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Özge Ülkem, M.Sc. Mathematik
aus Sabancı University, geboren in Simav, Türkei

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Uniformization of generalized $\mathcal{D}$-elliptic sheaves

Advisor: Prof. Dr. Gebhard Böckle

Abstract

Drinfeld defined the notion of *elliptic modules*, which are now called Drinfeld modules, as an analogue of elliptic curves in the function field setting. To prove the Langlands correspondence in this context, Drinfeld studied moduli spaces of elliptic sheaves. The categories of elliptic sheaves and Drinfeld modules are equivalent under certain conditions. Since then, many generalizations of elliptic sheaves have been studied, such as $\mathcal{D}$-elliptic sheaves defined by Laumon, Rapoport and Stuhler and Frobenius-Hecke sheaves defined by Stuhler. In this thesis, I introduce a new generalization of elliptic sheaves, called *generalized $\mathcal{D}$-elliptic sheaves* which can be thought of as a generalization of both $\mathcal{D}$-elliptic sheaves and Frobenius-Hecke sheaves. I study their moduli space and prove a uniformization theorem. This builds on work of Laumon-Rapoport-Stuhler, of Hartl and of Rapoport-Zink.

Zusammenfassung

Als Analogon zu elliptischen Kurven über Funktionenkörpern definierte Drinfeld den Begriff eines *elliptischen Moduls*, die man inzwischen unter dem Namen Drinfeld Moduln kennt. Um in diesem Kontext die Langlands Korrespondenzen zu beweisen, studierte Drinfeld Modulräume von elliptischen Garben. Die Kategorien der elliptischen Garben und die der Drinfeld Moduln sind unter bestimmten Voraussetzungen äquivalent. Inzwischen gibt es viele Verallgemeinerungen von elliptischen Garben, beispielsweise die $\mathcal{D}$-elliptischen Garben, definiert von Laumon, Rapoport und Stuhler sowie die Frobenius-Hecke Garben, definiert von Stuhler. In dieser Dissertation konstruiere ich eine neue Verallgemeinerung von elliptischen Garben, die sogenannten *verallgemeinerten $\mathcal{D}$-elliptischen Garben*, die als Verallgemeinerung sowohl von den $\mathcal{D}$-elliptischen Garben als auch von den Frobenius-Hecke Garben betrachtet werden können. Ich studiere deren Modulräume und beweise einen Uniformisierungs-Satz. Dies baut auf Arbeiten von Laumon-Rapoport-Stuhler, von Hartl und von Rapoport-Zink auf.

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Contents

1 Introduction

In the seminal papers [15], [16], Drinfeld introduced elliptic modules (nowadays called Drinfeld modules) as analogues of elliptic curves and abelian varieties in the function field setting. Drinfeld's main interest were the moduli spaces of these objects, the so called Drinfeld modular varieties. These are the analogues of modular curves in the function field setting and their l-adic cohomology realizes part of the Langlands correspondences over global function fields.

In the classical setting of number fields, modular curves are attached to the group $\mathrm{GL}_2/\mathbb{Q}$ and through the theory of Shimura varieties, there are generalizations of these moduli spaces to other groups, such as symplectic and unitary groups, but also to forms of GL_2 and to more general base fields. These generalizations are important objects in the Langlands program. In the function field setting generalizations of the Drinfeld modular varieties have also been constructed as moduli spaces of objects that generalize Drinfeld modules. Let us explain three main generalizations.

In [35], Laumon, Rapoport and Stuhler defined $\mathcal{D}$-elliptic sheaves and their moduli varieties to prove the local Langlands correspondence. Here $\mathcal{D}$ stands (essentially) for a maximal order of a division algebra. The moduli varieties mimick (and generalize) classical Shimura curves attached to an inner form of GL_2 over $\mathbb{Q}$. These varieties are smooth and compact; unlike those of Drinfeld.

In [46], Stuhler defined *Frobenius-Hecke sheaves*. The Frobenius-Hecke sheaves can be thought of elliptic sheaves that can have many "poles". Stuhler also constructed moduli spaces of Frobenius-Hecke sheaves, the analogue to Hilbert modular varieties attached to GL_2 over a totally real field.

In [23], Hartl defined *abelian sheaves* as an analogue of abelian varieties in the classical theory. Abelian sheaves are higher dimensional generalizations of elliptic sheaves. In the same paper it is proved that the stack of abelian sheaves is a Deligne-Mumford stack. Hartl studied uniformization of abelian sheaves at ∞. There is a uniformizable locus in the stack of abelian sheaves. In [23], a uniformization theorem is proved for this uniformizable locus. In order to prove this, Hartl introduced analogues of Rapoport-Zink spaces. The main guides for this thesis are [35] and [23].

What has been missing in the function field case so far, was the analogue of Shimura curves over totally real fields attached to inner forms of GL_2, that are split at all infinite places. In this thesis we provide such a generalization. We want to emphasize at this point that in the function field setting all the generalizations above as well as our results work for GL_d and not only for GL_2.

In order to construct these more general moduli spaces we first generalize elliptic sheaves in a new way. We call these new objects *generalized $\mathcal{D}$-elliptic sheaves*. They are a simultaneous generalization of both $\mathcal{D}$-elliptic sheaves and Frobenius-Hecke sheaves. They can be thought of as $\mathcal{D}$-elliptic sheaves that can have many poles or as Frobenius-Hecke sheaves with a $\mathcal{D}$-action.

In this thesis, we construct the moduli space of generalized $\mathcal{D}$-elliptic sheaves. We formulate the moduli problem of generalized $\mathcal{D}$-elliptic sheaves with level I-structure. We show that the moduli space of generalized $\mathcal{D}$-elliptic sheaves has nice algebraic properties as in the $\mathcal{D}$-elliptic sheaf case. More presicely, the stack of generalized $\mathcal{D}$-elliptic sheaves is a Deligne-Mumford stack and with a non-trivial level structure, it is even a scheme. We show that the characteristic morphism is proper over a suitable base. In order to prove these properties, we mainly follow [35]. One big difference to [35] is that in [35], Laumon, Rapoport and Stuhler assumed that a $\mathcal{D}$-elliptic sheaf has finite characteristic whereas in our case the characteristic can be infinite.

The second main result of this thesis is the proof of the uniformization theorem. We uniformize the moduli space of generalized $\mathcal{D}$-elliptic sheaves at the infinity place. This is analogous to the classical complex uniformization of Shimura varieties which is a very important result in the classical theory.

In future work, we hope to use our uniformization result to study arithmetic questions about the moduli space of generalized $\mathcal{D}$-elliptic sheaves. For example, the uniformization result is needed in order to build a definition of modular forms as holomorphic functions with certain transformation properties. We then plan to study the motives attached to these forms, their good reduction properties and their L-functions.

We will now give a more detailed overview of the results of this thesis.

1.1 Overview of the results

Let X, Y be smooth gemometrically irreducible projective curves over $\mathbb{F}_q$ and $\pi : X \longrightarrow Y$ be a morphism of degree t. Let $\infty \in |Y|$ be a closed place and $\{\infty_1, \cdots, \infty_t\}$ be the closed places in $|X|$ lying above ∞. Define F to be $\mathbb{F}_q(X)$. (This is the geometric analogue in the function field setting to passing from $\mathbb{Q}$ to a totally real field.)
Let $\mathcal{D}$ be an Azumaya $\mathcal{O}_X$-algebra such that its stalk at the generic point η of X is a division algebra over F. We denote by $\underline{Bad}$ the ramified places for $\mathcal{D}$, and put $\mathsf{B} := \pi(\underline{Bad})$. Our generalization of Drinfeld's notion of elliptic sheaf is provided by the following definition:

Definition 1.1. (Definition 2.2)

Let S be an $\mathbb{F}_q$-scheme and fix a closed immersion $\psi : S' \longrightarrow (X \setminus \underline{\mathrm{Bad}}) \times_{\mathbb{F}_q} S$ such that $pr_2 \circ \psi : S' \longrightarrow S$ is finite locally free of degree t.

