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Mirror Symmetry for Del Pezzo Surfaces

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Mirror Symmetry for Del Pezzo Surfaces

Abstract

We describe mirror symmetry as an equivalence of $\mathcal{D}$-modules. On the A-side we give an introduction to Gromov-Witten invariants, quantum cohomology and the Dubrovin connection. In particular we compute the small quantum cohomology for Del Pezzo surfaces in general and the Dubrovin connection for X_4 explicitly. On the B-side a mirror $\mathcal{D}$-module is constructed as some Fourier-Laplace transformed Gauß-Manin system. We consider its Brieskorn lattice and explicitly compute it for the toric variety X_4^o. Furthermore we derive a solution to Birkhoff's problem by determining concretely a good basis in the sense of M. Saito. Consequently we prove a mirror theorem for X_4.

Abstract

Wir beschreiben Spiegelsymmetry als eine Äquivalenz zwischen $\mathcal{D}$-Moduln. Auf der A-Seite geben wir eine Einfürung in Gromov-Witten Invarianten, Quanten Kohomologie und den Dubrovin Zusammenhang. Insbesondere berechnen wir die kleine Quanten Kohomologie von Del Pezzo Flächen generell und geben explizit den Dubrovin Zusammenhang von X_4 an. Auf der B-Seite wird ein Spiegel $\mathcal{D}$-Modul konstruiert als ein Fourier-Laplace transformiertes Gauß-Manin System. Darin betrachten wir das Brieskorn Gitter und berechnen es konkret im Falle der torischen Varietät X_4^o. Des Weiteren finden wir eine Lösung zu Birkhoffs Problem, indem wir explizit eine gute Basis im Sinne von M. Saito angeben. Daraus folgend beweisen wir ein Spiegeltheorem für X_4.

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1 Introduction

Mirror symmetry has attracted much mathematical attention ever since Candelas et al. [COGP91] correctly predicted the number of rational curves on a quintic hypersurface in $\mathbb{P}^4$. Their ideas were based on the physical equivalence of two $N = 2$ super-conformal field theories, related by a sign-change of the generators of their algebras. Mathematically this predicts a surprising connection between Gromov-Witten invariants and period integrals of a variation of Hodge structures. Rigorous proofs of the results of Candelas et al. were given independently in [Giv96] and [LLY97]. In the time since, mathematicians have tried in various ways to make this connection precise and generalise it. Amongst the most well-known mirror conjectures are Kontsevich's homological mirror symmetry conjecture [Kon95] and the SYZ conjecture [SYZ96].

Our approach to mirror symmetry is based on $\mathcal{D}$-modules. Intuitively, a $\mathcal{D}$-module is a generalisation of a vector bundle with connection. It turns out that the Gromov-Witten invariants of a variety can be used to define a ring, called the quantum cohomology ring, which in turn gives rise to a vector bundle with meromorphic connection, called the Dubrovin connection. This object is in general very complicated to compute. However, much information is retained when restricting both ring and connection to a smaller number of Gromov-Witten invariants. These objects are then called small quantum cohomology and small Dubrovin connection or quantum $\mathcal{D}$-module respectively. The latter object in particular will be the centre of our attention in this thesis. Mirror symmetry for us then consists in finding a mirror analogue to the quantum $\mathcal{D}$-module.

This has been done for nef toric varieties in a series of work [Iri09, RS15]. The proposed mirror is, inspired by the physical notion of non-linear σ-model, a Landau-Ginzburg model. The important input on the B-side is the Fourier-Laplace transformed Gauß-Manin system of this Landau-Ginzburg model. It admits a lattice, called the Brieskorn lattice, which can be extended by solving the Birkhoff problem to yield a vector bundle with connection isomorphic to the quantum $\mathcal{D}$-module.

In this thesis we will focus our attention on Del Pezzo surfaces. These can be either described as blow-ups of $\mathbb{P}^2$ in nine or fewer points in general position or, equivalently, as two-dimensional Fano varieties. On the A-side they have been extensively studied (c.f. [Vak00, AKO06]) and a description of their small quantum cohomology exists [CM95]. We will follow this last paper and make the computations explicit in the case of the Del Pezzo surface X_4. As mirror to X_4 we will propose the same mirror Landau-Ginzburg model as for the related toric variety X_4^o (up to a change of co-ordinates). Our major contribution consists in a concrete solution of the Birkhoff problem in this case.

Let us now give an overview of the structure of this thesis: in chapter 2 we shall give a general introduction into Gromov-Witten invariants from an algebro-geometric (in particular, not symplectic) point of view. This is foundational for our discussion of the small and big quantum cohomology in chapter 3. We then introduce Del Pezzo surfaces and discuss their small quantum cohomology in chapter 4. Combining the previous chapters, we shall then define the Dubrovin connection and explicitly compute it in the case of X_4. This is done in chapter 5. Thus chapters 2 through 5 can be considered an introduction to the A-side of mirror symmetry, combined with a computation in the case of X_4.

Consequently chapters 6 through 9 can be considered an introduction to the B-side, containing a computation for the toric variety X_4^o. We start by introducing $\mathcal{D}$-modules in chapter 6, as well as toric varieties in chapter 7. We proceed with a discussion of the Brieskorn lattice and its realisation as a Fourier-Laplace transformed Gauß-Manin system. This is done in chapter 8. In particular, we compute there the Brieskorn lattice for X_4^o, thereby closely following the presentation in [RS15].

Our main contribution consists of an explicit solution to the Birkhoff problem, which is done, after some introduction, in chapter 9.

Combining the previous chapters we shall then give a brief idea of mirror symmetry and prove our mirror theorem for the Del Pezzo surface X_4 (theorem 10.1).

2 Gromov-Witten Invariants

Gromov-Witten invariants were originally meant to count curves on some projective variety, resp. symplectic manifold X. In order for this to be a well-defined, finite number we need a couple of specifications. Firstly we need to specify the kind of curves that we want to count, i.e. we need a genus g and a homology class $\beta \in H_2(X, \mathbb{Z})$. Furthermore we need a number of incidence relations: cycles $Z_1, \ldots, Z_n \in H_*(X, \mathbb{Z})$ in general position, which our curves should meet. However, simply considering the set

$$\left\{ C \subset X \;\middle|\; g(C) = g \text{ and } [C] = \beta \text{ and } C \cap Z_i \neq \emptyset \text{ for all } i \right\} \tag{1}$$

turns out to be hopelessly naïve. To refine this idea, we shall make two important changes: firstly we will follow Kontsevich's idea of *stable maps*. This means essentially that we stop considering $C \subset X$ as an embedded curve and instead let it be an abstract curve together with a morphism $f : C \to X$. In order to express our incidence conditions we then *mark* C by specifying n distinct[1] points $p_1, \ldots, p_n \in C$ and require $f(p_i) \in Z_i$. Secondly we will interpret all our sets (set of stable curves/ stable maps) as moduli spaces ($\overline{M}_{\mathrm{g,n}}$/ $\overline{M}_{\mathrm{g,n}}(X, \beta)$) and endow them with additional structure thus regard them as orbifolds. (1) respectively its equivalent in our new sophisticated set-up, will then determine a subset in $\overline{M}_{\mathrm{g,n}}$ and thus a cohomology class of in $H^*(\overline{M}_{\mathrm{g,n}}, \mathbb{C})$.[2] Working in general with homology and cohomology classes turns out to be useful. For one thing, we can push them forward and pull them back along natural maps. But they also carry more information than just the numbers/ invariants, which are nothing but the order of the class if the class is finite.

2.1 The Relevant Moduli Spaces

We denote by $\overline{M}_{\mathrm{g,n}}$ the Deligne-Mumford coarse moduli space of n-pointed, genus g stable curves. A point in $\overline{M}_{\mathrm{g,n}}$ represents - up to isomorphisms - the data $(C, p_1, \ldots, p_n)$, called an *n-marked stable curve*, where C is a projective, connected, at worst nodal curve of genus g and $p_i \in C$ for all i, satisfying the following conditions:

- if $E \subset C$ is an irreducible component of C such that $E \simeq \mathbb{P}^1$, then E contains at least three special (marked or nodal) points,
- if C has only one irreducible component of genus $g(C) = 1$, then we have $n \geq 1$.

We need to include curves with nodal singularities in our description of $\overline{M}_{\mathrm{g,n}}$ in order to obtain a compact moduli space. Note how the two conditions on $(C, p_1, \ldots, p_n)$ are equivalent to demanding that each data point have only finitely many automorphisms (in the obvious sense). The finiteness of the automorphism groups of our stable curves is essential for $\overline{M}_{\mathrm{g,n}}$ to only have "well-behaved" singularities. Indeed for $n + 2g \geq 3$, $\overline{M}_{\mathrm{g,n}}$ exists and is an orbifold of dimension[3] $3g - 3 + n$. Both

[1] whenever we talk about marked curves we shall require the markings to be distinct.

[2] The switch from $\mathbb{Z}$-coefficients to $\mathbb{Q}$-coefficients is motived by the fact that $\overline{M}_{\mathrm{g,n}}$ is an orbifold in general. We go one step further and consider $\mathbb{C}$-coefficients in order to avoid generalisations at some later stage.

[3] Unless stated otherwise, *dimension* will always refer to the complex dimension in this work.

the existence as an orbifold (due to Deligne and Mumford) and the dimension count are non-trivial statements. In spite of that, the latter result was already known to Riemann in the 19th century [Rie57]. [FP97] gives an excellent introduction to $\overline{M}_{\mathrm{g,n}}$.

From now on let X be a projective algebraic variety and $\beta \in H_2(X, \mathbb{Z})$. As mentioned in the introduction, the moduli space $\overline{M}_{\mathrm{g,n}}$ - even with its obvious alteration to include only curves on of class β - is insufficient for us to define Gromov-Witten invariants. We need additionally the moduli space $\overline{M}_{\mathrm{g,n}}(X, \beta)$ of *n-marked stable maps.*

Definition 2.1. *An n-*marked stable map *is a connected, marked, at worst nodal curve* $(C, p_1, \ldots, p_n)$ *together with a morphism $f : C \to X$, satisfying the following two stability conditions:*

- *if $E \subset C$ is an irreducible component of C such that $E \simeq \mathbb{P}^1$ and $f(E) = \{pt\} \in X$ is a point, then E contains at least three special (marked or nodal) points,*
- *if C has only one irreducible component of genus $g(C) = 1$ and $f(C) = \{pt\} \in X$ is a point, then we have $n \geq 1$.*

Note how once again the stability condition ensures that the data $(C, p_1, \ldots, p_n, f)$ does not have infinitely many automorphisms. The coarse moduli space $\overline{M}_{\mathrm{g,n}}(X, \beta)$ for some $\beta \in H_2(X, \mathbb{Z})$ then exists and every point of it represents - up to isomorphism - $(C, p_1, \ldots, p_n, f)$ as before with the additional condition that $[f_*(C)] = \beta$. $\overline{M}_{\mathrm{g,n}}(X, \beta)$ can be shown to be a projective scheme over $\mathbb{C}$ - see for example [CK99].

Before we start working with $\overline{M}_{\mathrm{g,n}}$ and $\overline{M}_{\mathrm{g,n}}(X, \beta)$, let us point out some interesting maps between these spaces in form of a commutative diagram:

$$\begin{tikzcd}
& \overline{M}_{\mathrm{g,n}}(X,\beta) \arrow[r, "\mathrm{ev}_i"] \arrow[dl, "\pi"'] \arrow[dd, "\widetilde{\pi}"] \arrow[dr, "\mathrm{ev}"] & X \arrow[d, hook] \\
\overline{M}_{\mathrm{g,n}} & & X^n \\
& \overline{M}_{\mathrm{g,n}} \times X \arrow[ul, "\mathrm{proj}_1"] \arrow[ur, "\mathrm{proj}_2"'] &
\end{tikzcd}$$

Here $\mathrm{ev}_i(C, p_1, \ldots, p_n, f) = f(p_i)$ is the so-called *evaluation map.* proj_1 and proj_2 are the projection onto the first resp. second factor. π is the map, which ignores the data associated to X and β, i.e. f. However, a little care has to be taken when considering π. The natural idea of simply forgetting the additional data f does not work, since the underlying curve of a stable map need not be stable itself. We can remedy this problem by collapsing any unstable component to obtain a stabilised curve. Note how this only works for $n + 2g \geq 3$.

2.2 Gromov-Witten Invariants

We are now in a position to transfer our intuition from the introduction into precise mathematical terms. Firstly we convert our choice of cycles $Z_1, \ldots, Z_n$ into cohomology classes $\alpha_1, \ldots, \alpha_n$ utilising Poincaré duality. By Künneth's theorem, this describes a cohomology class $\alpha_1 \otimes \cdots \otimes \alpha_n \in H^*(X^n, \mathbb{Z})$. This class can now be pulled back to $\overline{M}_{\mathrm{g,n}}(X, \beta)$ via ev^*, thereby giving us all stable maps, which evaluate into our cycles. All we need to do now is to forget about the map into X and focus on the underlying stabilised curve. However, this requires a push-forward, which is why we need to first use Poincaré duality again, converting into a homology class, after which we can push this class forward to $\overline{M}_{\mathrm{g,n}}$ via π_* and re-convert it into a cohomology class. In other words, we would like to define the Gromov-Witten class $I_{g,n,\beta}$ to be

$$I_{g,n,\beta}(\alpha_1,\dots,\alpha_n) = PD^{-1}_{\overline{M}_{\mathrm{g,n}}} \circ \pi_* \circ PD_{\overline{M}_{\mathrm{g,n}}(X,\beta)} \circ \mathrm{ev}^*(\alpha_1 \otimes \cdots \otimes \alpha_n). \tag{2}$$

By considering deformations of the tangent bundle of $\overline{M}_{\mathrm{g,n}}(X,\beta)$ and their obstructions, the *expected dimension* of the moduli space of stable maps can be calculated [LT98] as

$$(1-g)(\dim(X)-3) - \int_\beta \omega_X + n,$$

where ω_X is the canonical class of X.

The above idea relies on Poincaré duality holding for both $\overline{M}_{\mathrm{g,n}}$ and $\overline{M}_{\mathrm{g,n}}(X,\beta)$. Since $\overline{M}_{\mathrm{g,n}}$ is a smooth orbifold, this was shown by [Beh05]. For $\overline{M}_{\mathrm{g,n}}(X,\beta)$, however, this may fail, as it can have singular components, whose dimension exceeds the expected dimension. Thus we have to modify (2) by expressing it slightly differently:

$$\begin{aligned}
&PD^{-1}_{\overline{M}_{\mathrm{g,n}}} \circ \pi_* \circ PD_{\overline{M}_{\mathrm{g,n}}(X,\beta)} \circ \mathrm{ev}^*(\alpha_1 \otimes \cdots \otimes \alpha_n) \\
&\qquad = PD^{-1}_{\overline{M}_{\mathrm{g,n}}} \circ (\mathrm{proj}_1)_* \circ \widetilde{\pi}_* \left(\widetilde{\pi}^* \circ (\mathrm{proj}_2)^*(\alpha_1 \otimes \cdots \otimes \alpha_n) \cap \left[\overline{M}_{\mathrm{g,n}}(X,\beta)\right]\right) \\
&\qquad = PD^{-1}_{\overline{M}_{\mathrm{g,n}}} \circ (\mathrm{proj}_1)_* \left((\mathrm{proj}_2)^*(\alpha_1 \otimes \cdots \otimes \alpha_n) \cap \widetilde{\pi}_* \left(\left[\overline{M}_{\mathrm{g,n}}(X,\beta)\right]\right)\right).
\end{aligned}$$

This is true, provided the expected dimension of the moduli space of stable maps is equal to the dimension of its highest-dimensional components. This expression then has the advantage of emphasising the role of the fundamental class $\left[\overline{M}_{\mathrm{g,n}}(X,\beta)\right]$ of $\overline{M}_{\mathrm{g,n}}(X,\beta)$.

Why is this an advantage? Well, it turns out that we can always define a so-called *virtual fundamental class* $\left[\overline{M}_{\mathrm{g,n}}(X,\beta)\right]^{\mathrm{vir}} \in H_*(\overline{M}_{\mathrm{g,n}}(X,\beta),\mathbb{C})$ which has the desirable properties. Virtual fundamental classes were first defined in [BF97] and [LT98] and their definition is rather involved. We shall not go into it and instead refer the reader to [Sie04] for a discussion and [CK99] for an introduction. We can now use this expression and our knowledge of the existence of a virtual fundamental class to finally define Gromov-Witten classes and invariants.

Definition 2.2. *Let the notation and all conventions remain as before.*

- *For $n+2g \geq 3$, the* Gromov-Witten class $I_{g,n,\beta}(\alpha_1,\dots,\alpha_n)$ *is the cohomology class in $H^*(\overline{M}_{\mathrm{g,n}},\mathbb{C})$ defined by*

$$PD^{-1}_{\overline{M}_{\mathrm{g,n}}} \circ (\mathrm{proj}_1)_* \left((\mathrm{proj}_2)^*(\alpha_1 \otimes \cdots \otimes \alpha_n) \cap \widetilde{\pi}_* \left(\left[\overline{M}_{\mathrm{g,n}}(X,\beta)\right]^{\mathrm{vir}}\right)\right).$$

- *For $n, g \geq 0$, the* Gromov-Witten invariant $\langle I_{g,n,\beta}\rangle(\alpha_1,\dots,\alpha_n)$ *is the complex number defined by*

$$\int_{\left[\overline{M}_{\mathrm{g,n}}(X,\beta)\right]^{\mathrm{vir}}} \mathrm{ev}_1^*(\alpha_1) \cup \cdots \mathrm{ev}_n^*(\alpha_n).$$

Remark 2.3. *It is not difficult to show that for $n+2g \geq 3$ the definition of a Gromov-Witten invariant expresses exactly our previously motivated intuition as in this case*

$$\langle I_{g,n,\beta}\rangle(\alpha_1,\dots,\alpha_n) = \int_{\overline{M}_{\mathrm{g,n}}} I_{g,n,\beta}(\alpha_1,\dots,\alpha_n).$$

Remark 2.4. *We will often interpret Gromov-Witten classes resp. invariants as maps*

$$\begin{aligned} I_{g,n,\beta} &: H^*(X,\mathbb{C})^{\otimes n} \longrightarrow H^*(\overline{M}_{\mathrm{g,n}},\mathbb{C}) \qquad \textit{resp.} \\ \langle I_{g,n,\beta}\rangle &: H^*(X,\mathbb{C})^{\otimes n} \longrightarrow \mathbb{C} \end{aligned}$$

and use the equivalent notation $I_{g,n,\beta}(\alpha_1 \otimes \cdots \otimes \alpha_n) = I_{g,n,\beta}(\alpha_1,\ldots,\alpha_n)$*, respectively* $\langle I_{g,n,\beta}\rangle(\alpha_1 \otimes \cdots \otimes \alpha_n) = \langle I_{g,n,\beta}\rangle(\alpha_1,\ldots,\alpha_n)$*.*

2.3 Gromov-Witten Axioms

In the previous chapter we have defined Gromov-Witten invariants algebraically. That is not the only way of approaching the theory. In fact, the earliest definitions of Gromov-Witten invariants appeared in the context of semi-positive symplectic manifolds [RT95]. Overall both symplectic and algebraic geometry provide sophisticated definitions of these invariants, which are believed to be equivalent in the domain of common validity. In some cases this has actually been proven, e.g. by [Sie99],[CK99]. Fortunately we need not concern us with this, since we only need Gromov-Witten invariants to fulfil a number of properties called the *Gromov-Witten axioms.* Both the algebraic as well as the symplectic definitions satisfy these axioms and one can equally well justify a third, axiomatic approach to Gromov-Witten theory as in [KM94]. For us it shall suffice to state the axioms, only in some cases giving an intuition as to why they might be true:

Linearity Axiom The map

$$I_{g,n,\beta} : H^*(X,\mathbb{C})^{\otimes n} \to H^*(\overline{M}_{\mathrm{g,n}},\mathbb{C})$$

is linear in all n arguments.

Effectivity Axiom $I_{g,n,\beta} = 0$ if β is not an effective class. This makes intuitive sense because $f_*([C])$ is effective, when $f : C \to X$ is holomorphic.

Equivariance Axiom The symmetric group S_n acts naturally on both $H^*(X,\mathbb{C})^{\otimes n}$, as well as $H^*(\overline{M}_{\mathrm{g,n}},\mathbb{C})$. On the former by permuting the factors, on the latter by permuting the marked points. It makes sense - and is indeed an axiom - that the map $I_{g,n,\beta}$ is equivariant with respect to these two S_n-actions.

Degree Axiom Provided that $\alpha_1,\ldots,\alpha_n$ are homogeneous classes, the degree axioms states that the degree of $I_{g,n,\beta}(\alpha_1,\ldots,\alpha_n)$ is

$$2(g-1)\dim(X) + 2\int_\beta \omega_X + \sum_{i=1}^n \deg(\alpha_i).$$

The Gromov-Witten invariant $\langle I_{g,n,\beta}\rangle$ is only non-zero, when $I_{g,n,\beta}$ is a top-degree class. Therefore, when considering Gromov-Witten invariants, we may assume that

$$\sum_{i=1}^n \deg(\alpha_i) = 2(1-g)\dim(X) - 2\int_\beta \omega_X + 2(3g-3+n).$$

Fundamental Class Axiom If $\beta \neq 0$ and $n \geq 1$ or else if $n + 2g > 3$, we can obtain a natural map $\pi_n : \overline{M}_{\mathrm{g,n}}(X,\beta) \to \overline{M}_{g,n-1}(X,\beta)$, given by forgetting about the last marked point and, if necessary, stabilising the map. If $[X] \in H^0(X,\mathbb{C})$ is the fundamental class of X, then this axiom asserts that

$$I_{g,n,\beta}(\alpha_1,\dots,\alpha_{n-1},[X]) = \pi_n^* I_{g,n-1,\beta}(\alpha_1,\dots,\alpha_{n-1}).$$

This axiom should be rather intuitive as it essentially tells us that the incidence relation "curve meets X" does not constitute a relevant condition. Note how this axiom implies that

$$\langle I_{g,n,\beta}\rangle(\alpha_1,\dots,\alpha_{n-1},[X]) = 0$$

whenever π_n exists.

Remark 2.5. *Here is a general recipe for how to find the image of $(C,p_1,\dots,p_n,f)$ under π_n : if $(C,p_1,\dots,p_{n-1},f)$ remains a stable map, then it is the image we are looking for. So assume that $(C,p_1,\dots,p_{n-1},f)$ is not stable any more. Note firstly that this implies that f is constant on the irreducible component of C, which contains p_n. Therefore the condition $\beta \neq 0$ is sufficient for π_n to exist. According to the definition of stable maps, we now distinguish two different cases: Suppose in the first case that p_n lies on the irreducible component $C_i \subset C$ and that $C_i \neq C$. Then the assumption that $(C,p_1,\dots,p_{n-1},f)$ is not a stable map, implies $C_i \simeq \mathbb{P}^1$ and that C_i contains exactly three special points, one of which is p_n and another of which is a singular node connecting C_i to some other irreducible component of C. The third special point stays special upon collapsing the component C_i. Thus we have obtained a new curve C' and since $f|_{C_i}$ was constant we have a well-defined induced map $f' : C' \to X$. Then $\pi_n((C,p_1,\dots,p_n,f)) = (C',p_1,\dots,p_{n-1},f')$. In the second case we have $C_i = C$ is irreducible. Since $(C,p_1,\dots,p_n,f)$ is stable and $(C,p_1,\dots,p_{n-1},f)$ is not, we conclude that either $n = 3$ and $C \simeq \mathbb{P}^1$ or that $n = 1$ and $g(C) = 1$. In this case we cannot find a natural map π_n. We conclude that for π_n to exist, we require $n \geq 1$ and either $\beta \neq 0$ or $n + 2g > 3$.*

Divisor Axiom Once again let π_n be as in the previous axiom, and assume that it exists. The divisor axiom concerns the special case of $\alpha_n \in H^2(X,\mathbb{C})$, when it states that

$$(\pi_n)_* I_{g,n,\beta}(\alpha_1,\dots,\alpha_n) = \left(\int_\beta \alpha_n\right) I_{g,n-1,\beta}(\alpha_1,\dots,\alpha_{n-1}).$$

To see why this might be true we return to our intuition about Gromov-Witten invariants: suppose we are given $I_{g,n-1,\beta}(\alpha_1,\dots,\alpha_{n-1})$, which we think about as a collection of $(n-1)$-marked curves such that each p_i gets mapped into Z_i, the Poincaré dual of α_i. If we want to make such a curve into an n-marked curve we need to specify one more marked point inside $f(C) \cap Z_n$. Since the image of the curve defines the homology class β, there should be $\int_\beta \alpha_n$ such choices.

Point Mapping Axiom This axioms describes the case of $\beta = 0$ and $g = 0$.[4] If all α_i are homogeneous, then we have the following formula:

$$I_{0,n,0}(\alpha_1,\dots,\alpha_n) = \begin{cases} \left(\int_X \alpha_1 \cup \cdots \cup \alpha_n\right) [\overline{M}_{\mathrm{g,n}}] & \text{if } \sum_{i=1}^n \deg(\alpha_i) = 2\dim(X) \\ 0 & \text{otherwise.} \end{cases}$$

[4] There can be similar formulae for $I_{g,n,0}$ for higher genus g, but they are rather complicated. See [KM94, Sect.2.2.5] for a discussion.

If we want to consider Gromov-Witten invariants we notice that - by the degree axiom - we need to assume $n = 3$. In other words:

$$\langle I_{0,n,0}\rangle(\alpha_1, \ldots, \alpha_n) = \begin{cases} \int_X \alpha_1 \cup \alpha_2 \cup \alpha_3 & \text{if } n = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Splitting Axiom In order to state the splitting axiom we need to introduce one more map between moduli spaces. Suppose we have a *splitting* of $n = n_1 + n_2$ and $g = g_1 + g_2$ such that both moduli spaces $\overline{M}_{g_i,n_i+1}$ exist. We now construct a map

$$\varphi : \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{\mathrm{g,n}}$$

by sending the two stable curves $(C_1, p_1, \ldots, p_{n_1+1})$ and $(C_2, q_1, \ldots, q_{n_2+1})$ to the curve $(C_1 \cup C_2, p_1, \ldots, p_{n_1}, q_1, \ldots, q_{n_2})$. Here we obtained $C_1 \cup C_2$ by identifying p_{n_1+1} with q_{n_2+1}. This has the nice side-effect of making the resulting curve stable too. In this situation the splitting axiom constitutes the following formula for $\varphi^* I_{g,n,\beta}(\alpha_1, \ldots, \alpha_n)$:

$$\sum_{\beta=\beta_1+\beta_2} \sum_{i,j} g^{ij} I_{g_1,n_1+1,\beta_1}(\alpha_1, \ldots, \alpha_{n_1}, T_i) \otimes I_{g_2,n_2+1,\beta_2}(T_j, \alpha_{n_1+1}, \ldots, \alpha_n),$$

where the T_i are a homogeneous basis of $H^*(X, \mathbb{C})$ and (g^{ij}) is the inverse of the matrix (g_{ij}) with $g_{ij} = \int_X T_i \cup T_j$.

Genus Reduction Axiom For this axiom we use the fact that we can obtain a natural map $\psi : \overline{M}_{g-1,n+2} \to \overline{M}_{\mathrm{g,n}}$ by gluing together the last two marked points of a stable map. Then - with the notation as in the previous axiom - we have

$$\psi^* I_{g,n,\beta}(\alpha_1, \ldots, \alpha_n) = \sum_{i,j} g^{ij} I_{g-1,n+2,\beta}(\alpha_1, \ldots, \alpha_n, T_i, T_j).$$

Deformation Axiom This axiom intuitively tells us that the Gromov-Witten classes should be invariant under the deformation of the complex structure of X. We will state it for a smooth proper map $F : \mathcal{X} \to T$ with connected base T and fibres X_t for $t \in T$. Now choose a locally constant section $\beta_t \in H^2(X_t, \mathbb{Z})$, so that for every $t \in T$ we get a map

$$I^{X_t}_{g,n,\beta_t} : H^*(X_t, \mathbb{C})^{\otimes n} \to H^*(\overline{M}_{\mathrm{g,n}}, \mathbb{C}).$$

The the deformation axiom asserts that, for locally constant sections $\alpha_1, \ldots, \alpha_n$ of $H^*(X_t, \mathbb{C})$, $I^{X_t}_{g,n,\beta_t}(\alpha_1, \ldots, \alpha_n)$ is constant.

3 Quantum Cohomology

It turns out that Gromov-Witten invariants can be used to define a *quantum product* and with it a *quantum cohomology ring*. This ring first arose in physics (as the *chiral ring*), where its mathematical properties are supported by physical intuition. One point of view considers it as a perturbation of the usual cohomology ring. However, different from the usual cohomology ring, the quantum cohomology is not functorial. On the other hand we find that the ring structure reveals often very subtle connections between various Gromov-Witten invariants. For example on $\mathbb{P}^2$ we have the following recursive formula for N_d, the number of degree d rational curves passing through $3d-1$ generic point:

$$N_d = \sum_{d_1+d_2=d} N_{d_1}N_{d_2}\left(d_1^2d_2^2\binom{3d-4}{3d_1-1} - d_1^3d_2\binom{3d-4}{3d_1-1}\right).$$

This turns out to be essentially equivalent to the fact that the quantum product is associative.

3.1 The Small Quantum Cohomology Ring

Let us begin by fixing some notation for this chapter: We shall assume a basis $\{T_0, T_1, \ldots, T_m\}$ of $H^*(X, \mathbb{C})$, where $T_0 = 1$ and all T_i are homogeneous elements. Furthermore let $g_{ij} = \int_X T_i \cup T_j$ and denote by (g_{ij}) the matrix with the respective entries. Similarly we write $(g^{ij}) = (g_{ij})^{-1}$ for its inverse. Last, but not least, let $T^0, \ldots, T^m$ denote the dual basis to $T_0, \ldots, T_m$, i.e. $T^i = \sum_j g^{ij}T_j$. Alternatively we view the T^i as the elements defined by $\int_X T^i \cup T_j = \delta_{ij}$.

Definition 3.1. *For $a, b \in H^*(X, \mathbb{C})$ we define their* small quantum product *as*

$$a * b = \sum_{i=0}^{m} \sum_{\beta \in H_2(X,\mathbb{Z})} \langle I_{0,3,\beta}\rangle(a, b, T_i)q^\beta T^i,$$

where $\omega \in H^2(X, \mathbb{C})$ is a complexified Kähler class and $q^\beta = e^{2\pi\sqrt{-1}\int_\beta \omega}$. Note the dependence on the class ω.

Remark 3.2. *We should pause here to make some general comments on the coefficients of our quantum cohomology. There are essentially two ways of dealing with the* quantum variable *q^β. The first one is the point of view of definition 3.1. It has the advantage of making q^β "computable", i.e. for given ω and β, q^β is simply a complex number. Additionally, this makes the quantum product $*$ into a binary operation on $H^*(X, \mathbb{C})$ and we do not have to consider more general coefficients. The second way of dealing with q^β is to regard it as a formal variable in some coefficient ring other than $\mathbb{C}$. Adopting this view has the advantage that we can simultaneously solve eventual convergence issues: note how the convergence of the sum in definition 3.1 is not a priori given. In fact, if we regard $q^\beta = e^{2\pi\sqrt{-1}\int_\beta \omega}$, then there will be examples, when the sum is divergent. Thus in practice it is often assumed that the coefficients of our quantum cohomology are in the so-called* Novikov ring, *which is a subring of the ring formal power series $\mathbb{C}[[H_2(X, \mathbb{Z})]]$. A general element of the Novikov ring is then a power series*

$$\sum_{\beta \in H_2(X,\mathbb{Z})} a_\beta q^\beta,$$

where $a_\beta \in \mathbb{C}$ and the set

$$\left\{ a_\beta \;\middle|\; a_\beta \neq 0 \text{ and } |\int_\beta \omega| < C \right\}$$

is finite for all values $C \in \mathbb{R}$. Multiplication in the Novikov ring is commutative and for all $\beta_1, \beta_2 \in H_2(X, \mathbb{Z})$ we have $q^{\beta_1}q^{\beta_2} = q^{\beta_1+\beta_2}$.

That being said, due to proposition 3.4, we shall not be concerned with convergence issues. Thus we will be free to consider q^β as a complex number or as a formal variable. Moreover, in the latter *case we do not have to resort to the use of the Novikov ring, but instead will be fine using the semi-group ring R of $H_2(X,\mathbb{Z})$:*

$$R = \frac{\mathbb{C}[q^\beta;\ \ \beta \in H_2(X,\mathbb{Z})]}{\langle q^{\beta_1}q^{\beta_2} - q^{\beta_1+\beta_2}\rangle_{\beta_1,\beta_2\in H_2(X,\mathbb{Z})}}.$$

Remark 3.3. *After a choice $\{\beta_1,\dots,\beta_r\}$ of basis of $H_2(X,\mathbb{Z})$, we can then identify $R = \mathbb{C}[q_1^\pm,\dots,q_r^\pm]$, where $q_i = q^{\beta_i}$. But note that due to the effectivity axiom (compare chapter 2.3), $\langle I_{g,n,\beta}\rangle \neq 0$ implies that $\beta \in H_2(X,\mathbb{Z})$ is the class of an effective curve. Therefore we will later choose the basis $\{\beta_1,\dots,\beta_r\}$ in such a way that the cone of effective curves $\mathrm{NE}(X)$ is contained in the cone $\sum_i \mathbb{R}_{\geq 0}\beta_i \subset H_2(X,\mathbb{R})$. This ensure that q^β will be a product of positive powers of the q_i, i.e. $*$ (and later $\star$) will be a product on the ring $\mathbb{C}[q_1,\dots,q_r]$ and there will be a well-defined limit $q_i \to 0$.*

Proposition 3.4. *Let X be a smooth, projective, Fano variety. Then the sum in definition 3.1 is finite for all $a,b \in H^*(X,\mathbb{C})$.*

Proof: It suffices to show that there are only finitely many $\beta \in H_2(X,\mathbb{Z})$ for which $\langle I_{0,3,\beta}\rangle(a,b,T_i) \neq 0$. We know from the degree axiom of the Gromov-Witten classes that this can only happen when

$$\deg(a) + \deg(b) + \deg(T_i) = 2\dim(X) - 2\int_\beta \omega_X.$$

By the effectivity axiom we can further restrict our consideration to effective classes $\beta \in H_2(X,\mathbb{Z})$, which form a lattice inside the cone of curves $\mathrm{NE}(X)$. Being Fano by definition means that ω_X^{-1} (the anti-canonical class) is ample, which by Kleiman's condition [Laz04] implies that $\int_{\widetilde{\beta}} \omega_X^{-1} > 0$ for all $\widetilde{\beta}$ inside the closure $\overline{\mathrm{NE}}(X)$ of the cone of curves. So for any real constant C the set

$$\overline{\mathrm{NE}}(X) \cap \left\{ \widetilde{\beta} \;\middle|\; \int_{\widetilde{\beta}} \omega_X^{-1} < C \right\} \subset H_2(X,\mathbb{R})$$

is bounded[5] and therefore contains only finitely many lattice points $\beta \in H_2(X,\mathbb{Z})$. $\square$

Returning to the definition of the quantum product, we now see that it is a well-defined product on $H^*(X,R)$. I.e. for every $\beta \in H_2(X,\mathbb{Z})$ we have a formal variable q^β with the property that $q^{\beta_1} * q^{\beta_2} = q^{\beta_1+\beta_2}$. So every element of $H^*(X,R)$ is a sum of finitely many q^β's with coefficients in $H^*(X,\mathbb{C})$ and $*$ defines a binary operation on $H^*(X,R)$, since the sum in definition 3.1 is finite. In fact, this quantum product gives $H^*(X,R)$ a ring structure as the following theorem shows.

Theorem 3.5. *$H^*(X,R)$ is a super-commutative[6] ring with identity under the small quantum product.*

Proof: The Equivariance Axiom implies that $\langle I_{0,3,\beta}\rangle(a,b,T_i) = (-1)^{\deg(a)\deg(b)}\langle I_{0,3,\beta}\rangle(b,a,T_i)$, which in turn implies super-commutativity. Therefore we only need to show that $T_0 = 1 = [X] \in H^0(X,\mathbb{C}) \subset H^0(X,R)$ is the identity element and that $*$ is associative. For the latter we can refer the reader to [CK99]. Alternatively, associativity of the small quantum product follows from the associativity of the big quantum product - see chapter 3.3. To see why T_0 should act as the identity

[5] equipping $H_2(X,\mathbb{R})$ with the Euclidean topology.

[6] by super-commutative we mean that $a * b = (-1)^{\deg(a)\deg(b)} b * a$.

under $*$ recall the Fundamental Class Axiom and the succeeding remark 2.5. There we saw that unless $\beta = 0$, $\langle I_{0,3,\beta}\rangle(a, T_0, T_i) = 0$. Furthermore, assuming $\beta = 0$, the Point Mapping Axiom provides us with a formula for $\langle I_{0,3,0}\rangle(a, T_0, T_i)$. Hence

$$\begin{aligned} a * T_0 &= \sum_{i=1}^{n} \langle I_{0,3,0}\rangle(a, T_0, T_i) T^i = \sum_{i=1}^{n} \left(\int_X a \cup T_0 \cup T_i \right) T^i \\ &= \sum_{i=1}^{n} \left(\int_X (\sum_{j=1}^{n} a_j T_j) \cup T_i \right) T^i = \sum_{i,j=1}^{n} a_j \left(\int_X T_j \cup T_i \right) T^i \\ &= \sum_{i,j=1}^{n} a_j g_{ij} T^i = \sum_{j=1}^{n} a_j T_j \\ &= a. \end{aligned}$$

□

Remark 3.6. *It is natural to ask, if the quantum product is related to our usual cup product. This is indeed the case. In a manner of speaking, the cup product is the limit of the small quantum product as $q \to 0$ (compare remark 3.3). Considering q in this limit is equivalent to discarding the inner sum in our definition (over $\beta \in H_2(X, \mathbb{Z})$) and working only with $\beta = 0$.*

$$\begin{aligned} a * b\big|_{q=0} &= \sum_{i=1}^{n} \langle I_{0,3,0}\rangle(a, b, T_i) T^i \\ &= \sum_{i=0}^{n} \left(\int_X a \cup b \cup T_i \right) T^i \\ &= \int_X a \cup b, \end{aligned}$$

where we made use of the Point Mapping Axiom.

3.2 Three-Point Function

Three-point functions constitute a slightly different, sometimes easier approach to the small quantum product.

Definition 3.7. *The three-point function $\langle a, b, c\rangle$ for $a, b, c \in H^*(X, R)$ is defined to be*

$$\langle a, b, c\rangle = \sum_{\beta \in H_2(X,\mathbb{Z})} \langle I_{0,3,\beta}\rangle(a, b, c) q^{\beta},$$

with q^β as in definition 3.1.

Had we started by defining the three-point function, we could now define the small quantum product by

$$a * b = \sum_{i=1}^{n} \langle a, b, T_i\rangle T^i.$$

Proposition 3.8. *Let $a, b, c \in H^*(X, R)$ and denote $g(a,b) = \int_X a \cup b$. Then*

(i) $g(a * b, c) = g(a, b * c) = \langle a, b, c\rangle$,

(ii) $\int_X a * b = g(a, b)$,

(iii) $\int_X a * b * c = \langle a, b, c\rangle$.

Proof:

$$\begin{aligned} g(a * b, c) &= \int_X \left(\sum_{i=1}^n \langle a, b, T_i\rangle T^i\right) \cup \left(\sum_{j=1}^n c_j T_j\right) \\ &= \sum_{i,j=1}^n \langle a, b, T_i\rangle c_j \int_X T^i \cup T_j \\ &= \sum_{i,j=1}^n \langle a, b, c_j T_i\rangle \delta_{ij} \\ &= \langle a, b, c\rangle. \end{aligned}$$

$\langle a, b, c\rangle = g(a, b * c)$ can be shown very similarly. The second part now follows from our previous observation that T_0 is the identity for $*$, as well as (i):

$$\int_X a * b = g(T_0, a * b) = g(T_0 * a, b) = g(a, b).$$

Last, but not least, (iii) follows straight from (i) and (ii). □

3.3 The Gromov-Witten Potential and Big Quantum Cohomology Ring

The fact that we introduced a *small* quantum cohomology ring has surely alerted the attentive reader to the potential presence of a *big* quantum cohomology ring. And indeed this exists.

To introduce the big quantum cohomology we first need to introduce the Gromov-Witten potential. This can be thought of as a generating function for the genus 0 Gromov-Witten invariants. It has physically a natural interpretation as the genus 0 free energy.

Definition 3.9. *The* Gromov-Witten potential *is the power series*

$$\Phi(\gamma) = \sum_{n=0}^\infty \sum_{\beta \in H_2(X,\mathbb{Z})} \frac{1}{n!} \langle I_{0,n,\beta}\rangle(\gamma^{\otimes n})\mathbf{q}^\beta,$$

where $\mathbf{q}^\beta$ is in the Novikov ring (see remark 3.2) and we set $\langle I_{0,n,\beta}\rangle = 0$ for $n \leq 2$.

Remark 3.10. *Just like in the case of the small quantum cohomology, there is the issue of convergence of Φ, which we have solved here by adopting the Novikov ring. However, if convergence is assumed there is no further need for the quantum variable $\mathbf{q}^\beta$ and we will assume it to be identically* 1. *Indeed, since we will be mostly concerned with the Fano case, we shall assume the convergence of Φ for now and drop $\mathbf{q}^\beta$ in the definition. This is justified by proposition 5.5*[7]

[7] Unfortunately, proposition 3.4 is not sufficient here as the degree condition becomes more complicated for $n > 3$.

Remark 3.11. *A little care has to be taken when defining* $\langle I_{0,n,\beta}\rangle(\gamma^{\otimes n})$*. First we introduce coordinates* $\{t_0, \ldots, t_m\}$ *for our basis* $T_0, \ldots, T_m$*. In these coordinates we can write* $\gamma = t_0T_0 + \cdots + t_mT_m$ *as a function of* $t_0, \ldots, t_m$*. Then expanding* $\gamma^{\otimes n}$ *with these coordinates and making repeated use of the Gromov-Witten Axioms we find*

$$\frac{1}{n!}\langle I_{0,n,\beta}\rangle(\gamma^{\otimes n}) = \sum_{a\in\mathbb{Z}^{m+1},|a|=n} \varepsilon(a)\langle I_{0,n,\beta}\rangle(T_a)\frac{t^a}{a!},$$

where $\varepsilon(a) = \pm 1$ *is a sign introduced by the Equivariance Axiom. Here we have used the standard monomial notation for* $T^a = T_0^{a_0}T_1^{a_1}\ldots T_m^{a_m}, a! = a_0!a_1!\ldots a_m!$*, etc. for* $a = (a_0, \ldots, a_m)$*. For the Gromov-Witten potential this means that we can write it depending on the* t_i *:*

$$\Phi(t_0, \ldots, t_m) = \sum_{a_0+\ldots+a_m\geq 3}\ \sum_{\beta\in H_2(X,\mathbb{Z})} \frac{\varepsilon(a_0, \ldots, a_m)}{a_0!\cdots a_m!}\langle I_{0,n,\beta}\rangle(T_0^{\otimes a_0}\otimes\cdots\otimes T_m^{\otimes a_m})t_0^{a_0}\cdots t_m^{a_m}\mathbf{q}^\beta. \quad (3)$$

Note that we have a partial derivative operator ∂_{t_i},[8] defined by

$$\partial_{t_i}(t_i^k t^a) = kt_i^{k-1}t^a,$$

for some monomial t^a not involving t_i. Note also that the super-commutativity implies

$$\partial_{t_i}\partial_{t_j}(\Phi) = (-1)^{\deg(t_i)\deg(t_j)}\partial_{t_j}\partial_{t_i}(\Phi).$$

Definition 3.12. *Let* Φ *be the Gromov-Witten potential of* X*. In the established notation we define*

$$T_i \star T_j = \sum_{k=0}^{m} \partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi)T^k.$$

Extending this linearly defines a product, called the big quantum product*.*

Proposition 3.13. $T_0 = [X]$ *is the unit for* $\star$*.*

Proof:

$$\partial_{t_0}\partial_{t_j}\partial_{t_k}(\Phi) = \langle I_{0,3,0}\rangle(T_0, T_j, T_k) = \int_X T_j \cup T_k = g_{jk}$$

$$\Rightarrow T_0 \star T_j = \sum_{k=0}^{m} g_{jk}T^k = T_j.$$

□

Proposition 3.14. *The Gromov-Witten potential* Φ *satisfies the WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equation*

$$\sum_{a,b=0}^{m} \partial_{t_i}\partial_{t_j}\partial_{t_a}(\Phi)g^{ab}\partial_{t_b}\partial_{t_k}\partial_{t_l}(\Phi) = (-1)^{\deg(t_i)(\deg(t_j)+\deg(t_k))}\sum_{a,b=0}^{m} \partial_{t_j}\partial_{t_k}\partial_{t_a}(\Phi)g^{ab}\partial_{t_b}\partial_{t_i}\partial_{t_l}(\Phi)$$

and this is equivalent to the associativity of the big quantum product.

[8]Throughout this thesis we shall denote all partial derivatives and partial derivative operators by $\frac{\partial}{\partial x} = \partial_x$.

Proof: See [FP97, theorem 8.4]. □

Lastly we should point out the relationship between the small quantum cohomology and the big quantum cohomology. Let $\gamma_2 = \sum_{i=1}^{r} t_i T_i$ and $\widetilde{\gamma} = t_0 T_0 + \sum_{i=r+1}^{m} t_i T_i$, where we assumed $T_1, \ldots, T_r$ span $H^2(X, \mathbb{C})$.

Proposition 3.15. *Then*

$$T_i \star T_j = \sum_{k=0}^{m} \sum_{n=0}^{\infty} \sum_{\beta \in H_2(X,\mathbb{Z})} \frac{1}{n!} \langle I_{0,n+3,\beta} \rangle (T_i, T_j, T_k, \widetilde{\gamma}^n) T^k \exp(\int_\beta \gamma_2) \mathbf{q}^\beta .$$

Proof: See [CK99, chapter 8.5.1]. □

In light of this proposition there are now two natural ways of comparing the small and the big quantum product. The most obvious way is to note that $\star$ is given as a sum in terms of the variables $t_0, \ldots, t_n$ and $\mathbf{q}^\beta$, whereas $*$ depends only on q^β. Thus we could set $\gamma = \gamma_2 = \widetilde{\gamma} = 0$ and identify $\mathbf{q}^\beta$ and q^β either as the same element of the Novikov ring or by using an appropriate isomorphism, mapping $\mathbf{q}^\beta$ to $\exp(2\pi\sqrt{-1}\int_\beta \omega)$, depending on our interpretation of q^β in definition 3.1 (compare remark 3.2). In other words:

$$T_i * T_j = T_i \star T_j \big|_{\gamma_2 = \widetilde{\gamma} = 0} .$$

However, the more enlightening way of interpreting the relationship between small and big quantum cohomology, comes from "restricting to $H^2(X, \mathbb{C})$". We achieve this by setting $\widetilde{\gamma} = 0 \Leftrightarrow \gamma \in H^2(X, \mathbb{C})$ and resorting to our convention that $\mathbf{q}^\beta \equiv 1$. Then we find that γ_2 takes on the role of ω in definition 3.1. Explicitly we either set $2\pi\sqrt{-1}\omega = \gamma_2$ or identify $q^\beta = \exp(\int_\beta \gamma_2)$ and find that the formulae for $*$ and $\star$ agree.

3.4 Homogeneity of the Gromov-Witten Potential

Before we move on to explicit computation we should point out that the Gromov-Witten potential Φ can be thought of as homogeneous. This requires a precise notion of degree and, chosen suitably, this will make the (big) quantum cohomology a degree-preserving product. A typical summand of the Gromov-Witten potential has the form

$$\pm \langle I_{0,n,\beta} \rangle (T_0^{a_0}, \ldots, T_m^{a_m}) \frac{t_0^{a_0} \cdots t_m^{a_m}}{a_0! \cdots a_m!} \mathbf{q}^\beta . \tag{4}$$

Let us assign the following degrees:

$$|t_i| := 2 - |T_i| \qquad \text{and} \qquad |\mathbf{q}^\beta| := -2 \int_\beta \omega_X , \tag{5}$$

where as usual ω_X denotes the canonical class of X. Then we find that the total degree of the term (4) is

$$\begin{aligned} \left| \pm \langle I_{0,n,\beta} \rangle (T_0^{a_0}, \ldots, T_m^{a_m}) \frac{t_0^{a_0} \cdots t_m^{a_m}}{a_0! \cdots a_m!} \mathbf{q}^\beta \right| &= \left| t_0^{a_0} \cdots t_m^{a_m} \mathbf{q}^\beta \right| \\ &= \sum_{i=0}^{m} a_i (2 - |T_i|) - 2 \int_\beta \omega_X . \end{aligned}$$

But $\langle I_{0,n,\beta}\rangle(T_0^{a_0},\ldots,T_m^{a_m})\neq 0$ implies by the degree axiom from chapter 2.3 that

$$\sum_{i=0}^{m} a_i|T_i| = 2\dim(X) - 2\int_\beta \omega_X + 2(n-3).$$

Therefore, using these notions of degree we find that Φ is homogeneous of total degree $6-2\dim(X)$.

Here is another way of looking at the notion of degree and homogeneity: we say that a function $f:\mathbb{C}^n\to\mathbb{C}$ is quasi-homogeneous of degree $a\in\mathbb{Z}\setminus\{0\}$, if there exist $a_1,\ldots,a_n\in\mathbb{Z}$ such that

$$f(\lambda^{a_1}x_1,\ldots,\lambda^{a_n}x_n) = \lambda^a f(x_1,\ldots,x_n)$$

for all $\lambda\in\mathbb{C}$. In this case we say that x_i is of degree a_i. We can now read the homogeneity of Φ off the next proposition.

