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Usually in integration theory the function space $\mathscr{E}$ on which the functional I is defined is assumed to be a lattice, and this property is used right from the beginning. We avoid this. $\mathscr{L}^1$ and the integral are constructed and the Beppo Levi Theorem is proved, without the lattice condition. With a weak extra assumption, the Monotone Convergence Theorem holds. If in addition $\mathscr{L}^1$ is a lattice (which is much weaker than requiring $\mathscr{E}$ to be a lattice), Fatou's Lemma and Lebesgue's Theorem on Dominated Convergence follow. In the classical case where $\mathscr{E}$ is a lattice, the construction is equivalent to the usual one and yields the same results. Proofs are simple. In a sense, the procedure just described makes integration more applicable. Even when there is an underlying measure, it is sometimes convenient and natural to start from a function space $\mathscr{E}$ that is not a lattice. Such a situation in Harmonic Analysis will be discussed in a subsequent note.

$\mathbb{N}$ denotes the natural numbers, $\mathbb{R}$ the real numbers, $\overline{\mathbb{R}}$ the extended real numbers $\mathbb{R} \cup \{-\infty, \infty\}$ with the usual ordering and operations (in particular $0 \cdot \infty = \infty \cdot 0 = 0$, whereas $\infty - \infty$ is not defined). The infimum of any subset of $\overline{\mathbb{R}}$ is taken in $\overline{\mathbb{R}}$ and so always exists. For instance $\inf \emptyset = \infty$. If $\mathscr{E}$ is a space of functions with values in $\overline{\mathbb{R}}$, we denote by $\mathscr{E}^+$ the positive part of $\mathscr{E}$, that is: $\mathscr{E}^+ = \{f \in \mathscr{E} \mid f \geqq 0\}$. For $\overline{\mathbb{R}}$-valued functions f, g we let $f \wedge g = \min(f, g)$, $f \vee g = \max(f, g)$ and $f^+ = f \vee 0$. By $|g|$ we denote the absolute value of g. By $\{|g| = \infty\}$, $\{g \neq 0\}$, $\{g > \alpha\}$ we mean the sets of all points x in the domain of g such that $|g(x)| = \infty$, $g(x) \neq 0$, $g(x) > \alpha$ respectively. We denote by χ_A, $A \setminus B$, $f|_A$ the characteristic function of A, the set-theoretic difference of A and B, and the restriction of the map f to the subset A of its domain.

1. Let X be a non-void set, $\mathscr{E}$ a vector space of real functions on X and let I be a *Daniell integral* on $\mathscr{E}$, that is: a map $I: \mathscr{E} \to \mathbb{R}$ satisfying

(1) $$f \geqq g \to I(f) \geqq I(g)$$ (Isotony)

(2) $$I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$$ (Linearity)

(3) $$f \in \mathscr{E}, \quad f_n \in \mathscr{E}^+, \quad \sum_1^\infty f_n \geqq f \to \sum_1^\infty I(f_n) \geqq I(f)$$ (Continuity from below).

Because of (2), we may replace (1) by

(1′) $f \geqq 0 \to I(f) \geqq 0$ (Positivity).

Letting $f = f_2 = f_3 = \cdots = 0$ in (3), we see that (1′) and hence (1) follows from (2) and (3).

Starting from $(X, \mathscr{E}, I)$, we now want to construct $\mathscr{L}^1 = \mathscr{L}^1(X, \mathscr{E}, I)$ and the integral $\int$ on $\mathscr{L}^1$. For the construction of $\mathscr{L}^1$ condition (3) is not needed, so let us first use only positivity and linearity of I. For arbitrary $f\colon X \to [0, \infty]$ let

$$\bar{I}(f) = \inf\{\sum I(f_n) \mid f_n \in \mathscr{E}^+,\ \sum f_n \geqq f\}.$$

Note that $\bar{I}(f)$ may be infinite. On the set $\mathscr{P}$ of all functions $f\colon X \to [0, \infty]$ the functional $\bar{I}$ is isotone and positive (as I is positive) and satisfies

(4) $\bar{I}(\alpha f) = \alpha \bar{I}(f)$ for $\alpha \geqq 0$ (Positive homogeneity).

