Big|_{\varphi^A(a,c)} = C_{:\bar{5}}{}^{yz}\Big|_{\varphi^A(a,c)} = \frac{1}{\sqrt{2}}\left(C_{:\bar{1}}{}^{xx}\Big|_{\varphi^A(a,c)} - C_{:\bar{1}}{}^{yy}\Big|_{\varphi^A(a,c)}\right).$$

These relations provide the last piece of information one can extract about either the derivative $C_{:A}{}^{\mu\nu}|_{\varphi^A(a,c)}$ or the expansion coefficient C^{cosmo}_{AB} to this level. This means one has to extract the relations required for the solution of the chain rule equations (6.4) and (6.5) for the two independent components $C^{\text{cosmo}}_{\overline{11}}$ and $C^{\text{cosmo}}_{\overline{14}}$.

For the component $C^{\text{cosmo}}_{\overline{11}}$, the relations obtained from the analysis of $(C10_2)|_{\varphi^A(a,c)}$ simplify the two chain rule equations to

$$\begin{aligned}
\frac{\partial C^{\text{cosmo}}_{\overline{11}}}{\partial a} &= \frac{7}{a}\, C^{\text{cosmo}}_{\overline{11}} \\
\frac{\partial C^{\text{cosmo}}_{\overline{11}}}{\partial c} &= -\frac{2\,c}{a^2}\left(C_{\overline{11}:\bar{1}}|_{\varphi^A(a,c)} + C_{\overline{11}:\bar{4}}|_{\varphi^A(a,c)} + C_{\overline{11}:\bar{6}}|_{\varphi^A(a,c)}\right) - \frac{14}{c}\, C^{\text{cosmo}}_{\overline{11}}
\end{aligned}$$

which can be solved to

$$C^{\text{cosmo}}_{\overline{11}} = f_2(c)\, a^7$$

with a third undetermined function f_2 of the second scale factor $c(t)$. In the same way, the component $C^{\text{cosmo}}_{\overline{14}}$ picks up a fourth undetermined function $f_3(c)$,

$$C^{\text{cosmo}}_{\overline{14}} = f_3(c)\, a^7\,.$$

So far, the two components $C^{\text{cosmo}}_{\overline{11}}$ and $C^{\text{cosmo}}_{\overline{14}}$ introduced two undetermined functions of the second scale factor $c(t)$ to the expansion coefficient C^{cosmo}_{AB}. As the analysis of closure equation $(C3)|_{\varphi^A(a,c)}$ showed, the other components of C^{cosmo}_{AB} contributing to the cosmological spacetime action are given by the two independent components and the two independent derivatives $C_{:\bar{1}}{}^{xx}\big|_{\varphi^A(a,c)}$ and $C_{:\bar{1}}{}^{yy}\big|_{\varphi^A(a,c)}$. However, these are also undetermined by the closure equations and thus introduce two new undetermined functions to the components $C^{\text{cosmo}}_{\overline{\overline{AB}}}$ and $C^{\text{cosmo}}_{\overline{\overline{\overline{AB}}}}$. Collecting all terms

$$C^{\text{cosmo}}_{AB}\,\dot{\varphi}^A\,\dot{\varphi}^B$$

contributing to the cosmological spacetime action, one finds

$$\begin{aligned}
C^{\text{cosmo}}_{AB}\,\dot{\varphi}^A\,\dot{\varphi}^B =& \left(12\,(f_2 + 2\,f_3)(c)\,a^3 - 66\ C_{:\bar{1}}{}^{xx}\Big|_{\varphi^A(a,c)}\right)\dot{c}^2 + 12\,\frac{\dot{a}^2\,c^2}{a^2}\left(C_{:\bar{1}}{}^{xx}\Big|_{\varphi^A(a,c)} - C_{:1}{}^{yy}\Big|_{\varphi^A(a,c)}\right) \\
&+ \frac{\dot{a}\,\dot{c}\,c}{a}\left(60\ C_{:\bar{1}}{}^{xx}\Big|_{\varphi^A(a,c)} + 36\ C_{:\bar{1}}{}^{yy}\Big|_{\varphi^A(a,c)}\right).
\end{aligned} \tag{6.20}$$

Dimensional analysis reveals that the two derivatives contain the first scale factor $a(t)$ cubicly together with two undetermined functions $\tilde{f}(c)$ and $\tilde{\tilde{f}}(c)$ of the second scale factor,

$$C_{:\bar{1}}{}^{xx}\Big|_{\varphi^A(a,c)} = \tilde{f}(c)\,a^3 \quad , \quad C_{:\bar{1}}{}^{yy}\Big|_{\varphi^A(a,c)} = \tilde{\tilde{f}}(c)\,a^3\,.$$

This implies to define three functions

$$\begin{aligned}
F_2(c) &:= 12\ (f_2 + 2\,f_3)\,(c) - 66\,\tilde{f}(c)\,, \\
F_3(c) &:= 12\,c^2\left(\tilde{f} - \tilde{\tilde{f}}\right)(c)\,, \\
F_4(c) &:= 60\,\tilde{f}(c) + 36\,\tilde{\tilde{f}}(c)
\end{aligned}$$

in order to write Eq. (6.20) more compactly as

$$C^{\text{cosmo}}_{AB}\,\dot{\varphi}^A\,\dot{\varphi}^B = F_2(c)\,\dot{c}^2\,a^3 + F_3(c)\,\dot{a}^2\,a + F_4(c)\,\dot{c}\,\dot{a}\,a^2\,.$$

This result confirms the one obtained from previous studies such as of Ref. [24]. While they might seem unsatisfying, the results presented here are the limit of what can be achieved by pen-and-paper calculations. While taking into account the first prolongation of closure equation ($C21_3$) reduced the amount of undetermined derivatives $C_{:A}{}^{\mu\nu}|_{\varphi^A(a,c)}$ to ultimately two, it was not strong enough to determine the components of C^{cosmo}_{AB} free of any undetermined functions of the second scale factor. To this level, it cannot be ruled out that further prolongations of closure equations ($C10_2$) would provide further information about the components C^{cosmo}_{AB}. These prolongations require the extensive use of suitable computer algebra systems as the amount of equations increases by a factor of 12 considering only the prolongation $(C10_2)_{:D}|_{\varphi^A(a,c)}$. This results in the task to evaluate more than 2000 single relations. Besides, it is not guaranteed that this prolongation does actually provide new information on the components C^{cosmo}_{AB} and reduce the amount of undetermined functions.

Attention is thus turned towards the next expansion coefficient $C^{\text{cosmo}}_{A_1A_2A_3}$ which is determined by a very compact calculation compared to the one carried out for C^{cosmo}_{AB}.

6.5 Solution for the fourth expansion coefficient

For the solution of higher order expansion coefficients $C^{\text{cosmo}}_{A_1\dots A_N}$, the sequences of closure equations become more and more important. The previous analysis of C^{cosmo}_{AB} already used the instance $(C10_2)|_{\varphi^A(a,c)}$ of the sequence ($C10_{N\geq 2}$) of closure equations. The solution for higher order expansion coefficients will also make use of sequence ($C16_{N\geq 2}$) generalizing equation $(C3)|_{\varphi^A(a,c)}$ used in the solution of C^{cosmo}_{AB}.

The role of coefficient $C^{\text{cosmo}}_{A_1A_2A_3}$ is however different than the one of the other higher order coefficients. This is due to the special structure of closure equation $(C16_2)|_{\varphi^A(a,c)}$. For $N = 2$, the last term of $(C16_2)|_{\varphi^A(a,c)}$ drops out of the equation which is left as

$$0 = 18\, C^{\text{cosmo}}_{AB_1B_2}\ p^{\rho(\mu|}\Big|_{\varphi^A(a,c)}\ F^A{}_\rho{}^{|\nu)}\Big|_{\varphi^A(a,c)}\ ,$$

as the input coefficient $M^{A\gamma}$ vanishes when symmetry is imposed. Evaluating these relations allows to express the components

$$C^{\text{cosmo}}_{\overline{\overline{A_1}}\,\overline{\overline{A_2}}\,\overline{\overline{A_3}}}\quad ,\quad C^{\text{cosmo}}_{\overline{\overline{A_1}}\,\overline{\overline{A_2}}\,\overline{A_3}}\quad \text{and}\quad C^{\text{cosmo}}_{\overline{\overline{A_1}}\,\overline{A_2}\,\overline{A_3}}$$

in terms of the purely single-overlined components $C^{\text{cosmo}}_{\overline{A_1}\,\overline{A_2}\,\overline{A_3}}$. Of these, only 10 components actually enter the cosmological action, namely

$$C^{\text{cosmo}}_{\overline{111}}\,,\ C^{\text{cosmo}}_{\overline{114}}\,,\ C^{\text{cosmo}}_{\overline{116}}\,,\ C^{\text{cosmo}}_{\overline{144}}\,,\ C^{\text{cosmo}}_{\overline{146}}\,,\ C^{\text{cosmo}}_{\overline{166}}\,,\ C^{\text{cosmo}}_{\overline{444}}\,,\ C^{\text{cosmo}}_{\overline{446}}\,,\ C^{\text{cosmo}}_{\overline{466}}\quad \text{and}\quad C^{\text{cosmo}}_{\overline{666}}\,. \tag{6.21}$$

These 10 components are the ones that will ultimately have to be determined by the closure equations and the differential equations (6.4) and (6.5). The cosmological spacetime action is of course constituted by other components of $C^{\text{cosmo}}_{A_1A_2A_3}$ which are expressed by the 10 independent ones from Eq. (6.21). The 10 desired components $C^{\text{cosmo}}_{\overline{\overline{A_1}}\,\overline{\overline{A_2}}\,\overline{\overline{A_3}}}$ are given by

$$\begin{aligned}
&C^{\text{cosmo}}_{\overline{\overline{111}}} = \frac{c^{18}}{a^{12}}\,C^{\text{cosmo}}_{\overline{111}}\,,\ C^{\text{cosmo}}_{\overline{\overline{114}}} = \frac{c^{18}}{a^{12}}\,C^{\text{cosmo}}_{\overline{114}}\,,\ C^{\text{cosmo}}_{\overline{\overline{116}}} = \frac{c^{18}}{a^{12}}\,C^{\text{cosmo}}_{\overline{116}}\,,\ C^{\text{cosmo}}_{\overline{\overline{144}}} = \frac{c^{18}}{a^{12}}\,C^{\text{cosmo}}_{\overline{144}}\,,\\
&C^{\text{cosmo}}_{\overline{\overline{146}}} = \frac{c^{18}}{a^{12}}\,C^{\text{cosmo}}_{\overline{146}}\,,\ C^{\text{cosmo}}_{\overline{\overline{166}}} = \frac{c^{18}}{a^{12}}\,C^{\text{cosmo}}_{\overline{166}}\,,\ C^{\text{cosmo}}_{\overline{\overline{444}}} = \frac{c^{18}}{a^{12}}\,C^{\text{cosmo}}_{\overline{444}}\,,\ C^{\text{cosmo}}_{\overline{\overline{446}}} = \frac{c^{18}}{a^{12}}\,C^{\text{cosmo}}_{\overline{446}}\,,\\
&C^{\text{cosmo}}_{\overline{\overline{466}}} = \frac{c^{18}}{a^{12}}\,C^{\text{cosmo}}_{\overline{466}}\,,\ C^{\text{cosmo}}_{\overline{\overline{666}}} = \frac{c^{18}}{a^{12}}\,C^{\text{cosmo}}_{\overline{666}}\,.
\end{aligned}$$

Additionally, the 18 relevant components of $C^{\mathrm{cosmo}}_{\bar{\bar{A}}_1\bar{\bar{A}}_2\bar{A}_3}$ are expressed as follows,

$$
\begin{aligned}
&C^{\mathrm{cosmo}}_{\overline{\overline{11}1}} = \frac{c^{12}}{a^8}\, C^{\mathrm{cosmo}}_{\overline{111}}\,,\; C^{\mathrm{cosmo}}_{\overline{\overline{44}4}} = \frac{c^{12}}{a^8}\, C^{\mathrm{cosmo}}_{\overline{444}}\,,\; C^{\mathrm{cosmo}}_{\overline{\overline{66}6}} = \frac{c^{12}}{a^8}\, C^{\mathrm{cosmo}}_{\overline{666}}\,,\; C^{\mathrm{cosmo}}_{\overline{\overline{14}6}} = C^{\mathrm{cosmo}}_{\overline{\overline{16}4}} = C^{\mathrm{cosmo}}_{\overline{\overline{46}1}} = \frac{c^{12}}{a^8}\, C^{\mathrm{cosmo}}_{\overline{146}}\,,\\
&C^{\mathrm{cosmo}}_{\overline{\overline{11}4}} = C^{\mathrm{cosmo}}_{\overline{\overline{14}1}} = \frac{c^{12}}{a^8}\, C^{\mathrm{cosmo}}_{\overline{114}}\,,\; C^{\mathrm{cosmo}}_{\overline{\overline{11}6}} = C^{\mathrm{cosmo}}_{\overline{\overline{16}1}} = \frac{c^{12}}{a^8}\, C^{\mathrm{cosmo}}_{\overline{116}}\,,\; C^{\mathrm{cosmo}}_{\overline{\overline{14}4}} = C^{\mathrm{cosmo}}_{\overline{\overline{44}1}} = \frac{c^{12}}{a^8}\, C^{\mathrm{cosmo}}_{\overline{144}}\,,\\
&C^{\mathrm{cosmo}}_{\overline{\overline{16}6}} = C^{\mathrm{cosmo}}_{\overline{\overline{66}1}} = \frac{c^{12}}{a^8}\, C^{\mathrm{cosmo}}_{\overline{166}}\,,\; C^{\mathrm{cosmo}}_{\overline{\overline{44}6}} = C^{\mathrm{cosmo}}_{\overline{\overline{46}4}} = \frac{c^{12}}{a^8}\, C^{\mathrm{cosmo}}_{\overline{446}}\,, C^{\mathrm{cosmo}}_{\overline{\overline{46}6}} = C^{\mathrm{cosmo}}_{\overline{\overline{66}4}} = \frac{c^{12}}{a^8}\, C^{\mathrm{cosmo}}_{\overline{466}}\,.
\end{aligned}
$$

