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Abstract

In this thesis we present a new method to construct modular forms using rational functions. It relies on contour integration and Weil's converse theorem. We give several applications, reaching from a relation between cotangent sums and L-functions, formulas for Eichler integrals and period polynomials and series representations for L-functions corresponding to products of Eisenstein series.
With similar ideas, based on contour integration, we move on to equations which were originally studied by Ramanujan and generalize his formulas to those containing L-functions at rational arguments. We work out a very general framework for finding new equations of the Ramanujan type that can be applied to a wide range of L-functions.

Zusammenfassung

In der vorliegenden Arbeit stellen wir eine neue Methode zur Konstruktion von Modulformen vor, die rationale Funktionen benutzt. Der Beweis dieser Technik beruht auf Kurvenintegration und dem Weilschen Umkehrsatz. Wir geben zahlreiche Anwendungen, wie die Herleitung einer natürlichen Beziehung zwischen Kotangenssummen und L-Funktionen, Formeln für Eichler-Integrale und Periodenpolynome und Reihendarstellungen für L-Funktionen, die zu Produkten von Eisensteinreihen korrespondieren.

Mit ähnlichen Ideen, ebenfalls basierend auf Kurvenintegration, arbeiten wir uns zu gewissen Gleichungen vor, die bereits von Ramanujan studiert wurden. Diese werden auf den Fall von L-Funktionen an rationalen Stellen verallgemeinert. Wir arbeiten indes einen sehr allgemeinen Rahmen aus, innerhalb dessen neuartige Gleichungen des Ramanujan Typs für eine große Klasse von L-Funktionen gefunden werden können.

Introduction

Modular forms belong to the most important objects in number theory. In short, these are holomorphic functions on the complex upper half plane that satisfy certain transformation laws. So at first view they are objects from complex analysis. Their discovery dates back to the 19th century, when they were considered in the context of elliptic functions. Gotthold Eisenstein defined the Eisenstein series

$$E_k(\tau) := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m\tau + n)^{-k}, \qquad k \geq 3,$$

named after him today as very fundamental examples of modular forms. It is a simple observation that each E_k is the constant zero function if and only if k is an odd number. Their enormous importance is given to modular forms in combination with numerous other theories in mathematics and physics. They are applied in string theory, the theory of quadratic forms (see [51]) and in widely generalized form in algebraic geometry and representation theory (see, for example, [14], and [19]). Particularly prominent and important is the long suspected relationship between modular forms and elliptic curves, for a very readable introduction to this topic see [22]. The proof of the Modularity theorem was given by Andrew Wiles and Richard Taylor and represents one of the highlights of 20th century mathematics. This was used in proving Fermat's last Theorem, which states that for all $n > 2$ the equation $x^n + y^n = z^n$ has only trivial integer solutions. For more information on this topic (including some background information on the history of the proof), the reader may wish to consult [17].

The concept of L-functions is of fundamental importance in this context. At first, these appear, like modular forms themselves, as analytical functions in a complex half plane. They can often be continued globally (except for a few possible singularities) and satisfy a certain functional equation. For a systematic approach, see for example [18]. When equipped with an Euler product over prime numbers, these also play a central role in number theory because they encode important arithmetic information. Of great importance in this context are so-called non-vanishing results of L-functions, see [41]. An example is the Riemann zeta function, which is given for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ by

$$\zeta(s) := \prod_p \frac{1}{1 - p^{-s}},$$

where the product is taken over all prime numbers. It can be shown that $\zeta(s)$ has a holomorphic continuation to $\mathbb{C} \setminus \{1\}$ with a simple pole in $s = 1$. Its non-vanishing in the region $\mathrm{Re}(s) \geq 1$ is equivalent to the prime number theorem, which states that

$$\#\{p \text{ prime number} \mid p \leq x\} \sim \frac{x}{\log(x)}, \qquad x \to \infty.$$

The Riemann hypothesis, which assumes that $\zeta(s) \neq 0$ for all $\mathrm{Re}(s) > \frac{1}{2}$, would result in an exceptional smoothness of the prime counting function, i.e., there is some computable constant $C > 0$, such that

$$\left| \#\{p \text{ prime number} \mid p \leq x\} - \int_2^x \frac{\mathrm{d}t}{\log(t)} \right| \leq C x^{\frac{1}{2}} \log(x), \qquad x > 2.$$

For more details on this topic the reader is advised to consult [41], [52] or [57] (Ch. II, 3–4).

According to theories first described in the nineteen-twenties by Erich Hecke, there is a very close relationship between modular forms and a whole class of L-functions (see [35], for a more general version [58]). In classical form it describes an isomorphism between certain classes of modular forms and L-functions. This phenomenon will play an important role in this thesis.

In Chapter I we obtain a new perspective on the theory of modular forms. To be more precise, we present a new construction path for Eisenstein series, which results from the theory of rational functions. This results in useful perspectives on well-known objects that emerge from modular forms. Two important fields of application are Eichler integrals and L-functions.

In the literature, besides the theta functions, Eisenstein series attached to Dirichlet characters χ and ψ,

$$E_k(\chi, \psi; \tau) := \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \chi(m)\psi(n)(m\tau + n)^{-k}, \qquad k \geq 3,$$

are typically the first examples of modular forms (see, for example, [22], and [53]). The reason is that the transformation properties of E_k are obvious by construction at least in the cases $k \geq 3$. For theta functions, for example the Jacobi theta function

$$\theta(\tau) := \sum_{n \in \mathbb{Z}} e^{2\pi i \tau n^2},$$

we need extra knowledge about the Poisson summation formula and Fourier analysis to verify modularity, see [43]. Theta functions have arithmetic applications in the theory of quadratic forms (see for example [32] or [38], where the number of representations

$n = x_1^2 + \cdots + x_k^2$ is investigated, and [13] (Sections 3.1 and 3.2)) and elliptic functions (see [29], Chapter VI).

We present a new elementary method to construct modular forms. This method seems to be natural in the sense, that functional equation and Fourier series are on an equal footing. It also does not distinguish between weights $k = 1, 2$ and $k \geq 3$, as the one described above does. The method builds on a class of very simple functions which we will call *weak functions*. Here, a weak function ω is a 1-periodic meromorphic function in the entire plane, which only has poles at points in $\mathbb{Q}$ that are all simple and tends rapidly to 0 as the imaginary part increases. By Liouville's theorem one quickly sees that each weak ω is essentially just a rational function $R \in \mathbb{C}(X)$ with (only simple) poles only in roots of unity, such that $R(0) = R(\infty) = 0$. The function R then transforms to a weak function by setting $X := e^{2\pi i z}$. There are no non-trivial weak functions with poles only in $\mathbb{Z}$. This later refers to the fact that there are no non-trivial cusp forms of weight $2 \leq k \leq 14$, $k \neq 12$, for the full modular group. It is possible to assign an integer level to each weak function ω. It is given by the smallest positive integer N such that $\omega(\frac{z}{N})$ only has poles in $\mathbb{Z}$. We collect all weak functions with level $N|M$ in the vector space W_M. The key construction theorem describes a way to write modular forms for the congruence subgroup $\Gamma_1(N_1, N_2)$ described below, where $N_1, N_2 \geq 1$ are integers, as a series involving a pair of weak functions. In Theorem 1.2.16 we specify homomorphisms between the vector spaces $W_{N_1} \otimes W_{N_2}$ and the weight k modular forms $M_k(\Gamma_1(N_1, N_2))$ where $k \geq 1$ is an integer. Here we assign to each element $\omega \otimes \eta$ an infinite sum defined on the upper (and lower) half plane, which is determined by the residues of ω and η. In fact one can show that these holomorphic functions have all the necessary transformation properties of modular forms for congruence subgroups. We call all modular forms, that are generated by rational functions in the above way, *weak modular forms*. Their spaces are classified in Theorem 1.2.21. The key ingredient for Theorem 1.2.16 foots on a transformation law, see Theorem 1.2.7, which is proved using contour integration. The proof then makes use of Weil's converse theorem 1.1.4, which provides sufficient conditions that a periodic holomorphic function on the upper half plane is modular.

Besides providing a new way of construction for modular forms, Theorem 1.2.16 has other applications. One of them are series representations for L-functions corresponding to several modular forms. Remember that for a modular form f for some congruence subgroup, say $\Gamma_0(N)$, with Fourier series expansion

$$f(\tau) = \sum_{n=0}^{\infty} a(n) q^n,$$

we can associate a L-function

$$L(f; s) := \sum_{n=1}^{\infty} a(n) n^{-s}, \qquad \mathrm{Re}(s) \gg 1.$$

The function $L(f;s)$ has a meromorphic continuation on the entire plane and satisfies a functional equation, but the Dirichlet series on the right will not converge for all values of s. With the help of weak functions, we present a different Dirichlet type series representation for $L(f;s)$ if f is a product of Eisenstein series that are not associated to the trivial character. For its proof we use a Dominated convergence theorem for Eisenstein series, see Theorem 1.3.14. In this theorem, the formula for modular forms in terms of rational functions is used to derive a principle of fast convergence. We are then allowed to switch integration and summation in the Mellin transform of products of Eisenstein series in larger half planes. This yields series representations for L-functions with improved convergence properties, see Corollary 1.3.33. The formula there, involving products of Eisenstein series, is a direct generalization of the single Eisenstein series case, given in [49] on p. 270.

A wider range of applications is achievable when relaxing the conditions on weak functions by allowing poles at arbitrary real values. We then consider the concept of "generalized Eisenstein series". These are given by Fourier type series that may include non-rational exponents of $e^{2\pi i\tau}$. A modular transformation law of these objects is stated in Theorem 1.4.8 which is a consequence of the generalized transformation law described in Theorem 1.4.5. When considering pre-weak functions, which only have to be bounded on vertical strips, we are able to develop a much wider framework for q-series.

An example is a method to construct functions $f(\tau) = g_0(\tau) + \tau g_1(\tau) + \cdots + \tau^n g_n(\tau)$, where the $g_j(\tau)$ are q-type-series, respectively, that own modular transformation properties. An explicit example is

$$f_k(\tau) = (k-1)\sum_{n=1}^{\infty} n^{k-2}\frac{q^n}{(1-q^n)^2} + 2\pi i\tau\sum_{n=1}^{\infty} n^{k-1}\frac{q^n(1+q^n)}{(1-q^n)^3},$$

that fulfills

$$f_k\left(-\frac{1}{\tau}\right) = -\tau^k f_k(\tau)$$

for all even $k > 4$. This follows by the version Theorem 1.4.43 of the main transformation law (Theorem 1.4.5) for weak functions with poles of arbitrary order. We can also apply Theorem 1.4.43 to prove a much more general result on the transformation of so called rational type q-series, stated in Theorem 1.4.47.

In the more specific situation of negative values of the weight k we can apply the construction formalism to cotangent sums. E.g., for a character χ modulo N we use the denotation

$$C(\chi;m) := \sum_{j=1}^{N-1}\chi(j)\cot^m\left(\frac{\pi j}{N}\right). \tag{0.0.0.1}$$

Cotangent sums have been studied in lots of different forms and areas: In the setting of period functions by Bettin and Conrey [8] and by Lewis and Zagier [39] referred to quantum modular forms and the generalized Riemann hypothesis. Generally, it turns out that

the arithmetic nature of such cotangent sums is strongly tied with the arithmetic nature of corresponding L-functions. For any 1-periodic function $\beta : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ with finite support one can assign cotangent sums $\sum_{x\in\mathbb{R}/\mathbb{Z}} \beta(x)\cot^m(\pi x)$ and a sequence of L-numbers $\widetilde{L}(\beta;k) := \sum_{x\in\mathbb{R}^\times} \beta(x)x^{-k}$ for $k \in \mathbb{N}$. E.g., in the case of (0.0.0.1) these are essentially Dirichlet L-functions of the characters χ. The key idea is to express the cotangent sums as rational combinations of the $\widetilde{L}$-values and hence to compare their arithmetic nature, see Theorem 1.4.24. For example, with $\zeta(2k) \in \mathbb{Q}\pi^{2k}$ an easy consequence of Theorem 1.4.24 is

$$C(\chi_{0,N};m) = \sum_{j=1}^{N-1} \cot^m\left(\frac{j\pi}{N}\right) \in \mathbb{Q}, \qquad \forall m, N \in \mathbb{N}.$$

This is well-known and was verified by Berndt and Yeap (see [4], p. 6). Furthermore, with our method it is possible to derive explicit formulas for the cotangent sums $C(\chi;m)$ where χ is an arbitrary primitive character. These will be stated in Corollary 1.4.29. Similarly, we can give (rather complicated) formulas for Dirichlet series with trigonometric coefficients at integer arguments, see Corollary 1.4.32 and Remark 1.4.33. Finally, using Fourier analysis and the generalized Clausen functions one can derive closed formulas for cotangent sums presented by Berndt and Yeap [4] involving sine and cosine functions. Here we use explicit terms (described in Theorem 1.4.25) of the rational isomorphisms briefly described above.

The rational function at the end of the transformation law refers to a period polynomial of a modular form. This implies that critical values of L-functions attached to Eisenstein series are just residues of elementary functions. If one formally studies this relationship, it leads to a duality principle, which we call *Eichler duality*. Here, the $(k-1)$-fold integral of the function $\vartheta_k(\omega \otimes \eta)$, which assigns to a pair of weak functions ω and η a modular form, is related to functions $\vartheta_j(\alpha \otimes \beta)$ where j takes on negative integers. This principle is described in Theorem 1.4.58.

In Chapter II we are inspired by the tools that have already been developed in Chapter I. The Eichler duality mentioned above can be used, for example, to immediately derive classical series representations for L-functions. A prominent example involves Apery's constant:

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n}-1)}.$$

We mainly use strong properties of several gamma factors and functional equations of L-functions to generalize these kinds of identities to products of L-functions at rational arguments. Although the techniques are mostly elementary, the given framework is useful and might comprise deeper information about values of L-functions.

First we continue the study of L-functions at rational points which was done in the case of Dirichlet L-functions in [25]. In that paper the author generalized some classical identities for Dirichlet L-functions by Ramanujan. An example is given by the following

formula involving values of the Riemann zeta function at integers

$$\begin{aligned}
&\alpha^{-N}\left(\frac{1}{2}\zeta(2N+1)+\sum_{k=1}^{\infty}\frac{1}{k^{2N+1}(e^{2\alpha k}-1)}\right)\\
&=(-\beta)^{-N}\left(\frac{1}{2}\zeta(2N+1)+\sum_{k=1}^{\infty}\frac{1}{k^{2N+1}(e^{2\beta k}-1)}\right)\\
&-2^{2N}\sum_{k=0}^{N+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{2N+2-2k}}{(2N+2-2k)!}\alpha^{N+1-k}\beta^k,
\end{aligned}\tag{0.0.0.2}$$

where $N>0$ is an integer and α,β are positive real numbers such that $\alpha\beta=\pi^2$. A proof for this relation is also given in [6]. In fact, one notes that the terms $\frac{1}{2}\zeta(2N+1)$ on both sides and the finite sum over the Bernoulli numbers come from the residues of the completion $\Lambda(s):=(2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s+2N+1)$ at the points $s=0$ and $s=-2N$ (note that $\zeta(0)=-\frac{1}{2}$), and $s=-2N-1,-2N+1,...,-1,1$, respectively. The infinite sums are of Lambert type but can be rearranged to power series in $z=e^{-2\alpha}$ (and $z=e^{-2\beta}$) with coefficients identical to those of the Dirichlet series $\zeta(s)\zeta(s+2N+1)$.
The formula (0.0.0.2) is associated to the number field $K=\mathbb{Q}$. However, the following new formula corresponds to the case where $K=\mathbb{Q}(\sqrt{D})$ is a real quadratic number field. Let $N\in\mathbb{N}$, d_K be the discriminant and χ_K the character associated to K. Let

$$c(n):=2\sum_{d|n}\chi_K(d)\sigma_{-2N-1}(d)\sigma_{-2N-1}\left(\frac{n}{d}\right)$$

be the coefficients of the Dirichlet series $2\zeta_K(s)\zeta_K(s+2N+1)$, where $\zeta_K(s)$ is the Dedekind zeta function associated to K. Then we have for all $\alpha,\beta>0$ with $\alpha\beta=4\pi^2d_K^{-1}$:

$$\begin{aligned}
&\alpha^{-2N}\left(-\zeta_K'(0)\zeta_K(2N+1)+\sum_{n=1}^{\infty}c(n)K_0(2\alpha\sqrt{n})\right)\\
&=\beta^{-2N}\left(-\zeta_K'(0)\zeta_K(2N+1)+\sum_{n=1}^{\infty}c(n)K_0(2\beta\sqrt{n})\right)\\
&\quad+\sum_{\substack{\ell=0\\ \ell\text{ even}}}^{2N-2}\alpha^{2N-2\ell-2}\left(\frac{\zeta_K'(\ell-2N+1)\zeta_K(\ell+2)+\zeta_K(\ell-2N+1)\zeta_K'(\ell+2)}{(2N-\ell-1)!^2}\right.\\
&\left.-\frac{2\zeta_K(\ell-2N+1)\zeta_K(\ell+2)\log(\alpha)}{(2N-\ell-1)!^2}+2\frac{H_{2N-\ell-1}-\gamma}{(2N-\ell-1)!^2}\zeta_K(\ell-2N+1)\zeta_K(\ell+2)\right)\\
&+R_K\zeta_K(2N+2)(\alpha^{-2N-2}-\beta^{-2N-2}),
\end{aligned}\tag{0.0.0.3}$$

where K_0 is the Bessel function, $H_n:=\sum_{j=1}^{n}\frac{1}{j}$ is the n-th *harmonic number*, $\gamma=0,57721...$ is the Euler-Mascheroni constant and R_K is given by

$$R_K=\frac{2\log(\varepsilon)h_K}{\sqrt{|d_K|}},$$

where h_K is the class number and ε is the fundamental unit. This new result is analogous to (0.0.0.2) in the following sense: Firstly, the infinite sums now involve functions of higher degree. Secondly, the terms $-\zeta(0)\zeta(2N+1)$ in (0.0.0.2) are replaced by $-\zeta_K'(0)\zeta_K(2N+1)$. And lastly, the finite sum now also involves values of ζ_K' at integer arguments and a logarithmic term since the degree of K is not $n=1$ but $n=2$ and the completion

$$\widehat{D}(s) = \left(\frac{4\pi^2}{d_K}\right)^{-s} \Gamma(s)^2 \zeta_K(s)\zeta_K(s+2N+1)$$

has also poles of order 2 in the critical strip. Note that there is a connection to Maass Eisenstein series. Indeed, the coefficients $a(n) := \chi_D(n)\sigma_0(n)$ generate the Dirichlet series $\zeta_{\mathbb{Q}(\sqrt{D})}(s)^2$ and

$$u_D(z) := y^{\frac{1}{2}} \sum_{n=1}^{\infty} a(n) K_0\left(\frac{2\pi n y}{|D|}\right) \sin\left(\frac{2\pi n x}{|D|}\right), \qquad z = x+iy, y>0,$$

is a corresponding Maass Eisenstein series on $\Gamma_D := \Gamma_0(D) \cup S\Gamma_0(D)$ with eigenvalue $\frac{1}{4}$ with respect to the hyperbolic Laplacian operator. Here $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is the inversion and

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$$

the usual congruence subgroup with level N. In this thesis we consider the situation that $K_0(*ny)\sin(*nx)$ is replaced by $K_0(*\sqrt{-ni\tau})$, which is (when looking at the corresponding gamma factor) a function of degree 2. Note that in [47] Lewis and Zagier study the exchange by $e^{-*in\tau}$, which is a function of degree 1. To arrive at this point, we consider generalized Dirichlet series $\sum_{n=1}^{\infty} a(n) n^{-\frac{s}{b}}$, collected in the vector space $\mathcal{D}((\gamma,\gamma^*),\sigma,k)$, with absolute abscissa σ and properties described in Definition 2.2.7 in detail, such as a functional equation under $s \mapsto k-s$. They are completed by gamma factors of the form

$$\gamma(s) = ab^s \prod_{j=1}^{n} \Gamma(a_j+s)^{c_j} \Gamma(b_j-s)^{d_j}$$

specified in Definition 2.2.3 with exponential decay in vertical strips. This is a very general situation and many important Dirichlet series do fit into this family. In Theorem 2.2.10 we describe the key method for "glueing" several Dirichlet series with transformation properties to a new Dirichlet series with transformation properties. From this starting point, we are able to give generalized Ramanujan identities. As examples we consider the case of Hecke L-functions and L-functions corresponding to modular forms of half integral weight. In Theorem 2.2.14 a quite general and explicit version for Dedekind zeta functions of quadratic fields is presented. At the end we will state some open questions regarding the generalized period polynomials.

Notation

In the following, we present a detailed list of notation used in this thesis. The reader is advised to look up the notation only when needed.

As usual we denote $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ as the sets of natural numbers, integer numbers, rational numbers, real numbers and complex numbers, respectively. Also we write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We set $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ as the real and imaginary part of s. We denote $\mathbb{H}$ as the upper half plane, i.e. $\tau \in \mathbb{H}$ if and only if $\mathrm{Im}(\tau) > 0$. We abbreviate $\mathbb{F}_d := \mathbb{Z}/d\mathbb{Z}$ for any number $d \in \mathbb{N}$. For positive integers N we sometimes write $\mathbb{F}_{N^{-1}} := \mathbb{Z}[N^{-1}]/\mathbb{Z}$. Several complex variables with different meanings will appear. We put $e(z) := e^{2\pi i z}$. However, in the case of variable τ we use the common notation $q := e^{2\pi i \tau}$.

We use $\#S$ to indicate the cardinality of a finite set S. Let S be a set (or class) and $o := (o_1, ..., o_n) \in S \times \cdots \times S$ be a tuple. We will then use the notation $o_{\mathrm{inv}} = (o_n, ..., o_1)$ several times. Sometimes we will use the notation $\exp_\mu(x) := \mu^x$.

Throughout, if not defined otherwise, k, ℓ, N, M, N_1 and N_2 are (positive) integers. We briefly define $\boldsymbol{k} = (k_1, ..., k_\ell) \in \mathbb{N}^\ell$ to be a vector of positive integers. We write $|\boldsymbol{k}| = k_1 + \cdots + k_\ell$. For real valued vectors $\boldsymbol{u} = (u_1, ..., u_\ell) \in \mathbb{R}^\ell$ we briefly write $\max(\boldsymbol{u}) := \max\{u_1, ..., u_\ell\}$. We use the notation ∞ in the context of the usual point at infinity on the Riemann sphere and $i\infty$ in the context of the cusp at infinity on a modular curve.

We use the notation $\mathrm{sgn}(f) = \pm 1$ to indicate that f is an even or odd function, respectively.

Several times we shall use differential operators. To avoid any confusing situation with possible inner functions we stress at this point that we always mean

$$\frac{\partial}{\partial x} f(g(x)) = g'(x) f'(g(x))$$

and $f^{(1)}(g(x)) = f'(g(x))$ (the same for n-th derivatives). We write $\partial_\tau := \frac{1}{2\pi i}\frac{\partial}{\partial \tau}$ and $\partial_z := \frac{1}{2\pi i}\frac{\partial}{\partial z}$. If the variable is clear we only write ∂.

Throughout we will denote $S(f) \subset U$ as the set of poles of a given meromorphic function $f : U \to \overline{\mathbb{C}}$.

For each set L (for example the real or the complex numbers) we define $L^{\mathbb{C}_0}$ as the space of all functions $f : L \to \mathbb{C}$, that are zero everywhere except *finitely* many $x \in L$. The subspace $L_0^{\mathbb{C}_0} \subset L^{\mathbb{C}_0}$ is given by all f satisfying $\sum_{x\in L} f(x) = 0$. In the case $0 \in L$ we write $L^{\mathbb{C}_{0,0}}$ for the subspace of functions with $f(0) = 0$.

In the context of weak and pre-weak functions there will be lots of notation for slightly different objects, so we are willing to provide an overview:

- For any positive integer N, W_N denotes the space of weak functions with level N.
- The spaces W_{weak} and W_{pre} collect all weak and pre-weak functions of degree 1, respectively.
- The spaces $W_{\text{weak},a}$ and $W_{\text{pre},a}$ collect all weak and pre-weak functions of degree a, respectively. In the case $a = \infty$, these spaces collect functions of arbitrary (but finite) degree.
- By $V^{\pm}$ we always mean the subspaces of a space V of complex functions consisting of all even and odd functions, respectively.
- By $W^{\pm i\infty}$ we denote the subspace of functions $f \in W$ with the property $f(\pm i\infty) = 0$.
- By W^0 we denote all functions $f \in W$ that have a removable singularity in $z = 0$.
- The set $W[\mathcal{T}]$ contains all (pre)-weak functions only having poles in $\mathcal{T} \subset \mathbb{R}/\mathbb{Z}$. We write $\mathcal{T}_N := \{0, \frac{1}{N}, ..., \frac{N-1}{N}\}$.

We will identify functions $f \in \mathbb{F}_N^{\mathbb{C}_0}$ with N-periodic functions $f : \mathbb{Z} \to \mathbb{C}$. For integers M we will set $f[M](x) := f(Mx)$ when $f : \mathbb{Z} \to \mathbb{C}$. Note that we will use the same notation in the context of Atkin-Lehner operators for weak functions.

We write $\mathfrak{C}_N$ for the (multiplicative) group of Dirichlet characters modulo N. We denote the principal character modulo N by $\chi_{0,N}$. In particular, the trivial character is given by $\chi_{0,1}$. For any Dirichlet character χ, we define its L-function by $L(\chi; s) := \sum_{n=1}^{\infty} \chi(n)n^{-s}$ (where the series only converges for $\text{Re}(s) > 0$ if χ is non-principal and for $\text{Re}(s) > 1$ else). Throughout $\zeta(s)$ and $\Gamma(s)$ denote the Riemann zeta function and the Euler Gamma function, respectively. As usual for an integer n we write $\mathcal{G}(\chi; n)$ for the Gauß sum $\sum_{j=1}^{m} \chi(j)e^{\frac{2\pi ijn}{m}}$, where χ is a character modulo m. We abbreviate $\mathcal{G}(\chi) = \mathcal{G}(\chi; 1)$. We use the short notation $\zeta_M := e^{\frac{2\pi i}{M}}$.

By GL_2 and SL_2 we mean the general and the special linear group. We write $\Gamma \subset \text{SL}_2(\mathbb{Z})$ for an arbitrary congruence subgroup. The space of modular (cusp) forms of weight k for Γ is denoted by $M_k(\Gamma)$ and $S_k(\Gamma)$, respectively. We will investigate modular

forms for some special congruence subgroups. These are

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$
$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\},$$
$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

and for integers $N_1, N_2 \geq 1$

$$\Gamma_0(N_1, N_2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| b \equiv 0 \pmod{N_1}, c \equiv 0 \pmod{N_2} \right\},$$
$$\Gamma_1(N_1, N_2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_1, N_2) \middle| a \equiv d \equiv 1 \pmod{N_1 N_2} \right\}.$$

We will use the common notation $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. In addition, we denote the *Fricke involution* by $w(N) := \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ for integers $N \geq 1$. As usual we define the *Petersson slash operator* for integers k by

$$f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) := (ad - bc)^{\frac{k}{2}} (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

We use K to denote a number field with ring of integers $\mathcal{O}_K$ and we write $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{f}$ for (fractional) ideals of K. Also we use the common notation $\mathfrak{p}$ for prime ideals.

We will use the symbol $\mathbf{1}$ to denote the vector $(1, 1, ..., 1) \in \mathbb{R}^n$, where n shall be clear from the context. Also, for arbitrary $\mathbf{a} \in \mathbb{R}^n$, we write $s_{\mathbf{a}} = \langle \mathbf{a}, \mathbf{1} \rangle = \sum_{\nu=1}^{n} a_\nu$ for the sum of all entries in $\mathbf{a}$.

Contents

Chapter 1

Rational functions, modular forms and series representations for L-functions

Note. The main results of this chapter have been accepted for publication, see [27], or submitted, see [24] and [28].

1.1 Preliminaries

1.1.1 Periodic functions and discrete Fourier analysis

In this short section we want to recall the most important notations and facts about discrete Fourier analysis. Especially when going over to Fourier series it will be useful to identify functions in $\mathbb{F}_{N^{-1}}^{\mathbb{C}_0}$ with those in $\mathbb{F}_N^{\mathbb{C}_0}$ via the obvious choice of mapping

$$\kappa_N : \mathbb{F}_{\frac{1}{N}}^{\mathbb{C}_0} \xrightarrow{\sim} \mathbb{F}_N^{\mathbb{C}_0}$$

$$(\kappa_N f)(x) := f\left(\frac{x}{N}\right).$$

For $d|N$ we also use the trivial injection

$$\iota_N^d : \mathbb{F}_d^{\mathbb{C}_0} \longrightarrow \mathbb{F}_N^{\mathbb{C}_0}$$

$$(\iota_N^d f)(x) := \begin{cases} f(\frac{xd}{N}), & \text{if } x \equiv 0 \pmod{\frac{N}{d}}, \\ 0, & \text{else,} \end{cases} \tag{1.1.1.1}$$

for purposes of notation. E.g., depending on the situation it will be more appropriate to describe a Dirichlet character χ modulo N as a function $\kappa_N^{-1}\chi : \mathbb{F}_{N^{-1}} \to \mathbb{C}$. A very important point here is how we understand the embedding of characters with modulus

$d|N$ in $\mathbb{F}_N^{\mathbb{C}_0}$ in this context. A legitime way would be to consider those as N-periodic function and just use κ_N^{-1}. But we use a different interpretation. Write $\overline{\mathfrak{C}_N}$ for the set of all characters modulo d, where d divides N. We give $\overline{\mathfrak{C}_N}$ the structure of a vector space by

$$\overline{\mathfrak{C}_N} \cong \bigoplus_{d|N} \kappa_N^{-1}\left(\iota_N^d \mathfrak{C}_d\right).$$

For example, the space $\overline{\mathfrak{C}_5}$ is generated by the functions $f_0(0) = 1$, $f_0(\frac{1}{5}) = \cdots = f_0(\frac{4}{5}) = 0$ and $f_j(\frac{k}{5}) = \chi_j(k)$ for $j = 1, ..., 4$, where the χ_j are the characters modulo 5. It is clear, that $\overline{\mathfrak{C}_N}$ just contains all functions on $\mathbb{F}_N^{\mathbb{C}_0}$, i.e., we have an isomorphism

$$\overline{\mathfrak{C}_N} \cong \mathbb{F}_N^{\mathbb{C}_0}.$$

Definition 1.1.1. *Let N be a positive integer. Consider the space of functions $\beta : \mathbb{F}_N \to \mathbb{C}$ (whose may be identified with functions $\mathbb{F}_{N^{-1}} \to \mathbb{C}$). Then we define the Fourier transform of β by*

$$\mathcal{F}_N(\beta)(n) := \sum_{j \in \mathbb{F}_N} \beta(j) e^{-\frac{2\pi i n j}{N}}.$$

Note that we will use exactly the same notation for corresponding functions $\widetilde{\beta} : \mathbb{F}_{N^{-1}} \to \mathbb{C}$, i.e., we put

$$\mathcal{F}_N(\widetilde{\beta})(y) := \sum_{x \in \mathbb{F}_{N-1}} \widetilde{\beta}(x) e^{-2\pi i N x y},$$

which is again in $\mathbb{F}_{N-1}^{\mathbb{C}_0}$. We obtain the following.

Proposition 1.1.2. *We have the following statements.*

(i) The function $\mathcal{F}_N$ is an automorphism on $\mathbb{F}_N^{\mathbb{C}_0}$ with inverse function

$$\mathcal{F}_N^{-1}(\beta)(n) := \frac{1}{N} \sum_{j \in \mathbb{F}_N} \beta(j) e^{\frac{2\pi i n j}{N}}.$$

(ii) Both the functions $\mathcal{F}_N$ and $\mathcal{F}_N^{-1}$ induce an isomorphism $\mathbb{F}_N^{\mathbb{C}_{0,0}} \to (\mathbb{F}_N)_0^{\mathbb{C}_0}$.

(iii) The function $\mathcal{F}_N$ preserves odd and even functional relations, i.e., if $\beta(-n) = \pm\beta(n)$ then we have $\mathcal{F}_N(\beta)(-n) = \pm\mathcal{F}_N(\beta)(n)$.

Proof. For (i) we note

$$\mathcal{F}_N^{-1}(\mathcal{F}_N(\beta))(n) = \frac{1}{N} \sum_{j \in \mathbb{F}_N} \left(\sum_{k \in \mathbb{F}_N} \beta(k) e^{-\frac{2\pi i j k}{N}} \right) e^{\frac{2\pi i n j}{N}} = \frac{1}{N} \sum_{k \in \mathbb{F}_N} N \delta_{k,n} \beta(k) = \beta(n).$$

The calculation $\mathcal{F}_N(\mathcal{F}_N^{-1}(\beta))(n) = \beta(n)$ works the same. To prove (ii), let β be a function in $\mathbb{F}_N^{\mathbb{C}_{0,0}}$, i.e., $\beta(0) = 0$. Then we have

$$\sum_{n\in\mathbb{F}_N} \mathcal{F}_N(\beta)(n) = \sum_{n\in\mathbb{F}_N}\sum_{j\in\mathbb{F}_N} \beta(j)e^{-\frac{2\pi inj}{N}} = N\beta(0) = 0.$$

This shows $\mathcal{F}_N(\beta) \in (\mathbb{F}_N)_0^{\mathbb{C}_0}$. On the other side, if $\widetilde{\beta} \in (\mathbb{F}_N)_0^{\mathbb{C}_0}$ we have

$$\mathcal{F}_N(\widetilde{\beta})(0) = \sum_{j\in\mathbb{F}_N} \widetilde{\beta}(j) = 0.$$

Since both spaces $\mathbb{F}_N^{\mathbb{C}_{0,0}}$ and $(\mathbb{F}_N)_0^{\mathbb{C}_0}$ have co-dimension 1 in $\mathbb{F}_N^{\mathbb{C}_0}$, (ii) follows with (i). To see (iii), let β fulfill $\beta(-n) = \pm\beta(n)$ for all $n \in \mathbb{F}_N$. Then we obtain for all $x \in \mathbb{F}_N$:

$$\mathcal{F}_N(\beta)(-x) = \sum_{j\in\mathbb{F}_N} \beta(j)e^{\frac{2\pi ixj}{N}} = \sum_{j\in\mathbb{F}_N} \beta(-j)e^{-\frac{2\pi ixj}{N}} = \pm\mathcal{F}_N(\beta)(x),$$

which concludes the proof of the proposition. □

Since $\overline{\mathfrak{C}_N}$ is isomorphic to $\mathbb{F}_N^{\mathbb{C}_0}$, we can apply the Fourier transform $\mathcal{F}_N$ on its elements too. In the next proposition, we introduce some useful calculation tools.

Proposition 1.1.3. *Let d be a divisor of N and $\psi \in \mathfrak{C}_d$ a character. We then have*

$$\mathcal{F}_N\left(\iota_N^d\psi\right) = \mathcal{F}_d\left(\psi\right)$$

as functions on $\mathbb{F}_d$.

Proof. We have for arbitrary $n \in \mathbb{F}_N$

$$\mathcal{F}_N\left(\iota_N^d\psi\right)(n) = \sum_{j\in\mathbb{F}_N} (\iota_N^d\psi)(j)e^{-\frac{2\pi inj}{N}} = \sum_{\substack{j\in\mathbb{F}_N \\ j\in\ker\left(\mathbb{F}_N\to\mathbb{F}_{Nd^{-1}}\right)}} \psi\left(\frac{jd}{N}\right)e^{-\frac{2\pi inj}{N}}$$

and since each element in $\ker\left(\mathbb{F}_N \to \mathbb{F}_{Nd^{-1}}\right)$, independent of the class representative, is a multiple of Nd^{-1}, this equals

$$\sum_{k\in\mathbb{F}_d} \psi(k)e^{-\frac{2\pi ink}{d}} = \mathcal{F}_d\left(\psi\right)(n).$$

Finally, the function $n \mapsto \mathcal{F}_N\left(\iota_N^d\psi\right)(n)$ is well defined as a function on $\mathbb{F}_d$, since the right side does not change under $n + dm$ for any $m \in \mathbb{Z}$. □

Note that this is unusual in the sense that normally there is no canonically restriction map from $\mathbb{F}_N^{\mathbb{C}_0}$ to $\mathbb{F}_d^{\mathbb{C}_0}$ if $d < N$.

1.1.2 Elliptic modular forms

General definition

Recall the fact that the group $\mathrm{GL}_2^+(\mathbb{R})$ of real matrices with positive determinant acts on the upper half plane via Möbius transformation. A *congruence subgroup* $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a subgroup such that $\Gamma(N) \subset \Gamma$ for some integer N. The minimal N with this property is called the *level* of Γ. Note that every congruence subgroup contains a translation element $\left(\begin{smallmatrix} 1 & h \\ 1 & 0 \end{smallmatrix}\right)$ for some minimal $h > 0$. An *elliptic modular form* f of weight k for a congruent subgroup Γ is a holomorphic function on the complex upper half plane, that is invariant under the Petersson operator for all $M \in \Gamma$ and holomorphic at the cusps $\mathbb{Q} \cup \{i\infty\}$. In other words, we have $f|_k M = f$ for all matrices $M \in \Gamma$ and for all $U \in \mathrm{SL}_2(\mathbb{Z})$ the function $f|_k U$ has a Fourier series expansion

$$f|_k U(\tau) = \sum_{n=0}^{\infty} a_U(n) q^{\frac{n}{h}}, \qquad \tau \in \mathbb{H}.$$

In the case $a_U(0) = 0$ for all U we say that f is a *cusp form*. We collect all modular forms of weight k for Γ in the space $M_k(\Gamma)$ and all cusp forms in the subspace $S_k(\Gamma)$. In a more general context, whenever $\chi : \Gamma \to \mathbb{C}^\times$ is an abelian character and a modular form f fulfills

$$f|_k M = \chi(M) f,$$

we write $f \in M_k(\Gamma, \chi)$. It is a well-known fact that the coefficients $a(n) := a_I(n)$ have the property $a(n) = O(n^k)$ (and $a(n) = O(n^{k-1})$ if $k > 2$). Hence, it is possible to attach f an L-function

$$L(f; s) := \sum_{n=1}^{\infty} a(n) n^{-s}.$$

Let $f \in M_k(\Gamma_0(N))$. If f vanishes in the cusps $\tau = 0$ and $\tau = i\infty$, it is well known that we can associate a L-function $L(f, s)$ to f by the Mellin transformation formula

$$\Lambda(f, s) := \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s) L(f, s) = \int_0^{\infty} f\left(\frac{ix}{\sqrt{N}}\right) x^{s-1} \mathrm{d}x, \tag{1.1.2.1}$$

where the integral converges for all $s \in \mathbb{C}$ and hence represents an entire function. Since f fulfills modular transformation properties of weight k, one can show that its L-function is related to another Dirichlet series by a functional equation under $s \mapsto k - s$. One can show that there are no non-constant modular forms for $k = 0$, no non-zero modular forms for $k < 0$, and that the spaces $M_k(\Gamma, \chi)$ are finite-dimensional. A fruitful tool for computing the exact value of the dimensions if $k > 1$ is the Riemann-Roch formula. Here, the connection between modular forms and differential forms on Riemann surfaces gives the key insights. For more explicit details see for example [22]. Many generalizations of the classical modular forms have been found, such as Siegel modular forms (see also [1]

and [31]) for matrix valued arguments that transform under congruence subgroups of the symplectic group Sp_n, and Hilbert modular forms (for a great introduction, the reader may wish to consult [30]) that transform under congruence subgroups of $\mathrm{SL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of a number field K.

Eisenstein series

We briefly sketch the theory of Eisenstein series associated to a pair of Dirichlet characters. For Dirichlet characters χ and ψ modulo positive integers M and N, respectively, and some integer $k \geq 3$, one defines, as already mentioned, the corresponding Eisenstein series for $\tau \in \mathbb{H}$ via

$$E_k(\chi, \psi; \tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \chi(m)\psi(n)(m\tau + n)^{-k}. \tag{1.1.2.2}$$

This series converges absolutely and uniformly on compact subsets of the upper half plane and defines a holomorphic function in that region. One can show that (1.1.2.2) gives a non-zero function if and only if $\chi(-1)\psi(-1) = (-1)^k$ and that the E_k are modular forms of weight k for the congruence subgroups $\Gamma_0(M, N)$ with Nebentypus character $\chi\overline{\psi}$ of $\Gamma_0(M, N)$. Proofs and a systematic approach to Eisenstein series can be found in [22] and [49].

However, in the case $k = 1, 2$ the above series will no longer give the desired convergence properties. But there might be also non-trivial modular forms of weight $k = 1$ and $k = 2$. We can remedy this using the non-holomorphic generalization

$$E_k(\chi, \psi; \tau, s) := \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \chi(m)\psi(n)(m\tau + n)^{-k}|m\tau + n|^{-2s} \tag{1.1.2.3}$$

if $\mathrm{Re}(s) \gg 1$, and analytic continuation in s. As a result, the functions E_k keep their modularity properties when considering the weights $k = 1, 2$. In this situation it is reasonable to define E_k via their Fourier expansion. For a detailed presentation of this Hecke trick see [49] on p. 274 ff.

Every Eisenstein series admits a Fourier series. The coefficients of $E_k(\chi, \psi; \tau)$ are well-known and given by

$$2L(\psi, k)\chi(0) + \frac{2(-2\pi i)^k \psi(-1)}{N^k(k-1)!} \sum_{m=1}^{\infty} \left(\sum_{d|m} d^{k-1} (\mathcal{F}_N \psi)(d) \chi\left(\frac{m}{d}\right) \right) q^{\frac{m}{N}}. \tag{1.1.2.4}$$

Note that in the case that ψ is primitive one has $(\mathcal{F}_N\psi)(a) = \overline{\psi(a)}(\mathcal{F}_N\psi)(1)$ and obtains the simpler expression $\sum_{d|n} d^{k-1}\overline{\psi}(d)\chi\left(\frac{n}{d}\right)$ for the coefficients up to a constant.

1.1.3 Twists and Weil's converse theorem

Weil's converse theorem is a technique to determine modularity of a given Fourier series using twists.

Theorem 1.1.4 (Weil)**.** *Let k and N be two positive integers and χ let be a Dirichlet chraracter modulo N such that $\chi(-1) = (-1)^k$. Additionally, let $a(n)$ and $b(n)$ be two complex sequences such that $a(n), b(n) = O(n^L)$ for $n \geq 0$ and some $L > 0$. If we now put*

$$f(\tau) := \sum_{n=0}^{\infty} a(n)q^n \qquad \text{and} \qquad g(\tau) := \sum_{n=0}^{\infty} b(n)q^n,$$

we have $f \in M_k(\Gamma_0(N), \chi)$ and $g \in M_k(\Gamma_0(N), \overline{\chi})$, if the following is satisfied:

(i) We have $g(\tau) = (\sqrt{N}\tau)^{-k} f(-\frac{1}{N\tau})$.

(ii) For any primitive Dirichlet character ψ whose conductor is prime to N we have

$$\chi(m_\psi)\psi(-N)\mathcal{G}(\psi)\mathcal{G}\left(\overline{\psi}\right)^{-1} g_{\overline{\psi}}(\tau) = \left(\sqrt{Nm_\psi^2}\tau\right)^{-k} f_\psi\left(-\frac{1}{Nm_\psi^2\tau}\right).$$

Here $\mathcal{G}(\psi)$ is the Gauß sum of the character ψ and f_ψ is a twist of f given by $f_\psi(\tau) = \sum_{n=0}^{\infty} \chi(n)a(n)q^n$. Proofs for Theorem 1.1.4 can be found in [49], p. 128 and also [14], p. 61.

Remark 1.1.5. *Let N and k be positive integers. Let $\Lambda_N(f;s) := \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s)L(f;s)$, where $L(f;s) := \sum_{n=1}^{\infty} a(n)n^{-s}$ denotes the L-series corresponding to f. Note that the conditions (i) and (ii) in Theorem 1.1.4 are equivalent to the following assertions (i)' and (ii)'.*

(i)' Both $\Lambda_N(f;s)$ and $\Lambda_N(g;s)$ can be analytically continued to the whole complex plane and satisfy the functional equation

$$\Lambda_N(f;s) = i^k \Lambda_N(g; k-s),$$

and

$$\Lambda_N(f;s) + \frac{a(0)}{s} + \frac{i^k b(0)}{k-s}$$

is entire and bounded on any vertical strip $-\infty < \sigma_0 < \mathrm{Re}(s) < \sigma_1 < \infty$.

(ii)' The function

$$\Lambda_N(f,\psi;s) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s)L(f,\psi;s),$$

where $L(f,\psi;s) = \sum_{n=1}^{\infty} \psi(n)a(n)n^{-s}$, has a holomorphic continuation to the whole complex plane, is bounded on any vertical strip, and satisfies the functional equation

$$\Lambda_N(f,\psi;s) = i^k \chi(m_\psi)\psi(-N)\mathcal{G}(\psi)\mathcal{G}\left(\overline{\psi}\right)^{-1} \Lambda_N(g,\overline{\psi}; k-s).$$

We will use the conditions stated in terms of Fourier series since our method does not require any use of L-functions.

1.1.4 Eichler integrals

To any modular form of weight $k \geq 2$ that vanishes in the cusps in $\tau = 0$ and $\tau = i\infty$, we can associate an Eichler integral. It has the form

$$\mathcal{E}(f;\tau) := \frac{(-2\pi i)^{k-1}}{(k-2)!} \int_{\tau}^{i\infty} f(z)(z-\tau)^{k-2}\mathrm{d}z.$$

This integral represents a holomorphic and periodic function on the upper half plane and is tied to the so-called *period polynomial* $p(f;\tau)$ of f via the functional equation

$$\mathcal{E}\left(f; -\frac{1}{\tau}\right) - \tau^{2-k}\mathcal{E}(f;\tau) =: p(f;\tau).$$

Explicitly, we have a correspondence to the critical values of the L-function associated to f via

$$p(f;\tau) = \sum_{n=0}^{k-2} \binom{k-2}{n} i^{1-n}\Lambda(f;n+1)\tau^{k-2-n}.$$

These period polynomials are highly important objects in number theory. For example, they appear in the context of a conjecture by Delinge-Beilinson-Scholl which asserts about the nature of values of derivatives of L-functions of Hecke cusp forms f, see also [21]. Moreover, an immediate implication of the Eichler-Shimura isomorphism, see [45], applied to the period polynomial is Manin's Periods Theorem [48]. It provides important information about the arithmetic nature of critical L-values. For a detailed investigation of the values of Eichler integrals at algebraic points, particularly in the context of Ramanujan identities for L-values at integer arguments, see [33]. Finally, a fairly good introduction to the so-called Riemann hypothesis for period polynomials attached to derivatives of L-functions is given in [21].

There are also cohomological approaches to Eichler integrals and period polynomials. More precisely, if $f \in S_k(\Gamma)$ is a cusp form for some congruence subgroup Γ we assign f a map σ_f that sends $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ to

$$\int_{z_0}^{\gamma^{-1}z_0} f(z)(z-\tau)^{k-2}\mathrm{d}z.$$

Here $z_0 \in \mathbb{H}\cup\mathbb{Q}\cup\{i\infty\}$ is some arbitrary value. We call σ_f a 1-cocycle and its cohomology class is independent of the choice of z_0. In particular, for $z_0 = 0$ and $\gamma = S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ we

obtain

$$\sigma_f(S)(\tau) = \int_0^{i\infty} f(z)(z-\tau)^{k-2}\mathrm{d}z. \tag{1.1.4.1}$$

Note that this integral converges since f is a cusp form. The polynomial $\sigma_f(S)$ is called the period polynomial of f. It is tied with values of the corresponding L-function L_f at critical values by the formula

$$\sigma_f(S)(\tau) = \sum_{n=0}^{k-2} \binom{k-2}{n} i^{1-n}\Lambda_f(n+1)\tau^{k-2-n}. \tag{1.1.4.2}$$

This relation is just the beginning of a story involving far-reaching consequences for the geometry and arithmetic of f. One example is Manin's Periods Theorem [48] providing important information about the arithmetic nature of critical L-values. On the other hand, note that

$$\sigma_f(S)(\tau) = F(\tau) - (-\tau)^{k-2}F\left(-\frac{1}{\tau}\right),$$

where

$$F(\tau) = \int_\tau^{i\infty} f(z)(z-\tau)^{k-2}\mathrm{d}z.$$

Note that F is essentially the $(k-1)$-fold integral of f.

Definition 1.1.6. *Let $m \geq 0$ be an integer. Then we define the m-fold integral map $\int_m$ by*

$$\int_m : \mathbb{C}_0^+[[q^{\frac{1}{M}}]] \longrightarrow \mathbb{C}_0^+[[q^{\frac{1}{M}}]]$$
$$f(\tau) = \sum_{n=1}^{\infty} a_f(n)q^{\frac{n}{M}} \longmapsto M^m \sum_{n=1}^{\infty} a_f(n)n^{-m}q^{\frac{n}{M}}.$$

Note that this is the inverse function of ∂_τ^m defined on $\mathbb{C}_0^+[[q]]$. This means if $f(\tau) = \sum_{n=1}^{\infty} a(n)q^n$, then we have

$$F(\tau) = \frac{(k-2)!}{(-2\pi i)^{k-1}}N^{k-1}\sum_{n=1}^{\infty} a(n)n^{1-k}q^{\frac{n}{N}} = \frac{(k-2)!}{(-2\pi i)^{k-1}}\int_{k-1} f(\tau).$$

However, for our purposes assuming f being a cusp form is too restrictive. We would prefer to investigate modular forms f for congruence subgroups vanishing in the cusps 0 and $i\infty$. In this more general case the integral (1.1.4.1) still exists for trivial reasons (for any calculations, we simply choose the straight line $\gamma(t) = it$ from 0 to $i\infty$ as the path of integration) and we also still obtain the identity (1.1.4.2). The next proposition gives the desired results in detail.

Proposition 1.1.7. *Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function with the following properties:*

(i) f is periodic and has a Fourier expansion of the form $f(\tau) = \sum_{n=1}^{\infty} a_f(n) q^{\frac{n}{\lambda}}$ for some $\lambda > 0$.

(ii) There exists an integer $k \geq 2$ and a dual function f^* with a Fourier expansion $f^*(\tau) = \sum_{n=1}^{\infty} a_{f^*}(n) q^{\frac{n}{\lambda^*}}$ with $\lambda^* > 0$ such that

$$f\left(-\frac{1}{\tau}\right) = \tau^k f^*(\tau).$$

(iii) The coefficients $a_f(n)$ and $a_{f^*}(n)$ are polynomially bounded, such that the corresponding L-functions $L_f(s) = \sum_{n=1}^{\infty} a_f(n) n^{-s}$ and $L_{f^*}(s)$ converge on some right half plane.

Then the functions L_f and L_{f^*} have meromorphic continuations to the entire plane. Put

$$F(\tau) = \frac{(k-2)!}{(-2\pi i)^{k-1}} \lambda^{k-1} \sum_{n=1}^{\infty} \frac{a_f(n)}{n^{k-1}} q^{\frac{n}{\lambda}},$$

and

$$F^*(\tau) = \frac{(k-2)!}{(-2\pi i)^{k-1}} (\lambda^*)^{k-1} \sum_{n=1}^{\infty} \frac{a_{f^*}(n)}{n^{k-1}} q^{\frac{n}{\lambda^*}}.$$

We obtain

$$F(\tau) - (-\tau)^{k-2} F^*\left(-\frac{1}{\tau}\right) = P_{f,f^*}(\tau) = (-1)^k \sum_{\ell=0}^{k-2} \binom{k-2}{\ell} i^{1-\ell} \Lambda_f(\ell+1) \tau^{k-2-\ell},$$

where

$$\Lambda_f(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) L_f(s).$$

Proof. Let $\tau \in \mathbb{H}$. Consider the integral

$$I = \int_0^{i\infty} f(z)(z-\tau)^{k-2} \mathrm{d}z$$

which converges absolutely since $f(iy) = O(\min(e^{-\varepsilon_1 y}, e^{-\frac{\varepsilon_2}{y}}))$ as $y \to 0$ and $y \to \infty$ for some constants $\varepsilon_1, \varepsilon_2 > 0$ due to assumption (ii). On one hand a calculation using $\int_0^{i\infty} f(iy) y^{s-1} \mathrm{d}y = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) L_f(s) = \Lambda_f(s)$ demonstrates

$$I = \sum_{\ell=0}^{k-2} \binom{k-2}{\ell} (-1)^{k-2-\ell} i^{\ell+1} \tau^{k-2-\ell} \int_0^{\infty} f(iy) y^{\ell} \mathrm{d}y = P_{f,f^*}(\tau).$$

On the other hand we find

$$\begin{aligned}
I &= \int_0^{\tau} f(z)(z-\tau)^{k-2}\mathrm{d}z + \int_{\tau}^{i\infty} f(z)(z-\tau)^{k-2}\mathrm{d}z \\
&= \int_{i\infty}^{-\frac{1}{\tau}} f\left(-\frac{1}{z}\right)\left(-\frac{1}{z}-\tau\right)^{k-2}\mathrm{d}\left(-\frac{1}{z}\right) + \int_{\tau}^{i\infty} f(z)(z-\tau)^{k-2}\mathrm{d}z \\
&= \int_{i\infty}^{-\frac{1}{\tau}} f^*(z)(-1-z\tau)^{k-2}\mathrm{d}z + \int_{\tau}^{i\infty} f(z)(z-\tau)^{k-2}\mathrm{d}z \\
&= -(-\tau)^{k-2}\int_{-\frac{1}{\tau}}^{i\infty} f^*(z)\left(z+\frac{1}{\tau}\right)^{k-2}\mathrm{d}z + \int_{\tau}^{i\infty} f(z)(z-\tau)^{k-2}\mathrm{d}z \\
&= F(\tau) - (-\tau)^{k-2}F^*\left(-\frac{1}{\tau}\right),
\end{aligned}$$

where in the last step we used

$$\begin{aligned}
\int_{\tau}^{i\infty} f(z)(z-\tau)^{k-2}\mathrm{d}z &= \sum_{n=1}^{\infty} a_f(n) q^{\frac{n}{\lambda}} \int_0^{i\infty} e^{\frac{2\pi i n z}{\lambda}} z^{k-2}\mathrm{d}z \\
&= \frac{(k-2)!}{(-2\pi i)^{k-1}} \sum_{n=1}^{\infty} a_f(n)\left(\frac{n}{\lambda}\right)^{1-k} q^{\frac{n}{\lambda}}
\end{aligned}$$

and the analogous result with f replaced by f^*. □

1.2 Construction of modular forms

1.2.1 Properties of weak functions

Let ω be a 1-periodic meromorphic function on $\mathbb{C}$ such that all poles of ω lie in $\mathbb{Q}$. We also want ω to be of rapid decay as the imaginary part of its arguments goes to $\pm\infty$. If we additionally assume that all poles are simple it is an easy consequence from Liouville's theorem that such an ω is given by

$$\omega(z) = \sum_{x \in \mathbb{Q}/\mathbb{Z}} \beta_\omega(x) h_x(z),$$

where $h_x(z) := \frac{e(z)}{(e(x)-e(z)}$ with some $\beta_\omega \in (\mathbb{Q}/\mathbb{Z})_0^{\mathbb{C}_0}$, see also notation. As already mentioned, we call such an ω a weak function. The level of ω is defined as the smallest positive integer N such that $\omega(\frac{z}{N})$ only has poles at integers. It is obvious that the set of all weak functions with level d such that $d|N$ form a finite dimensional vector space over the complex numbers which we will denote by W_N. The global vector space of all weak functions will be denoted by $W_\infty := \bigcup_{N=1}^{\infty} W_N$.

Remark 1.2.1. *We have $W_1 = 0$ since all weak functions with level 1 are multiples of $\cot(\pi z)$, which does not satisfy the growth condition. This elementary fact also has an interpretation using modular forms, see Remark 1.2.17.*

For a non-principal Dirichlet character χ modulo N we write

$$\omega_\chi(z) := \sum_{j \in \mathbb{F}_N} \chi(j) h_{\frac{j}{N}}(z).$$

In this section we use the complex analytic properties of weak functions and contour integration to construct modular forms. For further applications, such as the classification of all modular forms stemming from weak function, it is useful to know the precise structure of W_N. Let $\mathfrak{C}_L$ be the group of all Dirichlet characters modulo L. We define the principal part of W_N by

$$\mathfrak{P}_N := \left\{ \omega \in W_N \;\middle|\; \omega = \sum_{d|N} c_d \sum_{j \in \mathbb{F}_d} \chi_{0,d}(j) h_{\frac{j}{d}} \right\}.$$

Proposition 1.2.2. *We have a decomposition*

$$W_N = \mathfrak{P}_N \oplus \bigoplus_{d|N} \bigoplus_{\chi \in \mathfrak{C}_d \setminus \{\chi_{0,d}\}} \mathbb{C}\omega_\chi.$$

Proof. It is clear that W_N is isomorphic to $\{v \in \mathbb{C}^N \mid \sum_{j=1}^N v_j = 0\} \cong \mathbb{C}^{N-1}$ by $\omega \mapsto \left(\beta_\omega(0), \beta_\omega\left(\frac{1}{N}\right), ..., \beta_\omega\left(\frac{N-1}{N}\right)\right)$. We can now formally write with Euler's totient function

$$\mathbb{C}^{N-1} = \mathbb{C}^{\sigma_0(N)-1} \oplus \bigoplus_{d|N} \mathbb{C}^{\varphi(d)-1},$$

since

$$N - 1 = \sum_{d|N} \varphi(d) - 1 = \sum_{d|N} (\varphi(d) - 1) + \sum_{d|N} 1 - 1 = \sum_{d|N} (\varphi(d) - 1) + \sigma_0(N) - 1.$$

Recall that characters are linearly independent. Each summand $\mathbb{C}^{\varphi(d)-1}$ corresponds to a subspace of W_N given by the span of the ω_χ, where the χ are the non-principal characters modulo d. Therefore the quotient $\mathbb{C}^{\sigma_0(N)-1}$ is generated by the principal characters and since we have the vanishing condition of W_N this is given by $\mathfrak{P}_N$, as required. □

We use the same definition in the context of residue functions, i.e. we basically split them into non-principal characters and principal part elements. Let ϕ be a non-principal character or an element of the principal part $\phi = \sum_{d|N_\phi} c_d \iota_{N_\phi}^d \chi_{0,d}$ modulo N_ϕ. We then define the corresponding character by

$$\phi^*(n) = \begin{cases} \phi(n), & \text{if } \phi \text{ is non-principal,} \\ \chi_{0,N_\phi}(n), & \text{if } \phi \text{ is in the principal part.} \end{cases}$$

Note that we have $\phi(Mn) = \phi^*(M)\phi(n)$ for any M coprime to N_ϕ.

Definition 1.2.3. *Let N be a positive integer. For any positive integer M which is coprime to N, we define the Atkin-Lehner operator $[M] : W_N \to W_N$ by*

$$(\omega)[M] := \sum_{j \in \mathbb{F}_N} (\kappa_N \beta_\omega)(Mj) h_{\frac{j}{N}}.$$

Remark 1.2.4. *Each ω_ϕ is an eigenvector of $[M]$ with eigenvalue $\phi^*(M)$.*

Before we assign each element $\omega \otimes \eta \in W_N \otimes W_M$ a holomorphic function, we look carefully at even and odd subspaces, since there will be lots of trivial assignments by symmetry. On W_∞ we define an involution $\widehat{\cdot} : W_\infty \to W_\infty$ given by $\widehat{\omega}(z) := \omega(-z)$. One easily checks that this map is well-defined and level preserving. In particular, it restricts to maps $\widehat{\cdot} : W_N \to W_N$. We define $W_T^\pm \subset W_T$ for $T \in \mathbb{N} \cup \{\infty\}$ as the spaces of even and odd weak functions, respectively. This induces a canonical decomposition map $W_T \to W_T^+ \oplus W_T^-$ given by $\omega \mapsto \frac{\omega+\widehat{\omega}}{2} + \frac{\omega-\widehat{\omega}}{2}$. Hence we obtain multiplicative decompositions

$$W_{T_1} \otimes W_{T_2} \longrightarrow (W_{T_1}^+ \oplus W_{T_2}^-) \otimes (W_{T_1}^+ \oplus W_{T_2}^-),$$

and we define

$$(W_{T_1} \otimes W_{T_2})^+ := W_{T_1}^+ \otimes W_{T_2}^+ \oplus W_{T_1}^- \otimes W_{T_2}^-$$

and

$$(W_{T_1} \otimes W_{T_2})^- := W_{T_1}^+ \otimes W_{T_2}^- \oplus W_{T_1}^- \otimes W_{T_2}^+.$$

Remark 1.2.5. *Let $N \in \mathbb{N}$ be arbitrary. Together with the functional equation*

$$h_{\frac{j}{N}}(-z) = -1 - h_{-\frac{j}{N}}(z)$$

one easily sees that

$$\omega \in W_N^{\pm} = \left\{ \sum_{x \in \mathbb{F}_{\frac{1}{N}}} \beta_\omega(x) h_x(z) \;\middle|\; \beta_\omega(-x) = \mp \beta_\omega(x) \right\}.$$

Definition 1.2.6. *Fix an integer k. Every pair $\omega \otimes \eta$ in $W_M \otimes W_N$ induces a holomorphic function on the union of the upper and lower half plane $\underline{\mathbb{H}} := \mathbb{H}^+ \cup \mathbb{H}^-$ by*

$$\vartheta_k : W_M \otimes W_N \longrightarrow \mathcal{O}(\underline{\mathbb{H}})$$

$$\vartheta_k(\omega \otimes \eta; \tau) := -2\pi i \sum_{x \in \mathbb{Q}^\times} \mathrm{res}_{z=x}(z^{k-1}\eta(z)\omega(z\tau)).$$

It can be checked that this series converges absolutely and uniformly on compact subsets of $\underline{\mathbb{H}}$. So $\vartheta_k(\omega \otimes \eta; \tau)$ is indeed holomorphic in this region. By simple symmetry arguments one sees $(W_M \otimes W_N)^{\mp} \subseteq \ker(\vartheta_k)$ if $(-1)^k = \pm 1$.