(5) $\bar{I}(\sum f_n) \leqq \sum \bar{I}(f_n)$ (Countable subadditivity).

By the isotony of $\bar{I}$, we obtain from (5) that $\bar{I}$ is continuous from below on $\mathscr{P}$ (that is: (3) holds if I is replaced by $\bar{I}$ and $\mathscr{E}$, $\mathscr{E}^+$ are replaced by $\mathscr{P}$). Clearly $\bar{I} \leqq I$ on $\mathscr{E}^+$.

For arbitrary $g\colon X \to \overline{\mathbb{R}}$ let $\|g\| = \bar{I}(|g|)$. Let $\mathscr{E}' = \{f \in \mathscr{E} \mid \|f\| < \infty\}$ and $\mathscr{F} = \{g\colon X \to \overline{\mathbb{R}} \mid \|g\| < \infty\}$.

Remark. If $f = g - h$ with $g, h \in \mathscr{E}^+$, then $|f| \leqq g + h \in \mathscr{E}^+$ and so $\|f\| \leqq I(g) + I(h) < \infty$. Hence $\mathscr{E}' = \mathscr{E}$ if we have $\mathscr{E} = \mathscr{E}^+ - \mathscr{E}^+$.

Definition. A function $g\colon X \to \overline{\mathbb{R}}$ is called a *null function* if $\|g\| = 0$. A set $A \subset X$ is called a *null set* if χ_A is a null function. A property Q is said to hold *almost everywhere* (a.e.) if Q holds outside some null set.

Proposition. (i) $\|g\| = 0 \leftrightarrow g = 0$ a.e.

(ii) *The countable union of null sets is a null set.*

(iii) $g = f$ a.e. $\to \|g\| = \|f\|$.

(iv) $g \in \mathscr{F} \to \{|g| = \infty\}$ *is a null set.*

Proof. (i) $\to$: Let $\|g\| = 0$ and set $A = \{g \neq 0\}$. As $\chi_A \leqq \sum_1^\infty |g|$ we obtain $\|\chi_A\| = 0$ by isotony and countable subadditivity of $\bar{I}$, so A is a null set, and $g = 0$ outside A.

$\leftarrow$: Let $B \subset X$ be a null set such that $g = 0$ outside B. As $|g| \leqq \sum_1^\infty \chi_B$ we obtain $\|g\| = 0$.

(ii) Let $\{A_i\}$ be a sequence of null sets and let $A = \bigcup_1^\infty A_i$. Then $\chi_A \leqq \sum_1^\infty \chi_{A_i}$, so $\|\chi_A\| = 0$.

(iii) It suffices to show: if N is a null set and $M = X \backslash N$, then $\|g\chi_M\| = \|g\|$. This last assertion follows from $|g\chi_M| \leqq |g| \leqq |g\chi_M| + \sum_1^\infty \chi_N$.

(iv) Let $A = \{|g| = \infty\}$. We have $n\chi_A \leqq |g|$, so $n\|\chi_A\| \leqq \|g\| < \infty$ for all n, hence $\|\chi_A\| = 0$.

For convenience we identify functions that are equal almost everywhere. It then follows from the above Proposition that $\mathscr{F}$ with pointwise operations (where we define $(f+g)(x) = 0$ if $f(x) + g(x)$ does not exist) and norm $\| \ \|$ is a normed linear space. We can (and shall) admit the elements of $\mathscr{F}$ to be defined only almost everywhere, since this does not change $\mathscr{F}$ as a normed linear space. We note that, on the set of all a.e. defined functions, $\| \ \|$ is countably subadditive in the following sense: If $\sum f_n$ converges pointwise a.e. (so that $\left\| \sum_1^\infty f_n \right\|$ makes sense), then

$$\left\| \sum_1^\infty f_n \right\| \leqq \sum_1^\infty \| f_n \| .$$

This follows from the isotony and countable subadditivity of $\bar{I}$.