A *generalized $\mathcal{D}$-elliptic sheaf* of charachteristic ψ is a tuple $(\underline{\mathcal{E}}, \psi)$ where $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, j_{\underline{i},\underline{i}'}, t_{\underline{i}})_{i \in \mathbb{Z}^t}$ is a ladder (see Def. 2.1) of locally free sheaves $\mathcal{E}_{\underline{i}}$ of $\mathcal{O}_{X\times S}$-modules where S is an $\mathbb{F}_q$-scheme with a $\mathcal{D}$-action together with injective morphisms of $\mathcal{O}_{X\times S}$-modules

$$j_{\underline{i},\underline{i}'} : \mathcal{E}_{\underline{i}} \longrightarrow \mathcal{E}_{\underline{i}'} \text{ for } \underline{i} \leq \underline{i}'$$

$$t_{\underline{i}} : \sigma^* \mathcal{E}_{\underline{i}} \longrightarrow \mathcal{E}_{\underline{i}+\underline{1}}$$

which are compatible with the $\mathcal{D}$-action and which satisfy the following conditions:

(i) A periodicity condition: Put $\ell = \underline{d}.\deg \infty \cdot t$. We have

$$\mathcal{E}_{\underline{i}+\underline{\ell}} = \mathcal{E}_{\underline{i}}(\infty_1, \ldots, \infty_t)$$

where $\mathcal{E}_{\underline{i}}(\infty_1, \ldots, \infty_t) = \mathcal{E}_{\underline{i}} \otimes \mathcal{O}_X(\infty_1, \ldots, \infty_t)$.

(ii) A condition on the cokernel of $j_{\underline{i},\underline{i}'}$ around each ∞_j.

(iii) The cokernel of $t_{\underline{i}}$ has support on $\operatorname{Im}\psi$ and is locally free of rank d over S'.

We want to point out that if $t = 1$, generalized $\mathcal{D}$-elliptic sheaves are $\mathcal{D}$-elliptic sheaves defined in [35]. And if $\mathcal{D} = \mathbb{M}_d(\mathcal{O}_X)$, we show in Proposition 2.9 that the category of generalized $\mathcal{D}$-elliptic sheaves is Morita equivalent to the category of Frobenius-Hecke sheaves defined in [46]. We formulate the notion of level I-structures on generalized $\mathcal{D}$-elliptic sheaves in Definition 2.15. We also define adelic level structures in Section 7. However, for our main results we only need level I-structures.

Similarly to Stuhler ([46]), we will be working with generalized $\mathcal{D}$-elliptic sheaves that have certain characteristic (cf. Section 4). The generalized $\mathcal{D}$-elliptic sheaves with the condition on their characteristic are called *generalized $\mathcal{D}$-elliptic sheaves relative to X/Y*. We introduce those and define level I-structures on them in Section 4. We denote the stack of generalized $\mathcal{D}$-elliptic sheaves relative to X/Y with level I-structure by $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$. Let us put $\mathsf{J} := \pi(I)$. Our first main result in this thesis is the following:

Theorem 1.2. *(Theorem 5.3) The stack $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$ is an algebraic stack in the sense of Deligne-Mumford [11] which is smooth over $Y \smallsetminus (J \cup \mathsf{B} \cup \infty)$. Moreover, if $I \neq \emptyset$, it is a scheme that is a disjoint union of quasi-projective schemes.*

The idea to prove this theorem is to cover it with certain Deligne-Mumford stacks of I-stable generalized $\mathcal{D}$-elliptic sheaves. We follow [35] to prove that the I-stable generalized $\mathcal{D}$-elliptic sheaves form stacks. An important tool is the stack of I-stable vector bundles. To deal with the infinite characteristic we use the idea of Hartl in Proposition 5.19.

Next, we study the question of properness. We show that once we consider the translations of generalized $\mathcal{D}$-elliptic sheaves relative to X/Y as the same object, i.e, we consider the quotient $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}/\underline{1}\mathbb{Z}$, we have:

Theorem 1.3. *(cf. Theorem 6.1)*

The morphism

$$\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}/\underline{1}\mathbb{Z} \longrightarrow Y \smallsetminus \mathsf{B}$$

is proper.

We prove the properness by using the valuative criterion of properness.

After these global results, we now want to explain our uniformization theorem. For that let Z denote the fiber of $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$ over ∞ and $\widehat{\mathcal{GE}\ell\ell}^{Z}_{X/Y,\mathcal{D},I}$ denote the formal completion of $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$ along Z. Let G'_{gen} denote the moduli functor of generalized $\mathcal{D}$-elliptic sheaves relative to X/Y which are quasi-isogeneous to a fixed generalized $\mathcal{D}$-elliptic sheaves realtive to X/Y. We will explain some more details regarding this below. Let $\mathbb{A}_f$ denote the finite adeles of X and let H be a compact open subgroup of $D^\times(\mathbb{A}_f)$. Our uniformization theorem is the following:

Theorem 1.4. *(cf. Theorem 16.8): We have an isomorphism of formal schemes*

$$\widehat{\mathcal{GE}\ell\ell}^{Z}_{X/Y,\mathcal{D},I} \simeq D^\times \backslash G'_{gen} \times D^\times(\mathbb{A}_f)/H.$$

In order to prove the theorem, we largely follow the framework that was introduced by Hartl in [23](building on the work of Rapoport-Zink in [43]). We want to point out that unlike [23], our construction works for $\deg\infty > 1$ also.

A major part of the proof of the uniformization theorem consists of showing representability of the moduli space G'_{gen}. We also need a Serre-Tate theorem. For that we need an analogue of p-divisible groups and Dieudonné theory.

Hartl introduced the notions "z-divisible groups" and "Dieudonné $\mathbb{F}_q[\![z]\!]$-modules". These objects are the analogues of p-divisible groups and Dieudonné modules that were used by Rapoport and Zink to prove the p-adic uniformization of Shimura curves. We define analogues of these local objects first for the $\mathcal{D}$-elliptic sheaves over Y. Since these are local objects they carry a $\mathcal{D}_\infty := \mathcal{D} \otimes_{\mathcal{O}_Y} \mathcal{O}_\infty$- action where $\mathcal{O}_\infty$ denotes the completion of the stalk of the structure sheaf of Y at ∞. Following Hartl's steps, we define z-divisible $\mathcal{D}_\infty$-modules and Dieudonné $\mathcal{D}_\infty$-modules. Since $\mathcal{D}_\infty \simeq \mathbb{M}_d(\mathcal{O}_\infty)$, the category of z-divisible $\mathcal{D}_\infty$-modules are Morita equivalent to the category of z-divisible groups. Following [23] and [43], we work with the moduli space of z-divisible $\mathcal{D}_\infty$-module which are isogenous to a fixed z-divisible $\mathcal{D}_\infty$-module.

Generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-modules are then defined as t-tuples of z-divisible $\mathcal{D}_\infty$-modules at each ∞_i for $i = 1, \cdots, t$. Similarly, *generalized Dieudonné $\mathcal{D}_{\underline{\infty}}$-modules* are defined as t-tuples of Dieudonné $\mathcal{D}_\infty$-modules at each ∞_i for $i = 1, \cdots, t$. As in the classical case, there is a categorical anti-equivalence between the category of generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-modules and generalized Dieudonné $\mathcal{D}_{\underline{\infty}}$-modules. The rigidity of quasi-isogenies also holds for generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-modules as in the classical case (cf. Theorem 9.17).

Generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-modules are related to generalized $\mathcal{D}$-elliptic sheaves through their Dieudonné modules, namely *generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaves.* As before a generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf is t-tuple of the *formal $\mathcal{D}_{\infty_j}$-elliptic sheaves* where $j = 1, \cdots, t$. We will give the idea to construct formal $\mathcal{D}_\infty$-elliptic sheaf first for $t = 1$ case (cf. Construction 10.7). As we mentioned before, if $t = 1$ a generalized $\mathcal{D}$-elliptic sheaf is a $\mathcal{D}$-elliptic sheaf defined in [35]. Let $\underline{\mathcal{E}} = (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ be a $\mathcal{D}$-elliptic sheaf. We take the formal completion $\underline{\mathcal{E}}^\infty$ of $\underline{\mathcal{E}}$ along the fiber over ∞. Since the periodicity of $\underline{\mathcal{E}}$ is $d\deg\infty$, the periodicity of $\underline{\mathcal{E}}^\infty$ is also $d\deg\infty$. We take $\deg\infty$-jumps of this sheaves, denote it by $\widetilde{\mathcal{E}}^\infty$ and the latter will have periodicity d. We will use this to prove the representability of the moduli functor G'_{gen}. Now, if we consider a generalized $\mathcal{D}$-elliptic sheaf $\widehat{\underline{\mathcal{E}}}$, by following same path for each ∞_j, we get formal $\mathcal{D}_{\infty_j}$-elliptic sheaf for each $j = 1, \cdots, t$. Then the generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf associated to $\underline{\mathcal{E}}$ is the t-tuple of formal $\mathcal{D}_{\infty_j}$-elliptic sheaves.