Proposition 3.16. *Evaluate the formal variable $\mathbf{q}^\beta$ from the Gromov-Witten potential (3.9) on $\omega\in H^2(X,\mathbb{C})$ as*

$$\mathbf{q}^\beta(\omega) = \exp(\int_\beta \omega)$$

and denote the resulting Gromov-Witten potential by Φ_ω, thereby indicating the dependence on the class ω. Assuming the convergence of the Gromov-Witten potential, we have for any $t\in\mathbb{C}$

$$\Phi_{\omega-2\omega_X t}\left(e^{(2-|T_0|)t}t_0,\ldots,e^{(2-|T_m|)t}t_m\right) = e^{(6-2\dim(X))t}\Phi_\omega(t_0,\ldots,t_m). \tag{6}$$

Proof: According to the degree axiom (c.f. chapter 2.3) $\langle I_{0,n,\beta}\rangle(T_0^{\otimes a_0}\otimes\cdots\otimes T_m^{\otimes a_m})$ is unequal to zero only if

$$\sum_{i=0}^{m}|T_i|a_i = 2\dim(X) - 2\int_\beta\omega_X + 2(n-3)$$
$$\Longleftrightarrow \sum_{i=0}^{m}(2-|T_i|)a_i - 2\int_\beta\omega_X = 6-2\dim(X).$$

The claim then follows easily by simply expanding the left-hand side of (6) according to equation (3). $\square$

Proposition 3.17. *Both $\star$ and $*$ preserve the notion of degree introduced in (5).*

Proof: Clearly

$$\left|\partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi)\right| = |\Phi| - |t_i| - |t_j| - |t_k| = |T_i| + |T_j| + |T_k| - 2\dim(X).$$

Since T^k is such that $\int_X T_k \cup T^k = 1$, we have $|T_k| + |T^k| = 2\dim(X)$. Thus

$$|T_i \star T_j| = \left|\sum_{k=0}^{m} \partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi)T^k\right| = |T_i| + |T_j|,$$

which completes the proof for $\star$. In order to see that $*$ too preserves the degree define

$$\Phi_{\text{small}}(\gamma) = \frac{1}{6}\sum_{\beta\in H_2(X,\mathbb{Z})} \langle I_{0,3,\beta}\rangle(\gamma^{\otimes 3})q^\beta$$

for $\gamma \in H^*(X,\mathbb{C})$ and $q^\beta \in R$ (the semi-group ring of $H_2(X,\mathbb{Z})$). Note that up to the identification $q^\beta \leftrightarrow \mathbf{q}^\beta$, this is just the term of Φ with $n = 3$. Just as we did with Φ we can express $\gamma = t_0T_0 + \cdots + t_mT_m$ and expand Φ_{small} as a function in the variables $t_0, \ldots, t_m$. Φ_{small} is then cubic in $t_0, \ldots, t_m$ and thus $\partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi_{\text{small}})$ is independent of these variables for all choices of i, j, k. In fact, it is not difficult to see that

$$\sum_{k=0}^{m} \partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi_{\text{small}})T^k = T_i * T_j.$$

In other words, Φ_{small} is a potential for the small quantum cohomology. Given that it is contained as the summand for $n = 3$ in the Gromov-Witten potential, the same notion of degree makes Φ_{small} homogeneous and thus $*$ degree-preserving. $\square$

Remark 3.18. *Recall from before that $\{T_1, \ldots, T_r\}$ form a basis of $H^2(X,\mathbb{C})$. Assume now that $T_i \in H^2(X,\mathbb{Z}) \subset H^2(X,\mathbb{C})$. Then we have a natural dual basis $\{S^1, \ldots, S^r\}$ of the homology $H_2(X,\mathbb{Z})$ and we can express all q^β as monomials in the $q_i = q^{S^i}$ for $i = 1, \ldots r$. Now let us express the anti-canonical divisor $-K_X$ as*

$$-K_X = \sum_{i=1}^{r} \xi_i T_i.$$

In this case, by our definition of degree (5), we find that $|q_i| = 2\xi_i$. Alternatively this can be deduced from the homogeneity of the small quantum product. A typical term in $T_i * T_j$ has the following degree (compare definition 3.1):

$$\begin{aligned}\deg(T_i) + \deg(T_j) &= 2\dim(X) - \deg(T_k) - 2\int_\beta \omega_X \\ &= \deg(T^k) + 2\int_\beta \omega_X^{-1} \\ &= \deg(T^k) + \deg(q^\beta).\end{aligned}$$

4 Quantum Cohomology of Del Pezzo Surfaces

This chapter is meant to both introduce Del Pezzo surfaces, as well as concretely work out the quantum cohomology of them. We shall focus particularly on X_4, the blow-up of $\mathbb{P}^2$ in four points in general position, since this is the first new surface for which we will show mirror symmetry to hold. To this aim, chapter 4.1 will borrow much from [Man86] and [Dem80]. Section 4.2 is largely based on [CM95].

4.1 Del Pezzo Surfaces

Let us first introduce Del Pezzo surfaces. These are smooth, birationally trivial surfaces X, such that the anti-canonical divisor $-K_X$ is ample. Del Pezzo surfaces are generally classified by their degree, which is defined as the self-intersection number of its canonical divisor K_X. We have the following classical result:

Theorem 4.1 (Classification of Del Pezzo Surfaces)**.** *Let X be a Del Pezzo surface of degree d. Then we have $1 \leq d \leq 9$ and the following statements hold depending on d :*

$d = 9$ *Then X is isomorphic to $\mathbb{P}^2$,*

$d = 8$ *Then either $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or X is isomorphic to the blow-up of $\mathbb{P}^2$ in one point,*

$d \leq 7$ *Then X is isomorphic to the blow-up of $\mathbb{P}^2$ in $9 - d$ points in general position, i.e. no three points are collinear and no six points lie on one conic.*

Conversely, denote by X_r the surface obtained by blowing up $\mathbb{P}^2$ in r points in general position (in the same sense as before). Then for $1 \leq r \leq 9$, X_r is a Del Pezzo surface of degree $d = 9 - r$.

Proof: An excellent proof can be found e.g. in [Man86, Sect.24]. □

This classification theorem tells us that Del Pezzo surfaces can be thought of as blow-ups of $\mathbb{P}^2$. So in order to study them we should aim to understand blow-ups of $\mathbb{P}^2$. Recall that a blow-up of a surface T in a point $p \in T$ is a surface S together with a map $\mu : S \to T$, which is an isomorphism away from the inverse image of p, i.e. μ induces $S \setminus \{\mu^{-1}(p)\} \simeq T \setminus \{p\}$. The inverse image $E = \mu^{-1}(p)$ of p is called the *exceptional divisor* for this blow-up and is isomorphic to $\mathbb{P}^1$. Under the usual intersection product on S, we have that $E \cdot E = -1$. One form of the adjunction formula (proven in [Huy05, proposition 2.5.5]) tells us that $K_S = \mu^* K_T + E$, where K_S denotes the canonical divisor of S and K_T the one of T. Since $\mu_*(\mu^* K_T \cdot E) = \mu_* E \cdot K_S = [\{p\}] \cdot K_S = 0$, we intersect both side of the adjunction formula with E to see that $E \cdot K_S = -1$. In fact, the converse of these observations are true too, as can be seen by the following theorem:

Theorem 4.2 (Castelnuovo's Contractability Theorem)**.** *Let S be a non-singular projective surface and $E \subset S$ a (-1)-curve.*[9] *Then S is the blow-up $\mu : S \to T$ of a non-singular surface T in a point $\mu(E) \in T$.*

Proof: See for example [Mat02]. □

[9] A (-1)-curve is an irreducible, reduced curve, isomorphic to $\mathbb{P}^1$ with self-intersection $= -1$.

We can make this situation very concrete: let $\mu : S \to T$ be the blow-up of T in p and let us consider the Picard group of T. Once again by adjunction we know that $\text{Pic}(S) \simeq \text{Pic}(T) \oplus \mathbb{Z}[E]$. Now let X_r be the Del Pezzo surface obtained by blowing up $\mathbb{P}^2$ in r points in general position. It does not matter in which order we realise the blow-ups. By our generality condition on the points in $\mathbb{P}^2$ all possible orders produce naturally isomorphic surfaces with naturally isomorphic exceptional divisors. Therefore we shall think of this situation as $\mu : X_r \to \mathbb{P}^2$, which induces the isomorphism $X_r \setminus (E_1 \cup \cdots \cup E_r) \simeq \mathbb{P}^2 \setminus \{p_1, \ldots, p_r\}$, where $\mu(E_i) = p_i$, the exceptional divisors to the points p_i in the centre of the blow-up. Then we have

$$\text{Pic}(X_r) \simeq \mathbb{Z}[H] \oplus \mathbb{Z}[E_1] \oplus \cdots \oplus \mathbb{Z}[E_r],$$

where H is the pull-back of a class of a line, e.g. defined by the inverse image of a line in $\mathbb{P}^2$, which does not intersect any of the p_i. Moreover the intersection product on $\text{Pic}(X_r)$ is defined by

$$H \cdot H = 1 \qquad ; \qquad E_i \cdot E_j = -\delta_{ij} \qquad ; \qquad H \cdot E_i = 0$$

for all $1 \leq i, j \leq r$. In this notation, making repeated use of the adjunction formula $K_S = \mu^* K_T + E$ for a blow-up $\mu : S \to T$, as well as using the fact that $K_{\mathbb{P}^2} = -3H$, we obtain the following formula for the canonical divisor of X_r :

$$K_{X_r} = -3H + E_1 + \cdots + E_r.$$

Proposition 4.3. *For $r \leq 8$, X_r has only finitely many (-1)-curves.*

Proof: Let $D = \alpha H + \beta_1 E_1 + \cdots + \beta_r E_r$ be a (-1)-curve. Since $D \cdot D = D \cdot K_{X_r} = -1$ we have the following two conditions on α and the β_i :

$$\begin{aligned} -3\alpha - \beta_1 - \cdots - \beta_r &= -1, \\ \alpha^2 - \beta_1^2 - \cdots - \beta_r^2 &= -1. \end{aligned}$$

We can combine these equations appropriately and complete the squares to obtain the following expression:

$$\begin{aligned} -6\alpha(K_{X_r} \cdot D) - 9(D \cdot D) - 6\alpha + 9 &= 9\alpha^2 - r\alpha^2 - 6\alpha + 9 + \sum_{i=1}^{r} (\alpha^2 + 6\alpha\beta_i + 9\beta_i^2) \\ \Rightarrow 18 &= \alpha^2(8 - r) + (\alpha - 3)^2 + \sum_{i=0}^{r} (\alpha + 3\beta_i)^2. \end{aligned}$$

Clearly, for $r \leq 8$, there are only finitely many integer solutions to this and hence there can only be finitely many solutions to the original two equations. $\square$

Example 4.4. *We can use the previous proposition to find all (-1) curves. For example, let $r = 4$. Then we have to solve the equation*

$$4\alpha^2 + (\alpha - 3)^2 + (\alpha + 3\beta_1)^2 + (\alpha + 3\beta_2)^2 + (\alpha + 3\beta_3)^2 + (\alpha + 3\beta_4)^2 = 18. \tag{7}$$

By inspecting the first two terms we deduce immediately that $0 \leq \alpha \leq 2$, so let us consider the remaining cases separately:

$\alpha = 0$ *Then equation (7) simplifies to* $\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2 = 1$. *Since we know that* $D \cdot K_{X_4} = -1$, *we find four* (-1)*-curves, the four exceptional divisors of our blow-up.*

$\alpha = 1$ *In this case equation (7) simplifies to*

$$(1 + 3\beta_1)^2 + (1 + 3\beta_2)^2 + (1 + 3\beta_3)^2 + (1 + 3\beta_4)^2 = 10.$$

Now 10 *can be written in eight ways as the sum of four integer squares:* $(\pm 3)^2 + (\pm 1)^2 + 2 \cdot 0 = 2 \cdot (\pm 2)^2 + 2 \cdot (\pm 1)^2$. *Since we are looking for integer solutions, i.e.* $\beta_i \in \mathbb{Z}$, *only one of those sums of square works:* $10 = 2 \cdot (-2)^2 + 2 \cdot 1^2$ *with* $\beta_i = -1$ *for precisely two values of* i *and* $\beta_i = 0$ *for the remaining two* i. *Thus we end up with six* (-1)*-curves of the form* $H - E_i - E_j$ *for* $1 \leq i < j \leq 4$.

$\alpha = 2$ *Last, but not least we have equation (7) simplifying to* $\sum_i (2 + 3\beta_i)^2 = 1$. *However, this implies that for at least one* i *we have* $2 + 3\beta_i = 0$, *i.e. no integer solution.*

4.2 Quantum Cohomology of Del Pezzo Surfaces

Now in order to compute the quantum cohomology of X_r, the Del Pezzo surface of degree $9 - r$, we would have to make sense of the term

$$\sum_i \sum_{\beta \in H_2(X_r, \mathbb{Z})} \langle I_{0,3,\beta} \rangle (a, b, T_i) q^\beta T^i \tag{8}$$

for any $a, b \in H^*(X_r, \mathbb{C})$. So let us start by recalling the cohomology of $X_r : H^0(X_r, \mathbb{C}) = \mathbb{C}[X_r]$ and $H^4(X_r, \mathbb{C}) = \mathbb{C}[pt]$ are both one-dimensional and generated by the (Poincaré duals of the) fundamental class, respectively the class $[pt]$ of a point. Moreover, we know that $H^2(X_r, \mathbb{C}) = \mathrm{Pic}(X_r) \otimes \mathbb{C}$. So we can choose $H, E_1, \ldots, E_r$ (in the notation of the previous chapter) to be the generators of the second cohomology group. Since there is no odd-dimensional cohomology in our case, we have the following choice of basis of $H^*(X_r, \mathbb{C}) : T_0 = [X_r]$, $T_1 = H$, $T_i = E_{i-1}$ for $2 \leq i \leq r + 1$ and $T_{r+2} = [pt]$.

In practical terms the most difficult part of the expression (8) to make sense of, are the Gromov-Witten invariants $\langle I_{0,3,\beta} \rangle$. Luckily, [GP98] shows that we can simplify our situation by falling back on our intuition. It is shown there that in our case, the Del Pezzo case, all genus 0 Gromov-Witten invariants are enumerative in the following sense: Let $\beta = dH - \sum_{i=1}^r m_i E_i$ be a class in $H_2(X_r, \mathbb{Z})$.[10] In order to have a non-zero Gromov-Witten invariant $\langle I_{0,n,\beta} \rangle$ we need $n = 3d - \sum_{i=1}^r m_i - 1$. So suppose that is the case. Then $\langle I_{0,n,\beta} \rangle$ is an actual count of the irreducible, rational curves in X_r of class β, passing through n points in general position, which, moreover, is the same number as the irreducible, degree d rational plane curves in $\mathbb{P}^2$, which pass through each blown-up point p_i with multiplicity m_i.

So let $\beta \in H_2(X_r, \mathbb{Z})$ be a class, such that the corresponding complete linear system $|\beta|$ is non-empty and does not have any fixed components. We define $\mathcal{R}_\beta \subset |\beta|$ to be the locus of such curves in the system, which have at worst nodal singularities. We are now interested in the locus

$$\mathcal{S}_\beta := \left\{ (C, x_1, x_2, x_3) \in \mathcal{R}_\beta \times X_r^3 \;\middle|\; [C] = \beta \text{ and } x_i \in C \text{ are distinct, smooth} \right\}.$$

[10] By Poincaré duality we have a natural isomorphism $H_2(X_r, \mathbb{Z}) \simeq H^2(X_r, \mathbb{Z})$.

To be precise, we are interested in its closure $\overline{\mathcal{S}}_\beta \subset |\beta| \times X_r^3$. Because, denoting by π_i the respective projection $\overline{\mathcal{S}}_\beta \to X_r$, we can now write

$$\begin{aligned}\langle I_{0,3,\beta}\rangle(a,b,c) &= \int_{\left[\overline{M}_{0,3}(X_r,\beta)\right]^{\mathrm{vir}}} \mathrm{ev}_1^*(a) \cup \mathrm{ev}_2^*(b) \cup \mathrm{ev}_3^*(c) \\ &= \int_{\overline{\mathcal{S}}_\beta} \pi_1^*(a) \cup \pi_2^*(b) \cup \pi_3^*(c).\end{aligned}$$

Now let $\pi : \overline{\mathcal{S}}_\beta \to X_r^3$ be the obvious projection. Then we can simplify the previous expression further:

$$\begin{aligned}\langle I_{0,3,\beta}\rangle(a,b,c) &= \int_{\overline{\mathcal{S}}_\beta} \overline{\mathcal{S}}_\beta \cup \pi_1^*(a) \cup \pi_2^*(b) \cup \pi_3^*(c) \\ &= \int_{X_r^3} \pi_*(\overline{\mathcal{S}}_\beta) \cup (a \otimes b \otimes c). \end{aligned} \tag{9}$$

Now suppose that $a \in H^{2x}(X_r, \mathbb{C})$ and $b \in H^{2y}(X_r, \mathbb{C})$. Furthermore suppose that $\pi_*(\overline{\mathcal{S}}_\beta) \in H^{2w}(X_r^3, \mathbb{C})$. Provided that $4 \leq x + y + w \leq 6$, we find that the expression (9) defines a linear functional

$$\begin{aligned}H^{12-2x-2y-2w}&(X_r, \mathbb{C}) \to \mathbb{C} \\ &c \mapsto [\pi_*(\overline{\mathcal{S}}_\beta)] \cdot (a \otimes b \otimes c),\end{aligned}$$

where the intersection product is the one from X_r^3. By duality, this functional must be represented as the intersection with a cohomology class in $H^{2x+2y+2w-8}(X_r, \mathbb{C})$. Let us call this cohomology class $\varphi_\beta(a,b)$.

As in chapter 3.1, let T^i for $0 \leq i \leq r+2$ denote the dual basis to our basis of T_i's, with (g_{ij}) being the intersection matrix. Then

$$\begin{aligned}\sum_{i=0}^{r+2} \sum_{\beta \in H_2(X_r,\mathbb{Z})} \langle I_{0,3,\beta}\rangle(a,b,T_i) q^\beta T^i &= \sum_{\beta \in H_2(X_r,\mathbb{Z})} q^\beta \sum_{i=0}^{r+2} (\varphi_\beta(a,b) \cdot (T_i)) T^i \\ &= \sum_{\beta \in H_2(X_r,\mathbb{Z})} q^\beta \sum_{i=0}^{r+2} (\sum_{j=0}^{r+2} x_j T_j \cdot T_i) T^i \\ &= \sum_{\beta \in H_2(X_r,\mathbb{Z})} q^\beta \sum_{i,j=0}^{r+2} x_j g_{ij} T^i \\ &= \sum_{\beta \in H_2(X_r,\mathbb{Z})} q^\beta \sum_{j=0}^{r+2} x_j T_j \\ &= \sum_{\beta \in H_2(X_r,\mathbb{Z})} \varphi_\beta(a,b) q^\beta,\end{aligned}$$

giving us the very neat expression for the quantum product in X_r :

$$a * b = \sum_{\beta \in H_2(X_r,\mathbb{Z})} \varphi_\beta(a,b) q^\beta.$$

4.3 Quantum Cohomology of X_4

As an example let us calculate concretely the quantum cohomology of X_4. We have already seen a basis for the cohomology of X_4 in the previous chapter, which was

$$\Big\{[X_4], H, E_1, E_2, E_3, E_4, [pt]\Big\}.$$

To find a concrete formula for the quantum product, we now need to do two things: firstly there is the issue of understanding our simplified function $\varphi_\beta(a, b)$ and secondly, we should find out, which $\beta \in H_2(X_4, \mathbb{Z})$ are actually relevant, i.e. non-zero, in our sum.

Let us start with a simple example: suppose that $\pi_*(\overline{\mathcal{S}}_\beta)$ is the fundamental class of X_4^3. This happens when $\dim(\overline{\mathcal{S}}_\beta) > 6$, i.e. when our locus $\mathcal{R}_\beta$ has a dimension larger than three. Then

$$\pi_*(\overline{\mathcal{S}}_\beta) = [X_4] \otimes [X_4] \otimes [X_4] \in H^0(X_4^3, \mathbb{C}),$$

where we identified $H^*(X_4^3, \mathbb{C})$ with tensor products of $H^*(X_4, \mathbb{C})$ via the Künneth theorem. Now we have two conditions on (homogeneous) elements $a, b \in H^*(X_4, \mathbb{C})$, which ensure that $\varphi_\beta(a, b) \in H^*(X_4, \mathbb{C})$. Say (as before) they are of degree $2x$ and $2y$ respectively. By definition, for $c \in H^{2z}(X_4, \mathbb{C})$ we have

$$[\pi_*(\overline{\mathcal{S}}_\beta)] \cdot (a \otimes b \otimes c) = \varphi_\beta(a, b) \cdot c,$$

so that "intersecting with $\varphi_\beta(a, b)$" is a linear map from $H^{2z}(X_4, \mathbb{C})$ to $\mathbb{C}$. In order for the product to make sense on the left-hand side we need $z = 6 - x - y$, whilst on the right-hand we require $4 \leq x + y \leq 6$. The inequalities are due to the fact that we need $\varphi_\beta(a, b) \in H^{4-2z}(X_4, \mathbb{C})$, i.e. $0 \leq 4 - 2z \leq 4$. Therefore the only possibility is $x = y = z = 2$, i.e. a and b are multiples of the class of a point $[pt]$. By linearity it suffices to calculate

$$[\pi_*(\overline{\mathcal{S}}_\beta)] \cdot ([pt] \otimes [pt] \otimes [pt]) = 1$$

and therefore $\varphi_\beta([pt], [pt]) = [X_4]$ is the only non-trivial φ_β.

Remark 4.5. *By the linearity of the quantum product, we only need to know the products of our basis elements. But not even all of those are needed for a complete description of the quantum cohomology ring. Take for instance the basis element* $[pt]$*. We know that for two general elements* $a, b \in H^*(X, \mathbb{C})$*, the quantum product has the form* $a * b = a \cup b + ...$*, where the dots stand for some other terms, all of which involve the quantum variables* q*. In our example it turns out that* $H * H = [pt] + \ldots$*, where the quantum terms are independent of* $[pt]$*. Thus we can express* $[pt]$ *in terms of* $H * H$ *and the quantum variables. So again due to linearity, we do not need to include* $[pt]$ *in an eventual quantum multiplication table of our basis.*

The previous example provides already a strong restriction on which $\beta \in H_2(X_4, \mathbb{Z})$ are relevant for the quantum cohomology. We need $\dim(\mathcal{S}(\beta)) \leq 6$. Since also clearly $\dim(\mathcal{S}_\beta) = \dim(\mathcal{R}_\beta) + 3$ and $\mathcal{R}_\beta \neq \emptyset$, we need

$$3 \leq \dim(\mathcal{S}_\beta) \leq 6.$$

So how do we find $\dim(\mathcal{S}(\beta))$? We start by finding $\dim(|\beta|)$ for the complete linear system of divisors equivalent to β. Let us assume that $|\beta|$ has no fixed components and pick a general member D_β of $|\beta|$, i.e. $[D_\beta] = \beta \in H^2(X, \mathbb{Z})$. Since X_4 is rational, D_β is smooth and the dimension of the complete linear system is $\dim(|\beta|) = h^0(X_4, \mathcal{O}(D_\beta)) - 1$, where $\mathcal{O}(D_\beta)$ is the line bundle associated to the divisor β. [Har85] computes this dimension explicitly,[11] showing that

$$\dim(|\beta|) = \frac{D_\beta \cdot D_\beta - D_\beta \cdot K_{X_4}}{2} = D_\beta \cdot D_\beta + 1 - p_a(D_\beta).$$

Here, $p_a(D_\beta)$ is the arithmetic genus of D_β, which, by the Riemann-Roch theorem, can be calculated as $p_a(D_\beta) = \frac{1}{2}(2 + D_\beta \cdot D_\beta + D_\beta \cdot K_{X_4})$, where once again K_{X_4} denotes the canonical divisor of X_4. Elements of $\mathcal{R}_\beta$ are, a priori, not smooth, but have nodes. Imposing one such node is one condition, i.e. each node reduces the dimension by 1. This implies that the locus $\mathcal{R}_\beta$ has dimension $\dim(\mathcal{R}_\beta) = \dim(|\beta|) - p_a(\beta)$. Putting all of this together we get

$$\dim(\mathcal{S}_\beta) = 2 - D_\beta \cdot K_{X_4}.$$

Coming back to our special case, let $\beta = dH - \sum_{i=1}^4 m_i E_i$ and recall that $K_{X_4} = -3H + \sum_{i=1}^4 E_i$. Then

$$\dim(\mathcal{S}_\beta) = 3d + 2 - \sum_{i=1}^4 m_i.$$

Remark 4.6. *In fact, this result was to be expected, since we know that $\overline{\mathcal{S}}_\beta$ takes (in our example) on the role of $\overline{M}_{0,3}(X_4, \beta)$, for which we have the dimension formula:*

$$\begin{aligned} \operatorname{vir}\dim(\overline{M}_{0,3}(X_4, \beta)) &= (1-g)(\dim(X) - 3) - \int_\beta \omega_{X_4} + n \\ &= (1-0)(2-3) - D_\beta \cdot \omega_{X_4} + 3. \end{aligned}$$

So far we have found one necessary condition on β to be relevant for the quantum cohomology of X_4. Now also keep in mind that D_β has at worst nodal singularities, which means in particular that we have $p_a(D_\beta) \geq 0$. Therefore by Riemann-Roch:

$$\begin{aligned} 2p_a(D_\beta) &= 2 + \left(dH - \sum_{i=0}^4 m_i E_i\right) \cdot \left(dH - \sum_{i=0}^4 m_i E_i\right) + \left(dH - \sum_{i=0}^4 m_i E_i\right) \cdot \left(-3H + \sum_{i=0}^4 E_i\right) \\ &= 2 + d^2 - \sum_{i=0}^4 m_i^2 - 3d + \sum_{i=0}^4 m_i. \end{aligned}$$

Combining the bounds found so far, we end up with two inequalities, which d and m_i have to satisfy:

$$3d - 4 \leq m_1 + m_2 + m_3 + m_4 \leq 3d - 1, \tag{10}$$

since $3 \leq \dim(\mathcal{S}_\beta) \leq 6$ and

$$(m_1^2 - m_1) + (m_2^2 - m_2) + (m_3^2 - m_3) + (m_4^2 - m_4) \leq d^2 - 3d + 2, \tag{11}$$

since $p_a(D_\beta) \geq 0$.

[11] By the effectivity axiom we can assume that β is effective. Then corollary 2.4 in [Har85] shows that β is almost excellent since it has no fixed components. Now the fact that $-K_{X_4}$ (the anti-canonical divisor) is ample implies that β is excellent, whence the result follows from theorem 1.1 in [Har85].

Moreover, due to the effectivity axiom, we know a further condition on β : it has to be in the closure of the *cone of (effective) curves* $\overline{\mathrm{NE}}(X_4)$. This leaves us with the task of finding $\overline{\mathrm{NE}}(X_4)$. Fortunately, two strong theorems can help us with that, both of which are (in general form) explained and proven in [Deb01]. The first one is the *cone theorem*. In our case it states that there is a countable family of rational curves $(\Gamma_i)_{i\in I}$ on X_4, such that $0 < -\Gamma_i \cdot K_{X_4} \leq 3$ and

$$\overline{\mathrm{NE}}(X_4) = \left\{\beta \in \overline{\mathrm{NE}}(X_4) \;\middle|\; \beta \cdot K_{X_4} \geq 0\right\} + \sum_{i\in I} \mathbb{R}^+(\Gamma_i).$$

Observe here that the condition on the Γ_i, together with adjunction, tell us that the Γ_i are (-1)-curves on X_4. Moreover we find that

$$\left\{\beta \in \overline{\mathrm{NE}}(X_4) \;\middle|\; \beta \cdot K_{X_4} \geq 0\right\} = \emptyset, \tag{12}$$

due to the second strong theorem, called *Kleiman's criterion*: it states that D_β is an ample divisor on X_4, if and only if $D_\beta \cdot z > 0$ for all non-zero $z \in \overline{\mathrm{NE}}(X_4)$. But we know that $-K_{X_4}$ is ample, and since the ample cone is contained in $\overline{\mathrm{NE}}(X_4)$, $K_{X_4} \cdot z < 0$ for all non-zero z in the closure of the cone of curves, implying (12). In other words, if D_β is effective, then D_β is contained in the cone generated by positive multiples of the (-1)-curves of X_4, all of which we have found in example 4.4.

In practice, we shall take away mainly one observation from our computation of the cone of curves of X_4 : since it holds for all (-1)-curves, we have that

$$dH - \sum_{i=1}^{4} m_i E_i \in \overline{\mathrm{NE}}(X_4) \Rightarrow d \geq 0.$$

Now let us return to the inequalities (10) and (11). Since $m_i^2 - m_i \geq 0$ for all integers $m_i \in \mathbb{Z}$, we see straight away that given $d \geq 0$, there are only finitely many relevant curves for us. Now assume that $\sum_{i=0}^{4}(m_i^2 - m_i) = N \in \mathbb{N}$. Which $d \geq 0$ satisfy the inequalities (10) and (11)? Let us maximise the sum $m_1 + m_2 + m_3 + m_4$ under the given assumption. Reversing our view and setting $x_i = m_i^2 - m_i$, we want to maximise the expression

$$\sqrt{x_1 + \frac{1}{4}} + \sqrt{x_2 + \frac{1}{4}} + \sqrt{x_3 + \frac{1}{4}} + \sqrt{x_4 + \frac{1}{4}}.$$

Viewing this as as real function $(\mathbb{R}_{\geq 0})^4 \to \mathbb{R}$ under the additional constraint that $x_1 + x_2 + x_3 + x_4 = N \geq 0$, we can easily find that the expression reaches a maximum when $x_i = \frac{N}{4}$ for all i. Therefore we have a maximum of $m_1 + \cdots + m_4$, when $m_i = \sqrt{\frac{N}{4} + \frac{1}{4}}$ and in this case:

$$\begin{aligned} m_1 + m_2 + m_3 + m_4 &\leq 2 + 2\sqrt{N+1} \\ &\leq 2 + 2\sqrt{d^2 - 3d + 3} \\ &= 2 + \sqrt{3 + (2d-3)^2} \\ &< 2d - 1 < 3d - 4 \text{ for large } d. \end{aligned}$$

Therefore, in our case, we only need to check $d \in \{0, 1, 2, 3\}$. This is easily done and we can solve the inequalities (10) and (11), aided by the facts that firstly in all cases $d^2 - 3d + 2 \in \{0, 2\}$

and that secondly $m_i^2 - m_i \leq 2$ has its only integer solutions at $m_i^2 - m_i = 0 (\Rightarrow m_i \in \{0, 1\})$ and $m_i^2 - m_i = 2 (\Rightarrow m_i \in \{-1, 2\})$. Therefore we can write the quantum product of X_4 in the form

$$a * b = a \cup b + \sum_{\beta \in I} \varphi_\beta(a, b) q^\beta,$$

where

$$\begin{aligned}
I = &\{E_i\} \\
&\cup \{H\} \cup \{H - E_i\} \cup \{H - E_i - E_j\} \\
&\cup \{2H - E_i - E_j\} \cup \{2H - E_i - E_j - E_k\} \cup \{2H - E_1 - E_2 - E_3 - E_4\} \\
&\cup \{3H - 2E_i - E_j - E_k - E_l\}
\end{aligned}$$

for every $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

All we are left with now, is to understand the φ_β in those cases. In order to do this, we shall distinguish them by the self-intersection number of a general member D_β:

$D_\beta \cdot D_\beta = -1$ The exceptional divisors of X_4 are particularly easy to handle, since they are the only member of their respective complete linear system. In other words, let $\beta = [E]$ for some (-1)-curve $E \subset X_4$ Then we have $|E| = \{E\}$. Hence our locus $\mathcal{S}_\beta = \{E\} \times E \times E \times E \setminus \Delta$, where Δ is the large diagonal. In taking the closure of this we simply add Δ and find that $\overline{\mathcal{S}}_\beta = \{E\} \times E \times E \times E$ and therefore $\pi_*(\overline{\mathcal{S}}_\beta) = [E] \otimes [E] \otimes [E] \in H^6(X_4^3)$. Now say $a, b, c \in H^*(X_4, \mathbb{C})$ are of degree x, y, z respectively. Then we have

$$\pi_*(\overline{\mathcal{S}}_\beta) \cdot (a \otimes b \otimes c) = (D_\beta \cdot a)(D_\beta \cdot b)(D_\beta \cdot c),$$

which is only defined and non-zero, if a, b and c are all divisors on X_4. Therefore we have

$$\varphi_\beta(a, b) = \begin{cases} (D_\beta \cdot a)(D_\beta \cdot b)\beta & \text{if } a, b \in H^2(X_4, \mathbb{C}) \\ 0 & \text{otherwise.} \end{cases}$$

$D_\beta \cdot D_\beta = 0$ In this case the linear system $|\beta|$ is one-dimensional. $\mathcal{R}_\beta$ is the locus of smooth members of this pencil, since in all our cases $p_a(D_\beta) = 0$. Hence it is an open and dense subset of $|\beta|$ and $\dim(\mathcal{S}_\beta) = 4$. So, pushing forward, we have $\pi_*(\overline{\mathcal{S}}_\beta) \in H^4(X_4^3, \mathbb{C})$. Complementary classes are therefore of complex codimension four, i.e. they are in the linear span of $[pt] \otimes [pt] \otimes [X_4]$ and $a \otimes b \otimes [pt]$ (for divisor classes a, b) and all similar associated classes in $H^8(X_4^3, \mathbb{C})$, which can be obtained by symmetry.

However, $\pi_*(\overline{\mathcal{S}}_\beta) \cdot ([pt] \otimes [pt] \otimes [X_4]) = 0$ for the following reason: $D_\beta \cdot D_\beta = 0$ means that β defines a ruling of our surface X_4 and that members of the class β are fibres of this ruling. The intersection before answers mathematically the question of how many members of β go through two generic points of X_4. But two generic points will not lie on the same fibre of this ruling, meaning there is no such member of β.

Now consider the intersection product $\pi_*(\overline{\mathcal{S}}_\beta) \cdot ([a] \otimes [b] \otimes [pt])$. The last factor forces the curve to go through a point. Since β defines a ruling of X_4 this defines a unique member D_β of β. This member then intersects a and b in $D_\beta \cdot a$, resp. $D_\beta \cdot b$ points. Hence we have

$\pi_*(\overline{\mathcal{S}}_\beta) \cdot ([a] \otimes [b] \otimes [pt]) = (D_\beta \cdot a)(D_\beta \cdot b)$. Moreover, we can use the symmetry to see that $\pi_*(\overline{\mathcal{S}}_\beta) \cdot ([a] \otimes [pt] \otimes [b]) = (D_\beta \cdot a)(D_\beta \cdot b)$. Thus

$$\varphi_\beta(a,b) = \begin{cases} (D_\beta \cdot a)(D_\beta \cdot b)[X_4] & \text{if } a, b \in H^2(X_4, \mathbb{C}) \\ (D_\beta \cdot a)\beta & \text{if } a \in H^2(X_4, \mathbb{C}) \text{ and } b = [pt] \\ (D_\beta \cdot b)\beta & \text{if } b \in H^2(X_4, \mathbb{C}) \text{ and } a = [pt] \\ 0 & \text{otherwise.} \end{cases}$$

$D_\beta \cdot D_\beta = 1$ We argue similar to the previous case: we know that $\dim(|\beta|) = 2$, and since we only have cases with $p_a(D_\beta) = 0$, $\mathcal{R}_\beta$ is an open dense subset of $|\beta|$. Therefore $\dim(\mathcal{S}_\beta) = 5$ and the complementary classes to $\pi_*(\overline{\mathcal{S}}_\beta)$ in X_4^3 are of complex codimension five, i.e. in $H^{10}(X_4^3, \mathbb{C})$. This vector space is generated by the classes $a \otimes [pt] \otimes [pt]$ and its associates by symmetry. Since the dimension of $|\beta|$ has gone up by one, we now have a unique member D_β of β going through any two general points. This member will intersect a in $D_\beta \cdot a$ points, implying that $\pi_*(\overline{\mathcal{S}}_\beta) \cdot (a \otimes [pt] \otimes [pt]) = D_\beta \cdot a$. Now once again use the symmetry to find that

$$\varphi_\beta(a,b) = \begin{cases} (D_\beta \cdot a)[X_4] & \text{if } a \in H^2(X_4, \mathbb{C}) \text{ and } b = [pt] \\ (D_\beta \cdot b)[X_4] & \text{if } b \in H^2(X_4, \mathbb{C}) \text{ and } a = [pt] \\ \beta & \text{if } a = b = [pt] \\ 0 & \text{otherwise.} \end{cases}$$

$D_\beta \cdot D_\beta = 2$ In this case we have $\dim(|\beta|) = 3$ and since $p_a(D_\beta) = 0$ we have $\mathcal{R}_\beta$ as a dense open subset of $|\beta|$. Now $\dim(\mathcal{S}_\beta) = 6$ and we know that there is a unique member of $|\beta|$ intersecting three general points of X_4. We conclude that $\pi_*(\overline{\mathcal{S}}_\beta) = X_4^3$ and that these cases are not relevant for us after all.

We are now in a position to explicitly compute the quantum product of our basis elements. Recall that due to remark 4.5 and the fact that $[X_4]$ is the identity for the quantum product, we shall restrict ourselves to products of basis elements in $H^2(X_4, \mathbb{Z})$.

$$\begin{aligned} H * H = {} & [pt] + (H - E_1 - E_2)q^{H-E_1-E_2} + (H - E_1 - E_3)q^{H-E_1-E_3} \\ & + (H - E_1 - E_4)q^{H-E_1-E_4} + (H - E_2 - E_3)q^{H-E_2-E_3} \\ & + (H - E_2 - E_4)q^{H-E_2-E_4} + (H - E_3 - E_4)q^{H-E_3-E_4} \\ & + q^{H-E_1} + q^{H-E_2} + q^{H-E_3} + q^{H-E_4} + 4q^{2H-E_1-E_2-E_3-E_4} \\ H * E_1 = {} & (H - E_1 - E_2)q^{H-E_1-E_2} + (H - E_1 - E_3)q^{H-E_1-E_3} \\ & + (H - E_1 - E_4)q^{H-E_1-E_4} + q^{H-E_1} + 2q^{2H-E_1-E_2-E_3-E_4} \\ H * E_2 = {} & (H - E_1 - E_2)q^{H-E_1-E_2} + (H - E_2 - E_3)q^{H-E_2-E_3} \\ & + (H - E_2 - E_4)q^{H-E_2-E_4} + q^{H-E_2} + 2q^{2H-E_1-E_2-E_3-E_4} \\ H * E_3 = {} & (H - E_1 - E_3)q^{H-E_1-E_3} + (H - E_2 - E_3)q^{H-E_2-E_3} \\ & + (H - E_3 - E_4)q^{H-E_3-E_4} + q^{H-E_3} + 2q^{2H-E_1-E_2-E_3-E_4} \\ H * E_4 = {} & (H - E_1 - E_4)q^{H-E_1-E_4} + (H - E_2 - E_4)q^{H-E_2-E_4} \\ & + (H - E_3 - E_4)q^{H-E_3-E_4} + q^{H-E_4} + 2q^{2H-E_1-E_2-E_3-E_4} \end{aligned}$$

$$
\begin{aligned}
E_1 * E_1 = & -[pt] + E_1 q^{E_1} + (H - E_1 - E_2) q^{H-E_1-E_2} + (H - E_1 - E_3) q^{H-E_1-E_3} \\
& + (H - E_1 - E_4) q^{H-E_1-E_4} + q^{H-E_1} + q^{2H-E_1-E_2-E_3-E_4} \\
E_1 * E_2 = & (H - E_1 - E_2) q^{H-E_1-E_2} + q^{2H-E_1-E_2-E_3-E_4} \\
E_1 * E_3 = & (H - E_1 - E_3) q^{H-E_1-E_3} + q^{2H-E_1-E_2-E_3-E_4} \\
E_1 * E_4 = & (H - E_1 - E_4) q^{H-E_1-E_4} + q^{2H-E_1-E_2-E_3-E_4} \\
E_2 * E_2 = & -[pt] + E_2 q^{E_2} + (H - E_1 - E_2) q^{H-E_1-E_2} + (H - E_2 - E_3) q^{H-E_2-E_3} \\
& + (H - E_2 - E_4) q^{H-E_2-E_4} + q^{H-E_2} + q^{2H-E_1-E_2-E_3-E_4} \\
E_2 * E_3 = & (H - E_2 - E_3) q^{H-E_2-E_3} + q^{2H-E_1-E_2-E_3-E_4} \\
E_2 * E_4 = & (H - E_2 - E_4) q^{H-E_2-E_4} + q^{2H-E_1-E_2-E_3-E_4} \\
E_3 * E_3 = & -[pt] + E_3 q^{E_3} + (H - E_1 - E_3) q^{H-E_1-E_3} + (H - E_2 - E_3) q^{H-E_2-E_3} \\
& + (H - E_3 - E_4) q^{H-E_3-E_4} + q^{H-E_3} + q^{2H-E_1-E_2-E_3-E_4} \\
E_3 * E_4 = & (H - E_3 - E_4) q^{H-E_3-E_4} + q^{2H-E_1-E_2-E_3-E_4} \\
E_4 * E_4 = & -[pt] + E_4 q^{E_4} + (H - E_1 - E_4) q^{H-E_1-E_4} + (H - E_2 - E_4) q^{H-E_2-E_4} \\
& + (H - E_3 - E_4) q^{H-E_3-E_4} + q^{H-E_4} + q^{2H-E_1-E_2-E_3-E_4}
\end{aligned}
$$

5 The Dubrovin Connection

In this chapter we shall explore the main geometric consequence of the quantum product, called the *Dubrovin connection*. This is in fact a family of connections on the trivial bundle $H^*(X, \mathbb{C}) \times H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C})$, where the quantum product on the fibre over every point depends on the base point. Afterwards we shall find an explicit expression for the small Dubrovin connection in our example of X_4.

5.1 Definition and Properties

Once again let us recall the notation from previous chapters: we have a smooth projective variety X and a basis $\{T_0 = [X], T_1, \ldots, T_m\}$ of the cohomology ring $H^*(X, \mathbb{C})$. Denoting by $\{T^i\}_{i=0}^m$ the dual basis, i.e. $\int_X T^i \cup T_j = \delta_{ij}$, we have furthermore the quantum product $\star$ on $H^*(X, \mathbb{C})$, which is defined by

$$
T_i \star T_j = \sum_{k=0}^{m} \partial_{t_i} \partial_{t_j} \partial_{t_k} (\Phi) T^k
$$

for our basis and extended linearly. Here Φ is the Gromov-Witten potential as in definition 3.9.

We have previously regarded the factors $\mathbf{q}^\beta$ as formal variables in the Novikov ring. Let us now assume that the Gromov-Witten potential converges, in which case we will set $\mathbf{q}^\beta \equiv 1$ (compare remark 3.10). Then we can view the quantum product as a symmetric, bilinear operation

$$
\star : H^*(X, \mathbb{C}) \times H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C}).
$$

This is important when we consider the vector bundle $H^*(X, \mathbb{C}) \times H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C})$. Since

$H^*(X,\mathbb{C})$ is a finite-dimensional vector space, we can find a global trivialisation

$$\begin{aligned} \mathcal{T}_{H^*(X,\mathbb{C})} &\to H^*(X,\mathbb{C}) \times H^*(X,\mathbb{C}) \\ (\partial_{t_i}, \sum_{j=0}^{m} t_j T_j) &\mapsto (T_i, \sum_{j=0}^{m} t_j T_j), \end{aligned}$$

where we denoted by t_i the coordinates with respect to T_i. We can use this trivialisation to define the family of connections, collectively known as the Dubrovin connection by defining it on the global frame $\{\partial_{t_0}, \ldots, \partial_{t_m}\}$:

$$\nabla^\tau_{\partial_{t_i}}(\partial_{t_j}) := \tau(T_i \star T_j)$$

for any $\tau \in \mathbb{C}^*$.

Proposition 5.1.

- *The connections ∇^τ are torsion-free and this property is equivalent to $\star$ being commutative.*
- *The connections ∇^τ are flat and this property is equivalent to $\star$ being associative.*

Proof: For two vector fields $V, W \in \Gamma(\mathcal{T}_{H^*(X,\mathbb{C})})$, the torsion tensor of a connection ∇ is given by

$$\mathbf{T}(V,W) = \nabla_V(W) - \nabla_W(V) - [V,W].$$

Here $[V,W]$ denotes the Lie bracket of two vector fields. It suffices to compute the torsion on our global frame $\{\partial_{t_i}\}$:

$$\begin{aligned} \mathbf{T}^\tau(\partial_{t_i}, \partial_{t_j}) &= \nabla^\tau_{\partial_{t_i}}(\partial_{t_j}) - \nabla^\tau_{\partial_{t_j}}(\partial_{t_i}) - [\partial_{t_i}, \partial_{t_j}] \\ &= \tau(T_i \star T_j - T_j \star T_i). \end{aligned}$$

Similarly we have the curvature tensor for $U, V, W \in \Gamma(\mathcal{T}_{H^*(X,\mathbb{C})})$ given by

$$\mathbf{R}(U,V)(W) = \nabla_U \nabla_V(W) - \nabla_V \nabla_U(W) - \nabla_{[U,V]}(W).$$

Once again it suffices to compute this on $\{\partial_{t_i}\}$:

$$\begin{aligned} \mathbf{R}^\tau(\partial_{t_i}, \partial_{t_j})(\partial_{t_k}) &= \nabla^\tau_{\partial_{t_i}} \nabla^\tau_{\partial_{t_j}}(\partial_{t_k}) - \nabla^\tau_{\partial_{t_j}} \nabla^\tau_{\partial_{t_i}}(\partial_{t_k}) - \nabla^\tau_{[\partial_{t_i}, \partial_{t_j}]}(\partial_{t_k}) \\ &= \tau^2(T_i \star (T_j \star T_k)) - \tau^2(T_j \star (T_i \star T_k)) - 0 \\ &= \tau^2(T_i \star (T_k \star T_j) - (T_i \star T_k) \star T_j). \end{aligned}$$

□

Before we restrict ourselves to the small Dubrovin connection we should discuss two more manipulations/ extensions of ∇^τ : firstly, we would like to extend the range of τ from $\mathbb{C}^*$ to $\mathbb{P}^1$. Practically this is a formal step, where we introduce singularities at 0 and $\infty \in \mathbb{P}^1$. For the moment we shall not concern ourselves overly with this issue and will assume that τ is an affine coordinate on $\mathbb{C} \subset \mathbb{P}^1$. However, we will eventually pick up this trail and also consider the coordinate $z = \tau^{-1}$ on the other copy of $\mathbb{C} \subset \mathbb{P}^1$, when we discuss the nature of the singularities introduced.

Let us denote by π the projection:

$$\pi : \mathbb{P}^1 \times H^*(X,\mathbb{C}) \to H^*(X,\mathbb{C}).$$

Our second extension of ∇^τ, and the one we will be mostly concerned with, is the combination of all ∇^τ into a single connection $\widehat{\nabla}$ on the bundle $\widehat{\mathcal{T}} := \pi^*(\mathcal{T}_{H^*(X,\mathbb{C})})$. This is equivalent to defining $\widehat{\nabla}_{\partial_\tau}$, which we will do in such a way that $\widehat{\nabla}$ will remain flat. It will be a meromorphic connection with singularities at $0, \infty \in \mathbb{P}^1$.

For our definition of $\widehat{\nabla}_{\partial_\tau}$ we will closely follow the arguments of [KM94]. We begin our discussion with a short recap of the so-called *Euler vector field*. Recall our definition of a function $f : \mathbb{C}^n \to \mathbb{C}$ being quasi-homogeneous. We demanded that for all $\lambda \in \mathbb{C}$ and some collection $a_i, a \in \mathbb{Z}$ we have

$$f(\lambda^{a_1} x_1, \ldots, \lambda^{a_n} x_n) = \lambda^a f(x_1, \ldots, x_n). \tag{13}$$

Deriving both sides of (13) by the x_i and by λ, we find that f satisfies the following differential equation, independent of λ:

$$af = \sum_{i=1}^{n} a_i x_i \partial_{x_i}(f). \tag{14}$$

So let us define the differential operator $E = \sum_i a_i x_i \partial_{x_i}$. Then equation (14) is saying that f satisfies the equation $Ef = af$, i.e. that f is an eigenfunction of E. Note that the converse holds too: let $f : \mathbb{C}^n \to \mathbb{C}$ be an eigenfunction of E with eigenvalue a. Now let $g(\lambda) = f(\lambda^{a_1} x_1, \ldots, \lambda^{a_n} x_n)$ for some fixed $x_1, \ldots, x_n \in \mathbb{C}$. Then $Ef = af$ implies that $ag = \lambda \partial_\lambda(g)$ and thus that $g(\lambda) = c\lambda^a$ for some constant c. Now c is dependent on the x_i, but since $g(1) = f(x_1, \ldots, x_n)$ we find that in fact f satisfies equation (13), i.e. f is quasi-homogeneous. We consider E to be a vector field on $\mathbb{C}^n$ and call it the Euler vector field, since for $a_1 = \cdots = a_n$ we recover the classical notion of Euler vector field

$$x_1 \partial_{x_1} + \cdots + x_n \partial_{x_n}. \tag{15}$$

We now want to define an Euler vector field on our trivial bundle

$$H^*(X,\mathbb{C}) \times H^*(X,\mathbb{C}) \to H^*(X,\mathbb{C}).$$

In order to do so let us switch our point of view from vector fields or differential operators to local flows. The classical definition 15 produces the radial vector field, i.e. a flow in the direction of the scalar multiplication. For us, the vector space structure will be less important. Rather we want to define the Euler vector field taking into account our multiplication $\star$ on the fibres of the bundle, i.e. a flow "in the direction of Φ".

