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ISOMORPHISMS OF $AC(\sigma)$ SPACES

IAN DOUST AND MICHAEL LEINERT

ABSTRACT. Analogues of the classical Banach-Stone theorem for spaces of continuous functions are studied in the context of the spaces of absolutely continuous functions introduced by Ashton and Doust. We show that if $AC(\sigma_1)$ is algebra isomorphic to $AC(\sigma_2)$ then σ_1 is homeomorphic to σ_2. The converse however is false. In a positive direction we show that the converse implication does hold if the sets σ_1 and σ_2 are confined to a restricted collection of compact sets, such as the set of all simple polygons.

1. INTRODUCTION

In [3] Ashton and Doust defined the Banach algebra $AC(\sigma)$ consisting of 'absolutely continuous' functions with domain an arbitrary nonempty compact subset σ of $\mathbb{C}$ (or equivalently of $\mathbb{R}^2$). The motivation for their definition was to extend the spectral theory of well-bounded operators to cover operators whose spectra need not be contained in the real line. This led to the definition of an $AC(\sigma)$ operator being a bounded operator on a Banach space X which admits a bounded functional calculus $\Psi : AC(\sigma) \to B(X)$. Under some additional assumptions, the image of this map Ψ is an algebra of operators which is isomorphic to $AC(\sigma)$. Accordingly, one can recover certain aspects of the theories of normal operators and of scalar-type spectral operators, replacing algebras of continuous functions $C(\Omega)$ with algebras of absolutely continuous functions. Quite naturally then, underlying many of the open problems in this area are questions which ask for analogues of the classical topological results about $C(\Omega)$ spaces. (Details of the theory of $AC(\sigma)$ operators can be found in [5].)

One of the most classical of these topological results is the Banach-Stone theorem which says that two compact Hausdorff spaces Ω_1 and Ω_2 are homeomorphic if and only if the function algebras $C(\Omega_1)$ and $C(\Omega_2)$ are linearly isometric. There have been many generalizations and extensions of this result (see [9]). Work of Amir [1] shows that one may still deduce that Ω_1 and Ω_2 are homeomorphic if one only assumes that $C(\Omega_1)$ and $C(\Omega_2)$ are (linearly) isomorphic with Banach-Mazur distance less than 2. Cohen [6] has shown that the value 2 is sharp.

In a different direction, one might require that the spaces $C(\Omega_1)$ and $C(\Omega_2)$ be isomorphic as algebras. In this case one may argue using the maximal ideal spaces to get the same conclusion. In particular, as the next result shows, if $C(\Omega_1)$ and

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$C(\Omega_2)$ are algebra isomorphic, then they are isometrically isomorphic. This result was originally proved in [10]; a modern treatment is given in [9].

Theorem 1.1 (Gelfand and Kolmogoroff 1939). *Let Ω_1 and Ω_2 be compact Hausdorff spaces. Then $C(\Omega_1)$ and $C(\Omega_2)$ are isomorphic as algebras if and only if Ω_1 and Ω_2 are homeomorphic. Moreover, every algebra isomorphism $j : C(\Omega_1) \to C(\Omega_2)$ is of the form $j(f) = f \circ h$ where $h : \Omega_1 \to \Omega_2$ is a homeomorphism.*

The main issue that we shall address in this paper is the corresponding relationship between the topological structure of the set σ and the algebraic structure of $AC(\sigma)$. In Section 2 we shall give a simple proof that if $AC(\sigma_1)$ and $AC(\sigma_2)$ are algebra isomorphic, then σ_1 and σ_2 are homeomorphic. The converse of this is false however. In Section 3 we show that the algebra of absolutely continuous functions over the closed unit disk is not isomorphic to the algebra of absolutely continuous functions over a square.

If, however, one restricts the class of sets in which σ may lie, one can recover some sort of analogue of the Banach-Stone Theorem. In Theorem 6.3 we show that if P_1 and P_2 are simple polygons, then $AC(P_1)$ is algebra isomorphic to $AC(P_2)$. We then extend this result to cover more general sets based on polygons.

Notation. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras. We shall write $\mathcal{A} \simeq \mathcal{B}$ to mean that $\mathcal{A}$ is isomorphic to $\mathcal{B}$ (as a Banach algebra), and write $\mathcal{A} \cong \mathcal{B}$ to mean that $\mathcal{A}$ and $\mathcal{B}$ are isometrically isomorphic.

Throughout, we shall use the term polygon to refer to a simple polygon including its interior. In particular, every such polygon is homeomorphic to the closed unit disk.

2. Preliminaries

In this section we shall briefly outline the definition of the spaces $AC(\sigma)$. Here we follow [8] rather than the original definitions given in [3]. Throughout, σ, σ_1 and σ_2 will denote nonempty compact subsets of the plane. Although the original motivation for these definitions came from considering functions defined on subsets of the complex plane, for this paper it will be notationally easier consider the domains of the functions to be subsets of $\mathbb{R}^2$. We shall work throughout with algebras of complex-valued functions.

Suppose that $f : \sigma \to \mathbb{C}$. Let $S = [\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$ be a finite ordered list of elements of σ, where, for the moment, we shall assume that $n \geq 1$. Note that the elements of such a list do not need to be distinct.

We define the *curve variation of f on the set S* to be

$$(2.1) \qquad \mathrm{cvar}(f, S) = \sum_{i=1}^{n} |f(\boldsymbol{x}_i) - f(\boldsymbol{x}_{i-1})|.$$

We shall also need to measure the 'variation factor' of the list S. Loosely speaking, this is the greatest number of times that γ_S crosses any line in the plane, where γ_S denotes the piecewise linear curve joining the points of S in order. The following definition makes precise just what is meant by a crossing.

Definition 2.1. Suppose that ℓ is a line in the plane. We say that $\overline{\boldsymbol{x}_i\,\boldsymbol{x}_{i+1}}$, the line segment joining $\boldsymbol{x}_i$ to $\boldsymbol{x}_{i+1}$, is a *crossing segment* of $S = \left[\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\right]$ on ℓ if any one of the following holds:

(i) $\boldsymbol{x}_i$ and $\boldsymbol{x}_{i+1}$ lie on (strictly) opposite sides of ℓ.

(ii) $i = 0$ and $\boldsymbol{x}_i \in \ell$.

(iii) $i > 0$, $\boldsymbol{x}_i \in \ell$ and $\boldsymbol{x}_{i-1} \notin \ell$.

(iv) $i = n - 1$, $\boldsymbol{x}_i \notin \ell$ and $\boldsymbol{x}_{i+1} \in \ell$.

In this case we shall write $\overline{\boldsymbol{x}_i\,\boldsymbol{x}_{i+1}} \in X(S, \ell)$.

Definition 2.2. Let $\mathrm{vf}(S, \ell)$ denote the number of crossing segments of S on ℓ. The *variation factor* of S is defined to be

$$\mathrm{vf}(S) = \max_{\ell} \mathrm{vf}(S, \ell).$$

Clearly $1 \leq \mathrm{vf}(S) \leq n$. For completeness, in the case that $S = \left[\boldsymbol{x}_0\right]$ we set $\mathrm{cvar}(f, \left[\boldsymbol{x}_0\right]) = 0$ and let $\mathrm{vf}(\left[\boldsymbol{x}_0\right], \ell) = 1$ whenever $\boldsymbol{x}_0 \in \ell$.

Example 2.3. Consider the line ℓ and the list $S = [\boldsymbol{x}_i]_{i=0}^{8}$ as shown in Figure 1. Let $s_i = \overline{\boldsymbol{x}_i\,\boldsymbol{x}_{i+1}}$. Then the crossing segments for S on ℓ are s_0 (rule (ii)), s_2 (rule (i)), s_4 (rule (iii)) and s_7 (rule (iv)). All the other segments are not crossing segments of S on ℓ. Thus $\mathrm{vf}(S, \ell) = 4$.
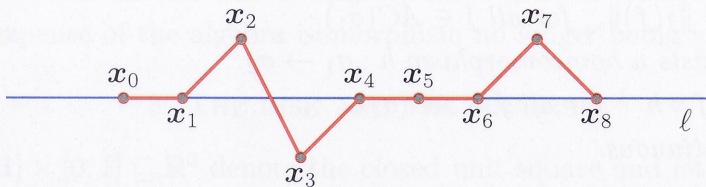

FIGURE 1. Examples of crossing segments.

The *two-dimensional variation* of a function $f : \sigma \to \mathbb{C}$ is defined to be

$$(2.2) \qquad \mathrm{var}(f, \sigma) = \sup_{S} \frac{\mathrm{cvar}(f, S)}{\mathrm{vf}(S)},$$

where the supremum is taken over all finite ordered lists of elements of σ. The *variation norm* is

$$\|f\|_{BV(\sigma)} = \|f\|_{\infty} + \mathrm{var}(f, \sigma)$$

and the set of functions of bounded variation on σ is

$$BV(\sigma) = \{f : \sigma \to \mathbb{C} : \|f\|_{BV(\sigma)} < \infty\}.$$

The space $BV(\sigma)$ is a Banach algebra under pointwise operations [3, Theorem 3.8]. If $\sigma = [0,1]$ then the above definition is equivalent to the more classical one.

Let $\mathcal{P}_2$ denote the space of polynomials in two real variables of the form $p(x,y) = \sum_{n,m} c_{nm} x^n y^m$, and let $\mathcal{P}_2(\sigma)$ denote the restrictions of elements on $\mathcal{P}_2$ to σ. The algebra $\mathcal{P}_2(\sigma)$ is always a subalgebra of $BV(\sigma)$ [3, Corollary 3.14].

Definition 2.4. The set of absolutely continuous functions on σ, denoted $AC(\sigma)$, is the closure of $\mathcal{P}_2(\sigma)$ in $BV(\sigma)$.

The set $AC(\sigma)$ forms a closed subalgebra of $BV(\sigma)$ and hence is a Banach algebra.

We shall say that $f \in C^1(\sigma)$ if there exists an open neighbourhood U of σ and an extension F of f to U such that the partial derivatives of F (of order one) are continuous on U. The space $CTPP(\sigma)$ consists of those functions f for which there is a triangulation of a neighbourhood U of σ and an extension of f to U which is continuous and piecewise planar on this triangulation. It was shown in [8] that both $C^1(\sigma)$ and $CTPP(\sigma)$ are dense subsets of $AC(\sigma)$.