By using the equivalence between generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-modules and generalized Dieudonné $\mathcal{D}_{\underline{\infty}}$-modules, we obtain a certain generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-module. These certain generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-modules are the generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-modules associated to the generalized $\mathcal{D}$-elliptic sheaves. This correspondence is similar to the relation between abelian varieties and their p-divisible groups in the classical world.

Similar to the classical case, we can look at deformations of generalized $\mathcal{D}$-elliptic sheaves and deformations of generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaves. Both categories are equivalent via a Serre-Tate theorem:

Theorem 1.5. *(Theorem 13.3) Let $\underline{\mathcal{E}}$ be a generalized $\mathcal{D}$-elliptic sheaf and let $\widehat{\underline{\mathcal{E}}}$ denote the generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf associated to $\underline{\mathcal{E}}$. Then, the category of deformations of $\underline{\mathcal{E}}$ is equivalent to the category of deformations of $\widehat{\underline{\mathcal{E}}}$.*

Let us finish this overview by explaining the uniformizing spaces from the uniformization theorem when $\mathcal{D}$ splits at all ∞_i's. As before consider the moduli problem G'_{gen} of generalized $\mathcal{D}$-elliptic sheaves which are quasi-isogenous to a fixed generalized $\mathcal{D}$-elliptic sheaf, say $\underline{\boldsymbol{\mathcal{E}}}$. To have a precise result on the field of definition of the uniformization, we use Genestier [19]. We define an equivalency of functors G'_{gen} and Genestier's functor G_O by using formal completion of generalized $\mathcal{D}$-elliptic sheaves over $\mathbb{T}$. Hence, as in [19], the functor G'_{gen} is representable by $(\mathbb{Z}\times\widehat{\Omega}^{(d)})^t$ where $\widehat{\Omega}^{(d)}$ is the Deligne-Mumford scheme (cf. [4], Section 4.3). This is higher dimension version of Drinfeld's upper plane plane which occurs in the uniformization of Shimura curves.

Over the algebraic closure of the residue field at ∞, the group of quasi-isogenies of $\underline{\boldsymbol{\mathcal{E}}}$ is $D^\times$. By using this we have an action of $D^\times$ on G'_{gen} and since $D^\times$ acts naturally on $D^\times(\mathbb{A}_f)$, we obtain a diagonal action on $G'_{gen}\times D^\times(\mathbb{A}_f)$. Putting in the level structures, we get an isomorphism

$$D^\times\backslash\prod(\mathbb{Z}\times\widehat{\Omega}^{(d)})\times D^\times(\mathbb{A}_f)/H \longrightarrow \widehat{\mathcal{GE}\ell\ell}_{X,\mathcal{D},I}.$$

And if $t=1$, we get a uniformization theorem of $\mathcal{D}$-elliptic sheaves as stated in [4]. I want to emphasise that in [4], the theorem is stated without a proof. So, the uniformization theorem for generalized $\mathcal{D}$-elliptic sheaves fills in this gap.

Outline of the thesis

The thesis contains three main parts: global part, local part, and the uniformization.

First part consists of Section 2 - 7. In these sections we define *generalized $\mathcal{D}$-elliptic sheaves* as a natural generalization of Frobenius-Hecke sheaves and $\mathcal{D}$-elliptic sheaves. We define (quasi-)isogenies between generalized $\mathcal{D}$-elliptic sheaves in Section 3 and give some examples. In Section 4, we look at generalized $\mathcal{D}$-elliptic sheaves of certain type, namely we put a condition on its characteristic, and consider their stack. This stack is a Deligne-Mumford stack and in fact is a scheme with non-trivial level structure. Since we have a $\mathcal{D}$-action the characteristic morphism is proper. We prove this in Section 6.

We work on the local part in Section 8-12. In Section 8-10, we define generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-modules, generalized Dieudonné $\mathcal{D}_{\underline{\infty}}$-modules and generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaves. The latter is the generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-module associated to a generalized $\mathcal{D}$-elliptic sheaves. We define some results like rigidity of quasi-isogenies analoguesly to the classical case. In Section 11-12, we give examples of $\mathcal{D}$-elliptic sheaves and generalized $\mathcal{D}$-elliptic sheaves. By using these examples we define the moduli functors of (generalized) formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaves.

We connect first and second part by Section 13. In Section 13 we give a Serre-Tate theorem in generalized $\mathcal{D}$-elliptic sheaf case. This theorem relates deformations of generalized $\mathcal{D}$-elliptic sheaves with the deformations of generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaves.

The aim of the last part is to uniformize the moduli space of generalized $\mathcal{D}$-elliptic sheaves. We prove the representability of the moduli functor in Section 15 by using its algebraization defined in Section 14. We state our uniformization theorem in Section 16 and give proof in Section 17.

This thesis also contains an appendix (Sections 18-20). In Section 18 we give Morita equivalence for rings, sheaves and stacks. Section 19 collects the background that we need from the theory of stacks. Section 20 we collect some facts about vector bundles.

2 Generalized $\mathcal{D}$-elliptic sheaves

In this section, we present simultaneous generalization of $\mathcal{D}$-elliptic sheaves([35]) and Frobenius-Hecke sheaves([46]). We will give the definition of *generalized $\mathcal{D}$-elliptic sheaves* following Stuhler [46]. In his paper [46], Stuhler defined *Frobenius-Hecke sheaves* as a generalization of elliptic sheaves to give a modular interpretation of

$$\mathrm{SL}_d(B) \diagdown (\prod_{j=1}^{t} \widehat{\Omega}^{(d)}_{\infty_j} / \operatorname{Spf}(\widehat{\mathcal{O}}_{\mathrm{Y},\infty}))$$

where $\widehat{\Omega}^{(d)}$ is Deligne-Mumford scheme(cf. [4], Section 4.3) around ∞_j for each $j = 1, \dots, t$.

Let X and Y be smooth projective geometrically irreducible curves over $\mathbb{F}_q$ with function fields F and L, respectively. Let $\pi : X \longrightarrow Y$ be a finite morphism of degree t. Let $\infty \in Y$ be a closed point which splits completely and $\infty_1, \dots, \infty_t$ be the points of X above ∞. Regard $\mathbb{T} = \{\infty_1, \dots, \infty_t\}$ as a closed irreducible subscheme of X. Let $A := \Gamma(Y - \infty, \mathcal{O}_Y)$ and $B := \Gamma(X - \mathbb{T}, \mathcal{O}_X)$. We have the following situation:

$$\begin{array}{cccc}
X & \mathbb{T} = \{\infty_1, \dots, \infty_t\} & B & F = \mathbb{F}_q(X) \\
\Big\downarrow{\scriptstyle \pi} & \Big| & \Big\uparrow & \Big\uparrow \\
Y & \infty & A & L = \mathbb{F}_q(Y)
\end{array}$$

We will denote the completion of the local ring $\mathcal{O}_{X,\infty_i}$ at ∞_i for each $i \in \{1, \dots, t\}$ by $\mathcal{O}_{\infty_i}$ and the completion of the local ring $\mathcal{O}_{Y,\infty}$ at ∞ by $\mathcal{O}_\infty$. Let z_i be a uniformizing element of $\mathcal{O}_{X,\infty_i}$ and z be a uniformizing element of $\mathcal{O}_{Y,\infty}$. Also, let k_{∞_i} be the residue field at ∞_i and k_∞ be the residue field at ∞.

Note that there are natural identifications of completions:

$$F_{\infty_j} \simeq L_\infty$$

and

$$\mathcal{O}_{\infty_j} \simeq \mathcal{O}_{Y,\infty}.$$

Let $\mathcal{O}_{X,\mathbb{T}} = \bigcap_j \mathcal{O}_{X,\infty_j}$ be the semilocal ring of X in $\mathbb{T}$ and

$$k(\mathbb{T}) := \mathcal{O}_{X,\mathbb{T}}/(rad(\mathcal{O}_{X,\mathbb{T}})) \simeq \prod k_{\infty_i}$$

.