Chapter 3.4 and in particular proposition 3.16 suggest we should be considering a flow of the form

$$\begin{aligned} t_i &\mapsto e^{(2-|T_i|)t} t_i, \\ \omega &\mapsto \omega - 2\omega_X t, \\ \Phi &\mapsto e^{(6-2\dim(X))t} \Phi. \end{aligned} \tag{16}$$

However, we can do better: In [KM94], the authors proved the following helpful proposition:

Proposition 5.2. *Express $\gamma = \gamma_0 + \gamma_2 + \gamma'$, where γ_i has degree i and γ' are the summands of all other degrees. Then:*

- $\Phi_\omega(\gamma) = \Phi_\omega(\gamma_2 + \gamma') +$ *"something quadratic in $\gamma_2 + \gamma'$",*
- $\Phi_\omega(\gamma_2 + \gamma') = \Phi_{\omega - \gamma_2}(\gamma') +$ *"something quadratic in γ and γ'".*

Proof: See [KM94, proposition 4.4]. □

Practically this means for us that we could replace the flow (16) by

$$\begin{aligned} &\omega \mapsto \omega, \\ &t_i \mapsto \begin{cases} e^{(2-|T_i|)t} t_i & \text{for } |T_i| \neq 2, \\ t_i + 2\xi_i t & \text{for } |T_i| = 2, \end{cases} \end{aligned}$$

where the ξ_i are defined by

$$-\omega_X = \sum_{i=1}^{r} \xi_i T_i,$$

assuming (as above) that $H^2(X, \mathbb{C})$ is spanned by $\{T_1, \ldots, T_r\}$. Φ however, does not have a uniform expression in this flow. Rather we find due to the chain rule that

$$\begin{aligned} \partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi) \mapsto & e^{((6-2\dim(X))-(2-|T_i|)-(2-|T_j|)-(2-|T_k|))t}\partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi) \\ &= e^{(|T_i|+|T_j|+|T_k|-2\dim(X))t}\partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi). \end{aligned} \tag{17}$$

This latest flow can be expressed as $\exp(tY)$, where Y is the vector field

$$Y = \sum_{i=0}^{m} (2 - |T_i|)\, t_i \partial_{t_i} + 2 \sum_{i=1}^{r} \xi_i \partial_{t_i}.$$

In light of (15) and since we will later want τ to have degree -2 so that $z = \tau^{-1}$ will have degree 2, it now makes sense to extend this to the Euler vector field E on the bundle $\widehat{\mathcal{T}} = \pi^*(\mathcal{T}_{H^*(X,\mathbb{C})})$ by defining:

$$E := Y - 2\tau\partial_\tau.$$

Then we can finally define the connection $\widehat{\nabla}$ on $\widehat{\mathcal{T}}$ by

$$\begin{aligned} \widehat{\nabla}_{\partial_{t_i}}(\partial_{t_j}) &:= \tau(T_i \star T_j), \\ \widehat{\nabla}_E(\partial_{t_i}) &:= (|T_i| - 2)T_i, \end{aligned}$$

where we make no notational difference between lifts of local sections of $\mathcal{T}_{H^*(X,\mathbb{C})}$ and the local sections themselves.

Remark 5.3. *How does that define $\widehat{\nabla}_{\partial_\tau}(\partial_{t_i})$? We have*

$$\begin{aligned} \widehat{\nabla}_E(\partial_{t_i}) &= \widehat{\nabla}_{Y-2\tau\partial_\tau}(\partial_{t_i}) \\ &= \tau(Y \star T_i) - 2\tau\widehat{\nabla}_{\partial_\tau}\partial_{t_i} \end{aligned}$$

and thus

$$\widehat{\nabla}_{\partial_\tau}(\partial_{t_i}) = \frac{1}{2}(Y \star T_i) + \frac{2 - |T_i|}{2\tau} T_i. \tag{18}$$

Proposition 5.4. *The connection $\widehat{\nabla}$ is flat with a simple pole at $\tau = 0$ and a pole of order two at $\tau = \infty$.*

Proof: Since ∇^τ is flat, it remains to show that

$$\widehat{\nabla}_{[\partial_{t_i}, E]}(\partial_{t_j}) = [\widehat{\nabla}_{\partial_{t_i}}, \widehat{\nabla}_E](\partial_{t_j}) \tag{19}$$

for all $i, j \in [0, m]$. The left-hand side of the above equation is easily calculated:

$$\begin{aligned}
[\partial_{t_i}, E](t_j) &= \partial_{t_i}(E(t_j)) - E(\partial_{t_i}(t_j)) \\
&= \begin{cases} \partial_{t_i}(2\xi_j) - E(\delta_{ij}) & \text{for } i \in [1, r] \\ \partial_{t_i}\left((2 - |T_j|)\, t_j\right) - E(\delta_{ij}) & \text{for } i \notin [1, r] \end{cases} \\
&= (2 - |T_j|)\, \delta_{ij}.
\end{aligned}$$

Since additionally

$$[\partial_{t_i}, E](\tau) = 0,$$

we can conclude that

$$[\partial_{t_i}, E] = (2 - |T_i|)\, \partial_{t_i}$$

and thus

$$\widehat{\nabla}_{[\partial_{t_i}, E]}(\partial_{t_j}) = \tau\, (2 - |T_i|)\, (T_i \star T_j).$$

Our next task consists in calculating the right-hand side of equation (19). Commence by calculating

$$\widehat{\nabla}_{\partial_{t_i}} \widehat{\nabla}_E(\partial_{t_j}) = \widehat{\nabla}_{\partial_{t_i}} \left((|T_j| - 2)\partial_{t_j}\right) = \tau\, (|T_j| - 2)\, (T_i \star T_j). \tag{20}$$

However,

$$\begin{aligned}
\widehat{\nabla}_E \widehat{\nabla}_{\partial_{t_i}}(\partial_{t_j}) &= \widehat{\nabla}_E\left(\tau (T_i \star T_j)\right) \\
&= \widehat{\nabla}_E \left(\tau \sum_{k=0}^{m} \partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi) T^k\right)
\end{aligned} \tag{21}$$

is a slightly more complicated expression. The crucial question being, how does E (as a vector field) act on $\partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi)T^k$. Recall from (17) that

$$Y(\partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi)) = (|T_i| + |T_j| + |T_k| - 2\dim(X))\, \partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi)$$

and that

$$\partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi)T^k = \sum_{l=0}^{m} \partial_{t_i}\partial_{t_j}\partial_{t_l}(\Phi) g^{lk} T_k,$$

where g^{ab} are entries of inverse matrix of the matrix, defined by its entries as $g_{ab} = \int_X T_a \cup T_b$. Therefore we have non-zero entries, only if $|T_a| + |T_b| = 2\dim(X)$, which implies that

$$Y(\partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi)T^k) = (|T_i| + |T_j| - |T_k|)\, \partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi)T^k.$$

We are now in a position to further simplify the right-hand side of equation (21):

$$\begin{aligned}
\widehat{\nabla}_E\left(\tau\sum_{k=0}^m \partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi)T^k\right) &= \widehat{\nabla}_E(\tau)\left(\sum_{k,l=0}^m \partial_{t_i}\partial_{t_j}\partial_{t_l}(\Phi)g^{lk}T_k\right) + \tau\sum_{k,l=0}^m \partial_{t_i}\partial_{t_j}\partial_{t_l}(\Phi)g^{lk}\widehat{\nabla}_E(\partial_{t_k}) \\
&\qquad + \tau\sum_{k,l=0}^m \widehat{\nabla}_E(\partial_{t_i}\partial_{t_j}\partial_{t_l}(\Phi)g^{lk})T_k \\
&= \sum_{k,l=0}^m \left(-2\tau\partial_\tau(\tau)+\tau(|T_k|-2)\right)\partial_{t_i}\partial_{t_j}\partial_{t_l}(\Phi)g^{lk}T_k \\
&\qquad + \tau\sum_{k,l=0}^m \big(Y(\partial_{t_i}\partial_{t_j}\partial_{t_l}(\Phi)g^{lk})T_k\big) \\
&= \sum_{k=0}^m \big(-4\tau+\tau|T_k|+\tau(|T_i|+|T_j|-|T_k|)\partial_{t_i}\partial_{t_j}\partial_{t_k}(\Phi)T^k\big) \\
&= \tau\left(|T_i|+|T_j|-4\right)(T_i\star T_j).
\end{aligned}$$

Therefore, combining this with (20) we have shown that

$$\widehat{\nabla}_{\partial_{t_i}}\widehat{\nabla}_E(\partial_{t_j}) - \widehat{\nabla}_E\widehat{\nabla}_{\partial_{t_i}}(\partial_{t_j}) = \tau\left(2-|T_i|\right)(T_i\star T_j),$$

which completes the proof of the flatness of $\widehat{\nabla}$. It remains to examine the singularities at $\tau = 0$ and $\tau = \infty$. We know from our analysis of ∇^τ and remark 5.3 that we can express $\widehat{\nabla}$ as

$$\widehat{\nabla} = \sum_{i=0}^m \tau\Omega^{(i)}dt_i - (B_0 + \frac{B_\infty}{\tau})d\tau,$$

where the matrices $\Omega^{(i)}$ encode the big quantum multiplication by T_i, B_0 encodes the big quantum multiplication by the Euler vector field E and B_∞ is the matrix of cohomological degrees of the basis. From this representation the simple pole at $\tau = 0$ is obvious. In order to see pole of order two at $\tau = \infty$, change coordinates from τ to $z = \tau^{-1}$. Then

$$\frac{d\tau}{\tau} = -\frac{dz}{z}$$

and therefore

$$\widehat{\nabla} = \sum_{i=0}^m \Omega^{(i)}\frac{dt_i}{z} + \left(\frac{B_0}{z^2}+\frac{B_\infty}{z}\right)dz,$$

which shows the pole of order two at $z = 0$, which is $\tau = \infty$, and concludes the proof. □

The Dubrovin connection also appears naturally in the context of Frobenius manifolds, where it is known as the *second structure connection*. This link allows us to establish the claim from remark 3.10.

Proposition 5.5. *In the case that X is a smooth H^2-generated Fano variety, the sum in definition 3.9 converges.*

Proof: This was shown in [Rei09]: the Dubrovin connection is sometimes known as a $(logD - trTLEP(\omega))$-structure. In proposition 1.10. of the cited paper, the author shows a $1-1$ correspondence of such $(logD - trTLEP(\omega))$-structures with Frobenius type structures with logarithmic pole. These in turn are shown in theorem 1.12. to be unfoldable to a unique (up to unique isomorphism) Frobenius manifold with logarithmic poles. The big quantum cohomology then underlies this logarithmic Frobenius manifold and is thus convergent. However, the entire argument is based on two further facts: firstly it assumes that the small quantum cohomology is convergent, which it is in our case due to proposition 3.4. Secondly it assumes that $H^2(X,\mathbb{C})$ generates $H^*(X,\mathbb{C})$ as an algebra, which is why we added that condition in our formulation of this proposition. This last condition is easily shown to be true in the case of X being a smooth nef (and hence Fano) toric variety (c.f. [Ful93]). $\square$

5.2 The Small Dubrovin Connection

The big quantum product is often difficult to compute. In contrast, the small quantum product is sometimes more accessible, whilst still retaining interesting geometric information. For this reason we would like to adjust our Dubrovin connection to reflect the small quantum product, rather than the big one. The way we are going to achieve this goal has already been alluded to in our discussion of big vs. small quantum cohomology from chapter 3.3. Namely, we shall restrict our base space to $H^2(X,\mathbb{C})$. So instead of considering the tangent bundle $\mathcal{T}_{H^*(X,\mathbb{C})}$, we will now consider the trivial $H^*(X,\mathbb{C})$-bundle over $H^2(X,\mathbb{C})$. If we call this bundle $\mathcal{T}$ and furthermore denote the projection

$$\pi : \mathbb{P}^1 \times H^2(X,\mathbb{C}) \to H^2(X,\mathbb{C}),$$

then we are interested in the bundle $\pi^*(\mathcal{T})$. Since $\star$ restricts to the small quantum product in our set-up, we now have a connection ∇^τ on $\mathcal{T}$ for every $\tau \in \mathbb{C}^*$, defined by

$$\nabla^\tau_{\partial_{t_i}}(T_j) = \tau(T_i * T_j),$$

where now $1 \leq i \leq r$ and $0 \leq j \leq m$. Even our generalisation to $\widehat{\nabla}$ restricts well. In light of (18) we find that

$$\widehat{\nabla}_{\partial_\tau}(T_i) = -\omega_X * T_i + \frac{2-|T_i|}{2\tau} T_i$$

makes $\widehat{\nabla}$ a flat, meromorphic connection on $\pi^*(\mathcal{T})$ with singularities along $\tau = 0$ and $\tau = \infty$.

Remark 5.6. *It should be mentioned that the entire argument of the previous section made no use of the precise form of the Gromov-Witten potential. We merely used the homogeneity of Φ and the definition of $\star$ in terms of third derivatives of Φ. Therefore one could develop the entire theory of the Dubrovin connection for a general homogeneous potential. This is called the Dubrovin formalism and applied to Φ_{small} from proposition 3.17, it would also yield the small Dubrovin connection.*

Here is a reason, why restricting to the small quantum product adds intrinsic value: we can now interpret ∇^τ as a connection on the trivial bundle $H^*(X,\mathbb{C}) \times H^2(X,\mathbb{C}) \to H^2(X,\mathbb{C})$, which encodes the small quantum product in its entirety. The fibre over every $\omega \in H^2(X,\mathbb{C})$ is a copy of $H^*(X,\mathbb{C})$ and thus comes intrinsically with a notion of small quantum product. Assuming convergence, this small quantum product $*_\omega$ on the fibre $H^*_\omega(X,\mathbb{C})$ is now dependent on ω by the factor $\exp(2\pi\sqrt{-1}\int_\beta \omega)$. In particular, $*$ is invariant under the natural map

$$H^2(X,\mathbb{C}) \to \frac{H^2(X,\mathbb{C})}{(2\pi\sqrt{-1})H^2(X,\mathbb{Z})} =: \mathcal{KM}_X.$$

We call the latter space the *Kähler moduli space*. Provided that $H^2(X, \mathbb{C})$ is torsion-free (which we shall assume), then

$$\mathcal{KM}_X \simeq H^2(X, \mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^r.$$

Therefore we have a natural change of coordinates

$$t_i \mapsto q_i = e^{t_i}$$

and $\widehat{\nabla}$ descends to a meromorphic connection on

$$H^*(X, \mathbb{C}) \times \mathbb{P}^1 \times \mathcal{KM}_X \to \mathbb{P}^1 \times \mathcal{KM}_X.$$

Moreover, we see that in q_i-coordinates, there is a well-defined limit for $q_i \to 0$. Since $*$ is a binary operation on $H^*(X, \mathbb{C})$, which is polynomial in the q_i in the case that X is Fano, we can extend $\widehat{\nabla}$ to a connection

$$H^*(X, \mathbb{C}) \times \mathbb{P}^1 \times \mathbb{C}^r \to \mathbb{P}^1 \times \mathbb{C}^r.$$

If X is not Fano, then we will not be able (in general) to extend $\widehat{\nabla}$ over all of $\mathbb{C}^r$, but can still extend it over some open neighbourhood of the origin in $\mathbb{C}^r$. Note how over $0 \in \mathbb{C}^r$, the quantum product just degenerates to the usual cup product on the cohomology due to remark 3.6. From now on this latest version of $\widehat{\nabla}$ is what we mean, when we refer to the Dubrovin connection.

5.3 The Dubrovin Connection of X_4

Let us make this explicit in our example of X_4. In other words, let us try to express $\widehat{\nabla} = d + \Omega$ for some matrix of one-forms Ω. In order to simplify our lengthy expressions let us also express all terms involving the quantum variable q in terms of the "basis"

$$q_1 = q^H \quad , \quad q_2 = q^{-E_1} \quad , \quad q_3 = q^{-E_2} \quad , \quad q_4 = q^{-E_3} \quad , \quad q_5 = q^{-E_4}.$$

Recall that in our expressions of the form q^β we had $\beta \in H_2(X_4, \mathbb{Z})$, i.e. in the dual (with respect to the ordinary intersection on X_4) of $H^2(X_4, \mathbb{Z})$, for which we have already chosen a basis, namely $\{H, E_1, E_2, E_3, E_4\}$. It seems therefore natural to choose the dual basis to our previous choice, i.e. $\{H, -E_1, -E_2, -E_3, -E_4\}$, which leads us naturally to choose the above q_is. Let us denote the basis $\{H, E_1, \ldots, E_4\}$ as before by $\{T_1, T_2, T_3, T_4, T_5\}$. As in chapter 4.3 extend this to a basis $\{T_0, \ldots, T_6\}$ of $H^*(X, \mathbb{C})$, by setting $T_0 = [X_4]$ and $T_6 = [pt]$. Given our trivialisation and the fact that $H^2(X_4, \mathbb{Z})$ and $H_2(X_4, \mathbb{Z})$ are dual, we have the natural identification $q_i = e^{t_i}$ and therefore $dt_i = \frac{dq_i}{q_i}$ for $1 \le i \le 5$. Therefore we will express

$$\widehat{\nabla} = d + \tau \sum_{i=1}^{5} \Omega^{(i)} \frac{dq_i}{q_i} - \left(B_0 + \frac{B_\infty}{\tau} \right) d\tau,$$

where, by definition of the Dubrovin connection, the columns of Ω_i are given as the coefficients of our basis vectors $\{T_0, \ldots, T_6\}$ in the expressions of $T_i * T_j$ (once again for $1 \le i \le 5$) Furthermore

$$B_0 = -3\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_5, \tag{22}$$

since $K_{x_4} = -3H + E_1 + E_2 + E_3 + E_4$, and

$$B_\infty = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}. \tag{23}$$

For instance, the fact that $T_0 * T_i = T_i$, tells us that the first column of $\Omega^{(i)}$ is the $(i+1)$th standard basis column vector.

Furthermore we know from chapter 4.3 the following expressions:

$$\begin{aligned}
T_1 * T_1 &= (q_1q_2 + q_1q_3 + q_1q_4 + q_1q_5 + 4q_1^2q_2q_3q_4q_5)T_0 \\
&\quad + (q_1q_2q_3 + q_1q_2q_4 + q_1q_2q_5 + q_1q_3q_4 + q_1q_3q_5 + q_1q_4q_5)T_1 \\
&\quad - (q_1q_2q_3 + q_1q_2q_4 + q_1q_2q_5)T_2 - (q_1q_2q_3 + q_1q_3q_4 + q_1q_3q_5)T_3 \\
&\quad - (q_1q_2q_4 + q_1q_3q_4 + q_1q_4q_5)T_4 - (q_1q_2q_5 + q_1q_3q_5 + q_1q_4q_5)T_5 + T_6, \\
T_1 * T_2 &= (q_1q_2 + 2q_1^2q_2q_3q_4q_5)T_0 + (q_1q_2q_3 + q_1q_2q_4 + q_1q_2q_5)T_1 \\
&\quad - (q_1q_2q_3 + q_1q_2q_4 + q_1q_2q_5)T_2 - q_1q_2q_3T_3 - q_1q_2q_4T_4 - q_1q_2q_5T_5, \\
T_1 * T_3 &= (q_1q_3 + 2q_1^2q_2q_3q_4q_5)T_0 + (q_1q_2q_3 + q_1q_3q_4 + q_1q_3q_5)T_1 \\
&\quad - q_1q_2q_3T_2 - (q_1q_2q_3 + q_1q_3q_4 + q_1q_3q_5)T_3 - q_1q_3q_4T_4 - q_1q_3q_5T_5, \\
T_1 * T_4 &= (q_1q_4 + 2q_1^2q_2q_3q_4q_5)T_0 + (q_1q_2q_4 + q_1q_3q_4 + q_1q_4q_5)T_1 \\
&\quad - q_1q_2q_4T_2 - q_1q_3q_4T_3 - (q_1q_2q_4 + q_1q_3q_4 + q_1q_4q_5)T_4 - q_1q_4q_5T_5, \\
T_1 * T_5 &= (q_1q_5 + 2q_1^2q_2q_3q_4q_5)T_0 + (q_1q_2q_5 + q_1q_3q_5 + q_1q_4q_5)T_1 \\
&\quad - q_1q_2q_5T_2 - q_1q_3q_5T_3 - q_1q_4q_5T_4 - (q_1q_2q_5 + q_1q_3q_5 + q_1q_4q_5)T_5, \\
T_2 * T_2 &= (q_1q_2 + q_1^2q_2q_3q_4q_5)T_0 + (q_1q_2q_3 + q_1q_2q_4 + q_1q_2q_5)T_1 \\
&\quad - (q_1q_2q_3 + q_1q_2q_4 + q_1q_2q_5 - q_2^{-1})T_2 - q_1q_2q_3T_3 \\
&\quad - q_1q_2q_4T_4 - q_1q_2q_5T_5 - T_6, \\
T_2 * T_3 &= q_1^2q_2q_3q_4q_5T_0 + q_1q_2q_3T_1 - q_1q_2q_3T_2 - q_1q_2q_3T_3, \\
T_2 * T_4 &= q_1^2q_2q_3q_4q_5T_0 + q_1q_2q_4T_1 - q_1q_2q_4T_2 - q_1q_2q_4T_4, \\
T_2 * T_5 &= q_1^2q_2q_3q_4q_5T_0 + q_1q_2q_5T_1 - q_1q_2q_5T_2 - q_1q_2q_5T_5, \\
T_3 * T_3 &= (q_1q_3 + q_1^2q_2q_3q_4q_5)T_0 + (q_1q_2q_3 + q_1q_3q_4 + q_1q_3q_5)T_1 \\
&\quad - q_1q_2q_3T_2 - (q_1q_2q_3 + q_1q_3q_4 + q_1q_3q_5 - q_3^{-1})T_3 \\
&\quad - q_1q_3q_4T_4 - q_1q_3q_5T_5 - T_6, \\
T_3 * T_4 &= q_1^2q_2q_3q_4q_5T_0 + q_1q_3q_4T_1 - q_1q_3q_4T_3 - q_1q_3q_4T_4, \\
T_3 * T_5 &= q_1^2q_2q_3q_4q_5T_0 + q_1q_3q_5T_1 - q_1q_3q_5T_3 - q_1q_3q_5T_5, \\
T_4 * T_4 &= (q_1q_4 + q_1^2q_2q_3q_4q_5)T_0 + (q_1q_2q_4 + q_1q_3q_4 + q_1q_4q_5)T_1 \\
&\quad - q_1q_2q_4T_2 - q_1q_3q_4T_3 \\
&\quad - (q_1q_2q_4 + q_1q_3q_4 + q_1q_4q_5 - q_4^{-1})T_4 - q_1q_4q_5T_5 - T_6,
\end{aligned}$$

$$
\begin{aligned}
T_4 * T_5 &= q_1^2 q_2 q_3 q_4 q_5 T_0 + q_1 q_4 q_5 T_1 - q_1 q_4 q_5 T_4 - q_1 q_4 q_5 T_5, \\
T_5 * T_5 &= (q_1 q_5 + q_1^2 q_2 q_3 q_4 q_5) T_0 + (q_1 q_2 q_5 + q_1 q_3 q_5 + q_1 q_4 q_5) T_1 \\
&\quad - q_1 q_2 q_5 T_2 - q_1 q_3 q_5 T_3 - q_1 q_4 q_5 T_4 \\
&\quad - (q_1 q_2 q_5 + q_1 q_3 q_5 + q_1 q_4 q_5 - q_5^{-1}) T_5 - T_6.
\end{aligned}
$$

So we know all $\Omega^{(i)}$ apart from their respective right-most columns. These are given by the coefficients of the expression $T_i * T_6$ for $1 \leq i \leq 5$. We previously computed that

$$
\begin{aligned}
T_6 = {} & T_1 * T_1 \\
& -(q_1 q_2 + q_1 q_3 + q_1 q_4 + q_1 q_5 + 4 q_1^2 q_2 q_3 q_4 q_5) T_0 \\
& -(q_1 q_2 q_3 + q_1 q_2 q_4 + q_1 q_2 q_5 + q_1 q_3 q_4 + q_1 q_3 q_5 + q_1 q_4 q_5) T_1 \\
& +(q_1 q_2 q_3 + q_1 q_2 q_4 + q_1 q_2 q_5) T_2 + (q_1 q_2 q_3 + q_1 q_3 q_4 + q_1 q_3 q_5) T_3 \\
& +(q_1 q_2 q_4 + q_1 q_3 q_4 + q_1 q_4 q_5) T_4 + (q_1 q_2 q_5 + q_1 q_3 q_5 + q_1 q_4 q_5) T_5 \\
& = - T_2 * T_2 \\
& +(q_1 q_2 + q_1^2 q_2 q_3 q_4 q_5) T_0 + (q_1 q_2 q_3 + q_1 q_2 q_4 + q_1 q_2 q_5) T_1 \\
& -(q_1 q_2 q_3 + q_1 q_2 q_4 + q_1 q_2 q_5 - q_2^{-1}) T_2 \\
& - q_1 q_2 q_3 T_3 - q_1 q_2 q_4 T_4 - q_1 q_2 q_5 T_5.
\end{aligned}
\tag{24}
$$

Hence we can simplify

$$
\begin{aligned}
T_1 * T_6 = {} & -(T_1 * T_2) * T_2 \\
& +(q_1 q_2 + q_1^2 q_2 q_3 q_4 q_5)(T_0 * T_1) + (q_1 q_2 q_3 + q_1 q_2 q_4 + q_1 q_2 q_5)(T_1 * T_1) \\
& -(q_1 q_2 q_3 + q_1 q_2 q_4 + q_1 q_2 q_5 - q_2^{-1})(T_1 * T_2) \\
& - q_1 q_2 q_3 (T_1 * T_3) - q_1 q_2 q_4 (T_1 * T_4) - q_1 q_2 q_5 (T_1 * T_5),
\end{aligned}
$$

by substituting the respective expressions which we already know. This yields the following identity:

$$
\begin{aligned}
T_1 * T_6 = {} & (q_1 + 2 q_1^2 q_2 q_3 q_4 + 2 q_1^2 q_2 q_3 q_5 + 2 q_1^2 q_2 q_4 q_5 + 2 q_1^2 q_3 q_4 q_5) T_0 \\
& +(q_1 q_2 + q_1 q_3 + q_1 q_4 + q_1 q_5 + 4 q_1^2 q_2 q_3 q_4 q_5) T_1 \\
& -(q_1 q_2 + 2 q_1^2 q_2 q_3 q_4 q_5) T_2 - (q_1 q_3 + 2 q_1^2 q_2 q_3 q_4 q_5) T_3 \\
& -(q_1 q_4 + 2 q_1^2 q_2 q_3 q_4 q_5) T_4 - (q_1 q_5 + 2 q_1^2 q_2 q_3 q_4 q_5) T_5.
\end{aligned}
$$

Similarly, by using the second equality in (24) we find

$$\begin{aligned}
T_2 * T_6 &= (q_1^2q_2q_3q_4 + q_1^2q_2q_3q_5 + q_1^2q_2q_4q_5)T_0 \\
&\quad + (q_1q_2 + 2q_1^2q_2q_3q_4q_5)T_1 - (q_1q_2 + q_1^2q_2q_3q_4q_5)T_2 \\
&\quad - q_1^2q_2q_3q_4q_5T_3 - q_1^2q_2q_3q_4q_5T_4 - q_1^2q_2q_3q_4q_5T_5, \\
T_3 * T_6 &= (q_1^2q_2q_3q_4 + q_1^2q_2q_3q_5 + q_1^2q_3q_4q_5)T_0 \\
&\quad + (q_1q_3 + 2q_1^2q_2q_3q_4q_5)T_1 - q_1^2q_2q_3q_4q_5T_2 \\
&\quad - (q_1q_3 + q_1^2q_2q_3q_4q_5)T_3 - q_1^2q_2q_3q_4q_5T_4 - q_1^2q_2q_3q_4q_5T_5, \\
T_4 * T_6 &= (q_1^2q_2q_3q_4 + q_1^2q_2q_4q_5 + q_1^2q_3q_4q_5)T_0 \\
&\quad + (q_1q_4 + 2q_1^2q_2q_3q_4q_5)T_1 - q_1^2q_2q_3q_4q_5T_2 - q_1^2q_2q_3q_4q_5T_3 \\
&\quad - (q_1q_4 + q_1^2q_2q_3q_4q_5)T_4 - q_1^2q_2q_3q_4q_5T_5, \\
T_5 * T_6 &= (q_1^2q_2q_3q_5 + q_1^2q_2q_4q_5 + q_1^2q_3q_4q_5)T_0 \\
&\quad + (q_1q_5 + 2q_1^2q_2q_3q_4q_5)T_1 - q_1^2q_2q_3q_4q_5T_2 - q_1^2q_2q_3q_4q_5T_3 \\
&\quad - q_1^2q_2q_3q_4q_5T_4 - (q_1q_5 + q_1^2q_2q_3q_4q_5)T_5.
\end{aligned}$$

Thus we can finish this chapter by writing

$$\widehat{\nabla} = d + \Omega^{(1)}\frac{\tau dq_1}{q_1} + \Omega(2)\frac{\tau dq_2}{q_2} + \Omega^{(3)}\frac{\tau dq_3}{q_3} + \Omega^{(4)}\frac{\tau dq_4}{q_4} + \Omega^{(5)}\frac{\tau dq_5}{q_5} - \left(B_0 + \frac{B_\infty}{\tau}\right)d\tau,$$

where the $\Omega^{(i)}$ are given on the following pages, B_∞ is given as (23) and B_0 can be calculated from the $\Omega^{(i)}$ according to equation (22). Furthermore it is a matter of simple matrix multiplication to verify $\widehat{\nabla}$ is indeed both flat and torsion-free. For this purpose it suffices to show that

$$[\Omega^{(i)}, \Omega^{(j)}] = 0 \qquad \text{and} \qquad \partial_{q_i}\left(\frac{\Omega^{(j)}}{q_j}\right) - \partial_{q_j}\left(\frac{\Omega^{(i)}}{q_i}\right) = 0 \tag{25}$$

for all distinct pairs of $1 \le i < j \le 5$ and that moreover

$$\left[\frac{\tau\Omega^{(i)}}{q_i}, \frac{-B_0\tau - B_\infty}{\tau}\right] + \partial_{q_i}\left(\frac{-B_0\tau - B_\infty}{\tau}\right) - \partial_\tau\left(\frac{\tau\Omega^{(i)}}{q_i}\right) = 0$$

for all $1 \le i \le 5$. Note that the latter equation is equivalent to the equation

$$[B_\infty, \Omega^{(i)}] = q_i\partial_{q_i}(B_0) + \Omega^{(i)},$$

since B_0 commutes with the $\Omega^{(i)}$ due to (22) and (25). All these equations are readily checked and (unsurprisingly) found to be true.

$$
\Omega^{(1)} = \begin{pmatrix}
0 & \substack{q_1q_2+q_1q_3+q_1q_4+q_1q_5\\+4q_1^2q_2q_3q_4q_5} & q_1q_2+2q_1^2q_2q_3q_4q_5 & q_1q_3+2q_1^2q_2q_3q_4q_5 & q_1q_4+2q_1^2q_2q_3q_4q_5 & q_1q_5+2q_1^2q_2q_3q_4q_5 & \substack{q_1+2q_1^2q_2q_3q_4+2q_1^2q_2q_3q_5\\+2q^2q_2q_4q_5+2q^2q_3q_4q_5} \\
1 & \substack{q_1q_2q_3+q_1q_2q_4+q_1q_2q_5\\+q_1q_3q_4+q_1q_3q_5+q_1q_4q_5} & q_1q_2q_3+q_1q_2q_4+q_1q_2q_5 & q_1q_2q_3+q_1q_3q_4+q_1q_3q_5 & q_1q_2q_4+q_1q_3q_4+q_1q_4q_5 & q_1q_2q_5+q_1q_3q_5+q_1q_4q_5 & \substack{q_1q_2+q_1q_3+q_1q_4+q_1q_5\\+4q_1^2q_2q_3q_4q_5} \\
0 & -q_1q_2q_3-q_1q_2q_4-q_1q_2q_5 & -q_1q_2q_3-q_1q_2q_4-q_1q_2q_5 & -q_1q_2q_3 & -q_1q_2q_4 & -q_1q_2q_5 & -q_1q_2-2q_1^2q_2q_3q_4q_5 \\
0 & -q_1q_2q_3-q_1q_3q_4-q_1q_3q_5 & -q_1q_2q_3 & -q_1q_2q_3-q_1q_3q_4-q_1q_3q_5 & -q_1q_3q_4 & -q_1q_3q_5 & -q_1q_3-2q_1^2q_2q_3q_4q_5 \\
0 & -q_1q_2q_4-q_1q_3q_4-q_1q_4q_5 & -q_1q_2q_4 & -q_1q_3q_4 & -q_1q_2q_4-q_1q_3q_4-q_1q_4q_5 & -q_1q_4q_5 & -q_1q_4-2q_1^2q_2q_3q_4q_5 \\
0 & -q_1q_2q_5-q_1q_3q_5-q_1q_4q_5 & -q_1q_2q_5 & -q_1q_3q_5 & -q_1q_4q_5 & -q_1q_2q_5-q_1q_3q_5-q_1q_4q_5 & -q_1q_5-2q_1^2q_2q_3q_4q_5 \\
0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\Omega^{(2)} = \begin{pmatrix}
0 & q_1q_2+2q_1^2q_2q_3q_4q_5 & q_1q_2+q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_4+q_1^2q_2q_3q_5+q_1^2q_2q_4q_5 \\
0 & q_1q_2q_3+q_1q_2q_4+q_1q_2q_5 & q_1q_2q_3+q_1q_2q_4+q_1q_2q_5 & q_1q_2q_3 & q_1q_2q_4 & q_1q_2q_5 & q_1q_2+2q_1^2q_2q_3q_4q_5 \\
1 & -q_1q_2q_3-q_1q_2q_4-q_1q_2q_5 & -q_1q_2q_3-q_1q_2q_4-q_1q_2q_5+q_2^{-1} & -q_1q_2q_3 & -q_1q_2q_4 & -q_1q_2q_5 & -q_1q_2-q_1^2q_2q_3q_4q_5 \\
0 & -q_1q_2q_3 & -q_1q_2q_3 & -q_1q_2q_3 & 0 & 0 & -q_1^2q_2q_3q_4q_5 \\
0 & -q_1q_2q_4 & -q_1q_2q_4 & 0 & -q_1q_2q_4 & 0 & -q_1^2q_2q_3q_4q_5 \\
0 & -q_1q_2q_5 & -q_1q_2q_5 & 0 & 0 & -q_1q_2q_5 & -q_1^2q_2q_3q_4q_5 \\
0 & 0 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\Omega^{(3)} = \begin{pmatrix}
0 & q_1q_3+2q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_4q_5 & q_1q_3+q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_4+q_1^2q_2q_3q_5+q_1^2q_3q_4q_5 \\
0 & q_1q_2q_3+q_1q_3q_4+q_1q_3q_5 & q_1q_2q_3 & q_1q_2q_3+q_1q_3q_4+q_1q_3q_5 & q_1q_3q_4 & q_1q_3q_5 & q_1q_3+2q_1^2q_2q_3q_4q_5 \\
0 & -q_1q_2q_3 & -q_1q_2q_3 & -q_1q_2q_3 & 0 & 0 & -q_1^2q_2q_3q_4q_5 \\
1 & -q_1q_2q_3-q_1q_3q_4-q_1q_3q_5 & -q_1q_2q_3 & -q_1q_2q_3-q_1q_3q_4-q_1q_3q_5+q_3^{-1} & -q_1q_3q_4 & -q_1q_3q_5 & -q_1q_3-q_1^2q_2q_3q_4q_5 \\
0 & -q_1q_3q_4 & 0 & -q_1q_3q_4 & -q_1q_3q_4 & 0 & -q_1^2q_2q_3q_4q_5 \\
0 & -q_1q_3q_5 & 0 & -q_1q_3q_5 & 0 & -q_1q_3q_5 & -q_1^2q_2q_3q_4q_5 \\
0 & 0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\Omega^{(4)} = \begin{pmatrix}
0 & q_1q_4 + 2q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_4q_5 & q_1q_4 + q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_4 + q_1^2q_2q_4q_5 + q_1^2q_3q_4q_5 \\
0 & q_1q_2q_4 + q_1q_3q_4 + q_1q_4q_5 & q_1q_2q_4 & q_1q_3q_4 & q_1q_2q_4 + q_1q_3q_4 + q_1q_4q_5 & q_1q_4q_5 & q_1q_4 + 2q_1^2q_2q_3q_4q_5 \\
0 & -q_1q_2q_4 & -q_1q_2q_4 & 0 & -q_1q_2q_4 & 0 & -q_1^2q_2q_3q_4q_5 \\
0 & -q_1q_3q_4 & 0 & -q_1q_3q_4 & -q_1q_3q_4 & 0 & -q_1^2q_2q_3q_4q_5 \\
1 & -q_1q_2q_4 - q_1q_3q_4 - q_1q_4q_5 & -q_1q_2q_4 & -q_1q_3q_4 & -q_1q_2q_4 - q_1q_3q_4 - q_1q_4q_5 + q_4^{-1} & -q_1q_4q_5 & -q_1q_4 - q_1^2q_2q_3q_4q_5 \\
0 & -q_1q_4q_5 & 0 & 0 & -q_1q_4q_5 & -q_1q_3q_5 & -q_1^2q_2q_3q_4q_5 \\
0 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}
$$

$$
\Omega^{(5)} = \begin{pmatrix}
0 & q_1q_5 + 2q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_4q_5 & q_1q_5 + q_1^2q_2q_3q_4q_5 & q_1^2q_2q_3q_5 + q_1^2q_2q_4q_5 + q_1^2q_3q_4q_5 \\
0 & q_1q_2q_5 + q_1q_3q_5 + q_1q_4q_5 & q_1q_2q_5 & q_1q_3q_5 & q_1q_4q_5 & q_1q_2q_5 + q_1q_3q_5 + q_1q_4q_5 & q_1q_5 + 2q_1^2q_2q_3q_4q_5 \\
0 & -q_1q_2q_5 & -q_1q_2q_5 & 0 & 0 & -q_1q_2q_5 & -q_1^2q_2q_3q_4q_5 \\
0 & -q_1q_3q_5 & 0 & -q_1q_3q_5 & 0 & -q_1q_3q_5 & -q_1^2q_2q_3q_4q_5 \\
0 & -q_1q_4q_5 & 0 & 0 & -q_1q_4q_5 & -q_1q_4q_5 & -q_1^2q_2q_3q_4q_5 \\
1 & -q_1q_2q_5 - q_1q_3q_5 - q_1q_4q_5 & -q_1q_2q_5 & -q_1q_3q_5 & -q_1q_4q_5 & -q_1q_2q_5 - q_1q_3q_5 - q_1q_4q_5 + q_5^{-1} & -q_1q_5 - q_1^2q_2q_3q_4q_5 \\
0 & 0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}
$$

6 $\mathcal{D}$-Modules

This chapter is mainly meant to introduce our notation in regards to $\mathcal{D}$-modules. We will avoid a deeper discussion on the theory. Excellent sources for that may be [HTT08, BGK$^+$87].

6.1 Direct and Inverse Images

For the following chapter let X, Y be smooth algebraic varieties of dimensions n, m respectively and let $f : X \to Y$ be a smooth map. We denote by $\mathcal{D}_X$ the sheaf of differential operators on X, i.e. the sheaf that is generated by $\mathcal{O}_X$ (the structure sheaf) and Θ_X (the sheaf of vector fields, a.k.a. the sheaf of derivations of $\mathcal{O}_X$, a.k.a. the tangent sheaf of X). For any $x \in X$ we can find an open neighbourhood $U \subset X$ containing x and a local coordinate system $\{x_i, \partial_{x_i}\}_{1 \le i \le n} \subset \mathcal{O}_X(U) \times \Theta_X(U)$ of $\mathcal{D}_X$ such that[12]

$$\Theta_X(U) = \bigoplus_{i=1}^{n} \mathcal{O}_X(U)\partial_{x_i} \text{ and } \quad [\partial_{x_i}, \partial_{x_j}] = 0, \quad [\partial_{x_i}, x_j] = \delta_{ij} \text{ and } \quad \mathcal{D}_U = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_X(U)\partial_x^\alpha.$$

Since $\mathcal{D}_X$ is a sheaf of non-Abelian rings, we need to distinguish between left and right $\mathcal{D}_X$-modules. The category of left $\mathcal{D}_X$-modules (resp. right $\mathcal{D}_X$-modules) will be denoted by $\mathrm{Mod}(\mathcal{D}_X)$ (resp. $\mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}})$). $\mathcal{O}_X$ has naturally the structure of a left $\mathcal{D}_X$-module, where elements of $\mathcal{O}_X$ act by multiplication and elements of Θ_X act by differentiation. We denote by ω_X the canonical sheaf of X. Then ω_X in turn has the structure of a right $\mathcal{D}_X$-module, given by

$$\omega\vartheta = -\operatorname{Lie}_\vartheta \omega \qquad (\omega \in \omega_X, \vartheta \in \Theta_X),$$

where $\operatorname{Lie}_\vartheta$ denotes the Lie derivative. It can be shown that the functor

$$\omega_X \otimes_{\mathcal{O}_X} (\bullet) : \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}})$$

is an equivalence of categories with its quasi-inverse given by

$$\begin{aligned} \omega_X^{-1} \otimes_{\mathcal{O}_X} (\bullet) = \mathscr{H}\!om_{\mathcal{O}_X}(\omega_X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} (\bullet) = \\ = \mathscr{H}\!om_{\mathcal{O}_X}(\omega_X, \bullet) : \mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}}) \to \mathrm{Mod}(\mathcal{D}_X). \end{aligned}$$

Thus we can use these functors to turn left into right modules and vice versa. This is particularly useful when defining the direct image, since the direct image is more easily defined for right $\mathcal{D}_X$-modules.

However, we cannot simply define direct and inverse images sheaf-theoretically and then endow them with a $\mathcal{D}$-module structure. Instead we need to use the language of derived categories: by $D(\mathcal{D}_X)$ (resp. $D^b(\mathcal{D}_X)$) we denote the derived category of the Abelian category $\mathrm{Mod}(\mathcal{D}_X)$ (resp. its derived category of bounded complexes). It is a fundamental fact that any object of $D^b(\mathcal{D}_X)$ has an injective resolution. Moreover, it is not difficult to show that any such object is represented by a bounded complex of flat $\mathcal{D}_X$-modules [HTT08, 1.5.6]. But beware that the injective and flat resolutions of any object in $D^b(\mathcal{D}_X)$ are not in general the same! Since the bifunctor $\otimes_{\mathcal{O}_X}$ is right

[12] By ∂_x^α for some $\alpha \in \mathbb{N}$ (non-negative integers) we mean $\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.

exact with respect to both factors in the category left $\mathcal{D}_X$-modules, we can use flat resolutions to define the *derived tensor product* by

$$\begin{aligned}(\bullet) \otimes^L_{\mathcal{O}_X} (\bullet) : D^b(\mathcal{D}_X) \times D^b(\mathcal{D}_X) &\to D^b(\mathcal{D}_X)\\ (M^\cdot, N^\cdot) &\mapsto M^\cdot \otimes^L_{\mathcal{O}_X} N^\cdot.\end{aligned}$$

In order to endow our inverse and direct images with a $\mathcal{D}$-module structure we initially define such a structure on the simplest of $\mathcal{D}$-modules, namely $\mathcal{D}_X$ itself. We denote the $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$ by $\mathcal{D}_{X\to Y}$. This is the inverse image in the category of $\mathcal{O}$-modules of $\mathcal{D}_Y$ with the obvious right-module structure of $f^{-1}\mathcal{D}_Y$, induced by the right multiplication of $\mathcal{D}_Y$ and the left $\mathcal{D}_X$-module structure, defined by the action of $\vartheta \in \Theta_X$:

$$\vartheta(\psi \otimes s) = \vartheta(\psi) \otimes s + \psi \sum_{i=1}^{m} \vartheta(y_i \circ f) \otimes \partial_{y_i} s,$$

where $\psi \otimes s \in \mathcal{D}_{X\to Y}$ and $\{y_i, \partial_{y_i}\}_{1\le i\le m}$ is a local coordinate system of Y. With a little care we can also define $\mathcal{D}_{Y\leftarrow X}$, which we will use for the direct image. However, we want $\mathcal{D}_{Y\leftarrow X}$ to be a $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodule, which is why we have to change right and left module structures in $\mathcal{D}_{X\to Y}$.

$$\begin{aligned}\mathcal{D}_{Y\leftarrow X} :&= \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^{-1}\\ &\simeq f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1}) \otimes_{f^{-1}\mathcal{O}_Y} \omega_X.\end{aligned}$$

The modules $\mathcal{D}_{X\to Y}$ and $\mathcal{D}_{Y\leftarrow X}$ are called the *transfer bimodules* for $f : X \to Y$.

We are now in a position to define the *derived inverse image functor* by

$$\begin{aligned}Lf^+ : D^b(\mathcal{D}_Y) &\to D^b(\mathcal{D}_X)\\ M^\cdot &\mapsto \mathcal{D}_{X\to Y} \otimes^L_{f^{-1}\mathcal{D}_Y} f^{-1}M^\cdot.\end{aligned}$$

The *derived direct image functor* is defined to be

$$\begin{aligned}Rf_+ : D^b(\mathcal{D}_X) &\to D^b(\mathcal{D}_Y)\\ M^\cdot &\mapsto Rf_*(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} M^\cdot).\end{aligned}$$

Proposition 6.1. *For $f : X \to Y$ and $g : Y \to Z$, smooth maps between smooth algebraic varieties, we have*

$$L(g \circ f)^+ = Lg^+ \circ Lf^+ \qquad R(g \circ f)_+ = Rf_+ \circ Rg_+.$$

Proof: [HTT08, chapter 1]. □

6.2 The Characteristic Variety

The following two chapters are meant to introduce a particularly well-behaved kind of $\mathcal{D}$-modules, so-called *holonomic* $\mathcal{D}$-modules. Despite holonomicity being by no means an obvious concept, it is a rather natural generalisation of ODEs in higher-dimensional cases (c.f. chapter 6.3). Furthermore holonomic modules turn out to have some nice properties, which make them invaluable in the study of differential equations as well as the answer to Hilbert's 21st problem, the *Riemann-Hilbert correspondence.*

Holonomicity is defined in terms of the dimensions of the *characteristic variety* of a $\mathcal{D}$-module. However, this variety is per se not defined for complexes of $\mathcal{D}$-modules. Therefore we will leave the derived category for now and define a holonomic $\mathcal{D}_X$-module for an algebraic variety X, only to return later to extend the definition for complexes of $\mathcal{D}_X$-modules.

We start by defining a filtration $F_0\mathcal{D}_X \subset \cdots \subset F_i\mathcal{D}_X \subset \cdots$ on the ring of differential operators by their order. In other words $F_0\mathcal{D}_X = \mathcal{O}_X$, $F_1\mathcal{D}_X = \mathcal{O}_X \oplus \Theta_X$ (the tangent sheaf) and any operator of the form $x^\alpha \partial_x^\beta = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$ is contained in $F_{|\beta|}\mathcal{D}_X$, where $|\beta| = \beta_1 + \cdots + \beta_n$. Clearly $\mathcal{D}_X$ is the union of all $F_i\mathcal{D}_X$ and if we set $F_{-1}\mathcal{D}_X = 0$, then we can define

$$\operatorname{Gr}\mathcal{D}_X = \bigoplus_{i=1}^{\infty} \operatorname{Gr}^i \mathcal{D}_X = \bigoplus_{i=0}^{\infty} \frac{F_i\mathcal{D}_X}{F_{i-1}\mathcal{D}_X}.$$

Now $\operatorname{Gr}\mathcal{D}_X$ is a sheaf of *commutative* algebras. If we choose an open set $U \subset X$ and a local trivialisation $\{x_i, \partial_{x_i}\}$ and furthermore denote by ξ_i the image of $\partial_{x_i} \in \operatorname{Gr}\mathcal{D}_X$, then we have

$$\operatorname{Gr}^i \mathcal{D}_U = \bigoplus_{|\alpha|=i} \mathcal{O}_U \xi^\alpha.$$

Moreover, by regarding $\xi_1, \ldots, \xi_n$ as the coordinate system of the cotangent space $\pi : \mathcal{T}_X^* \to X$, we obtain a canonical identification

$$\operatorname{Gr}\mathcal{D}_X = \mathcal{O}[\xi_1, \ldots, \xi_n] \simeq \pi_*\mathcal{O}_{\mathcal{T}_X^*}.$$

In many instances it is sensible to study $\mathcal{D}_X$ indirectly by studying $\operatorname{Gr}\mathcal{D}_X$, since the latter is commutative and many properties of one can be inferred from the other. This is particularly valid, since we can similarly pass from a $\mathcal{D}_X$-module M to a $\operatorname{Gr}\mathcal{D}_X$-module $\operatorname{Gr}^F M$. This is defined by using an exhaustive filtration by quasi-coherent $\mathcal{O}_X$-submodules F_iM of M, which is compatible with the order filtration on $\mathcal{D}_X$. In other words, we need $F_iM = 0$ for $i \ll 0$ and that $(F_i\mathcal{D}_X)(F_jM) \subset F_{i+j}M$.