The next theorem is one of the central statements of this thesis. It states that there is in some sense a "modular duality" induced by the isomorphism

$$W_M \otimes W_N \xrightarrow{\sim} W_N \otimes W_M,$$

$$\omega \otimes \eta \longmapsto \eta \otimes -\widehat{\omega}.$$

Theorem 1.2.7 (Main transformation law, see [27])**.** *Let $\omega \otimes \eta$ be in $W_M \otimes W_N$. Then we have*

$$\vartheta_k\left(\omega \otimes \eta; -\frac{1}{\tau}\right) = \tau^k \vartheta_k(\eta \otimes -\widehat{\omega}; \tau) + 2\pi i \ \mathrm{res}_{z=0}\left(z^{k-1}\eta(z)\widehat{\omega}\left(\frac{z}{\tau}\right)\right).$$

Remark 1.2.8. *Note that the second summand on the right is a rational function of τ which is holomorphic in $\mathbb{C}^\times$.*

Proof. Let $y > 0$ be arbitrary and fixed. Put $\tau := iy \in \mathbb{H}$ and define

$$g_\tau(z) := -2\pi i z^{k-1}\eta(z)\widehat{\omega}\left(\frac{z}{\tau}\right).$$

Then g_τ is a meromorphic function in the plane whose poles are simple and $S(g_\tau) \subset \frac{1}{M}\mathbb{Z} \cup \frac{1}{N}\mathbb{Z}\tau$. Note that all poles lie on the real and imaginary axes in this case. Now consider the closed contour integrals

$$I_n(\tau) = \frac{1}{2\pi i} \oint_{\gamma_n} g_\tau(z) \mathrm{d}z,$$

taken as usual counter-clockwise, where the γ_n denote rectangles crossing the axes at points lying exactly between the n-th and $n+1$-th consecutive pole of g_τ. Let $0 < x_1 < x_2 < x_3 < ...$ be all positive real and $iy_1, iy_2, iy_3, ...$, such that $0 < y_1 < y_2 < y_3...$, be all positive imaginary poles of g_τ. Then we conclude for $a_n := \frac{x_n + x_{n+1}}{2}$ and $b_n := \frac{y_n + y_{n+1}}{2}$

$$\int_{a_n - ib_n}^{a_n + ib_n} |g_\tau(z)||\mathrm{d}z| = O\left((a_n + b_n)^k b_n e^{-\delta |a_n|}\right)$$

for some $\delta > 0$, since $\eta(z)$ is periodic and $\omega(-iyz)$ of rapid decay in real direction. Hence this integral tends to zero as n increases. Given the symmetry of g_τ, we conclude that $I_n(\tau) = o(1)$. Hence by the Residue theorem

$$\sum_{\alpha \in \frac{1}{M}\mathbb{Z} \setminus \{0\}} \mathrm{res}_{z=\alpha}\left(g_\tau(z)\right) + \mathrm{res}_{z=0}(g_\tau(z)) + \sum_{\alpha \in \frac{1}{N}\mathbb{Z} \setminus \{0\}} \mathrm{res}_{z=\alpha\tau}\left(g_\tau(z)\right) = 0.$$

Since $\tau \in \mathbb{H}$ and the poles of ω are a subset of $\mathbb{Q}$, the poles of $z \mapsto \eta(z)\widehat{\omega}\left(\frac{z}{\tau}\right)$ in $\mathbb{Q}^\times$ are a subset of the poles of η, and hence the first sum clearly equals $\vartheta_k(\widehat{\omega} \otimes \eta; \frac{1}{\tau}) = \vartheta_k(\omega \otimes \eta; -\frac{1}{\tau})$. Since

$$\mathrm{res}_{z=\alpha\tau}\left(g_\tau(z)\right) = \tau^k \mathrm{res}_{z=\alpha}(g_\tau(\tau z)),$$

we obtain for the second sum

$$\sum_{\alpha \in \frac{1}{N}\mathbb{Z} \setminus \{0\}} \mathrm{res}_{z=\alpha\tau}\left(g_\tau(z)\right) = \tau^k \vartheta_k(\eta \otimes \widehat{\omega}; \tau).$$

The claim now follows with the Identity theorem. □

Remark 1.2.9. *This complex analytic philosophy is not new. For example, Siegel gave a short proof for the functional equation of the Dedekind eta function* $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ *using similar ideas, see [2] on p. 48 ff. They were already used similarly by Berndt and Straub in [7] when deducing an interesting functional equation for the secant series*

$$\psi_s(\tau) := \sum_{n=1}^{\infty} \frac{\sec(n\tau)}{n^s}.$$

A detailed description of the space $\vartheta_k(W_M \otimes W_N)$ will be given in Theorems 1.2.21 and 1.2.23.

1.2.2 Weak functions and modular forms

In this section we present an alternative proof that the $\vartheta_k(\omega \otimes \eta; \tau)$ define modular forms. We use the properties of weak functions and contour integration methods. The proof underlines the naturalness of the construction and gives modularity for all values $k \in$

$\mathbb{N}$ simultaneously without using the Hecke trick. Our main tools are the transformation law Theorem 1.2.7 and Weil's converse theorem, see Theorem 1.1.4. Note that we avoid the use of L-functions in the construction part throughout.

Let N_1, N_2 be positive integers and χ be a Dirichlet character modulo $N = N_1N_2$. Then we have an isomorphism

$$M_k(\Gamma_0(N_1, N_2), \chi) \xrightarrow{\sim} M_k(\Gamma_0(N), \chi)$$

$$f(\tau) \longmapsto f(N_2\tau). \tag{1.2.2.1}$$

In the same way we obtain an isomorphism $M_k(\Gamma_1(N_1, N_2)) \xrightarrow{\sim} M_k(\Gamma_1(N))$. Furthermore we have a useful decomposition

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_k(\Gamma_0(N), \chi),$$

where the sum runs over all Dirichlet characters modulo N. Together with (1.2.2.1) this gives the decomposition

$$M_k(\Gamma_1(N_1, N_2)) = \bigoplus_{\chi} M_k(\Gamma_0(N_1, N_2), \chi), \tag{1.2.2.2}$$

where the sum runs over all Dirichlet characters modulo N.

Let $M \geq 1$ be an integer and $f(\tau) = \sum_{n=0}^{\infty} a(n)q^{\frac{n}{M}}$ be a holomorphic function on the upper half plane. Let ψ a Dirichlet character modulo r. Then we put

$$f_\psi(\tau) = \sum_{n=0}^{\infty} \psi(n)a(n)q^{\frac{n}{M}}.$$

We say in this case that f is twisted by the character ψ. In the following and for all $\lambda \in \mathbb{R}$ we put $T(\lambda) = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$. The following result is well-known.

Proposition 1.2.10. *Let $f(\tau) = \sum_{n=0}^{\infty} a(n)q^{\frac{n}{M}}$ be holomorphic on the upper half plane such that $a(n) = O(n^L)$ for some $L > 0$. Let ψ be a primitive Dirichlet character with conductor m_ψ. Then for any integer $k > 0$ we have*

$$f_\psi(\tau) = \mathcal{G}\left(\overline{\psi}\right)^{-1} \sum_{u=1}^{m_\psi} \overline{\psi}(u) f|_k T\left(\frac{uM}{m_\psi}\right).$$

Now let $N > 1$ and $M > 1$ be coprime integers. We observe that two maps $\beta_N : \mathbb{F}_N \to \mathbb{C}$ and $\beta_M : \mathbb{F}_M \to \mathbb{C}$ induce a new map $\beta_N \times \beta_M : \mathbb{F}_{NM} \to \mathbb{C}$ when putting

$$(\beta_N \times \beta_M)(m) := \beta_N(v)\beta_M(u)$$

where $m = vM - uN$. We use this type of notation because it is more natural for later applications. According to the Chinese remainder theorem this is well-defined. Note that

$$(\beta_N \times \beta_M)(m) = \beta_N[M^{-1}](m)\beta_M[N^{-1}](-m),$$

where M^{-1} is the multiplicative inverse of M modulo N and N^{-1} is the multiplicative inverse of N modulo M.

Definition 1.2.11. *Let N and M be coprime. Then we define a bilinear map*

$$\times : W_N \times W_M \longrightarrow W_{NM}$$

by putting

$$(\omega \times \eta)(z) := \sum_{j \in \mathbb{F}_{NM}} (\beta_\omega \times \beta_\eta)(j) h_{\frac{j}{MN}}(z).$$

Note that this is well-defined since

$$\sum_{j \in \mathbb{F}_{NM}} (\beta_\omega \times \beta_\eta)(j) = \sum_{u \in \mathbb{F}_N} \sum_{v \in \mathbb{F}_M} \beta_\omega(u)\beta_\eta(v) = 0.$$

Lemma 1.2.12. *Let ψ be a primitive Dirichlet character modulo N and d a proper divisor of N. Then for all integers u we have*

$$\sum_{j=0}^{\frac{M}{d}-1} \psi(dj + u) = 0.$$

Proof. Let $d|N$ with $d < N$ and u be arbitrary. In the case $(u, d) > 1$ the assertion is clear. Therefore we may assume $(u, d) = 1$. Since ψ is primitive, there is some $a \in \mathbb{Z}$ such that $(a, N) = 1$ and $a \equiv 1 \pmod{d}$ such that $\psi(a) \neq 1$. Since we have a bijection

$$\left\{\overline{rd + u}, 0 \leq r \leq \frac{M}{d} - 1\right\} \longrightarrow \left\{\overline{rd + u}, 0 \leq r \leq \frac{M}{d} - 1\right\}$$
$$e \longmapsto ae$$

between subsets of $\mathbb{F}_N$, we obtain

$$(1 - \psi(a)) \sum_{j=0}^{\frac{M}{d}-1} \psi(dj + u) = 0.$$

This proves the lemma. □

The next lemma is a technical statement for rearranging sums over $\mathbb{F}_M \times \mathbb{F}_N$ over $\mathbb{F}_{MN}$ using the above cross product.

Lemma 1.2.13. *Let N and M be two coprime integers. Let $\beta : \mathbb{F}_N \to \mathbb{C}$ and $f : \mathbb{F}_{NM} \to \mathbb{C}$ be functions. Let also α be an integer and ψ a primitive Dirichlet character modulo M. Then we have the identity*

$$\sum_{u\in\mathbb{F}_M}\sum_{j\in\mathbb{F}_N}\psi(u)\beta(j)f(jM-\alpha uN) = \overline{\psi}(\alpha)\sum_{\ell\in\mathbb{F}_{NM}}(\psi\times\beta)(\ell)f(\ell).$$

Proof. We observe that $g_j(u) := \beta(j)f(jM-\alpha uN)$ is a function on $\mathbb{F}_M$ in the variable u. We first look at the case when $\frac{M}{d} := (\alpha, M) > 1$. Then g_j is a function on $\mathbb{F}_d$. Since ψ is primitive, we obtain with Lemma 1.2.12

$$\sum_{u\in\mathbb{F}_M}\psi(u)g_j(u) = \sum_{e=0}^{d-1}g_j(e)\sum_{r=0}^{\frac{M}{d}-1}\psi(dr+e) = 0.$$

Now let α and M be coprime. Then we obtain

$$F(\alpha) := \sum_{u\in\mathbb{F}_M}\sum_{j\in\mathbb{F}_N}\psi(u)\beta(j)f(jM-\alpha uN) = \overline{\psi}(\alpha)\sum_{u\in\mathbb{F}_M}\sum_{j\in\mathbb{F}_N}\psi(\alpha u)\beta(j)f(jM-\alpha uN)$$

With the bijection $\mathbb{F}_M \to \mathbb{F}_M$ given by $x \mapsto \alpha^{-1}x$ we can make the following rearrangement.

$$F(\alpha) = \overline{\psi}(\alpha)\sum_{u'\in\mathbb{F}_M}\sum_{j\in\mathbb{F}_N}\psi(u')\beta(j)f(jM-u'N) = \overline{\psi}(\alpha)\sum_{\ell\in\mathbb{F}_{NM}}(\psi\times\beta)(\ell)f(\ell).$$

This proves the lemma. □

The next theorem considers the twists of the functions ϑ_k.

Theorem 1.2.14. *Let N_1, N_2 and M be integers such that $(N_1, M) = (N_2, M) = 1$ and ψ a primitive Dirichlet character modulo M. Then for any $\omega \in W_{N_1}$ and $\eta \in W_{N_2}$ we have*

$$(\vartheta_k)_\psi(\omega\otimes\eta;\tau) = \mathcal{G}\left(\overline{\psi}\right)^{-1}M^{k-1}\psi(N_2)\vartheta_k\left(\left(\omega_{\overline{\psi}}\times\omega\right)\otimes\left(\omega_\psi\times\eta[M]\right);M\tau\right).$$

Proof. With Proposition 1.2.10 we obtain

$$\begin{aligned}
&(\vartheta_k)_\psi(\omega\otimes\eta;\tau)\\
&= 2N_2^{1-k}\mathcal{G}\left(\overline{\psi}\right)^{-1}\sum_{u\in\mathbb{F}_M}\overline{\psi}(u)\sum_{\alpha\in\mathbb{Z}\setminus\{0\}}\alpha^{k-1}\beta_\eta(\alpha)\omega\left(\frac{\alpha\tau}{N_2}+\frac{\alpha u}{M}\right)\\
&= 2N_2^{1-k}\mathcal{G}\left(\overline{\psi}\right)^{-1}\sum_{\alpha\in\mathbb{Z}\setminus\{0\}}\alpha^{k-1}\beta_\eta(\alpha)\sum_{u\in\mathbb{F}_M}\sum_{j\in\mathbb{F}_{N_1}}\overline{\psi}(u)\beta_\omega(j)\frac{e\left(\frac{\alpha\tau}{N_2}-\frac{jM-\alpha uN_1}{MN_1}\right)}{1-e\left(\frac{\alpha\tau}{N_2}-\frac{jM-\alpha uN_1}{MN_1}\right)}.
\end{aligned}$$

Note that $f(x) = e(\frac{\alpha\tau}{N_2} - \frac{x}{MN_1})(1 - e(\frac{\alpha\tau}{N_2} - \frac{x}{MN_1}))^{-1}$ is a function of $\mathbb{F}_{N_1M}$. Now with Lemma 1.4.3.7 this is

$$
\begin{aligned}
&= 2N_2^{1-k}\mathcal{G}\left(\overline{\psi}\right)^{-1} \sum_{\alpha\in\mathbb{Z}\setminus\{0\}} \alpha^{k-1}\beta_\eta(\alpha)\psi(\alpha) \sum_{\ell\in\mathbb{F}_{N_1M}} (\overline{\psi}\times\beta_\omega)(\ell)\frac{e\left(\frac{\alpha\tau}{N_2} - \frac{\ell}{MN_1}\right)}{1 - e\left(\frac{\alpha\tau}{N_2} - \frac{\ell}{MN_1}\right)} \\
&= 2N_2^{1-k}\psi(N_2)\mathcal{G}\left(\overline{\psi}\right)^{-1} \sum_{\alpha\in\mathbb{Z}\setminus\{0\}} \alpha^{k-1}\beta_\eta[M^{-1}](M\alpha)\psi[N_2^{-1}](\alpha) \\
&\quad\times \sum_{\ell\in\mathbb{F}_{N_1M}} (\overline{\psi}\times\beta_\omega)(\ell)\frac{e\left(\frac{\alpha\tau}{N_2} - \frac{\ell}{MN_1}\right)}{1 - e\left(\frac{\alpha\tau}{N_2} - \frac{\ell}{MN_1}\right)} \\
&= 2N_2^{1-k}\psi(N_2)\mathcal{G}\left(\overline{\psi}\right)^{-1} \sum_{\alpha\in\mathbb{Z}\setminus\{0\}} \alpha^{k-1}(\psi\times\beta_\eta[M])(\alpha) \sum_{\ell\in\mathbb{F}_{N_1M}} (\overline{\psi}\times\beta_\omega)(\ell)\frac{e\left(\frac{\alpha\tau}{N_2} - \frac{\ell}{MN_1}\right)}{1 - e\left(\frac{\alpha\tau}{N_2} - \frac{\ell}{MN_1}\right)} \\
&= 2M^{k-1}\psi(N_2)\mathcal{G}\left(\overline{\psi}\right)^{-1}(MN_2)^{1-k} \sum_{\alpha\in\mathbb{Z}\setminus\{0\}} \alpha^{k-1}(\psi\times\beta_\eta[M])(\alpha)(\omega_{\overline{\psi}}\times\omega)\left(\frac{\alpha M\tau}{N_2M}\right).
\end{aligned}
$$

This proves the theorem. □

Theorem 1.2.15 (see [27]). *Let χ and ϕ be two non-principal Dirichlet characters or principal elements modulo $N_\chi > 1$ and $N_\phi > 1$, respectively, and $k \geq 1$ an integer. Then if $f(\tau) = \vartheta_k(\omega_\chi \otimes \omega_\phi, N_\phi\tau)$ we have $f \in M_k(\Gamma_0(N_\chi N_\phi), \overline{\chi^*}\phi^*)$.*

Proof. We check the conditions of Weil's converse theorem. Here we use the equivalent version, which gets along without L-functions and uses the transformation properties of the twists of the Fourier series. For this we frequently use Theorem 1.2.7. Put $f(\tau) = \vartheta_k(\omega_\chi \otimes \omega_\phi; N_\phi\tau)$. It is clear by (1.2.3.1) that if we put $f(\tau) = \sum_{n=0}^{\infty} a(n)q^n$ we obtain $a(n) = O(n^L)$ for some $L > 0$. Now we set

$$
\begin{aligned}
g(\tau) &= \left(\sqrt{N_\phi N_\chi}\tau\right)^{-k} f\left(-\frac{1}{N_\phi N_\chi\tau}\right) = \left(\sqrt{N_\phi N_\chi}\tau\right)^{-k} \vartheta_k\left(\omega_\chi\otimes\omega_\phi; -\frac{1}{N_\chi\tau}\right) \\
&= -\left(\sqrt{N_\phi N_\chi}\tau\right)^{-k} N_\chi^k\tau^k\vartheta_k\left(\omega_\phi\otimes\widehat{\omega_\chi}; N_\chi\tau\right) = -\left(\frac{N_\chi}{N_\phi}\right)^{\frac{k}{2}} \chi^*(-1)\vartheta_k\left(\omega_\phi\otimes\omega_\chi; N_\chi\tau\right).
\end{aligned}
$$

From this it is clear that $g(\tau) = \sum_{n=0}^{\infty} b(n)q^n$ for some sequence $b(n)$ with $b(n) = O(n^L)$ for some $L > 0$. Let ψ be a primitive Dirichlet character with conductor M_ψ such that $(N_\chi, M_\psi) = (N_\phi, M_\psi) = 1$. We denote

$$
C_\psi = \overline{\chi^*}(M_\psi)\phi^*(M_\psi)\psi(-N_\chi N_\phi)\mathcal{G}(\psi)\mathcal{G}\left(\overline{\psi}\right)^{-1}.
$$

The theorem follows if we can show that

$$f_\psi|_k w(N_\phi N_\chi M_\psi^2) = C_\psi g_{\overline{\psi}}.$$

On the left hand side we find

$$f_\psi|_k w(N_\phi N_\chi M_\psi^2) = \left(\sqrt{N_\phi N_\chi M_\psi^2}\tau\right)^{-k} (\vartheta_k)_\psi \left(\omega_\chi \otimes \omega_\phi; -\frac{1}{N_\chi M_\psi^2 \tau}\right).$$

Since ψ is primitive we can apply Theorem 1.2.14 and obtain

$$\begin{aligned}
&\left(\sqrt{N_\phi N_\chi M_\psi^2}\tau\right)^{-k} \mathcal{G}\left(\overline{\psi}\right)^{-1} M_\psi^{k-1} \psi(N_\phi) \vartheta_k \left(\left(\omega_{\overline{\psi}} \times \omega_\chi\right) \otimes \left(\omega_\psi \times \omega_\phi[M_\psi]\right); -\frac{1}{N_\chi M_\psi \tau}\right) \\
&= -\left(\frac{N_\chi}{N_\phi}\right)^{\frac{k}{2}} \mathcal{G}\left(\overline{\psi}\right)^{-1} M_\psi^{k-1} \psi(N_\phi) \phi^*(M_\psi) \vartheta_k \left(\left(\omega_\psi \times \omega_\phi\right) \otimes \left(\widehat{\omega_{\overline{\psi}} \times \omega_\chi}\right); N_\chi M_\psi \tau\right) \\
&= -\psi(-1)\chi^*(-1) \left(\frac{N_\chi}{N_\phi}\right)^{\frac{k}{2}} \mathcal{G}\left(\overline{\psi}\right)^{-1} M_\psi^{k-1} \psi(N_\phi) \phi^*(M_\psi) \\
&\quad \times \vartheta_k \left(\left(\omega_\psi \times \omega_\phi\right) \otimes \left(\omega_{\overline{\psi}} \times \omega_\chi\right); N_\chi M_\psi \tau\right).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
g_{\overline{\psi}}(\tau) &= -\chi^*(-1) \left(\frac{N_\chi}{N_\phi}\right)^{\frac{k}{2}} (\vartheta_k)_{\overline{\psi}} (\omega_\phi \otimes \omega_\chi; N_\chi \tau) \\
&= -\chi^*(-1)\mathcal{G}(\psi)^{-1} \left(\frac{N_\chi}{N_\phi}\right)^{\frac{k}{2}} M_\psi^{k-1} \overline{\psi}(N_\chi) \vartheta_k \left(\left(\omega_\psi \times \omega_\chi\right) \otimes \left(\omega_{\overline{\psi}} \times \omega_\chi[M_\psi]\right); N_\chi M_\psi \tau\right) \\
&= -\chi^*(-1)\mathcal{G}(\psi)^{-1} \left(\frac{N_\chi}{N_\phi}\right)^{\frac{k}{2}} M_\psi^{k-1} \overline{\psi}(N_\chi) \chi^*(M_\psi) \\
&\quad \times \vartheta_k \left(\left(\omega_\psi \times \omega_\chi\right) \otimes \left(\omega_{\overline{\psi}} \times \omega_\chi\right); N_\chi M_\psi \tau\right).
\end{aligned}$$

Multiplying this by C_ψ clearly gives us $f_\psi|_k w(N_\phi N_\chi M_\psi^2)$. This proves the theorem. □

With this we are able to prove the main construction theorem.

Theorem 1.2.16 (see [27]). *Let $k \geq 3$ and $N_1, N_2 > 1$ be integers. There is a homomorphism*

$$\begin{aligned}
W_{N_1} \otimes W_{N_2} &\longrightarrow M_k(\Gamma_1(N_1, N_2)) \\
\omega \otimes \eta &\longmapsto \sum_{x \in \mathbb{Q}^\times} x^{k-1} \beta_\eta(x) \omega(x\tau).
\end{aligned}$$

In the case that $k = 1$ and $k = 2$ the map stays well-defined under the restriction that the function $z \mapsto z^{k-1}\eta(z)\omega(z\tau)$ is removable in $z = 0$.

Proof. First note that

$$-2\pi i \operatorname{res}_{z=x}(z^{k-1}\eta(z)\omega(z\tau)) = x^{k-1}\beta_\eta(x)\omega(x\tau).$$

Let N_1 and N_2 be positive integers with $\omega\otimes\eta \in W_{N_1}\otimes W_{N_2}$. This composes into elements $c_{ij}\omega_i\otimes\eta_j$, where both ω_i and η_j are either the principal part or correspond to non-principal characters modulo d_1 and d_2 respectively, where $d_i|N_i$. Here c_{ij} are proper constants. Hence $\vartheta_k(\omega\otimes\eta;\tau)$ decomposes into $c_{ij}\vartheta_k(\omega_i\otimes\eta_j;\tau)$, which belong to $M_k(\Gamma_0(d_1,d_2),\chi_{1,2})$ according to (1.2.2.1) and Theorem 1.2.15 with suitable characters $\chi_{1,2}$. But we have a canonical embedding $M_k(\Gamma_0(d_1,d_2),\chi_{1,2}) \to M_k(\Gamma_0(N_1,N_2),\chi_{1,2}\chi_{0,N_1N_2})$. Together with (1.2.2.2) this proves the theorem. □

Note that we may identify

$$W_{N_1}\otimes W_{N_2} \cong \left(\mathbb{F}_{\frac{1}{N_1}}\right)_0^{\mathbb{C}_0} \otimes \left(\mathbb{F}_{\frac{1}{N_2}}\right)_0^{\mathbb{C}_0}.$$

Hence, modular forms constructed via rational functions belong to "tuples" of periodic functions.

Remark 1.2.17. *All these modular forms vanish in the cusps $\tau \in \{0, i\infty\}$. So if there were non-trivial weak functions with level 1, they would be odd (since there is a simple pole in $z = 0$) and one could generate non-trivial cusp forms for any even weight $k \geq 2$ for $\mathrm{SL}_2(\mathbb{Z})$, which is impossible.*

In the case $N_1 = N_2 = N$ and $k \in 2\mathbb{N}$ we can even say a bit more. Let $\Gamma_S(N)$ be the group generated by $\Gamma_1(N,N)$ and S. Then we can define an abelian character χ_N on $\Gamma_S(N)$ given by

$$\chi_N(M) = \begin{cases} 1, & \text{if } M \equiv \pm E \pmod N, \\ -1, & \text{if } M \equiv \pm S \pmod N. \end{cases}$$

Corollary 1.2.18. *Let $\omega^\pm \otimes \omega^\pm \in W_N^\pm \otimes W_N^\pm$ and $k \geq 2$ an even integer. Then we have $\vartheta_k(\omega^+\otimes\omega^+) \in M_k(\Gamma_S(N),\chi_N)$ and $\vartheta_k(\omega^-\otimes\omega^-) \in M_k(\Gamma_S(N))$.*

Proof. With Theorem 1.2.16 we obtain $\vartheta_k(\omega^\pm\otimes\omega^\pm) \in M_k(\Gamma_1(N,N))$. Using Theorem 1.2.7 we additionally conclude

$$\vartheta_k(\omega^\pm\otimes\omega^\pm;\tau)|_k S = \mp\vartheta_k(\omega^\pm\otimes\omega^\pm;\tau).$$

Since $\Gamma_S(N)$ is generated by $\Gamma_1(N,N)$ and S, this proves the corollary. □

We give an example of quick construction. The theta group Γ_θ is a congruence subgroup generated by the elements $T^2 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and S.

Example 1.2.19. *Let $v_2(n)$ be the exponent of 2 in the prime decomposition of n. For any even $k \geq 4$ we then have that*

$$f(\tau) = \sum_{n=1}^{\infty} (-1)^{n-1} (2^{v_2(n)})^{k-1} \sigma_{k-1}\left(\frac{n}{2^{v_2(n)}}\right) q^{\frac{n}{2}}$$

is an entire modular form of weight k for Γ_θ.

Proof. The space $W_2^- \otimes W_2^-$ has one dimension and is generated by $\omega_2 \otimes \omega_2$, where

$$\omega_2(z) = \frac{e(z)}{e(\frac{1}{2}) - e(z)} - \frac{e(z)}{e(0) - e(z)} = -\frac{i}{\sin(2\pi z)}.$$

Hence due to Corollary 1.2.18 we obtain a modular form $f \in M_k(\Gamma_\theta)$ with

$$f(\tau) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{k-1} \frac{q^{\frac{n}{2}}}{1-q^n}.$$

Rearranging the Lambert sum shows

$$\begin{aligned} f(\tau) &= \sum_{m=1}^{\infty} \sum_{\substack{n,r \\ n(2r+1)=m}} (-1)^{\frac{m}{2r+1}-1} \left(\frac{m}{2r+1}\right)^{k-1} q^{\frac{m}{2}} \\ &= \sum_{m=1}^{\infty} \sum_{\substack{u|m \\ u \text{ odd}}} (-1)^{\frac{m}{u}-1} \left(\frac{m}{u}\right)^{k-1} q^{\frac{m}{2}}. \end{aligned}$$

With

$$\sum_{\substack{u|m \\ u \text{ odd}}} (-1)^{\frac{m}{u}-1} \left(\frac{m}{u}\right)^{k-1} = (-1)^{m-1} (2^{v_2(m)})^{k-1} \sigma_{k-1}\left(\frac{m}{2^{v_2(m)}}\right)$$

the claim follows. □

1.2.3 The space of weak modular forms

In this section we describe the structure of the spaces of modular forms coming from rational functions. Here we mainly use the Fourier series expansions of Eisenstein series introduced in Section 1.1.1, see (1.1.2.4). It is clear that every $\vartheta_k(\omega \otimes \eta; \tau)$ admits a Fourier expansion. Since we only focus on the non-trivial cases we assume $\omega \otimes \eta \in (W_M \otimes W_N)^{\pm}$ if $(-1)^k = \pm 1$.

Proposition 1.2.20. *We have the formula*

$$\vartheta_k(\omega \otimes \eta; \tau) = 2N^{1-k} \sum_{m=1}^{\infty} \sum_{d|m} \left(d^{k-1} (\kappa_N \beta_\eta)(d) \left(\mathcal{F}_M \kappa_M \beta_\omega\right)\left(\frac{m}{d}\right)\right) q^{\frac{m}{N}}. \qquad (1.2.3.1)$$

Proof. A calculation shows

$$\vartheta_k(\omega\otimes\eta;\tau)=-2\pi i\sum_{j\in\mathbb{Z}\setminus\{0\}}\operatorname{res}_{z=\frac{j}{N}}\left(z^{k-1}\eta(z)\omega(\tau z)\right)=\sum_{j\in\mathbb{Z}\setminus\{0\}}\left(\frac{j}{N}\right)^{k-1}\beta_\eta\left(\frac{j}{N}\right)\omega\left(\frac{j\tau}{N}\right).$$

By Remark 1.2.5 we have

$$\left(-\frac{j}{N}\right)^{k-1}\beta_\eta\left(-\frac{j}{N}\right)\omega\left(-\frac{j\tau}{N}\right)=\left(\frac{j}{N}\right)^{k-1}\beta_\eta\left(\frac{j}{N}\right)\omega\left(\frac{j\tau}{N}\right),$$

and so we can write the above sum as

$$2N^{1-k}\sum_{j=1}^{\infty}j^{k-1}\beta_\eta\left(\frac{j}{N}\right)\omega\left(\frac{j\tau}{N}\right)=2N^{1-k}\sum_{j=1}^{\infty}j^{k-1}\beta_\eta\left(\frac{j}{N}\right)\sum_{\alpha=1}^{M}\beta_\omega\left(\frac{\alpha}{M}\right)h_{\frac{\alpha}{M}}\left(\frac{j\tau}{N}\right).$$

We obtain, if $\operatorname{Im}(\tau)>0$, for the inner finite sum

$$\sum_{\alpha=1}^{M}\beta_\omega\left(\frac{\alpha}{M}\right)h_{\frac{\alpha}{M}}\left(\frac{j\tau}{N}\right)=\sum_{\alpha=1}^{M}\beta_\omega\left(\frac{\alpha}{M}\right)\sum_{\nu=1}^{\infty}e\left(-\frac{\nu\alpha}{M}\right)q^{\frac{\nu j}{N}}=\sum_{\nu=1}^{\infty}\mathcal{F}_M(\beta_\omega)(\nu)q^{\frac{\nu j}{N}}.$$

The proposition follows by sorting the terms via $m=\nu j$ and $(\kappa_N\beta_\eta)(j)=\beta_\eta\left(\frac{j}{N}\right)$ as well as $(\kappa_M\beta_\omega)(j)=\beta_\eta\left(\frac{j}{M}\right)$. □

According to (1.1.2.4) we conclude for non-principal characters modulo M and N:

$$E_k(\chi,\psi;\tau)=\frac{\psi(-1)(-2\pi i)^k}{N(k-1)!}\vartheta_k(\omega_{\mathcal{F}_M^{-1}(\chi)}\otimes\omega_{\mathcal{F}_N(\psi)};\tau).\tag{1.2.3.2}$$

In particular, if χ and ψ are primitive and hence conjugate up to a constant under the Fourier transform, this simplifies to

$$E_k(\chi,\psi;\tau)=\frac{\chi(-1)(-2\pi i)^k\mathcal{G}(\psi)}{N(k-1)!\mathcal{G}\left(\overline{\chi}\right)}\vartheta_k(\omega_{\overline{\chi}}\otimes\omega_{\overline{\psi}};\tau).\tag{1.2.3.3}$$

In this section we want to find generators for the space $\vartheta_k(W_{N_1}\otimes W_{N_2})$. We call their elements *weak modular forms*. In other words, the vector space $V_k(\Gamma_1(N_1,N_2))$ of all weak modular forms is the image of the linear map

$$W_{N_1}\otimes W_{N_2}\longrightarrow M_k(\Gamma_1(N_1,N_2)),$$

if we have $k\geq 3$. In the cases $k=1,2$ we have to explain weak modular forms via proper subspaces of $W_{N_1}\otimes W_{N_2}$. By Proposition 1.1.2 (ii) the transforms $\mathcal{F}_N^{\pm1}$ define isomorphisms between $\mathbb{F}_N^{\mathbb{C}_{0,0}}$ and $(\mathbb{F}_N)_0^{\mathbb{C}_0}$. With this and Proposition 1.2.2 we conclude that

$$\left(\omega_{\mathcal{F}_{N_1}^{-1}\iota_{N_1}^{d_1(\chi)}\chi}\otimes\omega_{\mathcal{F}_{N_2}\iota_{N_2}^{d_2(\psi)}\psi}\right)_{(\chi,\psi)\in\overline{\mathfrak{C}_{N_1}}\setminus\{\chi_{0,1}\}\times\overline{\mathfrak{C}_{N_2}}\setminus\{\chi_{0,1}\}}$$

is a basis for $W_{N_1}\otimes W_{N_2}$, where χ and ψ are characters modulo $d_1(\chi)$ and $d_2(\psi)$, respectively. The next theorem provides generators for the space $V_k(\Gamma_1(N_1,N_2))$.

Theorem 1.2.21 (see [24]). *Let $k \geq 3$. The space $V_k(\Gamma_1(N_1, N_2))$ is generated by the elements $E_k(\chi, \psi; \frac{N_1 d_2}{N_2 d_1}\tau)$ where χ and ψ run over all non-trivial characters modulo $d_1|N_1$ and $d_2|N_2$, respectively, such that $\chi(-1)\psi(-1) = (-1)^k$.*

Proof. By Proposition 1.1.2 (iii) the Fourier transform preserves the subspaces of odd and even functions. Hence, for characters satisfying $\chi(-1)\psi(-1) = (-1)^k$, we have by Proposition 1.2.20 the Fourier expansion

$$\begin{aligned}&\vartheta_k(\omega_{\mathcal{F}_{N_1}^{-1}\chi} \otimes \omega_{\mathcal{F}_{N_2}\psi}; \tau)\\ &= 2N_2^{1-k} \sum_{m=1}^{\infty} \sum_{d|m} \left(d^{k-1} (\mathcal{F}_{N_2} \iota_{N_2}^{d_2} \psi)(d) \left(\mathcal{F}_{N_1} \mathcal{F}_{N_1}^{-1} \iota_{N_1}^{d_1} \chi \right) \left(\frac{m}{d} \right) \right) q^{\frac{m}{N_2}},\end{aligned}$$

and by Proposition 1.1.3 this equals to

$$\begin{aligned}&2N_2^{1-k} \sum_{m=1}^{\infty} \sum_{d|m} \left(d^{k-1} (\mathcal{F}_{d_2}\psi)(d) \iota_{N_1}^{d_1} \chi \left(\frac{mN_1}{dd_1} \right) \right) q^{\frac{mN_1}{N_2 d_1}}\\ &= 2N_2^{1-k} \sum_{m=1}^{\infty} \sum_{d|m} \left(d^{k-1} (\mathcal{F}_{d_2}\psi)(d) \chi \left(\frac{m}{d} \right) \right) q^{\frac{mN_1 d_2}{N_2 d_1 d_2}}.\end{aligned}$$

From (1.1.2.4) the theorem follows. □

For our investigations we are especially interested in a subspace of V_k which we will denote by U_k and which contains all weak modular forms which arise from weak functions that have a removable singularity in $z = 0$. In the following we shall give generators for U_k. Let $H_{N_j} \subset W_{N_j}$ $(j = 1, 2)$ be the subspace of weak functions that have a removable singularity in $z = 0$. Then we have

$$W_{N_j} = \mathbb{C}\omega_{\mathcal{F}_{N_j}^{\pm 1}\chi_{0,N_j}} \oplus H_{N_j}.$$

In other words, the space H_N is given by weak elements $\omega(z)$ such that $\beta_\omega(0) = 0$, which is equivalent to the statement that $\omega(z)$ has a removable singularity in $z = 0$. On the periodic function side, we define the subspace of these coefficients by $(\mathbb{F}_{N_j^{-1}})_0^{\mathbb{C}_{0,0}}$. Note that, by Proposition 1.1.2 (ii), the Fourier transform $\mathcal{F}_{N_j}$ defines an automorphism on the subspace $(\mathbb{F}_{N_j})_0^{\mathbb{C}_{0,0}} = (\mathbb{F}_{N_j})_0^{\mathbb{C}_0} \cap (\mathbb{F}_{N_j})^{\mathbb{C}_{0,0}}$. So firstly, consider the basis $(\omega_{\mathcal{F}_{N_1}^{-1}\chi} \otimes \omega_{\mathcal{F}_{N_2}\psi})_{\chi,\psi}$ of $H_{N_1} \otimes H_{N_2}$, where χ and ψ are either non-principal characters modulo $d_1|N_1$ and $d_2|N_2$ or functions $\frac{\varphi(N_j)}{\varphi(d_j)} \iota_{N_j}^{d_j} \chi_{0,d_j} - \chi_{0,N_j}$ for $j = 1, 2$.

Theorem 1.2.22 (see [24]). *Let $k \geq 1$. The space $U_k = \vartheta_k(H_{N_1} \otimes H_{N_2})$ is generated by the elements $E_k(\chi, \psi; \frac{N_1 d_2}{N_2 d_1}\tau)$ and the linear combinations*

$$\frac{\varphi(N_1)}{\varphi(d_1)} E_k(\chi_{0,d_1}, \psi; \tfrac{N_1 d_2}{N_2 d_1}\tau) - E_k(\chi_{0,N_1}, \psi; \tfrac{d_2}{N_2}\tau),$$

$$\frac{\varphi(N_2)}{\varphi(d_2)}E_k(\chi,\chi_{0,d_2};\tfrac{N_1d_2}{N_2d_1}\tau)-\left(\frac{N_2}{d_2}\right)^k E_k(\chi,\chi_{0,N_2};\tfrac{N_1}{d_1}\tau),$$

and

$$\begin{aligned}&\frac{\varphi(N_1)}{\varphi(d_1)}\frac{\varphi(N_2)}{\varphi(d_2)}E_k(\chi_{0,d_1},\chi_{0,d_2};\tfrac{N_1d_2}{N_2d_1}\tau)-\frac{\varphi(N_1)}{\varphi(d_1)}\left(\frac{N_2}{d_2}\right)^k E_k(\chi_{0,d_1},\chi_{0,N_2};\tfrac{N_1}{d_1}\tau)\\&-\frac{\varphi(N_2)}{\varphi(d_2)}E_k(\chi_{0,N_1},\chi_{0,d_2};\tfrac{d_2}{N_2}\tau)+\left(\frac{N_2}{d_2}\right)^k E_k(\chi_{0,N_1},\chi_{0,N_2};\tau),\end{aligned}$$

where $1<d_j<N_j$ *and* χ,ψ *are non-principal characters modulo* d_1 *and* d_2*, respectively, such that* $\operatorname{sgn}(\chi\psi)=(-1)^k$.

Proof. Since all considered weak functions have a removable singularity in $z=0$, we can apply the theorem to all positive weights $k\in\mathbb{N}$. The proof works similar as the one of Theorem 1.2.21 and we omit it. □

Theorem 1.2.23 (see [24])**.** *We have the following.*

(i) The space of weak modular forms of weight $k=1$ *is given by* $V_1(\Gamma_1(N_1,N_2))=\vartheta_1(H_{N_1}\otimes H_{N_2})$*. In particular, it is generated by the elements given in Theorem 1.2.22 for* $k=1$*.*

(ii) The space of weak modular forms of weight $k=2$ *is given by* $V_2(\Gamma_2(N_1,N_2))=\vartheta_2(H_{N_1}\otimes H_{N_2}\oplus\mathbb{C}\omega_{\mathcal{F}_{N_1}^{-1}\chi_{0,N_1}}\otimes H_{N_2}\oplus H_{N_1}\otimes\mathbb{C}\omega_{\mathcal{F}_{N_2}\chi_{0,N_2}})$*. In particular, it is generated by the elements in Theorem 1.2.22 for* $k=2$ *and* $E_2(\chi_{0,N_1},\psi;\frac{d_2}{N_2}\tau),E_2(\chi,\chi_{0,N_2};\frac{N_1}{d_1}\tau)$*, where* χ *and* ψ *are non-principal characters modulo* $d_1|N_1$ *and* $d_2|N_2$*, respectively.*

Proof. Since for $k=1$ both ω and η must have a removable singularity in $z=0$, the claim follows easily in this case. In the case $k=2$ we are allowed that at most one function has a pole of degree 1 in $z=0$. The calculations are the same. □

In the last section we would like to investigate L-functions of products of weak functions. To formalize this, we give the following final definition.

Definition 1.2.24. *Let* $\boldsymbol{k}=(k_1,...,k_\ell)$ *be a vector of weights. We then define* $V_{\boldsymbol{k}}(\Gamma_1(N_1,N_2))$ *as the vector space of all modular forms that can be written as a sum* $\sum_j c_j f_{1,j}\cdots f_{\ell,j}$*, where each* $f_{r,j}$ *is an element of* $V_{k_r}(\Gamma_1(N_1,N_2))$*. Analogously, we define the subspace* $U_{\boldsymbol{k}}(\Gamma_1(N_1,N_2))\subset V_{\boldsymbol{k}}(N_1,N_2)$ *by demanding* $f_{r,j}\in U_{k_r}(\Gamma_1(N_1,N_2))$*. We will call the modular forms in* $U_{\boldsymbol{k}}(\Gamma_1(N_1,N_2))$ *higher weak modular forms.*

1.3 Series representations for L-functions

1.3.1 A Dominated convergence theorem

In this section we provide a Dominated convergence theorem, which will be applied to L-series associated to products of Eisenstein series in the following section. The idea is to investigate finite sums of the form

$$\sum_{n=1}^{T} n^{\alpha}\beta(n)\omega(n\tau) \tag{1.3.1.1}$$

on the upper half plane in detail, where $\alpha \geq 0$ is an integer, β is some N-periodic function ($N \in \mathbb{N}_{>1}$) and $\omega(z)$ is some weak function of level M with a removable singularity in $z = 0$. By Theorems 1.2.21, 1.2.22 and 1.2.23, expression (1.3.1.1) will converge to a linear combination of Eisenstein series as T tends to infinity, if $\beta = \beta_\eta$ comes from a weak function. The purpose of the Dominated convergence theorem is now to give a condition providing a non-trivial upper bound for the sum (1.3.1.1). In general, there will be no non-trivial "small" upper bound of (1.3.1.1) in terms of T, τ and α. However, when replacing T by NT and τ by iy, where $1 \geq y > 0$, it is possible, but quite technical, to give a "small" uniform upper-bound in the sense that it is independent of the choice of T. This upper bound is of the form Cy^w with some integer w. This is summarized in Theorem 1.3.14.

Before going into the proofs, we sketch the idea why dominated convergence of Eisenstein series is useful. When considering L-functions of modular forms (vanishing in the cusps $\tau \in \{0, i\infty\}$), we first look at the Mellin transform

$$\int_0^{\infty} f(iy)y^{s-1}\mathrm{d}y = \int_0^{\infty} \sum_{n=1}^{\infty} a(n)e^{-2\pi\frac{n}{N}y}y^{s-1}\mathrm{d}y.$$

While convergence of integral and sum is no problem on the interval $[1, \infty]$, the situation looks different for $(0, 1]$. A priori, we will only be allowed to switch integral and sum in the obvious region of absolute convergence. In this "trivial region" it is well-known that we end up with the ordinary Dirichlet series for the L-function. But if we can rearrange the Fourier series to a series of Lambert type and give "small" upper bounds for the partial sums (1.3.1.1), we may use Lebesgue's dominated convergence theorem to switch integral and sum also in non-trivial regions. As a result, we obtain a generalized form of Dirichlet series that also converges in a wider region to $L(f; s)$. All of this will be explained in great detail in Section 1.3.2.

We will start this section with a classical result.

Theorem 1.3.1 (Faulhaber's formula)**.** *We have for all $\alpha \in \mathbb{N}_0$ and $T \in \mathbb{N}$:*

$$\sum_{j=1}^{T} j^{\alpha} = \sum_{k=0}^{\alpha} \frac{B_k}{k!} \frac{\alpha!}{(\alpha-k+1)!} T^{\alpha-k+1}.$$

Here, the B_k denote the Bernoulli numbers.

It is a trivial but very important observation for us that the left sum defines a unique polynomial in T by interpolation, which is given on the right hand side. We will not prove Theorem 1.3.1. It can be verified, for example, by using Euler-MacLaurin summation. For more details on this topic, the reader is advised to consult [16] on p. 21–31.

Definition 1.3.2. *Let N be a positive integer and $\beta : \mathbb{Z} \to \mathbb{C}$ a function. We say that β has height d (with respect to N), if for all $\alpha \in \mathbb{N}_0$ and $T \in \mathbb{N}$:*

$$\sum_{j=1}^{NT} \beta(j) j^{\alpha} = \sum_{u=0}^{\alpha-d} \gamma_{\alpha,\beta}(u) T^{u} = O(T^{\alpha-d}), \qquad T \to \infty.$$

Here, the complex numbers $\gamma_{\alpha,\beta}(u)$ only depend on α, β and u. The height of the zero function is always defined to be ∞. We denote $[N, d]$ as the vector space of functions with height (with respect to N) at least d.

Like in Theorem 1.3.1, the key property of functions in Definition 1.3.2 is that the left side defines a polynomial. We easily see that the constant sequence $\beta(j) = 1$ and more generally, $\beta(j) = j^d$ will have heights -1 and $-d-1$, respectively, where $d \geq 0$ is some integer. But while here the negative height causes an increase in the growth of the considered sums, we are rather interested in the opposite phenomenon of a non-negative height. In this case we obtain a decrease in the growth. Periodic functions with this feature play the key role when looking for "small" upper bounds of partial sums (1.3.1.1). Of course, not all functions β do have a height.

Remark 1.3.3. *If $d_1 \leq d_2$ we have the natural embedding*

$$[N, d_2] \longrightarrow [N, d_1].$$

We are only interested in periodic functions. The next proposition guarantees that they have a height.

Proposition 1.3.4. *We have $\mathbb{F}_N^{\mathbb{C}_0} \subset [N, -1]$.*

Proof. Since β is periodic, we can rewrite the sum over $\beta(j)j^{\alpha}$ as

$$\sum_{j=1}^{NT} \beta(j) j^{\alpha} = \sum_{c=1}^{N} \beta(c) \sum_{j=0}^{T-1} (Nj + c)^{\alpha}.$$

It is clear by Theorem 1.3.1 that for any c the expressions

$$\beta(c)\sum_{j=0}^{T-1}(Nj+c)^{\alpha}$$

are polynomials in T with degree up to $\alpha+1$. This proves $\mathbb{F}_N^{\mathbb{C}_0}\subset[N,-1]$. □

Proposition 1.3.5. *Let $d\geq 0$ be an integer and $\beta:\mathbb{Z}\to\mathbb{C}$ be a N-periodic function, such that*

$$\sum_{j=1}^{N}\beta(j)j^{u}=0$$

for all $0\leq u\leq d$. Then $\beta\in[N,d]$.

Proof. Since β is N-periodic we know by Proposition 1.3.4 that the expressions

$$\sum_{j=1}^{NT}\beta(j)j^{\alpha}$$

define polynomials for all integers values $0\leq\alpha$. We need to show, that these have degree at most $\alpha-d$. We obtain

$$\begin{aligned}\sum_{j=1}^{NT}\beta(j)j^{\alpha}&=\sum_{\ell=0}^{T-1}\sum_{q=1}^{N}\beta(N\ell+q)(N\ell+q)^{\alpha}=\sum_{\ell=0}^{T-1}\beta(q)(N\ell+q)^{\alpha}\\&=\sum_{\ell=0}^{T-1}\sum_{q=1}^{N}\beta(q)\sum_{v=0}^{\alpha}\binom{\alpha}{v}(N\ell)^{\alpha-v}q^{v}=\sum_{\ell=0}^{T-1}\sum_{v=0}^{\alpha}\binom{\alpha}{v}(N\ell)^{\alpha-v}\sum_{q=1}^{N}\beta(q)q^{v}\\&=\sum_{\ell=0}^{T-1}\sum_{v=d+1}^{\alpha}\binom{\alpha}{v}(N\ell)^{\alpha-v}\sum_{q=1}^{N}\beta(q)q^{v}=\sum_{v=d+1}^{\alpha}\binom{\alpha}{v}N^{\alpha-v}\sum_{q=1}^{N}\beta(q)q^{v}\sum_{\ell=0}^{T-1}\ell^{\alpha-v}.\end{aligned}$$

Since the sum over v starts at $d+1$, by Theorem 1.3.1 this defines a polynomial of degree at most $\alpha-d$. Hence, $\beta\in[N,d]$. □

Example 1.3.6. *Each non-principal Dirichlet character mod N has height at least 0 with respect to N, since*

$$\sum_{j=1}^{N}\chi(j)=0,$$

and each (non-principal) even character has height at least 1, since then we additionally have

$$\sum_{j=1}^{N}\chi(j)j=0.$$

Proposition 1.3.7. *Let $\beta : \mathbb{Z} \to \mathbb{C}$ be in $[N, d]$ for $d \geq 0$. Then, for all $u \geq 0$, there are coefficients $\gamma_{\beta,u}$ such that*

$$(1-x)^{u-d} \sum_{p=1}^{N} \left(\sum_{r=1}^{p} \beta(r) r^u \right) x^p = \sum_{j=0}^{N+u-d} \gamma_{\beta,u}(j) x^j.$$

Proof. For $d \leq u$ the proposition is clear, so we assume $d \geq 1$ and $0 \leq u < d$. Let $0 \leq \ell \leq d-u-1$ be an integer. Let

$$P(x) := \sum_{p=1}^{N} \left(\sum_{r=1}^{p} \beta(r) r^u \right) x^p.$$

Then we obtain for the value $P^{(\ell)}(1)$:

$$\sum_{p=1}^{N} \left(\sum_{r=1}^{p} \beta(r) r^u p(p-1)\cdots(p-\ell+1) \right) = \sum_{r=1}^{N} \beta(r) r^u \sum_{p=r}^{N} \left[p^\ell + b_{\ell-1} p^{\ell-1} + \cdots + b_1 p \right]$$
$$= \sum_{r=1}^{N} \beta(r) r^u \left(Q_\ell(N) - Q_\ell(r-1) \right) = 0,$$

since Q_ℓ is some polynomial of degree $\ell + 1 \leq d - u$. This proves $P(x) = (1-x)^{d-u} Q(x)$ with some polynomial Q. □

Our investigations foot on the properties of some explicit polynomials. They are similar, but simpler as the sums in (1.3.1.1). For a fixed non-negative integer α we define a sequence by

$$p_T(\alpha; x) = (1-x)^{\alpha+1} \sum_{\ell=1}^{T} \ell^\alpha x^\ell, \qquad T = 1, 2, 3, \ldots.$$

For example we have $p_T(0; x) = x - x^{T+1}$ for $T = 1, 2, \ldots$.

Lemma 1.3.8. *The sequence $(p_T(\alpha; x))_{T \in \mathbb{N}}$ converges to some polynomial function on the interval $[0, 1)$ from below for all $\alpha \geq 0$. In particular, the terms p_T are uniformly bounded in the sense*

$$\sup_{T \in \mathbb{N}} \sup_{x \in [0,1]} |p_T(\alpha; x)| \leq C_\alpha$$

for some constant $C_\alpha > 0$.

This uniform boundedness is a very important property as we will see later.

Proof. It is clear that $p_T(\alpha; x)$ is increasing for fixed x. The power series

$$\sum_{\ell=1}^{\infty} \ell^\alpha x^\ell$$

converges for $x \in [0,1)$ to a rational function $\frac{Q_\alpha(x)}{(1-x)^{\alpha+1}}$, where $Q_\alpha(x)$ is some polynomial which is non-negative in $[0,1]$. This follows inductively by $\sum_{\ell=1}^{\infty} x^\ell = \frac{x}{1-x}$ and the fact that

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Q_{\alpha-1}(x)}{(1-x)^\alpha}\right) = \frac{Q_\alpha(x)}{(1-x)^{\alpha+1}}$$

with polynomials $Q_{\alpha-1}$ and Q_α. Put $C_\alpha = \sup_{x\in[0,1]} Q_\alpha(x)$. □

Remark 1.3.9. *In fact, one can give an explicit formula for the Q_α in terms of Eulerian numbers, but we will not need such a precise description for our applications.*

Lemma 1.3.10. *For each $T \geq 1$ there is some number $0 < \xi_{\alpha,T} < 1$ such that $p_T(\alpha; x)$ is increasing in the interval $[0, \xi_{\alpha,T}]$ and decreasing in the interval $[\xi_{\alpha,T}, 1]$.*

Proof. Since we have $p_T(\alpha; x) \geq 0$ for $0 \leq x \leq 1$ (with equality if $x = 0$ or $x = 1$), it is sufficient to show that $p_T'(\alpha; x) = 0$ has exactly one solution $0 < \xi_{\alpha,T} < 1$. For values $0 < x < 1$ we obtain

$$p_T'(\alpha; x) = -(\alpha+1)(1-x)^\alpha \sum_{\ell=1}^{T} \ell^\alpha x^\alpha + (1-x)^{\alpha+1} \sum_{\ell=1}^{T} \ell^{\alpha+1} x^{\ell-1} = 0$$

which is equivalent to

$$\sum_{\ell=1}^{T} \left(-(\alpha+1)x^\ell + \ell^{\alpha+1}x^{\ell-1} - \ell^{\alpha+1}x^\ell\right) = 0,$$

and after further manipulations

$$\frac{1}{x^T} + \sum_{\ell=1}^{T-1} \left(\sum_{j=2}^{\alpha+1} \binom{\alpha+1}{j} \ell^{\alpha+1-j} \right) x^{\ell-T} = (\alpha+1)T^\alpha + T^{\alpha+1}.$$

The right hand side is greater than the left hand side for $x = 1$, since

$$1 + \sum_{\ell=1}^{T-1} \left(\sum_{j=2}^{\alpha+1} \binom{\alpha+1}{j} \ell^{\alpha+1-j} \right) = 1 + \sum_{\ell=1}^{T-1} \left((1+\ell)^{\alpha+1} - (\alpha+1)\ell^\alpha - \ell^{\alpha+1} \right)$$

$$= 1 + (T^{\alpha+1} - 1) - (\alpha+1) \sum_{\ell=1}^{T-1} \ell^\alpha \leq T^{\alpha+1} < T^{\alpha+1} + (\alpha+1)T^\alpha.$$

On the other hand, the left hand side is unbounded and monotonically decreasing in the interval $(0,1]$. Hence, there is exactly one solution for the above equation in this area and the claim follows. □

Before we can go on to the next lemma of this section we recall:

Lemma 1.3.11. *Let a_k be a sequence of complex numbers and b_k and c_k sequences of positive real numbers such that $0 \leq b_{k+1} \leq b_k$ and $c_{k+1} \geq c_k \geq 0$ for all k. Then we have for all $n \geq 1$:*

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq b_1 \max_{r=1,\dots,n} \left|\sum_{k=1}^{r} a_k\right|$$

and

$$\left|\sum_{k=1}^{n} a_k c_k\right| \leq (2c_n - c_1) \max_{r=1,\dots,n} \left|\sum_{k=1}^{r} a_k\right|.$$

Proof. The first statement is called Abel's inequality, so we will only prove the second one. We set $A_n = \sum_{k=1}^{n} a_k$ and obtain by partial summation

$$\begin{aligned}\left|\sum_{k=1}^{n} a_k c_k\right| &= \left|A_n c_n + \sum_{k=1}^{n-1} A_k(c_k - c_{k+1})\right| \leq |A_n| c_n + \sum_{k=1}^{n-1} |A_k||c_k - c_{k+1}| \\ &\leq \max_{r=1,\dots,n} |A_r| \left(c_n + \sum_{k=1}^{n-1}(c_{k+1} - c_k)\right) = (2c_n - c_1) \max_{r=1,\dots,n} |A_r|.\end{aligned}$$

Hence the lemma is proved. □

Our strategy will be to expand $\omega(z)$ in (1.3.1.1) into a Fourier series. With this we will obtain a double series, which is on the one hand more complicated. On the other hand, this simplifies the occurring summands drastically. Partial summation and Abel's inequalities are then the key tools when estimating sums of this type, as the next boundedness lemma shows.

Lemma 1.3.12. *Let $M, L, T > 1$ and $w \geq 0$ be integers, $\zeta_M^j \neq 1$ be a root of unity, $0 \leq X, Y \leq 1$ be real numbers and c_k be a monotonically increasing (or decreasing) sequence (that may depend on X and Y), which is bounded by $0 \leq c_k \leq B$ and B does not depend on X, Y, L and j. Then we have uniformly for L, X, Y, j,*

$$\left|\sum_{k=1}^{L} (\zeta_M^j X)^k c_k p_T(w; Y^k)\right| \leq 6BC_w M,$$

where C_w is the constant defined in Lemma 1.3.8.

Proof. Without loss of generality, we assume c_k to be an increasing sequence. In the case that c_k is decreasing the proof works similar. By Lemma 1.3.11 we first obtain

$$\left|\sum_{k=1}^{L} (\zeta_M^j X)^k c_k p_T(w; Y^k)\right| \leq 2B \max_{1 \leq I \leq L} \left|\sum_{k=1}^{I} (\zeta_M^j X)^k p_T(w; Y^k)\right|. \tag{1.3.1.2}$$

In the case c_k is decreasing we could switch $2B$ by B, but since $B \leq 2B$ the estimate works in both cases. To estimate the inner sum for any value I with $1 \leq I \leq L$, we will use the fact, that the p_T are monotonically increasing first in some interval $[0, \xi_{w,T}]$ and then monotonically decreasing in $[\xi_{w,T}, 1]$, as it was shown in Lemma 1.3.10. For any I choose the unique $1 \leq I(w,T,Y) \leq I$ such that $Y^k > \xi_{w,T}$ for all $1 \leq k \leq I(w,T,Y)$ and $Y^k \leq \xi_{w,T}$ for $I(w,T,Y) < k \leq I$. Note that in the case $Y = 1$ the second condition is empty. Then, using the triangle inequality, we see

$$\left|\sum_{k=1}^{I}(\zeta_M^j X)^k p_T(w;Y^k)\right| \leq \left|\sum_{k=1}^{I(w,T,Y)}(\zeta_M^j X)^k p_T(w;Y^k)\right| + \left|\sum_{k=I(w,T,Y)+1}^{I}(\zeta_M^j X)^k p_T(w;Y^k)\right|.$$

We apply Lemma 1.3.11 on the first sum to obtain

$$\left|\sum_{k=1}^{I(w,T,Y)}(\zeta_M^j X)^k p_T(w;Y^k)\right| \leq 2C_w \max_{1\leq J\leq I(w,T,Y)}\left|\sum_{k=1}^{J}(\zeta_M^j X)^k\right|,$$

where C_w is the constant given in Lemma 1.3.8. The inner sum can be estimated again with Lemma 1.3.11, since $0 \leq X^{k+1} \leq X^k \leq 1$ by

$$\left|\sum_{k=1}^{J}(\zeta_M^j X)^k\right| \leq \max_{1\leq H\leq J}\left|\sum_{k=1}^{H}\zeta_M^{jk}\right| \leq M,$$

hence

$$\left|\sum_{k=1}^{I(w,T,Y)}(\zeta_M^j X)^k p_T(w;Y^k)\right| \leq 2C_w \max_{1\leq J\leq I(w,T,Y)} M = 2C_w M.$$

Similarly, we obtain with Lemma 1.3.11

$$\left|\sum_{k=I(w,T,Y)+1}^{I}(\zeta_M^j X)^k p_T(w;Y^k)\right| \leq C_w \max_{I(w,T,Y)+1\leq J\leq I}\left|\sum_{k=I(w,T,Y)+1}^{J}(\zeta_M^j X)^k\right| \leq C_w M.$$

Finally, with (1.3.1.2) we obtain

$$\left|\sum_{k=1}^{L}(\zeta_M^j X)^k c_k p_T(w;Y^k)\right| \leq 2B \max_{1\leq I\leq L} 3C_w M = 6BC_w M.$$

This proves the lemma. □

The next lemma can be seen as an analogous result to the previous lemma.