Finally, the 18 components of the form $C^{\mathrm{cosmo}}_{\bar{\bar{A}}_1\bar{A}_2\bar{A}_3}$ are given by

$$
\begin{aligned}
&C^{\mathrm{cosmo}}_{\overline{\overline{1}11}} = \frac{c^{6}}{a^4}\, C^{\mathrm{cosmo}}_{\overline{111}}\,,\; C^{\mathrm{cosmo}}_{\overline{\overline{4}44}} = \frac{c^{6}}{a^4}\, C^{\mathrm{cosmo}}_{\overline{444}}\,,\; C^{\mathrm{cosmo}}_{\overline{\overline{6}66}} = \frac{c^{6}}{a^4}\, C^{\mathrm{cosmo}}_{\overline{666}}\,,\; C^{\mathrm{cosmo}}_{\overline{\overline{1}46}} = C^{\mathrm{cosmo}}_{\overline{\overline{4}16}} = C^{\mathrm{cosmo}}_{\overline{\overline{6}14}} = \frac{c^{6}}{a^4}\, C^{\mathrm{cosmo}}_{\overline{146}}\,,\\
&C^{\mathrm{cosmo}}_{\overline{\overline{1}14}} = C^{\mathrm{cosmo}}_{\overline{\overline{4}11}} = \frac{c^{6}}{a^4}\, C^{\mathrm{cosmo}}_{\overline{114}}\,,\; C^{\mathrm{cosmo}}_{\overline{\overline{1}16}} = C^{\mathrm{cosmo}}_{\overline{\overline{6}11}} = \frac{c^{6}}{a^4}\, C^{\mathrm{cosmo}}_{\overline{116}}\,,\; C^{\mathrm{cosmo}}_{\overline{\overline{1}44}} = C^{\mathrm{cosmo}}_{\overline{\overline{4}14}} = \frac{c^{6}}{a^4}\, C^{\mathrm{cosmo}}_{\overline{144}}\,,\\
&C^{\mathrm{cosmo}}_{\overline{\overline{1}66}} = C^{\mathrm{cosmo}}_{\overline{\overline{6}16}} = \frac{c^{6}}{a^4}\, C^{\mathrm{cosmo}}_{\overline{166}}\,,\; C^{\mathrm{cosmo}}_{\overline{\overline{4}46}} = C^{\mathrm{cosmo}}_{\overline{\overline{6}44}} = \frac{c^{6}}{a^4}\, C^{\mathrm{cosmo}}_{\overline{446}}\,,\; C^{\mathrm{cosmo}}_{\overline{\overline{4}66}} = C^{\mathrm{cosmo}}_{\overline{\overline{6}46}} = \frac{c^{6}}{a^4}\, C^{\mathrm{cosmo}}_{\overline{466}}\,.
\end{aligned}
$$

After this reduction from 56 possible to ten independent components, it is now the task of closure equation $(C10_3)|_{\varphi^A(a,c)}$ to first reduce the number of independent components further to three and then provide the relations entering the differential equations (6.4) and (6.5) for these components. Already restricting the index range of the three free capital indices to the single-overlined range, $(C16_3)|_{\varphi^A(a,c)}$ is

$$
0 = C^{\mathrm{cosmo}}_{\bar{B}_1\bar{B}_2\bar{B}_3}\,\delta^{\gamma}_{\mu} + 3\,C^{\mathrm{cosmo}}_{\bar{A}(\bar{B}_1\bar{B}_2|}\,F^{\bar{A}}{}_{\mu}{}^{\gamma}{}_{:|\bar{B}_3)}\Big|_{\varphi^A(a,c)} + C_{\bar{B}_1\bar{B}_2\bar{B}_3:A}\Big|_{\varphi^A(a,c)}\,F^{A}{}_{\mu}{}^{\gamma}\Big|_{\varphi^A(a,c)}\,.
$$

From the 504 relations, one finds by straightforward evaluation that the previously ten independent components can be reduced further. Now, there are only three independent components, w. l. o. g.

$$
C^{\mathrm{cosmo}}_{\overline{111}} \quad , \quad C^{\mathrm{cosmo}}_{\overline{114}} \quad , \quad C^{\mathrm{cosmo}}_{\overline{146}}\,.
$$

The other seven components can be written in terms of these three as

$$
C^{\mathrm{cosmo}}_{\overline{111}} = C^{\mathrm{cosmo}}_{\overline{444}} = C^{\mathrm{cosmo}}_{\overline{666}} \quad , \quad C^{\mathrm{cosmo}}_{\overline{114}} = C^{\mathrm{cosmo}}_{\overline{116}} = C^{\mathrm{cosmo}}_{\overline{144}} = C^{\mathrm{cosmo}}_{\overline{166}} = C^{\mathrm{cosmo}}_{\overline{446}} = C^{\mathrm{cosmo}}_{\overline{466}}\,.
$$

There are also further non-trivial components which do not enter the cosmological spacetime action as they feature at least one index for which the associated $\dot{\varphi}^A$ vanishes. Yet, one should keep these components in the calculation as they might be needed in other closure equations or prolongations of e. g. $(C10_3)$. These components are expressed in terms of the three independent components by

$$
\begin{aligned}
&C^{\mathrm{cosmo}}_{\overline{122}} = C^{\mathrm{cosmo}}_{\overline{133}} = C^{\mathrm{cosmo}}_{\overline{224}} = C^{\mathrm{cosmo}}_{\overline{336}} = C^{\mathrm{cosmo}}_{\overline{455}} = C^{\mathrm{cosmo}}_{\overline{556}} = \frac{1}{2}\left(C^{\mathrm{cosmo}}_{\overline{111}} - C^{\mathrm{cosmo}}_{\overline{114}}\right),\\
&C^{\mathrm{cosmo}}_{\overline{155}} = C^{\mathrm{cosmo}}_{\overline{226}} = C^{\mathrm{cosmo}}_{\overline{334}} = C^{\mathrm{cosmo}}_{\overline{114}} - C^{\mathrm{cosmo}}_{\overline{146}} \quad \text{and} \quad C^{\mathrm{cosmo}}_{\overline{235}} = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\,C^{\mathrm{cosmo}}_{\overline{111}} - \frac{3}{2}\,C^{\mathrm{cosmo}}_{\overline{114}} + C^{\mathrm{cosmo}}_{\overline{146}}\right).
\end{aligned}
$$

Having arrived at this stage, the first chain rule (6.4) for the component $C^{\mathrm{cosmo}}_{\overline{111}}$ breaks down to

$$
\frac{\partial C^{\mathrm{cosmo}}_{\overline{111}}}{\partial a} = \frac{9}{a}\,C^{\mathrm{cosmo}}_{\overline{111}}
$$

which is solved as

$$
C^{\mathrm{cosmo}}_{\overline{111}} = f_4(c)\,a^9\,.
$$

With the second chain rule (6.5)

$$
\frac{\partial C^{\mathrm{cosmo}}_{\overline{111}}}{\partial c} = -\frac{2\,c}{a}\left(C_{\overline{111}:\bar{1}}\Big|_{\varphi^A(a,c)} + C_{\overline{111}:\bar{4}}\Big|_{\varphi^A(a,c)} + C_{\overline{111}:\bar{6}}\Big|_{\varphi^A(a,c)}\right) - \frac{18}{c}\,C^{\mathrm{cosmo}}_{\overline{111}}
$$

still containing undetermined derivatives of the expansion coefficient, the solution keeps a fifth undetermined function $f_4(c)$ of the second scale factor. In the same way, the two other independent components are

$$C^{\text{cosmo}}_{\overline{114}} = f_5(c)\, a^9 \quad \text{and} \quad C^{\text{cosmo}}_{\overline{146}} = f_6(c)\, a^9$$

with a sixth and seventh undetermined functions f_5 and f_6. The remaining components of the form

$$C^{\text{cosmo}}_{\overline{\overline{A}}_1\overline{A}_2\overline{A}_3} \quad , \quad C^{\text{cosmo}}_{\overline{\overline{A}}_1\overline{\overline{A}}_2\overline{A}_3} \quad \text{and} \quad C^{\text{cosmo}}_{\overline{\overline{A}}_1\overline{\overline{A}}_2\overline{\overline{A}}_3}$$

can be determined from the relations laid out above. Therefore, they also pick up no further undetermined functions.

While the expansion coefficient $C^{\text{cosmo}}_{A_1A_2A_3}$ is subject to more closure equations and prolongations, these turn out to be practically useless. Most prominently, closure equation $(C17_2)|_{\varphi^A(a,c)}$ contains the coefficient $C^{\text{cosmo}}_{A_1A_2A_3}$ directly as

$$0 = 9\, C^{\text{cosmo}}_{A(B_1B_2|}\ M^{A\mu}{}_{:|B_3)}\Big|_{\varphi^A(a,c)} + C_{B_1B_2:B_3}{}^{\mu}\Big|_{\varphi^A(a,c)} + C_{B_1B_3:B_2}{}^{\mu} + C_{B_2B_3:B_1}{}^{\mu}\Big|_{\varphi^A(a,c)}\ ,$$

but also derivatives of the expansion coefficient C_{AB}. Additionally, for one of the desired components (6.21) to appear in this equation, one of the three free capital indices has to be in the index range $\overline{\overline{\overline{m}}}$. This results in the appearance of the up to now undetermined components of the form $C^{\text{cosmo}}_{\overline{A}_1\overline{A}_2\overline{\overline{\overline{m}}}}$ and the unknown derivatives of the form $C_{\overline{A}_1\overline{A}_2:\overline{\overline{\overline{m}}}}{}^{\mu}\Big|_{\varphi^A(a,c)}$. These derivatives are again only contained in complicated prolongations of other closure equations and most likely not determined at all. This means that $(C17_2)|_{\varphi^A(a,c)}$ will not provide any new information on the three independent components of $C^{\text{cosmo}}_{A_1A_2A_3}$.

A similar result is obtained for the prolongation $(C6)_{:D}{}^{\lambda}|_{\varphi^A(a,c)}$ which also contains the expansion coefficient $C^{\text{cosmo}}_{A_1A_2A_3}$ free of any derivatives,

$$\begin{aligned} 0 = 18\, &\Big(C^{\text{cosmo}}_{B_1B_2D}\ p^{\mu\lambda}\Big|_{\varphi^A(a,c)} - C^{\text{cosmo}}_{AB_1B_2}\ p^{\mu\nu}{}_{:D}\Big|_{\varphi^A(a,c)}\ F^{A}{}_{\nu}{}^{\lambda}\Big|_{\varphi^A(a,c)}\Big) \\ &- 4\ C_{A(B_1|:D}{}^{\lambda}\Big|_{\varphi^A(a,c)}\ M^{A\mu}{}_{:|B_2)}\Big|_{\varphi^A(a,c)} - 2\ C_{B_1B_2:A}{}^{\lambda}\Big|_{\varphi^A(a,c)}\ M^{A\mu}{}_{:D}\Big|_{\varphi^A(a,c)} \\ &- C_{B_2:B_1}{}^{\mu}{}_{:D}{}^{\lambda}\Big|_{\varphi^A(a,c)} - C_{B_1:B_2}{}^{\mu}{}_{:D}{}^{\lambda}\Big|_{\varphi^A(a,c)} + 2\ C_{B_1:D:B_2}{}^{\mu\lambda}\Big|_{\varphi^A(a,c)}\ . \end{aligned}$$

Besides the three components of expansion coefficients $C^{\text{cosmo}}_{A_1A_2A_3}$, this equation contains five terms with involved derivatives of other expansion coefficients. In particular, if one restricts the attention to the actually desired components $C^{\text{cosmo}}_{\overline{A}_1\overline{A}_2\overline{A}_3}$, there are derivatives of the form $C_{\overline{B}_1\overline{B}_2:\overline{\overline{\overline{m}}}}{}^{\lambda}\Big|_{\varphi^A(a,c)}$ which is not determined yet and which is also likely not to be determined by a suitable prolongation of other closure equations. Thus, this equation will not provide new information about the components $C^{\text{cosmo}}_{A_1A_2A_3}$ either.

The most promising approach to finding additional information about the fourth expansion coefficient is the first prolongation $(C10_3)_{:D}|_{\varphi^A(a,c)}$ of the previously studied closure equation $(C10_3)$. However, even if one restricts the index range of the four capital indices to the single-overlined values,

$$\begin{aligned} 0 = &\ C_{\overline{B}_1\overline{B}_2\overline{B}_3:\overline{D}}\Big|_{\varphi^A(a,c)}\ \delta^{\gamma}_{\mu} + 3\ C_{\overline{A}(\overline{B}_1\overline{B}_2|:\overline{D}}\Big|_{\varphi^A(a,c)}\ F^{\overline{A}}{}_{\mu}{}^{\gamma}{}_{:|\overline{B}_3)}\Big|_{\varphi^A(a,c)} + C_{\overline{B}_1\overline{B}_2\overline{B}_3:\overline{A}}\Big|_{\varphi^A(a,c)}\ F^{\overline{A}}{}_{\mu}{}^{\gamma}{}_{:\overline{D}}\Big|_{\varphi^A(a,c)} \\ &+ C_{\overline{B}_1\overline{B}_2\overline{B}_3:\overline{D}:A}\Big|_{\varphi^A(a,c)}\ F^{A}{}_{\mu}{}^{\gamma}\Big|_{\varphi^A(a,c)} \end{aligned}$$

one has to analyze 3024 single relations and their combinations in order to identify possible further restrictions to the three independent components $C^{\text{cosmo}}_{\overline{111}}$ and $C^{\text{cosmo}}_{\overline{114}}$ and $C^{\text{cosmo}}_{\overline{146}}$. This amount of relations is too large to be trustfully handled by a pen-and-paper calculation. It shows the complexity of the closure equations as well as the need for their treatment by suitable computer algebra programs.