If we have such a filtration, the we define

$$\operatorname{Gr}^F M = \bigoplus_{i\in\mathbb{Z}} \frac{F_iM}{F_{i-1}M},$$

which is a graded module over the graded ring $\pi_*\mathcal{O}_{\mathcal{T}_X^*}$. Unfortunately, $\operatorname{Gr}^F M$ does in fact depend on the choice of filtration and not all those choices are suitable. We call F a good filtration if $\operatorname{Gr}^F M$ is coherent over $\pi_*\mathcal{O}_{\mathcal{T}_X^*}$. It is not difficult to show that any coherent $\mathcal{D}_X$-module M admits a globally defined good filtration and that vice versa any M admitting a good filtration is coherent (c.f. [HTT08]).

The coherency of M is thus a property which can theoretically be determined by considering the coherency of $\operatorname{Gr}^F M$. If we want to find more of these "linked" properties, we have to find some, which are independent of the choice of filtration. One of them is the support $\operatorname{Supp}(M)$, the set of $x \in X$, where the localisation of M is not trivial. However, we get a finer characteristic, when we change the setting from X to T^*X. We define the *characteristic variety*[13] of M by

$$\operatorname{Ch}(M) = \operatorname{Supp}(\mathcal{O}_{\mathcal{T}_X^*} \otimes_{\pi^{-1}\pi_*\mathcal{O}_{\mathcal{T}_X^*}} \pi^{-1}(\operatorname{Gr}^F M)).$$

[13] Also sometimes referred to as *singular support* of M.

In a way, $\mathrm{Ch}(M)$ can be seen as a local property: For an open subset $U \subset X$ we have

$$\mathrm{Ch}(M) \cap \mathcal{T}_U^* = \left\{ p \in \mathcal{T}_U^* \;\middle|\; f(p) = 0 \text{ for all } f \in \sqrt{\mathrm{Ann}_{\mathrm{Gr}\,\mathcal{D}_U}(\mathrm{Gr}^F M(U))} \right\},$$

where Ann denotes the annihilator. In fact, the radical of the annihilator of $\mathrm{Gr}^F M$ in $\mathrm{Gr}\,\mathcal{D}_X$ can be shown to be independent of the choice of good filtration, thereby making $\mathrm{Ch}(M) \subset \mathcal{T}_X^*$ independent of that choice.

Many interesting facts can be proven about the characteristic variety, but we shall restrict ourselves to simply stating a few of them and referring the reader to [HTT08] for full proofs.

Proposition 6.2. *Let M be a coherent $\mathcal{D}_X$-module. Then*

- $\mathrm{Ch}(M)$ *is conical, which means that* $(m, \xi) \in \mathrm{Ch}(M) \Rightarrow (m, \lambda\xi) \in \mathrm{Ch}(M)$ *for all* $\lambda \in \mathbb{C}$.
- *For $M \neq 0$ the* Bernstein inequality *holds:*

$$\dim(X) \leq \dim(\mathrm{Ch}(M)) \leq \dim(\mathcal{T}_X^*) = 2\dim(X).$$

- $\pi(\mathrm{Ch}(M)) = \mathrm{Supp}(M)$.
- *Let N_1, N_2 be coherent $\mathcal{D}_X$-modules, which fit in the short exact sequence*

$$0 \to N_1 \to M \to N_2 \to 0.$$

Then the characteristic varieties are related by

$$\mathrm{Ch}(M) = \mathrm{Ch}(N_1) \cup \mathrm{Ch}(N_2).$$

6.3 Holonomic $\mathcal{D}$-Modules

Definition 6.3. *A coherent $\mathcal{D}_X$-module M is called* holonomic *if either* $\dim(\mathrm{Ch}(M)) = \dim(X)$ *or $M = 0$. A complex of coherent $\mathcal{D}_X$-modules is called* holonomic *if all modules of its cohomology complex are holonomic.*

Why is this a sensible definition? Consider a system of differential equations $P_i(f) = 0$ for $1 \leq i \leq k$, where the P_i are differential operators acting on a function $f = f(x)$. We have $P_i \in \mathcal{D}_X$. So we might consider the module

$$M = \frac{\mathcal{D}_X}{\langle P_1, \ldots, P_k \rangle} = \frac{\mathcal{D}_X}{I}.$$

Then

$$\begin{aligned} \mathscr{H}\!om_{\mathcal{D}_X}(M, \mathcal{O}_X) &= \left\{ \varphi \in \mathscr{H}\!om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{O}_X) \;\middle|\; \varphi(I) = 0 \right\} \\ &\simeq \left\{ f \in \mathcal{O}_X \;\middle|\; P_i(f) = 0 \quad \forall \right\}, \end{aligned} \tag{26}$$

where the second line follows since $\varphi \in \mathscr{H}\!om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{O}_X)$ is determined by $\varphi(1) \in \mathcal{O}_X$ and since I is generated by the P_i.

In the one-dimensional case, we know that the solution space to a general ODE is always finite-dimensional. However, the solution space in (26) might not be finite-dimensional in general. If we want it to be "as small as possible", we should aim to make M small or equivalently I large. This makes intuitive sense: if we add more independent operators to our system of differential equations, then fewer functions should satisfy the entire system and the solution space should shrink. Now passing to our associated commutative situation, this means that we would like to increase $\operatorname{Gr} I$.[14] But by doing so we make our characteristic variety $\operatorname{Ch}(M)$ smaller, since the annihilator of $\operatorname{Gr}^F(M)$ grows. From this point of view, holonomic modules should be exactly those modules corresponding to systems of differential equations with minimal solution space. For precisely this reason, systems of differential equations, which give rise to holonomic $\mathcal{D}_X$-modules, are sometimes called *maximally overdetermined*.

These ideas are in fact the statement of the *Riemann-Hilbert correspondence*, the most sophisticated solution to Hilbert's 21st problem. It is stated not for holonomic, but instead for *regular* holonomic $\mathcal{D}$-modules:

Theorem 6.4 (Riemann-Hilbert correspondence)**.** *Let X be a smooth algebraic variety X and denote by $D^b_{rh}(\mathcal{D}_X)$ the derived category of bounded complexes of regular holonomic $\mathcal{D}_X$-modules. The de Rham functor*

$$\mathrm{DR}_X : D^b_{rh}(\mathcal{D}_X) \to D^b_c(X)$$

is an equivalence of categories. Here $D^b_c(X)$ denotes the subcategory of $D^b(\mathrm{Mod}(\mathbb{C}_X))$ consisting of those bounded complexes, whose cohomology sheaves are constructible. Furthermore it induces an equivalence of categories

$$\mathrm{DR}_X : \mathrm{Mod}_{rh}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Perv}(\mathbb{C}_X),$$

where $\mathrm{Mod}_{rh}(\mathcal{D}_X)$ denotes the category of regular holonomic $\mathcal{D}_X$-modules and $\mathrm{Perv}(\mathbb{C}_X)$ the full subcategory of perverse sheaves of $D^b_c(X)$.

Proof: See for example [Kas84]. □

The connection to our previous formulation is best seen in the following corollary:

Corollary 6.5. *The solution functor*

$$\begin{aligned} \mathrm{Sol}_X : D^b_{rh}(\mathcal{D}_X) &\to D^b_c(X)^{\mathrm{op}} \\ M^\cdot &\mapsto R\mathscr{H}om_{\mathcal{D}_X}(M^\cdot, \mathcal{O}_X) \end{aligned}$$

gives an equivalence of categories.

In addition to the Riemann-Hilbert correspondence there are a number of reasons to study holonomic $\mathcal{D}$-modules and in particular their derived category, which we denote by $D_h(\mathcal{D}_X)$ (respectively $D^b_h(\mathcal{D}_X)$ for bounded complexes). For one thing, proposition 6.2 shows that $D^b_h(\mathcal{D}_X)$ is a full triangulated subcategory of $D^b_c(\mathcal{D}_X)$. In fact, denoting by $\mathrm{Mod}_h(\mathcal{D}_X)$ the category of holonomic $\mathcal{D}_X$-modules, [Bei87] showed that

$$D^b(\mathrm{Mod}_h(\mathcal{D}_X)) \simeq D^b_h(\mathcal{D}_X).$$

[14] Defined in the obvious way by $F^iI = F^i\mathcal{D}_X \cap I$.

Moreover, even though Lf^+ and Rf_+ might not preserve coherency, they do preserve holonomicity! So for $f : X \to Y$ we have well-defined maps

$$Lf^+ : D_h^b(\mathcal{D}_Y) \to D_h^b(\mathcal{D}_X) \qquad \text{and} \qquad Rf_+ : D_h^b(\mathcal{D}_X) \to D_h^b(\mathcal{D}_Y).$$

All other standard operations on complexes of $\mathcal{D}$-modules preserve holonomicity too, e.g. the tensor product $(\bullet) \otimes^L_{\mathcal{O}_X} (\bullet)$, the exterior tensor product $(\bullet) \boxtimes (\bullet)$ or the duality functor $\mathbb{D} : D_h^b(\mathcal{D}_X) \xrightarrow{\sim} D_h^b(\mathcal{D}_X)^{\mathrm{op}}$.

Holonomic $\mathcal{D}$-modules also have strong links to the theory of flat (or integrable) connections. This is easy to see, as a $\mathcal{D}_X$-module is nothing but an $\mathcal{O}_X$-module, together with some linear map

$$\nabla : \Theta_X \to \mathscr{E}nd_{\mathbb{C}}(M),$$

extending the $\mathcal{O}_X$-module structure. ∇ obviously has to satisfy certain conditions, but these turn out to be exactly the same as those for a connection. So the left $\mathcal{D}_X$-module structure in terms of ∇ is given by

$$\vartheta m = \nabla_\vartheta(m) \qquad \text{for } \vartheta \in \Theta_X, m \in M$$

and obviously any such structure defines a connection by the very same formula. The following theorem details this connection.

Theorem 6.6. *Let $M \neq 0$ be a coherent $\mathcal{D}_X$-module. The following four conditions are equivalent:*

(i) *M defines a flat connection.*

(ii) *M is coherent over $\mathcal{O}_X$.*

(iii) $\mathrm{Ch}(M) = T^*_X X \simeq X$ *(the zero-section of T^*X).*

(iv) *M is holonomic.*

Proof: See [HTT08]. □

7 Toric Varieties

Ultimately, our aim is to show mirror symmetry (respectively our version of that) for the Del Pezzo surface X_4. The way to do this is to show that the same mirror works in our case as it does in the case of X_4^o, the toric surface obtained by blowing up the plane in four points (which are now *not* in general position). In order to achieve this goal we need to introduce toric varieties as well as construct their mirror, which is the aim of this chapter.

7.1 Torus Embeddings

First off let us clarify that throughout we will refer to the *algebraic torus* $T_n = (\mathbb{C}^*)^n$ simply as "torus", thereby creating no confusion with the geometric definition of torus as $(S^1)^n$, since we will never actually use the latter. Now how is a toric variety related to the torus? In general we can think of toric varieties as complex algebraic varieties, obtained by partially compactifying T_n with additional elements in the boundary. This explains the original, very suggestive name of "torus embeddings". But that name was dropped in favour of toric varieties, because these objects come not only with an embedding of the T_n, but also with an action of the torus on itself, which by definition is required to extend to the entire variety.

Definition 7.1. *A* toric variety *is an algebraic variety X such that the following two conditions hold:*

1. *X contains the algebraic torus T_n as a dense open subset* [15] *and*
2. *the natural action of T_n on itself extends to an action on X.*

The significance of the induced torus action is that we may think of the points in $X \setminus T_n$ as being determined by it. We shall do this by considering the limiting behaviour of certain subgroups of T_n. From the definition of a group action it is immediate that any subgroup of T_n also induces a group action on X, so maybe by considering the correct types of subgroup we can find a structure in $X \setminus T_n$. Indeed let us define

Definition 7.2. *A* one-parameter subgroup *of T_n is a group homomorphism*

$$\begin{array}{rccc} \lambda^{(a_1,\ldots,a_n)}: & \mathbb{C}^* & \to & T_n \\ & t & \mapsto & (t^{a_1},\ldots,t^{a_n}) \end{array}$$

for some $(a_1,\ldots,a_n) \in \mathbb{Z}^n$. [16]

Now every one-parameter subgroup gives us a way of observing the behaviour of T_n inside X by virtue of considering the $\lambda^{(a_1,\ldots,a_n)}(t)$ in the limit of t becoming small, i.e.

$$\lim_{t\to 0} \lambda^{(a_1,\ldots,a_n)}(t).$$

In fact, the group of one-parameter subgroups of T_n form a free abelian group, usually denoted N, of rank $n = \dim(X)$. As X is a (partial) compactification of T_n, we are mainly interested in the boundary $X \setminus T_n$, i.e. the above limit points. Any toric variety is uniquely determined by the limiting behaviour of its one-parameter subgroups. We can neatly describe these by considering $N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$, where one-parameter subgroups with the same limit define a *cone* and all these cones together form a *fan*. X is then uniquely defined by this fan.

However, there is also a dual way of thinking about toric varieties. Instead of one-parameter subgroups we may consider *characters* of X :

Definition 7.3. *A* character *of T_n is a group homomorphism*

$$\begin{array}{rccc} \chi^{(b_1,\ldots,b_n)}: & T_n & \to & \mathbb{C}^* \\ & t=(t_1,\ldots,t_n) & \mapsto & t_1^{b_1}\cdots t_n^{b_n} \end{array}$$

for some $(b_1,\ldots,b_n) \in \mathbb{Z}^n$. [17]

Characters of T_n too form a free abelian group of rank n, called the *character lattice* and denoted M. The duality becomes obvious when we start composing one-parameter subgroups with characters. For $a \in N$ and $b \in M$ we have a map $\chi^b \circ \lambda^a : \mathbb{C}^* \to \mathbb{C}^*$, given by $t \mapsto t^l$ for some $l \in \mathbb{Z}$. In other words, we have a bilinear pairing

$$\begin{array}{rccc} \langle\ \ ,\ \ \rangle: & M \times N & \to & \mathbb{Z} \\ & (a,b) & \mapsto & l. \end{array}$$

[15] In the Zariski topology.

[16] It can be shown (see e.g. [Hum75, §16]) that every group homomorphism $\lambda : \mathbb{C}^* \to T_n$ is of this form, thus justifying that the definition restricts to integers.

[17] Once again it can be shown (e.g. [Hum75, §16]) that every group homomorphism $\chi : T_n \to \mathbb{C}^*$ is of this form.

Once we have bases for N and M, and therefore isomorphisms of each with $\mathbb{Z}^n$, this pairing is easily shown to be the usual dot product. Therefore it is non-degenerate and induces identifications $M \simeq \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and $N \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.

7.2 An Elaborate Example

In order to clarify the above concepts let us consider the following, slightly elaborate example of a toric surface, which nonetheless will show up repeatedly in the coming chapters.

Let G be the group

$$G = \left\{ (t_1, \dots, t_7) \in (\mathbb{C}^*)^7 \;\middle|\; \frac{t_1 t_2}{t_4 t_5 t_6} = \frac{t_2 t_3 t_4}{t_6 t_7} = 1 \right\}.$$

As G is a subgroup of T_7 we have a natural action

$$\begin{aligned} G \times \mathbb{C}^7 &\to \mathbb{C}^7 \\ ((t_1, \dots, t_7), (x_1, \dots, x_7)) &\mapsto (t_1 x_1, \dots, t_7 x_7). \end{aligned}$$

Furthermore define $Z \subset \mathbb{C}^7$ to be the following subset:

$$Z = \bigcup_{i-j \neq \pm 1} \{x_i = x_j = 0\},$$

where the indices i, j are supposed to be taken modulo 7.

The surface we wish to consider is defined to be the quotient by the group action of G :

$$X = \frac{\mathbb{C}^7 \setminus Z}{G}.$$

Since G is a Lie group acting smoothly, freely and properly on the smooth manifold $\mathbb{C}^7 \setminus Z$, we know that X is in fact itself a smooth manifold. Each element $x \in X$ can be denoted $[x_1, \dots, x_7] = [t_1 x_1, \dots, t_7 x_7]$ for any $(t_1, \dots, t_7) \in G$.

Now let us consider the dense open subset of X, defined by $x \in X$ such that $x_1 \cdots x_7 \neq 0$. We can assign to every element x one representative in $\mathbb{C}^7$ of the equivalence class of x in the following way:

$$\begin{aligned} [x_1, \dots, x_7] &= \left[\frac{x_7}{x_3 x_4^2 x_5} x_1, \frac{x_3 x_4}{x_6 x_7} x_2, \frac{1}{x_3} x_3, \dots, \frac{1}{x_7} x_7 \right] \\ &= [y_1, y_2, 1, \dots, 1] \end{aligned}$$

with $y_1, y_2 \in \mathbb{C}^*$. Therefore we see that we have a natural embedding

$$\begin{aligned} T_2 &\hookrightarrow X \\ (y_1, y_2) &\mapsto [y_1, y_2, 1, \dots, 1]. \end{aligned}$$

Since the natural action of T_2 on itself extends in the obvious way to an action of T_2 on X, we have indeed a toric variety. So given $T_2 \simeq \{[y_1, y_2, 1, \dots, 1]\}_{y_i \in \mathbb{C}^*}$, what can we say about the boundary of T_2 in X? Every element of X has representatives in $\mathbb{C}^7$ and given the way we defined Z and the action of G, all elements in $X \setminus T_2$ have representatives, which have zeroes in exactly one or in two adjacent coordinates. Assume that $[x_1, \dots, x_7] \in X \setminus T_2$ is such that $x_i = 0$ for some $i \in \{1, \dots, 7\}$ and $x_j \neq 0$ for all $i \neq j$. Choose some fixed $j \in \{1, \dots, 7\} \setminus \{i\}$ and define $t_k = \frac{1}{x_k}$ for $k \notin \{i, j\}$. By using the equations, which define G as a subgroup of T_7, we can find unique $t_i, t_j \in \mathbb{C}^*$ with the property that $(t_1, \dots, t_7) \in G$. Then $[x_1, \dots, x_7] = [t_1 x_1, \dots, t_7 x_7]$ simplifies to give a representative of x in $\mathbb{C}^7$, which has some complex number $z \in \mathbb{C}^*$ in the jth coordinate, a 0 in the ith coordinate and 1s everywhere else. So the condition $x_i = 0$ defines a one-dimensional subset of X, isomorphic to $\mathbb{C}^*$. E.g.

$$\left\{[x_1, \dots, x_7] \;\middle|\; x_1 \cdots x_6 \neq 0, x_7 = 0\right\} = \left\{\frac{x_2}{x_4 x_5 x_6} x_1, \frac{1}{x_2} x_2, \dots, \frac{1}{x_6} x_6, \frac{x_6}{x_2 x_3 x_4} x_7\right\}$$
$$= \{[z, 1, \dots, 1, 0]\}_{z \in \mathbb{C}^*}.$$

Similarly we can handle the last remaining cases, when we have $x \in X$ with $x_i = x_{i+1} = 0$ and $x_j \neq 0$ for $j \notin \{i, i+1\}$ (once again we consider the indices modulo 7). In these cases we can find a unique element in $(t_1, \dots, t_7) \in G$ such that $[t_1 x_1, \dots, t_7 x_7]$ has ones in all coordinates apart from the ith and $(i+1)$th, where we find 0s. So these subsets just consist of a single point. Thus we have found X to have a nice stratification as depicted in figure 1.

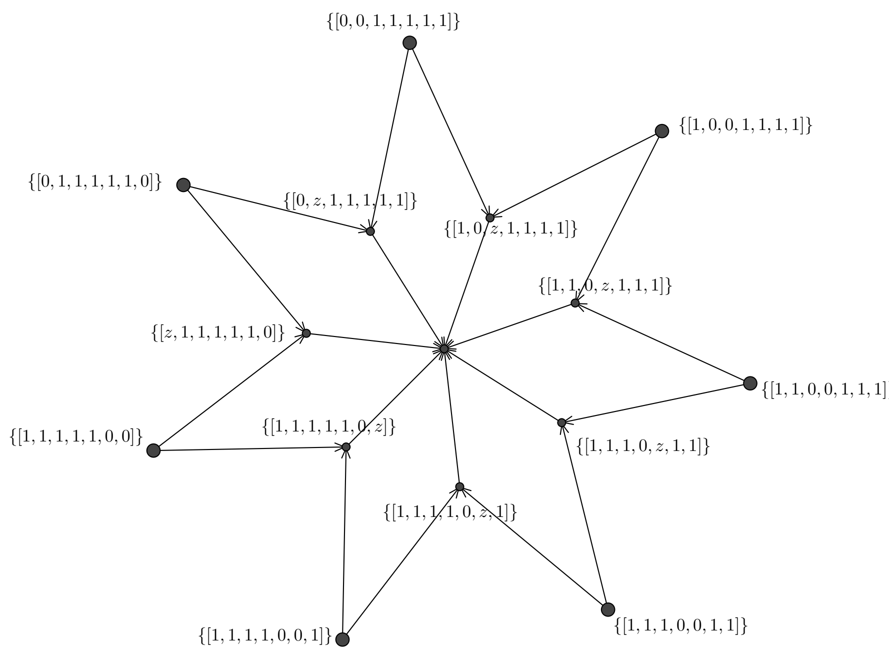

Figure 1: The stratification of X. Arrows indicate inclusion in the closure, all $z \in \mathbb{C}^*$ and the centre point represents $T_2 \in X$.

Now here is another way of looking at X, which uses our one-parameter subgroups to "probe" X. As we remarked earlier, the different limits of the one-parameter subgroups $\lambda^{(a,b)}(t) : \mathbb{C}^* \to X$

as $t \to 0$, should form a fan inside N. Note that

$$\lim_{t\to 0} t^a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{if } a > 0 \end{cases}$$

and that the limit does not exist for negative a. We therefore have the following situation:

$$\begin{aligned} \lim_{t\to 0} \lambda^{(a,b)}(t) &= \lim_{t\to 0} [t^a, t^b, 1, \dots, 1] \\ &= \lim_{t\to 0} [t^{s_1}, \dots, t^{s_7}], \end{aligned}$$

where

$$\begin{aligned} a &= s_1 + s_2 - s_4 - s_5 - s_6, \\ b &= s_2 + s_3 + s_4 - s_6 - s_7. \end{aligned}$$

The limit exists if we can find $s_1, \dots, s_7 \in \mathbb{R}_{\geq 0}$, which satisfy the above equations. Additionally we need to impose on the s_i that at most two of them are unequal to zero and if $s_i, s_j > 0$, then $i - j = \pm 1 \mod 7$. Given these conditions, we can distinguish all possible solutions, depending on which $s_i > 0$ and then reconstruct the (a, b), which correspond to those s_i. We find that:

$$\lim_{t\to 0} \lambda^{(a,b)}(t) = \begin{cases} [1,1,1,1,1,1,1] & \text{if } a = 0, b = 0, \\ [0,1,1,1,1,1,1] & \text{if } a > 0, b = 0, \\ [0,0,1,1,1,1,1] & \text{if } a > 0, b > 0, a > b, \\ [1,0,1,1,1,1,1] & \text{if } a > 0, b > 0, a = b, \\ [1,0,0,1,1,1,1] & \text{if } a > 0, b > 0, b > a, \\ [1,1,0,1,1,1,1] & \text{if } a = 0, b > 0, \\ [1,1,0,0,1,1,1] & \text{if } a < 0, b > 0, a + b > 0, \\ [1,1,1,0,1,1,1] & \text{if } a < 0, b > 0, a + b = 0, \\ [1,1,1,0,0,1,1] & \text{if } a < 0, b > 0, a + b < 0, \\ [1,1,1,1,0,1,1] & \text{if } a < 0, b = 0, \\ [1,1,1,1,0,0,1] & \text{if } a < 0, b < 0, b > a, \\ [1,1,1,1,1,0,1] & \text{if } a < 0, b < 0, a = b, \\ [1,1,1,1,1,0,0] & \text{if } a < 0, b < 0, a > b, \\ [1,1,1,1,1,1,0] & \text{if } a = 0, b < 0, \\ [0,1,1,1,1,1,0] & \text{if } a > 0, b < 0. \end{cases}$$

As we can identify the one-parameter subgroups of T_2 with $\mathbb{Z}^2$, we can picture all cases neatly in one diagram, separating the domains of different limiting behaviour. The result is shown in figure 2.

7.3 Toric Varieties

Here is a different way to think about toric varieties: notice that characters can almost be interpreted to be coordinate functions. As such we may even think of them as forming a ring of functions

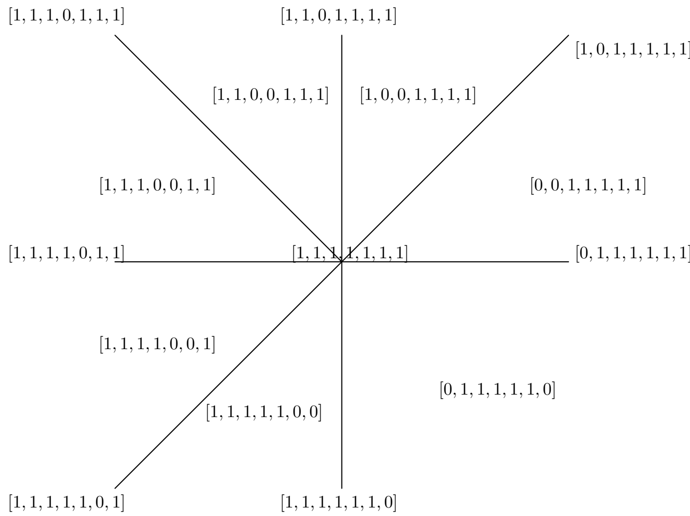

Figure 2: The subsets of $\mathbb{R}^2$ corresponding to different limiting behaviour of $\lim_{t\to 0} \lambda^{(a,b)}(t)$.

$T_n \to \mathbb{C}$. So in particular, we can think of them as $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$, where we need to include inverses, since for any $b \in \mathbb{Z}^n$ and its respective character χ^b, we have $\chi^b.\chi^{-b} = \chi^0$, the identity in this ring.

We have previously in this chapter seen, how a toric variety defines a fan in $N_{\mathbb{R}}$. By duality we therefore also obtain a collection of cones in $M_{\mathbb{R}}$ (they do not form a fan in general). Any cone of such a fan defines a subalgebra of $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$ and therefore an affine variety itself (by considering the spectrum). How are these varieties related to the original toric variety that we started with? The answer is that they are the same, so let us give a more detailed description of how to obtain a toric variety from a fan in $N_{\mathbb{R}}$.

Suppose we have a fan Σ in $N_{\mathbb{R}}$ and let us consider a face σ of it. By dualising we obtain a cone σ^* in $M_{\mathbb{R}}$. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$ and denote the characters $\chi^{e_i} = x_i$. Then all elements of σ^* form an algebra inside $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$, which we shall denote S_σ. Moreover, S_σ and thereby σ define the affine variety $\mathrm{Spec}(S_\sigma)$.

We have for instance $0 \hookrightarrow \sigma$ as a face of every strongly convex cone (we assume all our cones to be strongly convex). By duality this corresponds to the inclusion $\sigma^* \hookrightarrow M_{\mathbb{R}}$. Now the algebra defined by the entirety of $M_{\mathbb{R}}$ is exactly $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$, which we identify as the coordinate ring of the torus $(\mathbb{C}^*)^n$. So our inclusion corresponds to the affine inclusion $T_n \hookrightarrow \mathrm{Spec}(S_\sigma)$.

Here is another instructive example: Suppose we have an n-dimensional cone $\sigma \subset N_{\mathbb{R}} \simeq \mathbb{R}^n$, generated by the one-parameter subgroups corresponding to the elements $e_1, \ldots, e_n \in \mathbb{R}^n$. Then $\sigma^* \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$ is also generated by $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{R}^n$ (where ε_i is dual to e_i) and thus we have $S_\sigma = \mathbb{C}[x_1, \ldots, x_n]$ and the affine variety corresponding to σ is $\mathrm{Spec}(S_\sigma) \simeq \mathbb{C}^n$.

Note how a face $\tau \hookrightarrow \sigma$ of a cone $\sigma \subset N_{\mathbb{R}}$ now corresponds to a subvariety $\mathrm{Spec}(S_\tau) \hookrightarrow \mathrm{Spec}(S_\sigma)$. This is in fact the reason why it is often preferable to work with cones and fans in $N_{\mathbb{R}}$ rather than their equivalent counterparts in $M_{\mathbb{R}}$. Suppose now that we have two cones $\sigma_1, \sigma_2 \subset N_{\mathbb{R}}$ and that they share a common face $\tau = \sigma_1 \cap \sigma_2$. By our previous observation we now have two inclusions

$\text{Spec}(S_\tau) \hookrightarrow \text{Spec}(S_{\sigma_i})$. Thinking of the $\text{Spec}(S_{\sigma_i})$ as charts, these inclusion provide us with the gluing data, necessary to construct a variety. By this procedure a fan Σ defines an abstract toric variety X_Σ, which in turn defines a fan in $N_\mathbb{R}$ by considering the limits of one-parameter subgroups of X_Σ. This latter fan turns out to be precisely Σ.

In fact, this is part of a more general observation regarding the connection between fans and toric varieties. Many properties of the defining fan Σ "translate" to properties of X_Σ. This "dictionary" is part of the reason, why toric varieties have proven to be invaluable. Here are two more examples of this dictionary, the proves of which can be found in [CLS11].

Theorem 7.4. *Let X_Σ be defined by the fan $\Sigma \subset N_\mathbb{R}$.*

- *Then X_Σ is smooth if and only if all cones in Σ are regular, i.e. their minimal generators can be extended to a $\mathbb{Z}$-basis of N.*
- *Then X_Σ is compact (in the classical topology) if and only if the support of Σ, i.e. the union of all cones of Σ, is equal to $N_\mathbb{R}$.*

Theorem 7.5 (Orbit-Cone Correspondence)**.** *Let X_Σ be defined by the fan $\Sigma \subset N_\mathbb{R}$. Then there is a bijective correspondence*

$$\{\textit{cones } \sigma \textit{ in } \Sigma\} \longleftrightarrow \{T_n\textit{-orbits in } X_\Sigma\}.$$

Moreover, denoting the T_n-orbit corresponding to a cone σ by $O(\sigma)$, we have

$$\dim(O(\sigma)) = n - \dim(\sigma)$$

for all cones σ in Σ.

Moreover it can be shown that for a strongly convex cone $\sigma \subset N_\mathbb{R}$, the limit $\lim_{t\to 0} \lambda^u(t)$ exists in $\text{Spec}(S_\sigma)$ if and only if $u \in \sigma$. This, together with the orbit-cone correspondence explains how we recover the fan Σ as the limits of one-parameter subgroups in X_Σ.

Here are two illuminating examples:

Example 7.6. *Let Σ be the fan with three two-dimensional cones $\sigma_1, \sigma_2, \sigma_3$, which are generated by $\{e_1, e_2\}, \{e_2, -e_1 - e_2\}, \{-e_1 - e_2, e_1\}$ respectively. This fan is shown on the left in figure 3. Then the dual fans are generated by $\{\varepsilon_1, \varepsilon_2\}, \{-\varepsilon_1, -\varepsilon_1 + \varepsilon_2\}, \{\varepsilon_1 - \varepsilon_2, -\varepsilon_2\}$. So we have*

$$S_{\sigma_1} = \mathbb{C}[x, y], \qquad S_{\sigma_2} = \mathbb{C}\left[\frac{1}{x}, \frac{y}{x}\right], \qquad S_{\sigma_3} = \mathbb{C}\left[\frac{x}{y}, \frac{1}{y}\right]$$

and for all three corresponding affine varieties, which we call U_{σ_i}, we have $U_{\sigma_i} \simeq \mathbb{C}^2$. We can embed the U_{σ_i} in $\mathbb{P}^2$ as the three standard open subsets, i.e.

$$\begin{array}{ccc} U_{\sigma_1} & \hookrightarrow & \mathbb{P}^2 \\ (u,v) & \mapsto & (1:u:v) \end{array} \qquad \begin{array}{ccc} U_{\sigma_2} & \hookrightarrow & \mathbb{P}^2 \\ (u,v) & \mapsto & (u:1:v) \end{array} \qquad \begin{array}{ccc} U_{\sigma_3} & \hookrightarrow & \mathbb{P}^2 \\ (u,v) & \mapsto & (u:v:1) \end{array}$$

Let us now consider the glueing data, for example between σ_1 and σ_2. The glueing data here consists of a map $g_{12} : U_{\sigma_1} \to U_{\sigma_2}$ which maps $(x, y) \mapsto (\frac{1}{x}, \frac{y}{x})$. This induces a map on the intersection

$$\left\{(x:y:z) \;\middle|\; x \neq 0\right\} \cap \left\{(x:y:z) \;\middle|\; y \neq 0\right\} \subset \mathbb{P}^2.$$

But since $(1:u:v) = (\frac{1}{u}:1:\frac{v}{u})$, we find that the induced map is the identity. Similar computations can be done with g_{13} and g_{23} and they all induce the identity. In other words, $\mathbb{P}^2$ is obtained by glueing three copies of $\mathbb{C}^2$ in exactly the same way as we glue our U_{σ_i}, which shows that $X_\Sigma = \mathbb{P}^2$.

This example generalises easily into n dimensions. We define $e_0 = -e_1 - \cdots - e_n$ and let Σ be the fan, whose cones are generated by the proper subsets of $\{e_0, e_1, \ldots, e_n\}$. Using the same techniques as above we find that each n-dimensional cone σ corresponds to an affine patch $U_\sigma \simeq \mathbb{C}^n$ and those affine patches glue together precisely in the way that they do for $\mathbb{P}^n$, proving that $X_\Sigma \simeq \mathbb{P}^n$.

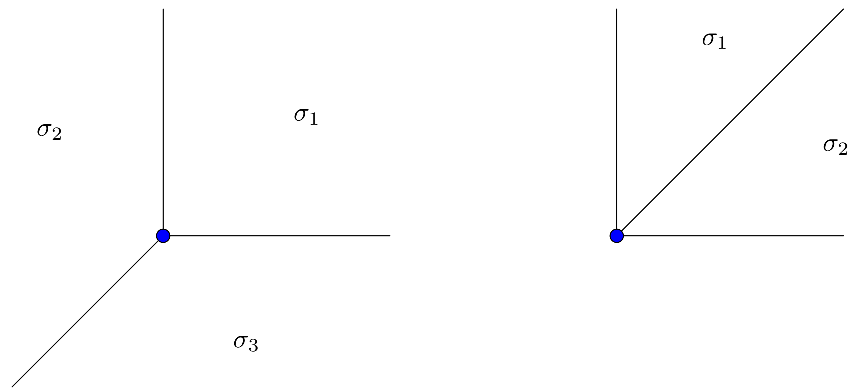

Figure 3: The fans from example one on the left (associated with $\mathbb{P}^2$) and from example two on the right (associated with $B_0(\mathbb{C}^2)$).

Example 7.7. *For our second example let Σ be the fan in $N_\mathbb{R}$, which is shown on the right of figure 3. Σ has two top-dimensional cones σ_1 and σ_2. σ_1 is generated by e_2 and $e_1 + e_2$, σ_2 by $e_1 + e_2$ and e_1. Therefore σ_1^* is generated by $\{\varepsilon_1, -\varepsilon_1 + \varepsilon_2\}$, $S_{\sigma_1} = \mathbb{C}[x, \frac{y}{x}]$ and $U_{\sigma_1} \simeq \mathbb{C}^2$. Similarly $\varepsilon_1 - \varepsilon_2$ and ε_2 generate σ_2^*, implying that $S_{\sigma_2} = \mathbb{C}[\frac{x}{y}, y]$ and $U_{\sigma_2} \simeq \mathbb{C}^2$. Now consider the blow-up of $\mathbb{C}^2$ at the origin, given as algebraic variety by*

$$B_0(\mathbb{C}^2) = \left\{ ((x_1, x_2), (y_1 : y_2)) \;\middle|\; x_1 y_2 = x_2 y_1 \right\} \subset \mathbb{C}^2 \times \mathbb{P}^1.$$

We can embed our affine spaces U_{σ_i} into $B_0(\mathbb{C}^2)$ in the following way:

$$\begin{array}{ccc} U_{\sigma_1} & \xrightarrow{\simeq} & B_0(\mathbb{C}^2) \setminus \{y_1 = 0\} \\ (u, v) & \mapsto & ((u, uv), (1 : v)) \end{array} \qquad \begin{array}{ccc} U_{\sigma_2} & \xrightarrow{\simeq} & B_0(\mathbb{C}^2) \setminus \{y_2 = 0\} \\ (u, v) & \mapsto & ((uv, u), (v : 1)) \end{array}$$

Interpreting $U_{\sigma_i} \subset B_0(\mathbb{C}^2)$, we have once again a map $g_{12} : U_{\sigma_1} \cap U_{\sigma_2} \to U_{\sigma_1} \cap U_{\sigma_2}$, given by the glueing data. The glueing data requires $(u, v) \mapsto (uv, \frac{1}{u})$, since it maps $\{\frac{y}{x}, x\}$ to $\{\frac{x}{y}, y\}$. Therefore *the induced map g_{12} is given by*

$$\begin{tikzcd}
U_{\sigma_1} \cap U_{\sigma_2} \arrow[rrr, dashed, "g_{12}"] \arrow[ddd, "\simeq"] & & & U_{\sigma_1} \cap U_{\sigma_2} \\
& ((x_1, x_2), (y_1 : y_2)) \arrow[r, mapsto] \arrow[d, mapsto] & \left(\left(x_1, x_1 \frac{y_2}{y_1}\right), \left(\frac{y_1}{y_2} : 1\right)\right) & \\
& \left(x_1, \frac{y_2}{y_1}\right) \arrow[r, mapsto] & \left(x_1 \frac{y_2}{y_1}, \frac{y_1}{y_2}\right) \arrow[u, mapsto] & \\
\mathbb{C} \times \mathbb{C}^* \arrow[rrr, "\simeq"] & & & \mathbb{C} \times \mathbb{C}^* \arrow[uuu, "\simeq"']
\end{tikzcd}$$

and we conclude that g_{12} is the identity on $U_{\sigma_1} \cap U_{\sigma_2}$, implying that $X_\Sigma = B_0(\mathbb{C}^2)$.

Recall now the fan corresponding to $\mathbb{C}^2$. It has only one two-dimensional cone, namely the cone σ generated by $e_1, e_2 \in N_\mathbb{R}$. Under the orbit-cone correspondence this cone corresponds to a T_2-orbit of dimension zero, i.e. to a point. It is easily seen that this point is in fact the origin. We can then interpret the fan of $B_0(\mathbb{C}^2)$ as the fan of $\mathbb{C}^2$, where we have replaced σ, i.e. the origin, by σ_1 and σ_2 and the ray generated by $e_1 + e_2$, i.e. by a one-dimensional T_2-orbit and two points. This is exactly what a blow-up does, replacing a point by some copy of $\mathbb{P}^1$ (at least for surfaces) and our second example is one instance of a more general procedure by which we blow up toric varieties:

Theorem 7.8. *Let $\Sigma \subset N_\mathbb{R} \simeq \mathbb{R}^n$ be the fan associated with X_Σ and suppose that σ is an n-dimensional cone of Σ, which is generated by a $\mathbb{Z}$-basis $\{u_1, \ldots, u_n\}$ of N. Furthermore let $u_0 = u_1 + \cdots + u_n$ and let $\Sigma'(\sigma)$ be the set of all cones generated by subsets of $\{u_0, u_1, \ldots, u_n\}$, which do not contain $\{u_1, \ldots, u_n\}$.*

- *Then we have a fan*
$$\Sigma^*(\sigma) = (\Sigma \setminus \{\sigma\}) \cup \Sigma'(\sigma),$$
which is in fact a refinement *of Σ.*
- *Furthermore we have a toric variety $X_{\Sigma^*(\sigma)}$ associated with this fan and an induced morphism*
$$\varphi : X_{\Sigma^*(\sigma)} \to X_\Sigma.$$
- *φ makes $X_{\Sigma^*(\sigma)}$ the blow-up of Σ in the point P_σ, corresponding under the orbit-cone correspondence to the T_n-orbit of σ.*

Proof: The first bullet point is a simple check. The second one is a little more involved and we refer the reader to [CLS11] for a detailed explanation and proof. The third bullet point becomes easy, once we accept the second. The key ingredient is noting that Σ and $\Sigma^*(\sigma)$ are the same on $N_\mathbb{R} \setminus |\sigma|$. We can therefore without loss of generality assume that X_Σ is the affine toric variety $\mathbb{C}^n$ and work the same example as above, just now in n dimensions (an easy generalisation). $\square$

Remark 7.9. *We can now characterise the toric variety first encountered in the previous chapter. From the orbit-cone correspondence we know that the fan from figure 2 defines the very variety,* *which generated it. Here is a way of constructing this fan as a blow-up of $\mathbb{P}^2$: Start with the fan of $\mathbb{P}^2$, i.e. then one with ray generators $e_1, e_2, -e_1-e_2$. For any three distinct points of $\mathbb{P}^2$, we can find an isomorphism sending them to the points $(1:0:0), (0:1:0), (0:0:1)$. So without loss of generality we may blow $\mathbb{P}^2$ up in those three points. Using the procedure described above we do this by adding the rays generated by e_1+e_2, $-e_1$ and $-e_2$. Then adding the additional ray generated by $-e_1+e_2$ amounts to further blowing up $\mathbb{P}^2$ in a fourth point, albeit a very special point, since it corresponds to a T_2-orbit, which we created by one of our earlier blow-ups. This is in fact the difference between the toric variety from the fan in figure 2 and the Del Pezzo surface X_4, since for the latter we had to blow up $\mathbb{P}^2$ in four* generic *points. For this reason we shall from now on denote the toric variety from figure* **??***Fan of X_4^o by X_4^o.*

7.4 The Kähler Cone of a Toric Variety

Since it will play an important role in chapter 8.4, let us now discuss the nef cone, and its closure the Kähler cone, of a toric variety. This is yet another example of how the fan Σ of a toric variety X_Σ determines its geometry.

In general, the Kähler cone $\mathrm{Nef}(X)$ of a smooth, complete variety X is the set of all nef divisors classes. Given that a divisor D on X is nef by definition, if $D \cdot C \geq 0$ for all irreducible curves $C \subset X$, it is clear, that $\mathrm{Nef}(X)$ is indeed a cone. Now define the *Mori cone* $\mathrm{NE}(X)$ of X to be

$$\mathrm{NE}(X) = \Big\{ \sum a_i [C_i] \;\Big|\; C_i \subset X \text{ an irreducible curve}, 0 \leq a_i \in \mathbb{R} \Big\}.$$

Then we see that the closure of the Kähler cone is the dual cone to the closure $\overline{\mathrm{NE}}(X)$ of the Mori cone, where the duality is with respect to the intersection pairing between curves and divisors. In fact, this provides us with a practical way of determining $\mathrm{Nef}(X)$: we start by finding $\overline{\mathrm{NE}}(X)$, usually with the help of the cone theorem. Then we use our knowledge of the intersection pairing to find its dual, i.e. $\mathrm{Nef}(X)$.

In our case of a toric variety X_Σ, stemming from a fan Σ, both those steps are considerably simplified by the combinatorics of Σ. Suppose that the support of Σ, i.e. the union of all its cones in $N_\mathbb{R} \simeq \mathbb{R}^n$, is convex of dimension n. Then the *toric cone theorem* [CLS11, Theorem 6.3.20] states that

$$\overline{\mathrm{NE}}(X_\Sigma) = \sum_{\tau \in \Sigma(n-1)} \mathbb{R}_{\geq 0}[D_\tau],$$

where $\Sigma(n-1)$ denotes the set of all codimension 1 cones in Σ and $[D_\tau]$ is the class of the orbit corresponding to such a cone τ under the orbit-cone correspondence. In particular, given that $\overline{\mathrm{NE}}(X_\Sigma)$ and $\mathrm{Nef}(X_\Sigma)$ are dual, we know that the Kähler cone of X_Σ is rational polyhedral. Thus we have an easy way of finding $\overline{\mathrm{NE}}(X_\Sigma)$ and in order to determine $\mathrm{Nef}(X_\Sigma)$ it only remains to dualise this cone. This is done using the intersection pairing of divisors and curves on X_Σ and is once again considerably simplified by our knowledge of Σ. Since we will focus on compact, smooth toric surfaces later on, let us demonstrate the technique in this case. Here Σ consists of the origin, some rays $\rho_1, \ldots, \rho_n$, which are generated by primitive vectors $a_1, \ldots, a_n \in N$ and the two-dimensional cones $\sigma_1, \ldots, \sigma_n$, where σ_i is generated by a_i and a_{i+1}.[18] By assumption of smoothness the set $\{a_i, a_{i+1}\}$ forms a $\mathbb{Z}$-basis of N for all i. So in particular there are four integers $\alpha, \beta, \gamma, \delta$

[18]The indices here and throughout the rest of this discussion are to be taken modulo n.

such that

$$a_{i-1} = \alpha a_i + \beta a_{i+1}$$
$$a_{i+1} = \gamma a_i + \delta a_{i-1}.$$

Upon plugging one equation into the other and using linear independence, we find that $\beta\delta = 1$, i.e. $\beta = \delta = \pm 1$. Since all our cones are assumed to be strongly convex, we know that a_{i-1} and a_{i+1} lie on two different sides of the line $\mathbb{R}a_i \subset N_{\mathbb{R}}$. Thus we have $\beta = \delta = -1$ and

$$a_{i-1} + a_{i+1} = ba_i$$

for some $b \in \mathbb{Z}$ and all i. Denote now by D_{ρ_i} the torus invariant divisor corresponding to the ray ρ_i. Then the intersection pairing on X_Σ is given by

$$D_{\rho_i} \cdot D_{\rho_j} = \begin{cases} 1 & \text{if } i - j = \pm 1 \mod n \\ -b & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

This is proven for instance in [CLS11, Theorem 10.4.4] and can easily be extended to higher dimensions and general simplicial toric varieties.

Example 7.10. *In our case of X_4^o the toric cone theorem writes*

$$\overline{\mathrm{NE}}(X_4^o) = \sum_{i=1}^{7} \mathbb{R}_{\geq 0}[D_i],$$

where D_i is the divisor corresponding to the ray ρ_i, which in turn has primitive generator a_i given in figure 2. Concretely:

$$(a_1, \ldots, a_7) = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 & -1 \end{pmatrix}.$$

In order to determine the Kähler cone we have to find the dual cone to $\overline{\mathrm{NE}}(X_4^o)$, where the pairing between two divisors is the intersection pairing. Let us choose the basis $\{[D_1], [D_2], [D_3], [D_4], [D_6]\}$ of $\mathrm{Pic}(X_4^o)$. Then according to the theory we just developed the intersection pairing is given by the following multiplication table:

	$[D_1]$	$[D_2]$	$[D_3]$	$[D_4]$	$[D_6]$
$[D_1]$	-1	1	0	0	0
$[D_2]$	1	-1	1	0	0
$[D_3]$	0	1	-2	1	0
$[D_4]$	0	0	1	-1	0
$[D_6]$	0	0	0	0	-1

Thus, writing a general element as $\alpha[D_1] + \beta[D_2] + \gamma[D_3] + \delta[D_4] + \varepsilon[D_6]$ we have to find all $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{R}$ which fulfil the following seven inequalities, arising from the intersection with each

$[D_i]$:

$$\begin{array}{rcrcrcrcrcl} -\alpha & + & \beta & & & & & & & \geq & 0, \\ \alpha & - & \beta & + & \gamma & & & & & \geq & 0, \\ & & \beta & - & 2\gamma & + & \delta & & & \geq & 0, \\ & & & & \gamma & - & \delta & & & \geq & 0, \\ & & & & & & \delta & + & \varepsilon & \geq & 0, \\ & & & & & & & - & \varepsilon & \geq & 0, \\ \alpha & & & & & & & + & \varepsilon & \geq & 0. \end{array}$$

In particular, we want to find the primitive generators of the rays of the cone defined by these inequalities. Though this task is technical and complex, it is not in fact difficult and we shall skip most of it here. Each inequality defines a half-space inside $\mathbb{R}^5$ and the rays which we are looking for, are simply the intersections of suitably many of those half-spaces. Thus the strategy is to successively equate some of these equalities until one is left with a one-dimensional solution space. For instance, assuming that the first two inequalities are in fact equalities (i.e. intersecting the first and the second half-space), we find that $\alpha = \beta$ and $\gamma = 0$. This simplifies the remaining inequalities immensely. Equalities four and six now read $-\delta \geq 0$ and $-\varepsilon \geq 0$ respectively. But since the fifth inequality assures us that the sum $\delta + \varepsilon$ is greater or equal to zero, we are led to believe that $\delta = \varepsilon = 0$ and hence we have found a ray generated by $[D_1] + [D_2]$. This way we can continue and eventually find all generators of the Kähler cone. They are

$$\begin{aligned} v_1 &= [D_1] + [D_2], \\ v_2 &= [D_1] + 2[D_2] + [D_3], \\ v_3 &= 2[D_1] + 2[D_2] + [D_3], \\ v_4 &= [D_1] + [D_2] + [D_3] + [D_4], \\ v_5 &= [D_1] + 3[D_2] + 2[D_3] + [D_4] - [D_6], \\ v_6 &= [D_1] + 2[D_2] + [D_3] + [D_4] - [D_6], \\ v_7 &= [D_1] + [D_2] + [D_3] + [D_4] - [D_6], \\ v_8 &= [D_2] + [D_3] + [D_4]. \end{aligned}$$

7.5 The Landau-Ginzburg Model of a Toric Variety

The aim of this chapter is to construct the mirror theoretic dual to a toric variety X_Σ. This dual is called *Landau-Ginzburg model* and was first proposed by Givental [Giv95, Giv98]. In order to construct this Landau-Ginzburg model we will need a particular short exact sequence, which will be introduced in theorem 7.11. In our proof of theorem 7.11 we will be closely following [CLS11].

Let us first set up some notation and convention: let $\mathrm{Div}_T(X_\Sigma)$ be the sub-group of the group of divisors[19] $\mathrm{Div}(X_\Sigma)$, which are invariant under the torus action on X_Σ. In other words, a divisor D is in $\mathrm{Div}_T(X_\Sigma)$ if and only if $t.D = D$ for all t in the torus T.