Generalized Beppo Levi Theorem. *If $f_n \in \mathscr{F}$, $\sum \| f_n \| < \infty$, then $\sum f_n$ converges pointwise a.e. and $\left\| \sum_1^\infty f_n - \sum_1^k f_n \right\| \to 0$. In particular, $\sum_1^\infty f_n$ is in $\mathscr{F}$.*

Proof. For every $n \in \mathbb{N}$ there are $h_{nk} \in \mathscr{E}^+$, such that $|f_n| \leqq \sum_k h_{nk}$ and $\sum_k I(h_{nk}) < \| f_n \| + 2^{-n}$. Since $\sum_n |f_n| \leqq \sum_{n,k} h_{nk}$ we obtain $\| \sum |f_n| \| \leqq \sum \| f_n \| + 1 < \infty$. By (iv) of the above Proposition we have $\sum |f_n| < \infty$ a.e. In particular $\sum f_n$ converges a.e. and the function $\sum_1^\infty f_n$ so defined satisfies $\left\| \sum_1^\infty f_n - \sum_1^k f_n \right\| \leqq \sum_{k+1}^\infty \| f_n \| \to 0$ by the countable subadditivity of $\| \ \|$.

Corollary 1. *If $\{g_k\}$ is a Cauchy sequence in $\mathscr{F}$, then there is $g \in \mathscr{F}$ with $\| g - g_k \| \to 0$ and $g_{k_n} \to g$ a.e. for a suitable subsequence $\{g_{k_n}\}$.*

Proof. Choose a subsequence $\{g_{k_n}\}$ such that $\| g_{k_n} - g_{k_{n+1}} \| < 2^{-n}$ for all n. As $f_n = g_{k_n} - g_{k_{n+1}}$ satisfies the hypothesis of the above theorem, the sequence $g_{k_n} = g_{k_1} - \sum_1^{n-1} f_m$ converges a.e. and in norm to $g = g_{k_1} - \sum_1^\infty f_m \in \mathscr{F}$.

Corollary 2. *$\mathscr{F}$ is complete.*

Definition. A function $g\colon X \to \bar{\mathbb{R}}$ is called *integrable* if for every $\varepsilon > 0$ there is $f \in \mathscr{E}'$ with $\| g - f \| < \varepsilon$. The space of all integrable functions is denoted by $\mathscr{L}^1$.

Remark. $\mathscr{L}^1$ is the norm closure of $\mathscr{E}'$ in $\mathscr{F}$, and $\mathscr{F}$ is complete, so $\mathscr{L}^1$ is complete.

From now on we also use condition (3). Then $\bar{I} \geqq I$ on $\mathscr{E}^+$ and, since we already have the reverse inequality, $\bar{I} = I$ on $\mathscr{E}^+$.

Proposition. $|I(f)| \leqq \| f \|$ *for* $f \in \mathscr{E}'$.

Proof. Let $g_n \in \mathscr{E}^+$ such that $f \leqq |f| \leqq \sum g_n$. By continuity from below we obtain $I(f) \leqq \sum I(g_n)$, hence $I(f) \leqq \| f \|$. Since also $-I(f) = I(-f) \leqq \| f \|$, the assertion is proved.

Definition. The extension of $I|_{\mathscr{E}'}$ to a norm-continuous linear functional on $\mathscr{L}^1$ is called the *I-integral*. Its value at f is denoted by $\int f\,dI$ or simply $\int f$ and called the *integral of* f (with respect to I).