These loose ends show that the undetermined functions f_4, f_5 and f_6 cannot be determined in a straightforward way – just as it was the case for the previous expansion coefficients as well. One should however note that the three functions f_4, f_5 and f_6 can be merged into one function F_5 by considering the contribution to the cosmological spacetime action,

$$C^{\text{cosmo}}_{A_1A_2A_3}\,\dot{\varphi}^{A_1}\,\dot{\varphi}^{A_2}\,\dot{\varphi}^{A_3}\,.$$

Collecting all terms, one finds that the sum breaks down to one term,

$$C^{\text{cosmo}}_{A_1A_2A_3}\,\dot{\varphi}^{A_1}\,\dot{\varphi}^{A_2}\,\dot{\varphi}^{A_3} = -24\,(f_4 + 6\,f_5 + 2\,f_6)(c)\,\dot{c}^3\,c^3\,a^3\,.$$

This implies to define a new function

$$F_5(c) := -24\,c^3\;(f_4 + 6\,f_5 + 2\,f_6)\,(c)$$

and write the contribution of the expansion coefficient $C^{\text{cosmo}}_{A_1A_2A_3}$ to the cosmological spacetime action (6.6) as

$$C^{\text{cosmo}}_{A_1A_2A_3}\,\dot{\varphi}^{A_1}\,\dot{\varphi}^{A_2}\,\dot{\varphi}^{A_3} = F_5(c)\,\dot{c}^3\,a^3\,.$$

Note that while this coefficient had three different undetermined functions at the level of the closure equations, the cosmological spacetime action sees only one undetermined function. This is special for the expansion coefficient $C^{\text{cosmo}}_{A_1A_2A_3}$. At the same time, expansion coefficient $C^{\text{cosmo}}_{A_1A_2A_3}$ provided the last instance whose solution could be constructed by simply evaluating all index combinations of the respective closure equations – evaluated on $\varphi^A(a,c)$. For all higher order expansion coefficients, the analysis of the equations becomes more involved due to the increasing amount of free capital indices.

This requires to investigate arguments leading to a recursive expression for the higher order expansion coefficients. It will turn out that the closure equations are so involved that meaningful statements about the solutions can hardly be made. Nevertheless, the closure equations contain information about the higher expansion coefficients which is revealed by performing a field redefinition and a change of variables concerning the configuration fields.

6.6 Higher order expansion coefficients & change of variables

One can summarize the closure equations involved in the solution of all expansion coefficients $C^{\text{cosmo}}_{A_1\dots A_N}$ with $N \geq 4$. For a fixed order $M \geq 4$, the expansion coefficient $C^{\text{cosmo}}_{A_1\dots A_M}$ is determined by the following closure equations:

1. Closure equation $(C10_M)|_{\varphi^A(a,c)}$ provides the relations required for the solution of the two chain rule equations (6.4) and (6.5) for each desired component.

2. Closure equation $(C16_{M-1})|_{\varphi^A(a,c)}$ relates different components $C^{\text{cosmo}}_{A_1\dots A_M}$ with each other, but also contains derivatives $C_{A_1\dots A_{M-2}:A_{M-1}}{}^{\mu\nu}\big|_{\varphi^A(a,c)}$ of lower order expansion coefficients.

3. These derivatives are (partially) determined by closure equation $(C12_{M-2})|_{\varphi^A(a,c)}$ and the prolongation $(C13_{M-2})_{\underset{:m}{\equiv}}|_{\varphi^A(a,c)}$. The analysis of these equations generalize the evaluation of closure equation $(C8_3)|_{\varphi^A(a,c)}$ and $(C21_3)_{\underset{:m}{\equiv}}|_{\varphi^A(a,c)}$ for the derivative $C_{:A}{}^{\mu\nu}|_{\varphi^A(a,c)}$. For every fixed component $C_{A_1\dots A_{M-2}}$, its 72 derivatives $C_{A_1\dots A_{M-2}:A_{M-1}}{}^{\mu\nu}\big|_{\varphi^A(a,c)}$ are expressed in terms of 10 independent ones.

4. In contrast to the analysis of the derivative $C_{:A}{}^{\mu\nu}|_{\varphi^A(a,c)}$, closure equation $(C15_{M-2})$ provides additional information on the derivatives $C_{A_1\dots A_{M-2}:A_{M-1}}{}^{\mu\nu}\big|_{\varphi^A(a,c)}$ by establishing an exchange symmetry. This implies that for different $A_1\dots A_{M-2}$, not all derivatives $C_{A_1\dots A_{M-2}:A_{M-1}}{}^{\mu\nu}\big|_{\varphi^A(a,c)}$ are actually independent. Inspect the following example for $M = 4$. Due to the exchange symmetry of $(C15_2)|_{\varphi^A(a,c)}$, the previously non-trivial derivative $C_{\overline{12}:\overline{5}}{}^{xx}\big|_{\varphi^A(a,c)}$ is equal to the trivial one $C_{\overline{15}:\overline{2}}{}^{xx}\big|_{\varphi^A(a,c)}$ and vanishes as well. This provides additional information about the components of $C^{\text{cosmo}}_{A_1\dots A_M}$ via closure equation $(C16_{M-1})|_{\varphi^A(a,c)}$.

In particular, the exchange symmetry $(C15_N)|_{\varphi^A(a,c)}$ in the capital indices makes it very difficult to translate the relation-by-relation calculations of the previous sections to recursive arguments. These are however needed in order to construct a general expression for the higher order expansion coefficients.

Nevertheless, one finds general arguments. From the analysis of $(C16_{M-1})|_{\varphi^A(a,c)}$, it becomes clear that by using the three aforementioned closure equations for the derivatives $C_{B_1\dots B_{M-2}:B_{M-1}}{}^{\mu\nu}\big|_{\varphi^A(a,c)}$, only the components $C^{\text{cosmo}}_{\overline{B}_1\dots\overline{B}_N}$ are the independent ones. The components with at least one double-overlined capital index are expressed by the purely single-overlined components and the independent derivatives $C_{B_1\dots B_{M-2}:B_{M-1}}{}^{\mu\nu}\big|_{\varphi^A(a,c)}$. Closure equation $(C10_M)|_{\varphi^A(a,c)}$ will provide additional relations between different components of $C^{\text{cosmo}}_{A_1\dots A_M}$, but only between those with the same amount of single and double-overlined indices. As already seen in the analysis of $(C10_2)|_{\varphi^A(a,c)}$ for the coefficient C^{cosmo}_{AB}, one will obtain additional information about the derivatives $C_{B_1\dots B_{M-2}:B_{M-1}}{}^{\mu\nu}\big|_{\varphi^A(a,c)}$ from the analysis of $(C10_M)|_{\varphi^A(a,c)}$. Due to the complexity and especially the exchange symmetry of the latter derivatives, it is hardly possible to quantify the number of independent derivatives left to each order M.

The dependence of the respective components on the first scale factor a will be determined by closure equation $(C10_M)|_{\varphi^A(a,c)}$ as the calculations from their first two instances $M = 2$ and $M = 3$ in the previous two sections directly generalize. However, due to the above arguments, the number of functions of the second scale factor c cannot be determined precisely. It is clear that the upper bound is $M + 1$. While the closure equations might in principle leave a larger number of coefficients and derivatives undetermined, it is the contribution of the expansion coefficient $C^{\text{cosmo}}_{A_1\dots A_M}$ to the spacetime action (6.6) given by

$$C^{\text{cosmo}}_{A_1\dots A_N}\,\dot{\varphi}^{A_1}\dots\dot{\varphi}^{A_M}$$

which can be written as

$$C^{\text{cosmo}}_{A_1\dots A_M}\,\dot{\varphi}^{A_1}\dots\dot{\varphi}^{A_M} = \sum_{I=0}^{M} B_I(c)\,\dot{c}^I\,\dot{a}^{M-I}\,a^{3-M+I}$$

with $M + 1$ undetermined functions $B_I(c)$ of the second scale factor c. This ansatz contains almost no information from the closure equations, but is rather the most general form of writing down the terms possibly contributing to the cosmological spacetime action. It turns out that this ansatz is too general and indeed, several of the functions $B_I(c)$ drop out. In order to see this, it is best to perform a field redefinition.

Field redefinition

The goal of the following field redefinition is to reduce the number of non-trivial configuration fields after imposing FLRW symmetries. Yet, the field redefinition has to be implemented before any kind of symmetry reduction and it is simply a different way of distributing the 17 canonical geometric degrees of freedom of general linear electrodynamics. To this end, define four fields

$$\begin{aligned}
h &:= \frac{1}{3}\,\overline{g}^{\alpha\beta}\,\overline{\overline{g}}_{\alpha\beta}\ ,\\
\overline{h}^{\alpha\beta} &:= \overline{g}^{\alpha\beta}\,,\\
\overline{\overline{h}}_{\alpha\beta} &:= \frac{1}{2}\left(\frac{1}{3}\,\overline{g}^{\lambda\omega}\,\overline{\overline{g}}_{\lambda\omega}\,\overline{g}_{\alpha\beta} - \overline{\overline{g}}_{\alpha\beta}\right),\\
\overline{\overline{\overline{h}}}_{\alpha\beta} &:= \overline{\overline{\overline{g}}}_{\alpha\beta}\,.
\end{aligned} \tag{6.22}$$

This redefinition removes the trace with respect to $\overline{g}$ from the field $\overline{\overline{g}}$ and promotes it to an independent field. The field $\overline{\overline{h}}$ remains as a traceless symmetric tensor field with respect to $\overline{h}$ which makes it similar to the third field $\overline{\overline{\overline{h}}}$ which inherits these properties from $\overline{\overline{\overline{g}}}$.

The redefinition of the hypersurface fields is bijective as the original fields are recovered by

$$\overline{g}^{\alpha\beta} = \overline{h}^{\alpha\beta} \quad , \quad \overline{\overline{g}}_{\alpha\beta} = h\,\overline{h}_{\alpha\beta} - 2\,\overline{\overline{h}}_{\alpha\beta} \quad , \quad \overline{\overline{\overline{g}}}_{\alpha\beta} = \overline{\overline{\overline{h}}}_{\alpha\beta}\,.$$

The fields h, $\bar{h}$, $\bar{\bar{h}}$ and $\bar{\bar{\bar{h}}}$ are also parametrized by configuration fields. Denote the new configuration fields by

$$\psi^A = \left(\psi, \bar{\psi}^1, \ldots, \bar{\psi}^6, \bar{\bar{\psi}}^1, \ldots, \bar{\bar{\psi}}^5, \bar{\bar{\bar{\psi}}}^1, \ldots, \bar{\bar{\bar{\psi}}}^5\right)$$

and define the parametrization maps

$$\begin{aligned}
\hat{h}(\psi) &:= \psi \quad , \quad \hat{\bar{h}}^{\alpha\beta}(\psi) := I^{\alpha\beta}{}_A\, \bar{\psi}^A \,, \\
\hat{\bar{\bar{h}}}_{\alpha\beta}(\psi) &:= I^A{}_{\alpha\beta} \left(\delta^B_A - \frac{n_A\, \bar{\psi}^B}{n_C\, \bar{\psi}^C}\right) \epsilon_{(m)\,B}\, \bar{\bar{\psi}}^m \,, \\
\hat{\bar{\bar{\bar{h}}}}_{\alpha\beta}(\psi) &:= I^A{}_{\alpha\beta} \left(\delta^B_A - \frac{n_A\, \bar{\psi}^B}{n_C\, \bar{\psi}^C}\right) \epsilon_{(m)\,B}\, \bar{\bar{\bar{\psi}}}^m \,,
\end{aligned}$$

where capital indices $A, B, \ldots$ again range from 1 to 6 and small latin indices $m, n, \ldots$ from 1 to 5. Also, the components of the $\mathbb{R}^6$-frame $\{t, e_1, \ldots, e_5\}$ and co-frame $\{n, \epsilon^1, \ldots, \epsilon^5\}$ remain the same as developed in Chapter 5. The inverse parametrization maps $\hat{\psi}^A$ are given by

$$\begin{aligned}
\hat{\psi} = h \quad &, \quad \hat{\bar{\psi}}^A = I^A{}_{\alpha\beta}\, \bar{h}^{\alpha\beta} \,, \\
\hat{\bar{\bar{\psi}}}^m = I^{\alpha\beta}{}_A\, e^{(m)\,A}\, \bar{\bar{h}}_{\alpha\beta} \quad &, \quad \hat{\bar{\bar{\bar{\psi}}}}^m = I^{\alpha\beta}{}_A\, e^{(m)\,A}\, \bar{\bar{\bar{h}}}_{\alpha\beta} \,.
\end{aligned}$$

Using this parametrization, one can calculate the three input coefficients entering the gravitational closure equations. The components with respect to the single configuration field ψ will be noted by $\cdot$. The first input coefficient $F^A{}_\mu{}^\gamma$ is

$$\begin{aligned}
F^{\cdot}{}_\mu{}^\gamma = 0 \quad &, \quad \bar{F}^A{}_\mu{}^\gamma = 2\, I^{\bar{A}}{}_{\mu\sigma}\, \bar{h}^{\gamma\sigma} \,, \\
\bar{\bar{F}}^m{}_\mu{}^\gamma &= -2\, I^{\gamma\alpha}{}_A\, e^{(m)A}\, I^B{}_{\mu\alpha} \left(\delta^C_B - \frac{n_B\, \bar{\varphi}^C}{n_F\, \bar{\varphi}^F}\right) \epsilon_{(n)C}\, \bar{\bar{\varphi}}^n \,, \\
\bar{\bar{\bar{F}}}^m{}_\mu{}^\gamma &= -2\, I^{\gamma\alpha}{}_A\, e^{(m)A}\, I^B{}_{\mu\alpha} \left(\delta^C_B - \frac{n_B\, \bar{\varphi}^C}{n_F\, \bar{\varphi}^F}\right) \epsilon_{(n)C}\, \bar{\bar{\varphi}}^n \,.
\end{aligned}$$