Lemma 1.3.13. *Let $M, N, L, T > 1$ be integers, $0 \leq y \leq 1$ any real number, $\zeta_M^j \neq 1$ a root of unity and $p(X) = \sum_{u=0}^{d} \gamma(u) X^u$ a polynomial of degree at most d, with coefficients independent of L, T and y. Then there is a constant $D_{j,M,N,p} > 0$ only depending on j, M, N and p such that uniformly in L, T and y:*

$$\left| y^d p(T) \sum_{k=1}^{L} \zeta_M^{jk} e^{-NTky} \right| \leq D_{j,M,N,p}.$$

Proof. The constant

$$U_{j,M} := \sup_{0 \leq x \leq \infty} \frac{1}{|1 - e^{-Nx} \zeta_M^j|}$$

exists and only depends on j and M. Put $x := yT$. We obtain with the geometric summation formula

$$y^d p(T) \sum_{k=1}^{L} \zeta_M^{jk} e^{-NTky} = \sum_{u=0}^{d} \gamma(u) x^u y^{d-u} \frac{e^{-Nx} \zeta_M^j - e^{-Nx(L+1)} \zeta_M^{j(L+1)}}{1 - e^{-Nx} \zeta_M^j}$$

and hence

$$\left| y^d p(T) \sum_{k=1}^{L} \zeta_M^{jk} e^{-NTky} \right| \leq U_{j,M} \cdot 2e^{-Nx} \sum_{u=0}^{d} |\gamma(u)| x^u.$$

The right hand side is obviously bounded for $0 \leq x \leq \infty$ and only depends on j, M, N and p, so we have found a possible $D_{j,M,N,p}$. □

We now have all the tools to prove the main theorem of this section.

Theorem 1.3.14 (Dominated convergence theorem, see [24])**.** *Let β be a N-periodic function in $[N, d]$, $d \geq 0$, and $\omega \in W_M$ be a weak function that has a removable singularity in $z = 0$. Then for all $\alpha \in \mathbb{N}_0$ there is a constant $C_{\beta,\omega,\alpha} > 0$ such that uniformly for all $T \in \mathbb{N}$ and $y \in [0, 1]$*

$$\left| \sum_{n=1}^{NT} n^\alpha \beta(n) \omega(niy) \right| \leq C_{\beta,\omega,\alpha} y^{d-\alpha}.$$

Remark 1.3.15. *Note that, by Theorem 1.3.14, in the case $\alpha \leq d$ the left hand side is bounded uniformly for values T and $y \in [0, 1]$. Since the series converges absolutely and uniformly on $[1, \infty]$, we obtain dominated convergence on $[0, \infty]$.*

Proof. For $y = 0$ the inequality holds since in the case $\alpha \leq d$ the left hand side is always zero (note that $\omega(0)$ exists) and otherwise the right hand side is $+\infty$ from the right. Let $y > 0$. We then have

$$\sum_{n=1}^{NT} n^\alpha \beta(n) \omega(niy) = \lim_{L \to \infty} \sum_{j \in \mathbb{F}_N} \beta_\omega(j) \sum_{k=1}^{L} \sum_{n=1}^{NT} n^\alpha \beta(n) \zeta_M^{kj} e^{-2\pi kny}. \tag{1.3.1.3}$$

In the first step we will only deal with the inner sums. We obtain with partial summation

$$\sum_{n=1}^{NT} n^\alpha \beta(n) e^{-2\pi kny} = e^{-2\pi kNTy} \sum_{n=1}^{NT} n^\alpha \beta(n) + \sum_{n=1}^{NT-1} \left(\sum_{r=1}^{n} \beta(r) r^\alpha \right) (e^{-2\pi kny} - e^{-2\pi k(n+1)y}).$$

Since β has height d, there is a polynomial $p_{\alpha,\beta}$ with degree at most $\alpha - d$ such that

$$e^{-2\pi kNTy} \sum_{n=1}^{NT} n^\alpha \beta(n) = e^{-2\pi kNTy} p_{\alpha,\beta}(T).$$

By Lemma 1.3.13 there is a constant $D_{\alpha,\beta,\omega} > 0$ only depending on α, β and ω (note that N belongs to β and M to ω, and that $\beta_\omega(0) = 0$ which implies $\zeta_M^j \neq 1$), such that

$$\left| \sum_{j \in \mathbb{F}_N} \beta_\omega(j) \sum_{k=1}^{L} \zeta_M^{jk} e^{-2\pi kNTy} \sum_{n=1}^{NT} n^\alpha \beta(n) \right| \leq y^{d-\alpha} \sum_{j \in \mathbb{F}_N} |\beta_\omega(j)| D_{j,M,N,p_{\alpha,\beta}}. \tag{1.3.1.4}$$

where we put

$$D_{\alpha,\beta,\omega} := \sum_{j \in \mathbb{F}_N} |\beta_\omega(j)| D_{j,M,N,p_{\alpha,\beta}}.$$

On the other hand, we have

$$\sum_{n=1}^{NT-1} \left(\sum_{r=1}^{n} \beta(r) r^\alpha \right) (e^{-2\pi kny} - e^{-2\pi k(n+1)y}) = \left(1 - e^{-2\pi ky}\right) \sum_{n=1}^{NT-1} \left(\sum_{r=1}^{n} \beta(r) r^\alpha \right) e^{-2\pi kny}$$
$$= \left(1 - e^{-2\pi ky}\right) \sum_{q=1}^{N} \sum_{\ell=0}^{T-1} \sum_{r=1}^{N\ell+q} \beta(r) r^\alpha e^{-2\pi k(N\ell+q)y} - \left(1 - e^{-2\pi ky}\right) \sum_{r=1}^{NT} \beta(r) r^\alpha e^{-2\pi kNTy}.$$

For the right sum we obtain with Lemma 1.3.11 and (1.3.1.4) (note that $1 - e^{-2\pi ky}$ is monotonous):

$$\sum_{j \in \mathbb{F}_N} |\beta_\omega(j)| \left| \sum_{k=1}^{L} \zeta_M^{jk} \left(1 - e^{-2\pi ky}\right) e^{-2\pi kNTy} \sum_{r=1}^{NT} \beta(r) r^\alpha \right| \leq 2y^{d-\alpha} D_{\alpha,\beta,\omega}. \tag{1.3.1.5}$$

So we are left to give an estimate for the left sum. Here we obtain

$$\left(1 - e^{-2\pi ky}\right) \sum_{q=1}^{N} \sum_{\ell=0}^{T-1} \sum_{r=1}^{N\ell+q} \beta(r) r^\alpha e^{-2\pi k(N\ell+q)y} \tag{1.3.1.6}$$
$$= \left(1 - e^{-2\pi ky}\right) \left(\sum_{q=1}^{N} \sum_{\ell=0}^{T-1} \sum_{r=1}^{N\ell} \beta(r) r^\alpha e^{-2\pi kN\ell y} e^{-2\pi kqy} + \sum_{q=1}^{N} \sum_{\ell=0}^{T-1} \sum_{r=N\ell+1}^{N\ell+q} \beta(r) r^\alpha e^{-2\pi kN\ell y} e^{-2\pi kqy} \right).$$

The final estimate will be given by the sum of two separate estimates of both of these sums. Without loss of generality we assume $\alpha > d$, since otherwise the left sum vanishes, which now equals to

$$\left(1-e^{-2\pi ky}\right)\sum_{q=1}^{N}\sum_{\ell=0}^{T-1}\sum_{r=1}^{N\ell}\beta(r)r^{\alpha}e^{-2\pi kN\ell y}e^{-2\pi kqy}=\left(e^{-2\pi ky}-e^{-2\pi k(N+1)y}\right)\sum_{\ell=0}^{T-1}p_{\alpha,\beta}(\ell)e^{-2\pi kN\ell y}$$
$$=\left(e^{-2\pi ky}-e^{-2\pi k(N+1)y}\right)\sum_{u=0}^{\alpha-d}\gamma_{\alpha,\beta}(u)\sum_{\ell=0}^{T-1}\ell^{u}e^{-2\pi kN\ell y}$$
$$=\left(e^{-2\pi ky}-e^{-2\pi k(N+1)y}\right)\sum_{u=0}^{\alpha-d}\gamma_{\alpha,\beta}(u)\frac{p_{T-1}(u;e^{-2\pi kNy})}{\left(1-e^{-2\pi kNy}\right)^{u+1}}.$$

After multiplying and dividing by $\left(1-e^{-2\pi kNy}\right)^{\alpha-d+1}$, this equals

$$\frac{\left(e^{-2\pi ky}-e^{-2\pi k(N+1)y}\right)}{\left(1-e^{-2\pi kNy}\right)^{\alpha-d+1}}\sum_{u=0}^{\alpha-d}\gamma_{\alpha,\beta}(u)\left(1-e^{-2\pi kNy}\right)^{\alpha-d-u}p_{T-1}(u;e^{-2\pi kNy}).$$

Put $Y := e^{-2\pi Ny}$. There is a constant $A > 0$ not depending on y and k such that $\left|y(1-Y^k)^{-1}\right| \le A$ for $0 < y \le 1$. Note that we have

$$\frac{\left(e^{-2\pi ky}-e^{-2\pi k(N+1)y}\right)}{\left(1-e^{-2\pi kNy}\right)^{\alpha-d+1}}=y^{d-\alpha}\frac{y^{\alpha-d}}{(1-Y^k)^{\alpha-d}}e^{-2\pi ky}.$$

For $k > 0$ the sequence

$$c_k := e^{-2\pi ky}\left(\frac{y}{1-Y^k}\right)^{\alpha-d}$$

is decreasing and bounded between 0 and $A^{\alpha-d}$. Also put

$$\sum_{u=0}^{\alpha-d}\gamma_{\alpha,\beta}(u)\left(1-Y^k\right)^{\alpha-d-u}=\sum_{u=0}^{\alpha-d}\widetilde{\gamma}_{\alpha,d}(u)Y^{ku}.$$

This gives us

$$y^{d-\alpha}\left|\sum_{k=1}^{L}\zeta_M^{kj}c_k\sum_{u=0}^{\alpha-d}\gamma_{\alpha,\beta}(u)\left(1-Y^k\right)^{\alpha-d-u}p_{T-1}(u;Y^k)\right| \tag{1.3.1.7}$$
$$\le y^{d-\alpha}\sum_{u=0}^{\alpha-d}|\widetilde{\gamma}_{\alpha,\beta}(u)|\left|\sum_{k=1}^{L}\left(\zeta_M^{j}Y^{u}\right)^{k}c_kp_{T-1}(u;Y^k)\right| \le 6y^{d-\alpha}MA^{\alpha-d}\sum_{u=0}^{\alpha-d}|\widetilde{\gamma}_{\alpha,\beta}(u)|C_u$$

when putting $X := Y^u$ and using Lemma 1.3.12.

On the other hand, when putting $Z := e^{-2\pi y}$, we obtain for the right sum in (1.3.1.6)

$$\begin{aligned}
&\left(1-Z^k\right)\sum_{q=1}^{N}\sum_{\ell=0}^{T-1}\sum_{r=N\ell+1}^{N\ell+q}\beta(r)r^{\alpha}Z^{\ell Nk}Z^{kq}\\
=\ &\left(1-Z^k\right)\sum_{\ell=0}^{T-1}\sum_{q=1}^{N}\sum_{r=1}^{q}\beta(N\ell+r)(N\ell+r)^{\alpha}Z^{\ell Nk}Z^{kq}
\end{aligned}$$

and since β is N-periodic this equals

$$\begin{aligned}
&\left(1-Z^k\right)\sum_{\ell=0}^{T-1}\sum_{q=1}^{N}\sum_{r=1}^{q}\beta(r)\sum_{u=0}^{\alpha}\binom{\alpha}{u}(N\ell)^u r^{\alpha-u}Z^{\ell Nk}e^{-2\pi kqy}\\
=\ &\left(1-Z^k\right)\sum_{u=0}^{\alpha}\binom{\alpha}{u}N^u\left(\sum_{q=1}^{N}\sum_{r=1}^{q}\beta(r)r^{\alpha-u}Z^{kq}\right)\frac{p_{T-1}\left(u;Z^{Nk}\right)}{\left(1-Z^{Nk}\right)^{u+1}}\\
=\ &\left(1-Z^k\right)\sum_{u=0}^{\alpha}\binom{\alpha}{u}N^u\left(\sum_{q=1}^{N}\sum_{r=1}^{q}\beta(r)r^{\alpha-u}Z^{kq}\right)\frac{\left(1-Z^k\right)^{u+1}p_{T-1}\left(u;Z^{Nk}\right)}{\left(1-Z^k\right)^{u+1}\left(1-Z^{Nk}\right)^{u+1}}\\
=\ &\sum_{u=0}^{\alpha}\binom{\alpha}{u}\frac{N^u}{(1-Z^k)^u}\left(\sum_{q=1}^{N}\sum_{r=1}^{q}\beta(r)r^{\alpha-u}Z^{kq}\right)c_k(u)p_{T-1}\left(u;Z^{Nk}\right),
\end{aligned}$$

with

$$c_k(u) := \left(\frac{1-Z^k}{1-Z^{Nk}}\right)^{u+1} = \frac{1}{\left(1+Z^k+Z^{2k}+\cdots+Z^{(N-1)k}\right)^{u+1}}.$$

Note that we always have $0 \le c_k(u) \le c_{k+1}(u) \le 1$. Since β has height d, by Lemma 1.3.7, there are coefficients $\delta_{\alpha,\beta,u}(w)$ such that

$$\left(1-Z^k\right)^{\alpha-d-u}\left(\sum_{q=1}^{N}\sum_{r=1}^{q}\beta(r)r^{\alpha-u}Z^{kq}\right) = \sum_{w=0}^{N+|\alpha-d|}\delta_{\alpha,\beta,u}(w)\left(Z^k\right)^w.$$

We conclude

$$\begin{aligned}
&\left|\sum_{k=1}^{L}\zeta_M^{kj}\sum_{u=0}^{\alpha}\binom{\alpha}{u}\frac{N^u}{(1-Z^k)^u}\left(\sum_{q=1}^{N}\sum_{r=1}^{q}\beta(r)r^{\alpha-u}Z^{kq}\right)c_k(u)p_{T-1}\left(u;Z^{Nk}\right)\right|\\
\le\ &y^{d-\alpha}\sum_{u=0}^{\alpha}\binom{\alpha}{u}N^u\left|\sum_{k=1}^{L}y^{\alpha-d}\left(1-Z^k\right)^{d-\alpha}\sum_{w=0}^{N+|\alpha-d|}\delta_{\alpha,\beta,u}(w)\left(Z^k\right)^w c_k(u)p_{T-1}\left(u;Z^{Nk}\right)\right|.
\end{aligned}$$

The sequence $y^{\alpha-d}\left(1-Z^k\right)^{d-\alpha}$ in k is bounded by some $V^{\alpha-d}$ and monotonous. Hence we obtain with Lemma 1.3.11 that the above estimate is smaller or equal to

$$2V^{\alpha-d}y^{d-\alpha}\sum_{u=0}^{\alpha}\binom{\alpha}{u}N^u\sum_{w=0}^{N+|\alpha-d|}|\delta_{\alpha,\beta}(w)|\max_{1\leq I\leq L}\left|\sum_{k=1}^{I}\left(\zeta_M^j Z^w\right)^k c_k(u)p_{T-1}(u;Z^w)\right|$$

and by Lemma 1.3.12 this is smaller or equal to

$$2V^{\alpha-d}y^{d-\alpha}\sum_{u=0}^{\alpha}\binom{\alpha}{u}N^u\sum_{w=0}^{N+|\alpha-d|}|\delta_{\alpha,\beta}(w)|\times 6C_uM\leq F_{\alpha,\beta,\omega}y^{d-\alpha}, \tag{1.3.1.8}$$

for some $F_{\alpha,\beta,\omega}>0$ only depending on α,β and ω. By considering (1.3.1.4), (1.3.1.5), (1.3.1.7) and (1.3.1.8) and using the triangle inequality in (1.3.1.3) (note that the constants do not depend on L), the theorem is proved. □

Since we have assumed β to be N-periodic it might come from a weak function $\eta\in W_N$, i.e., $\beta:=\beta_\eta$. The purpose of the next section will be to use the Dominated convergence theorem to improve regions of convergence of L-functions assigned to products of weak modular forms.

1.3.2 Application to L-functions of modular forms

Let $S=\{t_1,t_2,...\}$ be a countable, totally ordered set (the direction is simply given by $t_\ell\leq t_j$ if and only if $\ell\leq j$) equipped with an *integer map* $|\cdot|_S:S\to\mathbb{N}$ such that for some $L\geq 0$:

$$\#\{t\in S\mid |t|_S=n\}=O\left(n^L\right). \tag{1.3.2.1}$$

In the case the set S is clear, we simply write $|\cdot|$. For example, S could be the set of integral ideals of a number field and $|\cdot|$ their norm. Let $a(t_m)_{m\in\mathbb{N}}$ a sequence of complex numbers. We define the corresponding formal Dirichlet series by

$$F(s):=\sum_{t\in S}a(t)|t|^{-s}:=\sum_{m=1}^{\infty}a(t_m)|t_m|^{-s}.$$

In the case that the series

$$\sum_{n=1}^{\infty}\left|\frac{1}{|t_n|^s}-\frac{1}{|t_{n+1}|^s}\right|$$

converges for all $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$, one can check using partial summation that such Dirichlet series converge (if they do) on half planes and represent holomorphic functions in these regions. This is for example the case, if the $|t_n|$ increase monotonously. Since we have (1.3.2.1), one can show that $F(s)$ will converge in some point s_0 if and only if $a(t)=O(|t|^\nu)$ for some $\nu\in\mathbb{R}$.

Definition 1.3.16. *Let $F(s) = \sum_{t\in S} a(t)|t|^{-s}$ be a Dirichlet series, Q a totally ordered countable set together with a surjective map $w : Q \to S$ with finite fibres. We also assume that F converges to a holomorphic function on some half plane $\{\mathrm{Re}(s) > \sigma_0\}$. The order of Q shall respect the order of S, this means $u_1 \leq_Q u_2$ implies $w(u_1) \leq_S w(u_2)$ for all $u_1, u_2 \in Q$. We define an integer map on Q via $|u|_Q := |w(u)|_S$. In other words, all elements in the same fibre of a $t \in S$ are associated to the same integer. By a splitting of F we mean a Dirichlet series $\widetilde{F}(s) = \sum_{u\in Q} b(u)|u|_Q^{-s}$ that has the following properties:*

(i) $\widetilde{F}(s)$ converges to a holomorphic function in some half plane $\{\mathrm{Re}(s) > \widetilde{\sigma}_0\}$.

(ii) We have for all $t \in S$ the summation formula $\sum_{u\in w^{-1}(t)} b(u) = a(t)$.

We may think of splittings in the following way: we have $Q = \bigcup_{t\in S} \sigma^{-1}(t)$ and therefore

$$\sum_{t\in S} a(t)|t|^{-s} = \sum_{t\in S} \sum_{u\in w^{-1}(t)} b(u)|u|_Q^{-s}.$$

Example 1.3.17. *Consider the number theoretic function $A_4(n) := \#\{(x_1, x_2, x_3, x_4) \in \mathbb{N}_0^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = n\}$. Note that normally one considers tuples in $\mathbb{Z}^4$ but to keep things simple in this example we use $\mathbb{N}_0^4$. Then the ordinary Dirichlet series*

$$D_4(s) := \sum_{n=1}^{\infty} A_4(n)n^{-s}$$

converges for $\mathrm{Re}(s) > 2$. Here we have $S = \mathbb{N}$ and $|.|_{\mathbb{N}}$ is simply given by $|n|_{\mathbb{N}} := n$. Now put $Q := \mathbb{N}_0^4 \times \mathbb{N}_0^4 \setminus \{(\boldsymbol{x}, \boldsymbol{y}) \mid \langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0\}$ and consider the surjective map $w : Q \to \mathbb{N}$ with $w(\boldsymbol{x}, \boldsymbol{y}) := \langle \boldsymbol{x}, \boldsymbol{y}\rangle$. There are lots of orders we can define on Q as long as $(\boldsymbol{x}, \boldsymbol{y}) \leq_Q (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}})$ implies $\langle \boldsymbol{x}, \boldsymbol{y}\rangle \leq \langle \widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}\rangle$. Since $w^{-1}(n)$ consists of all $(\boldsymbol{x}, \boldsymbol{y})$ satisfying $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = n$ and $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = x_1^2 + x_2^2 + x_3^2 + x_4^2$, we obtain

$$\sum_{(\boldsymbol{x},\boldsymbol{y})\in w^{-1}(n)} \delta_{\boldsymbol{x},\boldsymbol{y}} = A_4(n)$$

with the Kronecker delta $\delta_{\boldsymbol{x},\boldsymbol{y}}$ which is 1 if $\boldsymbol{x} = \boldsymbol{y}$ and 0 else. As a result, the series

$$\sum_{(\boldsymbol{x},\boldsymbol{y})\in Q} \delta_{\boldsymbol{x},\boldsymbol{y}} \langle \boldsymbol{x}, \boldsymbol{y}\rangle^{-s}$$

is a possible splitting of D_4. Note, that this series also converges (independent from the chosen order) on $\mathrm{Re}(s) > 2$ and represents a holomorphic function in this region, which shows that also condition (i) of Definition 1.3.16 is satisfied.

Splittings that are obtained by maps $\mathbb{N}^\ell \times \mathbb{N}^\ell \to \mathbb{N}$ and $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \langle \boldsymbol{x}, \boldsymbol{y}\rangle$ will play the key role in the rest of this section. Throughout, we will omit the construction details as they were presented in the last example.

The next definition provides kind of an inverse concept for splittings.

Definition 1.3.18. *Let $S = \bigcup_{j=1}^{\infty} S_j$ be a disjoint covering with finite S_j. We say that a Dirichlet series $F(s) = \sum_{t\in S} a(t)|t|^{-s}$ respects the rearrangement $(S_j)_{j\in\mathbb{N}}$, if the series is given by the partial sums*

$$F_n(s) = \sum_{j=1}^{n} \sum_{t\in S_j} a(t)|t|^{-s}.$$

If there might be danger with confusion we simply write

$$(F, (S_j)_{j\in\mathbb{N}})(s) = \sum_{j=1}^{\infty} \sum_{t\in S_j} a(t)|t|^{-s}.$$

Obviously, $F(s)$ and $(F, (S_j)_{j\in\mathbb{N}})(s)$ coincide in all regions of absolute convergence. In the case of $S_j = \{t \in S \mid |t| = j\}$, $(F, (S_j)_{j\in\mathbb{N}})(s)$ is an ordinary Dirichlet series $\sum b(n)n^{-s}$ – we call this the standard rearrangement. The next proposition makes clear why rearrangements makes splitting undone in some situations.

Proposition 1.3.19. *Let $\widetilde{F}$ be a splitting of F over Q. Define the disjoint union $Q_j := \sigma^{-1}(t_j)$. If we now sum $\widetilde{F}$ with respect to $(Q_j)_{j\in\mathbb{N}}$ we obtain F.*

Proof. This follows directly from the definitions. □

Definition 1.3.20. *We call $(T_j)_{j\in\mathbb{N}}$ a sub-rearrangement of $(S_j)_{j\in\mathbb{N}}$, if there is a sequence of integers $0 < k_1 < k_2 < k_3 < \cdots$ such that $T_1 = S_1 \cup \cdots \cup S_{k_1}$, $T_2 = S_{k_1+1} \cup \cdots \cup S_{k_2}$ and so on.*

In the following we define for any rearrangement the abscissa of convergence $\sigma((F, (S_j)_{j\in\mathbb{N}}))$ to be the infimum real value σ_0, such that for all complex values $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \sigma_0$ the series converges and represents a holomorphic function in this region.

Remark 1.3.21. *One easily checks $\sigma((F, (T_j)_{j\in\mathbb{N}})) \leq \sigma((F, (S_j)_{j\in\mathbb{N}}))$. Hence Proposition 1.3.19 shows that splitting does not improve the area of convergence. However, when rearranging a split series the situation might look different.*

Let $\mathfrak{R}(F)$ the set of all rearrangements of F. We define an equivalence relation on $\mathfrak{R}(F)$ by putting two coverings in the same class if the resultant series have the same abscissa of convergence. We collect this data in $\mathfrak{R}(F)/\sim$. We would like to study $\mathfrak{R}(F)/\sim$, in particular, we are interested in the following question:

Question 1.3.22. *What is the value $\widetilde{\sigma}(F) := \inf_{G\in\mathfrak{R}(F)/\sim} \sigma(G)$?*

There is no simple answer to this question. It rather strongly depends on the Dirichlet series itself, as the next examples demonstrate.

(i) If $a(t) \geq 0$ globally, the region of convergence can not be improved by rearranging the Dirichlet series. Hence $|\mathfrak{R}(F)/\sim| = 1$ and $\widetilde{\sigma}(F) = \sigma(F)$.

(ii) Although the set $\mathfrak{G}$ is large, $-\widetilde{\sigma}(F)$ does not have to be unbounded even in the case that F is entire. If χ is an *even* real non-principal character modulo M, one can show that $\widetilde{\sigma}(L(\chi;s)) = -1$ if $L(\chi;-1) \notin \mathbb{Z}$. In this case the "best" rearrangement of $L(\chi;s)$ is given by $\mathbb{N} = \bigcup_{j\in\mathbb{N}}\{M(j-1)+k \mid 1 \leq k \leq M\}$ and we have

$$L(\chi;s) = \sum_{j=1}^{\infty}\left(\sum_{m=1}^{M}\chi(m)(M(j-1)+m)^{-s}\right), \qquad \mathrm{Re}(s) > -1.$$

We conclude $L(\chi,0) = 0$. Since all inner summands in the rearrangements are integers when $s = -1$, there is indeed no better choice if $L(\chi,-1) \notin \mathbb{Z}$, as the reader may easily check.
A similar argument shows $\widetilde{\sigma}(L(\chi;s)) = \sigma_0 = 0$ if χ is real, odd and $L(\chi,0) \notin \mathbb{Z}$.

(iii) The identity $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$ for $\mathrm{Re}(s) > 1$ is well-known and elementary. Here $\mu(n)$ is the Möbius function. Since $\mu(n)$ has sign changes, it makes sense to look at possible rearrangements. However, it seems to be extremely difficult to find improvements of $\sigma = 1$, since there is no progress in this area until today! We have $\frac{1}{2} \leq \widetilde{\sigma}(\zeta^{-1}) \leq 1$ and $\widetilde{\sigma}(\zeta^{-1}) = \frac{1}{2}$ implies the Riemann hypothesis.

Remark 1.3.23. *In the case of (ii), where the coefficients are well-studied, there are of course even more powerful tools for analytic continuation using series transformations, that can be seen as generalized rearrangements in the sense that we allow the splitting sets S_n to have infinite order. For example, when using Euler summation, we find the right hand series*

$$L(\chi;s) = \sum_{n=0}^{\infty}2^{-n-1}\sum_{\nu=0}^{n}\binom{n}{\nu}\chi(\nu+1)(\nu+1)^{-s},$$

will converge globally for non-principal characters χ.

Let $\boldsymbol{k} = (k_1,...,k_\ell)$ and $f \in U_{\boldsymbol{k}}(\Gamma_1(M,N))$ be a weak modular form. In the following we give a natural splitting for $L(f;s)$ in terms of the overset $Q = \mathbb{N}^\ell \times \mathbb{N}^\ell$. After this, when applying the Dominated convergence theorem from the last section we can find good rearrangements of these splittings to give estimates for the size defined in Question 1.3.22. Let $G_N^{(\ell)} = \mathbb{F}_N^\times \times \cdots \times \mathbb{F}_N^\times$ be the ℓ-fold product of the residue class groups modulo N. Then $G_N^{(\ell)}$ is a multiplicative group and there are $\varphi(N)^\ell$ characters $\psi : G_N^{(\ell)} \to \mathbb{C}^\times$ given by $\psi(\boldsymbol{n}) = \prod_{j=1}^{\ell}\psi_j(n_j)$, where $\psi_1,...,\psi_\ell$ are characters modulo N. We further call a character $\psi : G_N^{(\ell)} \to \mathbb{C}^\times$ *non-principal*, if no component ψ_j with $1 \leq j \leq \ell$ is principal and *principal* else. Analogously we say that ψ is *primitive* if and only if all components are primitive. Note that each ψ extends multiplicatively to a map $\psi : \mathbb{Z}^\ell \to \mathbb{C}^\times$. For $\boldsymbol{k} \in \mathbb{N}^\ell$ also define the (multiplicative) map $\Pi_{\boldsymbol{k}}(\boldsymbol{n}) = n_1^{k_1-1}\cdots n_\ell^{k_\ell-1}$.

Lemma 1.3.24. *Let h be a M-periodic function in $[M,d]$, where $0 \leq d$. Then there is*

some constant $C_h > 0$, only depending on h, such that we have uniformly for $x \in [0,1]$:

$$\left| \sum_{v=1}^{MT} h(v) x^v \right| \leq C_\beta (1-x)^d.$$

Proof. We use partial summation again. We obtain:

$$\begin{aligned}
&\sum_{v=1}^{MT} h(v)x^v = \sum_{v=1}^{MT} h(v)x^{MT} + \sum_{r=1}^{MT-1} \sum_{u=1}^{r} h(u)(x^r - x^{r+1}) \\
&= (1-x) \sum_{r=1}^{MT} \left(\sum_{u=1}^{r} h(u) \right) x^r = (1-x) \sum_{k=0}^{T-1} \sum_{u=1}^{M} \sum_{r=1}^{Mk+u} h(r) x^{Mk+u} \\
&= (1-x) \sum_{u=1}^{M} x^u \sum_{r=1}^{u} h(r) \sum_{k=0}^{T-1} x^{Mk} = (1-x) \sum_{u=1}^{M} \left(\sum_{r=1}^{u} h(r) \right) x^u \frac{1-x^{MT}}{1-x^M} \\
&= \frac{(1-x)(1-x^{MT})}{1-x^M} \sum_{u=1}^{M} \left(\sum_{r=1}^{u} h(r) \right) x^u.
\end{aligned}$$

By Proposition 1.3.7 there is a constant C_h such that uniformly on $[0,1]$:

$$\left| \sum_{u=1}^{M} \left(\sum_{r=1}^{u} h(r) \right) x^u \right| \leq C_h (1-x)^d.$$

On the other hand, we uniformly have

$$\frac{(1-x)(1-x^{MT})}{1-x^M} \leq 1.$$

This proves the lemma. □

In the following, consider the subsets $T_{M,p} \subset \mathbb{N}^\ell$ with $T_{M,p} = \{\boldsymbol{v} \in \mathbb{N}^\ell \mid (p-1)M < \max(\boldsymbol{v}) \leq Mp\}$. It is clear that we have a disjoint covering of $\mathbb{N}^\ell$ by all $T_{M,1}, T_{M,2}, ...$

Lemma 1.3.25. *Let $h_1, ..., h_\ell$ be functions in $(\mathbb{F}_{M^{-1}})_0^{\mathbb{C}_0}$, such that the associated weak functions ω_{h_j} have a removable singularity in $z = 0$, and $\mathcal{F}_M h_j \in [M, c_j]$ for some $c_j \geq 0$. Then we have for all vectors $\boldsymbol{u} \in \mathbb{N}^\ell$ and s with $\mathrm{Re}(s) > -\sum_{j=1}^{\ell} c_j$:*

$$\int_0^\infty \omega_{h_1}\left(\frac{u_1 x i}{N}\right) \cdots \omega_{h_\ell}\left(\frac{u_\ell x i}{N}\right) x^{s-1} \mathrm{d}x = \Gamma(s) \left(\frac{N}{2\pi}\right)^s \sum_{\boldsymbol{v} \in \mathbb{N}^\ell} \mathcal{F}_N^{(\ell)} h(\boldsymbol{v}) \left\langle \boldsymbol{u}, \boldsymbol{v} \right\rangle^{-s},$$

where $\mathcal{F}_N^{(\ell)} h(\boldsymbol{v}) = (\mathcal{F}_N h_1)(v_1) \cdots (\mathcal{F}_N h_\ell)(v_\ell)$ is the vector valued Fourier transform. Here, the order of summation respects the rearrangement $(T_{M,p})_{p \in \mathbb{N}}$.

Proof. We have

$$\begin{aligned}
\omega_{h_1}\left(\frac{u_1 xi}{N}\right)\cdots\omega_{h_\ell}\left(\frac{u_\ell xi}{N}\right) &= \sum_{\boldsymbol{q}\in G_M^{(\ell)}} h_1(q_1)\cdots h_\ell(q_\ell)\prod_{j=1}^{\ell}\frac{e(u_j\frac{ix}{N})}{e(\frac{q_j}{M})-e(u_j\frac{ix}{N})}\\
&= \sum_{\boldsymbol{q}\in G_M^{(\ell)}} h_1(q_1)\cdots h_\ell(q_\ell)\prod_{j=1}^{\ell}\sum_{v_j=1}^{\infty} e^{-\frac{2\pi u_j v_j x}{N}-\frac{2\pi i q_j v_j}{M}}\\
&= \lim_{T\to\infty}\sum_{\boldsymbol{q}\in G_M^{(\ell)}} h_1(q_1)\cdots h_\ell(q_\ell)\sum_{v_1=1}^{MT}\cdots\sum_{v_\ell=1}^{MT} e^{-\frac{2\pi(v_1u_1+\cdots+v_\ell u_\ell)x}{N}-\frac{2\pi i(v_1q_1+\cdots+v_\ell q_\ell)}{M}}\\
&= \prod_{j=1}^{\ell}\left(\sum_{v_j=1}^{MT}(\mathcal{F}_M h_j)(v_j)e^{-\frac{2\pi u_j v_j x}{M}}\right).
\end{aligned}$$

We obtain with Lemma 1.3.24:

$$\left|\prod_{j=1}^{\ell}\left(\sum_{v_j=1}^{MT}(\mathcal{F}_M h_j)(v_j)e^{-\frac{2\pi u_j v_j x}{M}}\right)x^{s-1}\right| \leq C_{\boldsymbol{h},\boldsymbol{u}}x^{\sigma-1+c_1+\cdots+c_\ell}$$

uniformly for $x\in[0,1]$, where $C_{\boldsymbol{h},\boldsymbol{u}}>0$ only depends on the functions $h_1,...,h_\ell$ and the vector $\boldsymbol{u}$. As a result, the integral

$$\int_0^\infty \omega_{h_1}\left(\frac{u_1 xi}{N}\right)\cdots\omega_{h_\ell}\left(\frac{u_\ell xi}{N}\right)x^{s-1}\mathrm{d}x$$

converges absolutely to a holomorphic function for $\mathrm{Re}(s)>-\sum_{j=1}^{\ell}c_j$ and me may switch it with summation in this region:

$$\begin{aligned}
&\int_0^\infty \omega_{h_1}\left(\frac{u_1 xi}{N}\right)\cdots\omega_{h_\ell}\left(\frac{u_\ell xi}{N}\right)x^{s-1}\mathrm{d}x = \int_0^\infty \lim_{T\to\infty}\prod_{j=1}^{\ell}\left(\sum_{v_j=1}^{MT}(\mathcal{F}_M h_j)(v_j)e^{-\frac{2\pi u_j v_j x}{M}}\right)x^{s-1}\mathrm{d}x\\
&= \lim_{T\to\infty}\sum_{\substack{v_j=1\\1\leq j\leq\ell}}^{MT}\prod_{j=1}^{\ell}(\mathcal{F}_M h_j)(v_j)\int_0^\infty e^{-\frac{2\pi(u_1v_1+u_2v_2+\cdots+u_\ell v_\ell)x}{N}}x^{s-1}\mathrm{d}x = \Gamma(s)\left(\frac{N}{2\pi}\right)^{-s}\sum_{\boldsymbol{v}\in\mathbb{N}^\ell}\frac{\mathcal{F}_N^{(\ell)}h(\boldsymbol{v})}{\langle\boldsymbol{u},\boldsymbol{v}\rangle^s},
\end{aligned}$$

where we respect the rearrangement $(T_{M,p})_{p\in\mathbb{N}}$ in the last sum. This proves the lemma. □

In the following, we will look at L-functions corresponding to higher weak modular forms. Let f be a modular form in $M_k(\Gamma_1(M,N))$ with Fourier expansion

$$f(\tau)=\sum_{n=0}^{\infty}a(n)q^{\frac{n}{N}}.$$

Then we remember that its corresponding L-function is given by

$$L(f;s) = \sum_{n=1}^{\infty} a(n) n^{-s}.$$

One can show that this series converges on some half plane and we have the relation

$$\left(\frac{2\pi}{N}\right)^{-s} \Gamma(s) L(f;s) = \int_0^{\infty} (f(ix) - a(0)) x^{s-1} \mathrm{d}x.$$

Following this type of Mellin transformation, one can show that each $L(f;s)$ has a meromorphic continuation to the entire complex plane and satisfies a functional equation. The next proposition makes a statement about the L-functions of certain products of weak modular forms defined in Definition 1.2.24.

Proposition 1.3.26. *Let $f \in U_{\boldsymbol{k}}(\Gamma_1(M,N))$ be a higher weak modular form, such that*

$$f = \sum_{\alpha=1}^{R} \mu_\alpha \vartheta_{k_1}(\omega_{h_{\alpha,1}} \otimes \omega_{t_{\alpha,1}}) \cdots \vartheta_{k_\ell}(\omega_{h_{\alpha,\ell}} \otimes \omega_{t_{\alpha,\ell}}).$$

Here we assume that $\mathrm{sgn}(h_{\alpha,j} t_{\alpha,j}) = (-1)^{k_j}$ for all $j = 1, ..., \ell$. Then, for all complex numbers s with $\mathrm{Re}(s) > |\boldsymbol{k}|$, we have

$$L(f;s) = \sum_{(\boldsymbol{u},\boldsymbol{v}) \in \mathbb{N}^\ell \times \mathbb{N}^\ell} a(\boldsymbol{u},\boldsymbol{v}) \left\langle \boldsymbol{u},\boldsymbol{v} \right\rangle^{-s}, \tag{1.3.2.2}$$

where the coefficients $a(\boldsymbol{u},\boldsymbol{v})$ are given by

$$a(\boldsymbol{u},\boldsymbol{v}) = 2^\ell N^{\ell - |\boldsymbol{k}|} \Pi_{\boldsymbol{k}}(\boldsymbol{u}) \sum_{\alpha=1}^{R} \mu_\alpha \prod_{j=1}^{\ell} t_{\alpha,j}(u_j) (\mathcal{F}_M h_{\alpha,j})(v_j). \tag{1.3.2.3}$$

Proof. The series on the right of (1.3.2.2) converges absolutely on the half plane $\{s \in \mathbb{C} \mid \sigma > |\boldsymbol{k}|\}$, since

$$|a(\boldsymbol{u},\boldsymbol{v})| \ll u_1^{k_1 - 1} \cdots u_\ell^{k_\ell - 1}$$

and on the other hand, for all $\varepsilon > 0$,

$$(u_1 v_1 + \cdots u_\ell v_\ell)^{|\boldsymbol{k}| + \varepsilon} = \prod_{j=1}^{\ell} (u_1 v_1 + \cdots u_\ell v_\ell)^{k_j + \frac{\varepsilon}{\ell}} \geq \prod_{j=1}^{\ell} (u_j v_j)^{k_j + \frac{\varepsilon}{\ell}},$$

and hence

$$\sum_{(\boldsymbol{u},\boldsymbol{v}) \in \mathbb{N}^\ell \times \mathbb{N}^\ell} \left| a(\boldsymbol{u},\boldsymbol{v}) \left\langle \boldsymbol{u},\boldsymbol{v} \right\rangle^{-|\boldsymbol{k}| - \varepsilon} \right| \ll \sum_{\substack{u_j = 1 \\ 1 \leq j \leq \ell}}^{\infty} \sum_{\substack{v_j = 1 \\ 1 \leq j \leq \ell}}^{\infty} \frac{u_1^{k_1 - 1} \cdots u_\ell^{k_\ell - 1}}{(u_1 v_1 + \cdots u_\ell v_\ell)^{|\boldsymbol{k}| + \varepsilon}}$$

$$\leq \sum_{\substack{u_j = 1 \\ 1 \leq j \leq \ell}}^{\infty} \sum_{\substack{v_j = 1 \\ 1 \leq j \leq \ell}}^{\infty} \frac{u_1^{k_1 - 1} \cdots u_\ell^{k_\ell - 1}}{\prod_{j=1}^{\ell} (u_j v_j)^{k_j + \frac{\varepsilon}{\ell}}} = \zeta\left(1 + \frac{\varepsilon}{\ell}\right)^\ell \prod_{j=1}^{\ell} \zeta\left(k_j + \frac{\varepsilon}{\ell}\right) < \infty.$$

Since $t_j(0) = h_j(0) = 0$ for all $1 \leq j \leq \ell$, all involved weak functions have a removable singularity in $z = 0$ and so have their product. We have for all $s \in \mathbb{C}$

$$\left(\frac{2\pi}{N}\right)^{-s} \Gamma(s) L(f;s) = \int_0^\infty \sum_{\alpha=1}^{R} \mu_\alpha(\vartheta_{k_1}(\omega_{h_{\alpha,1}} \otimes \omega_{t_{\alpha,1}}) \cdots \vartheta_{k_l}(\omega_{h_{\alpha,\ell}} \otimes \omega_{t_{\alpha,\ell}}))(ix) x^{s-1} \mathrm{d}x.$$

Hence, due to absolute convergence, we obtain for all s with $\sigma > |\boldsymbol{k}|$ according to Proposition 1.2.20:

$$\begin{aligned} L(f;s) = \lim_{T\to\infty} \frac{1}{\Gamma(s)} \left(\frac{2\pi}{N}\right)^s 2^\ell N^{\ell-|\boldsymbol{k}|} \sum_{\substack{u_j=1 \\ 1\leq j\leq \ell}}^{T} \sum_{\alpha=1}^{R} \mu_\alpha u_1^{k_{\alpha,1}-1} \cdots u_\ell^{k_{\alpha,\ell}-1} t_{\alpha,1}(u_1) \cdots t_{\alpha,\ell}(u_\ell) \\ \times \int_0^\infty \omega_{h_{\alpha,1}}\left(\frac{u_1 xi}{N}\right) \cdots \omega_{h_{\alpha,\ell}}\left(\frac{u_\ell xi}{N}\right) x^{s-1} \mathrm{d}x. \end{aligned}$$

Together with Lemma 1.3.25 we obtain, that this equals

$$\begin{aligned} & 2^\ell N^{\ell-|\boldsymbol{k}|} \sum_{\substack{u_j,v_j=1 \\ 1\leq j\leq \ell}}^{\infty} \sum_{\alpha=1}^{R} \mu_\alpha \left(\prod_{j=1}^{\ell} u_j^{k_j-1} t_{\alpha,j}(u_j) (\mathcal{F}_M h_{\alpha,j})(v_j) \right) (u_1 v_1 + \cdots + u_\ell v_\ell)^{-s} \\ = & 2^\ell N^{\ell-|\boldsymbol{k}|} \sum_{\substack{u_j,v_j=1 \\ 1\leq j\leq \ell}}^{\infty} \prod_{j=1}^{\ell} u_j^{k_j-1} \sum_{\alpha=1}^{R} \mu_\alpha \left(\prod_{j=1}^{\ell} t_{\alpha,j}(u_j) (\mathcal{F}_M h_{\alpha,j})(v_j) \right) (u_1 v_1 + \cdots + u_\ell v_\ell)^{-s}. \end{aligned}$$

This proves the proposition. □

Proposition 1.3.26 provides us coefficients $a(\boldsymbol{u}, \boldsymbol{v})$ that belong to splittings of $L(f;s)$ over $Q = \mathbb{N}^\ell \times \mathbb{N}^\ell$. We may use this to define a linear map from "splitting coefficients" to modular forms. Firstly, consider the vector space

$$A_{\boldsymbol{k}} := \left\{ a : \mathbb{N}^\ell \times \mathbb{N}^\ell \longrightarrow \mathbb{C} \,\middle|\, s \in \mathbb{C}, \mathrm{Re}(s) > |\boldsymbol{k}| : \sum_{(\boldsymbol{u},\boldsymbol{v}) \in \mathbb{N}^\ell \times \mathbb{N}^\ell} \left| a(\boldsymbol{u}, \boldsymbol{v}) \langle \boldsymbol{u}, \boldsymbol{v} \rangle^{-s} \right| < \infty \right\}.$$

Secondly, look at the subspace $B_{M,N,\boldsymbol{k}} \subset A_{\boldsymbol{k}}$ of functions that generate L-functions of higher weak modular forms in $U_{\boldsymbol{k}}(\Gamma_1(M,N))$. The linear map $B_{M,N,\boldsymbol{k}} \to \mathcal{O}(\{\mathrm{Re}(s) > |\boldsymbol{k}|\})$ with $a(\boldsymbol{u}, \boldsymbol{v}) \mapsto \sum a(\boldsymbol{u}, \boldsymbol{v}) \langle \boldsymbol{u}, \boldsymbol{v} \rangle^{-s}$ induces a linear map $\psi_{M,N,\boldsymbol{k}} : B_{M,N,\boldsymbol{k}} \to U_{\boldsymbol{k}}(\Gamma_1(M,N))$. Note that this map is well-defined, since the L-function of a modular form is uniquely determined, and of course surjective (Proposition 1.3.26 provides proper pre-images). However, this map does not have to be injective since there is obviously no Identity theorem for Dirichlet series of the form $\sum a(\boldsymbol{u}, \boldsymbol{v}) \langle \boldsymbol{u}, \boldsymbol{v} \rangle^{-s}$. This lack of uniqueness is measured

by the kernel $\Lambda_{M,N,\boldsymbol{k}} := \ker(\psi_{M,N,\boldsymbol{k}})$. So, when considering the coefficients $a(\boldsymbol{u},\boldsymbol{v})$ in (1.3.2.3), all coefficients $b(\boldsymbol{u},\boldsymbol{v})$ generating $L(f;s) = \sum b(\boldsymbol{u},\boldsymbol{v}) \langle \boldsymbol{u},\boldsymbol{v}\rangle^{-s}$ (assuming absolute convergence in $\mathrm{Re}(s) > |\boldsymbol{k}|$), are contained in the translated set $a + \Lambda_{M,N,\boldsymbol{k}}$. All of this can be summarized in the following proposition.

Proposition 1.3.27. *Let $f \in U_{\boldsymbol{k}}(\Gamma_1(M,N))$. Then $a + \Lambda_{M,N,\boldsymbol{k}}$ consists of all coefficient functions $b(\boldsymbol{u},\boldsymbol{v})$, such that*

$$L(f;s) = \sum_{(\boldsymbol{u},\boldsymbol{v})\in\mathbb{N}^\ell\times\mathbb{N}^\ell} b(\boldsymbol{u},\boldsymbol{v}) \langle \boldsymbol{u},\boldsymbol{v}\rangle^{-s}$$

and the series converges absolutely for $\mathrm{Re}(s) > |\boldsymbol{k}|$. In particular, all of them define splittings of $L(f;s)$ ($S = \langle \mathbb{N}^\ell, \mathbb{N}^\ell\rangle \subset \mathbb{N}$) over $Q = \mathbb{N}^\ell \times \mathbb{N}^\ell$ equipped with the integer map $|(\boldsymbol{u},\boldsymbol{v})|_Q := \langle \boldsymbol{u},\boldsymbol{v}\rangle$.

Of course, when using Proposition 1.3.19, one could reconstruct the original ordinary Dirichlet series with a standard rearrangement. However, in the following we study a completely different rearrangement $(U_{M,N,m})_{m\in\mathbb{N}}$ that arises from the results in the previous section. With this we want to extend the region of convergence of the series $L(f;s) = \sum a(\boldsymbol{u},\boldsymbol{v}) \langle \boldsymbol{u},\boldsymbol{v}\rangle^{-s}$ naturally. Fix an integer N. We define for $p,q \in \mathbb{N}$

$$T_{M,N,p,q} = \{(\boldsymbol{u},\boldsymbol{v}) \in \mathbb{N}^\ell \times \mathbb{N}^\ell \mid N(p-1) < \max(\boldsymbol{u}) \le Np, M(q-1) < \max(\boldsymbol{v}) \le Mq\}.$$

Note that the $T_{N,p,q}$ define a disjoint covering of $\mathbb{N}^\ell \times \mathbb{N}^\ell$. We then define the subrearrangement

$$\begin{aligned}
U_{M,N,1} &:= T_{M,N,1,1},\\
U_{M,N,2} &:= T_{M,N,1,2} \cup T_{M,N,2,1} \cup T_{M,N,2,2},\\
U_{M,N,3} &:= T_{M,N,1,3} \cup T_{M,N,2,3} \cup T_{M,N,3,1} \cup T_{M,N,3,2} \cup T_{M,N,3,3},
\end{aligned}$$

and so on. After Proposition 1.3.27 provided us some natural splittings (in fact, all Dirichlet series of $L(f;s)$ arising from products of weak functions for $\boldsymbol{k}$ and not from the usual Fourier series), we show that we can improve the region of convergence by rearranging the splittings by $U_{M,N,m}$.

Theorem 1.3.28 (see [24])**.** *Let $N > 1$ and $\ell \ge 1$ be integers and $h_j \in (\mathbb{F}_{M^{-1}})_0^{\mathbb{C}_0}$ with $\mathcal{F}_M h_j \in [M, c_j]$ and the $t_j \in [N, d_j]$ be even or odd N-periodic functions for $1 \le j \le \ell$ and some non-negative integers c_j and d_j. We further assume that we have $\mathrm{sgn}(h_j t_j) = (-1)^{k_j}$ for every $1 \le j \le \ell$. Consider the modular form*

$$f(\tau) = \prod_{j=1}^{\ell} \vartheta_{k_j}(\omega_{h_j} \otimes \omega_{t_j}; \tau) \in U_{\boldsymbol{k}}(\Gamma_1(M,N)).$$

For all values $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \max(|\boldsymbol{k}|-\ell-d, -c)$, where $c = \sum_{j=1}^{\ell} c_j$ and $d = \sum_{j=1}^{\ell} d_j$, we have the series representation

$$L(f;s) = 2^{\ell} N^{\ell-|\boldsymbol{k}|} \sum_{(\boldsymbol{u},\boldsymbol{v})\in\mathbb{N}^{\ell}\times\mathbb{N}^{\ell}} \Pi_{\boldsymbol{k}}(\boldsymbol{u}) t(\boldsymbol{u}) (\mathcal{F}_N^{(\ell)} h)(\boldsymbol{v}) \langle \boldsymbol{u}, \boldsymbol{v}\rangle^{-s},$$

where $t(\boldsymbol{u}) := t_1(u_1)\cdots t_{\ell}(u_{\ell})$ and $(\mathcal{F}_N^{(\ell)} h)(\boldsymbol{v}) := (\mathcal{F}_N h_1)(v_1)\cdots(\mathcal{F}_N h_{\ell})(v_{\ell})$ is the multidimensional Fourier transform. The summation respects the rearrangement $(U_{M,N,m})_{m\in\mathbb{N}}$. In particular, we have

$$\inf_{b\in a+\Lambda_{M,N,\boldsymbol{k}}} \widetilde{\sigma}\left(\sum_{(\boldsymbol{u},\boldsymbol{v})\in\mathbb{N}^{\ell}\times\mathbb{N}^{\ell}} b(\boldsymbol{u},\boldsymbol{v}) \langle \boldsymbol{u},\boldsymbol{v}\rangle^{-s}\right) \leq \max(|\boldsymbol{k}|-\ell-d, -c). \tag{1.3.2.4}$$

Here, $a(\boldsymbol{u},\boldsymbol{v})$ are the standard coefficients obtained in Proposition 1.3.26, for the set $a+\Lambda_{M,N,\boldsymbol{k}}$ see also Proposition 1.3.27.

Proof. The series on the right of (1.3.2.2) converges absolutely for all s with $\mathrm{Re}(s) > |\boldsymbol{k}|$. Since $t_j(0) = h_j(0) = 0$ for all $1 \leq j \leq \ell$, all involved weak functions have a removable singularity in $z = 0$ and so have their product. We have for all $s \in \mathbb{C}$

$$\left(\frac{2\pi}{N}\right)^{-s} \Gamma(s) L(f;s) = \int_0^{\infty} f(ix) x^{s-1} \mathrm{d}x = \int_0^{\infty} \prod_{j=1}^{\ell} \vartheta_{k_j}(\omega_{h_j} \otimes \omega_{t_j}; xi) x^{s-1} \mathrm{d}x.$$

The functions $t_1, ..., t_{\ell}$ have heights $d_1, ..., d_{\ell}$ which means by Theorem 1.3.14 that there is a constant $C > 0$ such that for all $T \in \mathbb{N}$ and $0 \leq x \leq 1$:

$$\begin{aligned}
&\left| \sum_{\substack{u_j=1 \\ 1\leq j\leq \ell}}^{NT} u_1^{k_1-1}\cdots u_{\ell}^{k_{\ell}-1} t_1(u_1)\cdots t_{\ell}(u_{\ell}) \omega_{h_1}\left(\frac{u_1 xi}{N}\right)\cdots\omega_{h_{\ell}}\left(\frac{u_{\ell} xi}{N}\right) x^{s-1}\right| \\
&= x^{\sigma-1} \prod_{j=1}^{\ell} \left| \sum_{u_j=1}^{NT} u_j^{k_j-1} t_j(u_j) \omega_{h_j}\left(\frac{u_j xi}{N}\right)\right| \leq C x^{\sigma+d-1-(|\boldsymbol{k}|-\ell)}
\end{aligned}$$

and the right hand side is an integrable majorant for $\sigma > |\boldsymbol{k}| - \ell - d$. For these values we therefore have dominated convergence on the interval $[0,1]$ and uniform convergence on

the interval $[1,\infty)$, hence we obtain for $\mathrm{Re}(s) > |\boldsymbol{k}| - \ell - d$

$$\begin{aligned}
L(f;s) =& \frac{1}{\Gamma(s)} \left(\frac{2\pi}{N}\right)^s \int_0^\infty \lim_{T\to\infty} 2^\ell N^{\ell-|\boldsymbol{k}|} \sum_{\substack{u_j=1\\1\le j\le \ell}}^{NT} u_1^{k_1-1}\cdots u_\ell^{k_\ell-1} t_1(u_1)\cdots t_\ell(u_\ell) \\
&\times \omega_{h_1}\left(\frac{ixu_1}{N}\right)\cdots\omega_{h_\ell}\left(\frac{ixu_\ell}{N}\right) x^{s-1}\mathrm{d}x \\
=& \lim_{T\to\infty} \frac{1}{\Gamma(s)} \left(\frac{2\pi}{N}\right)^s 2^\ell N^{\ell-|\boldsymbol{k}|} \sum_{\substack{u_j=1\\1\le j\le \ell}}^{NT} u_1^{k_1-1}\cdots u_\ell^{k_\ell-1} t_1(u_1)\cdots t_\ell(u_\ell) \\
&\times \int_0^\infty \omega_{h_1}\left(\frac{ixu_1}{N}\right)\cdots\omega_{h_\ell}\left(\frac{ixu_\ell}{N}\right) x^{s-1}\mathrm{d}x.
\end{aligned}$$

In the proof of the Dominated convergence theorem the upper bound was independent of the choice of the partial sums for the series of ω. Hence, together with Lemma 1.3.25 we obtain for $\mathrm{Re}(s) > -c$:

$$L(f;s) = \lim_{T\to\infty} 2^\ell N^{\ell-|\boldsymbol{k}|} \sum_{\substack{u_j=1\\1\le j\le \ell}}^{NT} u_1^{k_1-1}\cdots u_\ell^{k_\ell-1} t_1(u_1)\cdots t_\ell(u_\ell) \sum_{\substack{v_j=1\\1\le j\le \ell}}^{MT} \frac{\mathcal{F}_N^{(\ell)} h(\boldsymbol{v})}{(u_1v_1+\cdots+u_\ell v_\ell)^s}.$$

Since the order of summation in the partial sums respects the rearrangement $(U_{M,N,m})_{m\in\mathbb{N}}$, note that

$$\sum_{\substack{u_j=1\\1\le j\le \ell}}^{NT} \sum_{\substack{v_j=1\\1\le j\le \ell}}^{MT} - \sum_{\substack{u_j=1\\1\le j\le \ell}}^{N(T-1)} \sum_{\substack{v_j=1\\1\le j\le \ell}}^{M(T-1)} = \sum_{U_{M,N,T}} .$$

Since $\{\mathrm{Re}(s) > |\boldsymbol{k}|-\ell-d\}\cap\{\mathrm{Re}(s) > -c\} = \{\mathrm{Re}(s) > \max(|\boldsymbol{k}|-\ell-d, -c)\}$ and (1.3.2.4) follows with

$$\widetilde{\sigma}\left(\sum_{(\boldsymbol{u},\boldsymbol{v})\in\mathbb{N}^\ell\times\mathbb{N}^\ell} a(\boldsymbol{u},\boldsymbol{v})\,\langle \boldsymbol{u},\boldsymbol{v}\rangle^{-s}\right) \le \max(|\boldsymbol{k}|-\ell-d,-c),$$

the theorem is proved. □

From this we obtain a much more general result as (ii) presented in the above examples.

Corollary 1.3.29. *Let $t \neq 0$ be N-periodic and be an element of $[N,d]$. Then the series*

$$\lim_{T\to\infty} \sum_{n=1}^{NT} t(n) n^{-s} = \sum_{r=0}^{\infty} \sum_{\ell=1}^{N} t(\ell)(Nr+\ell)^{-s}$$

converges for all $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > -d$ *to a holomorphic function* $L(t,s)$*. In particular,* $L(t,-\alpha) = 0$ *for all* $0 \leq \alpha < d$*.*

Proof. Put $k = d+1$. Choose $h \neq 0$ such that $\mathrm{sgn}(t \cdot h) = (-1)^k$. Then we obtain with Theorem 1.3.28 that the series

$$\lim_{T\to\infty} N^{1-k} \sum_{u=1}^{NT} \sum_{\nu=1}^{\infty} \frac{u^{k-1} t(u) (\mathcal{F}_N h)(\nu)}{(u\nu)^s} = \lim_{T\to\infty} N^{1-k} \left(\sum_{u=1}^{NT} u^{d-s} t(u) \right) \left(\sum_{\nu=1}^{\infty} \frac{(\mathcal{F}_N h)(\nu)}{\nu^s} \right)$$

converges for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ to a holomorphic function. Since

$$\lim_{T\to\infty} \sum_{u=1}^{NT} u^{d-s} t(u) = \sum_{r=0}^{\infty} \sum_{\ell=1}^{N} t(\ell) (Nr+\ell)^{-s+d},$$

the claim follows. □

One consequence of this observation is an application to infinite products.

Example 1.3.30. *Consider the function*

$$a_4(n) = \begin{cases} -1, & \text{if } n \equiv \pm 1 \pmod 4, \\ 2, & \text{if } n \equiv 2 \pmod 4, \\ 0, & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Then a_4 *has height 1, since obviously* $\sum_{j=1}^{4} a_4(j) = \sum_{j=1}^{4} a_4(j) j = -1 + 4 - 3 = 0$*. One sees quickly that*

$$f(s) = \sum_{n=1}^{\infty} a_4(n) n^{-s} = \left(3 \cdot 2^{-s} - 2 \cdot 4^{-s} - 1\right) \zeta(s).$$

Together with Corollary 1.3.29 we conclude that

$$\sum_{n=0}^{\infty} \sum_{j=1}^{4} a_4(4n+j)(4n+j)^{-s}$$

converges to a holomorphic function for all $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > -1$ *and we find*

$$\sum_{n=0}^{\infty} (\log(4n+1) - 2\log(4n+2) + \log(4n+3)) = f'(0).$$

Since $\zeta(0) = -\frac{1}{2}$*, we obtain*

$$\prod_{n=0}^{\infty} \frac{(4n+1)(4n+3)}{(4n+2)^2} = \frac{1}{\sqrt{2}}.$$

Remark 1.3.31. *With a rearranged splitting*

$$\sum_{n=1}^{\infty}((2n-1)^{-s}-2(2n)^{-s}+(2n+1)^{-s})=2(1-2^{1-s})\zeta(s)-1,$$

that converges for $\mathrm{Re}(s)>-1$*, we similarly conclude (when using* $\zeta'(0)=-\frac{1}{2}\log(2\pi)$*) the Wallis product*

$$\prod_{n=1}^{\infty}\frac{(2n-1)(2n+1)}{(2n)^2}=\frac{2}{\pi}.$$

Example 1.3.32. *Let* χ *be a non-principal even character modulo* N*. Then, using the well-known Weierstraß product expansion*

$$\frac{1}{\Gamma(s)}=se^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}},$$

we find

$$\prod_{n=0}^{\infty}(Nn+1)(Nn+2)^{\chi(2)}(Nn+3)^{\chi(3)}\cdots(Nn+N-1)^{\chi(N-1)}=\prod_{m=1}^{N}\Gamma\left(\frac{m}{N}\right)^{\chi(m)}.$$

As a consequence, we obtain the following well-known identity

$$e^{L'(\chi,0)}=\prod_{m=1}^{N}\Gamma\left(\frac{m}{N}\right)^{\chi(m)}.$$

The next final corollary provides natural generalized Dirichlet series representations for L-functions associated to products of Eisenstein series for non-principal primitive Dirichlet characters.