Beppo Levi Theorem. *Let* $f_n \in \mathscr{L}^1$, $\sum \|f_n\| < \infty$. *Then* $\sum f_n$ *converges a.e. and* $\left\|\sum\limits_1^\infty f_n - \sum\limits_1^k f_n\right\| \to 0$, *in particular* $\sum\limits_1^\infty f_n$ *is in* $\mathscr{L}^1$ *and* $\int \sum\limits_1^\infty f_n = \sum\limits_1^\infty \int f_n$ *by the norm continuity of the integral.*

Proof. The assertion follows from the generalized Beppo Levi Theorem and the fact that $\mathscr{L}^1$ is a closed subspace of $\mathscr{F}$.

Let us consider the following conditions:

(6) Every $f \in (\mathscr{L}^1)^+$ can be approximated in $\|\ \|$ by elements of $\mathscr{E}^+$.

(7) $\mathscr{L}^1$ is a lattice, that is: $f, g \in \mathscr{L}^1 \to f \wedge g,\ f \vee g \in \mathscr{L}^1$.
(It is sufficient to require: $f \in \mathscr{L}^1 \to f^+ \in \mathscr{L}^1$.)

If $\mathscr{E}$ is a lattice, then (6) and (7) are satisfied because of $|f^+ - h^+| \leqq |f - h|$. Since $\|\ \| = \bar{I} = I = \int$ on $\mathscr{E}^+$ and since the integral $\int$ is $\|\ \|$-continuous, condition (6) implies

(8) $\int f = \|f\|$ for $f \in (\mathscr{L}^1)^+$.

Condition (8) implies in particular that $\int$ is a positive functional on $\mathscr{L}^1$. One might think that (8) is always true, but this is not so (see b) of the example in part 2). From now on we suppose (8) to be satisfied.

Monotone Convergence Theorem. *Let* $f_n \in \mathscr{L}^1$, $f_n \nearrow f$, $\int f_n \leqq \text{const} < \infty$. *Then* $\|f - f_n\| \to 0$, *in particular* $f \in \mathscr{L}^1$ *and* $\int f = \lim \int f_n$.

Proof. Consider $\sum g_n$ where $g_n = f_{n+1} - f_n \geqq 0$. As, by (8), $\int$ and $\|\ \|$ coincide on $(\mathscr{L}^1)^+$, the assertion follows from the Beppo Levi Theorem.

If $\mathscr{L}^1$ is a lattice (in particular if $\mathscr{E}$ is a lattice), then

(i) Fatou's Lemma and Lebesgue's Theorem on Dominated Convergence hold and are proved in the usual fashion by means of the Monotone Convergence Theorem.

(ii) The integrable sets (i.e. the sets whose characteristic function is in $\mathscr{L}^1$) form a ring because of $\chi_{A \cup B} = \chi_A \vee \chi_B$ and $\chi_{A \setminus B} = \chi_{A \cup B} - \chi_B$. The measurable sets (i.e. the sets whose intersection with integrable sets is integrable) then form a σ-algebra $\mathscr{M}$, and we obtain a measure μ on $\mathscr{M}$ by defining

$$\mu(A) = \begin{cases} \int \chi_A, & A \text{ integrable} \\ +\infty & \text{otherwise}. \end{cases}$$

If the Stone condition

(9) $f \in \mathscr{L}^1 \to f \wedge 1 \in \mathscr{L}^1$

is satisfied, it follows by monotone convergence that $\mathscr{L}^1$ is a lattice: for $f \in \mathscr{L}^1$ we have $f \wedge \frac{1}{n} = \frac{1}{n}(nf \wedge 1) \in \mathscr{L}^1$ and $f \wedge \frac{1}{n} \searrow f \wedge 0$. Stone's Theorem that (9) implies $\mathscr{L}^1 = \mathscr{L}^1(\mu)$ is proved as usual.