Let $\mathcal{D}$ be an Azumaya $\mathcal{O}_X$-algebra with $\dim_{\mathcal{O}_X} \mathcal{D} = d^2$. Assume $\mathcal{D}_x$ is a maximal order for each $x \in |X|$.

Denote the ramified places for $\mathcal{D}$ by $\underline{\mathrm{Bad}}$. Assume that no $\infty_i \in \underline{Bad}$ i.e, $\mathcal{D}_{\infty_i} = \mathcal{D} \otimes \mathcal{O}_{\infty_i} \simeq \mathbb{M}_d(\mathcal{O}_{\infty_i})$. Put $\mathcal{D}_{\underline{\infty}} = \prod \mathcal{D}_{\infty_i}$.

Let $\underline{i} = (i_1, \dots, i_t) \in \mathbb{Z}^t$ and $\underline{i}+\underline{1} = (i_1+1, \dots, i_t+1)$. There is a partial ordering on $\mathbb{Z}^t$:

$$\underline{i} \leq \underline{i}' \iff i_j \leq i'_j \text{ for all } j = 1, \dots, t$$

Let S be an $\mathbb{F}_q$-scheme. We denote the *Frobenius endomorphism* on S by $\sigma_S : S \longrightarrow S$ which is defined as the identitiy on points and as the q-power map on the structure sheaf. Let S' be a closed subscheme of $(X \smallsetminus \underline{Bad}) \times_k S$ such that $pr_2|_{S'} : S' \longrightarrow S$ is finite of degree t.

Definition 2.1. A *ladder* over S is a system $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ where $\mathcal{E}_{\underline{i}}$ are locally free $\mathcal{O}_{X\times S}$-modules of rank d^2 with right $\mathcal{D}$-action which is $\mathcal{O}_{X\times S}$-linear and the restriction of $\mathcal{D}$ to the scalars is same as the action of $\mathcal{O}_X$. And for $\underline{i}, \underline{i}' \in \mathbb{Z}^t$ with $\underline{i} \leq \underline{i}'$ there are injective $\mathcal{O}_{X\times S}$-module morphisms

$$j_{\underline{i},\underline{i}'} : \mathcal{E}_{\underline{i}} \hookrightarrow \mathcal{E}_{\underline{i}'}$$

$$t_{\underline{i}} : \sigma^* \mathcal{E}_{\underline{i}} \hookrightarrow \mathcal{E}_{\underline{i}+\underline{1}}$$

which are compatible with the $\mathcal{D}$-action such that for $\underline{i} \leq \underline{i}'$ the following diagram commutes:

$$\begin{tikzcd} & \mathcal{E}_{\underline{i}+\underline{1}} \arrow[r, hook] & \mathcal{E}_{\underline{i}'+\underline{1}} \\ \sigma^*\mathcal{E}_{\underline{i}} \arrow[ur, "t_{\underline{i}}"] \arrow[r, hook] & \sigma^*\mathcal{E}_{\underline{i}'} \arrow[ur, "t_{\underline{i}'}"] & \end{tikzcd}$$

Definition 2.2. A pair $(\underline{\mathcal{E}}, \psi)$ consisting of a ladder $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ and a closed immersion $\psi : S' \longrightarrow (X \smallsetminus \underline{\mathrm{Bad}}) \times_k S$ such that $pr_2 \circ \psi : S' \longrightarrow S$ is finite locally free of degree t is called a *generalized $\mathcal{D}$-elliptic sheaf over S* if the following conditions are satisfied:

(i) (*periodicity*) Put $\ell = \underline{d}.\deg\infty \cdot t$. We have

$$\mathcal{E}_{\underline{i}+\underline{\ell}} = \mathcal{E}_{\underline{i}}(\infty_1, \dots, \infty_t)$$

where $\mathcal{E}_{\underline{i}}(\infty_1, \dots, \infty_t) = \mathcal{E}_{\underline{i}} \otimes \mathcal{O}_X(\infty_1, \dots, \infty_t)$

(ii) Let $\underline{i}' = \underline{i} + (\delta_1, \dots, \delta_t)$ where $\delta_j = 0$ or 1. Then, the support of the quotient $\mathcal{E}_{\underline{i}'}/\mathcal{E}_{\underline{i}}$ is contained in $\mathbb{T} \times S$. Moreover,

$$(\mathcal{E}_{\underline{i}'}/\mathcal{E}_{\underline{i}})|_{\infty_j \times S} \simeq \mathcal{V}_{\underline{i},\underline{i}',j} \text{ is locally free of rank } d \cdot \delta_j \text{ over } \mathcal{O}_S$$

Assume $\delta_j = 1$. The induced action of $\mathcal{O}_{X,\mathbb{T}}$ on $\mathcal{V}_{\underline{i},\underline{i}',j}$ factorizes over an algebra morphism of the quotient

$$\eta^{(j)}_{\underline{i},\underline{i}'} : k(\mathbb{T}) \longrightarrow \mathrm{End}(\mathcal{V}_{\underline{i},\underline{i}',j}).$$

Put $\eta_{\underline{i},\underline{i}'} := (\eta^{(1)}_{\underline{i},\underline{i}'}, \ldots, \eta^{(t)}_{\underline{i},\underline{i}'})$. Then, for all $\underline{i} \in \mathbb{Z}^t$,

$$\chi_{\underline{i}+\underline{1},\underline{i}+\underline{2}} = \chi_{\underline{i},\underline{i}+\underline{1}} \circ \tau$$

where $\tau : k(\mathbb{T}) \longrightarrow k(\mathbb{T}), x \mapsto x^q$ is the Frobenius morphism.

(iii) The cokernel of $t_{\underline{i}}$ has support on $\operatorname{Im}\psi$ and is locally free of rank d over S'

Remark 2.3. 1. Let $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ be a generalized $\mathcal{D}$-elliptic sheaf and $\mathcal{V}_{\underline{i},\underline{i}',j}$ be a vector bundle defined as in condition ii. Define

$$\mathcal{L}_{\underline{i},\underline{i}',j} := \mathcal{V}_{\underline{i},\underline{i}',j} \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}.$$

Then, $\mathcal{V}_{\underline{i},\underline{i}',j} \simeq \mathcal{L}_{\underline{i},\underline{i}',j} \otimes_{\mathbb{F}_q} \mathbb{F}_q^{1\times d}$. Now, on $\mathcal{V}_{\underline{i},\underline{i}',j}$ there is a $\mathbb{M}_d(k(\mathbb{T})))$-action, denote it by $\eta^{(j)}_{\underline{i},\underline{i}'}$. Since $\mathbb{M}_d(k(\mathbb{T})) = k(\mathbb{T}) \otimes_{\mathbb{F}_q} \mathbb{M}_d(\mathbb{F}_q)$, the action $\eta^{(j)}_{\underline{i},\underline{i}'} = \chi^{(j)}_{\underline{i},\underline{i}'} \otimes r$ where $\chi^{(j)}_{\underline{i},\underline{i}'}$ is an action of $k(\mathbb{T})$ on $\mathcal{L}^{(j)}_{\underline{i},\underline{i}'}$ and r is the natural right action of $\mathbb{M}_d(\mathbb{F}_q)$ on $\mathbb{F}_q^{1\times d}$.

2. We want to point out that the vector bundle $\mathcal{V}_{\underline{i},\underline{i}',j}$ in the Definition 2.2 is isomorphic to $\mathcal{L}^{\oplus d}_{\underline{i},\underline{i}',j}$ where $\mathcal{L}_{\underline{i},\underline{i}',j}$ is the line bundle over $\mathcal{O}_S$.

3. The action $\chi^{(j)}_{\underline{i},\underline{i}'}$ of $k(\mathbb{T})$ on $\mathcal{V}_{\underline{i},\underline{i}',j}$ factors via the structure map

$$\mathcal{O}_S \longrightarrow \operatorname{End}_S(\mathcal{V}_{\underline{i},\underline{i}',j})$$

. So, the action morphism $\chi^{(j)}_{\underline{i},\underline{i}'}$ may be regarded as homomorphism $k(\mathbb{T}) \longrightarrow \mathcal{O}_S$. Indeed, let $\chi^{(j)}_{\underline{i},\underline{i}'} : k(\mathbb{T}) \longrightarrow \operatorname{End}_S(\mathcal{L}_{\underline{i},\underline{i}',j}) \simeq \mathcal{O}_S$ be an action(cf. Definition 2.8). By the previous item, we can define

$$k(\mathbb{T}) \xrightarrow{\chi^{(j)}_{\underline{i},\underline{i}'}} \mathcal{O}_S \longrightarrow \mathbb{M}_d(\mathcal{O}_S) \simeq \operatorname{End}_S(\mathcal{L}^{\oplus d}_{\underline{i},\underline{i}',j}) \simeq \operatorname{End}_S(\mathcal{V}_{\underline{i},\underline{i}',j}).$$

4. Let $\underline{i}' = \underline{i} + (\delta_1, \cdots, \delta_t)$ where $\delta_j = 0$ or 1. Note that the action of $\chi^{(j)}_{\underline{i},\underline{i}'}$ is independent of the components of $\underline{i}$ and $\underline{i}'$ with index different from j. By condition ii, all $\chi^{j}_{\underline{i},\underline{i}'}$ is determined by $\chi^{(j)}_{\underline{0},\underline{1}}$.