Corollary 1.3.33 (see [24])**.** *Let* $\chi,\psi:\mathbb{Z}^{\ell}\to\mathbb{C}^{\times}$ *be non-principal, primitive characters modulo* M *and* N*, respectively, such that* $\chi_j(-1)\psi_j(-1)=(-1)^{k_j}$ *for all* $j=1,...,\ell$*. For all* $s\in\mathbb{C}$ *with* $\mathrm{Re}(s)>\max\left(|\boldsymbol{k}|-\ell-\frac{1}{2}\sum_{j=1}^{\ell}(\psi_j(-1)+1),-\frac{1}{2}\sum_{j=1}^{\ell}(\chi_j(-1)+1)\right)$ *we have*

$$L\left(\prod_{j=1}^{\ell}E_{k_j}(\chi_j,\psi_j;\tau),s\right)=\left(-\frac{2\pi i}{N}\right)^{|\boldsymbol{k}|}\prod_{j=1}^{\ell}\frac{2\mathcal{G}(\psi_j)}{(k_j-1)!}\sum_{(\boldsymbol{u},\boldsymbol{v})\in\mathbb{N}^{\ell}\times\mathbb{N}^{\ell}}\Pi_{\boldsymbol{k}}(\boldsymbol{u})\overline{\psi}(\boldsymbol{u})\chi(\boldsymbol{v})\left\langle\boldsymbol{u},\boldsymbol{v}\right\rangle^{-s},$$

where the summation respects the rearrangement $(U_{M,N,m})_{m\in\mathbb{N}}$*.*

Proof. Since all characters are primitive, we have

$$E_{k_j}(\chi_j,\psi_j;\tau)=\frac{\chi_j(-1)(-2\pi i)^{k_j}\mathcal{G}(\psi_j)}{N(k_j-1)!\mathcal{G}(\overline{\chi_j})}\vartheta_{k_j}(\omega_{\overline{\chi_j}}\otimes\omega_{\overline{\psi_j}};\tau).$$

Hence we obtain with Theorem 1.3.28

$$L\left(\prod_{j=1}^{\ell} E_{k_j}(\chi_j,\psi_j;\tau),s\right)=\lambda_1\cdots\lambda_\ell 2^\ell N^{\ell-|\boldsymbol{k}|}\sum_{(\boldsymbol{u},\boldsymbol{v})\in\mathbb{N}^\ell\times\mathbb{N}^\ell}\Pi_{\boldsymbol{k}}(\boldsymbol{u})\overline{\psi}(\boldsymbol{u})(\mathcal{F}_N^{(\ell)}\overline{\chi})(\boldsymbol{v})\left\langle\boldsymbol{u},\boldsymbol{v}\right\rangle^{-s},$$

where

$$\lambda_j=\frac{\chi_j(-1)(-2\pi i)^{k_j}\mathcal{G}(\psi_j)}{N(k_j-1)!\mathcal{G}(\overline{\chi_j})}.$$

We can simplify the expression $(\mathcal{F}_N^{(\ell)})(\overline{\chi})$ by

$$\left(\mathcal{F}_N^{(\ell)}\right)(\overline{\chi})(\boldsymbol{v})=\chi(\boldsymbol{v})\left(\mathcal{F}_N^{(\ell)}\overline{\chi}\right)(\mathbf{1})=\chi(\boldsymbol{v})\prod_{j=1}^{\ell}\chi_j(-1)\mathcal{G}(\overline{\chi_j}),$$

so we obtain

$$\lambda_1\cdots\lambda_\ell N^{\ell-|\boldsymbol{k}|}\left(\mathcal{F}_N^{(\ell)}\overline{\chi}\right)(\boldsymbol{v})=\left(-\frac{2\pi i}{N}\right)^{|\boldsymbol{k}|}\prod_{j=1}^{\ell}\frac{\mathcal{G}(\psi_j)}{(k_j-1)!}\chi(\boldsymbol{v}).$$

The extended domain of convergence follows, because of the rearrangement, with Theorem 1.3.28 and the fact that the height of ψ_j is given by $\frac{1}{2}(\psi_j(-1)+1)$. □

Note that this representation of the L-function of the considered product is more natural since it is a direct generalization of the formula for $L(E_k(\chi,\psi;\tau),s)$ in the case $\ell=1$, where the series directly splits into a product of two Dirichlet L-functions:

$$\frac{2(-2\pi i)^k\mathcal{G}(\psi)}{N^k(k-1)!}\sum_{(u,v)\in\mathbb{N}\times\mathbb{N}}u^{k-1}\overline{\psi}(u)\chi(v)(uv)^{-s}=\frac{2(-2\pi i)^k\mathcal{G}(\psi)}{N^k(k-1)!}L\left(\overline{\psi};s-k+1\right)L(\chi;s).$$

The region of convergence may be improved when summing with respect to the rearrangement $(U_{M,N,m})_{m\in\mathbb{N}}$. In this case we end up with

$$\sum_{(u,v)\in\mathbb{N}\times\mathbb{N}}u^{k-1}\overline{\psi}(u)\chi(v)(uv)^{-s}=\lim_{T\to\infty}\sum_{u=1}^{NT}\sum_{v=1}^{MT}u^{k-1}\overline{\psi}(u)\chi(v)(uv)^{-s}.$$

By Corollary 1.3.29 this converges, if $k\geq 2$, for $\mathrm{Re}(s)>k-1$ if ψ is odd and for $\mathrm{Re}(s)>k-2$ if ψ is even (and of course, non-principal). In the case $k=1$ we have convergence in the region $\mathrm{Re}(s)>-1$ if and only if ψ and χ are both even and for $\mathrm{Re}(s)>0$ else. An important question, which is still unsolved in the very general case, if modular forms can be written as sums of products of Eisenstein series. But there is a lot of progress in this field. Dickson and Neururer have shown in [23], that, if $k\geq 4$, $N=p^aq^bN'$ where p^a,q^b are powers of primes and N' is square free, the space $M_k(\Gamma_0(N))$ is generated by $E_k(\Gamma_0(N),\chi_{0,N})$ and a subspace containing products of two Eisenstein series. A similar result for $M_k(p)$ and $k\geq 4$, where p is prime, is due to Imamoḡlu and Kohnen [38]. For a correspondence between values of L-functions for products of pairs of different Eisenstein series see [20]. Finally, we give an example.

Example 1.3.34. *Let χ be a primitive even Dirichlet character modulo $N > 1$. We then look on the Eisenstein series $E_2(\chi, \chi; \tau)$ of weight $k = 2$ and define $f(\tau) := E_2(\chi, \chi; \tau)^2$. Then f is a modular form of weight 4 for the group $\Gamma(N^2)$ and vanishes in the cusps $z = 0$ and $z = i\infty$, hence its L-function $L(f; s)$ is entire. We are especially interested in the critical value $L(f; 1)$. With Corollary 1.3.33 we obtain*

$$L(f;1) = \frac{64\pi^4 \mathcal{G}(\chi)^2}{N^4} \lim_{T\to\infty} \sum_{u_1,u_2,v_1,v_2=1}^{NT} \frac{u_1 u_2 \overline{\chi}(u_1)\overline{\chi}(u_2)\chi(v_1)\chi(v_2)}{u_1 v_1 + u_2 v_2}.$$

Note that this converges, since $|\boldsymbol{k}| = 4$, $\ell = 2$ and $\frac{1}{2}\sum_{j=1}^{2}(\chi(-1)+1) = 2$, and $4-2-2 = 0 < 1$.

1.4 Pre-weak functions of higher degree and applications

In this section we study the situation when some assumptions on weak functions are weakened. On the one hand, we would like to allow "weak functions" to have poles of arbitrary degree. On the other hand, we will consider "weak functions" with poles that are irrational. This will provide us generalized Eisenstein series with an "infinite level". Furthermore, we can continue our construction principle for functions with interesting transformation properties on a higher level. Other applications are formulas for cotangent sums and, when considering negative weights, a formalism which describes the $k-1$-fold integral of ϑ_k explicitly in the case of finite levels.

1.4.1 Generalized Eisenstein series and generalized periodic L-functions

We denote the vector space of all generalized weak functions of degree 1 (this means, that only poles os degree 1 are allowed) by W_{weak}. I.e., each function $\omega \in W_{\text{weak}}$ has period 1, is meromorphic in $\mathbb{C}$ and of rapid decay as $|\text{Im}(z)| \to \infty$ and only has poles of degree at most 1 at real values.

We now call a 1-periodic *pre-weak*, if it has all properties of a weak function except that it is just bounded as $y \to \pm\infty$ in the strip $\{0 \leq x < 1\}$. In other words, we have the exact sequence

$$0 \longrightarrow W_{\text{weak}} \longrightarrow W_{\text{pre}} \overset{f\mapsto(f(-i\infty),f(i\infty))}{\longrightarrow} \mathbb{C}^2 \longrightarrow 0.$$

The subspaces $W_{\text{pre}}^{\pm i\infty} \subset W_{\text{pre}}$ contain all pre-weak functions that additionally vanish in $z = \pm i\infty$. All introduced notations for weak functions will also apply to pre-weak functions, if appropriate. Note that each $\omega \in W_{\text{pre}}$ also has a representation

$$\omega(z) = \omega(i\infty) + \sum_{x\in\mathbb{R}/\mathbb{Z}} \beta_\omega(x) h_x(z), \qquad h_x(z) = \frac{e(z)}{e(x)-e(z)},$$

where the sum is of course finite. Now consider the homomorphism

$$(\mathbb{R}/\mathbb{Z})^{\mathbb{C}_{0,0}} \longrightarrow \mathcal{O}(\{s \in \mathbb{C} \mid \sigma > 1\}) \tag{1.4.1.1}$$

$$\beta \longmapsto L(\beta; s) := \sum_{x\in\mathbb{R}_{>0}} \beta(x) x^{-s}.$$

The holomorphic functions on the right will be called *periodic L-functions* (since the input function lives on the 1-torus). We have the decomposition

$$L(\beta; s) = \sum_{x\in(0,1]} \beta(x)\zeta(s, x), \tag{1.4.1.2}$$

where

$$\zeta(s,x) := \sum_{n=0}^{\infty}(n+x)^{-s}, \qquad x>0,$$

is the Hurwitz zeta function. By analytic continuation we may consider the subspace $\frac{1}{s-1}\mathcal{O}(\mathbb{C}) \subset \mathcal{O}(\{s\in\mathbb{C} \mid \sigma>1\})$ for the image in (1.4.1.1). The residue map $\beta \longmapsto \mathrm{res}_{s=1}L(\beta;s)$ has kernel $(\mathbb{R}/\mathbb{Z})_0^{\mathbb{C}_{0,0}}$. In the case that β has support on $\frac{1}{N}\mathbb{Z}\setminus\mathbb{Z}$ for some N, we obtain an ordinary Dirichlet series with an exponential factor.

$$L(\beta;s) = N^s\sum_{n=1}^{\infty}\beta\left(\frac{n}{N}\right)n^{-s}.$$

The aim of this section is to associate periodic L-functions with generalized Eisenstein series that satisfy certain transformation properties. These Eisenstein series $E_k(\omega\otimes\eta;\tau)$ arise from (generalized) weak functions $\omega\otimes\eta$ with real (but not necessarily rational) poles. Since we are not able to assign ω and η a meaningful finite integer level in the case they have irrational poles, the functions $E_k(\omega\otimes\eta;\tau)$ will not be modular forms (except of course they identically vanish).

Definition 1.4.1. *We will use the notation $W^{\pm}$ to indicate the sub-spaces spanned by odd and even functions. What we need is the following: for $k\in\mathbb{Z}$ we define*

$$W_{(k)}^{\otimes} := \begin{cases} W_{\mathrm{weak}}\otimes W_{\mathrm{weak}}, & \text{if } k>0,\\ \left\langle W_{\mathrm{pre}}\otimes W_{\mathrm{weak}}, W_{\mathrm{weak}}\otimes W_{\mathrm{pre}}, W_{\mathrm{pre}}^{+}\otimes W_{\mathrm{pre}}^{-}, W_{\mathrm{pre}}^{-}\otimes W_{\mathrm{pre}}^{+}\right\rangle, & \text{if } k=0,\\ W_{\mathrm{pre}}\otimes W_{\mathrm{pre}}, & \text{if } k<0. \end{cases}$$

Also we use the notation $W_{(k)}^{\otimes}[\mathcal{T}_1,\mathcal{T}_2]$ to indicate, that the first and the second space are associated to the subsets $\mathcal{T}_1,\mathcal{T}_2\subset\mathbb{R}/\mathbb{Z}$, e.g. $W_{(1)}^{\otimes}[\mathcal{T}_N,\mathcal{T}_M] = W_{\mathrm{weak}}[\mathcal{T}_N]\otimes W_{\mathrm{weak}}[\mathcal{T}_M]$.

Consider the following linear map between pairs of pre-weak functions and holomorphic functions

$$\vartheta_k : V_k \longrightarrow \mathcal{O}(\mathbb{H}^+\cup\mathbb{H}^-),$$

$$\omega\otimes\eta \longmapsto -2\pi i\lim_{R\to\infty}\sum_{\substack{x\in\mathbb{R}^\times\\ |x|\le R}}\mathrm{res}_{z=x}\left(z^{k-1}\eta(z)\omega(z\tau)\right) =: \vartheta_k(\omega\otimes\eta;\tau). \tag{1.4.1.3}$$

We explain V_k by $V_k := W_{\mathrm{weak}}\otimes W_{\mathrm{pre}}$ if $k>0$ and $V_k := W_{(k)}^{\otimes}$, else. A proof that this is well-defined is given in Proposition 1.4.4.

Remark 1.4.2. *If one considers the decomposition $W = W^+\oplus W^-$ into even and odd functions, respectively, one can easily show by symmetry that $(W^+\otimes W^+)\oplus(W^-\otimes W^-)\subset\ker(\vartheta_k)$ if $k\equiv 1 \pmod 2$, and $(W^+\otimes W^-)\oplus(W^-\otimes W^+)\subset\ker(\vartheta_k)$, else. We use for elements $\omega\in W^{\pm}\setminus\{0\}$ the notation $\mathrm{sgn}(\omega)=\pm 1$.*

Note that $W_{(0)}^{\otimes}$ is also spanned by the spaces $W_{\mathrm{pre}}^{+}\otimes W_{\mathrm{pre}}^{-}$ and $W_{\mathrm{pre}}^{-}\otimes W_{\mathrm{pre}}^{+}$ that entirely map to the constant zero function by Remark 1.4.2. But we will still use this notation for formal reasons.

Remark 1.4.3. *With the still valid functional equation*

$$h_x(-z) = -1 - h_{-x}(z)$$

one easily sees that

$$\omega \in W_{\mathrm{weak}}^{\pm} = \left\{ \sum_{x\in\mathbb{R}/\mathbb{Z}} \beta_\omega(x) h_x(z) \;\middle|\; \beta_\omega(-x) = \mp\beta_\omega(x) \right\}.$$

Proposition 1.4.4. *The map ϑ_k is well-defined.*

Proof. Let $x \in \mathbb{R}^\times$ and $K \subset \mathbb{H}^+ \cup \mathbb{H}^-$ be a compact subset. Then we have the estimate

$$|\mathrm{res}_{z=x} z^{k-1}\eta(z)\omega(z\tau)| \leq \max_{\tau\in K} |\omega(\tau x)| \cdot |\mathrm{res}_{z=x}\eta(z)| \cdot |x|^{k-1}.$$

We distinguish three cases.

1. In the case $k > 0$ the claim now follows easily since then $\omega \in W_{\mathrm{weak}}$ and hence there is a $\delta > 0$ (depending on K and ω), such that

$$\max_{\tau\in K} |\omega(\tau x)| = O\left(e^{-\delta|x|}\right).$$

On the other hand, the term $|\mathrm{res}_{z=x}\eta(z)|$ is bounded since η is periodic.

2. If $k < 0$ it follows that

$$\left|\mathrm{res}_{z=x} z^{k-1}\eta(z)\omega(z\tau)\right| \leq C|x|^{k-1}$$

where the constant $C > 0$ may be chosen as

$$C = \max_{w\in\bigcup_{0\neq t\in S(\eta)} tK} |\omega(w)| \cdot \max_{\lambda\in[0,1]} |\mathrm{res}_{z=\lambda}\eta(z)|.$$

Since the sum $\sum_{x\in S(\eta)\setminus\{0\}} |x|^{-1-|k|}$ converges the claim follows.

3. In the case $k = 0$ we note that the map is defined on the subspace $W_{\mathrm{weak}} \otimes W_{\mathrm{pre}}$ by the arguments of 1. It is clearly defined for $W_{\mathrm{pre}}^{+} \otimes W_{\mathrm{pre}}^{-}$ and $W_{\mathrm{pre}}^{-} \otimes W_{\mathrm{pre}}^{+}$ since then all summands cancel each other. So we are left to show that we can define it on $W_{\mathrm{pre}} \otimes W_{\mathrm{weak}}$. Without loss of generality we assume that $\omega \otimes \eta \in W_{\mathrm{pre}}^{\pm} \otimes W_{\mathrm{weak}}^{\pm}$. First let both functions be even. Then $\omega = c + \omega_{\mathrm{w}}$ with some constant c and $\omega_{\mathrm{w}} \in W_{\mathrm{weak}}$. In conclusion, we only have to show that the sequence

$$S := -2\pi i c \lim_{R\to\infty} \sum_{\substack{x\in\mathbb{R}^\times \\ |x|\leq R}} \mathrm{res}_{z=x}\eta(z) z^{-1}$$

converges. Let $0 < x_1 < x_2 < x_3 < \cdots$ the sequence of all positive poles of η. With partial summation we obtain

$$\sum_{j=1}^{N} \beta_\eta(x_j) x_j^{-1} = \left(\sum_{j=1}^{N} \beta_\eta(x_j) \right) x_N^{-1} + \sum_{u=1}^{N-1} \left(\sum_{j=1}^{u} \beta_\eta(x_j) \right) (x_{u+1}^{-1} - x_u^{-1}). \qquad (1.4.1.4)$$

Since η is weak, the term $\sum_{j=1}^{N} \beta_\eta(x_j)$ is bounded and hence the right hand side converges as N tends to infinity. The odd case works similarly, since then we have $\omega = c\cot(\pi z) + \omega_{\mathrm{w}}$ and hence (since the cotangent function is odd)

$$\mathrm{res}_{z=x} z^{-1}\eta(z)\omega(\tau z) = ic\beta_\eta(x)|x|^{-1} + O(e^{-\delta x}), \qquad \delta > 0,$$

so we are reduced to proving convergence of a series analogous to (1.4.1.4). Finally, since in both cases we obtain homomorphisms that coincide on the common subspace $W_{\mathrm{weak}} \otimes W_{\mathrm{weak}}$ we may extend it to the resultant space $\langle W_{\mathrm{pre}} \otimes W_{\mathrm{weak}}, W_{\mathrm{weak}} \otimes W_{\mathrm{pre}} \rangle$. □

We now obtain the following very general transformation law.

Theorem 1.4.5 (see [28]). *Let $\omega \otimes \eta \in W_{(k)}^{\otimes}$, then we have for all $k \in \mathbb{Z}$ and $\tau \in \mathbb{H}$*

$$\vartheta_k \left(\omega \otimes \eta; -\frac{1}{\tau} \right) = \tau^k \vartheta_k \left(\eta \otimes -\widehat{\omega}; \tau \right) + 2\pi i \ \mathrm{res}_{z=0} \left(z^{k-1} \eta(z) \widehat{\omega} \left(\frac{z}{\tau} \right) \right). \qquad (1.4.1.5)$$

Here $\widehat{\omega}(z) = \omega(-z)$.

Proof. Let $y > 0$ and $\tau = iy \in \mathbb{H}$. Define

$$g_y(z) := -2\pi i z^{k-1} \eta(z) \widehat{\omega} \left(\frac{z}{iy} \right).$$

Then g_y is a meromorphic function in the plane with simple poles at $S(g_y) = S(\eta) \cup S(\omega)iy \setminus \{0\}$ (all lying on the real and imaginary axes). Consider the closed contour integrals

$$I_n(y) = \frac{1}{2\pi i} \oint_{R_n(y)} g_y(z) \mathrm{d}z,$$

where $R_n(y)$ is a sequence of rectangles that cross the axes half between the respective poles x_n and x_{n+1}. We are left to show $I_n(y) \overset{n\to\infty}{\longrightarrow} 0$ since then the claim follows with the identity and residue theorem. Using periodicity of η, ω and the decay of g_y we find that this will certainly be the case for $k \neq 0$. So we are left to show it for $k = 0$.
We first consider the case $\omega \otimes \eta \in W_{\mathrm{pre}}^{\pm} \otimes W_{\mathrm{pre}}^{\mp}$. Then the functions $\vartheta_0(\omega \otimes \eta)$ and $\vartheta_0(\eta \otimes -\widehat{\omega})$ are constant zero. Since the product $\omega(z/iy)\eta(z)/z$ is an even function in this case, its residue at $z = 0$ will be 0. Hence the transformation law is trivially satisfied in this case. Now let $\omega \in W_{\mathrm{weak}}$. Then the integrals on the right and the left in the rectangle will go

to zero because of the exponential decay of ω and the periodicity of η. So we can express $I_n(y)$ in the form

$$I_n(y) = \int_{\sigma_n+it_n}^{-\sigma_n+it_n} g_y(z)\mathrm{d}z + \int_{-\sigma_n-it_n}^{\sigma_n-it_n} g_y(z)\mathrm{d}z + o(1) \tag{1.4.1.6}$$

where $0 < \sigma_n \to \infty$ and $0 < t_n \to \infty$ are chosen in the sense of $R_n(y)$. Now we divide the integrals into three parts:

$$\int_{-\sigma_n-it_n}^{\sigma_n-it_n} g_y(z)\mathrm{d}z = \int_{-\sigma_n-it_n}^{-c\sqrt{n}-it_n} g_y(z)\mathrm{d}z + \int_{-c\sqrt{n}-it_n}^{c\sqrt{n}-it_n} g_y(z)\mathrm{d}z + \int_{c\sqrt{n}-it_n}^{\sigma_n-it_n} g_y(z)\mathrm{d}z.$$

Here, $c > 0$ is some fixed constant (note that $\sqrt{n} = o(\sigma_n)$). There is a constant $C > 0$ such that we have $|\eta(z)| \leq C$ for all $|\mathrm{Im}(z)| \geq 1$. Also on the segments $[-\sigma_n \pm it_n, \sigma_n \pm it_n]$ the function $\widehat{\omega}(z/yi)$ is uniformly bounded (with respect to $n = 1, 2, 3, ...$) by some $D > 0$ since it is periodic along the imaginary axes. Hence for sufficiently large n we obtain

$$\int_{-c\sqrt{n}-it_n}^{c\sqrt{n}-it_n} g_y(z)\mathrm{d}z \ll \frac{\sqrt{n}}{t_n} = o(1).$$

On the other hand, since $\widehat{\omega}(z/yi)$ is of rapid decay as $\mathrm{Re}(z) \to \pm\infty$ we have $|g_y(z)| = O(e^{-\delta|\mathrm{Re}(z)|})$ uniformly on $\{z \in \mathbb{C} \mid |\mathrm{Re}(z)| > 1, |\mathrm{Im}(z)| > 1\}$ for some $\delta > 0$. Hence the integrals

$$\int_{\pm 1-it_n}^{\pm\infty-it_n} g_y(z)\mathrm{d}z$$

will certainly converge absolutely and also

$$\int_{\pm c\sqrt{n}-it_n}^{\pm\sigma_n-it_n} g_y(z)\mathrm{d}z = o(1).$$

The first integral in (1.4.1.6) tends to zero by the same argumentation. The case $\omega \otimes \eta \in W_{\mathrm{pre}} \otimes W_{\mathrm{weak}}$ works analogously. This proves the transformation formula. □

Definition 1.4.6. *Let β be any function in $(\mathbb{R}/\mathbb{Z})^{\mathbb{C}_0}$. Then we define its Fourier transform $\mathcal{F}(\beta) : \mathbb{R} \to \mathbb{C}$ by*

$$\mathcal{F}(\beta)(y) = \sum_{x \in \mathbb{R}/\mathbb{Z}} \beta(x) e^{-2\pi i x y}.$$

Definition 1.4.7. *Let $k \geq 3$ be an integer and β, γ be functions in $(\mathbb{R}/\mathbb{Z})_0^{\mathbb{C}_0}$, such that $\mathrm{sgn}(\beta)\mathrm{sgn}(\gamma) = (-1)^k$. We assign these data an Eisenstein series by*

$$E_k(\beta, \gamma; \tau) := \sum_{t \in \mathbb{R}_{>0}} a_k(\beta, \gamma; t) q^t$$

with the coefficients

$$a_k(\beta, \gamma; t) := \sum_{\substack{d_1 \in \mathbb{R}_{>0} \\ d_2 \in \mathbb{N} \\ d_1 d_2 = t}} d_1^{k-1} \beta(d_1) \mathcal{F}(\gamma)(d_2).$$

In the cases $k = 2$ and $k = 1$ we have the same definition under the restrictions $\beta(0)\gamma(0) = 0$ and $\beta(0) = \gamma(0) = 0$, respectively.

Note that the (non-trivial) exponents in the above Fourier series can be irrational numbers too.

Theorem 1.4.8 (see [28])**.** *Let all assumptions hold as above. The generalized Eisenstein series satisfies the modular identity*

$$E_k\left(\beta, \gamma; -\frac{1}{\tau}\right) = \tau^k E_k(\gamma, -\widehat{\beta}; \tau).$$

Proof. We find

$$\begin{aligned} \vartheta_k(\omega \otimes \eta; \tau) &= 2 \sum_{\alpha \in \mathbb{R}_{>0}} \alpha^{k-1} \beta(\alpha) \sum_{x \in \mathbb{R}/\mathbb{Z}} \gamma(x) \frac{e(\alpha\tau)}{e(x) - e(\alpha\tau)} \\ &= 2 \sum_{\alpha \in \mathbb{R}_{>0}} \sum_{\nu=1}^{\infty} \alpha^{k-1} \beta(\alpha) \left(\sum_{x \in \mathbb{R}/\mathbb{Z}} \gamma(x) e(-\nu x) \right) q^{\alpha\nu} = 2E_k(\beta, \gamma; \tau). \end{aligned}$$

The claim now follows by Theorem 1.4.5. Note that in the case $k = 2$ at least one and in the case $k = 1$ both of the functions ω_β and η_γ have a removable singularity in $z = 0$, such that in every case the rational part in (1.4.1.5) vanishes. □

Analogous to ordinary Eisenstein series we can assign a generalized L-function to $E_k(\beta, \gamma; \tau)$. The result is a generalized Dirichlet series

$$\sum_{t \in D} a(t) t^{-s},$$

where $D \subset \mathbb{R}_{>0}$ is a discrete subset and $a : D \to \mathbb{C}$ a sequence of complex numbers. Like in the classical case one can show (for example by Mellin transform, using the transformation law of the Eisenstein series) that these L-functions have a meromorphic continuation to the entire plane and satisfy a functional equation of the standard type.

Proposition 1.4.9. *The generalized L-function associated to $E_k(\beta,\gamma;\tau)$ is given by*

$$L(E_k(\beta,\gamma);s) = L(\beta;s+1-k)\sum_{x\in\mathbb{R}/\mathbb{Z}}\gamma(x)\mathrm{Li}_s(e^{-2\pi ix}),$$

where $\mathrm{Li}_s(z)$ denotes the polylogarithm. It converges on $\{s\in\mathbb{C}|\mathrm{Re}(s)>k\}$ and has a meromorphic continuation to the entire plane.

Note that $L(\beta;s)$ represents a holomorphic function on $\{s\in\mathbb{C}|\mathrm{Re}(s)>1\}$ by (1.4.1.2) (β is 1-periodic and zero at all but finitely many points) and has a holomorphic continuation to $\mathbb{C}\setminus\{1\}$ with a possible simple pole in $s=1$.

Proof. Starting with Definition 1.4.7 we obtain

$$\sum_{t\in\mathbb{R}_{>0}}\left(\sum_{\substack{d_1\in\mathbb{R}_{>0}\\ d_2\in\mathbb{N}\\ d_1d_2=t}} d_1^{k-1}\beta(d_1)\mathcal{F}(\gamma)(d_2)\right)t^{-s} = \left(\sum_{t\in\mathbb{R}_{>0}}\beta(t)t^{-s+k-1}\right)\sum_{n=1}^{\infty}\mathcal{F}(\gamma)(n)n^{-s}.$$

The function γ is zero almost everywhere. Since by

$$|\mathcal{F}(\gamma)(n)| \le \sum_{x\in\mathbb{R}/\mathbb{Z}}|\gamma(x)|$$

its Fourier transform $\mathcal{F}(\gamma)(n)$ is bounded and hence the corresponding Dirichlet series converges absolutely on $\{s\in\mathbb{C}|\mathrm{Re}(s)>1\}$. We now have

$$\sum_{n=1}^{\infty}\mathcal{F}(\gamma)(n)n^{-s} = \sum_{n=1}^{\infty}\sum_{x\in\mathbb{R}/\mathbb{Z}}\gamma(x)e^{-2\pi inx}n^{-s} = \sum_{x\in\mathbb{R}/\mathbb{Z}}\gamma(x)\mathrm{Li}_s(e^{-2\pi ix}).$$

The claim follows with the analytic properties of $s\mapsto\mathrm{Li}_s(e^{-2\pi ix})$ and $L(\beta;s)$. □

In the next theorem we prove a functional equation for the completed L-function associated to a generalized Eisenstein series.

Theorem 1.4.10. *The completed L-function*

$$\Lambda(\beta,\gamma;s) := (2\pi)^{-s}\Gamma(s)L(E_k(\beta,\gamma);s)$$

extends to an entire function and satisfies the functional equation

$$\Lambda(E_k(\beta,\gamma);k-s) = \Lambda\left(E_k\left(\gamma,-\widehat{\beta}\right);s\right).$$

Proof. By Mellin transformation we obtain

$$\Lambda(\beta,\gamma;s)=\int_0^\infty E_k(\beta,\gamma;iy)y^{s-1}\mathrm{d}y.$$

By splitting the integral in the intervals $[0,1]$ and $[1,\infty)$ and making the substitution $y\mapsto y^{-1}$ in the first integral we obtain

$$\begin{aligned}\Lambda(\beta,\gamma;s)&=\int_1^\infty E_k\left(\beta,\gamma;\frac{i}{y}\right)y^{-s-1}\mathrm{d}y+\int_1^\infty E_k(\beta,\gamma;iy)y^{s-1}\mathrm{d}y\\&=\int_1^\infty E_k(\gamma,-\widehat{\beta};iy)y^{k-s-1}\mathrm{d}y+\int_1^\infty E_k(\beta,\gamma;iy)y^{s-1}\mathrm{d}y.\end{aligned}$$

From this one sees that $\Lambda(\beta,\gamma;s)$ is entire. The symmetry on the right hand side leads to the desired functional equation. □

1.4.2 Cotangent sums

Besides periodic L-functions we may associate other objects to a pre-weak function. For integers $m=1,2,3,...$ we define the corresponding *cotangent sum*

$$C(\omega;m):=\sum_{x\in\mathbb{R}/\mathbb{Z}}\beta_\omega(x)\cot^m(\pi x).$$

The primary goal of this section is to develop a principle which helps to write cotangent sums as rational combinations of L-functions, and vice versa. With this we may conclude several results about cotangent sums using well known results about L-functions, and of course vice versa again.

A famous example for a cotangent sum is given in [34] on p. 262:

$$\sum_{j=1}^{N-1}\cot^2\left(\frac{\pi j}{N}\right)=\frac{(N-1)(N-2)}{3},\qquad N=2,3,... \tag{1.4.2.1}$$

Note that the sum is always rational independent of the choice of N. This was generalized by Chu and Marini in [15] and Berndt and Yeap [4] on p. 6.

Theorem 1.4.11. *Let N and n be positive integers. Then*

$$\sum_{j=1}^{N-1}\cot^{2n}\left(\frac{\pi j}{N}\right)=(-1)^nN-(-1)^n2^{2n}\sum_{j_0=0}^{n}\left(\sum_{\substack{j_1,...,j_{2n}\geq 0\\ j_0+j_1+\cdots+j_{2n}=n}}\prod_{r=0}^{2n}\frac{B_{2j_r}}{(2j_r)!}\right)N^{2j_0}.$$

In particular, we have

$$\sum_{j=1}^{N-1} \cot^{2n}\left(\frac{\pi j}{N}\right) \in \mathbb{Q}.$$

Note that the B_n denote the *Bernoulli numbers* defined by generating series

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \frac{x}{e^x - 1}.$$

The interesting identity in Theorem 1.4.11 can be proved by looking at

$$f(z) = \cot^{2n}(\pi z) \cot(\pi k z)$$

and using contour integration. Another more general result is presented in [4] on p. 17 (there is a mistake in the original paper) and looks as follows.

Theorem 1.4.12. *For positive integers $0 < a < k$ and n let*

$$s_n(k,a) := \sum_{j=1}^{k-1} \sin\left(\frac{2\pi a j}{k}\right) \cot^n\left(\frac{\pi j}{k}\right)$$

and

$$c_n(k,a) := \sum_{j=1}^{k-1} \cos\left(\frac{2\pi a j}{k}\right) \cot^n\left(\frac{\pi j}{k}\right).$$

Then we have for all positive integers m

$$s_{2m-1}(k,a) = (-1)^m 2^{2m-1} \sum_{\substack{j_1,\ldots,j_{2m-1},\mu,\nu \geq 0 \\ 2j_1+\cdots+2j_{2m-1}+\mu+\nu=2m-1}} a^\mu k^\nu \frac{1}{\mu!} \frac{B_\nu}{\nu!} \prod_{r=1}^{2m-1} \frac{B_{2j_r}}{(2j_r)!} \qquad (1.4.2.2)$$

and

$$c_{2m}(k,a) = (-1)^{m+1} 2^{2m} \sum_{\substack{j_1,\ldots,j_{2m},\mu,\nu \geq 0 \\ 2j_1+\cdots+2j_{2m}+\nu+\mu=2m}} a^\mu k^\nu \frac{1}{\mu!} \frac{B_\nu}{\nu!} \prod_{r=1}^{2m} \frac{B_{2j_r}}{(2j_r)!}. \qquad (1.4.2.3)$$

In particular, both s_m and c_m define sequences of elements in $\mathbb{Q}[k,a]$.

In other words, the theories of generalized periodic L-functions and cotangent sums are in some way equivalent. To understand this, we modify the definition (1.4.1.1) of a periodic L-function in the following way. In the entire section we denote W^0_{pre} as the subspace of pre-weak functions that have a removable singularity in $z = 0$, which is

equivalent to $\beta_\omega(0) = 0$. Consider now the homomorphism between the space of pre-weak functions and an infinite tuple of complete L-values at positive integers

$$W^0_{\text{pre}} \longrightarrow \mathbb{C}^{\mathbb{N}}$$

$$\omega \longmapsto \left(\widetilde{L}(\omega;1), \widetilde{L}(\omega;2), ...\right), \qquad \widetilde{L}(\omega;k) := \sum_{x\in\mathbb{R}^\times} \beta_\omega(x)x^{-k}.$$

In the case $k = 1$, we interpret the sum as

$$\widetilde{L}(\omega;1) = \lim_{N\to\infty} \sum_{-N\le x\le N, x\neq 0} \beta_\omega(x)x^{-1} = \sum_{x>0}(\beta_\omega(x) - \beta_\omega(-x))x^{-1}. \tag{1.4.2.4}$$

Remark 1.4.13. *Note that by Remark 1.4.3* $\operatorname{sgn}(\omega) = (-1)^k$ *implies* $L(\omega;k) = 0$ *for* $k > 1$ *(an even pre-weak function is weak up to a constant and an odd up to a cotangent function). If* $k = 1$ *this relation still holds if we restrict to weak functions or odd* ω*.*

Before we move on, we define a sequence of numbers which is of great importance in combinatorics.

Definition 1.4.14. *Let* $n \in \mathbb{N}_0$ *and* $k \in \mathbb{Z}$*. We define the Stirling numbers of the second kind by*

$$\left\{ {n \atop k} \right\} := \frac{1}{k!}\sum_{j=0}^{k}(-1)^j \binom{k}{j}(k-j)^n, \qquad 0 \le k \le n,$$

where $\left\{ {0 \atop 0} \right\} := 1$ *and* $\left\{ {n \atop k} \right\} := 0$ *whenever* $k > n$ *or* $k < 0$*.*

Put

$$\Delta(\ell, u) := \binom{\ell}{u} - \binom{\ell}{u-1}$$

and

$$S^*(n,k) := k!\left\{ {n \atop k} \right\} = \sum_{j=0}^{k}(-1)^j \binom{k}{j}(k-j)^n, \qquad k \le n.$$

To find the connection between (generalized) L-functions and cotangent sums we need the following lemma.

Lemma 1.4.15. *Define a sequence* $\delta : \mathbb{N}_0^2 \to \mathbb{C}$ *by*

$$\delta_0(0) = \delta_1(0) = \delta_0(1) := 0,$$

and for integers $\nu, u \ge 0$ *with* $\nu + u \ge 2$*:*

$$\delta_\nu(u) := \frac{i^{\nu+u}}{(\nu-1)!}\sum_{\ell=u-1}^{\nu-1}(-1)^{\nu+\ell-u-1}2^{\nu-1-\ell}S^*(\nu-1,\ell)\Delta(\ell,u).$$

Let $a \in \mathbb{C} \setminus \mathbb{Z}$. Then we have in an arbitrary small neighborhood of $z = 0$

$$\cot(\pi(z-a)) = \sum_{\nu=0}^{\infty} P_\nu(\cot(\pi a)) z^\nu = -\cot(\pi a) + (-\pi - \pi \cot^2(\pi a))z + \cdots,$$

where

$$P_\nu(X) = \pi^\nu \sum_{u=0}^{\nu+1} \delta_{\nu+1}(u) X^u.$$

Remark 1.4.16.

(i) The first polynomials P_ν are given by

$$\begin{aligned}
P_0(X) &= -X, \\
P_1(X) &= -\pi - \pi X^2, \\
P_2(X) &= -\pi^2 X - \pi^2 X^3, \\
P_3(X) &= -\frac{\pi^3}{3} - \frac{4\pi^3}{3} X^2 - \pi^3 X^4, \\
P_4(X) &= -\frac{2\pi^4}{3} X - \frac{5\pi^4}{3} X^3 - \pi^4 X^5.
\end{aligned}$$

(ii) We have for all $\nu \geq 1$ the formulas

$$\delta_\nu(\nu) = -1 \tag{1.4.2.5}$$

and for all $\nu \geq 2$

$$\delta_\nu(0) = \frac{i^\nu}{(\nu-1)!} \sum_{\ell=0}^{\nu-1} (-1)^{\nu+\ell-1} 2^{\nu-1-\ell} S^*(\nu-1, \ell),$$

since then $\Delta(\ell, 0) = 1$.

(iii) It is $\delta_\nu(u) = 0$ if $u > \nu$. Since the function $\cot(x)$ is odd, we obtain $\delta_\nu(u) = 0$ if $\nu + u \equiv 1 \pmod 2$.

Proof. It is clear that the function $f(z) = \cot(\pi(z-a))$ is holomorphic in a neighborhood of $z = 0$ in the case $a \in \mathbb{C} \setminus \mathbb{Z}$. For the constant term we find

$$\cot(\pi(-a)) = -\cot(\pi a) = \pi^0 \left(\delta_1(0) + \delta_1(1) \cot(\pi a)\right),$$

and indeed this coefficient is

$$\delta_1(1) = i^2 \cdot (-1) \cdot 2^0 \cdot S^*(0,0) \cdot \left(\binom{0}{1} - \binom{0}{0} \right) = -1.$$

Using the formula in [42] on p. 2,

$$\cot^{(n)}(x) = (2i)^n(\cot(x) - i)\sum_{v=0}^{n} \frac{v!}{2^v} \left\{ {n \atop v} \right\} (i\cot(x) - 1)^v, \qquad n \geq 1,$$

(note that in the paper, the sum starts at $v = 1$ but we have $n \geq 1$, hence $\{ {n \atop 0} \} = 0$) and the Binomial theorem, for $\nu \geq 1$, we end up with

$$f^{(\nu)}(0) = -(-2\pi i)^\nu \sum_{\ell=0}^{\nu} \sum_{u=0}^{\ell+1} (\alpha_{\nu,\ell}(u-1) - i\alpha_{\nu,\ell}(u)) \cot^u(\pi a),$$

where

$$\alpha_{\nu,\ell}(u) := \frac{S^*(\nu,\ell)}{2^\ell} \binom{\ell}{u} (-1)^{\ell-u} i^u.$$

Put

$$b_\nu(\ell,u) := \alpha_{\nu,\ell}(u-1) - i\alpha_{\nu,\ell}(u) = \frac{S^*(\nu,\ell) i^{u-1}(-1)^{\ell-u}}{2^\ell} \left(\binom{\ell}{u} - \binom{\ell}{u-1} \right), \quad (1.4.2.6)$$

and note that this implies $b_\nu(-1,0) = 0$. With the additional summand $b_\nu(-1,0)$ we obtain

$$\sum_{\ell=0}^{\nu} \sum_{u=0}^{\ell+1} b_\nu(\ell,u) = \sum_{u=0}^{\nu+1} \sum_{\ell=u-1}^{\nu} b_\nu(\ell,u)$$

and conclude

$$\frac{f^{(\nu)}(0)}{\nu!} = -\frac{(-2\pi i)^\nu}{\nu!} \sum_{u=0}^{\nu+1} \sum_{\ell=u-1}^{\nu} b_\nu(\ell,u) \cot^u(\pi a).$$

Together with (1.4.2.6) this proves the formula for $\delta_\nu(u)$, after the index shift $\nu \mapsto \nu - 1$. □

We can use Lemma 1.4.15 to determine the local Taylor expansion of $\omega(z)$ at $z = 0$. This will later help to explain the relationship between periodic L-functions and cotangent sums.

Lemma 1.4.17. *Let $\omega \in W^0_{\mathrm{pre}}$. Then we have*

$$\omega(z) = \omega(i\infty) - \frac{1}{2}C(\omega;0) + \frac{i}{2} \sum_{\nu=0}^{\infty} \left(\sum_{u=0}^{\nu+1} \delta_{\nu+1}(u) C(\omega;u) \right) (z\pi)^\nu.$$

Proof. With the behavior of the function $\cot(\pi z)$ at $z = i\infty$ we obtain the following canonical representation of ω:

$$\omega(z) = \omega(i\infty) + \sum_{x \in \mathbb{R}/\mathbb{Z}} \beta_\omega(x) \left(\frac{i}{2} \cot\left(\pi\left(z - x\right)\right) - \frac{1}{2} \right),$$

and with Lemma 1.4.15 we obtain

$$\frac{i}{2}\sum_{x\in\mathbb{R}/\mathbb{Z}}\beta_\omega(x)\cot\left(\pi\left(z-x\right)\right)=\frac{i}{2}\sum_{\nu=0}^{\infty}\sum_{x\in\mathbb{R}/\mathbb{Z}}\beta_\omega(x)P_\nu\left(\cot\pi x\right)z^\nu$$
$$=\frac{i}{2}\sum_{\nu=0}^{\infty}\sum_{x\in\mathbb{R}/\mathbb{Z}}\beta_\omega(x)\pi^\nu\sum_{u=0}^{\nu+1}\delta_{\nu+1}(u)\cot^u\left(\pi x\right)z^\nu=\frac{i}{2}\sum_{\nu=0}^{\infty}\left(\sum_{u=0}^{\nu+1}\delta_{\nu+1}(u)C(\omega;u)\right)(z\pi)^\nu.$$

The claim now follows with some simple rearrangements. □

At this point we stress the simple but important fact, that the coefficients $\delta_\nu(u)$ are independent of the choice of ω.

Lemma 1.4.18 (Generalized Abel's theorem)**.** *Let $f_n:\mathbb{E}\cup\{1\}\to\mathbb{C}$ be a sequence of continuous functions that are holomorphic in the unit disc $\mathbb{E}$, such that $f_n(z)\to f(z)$ as $n\to\infty$ for all $z\in\mathbb{E}$. We assume that f is bounded on $[0,1]$ and put $D:=\sup_{0\le t\le 1}|f(t)|$. Let $\sum_{n=1}^{\infty}a(n)$ be a converging series and $F(z)=\sum_{n=1}^{\infty}a(n)f_n(z)$ be holomorphic in $\mathbb{E}$. Assume that the f_n satisfy the Abelian condition: there is a constant $C>0$ such that uniformly for all $n>0$ and all $0\le t\le 1$:*

$$|f_n(t)-f_{n+1}(t)|\le C(1-t)t^n.$$

Then we have

$$\lim_{t\to 1^-}\sum_{n=1}^{\infty}a(n)f_n(t)=f(1)\sum_{n=1}^{\infty}a(n).$$

Note that the important case $f_n(z)=z^n$ is *Abel's theorem.*

Proof. We show that for each $\varepsilon>0$ there is an N_0 such that for all $N\ge N_0$:

$$\sup_{0\le t\le 1}\left|\sum_{n>N}a(n)f_n(t)\right|\le\varepsilon. \tag{1.4.2.7}$$

Let $\varepsilon>0$. Choose $\delta>0$ such that $\max\{|f_1(1)|\delta,\delta(C+D)\}\le\varepsilon$. We choose an integer N such that if $A_n=\sum_{k=N+1}^{n}a(k)$, we have

$$\sup_{n>N}|A_n|\le\delta.$$

This is possible since the series $\sum_{n=1}^{\infty}a(n)$ converges. By partial summation we obtain with $f_n(z)\to f(z)$ and $0\le t<1$:

$$\left|\sum_{n>N}a(n)f_n(t)\right|=\left|A_\infty f(t)-\sum_{n>N}A_n(f_n(t)-f_{n+1}(t))\right|$$
$$\le\delta D+\delta C(1-t)\sum_{n>N}t^n\le\delta(C+D)\le\varepsilon.$$

On the other hand we have

$$\left|\sum_{n>N} a(n)f_n(1)\right| = |f(1)|\left|\sum_{n>N} a(n)\right| \leq |f(1)|\delta \leq \varepsilon.$$

From this follows (1.4.2.7) and we conclude the lemma. □

We consider the following special case.

Lemma 1.4.19. *Let g be holomorphic on $\mathbb{E}$ and a neighborhood U of $z = 1$. Then $f_n(z) := g(z^n)$ satisfies the assertions of Lemma 1.4.18.*

Proof. Let $0 < b < a < 1$. To see the lemma one uses the Cauchy integral formula

$$\frac{g(a) - g(b)}{a - b} = \frac{1}{2\pi i}\oint_\gamma \frac{g(z)}{(z-a)(z-b)}\mathrm{d}z,$$

where the closed and smooth integration path $\gamma \subset \mathbb{E} \cup U$ with length $\mathcal{L}(\gamma)$ surrounds the compact line $[0, 1]$ once in positive direction. We find a minimum distance $\varepsilon > 0$ between γ and $[0, 1]$. Hence

$$\left|\frac{1}{2\pi i}\oint_\gamma \frac{g(z)}{(z-a)(z-b)}\mathrm{d}z\right| \leq \frac{1}{2\pi}\max_{z\in\gamma}\left|\frac{g(z)}{(z-a)(z-b)}\right|\mathcal{L}(\gamma) \leq C\frac{\max_{z\in\gamma}|g(z)|}{\varepsilon^2},$$

where $C > 0$ is independent from a and b. Put $a = t^n$ and $b = t^{n+1}$ for $0 < t < 1$. Since $g(t^n)$ converges to $g(0)$ if $0 \leq t < 1$ and to $g(1)$ if $t = 1$, one has $D := \max\{|g(0)|, |g(1)|\}$. □

We are now in the position to prove a result that ties values of L-functions with Taylor coefficients of pre-weak functions.

Proposition 1.4.20. *Let $k \geq 1$ be an integer and $\omega\otimes\eta \in W_{\mathrm{pre}}\otimes W_{\mathrm{pre}}$ if $k > 1$ and $\omega\otimes\eta \in \left\langle W_{\mathrm{pre}}\otimes W_{\mathrm{weak}}, W^+_{\mathrm{pre}}\otimes W^-_{\mathrm{pre}}, W^-_{\mathrm{pre}}\otimes W^+_{\mathrm{pre}}\right\rangle$ else, such that ω has a removable singularity in $z = 0$. We then have*

$$\lim_{y\to 0^+}\vartheta_{1-k}(\omega\otimes\eta; iy) = \omega(0)\widetilde{L}(\eta; k). \tag{1.4.2.8}$$

In particular, for $\omega\otimes\eta \in W_{\mathrm{pre}}\otimes W_{\mathrm{pre}}$ (and $\omega\otimes\eta \in \left\langle W_{\mathrm{pre}}\otimes W_{\mathrm{weak}}, W^+_{\mathrm{pre}}\otimes W^-_{\mathrm{pre}}\right\rangle$ if $k = 1$) we have the key identity

$$\widetilde{L}(\eta; k) = 2\pi i\ \mathrm{res}_{z=0}\left(z^{-k}\eta(z)\right). \tag{1.4.2.9}$$

Proof. First we note that in the case $k = 1$ (1.4.2.8) is trivial for elements $\omega\otimes\eta$ in $W^{\pm}_{\mathrm{pre}}\otimes W^{\mp}_{\mathrm{pre}}$, since then both the left hand side and the right hand side are zero (note that either $\omega(0) = 0$ or $\widetilde{L}(\eta; 1) = 0$). Also if $\omega\otimes\eta \in W^+_{\mathrm{pre}}\otimes W^-_{\mathrm{pre}}$ both sides vanish according

to Remark 1.4.13 and since η is odd. So we can assume η to be weak in this case. We have $\omega(z) = R(e(z))$ with a rational function R, which fulfills the conditions of Lemma 1.4.19 (note that ω has a removable singularity in $z = 0$). We obtain:

$$\vartheta_{1-k}(\omega \otimes \eta; iy) = \sum_{\alpha>0} \alpha^{-k}\beta_\eta(\alpha)\omega(\alpha iy) + \sum_{\alpha>0}(-1)^k\alpha^{-k}\beta_\eta(-\alpha)\omega(-\alpha iy).$$

Since η is weak for $k = 1$ both series will converge for $y = 0$ separately. Hence with Lemma 1.4.18 we conclude

$$\lim_{y\to 0^+} \vartheta_{1-k}(\omega \otimes \eta; iy) = \omega(0)\widetilde{L}(\eta; k).$$

Note that we have a homeomorphism between the segments $[0, i\infty]$ and $[0, 1]$ given by $z \mapsto e^{2\pi iz}$. On the other hand, with Theorem 1.4.5 we obtain

$$\begin{aligned}\lim_{\tau\to 0} \vartheta_{1-k}(\omega \otimes \eta; \tau) &= \lim_{\tau\to 0}\left[(-\tau)^{k-1}\vartheta_{1-k}\left(\eta \otimes -\widehat{\omega}; -\frac{1}{\tau}\right) + 2\pi i \mathrm{res}_{z=0}\left(z^{-k}\eta(z)\omega(z\tau)\right)\right] \\ &= 2\pi i\omega(0)\mathrm{res}_{z=0}\left(z^{-k}\eta(z)\right).\end{aligned}$$

In the case of $k = 1$, the first term on the right side vanishes because η is weak. The choice $\omega = 1$ finally proves (1.4.2.9). □

Throughout our analysis of cotangent sums we assume the first component of the $W_{\mathrm{pre}} \otimes W_{\mathrm{pre}}$ to be the function which is constant 1. It is trivial but crucial that this function is even. Since we want to consider all values of completed L-functions simultaneously, we only look at elements $1 \otimes \omega \in \left\langle W^+_{\mathrm{pre}} \otimes W^0_{\mathrm{weak}}, W^+_{\mathrm{pre}} \otimes W^{0,-}_{\mathrm{pre}}\right\rangle$. In other words, throughout, ω it is an odd pre-weak function or weak function - both have a removable singularity in $z = 0$. Together with Lemma 1.4.17 we can now suggest closed formulas for cotangent sums in terms of corresponding L-functions at integer arguments.

Proposition 1.4.21. *Let $k \geq 1$ and $\omega \in \left\langle W^0_{\mathrm{weak}}, W^{0,-}_{\mathrm{pre}}\right\rangle$. We have the formula*

$$\widetilde{L}(\omega; k) = \sum_{\alpha\in\mathbb{R}^\times} \beta_\omega(\alpha)\alpha^{-k} = -\pi^k \sum_{n=0}^{k} \delta_k(n)C(\omega; n),$$

which is equivalent to

$$\widetilde{L}^*(\omega; k) := -\frac{\widetilde{L}(\omega; k)}{\pi^k} - \delta_k(0)C(\omega; 0) = \sum_{n=1}^{k} \delta_k(n)C(\omega; n). \tag{1.4.2.10}$$

Proof. First note that $\delta_1(0) = 0$. In the case ω is odd it is trivial that $L(\omega; 1) = 0 = C(\omega; 1)$, which proves the formula in this case. So let ω be weak if $k = 1$. With Lemma 1.4.17 we see that the residue of $z^{-k}\omega(z)$ in $z = 0$ is given by

$$\mathrm{res}_{z=0}\left(z^{-k}\omega(z)\right) = \frac{i}{2}\pi^{k-1}\sum_{u=0}^{k} \delta_k(u)C(\omega; u).$$

Multiplying by $2\pi i$ proves the claim when using (1.4.2.9). $\square$

Definition 1.4.22. *For Dirichlet characters χ modulo N we put*

$$C(\chi; m) := \sum_{j=1}^{N-1} \chi(j) \cot^m\left(\frac{j\pi}{N}\right).$$

Remark 1.4.23. *Let $k > 0$ be an integer. In [5] a relation between the class number h_K of the field $K = \mathbb{Q}(\sqrt{-k})$ and cotangent sums is proved. If χ is an odd (real) character for K, we have*

$$C(\chi; 1) = 2\sqrt{k} h_K.$$

The present method now gives a further viewpoint to this equation since by Proposition (1.4.2.10) *we have*

$$\widetilde{L}(\omega_\chi; 1) = -\pi\delta_1(1) C(\beta_{\omega_\chi}; 1)$$

and by the class number formula $L(\chi; 1)$ is directly tied to h_K. Here we have put

$$\omega_\chi(z) := \sum_{j=1}^{N-1} \chi(j) h_{\frac{j}{N}}(z),$$

where χ is a character modulo N.

Let Δ_∞ be the linear operator

$$\Delta_\infty : \prod_{n\in\mathbb{N}} \mathbb{R} \longrightarrow \prod_{n\in\mathbb{N}} \mathbb{R}$$

$$(a_1, a_2, a_3, ...)^T \longmapsto \left(\sum_{j=1}^{m} \delta_m(j) a_j\right)_{m\in\mathbb{N}}.$$

We can write this formally as an infinite lower triangular matrix:

$$\Delta_\infty := \begin{pmatrix} \delta_1(1) & 0 & 0 & 0 & 0 & \cdots \\ \delta_2(1) & \delta_2(2) & 0 & 0 & 0 & \cdots \\ \delta_3(1) & \delta_3(2) & \delta_3(3) & 0 & 0 & \cdots \\ \delta_4(1) & \delta_4(2) & \delta_4(3) & \delta_4(4) & 0 & \cdots \\ \delta_5(1) & \delta_5(2) & \delta_5(3) & \delta_5(4) & \delta_5(5) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.4.2.11}$$

Proposition 1.4.21 provides us a linear system with countable many unknowns. In other words, we can find values for the cotangent sums recursively. We obtain:

$$\begin{pmatrix} \widetilde{L}^*(\omega;1) \\ \widetilde{L}^*(\omega;2) \\ \widetilde{L}^*(\omega;3) \\ \widetilde{L}^*(\omega;4) \\ \widetilde{L}^*(\omega;5) \\ \vdots \end{pmatrix} = \Delta_\infty \begin{pmatrix} C(\omega;1) \\ C(\omega;2) \\ C(\omega;3) \\ C(\omega;4) \\ C(\omega;5) \\ \vdots \end{pmatrix}. \tag{1.4.2.12}$$

Note that in the case that ω is weak we have $\widetilde{L}^*(\omega;k) = -\pi^{-k}\widetilde{L}(\omega;k)$. With $\delta_\nu(\nu) = -1$ (see (1.4.2.5)) we see that the system (1.4.2.12) is invertible, since we have a lower diagonal operator. In other words, for all positive integers m we have

$$\Delta_m^{-1}\boldsymbol{L}_m(\omega) = \boldsymbol{C}_m(\omega), \tag{1.4.2.13}$$

where $\boldsymbol{L}_m(\omega)$ and $\boldsymbol{C}_m(\omega)$ denote the first m rows vectors of (1.4.2.12) and Δ_m the regular major $m \times m$ block of the operator. Note that since $\Delta_m \in \mathbb{Q}^{m\times m}$ we have $\Delta_m^{-1} \in \mathbb{Q}^{m\times m}$. Therefore we obtain the following theorem.

Theorem 1.4.24 (see [28])**.** *Let $\omega \in \langle W_{\text{weak}}^0, W_{\text{pre}}^{0,-}\rangle$ be a pre-weak function. Let $K|\mathbb{Q}$ be a field extension (not necessarily finite) and $m \in \mathbb{N}$ be any positive integer. Assume that $C(\omega;0) \in K$. Then we have*

$$\frac{\widetilde{L}(\omega;1)}{\pi}, \frac{\widetilde{L}(\omega;2)}{\pi^2}, \cdots, \frac{\widetilde{L}(\omega;m)}{\pi^m} \in K \iff C(\omega;1), C(\omega;2), \cdots, C(\omega;m) \in K.$$

Proof. As (1.4.2.12) proves, we can express the terms $\widetilde{L}(\omega;k)\pi^{-k} + C(\omega;0)\delta_k(0)$ as rational combinations of $C(\omega;m)$, $1 \leq m \leq k$ and vice versa the terms $C(\omega;k)$ as rational combinations of $\widetilde{L}(\omega;m)\pi^{-m} + C(\omega;0)\delta_m(0)$. Since $\delta_m(0) \in \mathbb{Q}$ for all $m \geq 0$, the claim follows with $C(\omega;0) \in K$. □

We see that it turns out that there is an arithmetic connection between cotangent sums and generalized L-functions. Together with Theorems 1.4.11 and 1.4.12 we are able to find explicit formulas. Here, the key ingredient is the fact that expressions like

$$\sum_{j=1}^{N-1} \cot^m\left(\frac{j\pi}{N}\right)$$

are polynomials $P_m(N)$ for fixed m. Compare Theorem 1.4.11. For the next theorem we need the Euler numbers E_n that are defined by the generating series

$$\frac{2}{e^z + e^{-z}} = \sum_{n=0}^{\infty} \frac{\mathrm{E}_n}{n!} z^n.$$

Theorem 1.4.25 (see [28]). *Let $k \geq 1$ and $\omega \in \left\langle W^0_{\mathrm{weak}}, W^{0,-}_{\mathrm{pre}} \right\rangle$.*

(i) There are rational numbers $\delta_k(\ell)$ (given in 1.4.15) and $\delta_k^*(\ell)$, independent from the choice of ω, such that

$$-\frac{\tilde{L}(\omega;k)}{\pi^k} - \delta_k(0)C(\omega;0) = \sum_{\ell=1}^{k} \delta_k(\ell)C(\omega;\ell) \tag{1.4.2.14}$$

and

$$C(\omega;k) = \sum_{\ell=1}^{k} \delta_k^*(\ell)\left(-\frac{\tilde{L}(\omega;\ell)}{\pi^\ell} - \delta_\ell(0)C(\omega;0)\right). \tag{1.4.2.15}$$

(ii) Explicitly, we obtain $\delta_\nu^*(u) = 0$ if $\nu + u \equiv 1 \pmod 2$ and for $0 < \ell \leq k$

$$\delta_{2k}^*(2\ell) = (-1)^{k+\ell+1}2^{2k-2\ell} \sum_{\substack{j_1,\ldots,j_{2k}\geq 0 \\ \ell+j_1+\cdots+j_{2k}=k}} \prod_{r=1}^{2k} \frac{B_{2j_r}}{(2j_r)!} \tag{1.4.2.16}$$

and

$$\delta_{2k-1}^*(2\ell-1) = (-1)^{k+\ell+1}2^{2k-2\ell} \sum_{\substack{j_1,\ldots,j_{2k-1}\geq 0 \\ 2\ell-1+2j_1+\cdots+2j_{2k-1}=2k-1}} \prod_{r=1}^{2k-1} \frac{B_{2j_r}}{(2j_r)!}. \tag{1.4.2.17}$$

(iii) (Supplementary laws) We have for all positive integers k

$$(1) \qquad \sum_{\ell=1}^{k} \delta_{2k}^*(2\ell)\delta_{2\ell}(0) = (-1)^{k-1},$$

$$(2) \qquad \sum_{\ell=1}^{k} \delta_{2k}^*(2\ell)\zeta(2\ell)\pi^{-2\ell} = \frac{(-1)^k}{2}\left(1 - 2^{2k} \sum_{\substack{j_1,\ldots,j_{2k}\geq 0 \\ j_1+\cdots+j_{2k}=k}} \prod_{r=1}^{2k} \frac{B_{2j_r}}{(2j_r)!}\right).$$

Remark 1.4.26. *Supplementary law (1) reduces* (1.4.2.15) *to the formula*

$$C(\omega;k) + i^k\frac{1+(-1)^k}{2}C(\omega;0) = -\sum_{\ell=1}^{k} \delta_k^*(\ell)\tilde{L}(\omega;\ell)\pi^{-\ell}. \tag{1.4.2.18}$$

Proof.

(i) The formula (1.4.2.14) follows from Proposition 1.4.21. Let $k \leq m$ be arbitrarily chosen. Formula (1.4.2.15) follows with (1.4.2.13) and the fact that $\Delta_m^{-1} \in \mathbb{Q}^{m\times m}$ is again a lower triangular matrix, when denoting its coefficients by $\delta_\nu^*(u)$ (analogously as it was done in (1.4.2.11)). It is clear that all values $\delta_\nu^*(u)$ are independent of m and ω.