Our first step is to show that if $AC(\sigma_1)$ and $AC(\sigma_2)$ are isomorphic as algebras, then σ_1 and σ_2 must be homeomorphic. We note that one does not need to assume that the isomorphism is continuous.

Lemma 2.5. *Suppose that $f \in AC(\sigma)$. Then the spectrum of f is $\sigma(f) = f(\sigma)$ and hence the spectral radius of f is $r(f) = \|f\|_\infty$.*

Proof. This is more or less immediate from [3, Corollary 3.9]. $\qquad\qquad\square$

Theorem 2.6. *Suppose that $j : AC(\sigma_1) \to AC(\sigma_2)$ is an algebra isomorphism. Then*

    (1) $\|f\|_\infty = \|j(f)\|_\infty$ for all $f \in AC(\sigma_1)$.

    (2) there exists a homeomorphism $h : \sigma_1 \to \sigma_2$.

    (3) $j(f) = f \circ h^{-1}$ for all $f \in AC(\sigma_1)$.

    (4) j is continuous.

Proof. Since j preserves the identity element, it also preserves the spectrum of elements. Thus, using Lemma 2.5,

$$\|f\|_\infty = r(f) = r(j(f)) = \|j(f)\|_\infty$$

for all $f \in AC(\sigma)$. Since $AC(\sigma_1)$ is dense in $C(\sigma_1)$, this implies that j extends to an isometric isomorphism $\hat{j} : C(\sigma_1) \to C(\sigma_2)$ and hence, by the Banach-Stone Theorem, σ_1 is homeomorphic to σ_2. Indeed there exists a homeomorphism $h : \sigma_1 \to \sigma_2$ such that $\hat{j}(f) = f \circ h^{-1}$ for all $f \in C(\sigma_1)$. Restricting this to $AC(\sigma_1)$ gives part 3.

Suppose that $f_n \to f$ in $AC(\sigma_1)$. Then certainly $f_n \to f$ uniformly and hence pointwise. Suppose that $j(f_n) \to g$ in $AC(\sigma_2)$ (and hence also pointwise). Then for all $x \in \sigma_2$,

$$g(x) = \lim_n j(f_n)(x) = \lim_n f_n(h^{-1}(x)) = f(h^{-1}(x)) = j(f)(x)$$

and hence $j(f) = g$. Thus, by the Closed Graph Theorem, j is continuous.   □

It is easy to find homeomorphic sets σ_1 and σ_2 for which $AC(\sigma_1)$ and $AC(\sigma_2)$ are algebra isomorphic, but not isometrically. If the isomorphism preserves norms, then part 1 of the above theorem implies that it also preserves variation.

Corollary 2.7. *Suppose that $j : AC(\sigma_1) \to AC(\sigma_2)$ is an isometric Banach algebra isomorphism. Then $\mathrm{var}(f, \sigma_1) = \mathrm{var}(j(f), \sigma_2)$ for all $f \in AC(\sigma_1)$.*

Example 2.8. Let $\sigma_1 = \{0, 1, 2\}$ and $\sigma_2 = \{0, 1, i\}$. Since $\sigma_1 \subseteq \mathbb{R}$,

$$\|f\|_{BV(\sigma_1)} = \|f\|_\infty + |f(1) - f(0)| + |f(2) - f(1)|, \qquad f \in AC(\sigma_1).$$

On the other hand, any function defined on σ_2 can clearly be written in the form $f(x + iy) = g(y - x)$ for some function g of one real variable. Lemma 3.12 and Proposition 3.10 of [3] then imply that the norm for $f \in AC(\sigma_2)$ is given by

$$\|f\|_{BV(\sigma_2)} = \|f\|_\infty + \max\big(|f(1) - f(0)|, |f(i) - f(0)|, |f(i) - f(1)|\big).$$

Any isomorphism must map idempotents to idempotents. However it is easy to see that while all idempotents in $AC(\sigma_2)$ have variation at most 1, the algebra $AC(\sigma_1)$ contains the idempotent with $f(0) = f(2) = 0$ and $f(1) = 1$ which has variation 2. Thus these algebras can not be isometrically isomorphic.

In the other direction, suppose that $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is an invertible affine transformation. It is clear from the definition of variation that $\|f\|_{BV(\sigma)} = \|f \circ \alpha^{-1}\|_{BV(\alpha(\sigma))}$. Since affine transformations preserve polynomials, it is clear that $AC(\sigma)$ is isometrically isomorphic to $AC(\alpha(\sigma))$. (This is a very small extension of [3, Theorem 4.1].) In Section 4 we shall extend this to slightly more general transformations of the plane, at the expense of the algebra isomorphism no longer being isometric.

3. The disk and the square

Let $Q = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$ denote the closed unit square and let $D = \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\| \leq 1\}$ denote the closed unit disk. These sets are clearly homeomorphic. The aim of this section is to show that $AC(Q) \not\simeq AC(D)$.

Theorem 3.1. *$AC(Q)$ and $AC(D)$ are not isomorphic as algebras.*

Proof. Suppose that $j : AC(Q) \to AC(D)$ is an algebra isomorphism. By Theorem 2.6, the map j is continuous and hence $\|j(f)\|_{AC(D)} \leq \|j\| \, \|f\|_{AC(Q)}$ for all $f \in AC(Q)$.

Let $h : Q \to D$ be the homeomorphism associated with j. Then $h([0, 1] \times \{0\})$ is a closed arc on the unit circle ∂D.

Let $n \in \mathbb{N}$ be even. For $0 \leq k \leq n$, let $\boldsymbol{p}_k = h(\frac{k}{n}, 0)$. Now choose $\epsilon_n > 0$ small enough so that, for every odd k, the ϵ_n-disc with centre $\boldsymbol{p}_k$ does not meet the line segment $\overline{\boldsymbol{p}_{k-1} \, \boldsymbol{p}_{k+1}}$.

Now let $\delta_n > 0$ be chosen (using the uniform continuity of h) so that if $\boldsymbol{x}, \boldsymbol{x}' \in Q$ with $\|\boldsymbol{x} - \boldsymbol{x}'\| \leq \delta_n$ then $\|h(\boldsymbol{x}) - h(\boldsymbol{x}')\| < \epsilon_n$. Without loss we may assume that $\delta_n \leq 1$.
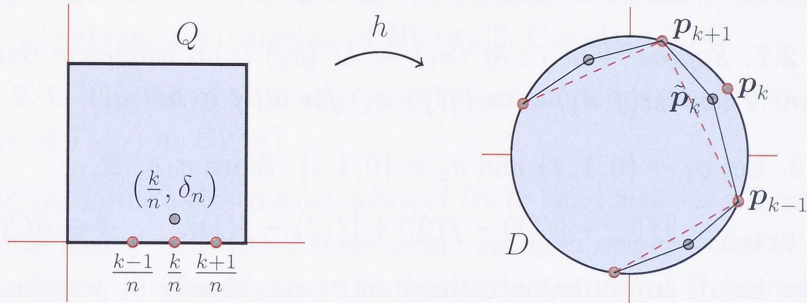

FIGURE 2. The construction of $\tilde{\boldsymbol{p}}_k$ in the proof of Theorem 3.1.

For each odd k, let $\tilde{\boldsymbol{p}}_k = h(\frac{k}{n}, \delta_n) \in B(\boldsymbol{p}_k, \epsilon_n)$. Let $S = [\boldsymbol{p}_0, \tilde{\boldsymbol{p}}_1, \boldsymbol{p}_2, \tilde{\boldsymbol{p}}_3, \ldots, \tilde{\boldsymbol{p}}_{n-1}, \boldsymbol{p}_n] \subseteq D$. It is easy to see that the points of S form the vertices of a convex subset of D and so, in particular, $\mathrm{vf}(S_n) = 2$.

Consider the map $f_n : Q \to \mathbb{R}$ defined by $f_n(x, y) = \min(y/\delta_n, 1)$. Clearly $f_n \in AC(Q)$ with $\|f_n\|_{AC(Q)} = 2$. Define $g_n : D \to \mathbb{R}$ by $g_n = f_n \circ h^{-1} = j(f_n)$. Then $g_n(\boldsymbol{p}_k) = 0$ for k even, and $g_n(\tilde{\boldsymbol{p}}_k) = 1$ for k odd. Thus

$$\|g_n\|_{AC(D)} \geq \mathrm{var}(g_n, D) \geq \frac{\mathrm{cvar}(g_n, S_n)}{\mathrm{vf}(S_n)} = \frac{n}{2}.$$

But for all n, $\|g_n\|_{AC(D)} \leq \|j\| \|f_n\|_{AC(Q)} \leq 2 \|j\|$ and hence we have a contradiction. $\qquad \square$

As we shall now show, there are severe restrictions on the behaviour of any algebra isomorphism which is associated with a C^2 homeomorphism from Q to another compact subset of the plane.

Lemma 3.2. *Suppose that $\Omega \subseteq \mathbb{R}^2$ is compact. Then a set $\ell \subseteq \Omega$ is a closed line segment if and only if it is closed, convex and can be disconnected by the removal of a single point.*

Proof. The forward implication is obvious.

Suppose now that the second condition holds. Let $\boldsymbol{x} \in \ell$ denote a point whose removal splits $\ell \setminus \{\boldsymbol{x}\}$ into disjoint sets ℓ_1 and ℓ_2. Choose points $\boldsymbol{y}_1 \in \ell_1$ and $\boldsymbol{y}_2 \in \ell_2$. Then $\overline{\boldsymbol{y}_1 \boldsymbol{y}_2}$ lies inside ℓ. and must pass through $\boldsymbol{x}$ as $\ell_1 \cup \ell_2$ is not connected. Since y_2 was an arbitrary element of ℓ_2, the line through $\boldsymbol{y}_1$ and $\boldsymbol{x}$ contains every element of ℓ_2 — and similarly every element of ℓ_1 must lie on the same line. Thus ℓ is a closed convex subset of a line, or in other words, a line segment. $\qquad \square$

Recall that a set U is mid-point convex if $\frac{1}{2}(\boldsymbol{x} + \boldsymbol{y}) \in U$ for all $\boldsymbol{x}, \boldsymbol{y} \in U$.