Example. Consider the space $\mathscr{E}$ of all real polynomials on $X = [0, 1]$ with $I =$ the Riemann integral. The conditions (6) and (9) and hence (8) and (7) are satisfied and in fact $\mathscr{L}^1$ is the usual $\mathscr{L}^1(0, 1)$. This follows from the Weierstraß Approximation Theorem.

The example shows that integration without the lattice condition on $\mathscr{E}$ makes good sense. In the case when $\mathscr{E}$ happens to be a lattice, we obtain the usual results, as the reader will have noticed, and the proofs appear to be simpler than the ones using the lattice property. However, integration in the sense of Bourbaki, using nets of functions, is not covered.

To finish this part let us mention a few simple facts without proof:

a) Denote by $\mathscr{L}^1_b$ the space of all bounded functions in $\mathscr{L}^1$ and suppose that every $g \in \mathscr{L}^1_b$ can be approximated in $\| \ \|$ by functions $f \in \mathscr{E}'$ with $|f| \leqq C < \infty$ where C is a constant depending on g. Now, if the implication $f \in \mathscr{E}' \to f^2 \in \mathscr{E}'$ holds (which is equivalent to $\mathscr{E}' \cdot \mathscr{E}' \subset \mathscr{E}'$), then $\mathscr{L}^1_b \cdot \mathscr{L}^1_b \subset \mathscr{L}^1_b$, so $\mathscr{L}^1_b$ with pointwise operations is an algebra. It follows that the integrable sets form a ring and the measurable sets form a σ-algebra, on which the integral defines a measure μ as described above.

b) For $\mathscr{L}^1$ to be a lattice it is necessary and sufficient that the implication $f \in \mathscr{E}' \to f \vee 0 \in \mathscr{L}^1$ holds. In particular, $\mathscr{L}^1$ is a lattice if for every $f \in \mathscr{E}'$ and $\varepsilon > 0$ there are $f_1, f_2 \in \mathscr{E}'$ such that $f_1 \leqq f \vee 0 \leqq f_2$ and $I(f_2 - f_1) < \varepsilon$. Note that in this case also (6) and hence (8) is satisfied, so all results except possibly Stone's Theorem hold.

c) $\mathscr{L}^1$ satisfies the Stone condition (9) if and only if the implication $f \in \mathscr{E}' \to f \wedge 1 \in \mathscr{L}^1$ holds. In particular, $\mathscr{L}^1$ satisfies the Stone condition, if for every $f \in \mathscr{E}'$ and $\varepsilon > 0$ there are $f_1, f_2 \in \mathscr{E}'$ such that $f_1 \leqq f \wedge 1 \leqq f_2$ and $I(f_2 - f_1) < \varepsilon$.

Appendix. Keep the above notation and let $X, \mathscr{E}, I$ be as in the beginning of part 1. We give an alternative description of the integral $\int$ on $\mathscr{L}^1$ and derive from this that $\int$ is a positive functional, without assuming (8).

For arbitrary $f\colon X \to \overline{\mathbb{R}}$ define

$$I'(f) = \inf\left\{\sum_1^\infty I(f_n) \;\middle|\; f_1 \in \mathscr{E}, \quad f_i \in \mathscr{E}^+ \text{ for } i \geqq 2, \ \sum_1^\infty f_n \geqq f\right\}.$$

Then I' is isotone, positive homogeneous, subadditive (more precisely: $I'(f + g) \leqq I'(f) + I'(g)$ if $f + g$ is defined and $I'(f) + I'(g)$ is defined) and $I' = I$ on $\mathscr{E}$ by continuity of I from below. Clearly $I' \leqq \bar{I}$ on $\mathscr{P}$. We assert

(10) $\quad |I'(f)| \leqq \|f\|$ for $f \in \mathscr{F}$. In particular, $I'(f)$ is finite.