(ii) We first show by induction that for $\nu, u \geq 1$ the $\delta_\nu^*(u)$ vanish if $\nu + u \equiv 1 \pmod 2$. This is clear for $\nu < u$, so we assume that $u \leq \nu$. Obviously, with the vanishing of the above triangle in mind, the statement is equivalent to the vanishing of all "odd" lower diagonals

$$\begin{aligned} D_1 &:= (\delta_\nu^*(\nu-1))_{\nu=2,3,\dots} \\ D_3 &:= (\delta_\nu^*(\nu-3))_{\nu=4,5,\dots} \\ &\vdots \\ D_{2k-1} &:= (\delta_\nu^*(\nu-2k+1))_{\nu=2k,2k+1,\dots} \\ &\vdots \end{aligned}$$

We formally write $\Delta_\infty^{-1}\Delta_\infty = I_\infty$. First we show the vanishing of D_1. Let $\nu \geq 2$. Then we obtain, multiplying the ν-th row of the operator Δ_∞^{-1} with the $\nu-1$-th column of Δ_∞:

$$\sum_{u=1}^{\infty} \delta_\nu^*(u)\delta_u(\nu-1) = \sum_{u=\nu-1}^{\nu} \delta_\nu^*(u)\delta_u(\nu-1) = \delta_\nu^*(\nu-1)\delta_{\nu-1}(\nu-1) = 0.$$

Hence $\delta_\nu^*(\nu-1) = 0$, since $\delta_{\nu-1}(\nu-1) = -1$ (note that $\delta_\nu(\nu-1) = 0$ – remember that $\delta_\nu(u) = 0$ if $\nu + u \equiv 1 \pmod 2$ by Remark 1.4.16 (iii)). Note that the sum could be reduced to two summands in the first step since we have multiplied two lower diagonal operators. For the induction step, we assume that we have proved vanishing for $D_1, D_3, ..., D_{2k-1}$. We show that under these circumstances we obtain the vanishing of D_{2k+1}. Let $\nu \geq 2k+2$, and multiply the ν-th row of Δ_∞^{-1} with the $\nu-2k-1$-th column of Δ_∞.

$$\sum_{u=1}^{\infty} \delta_\nu^*(u)\delta_u(\nu-2k-1) = \sum_{u=\nu-2k-1}^{\nu} \delta_\nu^*(u)\delta_u(\nu-2k-1) = 0. \qquad (1.4.2.19)$$

If $\nu - 2k \leq u \leq \nu$ is of the form $u = \nu - 2\ell$ for an integer ℓ, we have $\delta_\nu^*(u)\delta_u(\nu - 2k-1) = 0$ since $\delta_{\nu-2\ell}(\nu-2k-1) = 0$. Otherwise, if $u = \nu - 2\ell + 1$, we also have $\delta_\nu^*(u)\delta_u(\nu-2k-1) = 0$ since then $\delta_\nu^*(\nu-2\ell+1) = 0$ by assumption since $\ell \leq k$. Hence, (1.4.2.19) reduces to

$$\delta_\nu^*(\nu-2k-1)\delta_{\nu-2k-1}(\nu-2k-1) = 0.$$

Since $\delta_{\nu-2k-1}(\nu-2k-1) = -1$, we obtain $\delta_\nu^*(\nu-2k-1) = 0$.

To obtain the coefficients δ^* explicitly, we could of course simply use invert the operator Δ_∞, which would not be too bad, since all of its finite "blocks" are lower diagonal with determinant ± 1. However, there is even a quicker trick that uses a small subset of cotangent sums that are polynomials in the "period" variable N.

To prove the formula (1.4.2.16) for $\delta^*_{2k}(2\ell)$ with $1 \le \ell \le k$ choose

$$\omega_N(z) := \sum_{j=1}^{N-1} h_{\frac{j}{N}}(z) = \frac{i}{2}\left(N\cot(N\pi z) - \cot(\pi z)\right),$$

where $N > 1$ is a positive integer. A brief calculation shows $\omega_N \in W^{0,-}_{\mathrm{pre}}$. We have for integers $k > 0$

$$\widetilde{L}(\omega_N; k) = \sum_{r \not\equiv 0 \pmod N} \left(\frac{r}{N}\right)^{-k} = \begin{cases} 2\zeta(k)(N^k - 1), & \text{if } k \equiv 0 \pmod 2, \\ 0, & \text{else,} \end{cases}$$

and for $k = 1$ the right sum is understood as in (1.4.2.4). Since ω_N is not weak, we have to include the terms $C(\beta_{\omega_N}; 0) = N - 1$. From (1.4.2.12) and 1.4.11 we conclude for all even positive integers $2k$

$$\sum_{\ell=1}^{k} \delta^*_{2k}(2\ell)\pi^{-2\ell}\left(-2\zeta(2\ell)(N^{2\ell} - 1) - \pi^{2\ell}\delta_{2\ell}(0)(N-1)\right) \tag{1.4.2.20}$$

$$= (-1)^k N - (-1)^k 2^{2k} \sum_{j_0=0}^{k} \left(\sum_{\substack{j_1,\dots,j_{2k} \ge 0 \\ j_0 + j_1 + \cdots + j_{2k} = k}} \prod_{r=0}^{2k} \frac{B_{2j_r}}{(2j_r)!} \right) N^{2j_0}.$$

Both sides are a polynomial in N and since this identity is valid for all $N > 1$, we obtain

$$-2\delta^*_{2k}(2\ell)\zeta(2\ell)\pi^{-2\ell} = -(-1)^k 2^{2k} \frac{B_{2\ell}}{(2\ell)!} \sum_{\substack{j_1,\dots,j_{2k} \ge 0 \\ \ell + j_1 + \cdots + j_{2k} = k}} \prod_{r=1}^{2k} \frac{B_{2j_r}}{(2j_r)!}$$

by comparing coefficients. Note that by the classical result

$$\zeta(2\ell) = (-1)^{\ell-1} \frac{(2\pi)^{2\ell} B_{2\ell}}{2(2\ell)!}, \qquad \ell = 1, 2, 3, \dots,$$

this is equivalent to

$$\delta^*_{2k}(2\ell) 2^{2\ell} (-1)^{\ell} \frac{B_{2\ell}}{(2\ell)!} = (-1)^{k+1} 2^{2k} \frac{B_{2\ell}}{(2\ell)!} \sum_{\substack{j_1,\dots,j_{2k} \ge 0 \\ \ell + j_1 + \cdots + j_{2k} = k}} \prod_{r=1}^{2k} \frac{B_{2j_r}}{(2j_r)!}$$

and with $B_{2\ell} \neq 0$ formula (1.4.2.16) follows easily.

The proof of formula (1.4.2.17) works similar. Take a positive integer $N \equiv 0 \pmod 4$, set $a = \frac{N}{4}$ and

$$\eta_N(z) := \sum_{j=1}^{N-1} \sin\left(\frac{\pi j}{2}\right) h_{\frac{j}{N}}(z) = \sum_{j=1}^{N-1} \chi_4(j) h_{\frac{j}{N}}(z),$$

where χ_4 is the non-principal character modulo 4. Clearly η_N is weak with level N. Together with (1.4.2.12) and (1.4.2.2) we obtain for positive integers $2k-1$

$$-2\sum_{\ell=1}^{k} \delta^*_{2k-1}(2\ell-1)\pi^{1-2\ell} L(\chi_4; 2\ell-1) N^{2\ell-1} = \sum_{j=1}^{N-1} \sin\left(\frac{\pi j}{2}\right) \cot^{2k-1}\left(\frac{\pi j}{N}\right)$$

$$= (-1)^k 2^{2k-1} \sum_{\substack{j_1,\ldots,j_{2k-1},\mu,\nu \geq 0 \\ 2j_1+\cdots+2j_{2k-1}+\mu+\nu=2k-1}} \left(\frac{N}{4}\right)^{\mu} N^{\nu} \frac{1}{\mu!} \frac{B_\nu}{\nu!} \prod_{r=1}^{2k-1} \frac{B_{2j_r}}{(2j_r)!}$$

$$= (-1)^k 2^{2k-1} \sum_{\substack{j_0,j_1,\ldots,j_{2k-1} \geq 0 \\ j_0+2j_1+\cdots+2j_{2k-1}=2k-1}} \sum_{a=0}^{j_0} \frac{B_a 4^{a-j_0}}{(j_0-a)!a!} \prod_{r=1}^{2k-1} \frac{B_{2j_r}}{(2j_r)!} N^{j_0}.$$

Using the classical formula

$$L(\chi_4; 2\ell-1) = (-1)^{\ell-1} \frac{\mathrm{E}_{2\ell-2} \pi^{2\ell-1}}{4^\ell (2\ell-2)!}, \qquad \ell = 1, 2, 3, \ldots,$$

we obtain by comparing coefficients:

$$\frac{(-1)^\ell \delta^*_{2k-1}(2\ell-1)\mathrm{E}_{2\ell-2}}{2^{2\ell-1}(2\ell-2)!} = (-1)^k 2^{2k-1} 2^{2-4\ell} S_{2\ell-1} \sum_{\substack{j_1,\ldots,j_{2k-1} \geq 0 \\ 2\ell-1+2j_1+\cdots+2j_{2k-1}=2k-1}} \prod_{r=1}^{2k-1} \frac{B_{2j_r}}{(2j_r)!}$$

with

$$S_{2\ell-1} := \sum_{a=0}^{2\ell-1} \frac{B_a 4^a}{(2\ell-1-a)!a!}.$$

The identity

$$S_{2\ell-1} = -\frac{\mathrm{E}_{2\ell-2}}{(2\ell-2)!}$$

follows with the fact that

$$\left(\sum_{m=0}^{\infty} \frac{B_m 4^m}{m!} x^m\right)\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) + x\sum_{p=0}^{\infty} \frac{\mathrm{E}_{2p}}{(2p)!} x^{2p} = \frac{4xe^x}{e^{4x}-1} + \frac{2xe^x}{e^{2x}+1} = \frac{2x}{e^x - e^{-x}}$$

is an even function. The formula now follows after a simple rearrangement.

(iii) Looking again at (1.4.2.20) we obtain by comparing the coefficients belonging to N:

$$-\sum_{\ell=1}^{k}\delta_{2k}^{*}(2\ell)\delta_{2\ell}(0)=(-1)^{k}=-i^{2k}.$$

This proves supplementary law (1). On the other hand, making this comparison for the constant terms we find

$$2\sum_{\ell=1}^{k}\delta_{2k}^{*}(2\ell)\zeta(2\ell)\pi^{-2\ell}+\sum_{\ell=1}^{k}\delta_{2k}^{*}(2\ell)\delta_{2\ell}(0)=-(-1)^{k}2^{2k}\sum_{\substack{j_1,\dots,j_{2k}\geq 0\\ j_1+\cdots+j_{2k}=k}}\prod_{r=1}^{2k}\frac{B_{2j_r}}{(2j_r)!}$$

and using supplementary law (1) we immediately see (2).

This completes the proof. □

It is clear by the vanishing of δ^* and δ for arguments $\nu+u\equiv 1 \pmod 2$ that

$$\sum_{\ell=1}^{2k-1}\delta_{2k-1}^{*}(\ell)\delta_{\ell}(0)=0.$$

Hence

$$\sum_{\ell=1}^{k}\delta_{k}^{*}(\ell)\delta_{\ell}(0)=i^{k}\frac{1+(-1)^{k}}{2}$$

and with (1.4.2.15) we obtain (1.4.2.18).

We want to apply these theorems to make statements about cotangent sums using L-functions. What we need is the following classical result due to Leopoldt.

Theorem 1.4.27. *Let χ be a primitive character modulo N and k be a positive integer. Put $\delta:=\frac{1-\chi(-1)}{2}$. If $k\equiv\delta \pmod 2$, then*

$$L(\chi;k)=(-1)^{1+\frac{k-\delta}{2}}\frac{\mathcal{G}(\chi)}{2i^{\delta}}\frac{B_{k,\overline{\chi}}}{k!}\left(\frac{2\pi}{N}\right)^{k}.$$

Here the numbers $B_{k,\overline{\chi}}$ are the generalized Bernoulli numbers defined by the identity

$$\sum_{a=1}^{N}\frac{\chi(a)ze^{az}}{e^{Nz}-1}=\sum_{n=0}^{\infty}\frac{B_{n,\chi}}{n!}z^{n}.$$

Remark 1.4.28. *Let χ be a character modulo N. Note that we can express $B_{n,\chi}$ in terms of the standard Bernoulli numbers by the formula*

$$B_{n,\chi}=\sum_{j=1}^{N-1}\chi(j)\sum_{u=0}^{n}\binom{n}{u}B_{u}j^{n-u}N^{u-1}. \tag{1.4.2.21}$$

It follows that if χ is real we have $B_{n,\chi}\in\mathbb{Q}$.

We can use this to determine a closed formula for the character cotangent sums

$$C(\chi;m):=\sum_{j=1}^{N-1}\chi(j)\cot^m\left(\frac{\pi j}{N}\right).$$

Corollary 1.4.29. *Let χ^+ be an even and χ^- be an odd primitive character modulo $N>1$ and $m\geq 1$ be an integer. We have the explicit formulas*

$$C(\chi^+;2m)=\mathcal{G}(\chi^+)\sum_{\ell=1}^{m}(-1)^\ell 2^{2\ell}\delta^*_{2m}(2\ell)\frac{B_{2\ell,\overline{\chi^+}}}{(2\ell)!},\tag{1.4.2.22}$$

and

$$C(\chi^-;2m-1)=i\mathcal{G}(\chi^-)\sum_{\ell=1}^{m}(-1)^\ell 2^{2\ell-1}\delta^*_{2m-1}(2\ell-1)\frac{B_{2\ell-1,\overline{\chi^-}}}{(2\ell-1)!}.\tag{1.4.2.23}$$

In particular, independently of m, one has

$$\mathcal{G}(\chi^+)^{-1}C(\chi^+;2m)\in\mathbb{Q}(\chi^+(g_1),...,\chi^+(g_t))\subset\mathbb{Q}(e^{\frac{2\pi i}{\varphi(N)}})\tag{1.4.2.24}$$

and

$$i\mathcal{G}(\chi^-)^{-1}C(\chi^-;2m-1)\in\mathbb{Q}(\chi^-(g_1),...,\chi^-(g_t))\subset\mathbb{Q}(e^{\frac{2\pi i}{\varphi(N)}})\tag{1.4.2.25}$$

respectively, where the integers $g_1,...,g_t$ modulo N are generators of $\mathbb{F}_N^\times$ and $\varphi(N)$ is Euler's totient function.

Proof. Define

$$\omega_{\chi^\pm}(z):=\sum_{j=1}^{N-1}\chi^\pm(j)h_{\frac{j}{N}}(z).$$

Then $\omega_{\chi^\pm}(z)$ is weak and hence $C(\omega_{\chi^\pm};0)=0$. By Theorem 1.4.27 one obtains

$$\widetilde{L}(\omega_{\chi^+};2\ell)=(-1)^{\ell+1}\mathcal{G}(\chi)\frac{B_{2\ell,\overline{\chi^+}}}{(2\ell)!}(2\pi)^{2\ell}$$

and similarly

$$\widetilde{L}(\omega_{\chi^-};2\ell-1)=(-1)^{\ell+1}i\mathcal{G}(\chi)\frac{B_{2\ell-1,\overline{\chi^-}}}{(2\ell-1)!}(2\pi)^{2\ell-1}.$$

Note that we obtain an additionally factor 2 (by symmetry) and $N^{2\ell}$ and $N^{2\ell-1}$ (by the residues), respectively, in this calculation. The formulas (1.4.2.22) and (1.4.2.23) now follow with Theorem 1.4.25.
To see (1.4.2.24) and (1.4.2.25) we first note that the right inclusions follow from $g_j^{\varphi(N)}\equiv 1 \pmod N$. By (1.4.2.21) we see $B_{n,\overline{\chi}}\in\mathbb{Q}(\chi(g_1),...,\chi(g_t))$ and with (1.4.2.22) and (1.4.2.23) we are done. □

Corollary 1.4.30. *Let p be a prime and χ be the Legendre symbol modulo p. Then we have for all $m \in \mathbb{N}$*

$$\sqrt{p}C(\chi; m) \in \mathbb{Q}.$$

Proof. For the Legendre symbol χ we have the identity

$$\mathcal{G}(\chi) = \begin{cases} \sqrt{p}, & \text{if } p \equiv 1 \pmod 4 \\ i\sqrt{p}, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Since χ is real, it is rational, and the claim follows with Corollary 1.4.29. □

There has been lots of effort finding closed values for Gauss sums. The reader may wish to consult for example [40] for an elementary overview.

Example 1.4.31. *With Mathematica we obtain the identities*

$$\cot^2\left(\frac{\pi}{5}\right) - \cot^2\left(\frac{2\pi}{5}\right) - \cot^2\left(\frac{3\pi}{5}\right) + \cot^2\left(\frac{4\pi}{5}\right) = \frac{8}{\sqrt{5}},$$

$$\cot^6\left(\frac{\pi}{13}\right) - \cot^6\left(\frac{2\pi}{13}\right) + \cot^6\left(\frac{3\pi}{13}\right) + \cot^6\left(\frac{4\pi}{13}\right) - \cot^6\left(\frac{5\pi}{13}\right) + \cot^6\left(\frac{6\pi}{13}\right) - \cot^6\left(\frac{7\pi}{13}\right)$$
$$- \cot^6\left(\frac{8\pi}{13}\right) + \cot^6\left(\frac{9\pi}{13}\right) + \cot^6\left(\frac{10\pi}{13}\right) - \cot^6\left(\frac{11\pi}{13}\right) + \cot^6\left(\frac{12\pi}{13}\right) = \frac{31832}{\sqrt{13}},$$

and

$$\cot^{13}\left(\frac{\pi}{7}\right) + \cot^{13}\left(\frac{2\pi}{7}\right) - \cot^{13}\left(\frac{3\pi}{7}\right) + \cot^{13}\left(\frac{4\pi}{7}\right) - \cot^{13}\left(\frac{5\pi}{7}\right) - \cot^{13}\left(\frac{6\pi}{7}\right) = \frac{494370}{49\sqrt{7}}.$$

Also we can use the results about cotangent sums to derive properties about L-functions having trigonometric coefficients.

Corollary 1.4.32. *Let $\widetilde{\cot} = \cot$ except $\widetilde{\cot}(\pi n) := 0$ for all $n \in \mathbb{Z}$. Let $N > a \geq 1$ and $n_1, n_2, n_3 \geq 0$ be integers such that $n_1 n_2 = 0$. We then have for $k \geq 1$ with $n_1 + n_3 \equiv k \pmod 2$:*

$$\sum_{n=1}^{\infty} \frac{\sin^{n_1}\left(\frac{2\pi an}{N}\right) \cos^{n_2}\left(\frac{2\pi an}{N}\right) \widetilde{\cot}^{n_3}\left(\frac{\pi n}{N}\right)}{n^k} \in \mathbb{Q}\pi^k.$$

Proof. The condition $n_1 + n_3 \equiv r \pmod 2$ implies that the coefficients (when extended to $\mathbb{Z}$) define an even/odd function if and only if r is even/odd. The result now follows with the well-known expressions for $\sin^n$ and $\cos^n$ in terms of linear combinations of multiple arguments sin and cos functions and Theorems 1.4.12 and 1.4.24. □

Remark 1.4.33. *Again, using Theorems 1.4.12 and 1.4.25, one can find rather complicated explicit formulas for the above Dirichlet series in terms of the values $\delta_\nu(u)$.*

We can use this formalism to give a purely Fourier analytic proof for Theorem 1.4.12. Remember the modified *Clausen function*

$$\mathrm{Sl}_{2k-1}(\theta) := \sum_{n=1}^{\infty} \frac{\sin(2\pi\theta n)}{n^{2k-1}}$$

and

$$\mathrm{Sl}_{2k}(\theta) := \sum_{n=1}^{\infty} \frac{\cos(2\pi\theta n)}{n^{2k}}.$$

Using standard Fourier analysis one obtains for $0 \leq \theta < 1$:

$$\mathrm{Sl}_{2k-1}(\theta) = \frac{(-1)^k (2\pi)^{2k-1}}{2(2k-1)!} \sum_{j=0}^{2k-1} \binom{2k-1}{j} B_j \theta^{2k-1-j} \tag{1.4.2.26}$$

and

$$\mathrm{Sl}_{2k}(\theta) = \frac{(-1)^{k-1} (2\pi)^{2k}}{2(2k)!} \sum_{j=0}^{2k} \binom{2k}{j} B_j \theta^{2k-j}. \tag{1.4.2.27}$$

For a proof see also [16] on p. 16–17. We can now use Theorem 1.4.25 to find the closed formulas provided in Theorem 1.4.12. To see this, put $\theta = \frac{a}{k}$ for $0 < a < k$. Consider the function

$$\omega(z) = \sum_{j=1}^{k-1} \cos\left(\frac{2\pi a j}{k}\right) h_{\frac{j}{k}}(z)$$

which lies in $W_{\mathrm{pre}}^{0,-}$. Then, we have for even values $2\ell > 0$

$$\widetilde{L}(\omega; 2\ell) = \sum_{u \not\equiv 0 \pmod{k}} \cos\left(\frac{2\pi a u}{k}\right) \left(\frac{u}{k}\right)^{-2\ell} = 2\sum_{u=1}^{\infty} \cos\left(\frac{2\pi a u}{k}\right) \left(\frac{u}{k}\right)^{-2\ell} - 2\sum_{u=1}^{\infty} u^{-2\ell}$$

and by (1.4.2.27) this equals to

$$-2\zeta(2\ell) + \frac{(-1)^{\ell-1} (2\pi k)^{2\ell}}{(2\ell)!} \sum_{j=0}^{2\ell} \binom{2\ell}{j} B_j \left(\frac{a}{k}\right)^{2\ell-j}.$$

For odd values $2\ell - 1 > 0$ we find $\widetilde{L}(\omega; 2\ell - 1) = 0$. Note also that the sum $C(\omega; 0) = -1$ for obvious reasons. Hence, with Theorem 1.4.25 we find

$$C(\omega; 2m) + (-1)^m = -\sum_{\ell=1}^{m} \delta_{2m}^*(2\ell) \widetilde{L}(\omega; 2\ell) \pi^{-2\ell}. \tag{1.4.2.28}$$

By supplementary law (2) we have

$$2\sum_{\ell=1}^{m}\delta_{2m}^{*}(2\ell)\zeta(2\ell)\pi^{-2\ell}=(-1)^{m}+(-1)^{m+1}2^{2m}\sum_{\substack{j_1,\dots,j_{2m}\geq 0\\ 2j_1+\cdots+2j_{2m}=2m}}\prod_{r=1}^{2m}\frac{B_{2j_r}}{(2j_r)!}. \quad (1.4.2.29)$$

On the other hand, a straightforward calculation shows

$$\begin{aligned}&-\delta_{2m}^{*}(2\ell)\frac{(-1)^{\ell-1}(2\pi k)^{2\ell}}{(2\ell)!}\sum_{j=0}^{2\ell}\binom{2\ell}{j}B_j\left(\frac{a}{k}\right)^{2\ell-j}\\ &=(-1)^{m+1}2^{2m}\sum_{\mu=0}^{2\ell}\sum_{\substack{j_1,\dots,j_{2m}\geq 0\\ 2\ell+2j_1+\cdots+2j_{2m}=2m}}\frac{a^{2\ell-\mu}k^{\mu}B_{\mu}}{(2\ell-\mu)!\mu!}\prod_{r=1}^{2m}\frac{B_{2j_r}}{(2j_r)!}.\end{aligned}$$

The cosine formula of Theorem 1.4.12 follows now by summing this over $\ell=1,\dots,m$, making the substitution $2\ell=\nu+\mu$ and adding everything together. Note that the $(-1)^m$ in (1.4.2.28) will cancel with that of (1.4.2.29) and that the formula (1.4.2.29) is just the case $2\ell=\mu+\nu=0$, completing the sum in (1.4.2.3). Similarly, we can show the sine formula (1.4.2.2) in full generality.

Remark 1.4.34. *Note that we only have used the polynomials P_m and Q_m defined by*

$$P_m(N)=\sum_{j=1}^{N-1}\cot^m\left(\frac{\pi j}{N}\right)$$

$$Q_m(N)=\sum_{j=1}^{4N-1}\chi_4(j)\cot^m\left(\frac{\pi j}{4N}\right)$$

in the proof of Theorem 1.4.25.

1.4.3 The space $W_{\mathrm{pre},\infty}$ and applications

The proofs of Theorem 1.2.7 and 1.4.5 did not use the order of the poles that occurred, only their locations. This motivates us to generalize the concept of pre-weak functions in the sense, that we allow them to have poles of arbitrary order. In this section we investigate analogous transformation laws for this kind of situation and will apply this to specific types of q-series, see also Theorem 1.4.47.

Definition 1.4.35. *We call a meromorphic function ω pre-weak of degree d, if all conditions for pre-weak functions are satisfied except that ω has a pole of order d (and all other poles have order at most d). We denote the vector space of pre-weak functions with degree at most a with $W_{\mathrm{pre},a}$. We collect all pre-weak functions of arbitrary degree in the space*

$$W_{\mathrm{pre},\infty}=\bigcup_{a=1}^{\infty}W_{\mathrm{pre},a}.$$

Even in the higher degree situation, we will still use the notation

$$\vartheta_k(\omega \otimes \eta; \tau) := -2\pi i \sum_{x \in \mathbb{R}^\times} \mathrm{res}_{z=x} \left(z^{k-1} \eta(z) \omega(z\tau) \right).$$

Like in the special case $a = 1$ it is quite easy to classify all pre-weak functions of degree at most a using elementary complex analytic ideas. For this purpose we abbreviate

$$h_{x,\ell}(z) = \frac{e(z)}{(e(x) - e(z))^\ell}. \tag{1.4.3.1}$$

We now find that there are uniquely determined functions $\beta_j : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, $1 \leq j \leq a$, that are zero except finitely many arguments, such that

$$\omega(z) = \omega(i\infty) + \sum_{j=1}^{a} \sum_{x \in \mathbb{R}/\mathbb{Z}} \beta_j(x) h_{x,j}(z).$$

In other words, there is an isomorphism

$$W_{\mathrm{pre},\infty} \cong \mathbb{C} \oplus \bigoplus_{\ell \geq 1} (\mathbb{R}/\mathbb{Z})^{\mathbb{C}_0}.$$

As we will see later, it is natural to study transformations of rational functions when applying the differential $\partial = \frac{1}{2\pi i} \frac{\partial}{\partial z}$. Note that $h_{x,\ell}(z)$ satisfies the differential equation

$$\partial h_{x,\ell}(z) = (1 - \ell) h_{x,\ell}(z) + \ell e(x) h_{x,\ell+1}(z), \tag{1.4.3.2}$$

since $\partial e^{2\pi i z}(e^{2\pi i x} - e^{2\pi i z})^{-\ell}$ equals to

$$\begin{aligned}
&\frac{1}{2\pi i} \left(2\pi i e^{2\pi i z}(e^{2\pi i x} - e^{2\pi i z})^{-\ell} - 2\pi i e^{4\pi i z}(-\ell)(e^{2\pi i x} - e^{2\pi i z})^{-\ell-1} \right) \\
&= e^{2\pi i z}(e^{2\pi i x} - e^{2\pi i z})^{-\ell} + \ell e^{2\pi i z}(e^{2\pi i z} - e^{2\pi i x} + e^{2\pi i x})(e^{2\pi i x} - e^{2\pi i z})^{-\ell-1} \\
&= e^{2\pi i z}(e^{2\pi i x} - e^{2\pi i z})^{-\ell} + \ell e^{2\pi i z}(-(e^{2\pi i x} - e^{2\pi i z})^{-\ell} + e^{2\pi i x}(e^{2\pi i x} - e^{2\pi i z})^{-\ell-1}) \\
&= h_{x,\ell}(z) - \ell h_{x,\ell}(z) + \ell e^{2\pi i x} h_{x,\ell+1}(z) = (1 - \ell) h_{x,\ell}(z) + \ell e(x) h_{x,\ell+1}(z).
\end{aligned}$$

We define the projection $\pi_1 : W^{i\infty}_{\mathrm{pre},\infty} \to W^{i\infty}_{\mathrm{pre},1}$ (remember that the $i\infty$ in the exponent means that $\omega(i\infty) = 0$) by

$$\pi_1 \left(\sum_{\ell=1}^{N_\omega} \sum_{x \in \mathbb{R}/\mathbb{Z}} \beta_{\omega,\ell}(x) h_{x,\ell}(z) \right) = \sum_{x \in \mathbb{R}/\mathbb{Z}} \beta_{\omega,1}(x) h_{x,1}(z).$$

This implies

Proposition 1.4.36. *Let $a \geq 1$ be an integer. We have the exact sequences*

$$0 \longrightarrow W^{i\infty}_{\mathrm{pre},a} \xrightarrow{\partial} W^{i\infty}_{\mathrm{pre},a+1} \xrightarrow{\pi_1} W^{i\infty}_{\mathrm{pre},1} \longrightarrow 0,$$

and

$$0 \longrightarrow W^{i\infty}_{\mathrm{pre},\infty} \xrightarrow{\partial} W^{i\infty}_{\mathrm{pre},\infty} \xrightarrow{\pi_1} W^{i\infty}_{\mathrm{pre},1} \longrightarrow 0.$$

Proof. We only give a proof for the case of the integer a, since the exactness of the second sequence is immediate with the proof.
It is clear that π_1 is onto and that the extended homomorphism $W_{\mathrm{pre},a} \xrightarrow{\partial} W_{\mathrm{pre},a+1}$ has kernel $\mathbb{C}$. Since $W^{i\infty}_{\mathrm{pre},a} \cap \mathbb{C} = 0$, it follows that ∂ is injective.
To see $\mathrm{im}(\partial) \subset \ker(\pi_1)$ we observe by (1.4.3.2) that

$$\partial \sum_{\ell=1}^{a} \sum_{x \in \mathbb{R}/\mathbb{Z}} \beta_\ell(x) h_{x,\ell}(z) = \sum_{\ell=1}^{a} \sum_{x \in \mathbb{R}/\mathbb{Z}} \beta_\ell(x)((1-\ell)h_{x,\ell}(z) + \ell e(x) h_{x,\ell+1}(z))$$

has no non-vanishing term $h_{x,1}(z)$. Hence, $\pi_1(\partial\omega) = 0$ for all $\omega \in W^{i\infty}_{\mathrm{pre},\infty}$. On the other hand, if $\omega \in \ker(\pi_1)$, it is of the form

$$\omega(z) = \sum_{\ell=2}^{a} \sum_{x \in \mathbb{R}/\mathbb{Z}} \gamma_\ell(x) h_{x,\ell}(z).$$

We will show by induction over the maximal degree $2 \leq r \leq a+1$ that all expressions of the form

$$\sum_{\ell=2}^{r} \sum_{x \in \mathbb{R}/\mathbb{Z}} \gamma_\ell(x) h_{x,\ell}(z)$$

indeed have an integral in $W^{i\infty}_{\mathrm{pre},a}$. If $r = 2$ we are reduced to

$$\sum_{x \in \mathbb{R}/\mathbb{Z}} \gamma_\ell(x) h_{x,2}(z). \tag{1.4.3.3}$$

By (1.4.3.2) we find that

$$C + \frac{1}{2\pi i} \sum_{x \in \mathbb{R}/\mathbb{Z}} \gamma_\ell(x) e(-x) h_{x,1}(z)$$

is an integral of (1.4.3.3) and we may choose $C = 0$ to achieve that this is part of $W^{i\infty}_{\mathrm{pre},a}$.
Next, assume that we have proven that

$$\sum_{\ell=2}^{r} \sum_{x \in \mathbb{R}/\mathbb{Z}} \gamma_\ell(x) h_{x,\ell}(z)$$

has an integral in $W^{i\infty}_{\mathrm{pre},a}$ for a fixed number $2 \leq r \leq a$ (if we had $r = a+1$, we would be done). We then have

$$\omega_{r+1}(z) := \sum_{\ell=2}^{r} \sum_{x \in \mathbb{R}/\mathbb{Z}} \gamma_\ell(x) h_{x,\ell}(z) + \sum_{x \in \mathbb{R}/\mathbb{Z}} \gamma_{r+1}(x) h_{x,r+1}(z), \tag{1.4.3.4}$$

and the left sum on the right of the equation has an integral I_1 in $W^{i\infty}_{\mathrm{pre},a}$ by assumption. By (1.4.3.2) we obtain

$$\int h_{x,r+1}(z)\mathrm{d}z = \frac{1}{2\pi i r} e(-x) h_{x,r}(z) + \frac{r-1}{r} e(-x) \int h_{x,r}(z)\mathrm{d}z$$

and the integral on the right can be chosen to be in $W^{i\infty}_{\mathrm{pre},a}$ by assumption again. Hence, the right sum on the right side of (1.4.3.4) has an integral I_2 in $W^{i\infty}_{\mathrm{pre},a}$. Hence $2\pi i(I_1+I_2) \in W^{i\infty}_{\mathrm{pre},a}$ and $\partial(2\pi i(I_1+I_2)) = \omega_{r+1}$. The claim now follows by induction. □

With this we obtain the following.

Corollary 1.4.37. *Let $a \geq 1$ be an integer. We have the canonical isomorphisms*

$$W_{\mathrm{pre},a} \cong \mathbb{C} \oplus \bigoplus_{n=0}^{a-1} \partial^n W^{i\infty}_{\mathrm{pre},1},$$

and

$$W_{\mathrm{pre},\infty} \cong \mathbb{C} \oplus \bigoplus_{n=0}^{\infty} \partial^n W^{i\infty}_{\mathrm{pre},1}.$$

Proof. With Proposition 1.4.36 we see $W^{i\infty}_{\mathrm{pre},n} = W^{i\infty}_{\mathrm{pre},1} \oplus \partial W^{i\infty}_{\mathrm{pre},n-1}$. Hence, we inductively obtain

$$W^{i\infty}_{\mathrm{pre},n} = W^{i\infty}_{\mathrm{pre},1} \oplus \partial W^{i\infty}_{\mathrm{pre},n-1} = W^{i\infty}_{\mathrm{pre},1} \oplus \partial W^{i\infty}_{\mathrm{pre},1} \oplus \partial\partial W^{i\infty}_{\mathrm{pre},n-2} = \ \cdots,$$

hence

$$W^{i\infty}_{\mathrm{pre},a} = \bigoplus_{n=0}^{a-1} \partial^n W^{i\infty}_{\mathrm{pre},1}.$$

Since we have $W_{\mathrm{pre},a} \cong \mathbb{C} \oplus W^{i\infty}_{\mathrm{pre},a}$ and the proof is analogous in the infinite case, the corollary follows. □

We obtain a similar result for weak functions.

Corollary 1.4.38. *Let a be an integer. Then we have the decompositions*

$$W_{\mathrm{weak},a} \cong W_{\mathrm{weak},1} \oplus \bigoplus_{n=1}^{a-1} \partial^n W^{i\infty}_{\mathrm{pre},1} \cong W_{\mathrm{weak},1} \oplus \bigoplus_{n=0}^{a-2} \partial^n \left(\partial W_{\mathrm{weak},1} \oplus \mathbb{C} h_{0,2}\right).$$

Proof. First note that we can write each $\omega \in W_{\mathrm{weak},a}$ in the form

$$\omega(z) = \sum_{x\in\mathbb{R}/\mathbb{Z}} \beta_1(x)h_{x,1}(z) + \sum_{\ell=2}^{a}\sum_{x\in\mathbb{R}/\mathbb{Z}} \beta_\ell(x)h_{x,\ell}(z),$$

where

$$\sum_{x\in\mathbb{R}/\mathbb{Z}} \beta_1(x) = 0.$$

Hence, by Proposition 1.4.36, we obtain

$$W_{\mathrm{weak},a} \cong W_{\mathrm{weak},1} \oplus \partial W^{i\infty}_{\mathrm{pre},a-1} \cong W_{\mathrm{weak},1} \oplus \bigoplus_{n=1}^{a-1} \partial^n W^{i\infty}_{\mathrm{pre},1}.$$

Together with the obvious isomorphism

$$W^{i\infty}_{\mathrm{pre},1} \cong W_{\mathrm{weak},1} \oplus \mathbb{C}h_{0,1}$$

we quickly obtain

$$\partial W^{i\infty}_{\mathrm{pre},1} \cong \partial W_{\mathrm{weak},1} \oplus \mathbb{C}h_{0,2}.$$

Putting everything together shows the corollary. □

At some stage it will be crucial to change from $W_{\mathrm{weak},\infty}$ to $W_{\mathrm{pre},\infty}$ in the sense of decompositions into derivatives. This is done in the obvious way.

Proposition 1.4.39. *Let $\omega \in W_{\mathrm{weak},a}$. Then we have the following identity between decompositions provided by Corollary 1.4.38:*

$$\omega(z) = \lambda_0(z) + \sum_{j=1}^{a-1} \partial^j \lambda_j(z) = \lambda_0(z) + \sum_{j=1}^{a-1} \partial^{j-1}(\partial\omega_j(z) + c_j h_{0,2}(z)), \qquad (1.4.3.5)$$

where $\lambda_0, \omega_j \in W_{\mathrm{weak},1}$, $\lambda_j \in W^{i\infty}_{\mathrm{pre},1}$ for $1 \le j \le a-1$. As a result, we get $\beta_{\omega_0}(y) = \beta_{\lambda_0}(y)$ and for all $1 \le j \le a-1$ the corresponding coefficients

$$\beta_{\lambda_j}(y) = \begin{cases} \beta_{\omega_j}(y), & \text{if } y \neq 0, \\ \beta_{\omega_j}(0) + c_j, & \text{if } y = 0. \end{cases}$$

Proof. We have

$$\begin{aligned}
\omega(z) &= \lambda_0(z) + \sum_{j=1}^{a-1}\sum_{y\in\mathbb{R}/\mathbb{Z}} \beta_{\lambda_j}(y)\partial^j h_{y,1}(z) \\
&= \lambda_0(z) + \sum_{j=1}^{a-1}\left(c_j\partial^j h_{0,1}(z) + \sum_{y\in\mathbb{R}/\mathbb{Z}} \beta_{\omega_j}(y)\partial^j h_{y,1}(z)\right) \\
&= \lambda_0(z) + \sum_{j=1}^{a-1}\left((c_j + \beta_{\omega_j}(0))\partial^j h_{0,1}(z) + \sum_{0<y<1} \beta_{\omega_j}(y)\partial^j h_{y,1}(z)\right).
\end{aligned}$$

The result now follows by comparing coefficients, since the $\partial^j h_{y,1}(z)$ define a basis of $\partial W^{i\infty}_{\mathrm{pre},a-1}$. □

The next two lemmas will turn out to be very useful when going to Fourier series of ϑ_k, where symmetry in the sense of $\mathrm{sgn}(\omega)\mathrm{sgn}(\eta) = (-1)^k$ is required.

Lemma 1.4.40. *Let $\omega \in W_{\mathrm{weak},a}$ with $\mathrm{sgn}(\omega) = (-1)^n$ and decomposition*

$$\omega(z) = \lambda_0(z) + \sum_{j=1}^{a-1} \partial^j \lambda_j(z),$$

with $\lambda_0 \in W_{\mathrm{weak},1}$ and $\lambda_j \in W^{i\infty}_{\mathrm{pre},1}$, as in Proposition 1.4.39. Then we already have $\mathrm{sgn}(\lambda_j) = (-1)^{n+j}$.

Proof. Since we have $W_{\mathrm{weak},1} = W^+_{\mathrm{weak},1} \oplus W^-_{\mathrm{weak},1}$ and $W^{i\infty}_{\mathrm{pre},1} = W^{i\infty,+}_{\mathrm{pre},1} \oplus W^{i\infty,-}_{\mathrm{pre},1}$, we can always write $\lambda_j(z) = \lambda_j^+(z) + \lambda_j^-(z)$, where of course $\mathrm{sgn}(\lambda^\pm) = \pm 1$. We put $\lambda^{\pm 1} := \lambda^\pm$. With this we obtain

$$\omega(z) - \lambda_0^{(-1)^n}(z) - \sum_{j=1}^{a-1} \partial^j \lambda_j^{(-1)^{n+j}}(z) = \lambda_0^{(-1)^{n+1}}(z) + \sum_{j=1}^{a-1} \partial^j \lambda_j^{(-1)^{n+j+1}}(z). \quad (1.4.3.6)$$

The left side of (1.4.3.6) has signum $(-1)^n$ and the right side signum $(-1)^{n+1}$. Since it is an identity, both sides vanish identically. The claim now follows inductively. The term of highest degree $\partial^{a-1}\lambda_{a-1}^{(-1)^{n+a}}(z)$ has to vanish, since otherwise there would be a pole of degree a on the right side that does not cancel. Hence $\lambda_{a-1}^{(-1)^{n+a}}(z)$ is constant, and since $\lambda_{a-1}^{(-1)^{n+a}}(i\infty) = 0$, it is zero. Continue with the second highest term and so on and conclude $\lambda_j = \lambda_j^{(-1)^{n+j}}$, which proves $\mathrm{sgn}(\lambda_j) = (-1)^{n+j}$. □

The second lemma is stated and proved similarly.

Lemma 1.4.41. *Let $a \geq 1$ and $\omega \in W_{\mathrm{weak},a}$ satisfy $\mathrm{sgn}(\omega) = (-1)^n$ for some integer n. Then, if*

$$\omega(z) = \omega_0(z) + \sum_{j=1}^{a-1} \partial^j \omega_j(z) + \sum_{j=1}^{a-1} c_j \partial^{j-1} h_{0,2}(z),$$

as in Proposition 1.4.39, we have $\mathrm{sgn}(\omega_j) = (-1)^{n+j}$ and $c_j = 0$ if $(-1)^j = (-1)^n$.

Proof. Since we have $W_{\mathrm{weak},1} = W^+_{\mathrm{weak},1} \oplus W^-_{\mathrm{weak},1}$, we can write each $\omega_j(z)$ as a unique sum $\omega_j^+(z) + \omega_j^-(z)$, where of course $\mathrm{sgn}(\omega^\pm) = \pm 1$. For purpose of notation write

$\omega^{\pm 1}(z) := \omega^{\pm}(z)$. We now collect all terms with signum $(-1)^n$ on the left hand side.

$$\omega(z) - \sum_{j=0}^{a-1} \partial^j \omega_j^{(-1)^{n+j}}(z) - \sum_{\substack{j+n\equiv 1 \pmod 2 \\ 1\leq j\leq a-1}} c_j \partial^{j-1} h_{0,2}(z) \tag{1.4.3.7}$$

$$= \sum_{j=0}^{a-1} \partial^j \omega_j^{(-1)^{n+j+1}}(z) + \sum_{\substack{j+n\equiv 0 \pmod 2 \\ 1\leq j\leq a-1}} c_j \partial^{j-1} h_{0,2}(z).$$

Note that we have $\mathrm{sgn}(h_{0,2}) = 1$, and hence $\mathrm{sgn}(\partial^{j-1} h_{0,2}) = (-1)^{j-1}$. If $n + j \equiv 1 \mod 2$, it follows $\mathrm{sgn}(\partial^{j-1} h_{0,2}) = (-1)^{n+j-1-n} = (-1)^n$. Similarly, we find $\mathrm{sgn}\left(\partial^j \omega_j^{(-1)^{n+j}}\right) = (-1)^n$. Since both sides of (1.4.3.7) have different signums, both have to vanish. It is now easy to conclude the claim. Indeed, assume without loss of generality that

$$\partial^{a-1} \omega_{a-1}^{(-1)^{n+a}}(z) + c_{a-1} \partial^{a-2} h_{0,2}(z) \tag{1.4.3.8}$$

is part of the right side and its "highest" term. When assuming that $\omega_{a-1}^{(-1)^{n+a}}(z)$ does not vanish, it has to have at least two different poles mod $\mathbb{Z}$, since it is weak (compare also Remark 1.2.1). It follows, that its $a-1$-th derivative has at least two poles of order a. But no other term on the right of (1.4.3.7) has poles of order a except $c_{a-1}\partial^{a-2} h_{0,2}(z)$, but this only has one (mod $\mathbb{Z}$) single pole of degree a in $z = 0$. So both summands in (1.4.3.8) have to vanish separately, since otherwise the poles of degree a could never cancel each other. The lemma now follows when going over all pairs of summands on the right side in (1.4.3.7), from above, inductively. □

The next lemma provides some useful differential identities.

Lemma 1.4.42. *Let be $k \in \mathbb{Z}$ and $\omega \otimes \eta \in W_{(k)}^{\otimes}$.*

(i) We have $\vartheta_k(\partial_z \omega \otimes \eta; \tau) = \partial_\tau \vartheta_{k-1}(\omega \otimes \eta; \tau)$.

(ii) We have $\vartheta_k(\omega \otimes \partial_z \eta; \tau) = \frac{1}{2\pi i}(1 - k - \tau \frac{\partial}{\partial \tau})\vartheta_{k-1}(\omega \otimes \eta; \tau)$.

Proof. Since interchanging residue and differential operator is legitimated we easily see

$$\partial_\tau \sum_{\alpha \in \mathbb{R}/\mathbb{Z}} \mathrm{res}_{z=\alpha}\left(z^{k-2}\eta(z)\omega(\tau z)\right) = \sum_{\alpha \in \mathbb{R}/\mathbb{Z}} \mathrm{res}_{z=\alpha}\left(z^{k-1}\eta(z)\frac{1}{2\pi i}\omega'(\tau z)\right).$$

This proves (i).
For (ii) let $f(z) = z^{k-1}\omega(\tau z)$. Then we note

$$0 = \mathrm{res}_{z=z_0}((f(z)\eta(z))') = \mathrm{res}_{z=z_0} f(z)\eta'(z) + \mathrm{res}_{z=z_0} f'(z)\eta(z)$$

and hence

$$\begin{aligned}\vartheta_k(\omega\otimes\eta';\tau) &= 2\pi i\sum_{\alpha\in\mathbb{R}/\mathbb{Z}}\operatorname{res}_{z=\alpha}\left((k-1)z^{k-2}\omega(\tau z)\eta(z)+z^{k-1}\tau\omega'(\tau z)\eta(z)\right)\\ &= (1-k)\vartheta_{k-1}(\omega\otimes\eta;\tau)-\tau\vartheta_k(\omega'\otimes\eta;\tau)\\ &= \left((1-k)-\tau\frac{\partial}{\partial\tau}\right)\vartheta_{k-1}(\omega\otimes\eta;\tau),\end{aligned}$$

according to (i). □

As an application of the more general formalism we want to give a description of a special case of the main transformation law in the language of series of rational functions. To make things more explicit, we are going to use differentials of the form

$$w_0+w_1\tau\frac{\partial}{\partial\tau}+w_2\tau^2\frac{\partial^2}{\partial\tau^2}+\cdots+w_n\tau^n\frac{\partial^n}{\partial\tau^n},\qquad w_i\in\mathbb{C},$$

and apply the results of Lemma 1.4.42. Since the lemma tells us

$$\vartheta_k(\omega\otimes\partial_z\eta;\tau)=\frac{1}{2\pi i}\left(1-k-\tau\frac{\partial}{\partial\tau}\right)\vartheta_{k-1}(\omega\otimes\eta;\tau),$$

it seems reasonable to look at differentials

$$\begin{aligned}D_{k,n} &= (2\pi i)^{-n}\left(1-k-\tau\frac{\partial}{\partial\tau}\right)\left(2-k-\tau\frac{\partial}{\partial\tau}\right)\cdots\left(n-k-\tau\frac{\partial}{\partial\tau}\right)\\ &= (2\pi i)^{-n}\sum_{\ell=0}^{n}\left(\sum_{j=0}^{n}(-1)^n\left\{\begin{matrix}j\\ \ell\end{matrix}\right\}\kappa_{1-k,n-k}(j)\right)\tau^\ell\frac{\partial^\ell}{\partial\tau^\ell}\end{aligned}$$

to find that

$$\vartheta_k(\omega\otimes\partial_z^n\eta;\tau)=D_{k,n}\vartheta_{k-n}(\omega\otimes\eta;\tau).$$

Here $\left\{\begin{smallmatrix}j\\ \ell\end{smallmatrix}\right\}$ denote the Stirling numbers of the second kind and for integers $b\geq a-1$ the numbers $\kappa_{a,b}(j)$ are defined by

$$(X-a)(X-a-1)\cdots(X-b)=\sum_{j=0}^{b-a+1}\kappa_{a,b}(j)X^j.$$

We abbreviate $s(n,\ell):=(2\pi i)^{\ell-n-1}\sum_{j=0}^{n}(-1)^{n+1}\left\{\begin{smallmatrix}j\\ \ell\end{smallmatrix}\right\}\kappa_{1-k,n-k}(j)$.

It is remarkable that we still obtain a simple modular relationship between $\vartheta_k(\omega\otimes\eta;\tau)$ and $\vartheta_k(\eta\otimes\widehat{\omega};\tau)$, as it was the case in Theorem 1.4.5.

Theorem 1.4.43. *Let $\omega \otimes \eta \in W_{\mathrm{weak},\infty} \otimes W_{\mathrm{weak},\infty}$, where ω and η have the Laurent expansions*

$$\omega(z) = \sum_{n=-U}^{\infty} a_n z^n,$$
$$\eta(z) = \sum_{n=-V}^{\infty} b_n z^n.$$

We then have the identity

$$\vartheta_k\left(\omega \otimes \eta; -\frac{1}{\tau}\right) = (-1)^{k-1}\tau^k \vartheta_k(\eta \otimes \widehat{\omega}; \tau) + 2\pi i \sum_{c=0}^{U+V-k} b_{c-V} a_{V-k-c} (-1)^{V-c} \tau^{V-c}.$$

Proof. The proof is essentially the same as the one of Theorem 1.4.5. We may choose $\tau = iy$ with $y > 0$ and use the rapid decay of the functions ω and η for increasing imaginary parts to show

$$\frac{1}{2\pi i} \oint_{|z|=N+\varepsilon} z^{k-1}\eta(z)\omega(z\tau)\mathrm{d}z = \sum_{\substack{-N \le x \le N \\ x \neq 0}} \left(\mathrm{res}_{z=x} + \mathrm{res}_{z=-\frac{x}{\tau}}\right)\left(z^{k-1}\eta(z)\omega(z\tau)\right) + \mathrm{res}_{z=0}\left(z^{k-1}\eta(z)\omega(z\tau)\right) = o(1),$$

where $\varepsilon > 0$ is fixed and sufficiently small (note that ω and η are periodic and only have real poles).

Put $g_\tau(z) := z^{k-1}\eta(z)\widehat{\omega}(z\tau)$ and $h_\tau(z) := z^{k-1}\omega(z)\eta(z\tau)$. For each $\tau \in \mathbb{H}$ we obtain the functional equation

$$g_\tau\left(-\frac{z}{\tau}\right) = (-\tau)^{1-k} z^{k-1} \eta\left(-\frac{z}{\tau}\right)\widehat{\omega}(-z) = (-\tau)^{1-k} h_{-\frac{1}{\tau}}(z).$$

Hence

$$\mathrm{res}_{z=-\frac{x}{\tau}}\left(g_\tau(z)\right) = -\frac{1}{\tau}\mathrm{res}_{z=x}\left(g_\tau\left(-\frac{z}{\tau}\right)\right) = (-\tau)^{-k}\mathrm{res}_{z=x}\left(h_{-\frac{1}{\tau}}(z)\right) \tag{1.4.3.9}$$

by the linearity of the residue. For the residue in $z = 0$ we obtain

$$\begin{aligned}
\mathrm{res}_{z=0}\left(g_\tau(z)\right) &= \mathrm{res}_{z=0}\left(z^{k-U-V-1}\left(\sum_{\ell=0}^{\infty} b_{\ell-V} z^\ell\right)\left(\sum_{\ell=0}^{\infty}(-1)^{\ell-U} a_{\ell-U}\tau^{\ell-U} z^\ell\right)\right) \\
&= \mathrm{res}_{z=0}\left(z^{k-U-V-1}\left(\sum_{\ell=0}^{\infty}\left(\sum_{c=0}^{\ell} b_{c-V} a_{\ell-c-U}(-1)^{\ell-c-U}\tau^{\ell-c-U}\right) z^\ell\right)\right) \\
&= \sum_{c=0}^{U+V-k} b_{c-V} a_{V-k-c}(-1)^{V-k-c}\tau^{V-k-c}
\end{aligned}$$

and hence

$$-2\pi i \sum_{x\in\mathbb{R}^\times} \operatorname{res}_{z=x}(g_\tau(z)) - 2\pi i \sum_{x\in\mathbb{R}^\times} \operatorname{res}_{z=-\frac{x}{\tau}}(g_\tau(z)) - 2\pi i \operatorname{res}_{z=0}(g_\tau(z)) = 0,$$

and by (1.4.3.9) this implies

$$\vartheta_k(\eta\otimes\widehat{\omega};\tau) - 2\pi i \sum_{c=0}^{U+V-k} b_{c-V}a_{V-k-c}(-1)^{V-k-c}\tau^{V-k-c} = -(-\tau)^{-k}\vartheta_k\left(\omega\otimes\eta;-\frac{1}{\tau}\right).$$

Multiplying this by $(-1)^{k-1}\tau^k$ proves the claim. □

This framework can be used to derive transformation laws of "higher" functions $\vartheta_k(\omega\otimes\eta;\tau)$, where $\omega(z)$ and $\eta(z)$ are allowed to have poles of higher degree. The outcomes are functions of the form

$$f(\tau) = g_0(\tau) + \tau g_1(\tau) + \cdots + \tau^n g_n(\tau),$$

where the $g_j(\tau)$ are Fourier series on the upper half plane, such that the $f(\tau)$ possess non-trivial transformation properties. We will omit the details of this extremely technical setup but will give examples in order to convince the reader of its usefulness. We will not use Theorem 1.4.43 in full generality and show examples with rational poles and lower degrees.

Example 1.4.44. *Let $k \geq 6$ be an even integer. Put*

$$\omega(z) := \csc(2\pi z)$$

and

$$\eta(z) := i\cot(2\pi z)\csc(2\pi z).$$

Then we have

$$\partial_z(\omega(z)) = \eta(z)$$

and hence obtain

$$\vartheta_k(\partial_z\omega\otimes\partial_z\omega;\tau) = \partial_\tau\vartheta_{k-1}(\omega\otimes\partial_z\omega;\tau) = \frac{1}{2\pi i}\partial_\tau\left(2-k-\tau\frac{\partial}{\partial\tau}\right)\vartheta_{k-2}(\omega\otimes\omega;\tau).$$

This equals to

$$-\frac{1}{4\pi^2}\left((1-k)\frac{\partial}{\partial\tau}\vartheta_{k-2}(\omega\otimes\omega;\tau) - \tau\frac{\partial^2}{\partial\tau^2}\vartheta_{k-2}(\omega\otimes\omega;\tau)\right).$$

One could now use the transformation properties of $\vartheta_{k-2}(\omega\otimes\omega;\tau)$ given in Theorem 1.4.5 to make final conclusions. But we will use Theorem 1.4.43 to investigate $\vartheta_k(\eta\otimes\eta;\tau)$. Let n be a non-zero integer. We obtain with the series expansions

$$z^{k-1}\eta(z\tau) = A_0 + A_1\left(z-\frac{n}{2}\right) + O\left(\left(z-\frac{n}{2}\right)^2\right)$$

with

$$A_0 = i\left(\frac{n}{2}\right)^{k-1}\cot(n\pi\tau)\csc(n\pi\tau)$$

$$A_1 = i(k-1)\left(\frac{n}{2}\right)^{k-2}\cot(n\pi\tau)\csc(n\pi\tau) - 2i\left(\frac{n}{2}\right)^{k-1}\left(\pi\tau\csc(n\pi\tau) + 2\pi\tau\cot^2(n\pi\tau)\csc(n\pi\tau)\right)$$

and

$$\eta(z) = \frac{i(-1)^n}{4\pi^2(z-\frac{n}{2})^2} + O(1)$$

for $k \geq 6$:

$$\begin{aligned}4\pi^2\mathrm{res}_{x=\frac{n}{2}}\left(z^{k-1}\eta(z)\eta(z\tau)\right) &= (-1)^{n+1}(k-1)\left(\frac{n}{2}\right)^{k-2}\cot(n\pi\tau)\csc(n\pi\tau)\\ &\quad + 2(-1)^n\left(\frac{n}{2}\right)^{k-1}\left(\pi\tau\csc(n\pi\tau) + 2\pi\tau\cot^2(n\pi\tau)\csc(n\pi\tau)\right).\end{aligned}$$

Since we have

$$\begin{aligned}\cot(\pi n\tau)\csc(\pi n\tau) &= -\frac{2(q^n+1)q^{\frac{n}{2}}}{(q^n-1)^2},\\ \csc(n\pi\tau) &= \frac{2iq^{\frac{n}{2}}}{q^n-1},\\ \cot^2(n\pi\tau)\csc(n\pi\tau) &= -\frac{2i(q^n+1)^2q^{\frac{n}{2}}}{(q^n-1)^3},\end{aligned}$$

we obtain by symmetry and Theorem 1.4.43 (note that $\mathrm{sgn}(\eta) = 1$), that $f_k(\tau)$ with series representation

$$(k-1)\sum_{n=1}^{\infty}(-1)^n n^{k-2}\frac{(1+q^n)q^{\frac{n}{2}}}{(1-q^n)^2} + \pi i\tau\sum_{n=1}^{\infty}(-1)^n n^{k-1}\frac{(1+6q^n+q^{2n})q^{\frac{n}{2}}}{(1-q^n)^3}$$

satisfies

$$f_k\left(-\frac{1}{\tau}\right) = -\tau^k f_k(\tau).$$

Example 1.4.45. *This example is very similar to Example 1.4.44, we choose*

$$\omega(z) = \eta(z) := \csc^2(2\pi z)$$

this time. The main difference is that $\omega(z)$ has no integral function that is weak, since the integral is given by $-\frac{1}{2\pi}\cot(2\pi z) + C$, compare also the result of Corollary 1.4.38. Let $k \geq 6$ be even. Very similar to Example 1.4.44 we find for

$$f_k(\tau) := (k-1)\sum_{n=1}^{\infty}n^{k-2}\frac{q^n}{(1-q^n)^2} + 2\pi i\tau\sum_{n=1}^{\infty}n^{k-1}\frac{(1+q^n)q^n}{(1-q^n)^3},$$

the transformation law

$$f_k\left(-\frac{1}{\tau}\right) = -\tau^k f_k(\tau).$$

Definition 1.4.46. *We say that a holomorphic q-series*

$$f(\tau) := \sum_{n=0}^{\infty} a(n) q^{\frac{n}{N}}$$

on the upper half plane has rational type (M, N), if there is a N-periodic arithmetic function $\psi(n)$, a polynomial P and a rational function R with poles only in $\{z = \zeta_M^j, 0 \leq j < M\}$ and $R(\infty) = R(0) = 0$, such that

$$f(\tau) = \sum_{n \in \mathbb{Z}\setminus\{0\}} \psi(n) P(n) R\left(q^{\frac{n}{N}}\right).$$

Theorem 1.4.47 (Transformation law for rational type q-series, see [28])**.** *Let $f \neq 0$ be a (M, N)-rational type q-series with periodic function satisfying $\sum_{j=1}^{N} \psi(j) = 0$. Put $\delta(\psi) = 0$ if $\psi(0) = 0$ and $\delta(\psi) = 1$, else. Then there is a polynomial $Q_{-1}(X)$ of degree at most $-\mathrm{ord}_{X=1}(R) - \mathrm{ord}_{X=0}(P) - 1$, and complex numbers A_0 and A_1, such that*

$$f\left(-\frac{1}{\tau}\right) = Q_{-1}\left(-\frac{1}{\tau}\right) + A_0 + \delta(\psi) A_1 \tau + \sum_{j=\mathrm{ord}_{X=0}(P)+1}^{\deg(P)+a} \tau^j s_j(\tau),$$

where each $s_j(\tau)$ is a finite sum of q-series of rational type (N, M) and a is the degree of R, which is its maximal pole order.

Proof. We are able to present a constructive proof, but we will only sketch the ideas of construction. Without loss of generality, we assume $P(n) = n^{k-1}$ with an arbitrary integer $k > 0$. Hence $\mathrm{ord}_{X=0}(P) = k - 1$. For each (M, N)-rational type series with the additional assumption $\sum_{j=1}^{N} \psi(j) = 0$ we find weak functions

$$\eta(z) := N^{k-1} \sum_{j=1}^{N} \psi(j) h_{\frac{j}{N}}(z)$$

and

$$\omega(z) := R(e(z))$$

such that

$$f(\tau) = \vartheta_k(\omega \otimes \eta; \tau).$$

With Theorem 1.4.43 we find polynomials $Q_{-1}(X)$ and $Q_1(X)$ such that

$$f\left(-\frac{1}{\tau}\right) = Q_{-1}\left(-\frac{1}{\tau}\right) + Q_1(\tau) + (-1)^{k-1} \tau^k \vartheta_k(\eta \otimes \widehat{\omega}; \tau).$$

But since $V \leq 1$ and $V \leq 0$ if and only if $\delta(\psi) = 0$, we see that Q_1 has degree at most 1 and 1 only if $\delta(\psi) = 1$. On the other hand, also by Theorem 1.4.43, $Q_{-1}(X)$ has degree at most $U - k = -\mathrm{ord}_{X=1}(R) - \mathrm{ord}_{X=0}(P) - 1$ (note that $\mathrm{ord}_{X=0}(P)$ is the correct measure at this point, since if P had more higher degree terms then the degree of Q_{-1} would be smaller for these terms). For any fixed $x = \frac{j}{M} \neq 0$, consider the expansions

$$\widehat{\omega}(z) = \sum_{\nu=-V}^{\infty} b_\nu\left(\frac{j}{M}\right)\left(z - \frac{j}{M}\right)^\nu,$$
$$\eta(z\tau) = \sum_{u=0}^{\infty} \frac{\tau^u \eta^{(u)}\left(\frac{j\tau}{M}\right)}{u!}\left(z - \frac{j}{M}\right)^u.$$

With this we obtain, that $\mathrm{res}_{z=\frac{j}{M}}\left(z^{k-1}\widehat{\omega}(z)\eta(z\tau)\right)$ equals

$$\mathrm{res}_{z=\frac{j}{M}}\left(\sum_{\mu,\nu+V,u\geq 0}\binom{k-1}{\mu}\left(\frac{j}{M}\right)^{k-1-\mu} b_\nu\left(\frac{j}{M}\right)\frac{\tau^u\eta^{(u)}\left(\frac{j\tau}{M}\right)}{u!}\left(z-\frac{j}{M}\right)^{\mu+\nu+u}\right).$$

For any triple $\mu + \nu + u = -1$ with $0 \leq \mu \leq k-1$ this is essentially (up to a constant independent of j and τ) of the form

$$j^{k-1-\mu} b_\nu\left(\frac{j}{M}\right)\tau^u\eta^{(u)}\left(\frac{j\tau}{M}\right),$$

and since $\eta^{(u)}$ is weak of higher degree again, hence of the form $W(q^{\frac{j}{M}})$ (note that $W(0) = W(\infty) = 0$ and W may only have poles in roots of unity ζ_N^j) and $\beta_\nu(x)$ is 1-periodic, we may sum this over all $j' \equiv j \pmod M$ to obtain a (N, M)-rational type series

$$\tau^u \sum_{j'\in\mathbb{Z}\setminus\{0\}} j'^{k-1-\mu}\widetilde{b_\nu}(j')W\left(q^{\frac{j'}{M}}\right),$$

where the M-periodic $\widetilde{b_\nu}(j')$ takes the value $b_\nu\left(\frac{j'}{M}\right)$ if $j' \equiv j \pmod M$ and 0 else. It follows that the term $(-1)^{k-1}\tau^k\vartheta_k(\eta\otimes\widehat{\omega};\tau)$ is essentially a sum consisting of terms of the form

$$\tau^{u+k} \sum_{j'\in\mathbb{Z}\setminus\{0\}} j'^{k-1-\mu}\widetilde{b_\nu}(j')W\left(q^{\frac{j'}{M}}\right),$$

where $\deg_{X=0}(P) + 1 \leq k \leq \deg(P) + 1$ and $0 \leq u \leq a - 1$. Summing up all the terms shows the claim. □

Finally, we give one more example.