Lemma 3.3. *Suppose that $\Omega \subseteq \mathbb{R}^2$ is compact and that $\ell = \{\boldsymbol{x} + \lambda\boldsymbol{v} : 0 \leq \lambda \leq 1\} \subseteq \Omega$ is a closed line segment. Let $h : \Omega \to \sigma \subseteq \mathbb{R}^2$ be a homeomorphism. Then $h(\ell)$ is a line segment if and only if it is mid-point convex.*

Proof. Again the forward implication is clear.

Now suppose that $h(\ell)$ is mid-point convex. Since h is a homeomorphism, $h(\ell)$ is closed and can be disconnected by a point. Since $h(\ell)$ is closed and mid-point convex, it is convex, and so the result follows from the previous lemma. $\qquad\square$

Lemma 3.4. *Suppose that $\sigma \subseteq \mathbb{R}^2$ and that $h : Q \to \sigma$ is a homeomorphism. For $y \in [0,1]$ let $\ell_y = [0,1] \times \{y\}$. If $h(\ell_0)$ is not a line segment, then there exists $\delta > 0$ such that $h(\ell_y)$ is not a line segment for any $y \in [0, \delta]$.*

Proof. As $h(\ell_0)$ is not a line segment we may choose $x, x' \in [0,1]$ such that $\boldsymbol{v} = \frac{1}{2}(h(x,0) + h(x',0))$ is not an element of $h(\ell_0)$. Let $\epsilon = d(\boldsymbol{v}, h(\ell_0)) > 0$. Now choose $\delta > 0$ small enough so that if $\boldsymbol{u}, \boldsymbol{u}' \in Q$ with $\|\boldsymbol{u} - \boldsymbol{u}'\| \leq \delta$ then $\|h(\boldsymbol{u}) - h(\boldsymbol{u}')\| < \epsilon/3$.

Suppose that $0 \leq y \leq \delta$ and that $h(\ell_y)$ is a line segment. Since $h(\ell_y)$ is mid-point convex, there exists $t \in [0,1]$ such that $h(t,y) = \frac{1}{2}(h(x,y) + h(x',y))$. But then

$$\|h(t,0) - \boldsymbol{v}\| \leq \left\|h(t,0) - h(t,y)\right\| + \left\|h(t,y) - \frac{h(x,y) + h(x',y)}{2}\right\|$$
$$+ \left\|\frac{h(x,y) + h(x',y)}{2} - \frac{h(x,0) + h(x',0)}{2}\right\|$$
$$\leq \frac{\epsilon}{3} + 0 + \frac{\epsilon}{3} < \epsilon$$

contradicting that $d(\boldsymbol{v}, h(\ell_0)) = \epsilon$. $\qquad\square$

A consequence of this result is that if $h : Q \to \sigma$ is a homeomorphism and there exists a line segment $\ell \in Q$ such that $h(\ell)$ is not a line segment, then we may assume that both ℓ and $h(\ell)$ lie in the interiors of their respective sets.

Theorem 3.5. *Suppose that $\emptyset \neq \sigma \subseteq \mathbb{R}^2$ is compact and that $j : AC(Q) \to AC(\sigma)$ is an algebra isomorphism with associated homeomorphism $h : Q \to \sigma$. If h is C^2 then h maps line segments to line segments.*

Proof. Suppose that there exist σ, j and h as above such that for some line segment $\ell \subseteq Q$, $h(\ell)$ is not a line segment. We shall show that this leads to a contradiction. In order to streamline the proof, a number of simplifications can be made.

By the above remark we can assume that ℓ lies in the interior of Q. Since h is C^2, $h(\ell)$ has a tangent at each point and the curve can at least locally be considered as the graph of a C^2 function of a parametrization of this tangent line. Since $h(\ell)$ is not a line segment, one can therefore choose an invertible affine map $\beta : \mathbb{R}^2 \to \mathbb{R}^2$ such that there is a subsegment ℓ_0 of ℓ such that

(1) $(\beta \circ h)(\ell_0) = \{(s, t(s)) : 0 \leq s \leq 1\}$, and

(2) $t''(s) > 0$ for $0 < s < 1$.

Now choose an invertible affine map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ such that

(1) $\alpha(\ell_0) = [0, 1] \times \{0\} \subset [0, 1] \times [-1, 1] \subseteq \text{int}(\alpha(Q))$, and
(2) $(\beta \circ h \circ \alpha^{-1})([0, 1] \times [0, 1])$ lies above $(\beta \circ h)(\ell_0)$.

We shall write $h_1 = \beta \circ h \circ \alpha^{-1}$ for the homeomorphism from $\alpha(Q)$ to $\beta(\sigma)$, and C for the curve $(\beta \circ h)(\ell_0)$. (See Figure 3.)
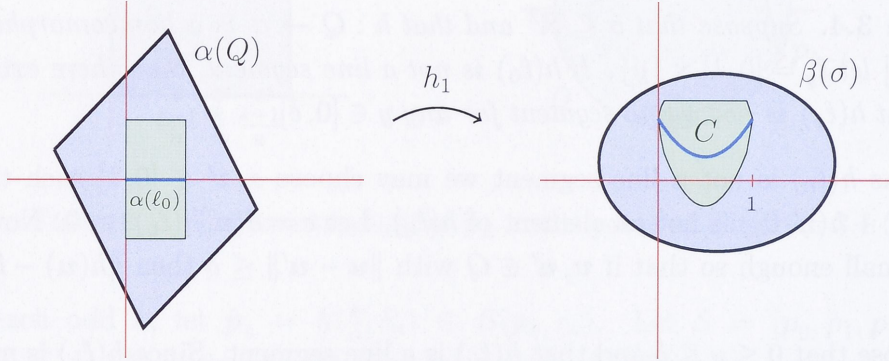

FIGURE 3. The homeomorphism $h_1 : \alpha(Q) \to \beta(\sigma)$ in the proof of Theorem 3.5.

The proof now mimics that of Theorem 3.1. Let $n \in \mathbb{N}$ be even. For $0 \leq k \leq n$, let $\boldsymbol{p}_k = (\frac{k}{n}, t(\frac{k}{n}))$ and let $x_k \in [0, 1]$ denote the (unique) number such that $h_1(x_k, 0) = \boldsymbol{p}_k$.

Choose ϵ_n small enough so that for all odd k, the ball $B(\boldsymbol{p}_k, \epsilon_n)$ lies beneath the line segment $\overline{\boldsymbol{p}_{k-1} \boldsymbol{p}_{k+1}}$. (This is of course possible by the convexity of the function t.)

Now choose $0 < \delta_n < 1$ so that if $\boldsymbol{x}, \boldsymbol{y} \in \alpha(Q)$ with $\|\boldsymbol{x} - \boldsymbol{x}'\| \leq \delta_n$ then $\|h_1(\boldsymbol{x}) - h_1(\boldsymbol{x}')\| < \epsilon_n$.

For each odd k, let $\tilde{\boldsymbol{p}}_k = h_1(x_k, \delta) \in B(\boldsymbol{p}_k, \epsilon)$. Then $\tilde{\boldsymbol{p}}_k$ lies above the curve C but below the chord $\overline{\boldsymbol{p}_{k-1} \boldsymbol{p}_{k+1}}$. Let $S_n = [\boldsymbol{p}_0, \tilde{\boldsymbol{p}}_1, \boldsymbol{p}_2, \tilde{\boldsymbol{p}}_3, \ldots, \tilde{\boldsymbol{p}}_{n-1}, \boldsymbol{p}_n]$, so that as in the proof of Theorem 3.1, $\text{vf}(S_n) = 2$.

Consider the map $f_n : \alpha(Q) \to \mathbb{R}$ defined by

$$f_n(x, y) = \begin{cases} 1, & \text{if } y \geq \delta_n, \\ y/\delta_n, & \text{if } 0 \leq y < \delta_n, \\ 0, & \text{if } y < 0. \end{cases}$$

Clearly $f_n \in AC(\alpha(Q))$ with $\|f_n\|_{AC(\alpha(Q))} = 2$. Again define $g_n : \beta(\sigma) \to \mathbb{R}$ by $g_n = f_n \circ h_1^{-1}$ to produce a function with $\|g_n\|_{AC(\beta(\sigma))} \geq n/2$. But, noting the remarks at the end of Section 2 about the invariance of variation norms under affine transformations, we have that for all n, $\|g_n\|_{AC(\beta(\sigma))} \leq \|j\| \|f_n\|_{AC(\alpha(Q))} \leq 2\|j\|$ which is the required contradiction. $\square$

Functions mapping line segments to line segments have been studied by various authors. We refer the reader to [7] for further details. It is worth noting that such maps need not be affine. For example $h(x, y) = ((x+1)/(y+1), 2y/(y+1))$, which maps the unit square to the trapezoid with vertices $(1, 0), (2, 0), (1, 1)$ and $(\frac{1}{2}, 1)$, is a nonaffine line-segment preserving function.

4. HALF-PLANE-AFFINE MAPS

Despite the results of the previous section there are some positive statements that can be made about when $AC(\sigma_1)$ and $AC(\sigma_2)$ are isomorphic. As we noted earlier, this is certainly the case if σ_2 is the image of σ_1 under an affine homeomorphism α. In this section we weaken this condition on the homeomorphism mapping σ_1 to σ_2.

Definition 4.1. A *half-plane splitting* of $\mathbb{R}^2$ is a pair of closed half-planes $\{H_1, H_2\}$ whose union is $\mathbb{R}^2$ and which only intersect along their shared boundary line.

Definition 4.2. An invertible map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be a *half-plane-affine map* if there exists a half-plane splitting $\{H_1, H_2\}$ of $\mathbb{R}^2$ and two affine maps $\alpha_1, \alpha_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\alpha(\boldsymbol{x}) = \alpha_j(\boldsymbol{x})$ whenever $\boldsymbol{x} \in H_j$. We shall write $\alpha = \{\alpha_1, \alpha_2\}_{H_1, H_2}$.

Any half-plane-affine map is clearly continuous. The assumption of invertibility ensures that $\{\alpha(H_1), \alpha(H_2)\}$ is a half-plane splitting of $\mathbb{R}^2$. We also have the following easy fact.