The component $p^{\alpha\beta}$ of the principal polynomial given by Eq. (5.5) becomes

$$p^{\alpha\beta} = \frac{1}{6}\left(-2\, h\, \bar{h}^{\alpha\beta} - 2\, \bar{h}^{\alpha\gamma}\, \bar{h}^{\beta\delta}\, \bar{\bar{h}}_{\gamma\delta} - 2\, \bar{h}^{\alpha\beta}\, \bar{h}^{\delta\mu}\, \bar{h}^{\gamma\nu}\, \bar{\bar{\bar{h}}}_{\mu\gamma}\, \bar{\bar{\bar{h}}}_{\nu\delta} + 3\, \bar{h}^{\gamma\delta}\, \bar{h}^{\alpha\mu}\, \bar{h}^{\beta\nu}\, \bar{\bar{\bar{h}}}_{\mu\gamma}\, \bar{\bar{\bar{h}}}_{\nu\delta}\right) .$$

The last input coefficient is $M^{A\gamma}$ which is constructed by using its definition (2.15) as

$$M^{A\gamma}(\psi) = \frac{\partial \hat{\psi}^A}{\partial g^{\mathcal{A}}}\, e^a_0\, \frac{\partial \mathbf{h}^{\mathcal{A}}}{\partial(\partial_\gamma X^a)} \,.$$

One can now apply the definition (6.22) of the new hypersurface fields to this expression. This will provide the possibility of constructing the input coefficient $M^{A\gamma}(\hat{h}(\psi))$ in terms of the components $M^{A\gamma}(\hat{g}(\varphi))$ derived in Chapter 5. Finally, one obtains

$$\begin{aligned}
M^{\cdot\gamma} &= -\frac{4}{3}\, \left(\det \bar{h}\right)^{\frac{1}{2}}\, \bar{h}^{\nu(\lambda}\, \epsilon^{\omega)\mu\gamma}\, \bar{\bar{h}}_{\lambda\omega}\, \bar{\bar{\bar{h}}}_{\mu\nu} \,, \\
\bar{M}^{A\gamma}(\psi) &= \bar{M}^{A\gamma}(g(h(\psi))) \,, \\
\bar{\bar{M}}^{m\gamma}(\psi) &= -\frac{2}{3}\, \left(\det \bar{h}\right)^{\frac{1}{2}}\, I^{\alpha\beta}{}_A\, e^{(m)\,A}\, \bar{h}_{\alpha\beta}\bar{h}^{\nu(\lambda}\, \epsilon^{\omega)\mu\gamma}\, \bar{\bar{\bar{h}}}_{\mu\nu}\, \bar{\bar{h}}_{\lambda\omega} + \frac{1}{2}\, \left(\det \bar{h}\right)^{\frac{1}{2}}\, I^{\alpha\lambda}{}_A\, \epsilon^{\nu\kappa\gamma}\, e^{(m)\,A}\, \bar{h}_{\nu\alpha}\, \bar{\bar{\bar{h}}}_{\lambda\kappa} \\
&\quad - 3\, \left(\det \bar{h}\right)^{-\frac{1}{2}}\, I^{\alpha\beta}{}_A\, e^{(m)\,A}\, \epsilon_{\alpha\mu\nu}\, \bar{\bar{\bar{h}}}_{\beta\lambda}\, \bar{h}^{\lambda\nu}\, p^{\mu\gamma}(h(\psi)) \,, \qquad (6.23) \\
\bar{\bar{\bar{M}}}^{m\gamma}(\psi) &= \bar{\bar{\bar{M}}}^{m\gamma}(g(h(\psi))) \,.
\end{aligned}$$

The general setup and the calculation of the three input coefficients for the gravitational closure equations allow to solve the latter using the configuration fields ψ^A. It will turn out that – after imposing the symmetry condition – these configuration fields give rise to closure equations for which a general recursive argument for all higher order expansion coefficients can be made.

Imposing FLRW symmetries on the redefined fields

The imposition of cosmological symmetries with flat spatial hypersurfaces restricts the redefined fields h to

$$h = \frac{1}{c^2(t)} \quad , \quad \bar{h}^{\alpha\beta} = \frac{c^2(t)}{a^2(t)}\,\mathrm{diag}\,(1,1,1)^{\alpha\beta} \quad \text{and} \quad \bar{\bar{h}}_{\alpha\beta} = \bar{\bar{\bar{h}}}_{\alpha\beta} = 0$$

with the two scale factors $a(t)$ and $c(t)$ as before. In contrast to the hypersurface fields $g^{\mathcal{A}}$, there are only four non-trivial configuration fields

$$\psi = \frac{1}{c^2(t)} \quad , \quad \bar{\psi}^1 = \bar{\psi}^4 = \bar{\psi}^6 = \frac{c^2}{a^2}$$

compared to the six non-trivial ones from before. As a direct consequence, the input coefficient $F^A{}_\mu{}^\gamma$ simplifies to the non-trivial components

$$\begin{aligned}\bar{F}^A{}_\mu{}^\gamma\big|_{\psi^A(a,c)} = {} & \frac{2\,c^2}{a^2}\left(\delta^A_1\,\delta^x_\mu\;\delta^\gamma_x + \delta^A_4\,\delta^y_\mu\;\;\delta^\gamma_y + \delta^A_6\,\delta^z_\mu\,\delta^\gamma_z\right)\\ & + \frac{\sqrt{2}\,c^2}{a^2}\left(\delta^A_2\left(\delta^x_\mu\,\delta^\gamma_y + \delta^y_\mu\,\delta^\gamma_x\right) + \delta^A_3\left(\delta^x_\mu\,\delta^\gamma_z + \delta^z_\mu\,\delta^\gamma_x\right) + \delta^A_5\left(\delta^y_\mu\,\delta^\gamma_z + \delta^z_\mu\,\delta^\gamma_y\right)\right)\end{aligned}$$

while all others vanish. The components $p^{\alpha\beta}$ of the principal polynomial are the same as before,

$$p^{xx}|_{\psi^A(a,c)} = p^{yy}|_{\psi^A(a,c)} = p^{zz}|_{\psi^A(a,c)} = -\frac{h}{3}\,\bar{h}^{\alpha\beta} = -\frac{1}{3\,a^2}\,.$$

Also as before, the components of the third input coefficient $M^{A\gamma}$ vanish after symmetry is imposed. Note that also for the redefined configuration fields ψ^A, there are derivatives of $M^{A\gamma}$ with respect to the configuration fields, such as $M^{\bar{A}\gamma}{}_{:\bar{\bar{\bar{m}}}}|_{\psi^A(a,c)}$, which are non-trivial. These derivatives can play an important role in the evaluation of the symmetry-reduced closure equations and their prolongations as this chapter has already demonstrated. For the higher order expansion coefficients, however, they will play only a minor role.

The last piece of information needed in order to construct the solution for the higher order expansion coefficients is the analog of the chain rule equations (6.4) and (6.5). These are now

$$\frac{\partial C^{\text{cosmo}}_{A_1\dots A_N}}{\partial a} = \left(C_{A_1\dots A_N:\bar{1}}\Big|_{\psi^A(a,c)} + C_{A_1\dots A_N:\bar{4}}\Big|_{\psi^A(a,c)} + C_{A_1\dots A_N:\bar{6}}\Big|_{\psi^A(a,c)}\right)\frac{-2\,c^2}{a^3}\,, \tag{6.24}$$

$$\frac{\partial C^{\text{cosmo}}_{A_1\dots A_N}}{\partial c} = \left(C_{A_1\dots A_N:\bar{1}}\Big|_{\psi^A(a,c)} + C_{A_1\dots A_N:\bar{4}}\Big|_{\psi^A(a,c)} + C_{A_1\dots A_N:\bar{6}}\Big|_{\psi^A(a,c)}\right)\frac{2\,c}{a^2} + C_{A_1\dots A_N:\cdot}\Big|_{\psi^A(a,c)}\,\frac{-2}{c^3}\,. \tag{6.25}$$

Note that especially the first equation looks similar to the chain rule equation known from the solution of the symmetry-reduced metric closure equations in Chapter 4. Consequently, it is again the second chain rule equation which will give rise to undetermined functions of the second scale factor. However, their number is no longer given by the upper bound of $M + 1$ as before. The closure equations can now be evaluated in a more systematic and compact way. This will reveal that for each expansion coefficient, there are at most four undetermined functions entering the cosmological spacetime action and the refined Friedmann equations.

Solution of the fifth expansion coefficient

In order to demonstrate how the redefined configuration fields allow to conduct more general arguments for the solution of the higher order expansion coefficients $C^{\text{cosmo}}_{A_1\ldots A_N}$, the solution of the fifth one will be carried out explicitly here. After that, the arguments are straightforwardly generalized for all higher order expansion coefficients. These will then complete the cosmological spacetime action leading up to the refined Friedmann equations following from general linear electrodynamics.

The closure equations which are involved in the solution are mostly the same as before, $(C10_M)|_{\psi^A(a,c)}$ and $(C16_{M-1})|_{\psi^A(a,c)}$. For $M = 4$, closure equation $(C16_3)|_{\psi^A(a,c)}$ is

$$\begin{aligned} 0 &= -12\,h\,C^{\text{cosmo}}_{\overline{A}B_1B_2B_3}\;\overline{F}^A{}_\rho{}^{(\mu|}\Big|_{\psi^A(a,c)}\,\overline{h}^{|\nu)\rho}\big|_{\psi^A(a,c)} + C_{B_1B_2:B_3}{}^{\mu\nu}\Big|_{\psi^A(a,c)}\,, \\ &= -24\,h\,C^{\text{cosmo}}_{\overline{A}B_1B_2B_3}\;I^{\overline{A}}{}_{\rho\sigma}\,\overline{h}^{\sigma(\mu}\,\overline{h}^{\nu)\rho} + C_{B_1B_2:B_3}{}^{\mu\nu}\Big|_{\psi^A(a,c)}\,, \end{aligned}$$

which contains a derivative of the third expansion coefficient C_{AB}. This derivative still needs to be determined. Before doing so, first rewrite $(C16_3)|_{\varphi^A(a,c)}$ as

$$I^{\overline{A}}{}_{\rho\sigma}\,C^{\text{cosmo}}_{\overline{A}B_1B_2B_3} = \frac{1}{24\,h}\,\overline{h}_{\rho\mu}\,\overline{h}_{\sigma\nu}\;C_{B_1B_2:B_3}{}^{\mu\nu}\Big|_{\psi^A(a,c)}\,, \tag{6.26}$$

Recall that the restricting part of the previous analysis of $(C16_3)|_{\psi^A(a,c)}$ was the determination of independent derivatives $C_{B_1\ldots B_2:B_3}{}^{\mu\nu}\Big|_{\varphi^A(a,c)}$. The field redefinition circumvents this difficulty.

The derivatives $C_{B_1B_2:B_3}{}^{\mu\nu}\Big|_{\psi^A(a,c)}$ are determined by closure equation $(C12_2)|_{\psi^A(a,c)}$. As the input coefficient $F^A{}_\mu{}^\gamma|_{\psi^A(a,c)}$ has only non-trivial components in the single-overlined index range, closure equation $(C12_2)|_{\psi^A(a,c)}$, one directly reads off that 15 derivatives vanish

$$\begin{aligned} 0 &= C_{B_1B_2:\overline{1}}{}^{xx}\Big|_{\psi^A(a,c)} = C_{B_1B_2:\overline{1}}{}^{xy}\Big|_{\psi^A(a,c)} = C_{B_1B_2:\overline{1}}{}^{xz}\Big|_{\psi^A(a,c)} = C_{B_1B_2:\overline{2}}{}^{xx}\Big|_{\psi^A(a,c)} = C_{B_1B_2:\overline{2}}{}^{yy}\Big|_{\psi^A(a,c)} \\ &= C_{B_1B_2:\overline{3}}{}^{xx}\Big|_{\psi^A(a,c)} = C_{B_1B_2:\overline{3}}{}^{zz}\Big|_{\psi^A(a,c)} = C_{B_1B_2:\overline{4}}{}^{xy}\Big|_{\psi^A(a,c)} = C_{B_1B_2:\overline{4}}{}^{yy}\Big|_{\psi^A(a,c)} = C_{B_1B_2:\overline{4}}{}^{yz}\Big|_{\psi^A(a,c)} \\ &= C_{B_1B_2:\overline{5}}{}^{yz}\Big|_{\psi^A(a,c)} = C_{B_1B_2:\overline{5}}{}^{zz}\Big|_{\psi^A(a,c)} = C_{B_1B_2:\overline{6}}{}^{xz}\Big|_{\psi^A(a,c)} = C_{B_1B_2:\overline{6}}{}^{yz}\Big|_{\psi^A(a,c)} = C_{B_1B_2:\overline{6}}{}^{zz}\Big|_{\psi^A(a,c)} \end{aligned}$$

while the other derivatives can be expressed in terms of five independent derivatives,

$$\begin{aligned} C_{B_1B_2:\overline{1}}{}^{yy}\Big|_{\psi^A(a,c)} &= -\sqrt{2}\;C_{B_1B_2:\overline{2}}{}^{xy}\Big|_{\psi^A(a,c)} = C_{B_1B_2:\overline{4}}{}^{xx}\Big|_{\psi^A(a,c)}\,, \\ C_{B_1B_2:\overline{1}}{}^{zz}\Big|_{\psi^A(a,c)} &= -\sqrt{2}\;C_{B_1B_2:\overline{3}}{}^{xz}\Big|_{\psi^A(a,c)} = C_{B_1B_2:\overline{6}}{}^{xx}\Big|_{\psi^A(a,c)}\,, \\ C_{B_1B_2:\overline{6}}{}^{xy}\Big|_{\psi^A(a,c)} &= \frac{1}{\sqrt{2}}\;C_{B_1B_2:\overline{2}}{}^{zz}\Big|_{\psi^A(a,c)} = -\sqrt{2}\;C_{B_1B_2:\overline{3}}{}^{yz}\Big|_{\psi^A(a,c)} = -\sqrt{2}\;C_{B_1B_2:\overline{5}}{}^{xz}\Big|_{\psi^A(a,c)}\,, \\ C_{B_1B_2:\overline{3}}{}^{yy}\Big|_{\psi^A(a,c)} &= \sqrt{2}\;C_{B_1B_2:\overline{4}}{}^{xz}\Big|_{\psi^A(a,c)} = -2\;C_{B_1B_2:\overline{2}}{}^{yz}\Big|_{\psi^A(a,c)} = -2\;C_{B_1B_2:\overline{5}}{}^{xy}\Big|_{\psi^A(a,c)}\,, \\ C_{B_1B_2:\overline{5}}{}^{xx}\Big|_{\psi^A(a,c)} &= \sqrt{2}\;C_{B_1B_2:\overline{1}}{}^{yz}\Big|_{\psi^A(a,c)} = -2\;C_{B_1B_2:\overline{2}}{}^{xz}\Big|_{\psi^A(a,c)} = -2\;C_{B_1B_2:\overline{3}}{}^{xy}\Big|_{\psi^A(a,c)}\,. \end{aligned}$$

These relations and especially the vanishing derivatives can now be used in the analysis of closure equation $(C16_3)|_{\varphi^A(a,c)}$ in the form given by Eq. (6.26). The strategy is to consider only the components possibly contributing to the cosmological spacetime action, that is, the components coupling exclusively to the non-vanishing derivatives $\dot{\psi}$, $\dot{\overline{\psi}}{}^1$, $\dot{\overline{\psi}}{}^4$ and $\dot{\overline{\psi}}{}^6$. In order to do so, it is best to consider a distinction of cases regarding the amount of free indices representing the degree of freedom ψ in Eq. (6.26).