Example 1.4.48. *Consider the weak functions* $\omega(z) := \csc^3(2\pi z)$ *and* $\eta(z) := \csc(2\pi z)$. *Put* $P(n) := n^{k-1}$ *with some even integer* $k \geq 6$. *This implies* $a = 3$, $\mathrm{ord}_{X=0}(P) = \deg(P) = k-1$, $V = 1$ *and* $U = 3$. *Following Theorem 1.4.47, the q-series*

$$f_k(\tau) := -2\pi i \sum_{n=1}^{\infty} \frac{1}{\pi}(-1)^n \left(\frac{n}{2}\right)^{k-1} \frac{(2i)^3 q^{\frac{3n}{2}}}{(q^n-1)^3} = 16 \sum_{n=1}^{\infty}(-1)^n \left(\frac{n}{2}\right)^{k-1} \frac{q^{\frac{3n}{2}}}{(1-q^n)^3},$$

which is essentially $\vartheta_k(\omega \otimes \eta; \tau)$, *satisfies the transformation law*

$$f\left(-\frac{1}{\tau}\right) = -\tau^k(g_1(\tau) + g_4(\tau)) - \tau^{k+1} g_2(\tau) - \tau^{k+2} g_3(\tau),$$

where

$$\begin{aligned}
g_1(\tau) &:= 2(-2\pi i) \sum_{n=1}^{\infty} \frac{(-1)^n}{4\pi}(-\csc(\pi n\tau)) \left(\frac{n}{2}\right)^{k-1} = 2 \sum_{n=1}^{\infty}(-1)^n \left(\frac{n}{2}\right)^{k-1} \frac{q^{\frac{n}{2}}}{1-q^n}, \\
g_2(\tau) &:= 2(-2\pi i) \sum_{n=1}^{\infty} \frac{(-1)^n}{8\pi^3} 2\pi \cot(\pi n\tau) \csc(\pi n\tau)(k-1) \left(\frac{n}{2}\right)^{k-2} \\
&= \frac{2i(k-1)}{\pi} \sum_{n=1}^{\infty}(-1)^n \left(\frac{n}{2}\right)^{k-2} \frac{(1+q^n) q^{\frac{n}{2}}}{(1-q^n)^2}, \\
g_3(\tau) &:= 2(-2\pi i) \sum_{n=1}^{\infty} \frac{(-1)^n}{8\pi^3}(-2) \left(\pi^2 \csc(\pi n\tau) + 2\pi^2 \cot^2(\pi n\tau) \csc(\pi n\tau)\right) \left(\frac{n}{2}\right)^{k-1} \\
&= 2 \sum_{n=1}^{\infty}(-1)^n \left(\frac{n}{2}\right)^{k-1} \left(\frac{q^{\frac{n}{2}}}{1-q^n} - \frac{2(1+q^n)^2 q^{\frac{n}{2}}}{(1-q^n)^3}\right), \\
g_4(\tau) &:= 2(-2\pi i) \sum_{n=1}^{\infty} \frac{(-1)^n}{8\pi^3}(-\csc(\pi n\tau)) \frac{(k-1)(k-2)}{2} \left(\frac{n}{2}\right)^{k-3} \\
&= \frac{(k-1)(k-2)}{2\pi^2} \sum_{n=1}^{\infty}(-1)^n \left(\frac{n}{2}\right)^{k-3} \frac{q^{\frac{n}{2}}}{1-q^n}.
\end{aligned}$$

Note that we were able to start summation at $n = 1$ *by symmetry.*

1.4.4 Eichler duality

When working with modular forms and generalized Eisenstein series we only considered positive weights k. We know that in the case of $\mathbb{R}/\mathbb{Z}$ we end up with generalized Fourier series, but in case $\mathbb{Q}/\mathbb{Z}$ they turn out to be modular forms. It is natural that there is also a theory for negative weights k, but of course, this will not provide non-trivial modular forms since they do not exist. Instead, in this case we find a direct access to the theory of Eichler integrals.

Let $k \geq 2$ be an integer. In this last section we develop an explicit formula for the $(k-1)$-fold integral of $\vartheta_k(\omega \otimes \eta; \tau)$ in the case $\omega \otimes \eta \in W_{\text{weak},a}[\mathcal{T}_N] \otimes W^{i\infty}_{\text{pre},1}[\mathcal{T}_M]$. On the rational function side it is given by a duality using Fourier transforms. Remember Definition 1.1.6 where we declared the notation $\int_m$ for the m-fold integral. Note that this is the inverse function of ∂_τ^m defined on $\mathbb{C}_0^+[[q]]$.

Before we start, we shortly introduce the Fourier transform of a pre-weak function with rational poles vanishing in $i\infty$. Let N be an integer. Then we define

$$\mathcal{F}_N : W^{i\infty}_{\text{pre},1}[\mathcal{T}_N] \xrightarrow{\sim} W^{i\infty}_{\text{pre},1}[\mathcal{T}_N]$$

$$\sum_{j=1}^{N} \beta\left(\frac{j}{N}\right) h_{\frac{j}{N}} \longmapsto \sum_{j=1}^{N} \mathcal{F}_N(\beta)\left(\frac{j}{N}\right) h_{\frac{j}{N}}.$$

A simple calculation verifies that the inverse of this isomorphism is given by

$$\mathcal{F}_N^{-1} : W^{i\infty}_{\text{pre},1}[\mathcal{T}_N] \xrightarrow{\sim} W^{i\infty}_{\text{pre},1}[\mathcal{T}_N]$$

$$\sum_{j=1}^{N} \beta\left(\frac{j}{N}\right) h_{\frac{j}{N}} \longmapsto \sum_{j=1}^{N} \mathcal{F}_N^{-1}(\beta)\left(\frac{j}{N}\right) h_{\frac{j}{N}}.$$

By Proposition 1.1.2 (ii) we see, that Fourier transforms induce isomorphisms $W^{i\infty,0}_{\text{pre},1}[\mathcal{T}_N] \to W_{\text{weak},1}[\mathcal{T}_N]$ between pre-weak functions vanishing in $i\infty$, that have a removable singularity in $z = 0$, and weak functions. In particular, they preserve the spaces $W^0_{\text{weak},1}[\mathcal{T}_N]$ of weak functions with removable singularities in $z = 0$. To prove Eichler duality we will introduce the following bracket notation which will simplify the Fourier series, that will occur frequently.

Definition 1.4.49. *Let β and γ be functions in $\mathbb{F}_N^{\mathbb{C}_0}$ and $\mathbb{F}_M^{\mathbb{C}_0}$. We put*

$$[\beta \otimes \gamma]_{k,\ell}(\tau) := 2M^{1-k} \sum_{m=1}^{\infty} \left(\sum_{d|m} d^{k-1} \gamma(d) \left(\frac{m}{d}\right)^{\ell} \beta\left(\frac{m}{d}\right) \right) q^{\frac{m}{M}}.$$

Note that $[\beta \otimes \gamma]_{k,\ell}$ always represents a holomorphic function on the upper half plane with a zero in $\tau = i\infty$.
In the following, we want to find the Fourier expansion of $\vartheta_k(\omega \otimes \eta; \tau)$ in the case that η has degree 1. This case is the most important one for most of our applications such as Eichler integrals. One of our main tool is a certain differential equation satisfied by the above introduced Fourier series.

Remark 1.4.50. *From now on, if not defined differently, we assume that if some $\omega \otimes \eta \in W_{\text{pre},\infty} \otimes W_{\text{pre},\infty}$ is used together with some integer k we have $\text{sgn}(\omega)\text{sgn}(\eta) = (-1)^k$. Remember that we say that a function ω has signum ± 1, if it is even or odd, respectively.*

In the applications of this section, we will use the term $[\beta\otimes\gamma]_{k,\ell}(\tau)$ for functions only referring to $\mathbb{F}_M^{\mathbb{C}_0}$ and $\mathbb{F}_N^{\mathbb{C}_0}$. This means, that if β and γ come from a pre-weak functions of the form

$$\omega(z)=\omega(i\infty)+\sum_{j=1}^{N}\beta\left(\frac{j}{N}\right)h_{\frac{j}{N}}$$

and

$$\eta(z)=\eta(i\infty)+\sum_{j=1}^{M}\gamma\left(\frac{j}{M}\right)h_{\frac{j}{M}}$$

we will find it useful to identify β with a N-periodic and γ with a M-periodic function on $\mathbb{Z}$, respectively, and put

$$[\beta\otimes\gamma]_{k,\ell}:=[\kappa_N\beta\otimes\kappa_M\gamma]_{k,\ell}.$$

Lemma 1.4.51. *Let β and γ be as above. We then have*

$$\partial_\tau[\beta\otimes\gamma]_{k,\ell}(\tau)=[\beta\otimes\gamma]_{k+1,\ell+1}(\tau).$$

Proof. Since we can differentiate termwise we obtain

$$\begin{aligned}
&\partial_\tau 2M^{1-k}\sum_{m=1}^{\infty}\left(\sum_{d|m}d^{k-1}\gamma(d)\left(\frac{m}{d}\right)^{\ell}\beta\left(\frac{m}{d}\right)\right)e^{2\pi i\tau\frac{m}{M}}\\
&=2M^{1-k}M^{-1}\sum_{m=1}^{\infty}\left(\sum_{d|m}d^{k-1}d\gamma(d)\left(\frac{m}{d}\right)^{\ell}\frac{m}{d}\beta\left(\frac{m}{d}\right)\right)e^{2\pi i\tau\frac{m}{M}}\\
&=2M^{-k}\sum_{m=1}^{\infty}\left(\sum_{d|m}d^{k}\gamma(d)\left(\frac{m}{d}\right)^{\ell+1}\beta\left(\frac{m}{d}\right)\right)e^{2\pi i\tau\frac{m}{M}}=[\beta\otimes\gamma]_{k+1,\ell+1}(\tau).
\end{aligned}$$

This proves the lemma. □

Proposition 1.4.52. *Let $k\equiv\frac{1\mp1}{2}\pmod 2$ be an integer and*

$$\omega\otimes\eta\in\begin{cases}W_N\otimes W_{\mathrm{pre},1}[\mathcal{T}_M], & \text{if } k>0,\\ \langle W_N\otimes W_{\mathrm{pre},1}[\mathcal{T}_M],W_{\mathrm{pre},1}[\mathcal{T}_N]\otimes W_M\rangle, & \text{if } k=0,\\ W_{\mathrm{pre},1}[\mathcal{T}_N]\otimes W_{\mathrm{pre},1}[\mathcal{T}_M], & \text{if } k<0,\end{cases}$$

such that $\mathrm{sgn}(\omega)\mathrm{sgn}(\eta)=(-1)^k$. Then the following assertions hold.

(i) For all τ on the upper half plane, the identity

$$\vartheta_k(\omega \otimes \eta; \tau) = A + [\mathcal{F}_N(\beta_\omega) \otimes \beta_\eta]_{k,0}(\tau)$$

holds, where

$$A = \begin{cases} \omega(i\infty)\widetilde{L}(\eta; 1-k), & \text{if } k \leq 0, \\ 0, & \text{if } k > 0. \end{cases}$$

(ii) For all τ on the upper half plane, the identity

$$\vartheta_k\left(h_{0,2} \otimes \eta; \tau\right) = [1 \otimes \beta_\eta]_{k,1}(\tau)$$

holds if $\operatorname{sgn}(\eta) = (-1)^k$. *Here* $1(x) = 1$ *for all* $x \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$.

Note that we use the convention of Remark 1.4.50 for all such assertions. A special case of Proposition 1.4.52 has been proven in Proposition 1.2.20.

Proof. We first observe that, given $\operatorname{sgn}(\omega\eta) = (-1)^k$, for all $\alpha \in \mathbb{Z} \setminus \{0\}$

$$\operatorname{res}_{z=\pm\frac{\alpha}{M}}\left(z^{k-1}\eta(z)\omega(z\tau)\right) = \frac{i}{2\pi}M^{1-k}\alpha^{k-1}\beta_\eta(\alpha)\omega\left(\frac{\alpha\tau}{M}\right).$$

Let $\omega = \omega(i\infty) + \omega_0$ with $\omega_0 \in W^{i\infty}_{\mathrm{pre},1}[\mathcal{T}_N]$ and note that $\beta_\omega = \beta_{\omega_0}$. Now we obtain by symmetry

$$\begin{aligned}
\vartheta_k(\omega \otimes \eta; \tau) &= 2M^{1-k}\sum_{\alpha=1}^{\infty}\alpha^{k-1}\beta_\eta(\alpha)\left(\omega(i\infty) + \omega_0\left(\frac{\alpha\tau}{M}\right)\right) \\
&= A + 2M^{1-k}\sum_{\alpha=1}^{\infty}\alpha^{k-1}\beta_\eta(\alpha)\sum_{j\in\mathbb{F}_N}\beta_\omega(j)\frac{e\left(\frac{\alpha\tau}{M} - \frac{j}{N}\right)}{1 - e\left(\frac{\alpha\tau}{M} - \frac{j}{N}\right)} \\
&= A + 2M^{1-k}\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty}\alpha^{k-1}\beta_\eta(\alpha)\sum_{j\in\mathbb{F}_N}\beta_\omega(j)e\left(-\frac{j\nu}{N}\right)q^{\frac{\alpha\nu}{M}} \\
&= A + 2M^{1-k}\sum_{m=1}^{\infty}\sum_{d|m}\left(d^{k-1}\beta_\eta(d)\sum_{j\in\mathbb{F}_N}\beta_\omega(j)e^{-\frac{2\pi imj}{Nd}}\right)q^{\frac{m}{M}}.
\end{aligned}$$

For convergence of the L-term in the case $k = 0$ see Proposition 1.4.4. In the case $\omega(z) = h_{0,2}(z)$ we find for $\operatorname{sgn}(h_{0,2}\eta) = \operatorname{sgn}(\eta) = (-1)^k$ by symmetry

$$\begin{aligned}
\vartheta_k\left(h_{0,2} \otimes \eta; \tau\right) &= 2M^{1-k}\sum_{\alpha=1}^{\infty}\alpha^{k-1}\beta_\eta(\alpha)\frac{e\left(\frac{\alpha\tau}{M}\right)}{\left(1 - e\left(\frac{\alpha\tau}{M}\right)\right)^2} \\
= 2M^{1-k}\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty}\alpha^{k-1}\beta_\eta(\alpha)\nu q^{\frac{\alpha\nu}{M}} &= 2M^{1-k}\sum_{m=1}^{\infty}\sum_{d|m}\left(d^{k-1}\beta_\eta(d)\left(\frac{m}{d}\right)\right)q^{\frac{m}{M}}.
\end{aligned}$$

This proves the theorem. □

Note that the inverse Fourier transform of $1(x)$ is given by

$$\mathcal{F}_N^{-1}(1)(x) = \delta_0(x),$$

where $\delta_0(x) = 1$ if $x = 0 \pmod{\mathbb{Z}}$ and $\delta_0(x) = 0$ for all other values $x \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$. So we can also write

$$\vartheta_k\left(h_{0,2} \otimes \eta; \tau\right) = [\mathcal{F}_N(\delta_0) \otimes \beta_\eta]_{k,1}(\tau).$$

The work we have done so far now provides

Theorem 1.4.53. *Let $k \geq 0$ be an integer and $\eta \in W_{\mathrm{pre},1}[\mathcal{T}_M]$. Let $\omega \in W_{\mathrm{weak},a}[\mathcal{T}_N]$ with decomposition*

$$\omega = \lambda_0 + \sum_{j=1}^{a-1} \partial^j \lambda_j$$

such that $\lambda_0 \in W_N$ and $\lambda_j \in W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N]$ (see also Proposition 1.4.39). Then, the following identity is valid on the upper half plane:

$$\vartheta_k(\omega \otimes \eta; \tau) = \sum_{j=0}^{a-1} [\mathcal{F}_N(\beta_{\lambda_j}) \otimes \beta_\eta]_{k,j}(\tau).$$

Proof. Starting with an expression $\omega = \omega_0 + \sum_{j=1}^{a-1} \partial^{j-1}(\partial\omega_j + c_j h_{0,2})$ with $\omega_j \in W_N$, we obtain

$$\begin{aligned}\vartheta_k(\omega \otimes \eta; \tau) &= \vartheta_k(\omega_0 \otimes \eta; \tau) + \sum_{j=1}^{a-1} \vartheta_k\left(\partial_z^j \omega_j \otimes \eta; \tau\right) + c_j \vartheta_k\left(\partial_z^{j-1} h_{0,2} \otimes \eta; \tau\right) \\ &= \vartheta_k(\omega_0 \otimes \eta; \tau) + \sum_{j=1}^{a-1} \partial_\tau^j \vartheta_{k-j}\left(\omega_j \otimes \eta; \tau\right) + \partial_\tau^{j-1} \vartheta_{k-j+1}\left(c_j h_{0,2} \otimes \eta; \tau\right)\end{aligned}$$

by Lemma 1.4.42 (i). By Lemma 1.4.41 we know that $\mathrm{sgn}(\omega_j) = (-1)^j \mathrm{sgn}(\omega)$ and $c_j = 0$ if $(-1)^j = \mathrm{sgn}(\omega)$. From this it follows $\mathrm{sgn}(\omega_j \eta) = (-1)^j \mathrm{sgn}(\omega\eta) = (-1)^{j+k} = (-1)^{k-j}$ and if $c_j \neq 0$, we necessarily have $(-1)^{-j+1} = \mathrm{sgn}(\omega) = (-1)^k \mathrm{sgn}(\eta)$. We conclude for this case

$$\mathrm{sgn}(h_{0,2}\eta) = \mathrm{sgn}(\eta) = (-1)^k \mathrm{sgn}(\omega) = (-1)^{k-j+1}.$$

It follows that in both (relevant) cases we are allowed to apply Proposition 1.4.52 (note that $\omega_0(i\infty) = 0$), so this simplifies to

$$[\mathcal{F}_N\left(\beta_{\omega_0}\right) \otimes \beta_\eta]_{k,0} + \sum_{j=1}^{a-1} \partial_\tau^j [\mathcal{F}_N(\beta_{\omega_j}) \otimes \beta_\eta]_{k-j,0}(\tau) + \partial_\tau^{j-1} [c_j \mathcal{F}_N(\delta_0) \otimes \beta_\eta]_{k-j+1,1}(\tau).$$

Note that we have added formally the vanishing c_j as well. With Lemma 1.4.51 we conclude that this equals to

$$[\mathcal{F}_N(\beta_{\omega_0}) \otimes \beta_\eta]_{k,0} + \sum_{j=1}^{a-1} [\mathcal{F}_N(\beta_{\omega_j} + c_j\delta_0) \otimes \beta_\eta]_{k,j}(\tau)$$

where $\delta_0(x) = 1$ if $x \in \mathbb{Z}$ and 0 else, and finally with Proposition 1.4.39, this equals to

$$\sum_{j=0}^{a-1} [\mathcal{F}_N(\beta_{\lambda_j}) \otimes \beta_\eta]_{k,j}(\tau).$$

Hence the theorem is proved. □

The next lemma imitates a classical result by Bol, see [10].

Lemma 1.4.54 (Weak Bol's identity)**.** *Let $k \geq 1$ and β and γ as above. Then we have*

$$\int_{k-1} ([\beta \otimes \gamma]_{k,\ell})(\tau) = N^{1+\ell-k}[\gamma \otimes \beta]_{2-k+\ell,0}\left(\frac{N\tau}{M}\right).$$

Note that the choice of $k-1$ is crucial for this kind of formula.

Proof. This can be followed by direct calculation and for the convenience of the reader we provide the details.

$$\begin{aligned}
\int_{k-1} ([\beta \otimes \gamma]_{k,\ell})(\tau) &= 2\int_{k-1} M^{1-k} \sum_{m=1}^{\infty} \left(\sum_{d|m} d^{k-1}\gamma(d) \left(\frac{m}{d}\right)^{\ell} \beta\left(\frac{m}{d}\right) \right) q^{\frac{m}{M}} \\
&= 2\sum_{m=1}^{\infty} \left(\sum_{d|m} \gamma(d) \left(\frac{m}{d}\right)^{\ell-k+1} \beta\left(\frac{m}{d}\right) \right) q^{\frac{m}{M}} \\
&= 2N^{1+\ell-k}N^{k-\ell-1} \sum_{m=1}^{\infty} \left(\sum_{d|m} \gamma\left(\frac{m}{d}\right) d^{(2-k+\ell)-1} \beta(d) \right) \left(q^{\frac{N}{M}}\right)^{\frac{m}{N}} \\
&= N^{1+\ell-k}[\gamma \otimes \beta]_{2-k+\ell,0}\left(\frac{N\tau}{M}\right).
\end{aligned}$$

This proves the claim. □

We apply the results to obtain a formula for multi-fold integrals of functions ϑ_k in terms of functions ϑ_j with $j \in \mathbb{Z}$.

Theorem 1.4.55. *Let $k \geq 2$ and $\omega \otimes \eta \in W_{\mathrm{weak},\infty}[\mathcal{T}_N] \otimes W_{\mathrm{pre},1}^{i\infty,0}[\mathcal{T}_M]$, where $\omega = \sum_{j=0}^{u} \partial_z^j \lambda_j$ with $\lambda_0 \in W_N$ and $\lambda_j \in W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N]$ as in Theorem 1.4.53. Then we have*

$$\int_{k-1} \vartheta_k(\omega \otimes \eta; \tau) = \sum_{j=0}^{u} N^{1+j-k} \vartheta_{2-k+j}\left(\mathcal{F}_M^{-1}\eta \otimes \mathcal{F}_N \lambda_j; \frac{N\tau}{M}\right).$$

Proof. First of all, Theorem 1.4.53 gives us

$$\vartheta_k(\omega \otimes \eta; \tau) = \sum_{j=0}^{u} [\mathcal{F}_N(\beta_{\lambda_j}) \otimes \beta_\eta]_{k,j}(\tau).$$

Now with Lemma 1.4.54 we conclude

$$\begin{aligned}\int_{k-1} \vartheta_k(\omega \otimes \eta; \tau) &= \sum_{j=0}^{u} \int_{k-1} [\mathcal{F}_N(\beta_{\lambda_j}) \otimes \beta_\eta]_{k,j}(\tau) \\ &= \sum_{j=0}^{u} N^{1+j-k} [\beta_\eta \otimes \mathcal{F}_N(\beta_{\lambda_j})]_{2-k+j,0}\left(\frac{N\tau}{M}\right).\end{aligned}$$

By Proposition 1.1.2 (iii) we know that Fourier transforms preserve odd and even functional relations, i.e., the signum of the function. Together with Lemma 1.4.40 we conclude, that $\operatorname{sgn}\left(\mathcal{F}_N \lambda_j \mathcal{F}_M^{-1}\eta\right) = \operatorname{sgn}(\lambda_j \eta) = (-1)^j \operatorname{sgn}(\omega\eta) = (-1)^{j+k} = (-1)^{2-k+j}$. Since $\mathcal{F}_M^{-1}\eta$ is weak, with Proposition 1.4.52 this equals

$$\sum_{j=0}^{u} N^{1+j-k} \vartheta_{2-k+j}\left(\mathcal{F}_M^{-1}\eta \otimes \mathcal{F}_N \lambda_j; \frac{N\tau}{M}\right).$$

This proves the claim. □

Remark 1.4.56. *Note that we can weaken the condition $\eta \in W_{\mathrm{pre},1}^{i\infty,0}[\mathcal{T}_M]$ to $\eta \in W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M]$ in Theorem 1.4.55, if $\omega \in W_{\mathrm{weak},a}[\mathcal{T}_N]$ such that $1 \leq a \leq k-2$. The reason for this is that if $1 \leq a \leq k-2$ it follows that $2-k+j \leq -1$ for all $0 \leq j \leq a-1$, which allows $\mathcal{F}_M^{-1}\eta$ to be a pre-weak function in the last step according to Proposition 1.4.52.*

To study Eichler duality we extend the $\mathbb{C}$ vector space $W_{\mathrm{pre},\infty}[\mathcal{T}_N]$ to a $\mathbb{C}[z, z^{-1}]$ module by putting

$$\mathfrak{M}_N = W_{\mathrm{pre},\infty}[\mathcal{T}_N] \otimes \mathbb{C}[z, z^{-1}].$$

In particular, we obtain a graded algebra

$$\mathfrak{M}_N = \bigoplus_{j=-\infty}^{\infty} z^j W_{\mathrm{pre},\infty}[\mathcal{T}_N],$$

whose elements naturally stand with the function ϑ_k in the sense that

$$\vartheta_k(z^\ell \cdot \omega \otimes \eta; \tau) = \vartheta_k(\omega \otimes z^\ell \cdot \eta; \tau) = \vartheta_{k+\ell}(\omega \otimes \eta; \tau).$$

Definition 1.4.57. *Let $k \geq 3$ be an integer. Then for each $1 \leq a \leq k-2$ we define the Eichler homomorphism*

$$\mathcal{E}_{k,a}^{N,M} : \bigoplus_{j=0}^{a-1} \partial_z^j W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N] \otimes W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M] \longrightarrow W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M] \otimes \bigoplus_{j=0}^{a-1} z^j W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N]$$

by

$$\omega \otimes \eta = \sum_{j=0}^{a-1} \partial_z^j \lambda_j \otimes \eta \longmapsto N^{1-k} \mathcal{F}_M^{-1} \eta \otimes \sum_{j=0}^{a-1} (zN)^j \mathcal{F}_N \lambda_j.$$

With these tools we are able to prove the main result on Eichler duality.

Theorem 1.4.58 (see [28])**.** *Let $k \geq 3$ and $1 \leq a \leq k-2$ be integers. We have the following assertions:*

(i) The map $\mathcal{E}_{k,a}^{N,M}$ is an isomorphism.

(ii) Consider the subspace $W_{\mathrm{weak},a}[\mathcal{T}_N] \otimes W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M] \subset \bigoplus_{j=0}^{a-1} \partial_z^j W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N] \otimes W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M]$. The diagram

$$\begin{array}{ccc} W_{\mathrm{weak},a}[\mathcal{T}_N] \otimes W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M] & \xrightarrow[\mathcal{E}_{k,a}^{N,M}]{} & W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M] \otimes \bigoplus_{j=0}^{a-1} z^j W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N] \\ \Big\downarrow \vartheta_k & & \Big\downarrow (\tau \mapsto \frac{N}{M}\tau) \circ \vartheta_{2-k} \\ \mathbb{C}_0^+[[q^{\frac{1}{M}}]] & \xrightarrow[\int_{k-1}]{} & \mathbb{C}_0^+[[q^{\frac{1}{M}}]] \end{array}$$

is commutative.

Proof. (i) We show that the map

$$\mathcal{I}_{k,a}^{M,N} : W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M] \otimes \bigoplus_{j=0}^{a-1} z^j W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N] \longrightarrow \bigoplus_{j=0}^{a-1} \partial_z^j W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N] \otimes W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M]$$

with

$$x \otimes y = x \otimes \sum_{\ell=0}^{a-1} z^\ell y_\ell \longmapsto N^{k-1} \sum_{\ell=0}^{a-1} N^{-\ell} \partial_z^\ell \mathcal{F}_N^{-1} y_\ell \otimes \mathcal{F}_M x$$

is an inverse of $\mathcal{E}_{k,a}^{N,M}$. We find for $\omega \otimes \eta \in \bigoplus_{j=0}^{a-1} \partial_z^j W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N] \otimes W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M]$:

$$\mathcal{I}_{k,a}^{M,N} \left(\mathcal{E}_{k,a}^{N,M}(\omega \otimes \eta) \right) = \mathcal{I}_{k,a}^{M,N} \left(N^{1-k} \mathcal{F}_M^{-1} \eta \otimes \sum_{j=0}^{a-1} (zN)^j \mathcal{F}_N \lambda_j \right)$$

$$= \sum_{j=0}^{a-1} \sum_{\ell=0}^{a-1} N^{j-\ell} (\delta_{j,\ell} \partial_z^\ell \mathcal{F}_N^{-1} \mathcal{F}_N \lambda_j \otimes \mathcal{F}_M \mathcal{F}_M^{-1} \eta) = \sum_{j=0}^{a-1} (\partial_z^j \lambda_j \otimes \eta) = \omega \otimes \eta.$$

The other way round we see for $a \otimes b \in W^{i\infty}_{\mathrm{pre},1}[\mathcal{T}_M] \otimes \bigoplus_{j=0}^{a-1} z^j W^{i\infty}_{\mathrm{pre},1}[\mathcal{T}_N]$:

$$\begin{aligned}\mathcal{E}^{N,M}_{k,a}\left(\mathcal{I}^{M,N}_{k,a}(a\otimes b)\right) &= \mathcal{E}^{N,M}_{k,a}\left(N^{k-1}\sum_{\ell=0}^{a-1} N^{-\ell}\partial_z^\ell \mathcal{F}_N^{-1} b_\ell \otimes \mathcal{F}_M a\right)\\ &= N^{k-1}N^{1-k}\left(\mathcal{F}_M^{-1}\mathcal{F}_M x \otimes \sum_{j=0}^{a-1} N^{-j}(zN)^j \mathcal{F}_N \mathcal{F}_N^{-1} b_j\right) = a\otimes b.\end{aligned}$$

(ii) We prove that the compositions give the same output. Let $\omega \otimes \eta \in W_{\mathrm{weak},a}[\mathcal{T}_N] \otimes W^{i\infty}_{\mathrm{pre},1}[\mathcal{T}_M]$ and ω be given by

$$\omega = \sum_{j=0}^{a-1} \partial^j \lambda_j, \qquad \lambda_0 \in W_N, \lambda_j \in W^{i\infty}_{\mathrm{pre},1}[\mathcal{T}_N].$$

Since $1 \le a \le k-2$, according to Theorem 1.4.55 and Remark 1.4.56 we have that

$$\int_{k-1} \vartheta_k(\omega\otimes\eta;\tau) = \sum_{j=0}^{a-1} N^{1+j-k}\vartheta_{2-k+j}\left(\mathcal{F}_M^{-1}\eta \otimes \mathcal{F}_N\lambda_j; \frac{N\tau}{M}\right). \tag{1.4.4.1}$$

On the other hand, we find

$$\begin{aligned}\vartheta_{2-k}\left(\mathcal{E}^{N,M}_{k,a}(\omega\otimes\eta);\tau\right) &= N^{1-k}\vartheta_{2-k}\left(\mathcal{F}_M^{-1}\eta \otimes \sum_{j=0}^{a-1}(Nz)^j \mathcal{F}_N\lambda_j;\tau\right)\\ &= \sum_{j=0}^{a-1} N^{1-k+j}\vartheta_{2-k+j}\left(\mathcal{F}_M^{-1}\eta\otimes\mathcal{F}_N\lambda_j;\tau\right).\end{aligned}$$

This proves the theorem. □

We can easily apply this formalism to the simplest case of Eisenstein series. It is well-known that if χ and ψ are primitive Dirichlet characters modulo $N_\chi > 1$ and $N_\psi > 1$ and $f(\tau) = E_k(\chi,\psi;\tau)$ the corresponding L-function is

$$L(E_k(\chi,\psi;\tau);s) = \frac{2(-2\pi i)^k \mathcal{G}(\psi)}{N_\psi^k (k-1)!} L(\chi;s) L\left(\overline{\psi}; s-k+1\right).$$

We assume that both χ and ψ are non-principal. From identity (1.2.3.3) and Theorem 1.4.58 we obtain

$$\begin{aligned}\int_{k-1} E_k(\chi,\psi;\tau) &= \frac{\chi(-1)(-2\pi i)^k\mathcal{G}(\psi)}{N_\psi(k-1)!\mathcal{G}\left(\overline{\chi}\right)}\int_{k-1}\vartheta_k(\omega_{\overline{\chi}}\otimes\omega_{\overline{\psi}};\tau)\\ &= \frac{\chi(-1)(-2\pi i)^k\mathcal{G}(\psi)}{N_\psi(k-1)!\mathcal{G}\left(\overline{\chi}\right)} \times \chi(-1)N_\chi^{1-k}N_\psi^{-1}\mathcal{G}\left(\overline{\psi}\right)\mathcal{G}\left(\overline{\chi}\right)\vartheta_{2-k}\left(\omega_\psi\otimes\omega_\chi;\frac{N_\chi\tau}{N_\psi}\right)\\ &= \frac{(-2\pi i)^k\psi(-1)}{N_\chi^{k-1}N_\psi(k-1)!}\vartheta_{2-k}\left(\omega_\psi\otimes\omega_\chi;\frac{N_\chi\tau}{N_\psi}\right).\end{aligned}$$

Since

$$\int_{k-1} E_k(\chi,\psi;\tau) = \frac{(-2\pi i)^{k-1}}{(k-2)!}F(\tau),$$

where

$$F(\tau) = \int_{\tau}^{i\infty} E_k(\chi,\psi;z)(z-\tau)^{k-2}\mathrm{d}z,$$

this combines to

$$F(\tau) = -\frac{2\pi i\psi(-1)}{N_\chi^{k-1}N_\psi(k-1)}\vartheta_{2-k}\left(\omega_\psi\otimes\omega_\chi;\frac{N_\chi\tau}{N_\psi}\right).$$

In the sense of Proposition 1.1.7 we have $E_k^*(\chi,\psi;\tau) = \chi(-1)E_k(\psi,\chi;\tau)$ and this provides

$$F^*(\tau) = -\frac{2\pi i}{N_\psi^{k-1}N_\chi(k-1)}\vartheta_{2-k}\left(\omega_\chi\otimes\omega_\psi;\frac{N_\psi\tau}{N_\chi}\right),$$

since $\chi(-1)^2 = 1$. Now according to Proposition 1.1.7 we have the functional equation

$$F(\tau) - (-\tau)^{k-2}F^*\left(-\frac{1}{\tau}\right) = (-1)^k\sum_{\ell=0}^{k-2}\binom{k-2}{\ell}i^{1-\ell}\Lambda_f(\ell+1)\tau^{k-2-\ell}. \qquad (1.4.4.2)$$

On the other hand, from Theorem 1.4.5, we obtain

$$\begin{aligned}
(-\tau)^{k-2}F^*\left(-\frac{1}{\tau}\right) &= -\tau^{k-2}(-1)^{k-2}\frac{2\pi i}{N_\psi^{k-1}N_\chi(k-1)}\vartheta_{2-k}\left(\omega_\chi\otimes\omega_\psi;-\frac{N_\psi}{N_\chi\tau}\right)\\
&= -\frac{2\pi i\tau^{k-2}(-1)^{k-2}}{N_\psi^{k-1}N_\chi(k-1)}\times 2\pi i\ (-\chi(-1))\mathrm{res}_{z=0}\left(z^{1-k}\omega_\psi(z)\omega_\chi\left(\frac{N_\psi z}{N_\chi\tau}\right)\right)\\
&\quad -\frac{2\pi i\tau^{k-2}(-1)^{k-2}}{N_\psi^{k-1}N_\chi(k-1)}\times\chi(-1)\ \left(\frac{N_\psi}{N_\chi\tau}\right)^{k-2}\vartheta_{2-k}\left(\omega_\psi\otimes\omega_\chi;\frac{N_\chi\tau}{N_\psi}\right)\\
&= -\frac{4\pi^2\tau^{k-2}\psi(-1)}{N_\psi^{k-1}N_\chi(k-1)}\mathrm{res}_{z=0}\left(z^{1-k}\omega_\psi(z)\omega_\chi\left(\frac{N_\psi z}{N_\chi\tau}\right)\right)\\
&\quad -\frac{2\pi i\tau^{k-2}(-1)^{k-2}}{N_\psi^{k-1}N_\chi(k-1)}\times\chi(-1)\ \left(\frac{N_\psi}{N_\chi\tau}\right)^{k-2}\vartheta_{2-k}\left(\omega_\psi\otimes\omega_\chi;\frac{N_\chi\tau}{N_\psi}\right)\\
&= -\frac{4\pi^2\tau^{k-2}\psi(-1)}{N_\psi^{k-1}N_\chi(k-1)}\mathrm{res}_{z=0}\left(z^{1-k}\omega_\psi(z)\omega_\chi\left(\frac{N_\psi z}{N_\chi\tau}\right)\right) + F(\tau). \qquad (1.4.4.3)
\end{aligned}$$

And this concludes the following theorem.

Theorem 1.4.59. *Let $k \geq 3$ be an integer, χ and ψ be two primitive Dirichlet characters with $\chi(-1)\psi(-1) = (-1)^k$ and $f(\tau) = E_k(\chi, \psi; \tau)$. We then have the following identity between rational functions:*

$$\sum_{\ell=0}^{k-2} \binom{k-2}{\ell} i^{1-\ell} \Lambda_f(\ell+1)\tau^{-\ell} = \frac{4\pi^2\chi(-1)}{N_\psi^{k-1}N_\chi(k-1)} \mathrm{res}_{z=0}\left(z^{1-k}\omega_\psi(z)\omega_\chi\left(\frac{N_\psi z}{N_\chi \tau}\right)\right).$$

Proof. With (1.4.4.2) and (1.4.4.3) we obtain

$$\frac{4\pi^2\tau^{k-2}\psi(-1)}{N_\psi^{k-1}N_\chi(k-1)} \mathrm{res}_{z=0}\left(z^{1-k}\omega_\psi(z)\omega_\chi\left(\frac{N_\psi z}{N_\chi \tau}\right)\right) = (-1)^k \sum_{\ell=0}^{k-2} \binom{k-2}{\ell} i^{1-\ell}\Lambda_f(\ell+1)\tau^{k-2-\ell}.$$

The claim now follows when dividing by τ^{k-2} and with $\psi(-1)\chi(-1) = (-1)^k$. □

Remark 1.4.60. *Note that Theorem 1.4.59 will initially work for $k = 2$ as well, since the considered characters all satisfy $\chi(0) = \psi(0) = 0$, so that we may assume that η is weak and has a removable singularity in $z = 0$ in the proof of Eichler duality. So we are allowed to fulfill step* (1.4.4.1) *even if $k - 2 < a$.*

Before looking at the much more difficult case of products of Eisenstein series, we give a numerical example.

Example 1.4.61. *Put $k = 6$, $N_\psi = 5$ and $N_\chi = 7$. Consider the Legendre symbol ψ_5 and the character χ_7 generated by $\chi_7(3) = e^{\frac{2\pi i}{3}}$. Note that we have $\psi_5(-1) = \chi_7(-1) = 1$. With this we obtain the data*

$$\Lambda_f(s) = \left(\frac{2\pi}{5}\right)^{-s} \Gamma(s) L\left(E_6(\chi, \psi; \tau), s\right) = -\frac{2^4\sqrt{5}\pi^6}{3 \cdot 5^7}\left(\frac{2\pi}{5}\right)^{-s} \Gamma(s) L(\chi_7; s) L(\psi_5; s-5).$$

With this we calculate

$$\sum_{\ell=0}^{4} \binom{4}{\ell} i^{1-\ell}\Lambda_f(\ell+1)\tau^{-\ell} = -\frac{2^4\sqrt{5}\pi^6}{3\cdot 5^7}\left(\frac{25L(\chi_7;2)L(\psi_5;-3)}{\pi^2\tau} - \frac{3\cdot 5^4 L(\chi_7;4)L(\psi_5;-1)}{2\pi^4\tau^3}\right)$$
$$= (-0.5941948118... + 0.0757203657...i)\,\tau^{-1} + (-0.5427628701... + 0.0234276364...i)\,\tau^{-3}$$

either by numerical calculation or by using the formulas. Again, using direct calculation or Mathematica one sees

$$\frac{4\pi^2\chi_7(-1)}{5^6\cdot 7}\mathrm{res}_{z=0}\left(z^{-5}\omega_{\psi_5}(z)\omega_{\chi_7}\left(\frac{5z}{7\tau}\right)\right)$$
$$= (-0.5941948118... + 0.0757203657...i)\,\tau^{-1} + (-0.5427628701... + 0.0234276364...i)\,\tau^{-3}.$$

We can now use this to give detailed expressions for the L-functions in the critical strip. For products of Eisenstein series, also studied in Section 1.3.2 and [24] in a bit

different context, the situation is naturally far more difficult. However, it is quite easy to write the Eichler integral as an infinite series of expressions ϑ_j in this case. We only need the following observation.

Proposition 1.4.62. *Let k and $g \geq 1$ be integers, $\chi_1, ..., \chi_g$ be a family of Dirichlet characters mod M, and $(\eta_j)_{1\leq j\leq g}$ the corresponding family of weak functions. Let $(\omega_j)_{1\leq j\leq g}$ be a family in $W_{\mathrm{weak},1}[\mathcal{T}_N]$. Put*

$$f(\tau) := \prod_{j=1}^{g} \vartheta_k(\omega_j \otimes \eta_j; \tau).$$

We then have the identity

$$f(\tau) = 2^{g-1} \sum_{(A_1,A_2,...,A_g)=1} \left(\prod_{j=1}^{g} A_j^{k-1} \chi_j(A_j) \right) \vartheta_{g(k-1)+1} \left(\prod_{j=1}^{g} \omega_j(A_j z) \otimes \eta_{\chi_1\chi_2\cdots\chi_g}; \tau \right).$$

Proof. Since all sums converge absolutely and uniformly on compact subsets $K \subset \mathbb{H}$, we may change the order of summands. We obtain

$$\begin{aligned}
\prod_{j=1}^{g} \vartheta_k(\omega_j \otimes \eta_j; \tau) &= 2^g \sum_{n_1=1}^{\infty} \cdots \sum_{n_g=1}^{\infty} \left(\frac{n_1 \cdots n_g}{M^g} \right)^{k-1} \chi_1(n_1) \cdots \chi_g(n_g) \prod_{j=1}^{g} \omega_j \left(\frac{n_j z}{M} \right) \\
&= 2^g \sum_{(A_1,...,A_g)=1} \sum_{n=1}^{\infty} \left(\frac{A_1 A_2 \cdots A_g n^g}{M^g} \right)^{k-1} \chi_1(A_1 n) \cdots \chi_g(A_g n) \prod_{j=1}^{g} \omega_j \left(\frac{A_j n z}{M} \right) \\
&= 2^g \sum_{(A_1,...,A_g)=1} \left(\prod_{j=1}^{g} A_j^{k-1} \chi_j(A_j) \right) \sum_{n=1}^{\infty} \left(\frac{n}{M} \right)^{g(k-1)} (\chi_1 \cdots \chi_g)(n) \prod_{j=1}^{g} \omega_j \left(\frac{A_j n z}{M} \right)
\end{aligned}$$

The inner sum is clearly $\frac{1}{2}\vartheta_{g(k-1)+1}(\prod_{j=1}^{g} \omega_j(A_j z) \otimes \eta_{\chi_1\cdots\chi_g}; \tau)$, hence

$$= 2^{g-1} \sum_{(A_1,A_2,...,A_g)=1} \left(\prod_{j=1}^{g} A_j^{k-1} \chi_j(A_j) \right) \vartheta_{g(k-1)+1} \left(\prod_{j=1}^{g} \omega_j(A_j z) \otimes \eta_{\chi_1\chi_2\cdots\chi_g}; \tau \right).$$

This shows the proposition. □

For integers $A_1, ..., A_g$ with $(A_1, ..., A_g) = 1$ we see that

$$\omega_1(A_1 z)\omega_2(A_2 z) \cdots \omega_g(A_g z) \in W_{\mathrm{weak},g}[\mathcal{T}_{A_1 A_2 \cdots A_g N}]$$

and we write

$$\omega_1(A_1 z)\omega_2(A_2 z) \cdots \omega_g(A_g z) = \sum_{j=0}^{g-1} \partial_z^j \lambda^j_{(A_1,...,A_g)}(z).$$

We can use this to prove the following identity for $g(k-1)$-fold integrals for products of g Eisenstein series.

Theorem 1.4.63. *Let $k \geq 3$ and $g \geq 1$ be integers, $\chi_1, ..., \chi_g$ be a family of Dirichlet characters mod M and $(\eta_j)_{1\leq j\leq g}$ the corresponding family of weak functions. Let $(\omega_j)_{1\leq j\leq g}$ be a family in $W_{\text{weak},1}[\mathcal{T}_N]$. We then have the following series representation for the $g(k-1)$-fold integral of the function $\prod_{j=1}^{g}\vartheta_k(\omega_j\otimes\eta_j;\tau)$:*

$$\int_{g(k-1)}\prod_{j=1}^{g}\vartheta_k(\omega_j\otimes\eta_j;\tau) = 2^{g-1}\sum_{(A_1,A_2,...,A_g)=1}\left(\prod_{j=1}^{g}A_j^{k-1}\chi_j(A_j)\right)$$
$$\times\sum_{j=0}^{g-1}(A_1\cdots A_gN)^{1-g(k-1)+j}\vartheta_{1-g(k-1)+j}(\mathcal{F}_M^{-1}\eta_{\chi_1\chi_2\cdots\chi_g}\otimes\mathcal{F}_{A_1\cdots A_gN}\lambda^j_{(A_1,...,A_g)};\tau).$$

Proof. Since the series converges absolutely and uniformly on compact subsets $K\subset\mathbb{H}$ we can switch integration and summation in Proposition 1.4.62 and the theorem now follows with Theorem 1.4.58. □

We can choose ω_j and η_j to correspond to several primitive characters. Note that the right hand side of this formula transforms in connection with the period polynomial of the product of Eisenstein series.

Chapter 2

Generalized Fourier series and L-functions at rational arguments

Note. The main results of this chapter have been published by the author in [25] and [26].

2.1 Preliminaries

Our main goal is to generalize several identities by Ramanujan to a wide range of L-functions. These identities give relations between generalized Fourier series and products of these L-functions at rational arguments. The theory behind these identities is a powerful framework which applies in many situations. Therefore, we will have a focus on a specific type of L-function, namely the Hecke L-function associated to a number field K.

In this section we will recall and develop the tools whose are required. In most cases we omit detailed proofs. Here we refer the reader to [14], [16], [36], [46] and [50]. In some cases, however, we sketch proofs to give the reader a feeling for the objects studied. In particular, the functional equation for the Hecke L-functions is proved at least in the case of narrow class number 1.

2.1.1 Basic algebraic number theory

The discriminant of a number field

Let K be an algebraic number field, i.e., a finite extension $K|\mathbb{Q}$ with degree $n = [K : \mathbb{Q}]$. We simply call its elements algebraic numbers. An algebraic number is called integral, or an algebraic integer, if it is a root of a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$. It can be shown that the subset $\mathcal{O}_K \subset K$ of algebraic integers in K is a ring, the so

called ring of integers of K. The ring $\mathcal{O}_K$ is integrally closed in K which means that each root of a polynomial $x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 \in \mathcal{O}_K[x]$, which lies in K, already lies in $\mathcal{O}_K$.

Let $L|K$ be a separable extension and $S = \{\sigma_j : L \to \overline{K}\}$ the set of different K-embeddings of L into an algebraic closure of K. We then define the trace and the norm of $x \in L$ with respect to K by

$$\mathrm{Tr}_{L|K}(x) = \sum_{\sigma_j \in S} \sigma_j x$$

and

$$N_{L|K}(x) = \prod_{\sigma_j \in S} \sigma_j x.$$

Let $\omega_1, ..., \omega_n$ be an integral basis of $\mathcal{O}_K$, i.e., $\mathcal{O}_K = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_n$. Such a basis exists by

Proposition 2.1.1. *Let $L|K$ be separable, A a principal ideal domain with quotient field K and B the integral closure of A in L. Then every finitely generated B-submodule $M \neq 0$ of L is a free A-module with rank $[L : K]$. In particular, B admits an integral basis over A.*

A very important invariant of a number field is its discriminant.

Definition 2.1.2. *We define the discriminant d_K of a number field K by*

$$d_K = d(\mathcal{O}_K) = \det(\sigma_j \omega_k)^2.$$

Since every base change has determinant ± 1, this is not dependent by the choice of the $\omega_1, ..., \omega_n$.

Example 2.1.3. *For every square-free $D \in \mathbb{Z} \setminus \{0\}$, the quadratic number field $K = \mathbb{Q}(\sqrt{D})$ has discriminant*

$$d_K = \begin{cases} D, & \text{if } D \equiv 1 \pmod 4, \\ 4D, & \text{if } D \equiv 2, 3 \pmod 4. \end{cases}$$

Ideals and the class number

The purpose of algebraic number theory is the study of "primes in $\mathcal{O}_K$". In the case that $\mathcal{O}_K$ is a principal ideal domain (PID) we know that it is factorial and every element of $\mathcal{O}_K$ can be uniquely factorized into prime elements up to a unit. However, in general the ring $\mathcal{O}_K$ will not be factorial and as a consequence one might lose the unique prime decompositions. But in fact, by an observation of Richard Dedekind we can remedy this situation.

Instead of looking at algebraic integers $a \in \mathcal{O}_K$ we consider integral ideals $\mathfrak{a} \subset \mathcal{O}_K$, that are finitely generated $\mathcal{O}_K$-modules. Since all finitely generated $\mathcal{O}_K$-submodules $\mathfrak{m}$ of K admit a $\mathbb{Z}$-basis

$$\mathfrak{m} = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n, \qquad n = [K : \mathbb{Q}],$$

so does every ideal $\mathfrak{a}$. We can add and multiply two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in the obvious ways

$$\mathfrak{a} + \mathfrak{b} = \{a + b \mid a \in \mathfrak{a}, b \in \mathfrak{b}\}$$

and

$$\mathfrak{a}\mathfrak{b} = \left\{ \sum_j a_j b_j \mid a_j \in \mathfrak{a}, b_j \in \mathfrak{b} \right\}.$$

We say that $\mathfrak{a}$ divides $\mathfrak{b}$ if $\mathfrak{b} \subset \mathfrak{a}$ and the smallest common multiple is the intersection $\mathfrak{a} \cap \mathfrak{b}$. The greatest common divisor is the largest ideal containing both $\mathfrak{a}$ and $\mathfrak{b}$. We have the following important theorem.

Theorem 2.1.4 (Chinese remainder theorem)**.** *Let $\mathfrak{a}_1, ..., \mathfrak{a}_r$ be ideals in the ring $\mathcal{O}_K$, such that $\mathfrak{a}_j + \mathfrak{a}_k = \mathcal{O}_K$ if $j \neq k$. Put $\mathfrak{a} = \bigcap_{j=1}^r \mathfrak{a}_j$. Then we have*

$$\mathcal{O}_K/\mathfrak{a} \cong \bigoplus_{j=1}^{r} \mathcal{O}_K/\mathfrak{a}_j.$$

Since $\mathfrak{a} \cong \mathbb{Z}^n$, we obtain that the quotients $\mathcal{O}_K/\mathfrak{a}$ are finite. Hence we can define a map

$$\boldsymbol{N} : \{\text{Ideals of } \mathcal{O}_K\} \longrightarrow \mathbb{N},$$

$$\boldsymbol{N}(0) = 0, \quad \boldsymbol{N}(\mathfrak{a}) = \#(\mathcal{O}_K/\mathfrak{a}).$$

We call $\boldsymbol{N}(\mathfrak{a})$ the norm of the ideal $\mathfrak{a}$. From the Chinese remainder theorem we conclude that the norm is multiplicative, i.e., $\boldsymbol{N}(\mathfrak{a}\mathfrak{b}) = \boldsymbol{N}(\mathfrak{a})\boldsymbol{N}(\mathfrak{b})$.

Theorem 2.1.5. *Every ideal $\mathfrak{a} \neq (0), \mathcal{O}_K$ of $\mathcal{O}_K$ admits a factorization*

$$\mathfrak{a} = \prod_{j=1}^{r} \mathfrak{p}_j^{\nu_j}, \qquad \nu_j \in \mathbb{N}_0,$$

into (non-zero) prime ideals which is unique up to the order of the factors.

Proof. See [50] for a complete proof of this statement. □

The reason behind this is that $\mathcal{O}_K$ is a so called Dedekind domain. This means, that it is a Noetherian and integrally closed ring such that each nonzero prime ideal is maximal.

Just as for nonzero numbers, we may extent the region of "integer ideals" by considering integer exponents $\nu_j \in \mathbb{Z}$ instead. With this we obtain so called fractional ideals, which are just finitely generated $\mathcal{O}_K$-submodules of K. More importantly we have

Proposition 2.1.6. *The fractional ideals form an abelian group, the so called ideal group J_K of K. The neutral element is $(1) = \mathcal{O}_K$ and the inverse of $\mathfrak{a} \neq (0)$:*

$$\mathfrak{a}^{-1} = \{x \in K \mid x\mathfrak{a} \subset \mathcal{O}_K\}.$$

Proof. For a proof of this statement, see for example [50] on p. 23. □

The fractional principal ideals $(a) = a\mathcal{O}_K$ with $a \in K^\times$ form a subgroup P_K of J_K. The quotient $\mathrm{Cl}_K = J_K/P_K$ is called the *class group* of K. We have the exact sequence

$$1 \longrightarrow \mathcal{O}_K^\times \longrightarrow K^\times \longrightarrow J_K \longrightarrow \mathrm{Cl}_K \longrightarrow 1,$$

so the class group can be interpreted as the expansion that takes place when we pass from "numbers" to ideals. Using Minkowski theory, one can show that Cl_K is a *finite* group. We call its number of elements the *class number* h_K of K. Similarly, the *narrow class group* is the quotient

$$\mathrm{Cl}_K^+ = J_K/P_K^+,$$

where P_K^+ is the group of all principal ideals, that are generated by an $a \in \mathcal{O}_K$ having exclusively positive embeddings. We will later demonstrate proofs for some results using totally real number fields with narrow class number 1, i.e., the narrow class group is trivial.

The class number turns out to be one of the most important invariants of a number field. Although a lot of effort has been put into finding relations to other mathematical areas (such as L-functions, cotangent sums, modular curves, etc.) it remains one of the bigger mysteries in modern mathematics. For example, it is conjectured, but still unsolved, if there is an infinite number of number fields with class number one. There even is a conjecture of Gauss saying there are infinitely many real quadratic fields with $h = 1$.

Minkowski theory

We say that K has signature (r_1, r_2), if there exist r_1 real and $2r_2$ non-real embeddings of K into $\mathbb{R}$ and $\mathbb{C}$, respectively. Note that $r_1 + 2r_2 = n = [K : \mathbb{Q}]$. Of course the number of non-real embeddings must be even since they occur in pairs by the complex conjugation. So in other words, we have *real* embeddings

$$\rho_1, ..., \rho_{r_1} : K \longrightarrow \mathbb{R}$$

and *complex* ones, which occur in pairs:

$$\sigma_1, \overline{\sigma_1}, ..., \sigma_{r_2}, \overline{\sigma_{r_2}} : K \longrightarrow \mathbb{C}.$$

We will sometimes also use the notation $x^{(j)}$ with $1 \leq j \leq r_1 + r_2$ for the embeddings $x^{(j)} := \rho_j x$ $(1 \leq j \leq r_1)$ and $x^{(j)} := \sigma_j x$ $(r_1 + 1 \leq j \leq r_1 + r_2)$ for some embeddings in the classes $(\sigma_{j-r_1}, \overline{\sigma_{j-r_1}})$. The idea of Minkowski is now to construct a "geometric object", which contains the field K. However, by modern research we know that this approach is outdated. But still, the resultant methods of Hecke are inspiring for our methods when generalizing formulas of Ramanujan, so we will keep the focus on them.

Definition 2.1.7. *Let $\{\tau\}$ denote all embeddings of K into $\mathbb{R}$ and $\mathbb{C}$, respectively. We call the euclidian vector space*

$$\boldsymbol{R} := \left\{ (z_\tau) \in \prod_\tau \mathbb{C} \;\middle|\; z_\rho \in \mathbb{R}; z_{\overline{\sigma}} = \overline{z_\sigma} \right\}$$

the Minkowski space of K.

One can show that $\boldsymbol{R}$ is canonically isomorphic to $\mathbb{R}^{r_1+2r_2}$ by the rule $(z_\tau) \mapsto (x_\tau)$, where $x_\rho = z_\rho$, $x_\sigma = \mathrm{Re}(z_\sigma)$ and $x_{\overline{\sigma}} = \mathrm{Im}(z_\sigma)$. Most importantly, there is a homomorphism of rings

$$i_K : K \longrightarrow \boldsymbol{R},$$

where addition and multiplication in $\boldsymbol{R}$ is defined component-wise. When considering the multiplicative subgroup $\boldsymbol{R}^\times$ (all components in the tuples are $\neq 0$) in $\boldsymbol{R}$, this restricts to a homomorphism of groups

$$\iota_\infty : K^\times \longrightarrow \boldsymbol{R}^\times.$$

The advantage of the Minkowski space is that we can embed K into a higher dimension object. This allows us, for example, to identify integral ideals $\mathfrak{a} \subseteq \mathcal{O}_K$ with lattices. When applying Fourier analysis on the Minkowski space, we are able to generalize the Poisson summation formula to "ideal lattices", from which transformation properties of the corresponding theta functions follow.

Embedded ideals $\mathfrak{a}$ of $\mathcal{O}_K$ in $\boldsymbol{R}$ will certainly be discrete subsets. We can say even more.

Proposition 2.1.8. *Let $\mathfrak{a} \neq (0)$ be an ideal of $\mathcal{O}_K$. Then, the embedding $i_K(\mathfrak{a})$ defines a complete lattice in $\boldsymbol{R}$. Let Γ be a fundamental mesh of this lattice. Then we have the volume formula*

$$\mathrm{Vol}(\Gamma) = \sqrt{|d_K|}\ (\mathcal{O}_K : \mathfrak{a}).$$

In particular, if $\mathcal{O}_K = \mathfrak{a}$, we obtain

$$\mathrm{Vol}(\Gamma) = \sqrt{|d_K|}.$$

Proof. For a proof of this statement, see for example [50] on p. 33. □

The Dirichlet unit theorem and the regulator

The following theorem describes the structure of the group of units $\mathcal{O}_K^\times$. It is a fundamental result and is important for the construction of Hecke characters.

Theorem 2.1.9 (Dirichlet unit theorem)**.** *Let K be a number field with signature (r_1, r_2). Let $r = r_1 + r_2 - 1$. Then the units $\mathcal{O}_K^\times$ form a finitely generated Abelian group. We have an isomorphism $\mathcal{O}_K^\times \cong \mu(K) \times \mathbb{Z}^r$ (the torsion part is given by the roots of unity $\mu(K)$ of K). In particular, we can write each unit $x \in \mathcal{O}_K^\times$ uniquely as*

$$x = \zeta u_1^{m_1} \cdots u_r^{m_r},$$

where ζ is a root of unity and $u_1, ..., u_r$ are fixed generators in $\mathcal{O}_K^\times$.

Proof. A proof of this statement is given in [46] on page 105 ff. □

We call an element $a \in K$ totally positive if all its embeddings are positive, i.e. $\tau a > 0$ for all $1 \leq j \leq n$. In this case we use the notation $a \gg 0$.

Corollary 2.1.10. *Let K be a totally real number field with narrow class number one. Let $\mathcal{O}_{K,+}^\times$ be the group of totally positive units. Then $\mathcal{O}_{K,+}^\times$ is free with rank $n - 1$.*

Finally, we introduce a way to measure the "density" of the the units of $\mathcal{O}_K$. Suppose that $u_1, ..., u_{r_1+r_2-1}$ are generators of the group $\mathcal{O}_K^\times/\mu(K)$. We define the matrix

$$A := \left(\log \left|u_p^{(q)}\right|\right)_{1 \leq p \leq r_1+r_2-1, 1 \leq q \leq r_1+r_2}.$$

Since the sum of each column is zero, the determinant of each $(r_1 + r_2 - 1) \times (r_1 + r_2 - 1)$-submatrix of A is independent of the removed column. We call its value the regulator R_K of the number field K. It does not depend on the choice of the generators. It can be shown that R_K is always a positive real number. In fact, the volume of a fundamental mesh of the lattice described in Dirichlet's unit theorem is equal to $R_K\sqrt{r_1 + r_2}$.

The discriminant and the different

The discriminant tells us, which prime numbers in $\mathbb{Z}$ ramify in $\mathcal{O}_K$. Originally, the different of a number field was constructed to obtain information about the prime ideals which are ramified in $\mathcal{O}_K$. It is an ideal which is divided by all prime ideals which ramify in $\mathcal{O}_K$.

Let $L|K$ be a separable extension. We consider the (nondegenerate) bilinear form $T(x, y) = \mathrm{Tr}(xy)$ on the K-vector space L. It induces a map from the fractional ideals of L to the $\mathcal{O}_L$-modules, given by

$$\mathfrak{a} \longmapsto {}^*\mathfrak{a} = \{x \in L \mid \mathrm{Tr}(x\mathfrak{a}) \subseteq \mathcal{O}_K\}.$$

Definition 2.1.11. *The fractional ideal*

$$\mathfrak{d}_{\mathcal{O}_L|\mathcal{O}_K} = \mathfrak{E}^{-1}_{\mathcal{O}_L|\mathcal{O}_K},$$

where

$$\mathfrak{E}_{\mathcal{O}_L|\mathcal{O}_K} = \{x \in L \mid \mathrm{Tr}(x\mathcal{O}_L) \subseteq \mathcal{O}_K\}$$

is called the (relative) different of $\mathcal{O}_L|\mathcal{O}_K$.

