Aspects of Positivity in Functional Analysis
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ON INTEGRATION WITH RESPECT TO A TRACE

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We propose an approach to noncommutative integration which is based on order. If φ is a faithful semifinite normal trace on the von Neumann algebra $\mathfrak{A}$, there is a natural upper integral on the (unbounded) positive self-adjoint operators affiliated with $\mathfrak{A}$. The upper integral together with interpolation provides an easy access to the usual results in noncommutative integration. This note grew out of an attempt to see whether the approach of [L] works in the noncommutative case.

For lack of space we only include some of the proofs. Complete details will be given elsewhere.

I should like to thank Michael Cowling for discussions and for introducing me to interpolation spaces.

Let H be a Hilbert space, T a linear operator on H, bounded or not. By D(T) we denote the domain of definition of T. If T is closable, $\bar{T}$ denotes its closure. Let $\mathfrak{A}$ be a von Neumann algebra on H, $\mathfrak{A}^+$ its positive part, $\mathfrak{A}'$ its commutant. A linear operator T is called affiliated with $\mathfrak{A}$ (in symbols: $T \sim \mathfrak{A}$), if $TU = UT$ for all unitary $U \in \mathfrak{A}'$. Let $\varphi = \mathfrak{A}^+ \longrightarrow [0,\infty]$ be a trace, i.e. a functional satisfying

(i) $\varphi(\lambda A) = \lambda\varphi(A)$ for $\lambda \geq 0$, $A \in \mathfrak{A}^+$ (with $0 \cdot \infty \stackrel{\text{def}}{=} 0$)

(ii) $\varphi(A+B) = \varphi(A) + \varphi(B)$ for $A,B \in \mathfrak{A}^+$

(iii) $\varphi(A^*A) = \varphi(AA^*)$ for $A \in \mathfrak{A}$.

For $B \in \mathfrak{A}^+$ and a partial isometry $u \in \mathfrak{A}$ one has by (iii)

$$(0.1) \qquad \varphi(uBu^*) = \varphi(B^{1/2}u^*uB^{1/2}) \leq \varphi(B)$$

since $B^{1/2}u^*uB^{1/2} \leq \|u^*u\| B \leq B$.

We suppose the trace φ to be faithful, i.e. $\varphi(A) = 0$ implies $A = 0$, semifinite, i.e. $\varphi(A) = \sup\{\varphi(B) \mid B \in \mathfrak{A}^+, B \leq A, \varphi(B) < \infty\}$, and normal, i.e. for any increasing net $\{T_\mu\}$ in $\mathfrak{A}^+$ with $T_\mu \nearrow T \in \mathfrak{A}^+$ in the weak operator topology, one has $\varphi(T_\mu) \nearrow \varphi(T)$. The normality condition may be restated as follows:

(0.2) For $T \in \mathfrak{A}^+$ and any increasing net $\{T_\mu\}$ in $\mathfrak{A}^+$ with $\lim(T_\mu x|x) \geq (Tx|x)$ for all $x \in H$, one has $\lim \varphi(T_\mu) \geq \varphi(T)$ (the limits being possibly infinite).

This condition clearly implies normality. It is equivalent to normality, because for every normal trace φ there is a family of vectors $\{x_i\}_{i\in I}$ in H such that $\varphi(A) = \sum_{i\in I}(Ax_i|x_i)$ (see [D2], p. 85).

The word "projection" always means "orthogonal projection". For $A, p \in \mathfrak{A}^+$, p a projection, letting $p^\perp = 1-p$ we have by (ii) and (iii)

$$\varphi(A) = \varphi(A^{1/2}pA^{1/2}) + \varphi(A^{1/2}p^\perp A^{1/2}) = \varphi(pAp) + \varphi(p^\perp Ap^\perp). \tag{0.3}$$

If T is a closed densely defined linear operator, it has a polar decomposition $T = u|T|$ where u is a partial isometry vanishing on $\operatorname{Ker}T$, and $|T| = (T^*T)^{1/2}$ is positive self-adjoint.

If $T \sim \mathfrak{A}$, then $u \in \mathfrak{A}$ and the spectral projections of $|T|$ are in $\mathfrak{A}$.

An equality $A = u|A|$ will usually mean that the right-hand side is the polar decomposition of A.

An equality $A = \int_0^\infty \lambda\, de_\lambda$ will usually mean that the right-hand side is the spectral representation of the positive self-adjoint operator A. An integral $\int_a^b$ will mean the integral over the half-open interval $[a,b)$.

