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REMARKS ON SEGAL ALGEBRAS

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Let B be an abstract Segal algebra in some Banach algebra A. There was some belief that in the commutative case A should be semi-simple, if B is, but this is not so (Section I). It is well known that a (proper) abstract Segal algebra does not have bounded right approximate units. It may however have a left unit. Pseudosymmetric Segal algebras in the sense of Reiter do not have bounded left approximate units (Section II). A nonfactorization proof is given for a class of algebras which contains most of the known examples of Segal algebras on abelian groups (Section III).

Throughout this note A denotes a Banach algebra with norm $|\ |_A$. We say that A factorizes, if any $a \in A$ can be written $a = bc$ with $b, c \in A$. We say that A has left (right) approximate units, if for any $a \in A$ and $\varepsilon>0$ there is some $b \in A$ with $|a - ba|_A < \varepsilon$ ($|a - ab|_A < \varepsilon$); if b can be chosen with norm less than some fixed constant, A is said to have bounded left (right) approximate units. This is equivalent (see [6]) to the existence of a bounded left (right) approximate identity, which is a net $\{e_\alpha\}$ in A, bounded in norm, with $\lim_\alpha e_\alpha a = a$ ($\lim_\alpha ae_\alpha = a$) for all $a \in A$. By the right operator norm on A we mean the norm which is induced by the right regular representation of A on itself. If G is a locally compact group and f is a function on G, we define ${}_xf$ for $x \in G$ by ${}_xf(y) = f(yx^{-1})$, $y \in G$, and call ${}_xf$ the (right) translate of f by x. We define R_xf by $R_xf(y) = \Delta(x)^{-1}\,{}_xf(y)$, where Δ is the modular function of G. The support of f is denoted by supp (f). For a subset K G we write χ_K for its characteristic function. If G is abelian and B $L^1(G)$ is a normed algebra with norm $|\ |_B$, we denote by $\hat{B}$ the image of B under the Fourier transformation $f \to \hat{f}$, and for $g \in B$ we define $|\hat{g}|_{\hat{B}} = |g|_B$. By $\mathbb{N}$ we denote the set of natural numbers, by $\mathbb{C}$ the set of complex numbers.

Definition: Let A be a Banach algebra with norm $|\ |_A$ and let B be a dense left ideal in A such that

(i) B is a Banach algebra with some norm $|\ |_B$

(ii) There is a constant D>0 such that

$$|b|_A \leq D \cdot |b|_B$$

for all $b \in B$.

Then B is called an abstract Segal algebra in A (or: left normed ideal in A).

For the definition of Segal algebra (on a locally compact group) see Reiter, [4]. An equivalent definition (see [2] and [3], p. 298) is:

Let G be a locally compact group. A Segal algebra on G is a dense left ideal S in $L^1(G)$ which is a Banach algebra with some norm $|\ |_S$ and has left approximate units (which are unbounded in all known cases).

1. Let S be the multiplicative semi-group of all integers $k \geq 2$. Consider the Banach algebra $\ell^1(S)$. We write e_k for the element in $\ell^1(S)$ corresponding to $k \in S$. Let $\mathbb{C}z$ be a one-dimensional algebra with $z^2 = 0$, and consider the direct product Banach algebra

$$A = \ell^1(S) \oplus \mathbb{C}z$$

with norm $|b + \lambda z|_A = |b|_{\ell^1(S)} + |\lambda|$.

For each integer $k \geq 2$ denote by p_k the smallest prime factor of k and define

$$b_k = \begin{cases} z + \frac{1}{k} e_k & \text{if } k \text{ is prime} \\ \frac{1}{p_k} e_k & \text{otherwise.} \end{cases}$$

Let $B = \{\sum_{i\geq 2} \lambda_i b_i \mid \sum_{i\geq 2} |\lambda_i| < \infty\}$. We have $B \subset A$, and for $b \in B$ the representation $b = \Sigma\lambda_i b_i$ is unique. Thus B is a Banach space with the norm

$$\left| \sum_{i\geq 2} \lambda_i b_i \right|_B = \sum_{i\geq 2} |\lambda_i| .$$

For $a = \zeta z + \Sigma \lambda_i e_i$ in A and $b = \Sigma \mu_j b_j$ in B we have