Lemma 4.3. *The inverse of a half-plane-affine map is a half-plane-affine map.*

Suppose for the remainder of this section that $\alpha = \{\alpha_1, \alpha_2\}_{H_1, H_2}$ is a half-plane-affine map.

We shall show below that for such maps $BV(\sigma) \simeq BV(\alpha(\sigma))$. The main point in proving this is showing that given any finite ordered list S of elements of σ, $\mathrm{vf}(S)$ is comparable to $\mathrm{vf}(\alpha(S))$. Heuristically, if the number of times that the curve γ_S crosses a line ℓ is k, then $\gamma_{\alpha(S)}$ should cross either $\alpha_1(\ell)$ or $\alpha_2(\ell)$ at least $\frac{k}{2}$ times. Proving this is a little delicate however because in the case that a segment $s_i = \overline{\boldsymbol{x}_i \boldsymbol{x}_{i+1}}$ has at least one of its endpoints on the line ℓ, it is possible that s_i is a crossing segment on ℓ but that $\overline{\alpha(\boldsymbol{x}_i) \alpha(\boldsymbol{x}_{i+1})}$ is not a crossing segment on either $\alpha_1(\ell)$ or $\alpha_2(\ell)$.

Lemma 4.4. *Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be a half-plane affine map. Suppose that $S = [\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n]$ is a finite ordered list of points in $\mathbb{R}^2$ and that $\hat{S} = [\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n]$ is the list of the images of these points under α. Then*

$$\frac{1}{2} \mathrm{vf}(S) \le \mathrm{vf}(\hat{S}) \le 2 \mathrm{vf}(S).$$

Proof. By the previous lemma it suffices to just prove the left-hand inequality.

Let $k = \mathrm{vf}(S)$ and fix a line ℓ such that $\mathrm{vf}(S, \ell) = k$. For $0 \leq i \leq n - 1$ let $s_i = \overline{\boldsymbol{x}_i \boldsymbol{x}_{i+1}}$ and let $\hat{s}_i = \overline{\boldsymbol{v}_i \boldsymbol{v}_{i+1}}$. Our aim is to find a correspondence between crossing segments of S on ℓ and crossing segments of $\hat{S}$ on either $\alpha_1(\ell)$ or $\alpha_2(\ell)$.

If $\hat{\ell} = \alpha(\ell)$ is also a line then $s_i \in X(S, \ell)$ if and only if $\hat{s}_i \in X(\hat{S}, \hat{\ell})$ and so $\mathrm{vf}(\hat{S}) \geq \mathrm{vf}(S)$ which certainly gives the required inequality. This occurs in particular if ℓ is parallel to the boundary line between H_1 and H_2.

If $\hat{\ell}$ is not a line, then $\alpha_1(\ell)$ and $\alpha_2(\ell)$ do not coincide, and there must exist a unique point $\boldsymbol{w} \in \ell$ that lies on the shared boundary of H_1 and H_2.

Suppose that $s_i \in X(S, \ell)$. If $\boldsymbol{x}_i$ and $\boldsymbol{x}_{i+1}$ lie strictly on opposite sides of ℓ, then $\boldsymbol{v}_i$ and $\boldsymbol{v}_{i+1}$ lie strictly on opposite sides of $\hat{\ell}$, and so $\hat{s}_i$ is a crossing segment for $\hat{S}$ for at least one of $\alpha_1(\ell)$ or $\alpha_2(\ell)$. The more difficult situation is if one of the endpoints of s_i lies on ℓ. Referring to Definition 2.1 we have the following possibilities:

(i) $i = 0$ and $\boldsymbol{x}_i \in \ell$. Then $\boldsymbol{v}_0 \in \hat{\ell}$ and hence $\hat{s}_0$ is a crossing segment of $\hat{S}$ on at least one of $\alpha_1(\ell)$ or $\alpha_2(\ell)$.

(ii) $i > 0$, $\boldsymbol{x}_i \in \ell$ and $\boldsymbol{x}_{i-1} \notin \ell$. Note that in this case $s_{i-1} \notin X(S, \ell)$.

Without loss of generality we may label the half-planes so that $\boldsymbol{x}_i \in H_1$. Now $\hat{s}_i$ is a crossing segment of $\hat{S}$ on $\alpha_1(\ell)$ except in the case that $\boldsymbol{v}_{i-1} \in \alpha_1(\ell)$ (see Figure 4). If $\boldsymbol{v}_{i-1} \in \alpha_1(\ell)$ then, as $\alpha_1(\ell)$ and $\alpha_2(\ell)$ do not coincide, $\boldsymbol{x}_{i-1} \in H_2$ and $\boldsymbol{v}_{i-1} \notin \alpha_2(\ell)$. We must now distinguish the case when $\boldsymbol{x}_i$ lies in the interior of H_1 and the case when $\boldsymbol{x}_i$ lies in the boundary of H_1.

If $\boldsymbol{x}_i = \boldsymbol{w}$ then $\boldsymbol{v}_i \in \alpha_2(\ell)$ so since $\boldsymbol{v}_{i-1} \notin \alpha_2(\ell)$ we have $\hat{s}_i \in X(\hat{S}, \alpha_2(\ell))$. If $\boldsymbol{x}_i \neq \boldsymbol{w}$, then (as in Figure 4) $\boldsymbol{v}_{i-1}$ and $\boldsymbol{v}_i$ lie on opposite sides of $\alpha_2(\ell)$ and hence $\hat{s}_{i-1} \in X(\hat{S}, \alpha_2(\ell))$.
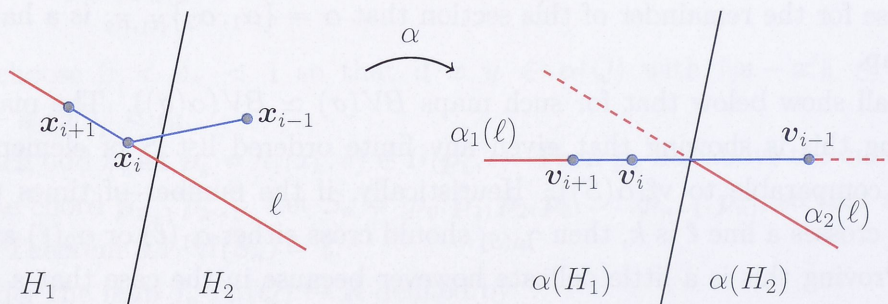

FIGURE 4. Mapping of crossing segments in the proof of Lemma 4.4.

Thus, while only one of s_{i-1} and s_i are crossing segments of S on ℓ, at least one of $\hat{s}_{i-1}$ and $\hat{s}_i$ is a crossing segment of $\hat{S}$ on either $\alpha_1(\ell)$ or $\alpha_2(\ell)$.

(iii) $i = n - 1$, $\boldsymbol{x}_i \notin \ell$ and $\boldsymbol{x}_{i+1} \in \ell$. Again we may assume that $\boldsymbol{x}_n \in H_1$ and hence that $\boldsymbol{v}_n \in \alpha_1(\ell)$.

If $\boldsymbol{v}_{n-1} \notin \alpha_1(\ell)$ then clearly $\hat{s}_{n-1} \in X(\hat{S}, \alpha_1(\ell))$. On the other hand, if $\boldsymbol{v}_{n-1} \in \alpha_1(\ell)$ then one may argue as in (ii) that $\hat{s}_{n-1} \in X(\hat{S}, \alpha_2(\ell))$. As before then, $\hat{s}_{n-1}$ is a crossing segment of $\hat{S}$ on at least one of $\alpha_1(\ell)$ or $\alpha_2(\ell)$.

Let k_1 and k_2 be the number of crossing segments of $\hat{S}$ on $\alpha_1(\ell)$ and $\alpha_2(\ell)$ respectively. Then the above discussion shows that $k_1 + k_2 \geq k$. It follows therefore that $\mathrm{vf}(\hat{S}) \geq \frac{k}{2}$ as required. □

Theorem 4.5. *Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be a half-plane affine map. Suppose that σ_1 is a nonempty compact subset of $\mathbb{R}^2$ and that $\sigma_2 = \alpha(\sigma_1)$. Then $BV(\sigma_1) \simeq BV(\sigma_2)$ and $AC(\sigma_1) \simeq AC(\sigma_2)$.*

Proof. Suppose that $\alpha = \{\alpha_1, \alpha_2\}_{H_1, H_2}$ and that ℓ_0 is the boundary line between H_1 and H_2. For $f \in BV(\sigma_1)$ let $\hat{f} : \sigma_2 \to \mathbb{R}$ be defined by

$$\hat{f}(\alpha(\boldsymbol{x})) = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \sigma_1.$$

The first step is to show that $\hat{f} \in BV(\sigma_2)$.

By the previous lemma

$$\frac{\mathrm{cvar}(\hat{f}, \hat{S})}{\mathrm{vf}(\hat{S})} \leq 2 \frac{\mathrm{cvar}(f, S)}{\mathrm{vf}(S)} \leq 2 \, \mathrm{var}(f, \sigma_1).$$

Taking the supremum over all such finite lists $\hat{S}$ shows that

$$\|\hat{f}\|_{BV(\sigma_2)} \leq 2 \|f\|_{BV(\sigma_1)}$$

and in particular that $\hat{f} \in BV(\sigma_2)$. Let $j : BV(\sigma_1) \to BV(\sigma_2)$ be defined by $j(f) = \hat{f}$. It is clear that j is a continuous algebra homomorphism. Lemma 4.3 can now be used to deduce that j is also onto and hence that j is a Banach algebra isomorphism.

Suppose that $g \in CTPP(\sigma_1)$. Then $j(g)$ will also be planar on polygonal regions (on a neighbourhood) of σ_2. Indeed j is a bijection from $CTPP(\sigma_1)$ to $CTPP(\sigma_2)$ and hence j is also a bijection between the closures of these sets, $AC(\sigma_1)$ and $AC(\sigma_2)$. □

As the example below shows, the factor of 2 in the above proof is necessary.

Example 4.6. Suppose that σ_1 is the nonconvex quadrilateral with vertices at $(1, 0)$, $(0, 4)$, $(-1, 0)$ and $(0, 2)$. Note that σ_1 is the image of the closed unit square under a half-plane-affine map. Theorem 4.5 then implies that $AC(Q) \simeq AC(\sigma_1)$.