Proof. Since $f \leqq |f|$, we have $I'(f) \leqq I'(|f|) \leqq \bar{I}(|f|) = \|f\|$. One is tempted to derive $I'(f) \geqq - \|f\|$ by using $I'(-f) = - I'(f)$, but this is not in general true for $f \in \mathscr{F}$. So let $f_1 \in \mathscr{E}$, $f_2, f_3, \ldots \in \mathscr{E}^+$ such that $\sum f_n \geqq f$ and let $h_1, h_2, \ldots \in \mathscr{E}^+$

such that $\sum h_n \geqq |f|$ and $\sum I(h_n) < \|f\| + \varepsilon$. Since $\sum f_n$ does not assume the value $-\infty$, the function $\sum f_n + \sum h_n$ is well defined. We have $\sum f_n + \sum h_n \geqq 0$, hence $\sum I(f_n) + \sum I(h_n) \geqq 0$ by continuity of I from below, which implies $\sum I(f_n) \geqq -\|f\| - \varepsilon$. This proves $I'(f) \geqq -\|f\|$.

Even though I' is usually not linear on $\mathscr{F}$, we have

(11) $$|I'(f) - I'(g)| \leqq \|f - g\| \quad \text{for} \quad f, g \in \mathscr{F}.$$

Proof. Suppose f and g assume only values in $\mathbb{R}$. Then $-|f-g| \leqq f - g \leqq |f-g|$. In particular $f \leqq g + |f-g|$ which implies $I'(f) \leqq I'(g) + I'(|f-g|)$, and $g \leqq f + |f-g|$ which implies $I'(g) \leqq I'(f) + I'(|f-g|)$. So we have $|I'(f) - I'(g)| \leqq I'(|f-g|) \leqq \|f-g\|$. This proof does not work if f, g are allowed to take infinite values, but this does not matter since we have

(12) $$f, g \in \mathscr{F}, \quad f = g \quad \text{a.e.} \rightarrow I'(f) = I'(g).$$

Proof. As I' is isotone, it suffices to prove the assertion for $f \wedge g$ and $f \vee g$, that is: we may assume $f \leqq g$. Let $N = \{f \neq g\}$. Then χ_N and hence $\infty \cdot \chi_N$ is a null function, so there are $h_n \in \mathscr{E}^+$ with $\sum h_n \geqq \infty \cdot \chi_N$ and $\sum I(h_n) < \varepsilon$. Now let $f_1 \in \mathscr{E}$, $f_2, f_3, \ldots \in \mathscr{E}^+$ such that $\sum f_n \geqq f$. As $\sum f_n$ does not assume the value $-\infty$, the function $\sum f_n + \sum h_n$ is well defined, and it dominates g, since it is $+\infty$ on N. So we have $I'(g) \leqq \sum I(f_n) + \sum I(h_n) < \sum I(f_n) + \varepsilon$ which implies $I'(g) \leqq I'(f)$. The opposite inequality is obvious, as I' is isotone.

Since, by (11), I' is continuous on $\mathscr{L}^1$ and coincides with I on $\mathscr{E}'$, we obtain

(13) $$\int f = I'(f) \quad \text{for} \quad f \in \mathscr{L}^1.$$

Since $I'(f) \geqq 0$ for $f \geqq 0$ (by continuity of I from below), we obtain from (13) that the integral is a positive functional on $\mathscr{L}^1$. So, (8) is not needed for this. Note that, by (13), assuming (8) is equivalent to assuming $I' = \bar{I}$ on $(\mathscr{L}^1)^+$.