Case 1: $B_1 \to \overline{B}_1$**,** $B_2 \to \overline{B}_2$**,** $B_3 \to \overline{B}_3$ In this case, Eq. (6.26) is

$$I^{\overline{A}}{}_{\rho\sigma}\,C^{\text{cosmo}}_{\overline{A}\overline{B}_1\overline{B}_2\overline{B}_3} = \frac{1}{24\,h}\,\overline{h}_{\rho\mu}\,\overline{h}_{\sigma\nu}\;C_{\overline{B}_1\overline{B}_2:\overline{B}_3}{}^{\mu\nu}\Big|_{\psi^A(a,c)}\,.$$

As the only relevant index values for $\overline{A}$ are $\overline{1}$, $\overline{4}$ and $\overline{6}$, one writes down the relations for the free indices $\rho = \sigma = x$, $\rho = \sigma = y$ and $\rho = \sigma = z$ as

$$\begin{aligned} C^{\text{cosmo}}_{\overline{1}\overline{B}_1\overline{B}_2\overline{B}_3} &= \frac{a^4}{24\,c^2}\; C_{\overline{B}_1\overline{B}_2:\overline{B}_3}{}^{xx}\Big|_{\psi^A(a,c)}\,, \\ C^{\text{cosmo}}_{\overline{4}\overline{B}_1\overline{B}_2\overline{B}_3} &= \frac{a^4}{24\,c^2}\; C_{\overline{B}_1\overline{B}_2:\overline{B}_3}{}^{yy}\Big|_{\psi^A(a,c)} \quad \text{and} \\ C^{\text{cosmo}}_{\overline{6}\overline{B}_1\overline{B}_2\overline{B}_3} &= \frac{a^4}{24\,c^2}\; C_{\overline{B}_1\overline{B}_2:\overline{B}_3}{}^{zz}\Big|_{\psi^A(a,c)}\,. \end{aligned}$$

Using these three relations together with the exchange symmetry

$$C_{B_2B_3:B_1}{}^{\mu\nu}\Big|_{\psi^A(a,c)} = C_{B_1B_3:B_2}{}^{\mu\nu}\Big|_{\psi^A(a,c)} = C_{B_1B_2:B_3}{}^{\mu\nu}\Big|_{\psi^A(a,c)}$$

of closure equation $(C15_2)|_{\psi^A(a,c)}$, one can show that all 15 components possibly contributing to the gravitational spacetime action vanish,

$$\begin{aligned} 0 &= C^{\text{cosmo}}_{\overline{1111}} = C^{\text{cosmo}}_{\overline{1114}} = C^{\text{cosmo}}_{\overline{1116}} = C^{\text{cosmo}}_{\overline{1144}} = C^{\text{cosmo}}_{\overline{1146}} = C^{\text{cosmo}}_{\overline{1166}} = C^{\text{cosmo}}_{\overline{1444}} = C^{\text{cosmo}}_{\overline{1446}} = C^{\text{cosmo}}_{\overline{1466}} = C^{\text{cosmo}}_{\overline{1666}} \\ &= C^{\text{cosmo}}_{\overline{4444}} = C^{\text{cosmo}}_{\overline{4446}} = C^{\text{cosmo}}_{\overline{4466}} = C^{\text{cosmo}}_{\overline{4666}} = C^{\text{cosmo}}_{\overline{6666}}\,. \end{aligned}$$

Case 2: $B_1 \to \cdot$, $B_2 \to \overline{B_2}$, $B_3 \to \overline{B}_3$ In analogy to the first case, Eq. (6.26) breaks down to the three relations

$$\begin{aligned} C^{\text{cosmo}}_{\cdot\overline{1}\overline{B}_2\overline{B}_3} &= \frac{a^4}{24\,c^2}\; C_{\cdot\overline{B}_2:\overline{B}_3}{}^{xx}\Big|_{\psi^A(a,c)}\,, \\ C^{\text{cosmo}}_{\cdot\overline{4}\overline{B}_2\overline{B}_3} &= \frac{a^4}{24\,c^2}\; C_{\cdot\overline{B}_2:\overline{B}_3}{}^{yy}\Big|_{\psi^A(a,c)} \quad \text{and} \\ C^{\text{cosmo}}_{\cdot\overline{6}\overline{B}_2\overline{B}_3} &= \frac{a^4}{24\,c^2}\; C_{\cdot\overline{B}_2:\overline{B}_3}{}^{zz}\Big|_{\psi^A(a,c)}\,. \end{aligned}$$

Evaluating these relations for the indices $\overline{B}_2$ and $\overline{B}_3$ taking values either $\overline{1}$, $\overline{4}$ or $\overline{6}$, one obtains that the only non-vanishing component is $C^{\text{cosmo}}_{\cdot\overline{146}}$. It needs to be determined by closure equation $(C10_4)|_{\psi^A(a,c)}$ given by

$$0 = C_{\cdot\overline{146}} + 3\,C^{\text{cosmo}}_{\cdot\overline{A}(\overline{14}|}\; F^{\overline{A}}{}_{\mu}{}^{\gamma}{}_{:|\overline{6})}\Big|_{\psi^A(a,c)} + C_{\cdot\overline{146}:\overline{A}}\Big|_{\psi^A(a,c)}\; \overline{F}^{A}{}_{\mu}{}^{\gamma}\Big|_{\psi^A(a,c)}\,.$$

From this equation, one determines the three terms

$$C_{\cdot\overline{146}:\overline{1}}\Big|_{\psi^A(a,c)} = C_{\cdot\overline{146}:\overline{4}}\Big|_{\psi^A(a,c)} = C_{\cdot\overline{146}:\overline{6}}\Big|_{\psi^A(a,c)} = -\frac{3\,a^2}{2\,c^2}\,C^{\text{cosmo}}_{\cdot\overline{146}}$$

which one needs in order to solve the chain rule equation (6.24). One obtains

$$C^{\text{cosmo}}_{\cdot\overline{146}} = f_7(c)\,a^9\,.$$

With the derivative $C_{\cdot\overline{146}:\cdot}\big|_{\psi^A(a,c)}$ undetermined from closure equation $(C10_4)|_{\psi^A(a,c)}$, the second chain rule equation (6.25) cannot be solved to remove the undetermined function $f_7(c)$ from the component. Also, considering the simplest prolongation $(C16_2)_{:\cdot}|_{\psi^A(a,c)}$ provides no information on the derivative $C_{\cdot\overline{146}:\cdot}\big|_{\psi^A(a,c)}$. Also, prolongations of other closure equations are not expected to reveal new information about this component as the discussion at the end of this section will demonstrate.

Case 3: $B_1 \to \cdot$, $B_2 \to \cdot$, $B_3 \to \overline{B_3}$ The procedure to solve this case is analogous to the previous one. Eq. (6.26) separates into three relevant relations,

$$
\begin{aligned}
C^{\text{cosmo}}_{\cdot\cdot\overline{1B_3}} &= \frac{a^4}{24\,c^2}\; C_{\cdot\cdot:\overline{B_3}}{}^{xx}\Big|_{\psi^A(a,c)}\,, \\
C^{\text{cosmo}}_{\cdot\cdot\overline{4B_3}} &= \frac{a^4}{24\,c^2}\; C_{\cdot\cdot:\overline{B_3}}{}^{y}\Big|_{\psi^A(a,c)} \quad \text{and} \\
C^{\text{cosmo}}_{\cdot\cdot\overline{6B_3}} &= \frac{a^4}{24\,c^2}\; C_{\cdot\cdot:\overline{B_3}}{}^{zz}\Big|_{\psi^A(a,c)}
\end{aligned}
$$

which leave the components $C^{\text{cosmo}}_{\cdot\cdot\overline{14}}$, $C^{\text{cosmo}}_{\cdot\cdot\overline{16}}$ and $C^{\text{cosmo}}_{\cdot\cdot\overline{46}}$ undetermined. The following analysis of closure equation $(C10_4)|_{\psi^A(a,c)}$ reveals that the three components are actually identical and one only needs to determine the component $C^{\text{cosmo}}_{\cdot\cdot\overline{14}}$ which is found to be

$$
C^{\text{cosmo}}_{\cdot\cdot\overline{14}} = f_8(c)\,a^7
$$

with a new undetermined function $f_8(c)$ of the second scale factor $c(t)$.

Case 4: $B_1 \to \cdot$, $B_2 \to \cdot$, $B_3 \to \cdot$ As Eq. (6.26) contains the undetermined derivative $C_{\cdot\cdot:\cdot}{}^{\mu\nu}|_{\varphi^A(a,c)}$ which is also not contained in any other closure equation, one has to determine the three components $C^{\text{cosmo}}_{\cdots\overline{1}}$, $C^{\text{cosmo}}_{\cdots\overline{4}}$ and $C^{\text{cosmo}}_{\cdots\overline{6}}$ directly from closure equation $(C10_4)|_{\psi^A(a,c)}$. One concludes from there that the three desired components are equal and one determines

$$
C^{\text{cosmo}}_{\cdots\overline{1}} = C^{\text{cosmo}}_{\cdots\overline{4}} = C^{\text{cosmo}}_{\cdots\overline{6}} = f_9(c)\,a^5
$$

with a new undetermined function f_9.

The last missing component of expansion coefficient $C^{\text{cosmo}}_{A_1A_2A_3A_4}$ is $C^{\text{cosmo}}_{\cdots\cdot}$ which does not appear in closure equation $(C16_3)|_{\psi^A(a,c)}$. Thus, the only way to determine this component is via closure equation $(C10_4)|_{\psi^A(a,c)}$ with $B_1 = B_2 = B_3 = B_4 = \cdot$. One finds a new undetermined function $f_{10}(c)$ as part of the solution

$$
C^{\text{cosmo}}_{\cdots\cdot} = f_{10}(c)\,a^3\,.
$$

This component completes the contribution of the expansion coefficient $C^{\text{cosmo}}_{A_1A_2A_3A_4}$ to the cosmological spacetime action. Collecting all terms, one finds

$$
C^{\text{cosmo}}_{A_1A_2A_3A_4}\,\dot\psi^{A_1}\dot\psi^{A_2}\dot\psi^{A_3}\dot\psi^{A_4} = F_6(c)\,\dot c^4\,a^3 + F_7(c)\,\dot c^3\,\dot a\,a^2 + F_8(c)\,\dot c^2\,\dot a^2\,a + F_9(c)\,\dot c\,\dot a^3
$$

with the four undetermined functions $F_6, \ldots, F_9$ defined as

$$
\begin{aligned}
F_6(c) &:= 16\,f_{10}(c)\,c^{-12} - 48\,f_9(c)\,c^{-8} + 96\,f_8(c)\,c^{-4} + 192\,f_7(c)\,, \\
F_7(c) &:= 48\,f_9(c)\,c^{-7} - 192\,f_8(c)\,c^{-3} - 576\,f_7(c)\,c\,, \\
F_8(c) &:= 96\,f_8(c)\,c^{-2} + 576\,f_7(c)\,c^2 \quad \text{and} \\
F_9(c) &:= -192\,f_7(c)\,c^3\,.
\end{aligned}
$$

Note that the expansion coefficient $C^{\text{cosmo}}_{B_1\ldots B_4}$ contributes only terms containing derivative orders of at most $\dot a^3$. A term containing the derivative $\dot a^4$ does not appear although it would be admissible in general. This result will be recovered for the further higher order expansion coefficients. In this fashion, the closure equations restrict the form of the cosmological spacetime action by dismissing terms with larger powers than $\dot a^3$ of the first scale factor $a(t)$.