Define

$$f(x, y) = \max(1 - y, 0), \qquad (x, y) \in \sigma_1.$$

Then, as f only varies in the y direction, it is clear that $\|f\|_\infty = 1$, that $\mathrm{var}(f, \sigma_1) = 1$ and hence that $\|f\|_{BV(\sigma_1)} = 2$. Now write $f = f_1 + f_2$ where $f_1(x, y) = f(x, y)\chi_{[-1,0]}(x)$ and $f_2(x, y) = f(x, y)\chi_{[0,1]}(x)$. Note that both f_1 and f_2 are in $CTPP(\sigma_1) \subseteq AC(\sigma_1)$ and that these functions have disjoint supports.

Suppose now that $j : AC(\sigma_1) \to AC(Q)$ is a Banach algebra isomorphism, with associated homeomorphism $h : \sigma_1 \to Q$. Let $g_1 = j(f_1)$, $g_2 = j(f_2)$ and $g = j(f)$. Using Theorem 2.6(1), we can choose points $\boldsymbol{z}_1, \boldsymbol{z}_2 \in Q$ such that $g_k(\boldsymbol{z}_\ell) = \delta_{k\ell}$. Let

γ denote the line segment joining z_1 and z_2. Then $h^{-1}(\gamma)$ is a continuous curve in σ_1 which necessarily passes though some point $(0, y) \in \sigma_1$. Let $w = h(0, y)$ be the corresponding point on γ. Then $g(w) = g_1(w) + g_2(w) = f_1(0, y) + f_2(0, y) = 0$. But this implies that $\mathrm{cvar}(g, \gamma) \geq 2$ and hence $\mathrm{var}(g, \sigma_2) \geq 2$. By Corollary 2.7 we see that j can not be isometric. In particular, this example shows that factor of 2 that appears in the proof of Theorem 4.5 is necessary.

Remark 4.7. A simple adjustment to the above example, replacing the vertex $(0, 2)$ with the point $(0, 0)$, shows that if T is a closed triangular region in $\mathbb{R}^2$, then $AC(T) \simeq AC(Q)$. Thus isomorphism class does not distinguish between the number of vertices in polygonal regions. We shall come back to this issue later in the paper.

5. LOCALLY PIECEWISE AFFINE MAPS

The results of the previous section can be extended to cover homeomorphisms of the plane made up from more than two affine maps.

Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible affine map, and let C be a convex n-gon. Then $\alpha(C)$ is also a convex n-gon. Denote the sides of C by $s_1, \ldots, s_n$. Suppose that $x_0 \in \mathrm{int}(C)$. The point x_0 determines a triangulation $T_1, \ldots, T_n$ of C, where T_j is the (closed) triangle with side s_j and vertex x_0. A point $y_0 \in \mathrm{int}(\alpha(C))$ determines a similar triangularization $\hat{T}_1, \ldots, \hat{T}_n$ of $\alpha(C)$, where the numbering is such that $\alpha(s_j) \subseteq \hat{T}_j$. The following fact is then clear.

Lemma 5.1. *With the notation as above, there is a unique map $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

(1) $h(x) = \alpha(x)$ for $x \notin \mathrm{int}(C)$,
(2) h maps T_j onto $\hat{T}_j$, for $1 \leq j \leq n$.
(3) $\alpha_j = h|T_j$ is affine, for $1 \leq j \leq n$.
(4) $h(x_0) = y_0$.

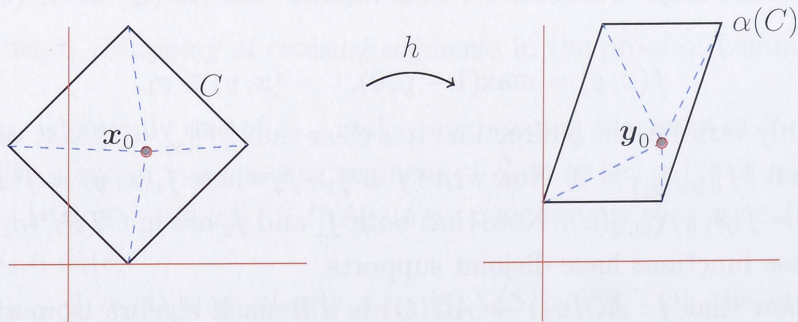We shall say that h is the *locally piecewise affine* map determined by (C, α, x_0, y_0).

FIGURE 5. The locally piecewise affine map h determined by (C, α, x_0, y_0).

It is clear that h is necessarily continuous and invertible. Indeed the following result is straightforward.

Lemma 5.2. *Let h be the locally piecewise affine map determined by $(C, \alpha, \boldsymbol{x}_0, \boldsymbol{y}_0)$. Then h^{-1} is the locally piecewise affine map determined by $(h(C), \alpha^{-1}, \boldsymbol{y}_0, \boldsymbol{x}_0)$.*

In the last section we shall repeatedly use the following special case of Lemma 5.1 applied with $\alpha = \mathrm{id}$, the identity mapping on $\mathbb{R}^2$.

Lemma 5.3. *Suppose that T and $\hat{T}$ are two triangles in $\mathbb{R}^2$ with vertices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\hat{\boldsymbol{a}}, \boldsymbol{b}, \boldsymbol{c}$ respectively. Suppose that Q is a convex quadrilateral in $\mathbb{R}^2$ which contains T and $\hat{T}$ and which has $\overline{\boldsymbol{bc}}$ as one side. Then the locally piecewise affine map determined by $(Q, \mathrm{id}, \boldsymbol{a}, \hat{\boldsymbol{a}})$ maps T onto $\hat{T}$ and fixes all points outside of Q.*
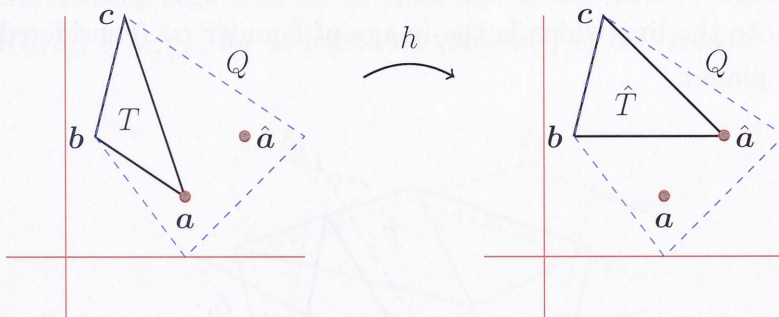

FIGURE 6. A locally piecewise affine map moving T to $\hat{T}$.

Our first aim is to show that for any locally piecewise affine map h, $AC(\sigma) \simeq AC(h(\sigma))$.

Lemma 5.4. *Suppose that h is a locally piecewise affine map determined by $(\alpha, C, \boldsymbol{x}_0, \boldsymbol{y}_0)$ where C is a convex n-gon. Let $S = [\boldsymbol{w}_0, \boldsymbol{w}_1, \dots, \boldsymbol{w}_m]$ be a list of elements in $\mathbb{R}^2$ and let $\hat{S} = [h(\boldsymbol{w}_0), h(\boldsymbol{w}_1), \dots, h(\boldsymbol{w}_m)]$. Then*

$$\frac{1}{c_n} \, \mathrm{vf}(S) \le \mathrm{vf}(\hat{S}) \le c_n \, \mathrm{vf}(S)$$

for some positive constant c_n which is independent of S.

Proof. By the previous lemma it suffices to prove either one of the inequalities. For notational simplicity, we shall write $s_i = \overline{\boldsymbol{w}_i \boldsymbol{w}_{i+1}}$, for $0 \le i \le m - 1$ and write $\boldsymbol{v}_i = h(\boldsymbol{w}_i)$ for $0 \le i \le m$.

Let $T_1, \dots, T_n$ denote the subsets of the plane defined at the start of this section, and let $T_0 = \mathbb{R}^2 \setminus \mathrm{int}(C)$.

Suppose then that $\mathrm{vf}(S) = k$ and that ℓ is a line such that $\mathrm{vf}(S, \ell) = \mathrm{vf}(S)$.

At least one of the regions $T_0, \dots, T_n$ has at least $k_1 = \lceil \frac{k}{n+1} \rceil$ crossing segments of S on ℓ with at least one of their endpoints in that region. Suppose that this region is T_r. Let K denote the set of indices i such that $s_i \in X(S, \ell)$ and s_i has at least one

endpoint in T_r. Our aim is to show that each of these crossing segments corresponds to a crossing segment of $\hat{S}$ on one of a finite number of lines. This will require a careful consideration of cases.

Let

$$\delta = \min\{d(\boldsymbol{v}_i, h(T_r)) \,:\, 0 \le i \le n \text{ and } \boldsymbol{v}_i \notin T_r\}.$$

We take the minimum of the empty set to be zero.

Suppose first that $1 \le r \le n$. Then $h(T_r)$ is a triangle. Choose a triangle $\hat{T}$ with sides parallel to those of $h(T_r)$, which contains $h(T_r)$ in its interior, and such that if $\boldsymbol{v} \in \hat{T}$ then $d(\boldsymbol{v}, h(T_r)) < \frac{\delta}{2}$. Let ℓ_1, ℓ_2 and ℓ_3 denote the three lines forming the sides of $\hat{T}$ (see Figure 7). From this construction, every segment $\hat{s}_i$ with $i \in K$ either lies entirely inside $h(T_r)$, or else it is a crossing segment for $\hat{S}$ on at least one of the lines ℓ_1, ℓ_2 or ℓ_3.

Let ℓ_0 denote the line which is the image of ℓ under α_r (considered as extended to the whole plane).
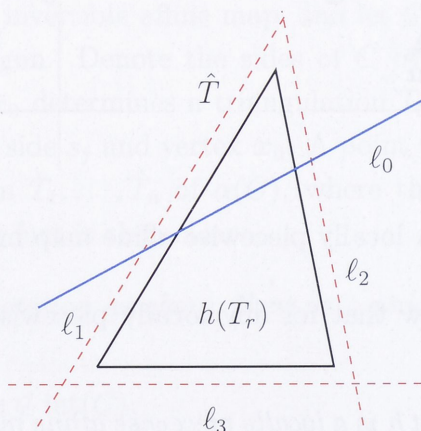

FIGURE 7. The construction of ℓ_0, ℓ_1 ,ℓ_2 and ℓ_3.