Definition 2.4. The map ψ in the definition of generalized $\mathcal{D}$-elliptic sheaf is called *the characteristic* of the generalized $\mathcal{D}$-elliptic sheaf.

Let $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})_{\underline{i}\in\mathbb{Z}^t}$ be a generalized $\mathcal{D}$-elliptic sheaf and $\underline{n} = (n_1, \ldots, n_t) \in \mathbb{Z}^t$. Then, one can define the $\mathbb{Z}^t$-action on $\underline{\mathcal{E}}$ as follows:

$$\underline{\mathcal{E}}[n] = (\mathcal{E}_{\underline{i}-\underline{n}}, t_{\underline{i}-\underline{n}})$$

where $\underline{i} - \underline{n} = (i_1 - n_1, \ldots, i_t - n_t)$. In particular,

$$\underline{\mathcal{E}}[1] = \underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}-\underline{1}}, t_{\underline{i}-\underline{1}})$$

.

Remark 2.5. 1. Let $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ be a generalized $\mathcal{D}$-elliptic sheaf over S. By condition (ii), $H^0((X \smallsetminus \mathbb{T}) \otimes \mathcal{O}_S, \mathcal{E}_{\underline{i}})$ is independent of $\underline{i}$. Moreover, if $S = \operatorname{Spec} K$ where K is a field it is a $K[\tau]$-module where the τ action comes from $t_{\underline{i}} : \sigma^*\mathcal{E}_{\underline{i}} \longrightarrow \mathcal{E}_{\underline{i}+\underline{1}}$.

2. Assume $t = 1$ and $X = Y$. Then, locally $\mathcal{O}_{S'} \simeq \mathcal{O}_S$ and, we get a *$\mathcal{D}$-elliptic sheaf* ([35], Definition 2.2). In this case the module in the previous item is called *Drinfeld-Stuher $\mathcal{O}_D$-module*. For more details, please see [41].

Remark 2.6. (1) Assume $\mathcal{D} = \mathbb{M}_d(\mathcal{O}_X)$ and $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ is a generalized $\mathcal{D}$-elliptic sheaf. Then, the generalized $\mathcal{D}$-elliptic sheaves are called *Frobenius-Hecke sheaves* of rank d that were defined in [46]. In Proposition 2.9, we will see that the category of Frobenius-Hecke sheaves and generalized $\mathcal{D}$-elliptic sheaves are Morita equivalent when $\mathcal{D} = \mathbb{M}_d(\mathcal{O}_X)$.

(2) In Frobenius-Hecke sheaf case, the module $H^0((X \smallsetminus \mathbb{T}) \otimes K, \mathcal{E}_{\underline{i}})$ is considered by Anderson in [3](*Hilbert-Blumenthal t-modules*) and by Stuhler in [46](*Abelian $\mathcal{O}_K$-module of Hilbert-Blumenthal type*) separately.

Remark 2.7. We want to point out that in [46], in the definition of Frobenius-Hecke sheaves item ii is different. Stuhler says $\operatorname{coker} j_{\underline{i},\underline{i}'}$ is free over $\mathcal{O}_S$. However, we want elliptic sheaves defined by Drinfeld as a special case of Frobenius-Hecke sheaves, so $\operatorname{coker} j_{\underline{i},\underline{i}'}$ should be locally free over $\mathcal{O}_S$. We write the definition of Frobenius-Hecke sheaves in the correct form below. Please also note that in this case a ladder consists of $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ where $\mathcal{E}_{\underline{i}}$ are locally free $\mathcal{O}_X$-modules of rank d such that the obvious diagrams commutes.

Definition 2.8. A pair $(\underline{\mathcal{E}}, \psi)$ consisting of a ladder $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ and a closed immersion $\psi : S' \longrightarrow (X \smallsetminus \underline{\mathrm{Bad}}) \times_k S$ such that $pr_2 \circ \psi : S' \longrightarrow S$ is finite locally free of degree t is called a *Frobenius-Hecke sheaf* if the following conditions are satisfied:

(i) (*periodicity*) Put $\ell = \underline{d}.\deg\infty$. We have

$$\mathcal{E}_{\underline{i}+\underline{\ell}} = \mathcal{E}_{\underline{i}}(\infty_1, \ldots, \infty_t)$$

where $\mathcal{E}_{\underline{i}}(\infty_1, \ldots, \infty_t) = \mathcal{E}_{\underline{i}} \otimes \mathcal{O}_X(\infty_1, \ldots, \infty_t)$

(ii) Let $\underline{i}' = \underline{i} + (\delta_1, \ldots, \delta_t)$ where $\delta_j = 0$ or 1. Then, the support of the quotient $\mathcal{E}_{\underline{i}'}/\mathcal{E}_{\underline{i}}$ is contained in $\mathbb{T} \times S$. Moreover,

$$(\mathcal{E}_{\underline{i}'}/\mathcal{E}_{\underline{i}})|_{\infty_j \times S} \simeq \mathcal{L}_{\underline{i},\underline{i}',j} \text{ is locally free of rank } \delta_j \text{ over } \mathcal{O}_S$$

Assume $\delta_j = 1$. The induced action of $\mathcal{O}_{X,\mathbb{T}}$ on $\mathcal{L}_{\underline{i},\underline{i}',j}$ factorizes over an algebra morphism of the quotient

$$\chi^{(j)}_{\underline{i},\underline{i}'} : k(\mathbb{T}) \longrightarrow \mathrm{End}(\mathcal{L}_{\underline{i},\underline{i}',j}).$$

Put $\chi_{\underline{i},\underline{i}'} := (\chi^{(1)}_{\underline{i},\underline{i}'}, \ldots, \chi^{(t)}_{\underline{i},\underline{i}'})$. Then, for all $\underline{i} \in \mathbb{Z}^t$,

$$\chi_{\underline{i}+\underline{1},\underline{i}+\underline{2}} = \chi_{\underline{i},\underline{i}+\underline{1}} \circ \tau$$

where $\tau : k(\mathbb{T}) \longrightarrow k(\mathbb{T}), x \mapsto x^q$ is the Frobenius morphism.

(iii) The cokernel of $t_{\underline{i}}$ has support on $\mathrm{Im}\,\psi$ and is locally free of rank 1 over S'.

In [35],p. 224 it is mentioned that $\mathcal{D}$-elliptic sheaves are Morita equivalent to elliptic sheaves when $\mathcal{D} = \mathbb{M}_d(\mathcal{O}_X)$. As for $\mathcal{D}$-elliptic sheaves, one can use Morita equivalence for generalized $\mathcal{D}$-elliptic sheaves also. Recall that generalized $\mathcal{O}_X$-elliptic sheaves are Frobenius-Hecke sheaves by definition. For a review of Morita equivalence, we refer to Section 18.