Results for all higher order expansion coefficients

The arguments developed for the solution of the fifth expansion coefficient $C^{\text{cosmo}}_{B_1B_2B_3B_4}$ translate directly to all higher order expansion coefficients. One always needs to evaluate closure equation $(C16_{M-1})|_{\psi^A(a,c)}$ in the form

$$I^A{}_{\rho\sigma}\, C^{\text{cosmo}}_{AB_1\ldots B_M} = \frac{3\,(M-3)}{4\,M(M-1))\,\psi}\,\hat{\bar{h}}_{\rho\mu}(\psi)\,\hat{\bar{h}}_{\sigma\nu}(\psi)\;\; C_{B_1\ldots B_{M-1}:B_{M-2}}{}^{\mu\nu}\Big|_{\psi^A(a,c)}$$

using a distinction of cases for the number of capital indices $B = \cdot$. The derivative $C_{B_1\ldots B_{M-1}:B_{M-2}}{}^{\mu\nu}\Big|_{\psi^A(a,c)}$ is determined by $(C12_{M-2})|_{\psi^A(a,c)}$ in the precise same way as above for $M = 4$ only with a larger number of indices. That means, for fixed indices $B_1 \ldots B_{M-2}$, there are 15 trivial derivatives and the remaining ones are expressed in terms of five independent ones. The evaluation of $(C16_{M-1})|_{\psi^A(a,c)}$ is then performed only for the possibly contributing components of $C^{\text{cosmo}}_{B_1\ldots B_M}$ by setting by evaluating the equation for the free spatial indices $\rho = \sigma = x$, $\rho = \sigma = y$ and $\rho = \sigma = z$ as well as the capital indices taking values $\overline{1}$, $\overline{4}$, $\overline{6}$ and $\cdot$.

The first non-trivial component of expansion coefficient $C^{\text{cosmo}}_{B_1\ldots B_M}$ is always the one with $M-3$ many indices attributed to the configuration field ψ,

$$C^{\text{cosmo}}_{\underbrace{\cdot\,\ldots\,\cdot}_{M-3\text{ many}}\,\overline{1}\overline{4}\overline{6}}\,.$$

The remaining non-trivial components are just as before

$$C^{\text{cosmo}}_{\underbrace{\cdot\,\ldots\,\cdot}_{M-2\text{ many}}\,\overline{1}\overline{4}}\quad,\quad C^{\text{cosmo}}_{\underbrace{\cdot\,\ldots\,\cdot}_{M-1\text{ many}}\,\overline{1}}\quad\text{and}\quad C^{\text{cosmo}}_{\underbrace{\cdot\,\ldots\,\cdot}_{M\text{ many}}}\,.$$

The contribution to the spacetime action subsequently contains four undetermined functions for each order M of the expansion coefficients,

$$C^{\text{cosmo}}_{A_1\ldots A_M}\,\dot{\psi}^{A_1}\ldots\dot{\psi}^{A_M} = F_{M0}(c)\,\dot{c}^M\,a^3 + F_{M1}(c)\,\dot{c}^{M-1}\,\dot{a}\,a^2 + F_{M2}(c)\,\dot{c}^{M-2}\,\dot{a}^2\,a + F_{M3}(c)\,\dot{c}^{M-3}\,\dot{a}^3\,.$$

This result achieves the goal of expressing all higher order derivative in a recursive formula which enables one to write the cosmological spacetime action in a compact way. Nevertheless, picking up four undetermined functions for each additional order of the expansion coefficients is unsatisfying.

One might wonder whether prolongations of the closure equations further restrict these undetermined functions and either reduce their number or determine them completely. While this conjecture cannot be completely dismissed, considering possible prolongations certainly requires to introduce a large computer algebra system – possibly so large that it is almost prohibitive to present day programs. In order to obtain derivatives with respect to the configuration field ψ, one e. g. has to use second prolongations of the input coefficient $M^{A\gamma}$. This however will result in very involved (and possibly not even determined) terms in the thus obtained equations.

Such calculations – especially if an index-by-index evaluation is required – will feature a huge number of single relations. As already mentioned, one possible way to address these issues is by a compatible computer algebra system which can perform the index-by-index evaluations. It is doubtful that the data storage and runtime of present-day computer algebra systems suffice here. The other – mathematical – way to gain information about the undetermined functions in the solution is to consider the gravitational closure equations as a system of partial differential equations and extract information about the dimensionality of their solution space. This is closely related to the question whether the closure equations are involutive, that is, whether all information required for their solution is already explicitly present. As this thesis already demonstrated, prolongations indeed provide additional information and therefore, the closure equations will most likely not be involutive in their form presented here. Thus, they would have to be made involutive by the Cartan-Kuranishi algorithm. This however points to a mathematically very involved topic which vastly exceeds the scope of this thesis. More remarks on involutivity of partial differential equations and first applications to the closure equations can be found in Ref. [20].

This discussion demonstrates that pen-and-paper calculations have limits when the system of partial differential equations becomes too large or too involved. Nevertheless, it was possible to construct a solution from the symmetry-reduced closure equations of general linear electrodynamics. The remaining task of this section is to collect all terms and write down the cosmological spacetime action by plugging the results into Eq. (6.6).

Cosmological spacetime action

After imposing the symmetry condition, the cosmological spacetime action could already be written in a compact form given by Eq. (6.6). To each power in the derivatives of the configuration fields $\dot{\varphi}^A$, the essential terms were the sums

$$C^{\text{cosmo}}_{A_1 \dots A_N} \dot{\varphi}^{A_1} \dots \dot{\varphi}^{A_N}$$

which constitute the cosmological spacetime action. Therefore, after obtaining the solution for every expansion coefficient, already those terms were calculcated in order to see how many undetermined functions F actually enter the cosmological spacetime action. Collecting all terms from the previous sections, the cosmological action following from general linear electrodynamics is

$$\begin{aligned} S_{\text{cosmo}} = \int \mathrm{d}t \Big(& N\, F_0(c)\, a^3 + F_1(c)\, \dot{c}\, a^3 + (F_2(c)\, \dot{c}^2\, a^3 + F_3(c)\, \dot{a}^2\, a + F_4(c)\, \dot{c}\, \dot{a}\, a^2)\, N^{-1} + F_5(c)\, \dot{c}^3\, a^3\, N^{-2} \\ & + \sum_{M=4}^{\infty} N^{1-M} \Big(F_{M0}(c)\, \dot{c}^M\, a^3 + F_{M1}(c)\, \dot{c}^{M-1}\, \dot{a}\, a^2 + F_{M2}(c)\, \dot{c}^{M-2}\, \dot{a}^2\, a + F_{M3}(c)\, \dot{c}^{M-3}\, \dot{a}^3 \Big) \Big) . \end{aligned}$$

This formula can be written even more compactly as

$$S_{\text{cosmo}} = \int \mathrm{d}t \sum_{K=0}^{\infty} \sum_{L=0}^{3} f_{KL}(c)\, \dot{c}^K\, \dot{a}^L\, a^{3-L}\, N^{1-K-L} , \tag{6.27}$$

where the four functions $f_{01}(c)$, $f_{03}(c)$, $f_{12}(c)$, $f_{21}(c)$ vanish. Note that the two scale factors $a(t)$ and $c(t)$ as well as the lapse function $N(t)$ depend only on the foliation parameter t. The two scale factors and the lapse are also the only symmetric d. o. f. s left in the action. Variation with respect to these three functions provides the geometric part of the refined Friedmann equations which will consist of one constraint (variation w. r. t. the lapse function N) and two evolution equations (variation w. r. t. the scale factors). In order to complete the refined Friedmann equations, one needs to impose the symmetry condition of spatial homogeneity and isotropy also on the matter sourcing the gravitational dynamics. This is covered by the next section.

6.7 Refined matter sources

In order to find the gravitational sources for the refined Friedmann equations of general linear electrodynamics, one can straightforwardly generalize the arguments used in Section 4.3. Again, the gravitational sources are in general given by the expression

$$\frac{\delta S_{\text{matter}}}{\delta G}$$

appearing on the right hand side of the gravitational field equations. For the geometry G^{abcd} of general linear electrodynamics, one reads off the source tensor density from this as

$$\widetilde{S}_{abcd} := -4\, \frac{\delta S_{\text{matter}}}{\delta G^{abcd}} .$$

The stress-energy-momentum tensor density a la Gotay and Marsden is still a $(1,1)$ tensor density $\widetilde{T}$ which using Ref. [25] can be related to the source tensor density as

$$\widetilde{T}^a{}_b = G^{ampq}\, \widetilde{S}_{bmpq} . \tag{6.28}$$

As already laid out in the case of standard cosmology in Section 4.3, cosmological symmetries have to hold on large scales only. Imposing them locally at all scales might not even be compatible with non-trivial solutions of the matter field equations. Thus, the gravitational sources for cosmological have to be modelled differently. As the symmetry condition has to hold at the respectively large scales, it is again the Killing condition which has to be satisfied by the two tensorial quantities $T = \widetilde{T}\,\omega_G^{-1}$ and $S = \widetilde{S}\,\omega_G^{-1}$. Since the Killing condition for the stress-energy-momentum tensor T is independent of the chosen spacetime geometry, it is written again as

$$T^m{}_n = \operatorname{diag}(\rho, -p, -p, -p)^m{}_n\,,$$

just as in Section 4.3. The source tensor S also satisfies the Killing condition which results in S containing three undetermined functions of the foliation time t. For the components of S, one can write

$$\begin{aligned} S_{0\alpha0\beta} &= -C_1(t)\,N^2\,a^2\,c^{-2}\,\gamma_{\alpha\beta}\,,\\ S_{0\alpha\beta\gamma} &= C_2(t)\,N\,a^3\,c^{-3}\,\epsilon_{\alpha\beta\gamma}\,\sqrt{\det\gamma}\,,\\ S_{\alpha\beta\gamma\delta} &= -2\,C_3(t)\,a^4\,c^{-2}\,\gamma_{\alpha[\gamma}\gamma_{\delta]\beta}\,, \end{aligned} \tag{6.29}$$

with $\gamma_{\alpha\beta} = \operatorname{diag}(1, 1, 1)_{\alpha\beta}$. Note that the determinant of γ is 1, but it is still denoted in the second equation in order to demonstrate that this component is indeed tensorial. By evaluating the connection (6.28) of stress-energy-momentum and gravitational sources, one finds

$$\rho(t) = 6\,(C_1(t) + C_2(t)) \quad \text{and} \quad p(t) = 2\,(2\,C_3(t) - C_1(t) - 3\,C_2(t))\,.$$

This means that there has to be a third quantity which is acting only as a gravitational source, but is not related to stress-energy-momentum. It can be identified by the covariant conservation of the source and stress-energy-momentum tensor densities on-shell,

$$0 = \partial_m(\omega_G\,T^m{}_n) + \frac{\omega_G}{4}\,S_{abcd}\,\partial_n G^{abcd}\,.$$

For $n = 0$, this conservation equation provides

$$0 = \dot\rho(t) + 3\,\frac{\dot a}{a}\,(\rho + p) + 3\,\frac{\dot c}{c}\,q(t)$$

with the function $q(t)$ defined as

$$q(t) := -12\,C_1(t) - 6\,C_3(t)\,.$$

The interpretation of this new type of gravitational source is not yet clear as this would require a proper study of perfect fluids on spacetimes with general linear electrodynamics. Also note that in the metric limit when $c = 1$ and $\dot c = 0$, one recovers the well-known conservation equation from textbook cosmology.

While the three quantities ρ, p and q are determined by the three undetermined functions from the source tensor (6.29), these relations can also be inverted as

$$C_1 = -\frac{q}{6} - \frac{\rho + p}{4} \quad , \quad C_2 = -\frac{\rho}{12} - \frac{p}{4} - \frac{q}{6} \quad , \quad C_3 = \frac{\rho + p}{2} + \frac{q}{6}\,.$$

By identifying the gravitational sources of the cosmological dynamics of general linear electrodynamics, everything has been prepared for the actual goal of this chapter – the derivation of the refined Friedmann equations. The cosmological spacetime action will be combined with the components of the gravitational source tensor developed in this section. Variation with respect to the symmetric geometric degrees of freedom will then give rise to the refined Friedmann equations.

6.8 The refined Friedmann equations

The calculations from the seven sections of this chapter were carried out in order to derive the refined Friedmann equations following from general linear electrodynamics. The first step was to set up the symmetry-reduced gravitational closure equations by imposing the FLRW symmetry on the three input coefficients from Chapter 5. After a quite involved calculation, the cosmological spacetime action S_{cosmo} was obtained as a solution of the closure equations. The gravitational sources described by three variables $\rho(t)$, $p(t)$ and $q(t)$ on a spacetime with general linear electrodynamics complete the ingredients of the refined Friedmann equations.