1. The upper integral $\overline{\varphi}$

Let N be the set of all densely defined closed linear operators affiliated to $\mathfrak{A}$. On $N^+ = \{T \in N \mid T \text{ positive self-adjoint}\}$ we define an upper integral $\overline{\varphi}$ by

$$\overline{\varphi}(T) = \inf\{\sum_1^\infty \varphi(A_n) \mid A_n \in \mathfrak{A}^+, \sum_1^\infty A_n \geq T\}$$

where $\sum_1^\infty A_n \geq T$ means $\sum_1^\infty (A_n x|x) \geq (Tx|x)$ for all $x \in D(T)$, the left side of this inequality being allowed to take the value ∞.

If $\bar{\varphi}(T)<\infty$, one can show that $D(T)$ is "big with respect to φ" or shorter "φ-dense" in the sense of the following definition:

(1.1) <u>Definition:</u> A subspace S of H is called φ-dense, if for every $\varepsilon>0$ there is a projection $p\in\mathfrak{A}$ with $pH\subset S$ and $\varphi(p^\perp)<\varepsilon$. We say that a property holds almost everywhere (a.e.) on H, if it holds on a φ-dense subspace.

Since φ is faithful, a φ-dense subspace is automatically norm dense in H.

The intersection of countably many φ-dense subspaces is φ-dense, because $\varphi((\bigcap p_i)^\perp)\leqslant\sum\varphi(p_i^\perp)$ which follows from $(\bigcap p_i)^\perp=\bigcup p_i^\perp\leqslant\sum p_i^\perp$ and (0.2).

The fact that $\bar{\varphi}(T)<\infty$ implies that $D(T)$ is φ-dense has two consequences:

(A) All operators we shall deal with will automatically belong to $M=\{T\in N\,|\,D(T)$ is φ-dense$\}$, in particular they will be measurable in the sense of I. Segal [S]. It seems reasonable to call M the set of strongly measurable operators. One can show that for $S,T\in M$ the operators $S+T$, ST, S^* are φ-densely defined and closable and that M is a $*$-algebra when equipped with the "strong sum" $\overline{S+T}$, the "strong product" $\overline{ST}$, and the adjoint operation $*$ (this is no problem and takes less than a page in print).

(B) $\bar{\varphi}$ can be written differently:

$$\bar{\varphi}(T)=\inf\{\textstyle\sum\varphi(A_i)\;|\;\sum A_i\geqslant T\quad\text{a.e.}\}$$

where $\sum A_i\geqslant T$ a.e. means $\sum(A_ix|x)\geqslant(Tx|x)$ for all x in a φ-dense subspace of $D(T)$.

Taking the above as motivation we now use (B) as the definition of the upper integral $\bar{\varphi}$.

(1.2) <u>Proposition:</u> $\bar{\varphi}=\varphi$ on $\mathfrak{A}^+$.

Proof: Clearly $\bar{\varphi}\leqslant\varphi$ on $\mathfrak{A}^+$, so we have to show $\bar{\varphi}\geqslant\varphi$. Let $A\in\mathfrak{A}^+$ and $A_i\in\mathfrak{A}^+$ with $\sum A_i\geqslant A$ a.e. Then there is a projection $p\in\mathfrak{A}$ with $\sum A_i\geqslant A$ on pH and $\varphi(p^\perp)<\varepsilon$. By (0.3) we have

$$\varphi(A)=\varphi(pAp)+\varphi(p^\perp Ap^\perp)\leqslant\varphi(pAp)+\|A\|\,\varphi(p^\perp)$$

and by (0.2) since $\sum pA_ip\geqslant pAp$:

$$\leq \sum \varphi(pA_i p) + \|A\| \cdot \varepsilon$$
$$\leq \sum \varphi(A_i) + \|A\| \cdot \varepsilon \qquad \text{by } (0.1).$$

So $\varphi(A) \leq \sum \varphi(A_i)$, hence $\varphi(A) \leq \bar{\varphi}(A)$.

(1.3) <u>Remark:</u> If $S,T,T_n \in M^+$, $\bar{\varphi}$ clearly satisfies

(i) $S \leq T$ a.e. implies $\bar{\varphi}(S) \leq \bar{\varphi}(T)$ (Isotony)

(ii) $\bar{\varphi}(\lambda T) = \lambda_\infty \bar{\varphi}(T)$ for $\lambda \geq 0$ (Positive homogeneity)

(iii) If $T = \sum_1^\infty T_n$ weakly a.e. (i.e. $(Tx|y) = \sum (T_n x|y)$ for all x,y in a φ-dense subspace) then $\bar{\varphi}(T) \leq \sum_1^\infty \bar{\varphi}(T_n)$ (Countable subadditivity)

In particular, $\bar{\varphi}(\overline{S+T}) \leq \bar{\varphi}(S) + \bar{\varphi}(T)$ for $S,T \in M^+$.