Recall that we previously used the description $\mathrm{Pic}(X) = H^2(X, \mathbb{Z})$ (which is true under provided that $H^1(X, \mathcal{O}_X) = 0$, in particular for Del Pezzo surfaces) and also described the Picard group as the group of invertible line bundles on X with the group operation being the tensor product. In this chapter, however, we will use the bijection between divisors and line bundles on X. Since two

[19]We shall mix up Weil and Cartier divisors freely in this chapter, since all varieties that we consider will be smooth.

linearly equivalent divisors define the same line bundle, we shall think of $\mathrm{Pic}(X)$ as the class of divisors modulo linear equivalence.

Theorem 7.11. *For a smooth, compact toric variety X_Σ we have a short exact sequence*

$$0 \longrightarrow M \longrightarrow \mathrm{Div}_T(X_\Sigma) \longrightarrow \mathrm{Pic}(X_\Sigma) \longrightarrow 0, \tag{27}$$

where M denotes the character lattice. The map $\mathrm{Div}_T(X_\Sigma) \to \mathrm{Pic}(X_\Sigma)$ is given naturally in our above interpretation as sending a divisor D to the equivalence class of its associated line bundle. Here the first map interprets $m \in M$, respectively its character χ^m, as a rational function on X_Σ and maps

$$m \mapsto \sum_{D \in \mathrm{Div}_T(X_\Sigma) \text{ prime}} \mathrm{ord}_D(\chi^m) D,$$

where $\mathrm{ord}_D(f)$ denotes the order (of vanishing or of poles) of a rational $f : X_\Sigma \dashrightarrow \mathbb{C}$ along D.

Let us start off by investigating the structure of $\mathrm{Div}_T(X_\Sigma)$:

Lemma 7.12. *Denote the set of rays of Σ by $\Sigma(1)$ and the closure of the orbit associated to $\rho \in \Sigma(1)$ by D_ρ. Then*

$$\mathrm{Div}_T(X_\Sigma) = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho.$$

Proof: Suppose that we have $D \in \mathrm{Div}_T(X_\Sigma)$ and assume furthermore that D is supported on $T \subset X_\Sigma$, i.e. that there exists $x \in \mathrm{Supp}(D) \cap T_n$. Now let $y \in T_n$ be arbitrary. Since $(yx^{-1}).D = D$, we know that $y \in \mathrm{Supp}(D)$ and therefore $T_n \subset \mathrm{Supp}(D)$. However, this is clearly a contradiction as $\mathrm{codim}(\mathrm{Supp}(D)) = 1$ and $\mathrm{codim}(T_n) = 0$. We conclude that a torus-invariant divisor D is only supported on $X_\Sigma \setminus T_n$.

Now let $x \in X_\Sigma \setminus T_n$. Then x determines a T_n-orbit, which corresponds to some face σ of Σ. If $\sigma = \rho \in \Sigma(1)$, then $x \in D_\rho$. Otherwise σ contains a one-dimensional ray ρ and by our construction of X_Σ we know that $x \in U_\sigma$ is in the closure of U_ρ and so once again we have $x \in D_\rho$. Thus

$$X_\Sigma \setminus T_n = \bigcup_{\rho \in \Sigma(1)} D_\rho$$

and the assertion follows immediately. □

Lemma 7.13. *In the same notation as before let $m \in M$ have an associated character $\chi^m \in \mathbb{C}(X_\Sigma)$, the rational functions on X_Σ. For $\rho \in \Sigma(1)$ let $a_\rho \in N$ denote its primitive generator in N. Then*

$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, a_\rho \rangle D_\rho.$$

Proof: This is trivially true if $m = 0$, so assume $m \neq 0$. Since χ^m is both defined and does not vanish on $T_n \subset X_\Sigma$

$$\mathrm{Supp}(\mathrm{div}(\chi^m)) \subset X_\Sigma \setminus T_n = \bigcup_{\rho \in \Sigma(1)} D_\rho$$

and we are only interested in $\mathrm{ord}_{D_\rho}(\chi^m)$. Because a_ρ is primitive, we have an isomorphism $N \simeq \mathbb{Z}^n$, where a_ρ maps to the standard basis vector e_1. Thus we know that

$$U_\rho = \mathrm{Spec}(\mathbb{C}[x_1, x_2^{\pm}, \ldots, x_n^{\pm}]) \simeq \mathbb{C} \times (\mathbb{C}^*)^{n-1}.$$

$m \neq 0$ implies that $\chi^m \in \mathbb{C}(x_1, \dots, x_n)$ is invertible. We can write

$$\chi^m = x_1^l \frac{f}{g} \tag{28}$$

for some $l \in \mathbb{Z}$ and $f, g \in \frac{\mathbb{C}[x_1,\dots,x_n]}{\langle x_1 \rangle}$. Now the e_i are one-parameter sub-groups and the x_i are their dual characters. In other words, $x_i \circ e_j : \mathbb{C}^* \to T \to \mathbb{C}^*$ and $x_i \circ e_j = \delta_{ij}$. Thus we have a decomposition

$$\chi^m = \chi^{(m_1,\dots,m_n)} = \chi^{\langle m,e_1\rangle\varepsilon_1} \cdots \chi^{\langle m,e_n\rangle\varepsilon_n} = x_1^{\langle m,e_1\rangle} \cdots x_n^{\langle m,e_n\rangle}. \tag{29}$$

By comparison of equations (28) and (29) we see that

$$\mathrm{ord}_{D_\rho}(\chi^m) = \langle m, e_1 \rangle = \langle m, a_\rho \rangle$$

and therefore

$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, a_\rho \rangle D_\rho.$$

□

Proof of Theorem 7.11: We have already defined all maps of our short exact sequence so it remains only to prove exactness.

Exactness at M is equivalent to the injectivity of $M \to \mathrm{Div}_T(X_\Sigma)$. By theorem 7.4 we know that compactness and smoothness of X_Σ imply that $\{a_\rho\}_{\rho\in\Sigma(1)}$ contains a basis of $N_\mathbb{R}$. Injectivity is then a trivial corollary from lemma 7.13.

Exactness at $\mathrm{Pic}(X_\Sigma)$ boils down to the surjectivity of the map $\mathrm{Div}_T(X_\Sigma) \to \mathrm{Pic}(X_\Sigma)$. So let $[D] \in \mathrm{Pic}(X_\Sigma)$. Then D is a divisor on X_Σ (though not necessarily torus-invariant). Consider $D|_{T_n}$, its restriction to $T_n \simeq (\mathbb{C}^*)^n$. It is well-known that $\mathrm{Pic}((\mathbb{C}^*)^n) = 0$, implying that $D|_{T_n} = \mathrm{div}(f)$ for some rational f on T_n. Since X_Σ is just the closure of T_n, f also defines a rational $f : X_\Sigma \dashrightarrow \mathbb{C}$ and $D|_{T_n} = \mathrm{div}(f)|_{T_n}$. Then

$$\mathrm{Supp}(D - \mathrm{div}(f)) \subset X_\Sigma \setminus T_n,$$

implying that $D - \mathrm{div}(f) \in \mathrm{Div}_T(X_\Sigma)$.

So we are left to show exactness at $\mathrm{Div}_T(X_\Sigma)$. Obviously we have that $[\mathrm{div}(\chi^m)] = 0$, so the composition of maps $M \to \mathrm{Pic}(X_\Sigma)$ is zero. So suppose that we have $D \in \mathrm{Div}_T(X_\Sigma)$ with $[D] = 0 \in \mathrm{Pic}(X_\Sigma)$. This says that $D = \mathrm{div}(f)$ for some rational $f \in \mathbb{C}(X_\Sigma)$. As D is torus-invariant, it can only be supported outside of T_n, i.e. $\mathrm{Supp}(D) \subset X_\Sigma \setminus T_n$ Hence we know that $\mathrm{div}(f)|_{T_n} = 0$. This in turn implies that $f|_{T_n} : T_n \to \mathbb{C}^*$ is an invertible morphism and thus has the form $f|_{T_n} = c\chi^m$ for some non-zero $c \in \mathbb{C}$ and $m \in M$ (c.f. [Hum75]). This shows that D is in the image of $M \to \mathrm{Div}_T(X_\Sigma)$ and thus concludes the proof. □

We can now start our construction of the Landau-Ginzburg model of X_Σ. In general a Landau-Ginzburg model is the data $f : Y \to Y'$, a dominant morphism between two varieties, and a holomorphic $\kappa : Y \to \mathbb{C}$, sometimes called the *superpotential*. Let us start by finding the morphism f. Given the short exact sequence (27), we have a surjective map

$$\mathrm{Div}_T(X_\Sigma) \to \mathrm{Pic}(X_\Sigma),$$

which, after a choice of basis, becomes a surjective map

$$\mathbb{Z}^{|\Sigma(1)|} \to \mathbb{Z}^{|\Sigma(1)|-\dim(X_\Sigma)}.$$

Denote $|\Sigma(1)| = n$, $\dim(X_\Sigma) = d$ and $|\Sigma(1)| - \dim(X_\Sigma) = n - d = r$. Upon tensoring over $\mathbb{Z}$ with $\mathbb{C}^*$ our short exact sequence (27) becomes

$$1 \longrightarrow S_0 = (\mathbb{C}^*)^d \longrightarrow S_1 = (\mathbb{C}^*)^n \xrightarrow{\vartheta} S_2 = (\mathbb{C}^*)^r \longrightarrow 1.$$

In particular, we have a torus fibration $\vartheta : S_1 \to S_2$. This fibration will play the role of our $f : Y \to Y'$, as introduced earlier. Furthermore we define the superpotential of this Landau-Ginzburg model to be

$$\begin{aligned} \kappa : S_1 &\to \mathbb{C}, \\ (x_1, \ldots, x_n) &\mapsto \sum_{i=1}^{n} x_i. \end{aligned}$$

Let us denote this Landau-Ginzburg model succinctly by writing

$$W = (\kappa, \vartheta) : S_1 \longrightarrow \mathbb{C}_t \times S_2, \tag{30}$$

where the index t denotes our coordinate on $\mathbb{C}$.

Example 7.14. *Let us construct the Landau-Ginzburg model in the concrete case of our toric variety X_4^o from chapter 7. Recall the specific form of $\Sigma \subset N$ from figure* **??***Fan of X_4^o. Choosing a basis $\{e_1, e_2\}$ of N determines the concrete form of the rays in $\Sigma(1)$ and in particular their primitive generators $a_1, \ldots, a_7$:*

$$(a_1, \ldots, a_7) = (e_1, e_1 + e_2, e_2, -e_1 + e_2, -e_1, -e_1 - e_2, -e_2).$$

This in turn, combined with lemma 7.13, determines the first map in our short exact sequence:

$$\begin{aligned} &M \to \mathrm{Div}_T(X_4^o), \\ &\varepsilon_1 \mapsto D_1 + D_2 - D_4 - D_5 - D_6, \\ &\varepsilon_2 \mapsto D_2 + D_3 + D_4 - D_6 - D_7. \end{aligned}$$

Having identified $M \simeq \mathbb{Z}^2$ via the basis $\{\varepsilon_1, \varepsilon_2\}$, let us also identify $\mathrm{Div}_T(X_4^o) \simeq \mathbb{Z}^7$ by choosing the basis $\{D_1, \ldots, D_7\}$. Then, assuming row vectors, the above map is represented by the matrix

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 & -1 \end{pmatrix}.$$

So let us choose a basis for $\mathrm{Pic}(X_4^o) \simeq \mathbb{Z}^5 : \{[D_1], [D_2], [D_3], [D_4], [D_6]\}$. Then the interesting map $\mathrm{Div}_T(X_4^o) \to \mathrm{Pic}(X_4^o)$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

After tensoring with $\mathbb{C}^*$ *we can now summarise our Landau-Ginzburg model as before in the following form:*

$$W : (\mathbb{C}^*)^7 \to \mathbb{C} \times (\mathbb{C}^*)^5,$$
$$(x_1, \ldots, x_7) \to \left(x_1 + \cdots + x_7, \left(x_1x_5, x_2x_5x_7, x_3x_7, \frac{x_4x_7}{x_5}, \frac{x_6}{x_5x_7} \right) \right).$$

There is, by no means, a canonical choice of basis for $\mathrm{Pic}(X_4^o)$ *and the above mechanism works for any basis, but always depends on that basis. For reasons that will become clear later (chapter 8.4 and in particular example 8.16), choose the following basis for* $\mathrm{Pic}(X_4^o)$:

$$\begin{aligned}
u_1 &= [D_1] + [D_2], \\
u_2 &= [D_1] + 2[D_2] + [D_3], \\
u_3 &= 2[D_1] + 2[D_2] + [D_3], \\
u_4 &= [D_1] + [D_2] + [D_3] + [D_4], \\
u_5 &= [D_1] + [D_2] + [D_3] + [D_4] - [D_6].
\end{aligned}$$

With respect to this basis $\mathrm{Div}_T(X_4^o) \to \mathrm{Pic}(X_4^o)$ *is represented by*

$$\begin{pmatrix}
0 & -1 & 1 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 \\
-2 & 0 & 1 & 0 & 0 \\
1 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 1 & -1 \\
0 & 1 & -1 & 0 & 1
\end{pmatrix}.$$

In other words, we have

$$W : (\mathbb{C}^*)^7 \longrightarrow \mathbb{C} \times (\mathbb{C}^*)^5,$$
$$(x_1, \ldots, x_7) \mapsto \left(x_1 + \cdots + x_7, \left(\frac{x_2x_4}{x_3^2}, \frac{x_2x_7}{x_1}, \frac{x_1x_3x_5}{x_2x_4x_7}, \frac{x_4x_6}{x_5^2}, \frac{x_5x_7}{x_6} \right) \right).$$

We can actually apply this example's construction to the general case: let X_Σ be a smooth, compact toric variety. Choose a basis $\{e_1, \ldots, e_d\}$ of N. If $\Sigma(1) = \{\rho_1, \ldots, \rho_n\}$, then we can describe the corresponding primitive integral generators a_i of the ρ_i as (column) vectors

$$a_i = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{id} \end{pmatrix} = a_{i1}e_1 + \cdots + a_{id}e_d.$$

With the natural basis $\{D_1, \ldots, D_n\}$ of $\mathrm{Div}_T(X_\Sigma)$, where D_i is the divisor corresponding to ρ_i, we can express the first map of the sequence (27) as

$$M \to \mathrm{Div}_T(X_\Sigma),$$
$$m_1e_1 + \cdots + m_de_d \mapsto \sum_{i=1}^{n} \sum_{j=1}^{d} a_{ij}m_jD_i,$$

where once again, $\{\varepsilon_i\} \subset M$ is the dual basis to $\{e_i\} \subset N$. In other words:

$$(m_1, \ldots, m_d) \mapsto (m_1, \ldots, m_d) \cdot (a_1, \ldots, a_n).$$

In order to similarly express the second map of (27), we need to choose a basis $\{u_1, \ldots, u_r\}$ of $\mathrm{Pic}(X_\Sigma)$. Once we have done so, we can write

$$\begin{aligned} \mathrm{Div}_T(X_\Sigma) &\to \mathrm{Pic}(X_\Sigma), \\ \sum_{i=1}^n d_i D_i &\to \sum_{j=1}^r \sum_{i=1}^n b_{ij} d_i u_j, \end{aligned}$$

where

$$B = (b_1, \ldots, b_r) = \begin{pmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nr} \end{pmatrix}$$

is a $n \times r$ integer matrix, satisfying

$$(a_1, \ldots, a_n) \cdot B = 0.$$

On tensoring our short exact sequence over $\mathbb{Z}$ with $\mathbb{C}^*$, we thus gain the following explicit description of ϑ :

$$\begin{aligned} \vartheta : S_1 &\to S_2 \\ (x_1, \ldots, x_n) &\mapsto (x^{b_1}, \ldots, x^{b_r}), \end{aligned}$$

where we used the short-hand notation $x^{b_i} = x_1^{b_{1i}} \cdots x_n^{b_{ni}}$.

8 Brieskorn's Lattice and the Gauß-Manin Connection

In this thesis we will not be concerned with mirror symmetry in its original form as in chapter 10.1. Instead, we shall prove a mirror theorem for $\mathcal{D}$-modules. Simply put, this will involve the equivalence of two $\mathcal{D}$-modules: the A-model $\mathcal{D}$-module and the B-model $\mathcal{D}$-module. The former is also known as the *quantum $\mathcal{D}$-module*. Being on the A-side of mirror symmetry, it is essentially given by the Dubrovin connection, which we discussed earlier. The upcoming two chapters are concerned with finding the analogue to the Dubrovin connection on the B-side. We defined the Dubrovin connection to be a flat, meromorphic connection on a holomorphic vector bundle over $\mathbb{P}^1 \times U$, where $U \subset \mathbb{C}^r$ was some open neighbourhood of the origin in some local compactification of a parameter space. Our first step in finding the B-model analogue will be to construct a holomorphic vector bundle with meromorphic connection on $\mathbb{C} \times U$. For historical reasons this is called the Brieskorn lattice and is the subject of the next chapter. The ensuing chapter will then consequently be concerned with the Birkhoff problem, which is the attempt to extend this vector bundle with connection to $\mathbb{P}^1 \times U$.

8.1 Brieskorn's Lattice According to Himself

The Brieskorn Lattice has its origins, as suggested by the name, in a groundbreaking paper by Brieskorn [Bri70] on the monodromy of an isolated hypersurface singularity. The set-up is the following situation: we are given a holomorphic map $f : X \to \Delta$, where $X \subset \mathbb{C}^{n+1}$ (containing the origin) and $\Delta \subset \mathbb{C}$ is a small disc centred at the origin. We assume that f has an isolated hypersurface singularity at $0 \in X$ and that $f(0) = 0$; hence (in local co-ordinates $x_0, \dots, x_n$ on $\mathbb{C}^{n+1}$) $\partial_{x_i}(f) = 0$ for all i. Brieskorn was originally motivated by the study of the monodromy of the germ of an isolated hypersurface singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. In choosing sufficiently small balls around both origins and restricting to the preimage, we can arrive in precisely the above situation $f : X \to \Delta$.

Milnor had proven earlier in [Mil68] that (for X and Δ sufficiently small), for each $t \in \Delta' = \Delta \setminus \{0\}$ the fibre $X_t = f^{-1}(t)$ has the homotopy type of a bouquet of μ n-spheres, where μ is the aptly called *Milnor number*. Moreover, the restriction (still denoted by f) $f : X' = X \setminus X_0 \to \Delta'$, is a locally trivial fibre bundle. The monodromy of the singularity is given, as the action of $\pi_1(\Delta', t_0) \simeq \mathbb{Z}$ on the cohomology group $H^n(X_{t_0}, \mathbb{C}) \simeq \mathbb{C}^\mu$. Define a vector bundle $\underline{H} = \cup_{t \in \Delta'} H^n(X_t, \mathbb{C})$ on Δ'. There is an equivalence of categories between the category of vector bundles on Δ' and the category of locally free sheaves on Δ'. This equivalence assigns to $\underline{H}$ the sheaf of sections

$$\mathcal{H} = \underline{H} \otimes_{\mathbb{C}_{\Delta'}} \mathcal{O}_{\Delta'} = R^p f_* \mathbb{C}_{X'} \otimes_{\mathbb{C}_{\Delta'}} \mathcal{O}_{\Delta'}.$$

Furthermore we have a natural connection ∇ on $\mathcal{H}$, defined by $\nabla(\sigma \otimes g) = \sigma \otimes dg$ for $\sigma \in \underline{H}$ and g a local section of $\mathcal{O}_{\Delta'}$. We will think about ∇ as being defined by the flat sections of $\mathcal{H}$, which we interpret as elements of $\underline{H} = R^p f_* \mathbb{C}_{X'}$. This connection on $\mathcal{H}$ is called the *Gauß-Manin connection*. The presence of the Gauß-Manin connection enables us to talk about parallel transport and thus we can describe the monodromy action of an element $\gamma \in \pi_1(\Delta', t_0)$ in the following manner: choose an element of $H^n(X_{t_0}, \mathbb{C})$ and transport it parallel along γ. This defines a map $h_\gamma : H^n(X_{t_0}, \mathbb{C}) \to H^n(X_{t_0}, \mathbb{C})$ and the monodromy of f is defined to be the representation $h : \pi_1(\Delta', t_0) \to \mathrm{Aut}(H^n(X_{t_0}, \mathbb{C}))$ given by $\gamma \mapsto h_\gamma$.

Brieskorn started his observations by studying the complex of relative de Rham differentials $\Omega^*_{Y/Z}$ for a morphism of smooth varieties $g : Y \to Z$. Let Ω^p_Y denote the sheaf of holomorphic p-forms on Y, then

$$\Omega^p_{Y/Z} := \frac{\Omega^p_Y}{dg \wedge \Omega^{p-1}_Y}.$$

This complex gives rise to sheaves $\mathcal{H}^p_{\mathrm{DR}}(Y/Z) = R^p g_* \Omega^\bullet_{Y/Z}$ on Z, which have particularly nice properties, when applied to our case of $g = f$ being the Milnor fibration. For instance, Brieskorn proved that $\mathcal{H}^p_{\mathrm{DR}}(X/\Delta)$ are coherent sheaves of $\mathcal{O}_\Delta$-modules and [Seb70] proved that they are torsion-free. Moreover, it is not difficult to see that

$$\mathcal{H}^n_{\mathrm{DR}}(X/\Delta)\big|_{\Delta'} \simeq \mathcal{H}.$$

Therefore we can interpret $\mathcal{H}^n_{\mathrm{DR}}(X/\Delta)$ as an extension of $\mathcal{H}$ over the origin. Since we have the Gauß-Manin connection on $\mathcal{H}$, we now want to transfer it to $\mathcal{H}^n_{\mathrm{DR}}(X/\Delta)$ and hope that it has a nice extension over the origin.

In order to do this, let us consider the homological Milnor fibration $\underline{H}^* = \cup_{t \in \Delta'} H_n(X_t, \mathbb{C})$ and the associated, locally free sheaf $\mathcal{H}^*$. We have a non-degenerate pairing $\langle , \rangle : \underline{H} \times \underline{H}^* \to \mathbb{C}$ which extends to a non-degenerate pairing $\langle , \rangle : \mathcal{H} \times \mathcal{H}^* \to \mathcal{O}_{\Delta'}$. Similarly we have a non-degenerate pairing $\mathcal{H}^n_{\mathrm{DR}}(X'/\Delta') \times \mathcal{H}^* \to \mathcal{O}_{\Delta'}$. This is given in the following manner: let $\sigma \in \underline{H}^*$ be a flat local section of $\mathcal{H}^*$. We think of σ as $\sigma(t) \in H_n(X_t, \mathbb{C})$, a family of cycles. Now let ω be a local section of $\mathcal{H}^n_{\mathrm{DR}}(X'/\Delta')$. Then ω is represented by an n-form (still denoted ω) such that $d\omega = df \wedge \eta$ for some n-form η. Our pairing is then given by

$$\begin{aligned} I(t) : \mathcal{H}^n_{\mathrm{DR}}(X'/\Delta') \times \mathcal{H}^* &\to \mathcal{O}_{\Delta'} \\ (\omega, \sigma) &\mapsto \int_{\sigma(t)} \omega. \end{aligned}$$

[Kul98] then shows two facts about these pairings:

- with ω, σ as before:
$$\frac{d}{dt}(I(t)) = \int_{\sigma(t)} \eta.$$
- With ω' a local section of $\mathcal{H}$ and σ as before (i.e. a local section in $\underline{H}^*$):
$$\frac{d}{dt}\langle \omega', \sigma \rangle = \langle \nabla_{\partial_t}(\omega'), \sigma \rangle.$$

Thus we see that the flat connection ∇ on $\mathcal{H}$ carries over via our identification of $\mathcal{H} \simeq \mathcal{H}^n_{\mathrm{DR}}(X/\Delta)|_{\Delta'}$ to a connection

$$\begin{aligned} \nabla_{\partial_t} : \mathcal{H}^n_{\mathrm{DR}}(X/\Delta)_0 &\to \mathcal{H}^n_{\mathrm{DR}}(X/\Delta)_0 \\ \omega &\mapsto \eta, \end{aligned} \tag{31}$$

where $d\omega = df \wedge \eta \in \Omega^{n+1}_X$. However, there is a problem with this: we do not know a priori, whether η represents an element of $\mathcal{H}^n_{\mathrm{DR}}(X/\Delta)_0$, i.e. whether there exists $\xi \in \Omega_X, 0^n$ with $d\eta = df \wedge \xi$. Indeed, it might not. This is expression of the fact that f has a critical value at 0, i.e. the fibre at the origin is singular. Therefore we expect the connection ∇ to also have a singularity at $0 \in \Delta$. So we have to expand our definition of connection to include meromorphic connections:

Definition 8.1. *Let $\mathcal{K} = \mathcal{O}_{\Delta,0}[t^{-1}]$ be the field of germs of meromorphic functions at $0 \in \Delta$. Let M be a finite-dimensional vector space over $\mathcal{K}$. A* meromorphic connection *on M is a $\mathbb{C}$-linear map*

$$\nabla_{\partial_t} : M \to M$$

satisfying the Leibniz rule, i.e. $\nabla_{\partial_t}(gm) = \frac{dg}{dt}m + g\nabla_{\partial_t}(m)$. For a meromorphic connection on M we define a lattice *of M to be a finitely-generated $\mathcal{O}_{\Delta,0}$-submodule $E \subset M$ such that $\mathcal{K}E = M$.*

Remark 8.2. *Given an $\mathcal{O}_{\Delta,0}$-module E of finite rank with a connection ∇_{∂_t}, we can extend this connection naturally to a connection on $M = E \otimes \mathcal{K}$, by defining*

$$\nabla_{\partial_t}(e \otimes t^{-k}) = \nabla_{\partial_t}(e) \otimes t^{-k} - e \otimes kt^{-k-1}$$

for any $e \in E$.

So in this new terminology we can express our map ∇_{∂_t} (which, as we saw, was not in fact a connection, not even well-defined) from (31) as a map

$$\nabla_{\partial_t} : \mathcal{H}^n_{\mathrm{DR}}(X/\Delta)_0 \otimes \mathcal{K} \to \mathcal{H}^n_{\mathrm{DR}}(X/\Delta)_0 \otimes \mathcal{K}$$

and we find that it now defines a connection, which we will still refer to as Gauß-Manin connection. Furthermore $\mathcal{H}^n_{\mathrm{DR}}(X/\Delta)_0$ is a lattice. Brieskorn in [Bri70] showed a number of properties of this Gauß-Manin connection, chief amongst are the fact that it is regular singular, meaning that flat sections have moderate growth towards $0 \in \Delta$ and that the monodromy is neatly encoded in the differential equation that these flat sections have to fulfil.

Moreover, he was able to explicitly compute the monodromy by giving a formula for the Gauß-Manin connection: he did so by considering two more natural lattices in $\mathcal{H}^n_{\mathrm{DR}}(X/\Delta)_0 \otimes \mathcal{K}$.

$$\mathcal{H}' := \frac{f_* \Omega^n_{X/\Delta}}{d(f_* \Omega^{n-1}_{X/\Delta})}$$

is an interesting sheaf for two reasons: firstly we have a natural inclusion $\mathcal{H}^n_{\mathrm{DR}}(X/\Delta) \hookrightarrow \mathcal{H}'$, which is in fact an isomorphism when restricted to Δ'. Secondly the connection operator ∇_{∂_t} restricts to $\mathcal{H}^n_{\mathrm{DR}}(X/\Delta)$ nicely:

$$\begin{aligned} \nabla_{\partial_t} : \mathcal{H}^n_{\mathrm{DR}}(X/\Delta) &\to \mathcal{H}' \\ \omega &\mapsto \eta, \end{aligned} \tag{32}$$

where $d\omega = df \wedge \eta \in \Omega^{n+1}_X$. This can be seen easily when considering the problem we had with the original definition of ∇_{∂_t} in (31). Moreover, the stalks at 0 of both sheaves define lattices of our connection and in fact

$$\nabla_{\partial_t} : \mathcal{H}^n_{\mathrm{DR}}(X/\Delta)_0 \to \mathcal{H}'_0$$

provides an isomorphism (c.f. [Kul98]). The second natural sheaf, which Brieskorn considered, is

$$\mathcal{H}'' := \frac{f_* \Omega^{n+1}_X}{df \wedge d(f_* \Omega^{n-1}_X)}.$$

Once again there is an isomorphism

$$\mathcal{H}^n_{\mathrm{DR}}(X/\Delta)|_{\Delta'} \simeq \mathcal{H}'|_{\Delta'} \simeq \mathcal{H}''|_{\Delta'}.$$

Moreover, we also have an inclusion

$$df \wedge : \mathcal{H}' \hookrightarrow \mathcal{H}''.$$

This, combined with the Gauß-Manin connection as in (32) imply that

$$\begin{aligned} \nabla_{\partial_t} : \mathcal{H}' &\to \mathcal{H}'' \\ \omega &\to d\omega. \end{aligned} \tag{33}$$

$\mathcal{H}''_0$ is called the *Brieskorn lattice*.

One may ask, how the Brieskorn lattice is special enough to merit its own name. For Brieskorn the answer lay in the computability of the Gauß-Manin connection using the Brieskorn lattice: note that an element of $\mathcal{H}''$ is represented by an $(n+1)$-form $\omega = g(x)dx$.[20] It is not difficult to show that there exists $k \in \mathbb{N}$ such that $f^k\mathcal{H}'' \subset \mathcal{H}'$ (c.f. [Kul98]). So once we have found some $\xi \in \Omega^n_X$ such that $f^k dx = df \wedge \xi$, we can combine (32) and (33) to

$$\nabla_{\partial_t}(f^k\omega) = kf^{k-1}\omega + f^k\nabla_{\partial_t}(\omega) = \nabla_{\partial_t}(df \wedge g\xi) = d(g\xi)$$

and thus obtain the explicit formula for the Gauß-Manin connection on the Brieskorn lattice as

$$\begin{aligned} \nabla_{\partial_t} : \mathcal{H}''_0 \otimes \mathcal{K} &\to \mathcal{H}''_0 \otimes \mathcal{K} \\ \omega &\mapsto f^{-k}d(g\xi) - kf^{-1}\omega. \end{aligned}$$

But the Brieskorn lattice has more interesting properties: [Seb70] showed that $\mathcal{H}''_0$ is torsion free and [Mal74] later refined this to show that $\mathcal{H}''_0 \otimes \mathcal{K}$ is a vector space of dimension μ (the Milnor number) over $\mathcal{K}$.[21] Moreover, Malgrange showed in [Mal74, Mal75] that the Brieskorn lattice has another nice module structure:

Theorem 8.3.

- *The action of ∂_t induced by ∇ has an inverse ∂_t^{-1}.*
- *The action of ∂_t^{-1} makes $\mathcal{H}''_0$ into a module over the ring of micro-local differential operators*

$$\mathbb{C}\{\{\partial_t^{-1}\}\} = \left\{ \sum_{k\geq 0} a_k\partial_t^{-k} \;\middle|\; \sum_{k\geq 0} |a_k|\frac{r^k}{k!} \textit{ converges for some } r > 0 \right\}.$$

- *$\mathcal{H}''_0$ is also free of rank μ over the ring $\mathbb{C}\{\{\partial_t^{-1}\}\}$.*

However, a priori, the Brieskorn lattice is just one lattice of many of the Gauß-Manin connection. In the original case of an isolated hypersurface singularity, M. Saito gave a characterisation of $\mathcal{H}''_0$ in terms of Hodge theory. In our case of a non-isolated singularity, [DS03, Sab06] suggested that instead of looking at the Gauß-Manin system, we should consider the Fourier-Laplace transformed Gauß-Manin system instead. They showed that it is of the form

$$\frac{f_*(\Omega^n_{X/\Delta})[z^\pm]}{zd(f_*\Omega^{n-1}_{X/\Delta}) - df \wedge f_*\Omega^{n-1}_{X/\Delta}},$$

where $z = \partial_t^{-1}$. Moreover, we can express the Brieskorn lattice canonically as the stalk at 0 of the sheaf

$$\frac{f_*(\Omega^n_{X/\Delta})[z]}{zd(f_*\Omega^{n-1}_{X/\Delta}) - df \wedge f_*\Omega^{n-1}_{X/\Delta}}. \tag{34}$$

[20]Here $dx = dx_0 \wedge \cdots \wedge dx_n$

[21]Both these results are in fact also true for $\mathcal{H}^n_{\mathrm{DR}}(X/\Delta)_0$ and $\mathcal{H}'_0$

8.2 The $\mathcal{D}$-module Perspective

How can we translate this situation and in particular the Brieskorn lattice into our language of $\mathcal{D}$-modules?

Recall from chapter 6.3 a $\mathcal{D}_X$-module is a natural generalisation of the notion of a connection. A vector bundle on a smooth variety X is a locally free, coherent sheaf of $\mathcal{O}_X$-modules. Equipping it with a flat connection is the same as equipping the sheaf of $\mathcal{O}_X$-modules with a $\mathcal{D}_X$-module structure. But there are natural generalisations of this concept in $\mathcal{D}$-module language.

Definition 8.4. *Let D be a normal crossing divisor of a smooth algebraic variety X. Denote by $\mathcal{O}_X(*D)$ the sheaf of holomorphic functions on $X \setminus D$, which are meromorphic along D. A* meromorphic vector bundle *is a locally free, coherent sheaf of $\mathcal{O}_X(*D)$-modules.*

Once again we see that equipping a meromorphic vector bundle with a connection is the same as giving the corresponding sheaf of $\mathcal{O}_X(*D)$-modules a $\mathcal{D}_X$-module structure. Parallel to definition 8.1 we can define a lattice inside a meromorphic vector bundle with connection.

Definition 8.5. *Let M be a coherent $\mathcal{O}_X(*D)$-module with connection. A* lattice *$N \subset M$ is a coherent $\mathcal{O}_X$-submodule, which generates M over $\mathcal{O}_X(*D)$.*

So from the point of view of $\mathcal{D}$-modules, the Brieskorn lattice is the germ of a coherent $\mathcal{O}_\Delta$-submodule, which generates the Gauß-Manin connection over $\mathcal{O}_\Delta(*0)$. So how can we express the Gauß-Manin connection as $\mathcal{D}$-module? Assume for a moment that $g : Y \to Z$ is smooth. Defining $\Omega^\bullet_{Y/Z}$ as before we had defined $\mathcal{H}^n_{\mathrm{DR}}(Y/Z) = R^n f_*(\Omega^\bullet_{Y/Z})$. Similarly we can define $\mathcal{H}^p_{\mathrm{DR}}(Y/Z)$ for any $p \in \mathbb{Z}$. By the relative Poincaré lemma, we know that $\Omega^\bullet_{Y/Z}$ is a resolution of the sheaf $g^{-1}\mathcal{O}_Z$ and thus

$$\mathcal{H}^p_{\mathrm{DR}}(Y/Z) = R^p g_*(g^{-1}\mathcal{O}_Z) \simeq R^p f_*(\mathbb{C}_Y) \otimes_{\mathbb{C}_Z} \mathcal{O}_Z.$$

Using e.g. [Kul98, II.5.1,2] we find that $\mathcal{H}^\bullet_{\mathrm{DR}}(Y/Z) \simeq g_+(\mathcal{O}_Y)$. For general $g : Y \to Z$, we call $g_+(\mathcal{O}_Y)$ the *Gauß-Manin system.*

Now in order to find the Brieskorn lattice inside the Gauß-Manin system we need to use some other of its defining properties. In particular, we stated in theorem 8.3 the structure of the Brieskorn lattice as module over micro-local differential operators. So in the algebraic study of the Brieskorn lattice, one also needs an algebraic equivalent to the notion of a micro-local differential operator. Luckily this turns out to be a well-known phenomenon, called the *Fourier-Laplace transformation.* Let M be a $\mathcal{D}_{\mathbb{C}^r}$-module with coordinates $(t_1, \ldots, t_r)$ on $\mathbb{C}^r$. Then the Fourier-Laplace transformation of M, in the sense of algebraic $\mathcal{D}$-modules, is denoted by $\mathrm{FL}^{\tau_1,\ldots,\tau_r}_{t_1,\ldots,t_r}(M)$. It is an $\mathcal{D}_{(\mathbb{C}^r)^*}$-module, where $(\mathbb{C}^r)^*$, the dual vector space to $\mathbb{C}^r$, has dual coordinates $\tau_1, \ldots, \tau_r$. $\mathrm{FL}^{\tau_1,\ldots,\tau_r}_{t_1,\ldots,t_r}(M)$ is the same vector space as M, but with an action of $\tau_1, \ldots, \tau_r$ and $\partial_{\tau_1}, \ldots, \partial_{\tau_r}$. The action of τ_i is defined to be the same as the action of $-\partial_{t_i}$ on M, whereas ∂_{τ_i} acts like t_i. Pham showed in [Pha85] that indeed switching from the $\mathbb{C}\{t\}$-module structure to the $\mathbb{C}\{\{\partial_t^{-1}\}\}$-structure of $\mathcal{H}''_0$ is the analytic equivalent of the Fourier-Laplace transformation.

Now let us return to our situation of the Milnor fibration $f : X \to \Delta$. Realising the Brieskorn lattice as a lattice in the stalk of $\mathcal{H}''$ at $0 \in \Delta$, it remains only to localise our Fourier-Laplace transformed Gauß-Manin system there. Thus

$$\mathrm{FL}^\tau_t(\mathcal{H}^0 f_+ \mathcal{O}_X)[\tau^{-1}] \tag{35}$$

is the object, which we are interested in and in which we aim to find a lattice corresponding algebraically to $\mathcal{H}''_0$. This can be done given the characterisation of the Brieskorn lattice as in (34).

For this reason too we shall frequently change co-ordinates $\tau^{-1} = z$. However, before we proceed let us generalise the situation a bit and adapt it to our Landau-Ginzburg model.

First of all, let us leave the local situation of a germ of a hypersurface singularity. We want to consider $f : X \to \mathbb{C}_t$, an algebraic function from some complex, quasi-projective variety X to $\mathbb{C}$. We assume that f has only finitely many isolated hypersurface singularities. This leads to a number of isolated critical values in the complex plane, above which the hypersurface in X described as the fibre of f has a finite number of singularities. Fortunately, the definition of the Gauß-Manin system carries over from the local situation. The same is true for the Fourier-Laplace transformation, which means that we are still interested in basically the same object (35).

How is this connected to our Landau-Ginzburg model? The Landau-Ginzburg model can be interpreted as precisely the above situation for a family of maps f. Let us compare this to our Landau-Ginzburg model from (30). In the same notation as in chapter 7.5, we can restrict the superpotential κ to fibres of the map ϑ : For $q \in S_2$ we have a hypersurface $X_q = \vartheta^{-1}(q) \subset S_1$. Moreover, given the short exact sequence

$$1 \longrightarrow S_0 \longrightarrow S_1 \xrightarrow{\vartheta} S_2 \longrightarrow 1,$$

we see that the fibres are canonically isomorphic to S_0. So for each $q \in S_2$ we have a map

$$\kappa\big|_{X_q} : S_0 \to \mathbb{C}.$$

Here is another way of looking at this: let us begin by noting that the short exact sequence (27) splits. In particular, even after tensoring with $\mathbb{C}^*$, we can still find a section $g : S_2 \to S_1$ of the map ϑ. For completeness sake let us denote the short exact sequence by

$$\begin{tikzcd} 1 \arrow[r] & S_0 \arrow[r, "\zeta"] & S_1 \arrow[r, bend left, "\vartheta"] & S_2 \arrow[l, bend left, dashed, "g"] \arrow[r] & 1. \end{tikzcd} \tag{36}$$

As a result we have an isomorphism $\zeta.g : S_0 \times S_2 \to S_1$, where $\zeta.g(y,q) = \zeta(y).g(q)$ and the product is the usual product inside the multiplicative group S_1. In other words, when we consider $S_1 \simeq S_0 \times S_2$, then ϑ simply becomes the projection onto the S_2 factor and we obtain a family $\kappa : S_0 \to \mathbb{C}$ indexed by $q \in S_2$.

Example 8.6. *Recall our running example:*

$$\begin{aligned} W : S_1 &\to \mathbb{C} \times S_2 \\ (x_1, \ldots, x_7) &\to \left(x_1 + \cdots + x_7, \left(\frac{x_2 x_4}{x_3^2}, \frac{x_2 x_7}{x_1}, \frac{x_1 x_3 x_5}{x_2 x_4 x_7}, \frac{x_4 x_6}{x_5^2}, \frac{x_5 x_7}{x_6} \right) \right). \end{aligned}$$

Now fix $q = (q_1, \ldots, q_5) \in S_2$ and a section

$$\begin{aligned} g : S_2 &\to S_1 \\ q &\mapsto \left(1, \frac{1}{q_3 q_4 q_5}, 1, q_1 q_3 q_4 q_5, q_1 q_2 q_3^2 q_4 q_5, q_1 q_2^2 q_3^3 q_4^2 q_5, q_2 q_3 q_4 q_5 \right). \end{aligned}$$

Then we have an isomorphism

$$\begin{aligned} S_0 &\longrightarrow X_q = \vartheta^{-1}(q) \\ (y_1, y_2) &\mapsto \left(y_1, \frac{1}{q_3 q_4 q_5} y_1 y_2, y_2, q_1 q_3 q_4 q_5 \frac{y_2}{y_1}, q_1 q_2 q_3^2 q_4 q_5 \frac{1}{y_1}, q_1 q_2^2 q_3^3 q_4^2 q_5 \frac{1}{y_1 y_2}, q_2 q_3 q_4 q_5 \frac{1}{y_2} \right). \end{aligned}$$

Thus our Landau-Ginzburg model can be interpreted to encode the following family of Laurent polynomials:

$$\begin{aligned} &W_q : S_0 \longrightarrow \mathbb{C} \\ &(y_1, y_2) \mapsto y_1 + \frac{1}{q_3q_4q_5}y_1y_2 + y_2 + q_1q_3q_4q_5\frac{y_2}{y_1} + q_1q_2q_3^2q_4q_5\frac{1}{y_1} + q_1q_2^2q_3^3q_4^2q_5\frac{1}{y_1y_2} + q_2q_3q_4q_5\frac{1}{y_2}. \end{aligned}$$

Note that this representation is by no means unique. It depends, for instance, on the choice of section g, or more generally on the choice of isomorphism $S_0 \simeq X_q$. Note also that each $x = g(q) \in X_q$ has an associated, natural isomorphism mapping $1 \in S_0$ to x :

$$\begin{aligned} S_0 &\xrightarrow{\sim} X_q \\ y &\mapsto \zeta(y).x = (x_1y^{a_1}, \dots, x_ny^{a_n}), \end{aligned}$$

where the a_i are as in chapter 7.5. It is often more convenient to think of our family of Laurent polynomials as

$$\begin{aligned} \varphi_A : S_0 \times S_1 &\longrightarrow \mathbb{C} \times S_1 \\ (y, x) &\mapsto (x_1y^{a_1} + \dots + x_ny^{a_n}, x) = (\varphi_A(x), x), \end{aligned}$$

indexed by $x \in S_1$.

Remark 8.7. *Note how S_1 appears in two different roles: in our Landau-Ginzburg model, it is the domain of W. For the closely related family of Laurent polynomials it is a parameter space.*

Thus interpreting our Landau-Ginzburg model as a family of polynomials on the fibres of ϑ, we are still interested in the same algebraic object: the Fourier-Laplace transformed Gauß-Manin system. Only now this system is also dependent on the parameter $q \in S_2$, respectively $x \in S_1$. In other words, we are interested in

$$\mathrm{FL}_t^\tau(\mathcal{H}^0 W_+ \mathcal{O}_{S_1})[\tau^{-1}].$$

However, we might run into a problem here: recall that for a fixed $q \in S_2$, our Landau-Ginzburg model defines (via the restriction of κ) a polynomial $S_0 \to \mathbb{C}$. This polynomial can have a number of critical values in $\mathbb{C}$, each coming from one or more critical points in S_0. Letting q vary in S_2, these critical values vary in $\mathbb{C}$. The problem now occurs when, for some $q_0 \in S_2$, one of the critical values in $\mathbb{C}$ approaches infinity. We can still build up our theory, considering the direct image (and then its Fourier-Laplace transform), but the Brieskorn lattice might not be coherent any more. In this case our Fourier-Laplace transformed Brieskorn lattice would not be coherent either and thus we have to discard this bad point $q_0 \in S_2$. Clearly equivalently we can discard the bad parameters $x \in \vartheta^{-1}(q_0)$. These parameters x together define the so-called *non-tame locus* $\Delta^\infty \subset S_1$. For the entire theory to work, we have to restrict our Landau-Ginzburg model to tame points, i.e. to $S_1^o = S_1 \setminus \Delta^\infty$ and

$$W\big|_{S_1^o} : S_1^o \to \mathbb{C}_t \times S_2^o = \mathbb{C} \times \vartheta(S_1^o).$$

So what exactly does it mean for a parameter $x \in S_1$ to be bad? Respectively, can we find an easy criterion to tell if the superpotential is non-tame at x? We can. Being bad in this sense means that $\varphi_A(x)$ is degenerate with respect to its Newton polygon. Recall that the Newton polygon of a Laurent polynomial $\sum_{a \in \mathbb{Z}^d} f_a y^a$ is the polygon

$$Q := \mathrm{Conv}\left(\left\{a \in \mathbb{Z}^d \;\middle|\; f_a \neq 0\right\}\right),$$

where $\mathrm{Conv}(S)$ denotes the convex hull of a set S. Let now Γ be a proper face of Q. We define the critical locus of $x^o \in S_1$ with respect to Γ by

$$S_\Gamma^{\mathrm{crit},x^o} := \left\{ (y_1,\ldots,y_d) \in S_0 \;\middle|\; y_k \partial_{y_k} \Big(\sum_{a_i \in \Gamma} x_i^o y^{a_i} \Big) = 0 \text{ for all } k \in [1,d] \right\}.$$

Then we have the following definition:

Definition 8.8. *The* non-tame locus *of φ_A is the set*

$$\Delta^\infty := \left\{ x \in S_1 \;\middle|\; \exists \Gamma \neq Q \textit{ with } S_\Gamma^{\mathrm{crit},x} \neq \emptyset \right\}.$$

We also say that the fibre of φ_A at x has a singularity at infinity*.*

Example 8.9. *Let us calculate the non-tame locus in our Landau-Ginzburg model from before. In our case we have that*

$$Q = \mathrm{Conv}\left(\{a_1,\ldots,a_7\}\right) = \mathrm{Conv}\left(\{a_1,a_2,a_4,a_6,a_7\}\right),$$

where

$$A = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 & -1 \end{pmatrix}.$$

Our φ_A is given by

$$\varphi_A(x)(y_1,y_2) = x_1 y_1 + x_2 y_1 y_2 + x_3 y_2 + x_4 \frac{y_2}{y_1} + x_5 \frac{1}{y_1} + x_6 \frac{1}{y_1 y_2} + x_7 \frac{1}{y_2}.$$

The proper faces of Q are given by the single vertices a_i, as well as the five line segments

$$\overline{a_1a_2}, \quad \overline{a_2a_4}, \quad \overline{a_4a_6}, \quad \overline{a_6a_7}, \quad \overline{a_7a_1}.$$

Clearly, when we restrict φ_A to a single vertex, we will be left with only one summand. Since $y_1, y_2, x_i \in \mathbb{C}^*$, the resulting two equations (after applying $y_1\partial_{y_1}$ and $y_2\partial_{y_2}$ have no solutions. So let us consider $\Gamma = \overline{a_1a_2}$. In this case

$$\begin{aligned} S_\Gamma^{\mathrm{crit},x} &= \left\{ (y_1,y_2) \in (\mathbb{C}^*)^2 \;\middle|\; x_1y_1 + x_2y_1y_2 = 0 \textit{ and } x_2y_1y_2 = 0 \right\} \\ &= \emptyset. \end{aligned}$$

Similarly we have $S_\Gamma^{\mathrm{crit},x} = \emptyset$ for $\Gamma = \overline{a_6a_7}$ and $\Gamma = \overline{a_7a_1}$. Next let $\Gamma = \overline{a_2a_4}$. Then

$$\begin{aligned} S_\Gamma^{\mathrm{crit},x} &= \left\{ (y_1,y_2) \;\middle|\; x_2y_1y_2 - x_4\frac{y_2}{y_1} = 0 \textit{ and } x_2y_1y_2 + x_3y_2 + x_4\frac{y_2}{y_1} = 0 \right\} \\ &= \left\{ (y_1,y_2) \;\middle|\; x_2y_1^2 - x_4 = 0 \textit{ and } x_3y_1 + 2x_4 = 0 \right\} \\ &= \begin{cases} \{(-\frac{2x_4}{x_3}, y_2)\} & \textit{if } 4x_2x_4 - x_3^2 = 0 \\ \emptyset & \textit{otherwise.} \end{cases} \end{aligned}$$

Therefore we have that for $x \in S_1$ *with* $4x_2x_4 - x_3^2 = 0$, $S_\Gamma^{\mathrm{crit},x} \neq \emptyset$. *In other words:*

$$\{4x_2x_4 - x_3^2 = 0\} \subset \Delta^\infty.$$

Similarly we have

$$S_\Gamma^{\mathrm{crit},x} = \begin{cases} \{(y_1, -\frac{2x_6}{x_4})\} & \text{if } 4x_4x_6 - x_5^2 = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

in the case of $\Gamma = \overline{a_4a_6}$. *We define*

$$S_1^o := (\mathbb{C}^*)^7 \setminus \left(\{4x_2x_4 - x_3^2 = 0\} \cup \{4x_4x_6 - x_5^2 = 0\}\right).$$

Observe that

$$4x_2x_4 - x_3^2 = 0 \Leftrightarrow 4q_1 - 1 = 0$$

and

$$4x_4x_6 - x_5^2 = 0 \Leftrightarrow 4q_4 - 1 = 0,$$

where we used that

$$\vartheta(x_1, \ldots, x_7) = (q_1, \ldots, q_5) = \left(\frac{x_2x_4}{x_3^2}, \frac{x_2x_7}{x_1}, \frac{x_1x_3x_5}{x_2x_4x_7}, \frac{x_4x_6}{x_5^2}, \frac{x_5x_7}{x_6}\right).$$

Accordingly we define

$$S_2^o := S_2 \setminus (\{q_1 = 1/4\} \cup \{q_4 = 1/4\})$$

and will henceforth mean

$$W = (\kappa, \vartheta) : S_1^o \to \mathbb{C} \times S_2^o,$$

whenever we refer to "our Landau-Ginzburg model".