The different $\mathfrak{d}_K := \mathfrak{d}_{\mathcal{O}_K|\mathcal{O}_\mathbb{Q}}$ plays a role when defining the Gauss sum and proving the functional equation for the Hecke L-functions: let $\mathfrak{f} \subset \mathcal{O}_K$ be a non-zero ideal and χ be a character of $(\mathcal{O}_K/\mathfrak{f})^\times$ and $y \in \mathfrak{f}^{-1}\mathfrak{d}_K^{-1}$. Then we define the *Gauss sum* of χ by

$$\mathcal{G}_\mathfrak{f}(\chi; y) := \sum_{x \in (\mathcal{O}_K/\mathfrak{f})^\times} \chi(x) e^{2\pi i \mathrm{Tr}(xy)}. \tag{2.1.1.1}$$

2.1.2 Fourier transforms and the Poisson summation formula

In this small section we recall the most important basic facts about (high-dimensional) Fourier transforms. Amongst others, we will state the Poisson summation formula. This section is mostly inspired by [46], there the reader can also find more detailed information and examples.

Let f be a function on $\mathbb{R}^n$. We say that f tends rapidly to 0 at infinity if for each positive integer N we have $f(x) = O((1 + ||x||)^{-N})$.

Definition 2.1.12. *We define the Schwartz space* S_n *to be the set of functions* $f : \mathbb{R}^n \to \mathbb{C}$ *with the following properties:*

(i) The function f *is smooth in each variable (i.e. partial derivatives of all orders exist and are continuous).*

(ii) The function f *and all its partial derivatives tend rapidly to zero at infinity.*

An important example for a function in the Schwartz space S_n is $f(x) = e^{-(x_1^2+\cdots+x_n^2)}$. Clearly, the Schwartz space forms a vector space, but we can even say more. Let ∂_j be the partial derivative in the x_j direction. Then, for each $p = (p_1, ..., p_n) \in \mathbb{N}_0^n$ the operator

$$\partial^p := \partial_1^{p_1} \cdots \partial_n^{p_n}$$

defines and endomorphism of S_n. It is also convenient to use the notations $|p| = p_1 + \cdots + p_n$ and

$$(M_j f)(x) = x_j f(x).$$

Note that $M^p f = M^{p_1} \cdots M^{p_n} f$ maps S_n to itself as well.

Definition 2.1.13. *Let $f \in S_n$. We define the Fourier transform $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ of f by*

$$\widehat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, y \rangle} \mathrm{d}x,$$

where $\mathrm{d}x = \mathrm{d}x_1 \cdots \mathrm{d}x_n$.

Since $f(x) = O((1 + x_1^2)^{-1} \cdots (1 + x_n^2)^{-1})$ it is obvious that $\widehat{f}$ is well-defined. Less obvious is the following.

Proposition 2.1.14. *The Fourier transform $f \mapsto \widehat{f}$ defines an endomorphism of S_n.*

Proof. We first observe that $\widehat{f}$ is bounded by

$$\left| \widehat{f}(y) \right| \leq \int_{\mathbb{R}^n} |f(x)| \mathrm{d}x.$$

Note that the right hand side is bounded since f is decreasing rapidly. A simple observation shows

$$\partial^p \widehat{f} = (-2\pi i)^{|p|} \widehat{M^p f}$$

and

$$M^p \widehat{f} = (2\pi i)^{-|p|} \widehat{D^p f}.$$

Note that switching integral and differential operator is justified by absolute integrability. As a consequence, for each p, we find that $M^p \widehat{f}$ is bounded at infinity. Hence $\widehat{f}$ tends rapidly to 0 at infinity. Similarly, we see that $M^p D^q$ is bounded at infinity since

$$M^p D^q \widehat{f} = (-2\pi i)^{|q|} (2\pi i)^{-|p|} \widehat{D^p M^q f}.$$

This proves the proposition. □

We call a function $g : \mathbb{R}^n \to \mathbb{C}$ periodic with respect to the lattice L if $g(x+k) = g(x)$ for all $k \in L$. Note that g is also defined on the hyper rectangle $\mathbb{R}^n / L$. Let g be a smooth periodic function. Then we have a representation as a Fourier series

$$g(x) = \sum_{m \in L^*} c_m e^{2\pi i \langle m, x \rangle},$$

which converges uniformly, where the dual lattice L^* to L is the set of all $x \in \mathbb{R}^n$ such that $\langle x, y \rangle \in \mathbb{Z}$ for all $y \in L$. Note that we obtain the m-th Fourier coefficient for $m \in L^*$ by

$$c_m = \frac{1}{\mathrm{Vol}(\mathbb{R}^n / L)} \int_{\mathbb{R}^n / L} g(x) e^{-2\pi i \langle m, x \rangle} \mathrm{d}x.$$

Theorem 2.1.15 (Poisson summation formula)**.** *Let f be in the Schwartz space S_n. Then we have*

$$\sum_{m\in L} f(m) = \frac{1}{\mathrm{Vol}(\Gamma_L)} \sum_{m\in L^*} \widehat{f}(m)$$

for any lattice $L \subset \mathbb{R}^n$.

Proof. The function

$$g(x) = \sum_{k\in L} f(x+k)$$

is well-defined (since f is decreasing rapidly), smooth and periodic with respect to L. If c_m is the m-the Fourier coefficient, then

$$g(0) = \sum_{m\in L^*} c_m = \sum_{k\in L} f(k).$$

Therefore it is sufficient to prove $c_m = \frac{1}{\mathrm{Vol}(\Gamma_L)}\widehat{f}(m)$. Indeed, we have

$$\begin{aligned} c_m &= \frac{1}{\mathrm{Vol}(\Gamma_L)} \int_{\Gamma_L} g(x) e^{-2\pi i\langle m,x\rangle} \mathrm{d}x = \frac{1}{\mathrm{Vol}(\Gamma_L)} \sum_{k\in L} \int_{\Gamma_L} f(x+k) e^{-2\pi i\langle m,x\rangle} \mathrm{d}x \\ &= \frac{1}{\mathrm{Vol}(\Gamma_L)} \sum_{k\in L} \int_{\Gamma_L} f(x+k) e^{-2\pi i\langle m,x+k\rangle} \mathrm{d}x = \frac{1}{\mathrm{Vol}(\Gamma_L)} \int_{\mathbb{R}^n} f(x) e^{-2\pi i\langle m,x\rangle} \mathrm{d}x \\ &= \frac{1}{\mathrm{Vol}(\Gamma_L)} \widehat{f}(m). \end{aligned}$$

This proves the theorem. □

2.1.3 Hecke characters

Hecke characters are a generalization of Dirichlet characters to arbitrary number fields. They were first introduced by Erich Hecke, who proved the functional equation of their L-functions, see also [36]. In this section we want to recall the ideas of Hecke and - for the reader's convenience - we include proofs for the functional equations in some special cases.

In general, let K be a number field with signature (r_1, r_2), $\mathfrak{f} \subseteq \mathcal{O}_K$ be an integral ideal and $J_K^{\mathfrak{f}} \subseteq J_K$ be the subgroup of all ideals $\mathfrak{a}$, such that $\mathfrak{f}$ and $\mathfrak{a}$ have no prime factor in common. Let $\chi : J_K^{\mathfrak{f}} \to \mathbb{C}^\times$ be a character. We may form the L-series

$$L(\chi; s) := \sum_{\mathfrak{a}} \chi(\mathfrak{a}) \boldsymbol{N}(\mathfrak{a})^{-s},$$

with respect to χ, where the sum is taken over all integral ideals $\mathfrak{a}$ of K and we put $\chi(\mathfrak{a}) = 0$ whenever $(\mathfrak{a}, \mathfrak{f}) \neq 1$. Hecke was searching for the most comprehensive class

of such characters $\chi \in \mathrm{Hom}(J_K^{\mathfrak{f}}, \mathbb{C}^\times)$ for which the corresponding L-series has analytic continuation and satisfies a functional equation.

Define also the mutiplicative subgroup $K^{(\mathfrak{f})} \subseteq K$ consisting of all numbers $x \in K$ such that $(x, \mathfrak{f}) = 1$. This means that the ideals (x) and $\mathfrak{f}$ have distinct factorization into prime ideals. Note that we always have a homomorphism

$$\iota_{\mathfrak{f}} : K^{(\mathfrak{f})} \longrightarrow (\mathcal{O}_K/\mathfrak{f})^\times, \qquad \frac{a}{b} \longmapsto \frac{[a+\mathfrak{f}]}{[b+\mathfrak{f}]}.$$

This leads us to the notion of a Hecke character.

Definition 2.1.16. *A Hecke character mod $\mathfrak{f}$ is a character $J_K^{\mathfrak{f}} \to \mathbb{C}^\times$ for which there exists a pair $(\chi_{\mathfrak{f}}, \chi_\infty)$ of characters,*

$$\chi_{\mathfrak{f}} : (\mathcal{O}_K/\mathfrak{f})^\times \longrightarrow \mathbb{C}^\times, \qquad \chi_\infty : \boldsymbol{R}^\times \longrightarrow \mathbb{C}^\times,$$

such that

$$\chi((a)) = \chi_{\mathfrak{f}}(\iota_{\mathfrak{f}}(a))\chi_\infty(\iota_\infty(a))$$

for every algebraic integer $a \in \mathcal{O}_K$ coprime to $\mathfrak{f}$. Recall that $\iota_\infty : K^\times \to \boldsymbol{R}^\times$ was the embedding of multiplicative groups.

In future applications, we will abbreviate $\chi_{\mathfrak{f}}(a)$ and $\chi_\infty(a)$. Let $\mathfrak{f}' \mid \mathfrak{f}$ be any proper divisor of $\mathfrak{f}$. We call a Hecke character *primitive*, if it is not the restriction of a Hecke character $\chi' : J_K^{\mathfrak{f}'} \to \mathbb{C}^\times$. This is equivalent to the assertion that $\chi_{\mathfrak{f}}$ does not factor through $(\mathcal{O}_K/\mathfrak{f})^\times \to (\mathcal{O}_K/\mathfrak{f}')^\times \to \mathbb{C}^\times$.

The next step is to identify all possible characters

$$\chi_\infty : \boldsymbol{R}^\times \longrightarrow \mathbb{C}^\times.$$

Note that the terminus of a character ψ always means, that, besides multiplicativity, $|\psi(x)| = 1$ for all arguments x. We have the following

Proposition 2.1.17. *There is a 1-1 correspondence between characters*

$$\chi_\infty : \boldsymbol{R}^\times \longrightarrow \mathbb{C}^\times,$$

and numbers $\varepsilon_{\rho_1}, \varepsilon_{\rho_2}, ..., \varepsilon_{\rho_{r_1}}, \varepsilon_{\sigma_1}, \varepsilon_{\overline{\sigma_1}}, ..., \varepsilon_{\sigma_{r_2}}, \varepsilon_{\overline{\sigma_{r_2}}} \in \mathbb{Z}$ with $\varepsilon_\rho \in \{0, 1\}$, $\varepsilon_\sigma, \varepsilon_{\overline{\sigma}} \geq 0$, $\varepsilon_\sigma \varepsilon_{\overline{\sigma}} = 0$ and $\nu_{\rho_1}, \nu_{\rho_2}, ..., \nu_{\rho_{r_1}}, \nu_{\sigma_1}, \nu_{\overline{\sigma_1}}, ..., \nu_{\sigma_{r_2}}, \nu_{\overline{\sigma_{r_2}}} \in \mathbb{R}$ with $\nu_\sigma = \nu_{\overline{\sigma}} =: \nu_{\sigma,\overline{\sigma}}$. We put $\varepsilon_{\sigma,\overline{\sigma}} = \varepsilon_\sigma + \varepsilon_{\overline{\sigma}}$. The correspondence is then given by

$$(\varepsilon, i\nu) \longmapsto \left((t_1, ..., t_{r_1}, s_1, \overline{s_1}, ..., s_{r_2}, \overline{s_{r_2}}) \longmapsto \prod_{j=1}^{r_1} |t_j|^{i\nu_{\rho_j}} \left(\frac{t_j}{|t_j|}\right)^{\varepsilon_{\rho_j}} \prod_{k=1}^{r_2} |s_k|^{i\nu_{\sigma_k,\overline{\sigma_k}}} \left(\frac{s_k}{|s_k|}\right)^{\varepsilon_{\sigma_k,\overline{\sigma_k}}} \right).$$

A proof of Proposition 2.1.17 can be found in [50].

In other words, we can associate to each Hecke character a uniquely determined tuple $(\varepsilon, i\nu)$ like in Proposition 2.1.17.

Throughout we denote

$$\tilde{\varepsilon}_\rho := (\varepsilon_{\rho_1}, ..., \varepsilon_{\rho_{r_1}}), \qquad \tilde{\nu}_\rho := (\nu_{\rho_1}, ..., \nu_{\rho_{r_1}}).$$

and

$$\tilde{\varepsilon}_\sigma := (\varepsilon_{\sigma_1,\overline{\sigma_1}}, ..., \varepsilon_{\sigma_{r_2},\overline{\sigma_{r_2}}}), \qquad \tilde{\nu}_\sigma := (\nu_{\sigma_1,\overline{\sigma_1}}, ..., \nu_{\sigma_{r_2},\overline{\sigma_{r_2}}}).$$

The case of totally real number fields with narrow class number one

The previous definition of a Hecke character does not give any intuition on the construction. In this section we give the reader a detailed description, starting from a different direction, which are the characters of $(\mathcal{O}_K/\mathfrak{f})^\times$ and how to lift them to "characters of ideals". We reduce the situation to totally real number fields with narrow class number one. We abbreviate the real embedding $\rho_j(x)$ by $x^{(j)}$.
To understand the idea of Hecke characters properly one should first look at a classical Dirichlet character χ mod N. We are able to interpret χ as a function of ideals described as follows. Let $\mathfrak{a} \subset \mathbb{Z}$ be an ideal. Then $\mathfrak{a}$ is generated by a (unique) non-negative integer a. We put $\chi(\mathfrak{a}) = \chi(a)$ and thus $\chi(\mathfrak{a}) \neq 0$ if and only if $(\mathfrak{a}, N) = 1$, in which case we have $|\chi(\mathfrak{a})| = 1$. Now, the Dirichlet L-function is given by

$$L(\chi; s) = \sum_{(0)\neq\mathfrak{a}\subset\mathbb{Z}} \chi(\mathfrak{a})\boldsymbol{N}(\mathfrak{a})^{-s} = \sum_{n=1}^{\infty} \chi(n)n^{-s}.$$

The reason why it is more useful to think of characters of ideals is obvious, since we do not have a unique factorization into primes in the ring $\mathcal{O}_K$ (at least when the class number of K is greater than 1). As a consequence, as it was shown in the previous sections, it is reasonable to rather think in decompositions of ideals since here the unique factorization is recovered. However, if K is any totally real number field the situation is more difficult. Let $\mathfrak{f}$ be an ideal of $\mathcal{O}_K$ and $\chi_\mathfrak{f}$ be a character of the multiplicative group $(\mathcal{O}_K/\mathfrak{f})^\times$. We assume that $\chi_\mathfrak{f}$ is primitive, hence it does not factor through $(\mathcal{O}_K/\mathfrak{f})^\times \to (\mathcal{O}_K/\mathfrak{f}_1)^\times$ for any proper divisor $\mathfrak{f}_1$ of $\mathfrak{f}$. Our goal is to identify $\chi_\mathfrak{f}$ with a "character" on the ideals of $\mathcal{O}_K$, as we have done it with Dirichlet characters in the case $K = \mathbb{Q}$. First, we extend $\chi_\mathfrak{f}$ to a function on $\mathcal{O}_K$ via

$$\chi_\mathfrak{f}(a) = \begin{cases} \chi_\mathfrak{f}(a \bmod \mathfrak{f}), & \text{if } (a, \mathfrak{f}) = 1, \\ 0 & \text{else.} \end{cases}$$

We would like to obtain a character on ideals, or at least principal ideals. However, $\chi_\mathfrak{f}$ may not be trivial on units, in other words, $\chi_\mathfrak{f}$ might take values $\neq 1$ on $\mathcal{O}_K^\times$. The

problem is that under these circumstances the extension to (principal) ideals may not be well-defined. We can remedy this by including a further character χ_∞, the "infinite" part of our Hecke character. Let $\nu_1, ..., \nu_n$ be real parameters such that

$$\nu_1 + \cdots + \nu_n = 0.$$

Let ε_j be numbers each equal either 0 or 1. For each $1 \leq j \leq n$ we define a character χ_j of $\mathbb{R}^\times$ by $\chi_j(x) = \operatorname{sgn}(x)^{\varepsilon_j}|x|^{i\nu_j}$. Then the corresponding character χ_∞ of $\mathcal{O}_K^\times$ is defined by

$$\chi_\infty(x) = \prod_{j=1}^{n} \chi_j\left(x^{(j)}\right).$$

Note that χ_∞ is indeed a character since the embeddings are homomorphisms of fields. We may choose these data so that $\chi(x) := \chi_{\mathfrak{f}}(x)\chi_\infty(x)$ is trivial on $\mathcal{O}_K^\times$. Now χ_∞ can be interpreted as the "infinite" part of the character χ. As we will see, this is a consequence of the unit theorem. Indeed, by Corollary 2.1.10 the group $\mathcal{O}_{K,+}^\times$ of totally positive units is free of rank $n-1$. Let $u_1, ..., u_{n-1}$ be a basis of $\mathcal{O}_{K,+}^\times$. Let $m_1, ..., m_{n-1}$ be integers. We choose the numbers $\nu_1, ..., \nu_n$ to satisfy the linear equations

$$i\nu_1 + \cdots + i\nu_n = 0$$

$$\sum_{j=1}^{n} i\nu_j \log(u_k^{(j)}) = 2\pi i m_k - \log \chi_{\mathfrak{f}}(u_k).$$

The determinant of the corresponding matrix is essentially the regulator of K and therefore nonzero. Obviously, the branch of the logarithm is not important, since we may choose other values for the m_j. Given that $|\chi_{\mathfrak{f}}(u_k)| = 1$, we conclude that the right hand side is purely imaginary and hence the numbers $\nu_1, ..., \nu_n$ are all real. We now may choose the $\nu_1, ..., \nu_n$ so that χ is trivial on $\mathcal{O}_{K,+}^\times$. Indeed, if $u = u_1^{r_1} \cdots u_{n-1}^{r_{n-1}}$ for some $r_j \in \mathbb{Z}$ is a unit in $\mathcal{O}_{K,+}^\times$, we obtain

$$\begin{aligned}
\chi(u) = \chi_{\mathfrak{f}}(u)\chi_\infty(u) &= \chi_{\mathfrak{f}}(u_1)^{r_1} \cdots \chi_{\mathfrak{f}}(u_{n-1})^{r_{n-1}} \prod_{j=1}^{n} \chi_j(u^{(j)}) \\
&= \chi_{\mathfrak{f}}(u_1)^{r_1} \cdots \chi_{\mathfrak{f}}(u_{n-1})^{r_{n-1}} \prod_{j=1}^{n} \operatorname{sgn}(u^{(j)})^{\varepsilon_j}|u^{(j)}|^{i\nu_j} \\
&= \chi_{\mathfrak{f}}(u_1)^{r_1} \cdots \chi_{\mathfrak{f}}(u_{n-1})^{r_{n-1}} \prod_{j=1}^{n} |(u_1^{(j)})^{r_1} \cdots (u_{n-1}^{(j)})^{r_{n-1}}|^{i\nu_j} \\
&= \chi_{\mathfrak{f}}(u_1)^{r_1} \cdots \chi_{\mathfrak{f}}(u_{n-1})^{r_{n-1}} \prod_{j=1}^{n} (u_1^{(j)})^{ir_1\nu_j} \cdots \prod_{j=1}^{n} (u_{n-1}^{(j)})^{ir_{n-1}\nu_j} \\
&= \chi_{\mathfrak{f}}(u_1)^{r_1} \cdots \chi_{\mathfrak{f}}(u_{n-1})^{r_{n-1}} \chi_{\mathfrak{f}}(u_1)^{-r_1} \cdots \chi_{\mathfrak{f}}(u_{n-1})^{-r_{n-1}} = 1.
\end{aligned}$$

Moreover, if $w \in \mathcal{O}_K^\times$, $\chi(w)$ only depends on the signs of the $w^{(j)}$, so the ε_j may be chosen such that χ is trivial on units. In other words, since χ is trivial on $\mathcal{O}_{K,+}^\times$ for each choice of the ε_j we obtain a real character of the finite quotient $\mathcal{O}_K^\times/\mathcal{O}_{K,+}^\times$. By adjusting the ε_j we may make this character trivial (for example using generators $\pm w_1, ..., \pm w_{n-1}$ modulo $\mathcal{O}_{K,+}^\times$). Thus, we obtain a character χ of principal ideals prime to $\mathfrak{f}$. Finally, a character on $J_K^{\mathfrak{f}}$ whose restriction to $P_K^{\mathfrak{f}}$ arises in such a way is called a Hecke character.

2.1.4 Hecke L-functions of number fields

Let K be a totally real number field of degree n and χ a Hecke character. Then we define its *Hecke L-function* by

$$L(\chi; s) := \sum_{\mathfrak{a}} \chi(\mathfrak{a}) \boldsymbol{N}(\mathfrak{a})^{-s}.$$

Here the sum is taken over all non-zero ideals of $\mathcal{O}_K$. The unique factorization into prime ideals of $\mathcal{O}_K$ is summarized by the identity

$$L(\chi; s) = \prod_{\mathfrak{p}} \frac{1}{1 - \chi(\mathfrak{p}) \boldsymbol{N}(\mathfrak{p})^{-s}}.$$

This Euler product converges absolutely on the half plane $\mathrm{Re}(s) > 1$. An important special case, if $\mathfrak{f} = 1$ and χ is the trivial character, is the *Dedekind zeta function:*

$$\zeta_K(s) = \sum_{\mathfrak{a}} \boldsymbol{N}(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} \frac{1}{1 - \boldsymbol{N}(\mathfrak{p})^{-s}}.$$

In this section we prove that the Hecke L-functions have meromorphic continuation to the entire plane and satisfy a functional equation. These are the important foundational results we need for the generalization of Ramanujan's identities.
Before we can state the functional equation, we have to introduce some more notation. Denote the "infinite part" of the complete Hecke L-function, which is defined below, by

$$\begin{aligned}
L_\infty(\chi; s) &:= \prod_{j=1}^{r_1} \pi^{-\frac{s - i\nu_{\rho_j} + \varepsilon_{\rho_j}}{2}} \Gamma\left(\frac{s - i\nu_{\rho_j} + \varepsilon_{\rho_j}}{2}\right) \\
&\quad \times \prod_{k=1}^{r_2} 2(2\pi)^{-\left(s - i\nu_{\sigma_k,\overline{\sigma_k}} + \frac{\varepsilon_{\sigma_k,\overline{\sigma_k}}}{2}\right)} \Gamma\left(s - i\nu_{\sigma_k,\overline{\sigma_k}} + \frac{\varepsilon_{\sigma_k,\overline{\sigma_k}}}{2}\right) \\
&= C_\chi \left(2^{r_2} \pi^{\frac{n}{2}}\right)^{-s} \prod_{j=1}^{r_1} \Gamma\left(\frac{s - i\nu_{\rho_j} + \varepsilon_{\rho_j}}{2}\right) \prod_{k=1}^{r_2} \Gamma\left(s - i\nu_{\sigma_k,\overline{\sigma_k}} + \frac{\varepsilon_{\sigma_k,\overline{\sigma_k}}}{2}\right) \\
&= C_\chi \left(2^{r_2} \pi^{\frac{n}{2}}\right)^{-s} \prod_{j=1}^{r_1} \Gamma\left(\frac{s - i\nu_{\rho_j} + \varepsilon_{\rho_j}}{2}\right) \gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2} - i\tilde{\nu}_\sigma, \mathbf{0}, \mathbf{1}, \mathbf{0}}(s),
\end{aligned}$$

using the notation from Definition 2.2.3, where

$$C_\chi := \pi^{\frac{\mathrm{Tr}(i\nu-\varepsilon)}{2}} 2^{\mathrm{Tr}(i\tilde{\nu}_\sigma+\mathbf{1}-\frac{\tilde{\varepsilon}_\sigma}{2})}. \tag{2.1.4.1}$$

For this definition, see also [50] on p. 518.

Definition 2.1.18. *Let χ be a primitive Hecke character with conductor $\mathfrak{f}$. We define the complete Hecke L-function by*

$$\Lambda_K(\chi; s) := (|d_K|\boldsymbol{N}(\mathfrak{f}))^{\frac{s}{2}} L_\infty(\chi; s) L(\chi; s).$$

Definition 2.1.19. *Let K be a totally real number field and $d \gg 0$ be a generator of the different $\mathfrak{d}$ of K and $f \gg 0$ be a generator of the ideal $\mathfrak{f}$. Then we define the Gauss sum of $\chi_\mathfrak{f}$ with respect to the choice of the tuple (d, f) by*

$$\mathcal{G}_{K,d,f}(\chi_\mathfrak{f}) := \sum_{\alpha \in \mathcal{O}_K/\mathfrak{f}} \chi_\mathfrak{f}(\alpha) e^{2\pi i \mathrm{Tr}(\frac{\alpha}{fd})}.$$

Note that by (2.1.1.1) *we have $\mathcal{G}_{K,d,f}(\chi_\mathfrak{f}) = \mathcal{G}_\mathfrak{f}(\chi_\mathfrak{f}; \frac{1}{fd})$.*

We have the following identity for primitive $\chi_\mathfrak{f}$:

$$\mathcal{G}_{K,d,f}(\overline{\chi_\mathfrak{f}})\chi_\mathfrak{f}(\alpha) = \sum_{\beta \in \mathcal{O}_K/\mathfrak{f}} \overline{\chi_\mathfrak{f}(\beta)} e^{2\pi i \mathrm{Tr}(\frac{\alpha\beta}{fd})}. \tag{2.1.4.2}$$

From this one can derive

$$\mathcal{G}_{K,d,f}(\overline{\chi_\mathfrak{f}})\mathcal{G}_{K,d,f}(\chi_\mathfrak{f}) = \chi_\mathfrak{f}(-1)\boldsymbol{N}(\mathfrak{f}),$$

and in particular

$$|\mathcal{G}_{K,d,f}(\chi_\mathfrak{f})| = \sqrt{\boldsymbol{N}(\mathfrak{f})}. \tag{2.1.4.3}$$

Theorem 2.1.20 (Functional equation for Hecke L-functions, see also [50] on p. 524)**.** *Let χ be a primitive character with conductor $\mathfrak{f}$. The function $\Lambda_K(\chi; s)$ has meromorphic continuation to the entire plane, is bounded on vertical strips (except near possible poles) and fulfills the functional equation*

$$\Lambda_K(\chi; s) = W(\chi)\Lambda_K(\overline{\chi}; 1-s),$$

where $W(\chi)$ satisfies $|W(\chi)| = 1$. The function $\Lambda_K(\chi; s)$ (and so $L(\chi; s)$) is holomorphic except for possible poles of order at most one at $s = \frac{\mathrm{Tr}(-\varepsilon+i\nu)}{n}$ and $s = 1 + \frac{\mathrm{Tr}(\varepsilon+i\nu)}{n}$. If $\mathfrak{f} \neq 1$ or $\varepsilon \neq 0$, $\Lambda_K(\chi; s)$ is an entire function.

One can give an explicit formula for the factor $W(\chi)$ in terms of χ.

Proofs for the very general case can be found in [36], [50] and *Tate's thesis* [56]. Here, the functional equations are proved using a purely adelic approach. A good exposition can be found in [46]. We will not present the proof here, since it uses techniques that will not be required for the applications.

Proof. We only sketch the main ideas in the case that K is totally positive with narrow class number one. Consider the lattice $\mathcal{O}_K \subset \mathbb{R}^n$. For any function F in the Schwartz space S_n we have the Poisson summation formula

$$\sum_{\alpha \in \mathcal{O}_K} F(\alpha) = \frac{1}{\sqrt{d_K}} \sum_{\alpha \in \mathcal{O}_K} \widehat{F}\left(\frac{\alpha}{d}\right).$$

This is the case as the dual $\mathcal{O}_K^*$ is just given by the inverse different $\mathfrak{d}^{-1}$ which is generated by d^{-1}, hence $\mathcal{O}_K^* = d^{-1}\mathcal{O}_K$. Also note that $\mathrm{Vol}(\Gamma_{\mathcal{O}_K}) = \sqrt{d_K}$ by Proposition 2.1.8, hence the claim follows by applying Theorem 2.1.15. Using (2.1.4.2) we obtain the twisted Poisson summation formula

$$\mathcal{G}_{K,d,f}(\overline{\chi_{\mathfrak{f}}}) \sum_{\alpha \in \mathcal{O}_K} \chi_{\mathfrak{f}}(\alpha) F(\alpha) = \frac{1}{\sqrt{d_K}} \sum_{\alpha \in \mathcal{O}_K} \overline{\chi_{\mathfrak{f}}(\alpha)} \widehat{F}\left(\frac{\alpha}{df}\right). \tag{2.1.4.4}$$

This is a series of functions and for suitably chosen F the right side is non-zero. Therefore, in the following we assume $\mathcal{G}_{K,d,f}(\overline{\chi_{\mathfrak{f}}}) \neq 0$.
Let $t = (t_1, ..., t_n) \gg 0$ be a vector. We shall calculate the Fourier transform of the function $F_t : \mathbb{R}^n \to \mathbb{C}$ with

$$F_t(x) = N(x^{\varepsilon}) e^{-\pi \mathrm{Tr}(\frac{tx^2}{d})},$$

where $N(x^{\varepsilon}) = \prod_{j=1}^n x_j^{\varepsilon_j}$. In our notation we also have $d = (d_j) = (d^{(j)})$. For each $1 \leq j \leq n$ we have the integral

$$\frac{1}{\sqrt{d^{(j)}}} \int_{-\infty}^{\infty} e^{-\frac{\pi t_j x_j^2}{d^{(j)}} - \frac{2\pi i y_j x_j}{d^{(j)}}} \mathrm{d}x_j = \frac{1}{\sqrt{t_j}} e^{-\frac{\pi y_j^2}{t_j d^{(j)}}}. \tag{2.1.4.5}$$

This well-known fact can be proved for example using the Residue theorem. By differentiating we obtain the second identity

$$-\frac{1}{\sqrt{d^{(j)}}} \int_{-\infty}^{\infty} x_j e^{-\frac{\pi t_j x_j^2}{d^{(j)}} - \frac{2\pi i y_j x_j}{d^{(j)}}} \mathrm{d}x_j = \frac{i y_j}{t_j^{\frac{3}{2}}} e^{-\frac{\pi y_j^2}{t_j d^{(j)}}}. \tag{2.1.4.6}$$

Putting (2.1.4.5) and (2.1.4.6) together shows

$$\widehat{F_t}(x) = \frac{1}{\sqrt{N(t)}} N\left(\left(\frac{i}{t}\right)^{\varepsilon}\right) F_{\frac{1}{t}}(x). \tag{2.1.4.7}$$

Of course we want to use the Poisson summation formula. We apply this to the theta function $\Theta : \mathbb{R}^n_{>0} \to \mathbb{C}$ defined by

$$\Theta(\chi_{\mathfrak{f}}, t) = \sum_{\alpha \in \mathcal{O}_K} \chi_{\mathfrak{f}}(\alpha) N(\alpha^{\varepsilon}) e^{-\pi \mathrm{Tr}(\frac{t\alpha^2}{d})}.$$

By (2.1.4.4) and (2.1.4.7) we obtain the functional equation

$$\Theta(\chi_{\mathfrak{f}}, t) = \frac{1}{\mathcal{G}_{K,d,f}(\overline{\chi_{\mathfrak{f}}})} \frac{1}{\sqrt{N(t)}} N\left(\left(\frac{i}{ft}\right)^{\varepsilon}\right) \Theta\left(\overline{\chi_{\mathfrak{f}}}, \frac{1}{f^2 t}\right).$$

As usual, we prove the analytic continuation and functional equation of $\Lambda(\chi, s)$ by expressing it as an integral of the corresponding automorphic function, in this case $\Theta(\chi_{\mathfrak{f}}, t)$. First assume that $\mathfrak{f} \neq \mathcal{O}_K$. Since $\chi_{\mathfrak{f}}$ is primitive, it must be non-trivial and the series $\Theta(\chi_{\mathfrak{f}}, t)$ then has no constant term and hence is of rapid decay as $t_j \to \infty$. For complex values s (with sufficiently large real part) we show the relation

$$\int_{\mathbb{R}^n_{>0}/\mathcal{O}^{\times}_{K,+}} \Theta(\chi_{\mathfrak{f}}, t) N(t^{\frac{s+\varepsilon-\nu i}{2}}) \frac{\mathrm{d}t}{N(t)} = 2\pi^{-\mathrm{Tr}(\varepsilon)} N(d^{\frac{\varepsilon-i\nu}{2}}) \Lambda(\chi, s). \tag{2.1.4.8}$$

This readily implies the analytic continuation of $\Lambda(\chi, s)$ and by replacing t by $\frac{1}{f^2 t}$, we obtain

$$N(d^{\frac{\varepsilon-i\nu}{2}}) \Lambda(\chi, s) = \mathcal{G}_{K,d,f}(\overline{\chi_{\mathfrak{f}}})^{-1} N\left(\left(\frac{i}{f}^{\varepsilon}\right)\right) N(f^{-s1+1+\varepsilon+i\nu}) N(d^{\frac{\varepsilon+i\nu}{2}}) \Lambda(\overline{\chi}, 1-s).$$

Then the claim follows by applying (2.1.4.3). So we are left to show (2.1.4.8). Of course it has to be checked that the integrand on the left is invariant under the action of $\mathcal{O}^{\times}_{K,+}$, but this will be clear after the following arguments. Since every ideal has a single totally positive generator, every element in $\mathcal{O}_K$ can be written as a product of a totally positive number and a unit. Hence there is a bijection $\mathcal{O}_K \cong \mathcal{O}_{K,+}/\mathcal{O}^{\times}_{K,+} \times \mathcal{O}^{\times}_K$, and the left hand side of (2.1.4.8) is equal to

$$\begin{aligned} &\sum_{\alpha \in \mathcal{O}_K} \chi_{\mathfrak{f}}(\alpha) N(\alpha^{\varepsilon}) \int_{\mathbb{R}^n_+/\mathcal{O}^{\times}_{K,+}} e^{-\pi \mathrm{Tr}\left(\frac{t\alpha^2}{d}\right)} N(t^{\frac{s+\varepsilon-\nu i}{2}}) \frac{\mathrm{d}t}{N(t)} \\ = &\sum_{\alpha \in \mathcal{O}_{K,+}/\mathcal{O}^{\times}_{K,+}} \sum_{\eta \in \mathcal{O}^{\times}_K} \chi_{\mathfrak{f}}(\eta\alpha) N((\eta\alpha)^{\varepsilon}) \int_{\mathbb{R}^n_+/\mathcal{O}^{\times}_{K,+}} e^{-\pi \mathrm{Tr}\left(\frac{t\eta^2\alpha^2}{d}\right)} N(t^{\frac{s+\varepsilon-i\nu}{2}}) \frac{\mathrm{d}t}{N(t)}. \end{aligned}$$

The next step shows that all of this is well-defined. Since χ is trivial on units we have $\chi_{\mathfrak{f}}(\eta) = N(\mathrm{sgn}(\eta)^{-\varepsilon})N(|\eta|^{-i\nu})$. Therefore, we see $\chi_{\mathfrak{f}}(\eta)N(\eta^{\varepsilon}) = \chi_{\mathfrak{f}}(\eta)N(\mathrm{sgn}(\eta)^{\varepsilon})N(|\eta|^{\varepsilon}) = N(|\eta|^{\varepsilon-i\nu})$. Hence

$$\begin{aligned}
&= \sum_{\alpha\in\mathcal{O}_{K,+}/\mathcal{O}_{K,+}^{\times}} \sum_{\eta\in\mathcal{O}_K^{\times}} \chi_{\mathfrak{f}}(\alpha)N(\alpha^{\varepsilon}) \int_{\mathbb{R}_+^n/\mathcal{O}_{K,+}^{\times}} e^{-\pi\mathrm{Tr}\left(\frac{t\eta^2\alpha^2}{d}\right)} N(t^{\frac{s+\varepsilon-i\nu}{2}})\frac{\mathrm{d}t}{N(t)} \\
&= 2\sum_{\alpha\in\mathcal{O}_{K,+}/\mathcal{O}_{K,+}^{\times}} \chi_{\mathfrak{f}}(\alpha)N(\alpha^{\varepsilon}) \int_{\mathbb{R}_+^n} e^{-\pi\mathrm{Tr}\left(\frac{t\alpha^2}{d}\right)} N(t^{\frac{s+\varepsilon-i\nu}{2}})\frac{\mathrm{d}t}{N(t)}.
\end{aligned}$$

The last equality holds since η runs through $\mathcal{O}_K^{\times}$, so clearly η^2 runs through $\mathcal{O}_{K,+}^{\times}$ twice. The resultant integral is just the gamma factor introduced in Definition 2.1.18. Hence (2.1.4.8) follows.

The case $\mathfrak{f} = \mathcal{O}_K$ is treated similarly. We omit the details in this case and refer to a very detailed proof presented in [46] in Chapter XIII. □

A very important class of Hecke L-functions is the Dedekind zeta function $\zeta_K(s)$ of a number field K. It corresponds to the trivial character $\chi_{\mathfrak{f}}$ when $\mathfrak{f} = \mathcal{O}_K$ and is a generalization of the Riemann zeta function $\zeta(s)$, which corresponds to the number field $\mathbb{Q}$.

For example, in the case that K is a totally real quadratic extension, its Dedekind zeta function is given by

$$\zeta_K(s) = \zeta(s)L(\chi_K; s),$$

where χ_K is the quadratic Dirichlet character modulo d_K such that

$$\chi_K(p) = \begin{cases} 1, & \text{if } p \text{ splits in } K, \\ -1, & \text{if } p \text{ remains prime in } K, \\ 0, & \text{if } p \text{ ramifies in } K. \end{cases}$$

Finally, we provide a detailed version of Theorem 2.1.20 in the case of the Dedekind zeta function.

Theorem 2.1.21. *Let K be a number field with degree n and signature (r_1, r_2). Denote by d_K, h_K, R_K and w_K the discriminant, class number, regulator and number of roots of unity in K. Then we have the following.*

(i) The function

$$\Lambda_K(s) = \left(\frac{\sqrt{|d_K|}}{2^{r_2}\pi^{\frac{n}{2}}}\right)^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2}\zeta_K(s)$$

has a holomorphic continuation to $\mathbb{C}\setminus\{0,1\}$ with simple poles for $s \in \{0,1\}$ and fulfills the functional equation

$$\Lambda_K(1-s) = \Lambda_K(s).$$

(ii) If we set $r = r_1 + r_2 - 1$, which is the rank of the unit group of K, then $\zeta_K(s)$ has a zero in $s = 0$ of order r and we have

$$\lim_{s\to 0} s^{-r}\zeta_K(s) = -\frac{h_K R_K}{w_K}.$$

(iii) Equivalently, by the functional equation, we have the so called class number formula

$$\mathrm{res}_{s=1}\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K\sqrt{|d_K|}}.$$

2.1.5 Modular forms of half integral weight

In this section we sketch the basic theory of modular forms of half integral weight, in particular their L-functions. For more details on this topic, the reader is advised to consult the basic article [55] by Shimura. In the next section we will also apply the framework of generalized Ramanujan identities to those.

Modular forms of half integral weight differ from those of integer weight in many points. Therefore we anticipate some explanations before giving a final definition.
We start with an example of a modular form of half integral weight for the subgroup $\Gamma_0(4)$ and remember the classical theta function

$$\theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots .$$

Indeed, besides the obvious identity $\theta(\tau + 1) = \theta(\tau)$ and we have verified by the Poisson summation formula that

$$\theta\left(-\frac{1}{4\tau}\right) = (-2i\tau)^{\frac{1}{2}}\theta(\tau),$$

where consider the principal branch of the square root. By the Jacobi tripel product identity

$$\theta(\tau) = \prod_{n=1}^{\infty}(1 - q^{2n})(1 + q^{2n-1})^2$$

we further conclude that θ has no zero on the entire upper half plane. Let $N \equiv 0 \pmod 4$ be a positive integer. We then define an *automorphic factor* j by

$$j(\gamma, \tau) := \frac{\theta(\gamma\tau)}{\theta(\tau)}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N), \tau \in \mathbb{H}.$$

One can show that

$$j(\gamma, \tau)^2 = \left(\frac{-1}{d}\right)(c\tau + d),$$

where $(\frac{\cdot}{d})$ is the quadratic residue symbol modulo d. Furthermore, in [54] Serre and Stark give the explicit formula

$$j(\gamma,\tau) = \varepsilon_d^{-1}\left(\frac{c}{d}\right)(c\tau+d)^{\frac{1}{2}}$$

with

$$\varepsilon_d = \begin{cases} 1, & \text{if } d \equiv 1 \pmod 4, \\ i, & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

Again, we choose the branch of $\sqrt{z} = z^{\frac{1}{2}}$ to be $-\frac{\pi}{2} < \arg(z^{\frac{1}{2}}) \leq \frac{\pi}{2}$. Let $\mathbb{T} := \{w \in \mathbb{C} \mid |w| = 1\}$. First it is convenient to extend the group $\mathrm{GL}_2^+(\mathbb{R})$ to a larger set $\mathcal{G}$. An element of $\mathcal{G}$ is a tuple (γ,ϕ), such that $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is holomorphic such that $\phi(\tau)^2 = v\det(\gamma)^{-\frac{1}{2}}(c\tau+d)$. Here, $v \in \mathbb{T}$ denotes a constant independent of τ. Since $c\tau + d \neq 0$ for all $\tau \in \mathbb{H}$, the function $\tau \mapsto c\tau + d$ will have a holomorphic square root on the upper half plane, i.e., the canonical map

$$\mathcal{G} \longrightarrow \mathrm{GL}_2^+(\mathbb{R})$$

surjects indeed. Note that here the fibre over an element $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ is isomorphic to $\mathbb{T}$. Most importantly, $\mathcal{G}$ carries the structure of a group by

$$(\gamma,\phi)(\delta,\psi) := (\gamma\delta, \tau \longmapsto \phi(\delta\tau)\psi(\tau)),$$

with neutral element $(I,1)$. We can now define an action of $\mathcal{G}$ on the space of holomorphic functions on the upper half plane for $t \in \mathbb{R}$ by

$$(f|_t\xi)(\tau) := f(\gamma\tau)\phi(\tau)^{-2t}, \qquad \xi := (\gamma,\phi) \in \mathcal{G},$$

where we consider the principal branch of the power map that is holomorphic on the upper half plane. Let N be divisible by 4 and κ be an odd integer. A modular form f of weight $\frac{\kappa}{2}$ for some congruence subgroup $\Gamma_1(N)$ is a holomorphic function on $\mathbb{H}$ that satisfies

$$(f|_{\frac{\kappa}{2}}\gamma^*)(z) = f, \qquad \gamma^* := (\gamma, j(\gamma,z)), \forall\gamma \in \Gamma_1(N),$$

and is also holomorphic at the cusps (i.e., for each $s \in \mathbb{Q}\cup\{i\infty\}$ choose some $M \in \mathrm{SL}_2(\mathbb{Z})$ such that $M(i\infty) = s$, lift it to $\tilde{M} \in \mathcal{G}$, and demand that $f|_{\frac{\kappa}{2}}\tilde{M}$ has a Fourier expansion with no terms of negative exponent). We denote their vector space by $M_{\frac{\kappa}{2}}(\Gamma_1(N))$ and the space of cusp forms as usual by $S_{\frac{\kappa}{2}}(\Gamma_1(N))$. We can of course extend this definition to the larger groups $\Gamma_0(N)$ involving characters. We say that f is a modular form of weight $\frac{\kappa}{2}$ for $\Gamma_0(N)$ with character χ if for all $\gamma \in \Gamma_0(N)$ we have

$$(f|_{\frac{\kappa}{2}}\gamma^*)(\tau) = \chi(d)f.$$

As in the case of integral weight, we obtain a representation of $M_{\frac{\kappa}{2}}(\Gamma_1(N))$ in terms of a direct sum

$$M_{\frac{\kappa}{2}}(\Gamma_1(N)) = \bigoplus_{\chi} M_{\frac{\kappa}{2}}(\Gamma_0(N), \chi).$$

Similarly for cusp forms. It is clear that we can expand each modular form of half integral weight as a Fourier series

$$f(\tau) = \sum_{n=0}^{\infty} a(n) q^n,$$

this will be also very important for the construction of L-functions.
In the following, we shall also introduce further very important operators on the spaces $M_{\frac{\kappa}{2}}(\Gamma_0(N), \chi)$, see also [12]. Put

$$w(N) := \left(\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}, N^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}} \right) \in \mathcal{G},$$

where we take the principal branch in the square root. Then the *Fricke involution* on $M_{\frac{\kappa}{2}}(\Gamma_0(N), \chi)$ is defined by $f \mapsto f|_{\frac{\kappa}{2}} w(N)$. Of course we have

$$f|_{\frac{\kappa}{2}} w(N)(\tau) = \left(-i\sqrt{N}\tau\right)^{-\frac{\kappa}{2}} f\left(-\frac{1}{N\tau}\right). \tag{2.1.5.1}$$

One can show that this defines a linear map

$$M_{\frac{\kappa}{2}}(\Gamma_0(N), \chi) \overset{f \mapsto f|_{\frac{\kappa}{2}} w(N)}{\longrightarrow} M_{\frac{\kappa}{2}}(\Gamma_0(N), (\tfrac{N}{\cdot})\overline{\chi}). \tag{2.1.5.2}$$

In 1973, Shimura proved a correspondence between modular forms of half integral weight and even integral weight. In particular, he proved that if f is a cusp form of weight $\frac{\kappa}{2}$ for $\Gamma_0(N)$ with character χ and

$$\sum_{n=1}^{\infty} \Lambda(n) n^{-s} := \prod_{p} \left(1 - \omega_p p^{-s} + \chi(p^2) p^{\kappa-2-2s}\right)^{-1},$$

where the ω_p are the eigenvalues of f of the Hecke operators $T_{p^2} : S_{\frac{\kappa}{2}}(\Gamma_0(N), \chi) \to S_{\frac{\kappa}{2}}(\Gamma_0(N), \chi)$, then

$$F(\tau) = \sum_{n=1}^{\infty} \Lambda(n) q^n \in S_{\kappa-1}(\Gamma_0(N), \chi^2).$$

One gets cusp forms for weight at least $\frac{5}{2}$, in weight $\frac{3}{2}$ unary theta functions are mapped to Eisenstein series.

Examples. Important examples of modular forms of half integral weight are the Shimura theta functions

$$\theta_{\psi,m}(\tau) := \sum_{n=-\infty}^{\infty} \psi(n) n^a q^{mn^2}$$

where ψ is a primitive Dirichlet character modulo $r > 0$, $m > 0$ is an integer, $a := \frac{1-\psi(-1)}{2}$ and the Dedekind eta function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

It is shown in [55] that

$$\theta_{\psi,m} \in \begin{cases} M_{\frac{1}{2}}(\Gamma_0(4r^2m), \chi_m\psi), & \text{if } a = 0, \\ S_{\frac{3}{2}}(\Gamma_0(4r^2m), \chi_{-m}\psi), & \text{if } a = 1. \end{cases}$$

where χ_d denotes the quadratic character corresponding to $\mathbb{Q}(\sqrt{d})$.

L-functions associated to modular forms of half integral weight

Let $f \in M_{\frac{\kappa}{2}}(\Gamma_0(N), \chi)$ with

$$f(\tau) = \sum_{n=0}^{\infty} a(n) q^n.$$

Then we can attach to f a Dirichlet series

$$L(f; s) := \sum_{n=1}^{\infty} a(n) n^{-s}$$

that will converge on some right half plane. It is known that the L-function $L(f; s)$ has a continuation to a meromorphic function in the entire complex plane. When putting

$$\Lambda(f; s) := \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(f; s)$$

we find the functional equation

$$\Lambda\left(f; \frac{\kappa}{2} - s\right) = \Lambda(f|_{w(N)}; s).$$

To see this one can use the standard Hecke converse argument in the setting of a different multiplier system. Consider the Mellin transform

$$\Lambda(f; s) = \int_0^{\infty} \left(f\left(\frac{ix}{\sqrt{N}}\right) - a(0)\right) x^{s-1} \mathrm{d}x.$$

Then we obtain with a substitution in the first integral

$$\begin{aligned}\Lambda(f;s) &= \int_0^1 \left(f\left(\frac{ix}{\sqrt{N}}\right) - a(0)\right) x^{s-1}\mathrm{d}x + \int_1^\infty \left(f\left(\frac{ix}{\sqrt{N}}\right) - a(0)\right) x^{s-1}\mathrm{d}x \\ &= \int_1^\infty \left(f\left(\frac{i}{x\sqrt{N}}\right) - a(0)\right) x^{-s-1}\mathrm{d}x + \int_1^\infty \left(f\left(\frac{ix}{\sqrt{N}}\right) - a(0)\right) x^{s-1}\mathrm{d}x\end{aligned}$$

and with (2.1.5.1) and (2.1.5.2)

$$\begin{aligned}&= \int_1^\infty \left(x^{\frac{\kappa}{2}} f|_{\frac{\kappa}{2}} w(N)\left(\frac{ix}{\sqrt{N}}\right) - a(0)\right) x^{-s-1}\mathrm{d}x + \int_1^\infty \left(f\left(\frac{ix}{\sqrt{N}}\right) - a(0)\right) x^{s-1}\mathrm{d}x \\ &= \int_1^\infty \left(f|_{\frac{\kappa}{2}} w(N)\left(\frac{ix}{\sqrt{N}}\right) - a^*(0)\right) x^{\frac{\kappa}{2}-s-1}\mathrm{d}x + \int_1^\infty \left(f\left(\frac{ix}{\sqrt{N}}\right) - a(0)\right) x^{s-1}\mathrm{d}x \\ &\quad + \frac{a^*(0)}{\frac{\kappa}{2}-s} - \frac{a(0)}{s},\end{aligned}$$

where $f|_{\frac{\kappa}{2}} w(N)(\tau) =: \sum_{n=0}^\infty a^*(n) q^n$. This is a holomorphic function in $\mathbb{C} \setminus \{0, \frac{\kappa}{2}\}$. When interchanging the role of f and $f|_{\frac{\kappa}{2}} w(N)$ in the last equation the functional equation becomes clear.

2.2 Generalized Ramanujan identities

2.2.1 Dirichlet series and general modular relations

The theory behind the formula (0.0.0.2) can be explained by the fact that the values $\zeta(2N+1)$ appear as coefficients of certain period polynomials (or rational functions more generally speaking) of Eichler integrals $\mathcal{E}(f;\tau)$, where f is a modular form of weight k with Fourier expansion $f(\tau)=\sum_{n=0}^{\infty}a(n)q^n$. An example of this situation looks as follows. Let χ be a primitive character with conductor $m>1$ and Gauss sum $\mathcal{G}(\chi)$. We now find that if

$$F_k(\tau,\chi):=\frac{\mathcal{G}(\chi)}{m}\sum_{n=1}^{\infty}\sum_{\ell=1}^{m}\frac{\overline{\chi}(\ell n)q^{\frac{n}{M}}}{n^k(e^{\frac{2\pi i\ell}{m}}-q^{\frac{n}{M}})},$$

we have

$$\left(F_k-CF_k|_{1-k}S\right)(\tau)=P(\tau), \tag{2.2.1.1}$$

where $C=\chi(-1)$ and P is a polynomial with degree at most k. The coefficients of P are related to values of L-functions at integer arguments. In particular, one can compute

$$P(\tau)=\sum_{\ell=0}^{k}\frac{(-1)^\ell}{\ell!}L(\chi;-\ell)L(\overline{\chi};k-\ell)\left(-\frac{2\pi i\tau}{m}\right)^\ell.$$

As easy corollaries we obtain identities in the spirit of Ramanujan, e.g.

$$L(\chi_5;2)=\frac{5\sqrt{5}}{2\pi}\sum_{n=1}^{\infty}\frac{\chi_5(n)}{n^3}\left(\frac{1}{e^{\frac{2\pi n}{5}}\zeta_5-1}-\frac{1}{e^{\frac{2\pi n}{5}}\zeta_5^2-1}-\frac{1}{e^{\frac{2\pi n}{5}}\zeta_5^3-1}+\frac{1}{e^{\frac{2\pi n}{5}}\zeta_5^4-1}\right).$$

To receive this formula one chooses χ to be the Legendre symbol modulo 5 and substitutes $\tau=i$ into (2.2.1.1). For more details about the general theory of Eichler integrals and period polynomials the reader is referred to [11]. In [33] transcendental values of Eichler integrals are investigated.

In [25] the author generalized the above identities to the case of not only integer but rational arguments. We proved the following: let χ be a primitive character modulo m, k and b be positive integers and $k\equiv 1 \pmod 2$. We define $M_{k,b}(\tau,\chi)$ as a holomorphic function on the upper half plane given by a *generalized Fourier series*

$$M_{k,b}(\tau,\chi)=\sum_{n=1}^{\infty}\lambda_{k,b}(n,\chi)q^{\frac{bn^{\frac{1}{b}}}{m}}, \tag{2.2.1.2}$$

where the coefficients $\lambda_{k,b}(n,\chi)$ are defined by the identity

$$\sum_{n=1}^{\infty}\lambda_{k,b}(n,\chi)n^{-s}=\prod_{j=1}^{b}L\left(\chi;s+\frac{j-1}{b}\right)L\left(\overline{\chi};s+\frac{j-1}{b}+k\right).$$

Then we have for the unit character:

Theorem 2.2.1 (see [25], p. 93 – 94)**.** *Let $k > 1$ be an odd integer and*

$$\gamma_\ell = (2\pi b)^{-\ell} b(\ell-1)! \prod_{b-\ell+1 \neq j=1}^{b} \zeta\left(\frac{\ell+j-1}{b}\right) \prod_{j=1}^{b} \zeta\left(\frac{\ell+j-1}{b}+k\right)$$

if $1 \leq \ell \leq b$,

$$\gamma_\ell = (2\pi b)^{-\ell} \frac{(-1)^\ell}{(-\ell)!} \zeta'(1-k) \prod_{(1-k)b+1-\ell \neq j=1}^{b} \zeta\left(\frac{\ell+j-1}{b}\right) \zeta\left(\frac{\ell+j-1}{b}+k\right)$$

if $1-bk \leq \ell \leq b-bk$ and

$$\gamma_\ell = (2\pi b)^{-\ell} \frac{(-1)^\ell}{(-\ell)!} \prod_{j=1}^{b} \zeta\left(\frac{\ell+j-1}{b}\right) \zeta\left(\frac{\ell+j-1}{b}+k\right)$$

else. Then we have the modular identity

$$M_{k,b}(\tau, \chi_0) - (-1)^A (-i\tau)^{bk-1} M_{k,b}\left(-\frac{1}{\tau}, \chi_0\right) = \sum_{\ell=1-bk-b}^{b} \gamma_\ell (-i\tau)^{-\ell},$$

where $A = \frac{b(k-\chi_0(-1))}{2} = \frac{b(k-1)}{2}$.

For non-principal primitive characters χ we obtain the following.

Theorem 2.2.2 (see [25], p. 94)**.** *Let*

$$\gamma_\ell = \left(\frac{2\pi b}{m}\right)^{-\ell} \frac{(-1)^\ell}{(-\ell)!} \prod_{j=1}^{b} L\left(\chi; \frac{\ell+j-1}{b}\right) L\left(\overline{\chi}; \frac{\ell+j-1}{b}+k\right)$$

if $1-bk-b \leq \ell \leq 0$ and $\gamma_\ell = 0$ otherwise. Then we have the modular identity

$$M_{k,b}(\tau, \chi) - (-1)^B (-i\tau)^{bk-1} M_{k,b}\left(-\frac{1}{\tau}, \chi\right) = \sum_{\ell=1-bk-b}^{b} \gamma_\ell (-i\tau)^{-\ell},$$

where $B = \frac{b(k-\chi(-1))}{2}$.

As a result, products of values of Dirichlet L-functions at rational arguments are linked with objects which have similar properties like classical modular forms. For example, when considering the unit character, we obtain

$$\sum_{n=1}^{\infty} \left(\sum_{d|n} \sigma_{-3}\left(\frac{n}{d}\right) \sigma_{-3}(d) d^{-\frac{1}{2}} \right) \left(e^{-2\pi\sqrt{n}} - \frac{1}{32} e^{-8\pi\sqrt{n}} \right) = \sum_{j=1}^{5} A_j \qquad (2.2.1.3)$$

where

$$A_1 = \tfrac{511}{92160}\pi^2\zeta(\tfrac{3}{2})\zeta(\tfrac{9}{2}), \qquad A_2 = \tfrac{1}{288}\pi^3\zeta(\tfrac{3}{2})\zeta(\tfrac{5}{2}),$$
$$A_3 = -\tfrac{7}{32}\zeta(\tfrac{3}{2})\zeta(\tfrac{5}{2})\zeta(3), \qquad A_4 = \tfrac{127}{11520}\pi^3\zeta(\tfrac{1}{2})\zeta(\tfrac{7}{2}),$$
$$A_5 = -\tfrac{31}{64}\zeta(\tfrac{1}{2})\zeta(\tfrac{7}{2})\zeta(3).$$

This identity is the case $b = 2$, $k = 3$ and $\tau = \frac{i}{2}$ of Theorem 2.2.1. Note that we used the functional equation $\zeta(1-s) = 2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)$ to eliminate negative arguments on the right hand side of the identity.
The purpose now is to generalize this concept to a much wider class of L-functions. The main problem here is that the gamma factor $\gamma(s)$ of L-functions in the completion $\Lambda(s, L) := \gamma(s)L(s)$ (which continues to a meromorphic function on the complex plane and satisfies a functional equation of the standard type) is not of he form $\gamma(s) = A^s\Gamma(s)$ in general. Consequently, the exponential terms in (2.2.1.2) are replaced by functions which arise as special cases of the *Meijer G-function*

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \,\middle|\, z\right) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + s)\prod_{j=n+1}^{p}\Gamma(a_j - s)}\, z^s \mathrm{d}s,$$

where $0 \leq n < p$, $0 \leq m < q$ are integers and $\mathcal{L}$ describes a suitable path of integration in the sense of an inverse Mellin transformation. For any further details the reader may wish to consult [3], p. 374.

The matter of this section is to explain the term "Ramanujan identity" and to summarize the concept in a formal definition. Like in the special case of modular forms there is a one-to-one correspondence between Dirichlet series with certain properties (such as a functional equation) and functions which are holomorphic on the upper half plane and are related to interesting rational functions. The examples given by Ramanujan only referred to values of L-functions at integer arguments. However, by including generalized Dirichlet series of the form

$$D(s) = \sum_{\nu=1}^{\infty} a(\nu)\nu^{-\frac{s}{b}}$$

for some $b \in \mathbb{N}$ it is possible to develop an analogous theory for L-functions at rational arguments. To formalize this theory we need the following.

Definition 2.2.3. *Let $a \in \mathbb{C}^\times$, $b \in \mathbb{R}_{>0}$, $\mathbf{a} = (a_1, ..., a_n), \mathbf{b} = (b_1, ..., b_n)$ be in $\mathbb{C}^n$ and $\mathbf{c} = (c_1, ..., c_n), \mathbf{d} = (d_1, ..., d_n)$ be in $\mathbb{Z}^n$. We define the corresponding gamma factor by*

$$\gamma^{(n)}_{a,b,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}(s) = \gamma_{a,b,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}(s) = ab^s\prod_{j=1}^{n}\Gamma\left(a_j + s\right)^{c_j}\Gamma\left(b_j - s\right)^{d_j}.$$

Observe that in the case $n = 1$ we have Euler's formula:

$$\gamma_{1,1,0,1,1,1}(s) = \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

It is obvious that products of gamma factors are again gamma factors and we obtain that the set $\mathfrak{W}$ of all gamma factors carries the structure of a multiplicative abelian group. We will simply write $\gamma(s)$ instead of $\gamma_{a,b,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}(s)$ when the parameters are clear. We have the following formal trick.

Proposition 2.2.4. *For $\mu = 1, 2, 3, ...$ we have multiplicative operators*

$$G_\mu : \mathfrak{W} \longrightarrow \mathfrak{W}, \qquad \gamma(s) \longmapsto \prod_{j=1}^{\mu} \gamma\left(\frac{s+j-1}{\mu}\right).$$

Proof. It is well-known that

$$G_\mu(\Gamma(a_j + s)) = (2\pi)^{\frac{\mu-1}{2}} \mu^{\frac{1}{2}-s-\mu a_j} \Gamma(\mu a_j + s) \in \mathfrak{W},$$

and similarly we obtain

$$G_\mu(\Gamma(b_j - s)), G_\mu(ab^s) \in \mathfrak{W}.$$

Since G_μ is a multiplicative map (in fact, a group homomorphism) our assertion follows. □

If we fix complex vectors $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ in the expression above, we obtain the subgroup $\mathfrak{W}_{\mathbf{a},\mathbf{b}}$. We then have restricted homomorphisms

$$G_\mu : \mathfrak{W}_{\mathbf{a},\mathbf{b}} \longrightarrow \mathfrak{W}_{\mu\mathbf{a},\mu\mathbf{b}-(\mu-1)\mathbf{1}}$$

by Remark 2.2.6. As we will see later, for some applications the above Gamma trick is still too restrictive. But by fixing $\mathbf{c} = (c_1, ..., c_n)$ and $\mathbf{d} = (d_1, ..., d_n)$ in $\mathbb{Z}^n$, we eventually obtain mappings

$$\times_{j=1}^{\mu} \mathfrak{W}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} \longrightarrow \mathfrak{W}_{\mu\mathbf{a},\mu\mathbf{b}-(\mu-1)\mathbf{1},\mathbf{c},\mathbf{d}}.$$

This is explained in greater detail in the following proposition. Note that $\mathfrak{W}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}$ is not a group with the operation declared above.