(1.4) <u>Proposition:</u> Let $T \in M^+$ and $T_n \in \mathfrak{A}^+$ with $T = \sum T_n$ weakly a.e. Then $\bar{\varphi}(T) = \sum \varphi(T_n)$.

Proof: Clearly $\bar{\varphi}(T) \leq \sum_1^\infty \varphi(T_n)$. But $T \geq \sum_1^k T_n$ a.e., so $\bar{\varphi}(T) \geq \bar{\varphi}(\sum_1^k T_n) = \varphi(\sum_1^k T_n) = \sum_1^k \varphi(T_n)$. Hence $\bar{\varphi}(T) \geq \sum_1^\infty \varphi(T_n)$.

(1.5) <u>Corollary:</u> For $T = \int_0^\infty \lambda de_\lambda \in M^+$ and $p \geq 1$ we have

$$\bar{\varphi}(T^p) = \int_0^\infty \lambda^p d\varphi(e_\lambda).$$

Proof: Taking $T_n = \int_{n-1}^n \lambda^p de_\lambda$ we obtain from the last Proposition

$$\bar{\varphi}(T^p) = \sum \varphi(T_n) = \sum \int_{n-1}^n \lambda^p d\varphi(e_\lambda) = \int_0^\infty \lambda^p d\varphi(e_\lambda).$$

(1.6) <u>Corollary:</u> $\bar{\varphi}$ is positive linear on M^+.

Proof: Let $S,T \in M^+$ and choose $S_n,T_n \in \mathfrak{A}^+$ such that $S = \sum S_n$ a.e., $T = \sum T_n$ a.e. Then $S+T = \sum (S_n + T_n)$ a.e. and by Proposition (1.4) we have

$$\bar{\varphi}(\overline{S+T}) = \sum \varphi(S_n + T_n) = \sum \varphi(S_n) + \sum \varphi(T_n) = \bar{\varphi}(S) + \bar{\varphi}(T).$$

By the way, the argument of the above proof shows that $\bar{\varphi}$ is even countably additive on M^+:

(1.7) If $T,T_n \in M^+$ with $T = \sum T_n$ weakly a.e., then $\bar{\varphi}(T) = \sum \bar{\varphi}(T_n)$.

The following is a noncommutative Beppo Levi theorem:

(1.8) Theorem: Let $T_i \in M^+$ with $\sum \bar{\varphi}(T_i) < \infty$. There is $T \in M^+$ such that $\sum T_i$ converges weakly a.e. to T, and we have $\bar{\varphi}(T) = \sum \bar{\varphi}(T_i)$.

Proof: Let $A_{ik} \in \mathfrak{A}^+$ with $T_i = \sum_k A_{ik}$ weakly on $D(T_i)$. Let $D_o = \{x \in H \mid \sum_{i,k} (A_{ik}x|x) = \sum \|A_{ik}^{1/2}x\|^2 < \infty\}$. Clearly, D_o is a linear subspace and $UD_o \subset D_o$ for unitary $U \in \mathfrak{A}'$, so the projection p of H onto $\bar{D}_o$ is in $\mathfrak{A}$. Since $\sum (A_{ik}x|x) = \infty$ outside D_o, we have for $n \in \mathbb{N}$

$$np^{\perp} \leqslant \sum A_{ik}$$

$$\text{hence} \quad n\varphi(p^{\perp}) \leqslant \sum \varphi(A_{ik}) < \infty ,$$

so $\varphi(p^{\perp}) = 0$, that is: D_o is dense. The form $\sum A_{ik}$ on D_o is closed (easy to check), so there is a self-adjoint positive operator T on $D \subset D_o$ with $T = \sum A_{ik}$ weakly on D (see e.g. [RS], p. 278). We have $T \sim \mathfrak{A}$, so $T \in N$, and assuming for the moment that $D = D(T)$ is φ-dense, we have by (1.4) $\bar{\varphi}(T) = \sum \varphi(A_{ik}) = \sum \bar{\varphi}(T_i)$. Clearly, $T = \sum T_i$ weakly on the φ-dense space $(\bigcap_i D(T_i)) \cap D$.