As there are two scale factors and the lapse function as the symmetric geometric degrees of freedom, it is clear that there will be one constraint equation and two evolution equations. The simplest of the three equations is the constraint equation. The procedure is the same as in Section 4.3 in which the Friedmann equations of standard model cosmology were derived. First, perform the variation of S_{cosmo} from Eq. (6.27) with respect to the lapse and then choose a parametrization where the lapse is constant everywhere. This yields

$$\left.\frac{\delta S_{\text{cosmo}}}{\delta N(t)}\right|_{N=1} = \sum_{K=0}^{\infty}\sum_{L=0}^{3} f_{KL}(c)\,\dot{c}^K\,\dot{a}^L\,a^{3-L}\,(1-K-L)\,.$$

By introducing the source tensor S_{abcd} in the variation

$$\left.\frac{\delta S_{\text{matter}}}{\delta N(t)}\right|_{N=1} = -\left.\left(\frac{\omega_G}{4}\,S_{abcd}\,\frac{\partial G^{abcd}}{\partial N}\right)\right|_{N=1} = \frac{a^3}{c^3}\,\rho$$

one obtains – after dividing out the factor a^3 – the constraint equation

$$0 = \rho + c^3\sum_{K=0}^{\infty}\sum_{L=0}^{3}(1-K-L)\,f_{KL}(c)\,\dot{c}^K\left(\frac{\dot{a}}{a}\right)^L\,. \tag{6.30}$$

For the first evolution equation, one proceeds in the same way as in Section 4.3 and performs the variation

$$\begin{aligned}\left.\frac{\delta S_{\text{cosmo}}}{\delta a(t)}\right|_{N=1} &= \sum_{K=0}^{\infty}\sum_{L=0}^{2}(3-L)\,(1-L)\,f_{KL}(c)\,\dot{c}^K\,\dot{a}^L a^{2-L}\\ &\quad-\sum_{K=0}^{\infty}\sum_{L=1}^{3} L\,a^{3-L}\left[f'_{KL}(c)\,\dot{c}^{K+1}\,\dot{a}^{L-1}+K\,f_{KL}(c)\,\ddot{c}\,\dot{c}^{K-1}\,\dot{a}^{L-1}+(L-1)\,f_{KL}(c)\,\dot{c}^K\,\dot{a}^{L-2}\,\ddot{a}\right].\end{aligned}$$

Complementing this by the gravitational sources

$$\left.\frac{\delta S_{\text{matter}}}{\delta a(t)}\right|_{N=1} = -3\,\frac{a^2}{c^3}\,p(t)\,,$$

one obtains the first evolution equation to be

$$\begin{aligned}0 &= -3\,p(t) + c^3\sum_{K=0}^{\infty}\sum_{L=0}^{2}(3-L)\,(1-L)\,f_{KL}(c)\,\dot{c}^K\left(\frac{\dot{a}}{a}\right)^L\\ &\quad-\sum_{K=0}^{\infty}\sum_{L=1}^{3} L\left[f'_{KL}(c)\,\dot{c}^{K+1}\left(\frac{\dot{a}}{a}\right)^{L-1}+K\,f_{KL}(c)\,\ddot{c}\,\dot{c}^{K-1}\left(\frac{\dot{a}}{a}\right)^{L-1}+(L-1)\,f_{KL}(c)\,\dot{c}^K\left(\frac{\dot{a}}{a}\right)^{L-2}\frac{\ddot{a}}{a}\right].\end{aligned} \tag{6.31}$$

The second evolution equation is then derived by first varying the cosmological action (6.27) with respect to the second scale factor $c(t)$ which results in

$$\begin{aligned}\left.\frac{\delta S_{\text{cosmo}}}{\delta c(t)}\right|_{N=1} &= \sum_{K=0}^{\infty}\sum_{L=0}^{3}(1-K)\,f'_{KL}(c)\,\dot{c}^K\,\dot{a}^L\,a^{3-L}\\ &\quad-\sum_{K=0}^{\infty}\sum_{L=0}^{3} K\,f_{KL}(c)\left[(K-1)\,\ddot{c}\,\dot{c}^{K-2}\,\dot{a}^L\,a^{3-L}+L\,\dot{c}^{K-1}\,\dot{a}^{L-1}\,\ddot{a}\,a^{3-L}+(3-L)\,\dot{c}^{K-1}\,\dot{a}^{L+1}\,a^{2-L}\right].\end{aligned}$$

The gravitational sources introduce the new third variable $q(t)$ to the refined Friedmann equations by

$$\left.\frac{\delta S_{\text{matter}}}{\delta c(t)}\right|_{N=1} = -\frac{a^3}{c^4}\,(3\,\rho + q)\,.$$

Combining these two expressions and multiplying with the factor $c^4\,a^{-3}$, the third refined Friedmann equation is

$$\begin{aligned}0 = &-(q + 3\,\rho) + c^4 \sum_{K=0}^{\infty}\sum_{L=0}^{3}(1-K)\,f'_{KL}(c)\,\dot{c}^K\left(\frac{\dot{a}}{a}\right)^L \\ &- c^4 \sum_{K=0}^{\infty}\sum_{L=0}^{3} K\,f_{KL}(c)\left[(K-1)\,\ddot{c}\,\dot{c}^{K-2}\left(\frac{\dot{a}}{a}\right)^L + L\,\dot{c}^{K-1}\left(\frac{\dot{a}}{a}\right)^{L-1}\frac{\ddot{a}}{a} + (3-L)\,\dot{c}^{K-1}\left(\frac{\dot{a}}{a}\right)^{L+1}\right]. \qquad (6.32)\end{aligned}$$

The three equations (6.30) – (6.32) present the refined Friedmann equations for a spacetime with general linear electrodynamics. Although the four functions f_{01}, f_{03}, f_{12} and f_{21} vanish, the three refined Friedmann equation still contain countably many undetermined functions f_{KL} of the second scale factor c. This prohibits the theory from being predictive as one would first have to conduct infinitely many experiments in order to determine all functions – supposing they are analytical. While this might be unsatisfying at first, the three refined Friedmann equations still provide a basis for valuable discussions about possible fingerprints of general linear electrodynamics in cosmology. Possible topics include the case of weak deviations from the metric limit which is given by $c(t) = 1$. This will be discussed in the next chapter which will also summarise the results obtained in this work and present future perspectives.

Chapter 7

Summary, results and future topics

It is the central result of the constructive gravity program that prescribed matter dynamics already contain all information about the gravitational dynamics if three physically weak conditions – called the matter conditions – are satisfied [3]. Conceptually, this is achieved by requiring causal compatibility between matter and gravitational dynamics. This means, both theories share initial data surfaces and evolution of those surfaces. This physically rather weak requirement turns out to be technically so strong that the gravitational action can, and hence has to be derived from given matter dynamics. Technically, the key is to solve a countable system of partial differential equations, the gravitational closure equations.

The procedure to gravitationally close matter dynamics satisfying the three matter conditions from Section 2.1 can be summarized in three steps.

1. Calculate the three input coefficients ${F^A}_\mu{}^\gamma$, $p^{\mu\nu}$ and $M^{A\gamma}$ from the matter dynamics; these are the coefficient functions entering the gravitational closure equations.

2. Solve the gravitational closure equations for the expansion coefficients which constitute the gravitational Lagrangian.

3. Construct the gravitational spacetime action S_{gravity} and derive the gravitational field equations by variation with respect to the geometric degrees of freedom.

The probably simplest, but at the same time indispensable test of the constructive gravity program is the gravitational closure of the standard model of particle physics (or any subsector thereof). The resulting gravity theory is general relativity as known from the textbook. The calculations carried out in Chapter 3 of this thesis showed how to solve the gravitational closure equations which resulted in the two-parameter family of Einstein-Hilbert actions. These calculations improve and exceed the calculations laid out in past work such as Ref. [8, 12]. Most prominently, due to the introduction of the configuration fields, the previous arguments which mostly dated back to the ones developed by Kuchar et al. [10, 11] are no longer valid and had to be replaced by suitable solution methods. These were successfully developed in this thesis.

For matter models beyond the standard model operating on tensorial geometries different from a metric, the second step – solving the gravitational closure equations – is the most difficult and sometimes even a prohibitively hard task. For general linear electrodynamics – a slight generalization of Maxwell theory allowing for birefringence of light in vacuo – the gravitational closure equations are already so involved that a general solution seems to be out of reach. This was demonstrated explicitly in Chapter 5 of this thesis. This is also true for other matter models one could come up with, e. g. for two scalar fields coupling to two metric tensor fields [3].

A significant simplification can be achieved by imposing spacetime symmetries already at the level of the closure equations. This of course comes at the price of obtaining only the symmetry-reduced gravitational dynamics which is usually acceptable. The full gravitational field equations often have to be symmetry-reduced anyway to find according solutions. In this thesis, the symmetry reduction of the gravitational closure equations was implemented and demonstrated for spatially maximal symmetry

– spatial homogeneity and isotropy. Starting from standard model matter, inserting these symmetries into the closure equations simplified them in such a way that one could directly derive the Friedmann equations of standard cosmology without ever having to know the Einstein equations in Chapter 4.

This method was then applied to general linear electrodynamics with the fourth rank tensor field G acting as the spacetime geometry. For FLRW symmetries, one obtains two scale factors $a(t)$ and $c(t)$. The refined Friedmann equations for them arise from a solution of the respective symmetry-reduced closure equations.

The calculations carried out in Chapter 6 show on the one hand that the symmetry assumption indeed simplifies the closure equations to the point where a solution by hand becomes manageable. On the other hand, the refined Friedmann equations (6.30) – (6.32) contained a countably infinite set of undetermined functions f_{KL} of the new, second scale factor. Thus, this theory is not predictive as one first needs to determine the countable set of functions (or their coefficients in a series expansion) by experiments.

A potential future topic is to consider small deviation from the metric limit which is given by the second scale factor being identically one.

Weak deviations from the metric limit

For constant second scale factor $c(t) = 1$, the cosmologically symmetric G^{abcd} given by Eq. (6.2) is induced by the usual FLRW metric g,

$$G^{abcd}(a, c = 1) = 2\, g^{a[c}(a) g^{d]b}(a) - \sqrt{-\det g(a)}\, \epsilon^{abcd}\,.$$

One can investigate configurations for which the second scale factor c deviates only weakly from the metric limit and the rate of change – the derivatives of c – is also slow. Thus, the refined Friedmann equations can be expanded to linear order in the deviation γ from the metric limit of the second scale factor $c = 1 + \gamma$. This will also provide a finite set of parameters in the γ-linearized refined Friedmann equations.

Assuming the functions $f_{KL}(c)$ appearing in the cosmological spacetime action are analytic, they can be expanded as

$$f_{KL}(1 + \gamma) = \sum_{M=0}^{\infty} \frac{1}{M!}\, f_{KLM}\, \gamma^M$$

with constants f_{KLM}. Applying this to the three refined Friedmann equations (6.30) – (6.32), one finds the linearized constraint equation to be

$$0 = \rho + f_{000} - f_{020} \left(\frac{\dot a}{a}\right)^2 + \gamma \left[f_{001} + 3\, f_{000} - (f_{021} + 3\, f_{020}) \left(\frac{\dot a}{a}\right)^2 \right] - \frac{\dot\gamma\, \dot a}{a} \left[f_{110} + f_{130} \left(\frac{\dot a}{a}\right)^2 \right].$$

The first three terms provide the precise metric limit with the two integration constants later identified as gravitational and cosmological constant. All additional terms are non-metric terms arising from the richer spacetime structure behind general linear electrodynamics. This can also be seen in the first evolution equation (6.31) which in γ-linearized form is

$$\begin{aligned}
0 = {} & -3\, p + 3\, f_{000} - f_{020} \left(\frac{\dot a}{a}\right)^2 - 2\, f_{020}\, \frac{\ddot a}{a} \\
& + \gamma \left[3\, f_{001} + 9\, f_{000} - f_{021} \left(\frac{\dot a}{a}\right)^2 - 2\, f_{021}\, \frac{\ddot a}{a} - 3\, f_{020} \left(\frac{\dot a}{a}\right)^2 - 6\, f_{020}\, \frac{\ddot a}{a} \right] \\
& + \dot\gamma \left[3\, f_{100} - f_{011} - 2\, f_{021}\, \frac{\dot a}{a} - 6\, f_{130}\, \frac{\dot a\, \ddot a}{a^2} \right] - \ddot\gamma \left[f_{110} + 3\, f_{130} \left(\frac{\dot a}{a}\right)^2 \right].
\end{aligned}$$

If one wishes to follow the way of standard FLRW cosmology, one replaces the term $f_{020} \left(\frac{\dot a}{a}\right)^2$ by its expression from the constraint equation. This introduces the parameter ρ to the γ-linearized first evolution equation which then is

$$
\begin{aligned}
0 = &-(\rho + 3\,p) + 2\,f_{000} - 2\,f_{020}\,\frac{\ddot a}{a} + \gamma\,\left[2\,f_{001} + 6\,f_{000} - (2\,f_{021} + 6\,f_{020})\,\frac{\ddot a}{a}\right] \\
&+ \dot\gamma\left[3\,f_{100} - f_{011} - 2\,f_{021}\,\frac{\dot a}{a} + f_{110}\,\frac{\dot a}{a} + f_{130}\,\left(\frac{\dot a}{a}\right)^3 - 6\,f_{130}\,\frac{\ddot a\dot a}{a^2}\right] - \ddot\gamma\left[f_{110} + 3\,f_{130}\,\left(\frac{\dot a}{a}\right)^2\right].
\end{aligned}
$$

One again obtains the three terms of the pure metric limit in which γ and all of its derivatives vanish. The additional terms in this equation are again specific to the generalized spacetime structure with two scale factors.

This second scale factor also provides the second evolution equation which is completely new compared to FLRW cosmology. Again, expanding the undetermined functions f_{KL} and keeping terms up to linear order in the deviation γ from the metric limit in the second evolution equation (6.32) provides

$$
\begin{aligned}
0 = &\,q + 3\,\rho + f_{001} + 3\,f_{100}\,\frac{\dot a}{a} + (f_{021} + 3\,f_{130} + 2\,f_{110})\,\left(\frac{\dot a}{a}\right)^2 + f_{110}\,\frac{\ddot a}{a} \\
&- \gamma\left[f_{002} + 4\,f_{001} + (3\,f_{101} + 12\,f_{100})\frac{\dot a}{a} + (f_{022} + 2\,f_{111} + 4\,f_{021} + 8\,f_{110})\left(\frac{\dot a}{a}\right)^2\right. \\
&\qquad\left. + (f_{111} + 4\,f_{110})\,\frac{\ddot a}{a} + (3\,f_{131} + 12\,f_{130})\,\left(\frac{\dot a}{a}\right)^2\,\frac{\ddot a}{a}\right] \\
&- \dot\gamma\left[6\,f_{200}\,\frac{\dot a}{a} + 2\,f_{220}\,\left(\frac{\dot a}{a}\right)^3 + 4\,f_{220}\,\frac{\dot a\,\ddot a}{a^2} + 6\,f_{230}\,\left(\frac{\dot a}{a}\right)^2\,\frac{\ddot a}{a}\right] - \ddot\gamma\,\left[2\,f_{200} + 2\,f_{220}\,\left(\frac{\dot a}{a}\right)^2 + 2\,f_{230}\,\left(\frac{\dot a}{a}\right)^3\right].
\end{aligned}
$$

This equation, for which there is also no metric equivalent, is the probably most involved of three γ-linearized refined Friedmann equations.

Nevertheless, by restricting oneself to the case of small and slowly changing deviations from the metric limit, the number of undetermined constants in the refined Friedmann equations was reduced to a finite one. Still, the γ-linearized refined Friedmann equations feature 15 constants of integration which first need to be determined by experiment before the theory can become predictive. Besides, it is to expect that a solution of the three coupled differential equations is still difficult to find. If it is found, one would be able to conclude how deviations of a spacetime with general linear electrodynamics from a metric one would look like in cosmology.