Proposition 2.9. *The category of Frobenius-Hecke sheaves of rank d is Morita equivalent to the category of generalized $\mathbb{M}_d(\mathcal{O}_X)$-sheaves.*

Proof. (of Proposition 2.9)
Let $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ be a Frobenius-Hecke sheaf over S. Define

$$\underline{\mathcal{F}} = (\mathcal{F}_{\underline{i}}, \tau_{\underline{i}}) := (\mathcal{E}_{\underline{i}} \otimes_{\mathcal{O}_X} N, t_{\underline{i}} \otimes_{\mathcal{O}_X} N)$$

where $N = \mathcal{O}_X^{1\times d}$ is a simple right $\mathbb{M}_d(\mathcal{O}_X)$-module. Note that $\mathcal{E}_{\underline{i}} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{1\times d} \simeq \mathcal{E}_{\underline{i}} \oplus \cdots \oplus \mathcal{E}_{\underline{i}}$ where the direct sum is taken d-copies. Let us focus on the condition (i) in Definition 2.8. Let $\underline{i}' = \underline{i} + (\delta_1, \cdots, \delta_j)$ where each $\delta_j = 0$ or 1. Then we have,

$$\mathcal{F}_{\underline{i}'}/\mathcal{F}_{\underline{i}} = (\mathcal{E}_{\underline{i}'} \oplus \cdots \oplus \mathcal{E}_{\underline{i}'})/(\mathcal{E}_{\underline{i}} \oplus \cdots \oplus \mathcal{E}_{\underline{i}}) \simeq \mathcal{E}_{\underline{i}'}/\mathcal{E}_{\underline{i}} \oplus \cdots \oplus \mathcal{E}_{\underline{i}'}/\mathcal{E}_{\underline{i}}$$

Hence $\left(\mathcal{F}_{\underline{i}'}/\mathcal{F}_{\underline{i}}\right)|_{\infty_j \times S} \simeq (\mathcal{E}_{\underline{i}'}/\mathcal{E}_{\underline{i}})^{\oplus d}|_{\infty_j \times S} \simeq \mathcal{L}^{\oplus d}_{\underline{i},\underline{i}',j}$ where $\mathcal{L}_{\underline{i},\underline{i}',j}$ is locally free over $\mathcal{O}_S$ of rank δ_j. For simplicity, put $\mathcal{V}_{\underline{i},\underline{i}',j} := \mathcal{L}^{\oplus d}_{\underline{i},\underline{i}',j}$. Then, $\mathcal{V}_{\underline{i},\underline{i}',j}$ is locally free of rank $d \cdot \delta_j$ over $\mathcal{O}_S$.

By condition (ii) in Definition 2.8, we have an action homomorphism

$$\chi^{(j)}_{\underline{i},\underline{i}'} : k(\mathbb{T}) \longrightarrow \mathrm{End}_S(\mathcal{L}_{\underline{i},\underline{i}',j}) \simeq \mathcal{O}_S.$$

Thus on

$$\mathcal{F}_{\underline{i}'}/\mathcal{F}_{\underline{i}} \simeq \mathcal{L}_{\underline{i},\underline{i}',j} \otimes_{k(\mathbb{T})} k(\mathbb{T})^{1\times d} \simeq \mathcal{L}_{\underline{i},\underline{i}',j} \otimes_{\mathbb{F}_q} \mathbb{F}_q^{1\times d}$$

we have an induced action

$$k(\mathbb{T}) \xrightarrow{\chi^{(j)}_{\underline{i},\underline{i}'}} \mathcal{O}_S \xrightarrow{can} \mathbb{M}_d(\mathcal{O}_S) \simeq \mathrm{End}(\mathcal{L}^{\oplus d}_{\underline{i},\underline{i}',j})$$

as $\mathcal{O}_S$-algebra where can is the natural action of $\mathcal{O}_S$ on $\mathbb{M}_d(\mathcal{O}_S)$. Define $\eta^{(j)}_{\underline{i},\underline{i}'} := can \circ \chi^{(j)}_{\underline{i},\underline{i}'}$. Now, let $\overline{a} \in k(\mathbb{T})$ be any. We have

$$\eta^{(j)}_{\underline{i}+\underline{1},\underline{i}+\underline{2}}(\overline{a}) = can \circ \chi^{(j)}_{\underline{i}+\underline{1},\underline{i}+\underline{2}}(\overline{a}) = can \circ \chi^{(j)}_{\underline{i},\underline{i}+\underline{1}} \circ \tau(\overline{a}) = \eta^{(j)}_{\underline{i},\underline{i}+\underline{1}} \circ \tau(\overline{a})$$

where $\tau : k(\mathbb{T}) \longrightarrow k(\mathbb{T})$ is the Frobenius endomorphism.

Now, let us consider the morphisms $\tau_{\underline{i}} : \mathcal{F}_{\underline{i}} \longrightarrow \mathcal{F}_{\underline{i}+\underline{1}}$. As before we have $\mathrm{coker}\,\tau_{\underline{i}} \simeq \mathrm{coker}\, t^{\oplus d}_{\underline{i}}$, which is locally free of rank d over S'. Hence, $\underline{\mathcal{F}} = (\mathcal{F}_{\underline{i}}, \tau_{\underline{i}})$ is a generalized $\mathcal{D}$-elliptic sheaf.

Conversely, given a generalized $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{F}} = (\mathcal{F}_{\underline{i}}, \tau_{\underline{i}})$ for $\mathcal{D} = \mathbb{M}_d(\mathcal{O}_X)$. Suppose $0 \le \underline{\delta} \le \underline{i}' - \underline{i} \le \underline{1}$. We know that

$$\big(\mathcal{F}_{\underline{i}'}/\mathcal{F}_{\underline{i}}\big)|_{\infty_j \times \mathcal{O}_S} \simeq \mathcal{V}_{\underline{i},\underline{i}',j}$$

is locally free over $\mathcal{O}_S$ of rank $d\delta_j$. It comes with an action of $\mathbb{M}_d(k(\mathbb{T})) = k(\mathbb{T}) \otimes_{\mathbb{F}_q} \mathbb{M}_d(\mathbb{F}_q)$. Define

$$\mathcal{L}_{\underline{i},\underline{i}',j} := \mathcal{V}_{\underline{i},\underline{i}',j} \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}.$$

Then, $\mathcal{V}_{\underline{i},\underline{i}',j} \simeq \mathcal{L}_{\underline{i},\underline{i}',j} \otimes_{\mathbb{F}_q} \mathbb{F}_q^{1\times d}$.

Now, define $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}}) := (\mathcal{F}_{\underline{i}} \otimes_{\mathbb{M}_d(\mathcal{O}_X)} \mathcal{O}_X^{1\times d}, \tau_{\underline{i}} \otimes_{\mathbb{M}_d(\mathcal{O}_X)} \mathcal{O}_X^{1\times d})$. Then,

$$\begin{aligned}\big(\mathcal{E}_{\underline{i}'}/\mathcal{E}_{\underline{i}}\big)|_{\infty_j \times S} &\simeq \mathcal{V}_{\underline{i},\underline{i}',j} \otimes_{\mathbb{M}_d(\mathcal{O}_X)} \mathcal{O}_X^{1\times d} \\ &\simeq \mathcal{L}_{\underline{i},\underline{i}',j} \otimes_{\mathbb{F}_q} \mathbb{F}_q^{1\times d} \otimes_{\mathbb{M}_d} \mathcal{O}_X^{1\times d}\end{aligned}$$

By Remark 2.3 (i), the $\chi^{(j)}_{\underline{i},\underline{i}'} \otimes r$-action on $\mathcal{L}_{\underline{i},\underline{i}',j} \otimes_{\mathbb{F}_q} \mathbb{F}_q^{1\times d}$ induces $\chi_{\underline{i},\underline{i}'}$-action of $k(\mathbb{T})$ on

$$\mathcal{L}_{\underline{i},\underline{i}',j} \otimes_{\mathbb{F}_q} \mathbb{F}_q^{1\times d} \otimes_{\mathbb{M}_d(\mathcal{O}_X)} \mathcal{O}_X^{1\times d} \simeq \mathcal{L}_{\underline{i},\underline{i}',j}.$$

Then by definition, it follows that

$$\chi_{\underline{i}+\underline{1},\underline{i}+\underline{2}} = \chi_{\underline{i},\underline{i}+\underline{1}} \circ \tau$$

where $\tau : k(\mathbb{T}) \longrightarrow k(\mathbb{T})$ is the Frobenius endomorphism.

Now, let us consider the morphisms $t_{\underline{i}} : \sigma^* \mathcal{E}_{\underline{i}} \longrightarrow \mathcal{E}_{\underline{i}+\underline{1}}$. As before, we have coker $t_{\underline{i}}$ is locally free of rank 1. Hence $\underline{\mathcal{E}}$ is a Frobenius-Hecke sheaf.

□

Definition 2.10. A *morphism* between two generalized $\mathcal{D}$-elliptic sheaves $\underline{\mathcal{E}}$ and $\underline{\mathcal{E}}'$ is a morphism between locally free $\mathcal{O}_{X\times S}$-modules $f_{\underline{i}} : \mathcal{E}_{\underline{i}} \longrightarrow \mathcal{E}'_{\underline{i}}$ which respects the $\mathcal{D}$-action and compatible with $j_{\underline{i},\underline{i}'}$'s and $t_{\underline{i}}$'s.