That D is φ-dense follows from the fact that the two seemingly different definitions of $\bar{\varphi}$ given at the beginning coincide. But we can also use a direct argument similar to the one just used above: If $T = \int \lambda de_{\lambda}$ and $t > 0$, we have $e_t H \subset D = D(T)$ and $te_t^{\perp} \leqslant T$ on D. Since D is a core for the form $\sum A_{ik}$ on D_o we have $te_t^{\perp} \leqslant \sum A_{ik}$ on D_o. Outside D_o this last inequality holds, too, the form on the right being ∞ there. So $te_t^{\perp} \leqslant \sum A_{ik}$ a.e. (in fact: everywhere on H) which implies

$$t\varphi(e_t^{\perp}) \leqslant \sum \varphi(A_{ik}) < \infty.$$

Thus we have $\varphi(e_t^{\perp}) < \varepsilon$ for t sufficiently large. Hence D is φ-dense.

(1.9) Proposition: For $T \in M^+$ and $u \in \mathfrak{A}$ a partial isometry we have $\bar{\varphi}(u^*Tu) \leqslant \bar{\varphi}(T)$.

Proof: Let $A_i \in \mathfrak{A}^+$ with $\sum A_i \geqslant T$ a.e. We have $\sum u^*A_i u \geqslant u^*Tu$ a.e., hence $\bar{\varphi}(u^*Tu) \leqslant \sum \varphi(u^*A_i u) \leqslant \sum \varphi(A_i)$ by (0.1). So $\bar{\varphi}(u^*Tu) \leqslant \bar{\varphi}(T)$.

2. The Banach space L^1 and the normed sets L^p

For $1 \leqslant p < \infty$ we define $L^p = \{T \in M \mid \bar{\varphi}(|T|^p) < \infty\}$ and $\|T\|_p = \bar{\varphi}(|T|^p)^{1/p}$ for $T \in L^p$. If $|T| = \int \lambda de_\lambda$, then $\|T\|_p = (\int \lambda^p d\varphi(e_\lambda))^{1/p}$ by (1.5). Using the properties of $\bar{\varphi}$ and classical arguments we obtain that L^p is $*$-invariant and $\|T^*\|_p = \|T\|_p$, that L^1 is a normed $\mathfrak{A}$-module and $\bar{\varphi}$ extends to a linear functional on L^1 satisfying

$$\|T\|_1 = \sup\{\bar{\varphi}(ST) \mid S \in \mathfrak{A}, \ \|S\| \leqslant 1\}$$

for $T \in L^1$, and that $(S|T) = \bar{\varphi}(T^*S)$ defines an inner product on L^2. We have $\bar{\varphi}(ST) = \bar{\varphi}(TS)$ for $S, T \in L^2$ or $S \in \mathfrak{A}$, $T \in L^1$.

(2.1) <u>Theorem:</u> L^1 is complete.

Proof: It suffices to show that $\sum \|T_n\|_1 < \infty$ implies convergence of $\sum T_n$ in L^1. Decomposing into positive parts and applying Theorem (1.8) we obtain the desired result.

(2.2) <u>Monotone Convergence Theorem:</u> Let $\{T_n\}$ be a sequence in L^1 with $T_{n+1} \geqslant T_n$ on $\bigcap_1^\infty D(T_k)$ and suppose $\sup_n \bar{\varphi}(T_n) < \infty$. Then there is $T \in L^1$ such that $T_n \to T$ weakly a.e. and $T_n \to T$ in L^1.

Proof: We have $S_n = T_{n+1} - T_n \geqslant 0$ a.e. and this readily implies $S_n \in (L^1)^+$. By Theorem (1.8) the series $\sum S_n$ converges weakly a.e. and in L^1, hence so does $T_n = T_1 + \sum_1^{n-1} S_i$.

(2.3) <u>Egoroff's Theorem:</u> Let $T, T_n \in M$ with $T_n \to T$ strongly a.e. and let $q \in \mathfrak{A}$ be a projection with $\varphi(q) < \infty$. Then there is a subsequence $\{T_{n_i}\}$ which converges to T "almost uniformly on qH", i.e. for every $\varepsilon > 0$ there is a projection $p \in \mathfrak{A}$ with $p \leqslant q$, $\varphi(q-p) < \varepsilon$ and such that $T_{n_i}|_{pH} \to T|_{pH}$ in the uniform norm.

The reader should be warned that (unlike the classical Egoroff Theorem for functions) in the above theorem the conclusion does not hold for the full sequence $\{T_n\}$, even in the commutative case! This is due to the fact that for $H = L^2(X, \mu)$, $\mathfrak{A} = L^\infty(X, \mu)$, $\varphi(f) = \int f d\mu$ for $f \in \mathfrak{A}^+$ "a.e. on H" is almost but not precisely the same as "a.e. on X".

(2.4) <u>Dominated Convergence Theorem:</u> Let $T, T_n \in M$ with $T_n \to T$ strongly a.e. and suppose there is $A \in (L^1)^+$ with $|T-T_n| \leqslant A$ a.e. Then $T_n \to T$ in L^1.