Suppose then that $i \in K$, and that both $\boldsymbol{w}_i$ and $\boldsymbol{w}_{i+1}$ lie in T_r. Referring to Definition 2.1 there are four possibilities:

(i) $\boldsymbol{w}_i$ and $\boldsymbol{w}_{i+1}$ lie on strictly opposite sides of ℓ. In this case $\boldsymbol{v}_i$ and $\boldsymbol{v}_{i+1}$ lie on strictly opposite sides of ℓ_0 and so $\hat{s}_i \in X(\hat{S}, \ell_0)$.

(ii) $i = 0$ and $\boldsymbol{w}_i \in \ell$. Clearly then $\boldsymbol{v}_i \in \ell_0$ and so $\hat{s}_i \in X(\hat{S}, \ell_0)$.

(iii) $i > 0$, $\boldsymbol{w}_i \in \ell$ and $\boldsymbol{w}_{i-1} \notin \ell$. In this case $s_{i-1} \notin X(S, \ell)$ and either:

    (a) $\boldsymbol{w}_{i-1} \in T_r$. In this case $\boldsymbol{v}_i \in \ell_0$ and $\boldsymbol{v}_{i-1} \notin \ell_0$ and hence $\hat{s}_i \in X(\hat{S}, \ell_0)$.

    (b) $\boldsymbol{w}_{i-1} \notin T_r$. In this case $\hat{s}_i$ need not be in $X(\hat{S}, \ell_0)$ since $\boldsymbol{v}_{i-1}$ might lie on ℓ_0. However, $\hat{s}_{i-1}$ must be a crossing segment for $\hat{S}$ on one of the boundary lines ℓ_1, ℓ_2 or ℓ_3.

(iv) $i = m - 1$, $\boldsymbol{w}_i \notin \ell$ and $\boldsymbol{w}_{i+1} \in \ell$. Again $\hat{s}_i \in X(\hat{S}, \ell_0)$.

Suppose next that $i \in K$ and that one of $\boldsymbol{w}_i$ and $\boldsymbol{w}_{i+1}$ does not lie in T_r. As noted above, in this case $\hat{s}_i \in X(\hat{S}, \ell_j)$ for some $j = 1, 2, 3$.

At this stage we have shown that if $1 \leq r \leq n$, then there are at least k_1 segments of $\hat{S}$ which are crossing segments for at least one of the lines ℓ_j with $0 \leq j \leq 4$. Thus, for at least one of these values of j, $\mathrm{vf}(\hat{S}, \ell_j) \geq \lceil \frac{k_1}{4} \rceil$.

The remaining case is where $r = 0$ and so T_r is not a triangle. The proof in this case is almost identical except that now one must work with lines $\ell_1, \ldots, \ell_n$ chosen close to the boundary of $h(T_0)$ so that all the endpoints $\boldsymbol{v}_i$ which are not in T_0 lie inside the smaller n-gon determined by these lines, and the line $\ell_0 = \alpha(\ell)$ (see Figure 8). Every segment $\hat{s}_i = \overline{\boldsymbol{v}_i \boldsymbol{v}_{i+1}}$ with only one endpoint in T_0 lies in $X(\hat{S}, \ell_j)$ for at least one j with $1 \leq j \leq n$.

Following the proof above one can then show that there are (at least) k_1 segments of $\hat{S}$ which are crossing segments for at least one of the lines ℓ_j where $0 \leq j \leq n$, and hence $\mathrm{vf}(\hat{S}, \ell_j) \geq \lceil \frac{k_1}{n+1} \rceil$ for at least one value of j in this range.
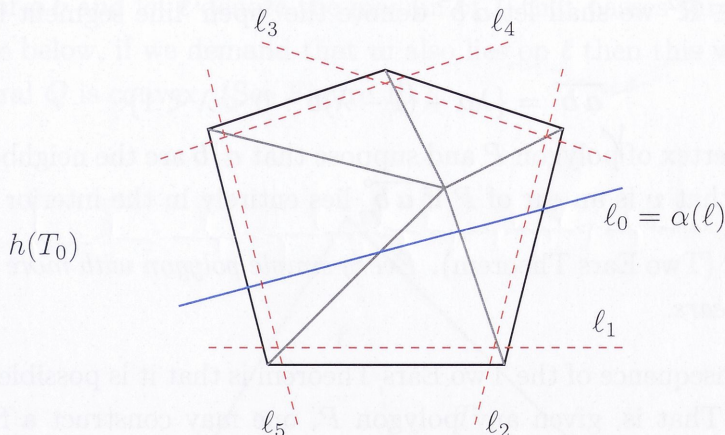

FIGURE 8. Choosing $\ell_0, \ldots, \ell_n$ when $r = 0$ (and $n = 5$).

In either case then

$$\mathrm{vf}(\hat{S}) \geq \left\lceil \frac{k_1}{n+1} \right\rceil = \left\lceil \frac{\lceil \mathrm{vf}(S)/(n+1) \rceil}{n+1} \right\rceil \geq \frac{\mathrm{vf}(S)}{(n+1)^2}.$$

$\square$

The above proof of course shows that $c_n \leq (n+1)^2$. Heuristically we expect that $c_n = n + 1$ but we are unable to prove this.

Theorem 5.5. *Suppose that σ is a nonempty compact subset of the plane, and that h is a locally piecewise affine map. Then $BV(\sigma) \simeq BV(h(\sigma))$ and $AC(\sigma) \simeq AC(h(\sigma))$*

Proof. For $f \in BV(\sigma)$, let $\hat{f} : h(\sigma) \to \mathbb{C}$ be defined by $\hat{f}(h(\boldsymbol{x})) = f(\boldsymbol{x})$. Suppose that $f \in BV(\sigma)$ and that $\hat{S} = [\boldsymbol{v}_0, \ldots, \boldsymbol{v}_m]$ is a list of points in $h(\sigma)$. Let $S =$

$[\boldsymbol{w}_0, \ldots, \boldsymbol{w}_m] \subset \sigma$ denote the list of preimages of the points in $\hat{S}$. Then, using the notation of Lemma 5.4,

$$\frac{\mathrm{cvar}(\hat{f}, \hat{S})}{\mathrm{vf}(\hat{S})} = \frac{\mathrm{cvar}(f, S)}{\mathrm{vf}(\hat{S})} \leq C_n \frac{\mathrm{cvar}(f, S)}{\mathrm{vf}(S)} \leq C_n \, \mathrm{var}(f, \sigma).$$

Thus, $\hat{f}$ is of bounded variation with $\|\hat{f}\|_{BV(h(\sigma))} \leq (1 + C_n) \|f\|_{BV(\sigma)}$. It follows, using Lemma 5.2 that the map $j : f \mapsto \hat{f}$ is a bounded isomorphism from $BV(\sigma)$ onto $BV(h(\sigma))$.

It is clear that j maps $CTPP(\sigma)$ onto $CTPP(h(\sigma))$ and hence that j provides an isomorphism from $AC(\sigma)$ onto $AC(h(\sigma))$. $\square$

6. Polygons and ears

The main result from this section is that given any two simple polygons P_1 and P_2 we have that $AC(P_1) \simeq AC(P_2)$. The proof requires a nice fact from computational geometry called the 'Two Ears Theorem' which was proven by Meisters [11].

Given $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$ we shall let $\overline{\boldsymbol{a}\,\boldsymbol{b}}^{\,\circ}$ denote the 'open' line segment between $\boldsymbol{a}$ and $\boldsymbol{b}$, that is

$$\overline{\boldsymbol{a}\,\boldsymbol{b}}^{\,\circ} = \{\lambda \boldsymbol{a} + (1 - \lambda)\boldsymbol{b} : 0 < \lambda < 1\}.$$

Let $\boldsymbol{v}$ be a vertex of the polygon P and suppose that $\boldsymbol{a}, \boldsymbol{b}$ are the neighbouring vertices to $\boldsymbol{v}$. We say that $\boldsymbol{v}$ is an ear of P if $\overline{\boldsymbol{a}\,\boldsymbol{b}}^{\,\circ}$ lies entirely in the interior of P.

Theorem 6.1 (Two Ears Theorem). *Every simple polygon with more than 3 vertices has at least 2 ears.*

A simple consequence of the Two Ears Theorem is that it is possible to triangulate any polygon. That is, given any polygon P, one may construct a finite family of 'disjoint' triangles $\{T_n\}$ whose vertices are all vertices of P and whose union equals P.

Lemma 6.2. *Suppose that $\boldsymbol{v}$ is an ear of a polygon P. Let $T = \triangle \boldsymbol{a}\boldsymbol{v}\boldsymbol{b}$ denote the triangle formed by the two sides of P which meet at $\boldsymbol{v}$ and the corresponding diagonal $\overline{\boldsymbol{a}\,\boldsymbol{b}}$. Then there exists a convex quadrilateral Q with vertices $\boldsymbol{a}$, $\boldsymbol{u}$, $\boldsymbol{b}$ and $\boldsymbol{w}$ such that*

    (1) $T \setminus \{\boldsymbol{a}, \boldsymbol{b}\} \subseteq \mathrm{int}(Q)$,
    (2) $\overline{\boldsymbol{a}\,\boldsymbol{u}}^{\,\circ}$ and $\overline{\boldsymbol{b}\,\boldsymbol{u}}^{\,\circ}$ lie in the complement of P, and
    (3) $\overline{\boldsymbol{a}\,\boldsymbol{w}}^{\,\circ}$ and $\overline{\boldsymbol{b}\,\boldsymbol{w}}^{\,\circ}$ lie in the interior of P.