8.3 Computing the Structure of the Brieskorn Lattice

Recall that we are interested in the following object:

$$\mathrm{FL}_t^\tau(\mathcal{H}^0(W_+\mathcal{O}_{S_1^o}))[\tau^{-1}], \tag{37}$$

where

$$W = (\kappa, \vartheta) : S_1^o \longrightarrow \mathbb{C} \times S_2^o$$

is our Landau-Ginzburg model. This module has no obvious, concrete description, so we want to use this chapter to compute the structure of it explicitly.

Given our discussion around the split sequence (36) we have the following commutative diagram:

$$\begin{tikzcd} & S_1 \arrow[r, "\kappa"] \arrow[dr, "\vartheta"] & \mathbb{C}_t \\ S_0 \times S_2 \arrow[ur, "\zeta.g"] \arrow[rr, "\pi"] & & S_2 \end{tikzcd}$$

Write $W' = (\kappa \circ (\zeta.g), \pi) : S_0 \times S_2 \to \mathbb{C} \times S_2$. Then we can express (37) equivalently as

$$\mathrm{FL}_t^\tau(\mathcal{H}^0(W'_+\mathcal{O}_{S_0 \times S_2^o}))[\tau^{-1}].$$

Moreover, consider yet another commutative diagram:

$$\begin{tikzcd} S_0 \times S_2^o \arrow[r, "\widetilde{g}"] \arrow[d, "W'"] & S_0 \times S_1^o \arrow[d, "\widetilde{W}"] \\ \mathbb{C}_t \times S_2^o \arrow[r, "\widehat{g}=\mathrm{Id}_{\mathbb{C}_t} \times g"] & \mathbb{C}_t \times S_1^o \end{tikzcd}$$

Using the base change properties of the direct image (c.f [HTT08]), we obtain:

$$\begin{aligned} \mathrm{FL}_t^\tau(\mathcal{H}^0(W'_+\mathcal{O}_{S_0\times S_2^o}))[\tau^{-1}] &= \mathrm{FL}_t^\tau(\widehat{g}^+(\mathcal{H}^0(\widetilde{W}_+\mathcal{O}_{S_0\times S_1^o})))[\tau^{-1}] \\ &= \widehat{g}^+\,\mathrm{FL}_t^\tau(\mathcal{H}^0(\widetilde{W}_+\mathcal{O}_{S_0\times S_1^o}))[\tau^{-1}]. \end{aligned}$$

The latter object has a very nice description, found in [RS15][theorem 2.4 and proposition 3.2], which we are now going to describe.

We start with an abuse of notation: denote by $\widetilde{W}$ the map

$$\begin{aligned} S_0 \times \mathbb{C}^n &\longrightarrow \mathbb{C}_t \times \mathbb{C}^n \\ ((y_1,\dots,y_d),(x_1,\dots,x_n)) &\mapsto \left(-\left(\sum_{i=1}^n x_i y^{a_i}\right),(x_1,\dots,x_n)\right) \end{aligned} \tag{38}$$

where the a_i are as in chapter 7.5, i.e. they are the primitive integral generators of the fan Σ of the toric variety X_Σ. However, they need not be defined in terms of the toric data, we can equally well assume that they are simply n vectors $a_1,\dots,a_n \in \mathbb{Z}^d$, which generate $\mathbb{Z}^d$ over $\mathbb{Z}$. In [DS03] it was shown that then

$$\frac{\widetilde{\pi}_*\Omega^n_{S_0\,\times\,\mathbb{C}^n/\mathbb{C}^n}[z^\pm]}{(zd-dF\wedge)\widetilde{\pi}_*\Omega^{n-1}_{S_0\,\times\,\mathbb{C}^n/\mathbb{C}^n}[z^\pm]} \simeq \mathrm{FL}_t^\tau(\mathcal{H}^0(\widetilde{W}_+\mathcal{O}_{S_0\times\mathbb{C}^n}))[\tau^{-1}], \tag{39}$$

where we wrote $\widetilde{W} = F\times\widetilde{\pi}$. We will compute the right-hand side of this equation, but in doing so should keep track of what happens to the left-hand side, in order to obtain the Brieskorn lattice as (the restriction to the tame locus of) the lattice

$$\frac{\widetilde{\pi}_*\Omega^n_{S_0\,\times\,\mathbb{C}^n/\mathbb{C}^n}[z]}{(zd-dF\wedge)\widetilde{\pi}_*\Omega^{n-1}_{S_0\,\times\,\mathbb{C}^n/\mathbb{C}^n}[z]}.$$

In [RS15] the authors described (39) explicitly as a *GKZ system.*

Definition 8.10 (GKZ system). *Let $c_1,\dots,c_s$ be vectors in $\mathbb{Z}^t$ and write $C=(c_1,\dots,c_s)$ for the corresponding $t\times s$ matrix. Write $\mathbb{L}$ for the module of relations of C, i.e. $l=(l_1,\dots,l_s)\in\mathbb{L}\subset\mathbb{Z}^s$ if and only if $l_1c_1+\dots+l_sc_s=0$. Furthermore choose coordinates $\lambda_1,\dots,\lambda_s$ on $\mathbb{C}^s$ and denote the sheaf of differential operators on $\mathbb{C}^s$ by $\mathcal{D}_{\mathbb{C}^s}$. Lastly let $\gamma=(\gamma_1,\dots,\gamma_t)\in\mathbb{C}^t$. We define*

$$\mathcal{M}_C^\gamma := \frac{\mathcal{D}_{\mathbb{C}^s}}{\mathcal{D}_{\mathbb{C}^s}(\Box_l)_{l\in\mathbb{L}} + \mathcal{D}_{\mathbb{C}^s}(Z_k)_{k=1,\dots,t}},$$

where

$$\Box_l = \prod_{i|l_i<0} \partial_{\lambda_i}^{-l_i} - \prod_{i|l_i>0} \partial_{\lambda_i}^{l_i}$$
$$Z_k = \sum_{i=1}^{s} c_{ki}\lambda_i\partial_{\lambda_i} + \gamma_k.$$

We shall only consider $\mathcal{M}_C^\gamma$ for $\gamma = (1, 0, \ldots, 0)$ and for such special γ we will denote the GKZ system simply by $\mathcal{M}_C$.

Let us adapt this to our situation. We do not actually need the most general definition here, as we will always assume the matrix C to be the one given by toric data. However, it will not be $A = (a_1, \ldots, a_n)$, but rather the enhanced, homogenised matrix $\widetilde{A}$, given as

$$\widetilde{A} = \left(\begin{array}{c|ccc} 1 & 1 & \ldots & 1 \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array} \right).$$

This is then a $(n-r+1)\times(n+1)$ matrix. We denote its columns by $\widetilde{a}_0, \ldots, \widetilde{a}_n$. Also, staying true to our previous notation, we shall give $\mathbb{C}^{n+1}$ coordinates $t, x_1, \ldots, x_n$. Thus $\mathcal{M}_{\widetilde{A}}$ is the $\mathcal{D}_{\mathbb{C}^{n+1}}$-module

$$\mathcal{M}_{\widetilde{A}} = \frac{\mathcal{D}_{\mathbb{C}^{n+1}}}{\mathcal{I}},$$

where $\mathcal{I}$ is the sheaf of left ideals of $\mathcal{D}_{\mathbb{C}^{n+1}}$, generated by

$$\Box_l = \begin{cases} \partial_t^{l_1+\cdots+l_n} \prod_{i|l_i<0} \partial_{x_i}^{-l_i} - \prod_{i|l_i>0} \partial_{x_i}^{l_i} & \text{if } l_1+\cdots+l_n \geq 0, \\ \prod_{i|l_i<0} \partial_{x_i}^{-l_i} - \partial_t^{-l_1-\cdots-l_n} \prod_{i|l_i>0} \partial_{x_i}^{l_i} & \text{if } l_1+\cdots+l_n < 0, \end{cases}$$
$$Z_k = \sum_{i=1}^{n} a_{ki}x_i\partial_{x_i},$$
$$Z_0 = E = \sum_{i=1}^{n} x_i\partial_{x_i} + t\partial_t + 1.$$

Here k runs from 1 to d and the generators of $\mathcal{I}$ include all box operators $l \in \mathbb{L}$. Note that $\mathbb{L}$ is isomorphic for A and $\widetilde{A}$: given a relation $(l_1, \ldots, l_n)$ of the columns of A, $(-l_1 - \cdots - l_n, l_1, \ldots, l_n)$ is a relation on $\widetilde{A}$ and vice versa, given a relation $(l_0, \ldots, l_n)$ of $\widetilde{A}$, we see that

$$l_0\widetilde{a}_0 + \cdots + l_n\widetilde{a}_n = 0 \qquad \Rightarrow \qquad l_1a_1 + \cdots + l_na_n = 0.$$

Theorem 8.11 ([RS15]-theorem 2.4)**.** *Given $\widetilde{W} : S_0 \times \mathbb{C}^n \to \mathbb{C}_t \times \mathbb{C}^n$ as in (38), there exists an isomorphism*

$$\varphi : \mathrm{FL}_t^\tau(\mathcal{M}_{\widetilde{A}})[\tau^{-1}] \longrightarrow \mathrm{FL}_t^\tau(\mathcal{H}^0(\widetilde{W}_+\mathcal{O}_{S_0\times\mathbb{C}^n}))[\tau^{-1}]$$

of $\mathcal{D}_{\mathbb{C}_\tau\times\mathbb{C}^n}$-modules.

This is important as we can compute the module $\mathrm{FL}_t^\tau(\mathcal{M}_{\widetilde{A}})[\tau^{-1}]$. If we denote it by $\widehat{\mathcal{M}}_{\widetilde{A}}$, then

$$\widehat{\mathcal{M}}_{\widetilde{A}} = \frac{\mathcal{D}_{\mathbb{C}_\tau \times \mathbb{C}^n}[\tau^{-1}]}{\widehat{\mathcal{I}}},$$

where $\widehat{\mathcal{I}}$ is the sheaf of left ideals generated by $\widehat{\Box}_l = \mathrm{FL}_t^\tau(\Box_l)$ for $l \in \mathbb{L}$, as well as $\widehat{Z}_k = \mathrm{FL}_t^\tau(Z_k)$ for $k = 1, \ldots, n$ and $\widehat{E} = \mathrm{FL}_t^\tau(E)$. Explicitly:

$$\begin{aligned}
\widehat{\Box}_l &= \tau^{l_1+\cdots+l_n} \prod_{i|l_i<0} \partial_{x_i}^{-l_i} - \prod_{i|l_i>0} \partial_{x_i}^{l_i}, \\
\widehat{Z}_k &= \sum_{i=1}^n a_{ki} x_i \partial_{x_i}, \\
\widehat{E} &= \sum_{i=1}^n x_i \partial_{x_i} - \tau \partial_\tau.
\end{aligned}$$

Now for this to be applicable to the Landau-Ginzburg model, we still have to restrict the entire situation to $\mathbb{C}_t \times S_1$, i.e. return to $\widetilde{W} : S_0 \times S_1 \to \mathbb{C}_t \times S_1$. Denote by j the open inclusion $S_0 \times S_1 \hookrightarrow S_0 \times \mathbb{C}^n$. Then in particular, the left-hand side of theorem 8.11 restricts to $\widehat{\mathcal{M}}_{\widetilde{A}}^{\mathrm{loc}} := j^*(\widehat{\mathcal{M}}_{\widetilde{A}})$. Informally speaking, this simply amounts to inverting the x_i. The right-hand side of theorem 8.11 restricts to

$$\mathrm{FL}_t^\tau(\mathcal{H}^0(\widetilde{W}_+ \mathcal{O}_{S_0 \times S_1}))[\tau^{-1}].$$

Given the identifications so far, how do we identify the Brieskorn lattice? In order to answer this let us consider the ring

$$R = \mathbb{C}[z, x_1^\pm, \ldots, x_n^\pm]\langle z^2\partial_z, z\partial_{x_1}, \ldots, z\partial_{x_n}\rangle$$

and the associated sheaf $\mathcal{R}$ of quasi-coherent $\mathcal{O}_{\mathbb{C}_z \times S_1}$-algebras. $\mathcal{R}$ is not the same as $\mathcal{D}_{\mathbb{C}_z \times S_1}$ or $\mathcal{D}_{\mathbb{C}_\tau \times S_1}$, however on $\{z \neq 0\}$ $\mathcal{R}$ restricts to $\mathcal{D}_{\mathbb{C}_\tau^* \times S_1}$. On this locus we can replace $\widehat{\Box}_l$ in our definition of $\widehat{\mathcal{I}}$ by

$$\widehat{\Box}'_l = z^{\sum_{i|l_i>0}} \widehat{\Box}_l = \prod_{i|l_i<0} (z\partial_{x_i})^{-l_i} - \prod_{i|l_i>0} (z\partial_{x_i})^{l_i}$$

and $\widehat{Z}_k$ and $\widehat{E}$ by $\widehat{Z}'_k = z\widehat{Z}_k$ and $\widehat{E}' = z\widehat{E}$ respectively. So define the sheaf $\mathcal{J}$ of left ideals of $\mathcal{R}$ by

$$\mathcal{J} = \mathcal{R}(\widehat{\Box}'_l)_{l \in \mathbb{L}} + \mathcal{R}(\widehat{Z}'_k)_{k=1,\ldots,n} + \mathcal{R}\widehat{E}'. \tag{40}$$

Still following [RS15] we denote ${}_0\widehat{\mathcal{M}}_{\widetilde{A}}^{\mathrm{loc}} = \frac{\mathcal{R}}{\mathcal{J}}$.

Proposition 8.12 ([RS15]-Corollary 2.12)**.** *The restriction of the isomorphism φ from theorem 8.11 to $\mathbb{C}_z \times S_1$ maps ${}_0\widehat{\mathcal{M}}_{\widetilde{A}}^{\mathrm{loc}}$ isomorphically to*

$$\frac{\widetilde{\pi}_* \Omega^n_{S_0 \times S_1/S_1}[z]}{(zd - dF\wedge)\widetilde{\pi}_* \Omega^{n-1}_{S_0 \times S_1/S_1}[z]}.$$

Let us return to our object of interest (37):

$$\mathrm{FL}_t^\tau(\mathcal{H}^0(W_+\mathcal{O}_{S_1^\circ}))[\tau^{-1}] = \widehat{g}^+\widehat{\mathcal{M}}^{\mathrm{loc}}_{\widetilde{A}}.$$

We need to figure out, how $\widehat{g}^+$ acts on $\widehat{\mathcal{M}}^{\mathrm{loc}}_{\widetilde{A}}$. Once again we are being helped by [RS15, proposition 3.2]. The authors show that applying $\widehat{g}^+$ is essentially the same as pushing forward the vector fields ∂_{x_i} along ϑ to vector fields on S_2. In particular, the result is independent of the choice of section g!

How is that described in practical terms? Let us recall our set-up: we have a short exact sequence

$$0 \longrightarrow M \xhookrightarrow{A} \mathrm{Div}_T(X_\Sigma) \xrightarrow{B} \mathrm{Pic}(X_\Sigma) \longrightarrow 0, \tag{41}$$

where the matrices A and B depend on the chosen bases. We have chosen so far $\{\varepsilon_1, \ldots, \varepsilon_d\}$ as standard basis for $M \simeq \mathbb{Z}^d$, $\{D_1, \ldots, D_n\}$ as basis for $\mathrm{Div}_T(X_\Sigma) \simeq \mathbb{Z}^n$ and $\{u_1, \ldots, u_r\}$ as basis for $\mathrm{Pic}(X_\Sigma) \simeq \mathbb{Z}^r$. With this choice we found the matrix $A = (a_1, \ldots, a_n)$ to be comprised of the n integral generators of $\Sigma(1) \subset N$, where the a_i are expressed in the basis $\{e_1, \ldots, e_d\}$ dual to our choice of basis of M and where each a_i generates a ray ρ_i over $\mathbb{R}_{\geq 0}$, which corresponds under the orbit cone correspondence to a divisor D_i. Upon tensoring with $\mathbb{C}^*$ we obtained another short exact sequence

$$1 \longrightarrow S_0 \simeq (\mathbb{C}^*)^d \xhookrightarrow{\zeta} S_1 \simeq (\mathbb{C}^*)^n \xrightarrow{\vartheta} S_2 \simeq (\mathbb{C}^*)^r \longrightarrow 1.$$

If we write the matrix $B = (b_{ij}) = (b_1, \ldots, b_r)$, then we could express $\vartheta(x) = (x^{b_1}, \ldots, x^{b_r})$. In particular, giving S_2 coordinates $q_1, \ldots, q_r$ we see that $q_i = x^{b_i} = x_1^{b_{1i}} \cdots x_n^{b_{ni}}$. We can now push-forward the vector fields $x_i\partial_{x_i}$:

$$\vartheta_*(x_i\partial_{x_i}) = x_i \sum_{j=1}^r \partial_{x_i}\left(x^{b_j}\right)\partial_{q_j} = x_i \sum_{j=1}^r b_{ij}x^{b_j - e_i}\partial_{q_j} = \sum_{j=1}^r b_{ij}q_j\partial_{q_j}. \tag{42}$$

This is especially useful, when we combine it with the observation that we can express the ideal $\widehat{\mathcal{I}} \subset \mathcal{D}_{\mathbb{C}_\tau \times S_1}[\tau^{-1}]$ using generators, which are purely in terms of $zx_i\partial_{x_i}$ and $z^2\partial_z$: clearly the $\widehat{Z}'_k$ and $\widehat{E}'$ from (40) are already of this form. So let us replace the operators $\widehat{\Box}'_l$ from before with

$$\widehat{\Box}''_l = \left(\prod_{i|l_i>0} x_i^{l_i}\right)\widehat{\Box}'_l = \left(\prod_{i=1}^n x_i^{l_i}\right)\prod_{i|l_i<0}\prod_{j=0}^{-l_i-1}(zx_i\partial_{x_i} - jz) - \prod_{i|l_i>0}\prod_{j=0}^{l_i-1}(zx_i\partial_{x_i} - jz).$$

Here we made explicit use of the fact that

$$x_i^k\partial_{x_i}^k = \prod_{j=0}^{k-1}(x_i\partial_{x_i} - k).$$

Upon pushing forward the operators $\widehat{\Box}''_l$, $\widehat{Z}'_k$ and $\widehat{E}'$, we simply replace the $zx_i\partial_{x_i}$ with their respective expressions according to (42). Conveniently, all $\widehat{Z}'_k$ then vanish, since

$$\widehat{Z}'_k = \sum_{j=1}^n a_{kj}\vartheta_*(x_j\partial_{x_j}) = \sum_{i=1}^r\sum_{j=1}^n a_{kj}b_{ji}q_i\partial_{q_i} = \sum_{i=1}^r (A.B)_{ki}q_i\partial_{q_i}.$$

Now recall the map $(\mathrm{Pic}(X_\Sigma))^\vee \to (\mathrm{Div}_T(X_\Sigma))^\vee$, dual to the map $\mathrm{Div}_T(X_\Sigma) \to \mathrm{Pic}(X_\Sigma)$ represented in our bases by B. Choosing the natural dual bases in all involved vector spaces, we can then represent this map by B^T, the transpose of B. Note that we can identify $\mathbb{L} \subset (\mathrm{Div}_T(X_\Sigma))^\vee$ with $\mathbb{L} = (\mathrm{Pic}(X_\Sigma))^\vee$, since the columns of B and hence the rows of B^T naturally generate all the relations between the rows of A, i.e. the vectors $a_1, \ldots, a_n$. Under this identification we have

$$l = l_1 D_1^* + \cdots + l_n D_n^* = m_1 u_1^* + \ldots + m_r u_r^*$$
$$(l_1, \ldots, l_n) = m_1 b_1^T + \cdots + m_r b_r^T.$$

Thus

$$\prod_{i=1}^n x_i^{l_i} = x^{(l^T)} = \prod_{i=1}^r x^{m_i b_i} = \prod_{i=1}^r \left(x^{b_i}\right)^{m_i} = \prod_{i=1}^r q_i^{m_i} = \prod_{i=1}^r q_i^{u_i(l)}.$$

Lastly note that the anti-canonical divisor $-K_{X_\Sigma}$ xan be expressed as $\sum_{i=1}^n D_i$ on a toric variety. In particular, this yields a convenient description for $\widehat{E}'$:

$$\widehat{E}' = z^2 \partial_z + \sum_{i=1}^r u_i^*(\omega_{X_\Sigma}^{-1}) z q_i \partial_{q_i},$$

where, as usual, $\omega_{X_\Sigma}^{-1}$ denoted the anti-canonical bundle. This implies the following proposition:

Proposition 8.13 ([RS15]-Prop 3.2.)**.** *For $\widehat{g}$ and $\widehat{\mathcal{M}}_{\widetilde{A}}^{\mathrm{loc}}$ as before we have*

$$\widehat{g}^+ \widehat{\mathcal{M}}_{\widetilde{A}}^{\mathrm{loc}} = \frac{\mathcal{D}_{\mathbb{C}_\tau \times S_2}[\tau^{-1}]}{\widetilde{\mathcal{I}}},$$

where $\widetilde{\mathcal{I}}$ is the sheaf of left ideals generated by

$$\widetilde{\Box}_l = \prod_{a | u_a(l) > 0} q_a^{u_a(l)} \prod_{i | l_i < 0} \prod_{j=0}^{-l_i - 1} \left(\sum_{k=1}^{n-r} m_{ik} z q_k \partial_{q_k} - jz \right)$$
$$- \prod_{a | u_a(l) < 0} q_a^{-u_a(l)} \prod_{i | l_i > 0} \prod_{j=0}^{l_i - 1} \left(\sum_{k=1}^{n-r} m_{ik} z q_k \partial_{q_k} - jz \right)$$

for all $l \in \mathbb{L}$ and by the operator

$$\widetilde{E} = z^2 \partial_z + \sum_{i=1}^r u_i^*(\omega_{X_\Sigma}^{-1}) z q_i \partial_{q_i}. \tag{43}$$

Now that we have the structure of (37), it is not difficult to find explicitly our Brieskorn lattice. Define

$$\mathcal{QM}_{\widetilde{A}}^{\mathrm{loc}} := \mathcal{O}_{\mathbb{C}_\tau \times S_2^\circ} \otimes_{\mathcal{O}_{\mathbb{C}_\tau \times S_2}} \frac{\mathcal{D}_{\mathbb{C}_\tau \times S_2}[\tau^{-1}]}{\widetilde{\mathcal{I}}}$$

with $\widetilde{\mathcal{I}}$ as before. Then [RS15] show that this $\mathcal{D}_{\mathbb{C}_\tau \times S_2^\circ}$-module is equipped with an increasing filtration $G_\bullet$ by $\mathcal{O}_{\mathbb{C}_z \times S_2^\circ}$-modules, such that $G_k \mathcal{QM}_{\widetilde{A}}^{\mathrm{loc}}$ is locally free of rank $k! \,\mathrm{Vol}(\mathrm{Conv}(a_1, \ldots, a_n))$.

The Brieskorn lattice is then the zeroth term of this filtration ${}_0\mathcal{QM}^{\text{loc}}_{\widetilde{A}} = G_0\mathcal{QM}^{\text{loc}}_{\widetilde{A}}$, which is the restriction to $\mathbb{C}_z \times S_2^o$ of the sheaf associated to the module

$$\frac{\mathbb{C}[z, q_1^{\pm}, \dots, q_r^{\pm}]\langle z^2\partial_z, zq_1\partial_{q_1}, \dots, zq_r\partial_{q_r}\rangle}{\langle \widetilde{E}, \widetilde{\Box}_l\rangle_{l\in\mathbb{L}}}.$$

Example 8.14. *Let us now apply the previous constructions to our running example. Recall our choice of basis* $\{u_1, u_2, u_3, u_4, u_5\}$ *with the* u_i *being defined in example 7.14. So we have*

$$-K_{X_4^o} = [D_1] + \cdots + [D_7] = 2[D_1] + 3[D_2] + 2[D_3] + [D_4] - [D_6] = u_2 + u_5,$$

which implies that $\widetilde{E}$ *is the operator*

$$\widetilde{E} = z^2\partial_z + zq_2\partial_{q_2} + zq_5\partial_{q_5}. \tag{44}$$

In order to describe the Brieskorn lattice we need to find suitable box operators. Clearly we do not need all of them, but rather enough to make sure that ${}_0\mathcal{QM}^{\text{loc}}_{\widetilde{A}}$ *has rank* $\mathrm{Vol}(\mathrm{Conv}(a_1, \dots, a_n)) = 7$*. The following relations* $l \in \mathbb{L}$ *will suffice for this purpose:*

$$\begin{aligned}
l_1 &= (0,0,-1,0,0,0,-1) &&= -u_2^* - u_3^* - u_4^* - u_5^*\\
l_2 &= (0,-1,0,0,0,-1,0) &&= -u_1^* - 2u_2^* - 2u_3^* - u_4^*\\
l_3 &= (-1,0,0,0,-1,0,0) &&= -u_1^* - u_2^* - 2u_3^* - u_4^* - u_5^*\\
l_4 &= (1,-1,0,0,0,0,-1) &&= -u_2^*\\
l_5 &= (0,0,0,0,-1,1,-1) &&= -u_5^*\\
l_6 &= (0,0,0,-1,1,0,-1) &&= -u_4^* - u_5^*\\
l_7 &= (0,0,-1,1,-1,0,0) &&= -u_2^* - u_3^*\\
l_8 &= (0,0,-1,0,1,-1,0) &&= -u_2^* - u_3^* - u_4^*\\
l_9 &= (0,-1,1,0,-1,0,0) &&= -u_1^* - u_2^* - u_3^*\\
l_{10} &= (-1,1,-1,0,0,0,0) &&= -u_3^* - u_4^* - u_5^*\\
l_{11} &= (-1,0,1,-1,0,0,0) &&= -u_1^* - u_3^* - u_4^* - u_5^*\\
l_{12} &= (-1,0,0,0,0,-1,1) &&= -u_1^* - u_2^* - 2u_3^* - u_4^*\\
l_{13} &= (0,1,-2,1,0,0,0) &&= u_1^*\\
l_{14} &= (0,0,0,0,1,-2,1,0) &&= u_4^*.
\end{aligned}$$

Upon "translating" them into box operators an clearing the denominators, they yield the following *fourteen operators on* $\mathbb{C}[z, q_1^{\pm}, \ldots, q_5^{\pm}]\langle z^2\partial_z, zq_1\partial_{q_1}, \ldots, zq_5\partial_{q_5}\rangle$:

$$\begin{aligned}
\widetilde{\Box}_{l_1} &= (-2zq_1\partial_{q_1} + zq_3\partial_{q_3})(zq_2\partial_{q_2} - zq_3\partial_{q_3} + zq_5\partial_{q_5}) - q_2q_3q_4q_5,\\
\widetilde{\Box}_{l_2} &= (zq_1\partial_{q_1} + zq_2\partial_{q_2} - zq_3\partial_{q_3})(zq_4\partial_{q_4} - zq_5\partial_{q_5}) - q_1q_2^2q_3^2q_4,\\
\widetilde{\Box}_{l_3} &= (-zq_2\partial_{q_2})(zq_3\partial_{q_3} - 2zq_4\partial_{q_4} + zq_5\partial_{q_5}) - q_1q_2q_3^2q_4q_5,\\
\widetilde{\Box}_{l_4} &= (zq_1\partial_{q_1} + zq_2\partial_{q_2} - zq_3\partial_{q_3})(zq_2\partial_{q_2} - zq_3\partial_{q_3} + zq_5\partial_{q_5}) - q_2(-zq_2\partial_{q_2}),\\
\widetilde{\Box}_{l_5} &= (zq_3\partial_{q_3} - 2zq_4\partial_{q_4} + zq_5\partial_{q_5})(zq_2\partial_{q_2} - zq_3\partial_{q_3} + zq_5\partial_{q_5}) - q_5(zq_4\partial_{q_4} - zq_5\partial_{q_5}),\\
\widetilde{\Box}_{l_6} &= (zq_1\partial_{q_1} - zq_3\partial_{q_3} + zq_4\partial_{q_4})(zq_2\partial_{q_2} - zq_3\partial_{q_3} + zq_5\partial_{q_5}) - q_4q_5(zq_3\partial_{q_3} - 2zq_4\partial_{q_4} + zq_5\partial_{q_5}),\\
\widetilde{\Box}_{l_7} &= (-2zq_1\partial_{q_1} + zq_3\partial_{q_3})(zq_3\partial_{q_3} - 2zq_4\partial_{q_4} + zq_5\partial_{q_5}) - q_2q_3(zq_1\partial_{q_1} - zq_3\partial_{q_3} + zq_4\partial_{q_4}),\\
\widetilde{\Box}_{l_8} &= (-2zq_1\partial_{q_1} + zq_3\partial_{q_3})(zq_4\partial_{q_4} - zq_5\partial_{q_5}) - q_2q_3q_4(zq_3\partial_{q_3} - 2zq_4\partial_{q_4} + zq_5\partial_{q_5}),\\
\widetilde{\Box}_{l_9} &= (zq_1\partial_{q_1} + zq_2\partial_{q_2} - zq_3\partial_{q_3})(zq_3\partial_{q_3} - 2zq_4\partial_{q_4} + zq_5\partial_{q_5}) - q_1q_2q_3(-2zq_1\partial_{q_1} + zq_3\partial_{q_3}),\\
\widetilde{\Box}_{l_{10}} &= (-zq_2\partial_{q_2})(-2zq_1\partial_{q_1} + zq_3\partial_{q_3}) - q_3q_4q_5(zq_1\partial_{q_1} + zq_2\partial_{q_2} - zq_3\partial_{q_3}),\\
\widetilde{\Box}_{l_{11}} &= (-zq_2\partial_{q_2})(zq_1\partial_{q_1} - zq_3\partial_{q_3} + zq_4\partial_{q_4}) - q_1q_3q_4q_5(-2zq_1\partial_{q_1} + zq_3\partial_{q_3}),\\
\widetilde{\Box}_{l_{12}} &= (-zq_2\partial_{q_2})(zq_4\partial_{q_4} - zq_5\partial_{q_5}) - q_1q_2q_3^2q_4(zq_2\partial_{q_2} - zq_3\partial_{q_3} + zq_5\partial_{q_5}),\\
\widetilde{\Box}_{l_{13}} &= q_1(-2zq_1\partial_{q_1} + zq_3\partial_{q_3})^2 - zq_1(-2zq_1\partial_{q_1} + zq_3\partial_{q_3})\\
&\qquad - (zq_1\partial_{q_1} + zq_2\partial_{q_2} - zq_3\partial_{q_3})(zq_1\partial_{q_1} - zq_3\partial_{q_3} + zq_4\partial_{q_4}),\\
\widetilde{\Box}_{l_{14}} &= q_4(zq_3\partial_{q_3} - 2zq_4\partial_{q_4} + zq_5\partial_{q_5})^2 - zq_4(zq_3\partial_{q_3} - 2zq_4\partial_{q_4} + zq_5\partial_{q_5})\\
&\qquad - (zq_1\partial_{q_1} - zq_3\partial_{q_3} + zq_4\partial_{q_4})(zq_4\partial_{q_4} - zq_5\partial_{q_5}).
\end{aligned}$$

How can we be sure that these few box operators, together with $\widetilde{E}$ *generate* $\widetilde{\mathcal{I}}$*? For this purpose let us rewrite* $\widetilde{\Box}_{l_1}, \ldots, \widetilde{\Box}_{l_{14}}$ *as a column vector in the form*

$$\begin{pmatrix} \widetilde{\Box}_{l_1} \\ \vdots \\ \widetilde{\Box}_{l_{14}} \end{pmatrix} = M. \begin{pmatrix} (zq_1\partial_{q_1})^2 \\ \vdots \\ (zq_5\partial_{q_5})^2 \end{pmatrix} + \begin{pmatrix} r_1 \\ \vdots \\ r_{14} \end{pmatrix}, \tag{45}$$

where the "column vector of squares" is the transpose of

$$\begin{aligned}
\Big((zq_1\partial_{q_1})^2, (zq_1\partial_{q_1})(zq_2\partial_{q_2}), (zq_1\partial_{q_1})(zq_3\partial_{q_3}), (zq_1\partial_{q_1})(zq_4\partial_{q_4}), (zq_1\partial_{q_1})(zq_5\partial_{q_5}),\\
(zq_2\partial_{q_2})^2, (zq_2\partial_{q_2})(zq_3\partial_{q_3}), (zq_2\partial_{q_2})(zq_4\partial_{q_4}), (zq_2\partial_{q_2})(zq_5\partial_{q_5}),\\
(zq_3\partial_{q_3})^2, (zq_3\partial_{q_3})(zq_4\partial_{q_4}), (zq_3\partial_{q_3})(zq_5\partial_{q_5}), (zq_4\partial_{q_4})^2, (zq_4\partial_{q_4})(zq_5\partial_{q_5}), (zq_5\partial_{q_5})^2\Big)
\end{aligned} \tag{46}$$

and where the r_i *are the "square-free terms", i.e. terms only involving* $(zq_i\partial_{q_i})^k$ *for* $k = 0, 1$*. We know that* ${}_0\mathcal{QM}^{\mathrm{loc}}_{\widetilde{A}}$ *is isomorphic to the module*

$$\frac{\mathbb{C}[z, q_1^{\pm}, \ldots, q_5^{\pm}]\langle z^2\partial_z, zq_1\partial_{q_1}, \ldots, zq_5\partial_{q_5}\rangle}{\langle \widetilde{E}, \widetilde{\Box}_l\rangle_{l\in\mathbb{L}}}$$

and that it should be free over $\mathbb{C}[z, q_1^{\pm}, \dots, q_5^{\pm}]$ *of rank* 7*. In fact, a basis of this module is given by*

$$\{1, zq_1\partial_{q_1}, \dots, zq_5\partial_{q_5}, (zq_1\partial_{q_1})(zq_2\partial_{q_2})\}. \tag{47}$$

In order to show this we realise firstly that $\widetilde{E}$ *gives us a relation to express* $z^2\partial_z$ *in terms of this basis. Furthermore we have fifteen entries in our matrix of squares. We would like to express them all in terms of our basis, which means in practice that the matrix* M *needs to have rank fourteen. In our example we readily write down* M *as*

$$\begin{pmatrix}
0 & -2 & 2 & 0 & -2 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & 1 & -2 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & -1 & 2 & 0 & 0 & -2 & 1 \\
0 & 1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & 0 \\
0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 1 & -2 & 1 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 2 & -2 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 \\
-1+4q_1 & -1 & 2-4q_1 & -1 & 0 & 0 & 1 & -1 & 0 & -1+q_1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & q_4 & 1-4q_4 & -1+2q_4 & -1+4q_4 & 1-4q_4 & q_4
\end{pmatrix}.$$

This matrix does indeed have rank 14*, as we can see by reducing it to echelon form:*

$$\begin{pmatrix}
1 & \frac{2q_1}{1-4q_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -\frac{1-6q_4}{1-4q_4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

where, as we recall from (46), the second column corresponds to the coefficients of $(zq_1\partial_{q_1})(zq_2\partial_{q_2})$*. Thus we know that our Brieskorn lattice*

$$\frac{\mathbb{C}[z, q_1^{\pm}, \dots, q_5^{\pm}]\langle z^2\partial_z, zq_1\partial_{q_1}, \dots, zq_5\partial_{q_5}\rangle}{\langle \widetilde{E}, \widetilde{\Box}_l\rangle_{l\in\mathbb{L}}} = \frac{\mathbb{C}[z, q_1^{\pm}, \dots, q_5^{\pm}]\langle z^2\partial_z, zq_1\partial_{q_1}, \dots, zq_5\partial_{q_5}\rangle}{\langle \widetilde{E}, \widetilde{\Box}_{l_i}\rangle_{1\le i\le 14}}$$

is free of rank 7 *over* $\mathbb{C}[z, q_1^{\pm}, \dots, q_5^{\pm}]$*, with a basis is given, for instance, by (47).*

8.4 Extending the Brieskorn Lattice

As we mentioned earlier, the Brieskorn lattice is supposed to be a lattice of a $\mathcal{D}$-module on $\mathbb{C} \times U$, where U is a suitably small neighbourhood of the origin in $\mathbb{C}^r$. Only then can we attempt to solve the Birkhoff problem by extending it to $\mathbb{P}^1 \times U$ and thus get the B-model equivalent of our Dubrovin connection. However, so far we have ${}_0\mathcal{QM}^{\text{loc}}_{\widetilde{A}}$ being a sheaf on $\mathbb{C}_z \times S_2^o$. Therefore we have to extend the Brieskorn lattice through the inclusion $S_2^o \hookrightarrow S_2 \hookrightarrow \mathbb{C}^r$ and afterwards restrict to a suitable $U \subset \mathbb{C}^r$.

Realising the inclusion $S_2^o \hookrightarrow S_2$ is particularly easy. Since we will eventually restrict to U, a ball around the origin of $\mathbb{C}^r$, it suffices to show that there exists such a ball, which does not intersect a compactification of the non-tame locus. In mathematical terms: let $\overline{S_2^o} := \mathbb{C}^r \setminus (\overline{S_2 \setminus S_2^o})$, where $\overline{S_2 \setminus S_2^o}$ denotes the closure of $S_2 \setminus S_2^o$ in $\mathbb{C}^r$. We want $0 \in \overline{S_2^o}$, which then immediately implies the existence of a ball $B_\varepsilon(0)$ of radius $\varepsilon = \inf \left\{ |q| \;\middle|\; q \notin \overline{S_2 \setminus S_2^o} \right\} > 0$, such that $B_\varepsilon(0) \subset \overline{S_2^o}$. However, this might not be true as the following example shows.

Example 8.15. *Given our calculations in example 8.9, we see that in our case any* $\varepsilon < 1/4$ *will suffice. Here is an example of how this might fail, i.e. how the non-tame locus might intersect the origin: recall our initial choice of basis for* $\operatorname{Pic}(X_4^o)$ *from example 7.14. Since in this basis*

$$\vartheta(x_1, \dots, x_7) = \left(x_1x_5, x_2x_5x_7, x_3x_7, \frac{x_4x_7}{x_5}, \frac{x_6}{x_5x_7} \right)$$

we see that our non-tame locus is given not only by

$$4x_4x_6 - x_5^2 = 0 \Leftrightarrow 4\left(\frac{x_4x_7}{x_5}\right)\left(\frac{x_6}{x_5x_7}\right) - 1 = 0 \Leftrightarrow 4q_4q_5 - 1 = 0,$$

but also by

$$4x_2x_4 - x_3^2 = 0 \Leftrightarrow 4\left(x_2x_5x_7\right)\left(\frac{x_4x_7}{x_5}\right) - (x_3x_7)^2 = 0 \Leftrightarrow 4q_2q_4 - q_3^2 = 0.$$

So in this example, $0 \in \Delta^\infty$.

The reason for the problem in this example is that our initial choice of basis of $\operatorname{Pic}(X_4^o)$, $\{[D_1], [D_2], [D_3], [D_4], [D_6]\}$, was not contained in the Kähler cone of X_4^o. Recall the short exact sequence (41):

$$0 \longrightarrow M \xhookrightarrow{A} \operatorname{Div}_T(X_\Sigma) \xrightarrow{B} \operatorname{Pic}(X_\Sigma) \longrightarrow 0.$$

The ensuing calculations of our Landau-Ginzburg model relied on the matrix B and hence on the choice of basis $\{u_1, \dots, u_r\}$ of $\operatorname{Pic}(X)$. Iritani showed in [Iri09, appendix 6.1] that if $u_1, \dots, u_r \in \operatorname{Nef}(X)$ are in the Kähler cone and that furthermore $-K_X$ is contained in the cone generated by $u_1, \dots, u_r$, then $0 \in \overline{S_2^o}$. We will henceforth assume that the u_i were chosen to fulfil these two conditions.

Example 8.16. *Let us see how this affects our running example of* X_4^o *: start by recalling the Kähler cone of* X_4^o *from example 7.10.*

Claim 8.17. *There exists no* $\mathbb{Z}$*-basis* $\{u_1, \dots, u_5\}$ *of* $\operatorname{Pic}(X_4^o)$*, such that* $-K_{X_4^o}$ *is strictly on the inside of the cone generated by* $u_1, \dots, u_5$.

Proof: Assume such a basis existed, i.e. that there are $\kappa_1, \ldots, \kappa_5 > 0$ such that $-K_{X_4^o} = \kappa_1 u_1 + \cdots \kappa_5 u_5$. As $\{u_1, \ldots, u_5\}$ is a $\mathbb{Z}$-basis, this implies that the $\kappa_i \in \mathbb{Z}$. Thus each $\kappa_i \geq 1$. Now let us identify $\mathrm{Pic}(X_4^o)_{\mathbb{R}} \simeq \mathbb{R}^5$ by choosing coordinates with respect to the basis $\{D_1, D_2, D_3, D_4, -D_6\}$. Note that the Kähler cone is contained in the non-negative orthant, as all its generators are. This implies that $u = \alpha_1 D_1 + \alpha_2 D_2 + \alpha_3 D_3 + \alpha_4 D_4 - \alpha_5 D_6 \in \mathrm{Nef}(X_4^o)$ can be part of our basis only if $\alpha_1 \leq 2$, $\alpha_2 \leq 3$, $\alpha_3 \leq 2$, $\alpha_4 \leq 1$ and $\alpha_5 \leq 1$, since

$$-K_{X_4^o} = \sum_{i=1}^{7} [D_i] = 2[D_1] + 3[D_2] + 2[D_3] + [D_4] - [D_6].$$

So let us find all such points. Once again this is a straightforward, but tedious task: start off by writing $u = \beta_1 v_1 + \cdots + \beta_8 v_8$,where the v_i are the generators of the Kähler cone as determined in example 7.10. Then $(\beta_1, \ldots, \beta_8) = \beta^T$ relates to $(\alpha_1, \ldots, \alpha_5) = \alpha^T$ by

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 & 3 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \beta = \alpha.$$

This is equivalent to

$$\begin{aligned} \beta_1 + \beta_6 &= \alpha_2 - 2\alpha_3 + \alpha_4, \\ \beta_2 - \beta_6 - 2\beta_7 + \beta_8 &= -\alpha_1 + \alpha_2 - 2\alpha_5, \\ \beta_3 + \beta_7 - \beta_8 &= \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \alpha_5, \\ \beta_4 + \beta_8 &= \alpha_4 - \alpha_5, \\ \beta_5 + \beta_6 + \beta_7 &= \alpha_5. \end{aligned}$$

Given our conditions on the α_i and the fact that $\beta_i \geq 0$ we end up with only a few possible values of α, of which only fewer are realised by a possible β. As a result we have 16 possible candidates for our basis, excluding $2D_1 + 2D_2 = 2v_1$, the origin and $-K_{X_4^o}$. In $\{D_1, D_2, D_3, D_4, -D_6\}$-coordinates they are

$$\begin{array}{llll} (1,1,0,0,0), & (2,1,0,0,0), & (1,2,1,0,0), & (2,2,1,0,0), \\ (2,3,1,0,0), & (0,1,1,1,0), & (1,1,1,1,0), & (2,1,1,1,0), \\ (1,2,1,1,0), & (2,2,1,1,0), & (2,3,1,1,0), & (1,3,2,1,0), \\ (1,2,1,1,1), & (2,2,1,1,1), & (2,3,1,1,1), & (1,3,2,1,1). \end{array}$$

Clearly

$$\begin{aligned} -K_{X_4^o} = (2,3,2,1,1) &= (1,2,1,0,0) + (1,1,1,1,1) \\ &= (2,2,1,0,0) + (0,1,1,1,1) \end{aligned}$$

are the only ways of describing $-K_{X_4^o}$ as the sum elements of the Kähler cone. In particular, there exists no choice of $\mathbb{Z}$-basis $\{u_1, \ldots, u_5\}$, in which $-K_{X_4^o} = \kappa_1 u_1 + \cdots \kappa_5 u_5$ with all $\kappa_i > 0$. In fact, we see that at most two κ_i can be non-trivial. $\square$

In light of this claim we choose $\{v_1, v_2, v_3, v_4, v_7\}$ *as in example 7.10 as our basis for* $\mathrm{Pic}(X_4^o)$, *thus explaining the initially strange choice in examples 7.14 and 8.14.*

So provided we have chosen a suitable basis of $\mathrm{Pic}(X_4^o)$, which we are assuming, we have a small open neighbourhood $U \subset \mathbb{C}^r$ of the origin such that $S_2^o \cap U = S_2 \cap U$. By a simple restriction of the sheaves we worked with before, we now interpret $\mathcal{QM}_{\widetilde{A}}^{\mathrm{loc}}$ as a sheaf of $\mathcal{D}_{\mathbb{C}_\tau \times (S_2 \cap U)}$-module and ${}_0\mathcal{QM}_{\widetilde{A}}^{\mathrm{loc}}$ consequently as a $\mathcal{O}_{\mathbb{C}_z \times (S_2 \cap U)}$-module. Now let $\widetilde{\mathcal{R}}$ be the sheaf of $\mathcal{O}_{\mathbb{C}_z \times \mathbb{C}^r}$-algebras associated to the ring $\mathbb{C}[z, q_1, \ldots, q_r]\langle z^2\partial_z, zq_1\partial_{q_1}, \ldots, zq_r\partial_{q_r}\rangle$. As we have seen in chapter 8.3, both $\widetilde{E}$ and $\widetilde{\Box}_l$ can be expressed in terms of z and $zq_i\partial_{q_i}$. Seeing now that $\widetilde{\mathcal{R}}$ restricts to $\mathcal{D}_{\mathbb{C}_\tau^* \times (U \cap S_2)}$ on $\mathbb{C}_z^* \times (U \cap S_2)$, we find that $\widehat{E}$ and $\widetilde{\Box}_l$ for $l \in \mathbb{L}$ define the ideal $\widetilde{\mathcal{I}}$ from proposition 8.13 in $\mathcal{D}_{\mathbb{C}_\tau^* \times (U \cap S_2)}$. In particular we have $\widetilde{\mathcal{R}}/\widetilde{\mathcal{I}}|_{\mathbb{C}_z \times (U \cap S_2)} = {}_0\mathcal{QM}_{\widetilde{A}}^{\mathrm{loc}}|_{\mathbb{C}_z \times (U \cap S_2)}$, which implies that $\widetilde{\mathcal{R}}/\widetilde{\mathcal{I}}|_{\mathbb{C}_z^* \times (U \cap S_2)} = \mathcal{QM}_{\widetilde{A}}^{\mathrm{loc}}|_{\mathbb{C}_\tau^* \times (U \cap S_2)}$. The important step in extending the Brieskorn lattice to the entirety of U was proven in [RS15, Theorem 3.7]: define ${}_0\mathcal{QM}_{\widetilde{A}} := \mathcal{O}_{\mathbb{C}_z \times U} \otimes_{\mathcal{O}_{\mathbb{C}_z \times \mathbb{C}^r}} \widetilde{R}/\widetilde{I}$. The authors showed that a neighbourhood U of the origin exists on which ${}_0\mathcal{QM}_{\widetilde{A}}$ is $\mathcal{O}_{\mathbb{C}_z \times U}$-coherent and that furthermore there exists a meromorphic connection with poles along $\{z = 0\} \times U$ and $\mathbb{C}_z \times D$ (D being the normal crossing divisor defined by $q_1 \cdots q_r = 0$), which extends the $\mathcal{D}_{\mathbb{C}_\tau^* \times (U \cap S_2)}$-module structure of $\mathcal{QM}_{\widetilde{A}}^{\mathrm{loc}}|_{\mathbb{C}_\tau^* \times (U \cap S_2)}$. Thus we have extended our Brieskorn lattice to $\mathbb{C}_z \times U$ as a sheaf of modules associated to

$$F := \frac{\mathbb{C}[z, q_1, \ldots, q_r]\langle z^2\partial_z, zq_1\partial_{q_1}, \ldots, zq_r\partial_{q_r}\rangle}{\langle \widetilde{E}, \widetilde{\Box}_l\rangle_{l \in \mathbb{L}}}. \tag{48}$$

9 Birkhoff's Problem

As mentioned in the introduction to the previous chapter, we now need to extend our Brieskorn lattice from $\mathbb{C} \times U$ to $\mathbb{P}^1 \times U$. Only then can we compare it directly to the Dubrovin connection from chapter 5. A much more thorough introduction to the Birkhoff problem, and in particular its many connections to the Riemann-Hilbert problem, can be found in [Sab08].

9.1 Birkhoff's Problem Historically

Birkhoff was not trying to extend any bundles with connections from $\mathbb{C}$ to $\mathbb{P}^1$. At least he would not have formulated it that way. From his point of view he was studying systems of linear differential equations of the form

$$\frac{dX}{d\tau} = \tau^k A(\tau)X \tag{49}$$

for some $d \times d$ matrix $A(\tau) = A_0 + \tau^{-1}A_1 + \ldots$, which is assumed to be analytic for large enough (complex) $|\tau|$. His aim was to put (49) into as simple a form as possible, by using a gauge transformation $X = P(\tau)Y$. Here, $P(\tau)$ is also required to be analytic for large enough $|\tau|$ and in particular to be analytic wherever A is. His transformed equation would then read

$$\frac{dY}{d\tau} = \tau^k B(\tau)Y,$$

where $B(\tau) = B_0 + \tau^{-1}B_1 + \ldots + \tau^{-s}B_s$ for some minimal natural number s. Birkhoff's claim in [Bir09] was that $s = k + 1$ suffices. However, counter-examples were found in the 1950s [Gan59, Mas59] and Turrittin [Tur63] later pointed out Birkhoff's mistake and gave a revised analysis. In the case $k = -1$ for example he found $s = 1$ to be minimal.