Proof: In the case of functions on a measure space X, to estimate $\|T-T_n\|_1$, one would for instance choose a set Q of finite measure such that the integral of A over $X\setminus Q$ is less than ε and A is bounded on Q, choose a slightly smaller set $P \subset Q$ according to Egoroff's Theorem (for $\varepsilon' = \varepsilon$ divided by the bound of A on Q), split the integral $\int|T-T_n|$ into three integrals (over P, $Q\setminus P$, and $X\setminus Q$) and estimate, ending up with $\|T-T_n\|_1 < 3\varepsilon$ for $n > n_o$. This proof carries over to the noncommutative case. Because of (2.3) we obtain the assertion for a subsequence $\{T_{n_i}\}$, but this suffices.

(2.5) <u>Theorem:</u> For $B \in L^1$ let $f_B \in \mathfrak{A}^*$ be the functional $A \mapsto \bar{\varphi}(AB)$. The map $B \mapsto f_B$ is an isometric isomorphism of L^1 onto the predual $\mathfrak{A}_*$, the space of ultraweakly continuous linear functionals on $\mathfrak{A}$.

Proof: by standard arguments.

3. Identification of L^p as interpolation space

M is a Hausdorff topological linear space when equipped with the topology of convergence in measure which has the following sets $N(\varepsilon)$ as a neighbourhood basis at 0:

$N(\varepsilon) = \{T \in M|$ there is a projection $p \in \mathfrak{A}$ with $pH \subset D(T)$, $\|Tp\|<\varepsilon$, and $\varphi(p^\perp)<\varepsilon\}$. The inclusions $\mathfrak{A} \subset M$ and $L^1 \subset M$ are continuous. For $0<\theta<1$ let $(\mathfrak{A}, L^1)_\theta$ be the interpolation space constructed by the complex method.

(3.1) <u>Theorem:</u> $L^p = (\mathfrak{A}, L^1)_{1/p}$ with equal norms. In particular, L^p is a linear space, $\| \ \|_p$ is a norm, and L^p is complete under $\| \ \|_p$.

Sketch of proof: With a bit of care we may follow the classical proof for functions ([BL], p. 106).

a) For $A = u|A| \in L^p$ with $|A| = \int_a^b \lambda de_\lambda$ $(0<a<b<\infty)$ the function $f(z) = u|A|^{pz}$ (with a factor $\exp(\varepsilon z^2 - \varepsilon p^{-2})$ depending on which definitions one uses) is in the function space F used for the construction of $(\mathfrak{A}, L^1)_{1/p}$, and this together with a simple approximation argument shows that $L^p \subset (\mathcal{A}, L^1)_{1/p}$ (norm-decreasing inclusion).

b) The classical proof of $(L^\infty, L^1)_{1/p} \subset L^p$ seems to use duality between L^p and L^q, but really uses much less, so we can easily imitate it. Let $A \in (\mathcal{A}, L^1)_{1/p}$. Then in particular $A \in M$. We can approximate $\|A\|_p$ (be it finite or infinite) by expressions $\bar{\varphi}(B|A|)$ where B has $\|B\|_q = 1$ and is a linear combination of pairwise orthogonal projections commuting with $|A|$.
If $f \in F$ with $f(1/p) = A = u|A|$, consider the function $h(z) = \bar{\varphi}(B^{q(1-z)}u^* f(z))$. Applying the three lines theorem to h, we get $|h(1/p)| = \bar{\varphi}(B|A|) \leq \|f\|$, hence $\|A\|_p \leq \|f\|$ which proves the norm-decreasing inclusion $(\mathcal{A}, L^1)_{1/p} \subset L^p$.

The rest is an easy consequence of well-known facts from complex interpolation:

(3.2) <u>Corollary:</u> (i) L^2 is a Hilbert space.
(ii) L^p is reflexive for $1<p<\infty$.
(iii) $(L^p)^* = L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: (i) Being an interpolation space, L^2 is a Banach space and we have already seen that its norm comes from an inner product.

(ii) This is obtained by interpolation between L^1 and L^2, and between L^2 and $\mathcal{A}$, using that L^2 is reflexive (see [C], 2.12).

(iii) is true for $1 \leq p \leq 2$ by interpolation since $(L^1)^* = \mathcal{A} \stackrel{\text{def}}{=} L^\infty$ and $(L^2)^* = L^2$ (see [BL], Corollary 4.5.2). Since L^p is reflexive by (ii), the result holds for $2 \leq p < \infty$, too.

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