Proof. We begin by triangulating P using the standard algorithm of removing one ear at a time. We may clearly start by dealing with the ear at $\boldsymbol{v}$ and so T is one of the triangles in our triangulation. The diagonal $\overline{\boldsymbol{a}\,\boldsymbol{b}}$ must form an edge of two of the triangles, namely T, and another which we shall denote by T_1. One can choose $\boldsymbol{w}$ to be any interior point of T_1 and this will clearly have property 3. Let $\boldsymbol{m}$ denote
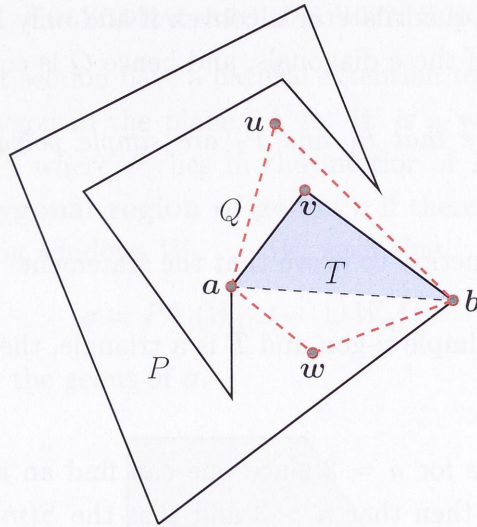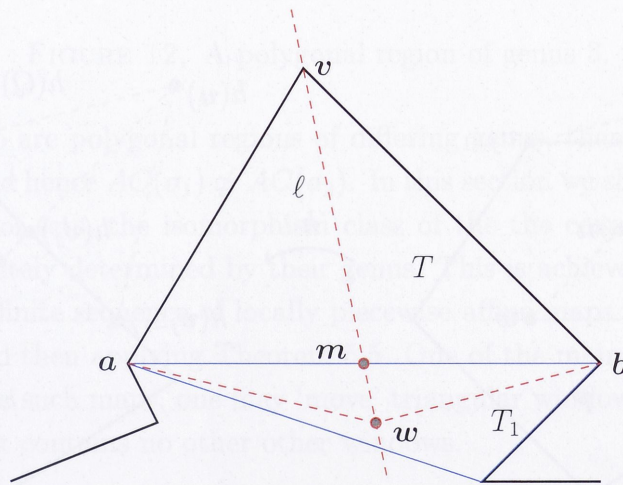
FIGURE 9. Lemma 6.2.

the midpoint of $\overline{a\,b}$ and let ℓ denote the median of T that passes through v and m. As we shall see below, if we demand that w also lies on ℓ then this will ensure that the quadrilateral Q is convex. (See Figure 10.)

FIGURE 10. Construction of w.

Finding a suitable point u is slightly more delicate. For small $t > 0$, $u(t) = (1 + t)v - tw$ lies in the complement of P. If $\overline{a\,u(t)}^{\,\circ}$ or $\overline{b\,u(t)}^{\,\circ}$ does not lie in the complement of P then it must be the case that some vertices of P lie in the interior of the quadrilateral $avbu(t)$ or on one of the boundary lines $\overline{a\,u(t)}$ or $\overline{b\,u(t)}$. Since there can be only finitely many such vertices, by choosing $t_0 > 0$ sufficiently small we can ensure that $u = u(t_0)$ satisfies condition (2).

Property (1) is clear so it remains to check convexity. By the construction, the two diagonals of Q will meet at m and clearly $m \in \overline{a\,b}^{\,\circ}$ and $m \in \overline{u\,w}^{\,\circ}$. But by

Theorem 6.7.9 of [13] a quadrilateral is convex if and only if the diagonals meet at a point in the interior of these diagonals, and hence Q is convex.   □

Theorem 6.3. *Suppose that P_1 and P_2 are simple polygons. Then $AC(P_1) \simeq AC(P_2)$.*

Proof. We shall use induction to prove that the statement

$$S(n): \text{ if } P \text{ is any simple } n\text{-gon and } T \text{ is a triangle, then } AC(P) \simeq AC(T)$$

holds for all $n \geq 3$.

The statement is true for $n = 3$ since one can find an affine map between any two triangles. Suppose then that $n > 3$ and that the $S(m)$ is true for all m with $3 \leq m < n$. Let P be an n-gon with vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. By the Two Ears Theorem, there exists an ear $\boldsymbol{v}_j$. Let $T_{\boldsymbol{v}_j}$ be the triangle with vertices at $\boldsymbol{v}_{j-1}$, $\boldsymbol{v}_j$ and $\boldsymbol{v}_{j+1}$.

Using Lemma 6.2 fix a convex quadrilateral Q with vertices at $\boldsymbol{v}_{j-1}$ and $\boldsymbol{v}_{j+1}$ and two additional points $\boldsymbol{u}, \boldsymbol{w}$ chosen so that $T_{\boldsymbol{v}_j} \setminus \{\boldsymbol{v}_{j-1}, \boldsymbol{v}_{j+1}\}$ lies in the interior of Q and so that $\overline{\boldsymbol{v}_{j-1}\boldsymbol{w}}^{\circ}$ and $\overline{\boldsymbol{v}_{j+1}\boldsymbol{w}}^{\circ}$ lie in the interior of P. As Q is convex, the point $\boldsymbol{v}_j$ and the midpoint $\boldsymbol{m}$ of $\overline{\boldsymbol{v}_{j-1}\boldsymbol{v}_{j+1}}$ are both interior points of Q.
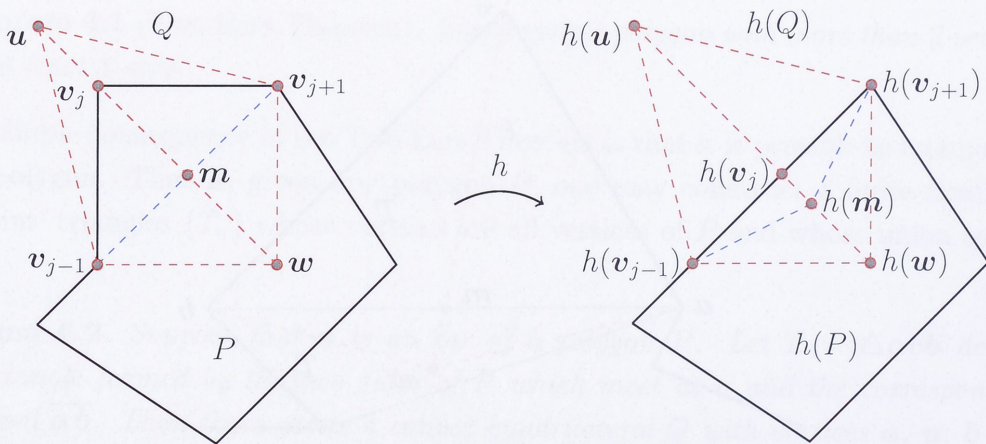

FIGURE 11. The action of h.

Let α denote the identity map on $\mathbb{R}^2$, and let h denote the unique locally piecewise affine map determined by $(Q, \alpha, \boldsymbol{v}_j, \boldsymbol{m})$. This map sends $\overline{\boldsymbol{v}_{j-1}\boldsymbol{v}_j}$ to $\overline{\boldsymbol{v}_{j-1}\boldsymbol{m}}$ and $\overline{\boldsymbol{v}_j\boldsymbol{v}_{j+1}}$ to $\overline{\boldsymbol{m}\boldsymbol{v}_{j+1}}$ (see Figure 11). All the other edges of P lie in the complement of Q and are therefore fixed. It follows therefore that the image of P under h is an m-gon for some $m < n$. By Theorem 5.5, $AC(P) \simeq AC(h(P))$. But by the induction hypothesis, $AC(h(P)) \simeq AC(T)$ for any triangle T and so the proof is complete.   □

7. POLYGONAL REGIONS WITH HOLES

The results of the last section have a natural extension to a wider class of regions. Let P be a simple polygon in the plane. A set W is a **window** in P if it is the interior of a polygon P' where P' lies in the interior of P. We shall say that a compact set σ is a **polygonal region of genus** n if there exists a simple polygon P with n nonoverlapping windows $W_1, \ldots, W_n$ such that

$$\sigma = P \setminus (W_1 \cup \cdots \cup W_n)$$
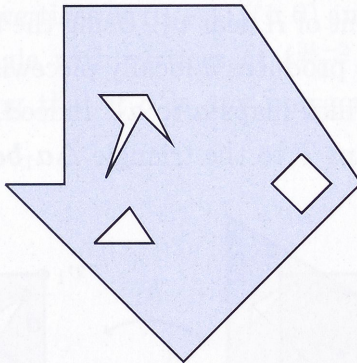
and write $G(\sigma) = n$ for the genus of σ.

FIGURE 12. A polygonal region of genus 3.

If σ_1 and σ two are polygonal regions of differing genus, then these sets are not homeomorphic and hence $AC(\sigma_1) \not\simeq AC(\sigma_2)$. In this section we shall show that show within this class of sets, the isomorphism class of the the corresponding function algebras is completely determined by their genus. This is achieved by showing that there is always a finite sequence of locally piecewise affine maps whose composition sends σ_1 to σ, and then applying Theorem 5.5. One of the main tools in doing this is to show that via such maps, one may 'move' triangular windows anywhere within any rectangle that contains no other other windows.

Lemma 7.1. *Suppose that R is a rectangle and that $T = \triangle \boldsymbol{abc}$ and $T' = \triangle \boldsymbol{a'b'c'}$ are two triangles in the interior of R. Then there is a continuous bijection $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

    (1) h can be written as a composition of finitely many locally piecewise affine maps,

    (2) $h(\boldsymbol{x}) = \boldsymbol{x}$ for all $\boldsymbol{x} \notin R$,

    (3) $h(R) = R$, and

    (4) $h(T) = T'$.

Proof. Choose $\epsilon > 0$ such that no point of T or T' lies within distance 2ϵ of the boundary of R. We shall call the four interior points of R which lie at distance ϵ

along the diagonals from the vertices of R, the ϵ-corner points of R. Fix any three of these ϵ-corner points and let T_0 denote the triangle with these points as vertices. We shall show that there is a function h satisfying (1), (2) and (3) and such that $h(T) = T_0$. The same proof of course would construct a corresponding map sending T' to T_0. Since the inverse of a locally piecewise affine map is also locally piecewise affine, this produces a finite sequence of locally piecewise affine maps which has properties (1) – (4).

The line through b and c splits R into two convex polygons. Let P denote the polygon which contains a. At least one of the vertices of P, say v_1, is a vertex of R not lying on the line through b and c.