$$ab = \sum_{i,j} \lambda_i \frac{1}{p_j} \mu_j e_{ij} .$$

Since $p_{ij} \leq p_j$ we obtain $ab \in B$ and

$$|ab|_B \leq \sum_{i,j} |\lambda_i \mu_j| \leq |a|_A |b|_B .$$

So B is a left ideal, and since $|a|_A \leq 2|a|_B$ for $a \in B$, it is a Banach algebra. Since the closure of B in A contains z, B is dense in A. So B is an abstract Segal algebra in A. For $b \in B$ the spectrum in B and the spectrum in A coincide, B being a left ideal in A. In particular the spectral radius of b in B is the spectral radius of its $\ell^1(S)$ - component in $\ell^1(S)$. So B is semi-simple, since $z \notin B$ and $\ell^1(S)$ is semi-simple (being a subalgebra of a commutative group algebra). We have $zA = \{0\}$, so A is not semi-simple.

This example contradicts the second part of Theorem 2.1 in [1] and shows that the condition given in [3], p. 3o3, is non-void.

II. It is well known that an abstract Segal algebra B in A cannot have bounded right approximate units unless B = A. This is not so for left approximate units.

Let $A = \ell^2(\mathbb{N})$, $B = \ell^1(\mathbb{N})$. Choose $\xi \in B$ with $|\xi|_A = 1$ and define for a, b $\in$ A

$$a \cdot b = \langle a, \xi \rangle\, b$$

where $\langle , \rangle$ denotes the scalar product in $\ell^2(\mathbb{N})$. We thus obtain an assoziative multiplication for A, with $|ab|_A \leq |a|_A |b|_A$ for a, b $\in$ A. Obviously we have $\xi b = b$ for all b $\in$ A. Since B is a linear subspace, it is a left ideal in A. For a, b $\in$ B we have

$$|ab|_B \leq |a|_A\, |\xi|_A\, |b|_B \leq |a|_B\, |b|_B\, .$$

So B is an abstract Segal algebra in A having a left unit.

It is not known, if Segal algebras (in the sense of Reiter) can have bounded left approximate units. For Segal algebras on compact groups this is not possible (B.E. Johnson, unpublished). The same holds for pseudosymmetric Segal algebras.

<u>Proposition 1.</u> Let G be a locally compact group and S be a pseudosymmetric Segal algebra on G. If S has bounded left approximate units, then $S = L^1(G)$.

Proof. Suppose S has a left approximate identity $\{e_\alpha\}$ bounded in norm by C>0. Let $K \subset G$ be a fixed compact neighbourhood of the identity in G. There is a (two-sided) approximate identity $\{f_\beta\}$ of functions $f_\beta \in S$, $f_\beta \geq 0$, $\int f_\beta = 1$ with supp $(f_\beta) \subset K$ (see [4], p. 34). We have for $f \in S$

$$|f*f_\beta|_S = |\int f_\beta(y)\, R_y f\, dy|_S$$

$$\leq 1 \cdot \sup_{y\in K} |R_y f|_S < +\infty \quad .$$

By the principle of uniform boundedness the f_β are uniformly bounded in right operator norm on S, say by C'. We have

$$|f_\beta|_S = |\lim_\alpha e_\alpha f_\beta|_S \leq C \cdot C' .$$

So $\{f_\beta\}$ is bounded in S, hence $S = L^1(G)$.

III. The example given in II. shows that abstract Segal algebras may factorize. It is not known if this is possible in the case of Segal algebras on o locally compact group.

Proposition 2. Let G be a locally compact nondiscrete abelian group with dual group $\hat{G}$. Suppose $\hat{G}$ to be compactly generated and let $U = U^{-1}$ be a generating compact subset of $\hat{G}$. Let $B \subset L^1(G)$ be a subalgebra which is a Banach algebra with respect to some norm $|\ |_B$ and suppose that $B \subset L^p(\hat{G})$ for some finite p. Let $f \neq 0 \in B$ be such that $\hat{f}$ is nonnegative or has compact support in $\hat{G}$ and such that all translates of $\hat{f}$ by elements of $\hat{G}$ are in $\hat{B}$. If there is a real polynomial P such that

(1) $$|{}_a\hat{f}|_{\hat{B}} \leq P(n)$$

for all $a \in U^n$ and all $n \in \mathbb{N}$, then B does not factorize.