Proposition 2.2.5 (Generalized Gauß formula)**.** *We have mappings*

$$G^{\mu}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} : \times_{j=1}^{\mu} \mathfrak{W}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} \longrightarrow \mathfrak{W}_{\mu\mathbf{a},\mu\mathbf{b}-(\mu-1)\mathbf{1},\mathbf{c},\mathbf{d}}$$

given by

$$(\gamma_{\alpha_j,\beta_j,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}(s))_{j=1}^{\mu} \longmapsto \prod_{j=1}^{\mu} \gamma_{\alpha_j,\beta_j,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}\left(\frac{s+j-1}{\mu}\right).$$

We explicitly have

$$\prod_{j=1}^{\mu} \gamma_{\alpha_j,\beta_j,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}\left(\frac{s+j-1}{\mu}\right) = \gamma_{A,B,\mu\mathbf{a},\mu\mathbf{b}-(\mu-1)\mathbf{1},\mathbf{c},\mathbf{d}}(s),$$

where the real numbers A, B are given by

$$A = \left(\prod_{j=1}^{\mu} \alpha_j \beta_j^{\frac{j-1}{\mu}}\right) (2\pi)^{\frac{\mu-1}{2}\cdot(s_{\mathbf{c}}+s_{\mathbf{d}})} \exp_\mu\left(\frac{s_{\mathbf{c}}+s_{\mathbf{d}}}{2} - (\langle \mathbf{a},\mathbf{c}\rangle + \langle \mathbf{b},\mathbf{d}\rangle)\mu + (\mu-1)s_{\mathbf{d}}\right)$$

and

$$B = \left(\prod_{j=1}^{\mu} \beta_j\right)^{\frac{1}{\mu}} \mu^{s_{\mathbf{d}}-s_{\mathbf{c}}},$$

respectively. Recall that $s_{\mathbf{v}} = \langle \mathbf{v}, \mathbf{1}\rangle$.

Proof. Expanding the product shows

$$\begin{aligned}
&\prod_{j=1}^{\mu} \gamma_{\alpha_j,\beta_j,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}\left(\frac{s+j-1}{\mu}\right) \\
&= \prod_{j=1}^{\mu} \alpha_j \beta_j^{\frac{s+j-1}{\mu}} \prod_{\ell=1}^{n} \Gamma\left(a_\ell + \frac{s+j-1}{\mu}\right)^{c_\ell} \Gamma\left(b_\ell - \frac{s+j-1}{\mu}\right)^{d_\ell}
\end{aligned}$$

and we use $\prod_{j=1}^{\mu} \Gamma(b_j - \frac{s+j-1}{\mu}) = \prod_{j=1}^{\mu} \Gamma(b_\ell - \frac{s}{\mu} - \frac{\mu-1}{\mu} + \frac{j-1}{\mu})$ to obtain

$$\begin{aligned}
&= \prod_{j=1}^{\mu} \alpha_j \beta_j^{\frac{s+j-1}{\mu}} \times \prod_{\ell=1}^{n} \left((2\pi)^{\frac{\mu-1}{2}} \mu^{\frac{1}{2}-\mu a_\ell - s} \Gamma(\mu a_\ell + s)\right)^{c_\ell} \\
&\qquad \times \left((2\pi)^{\frac{\mu-1}{2}} \mu^{\frac{1}{2}-\mu b_\ell + s + \mu - 1} \Gamma(\mu b_\ell - s - \mu + 1)\right)^{d_\ell}.
\end{aligned}$$

Sorting the terms shows that this equals $\gamma_{A,B,\mu\mathbf{a},\mu\mathbf{b}-(\mu-1)\mathbf{1},\mathbf{c},\mathbf{d}}(s)$, as required. □

Sometimes we will leave out the indices of G when the parameters should be clear.

Remark 2.2.6. *Proposition 2.2.5 provides us with the explicit formula*

$$G_\mu(\gamma_{a,b,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}(s)) = \gamma_{A_\mu, b\mu^{-(s_{\mathbf{c}}+s_{\mathbf{d}})}, \mu\mathbf{a}, \mu\mathbf{b}-(\mu-1)\mathbf{1}}(s),$$

where

$$A = a^\mu b^{\frac{\mu-1}{2}} (2\pi)^{\frac{\mu-1}{2}\cdot(s_{\mathbf{c}}+s_{\mathbf{d}})} \exp_\mu\left(\frac{s_{\mathbf{c}}+s_{\mathbf{d}}}{2} - (\langle \mathbf{a},\mathbf{c}\rangle + \langle \mathbf{b},\mathbf{d}\rangle)\mu + (\mu-1)s_{\mathbf{d}}\right).$$

The next definition comprises all relevant Dirichlet series for our purposes.

Definition 2.2.7. *Let $b, b^* \in \mathbb{N}$. We say that a (generalized) Dirichlet series of the form*

$$D(s) = \sum_{\nu=1}^{\infty} a(\nu)\nu^{-\frac{s}{b}}$$

has signature $((\gamma, \gamma^*), \sigma, k)$ where $(\gamma, \gamma^*) \in \mathfrak{W}^2$ and $k \in \mathbb{R}$, if the following conditions are all satisfied:

(i) $D(s)$ is absolutely convergent in the right half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \sigma\}$, its coefficients satisfy $\alpha(\nu) \ll \nu^{\frac{\sigma}{b}-1}$ and has a meromorphic continuation to the entire complex plane, such that in the "critical strip" $\{s \in \mathbb{C} \mid k - \sigma \leq \mathrm{Re}(s) \leq \sigma\}$ there are only finitely many poles.

(ii) There is a dual (generalized) Dirichlet series

$$D^*(s) := \sum_{\nu=1}^{\infty} \alpha^*(\nu)\nu^{-\frac{s}{b^*}}$$

also absolutely convergent in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \sigma\}$ with coefficients $\alpha^*(\nu) \ll \nu^{\frac{\sigma}{b^*}-1}$ and a meromorphic continuation to the entire plane such that the completions

$$\widehat{D}(s) := \gamma(s)D(s)$$

and

$$\widehat{D^*}(s) := \gamma^*(s)D^*(s)$$

are related by the functional equation

$$\widehat{D}(k-s) = \widehat{D^*}(s).$$

(iii) There is a $C > 0$, such that the function $\widehat{D}(s)$ is bounded on every vertical strip $\{-\infty < \sigma_1 < \mathrm{Re}(s) < \sigma_2 < \infty\} \cap \{|\mathrm{Im}(s)| \geq C\}$.

We denote the space of such generalized Dirichlet series $D(s)$ by $\mathcal{D}((\gamma, \gamma^*), \sigma, k)$. In the case $\gamma = \gamma^*$, we simply write $\mathcal{D}(\gamma, \sigma, k)$.

From now on let us fix some gamma factors γ, γ^* with the property

$$\gamma(s), \gamma^*(s) \ll_{\sigma_1,\sigma_2} |s|^{\nu_{\sigma_1,\sigma_2}} e^{-\frac{\pi}{2}|\mathrm{Im}(s)|}, \qquad \nu_{\sigma_1,\sigma_2} > 0, \tag{2.2.1.4}$$

on every vertical strip $\sigma_1 < \mathrm{Re}(s) < \sigma_2$. For lots of applications this follows by application of Stirling's formula

$$\Gamma(s) = \sqrt{2\pi}s^{s-\frac{1}{2}}e^{-s+H(s)}$$

in $\mathbb{C}_-$ with a holomorphic H with the property

$$\lim_{\substack{|s|\to\infty \\ -\pi+\delta<\mathrm{Arg}(s)<\pi-\delta}} H(s) = 0$$

for all fixed values $\delta \in (0,\pi)$. Let $S(f) \subset U$ denote the set of poles of the meromorphic function $f : U \to \overline{\mathbb{C}}$. For fixed σ we define

$$\theta_1 := \min\{x \in \mathrm{Re}(S(\gamma) \cup S(\gamma^*)) \mid x > \sigma\}.$$

Note that θ_1 is well-defined, since γ and γ^* consist of a finite number of relevant factors of the form $\Gamma(s-a)$ and $\Gamma(b-s)$, from which follows that there exists a pole and also a zero with maximum imaginary part. In the case that γ, γ^* have no pole s with $\mathrm{Re}(s) > \sigma$, we simply set $\theta_1 = \infty$. Note that we have a holomorphic inverse Mellin transform of γ

$$\mathcal{M}_\sigma^{-1}(\gamma, x) := \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \gamma(s) x^{-s} \mathrm{d}s, \qquad \sigma < c < \theta_1, \tag{2.2.1.5}$$

on the half plane $\mathrm{Re}(x) > 0$. By the usual argument including contour integration we see that (2.2.1.5) is independent from the choice of c. We can estimate the integral (2.2.1.5) uniformly for all $-i\tau \in W_\delta := \{z \in \mathbb{C}^\times \mid |\mathrm{Arg}(z)| \leq \frac{\pi}{2} - \delta\}$ by

$$|\mathcal{M}_\sigma^{-1}(\gamma, -i\tau)| \leq \frac{|\tau|^{-c}}{2\pi} \int\limits_{-\infty}^{\infty} |\gamma(c+it)| e^{\mathrm{Arg}(-i\tau)t} \mathrm{d}t \ll_{\gamma,c,\delta} |\tau|^{-c}, \tag{2.2.1.6}$$

where $\sigma < c < \theta_1$ is arbitrarily chosen.

Definition 2.2.8. *Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function. We say that f induces a modular identity (of the Ramanujan type) of signature $((\gamma, \gamma^*), \sigma, k)$ (where $k \in \mathbb{R}$ and $\gamma, \gamma^* \in \mathfrak{W}$ satisfies condition (2.2.1.4)) if the following conditions are satisfied:*

(i) We can expand f in series of the form

$$f(\tau) = \sum_{\nu=1}^{\infty} \alpha(\nu) \mathcal{M}_\sigma^{-1}(\gamma, -i\tau\nu^{\frac{1}{b}}), \qquad b \in \mathbb{N}, \tag{2.2.1.7}$$

such that $\alpha(\nu) \ll \nu^{\frac{\sigma}{b}-1}$ in this case.

(ii) There is a dual function f^* with expansion

$$f^*(\tau) = \sum_{\nu=1}^{\infty} \alpha^*(\nu) \mathcal{M}_\sigma^{-1}(\gamma^*, -i\tau\nu^{\frac{1}{b^*}}), \qquad b^* \in \mathbb{N},$$

$\alpha^*(\nu) \ll \nu^{\frac{\sigma}{b^*}-1}$, and also complex numbers $u_1, ..., u_\ell$ with $k - \sigma \leq \mathrm{Re}(u_1), ..., \mathrm{Re}(u_\ell) \leq \sigma$ and polynomials $P_1, ..., P_\ell$ with transformation property

$$f\left(-\frac{1}{\tau}\right) = (-i\tau)^k f^*(\tau) + \sum_{j=1}^{\ell} P_j(\log(-i\tau))(-i\tau)^{u_j}.$$

We denote the space of such functions by $\mathcal{R}((\gamma,\gamma^*),\sigma,k)$. Again, if $\gamma=\gamma^*$ we write $\mathcal{R}(\gamma,\sigma,k)$. As in classical theory, we will sometimes call k the weight of f.

The next theorem provides a converse theorem, which is the main framework we are going to work with. It will be used to construct Ramanujan identities from a given completed generalized Dirichlet series with functional equation.

Theorem 2.2.9 (see [26]). *Let γ,γ^* be gamma factors which satisfy* (2.2.1.4)*, and $k\in\mathbb{R}$ with $k<2\sigma$ (non-emtpy critical strip). Then we have an isomorphism between spaces*

$$\Upsilon:\mathcal{D}((\gamma,\gamma^*),\sigma,k)\xrightarrow{\sim}\mathcal{R}((\gamma,\gamma^*),\sigma,k)$$

given by

$$\Upsilon:D\longmapsto\left(\tau\longmapsto\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\gamma(s)D(s)(-i\tau)^{-s}\mathrm{d}s,\qquad\sigma<c<\theta_1\right),$$

with inverse

$$\Upsilon^{-1}:f\longmapsto\left(s\longmapsto\frac{1}{\gamma(s)}\int\limits_0^{\infty}f(ix)x^{s-1}\mathrm{d}x,\qquad\sigma<\mathrm{Re}(s)<\theta_1\right).$$

Note that the representation of $\Upsilon^{-1}(f)$ is not defined for all s but its meromorphic continuation is an element of $\mathcal{D}((\gamma,\gamma^*),\sigma,k)$.

Proof. First we consider the map Υ. Let $\tau=iy$ with $y>0$. Consider the closed contour integral

$$\frac{1}{2\pi i}\oint_R\widehat{D}(s)y^{-s}\mathrm{d}s,$$

where R is the rectangle with vertices $\sigma+\varepsilon\pm iT$ and $k-\sigma-\varepsilon\pm iT$, where $0<\varepsilon<\theta_1-\sigma$, taken anti-clockwise. As T goes to ∞, this will converge to the following expression

$$\frac{1}{2\pi i}\int\limits_{\sigma+\varepsilon-i\infty}^{\sigma+\varepsilon+i\infty}\widehat{D}(s)y^{-s}\mathrm{d}s-\frac{1}{2\pi i}\int\limits_{k-\sigma-\varepsilon-i\infty}^{k-\sigma-\varepsilon+i\infty}\widehat{D}(s)y^{-s}\mathrm{d}s,\tag{2.2.1.8}$$

since the boundedness condition for the complete Dirichlet series on vertical strips allows us to apply the Phragmen-Lindelöf principle (for details see for example [49] on p. 118) to the function

$$s\longmapsto(s-u_1)^{n_1}(s-u_2)^{n_2}\cdots(s-u_\ell)^{n_\ell}\widehat{D}(s)\in\mathcal{O}(\{k-\sigma-\varepsilon<\mathrm{Re}(s)<\sigma+\varepsilon\})$$

for some natural numbers n_j, which means that the horizontal parts will vanish. Note that for this argument one uses the functional equation $\widehat{D}(s) = \widehat{D}^*(k-s)$, the growth properties of $\gamma^*(k-s)$, the absolute convergence of $D^*(k-s)$ on the left line of R and the fact that both γ and γ^* only have a finite amount of poles in the critical strip. After the substitution $s \mapsto k-s$ in the right integral, expression (2.2.1.8) equals

$$\sum_{\nu=1}^{\infty} \alpha(\nu) \mathcal{M}_{\sigma}^{-1}(\gamma, y\nu^{\frac{1}{b}}) - y^{-k} \sum_{\nu=1}^{\infty} \alpha^*(\nu) \mathcal{M}_{\sigma}^{-1}\left(\gamma^*, \frac{\nu^{\frac{1}{b^*}}}{y}\right).$$

Exchanging summation and integration is justified due to absolute convergence and Lebesgue's theorem, since we have

$$\int_{\sigma+\varepsilon-i\infty}^{\sigma+\varepsilon+i\infty} \sum_{\nu=1}^{\infty} \left|a(\nu)\nu^{-\frac{s}{b}}\gamma(s)y^{-s}\right| |\mathrm{d}s| \ll \int_{-\infty}^{\infty} \sum_{\nu=1}^{\infty} \nu^{\frac{-\sigma-\varepsilon}{b}} \nu^{\frac{\sigma}{b}-1} |\gamma(\sigma+\varepsilon+it)| y^{-\sigma-\varepsilon} \mathrm{d}t < \infty.$$

Finally, the Residue theorem gives the desired error terms

$$\sum_{j=1}^{\ell} \mathrm{res}_{s=u_j}\left(\widehat{D}(s)y^{-s}\right) = \sum_{j=1}^{\ell} P_j(\log(y)) y^{-u_j},$$

and the result follows (after adjusting the objects notation) by analytic continuation.
For the other direction one obtains the Dirichlet series by construction (since the coefficients grow not too fast) by

$$\int_0^{\infty} f(ix)x^{s-1}\mathrm{d}x = \sum_{\nu=1}^{\infty} \alpha(\nu) \int_0^{\infty} \mathcal{M}_{\sigma}^{-1}(\gamma, x\nu^{\frac{1}{b}})x^{s-1}\mathrm{d}x = \gamma(s)D(s)$$

on the strip $\sigma < \mathrm{Re}(s) < \theta_1$. Note that switching integration and summation is again allowed using absolute convergence and Lebesgue's theorem. Indeed, we have

$$\begin{aligned}
&\int_0^{\infty} \sum_{\nu=1}^{\infty} \left|\alpha(\nu)\mathcal{M}_{\sigma}^{-1}(\gamma, x\nu^{\frac{1}{b}})x^{s-1}\right| \mathrm{d}x \\
&= \int_0^{1} \sum_{\nu=1}^{\infty} \left|\alpha(\nu)\mathcal{M}_{\sigma}^{-1}(\gamma, x\nu^{\frac{1}{b}})x^{s-1}\right| \mathrm{d}x + \int_1^{\infty} \sum_{\nu=1}^{\infty} \left|\alpha(\nu)\mathcal{M}_{\sigma}^{-1}(\gamma, x\nu^{\frac{1}{b}})x^{s-1}\right| \mathrm{d}x
\end{aligned}$$

and now choose values $\theta_1 > \mathrm{Re}(s) > c_1 > \sigma$ and $\theta_1 > c_2 > \mathrm{Re}(s) > \sigma$ satisfying (2.2.1.6), to finally obtain

$$\begin{aligned}
&\ll \sum_{\nu=1}^{\infty} \nu^{\frac{\sigma}{b}-1}\nu^{-\frac{c_1}{b}} \int_0^{1} x^{\mathrm{Re}(s)-c_1-1}\mathrm{d}x + \sum_{\nu=1}^{\infty} \nu^{\frac{\sigma}{b}-1}\nu^{-\frac{c_2}{b}} \int_1^{\infty} x^{\mathrm{Re}(s)-c_2-1}\mathrm{d}x \\
&\leq \frac{\zeta\left(1+\frac{c_1-\sigma}{b}\right)}{\mathrm{Re}(s)-c_1} + \frac{\zeta\left(1+\frac{c_2-\sigma}{b}\right)}{c_2-\mathrm{Re}(s)} < \infty.
\end{aligned}$$

The dual integral is defined analogously and with the transformation property one gets back the functional equation. In particular, since $\mathrm{Re}(u_j) \leq \sigma$ for all $1 \leq j \leq \ell$, one has with the notation I_∞ for the integral of $f(ix)x^{s-1}\mathrm{d}x$ between 1 and ∞

$$\begin{aligned}\int_0^\infty f(ix)x^{s-1}\mathrm{d}x &= \int_1^\infty f\left(\frac{i}{x}\right)x^{-s-1}\mathrm{d}x + I_\infty(s)\\ &= \int_1^\infty \left(x^k f^*(ix) + \sum_{j=1}^{\ell} P_j(\log(x))x^{u_j}\right)x^{-s-1}\mathrm{d}x + I_\infty(s)\\ &= I_\infty^*(k-s) + \sum_{j=1}^{\ell}\tilde{P}_j\left(\frac{1}{s-u_j}\right) + I_\infty(s),\end{aligned}$$

where the dual integral converges absolutely since $k - \mathrm{Re}(s) < k - \sigma < \sigma$ and the $\tilde{P}_j$ are some polynomials. Hence

$$\gamma(s)D(s) - I_\infty^*(k-s) - \sum_{j=1}^{\ell}\tilde{P}_j\left(\frac{1}{s-u_j}\right) = I_\infty(s). \tag{2.2.1.9}$$

With the same arguments (note that we have $k - \sigma \leq \mathrm{Re}(u_j)$ too) one obtains

$$\gamma^*(s)D^*(s) - I_\infty(k-s) - \sum_{j=1}^{\ell}\tilde{P}_j\left(\frac{1}{k-s-u_j}\right) = I_\infty^*(s). \tag{2.2.1.10}$$

Note that both functions $I(s)$ and $I^*(s)$ are holomorphic in the left half plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) < \theta_1\}$, since for all s in this area we can find some $\mathrm{Re}(s) < c_2 < \theta_1$ such that

$$\int_1^\infty \sum_{\nu=1}^{\infty} |\alpha(\nu)| \left|\mathcal{M}_\sigma^{-1}\left(\gamma; x\nu^{\frac{1}{b}}\right)\right| x^{\mathrm{Re}(s)-1}\mathrm{d}x \ll \sum_{\nu=1}^{\infty}\nu^{\frac{\sigma}{b}-\frac{c_2}{b}-1}\int_1^\infty x^{\mathrm{Re}(s)-c_2-1}\mathrm{d}x$$

and the last integral converges for all $\mathrm{Re}(s) < c_2$, similarly for $I^*(s)$. It follows by (2.2.1.9) and (2.2.1.10) that $I_\infty(s)$ and $I_\infty^*(s)$ and hence $D(s)$ and $D^*(s)$ have meromorphic continuations to the entire plane, since the vertical half planes $\{\mathrm{Re}(s) > \max\{k-\theta_1, \sigma\}\}$ and $\{\mathrm{Re}(s) < \theta_1\}$ have non-empty intersection. The function $D(s)$ has only finitely many poles in the critical strip, since I and I^* are holomorphic in this area and all poles and zeros of γ, γ^* do have bounded imaginary parts in absolute values. The functional equation becomes clear with (2.2.1.9) and (2.2.1.10).
The growth conditions are clear for vertical strips in $\{\mathrm{Re}(s) > \sigma\}$ and $\{\mathrm{Re}(s) < k - \sigma\}$ due to the functional equation (again using that fact that the poles of γ and γ^* have bounded imaginary parts). For the critical strip $\{k - \sigma \leq \mathrm{Re}(s) \leq \sigma\}$ one uses the standard estimate of the integrals I_∞ and I_∞^* along vertical lines. □

For more about converse theorems the reader is referred to, e.g., [9] (p. 336 – 338: Lemma 1 and Theorems 2 and 3) where general Dirichlet series $\sum_{n=1}^{\infty} a(n)e^{-\lambda_n s}$ (as usual, λ_n is a real increasing sequence with $\lambda_n \to \infty$) and modular relations of the type

$$\sum_{n=0}^{\infty} a(n)\exp(-\lambda_n x) = x^{-\delta}\sum_{n=0}^{\infty} b(n)\exp\left(-\frac{\mu_n}{x}\right)$$

are investigated. In Lemma 1, the effect of the residue integral on the modular error term is described in detail. Although Bochner assumes $\theta_1 = \infty$ for the Mellin integrals the arguments are similar.
We can now use the generalized Gauß formula to introduce a general method to extract analytic objects related to L-functions at rational arguments from those related to integer arguments. This is summed up in the next theorem.

Theorem 2.2.10 (see [26])**.** *Let $\mu \in \mathbb{N}$ and $\tilde{\gamma} = (\gamma_j)_{1\leq j\leq \mu}$ and $\tilde{\gamma}^* = (\gamma_j^*)_{1\leq j\leq \mu}$ be collections of gamma factors in $\mathfrak{W}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}$. We then have a map*

$$\mathcal{T}_\mu : \times_{j=1}^{\mu}\mathcal{D}((\gamma_j,\gamma_j^*),\sigma,k) \longrightarrow \mathcal{D}\left(\left(G^{\mu}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}(\tilde{\gamma}), G^{\mu}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}(\tilde{\gamma}^*_{\mathrm{inv}})\right), \mu\sigma, \mu k-\mu+1\right)$$

$$(D_1,...,D_\mu) \longmapsto \left(s \longmapsto \prod_{j=1}^{\mu} D_j\left(\frac{s+j-1}{\mu}\right)\right).$$

Proof. Firstly, we show that the above map is indeed well-defined. To do so, we have to check that the image of some tuple $(D_1,...,D_\mu)$ is a generalized Dirichlet series with signature $\left(\left(G^{\mu}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}(\tilde{\gamma}), G^{\mu}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}(\tilde{\gamma}^*_{\mathrm{inv}})\right), \mu\sigma, \mu k-\mu+1\right)$ as introduced in Definition 2.2.7. Since

$$\mu\sigma < \mathrm{Re}(s) \;\text{ implies }\; \sigma < \mathrm{Re}\left(\frac{s}{\mu}\right) \leq \mathrm{Re}\left(\frac{s+j-1}{\mu}\right),$$

the convergence part of condition (i) is clearly satisfied. As a product of meromorphic functions in the complex plane the resultant function is meromorphic too and still has poles only in $\mathbb{R}$ as every factor does. For part (ii) we use the functional equations of the individual factors:

$$\widehat{\mathcal{T}_\mu((D_j))}(\mu k-(\mu-1)-s) = \prod_{j=1}^{\mu}\widehat{D_j}\left(k-\frac{s+\mu-j}{\mu}\right) = \prod_{j=1}^{\mu}\widehat{D^*_{\mu-j+1}}\left(\frac{s+j-1}{\mu}\right).$$

With

$$\mathcal{T}_\mu((D_j))^*(s) = \prod_{j=1}^{\mu} D^*_{\mu-j+1}\left(\frac{s+j-1}{\mu}\right)$$

we have found the dual which also converges absolutely for all s with $\mathrm{Re}(s) > \sigma$, with corresponding gamma factor $G^{\mu}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}(\tilde{\gamma}^*_{\mathrm{inv}})$. It is plain that (iii) is satisfied and this proves the theorem. □

2.2.2 Application to Hecke L-functions of number fields

Let K_1 and K_2 be number fields of the same signature (r_1, r_2). Let χ_1 and χ_2 be Hecke characters modulo $\mathfrak{f}_1$ and $\mathfrak{f}_2$, respectively, with identical exponents. This means, that

$$\chi_j(x) = \chi_{\mathfrak{f}_j}(x)\chi_\infty(x), \qquad j = 1, 2,$$

for the same $\chi_\infty(x)$ (and in particular the same data ε and ν). For example, this is the case when χ_1 and χ_2 are both trivial. The aim of this section is to construct non-trivial functions in $\mathcal{D}((\gamma, \gamma), 1-w, 1)$ involving Hecke L-functions to generalize identities of Ramanujan. Here, w is some odd integer. This is done in several steps. In the easy case that $w = 1$ we may simply consider the function

$$s \longmapsto \Lambda_{K_1}(\chi_1; s)\Lambda_{K_2}(\chi_2; s+1),$$

which is related to $W(\chi_1)W(\chi_2)\Lambda_{K_2}(\overline{\chi_2}; s)\Lambda_{K_1}(\overline{\chi_1}; s+1)$ under $s \mapsto -s$. From this it is easy to see that the Dirichlet series

$$D(s) = L(\chi_1; s)L(\chi_2; s+1)$$

may be an element of $\mathcal{D}((\gamma_{1,2}, \gamma_{1,2}^*), 0, 1)$ in case we find a proper gamma factor $\gamma_{1,2}$ in terms of the ideals $\mathfrak{f}_1$ and $\mathfrak{f}_2$. Using Definitions 2.2.3 and 2.1.18, we write

$$\Lambda_K(\chi; s) = C_\chi A_\chi^s \prod_{j=1}^{r_1} \Gamma\left(\frac{s - i\nu_{\rho_j} + \varepsilon_{\rho_j}}{2}\right) \gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2} - i\tilde{\nu}_\sigma, \mathbf{0},\mathbf{1},\mathbf{0}}(s) L(\chi; s),$$

where C_χ was introduced in (2.1.4.1) and

$$A_\chi := \frac{\sqrt{|d_K|\boldsymbol{N}(\mathfrak{f})}}{\pi^{\frac{n}{2}} 2^{r_2}}.$$

Indeed, using the duplication formula we find

$$\begin{aligned}
\Lambda_{K_1}(\chi_1; s)\Lambda_{K_2}(\chi_2; s+1) &= C_{\chi_1}C_{\chi_2}A_{\chi_1}^s A_{\chi_2}^{s+1} \prod_{j=1}^{r_1} \Gamma\left(\frac{s - \nu_{\rho_j} i + \varepsilon_{\rho_j}}{2}\right) \Gamma\left(\frac{s - \nu_{\rho_j} i + \varepsilon_{\rho_j} + 1}{2}\right) \\
&\quad \times \gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2} - i\tilde{\nu}_\sigma, \mathbf{0},\mathbf{1},\mathbf{0}}(s) \gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2} - i\tilde{\nu}_\sigma, \mathbf{0},\mathbf{1},\mathbf{0}}(s+1) L(\chi_1; s)L(\chi_2; s+1) \\
&= D_{\chi_1,\chi_2} E^s_{\chi_1,\chi_2} \gamma^{(r_1)}_{1,1,\tilde{\varepsilon}_\rho - i\tilde{\nu}_\rho, \mathbf{0},\mathbf{1},\mathbf{0}}(s) \gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2} - i\tilde{\nu}_\sigma, \mathbf{0},\mathbf{1},\mathbf{0}}(s) \gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2} - i\tilde{\nu}_\sigma, \mathbf{0},\mathbf{1},\mathbf{0}}(s+1) L(\chi_1; s)L(\chi_2; s+1),
\end{aligned}$$

with

$$\begin{aligned}
D_{\chi_1,\chi_2} &:= (4\pi)^{r_{\frac{1}{2}}} 2^{\mathrm{Tr}(i\tilde{\nu}_\rho - \tilde{\varepsilon}_\rho)} C_{\chi_1}C_{\chi_2}A_{\chi_2}, \\
E_{\chi_1,\chi_2} &:= 2^{-r_1} A_{\chi_1}A_{\chi_2}.
\end{aligned}$$

We conclude that $\gamma_{1,2}$ is given by

$$\gamma_{1,2}(s) = D_{\chi_1,\chi_2} E^s_{\chi_1,\chi_2} \gamma^{(r_1)}_{1,1,\tilde{\varepsilon}_\rho - i\tilde{\nu}_\rho,\mathbf{0},\mathbf{1},\mathbf{0}}(s) \gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2} - i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s) \gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2} - i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s+1).$$

Clearly, the dual of $L(\chi_1;s)L(\chi_2;s+1)$ may be chosen as $W(\chi_1)W(\chi_2)L(\overline{\chi_2};s)L(\overline{\chi_1};s+1)$ and then we have

$$\gamma^*_{1,2}(s) = D_{\overline{\chi_2},\overline{\chi_1}} E^s_{\overline{\chi_2},\overline{\chi_1}} \gamma^{(r_1)}_{1,1,\tilde{\varepsilon}_\rho + i\tilde{\nu}_\rho,\mathbf{0},\mathbf{1},\mathbf{0}}(s) \gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2} + i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s) \gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2} + i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s+1).$$

In the case $w > 1$ the situation is more difficult. The problem that occurs now is that we will not be able to find a suitable gamma factor without making further assumptions for χ_1 and χ_2. But we can remedy this by assuming that the ε_ρ-factor of χ_1 and χ_2 is trivial, i.e. $(\varepsilon_{\rho_1},...,\varepsilon_{\rho_{r_1}}) = (0,...,0)$. This holds true for example if χ_1,χ_2 are squares of another character. In this case we achieve an analogous result by looking at the function

$$f_w(K_1,\chi_1;K_2,\chi_2;s) := \Lambda_{K_1}(\chi_1;s)\Lambda_{K_1}(\overline{\chi_1};s)\Lambda_{K_2}(\chi_2;s+w)\Lambda_{K_2}(\overline{\chi_2};s+w). \quad (2.2.2.1)$$

We have the following result.

Proposition 2.2.11. *Let χ_1 and χ_2 be two Hecke characters with the same multipliers ε, ν and trivial ε_ρ-factor, i.e. $\varepsilon_\rho = 0$. The Dirichlet series*

$$D_w(\chi_1,\chi_2;s) := L(\chi_1;s)L(\overline{\chi_1};s)L(\chi_2;s+w)L(\overline{\chi_2};s+w)$$

is an element of $\mathcal{D}((\gamma_{w,\chi_1,\chi_2},\gamma_{w,\chi_2,\chi_1}),1,1-w)$ with dual $D^*_w(\chi_1,\chi_2;s) = D_w(\chi_2,\chi_1;s)$ and corresponding gamma factor

$$\gamma_{w,\chi_1,\chi_2}(s) := \widetilde{D}_{\chi_1,\chi_2} \widetilde{E}^s_{\chi_1,\chi_2} \gamma^{(r_1)}_{1,1,i\tilde{\nu}_\rho,\mathbf{0},\mathbf{1},\mathbf{0}}(s) \gamma^{(r_1)}_{1,1,-i\tilde{\nu}_\rho,\mathbf{0},\mathbf{1},\mathbf{0}}(s)$$

$$\times \gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2} - i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s) \gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2} - i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s+w) \gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2} + i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s) \gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2} + i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s+w).$$

The constants $\widetilde{D}_{\chi_1,\chi_2}$ and $\widetilde{E}_{\chi_1,\chi_2}$ are defined in (2.2.2.3) *and* (2.2.2.4)*, respectively.*

Proof. It is clear that one may choose $\sigma = 1$ as the abscissa of convergence. Furthermore, Theorem 2.1.20 tells us that $D_w(\chi_1,\chi_2;s)$ only has a finite number of possible poles in the critical strip. Let

$$\psi_w(K_1,\chi_1;K_2,\chi_2;s) := \gamma_{w,\chi_1,\chi_2}(s) D_w(\chi_1,\chi_2;s).$$

We have to prove the functional equation

$$\psi_w(K_1,\chi_1;K_2,\chi_2;1-w-s) = \psi_w(K_2,\chi_2;K_1,\chi_1;s). \quad (2.2.2.2)$$

Firstly, it is easily verified that f_w in (2.2.2.1) also satisfies (2.2.2.2). Writing the expression $f_w(K_1,\chi_1;K_2,\chi_2;s)$ out, we see that it equals

$$\widetilde{C}_{\chi_1}\widetilde{C}_{\chi_2}\widetilde{A}^w_{\chi_2}(\widetilde{A}_{\chi_1}\widetilde{A}_{\chi_2})^s \prod_{j=1}^{r_1} \Gamma\left(\frac{s-\nu_{\rho_j}i}{2}\right)\Gamma\left(\frac{s-\nu_{\rho_j}i+w}{2}\right)\Gamma\left(\frac{s+\nu_{\rho_j}i}{2}\right)\Gamma\left(\frac{s+\nu_{\rho_j}i+w}{2}\right)$$

$$\times\gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2}-i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s)\gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2}-i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s+w)$$
$$\times\gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2}+i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s)\gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2}+i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s+w)D_w(\chi_1,\chi_2;s),$$

where

$$\widetilde{C}_\chi := C_\chi C_{\overline{\chi}}, \qquad \widetilde{A}_\chi := A_\chi A_{\overline{\chi}}.$$

By the duplication formula this equals

$$\widetilde{D}_{\chi_1,\chi_2}\widetilde{E}^s_{\chi_1,\chi_2}P_w(s)\prod_{j=1}^{r_1}\Gamma\left(s+\nu_{\rho_j}i\right)\Gamma\left(s-\nu_{\rho_j}i\right)$$
$$\times\gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2}-i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s)\gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2}-i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s+w)$$
$$\times\gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2}+i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s)\gamma^{(r_2)}_{1,1,\frac{\tilde{\varepsilon}_\sigma}{2}+i\tilde{\nu}_\sigma,\mathbf{0},\mathbf{1},\mathbf{0}}(s+w)D_w(\chi_1,\chi_2;s)$$

with

$$\widetilde{D}_{\chi_1,\chi_2} := (4\pi)^{r_1}\widetilde{C}_{\chi_1}\widetilde{C}_{\chi_2}\widetilde{A}^w_{\chi_2}, \tag{2.2.2.3}$$

$$\widetilde{E}_{\chi_1,\chi_2} := 4^{-r_1}\widetilde{A}_{\chi_1}\widetilde{A}_{\chi_2}, \tag{2.2.2.4}$$

and the polynomial factor

$$P_w(s) := \prod_{j=1}^{r_1}\prod_{d=0}^{\frac{w-3}{2}}\left(\frac{s+1-\nu_{\rho_j}i}{2}+d\right)\left(\frac{s+1+\nu_{\rho_j}i}{2}+d\right).$$

The polynomial $P_w(s)$ is symmetric under $s\mapsto 1-w-s$. Indeed,

$$\begin{aligned}P_w(1-w-s) &= \prod_{j=1}^{r_1}\prod_{d=0}^{\frac{w-3}{2}}\left(\frac{1-w-s+1-\nu_{\rho_j}i}{2}+d\right)\left(\frac{1-w-s+1+\nu_{\rho_j}i}{2}+d\right)\\ &= \prod_{j=1}^{r_1}\prod_{d=0}^{\frac{w-3}{2}}\left(\frac{s+1+\nu_{\rho_j}i}{2}+\frac{w-3}{2}-d\right)\left(\frac{s+1-\nu_{\rho_j}i}{2}+\frac{w-3}{2}-d\right)\\ &= \prod_{j=1}^{r_1}\prod_{d=0}^{\frac{w-3}{2}}\left(\frac{s+1+\nu_{\rho_j}i}{2}+d\right)\left(\frac{s+1-\nu_{\rho_j}i}{2}+d\right) = P_w(s).\end{aligned}$$

But

$$\psi_w(K_1,\chi_1;K_2,\chi_2;s) = \frac{f_w(K_1,\chi_1;K_2,\chi_2;s)}{P_w(s)}.$$

Hence the theorem is proved. □

Proposition 2.2.11 shows that $D_w(\chi_1, \chi_2; s)$ are Dirichlet series in

$$\mathcal{D}((\gamma_{w,\chi_1,\chi_2}, \gamma_{w,\chi_2,\chi_1}), 1, 1-w)$$

and so we may apply Theorem 2.2.10 to obtain identities for their rational values of the Ramanujan type using the isomorphism Theorem 2.2.9. We omit any further calculations in the very general case. In the next section we present some results for the case of quadratic number fields.

Application to Dedekind zeta functions of number fields

In this section, we will construct identities of higher degree by looking at the specific gamma factors $\gamma(s) = ab^s\Gamma^n(s)$ for integers $n = 1, 2, 3, ...$ We can define the holomorphic functions

$$W_n(\tau) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)^n(-i\tau)^{-s}\mathrm{d}s,$$

on the upper half plane, where $c > 0$ is some real number. Mellin transforms of (products of) completed Dedekind zeta functions are now given by series of the form

$$\sum_{k=1}^{\infty} a(k)W_n(b\tau k^q).$$

Let K_1 and K_2 be two number fields of degree n and the same signature (r_1, r_2), where r_1 and r_2 denote the numbers of real and complex embedding of K_1 and K_2, respectively. We now consider the special gamma factor

$$\gamma_{K_1,K_2}(s)_n = \left(\frac{\sqrt{|d_{K_1}d_{K_2}|}}{(2\pi)^n}\right)^s \Gamma(s)^n.$$

Here, d_{K_1} and d_{K_2} denote the discriminants of K_1 and K_2, respectively. We are interested in the space $\mathcal{D}(\gamma_{K_1,K_2}, 1, k)$.

Proposition 2.2.12. *Let K_1 and K_2 be two number fields of degree n and same signature (r_1, r_2), $w > 0$ an odd integer and*

$$\psi_w(s; K_1, K_2) := \gamma_{K_1,K_2}(s)_n\zeta_{K_1}(s)\zeta_{K_2}(s+w).$$

Then we have the functional equation

$$\psi_w(1-w-s; K_1, K_2) = (-1)^{\frac{r_1(w-1)}{2}+r_2}\psi_w(s; K_2, K_1). \tag{2.2.2.5}$$

In other words, we have $\zeta_{K_1}(s)\zeta_{K_2}(s+w) \in \mathcal{D}(\gamma_{K_1,K_2}, 1, 1-w)$, and

$$(\zeta_{K_1}(s)\zeta_{K_2}(s+w))^* = (-1)^{\frac{r_1(w-1)}{2}+r_2}\zeta_{K_2}(s)\zeta_{K_1}(s+w). \tag{2.2.2.6}$$

Proof. The proof works very similar to the one of Proposition 2.2.11. We will first show that

$$\psi_w(s;K_1,K_2)=\frac{1}{(2\sqrt{\pi})^{r_1}}\left(\frac{\sqrt{|d_{K_2}|}}{2^{r_2}\pi^{\frac{n}{2}}}\right)^{-w}\prod_{j=0}^{\frac{w-3}{2}}\left(\frac{s+1}{2}+j\right)^{-r_1}\prod_{\ell=0}^{w-1}(s+\ell)^{-r_2}\times\xi_{K_1}(s)\xi_{K_2}(s+w).$$

This is a simple calculation involving the duplication formula $\Gamma(s)\Gamma(s+\frac{1}{2})=\Gamma(2s)2^{1-2s}\sqrt{\pi}$. We obtain

$$\begin{aligned}
&\frac{1}{(2\sqrt{\pi})^{r_1}}\left(\frac{\sqrt{|d_{K_2}|}}{2^{r_2}\pi^{\frac{n}{2}}}\right)^{-w}\prod_{j=0}^{\frac{w-3}{2}}\left(\frac{s+1}{2}+j\right)^{-r_1}\prod_{\ell=0}^{w-1}(s+\ell)^{-r_2}\times\xi_{K_1}(s)\xi_{K_2}(s+w)\\
&=\frac{1}{(2\sqrt{\pi})^{r_1}}\left(\frac{\sqrt{|d_{K_1}d_{K_2}|}}{2^{2r_2}\pi^{n}}\right)^{s}\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma\left(\frac{s+1}{2}\right)^{r_1}\Gamma(s)^{2r_2}\zeta_{K_1}(s)\zeta_{K_2}(s+w)\\
&=\frac{1}{(2\sqrt{\pi})^{r_1}}\left(\frac{\sqrt{|d_{K_1}d_{K_2}|}}{2^{r_1+2r_2}\pi^{n}}\right)^{s}(2\sqrt{\pi})^{r_1}\Gamma(s)^{r_1+2r_2}\zeta_{K_1}(s)\zeta_{K_2}(s+w)
\end{aligned}$$

and since $n=r_1+2r_2$ we conclude that this equals

$$\left(\frac{\sqrt{|d_{K_1}d_{K_2}|}}{(2\pi)^{n}}\right)^{s}\Gamma(s)^{n}\zeta_{K_1}(s)\zeta_{K_2}(s+w),$$

as required.
Now we show the functional equation by using the above representation in terms of ξ_{K_j} with $j=1,2$. Obviously, the term $\xi_{K_1}(s)\xi_{K_2}(s+w)$ changes to $\xi_{K_2}(s)\xi_{K_1}(s+w)$ under the transformation $s\mapsto 1-w-s$. We have

$$\begin{aligned}
\prod_{j=0}^{\frac{w-3}{2}}\left(\frac{1-w-s+1}{2}+j\right)^{-r_1}&=(-1)^{\frac{r_1(w-1)}{2}}\prod_{j=0}^{\frac{w-3}{2}}\left(\frac{s+1}{2}+\frac{w-3}{2}-j\right)^{-r_1}\\
&=(-1)^{\frac{r_1(w-1)}{2}}\prod_{j=0}^{\frac{w-3}{2}}\left(\frac{s+1}{2}+j\right)^{-r_1}
\end{aligned}$$

and similarly

$$\prod_{\ell=0}^{w-1}(1-w-s+\ell)^{-r_2}=(-1)^{wr_2}\prod_{\ell=0}^{w-1}(s+w-1-\ell)^{-r_2}=(-1)^{wr_2}\prod_{\ell=0}^{w-1}(s+\ell)^{-r_2}.$$

Since $(-1)^w=-1$, the claim follows. □

We are interested in formulas for L-functions at rational arguments. To obtain those, we have to construct a proper generalized complete Dirichlet series.

Definition 2.2.13. *Let $w > 1$ and $b > 0$ be integers with $w \equiv 1 \pmod 2$. Also let $\tilde{K} = (K_1, K_1', K_2, K_2', ..., K_b, K_b')$ be a collection of number fields with the same degree n, such that K_j and K_j' have the same signature $(r_{1,j}, r_{2,j})$ for all $1 \leq j \leq b$. Then we define*

$$\Phi_{w,b}(s; \tilde{K}) = \prod_{j=1}^{b} \psi_w \left(\frac{s+j-1}{b}; K_j, K_j' \right).$$

For the sake of simplicity, we write $D^w_{K_1,K_2}(s) := \zeta_{K_1}(s)\zeta_{K_2}(s+w)$. Now one can apply Theorem 2.2.10 to the data $(D^w_{K_1,K_1'}, ..., D^w_{K_b,K_b'}) \in \times_{j=1}^{b} \mathcal{D}(\gamma_{K_j,K_j'}, 1, 1-w)$ to obtain the functional equation

$$\Phi_{w,b}(1-bw-s; \tilde{K}) = (-1)^{\sum_{j=1}^{b} \frac{r_{1,j}(w-1)}{2} + r_{2,j}} \Phi_{w,b}(s; \tilde{K}_{\text{inv}}). \tag{2.2.2.7}$$

For the convenience of the reader we want to demonstrate this general principle by the explicit case of real quadratic number fields. So assume that the above collection now only contains real quadratic number fields. A calculation shows

$$G^b_{0,0,2,0}((\gamma_{K_j,K_j'})_{1\leq j\leq b}) = \Delta b \left(\frac{2\pi b}{\sqrt[2b]{D}} \right)^{-2s} \Gamma(s)^2,$$

where $D = \sqrt{|d_{K_1} \cdot d_{K_1'} \cdots d_{K_b} \cdot d_{K_b'}|}$ and $\Delta = \prod_{j=1}^{b} |d_{K_j} \cdot d_{K_j'}|^{\frac{j-1}{2b}}$ and hence we have

$$\Phi_{w,b}(s; \tilde{K}) = \Delta b \left(\frac{2\pi b}{\sqrt[2b]{D}} \right)^{-2s} \Gamma(s)^2 \prod_{j=1}^{b} \zeta_{K_j} \left(\frac{s+j-1}{b} \right) \zeta_{K_j'} \left(\frac{s+j-1}{b} + w \right) \tag{2.2.2.8}$$

Obviously, the central object of studying yet is the generalized Dirichlet series

$$D^{w,b}_{\tilde{K}}(s) := \prod_{j=1}^{b} \zeta_{K_j} \left(\frac{s+j-1}{b} \right) \zeta_{K_j'} \left(\frac{s+j-1}{b} + w \right) = \sum_{\nu=1}^{\infty} c_{w,b}(\nu; \tilde{K}) \nu^{-\frac{s}{b}}, \tag{2.2.2.9}$$

where the generating coefficients $c_{w,b}(\nu; \tilde{K})$ are defined by the product in the above equation.

Theorem 2.2.14 (see [26])**.** *Let $w > 1$ and $b > 0$ be integers with $w \equiv 1 \pmod 2$. Let $\tilde{K} = (K_1, K_1', ..., K_b, K_b')$ be a collection of real quadratic number fields as above. For all $\tau \in \mathbb{H}$ we define the (holomorphic) function*

$$E_{w,b}\left(\tau; \tilde{K}\right) = 2\Delta b \sum_{\nu=1}^{\infty} c_{w,b}\left(\nu; \tilde{K}\right) K_0 \left(4\pi b \sqrt[2b]{\frac{\nu}{D}} \sqrt{-i\tau} \right).$$

Then, for all $\tau \in \mathbb{H}$, we have an identity

$$E_{w,b}\left(\tau; \tilde{K}\right) - (-i\tau)^{bw-1} E_{w,b}\left(-\frac{1}{\tau}; \tilde{K}_{\text{inv}}\right) = \sum_{\alpha=0}^{w+1} P_\alpha(\tau) + \log(-i\tau) Q_\alpha(\tau),$$

where the P_j and Q_k are rational functions with $P_\alpha \equiv Q_\alpha \equiv 0$ whenever $1 < \alpha$ and $\alpha \equiv 0 \pmod 2$. The functions P_j are explicitly given by

$$P_0(\tau) = \sum_{\ell=1}^{b} (-i\tau)^{-\ell} A(\ell) R_{K_{b-\ell+1}} (\ell-1)!^2 \prod_{\substack{j=1 \\ j\neq b-\ell+1}}^{b} \zeta_{K_j}\left(\frac{\ell+j-1}{b}\right) \prod_{j=1}^{b} \zeta_{K'_j}\left(\frac{\ell+j-1}{b}+w\right),$$

$$P_1(\tau) = \sum_{\ell=1-b}^{0} (-i\tau)^{-\ell} A(\ell) \frac{\zeta'_{K_{1-\ell}}(0)\zeta_{K'_{1-\ell}}(w)}{b} \prod_{\substack{j=1 \\ j\neq 1-\ell}}^{b} \zeta_{K_j}\left(\frac{\ell+j-1}{b}\right) \zeta_{K'_j}\left(\frac{\ell+j-1}{b}+w\right),$$

$$P_w(\tau) = \sum_{\ell=1-bw}^{b-bw} (-i\tau)^{-\ell} A(\ell) R_{K'_{b-bw+1-\ell}} \frac{\zeta''_{K_{b-bw+1-\ell}}(\ell)}{2b^2(-\ell)!^2} \times \prod_{\substack{j=1 \\ j\neq b-bw+1-\ell}}^{b} \zeta_{K_j}\left(\frac{\ell+j-1}{b}\right) \zeta_{K'_j}\left(\frac{\ell+j-1}{b}+w\right),$$

where $R_K := \operatorname{res}_{s=1}\zeta_K(s)$, and for all $2 \le \alpha \le w+1$ with $\alpha \equiv 1 \pmod 2$ and $\alpha \neq w$

$$P_\alpha(\tau) = \sum_{\ell=1-b\alpha}^{b-b\alpha} (-i\tau)^{-\ell} \left[\Psi_1(\ell) + \Psi_2(\ell) + \Psi_3(\ell)\right]$$

where

$$\Psi_1(\ell) = A(\ell)(-1)^\ell \frac{H_{-\ell}-\gamma}{(-\ell)!^2} \prod_{j=1}^{b} \zeta_{K_j}\left(\frac{\ell+j-1}{b}\right) \zeta_{K'_j}\left(\frac{\ell+j-1}{b}+w\right),$$

$$\Psi_2(\ell) = A(\ell) \frac{1}{b(-\ell)!^2} \left(\sum_{\mu=1}^{b} \left(\zeta'_{K_\mu}\left(\frac{\ell+j-1}{b}\right) + \zeta'_{K'_\mu}\left(\frac{\ell+j-1}{b}+w\right)\right) Z_\mu(\ell)\right),$$

$$\Psi_3(\ell) = -2\log\left(\frac{2\pi b}{\sqrt[2b]{D}}\right) A(\ell) \frac{1}{(-\ell)!^2} \prod_{j=1}^{b} \zeta_{K_j}\left(\frac{\ell+j-1}{b}\right) \zeta_{K'_j}\left(\frac{\ell+j-1}{b}+w\right),$$

with

$$A(s) = \Delta b \left(\frac{2\pi b}{\sqrt[2b]{D}}\right)^{-2s}$$

and

$$Z_\mu(\ell) = \prod_{\substack{j=1 \\ j\neq\mu}}^{b} \zeta_{K_j}\left(\frac{s+j-1}{b}\right) \zeta_{K'_j}\left(\frac{s+j-1}{b}+w\right).$$

The functions Q_j satisfy $Q_0 \equiv Q_1 \equiv 0$ and

$$Q_\alpha(\tau) = -\sum_{\ell=1-b\alpha}^{b-b\alpha} (-i\tau)^{-\ell} A(\ell) \frac{1}{(-\ell)!^2} \prod_{j=1}^{b} \zeta_{K_j}\left(\frac{\ell+j-1}{b}\right) \zeta_{K'_j}\left(\frac{\ell+j-1}{b}+w\right)$$

as $\alpha \geq 2$ *and* $\alpha \equiv 1 \pmod 2$.

Proof. We remember that $2K_0(2\sqrt{x}) = W_2(x)$. From (2.2.2.7) we conclude

$$D_{\tilde{K}}^{w,b}(s) \in \mathcal{D}((G_{0,0,2,0}^{b}((\gamma_{K_j,K_j'})_{1\leq j\leq b}), G_{0,0,2,0}^{b}((\gamma_{K_{b-j+1}',K_{b-j+1}})_{1\leq j\leq b})), b, 1-bw)$$

with dual series

$$\left(D_{\tilde{K}}^{w,b}\right)^{*}(s) = D_{\tilde{K}_{\mathrm{inv}}}^{w,b}(s).$$

Since we clearly have $1 - bw < 2b$, we can use Theorem 2.2.9. The calculations all base on investigating the residues of the completed Dirichlet series. For example, poles of first order are given in $s = 1, 2, ..., b$. The residues here are given by the summands of $P_0(\tau)$. The details and the further calculations are omitted. □

Note that (0.0.0.3) follows by this new identity by setting $b = 1$ and $\alpha = 2\pi d_K^{-\frac{1}{2}}\sqrt{-i\tau}$ and $\beta = 2\pi d_K^{-\frac{1}{2}} \cdot \frac{1}{\sqrt{-i\tau}}$.

2.2.3 Application to L-functions for modular forms of half-integral weight

We can apply the developed methods to find new identities for L-functions assigned to modular forms of half-integral weight. We consider the Hecke group $\mathcal{H}(\lambda) \subset \mathrm{SL}_2(\mathbb{R})$, which is by definition generated by the elements $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T_\lambda = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ where $\lambda > 0$ is some real number. It was shown by Hecke that $\mathcal{H}(\lambda)$ is discrete if and only if $\lambda \geq 2$ or $\lambda = 2\cos\left(\frac{\pi}{m}\right)$ with an integer $m \geq 3$. Let f be a cusp form of weight $k \in \frac{1}{2} + \mathbb{N}_0$ for $\mathcal{H}(\lambda)$. Then f has a Fourier expansion $f(\tau) = \sum_{n=1}^{\infty} a(n) q^{\frac{n}{\lambda}}$ and satisfies the functional equation $f\left(-\frac{1}{\tau}\right) = (-i\tau)^k f(\tau)$. Furthermore, the coefficients $a(n)$ shall be bounded by $a(n) \ll_f n^{\frac{k}{2}}$ (for $\lambda < 2$ this is always the case, for $\lambda \geq 2$ we assume it). The corresponding Hecke L-function $L_f(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$ is absolutely convergent in the half-plane $\{\mathrm{Re}(s) > \frac{k}{2} + 1\}$ and extends to a meromorphic function on all of $\mathbb{C}$. Kohnen and Raji show in [44], that $\phi_f(s) = L_f(s + k - 1)$ is an element of $\mathcal{D}(\gamma_{\mathrm{half}}, 2 - \frac{k}{2}, 2 - k)$ with corresponding gamma factor

$$\gamma_{\mathrm{half}}(s) = \frac{1}{\pi}\left(\frac{2\pi}{\lambda}\right)^{-s}\Gamma(s)\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-s\right) = \left(\frac{2\pi}{\lambda}\right)^{-s}\frac{\Gamma(s)}{\cos(\pi s)}.$$

Now we can assign to f the series

$$\mathcal{E}_f^*(\tau) = \sum_{n=1}^{\infty} a(n) n^{1-k} H\left(\frac{2\pi i n \tau}{\lambda}\right),$$

where the function H is given by

$$H(\tau) := \frac{1}{\sqrt{\pi}}\left(e^{\tau}\Gamma\left(\frac{1}{2}, \tau\right) - \frac{1}{\sqrt{\tau}}\right),$$

and here,

$$\Gamma(\sigma,\tau) := \int_{\sigma}^{\infty} e^{-t}t^{\tau-1}\mathrm{d}t$$

denotes the incomplete Gamma function. Note that H is a holomorphic function on the upper half-plane and $H(\tau) = O(|\tau|^{-\frac{3}{2}})$. Given $a(n) \ll_f n^{\frac{k}{2}}$ it is easy to see that $\mathcal{E}_f^*$ is a holomorphic function on the upper half-plane. In [44], Kohnen and Raji used this series to start a cohomology theory in the case of half-integral weight. It is shown that

$$\mathcal{E}_f^*(\tau) - (i\tau)^{k-2}\mathcal{E}_f^*\left(-\frac{1}{\tau}\right) = P_f(\tau) + \left(\frac{2\pi i\tau}{\lambda}\right)^{-\frac{1}{2}} Q_f(\tau),$$

where P_f and Q_f are polynomials of degree at most $k-\frac{1}{2}$. In the case $k \geq 3$ this result follows also by $D \in \mathcal{D}(\gamma_{\mathrm{half}}, 2-\frac{k}{2}, 2-k)$,

$$\mathcal{M}_{\frac{1}{2}}^{-1}\left(\Gamma(s)\sec(\pi s), x\right) = H(x) \tag{2.2.3.1}$$

and Theorem 2.2.9 by studying the poles of $\widehat{D}(s)$ at half-integral values. Note that the natural embedding

$$\mathcal{D}\left(\gamma_{\mathrm{half}}, 2-\frac{k}{2}, 2-k\right) \hookrightarrow \mathcal{D}\left(\gamma_{\mathrm{half}}, \frac{1}{2}, 2-k\right)$$

and hence the values $\sigma = \frac{1}{2}$ and $\theta_1 = \frac{3}{2}$ are used. We want to apply the main theorem to construct curious formulas for the functions L_f at rational arguments.

Lemma 2.2.15. *Let $\mu \in \mathbb{N}$. We obtain*

$$G^{\mu}(\gamma_{\mathrm{half}})(s) = \sqrt{\mu}\left(2\pi\sqrt{\lambda}\right)^{\mu-1}\left(\frac{2\pi\mu}{\lambda}\right)^{-s}\frac{\Gamma(s)}{\cos^{(\mu-1)}(\pi s)}.$$

Proof. This is routine, observe that

$$\prod_{j=1}^{\mu}\cos\left(\pi\frac{s+j-1}{\mu}\right) = (2\pi)^{1-\mu}\cos^{(\mu-1)}(\pi s).$$

The rest follows by straight calculations. □

Let $(f_j)_{1\leq j\leq \mu}$ be a finite collection of cusp forms with same weight $k \in \frac{1}{2}+\mathbb{N}_0$. One can now use Theorem 2.2.10 to show that

$$\prod_{j=1}^{\mu}\phi_{f_\mu}\left(\frac{s+j-1}{\mu}\right) \in \mathcal{D}\left(G^{\mu}(\gamma_{\mathrm{half}}), \mu\left(2-\frac{k}{2}\right), \mu-\mu k+1\right).$$

At this point we obtain an infinite number of new identities, the details are omitted.

Example 2.2.16. *Let $\mu = 3$. We consider the Dedekind eta function*

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n),$$

which is well known to be a holomorphic modular form of weight $k = \frac{1}{2}$ for $\mathrm{SL}_2(\mathbb{Z})$ with certain Nebentypus character. Due to the above discussed results we find that

$$\phi_\eta = L_\eta\left(s-\frac{1}{2}\right) \in \mathcal{D}\left(\gamma_{\mathrm{half}}, \frac{7}{4}, \frac{3}{2}\right).$$

Hence

$$\mathcal{T}_3(\phi_\eta) \in \mathcal{D}\left(G^3(\gamma_{\mathrm{half}}), \frac{21}{4}, \frac{5}{2}\right),$$

where

$$\gamma_3(s) := G^3(\gamma_{\mathrm{half}})(s) = -96\sqrt{3}\pi^2\left(\frac{\pi}{4}\right)^{-s}\Gamma(s)\sec(\pi s).$$

With residue calculus and (2.2.3.1) *we find*

$$\mathcal{M}^{-1}_{\frac{21}{4}}(\gamma_3, x) = -96\sqrt{3}\pi^{\frac{3}{2}}e^{\frac{\pi}{4}x}\Gamma\left(\frac{1}{2}, \frac{\pi x}{4}\right) + E(x),$$

where

$$E(x) = -\frac{192\sqrt{3}\pi}{x^{\frac{1}{2}}} - \frac{384\sqrt{3}}{x^{\frac{3}{2}}} + \frac{2304\sqrt{3}}{\pi x^{\frac{5}{2}}} - \frac{23040\sqrt{3}}{\pi^2 x^{\frac{7}{2}}} + \frac{322560\sqrt{3}}{\pi^3 x^{\frac{9}{2}}}.$$

Put

$$\phi_\eta\left(\frac{s}{3}\right)\phi_\eta\left(\frac{s+1}{3}\right)\phi_\eta\left(\frac{s+2}{3}\right) = \sum_{m=1}^{\infty}\lambda_{\eta,3}(m)m^{-\frac{s}{3}},$$

then we obtain

$$f(\tau) = \sum_{m=1}^{\infty}\lambda_{\eta,3}(m)\mathcal{M}^{-1}_{\frac{21}{4}}(\gamma_3, -i\tau m^{\frac{1}{3}}) \in \mathcal{R}\left(\gamma_3, \frac{21}{4}, \frac{5}{2}\right).$$

The error term in the transformation law of $f\left(-\frac{1}{\tau}\right)$ is now related to products of values of L_η at arguments $s \in \frac{1}{6}\mathbb{Z}$. The calculations are analogous to those made in [25] when proving Theorem 2.2.2.

2.3 Questions and outlook

It is natural to ask the following question at this stage.

Question 2.3.1. *Is there a possibility to extract more detailed information about (products of) L-functions at rational arguments using the introduced techniques?*

The most promising way is probably finding a cohomology theory just as in the case of modular forms of integer and half-integer weight to describe the period polynomials which have occurred.

A second question refers to results of Jin, Ma, Ono and Soundararajan in [37], who proved that the zeros of the period polynomial of a newform $f \in S_k(\Gamma_0(N))$ lie on the circle $|z| = \frac{1}{\sqrt{N}}$.

Question 2.3.2. *What can we say about the zeros of the error polynomials related to L-functions at rational arguments?*

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