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Another proof of the Shirali-Ford theorem

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ABSTRACT. Shirali and Ford showed that every hermitian Banach $*$-algebra is symmetric. Meanwhile there have been several proofs of this theorem. We give another proof, a fairly conceptual one. It actually shows that every $*$-algebra which admits a spectral C^*-seminorm is completely symmetric.

Let A be a Banach $*$-algebra, i.e. a Banach algebra with a (not necessarily continuous) involution. Suppose that A is hermitian, i.e. the spectrum of selfadjoint elements $a = a^* \in A$ is real. The famous theorem of Shirali and Ford then states that A is symmetric, i.e. every element a^*a, where $a \in A$, has its spectrum $\sigma(a^*a)$ contained in $[0, \infty)$. There are several proofs of this theorem (see [DB; Theorem 33.2] and the comments before it as well as [Pt], [B1], [B2], [F]). We give another proof starting from the fact that, on hermitian algebras, the Pták functional $s(a) = r(a^*a)^{1/2}$, where $r(b)$ denotes the spectral radius of $b \in A$, is a spectral C^*-seminorm, i.e. a C^*-seminorm which dominates the spectral radius on A (see [DB; Theorem 33.1(a), (d), (j), (k)] for a proof of this fact). Let A_1 denote the algebra with unit adjoined.

THEOREM (Shirali and Ford). Every hermitian Banach $*$-algebra is symmetric.

PROOF. A is hermitian if and only if A_1 is hermitian, as is easily checked, so we may suppose $1 \in A$. Let $a \in A$ and $\lambda \in \sigma_A(a^*a)$. If B is the closed subalgebra of A generated by 1 and a^*a, by Gelfand's theorem there is an algebra homomorphism $\varphi : B \to \mathbb{C}$ with $\varphi(a^*a) = \lambda$. For $b \in B$ we have $|\varphi(b)| \le r_B(b) = r_A(b) \le s(b)$, the last inequality holding because A is hermitian ([DB; 33.1(a)]). Since s is a seminorm on A, by Hahn-Banach there is a linear extension $f : A \to \mathbb{C}$ of φ with $|f(c)| \le s(c)$ for all $c \in A$. Since $f(1) = \varphi(1) = 1$, the following proposition implies positivity of f, hence $\lambda = \varphi(a^*a) = f(a^*a) \ge 0$. □

PROPOSITION. Let q be a C^*-seminorm on a complex $*$-algebra A with unit. Let $f : A \to \mathbb{C}$ be linear with $|f(a)| \le q(a)$ for all $a \in A$ and $f(1) = 1$. Then f is positive.

PROOF. For a C^*-algebra this is well known (see for instance [DB; Corollary 22.18]). So the proof is a reduction to this case. For the reader's convenience, we

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write the argument down. The set $N = \{a \in A | q(a) = 0\}$ is a $*$-ideal in A. On the $*$-algebra A/N define $\dot{f}$ and $\dot{q}$ by $\dot{f}(\dot{a}) = f(a)$, $\dot{q}(\dot{a}) = q(a)$ where $\dot{a} = a + N$, $a \in A$. Then $\dot{q}$ is a C^*-norm on A/N, and $|\dot{f}(\dot{a})| \leq \dot{q}(\dot{a})$ for $\dot{a} \in A/N$. Denote the completion of $(A/N, \dot{q})$ by $(C, \| \|)$, and the continuous extension of $\dot{f}$ to C by F. We have $|F(b)| \leq \|b\|$ for $b \in C$ and $F(\dot{1}) = 1$, hence $\|F\| = F(\dot{1})$ where $\dot{1}$ is the unit of C. Since C is a C^*-algebra, F is positive. So, for $a \in A$, we have $f(a^*a) = F(\dot{a}^*\dot{a}) \geq 0$, i.e. f is positive. □

Remark 1. If we replace a^*a in the proof of the theorem by $a_1^*a_1 + \ldots + a_k^*a_k$, we obtain complete symmetry of A (i.e. the spectrum of elements $a_1^*a_1 + \ldots + a_k^*a_k$ is contained in $[0, \infty)$). The concept of complete symmetry is due to Wichmann [W].

Remark 2. If one wants to use a more elementary argument (without use of Gelfand's theorem), the third and fourth sentence of the theorem's proof should be replaced by "The map $\varphi : \sum_0^n \alpha_k (a^*a)^k \mapsto \sum_0^n \alpha_k \lambda^k$ from the subalgebra B of all polynomials in a^*a to the complex numbers $\mathbb{C}$ is well defined, linear, and satisfies $|\varphi(b)| \leq r_A(b) \leq s(b)$ for $b \in B$. The last inequality holds because A is hermitian."

From the above remarks and the theorem's proof we obtain the following

Corollary. Every complex $*$-algebra which admits a spectral C^*-seminorm is completely symmetric.

Proof. (i) If A is a $*$-algebra with unit, q a spectral C^*-seminorm on it, let $x_1, \ldots, x_n \in A$ and $y = \sum x_i^* x_i$. If $\lambda \in \sigma(y)$, the map $\varphi : p(y) \mapsto p(\lambda)$ is well defined linear from the subalgebra B of all polynomials in y to $\mathbb{C}$ satisfying $|\varphi(b)| \leq r(b) \leq q(b)$ for all $b \in B$. By Hahn-Banach there is a linear extension $f : A \to \mathbb{C}$ with $|f(a)| \leq q(a)$ for all $a \in A$. Since $f(1) = \varphi(1) = 1$, f is positive (see the Proposition), so $\lambda = \varphi(y) = f(y) = f(\sum x_i^* x_i) \geq 0$.
(ii) If A has no unit, let r_1 denote the spectral radius in A_1, q_1 the canonical C^*-seminorm extension of q to A_1. Since $(\mu + a) \mapsto |\mu|$ is a C^*-seminorm on A_1, so is $q' : \mu + a \mapsto \max\{|\mu|, q_1(a)\}$. For $\mu + a \in A_1$ one has $r_1(\mu + a) \leq |\mu| + r_1(a) = |\mu| + r(a) \leq |\mu| + q(a) \leq 2\max\{|\mu|, q_1(a)\} = 2q'(\mu + a)$. For $c = \mu + a$ this implies $r_1(c) = r_1(c^n)^{1/n} \leq 2^{1/n} q'(n) \to q'(n)$, so q' is a spectral seminorm on A_1. By (i), A_1 and hence A is completely symmetric. □

D.Birbas states in [B1,Theorem 3.2(i)] that every involutive algebra with realvalued subadditive Pták function (which then is a spectral C^*-seminorm, see [B1,Lemma 3.1]) is symmetric. His proof actually shows complete symmetry. At first sight, this seems to be a rather special case of the above Corollary, but on the other hand, any nonzero spectral C^*-seminorm has to coincide with the Pták function.

Let us also mention that the Corollary provides a more direct proof for the main part of [P,Proposition 10.4.2].

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