Proof. a) Let X be a quasi-normed linear space i.e. a normed linear space except for the fact that

$$|a+b|_X \leq C\,(|a|_X + |b|_X)$$

with some constant C. Define a subset $S\subset X$ to be open, if for any $s \in S$ there is an "open ball" $U_\varepsilon(s) = \{ x \in X |\ |x - s|_X < \varepsilon \}$ with $U_\varepsilon(s) \subset S$. Then an "open ball" need not be open but has non-void interior containing 0. Hence a linear map T from X to some quasi-normed space Y is continuous if and only if it is bounded. If we denote by $|T|$ the least constant C with $|Ta|_Y \leq C|a|_X$ for all $a \in X$, the space of all continuous linear maps from X to Y becomes a quasi-normed linear space (and a normed space, if Y is normed). It is easy to check that nearly everything works as in the normed linear case. In particular the Principle of uniform boundedness, the Open mapping theorem, and the Closed graph theorem are valid for complete quasi-normed linear spaces.

b) Let B, f, and U be as in the assumption of the proposition and suppose $\hat{f} \geq 0$ (the modification of the proof for $\hat{f}$ complexvalued with compact support will be obvious). We assume that B factorizes. This implies, since $\hat{B} \subset L^p(\hat{G})$, that $\hat{B} \subset L^{\frac{p}{2}}(\hat{G})$ and hence $\hat{B} \subset L^{p\cdot 2^{-n}}(\hat{G})$ for all $n \in \mathbb{N}$. So $\hat{B} \subset L^q(\hat{G})$ for all $q>0$. By means of the Closed graph theorem (it may be applied, since $L^q(\hat{G})$ is a complete quasi-normed linear space) we obtain that the inclusion map $i_q : \hat{B} \to L^q(\hat{G})$ is continuous. Hence we have

$$(2) \qquad |\hat{b}|_q \leq C_q \cdot |\hat{b}|_{\hat{B}}$$

for all $b \in B$. Let P be a polynomial of degree m say, such that (1) holds. Choose $p < \frac{1}{m+1}$ and $\varepsilon>0$. There is some sompact set K in $\hat{G}$ with $K = K^{-1} \supset U$ and such that

$$\int_{\hat{G}\setminus K} |\hat{f}(x)|^p \, dx < \varepsilon .$$

Choose $x_k \in K^{3k} \setminus K^{3k-1}$ for all $k \in \mathbb{N}$. We have

$$x_k K \cap x_{k'} K = \emptyset$$

for $k \neq k'$. For $n \in \mathbb{N}$ let $g_n = \sum_{k=1}^{n} {}_{x_k}\hat{f}$. We have $K \subset U^i$ for some $i \in \mathbb{N}$. Hence

(3) $$|g_n|_{\hat{B}} \leq \sum_{k=1}^{n} |{}_{x_k}\hat{f}|_{\hat{B}} \leq \sum_{k=1}^{n} P(3ik)$$

$$\leq Q(n) ,$$

where Q is a suitable polynomial of degree $n + 1$. We also have

$$|g_n|_p^p \geq \int \Big| \sum_{k=1}^{n} {}_{x_k}(\hat{f}\, \chi_K) \Big|^p$$

since $\hat{f}$ is nonnegative,

$$= \int \sum_{k=1}^{n} |{}_{x_k}(\hat{f}\chi_K)|^p \geq n \, (|\hat{f}|_p^p - \varepsilon),$$

hence

$$|g_n|_p \geq n^{\frac{1}{p}} \text{ const.}$$

This and (3) contradict (2), since $\frac{1}{p} > m + 1$. So B does not factorize.

<u>Corollary.</u> If $\hat{B}$ is translation-invariant and $|{}_a g|_{\hat{B}} = |g|_{\hat{B}}$ for all $a \in \hat{G}$, $g \in \hat{B}$, then B does not factorize.

Remark. As H.G. Feichtinger has pointed out to me, if the algebra B in Proposition 2 is a Segal algebra, the assumption on f to have positive or compactly supported Fourier transform is not necessary. For, let $f \neq 0 \in B$ satisfy equation (1) and $k \in B$ be such that $\hat{k}$ has compact support and $f*k \neq 0$. Then

$$|_a \widehat{(k*f)}|_{\hat{B}} \leq |_a \hat{k}|_{\widehat{L^1(G)}} \cdot |_a \hat{f}|_{\hat{B}}$$

$$\leq |k|_1 \cdot P(n) .$$

So $f' = k*f \neq 0$ satisfies (1) and its Fourier transform has compact support.

Proposition 2 has some overlap with [5], Theorem 4.1.

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