Both Birkhoff's and Turrittin's arguments are based on a generalisation of the following fact: let a complex-valued function $f(t)$ be analytic for large enough $|t|$. Then $f(t) = g(t)t^k h(t)$, where g is analytic for large $|t|$ and invertible at ∞, h is an entire invertible function on $\mathbb{C}_t$ and $k \in \mathbb{Z}$. Why would this be true? For example by utilising a Laurent series expansion or otherwise, we can write $\widetilde{f}(t) = \widetilde{g}(t) + \widetilde{h}(t)$ for any complex-valued function $\widetilde{f}$, which is analytic for large $|t|$, and where $\widetilde{g}$ is analytic for large $|t|$, does not vanish at ∞ and where $\widetilde{h}$ is entire. Now observe that the multivalued function $\log(f(t))$ is analytic for large $|t|$ and its monodromy (with respect to a single large loop counter-clockwise around the origin) is given by addition of $2k\pi\sqrt{-1}$. Thus $\log(f(t)) - k\log(t)$ is single-valued and analytic for large $|t|$ and we can express it as $\widetilde{g}(t) + \widetilde{h}(t)$. Afterwards apply the exponential function on both sides to obtain the desired decomposition. The generalisation to matrices used by Birkhoff is given by the following proposition.

Proposition 9.1 (Birkhoff Factorisation)**.** *Let $L(\tau)$ be a $d \times d$ matrix depending on $\tau \in \mathbb{C}$. Assume that on the annulus $0 \le r < |\tau| < R \le \infty$ $L(\tau)$ is analytic and has nowhere vanishing determinant. Then there exist two $d \times d$ matrices $P(\tau), Q(\tau)$ and a unique sequence $a_1 \ge \ldots \ge a_d$ of integers such that P is analytic on $|\tau| < R$, Q on $|\tau| > r$ and on the annulus we have*

$$L(\tau) = P(\tau)\,\mathrm{Diag}\{\tau^{a_1}, \ldots, \tau^{a_d}\}Q(\tau).$$

Let us now translate Birkhoff's problem into a language more suitable for our purposes. Start by performing the co-ordinate change $z = \tau^{-1}$ in equation (49). We obtain

$$\frac{dX}{dz} + z^{-r}B(z)X = 0, \tag{50}$$

where $r = k + 2$ and $B(z) = A(z^{-1})$. This is clearly the differential equation fulfilled by constant sections of the trivial bundle on U with connection ∇, where $U \subset \mathbb{C}_z$ is a small neighbourhood of the origin and ∇ is given by the connection matrix $z^{-r}B(z)$. Birkhoff's claim was that for any such connection on U there exists a holomorphic change of coordinates after which the connection matrix takes the form

$$C(z) = \frac{C_r}{z^r} + \cdots + \frac{C_1}{z}.$$

Now let us be more specific and consider the situation, where ∇ is given algebraically as a flat connection on a trivial meromorphic bundle on $\mathbb{P}^1$. Then $A(\tau) \in \mathrm{Mat}_d(\mathbb{C}[\tau^{\pm}])$ and we want a gauge transformation with $P \in \mathrm{GL}_d(\mathbb{C}[z])$. Even more, we shall assume that $r = 2$, i.e. we have at most a regular singularity at ∞. Note that this gives no information on the germ of the bundle with connection at $\tau = 0$, where it may well be irregular.

Remark 9.2. *Here is an important observation about bundles on $\mathbb{P}^1$: say $E \to \mathbb{P}^1$ is such a bundle of rank d. We will always be thinking in the equivalent sheaf of sections $\mathcal{E}$ of E. Covering $\mathbb{P}^1$ using the two open sets $\{|\tau| < R\}$ and $\{|\tau| > r\} \cup \{\infty\}$, we see $\mathcal{E}$ is uniquely defined by a cocycle, which in this case is the equivalence class of an invertible matrix $L(\tau)$, holomorphic on $\{r < |\tau| < R\}$. By the Birkhoff factorisation 9.1, the same cocycle is defined by the diagonal matrix* $\mathrm{Diag}\{\tau^{a_1}, \ldots, \tau^{a_d}\}$. *Thus we see that $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}^1}(a_d)$ for some sequence of integers $a_1 \le \ldots \le a_d$. This is known as the* Birkhoff-Grothendieck theorem. *Moreover, assume now that we have a meromorphic bundle on $\mathcal{M}$ with singularities in a discrete set of points $\Sigma \subset \mathbb{P}^1$. It is not difficult to find locally around every point of Σ a lattice of $\mathcal{M}$ and since they all agree with $\mathcal{M}$ on $\mathbb{P}^1 \setminus \Sigma$, they can be glued to a global lattice $\mathcal{E}$ of $\mathcal{M}$. This lattice is a holomorphic bundle on $\mathbb{P}^1$ and hence of the form $\mathcal{O}_{\mathbb{P}^1}(a_1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}^1}(a_d)$. Now tensoring both sides with $\mathcal{O}_{\mathbb{P}^1}(*\Sigma)$, we find that $\mathcal{M}$ is necessarily isomorphic to the trivial bundle $\mathcal{O}_{\mathbb{P}^1}(*\Sigma)^d$.*

So let us return to our previous set-up: we have a meromorphic bundle $\mathcal{M}$ with flat connection ∇, which has singularities at 0 and ∞ only. ∇ is given by the connection matrix $A(\tau) \in \mathrm{Mat}_d(\mathbb{C}[\tau^{\pm}])$. We shall also assume that the singularity at ∞ is regular and that $\mathcal{M}$ admits a lattice $\mathcal{E}$ of order r in a neighbourhood of 0. The Birkhoff problem thus translated asks whether there exists a global trivial lattice $\mathcal{E}'$, extending $\mathcal{E}$, which is logarithmic at ∞.

9.2 The Problem in Our Case

Let us now take a closer look at our situation. From the algebraic point of view we have a trivial vector bundle over $\mathbb{C}_z \times U$, given by the module F as in (48) over the ring $\mathbb{C}[z, q_1, \ldots, q_r]$. Say a basis of this module is given by monomials $(zq_1\partial_{q_1})^{i_1} \cdots (zq_r\partial_{q_r})^{i_r} = (zq_I\partial_{q_I})^I$. This can be done, since $\widetilde{E}$ is of the form (43) and thus we can commute terms involving $z^2\partial_z$ to the right and there replace them by $-\sum u_j^*(\omega_{X_\Sigma}^{-1})zq_j\partial_{q_j}$. Then the (rational) connection on this module is given by $\nabla = d + \Omega$, where

$$\Omega = \sum_{j=1}^{r} \Omega^{(j)} \frac{dq_j}{zq_j} + \Omega^q \frac{dz}{z^2}.$$

Here $\Omega^{(j)}$ is the matrix representing the action of $zq_j\partial_{q_j}$ on the module and Ω^q as the one representing the action of $z^2\partial_z$. Both $\Omega^{(j)}$ and Ω^q are with respect to our chosen basis of F.

Clearly this situation differs from the one originally considered by Brieskorn mainly in the way that we have a family of connections on $\mathbb{C}_z$ parametrised by the set $U \subset \mathbb{C}^r$. Suppose we restrict our attention to some particular $q^o \in U$. Then we have a well-defined restriction to a connection ∇^o on $\mathbb{C}_z$. Denoting the dependence of $\Omega^q = \Omega^q(z,q)$, ∇^o is given by the connection matrix $\Omega^o = z^{-2}\Omega^q(z, q^o)$. Fortunately, it turns out that solutions to the Birkhoff problem admit local extensions, a statement which the following proposition aims to make precise in our particular case.

Proposition 9.3. *Let $F \to \mathbb{C}_z \times U$ be a holomorphic bundle with a meromorphic connection ∇. Suppose we have solved the Birkhoff problem for some particular value $q^o \in U$, i.e. we have found a basis ε^o of F^o (the fibre at q^o), such that the restricted connection ∇^o is given by the connection matrix $\Omega^o = (B_\infty/z + B_0^o/z^2)dz$. Then there exists an open set $V \subset U$ containing q^o and a local frame ε of F over V such that $\varepsilon|_{q^o} = \varepsilon^o$ and in the basis ε, ∇ is given by the connection matrix*

$$\Omega(z,q) = \left(\frac{B_\infty}{z} + \frac{B_0(q)}{z^2}\right) dz + \frac{C(q)}{z},$$

and where $B_0(q^o) = B_0^o$ and C is a matrix of 1-forms.

Proof: See [Sab08, theorem VI.2.1]. □

In our case we find $z^2\Omega^o$ as the matrix representing the action of $z^2\partial_z$ on our basis elements, which as we recall, are of the form $(zq_I\partial_{q_I})^I$ for some $I \in \mathbb{N}^r$. Now in F we have

$$\begin{aligned} z^2\partial_z(zq_I\partial_{q_I})^I &= [z^2\partial_z, (zq_I\partial_{q_I})^I] + (zq_I\partial_{q_I})^I z^2\partial_z \\ &= \left(z^2 \sum_{j=1}^{r} (zq_1\partial_{q_1})^{i_1} \cdots [\partial_z, zq_j\partial_{q_j}^{i_j}] \cdots (zq_r\partial_{q_r})^{i_r}\right) + (zq_I\partial_{q_I})^I \left(-\sum_{j=1}^{r} u_j^*(\omega_{X_\Sigma}^{-1}) zq_j\partial_{q_j}\right) \\ &= |I| z (zq_I\partial_{q_I})^I - \left(\sum_{j=1}^{r} u_j^*(\omega_{X_\Sigma}^{-1})\Omega^{(j)}\right)(zq_I\partial_{q_I})^I, \end{aligned}$$

where the last equality follows since $zq_i\partial_{q_i}$ and $zq_j\partial_{q_j}$ commute for all i, j and since $\Omega^{(j)}$ is precisely the matrix representing the action of $zq_j\partial_{q_j}$ on our basis elements. Thus we see that in our case B_∞ is a diagonal matrix with $|I|$ at the diagonal place corresponding to the basis element $(zq_I\partial_{q_I})^I$. However, B_0^o might contain terms involving z, so it is not in the correct form yet and solving the Birkhoff problem means finding a basis, in which it is independent of z.

Here is an important observation: Provided we have chosen a basis of F, then the connection matrix Ω is given in terms of the matrices $B_0(z,q)$ and B_∞ as above, as well as the matrices $\Omega^{(j)}$, representing the action of $zq_j\partial_{q_j}$ on our basis. Instead of restricting to some fixed parameter $q^o \in U$, let us restrict to some fixed $z \in \mathbb{C}$ (as we did when introducing the Dubrovin connection in chapter 5). Then our restricted connection matrix takes on the form

$$\Omega^z = \sum_{j=1}^{r} \Omega^{(j)} \frac{dq_j}{zq_j},$$

where the only difference to our original form of Ω is the missing dz term and the fact that we are assuming z to be fixed in $\Omega^{(j)}(z,q)$. Assume we found a basis of F in which each $\Omega^{(j)}$ is independent of z. Then by our previous observation on the form of $B_0(z,q)$ as a weighted sum of the $\Omega^{(j)}$, we would automatically have B_0 being independent of z. Thus we would simultaneously solve the Birkhoff problem at any point $q^o \in U$ and construct the basis ε of F over $\mathbb{C}_z \times U$ from proposition 9.3.

Following [Gue08], here is an argument why we should always be able to find a solution to our modified Birkhoff problem, i.e. locally around some q^o find a basis such that $z\Omega^z$ is independent of z. Note first that the action of $zq_j\partial_{q_j}$ on a basis element $(zq_I\partial_{q_I})^I$ does not involve negative powers of z. Thus our connection matrix Ω^z is of the form

$$\Omega^z = \frac{1}{z}\omega + \vartheta_0 + z\vartheta_2 + \cdots + z^p\vartheta_p,$$

where ω and the ϑ_j are matrix-valued 1-forms, depending on the q_j. Since the connection $\nabla^z = d + \Omega^z$ is flat (c.f. [Sab08, chapter 0.12.]), its dual connection $\nabla^{z,*} = d - (\Omega^z)^T$ is flat too. We can find (at least locally) a fundamental solution to the differential equation defined by $\nabla^{z,*}$, i.e. an invertible matrix X with entries in $\mathbb{C}[z, q_1, \ldots, q_r]$, which are holomorphic in a neighbourhood of q^o, such that

$$dX = (\Omega^z)^T X \qquad \Rightarrow \qquad (\Omega^z)^T = dXX^{-1}.$$

Now letting $P = X^T$ we have expressed

$$\Omega^z = P^{-1}dP.$$

By proposition 9.1, we know that P has a Birkhoff factorisation as $P = NM$, where we have included the diagonal term into the right-most factor:

$$\begin{aligned} N(z,q) &= I + \frac{1}{z}N_1(q) + \frac{1}{z^2}N_2(q) + \cdots \\ M(z,q) &= M_0(I + zM_1(q) + z^2M_2(q) + \cdots). \end{aligned}$$

Now using the gauge transformation $P \mapsto PL = N$, where $L = M^{-1}$, we have

$$\Omega^z = P^{-1}dP \mapsto \widehat{\Omega}^z = N^{-1}dN.$$

However,

$$N^{-1}dN = (PL)^{-1}d(PL)$$
$$= L^{-1}\Omega^z L + L^{-1}dL.$$

Since the left-hand sight contains only negative powers of z, whereas the right-hand side contains mostly positive ones, we see that

$$\widehat{\Omega}^z = \frac{1}{z}M_0^{-1}\omega(M_0)^{-1}.$$

Furthermore, imposing that

$$\lim_{q_i\to 0} M_0 = \lim_{z,q_i\to 0} M = \mathrm{Id}$$

determines M and hence L uniquely. This would show the general existence of a solution to the Birkhoff problem in our setting, were it not for the fact that [Gue08] assumes that Ω^z depends (locally) holomorphically on the q_i, which is given on S_2, but unfortunately not on our extension to $\mathbb{C}^r$. Be that as it may, [Sab06] shows that in general a solution exists to the Birkhoff problem at every point $q \in S_2^o$ and based on [Gue08] it was shown in [RS15] that one can indeed construct a solution to the Birkhoff problem even over our extension to $U \subset \mathbb{C}^r$. The strategy is to restrict to the fibre $\mathbb{C}_z \times \{q = 0\}$ and solve the Birkhoff problem there, i.e. construct a good basis (in the sense of [Sai89]) in which the restricted connection takes the desired form. Since such a basis extends locally by theorem 9.3, we have solved the Birkhoff problem on $\mathbb{P}^1 \times V \subset \mathbb{P}^1 \times U$ and by restriction assume that $V = U$.

Remark 9.4. *We have in this chapter happily mixed the algebraic and analytic points of view. At this last stage, however, we should stress that the extension of a solution to the Birkhoff problem from $\mathbb{P}^1 \times \{q = 0\}$ to $\mathbb{P}^1 \times U$ is purely analytic. This is since the non-tameness along $S_2 \setminus S_2^o$ may introduce new monodromy phenomena. See [RS15] for more details.*

The draw-back for us is that these proofs are all existential rather than constructive proofs. In order to explicitly solve the Birkhoff problem we have to find a change of basis matrix L, such that the new connection matrix $L^{-1}\Omega^z L + L^{-1}dL$ is of the form $\frac{1}{z}\omega$ with ω independent of z. We will be aided by the natural notion of degree on the quantum cohomology ring (see chapter 3.4)and the assumption that an equivalent exists on the B-side. If, as in equation (43)

$$\widetilde{E} = z^2\partial_z + \sum_{i=1}^{r} u_i^*(\omega_{X_\Sigma}^{-1})zq_i\partial_{q_i},$$

then we define the degree of z to be 2 and of q_i to be $2u_i^*(\omega_{X_\Sigma}^{-1})$ and we assume that our change of basis matrix L preserves this notion of degree.

9.3 Solving the Birkhoff Problem for X_4^o

Let us now solve the Birkhoff problem in our specific case. In order to express our connection ∇^z as $d + \Omega^z$ and explicitly calculate Ω^z we have to consider the action of $zq_i\partial_{q_i}$ on the Brieskorn lattice. In our chosen basis $\{1, (zq_i\partial_{q_i}), (zq_1\partial_{q_1})(zq_2\partial_{q_2})\}_{i=1,\dots,5}$ we write a general element $S \in F$ as row vector

$$S_0 + \sum_{i=1}^{5} S_i(zq_i\partial_{q_i}) + S_6(zq_1\partial_{q_1})(zq_2\partial_{q_2}) = \begin{pmatrix} S_0 \\ \vdots \\ S_6 \end{pmatrix}.$$

Then express this action in matrix form as

$$\begin{aligned}
zq_i\partial_{q_i} : F &\longrightarrow F \\
S &\mapsto \Omega^{(i)} \begin{pmatrix} S_0 \\ \vdots \\ S_6 \end{pmatrix} \\
&= \left(\sum_{j=0}^{6} \Omega^{(i)}_{1j} S_j\right) + \left(\sum_{j=0}^{6} \Omega^{(i)}_{2j} S_j\right)(zq_1\partial_{q_1}) + \cdots \\
&\quad + \left(\sum_{j=0}^{6} \Omega^{(i)}_{6j} S_j\right)(zq_5\partial_{q_5}) + \left(\sum_{j=0}^{6} \Omega^{(i)}_{7j} S_j\right)(zq_1\partial_{q_1})(zq_2\partial_{q_2}),
\end{aligned}$$

where $\Omega^{(i)} = (\Omega^{(i)}_{jk})$. In other words:

- The first column of $\Omega^{(i)}$ is simply the $(i+1)$th standard basis column vector.
- For $k = 2, \ldots, 6$, the kth column of $\Omega^{(i)}$, $(\Omega^{(i)}_{1k}, \ldots, \Omega^{i}_{7k})^T$ is such that

$$(zq_i\partial_{q_i})(zq_{k-1}\partial_{q_{k-1}}) = \Omega^{(i)}_{1k} + \sum_{j=2}^{6} \Omega^{(i)}_{jk}(zq_{j-1}\partial_{q_{j-1}}) + \Omega^{(i)}_{7k}(zq_1\partial_{q_1})(zq_2\partial_{q_2}).$$

- The last column of $\Omega^{(i)}$, $(\Omega^{(i)}_{17}, \ldots, \omega^{(i)}_{77})^T$ is such that

$$(zq_i\partial_{q_i})(zq_1\partial_{q_1})(zq_2\partial_{q_2}) = \Omega^{(i)}_{17} + \sum_{j=2}^{6} \Omega^{(i)}_{j7}(zq_{j-1}\partial_{q_{j-1}}) + \Omega^{(i)}_{77}(zq_1\partial_{q_1})(zq_2\partial_{q_2}).$$

The $\Omega^{(i)}$ thus expressed we can write

$$\Omega^z = \sum_{i=1}^{5} \Omega^{(i)} \frac{dq_i}{zq_i}.$$

So our first task is to compute the $\Omega^{(i)}$, which we do by solving equation (45). Equation (45) expresses our box operators $\widetilde{\Box}_{l_i}$ as sums of "squares" (elements of the form $(zq_i\partial_{q_i})(zq_j\partial_{q_j})$) and "lower degree terms" (elements of the form $(zq_i\partial_{q_i})$ or constant elements). Since the box operators vanish in F, we thereby obtain formulae for all elements of the form $(zq_i\partial_{q_i})(zq_j\partial_{q_j})$, i.e. we obtain columns two through six of each matrix $\Omega^{(i)}$. Note that we have already reduced the matrix M of equation (45) to echelon form, so it only remains to carefully repeat the same row operations which reduce M to echelon form, to the vector $(r_1, \ldots, r_{14})^T$ from equation (45). This is a matter of simple linear algebra and we find the following formulae for $(zq_i\partial_{q_i})(zq_j\partial_{q_j})$ for different values of $1 \leq i, j \leq 5$:

$$
\begin{aligned}
(zq_1\partial_{q_1})^2 &= q_1q_2q_3q_4\frac{q_5+q_2q_3+q_3q_5-2q_1q_3q_5}{1-4q_1} - \left(q_1q_3\frac{q_2+(1-4q_1)q_4q_5}{1-4q_1} - z\frac{2q_1}{1-4q_1}\right)(zq_1\partial_{q_1}) + q_1q_3q_4\frac{q_2q_3-2q_1q_2q_3+q_5}{1-4q_1}(zq_2\partial_{q_2})\\
&\quad + q_1q_3q_4\frac{q_2q_3-2q_1q_2q_3+q_5}{1-4q_1}(zq_2\partial_{q_2}) + \left(q_1q_3q_4\frac{2q_2-q_2q_3+2q_1q_2q_3-2q_1q_5}{1-4q_1} - z\frac{q_1}{1-4q_1}\right)(zq_3\partial_{q_3})\\
&\quad + q_1q_2q_3\frac{1-4q_4}{1-4q_1}(zq_4\partial_{q_4}) + q_1q_2q_3q_4\frac{2+q_3-2q_1q_3}{1-4q_1}(zq_5\partial_{q_5}) - \frac{2q_1}{1-4q_1}(zq_1\partial_{q_1})(zq_2\partial_{q_2})\\
(zq_1\partial_{q_1})(zq_3\partial_{q_3}) &= q_1q_2q_3^2q_4q_5 - 2q_1q_3q_4q_5(zq_1\partial_{q_1}) + q_1q_2q_3^2q_4(zq_2\partial_{q_2}) + q_1q_3q_4(q_5-q_2q_3)(zq_3\partial_{q_3}) + q_1q_2q_3^2q_4(zq_5\partial_{q_5}) + (zq_1\partial_{q_1})(zq_2\partial_{q_2})\\
(zq_1\partial_{q_1})(zq_4\partial_{q_4}) &= -q_1q_2^2q_3^2q_4 + 2q_1q_3(q_2-q_4q_5)(zq_1\partial_{q_1}) - q_1q_3(q_2-q_4q_5)(zq_3\partial_{q_3}) + (zq_1\partial_{q_1})(zq_2\partial_{q_2})\\
(zq_1\partial_{q_1})(zq_5\partial_{q_5}) &= -2q_1q_2^2q_3^2q_4 + 2q_1q_3(q_2-q_4q_5)(zq_1\partial_{q_1}) - q_1q_2q_3^2q_4(zq_2\partial_{q_2}) + q_1q_3(q_2q_3q_4-q_2+q_4q_5)(zq_3\partial_{q_3}) - q_1q_2q_3^2q_4(zq_5\partial_{q_5}) + (zq_1\partial_{q_1})(zq_2\partial_{q_2})\\
(zq_2\partial_{q_2})^2 &= q_2q_3q_4(2q_1q_2q_3+q_5) + q_2q_3(1-2q_1)(zq_1\partial_{q_1}) - q_2(zq_2\partial_{q_2}) + q_2(1-q_3+q_1q_3+2q_3q_4)(zq_3\partial_{q_3}) + q_2q_3(1-4q_4)(zq_4\partial_{q_4})\\
&\quad + 2q_2q_3q_4(zq_5\partial_{q_5}) + (zq_1\partial_{q_1})(zq_2\partial_{q_2})\\
(zq_2\partial_{q_2})(zq_3\partial_{q_3}) &= q_2q_3q_4q_5 + q_2q_3(zq_1\partial_{q_1}) - q_2q_3(1-2q_4)(zq_3\partial_{q_3}) + q_2q_3(1-4q_4)(zq_4\partial_{q_4}) + 2q_2q_3q_4(zq_5\partial_{q_5}) + 2(zq_1\partial_{q_1})(zq_2\partial_{q_2})\\
(zq_2\partial_{q_2})(zq_4\partial_{q_4}) &= q_2q_3q_4(q_5-2q_1q_2q_3-q_1q_3q_5) + 4q_1q_2q_3(zq_1\partial_{q_1}) - q_1q_2q_3^2q_4(zq_2\partial_{q_2}) + q_2q_3(q_4-2q_1+q_1q_3q_4)(zq_3\partial_{q_3}) - 2q_2q_3q_4(zq_4\partial_{q_4})\\
&\quad + q_2q_3q_4(1-q_1q_3)(zq_5\partial_{q_5}) + 2(zq_1\partial_{q_1})(zq_2\partial_{q_2})\\
(zq_2\partial_{q_2})(zq_5\partial_{q_5}) &= q_2q_3q_4(q_5-4q_1q_2q_3-q_1q_3q_5) + 4q_1q_2q_3(zq_1\partial_{q_1}) - 2q_1q_2q_3^2q_4(zq_2\partial_{q_2}) - 2q_1q_2q_3(1-q_3q_4)(zq_3\partial_{q_3}) - 2q_1q_2q_3^2q_4(zq_5\partial_{q_5}) + 2(zq_1\partial_{q_1})(zq_2\partial_{q_2})\\
(zq_3\partial_{q_3})^2 &= q_2q_3q_4q_5(1+2q_1q_3) + q_3(q_2+q_4q_5-4q_1q_4q_5)(zq_1\partial_{q_1}) + q_3q_4(2q_1q_2q_3+q_5)(zq_2\partial_{q_2}) + q_3(2q_2q_4-2q_1q_2q_3q_4-q_2+2q_1q_4q_5-q_4q_5)(zq_3\partial_{q_3})\\
&\quad + q_2q_3(1-4q_4)(zq_4\partial_{q_4}) + 2q_2q_3q_4(1+q_1q_3)(zq_5\partial_{q_5}) + 2(zq_1\partial_{q_1})(zq_2\partial_{q_2})\\
(zq_3\partial_{q_3})(zq_4\partial_{q_4}) &= q_2q_3q_4(q_5-2q_1q_2q_3) + q_3(4q_1q_2+q_4q_5-4q_1q_4q_5)(zq_1\partial_{q_1}) + q_3q_4q_5(zq_2\partial_{q_2}) + q_3(2q_1q_4q_5-2q_1q_2+q_2q_4-q_4q_5)(zq_3\partial_{q_3}) - 2q_2q_3q_4(zq_4\partial_{q_4})\\
&\quad + q_2q_3q_4(zq_5\partial_{q_5}) + 2(zq_1\partial_{q_1})(zq_2\partial_{q_2})\\
(zq_3\partial_{q_3})(zq_5\partial_{q_5}) &= q_2q_3q_4(q_5-4q_1q_2q_3) + q_3(4q_1q_2+q_4q_5-4q_1q_4q_5)(zq_1\partial_{q_1}) + q_3q_4(q_5-2q_1q_2q_3)(zq_2\partial_{q_2}) - q_3(2q_1q_2-2q_1q_2q_3q_4+q_4q_5-2q_1q_4q_5)(zq_3\partial_{q_3})\\
&\quad - 2q_1q_2q_3^2q_4(zq_5\partial_{q_5}) + 2(zq_1\partial_{q_1})(zq_2\partial_{q_2})\\
(zq_4\partial_{q_4})^2 &= q_2q_3q_4\frac{8q_1q_2q_3q_4-q_1q_2q_3+q_5-4q_4q_5+3q_1q_3q_4q_5}{1-4q_4} + \left(q_2q_3\frac{q_4+2q_1-16q_1q_4}{1-4q_4} + q_3q_4q_5(1-4q_1)\right)(zq_1\partial_{q_1}) + q_3q_4\left(\frac{2q_1q_2q_3q_4}{1-4q_4}+q_5\right)(zq_2\partial_{q_2})\\
&\quad - \left(\frac{q_1q_2q_3-8q_1q_2q_3q_4+2q_1q_2q_3^2q_4^2+2q_2q_3q_4^2-q_4q_5+2q_4^2q_5}{1-4q_4} - 2q_1q_3q_4q_5 + q_3q_4q_5 + z\frac{q_4}{1-4q_4}\right)(zq_3\partial_{q_3})\\
&\quad - \left(q_2q_3q_4+q_4q_5-z\frac{2q_4}{1-4q_4}\right)(zq_4\partial_{q_4}) - \left(q_4\frac{2q_4q_5-q_2q_3+2q_2q_3q_4-2q_1q_2q_3^2q_4}{1-4q_4} + z\frac{q_4}{1-4q_4}\right)(zq_5\partial_{q_5}) + \frac{1-6q_4}{1-4q_4}(zq_1\partial_{q_1})(zq_2\partial_{q_2})\\
(zq_4\partial_{q_4})(zq_5\partial_{q_5}) &= q_2q_3q_4(q_5-2q_1q_2q_3) + q_3(2q_1q_2+q_4q_5-4q_1q_4q_5)(zq_1\partial_{q_1}) + q_3q_4(q_5-q_1q_2q_3)(zq_2\partial_{q_2})\\
&\quad + (q_1q_2q_3^2q_4-q_1q_2q_3+q_4q_5-q_3q_4q_5+2q_1q_3q_4q_5)(zq_3\partial_{q_3}) - 2q_4q_5(zq_4\partial_{q_4}) + q_4(q_5-q_1q_2q_3^2)(zq_5\partial_{q_5}) + (zq_1\partial_{q_1})(zq_2\partial_{q_2})\\
(zq_5\partial_{q_5})^2 &= q_2q_3q_4q_5(1+q_1q_3) + q_3q_4q_5(1-4q_1)(zq_1\partial_{q_1}) + q_3q_4q_5(zq_2\partial_{q_2}) + q_4q_5(2-q_3+2q_1q_3)(zq_3\partial_{q_3}) + q_5(1-4q_4)(zq_4\partial_{q_4}) - q_5(1-2q_4)(zq_5\partial_{q_5})
\end{aligned}
$$

It remains to compute

$$(zq_i\partial_{q_i})(zq_1\partial_{q_1})(zq_2\partial_{q_2})$$

for $1 \le i \le 5$ in terms of our basis. To make our live easier, note that $(zq_5\partial_{q_5})^2$ does not contain a $(zq_1\partial_{q_1})(zq_2\partial_{q_2})$ term. Hence

$$(zq_1\partial_{q_1})(zq_5\partial_{q_5})^2 = [(zq_1\partial_{q_1}),(zq_5\partial_{q_5})^2] + S_0(zq_1\partial_{q_1}) + S_1(zq_1\partial_{q_1})^2 + \cdots + S_5(zq_1\partial_{q_1})(zq_5\partial_{q_5}),$$

where $(zq_5\partial_{q_5})^2$ expressed as row vector in our basis is $(S_0,\ldots,S_5,0)$ and $[,]$ denotes the usual commutator. So we compute

$$\begin{aligned}(zq_1\partial_{q_1})(zq_5\partial_{q_5})^2 = {} & q_1q_2q_3^2q_4q_5(q_2+q_2q_3q_4+3q_4q_5+z)\\ & + q_3q_4q_5(q_2-4q_1q_2+q_1q_3q_4q_5-2q_1z)(zq_1\partial_{q_1})\\ & + q_1q_3^2q_4q_5(q_2+q_4q_5)(zq_2\partial_{q_2})\\ & + q_1q_3q_4q_5(2q_2-q_2q_3+2q_2q_3q_4-q_3q_4q_5+z)(zq_3\partial_{q_3})\\ & + q_1q_2q_3^2q_4q_5(1-4q_4)(zq_4\partial_{q_4}) + q_1q_2q_3^2q_4q_5(1+2q_4)(zq_5\partial_{q_5}).\end{aligned}$$

This helps as we can now express

$$\begin{aligned}(zq_1\partial_{q_1})(zq_5\partial_{q_5})^2 &= (zq_5\partial_{q_5})((zq_1\partial_{q_1})(zq_5\partial_{q_5}))\\ &= [zq_5\partial_{q_5},(zq_1\partial_{q_1})(zq_5\partial_{q_5})]\\ &\quad + R_0(zq_5\partial_{q_5}) + R_1(zq_1\partial_{q_1})(zq_5\partial_{q_5}) + \cdots + R_5(zq_5\partial_{q_5})^2\\ &\quad + (zq_1\partial_{q_1})(zq_2\partial_{q_2})(zq_5\partial_{q_5}),\end{aligned}$$

where this time $(zq_1\partial_{q_1})(zq_5\partial_{q_5})$ as row vector is $(R_0,\ldots,R_5,1)$ in our basis. Thus we find

$$\begin{aligned}(zq_1\partial_{q_1})(zq_2\partial_{q_2})(zq_5\partial_{q_5}) = {} & 2q_1q_2q_3^2q_4q_5(q_2+q_2q_3q_4+q_4q_5) + z(q_1q_2q_3^2q_4q_5)\\ & + q_2q_3q_4q_5(1-4q_1+q_1q_3)(zq_1\partial_{q_1})\\ & + 2q_1q_2q_3^2q_4q_5(zq_2\partial_{q_2}) + 2q_1q_2q_3q_4q_5(1-q_3+2q_3q_4)(zq_3\partial_{q_3})\\ & + 2q_1q_2q_3^2q_4q_5(1-4q_4)(zq_4\partial_{q_4}) + 2q_1q_2q_3^2q_4(q_2+2q_4q_5)(zq_5\partial_{q_5}).\end{aligned}$$

Similarly we calculate the remaining terms, the result of which is shown on the next page. We could now write down the connection matrices $\Omega^{(i)}$ like in chapter 5.3. However, due to the involved terms being even more complicated than there, one would probably need a larger page format to do so and we will refrain from it here.

$$
\begin{aligned}
(zq_1\partial_{q_1})^2(zq_2\partial_{q_2}) = {} & \left(q_1q_2q_3^2q_4\frac{q_2q_5(1-6q_1)-q_1q_2q_3(2q_2+q_5)}{1-4q_1} + q_1q_2q_3^2q_4^2(q_5^2+q_2q_3q_5-2q_1q_2^2q_3^2-q_1q_2q_3^2q_5) + z(2q_1q_2^2q_3^2q_4+q_1q_2q_3^2q_4q_5)\right) \\
& + \left(\frac{2q_1^2q_2^2q_3^2}{1-4q_1} + q_1q_2q_3q_4(2q_2q_3+2q_1q_2q_3^2+2q_5-q_3q_5) - z\left(2q_1q_2q_3\frac{1-2q_1}{1-4q_1}\right)\right)(zq_1\partial_{q_1}) \\
& \left(q_1q_2q_3^2q_4\frac{q_5-6q_1q_5-q_1q_2q_3}{1-4q_1} - 2q_1^2q_2^2q_3^4q_4^2 + z(q_1q_2q_3^2q_4)\right)(zq_2\partial_{q_2}) \\
& + \left(q_1q_2q_3^2q_4\frac{4q_1^2q_2q_3-4q_1q_2+5q_1q_5-q_5}{1-4q_1} + q_1q_2q_3(2q_1q_2q_3^3q_4^2+q_4q_5+2q_3q_4^2q_5) + z\left(q_1q_2q_3\frac{1-2q_1}{1-4q_1} - q_1q_2q_3^2q_4\right)\right)(zq_3\partial_{q_3}) \\
& - q_1q_2q_3^2(1-4q_4)\left(\frac{2q_1q_2}{1-4q_1} - q_4q_5\right)(zq_4\partial_{q_4}) \\
& + \left(2q_1q_2q_3^2q_4(q_4q_5-q_1q_2q_3^2q_4+q_2) - q_1q_2^2q_3^2q_4\frac{1+q_1q_3}{1-4q_1} + z(q_1q_2q_3^2q_4)\right)(zq_5\partial_{q_5}) \\
& + q_1q_2q_3\left(\frac{1}{1-4q_1}+q_3q_4\right)(zq_1\partial_{q_1})(zq_2\partial_{q_2})
\end{aligned}
$$

$$
\begin{aligned}
(zq_1\partial_{q_1})(zq_2\partial_{q_2})^2 = {} & q_1q_2q_3^2q_4(3q_2q_5+q_4q_5^2+3q_2q_3q_4q_5-2q_1q_2^2q_3^2q_4-q_1q_2q_3^2q_4q_5) + z(q_1q_2q_3^2q_4(4q_2+q_5)) \\
& + \left(q_2q_3(q_1q_2q_3+2q_1^2q_2q_3^2q_4+q_4q_5-4q_1q_4q_5) - z(2q_1q_2q_3)\right)(zq_1\partial_{q_1}) \\
& + \left(q_1q_2q_3^2q_4(q_2+2q_5-2q_1q_2q_3^2q_4) + z(q_1q_2q_3^2q_4)\right)(zq_2\partial_{q_2}) \\
& + \left(q_1q_2q_3(3q_2q_3q_4-q_2q_3-q_1q_2q_3^2q_4+2q_1q_2q_3^3q_4^2+2q_4q_5-q_3q_4q_5+2q_3q_4^2q_5) + z(q_1q_2q_3-q_1q_2q_3^2q_4)\right)(zq_3\partial_{q_3}) \\
& + q_1q_2q_3^2(1-4q_4)(q_2+2q_4q_5)(2q_4\partial_{q_4}) \\
& + (2q_1q_2q_3^2q_4(2q_2-q_1q_2q_3^2q_4+q_4q_5) + z(q_1q_2q_3^2q_4))(zq_5\partial_{q_5}) \\
& + q_1q_2q_3^2q_4(zq_1\partial_{q_1})(zq_2\partial_{q_2})
\end{aligned}
$$

$$
\begin{aligned}
(zq_1\partial_{q_1})(zq_2\partial_{q_2})(zq_3\partial_{q_3}) = {} & q_1q_2q_3^2q_4(3q_2q_5+2q_4q_5^2+4q_2q_3q_4q_5-4q_1q_2^2q_3^2q_4-2q_1q_2q_3^2q_4q_5) + z(2q_1q_2q_3^2q_4(2q_2+q_5)) \\
& + \left(q_2q_3(q_1q_2q_3+4q_1^2q_2q_3^2q_4+q_4q_5-4q_1q_4q_5+q_1q_3q_4q_5) - z(2q_1q_2q_3)\right)(zq_1\partial_{q_1}) \\
& + (q_1q_2q_3^2q_4(3q_5-4q_1q_2q_3^2q_4) + z(2q_1q_2q_3^2q_4))(zq_2\partial_{q_2}) \\
& + \left(q_1q_2q_3(2q_4q_5-q_2q_3+4q_2q_3q_4-2q_3q_4q_5+4q_3q_4^2q_5-2q_1q_2q_3^2q_4+4q_1q_2q_3^3q_4^2) + z(q_1q_2q_3-2q_1q_2q_3^2q_4)\right)(zq_3\partial_{q_3}) \\
& + q_1q_2q_3^2(1-4q_4)(q_2+2q_4q_5)(2q_4\partial_{q_4}) \\
& + \left(4q_1q_2q_3^2q_4(q_2-q_1q_2q_3^2q_4+q_4q_5) + z(2q_1q_2q_3^2q_4)\right)(zq_5\partial_{q_5}) \\
& + 2q_1q_2q_3^2q_4(zq_1\partial_{q_1})(zq_2\partial_{q_2})
\end{aligned}
$$

$$
\begin{aligned}
(zq_1\partial_{q_1})(zq_2\partial_{q_2})(zq_4\partial_{q_4}) = {} & q_1q_2q_3^2q_4(2q_2q_5-2q_1q_2^2q_3^2q_4+3q_2q_3q_4q_5-q_1q_2q_3^2q_4q_5+2q_4q_5^2) + z(q_1q_2q_3^2q_4(2q_2+q_5)) \\
& + q_2q_3q_4(2q_1^2q_2q_3^2+q_5-4q_1q_5+q_1q_3q_5)(zq_1\partial_{q_1}) \\
& + \left(2q_1q_2q_3^2q_4(q_5-q_1q_2q_3^2q_4) + z(q_1q_2q_3^2q_4)\right)(zq_2\partial_{q_2}) \\
& + \left(q_1q_2q_3q_4(q_2q_3-q_1q_2q_3^2+2q_1q_2q_3^3q_4+2q_5-2q_3q_5+4q_3q_4q_5) - z(q_1q_2q_3^2q_4)\right)(zq_3\partial_{q_3}) \\
& + 2q_1q_2q_3^2q_4q_5(1-4q_4)(zq_4\partial_{q_4}) \\
& + \left(2q_1q_2q_3^2q_4(q_2-q_1q_2q_3^2q_4+2q_4q_5) + z(q_1q_2q_3^2q_4)\right)(zq_5\partial_{q_5}) \\
& + q_1q_2q_3^2q_4(zq_1\partial_{q_1})(zq_2\partial_{q_2})
\end{aligned}
$$

Now that we have found our connection matrix

$$\Omega^z = \Omega^{(1)} \frac{dq_1}{zq_1} + \cdots + \Omega^{(5)} \frac{dq_5}{zq_5},$$

we can attempt to solve the Birkhoff problem. In other words we need to find the unique change-of-basis matrix L such that:

- L preserves the degree of the operators $zq_i\partial_{q_i}$. Recall that due to the form of our Euler operator (44), z, q_2 and q_5 have all degree 2, whereas q_1, q_3 and q_4 have degree 0.
- The new connection matrix $\widehat{\Omega}^z$ is of the form

$$\widehat{\Omega}^z = L^{-1}\Omega^z L + L^{-1}dL = \sum_{i=1}^{5} \widehat{\Omega}^{(i)} \frac{dq_i}{zq_i},$$

  where for all i, $\widehat{\Omega}^{(i)}$ is independent of the parameter z.
- In the limit of z and the q_i approaching 0 we have

$$\lim_{(z,q_1,\ldots,q_5)\to 0} L = \mathrm{Id}(7), \tag{51}$$

  the 7×7 identity matrix.

How does this translate into practice? Note the degrees of our basis vectors: since $\deg((zq_i\partial_{q_i})^k) = 2k$, we have that L has to have the shape of a block upper diagonal matrix:

$$L = \begin{pmatrix} a & b & c \\ 0 & \widetilde{L} & d \\ 0 & 0 & e \end{pmatrix}, \tag{52}$$

where the blocks are square of respective sizes 1, 5 and 1. The preservation of degree forces us furthermore to assume that a, $\widetilde{L}$ and e are of degree 0, b and d are of degree at most 2 and c is of degree at most 4. Thus we know that $a = a(q_1, q_3, q_4)$, $\widetilde{L} = \widetilde{L}(q_1, q_3, q_4)$ and $e = e(q_1, q_3, q_4)$ are all independent of z, q_2 and q_5. Moreover,

$$\begin{aligned} b &= q_2 b_2(q_1, q_3, q_4) + q_5 b_5(q_1, q_3, q_4) + z b_z(q_1, q_3, q_4), \\ c &= q_2^2 c_{22}(q_1, q_3, q_4) + q_2 q_5 c_{25}(q_1, q_3, q_4) + q_2 z c_{2z}(q_1, q_3, q_4) \\ &\quad + q_5^2 c_{55}(q_1, q_3, q_4) + q_5 z c_{5z}(q_1, q_3, q_4) + z^2 c_{zz}(q_1, q_3, q_4), \\ d &= q_2 d_2(q_1, q_3, q_4) + q_5 d_5(q_1, q_3, q_4) + z d_z(q_1, q_3, q_4). \end{aligned} \tag{53}$$

Throughout this computation we should keep in mind that a, c and e are scalars, $\widetilde{L}$ is an invertible 5×5 matrix and b and d are 1×5, respectively 5×1 matrices.

Now paying careful attention to the order of multiplication (since we are dealing with a block matrix of different-sized blocks) we can express the inverse of L :

$$L^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}(b.\widetilde{L}^{-1}) & (ae)^{-1}(b.\widetilde{L}^{-1}.d - c) \\ 0 & \widetilde{L}^{-1} & -e^{-1}(\widetilde{L}^{-1}.d) \\ 0 & 0 & e^{-1} \end{pmatrix}.$$

In order to find $\widehat{\Omega}^z$, we need to express the $\Omega^{(i)}$ in the same block shape as L :

$$\Omega^{(i)} = \begin{pmatrix} 0 & Q_{12}^{(i)} & Q_{13}^{(i)} \\ Q_{21}^{(i)} & Q_{22}^{(i)} & Q_{23}^{(i)} \\ 0 & Q_{32}^{(i)} & Q_{33}^{(i)} \end{pmatrix},$$

where we note that $Q_{21}^{(i)}$ is the ith standard column vector.

Concretely we have

$$\widehat{\Omega}^z = \sum_{i=1}^{5} \widehat{\Omega}^{(i)} \frac{dq_i}{zq_i},$$

where

$$\widehat{\Omega}^{(i)} = L^{-1}\Omega^{(i)}L + zL^{-1}\partial_{q_i}(L).$$

This is easy enough to compute and we find that $\widehat{\Omega}^{(i)} = (\widehat{\Omega}_{rs}^{(i)})$ has the following entries:

$$\begin{aligned}
\widehat{\Omega}_{11}^{(i)} &= \frac{za_i}{a} - b.\widetilde{L}^{-1}.Q_{21}^{(i)}, \\
\widehat{\Omega}_{12}^{(i)} &= \frac{1}{a}\left(zb_i - zb.\widetilde{L}^{-1}.\widetilde{L}_i - b.\widetilde{L}^{-1}.Q_{21}^{(i)}.b + Q_{12}^{(i)}.\widetilde{L} - b.\widetilde{L}^{-1}.Q_{22}^{(i)}.\widetilde{L}\right), \\
&\qquad\qquad + \frac{1}{ae}\left(b.\widetilde{L}^{-1}.d.Q_{32}^{(i)}.\widetilde{L} - cQ_{32}^{(i)}.\widetilde{L}\right), \\
\widehat{\Omega}_{13}^{(i)} &= \frac{1}{a}\left(zc_i - zb.\widetilde{L}^{-1}.d_i - cb.\widetilde{L}^{-1}.Q_{21}^{(i)} + Q_{12}^{(i)}.d - b.\widetilde{L}^{-1}.Q_{22}^{(i)}.d + eQ_{13}^{(i)} - eb.\widetilde{L}^{-1}.Q_{23}^{(i)}\right. \\
&\qquad\qquad \left. + Q_{33}^{(i)}b.\widetilde{L}^{-1}.d - cQ_{33}^{(i)}\right) + \frac{1}{ae}\left(ze_ib.\widetilde{L}^{-1}.d - zce_i + b.\widetilde{L}^{-1}.d.Q_{32}^{(i)}.d - cQ_{32}^{(i)}.d\right), \\
\widehat{\Omega}_{21}^{(i)} &= a\widetilde{L}^{-1}.Q_{21}^{(i)}, \\
\widehat{\Omega}_{22}^{(i)} &= z\widetilde{L}^{-1}.\widetilde{L}_i + \widetilde{L}^{-1}.Q_{21}^{(i)}.b + \widetilde{L}^{-1}.Q_{22}^{(i)}.\widetilde{L} - \frac{1}{e}\widetilde{L}^{-1}.d.Q_{32}^{(i)}.\widetilde{L}, \\
\widehat{\Omega}_{23}^{(i)} &= z\widetilde{L}^{-1}.d_i - z\frac{e_i}{e}\widetilde{L}^{-1}.d + \widetilde{L}^{-1}.Q_{22}^{(i)}.d + c\widetilde{L}^{-1}.Q_{21}^{(i)} + e\widetilde{L}^{-1}.Q_{23}^{(i)} - \frac{1}{e}\widetilde{L}^{-1}.d.Q_{32}^{(i)}.d - Q_{33}^{(i)}\widetilde{L}^{-1}.d, \\
\widehat{\Omega}_{31}^{(i)} &= 0, \\
\widehat{\Omega}_{32}^{(i)} &= \frac{1}{e}Q_{32}^{(i)}.\widetilde{L}, \\
\widehat{\Omega}_{33}^{(i)} &= Q_{33}^{(i)} + \frac{1}{e}\left(ze_i + Q_{32}^{(i)}.d\right),
\end{aligned}$$

where a_i, b_i, c_i, d_i, e_i and $\widetilde{L}_i$ denote the partial derivative with respect to q_i of the respective block. Note that this notation is consistent with (53). Before we start the analysis, observe that the only blocks with z-dependence are b, c, d and $Q_{13}^{(i)}, Q_{22}^{(i)}, Q_{23}^{(i)}$. All of these, save c, are linear in z, whilst c is quadratic.