Let us finish the Appendix with a general remark to point out that we could have chosen a different approach. Starting with I' as above, we can define $\|\ \|'$ by $\|f\|' = I'(|f|)$ and then proceed as we have done with $\bar{I}$ and $\|\ \|$. Whenever I' is countably subadditive on positive functions (or at least when $I'(\sum |f_n|) \leqq \sum I'(|f_n|)$ for $f_n \in \mathscr{L}^1$, $\mathscr{L}^1$ now being the $\|\ \|'$-closure of $\mathscr{E}' = \{g \in \mathscr{E} \mid \|g\|' < \infty\}$), then the results of part 1 up to (and including) the Monotone Convergence Theorem hold, without additional assumptions. For the rest we need the same assumptions as before. Unfortunately, I' need not in general be countably subadditive, not even on sums $\sum f_n$ with $f_n \in (\mathscr{L}^1)^+$. (This can be seen by looking at a countable sum of things like the example in part 2, but where we let the length of the middle interval J_1 tend to 2 and define $I(f) = -$ length of J_1 whereas the $I(\chi_n)$ remain unchanged.) Since it seems that the countable subadditivity of I' is not an easily tested property (even though it almost always holds) we have not adopted this approach but chosen the approach described in part 1.

2. The following is a variant of part 1 with slightly weaker assumptions. We suppose that I is defined on $\mathscr{E}^+$, taking possibly infinite values and satisfying a slightly weaker continuity condition than before. Under the assumptions of part 1 we obtain a possibly smaller $\mathscr{L}^1$ space than before.

Let X be a non-void set, $\mathscr{E}$ a vector space of real functions and let $I\colon \mathscr{E}^+ \to [0, \infty]$ be isotone, positive linear (that is: (2) holds for $\alpha, \beta \in \mathbb{R}^+$, $f, g \in \mathscr{E}^+$) and continuous from below on $\mathscr{E}^+$ (that is: (3) holds for $f \in \mathscr{E}^+$). As before, for $f\colon X \to [0, \infty]$ we define

$$\bar{I}(f) = \inf\{\sum I(f_n) \mid f_n \in \mathscr{E}^+, \sum f_n \geqq f\}.$$

Then on the set $\mathscr{P}$ of all $f\colon X \to [0, \infty]$ the functional $\bar{I}$ is isotone, positive homogeneous, countably subadditive, and $\bar{I} = I$ on $\mathscr{E}^+$ by continuity from below. Again, for arbitrary $g\colon X \to \overline{\mathbb{R}}$ let $\|g\| = \bar{I}(|g|)$ and let $\mathscr{F} = \{g\colon X \to \overline{\mathbb{R}} \mid \|g\| < \infty\}$. Now let $\mathscr{E}_0^+ = \{f \in \mathscr{E}^+ \mid \|f\| < \infty\} = \{f \in \mathscr{E}^+ \mid I(f) < \infty\}$ and let $\mathscr{E}_0 = \mathscr{E}_0^+ - \mathscr{E}_0^+$. Then $\mathscr{E}_0$ is a linear space and $I|_{\mathscr{E}_0^+}$ extends in a unique way to a linear functional on $\mathscr{E}_0$. Since $(\mathscr{E}_0)^+ = \mathscr{E}_0^+$ this linear functional on $\mathscr{E}_0$, call it I again, is positive. Things now work the same way as in part 1, if we replace $\mathscr{E}'$ by $\mathscr{E}_0$. There is a slight change in the proof of $|I(f)| \leqq \|f\|$ for $f \in \mathscr{E}_0$: Let $f \in \mathscr{E}_0$. Then $f = f_1 - f_2$ with $f_1, f_2 \in \mathscr{E}_0^+$. Let $g_n \in \mathscr{E}^+$ such that $f = f_1 - f_2 \leqq |f| \leqq \sum_1^\infty g_n$. Since $f_1 \leqq f_2 + \sum_1^\infty g_n$ we obtain $I(f_1) \leqq I(f_2) + \sum_1^\infty I(g_n)$ by continuity from below of I on $\mathscr{E}^+$. So

$$I(f) = I(f_1) - I(f_2) \leqq \sum_1^\infty I(g_n),$$

which implies $I(f) \leqq \|f\|$. Also $-I(f) = I(-f) \leqq \|f\|$, hence $|I(f)| \leqq \|f\|$.