It is, however, clear that the cosmological symmetries are not the only symmetry assumption one can come up with. While it is the maximal symmetry and thus simplifies the closure equations maximally, there are also other symmetries one could pick, for example spherical symmetry alone.

Spherically symmetric spacetimes

Spherically symmetric spacetimes are of course spacetimes with weaker symmetry assumptions than the cosmological ones. Physically speaking, the gravitational dynamics of such spacetimes correspond to the gravitational field around a point mass – if the remaining spacetime is vacuum. For a metric spacetime, this is the Schwarzschild solution.

The starting point is given by the three Killing vector fields in spherical coordinates r, θ and ϕ,

$$
\begin{aligned}
K_1 &= \sin\phi\,\partial_\theta + \cot\theta\,\cos\phi\,\partial_\phi\,, \\
K_2 &= \cos\phi\,\partial_\theta + \cot\theta\,\sin\phi\,\partial_\phi\,, \\
K_3 &= -\partial_\phi\,,
\end{aligned}
$$

which provided the first half of the six cosmological Killing vector fields in Chapter 4. Due to Birkhoff's theorem, the Schwarzschild solution also features a timelike Killing vector field $K_4 = \partial_t$ which one may impose as well. While these four Killing vectors constitute a non-compact symmetry group, it is still admissible to insert this symmetry into the action and the closure equations without any problems as Fels and Torre point out in Ref. [4]. At this stage, the situation is different for Maxwell theory and general linear electrodynamics as the underlying matter theory.

Maxwell electrodynamics

Starting with Maxwell electrodynamics on a spactime with a Lorentzian metric, the general setup of the gravitational closure equation was laid out in Chapter 3 and Ref. [3]. Applying the four Killing vector fields $K_1, \dots, K_4$ to the spacetime metric by requiring the Killing condition[1]

$$(\mathcal{L}_{K_i} g)^{ab} \overset{!}{=} 0$$

provides four independent components of the spacetime metric which all depend at most on the radial coordinate r. Two of these components are the lapse function $N = N(r)$ and one component of the shift vector field $N^r = N^r(r)$. The spatial metric $g^{\alpha\beta}$ has diagonal form with two further functions $A(r)$ and $B(r)$ as

$$g^{\alpha\beta} = \operatorname{diag}\left(A(r), B(r), B(r)\, \sin^{-2}\theta\right)^{\alpha\beta} .$$

Thus, one also has three non-vanishing configuration fields $\varphi^1(A, B, \theta)$, $\varphi^4(A, B, \theta)$ and $\varphi^6(A, B, \theta)$. The evaluation of the closure equations thus runs along the same lines as for the FLRW symmetries in Chapter 4. The only large difference is the number of chain rule equations that need to be solved. While there were three equations for FLRW symmetries, the spherical symmetries feature seven such equations – one with respect to θ and three for both A and B as one also needs to solve for the dependence of the expansion coefficients on the first and second derivatives of A and B.

The solution of the symmetry-reduced closure equations and the construction of both spacetime action and the four symmetric gravitational field equations is straightforward. These field equations are however quite involved and there is no straightforward way to *construct* the solution to them. While one can show that the Schwarzschild solution actually *is* a solution to the four field equations, one needs to investigate ways how to solve the four symmetric field equations without using knowledge about the solution in the first place. This will become important for the refinements obtained for the geometry of general linear elctrodynamics.

General linear electrodynamics

The general setup of the gravitational closure equations for general linear electrodynamics was laid out in Chapter 5 which extends the results presented in Ref. [3]. For any symmetry reduction of this setup, the spacetime geometry G needs to satisfy the Killing condition

$$(\mathcal{L}_{K_i} G)^{abcd} \overset{!}{=} 0$$

for the four Killing vector fields $K_1 \dots K_4$. It is clear that all components of the spacetime geometry as well as of the canonical geometry do not depend on the foliation parameter t anymore. By evaluating all index combinations for the four spacetime indices a, b, c, d, one obtains eight functions of the radial coordinate $A_1(r), \dots, A_8(r)$. The solution of the symmetry-reduced closure equation will provide an action containing these functions and their derivatives. By variation, one will obtain differential equations for the eight functions $A_1 \dots A_8$ as the symmetric field equations. Of these eight functions, one will be the lapse functions and at least one component of the shift vector field will also remain non-trivial. How these eight functions are distributed concretely is subject to possible future work on this topic. The strategy is of course clear; one needs to decompose the spacetime tensor field G into the canonical geometry using the frame fields $\{N\, e_0 + N^\alpha\, e_\alpha, e_1, e_2, e_3\}$ and the respective co-frame. One can then read off the non-trivial components of the shift vector field and the three fields $\overline{g}$, $\overline{\overline{g}}$ and $\overline{\overline{\overline{g}}}$ of the canonical geometry – which is however quite involved.

While this certainly can be solved, it is the lacking solution of the closure equations for the maximal cosmological symmetries which up to now discouraged the consideration of spherically symmetric

[1] Recall that the Killing condition is formulated with the Lie derivative which is defined independently of any tensorial geometry. In particular, there is no need for a covariant derivative. Thus, the Killing condition can be applied to any tensorial quantity (or a tensor density) on any spacetime geometry.

spacetimes. While the case of flat $k = 0$ cosmology provided an infinity of undetermined functions, one is not even able to construct such a solution for the case of arbitrary k as of now. As long as this case of a spatially maximal symmetry reduction is not resolved, one is not advised to consider weaker spacetime symmetries. The solution of the symmetry-reduced closure equations will only be harder then, if not even prohibitively hard. Nevertheless, spherically symmetric spacetimes are to be kept in mind if one wishes to consider symmetries not as strong as the FLRW ones.

Of course, instead of different spacetime symmetries, one might equally well study other matter models on different geometric backgrounds.

Refined Friedmann equations for other matter models

As Ref. [3] shows, one might very well come up e. g. with theories featuring two (massless) scalar fields ϕ and ψ coupling to two spacetime metrics g and h. The according matter action is given by a composition of two Klein-Gordon terms,

$$S_{\text{matter}}[\phi, \psi; g, h) = \int_M \mathrm{d}^4x \left[\sqrt{-\det g}\, g^{ab}\, \partial_a\phi\, \partial_b\phi + \sqrt{-\det h}\, h^{ab}\, \partial_a\psi\, \partial_b\psi \right] .$$

The general setup of the closure equations from this matter theory is again quite involved and a general solution of the closure equations seems out of reach [3]. As perturbative solutions are available [14, 35], one might wonder whether imposing FLRW symmetries simplify the closure equations so much that a solution with only a finite set of undetermined integration constants remain. This provides another topic and another matter model beyond the standard model which could be cosmologically closed. At first glance, evaluating the Killing condition

$$(\mathcal{L}_{K_i} g)^{ab} \stackrel{!}{=} 0 \quad \text{and} \quad (\mathcal{L}_{K_i} h)^{ab} \stackrel{!}{=} 0$$

for the six Killing vector fields (6.1) leaves one with three scale factors next to a lapse function depending on the foliation time t. One will have to investigate whether this additional scale factor complicates the analysis of the symmetry-reduced gravitational closure equations even more or whether actually a solution with a finite set of constants can be constructed.

Bibliography

[1] A. Einstein. Zur Elektrodynamik bewegter Körper. *Annalen der Physik*, pages 891–901, 1905.

[2] A. Einstein. Die Feldgleichungen der Gravitation. *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin*, pages 844–847, 1915.

[3] Maximilian Düll, F. P. Schuller, N. Stritzelberger, and F. Wolz. Gravitational closure of matter field equations. *Phys. Rev.*, D97(8):084036, 2018.

[4] Mark E. Fels and Charles G. Torre. The Principle of symmetric criticality in general relativity. *Class. Quant. Grav.*, 19:641–676, 2002.

[5] Richard S. Palais. The principle of symmetric criticality. *Commun. Math. Phys.*, 69(1):19–30, 1979.

[6] Maximilian Düll. Refinement of Friedmann equations implied by electromagnetic birefringence. In *Proceedings of the Fifteenth Marcel Grossman Meeting on General Relativity*. World Scientific, 2020. to be published.

[7] Maximilian Düll, Nils L. Fischer, Björn Malte Schaefer, and Frederic P. Schuller. Symmetric gravitational closure. *arXiv:2003.07109*, 2020.

[8] Kristina Giesel, Frederic P. Schuller, Christof Witte, and Mattias N. R. Wohlfarth. Gravitational dynamics for all tensorial spacetimes carrying predictive, interpretable and quantizable matter. *Phys. Rev.*, D85:104042, 2012.

[9] Christof Witte. *Gravity actions from matter actions*. PhD thesis, Humboldt-Universität zu Berlin, 2014.

[10] S. A. Hojman, K. Kuchar, and C. Teitelboim. Geometrodynamics Regained. *Annals Phys.*, 96:88–135, 1976.

[11] K. Kuchar. Geometrodynamics regained - a lagrangian approach. *J. Math. Phys.*, 15:708–715, 1974.

[12] Frederic P. Schuller and Christof Witte. How quantizable matter gravitates: A practitioner's guide. *Phys. Rev.*, D89(10):104061, 2014.

[13] Sergio Rivera Hernandez. *Tensorial spacetime geometries carrying predictive, interpretable and quantizable matter dynamics*. PhD thesis, Universität Potsdam, 2012.

[14] Alexander Wierzba. Parametrodynamics. Master's thesis, Ludwig-Maximilians-Universität München, 2018.

[15] Dennis Raetzel, Sergio Rivera, and Frederic P. Schuller. Geometry of physical dispersion relations. *Phys. Rev.*, D83:044047, 2011.

[16] Yakov Itin. On light propagation in premetric electrodynamics. Covariant dispersion relation. *J. Phys.*, A42:475402, 2009.

[17] Lars Gårding. Linear hyperbolic partial differential equations with constant coefficients. *Acta Math.*, 85:1–62, 1951.

[18] Nadine Stritzelberger. Perturbative canonical gravitational dynamics for birefringent spacetimes. Master's thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg, 2016.

[19] Werner M. Seiler. *Involution – The Formal Theory of Differential Equations and its Applications in Computer Algebra*. Springer, Berlin Heidelberg, 2010.

[20] Florian Wolz. *in preparation*. PhD thesis, Leibniz-Universität Hannover, 2020.

[21] Jonas Schneider, Frederic P. Schuller, Nadine Stritzelberger, and Florian Wolz. Gravitational closure of weakly birefringent electrodynamics. 2017.

[22] D. Lovelock. The four-dimensionality of space and the Einstein tensor. *J. Math. Phys.*, 13:874–876, 1972.

[23] Manuel Hohmann and Christian Pfeifer. Geodesics and the magnitude-redshift relation on cosmologically symmetric Finsler spacetimes. *Phys. Rev.*, D95(10):104021, 2017.

[24] Nils Fischer. Cosmological closure of metric and area metric ideal fluids. Master's thesis, Ruprecht-Karls-Universität Heidelberg, 2017.

[25] M. J. Gotay and J. E. Marsden. Stress-Energy-Momentum Tensors and the Belinfante-Rosenfeld Formula. *Contemp. Math.*, 132:367–392, 1992.

[26] Friedrich W. Hehl and Yuri N. Obukhov. Spacetime metric from local and linear electrodynamics: A New axiomatic scheme. *Lect. Notes Phys.*, 702:163–187, 2006. [,163(2005)].

[27] Friedrich W. Hehl, Yuri N. Obukhov, and Guillermo F. Rubilar. Space-time metric from linear electrodynamics. 2. In *Gravitation. Proceedings, International European Conference, 27th Session of the Journees Relativistes, Weimar, Germany, September 12-17, 1999*, 1999. [Submitted to: Annals Phys.(1999)].

[28] Yuri N. Obukhov and Friedrich W. Hehl. Space-time metric from linear electrodynamics. *Phys. Lett.*, B458:466–470, 1999.

[29] F.W. Hehl and Y.N. Obukhov. *Foundations of Classical Electrodynamics: Charge, Flux, and Metric*. Progress in Mathematical Physics. Birkhäuser Boston, 2003.

[30] Frederic P. Schuller, Christof Witte, and Mattias N. R. Wohlfarth. Causal structure and algebraic classification of area metric spacetimes in four dimensions. *Annals Phys.*, 325:1853–1883, 2010.

[31] Sergio Rivera and Frederic P. Schuller. Quantization of general linear electrodynamics. *Phys. Rev.*, D83:064036, 2011.

[32] Simon Grosse-Holz, Frederic P. Schuller, and Roberto Tanzi. Gravitational effects in birefringent quantum electrodynamics. 2017.

[33] G. F. Rubilar. Linear pre-metric electrodynamics and deduction of the light cone. *Annalen Phys.*, 11:717–782, 2002.

[34] Guillermo F. Rubilar, Yuri N. Obukhov, and Friedrich W. Hehl. Space-time metric from linear electrodynamics. 3. *Int. J. Mod. Phys.*, D11:1227, 2002.

[35] Udo Beier. Perturbative gravitational closure of a bi-metric matter theory. Master's thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg, 2018.

Own publications

Parts of the results of this thesis have already been published in the following articles and conference proceedings. These are

M. Düll, F. P. Schuller, N. Stritzelberger and F. Wolz
Gravitational closure of matter field equations
Phys. Rev. D97 (2018), 084036,

M. Düll, N. L. Fischer, B. M. Schäfer, F. P. Schuller
Symmetric gravitational closure
arXiv:2003.07109 (2020)

M. Düll
Refinement of Friedmann equations implied by electromagnetic birefringence
to appear in: Proceedings of the 15th Marcel Grossmann Meeting on General Relativity

Parts of chapter 2 and in particular, section 2.4 of this thesis are joint work with my colleague Florian Wolz. Next to the indicated publications, these results will also be presented in his PhD thesis,

Florian Wolz
PhD thesis in prepration
Leibniz-Universität Hannover.

It has been indicated at the beginning of each chapter or section if its content is also subject to one of the publications listed here.

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