Definition 2.11. Let $\mathcal{GE}\ell\ell_{X,\mathcal{D}}(S)$ denote the category whose objects are the generalized $\mathcal{D}$-elliptic sheaves over S and morphisms are isomorphisms of generalized $\mathcal{D}$-elliptic sheaves.

Now, we are ready to define stack of generalized $\mathcal{D}$-elliptic sheaves. For a review of stacks, please see Section 19.

Proposition 2.12. *Let $Sch_{\mathbb{F}_q}$ denote the category of schemes over $\mathbb{F}_q$. Then the assignement*

$$S \longrightarrow \mathcal{GE}\ell\ell_{X,\mathcal{D}}(S)$$

defines a fibered category over $Sch_{\mathbb{F}_q}$. Moreover, it is a stack with respect to fppf-topology.

Proof. First we will show that $\mathcal{GE}\ell\ell_{X,\mathcal{D}}(S)$ is a fibered category for an $\mathbb{F}_q$-scheme S by using Definition 19.10. Let $T \longrightarrow S$ be a morphism of $\mathbb{F}_q$-schemes and let $\underline{\mathcal{E}}$ be a generalized $\mathcal{D}$-elliptic sheaf over S. Then, the pullback $f^*(\underline{\mathcal{E}})$ is a generalized $\mathcal{D}$-elliptic sheaf over T. So, we get a functor $f^* : \mathcal{GE}\ell\ell_{X,\mathcal{D}}(T) \longrightarrow \mathcal{GE}\ell\ell_{X,\mathcal{D}}(S)$ defined by taking pullback. Then, the conditions in Definition 19.10 are satisfied by defition of pullback and by the fact that $(f \circ g)^* = g^* \circ f^*$ for any two morphisms f, g in $Sch_{\mathbb{F}_q}$.

To show that this fibered category is in fact a stack, it remains to show the two conditions in Definition 19.23. More precisely, let S be an $\mathbb{F}_q$-scheme and $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{k}}, j_{\underline{k},\underline{k}'}, t_{\underline{k}})$ and $\underline{\mathcal{F}} = (\mathcal{F}_{\underline{k}}, \Pi_{\underline{k},\underline{k}'}, \tau_{\underline{k}})$ be two generalized $\mathcal{D}$-elliptic sheaf over S. We need to show that

- The functor
$$\mathrm{Iso}_S(\underline{\mathcal{E}}, \underline{\mathcal{F}}) : Sch_S \longrightarrow Sets$$
defined by $(f : T \longrightarrow S) \mapsto \{\varphi : f^*\underline{\mathcal{E}} \xrightarrow{\sim} f^*\underline{\mathcal{F}}$ isomorphism in $\mathcal{GE}\ell\ell_{X,\mathcal{D}}(T)\}$ is a sheaf in the étale topology.

- Suppose $T \longrightarrow S$ is a covering in $Sch_{\mathbb{F}_q}$ for the fppf topology. Suppose we have descent datum in $\mathcal{GE}\ell\ell_{X,\mathcal{D}}$. Then, these datum is effective.

We start with the first item. For simplicity we will denote the functor $\mathrm{Iso}_S(\underline{\mathcal{E}}, \underline{\mathcal{F}})$ by F. Let $f : T \longrightarrow S \in Sch_S$ and $f_i : T_i \longrightarrow T$ be an étale covering in Sch_S. Given $\varphi_i \in F(T_i)$. Let $pr_1 : T_i \times_T T_j \longrightarrow T_i$ and $pr_2 : T_i \times_T T_j \longrightarrow T_j$ denote the natural projections. Assume $pr_1^*(\varphi_i) = pr_2^*(\varphi_j) \in F(T_i \times_T T_j)$ for any i, j.

It is known that quasi-coherent sheaves over Sch_S form a stack ([52], Section 4.2.). Via this fact, there is an isomorphism $\varphi_{\underline{k}} : f^*\mathcal{E}_{\underline{k}} \xrightarrow{\sim} f^*\mathcal{F}_{\underline{k}}$ of locally free sheaves. We need to show that this isomorphism $\varphi : f^*\underline{\mathcal{E}} \longrightarrow f^*\underline{\mathcal{F}}$ is an isomorphism of generalized $\mathcal{D}$-elliptic sheaves. Write $f^*\underline{\mathcal{E}} = (f^*\mathcal{E}_{\underline{k}}, f^*j_{\underline{k},\underline{k}'}, f^*t_{\underline{k}})$ and $f^*\underline{\mathcal{F}} = (f^*\mathcal{F}_{\underline{k}}, f^*\Pi_{\underline{k},\underline{k}'}, f^*\tau_{\underline{k}})$. By the following diagram

$$\begin{tikzcd} f^*\mathcal{E}_{\underline{k}} \arrow[r, hook, "f^*j_{\underline{k},\underline{k}'}"] \arrow[d, "\varphi_{\underline{k}}"', "\simeq"] & \mathcal{E}_{\underline{k}'} \arrow[d, "\varphi_{\underline{k}'}", "\simeq"'] \\ f^*\mathcal{F}_{\underline{k}} \arrow[r, hook, "f^*\Pi_{\underline{k},\underline{k}'}"] & f^*\mathcal{F}_{\underline{k}'} \end{tikzcd}$$

we have two morphisms $\varphi_{\underline{k}} \circ f^*j_{\underline{k},\underline{k}'}$ and $f^*\Pi_{\underline{k},\underline{k}'} \circ \varphi_{\underline{k}}$ from $f^*\mathcal{E}_{\underline{k}} \longrightarrow f^*\mathcal{F}_{\underline{k}'}$. Since Hom is a sheaf for quasi-coherent sheaves, we have $\varphi_{\underline{k}} \circ f^*j_{\underline{k},\underline{k}'} = f^*\Pi_{\underline{k},\underline{k}'} \circ \varphi_{\underline{k}}$.

We have a similar diagram for the morphisms $t_{\underline{i}}$ and $\tau_{\underline{i}}$ for each $\underline{i} \in \mathbb{Z}^t$. Hence, the isomorphism φ commutes with the morphism $j_{\underline{i}}$'s and $t_{\underline{i}}$'s for each $\underline{i} \in \mathbb{Z}^t$. For $\mathcal{D}$-action, we have the following diagram

$$\begin{tikzcd} f^*\mathcal{E}_{\underline{k}} \arrow[r, hook, "\varphi_{\underline{k}}", "\simeq"'] \arrow[d, "\cdot\mathcal{D}"'] & \mathcal{F}_{\underline{k}} \arrow[d, "\cdot\mathcal{D}"'] \\ f^*\mathcal{E}_{\underline{k}} \arrow[r, hook, "\varphi_{\underline{k}}"] & f^*\mathcal{F}_{\underline{k}'} \end{tikzcd}$$

Similar as before, we get two morphisms $\cdot \circ \varphi_{\underline{k}}$ and $\varphi_{\underline{k}} \circ \cdot\mathcal{D}$ from $\mathcal{E}_{\underline{k}} \longrightarrow \mathcal{F}_{\underline{k}}$. Since Hom is a sheaf for quasi-coherent sheaves, we have $\cdot \circ \varphi_{\underline{k}} = \varphi_{\underline{k}} \circ \cdot\mathcal{D}$.