Let a_1 be the ϵ-corner point of R near v_1. Using the triangulations of P generated by a and by a_1, Lemma 5.1 produces a locally piecewise affine map h_1 which is the identity outside of P, and which maps a to a_1. Indeed, as T lies entirely in a region on which h_1 is affine, h_1 maps T to the triangle $\triangle a_1 bc$ (see Figure 13).
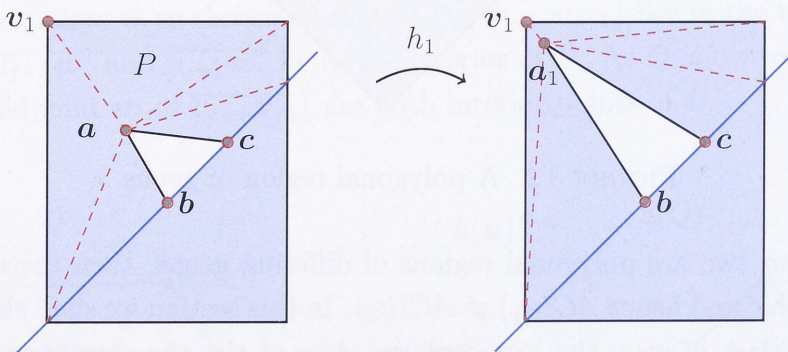

FIGURE 13. Moving the first vertex in Lemma 7.1.

Consider now the quadrilateral Q with vertices at a_1 and the three vertices of R other than v_1. Note that the position of a_1 ensures that Q is convex, and hence that the line through a_1 and c splits Q into two convex polygons. Let P_1 denote the polygon containing b. One of the vertices of Q adjacent to a_1 (which is therefore also a vertex of R) must lie in P_1. Denote this vertex by v_2 and let b_1 be the ϵ-corner point of R near v_2. (Note that b_1 must lie in P_1.) Applying Lemma 5.1 again we produce a locally piecewise affine map h_2 which is the identity outside of Q and which maps $\triangle a_1 bc$ onto $\triangle a_1 b_1 c$.

Finally, consider the convex quadrilateral Q_1 with vertices a_1, b_1 and the two remaining vertices of R. Let c_1 be the ϵ-corner point of R near one of these remaining vertices of R. Noting that c and c_1 are both in the interior of Q_1 we can find a locally piecewise affine map h_3 which is the identity outside of Q_1 and which maps $\triangle a_1 b_1 c$ onto $\triangle a_1 b_1 c_1$.

The vertices of this final triangle are all ϵ-corner points. With one or two further applications of locally piecewise affine maps we can arrange that the image of T under this composition of maps is T_0. $\square$

Theorem 7.2. *Suppose that σ_1 and σ_2 are polygonal regions of genus n_1 and n_2. Then $AC(\sigma_1) \simeq AC(\sigma_2)$ if and only if $n_1 = n_2$.*

Proof. As noted above it only remains to show the 'if' part of the theorem. Fix a genus n. Let τ denote the polygonal region of genus n

$$\tau = T \setminus (T_1 \cup \cdots \cup T_n)$$

where T is the triangle with vertices at $(0, -1)$, $(1, 0)$ and $(0, 1)$ and, for $k = 1, \ldots, n$, the window T_n is the triangle with vertices at $(\frac{3k-2}{3n}, 0)$, $(\frac{3k-1}{3n}, 0)$ and $(\frac{2k-1}{2n}, \frac{1}{3n})$. We shall proceed by showing that if σ is any polygonal regions of genus n, then $AC(\sigma) \simeq AC(\tau)$.
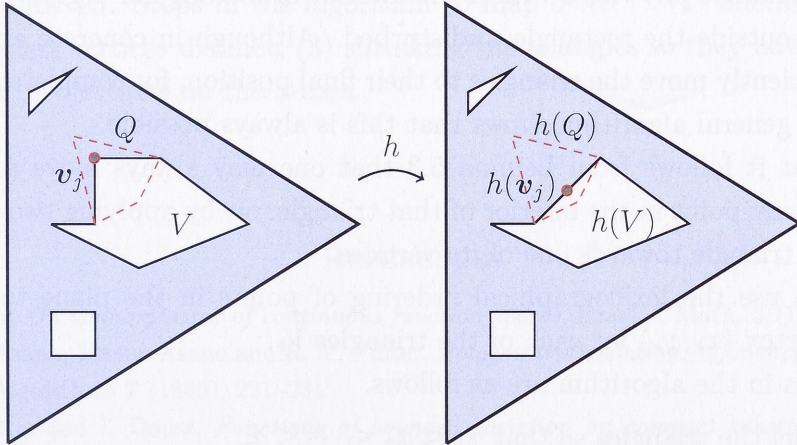

FIGURE 14. Reducing the number of edges in a window.

Suppose then that

$$\sigma = P \setminus (W_1 \cup \cdots \cup W_n).$$

The image of σ under any locally piecewise affine map is also a polygonal region of genus n and as before, the isomorphism class of the corresponding AC function space is preserved under such maps. By applying a finite sequence of locally piecewise affine maps as in the proof of Theorem 6.3 we may reduce the number of vertices in P to 3. Note that the effect of these maps might be to increase the number of vertices in some of the windows. By applying a suitable affine map then, we see that $AC(\sigma) \simeq AC(\sigma')$ where

$$\sigma' = T \setminus (V_1 \cup \cdots \cup V_n),$$

and where $V_1, \ldots, V_n$ are windows in T.

The same algorithm can now be used to reduce the windows $V_1, \ldots, V_n$ to triangles. Specifically, suppose that V is a window in T with at least 4 vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$. By the Two Ears Theorem we can choose an ear $\boldsymbol{v}_j$ in V. Since σ' can be triangulated, the proof of Theorem 6.2 allows us to choose a convex quadrilateral Q containing the triangular region $\boldsymbol{v}_{j-1} \boldsymbol{v}_j \boldsymbol{v}_{j+1}$ but not intersecting any of the other windows of σ'. Applying a suitable locally piecewise affine map h which fixes the complement of Q and maps $\boldsymbol{v}_j$ to the midpoint of $\overline{\boldsymbol{v}_{j-1} \boldsymbol{v}_{j+1}}$ we reduce the number of vertices in V while leaving all the other windows unchanged. (See Figure 14.)

It just remains to prove that if

$$\sigma' = T \setminus (V_1 \cup \cdots \cup V_n)$$

where each window is a triangle, then we can apply a finite sequence of locally piecewise affine maps to move the triangles $\{V_k\}$ to the corresponding triangles $\{T_k\}$ in the description of our standard set τ. Our main tool is Lemma 7.1 which allows us to move a triangle anywhere within the interior of rectangular region while leaving everything outside the rectangle undisturbed. Although in concrete examples it is easy to efficiently move the triangles to their final position, for completeness we shall now give a general algorithm shows that this is always possible.

Note that it follows from Lemma 5.3 that one may always move a vertex of a triangle to any point in the interior of that triangle, or, by applying two such moves, shrink any triangle towards one of its vertices.

We shall use the lexicographical ordering of points in the plane to choose the smallest vertex (x_k, y_k) for each of the triangles V_k.

The steps in the algorithm are as follows.

(1) Label the triangles so that $x_1 \le x_2 \le \cdots \le x_n$.

(2) Starting from the right, shrink as many triangles (toward one vertex say) as is necessary to ensure that the x-coordinates of the smallest vertex of each of the triangles are distinct.

(3) Starting from the left, shrink each triangle towards its smallest vertex. If the triangles are shrunk to a sufficiently small size then the projections of these triangles onto the x-axis will form disjoint intervals $[a_k, b_k]$. Indeed, after sufficient shrinking the triangles will sit within the interiors of disjoint rectangles R_k as in Figure 15.

(4) Using Lemma 7.1 use a sequence of locally piecewise affine maps to move the kth triangle (within rectangle R_k to one with vertices at $(a_k, 0)$, $(b_k, 0)$ and (a_k, e_k) where e_k is chosen small enough so that this triangle sits in R_k

(5) It remains to move the triangles to the correct positions to form the standard configuration τ. Choose $\delta < \min(a_1, \frac{1}{3n})$. Starting from the left, move each triangle in turn (using Lemma 7.1) so that it has vertices $(\frac{\delta(3k-1)}{3n}, 0)$, $(\frac{\delta(3k-2)}{3n}, 0)$ and $(\frac{\delta(2k-1)}{2n}, \frac{1}{3n})$.

(6) Now starting from the right, one can move the k-th triangle to the standard triangle T_k.

Since we have only applied a finite sequence of locally piecewise affine maps, $AC(\sigma') \simeq AC(\tau)$, and this completes the proof.
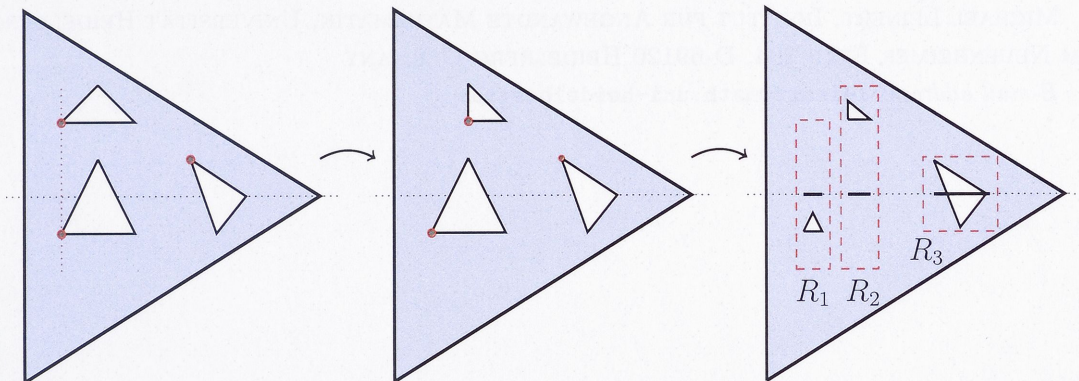

FIGURE 15. Steps in the algorithm to map σ' to τ: (2) making the smallest vertices distinct; (3) shrinking the triangles so they have disjoint projections on the x-axis.

□

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IAN DOUST, SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW SOUTH WALES, UNSW SYDNEY 2052 AUSTRALIA

E-mail address: i.doust@unsw.edu.au

MICHAEL LEINERT, INSTITUT FÜR ANGEWANDTE MATHEMATIK, UNIVERSITÄT HEIDELBERG, IM NEUENHEIMER FELD 294, D-69120 HEIDELBERG GERMANY

E-mail address: leinert@math.uni-heidelberg.de