This proof also shows that, on $\mathscr{E}_0$, I is continuous from below in the sense of (3). So we could consider $(X, \mathscr{E}_0, I)$ and apply part 1 to it. The results would be identical, since, by continuity from below, $\bar{I}$ constructed from $\mathscr{E}_0^+$ is the same as $\bar{I}$ constructed from $\mathscr{E}^+$. Under the assumptions of part 1, we have $\mathscr{E}_0 \subset \mathscr{E}'$ and so $\mathscr{L}^1$ is contained in the $\mathscr{L}^1$ space of part 1. Of course, equality holds, if $\mathscr{E} = \mathscr{E}^+ - \mathscr{E}^+$, in particular if $\mathscr{E}$ is a lattice, but in general the inclusion may be strict, as we see from the following

Example. a) Let

$$X = (-1, 1), \quad J_1 = \left[-\frac{1}{2}, \frac{1}{2}\right],$$

$$J_n = \left(1 - \frac{1}{n},\ 1 - \frac{1}{n+1}\right] \cup \left[-1 + \frac{1}{n+1},\ -1 + \frac{1}{n}\right)$$

for natural $n \geqq 2$. Let χ_n be the characteristic function of J_n, $n \geqq 1$. Denote by f the identity map of $(-1, 1)$. Define $\mathscr{E}$ to be the space of all finite linear combinations $\alpha f + \sum_1^k \alpha_i \chi_i$. Note that such a linear combination is positive if and only if $\alpha = 0$ and all $\alpha_i \geqq 0$. Let I be the Lebesgue integral on $\mathscr{E}$. Then (1), (2), (3) hold as follows from the properties of the Lebesgue integral on all of $\mathscr{L}^1(-1, 1)$. Since $|f| \leqq 1$ and $\sum_1^\infty \chi_n = 1$ we have $\|f\| \leqq \sum I(\chi_n) = 2 < \infty$, so $f \in \mathscr{E}' \subset \mathscr{L}^1(\mathscr{L}^1$ as in part 1). Let $g \in \mathscr{E}_0$. Then g is of the form $\sum_1^k \alpha_i \chi_i$, and we have

$$\sup\{|f(x) - g(x)| \mid x \in [-\tfrac{1}{2}, \tfrac{1}{2}]\}$$
$$= \sup\{|f(x) - \alpha_1 \chi_1(x)| \mid x \in [-\tfrac{1}{2}, \tfrac{1}{2}]\} \geqq \tfrac{1}{2}.$$

Therefore $\sum_1^\infty \alpha_n \chi_n \geqq |f - g|$ implies $\alpha_1 \geqq \frac{1}{2}$, hence $\|f - g\| \geqq \alpha_1 I(\chi_1) \geqq \frac{1}{2}$. This shows that f does not belong to $\mathscr{L}^1$ ($\mathscr{L}^1$ of this part 2).

b) We now show that for the example just given condition (8) is not satisfied. Let $\mathscr{L}^1$ denote the $\mathscr{L}^1$ of part 1. Consider $g = f + 1 \geqq 0$. For $g_n = f + \sum_1^n \chi_k$ we have $|g - g_n| = \sum_{n+1}^\infty \chi_k$, hence $\|g - g_n\| \leqq 2 \cdot \dfrac{1}{n+1} \to 0$. So $g \in (\mathscr{L}^1)^+$. Since $g = f + \sum_1^\infty \chi_n$, we have $I'(g) \leqq I(f) + \sum_1^\infty I(\chi_n) = 2$ (in fact $I'(g) = 2$). If $\sum_1^\infty \alpha_n \chi_n \geqq g$ then $\alpha_n \geqq \frac{3}{2}$ for all n (look at the value of g at the right endpoint of J_n for every n). This implies $\bar{I}(g) = \|g\| \geqq 2 \cdot \frac{3}{2} = 3$. So $I'(g) \neq \bar{I}(g)$ which by (13) shows that (8) is not satisfied for this example.

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