So let us start by considering $\widehat{\Omega}_{33}^{(i)}$: Since $\widehat{\Omega}^{(i)}$ is independent of z, we must have

$$\partial_z(\widehat{\Omega}_{33}^{(i)}) = \frac{1}{e}\left(e_i + Q_{32}^{(i)}.d_z\right) = 0. \tag{54}$$

Writing

$$d_z = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{pmatrix}$$

this gives us one (scalar) equation for each i. Noting that $e_2 = e_5 = 0$ and reading off our expressions for the $Q^{(i)}$, these five equations can be written as

$$\begin{pmatrix} -q_1e_1 \\ 0 \\ -q_3e_3 \\ -q_4e_4 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{2q_1}{1-4q_1} & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & \frac{1-6q_4}{1-4q_4} & 1 \\ 1 & 2 & 2 & 0 & \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{pmatrix}.$$

Being a linear system we can solve it for d_z :

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{pmatrix} = \begin{pmatrix} 2q_1(1-4q_1) \\ 0 \\ -q_1(1-4q_1) \\ 0 \\ 0 \end{pmatrix} e_1 + \begin{pmatrix} -(1-4q_1)q_3 \\ -q_3 \\ 2(1-q_1-q_4) \\ -(1-4q_4)q_3 \\ -2q_3q_4 \end{pmatrix} e_3 + \begin{pmatrix} 0 \\ 0 \\ -q_4(1-4q_4) \\ 2q_4(1-4q_4) \\ -(1-4q_4) \end{pmatrix} e_4. \tag{55}$$

However, in order to determine d_z we need more information. Conveniently, the expression for $\widehat{\Omega}_{23}^{(i)}$ also contains d. This time we consider the second derivative:

$$\partial_z^2(\widehat{\Omega}_{23}^{(i)}) = 2\widetilde{L}^{-1}.\partial_{q_i}(d_z) - \frac{2e_i}{e}\widetilde{L}^{-1}.d_z + 2\widetilde{L}^{-1}.\partial_z(Q_{22}^{(i)}).d_z - \frac{2}{e}\widetilde{L}^{-1}.d_z.Q_{32}^{(i)}.d_z + 2c_{zz}\widetilde{L}^{-1}.Q_{21}^{(i)}.$$

Since this equals to zero and as $\widetilde{L}$ is invertible, we obtain a matrix equation for each i :

$$\partial_{q_i}(d_z) + \partial_z(Q_{22}^{(i)}).d_z - \frac{1}{e}d_z.\left(e_i + Q_{32}^{(i)}.d_z\right) + c_{zz}Q_{21}^{(i)} = 0.$$

Note now two things: firstly that the third term, the one in brackets, vanishes due to (54). Secondly note that for $i = 2$ the entire equation reduces to $\partial_z^2(c)Q_{21}^{(2)} = 0$. Since $Q_{21}^{(2)} \neq 0$ (it is the second standard column vector), we must have

$$c_{zz} = \frac{1}{2}\partial_z^2(c) = 0.$$

So consider

$$\partial_{q_i}(d_z) + \partial_z(Q_{22}^{(i)}) = 0. \tag{56}$$

Clearly these equations are trivially true for $i \in \{2, 5\}$. $Q_{22}^{(3)}$ is independent of z, which implies that d_z must be too. By equation (55) this yields that q_3e_3 is independent of q_3 and consequently also e_1 and e_4. Thus we have the following equation for e :

$$e(q_1, q_3, q_4) = e'(q_1, q_4)\log(q_3) + \widetilde{e}(q_1, q_4).$$

Now deriving e by q_1 and q_4, keeping in mind that e_1 and e_4 are independent of q_3, we see that e' can only have trivial dependence on q_1, q_4, i.e. it is a constant. This is now, where the third condition (51) from before becomes important:

$$\lim_{z,q\to 0} e = 1$$

exists. We conclude that $e' = 0$, which is equivalent to e having no q_3 dependence, i.e. $e = e(q_1, q_4)$.

Let us return to the equations (56). For $i = 1$ these read:

$$0 = \frac{2}{1-4q_1}\delta_1 + \partial_{q_1}(\delta_1),$$
$$0 = \frac{-1}{1-4q_1}\delta_1 + \partial_{q_1}(\delta_3).$$

A first observation now is that

$$\partial_{q_1}(\delta_1 + 2\delta_3) = -2q_4(1-4q_4)\partial_{q_1}(e_4) = 0,$$

which implies that

$$\partial_{q_1}(e_4) = \partial_{q_1}\partial_{q_4}(e) = \partial_{q_4}(e_1) = 0.$$

Moreover, we can explicitly calculate the derivative of δ_1 according to (55):

$$\partial_{q_1}(\delta_1) = 2(1-8q_1)e_1 + 2q_1(1-4q_1)\partial_{q_1}(e_1)$$
$$\Rightarrow \qquad 2(1-6q_1)e_! + 2q_1(1-4q_1)\partial_{q_1}(e_1) = 0.$$

Then this is equivalent to

$$\frac{1-6q_1}{\sqrt{1-4q_1}}e_1 + q_1\sqrt{1-4q_1}\partial_{q_1}(e_1) = \partial_{q_1}\left(q_1\sqrt{1-4q_1}e_1\right) = 0.$$

Since e has no q_4 dependence, we conclude

$$e_1 = \frac{\varepsilon}{q_1\sqrt{1-4q_1}} \Rightarrow \qquad e = \varepsilon\left(\log(1-\sqrt{1-4q_1}) - \log(1+\sqrt{1-4q_1})\right) + e'(q_4).$$

But once again using the limit argument from before we find $\varepsilon = 0$, i.e. $e = e(q_4)$ only depends on q_4. Last, but not least, notice that (56) with $i = 4$ yields (amongst others) the equation

$$\frac{2}{1-4q_4}\delta_3 + \partial_{q_4}(\delta_3) = 0.$$

Thus we can argue as before showing that in fact e has no q_4 dependence either. In other words, e is a constant, and due to our limit argument we can conclude

$$e = 1.$$

Notice that now going back to our formula for d_z we also find

$$d_z = 0.$$

Armed with this knowledge, we can now consider

$$\partial_z(\widehat{\Omega}^{(i)}_{22}) = \widetilde{L}^{-1}.\widetilde{L}_i + \widetilde{L}^{-1}.Q^{(i)}_{21}.b_z + \widetilde{L}^{-1}.\partial_z(Q^{(i)}_{22}).\widetilde{L} - \frac{1}{e}\widetilde{L}^{-1}.d_z.Q^{(i)}_{32}.\widetilde{L}.$$

As this expression has to equate to zero and simplifying according to what we just learned, we obtain the following matrix equations for each i :

$$\widetilde{L}_i + Q^{(i)}_{21}.b_z + \partial_z(Q^{(i)}_{22}).\widetilde{L} = 0.$$

Now for $i = 2$ this reduces to the simple $Q^{(2)}_{21}.b_z = 0$, which implies (since $Q^{(2)}_{21} \neq 0$)

$$b_z = 0.$$

So far we have found $b_z = d_z = c_{zz} = 0$, so the only block with z dependence in L is c, which is linear in z. Therefore, there is only one remaining term in our expression for $\widehat{\Omega}^{(i)}_{13}$, which is quadratic in z : $\frac{1}{a}zc_i$. But all terms, which are quadratic in z, sum up to 0, implying that

$$c_{2z} = c_{5z} = 0 \Rightarrow \qquad \partial_z(L) = 0.$$

This makes life a lot easier for us. For instance consider now $\widehat{\Omega}^{(i)}_{11}$:

$$\partial_z(\widehat{\Omega}^{(i)}_{11}) = \frac{a_i}{a} = 0.$$

Thus a is a constant and once again due to (51) we have

$$a = 1.$$

Now let us consider another block of $\widehat{\Omega}^{(i)}$, this time $\widehat{\Omega}^{(i)}_{22}$:

$$\partial_z(\widehat{\Omega}^{(i)}_{22}) = \widetilde{L}^{-1}.\widetilde{L}_i + \widetilde{L}^{-1}.\partial_z(Q^{(i)}_{22}).\widetilde{L} = 0 \qquad \Rightarrow \qquad \widetilde{L}_i + \partial_z(Q^{(i)}_{22}).\widetilde{L} = 0.$$

This is helpful when considering $\widehat{\Omega}^{(i)}_{12}$:

$$\begin{aligned}\partial_z(\widehat{\Omega}^{(i)}_{12}) =& b_i - b.\widetilde{L}^{-1}.\widetilde{L}_i - b.\widetilde{L}^{-1}.\partial_z(Q^{(i)}_{22}).\widetilde{L}\\ =& b_i - b.\widetilde{L}^{-1}.\left(\widetilde{L}_i + \partial_z(Q^{(i)}_{22}).\widetilde{L}\right)\\ =& b_i = 0\end{aligned}$$

for all i. Since $b_z = 0$, we deduce that b is constant and hence

$$b = 0.$$

So far we have determined blocks of L in such a way that $\widehat{\Omega}^{(i)}_{rs}$ has no z dependence for most values of r and s. The remaining equations are

$$\begin{aligned}\partial_z(\widehat{\Omega}^{(i)}_{13}) = 0 \qquad &\Rightarrow \qquad c_i + \partial_z(Q^{(i)}_{13}) = 0,\\ \partial_z(\widehat{\Omega}^{(i)}_{23}) = 0 \qquad &\Rightarrow \qquad d_i + \partial_z(Q^{(i)}_{22}).d + \partial_z(Q^{(i)}_{23}) = 0,\\ \partial_z(\widehat{\Omega}^{(i)}_{22}) = 0 \qquad &\Rightarrow \qquad \widetilde{L}_i + \partial_z(Q^{(i)}_{22}).\widetilde{L} = 0.\end{aligned} \tag{57}$$

Let us start with the first one, since it is particularly easy: for $i = 5$ this equation reads

$$q_2 c_{25} + 2q_5 c_{55} = -q_1 q_2 q_3^2 q_4.$$

Upon deriving this by q_5 we find that

$$\begin{aligned} & c_{55} = 0 \\ \Rightarrow \qquad & c_{25} = -q_1 q_3^2 q_4. \end{aligned}$$

Then we use the case $i = 2$:

$$2q_2 c_{22} + q_5 c_{25} = -q_1 q_3^2 q_4 (4q_2 + q_5),$$

from which we deduce

$$c_{22} = -2q_1 q_3^2 q_4.$$

Similarly easy is the second equation amongst (57). Since

$$\partial_z(Q_{23}^{(5)}) = 0 \qquad \text{and} \qquad \partial_z(Q_{22}^{(5)}) = 0$$

we have

$$d_5 = 0.$$

Now for $i = 2$ we still have no z dependence in $Q_{22}^{(i)}$. $Q_{23}^{(i)}$ however is another matter:

$$d_2 = -\partial_z(Q_{23}^{(2)}) = \begin{pmatrix} 2q_1 q_3 \\ -q_1 q_3^2 q_4 \\ -(1 - q_3 q_4) q_1 q_3 \\ 0 \\ -q_1 q_3^2 q_4 \end{pmatrix}.$$

And thus we have

$$d = \begin{pmatrix} 2q_1 q_2 q_3 \\ -q_1 q_2 q_3^2 q_4 \\ -(1 - q_3 q_4) q_1 q_2 q_3 \\ 0 \\ -q_1 q_2 q_3^2 q_4 \end{pmatrix}.$$

Solving the third equation of (57) is slightly more intricate. However, this is mainly due to the fact that much more unknowns are involved. A first glance at

$$\widetilde{L}_i + \partial_z(Q_{22}^{(i)}).\widetilde{L} = 0 \tag{58}$$

reveals that this equation is trivially true for $i = 2, 5$. For $i = 3$ we have that the right-hand term vanishes, implying that $\widetilde{L}_3 = 0$. So let us denote

$$\widetilde{L} = \begin{pmatrix} \lambda_{11} & \dots & \lambda_{15} \\ \vdots & \ddots & \vdots \\ \lambda_{51} & \dots & \lambda_{55} \end{pmatrix},$$

where $\lambda_{ij} = \lambda_{ij}(q_1, q_4)$ depends a priori solely on q_1 and q_4. For $i = 1$ equation (58) is

$$\widetilde{L}_1 = \partial_{q_1} \begin{pmatrix} \lambda_{11} & \dots & \lambda_{15} \\ \vdots & \ddots & \vdots \\ \lambda_{51} & \dots & \lambda_{55} \end{pmatrix} = \frac{1}{1-4q_1} \begin{pmatrix} -2\lambda_{11} & \dots & -2\lambda_{15} \\ 0 & \dots & 0 \\ \lambda_{11} & \dots & \lambda_{15} \\ 0 & \dots & 0 \\ 0 & \dots & 0 \end{pmatrix},$$

whereas for $i = 4$ it becomes

$$\widetilde{L}_4 = \partial_{q_4} \begin{pmatrix} \lambda_{11} & \dots & \lambda_{15} \\ \vdots & \ddots & \vdots \\ \lambda_{51} & \dots & \lambda_{55} \end{pmatrix} = \frac{1}{1-4q_4} \begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \lambda_{31} & \dots & \lambda_{35} \\ -2\lambda_{31} & \dots & -2\lambda_{35} \\ \lambda_{31} & \dots & \lambda_{35} \end{pmatrix}.$$

As a first observation note that the second row of $\widetilde{L}$ depends neither on q_1 nor on q_4, meaning that it is constant:

$$(\lambda_{21}, \dots, \lambda_{25}) = (0, 1, 0, 0, 0).$$

We denote the resulting dependences by

$$\widetilde{L} = \begin{pmatrix} \lambda_{11}(q_1) & \lambda_{12}(q_1) & \lambda_{13}(q_1) & \lambda_{14}(q_1) & \lambda_{15}(q_1) \\ 0 & 1 & 0 & 0 & 0 \\ \lambda_{31}(q_1, q_4) & \lambda_{32}(q_1, q_4) & \lambda_{33}(q_1, q_4) & \lambda_{34}(q_1, q_4) & \lambda_{35}(q_1, q_4) \\ \lambda_{41}(q_4) & \lambda_{42}(q_4) & \lambda_{43}(q_4) & \lambda_{44}(q_4) & \lambda_{45}(q_4) \\ \lambda_{51}(q_4) & \lambda_{52}(q_4) & \lambda_{53}(q_4) & \lambda_{54}(q_4) & \lambda_{55}(q_4) \end{pmatrix}.$$

Let us take a closer look at the first row of $\widetilde{L}_1$. We find that for $1 \leq i \leq 5$:

$$\partial_{q_1}(\lambda_{1i}(q_1)) + \frac{2}{1-4q_1}\lambda_{1i}(q_1) = 0.$$

We solve this the following way:

$$\begin{aligned} 0 =& \frac{1}{\sqrt{1-4q_1}}\partial_{q_1}(\lambda_{1i}) + \frac{1}{(1-4q_1)\sqrt{1-4q_1}}\lambda_{1i} \\ =& \partial_{q_1}\left(\frac{1}{\sqrt{1-4q_1}}\lambda_{1i}\right) \\ &\Rightarrow \lambda_{1i}(q_1) = \lambda'\sqrt{1-4q_1}, \end{aligned}$$

where λ' is a constant, defined by the limit of $z, q \to 0$ of λ_{1i} :

$$(\lambda_{11}, \dots, \lambda_{15}) = \sqrt{1-4q_1}(1, 0, 0, 0, 0).$$

In precisely the same manner we also find

$$(\lambda_{41}, \dots, \lambda_{45}) = \sqrt{1-4q_4}(0, 0, 0, 1, 0).$$

This determines the last row of our equation in the case $i = 4$:

$$\partial_{q_4}(\lambda_{15}, \ldots, \lambda_{55}) = \frac{1}{\sqrt{1-4q_4}}(0, 0, 0, 1, 0).$$

Simple integration and a look at the limit behaviour then shows that

$$(\lambda_{15}, \ldots, \lambda_{55}) = \left(0, 0, 0, -\frac{1}{2}\sqrt{1-4q_4} + \frac{1}{2}, 1\right).$$

Remains to find the third row of $\widetilde{L}$. This time we have to solve the two equations

$$\begin{aligned}\partial_{q_1}(\lambda_{13}, \ldots, \lambda_{53}) =& \frac{1}{\sqrt{1-4q_1}}(1, 0, 0, 0, 0)\\ \partial_{q_4}(\lambda_{13}, \ldots, \lambda_{53}) =& \frac{1}{\sqrt{1-4q_4}}(0, 0, 0, 1, 0).\end{aligned}$$

Thus we have

$$(\lambda_{13}, \ldots, \lambda_{53}) = (\lambda_{13}(q_1), \lambda_{23}, \lambda_{33}, \lambda_{43}(q_4), \lambda_{53})$$

and applying the same methods as before to the individual equations:

$$(\lambda_{13}, \ldots, \lambda_{53}) = \left(-\frac{1}{2}\sqrt{1-4q_1} + \frac{1}{2}, 0, 1, -\frac{1}{2}\sqrt{1-4q_4} + \frac{1}{2}, 0\right).$$

In conclusion: we have determined the matrix L to be

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -(2q_2+q_5)q_1q_2q_3^2q_4 \\ 0 & \sqrt{1-4q_1} & 0 & 0 & 0 & 0 & 2q_1q_2q_3 \\ 0 & 0 & 1 & 0 & 0 & 0 & -q_1q_2q_3^2q_4 \\ 0 & -\frac{1}{2}\sqrt{1-4q_1}+\frac{1}{2} & 0 & 1 & -\frac{1}{2}\sqrt{1-4q_4}+\frac{1}{2} & 0 & -(1-q_3q_4)q_1q_2q_3 \\ 0 & 0 & 0 & 0 & \sqrt{1-4q_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}\sqrt{1-4q_4}+\frac{1}{2} & 1 & -q_1q_2q_3^2q_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (59)$$

thereby concretely solving the Birkhoff problem. Indeed this L works, as a simple calculation shows that

$$\partial_z\left(L^{-1}.\Omega^{(i)}.L + zL^{-1}\partial_{q_i}(L)\right) = 0$$

for all i.

10 Mirror Symmetry

The climax of this thesis consists of chapter 10.3, in which we will explicitly prove a mirror symmetry type theorem for the Del Pezzo surface X_4. So in preparation we shall give some general introduction to the topic of mirror symmetry and discuss all the necessary prerequisites for theorem 10.1.

10.1 Some Brief Remarks on Mirror Symmetry

Mirror symmetry originated in the realm of theoretical physics. More specifically, in the realm of string theory. String theory aims to unify the two big theories of the 20th century, general relativity and quantum mechanics, by providing one consistent quantum theory of gravity. It does so by posing that elementary particles are not zero-dimensional, but one-dimensional, i.e. they are strings propagating through spacetime. In doing so a particle (or string) traces out a surface Σ, called the world sheet, and classical fields are realised in string theory as functions (or vector bundles etc.) on Σ.

For consistency of the theory, physicists imposed a condition called supersymmetry on the strings, which is essentially a duality between fermionic particles (essentially matter particles) and bosonic particles (essentially particles carrying forces). With this condition in place, it become clear that (at least in heterotic string theory) these superstrings can predict the dimension of spacetime, in which they live. These are typically ten dimensions, so the clear discrepancy to our observed four dimensions must be resolved. The way this is done, is by assuming that spacetime is locally the product of the four-dimensional manifold we observe and some six-dimensional compact manifold, which has to be extremely small. This internal manifold must be a Calabi-Yau manifold in order to preserve supersymmetry. Thus physicists were led to study so-called *super-conformal field theories*[22] (SCFTs).

There are two types of interesting SCFTs, $N = 1$ SCFTs and $N = 2$ SCFTs. The N here denotes the number of imposed supersymmetries. Hence $N = 1$ SCFTs are more general and a $N = 2$ SCFT is simply an $N = 1$ SCFT together with an action of a $N = 2$ *super-conformal algebra*. It was observed that in heterotic string theory, if one compactifies spacetime using a Calabi-Yau manifold, then one would obtain a $N = 2$ SCFT, in which the equations of motion for fermions decouple into right- and left-moving parts. At the level of super-conformal algebras this meant that the $N = 2$ super-conformal algebra associated to this $N = 2$ SCFT contained two copies of ordinary ($N = 1$) super-conformal algebras. However, the generators of these two algebras are only defined up to a sign and thus the $N = 2$ super-conformal algebra has an interesting automorphism, called the *mirror automorphism*, which consequently induces the notion of a mirror morphism between two $N = 2$ SCFTs. In fact, to any $N = 2$ SCFT we can thus find a related $N = 2$ SCFT such that they are isomorphic as $N = 1$ SCFTs, but not in general as $N = 2$ SCFT.

Now the most important way to obtain an $N = 2$ SCFT is by taking a Calabi-Yau manifold X and realising the $N = 2$ SCFT as a so-called *non-linear σ-model* of X. However, $N = 2$ SCFTs are immensely complicated objects, for which in fact no general mathematical definition exists yet. To make them more accessible Witten in [Wit88] and independently [EY90] introduced a *topological twist*. Our input data at this point is a Calabi-Yau manifold, in particular, a choice of complex structure and a choice of Kähler class. Witten's topological twist now (informally) identifies a subsector of $SCFT$s and restricts to that. The two important ways in which this can be done and the resulting models are called A- and B-model respectively. The A-model considers the complex structure constant and thus "probes" the Kähler structure, whereas the B-model does just the opposite.

Now recall that two $N = 2$ SCFTs can be related by the sign change in the generators of their isomorphic super-conformal algebras. We called this the mirror morphism and if the two $N = 2$ SCFTs are realised as non-linear σ-models of two Calabi-Yau manifolds X and X°, we say that

[22] The conformity comes since Σ carries a conformal structure and thus supersymmetric string theory is supposed to be invariant under conformal equivalence.

X and X° form a mirror pair. The important point of Witten's twist (from a mirror symmetry perspective) is that the mirror automorphism of the $N = 2$ super-conformal algebra exchanges the A- and the B-model. Thus we have that the A-model of X should be equivalent to the B-model of X° and vice versa. We can summarise the situation in the following diagram

$$\begin{tikzcd}
 & & \text{A-model} \arrow[r, leftrightarrow, "\sim"] & \text{B-model} & & \\
\text{CY} \arrow[r, dotted] \arrow[urr, bend left] \arrow[drr, bend right] & N=2 \text{ SCFT} \arrow[ur, dotted] \arrow[dr, dotted] \arrow[rrr, leftrightarrow, dashed, "\text{mirror}" description] & & & N=2 \text{ SCFT} \arrow[ul, dotted] \arrow[dl, dotted] & \text{CY.} \arrow[l, dotted] \arrow[ull, bend right] \arrow[dll, bend left] \\
 & & \text{B-model} \arrow[r, leftrightarrow, "\sim"] & \text{A-model} & &
\end{tikzcd}$$

Why is this interesting for mathematicians? For one thing, the equivalence of the A-model of X and the B-model of X° suggests a deep connection between two seemingly unrelated types of data. The A-model side encodes counts of curves on X, i.e. Gromov-Witten invariants, which appear as coefficients of correlation functions. On the B-model side, classical period integrals are encoded the same way. However, mirror symmetry burst into the mathematicians' stage with the paper of Candelas et al. [COGP91] in 1991. The authors used the above equivalence and computed the B-model data associated to a variation of Hodge structure on the mirror manifold of a quintic hypersurface in $\mathbb{P}^4$. Thus they were able to predict Gromov-Witten invariants on this quintic hypersurface, the computation of which were far beyond anything mathematicians could do at the time. Since then mirror symmetry has been an active part of mathematical research. In fact, Givental in [Giv96] and independently [LLY97] were able to prove the predictions made in [COGP91] rigorously.

Last, but not least, let us mention that Hori and Vafa [VH00] found an interpretation of X° as a fibration together with a holomorphic map on the fibres. This data is in fact our Landau-Ginzburg model and it works in greater generality than just Calabi-Yau manifold0s as mirror partner for general complete intersections in toric manifolds [RS15].

10.2 Mirror symmetry from a $\mathcal{D}$-module perspective

As mentioned before, we are mainly interested in mirror symmetry as a tool for computing Gromov-Witten invariants. In particular, the significance of our main theorem 10.1 stems not from a mirror theorem in the sense of chapter 10.1, but rather from its enumerative implications: it allows us to count curves, i.e. compute GW invariants, on X_4 without having to apply the complicated machinery, developed in chapters 2-5, only using our Landau-Ginzburg model and some algebra.

We start off with a manifold X, in our case a Del Pezzo surface. Then we postulate a mirror partner X^o, which in our case is a Landau-Ginzburg model, as we have done in chapter 7.5. Associated to X we have an A-model, which we define here to simply be the Dubrovin connection. As such it encodes all (genus 0, 3-point) GW invariants. We mentioned in chapter 6 that a vector bundle with connection can be interpreted as a $\mathcal{D}$-module. From this point of view, our A-model is simply the Dubrovin connection, now called the *quantum $\mathcal{D}$-module*. Mirror symmetry for us now is the construction of a B-model $\mathcal{D}$-module from X^o, which recovers the Dubrovin connection and hence the GW invariants. And we have already done most of the work! The B-model equivalent to the quantum $\mathcal{D}$-module is essentially the Brieskorn lattice extended to $\mathbb{P}^1 \times U$ as defined in the previous chapters.

We say "essentially" and not "precisely", because we still have to change co-ordinates on the B-side. This change of coordinates $\rho : V \to U$, where $V \subset \mathbb{C}^r$ is some suitably small open neighbourhood of the origin, is called the *mirror map.* The mirror map is the isomorphism relating the A- and the B-model of mirror symmetry. It originally appeared in Givental's formulation of mirror symmetry [Giv98]. He defined two cohomology-valued formal functions I and J. The J-function incorporates the Gromov-Witten invariants (or rather *gravitational correlators*) and summarises the enumerative information of the A-model. The I-function on the other hand is determined by periods on the mirror manifolds and thus contains information on the moduli of the mirror. Both I- and J-function can be expressed in our by-now familiar z, q-coordinates. Givental's version of mirror symmetry then states that

$$I(q, z) = J(\rho(q), z).$$

In fact, for toric Fano varieties ρ was shown to be the identity [RS15, proposition 3.12.3] (note that X_4^o is not Fano, only nef). Interpreted in terms of $\mathcal{D}$-modules, the J-function is a solution to the quantum $\mathcal{D}$-module, whereas the I-function is a solution to its mirror $\mathcal{D}$-module.

10.3 Mirror Symmetry for X_4

The aim of this chapter is to prove the following theorem:

Theorem 10.1. *The Landau-Ginzburg model of X_4^o as defined in chapter 7.5 is a suitable mirror partner for X_4. In other words, the following two bundles with connection are isomorphic:*

A *The trivial bundle $H^*(X_4, \mathbb{C}) \times \mathbb{P}^1 \times U \to \mathbb{P}^1 \times U$, where $U \subset \mathbb{C}^r$ is a suitably small, open neighbourhood of the origin, equipped with the Dubrovin connection $\widehat{\nabla}$ as defined in chapter 5.*

B *After a change of coordinates $\rho : V \to U$, where $V, U \subset \mathbb{C}^r$ are small open neighbourhoods of the origin, the modified version F of ${}_0\mathcal{QM}_{\widetilde{A}}$ as described in chapters 8.4 and 9. In other words, the connection defined by the $\mathcal{D}$-module*

$$\frac{\mathbb{C}[z, q_1, \dots, q_r]\langle z^2\partial_z, zq_1\partial_{q_1}, \dots, zq_r\partial_{q_r}\rangle}{\langle \widetilde{E}, \widetilde{\Box}_l\rangle_{l\in\mathbb{L}}}, \tag{60}$$

where

$$\widetilde{E} = z^2\partial_z + \sum_{i=1}^{r} u_i^*(\omega_{X_4^o}^{-1}) zq_i\partial_{q_i}$$

and

$$\widetilde{\Box}_l = \prod_{a|u_a(l)>0} q_a^{u_a(l)} \prod_{i|l_i<0} \prod_{j=0}^{-l_i-1} \left(\sum_{k=1}^{n-r} m_{ik} zq_k\partial_{q_k} - jz\right) - \prod_{a|u_a(l)<0} q_a^{-u_a(l)} \prod_{i|l_i>0} \prod_{j=0}^{l_i-1} \left(\sum_{k=1}^{n-r} m_{ik} zq_k\partial_{q_k} - jz\right)$$

for all $l \in \mathbb{L} = (\mathrm{Pic}(X_\Sigma))^\vee$.

Proof: We have already done most of the work involved. For example, we have found a concrete description of the quantum $\mathcal{D}$-module, i.e. the Dubrovin connection, in chapter 5. Furthermore we have found an expression of the connection defined by (60) in chapter 9.3. Moreover, we have additionally solved the Birkhoff problem in the same chapter, i.e. we have chosen suitable coordinates in which to express our B-model $\mathcal{D}$-module. So let us now find the mirror map ρ concretely: the change-of-basis matrix L from (59) maps

$$\begin{aligned}
zq_1\partial_{q_1} &\mapsto \sqrt{1-4q_1}zq_1\partial_{q_1} + \frac{1}{2}(1-\sqrt{1-4q_1})zq_3\partial_{q_3},\\
zq_2\partial_{q_2} &\mapsto zq_2\partial_{q_2},\\
zq_3\partial_{q_3} &\mapsto zq_3\partial_{q_3},\\
zq_4\partial_{q_4} &\mapsto \frac{1}{2}(1-\sqrt{1-4q_4})zq_3\partial_{q_3} + \sqrt{1-4q_4}zq_4\partial_{q_4} + \frac{1}{2}(1-\sqrt{1-4q_4})zq_5\partial_{q_5},\\
zq_5\partial_{q_5} &\mapsto zq_5\partial_{q_5}.
\end{aligned}$$

ρ is now a map $V \to U$, where $V, U \subset \mathbb{C}^r$ are small open neighbourhoods of the origin. It sends $q_i \mapsto p_i$ such that

$$\begin{aligned}
p_1\partial_{p_1} &= \sqrt{1-4q_1}q_1\partial_{q_1} + \frac{1}{2}(1-\sqrt{1-4q_1})q_3\partial_{q_3},\\
p_2\partial_{p_2} &= q_2\partial_{q_2},\\
p_3\partial_{p_3} &= q_3\partial_{q_3},\\
p_4\partial_{p_4} &= \frac{1}{2}(1-\sqrt{1-4q_4})q_3\partial_{q_3} + \sqrt{1-4q_4}q_4\partial_{q_4} + \frac{1}{2}(1-\sqrt{1-4q_4})q_5\partial_{q_5},\\
p_5\partial_{p_5} &= q_5\partial_{q_5}.
\end{aligned}$$

For the moment write $q_i = \exp(t_i)$ and $p_i = \exp(s_i)$. Then clearly $q_i\partial_{q_i} = \partial_{t_i}$ and $p_i\partial_{p_i} = \partial_{s_i}$. Due to the chain rule we now get a system of linear differential equations:

$$\begin{pmatrix}\partial_{s_1}\\ \vdots \\ \partial_{s_5}\end{pmatrix} = \begin{pmatrix}\partial_{s_1}(t_1) & \dots & \partial_{s_1}(t_5)\\ \vdots & \ddots & \vdots\\ \partial_{s_5}(t_1) & \dots & \partial_{s_5}(t_5)\end{pmatrix} \cdot \begin{pmatrix}\partial_{t_1}\\ \vdots \\ \partial_{t_5}\end{pmatrix} = \widetilde{L}.\begin{pmatrix}\partial_{t_1}\\ \vdots \\ \partial_{t_5}\end{pmatrix},$$

with $\widetilde{L}$ as in (52), respectively in (59) in concrete form. To start us off, notice that ∂_{t_2} is independent of s_1, s_3, s_4 and s_5. Choosing 0 as constant of integration we then have

$$t_2 = s_2 \qquad \Rightarrow \qquad q_2 = p_2.$$

Next note now that

$$\partial_{s_1}(t_1) = \sqrt{1-4e^{t_1}},$$

whilst

$$\partial_{s_2}(t_1) = \partial_{s_3}(t_1) = \partial_{s_4}(t_1) = \partial_{s_5}(t_1) = 0,$$

implying that t_1 depends on s_1 only. We can solve these differential equations by elementary means, showing that

$$s_1 = c - \log\left(\frac{1+\sqrt{1-4e^{t_1}}}{1-\sqrt{1-4e^{t_1}}}\right)$$

for some constant $c \in \mathbb{C}$. For simplicity let us choose $c = 0$, yielding

$$p_1 = \exp(s_1) = \frac{1 - \sqrt{1 - 4q_1}}{1 + \sqrt{1 - 4q_1}}.$$

Solving for q_1 we obtain

$$q_1 = \frac{p_1}{(1 + p_1)^2}$$

and in particular

$$\sqrt{1 - 4q_1} = \frac{1 - p_1}{1 + p_1}$$

and

$$\frac{1}{2}(1 - \sqrt{1 - 4q_1}) = \frac{p_1}{1 + p_1}.$$

In precisely the same manner we find

$$\begin{aligned} q_4 &= \frac{p_4}{(1 + p_4)^2}, \\ \sqrt{1 - 4q_4} &= \frac{1 - p_4}{1 + p_4}, \\ \frac{1}{2}(1 - \sqrt{1 - 4q_4}) &= \frac{p_4}{1 + p_4}. \end{aligned}$$

So we are left with finding q_3 and q_5 in terms of the p_i. Once again this is a matter of elementary integration:

$$\partial_{s_4}(t_5) = p_4 \partial_{p_4}(t_5) = \frac{p_4}{1 + p_4} \qquad \Rightarrow \qquad q_5 = f(p_5)(1 + p_4),$$

where due to

$$\partial_{s_5}(t_5) = 1$$

we find

$$q_5 = c(1 + p_4)p_5$$

for some non-zero constant c. Again, for simplicity choosing $c = 1$, and performing the respective steps to find q_3, we obtain a concrete form of the mirror map. And indeed this map is a well-defined change of coordinates in a small neighbourhood of the origin. It changes q_i-coordinates to p_i-coordinates, where

$$\begin{aligned} q_1 &= \frac{p_1}{(1 + p_1)^2}, \\ q_2 &= p_2, \\ q_3 &= (1 + p_1)(1 + p_4)p_3, \\ q_4 &= \frac{p_4}{(1 + p_4)^2}, \\ q_5 &= (1 + p_4)p_5. \end{aligned}$$

In particular, we have

$$\begin{aligned}\sqrt{1-4q_i} &= \frac{1-p_i}{1+p_i},\\ \frac{1}{2}(1-\sqrt{1-4q_i}) &= \frac{p_i}{1+p_i}\end{aligned}$$

for $i = 1, 4$. The mirror map is applied to our $\mathcal{D}$-module, respectively our connection, by replacing the q_i with their respective expressions as p_i, as well as noting that

$$\begin{aligned}\frac{dq_1}{q_1} &= \frac{1-p_1}{1+p_1}\frac{dp_1}{p_1},\\ \frac{dq_2}{q_2} &= \frac{dp_2}{p_2},\\ \frac{dq_3}{q_3} &= \frac{p_1}{1+p_1}\frac{dp_1}{p_1} + \frac{dp_3}{p_3} + \frac{p_4}{1+p_4}\frac{dp_4}{p_4},\\ \frac{dq_4}{q_4} &= \frac{1-p_4}{1+p_4}\frac{dp_4}{p_4},\\ \frac{dq_5}{q_5} &= \frac{p_4}{1+p_4}\frac{dp_4}{p_4} + \frac{dp_5}{p_5}.\end{aligned}$$

Thus writing our connection as

$$\widehat{\Omega}^z = \widehat{\Omega}^{(1)}\frac{dp_1}{zp_1} + \cdots + \widehat{\Omega}^{(5)}\frac{dp_5}{zp_5}$$

yields the following expressions for the connection matrices $\widehat{\Omega}^{(i)}$:

$$
\widehat{\Omega}^{(1)} = \begin{pmatrix}
0 & p_1p_2p_3p_4(p_2p_3+p_5+p_3p_5+p_3p_4p_5) & p_1p_2p_3p_4(2p_2p_3+p_5+p_3p_5+p_3p_4p_5) & p_1p_2p_3p_4(2p_2p_3+p_5+2p_3p_5+2p_3p_4p_5) & p_1p_2p_3p_4(p_2p_3+p_5+p_3p_5+2p_3p_4p_5) & p_1p_2p_3p_4p_5(p_3+1+p_3p_4) & p_1p_2p_3^2p_4p_5(p_4p_5+p_2+p_2p_4+p_2p_3p_4+2p_1p_2p_3p_4) \\
1 & -p_1p_3(p_2+p_2p_4+p_4p_5) & -p_1p_2p_3(1+p_4) & -p_1p_3(p_2+p_2p_4+p_4p_5) & -p_1p_3p_4(p_2+p_5) & -p_1p_3p_4p_5 & -p_1p_2p_3p_4p_5 \\
0 & p_1p_3p_4(p_2p_3+p_5) & p_1p_2p_3^2p_4 & p_1p_3p_4(2p_2p_3+p_5) & p_1p_3p_4(p_2p_3+p_5) & p_1p_3p_4p_5 & p_1p_2p_3^2p_4p_5(1+p_4) \\
0 & p_1p_2p_3p_4(1-p_3) & p_1p_2p_3p_4(1-p_3) & p_1p_2p_3p_4(1-2p_3) & p_1p_2p_3p_4(1-p_3) & 0 & p_1p_2p_3p_4p_5(1-p_3) \\
0 & p_1p_2p_3(1-p_4) & p_1p_2p_3(1-p_4) & p_1p_2p_3(1-p_4) & -p_1p_2p_3p_4 & 0 & p_1p_2p_3^2p_4p_5(1-p_4) \\
0 & p_1p_2p_3p_4(1+p_3) & p_1p_2p_3p_4(1+p_3) & p_1p_2p_3p_4(1+2p_3) & p_1p_2p_3p_4(1+p_3) & 0 & p_1p_2p_3^2p_4(p_2+p_4p_5) \\
0 & 0 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}
$$

$$
\widehat{\Omega}^{(2)} = \begin{pmatrix}
0 & p_1p_2p_3p_4(2p_2p_3+p_5+p_3p_5+p_3p_4p_5) & p_2p_3p_4(4p_1p_2p_3+p_5+p_1p_5+p_1p_3p_5+p_1p_3p_4p_5) & p_2p_3p_4(p_5+4p_1p_2p_3+p_1p_5+2p_1p_3p_5+2p_1p_3p_4p_5) & p_2p_3p_4(2p_1p_2p_3+p_5+p_1p_5+p_1p_3p_5+2p_1p_3p_4p_5) & p_2p_3p_4p_5(1+p_1+p_1p_3+p_1p_3p_4) & p_1p_2p_3^2p_4p_5(2p_2+2p_2p_4+2p_2p_3p_4+2p_1p_2p_3p_4+p_4p_5) \\
0 & -p_1p_2p_3(1+p_4) & p_2p_3(1-p_1)(1+p_4) & p_2p_3(1-p_1)(1+p_4) & p_2p_3p_4(1-p_1) & 0 & p_2p_3p_4p_5(1-p_1) \\
1 & p_1p_2p_3^2p_4 & p_1p_2p_3^2p_4-p_2 & 2p_1p_2p_3^2p_4 & p_1p_2p_3^2p_4 & 0 & p_1p_2p_3^2p_4p_5(1+p_4) \\
0 & p_1p_2p_3p_4(1-p_3) & p_2(1-p_3)(1+p_1p_3p_4) & p_2p_3(p_1p_4-1-2p_1p_3p_4) & p_1p_2p_3p_4(1-p_3) & 0 & p_1p_2p_3p_4p_5(1-p_3) \\
0 & p_1p_2p_3(1-p_4) & p_2p_3(1+p_1)(1-p_4) & p_2p_3(1+p_1)(1-p_4) & -p_2p_3p_4(1+p_1) & 0 & p_1p_2p_3^2p_4p_5(1-p_4) \\
0 & p_1p_2p_3p_4(1+p_3) & p_2p_3p_4(1+p_1+p_1p_3) & p_2p_3p_4(1+p_1+2p_1p_3) & p_2p_3p_4(1+p_1+p_1p_3) & 0 & p_1p_2p_3^2p_4(2p_2+p_4p_5) \\
0 & 1 & 1 & 2 & 2 & 2 & 0
\end{pmatrix}
$$

$$
\widehat{\Omega}^{(3)} = \begin{pmatrix}
0 & p_1p_2p_3p_4(2p_2p_3+p_5+2p_3p_5+2p_3p_4p_5) & p_2p_3p_4(p_5+4p_1p_2p_3+p_1p_5+2p_1p_3p_5+2p_1p_3p_4p_5) & p_2p_3p_4(p_5+4p_1p_2p_3+p_1p_5+4p_1p_3p_5+4p_1p_3p_4p_5) & p_2p_3p_4(p_5+2p_1p_2p_3+p_1p_5+2p_1p_3p_5+4p_1p_3p_4p_5) & p_2p_3p_4p_5(1+p_1+2p_1p_3+2p_1p_3p_4) & p_1p_2p_3^2p_4p_5(2p_4p_5+2p_2+2p_2p_4+3p_2p_3p_4+3p_1p_2p_3p_4) \\
0 & -p_1p_3(p_2+p_2p_4+p_4p_5) & p_2p_3(1-p_1)(1+p_4) & p_3(1-p_1)(p_2+p_2p_4+p_4p_5) & p_3p_4(1-p_1)(p_2+p_5) & p_3p_4p_5(1-p_1) & p_2p_3p_4p_5(1-p_1) \\
0 & p_1p_3p_4(2p_2p_3+p_5) & 2p_1p_2p_3^2p_4 & p_3p_4(p_5+p_1p_5+4p_1p_2p_3) & p_3p_4(p_5+p_1p_5+2p_1p_2p_3) & p_3p_4p_5(1+p_1) & 2p_1p_2p_3^2p_4p_5(1+p_4) \\
1 & p_1p_2p_3p_4(1-2p_3) & p_2p_3(p_1-1-2p_1p_3p_4) & p_3(p_1p_2p_4-p_2-p_4p_5-4p_1p_2p_3p_4) & p_3p_4(p_1p_2-2p_1p_2p_3-p_5) & -p_3p_4p_5 & p_1p_2p_3p_4p_5(1-2p_3) \\
0 & p_1p_2p_3(1-p_4) & p_2p_3(1+p_1)(1-p_4) & p_2p_3(1+p_1)(1-p_4) & -p_2p_3p_4(1+p_1) & 0 & 2p_1p_2p_3^2p_4p_5(1-p_4) \\
0 & p_1p_2p_3p_4(1+2p_3) & p_2p_3p_4(1+p_1+2p_1p_3) & p_2p_3p_4(1+p_1+4p_1p_3) & p_2p_3p_4(1+p_1+2p_1p_3) & 0 & 2p_1p_2p_3^2p_4(p_2+p_4p_5) \\
0 & 1 & 2 & 2 & 2 & 2 & 0
\end{pmatrix}
$$

$$\widehat{\Omega}^{(4)} = \begin{pmatrix} 0 & \substack{p_1p_2p_3p_4(p_2p_3+p_5\\+p_3p_5+2p_3p_4p_5)} & \substack{p_2p_3p_4(2p_1p_2p_3+p_5\\+p_1p_5+p_1p_3p_5+2p_1p_3p_4p_5)} & \substack{p_2p_3p_4(2p_1p_2p_3+p_5\\+p_1p_5+2p_1p_3p_5+4p_1p_3p_4p_5)} & \substack{p_2p_3p_4(p_1p_2p_3+p_5\\+p_1p_5+p_1p_3p_5+4p_1p_3p_4p_5)} & \substack{p_2p_3p_4p_5(1+p_1\\+p_1p_3+2p_1p_3p_4)} & \substack{p_1p_2p_3^2p_4p_5(p_2+2p_2p_4\\+2p_2p_3p_4+2p_1p_2p_3p_4+2p_4p_5)} \\ 0 & -p_1p_3p_4(p_2+p_5) & p_2p_3p_4(1-p_1) & p_3p_4(1-p_1)(p_2+p_5) & p_3p_4(1-p_1)(p_2+p_5) & p_3p_4p_5(1-p_1) & p_2p_3p_4p_5(1-p_1) \\ 0 & p_1p_3p_4(p_2p_3+p_5) & p_1p_2p_3^2p_4 & p_3p_4(p_5+p_1p_5+2p_1p_2p_3) & p_3p_4(p_5+p_1p_5+p_1p_2p_3) & p_3p_4p_5(1+p_1) & p_1p_2p_3^2p_4p_5(1+2p_4) \\ 0 & p_1p_2p_3p_4(1-p_3) & p_1p_2p_3p_4(1-p_3) & p_3p_4(p_1p_2-2p_1p_2p_3-p_5) & p_4(1-p_3)(p_1p_2p_3+p_5) & p_4p_5(1-p_3) & p_1p_2p_3p_4p_5(1-p_3) \\ 1 & -p_1p_2p_3p_4 & -p_2p_3p_4(1+p_1) & -p_2p_3p_4(1+p_1) & -p_4(p_2p_3+p_1p_2p_3+p_5) & -p_4p_5 & p_1p_2p_3^2p_4p_5(1-2p_4) \\ 0 & p_1p_2p_3p_4(1+p_3) & p_2p_3p_4(1+p_1+p_1p_3) & p_2p_3p_4(1+p_1+2p_1p_3) & p_2p_3p_4(1+p_1+p_1p_3) & 0 & p_1p_2p_3^2p_4(p_2+2p_4p_5) \\ 0 & 1 & 2 & 2 & 1 & 1 & 0 \end{pmatrix}$$

$$\widehat{\Omega}^{(5)} = \begin{pmatrix} 0 & p_1p_2p_3p_4p_5(1+p_3+p_3p_4) & \substack{p_2p_3p_4p_5(1+p_1\\+p_1p_3+p_1p_3p_4)} & \substack{p_2p_3p_4p_5(1+p_1\\+2p_1p_3+2p_1p_3p_4)} & \substack{p_2p_3p_4p_5(1+p_1\\+p_1p_3+2p_1p_3p_4)} & \substack{p_2p_3p_4p_5(1+p_1\\+p_1p_3+p_1p_3p_4)} & \substack{p_1p_2p_3^2p_4p_5(p_2+p_2p_4\\+p_2p_3p_4+p_1p_2p_3p_4+2p_4p_5)} \\ 0 & -p_1p_3p_4p_5 & 0 & p_3p_4p_5(1-p_1) & p_3p_4p_5(1-p_1) & p_3p_4p_5(1-p_1) & p_2p_3p_4p_5(1-p_1) \\ 0 & p_1p_3p_4p_5 & 0 & p_3p_4p_5(1+p_1) & p_3p_4p_5(1+p_1) & p_3p_4p_5(1+p_1) & p_1p_2p_3^2p_4p_5(1+p_4) \\ 0 & 0 & 0 & -p_3p_4p_5 & p_4p_5(1-p_3) & p_4p_5(1-p_3) & p_1p_2p_3p_4p_5(1-p_3) \\ 0 & 0 & 0 & 0 & -p_4p_5 & p_5(1-p_4) & p_1p_2p_3^2p_4p_5(1-p_4) \\ 1 & 0 & 0 & 0 & 0 & -p_5 & p_1p_2p_3^2p_4^2p_5 \\ 0 & 1 & 2 & 2 & 1 & 0 & 0 \end{pmatrix}$$

Evidently these are not the exact same connection matrices as computed in chapter 5.3. This is ultimately due to different choices of bases in $\mathrm{Pic}(X_4)$ and $\mathrm{Pic}(X_4^o)$. To address this issue, notice that we have in fact a natural isomorphism, which respects the intersection product:

$$\begin{aligned}
\mathrm{Pic}(X_4^o) &\to \mathrm{Pic}(X_4), \\
[D_1] &\mapsto H - E_1 - E_4, \\
[D_2] &\mapsto E_1, \\
[D_3] &\mapsto H - E_1 - E_2 - E_3, \\
[D_4] &\mapsto E_2, \\
[D_5] &\mapsto E_3 - E_2, \\
[D_6] &\mapsto H - E_3 - E_4, \\
[D_7] &\mapsto E_4.
\end{aligned} \tag{61}$$

Importantly, this isomorphism identifies the anti-canonical divisors:

$$-K_{X_4^o} = \sum_{i=1}^{7} [D_i] \mapsto 3H - \sum_{i=1}^{4} E_i = -K_{X_4}. \tag{62}$$

Thus we have to make one last change of coordinates: in chapter 4.3 we chose the basis $\{[X_4], H, E_1, \dots, E_5, [pt]\}$ of $H^*(X_4, \mathbb{C})$. Thus, using (61), we change

$$(H, E_1, E_2, E_3, E_4, E_5) = (v_1, v_2, v_3, v_4, v_7).L = (v_1, v_2, v_3, v_4, v_7).\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix},$$

where the v_i are the generators of the Kähler cone of X_4^o as determined in example 7.10. $\{v_1, v_2, v_3, v_4, v_7\}$ is precisely our original basis for $\mathrm{Pic}(X_4^o)$, first defined in example 7.14. Note that we have to change the p_i variable to q_i variables as well, but that these change dually to our chosen bases:

$$\begin{array}{ccccccc}
p_1 & = & e^{v_1^*} & = & e^{H-E_1-E_2-E_3} & = & q_1 q_2 q_3 q_4 \\
p_2 & = & e^{v_2^*} & = & e^{H-E_1-E_4} & = & q_1 q_2 q_5 \\
p_3 & = & e^{v_3^*} & = & e^{-H+E_1+E_2+E_4} & = & \frac{1}{q_1 q_2 q_3 q_5} \\
p_4 & = & e^{v_4^*} & = & e^{-E_2+E_3} & = & \frac{q_3}{q_4} \\
p_5 & = & e^{v_7^*} & = & e^{H-E_3-E_4} & = & q_1 q_4 q_5.
\end{array}$$

Therefore

$$\begin{aligned}
\frac{dp_1}{p_1} &= \frac{dq_1}{q_1} + \frac{dq_2}{q_2} + \frac{dq_3}{q_3} + \frac{dq_4}{q_4} \\
\frac{dp_2}{p_2} &= \frac{dq_1}{q_1} + \frac{dq_2}{q_2} + \frac{dq_5}{q_5} \\
\frac{dp_3}{p_3} &= -\frac{dq_1}{q_1} - \frac{dq_2}{q_2} - \frac{dq_3}{q_3} - \frac{dq_5}{q_5} \\
\frac{dp_4}{p_4} &= \frac{dq_3}{q_3} - \frac{dq_4}{q_4} \\
\frac{dp_5}{p_5} &= \frac{dq_1}{q_1} + \frac{dq_4}{q_4} + \frac{dq_5}{q_5}.
\end{aligned}$$

Now changing our p_i-coordinates to q_i ones and using the change of basis for the connection $\nabla^z \mapsto \widehat{\nabla}^z$ given by

$$\widehat{\nabla}^z = d + L^{-1}\Omega^z L + L^{-1}dL = d + L^{-1}\Omega^z L,$$

we find that indeed $\widehat{\nabla}^z$ is precisely the restricted Dubrovin connection as computed for X_4 in chapter 5. Moreover, due to (62) and since $L^{-1}DL = D$ for any diagonal matrix, we have in fact found an isomorphism of bundles with connection between F (after the mirror map) and the Dubrovin connection $\widehat{\nabla}$ as defined on $\mathbb{P}^1 \times U$ in chapter 5. We have thus proven the equivalence of the quantum $\mathcal{D}$-module of X_4 and the Brieskorn lattice of the Landau-Ginzburg mirror of X_4^o. $\square$

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