Let $f : T_i \longrightarrow T$ be a covering in Sch_S. Let $(\{\underline{\mathcal{E}}^{(i)}\}, \{\psi_{i,j}\})$ be a descent data, i.e $\underline{\mathcal{E}}^{(i)} = (\mathcal{E}^{(i)}_{\underline{k}}, j^{(i)}_{\underline{k},\underline{k}'}, t^{(i)}_{\underline{k}})$ is a generalized $\mathcal{D}$-elliptic sheaf over T_i for each i. Since the category of quasi-coherent sheaves form a stack over Sch_S, there are locally free sheaves $\mathcal{F}^{(i)}_{\underline{k}}$ over T_i for each $\underline{k}$ and i, together with cartesian arrows $\mathcal{E}^{(i)}_{\underline{k}} \longrightarrow \mathcal{F}^{(i)}_{\underline{k}}$ such that the diagram in the definition commutes. Let $\underline{\mathcal{E}}^{(i)}$ be fixed but any. Then, we have the following diagram:

$$\begin{tikzcd} \cdots \arrow[r, hook] & \mathcal{E}^{(i)}_{\underline{k}} \arrow[r, hook, "j^{(i)}_{\underline{k},\underline{k}'}"] \arrow[d, "\varphi^{(i)}_{\underline{k}}"'] & \mathcal{E}^{(i)}_{\underline{k}'} \arrow[r, hook] \arrow[d, "\varphi^{(i)}_{\underline{k}'}"] & \cdots \\ \cdots & \mathcal{F}^{(i)}_{\underline{k}} & \mathcal{F}^{(i)}_{\underline{k}'} & \cdots \end{tikzcd}$$

where the vertical arrows are cartesian. Let $\Pi^{(i)}_{\underline{k},\underline{k}'} : \mathcal{F}^{(i)}_{\underline{k}} \longrightarrow \mathcal{F}^{(i)}_{\underline{k}'}$ be the morphism of locally free sheaves that makes the diagram above commutative, i.e, which satisfies

$\varphi'_{\underline{k}} \circ j^{(i)}_{\underline{k},\underline{k}'} = \Pi^{(i)}_{\underline{k}} \circ \varphi^{(i)}_{\underline{k}}$. Similarly, define $\tau^{(i)}_{\underline{k}} : \sigma^*\mathcal{F}^{(i)}_{\underline{k}} \longrightarrow \mathcal{F}^{(i)}_{\underline{k}+\underline{1}}$. Also, by a similar diagram we will have a $\mathcal{D}$-action on each $\mathcal{F}^{(i)}_{\underline{k}}$.

By Example 19.22, we get locally free sheaves $\mathcal{F}_{\underline{k}}$ for each k. By descent of morphisms(cf. Section 023R), we get morphism $\Pi_{\underline{k},\underline{k}'} : \mathcal{F}_{\underline{k}} \longrightarrow \mathcal{F}_{\underline{k}'}$ and $\tau_{\underline{k}} : \sigma^*\mathcal{F}_{\underline{k}} \longrightarrow \mathcal{F}_{\underline{k}+\underline{1}}$. For each $\underline{\mathcal{E}}^{(i)}$ there is a closed immersion $\psi^{(i)} : S' \longrightarrow (X \smallsetminus \underline{Bad}) \times_k T_i$ since each $\underline{\mathcal{E}}^{(i)}$ is generalized $\mathcal{D}$-elliptic sheaf. Similarly by descent of morphisms we have a $\psi : S' \longrightarrow (X \smallsetminus \underline{Bad}) \times T$. Hence, we get a ladder $\underline{\mathcal{F}}$ together with $\psi : S' \longrightarrow (X \smallsetminus \underline{Bad}) \times_k T$. We need to check that $(\underline{\mathcal{F}}, \psi) = (\mathcal{F}_{\underline{k}}, \Pi_{\underline{k},\underline{k}'}, \tau_{\underline{k}})$ satisfies the conditions in Definition 2.2. Let us check the condition (**??**). Let $C^{(i)}_{\underline{k}}$ denote the cokernel of $t^{(i)}_{\underline{k}} : \mathcal{E}^{(i)}_{\underline{k}} \longrightarrow \mathcal{E}^{(i)}_{\underline{k}+\underline{1}}$. Again, by the descent for quasi-coherent sheaves, we get $C_{\underline{k}}$ which is the cokernel of $\tau_{\underline{k}} : \mathcal{F}_{\underline{k}} \longrightarrow \mathcal{F}_{\underline{k}+\underline{1}}$ which is locally free of rank d over S'. Similarly by using the descent for quasi-coherent sheaves, we see that the other conditions in Definition 2.2 are satisfied. Hence, $(\underline{\mathcal{F}}, \psi)$ is a generalized $\mathcal{D}$-elliptic sheaf over T.

The commutativity of the diagram in Definition 19.17 is equivalent to the cocycle condition in the definition of the object with descent data.

□

Remark 2.13. Note that sending a generalized $\mathcal{D}$-elliptic sheaf $(\underline{\mathcal{E}}, \psi)$ to its characteristic ψ gives us a morphism of stacks

$$\mathcal{G}\mathcal{E}\ell\ell_{X,\mathcal{D}} \longrightarrow X \smallsetminus \underline{Bad}.$$

Now, we will define level structures on generalized $\mathcal{D}$-elliptic sheaves. Let $I \subset X$ be a closed subscheme such that $I \cap \psi(S') = \emptyset$. Then, the restrictions $\mathcal{E}_{\underline{i}}|_{I\times S}$ are all isomorphic via the morphisms $j_{\underline{i},\underline{i}'}$. We will denote this restriction by $\mathcal{E}|_{I\times S}$. Note also that the morphism $t|_{I\times S} : \sigma^*\mathcal{E}|_{I\times S} \longrightarrow \mathcal{E}|_{I\times S}$ are also isomorphisms since $I \cap \psi(S') = \emptyset$.

Definition 2.14. Let X and Y be two schemes and let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module and $\mathcal{G}$ be a locally free $\mathcal{O}_Y$-module. Let $pr_1 : X \times Y \longrightarrow X$ and $pr_2 : X \times Y \longrightarrow Y$ be the natural projections. We define the *external tensor product* of $\mathcal{F}$ and $\mathcal{G}$ as follows:

$$\mathcal{F} \boxtimes \mathcal{G} := pr_1^*(\mathcal{F}) \otimes_{\mathcal{O}_{X\times Y}} pr_2^*(\mathcal{G})$$

which is naturally a locally free $\mathcal{O}_{X\times Y}$-module.

Definition 2.15. Let $I \subset X \smallsetminus Bad$ be a finite closed subscheme different than $\mathbb{T} \cup \psi(S')$. A *level I-structure* on a generalized $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ is an isomorphism of $\mathcal{O}_{I\times_{\mathbb{F}_q} S}$-modules

$$\iota : \mathcal{D}_I \boxtimes \mathcal{O}_S \xrightarrow{\sim} \mathcal{E}|_{I \times_{\mathbb{F}_q} S}$$

compatible with $\mathcal{D}$-action and the Frobenius structure on S given by $\sigma_S : \sigma_S^* \mathcal{O}_S \longrightarrow \mathcal{O}_S$.

Remark 2.16. Fron now on, when we define a level I-structure, it is meant that I is disjoint from $\mathbb{T} \cup \psi(S')$.

Remark 2.17. We want to mention that we can define level structures for ideals of X, as in the case of Hilbert modular forms in the classical world.

Definition 2.18. Let $\mathcal{GE}\ell\ell_{X,\mathcal{D},I}(S)$ denote the category of generalized $\mathcal{D}$-elliptic sheaves over S with level I-structure and whose morphisms are morphisms of generalized $\mathcal{D}$-elliptic sheaves that respects the level I-structure.

Proposition 2.19. *Let S be an $\mathbb{F}_q$-scheme. Then, $S \mapsto \mathcal{GE}\ell\ell_{X,\mathcal{D},I}(S)$ defines a stack which we will denote by $\mathcal{GE}\ell\ell_{X,\mathcal{D},I}$.*

Proof. The proof is similar as to the proof of Proposition 2.12. □

Remark 2.20. In Section 5, we will show that after putting a condition on the characteristic of the generalized $\mathcal{D}$-elliptic sheaf, we will have Deligne-Mumford stack. Moreover, with nontrivial level structure we will have scheme.

Let $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ be a generalized $\mathcal{D}$-elliptic sheaf over S with level I-structure. Let Sch_S denote the category of schemes over S. As in the case of $\mathcal{D}$-elliptic sheaf [[35], (2.6)], we define the t-invariant elements functor

$$E_I : Sch_S \longrightarrow H^0(I, \mathcal{D}_I) - modules$$

by $T \mapsto (H^0(I \times T, \mathcal{E}|_{I \times T}))^{t=id}$

Remark 2.21. Note that $(H^0(I \times T, \mathcal{E}|_{I \times T}))^{t=id} = (H^0(T, pr_*(\mathcal{E}|_{I \times T}))^{t=id}$ where $pr : I \times T \longrightarrow T$ is the natural projection.

Proof. This follows from the fact that the global section of the direct image of a sheaf is isomorphic to the global sections of the given sheaf. □

Theorem 2.22. *The functor E_I is represented by a finite étale scheme over S which is free over $H^0(I, \mathcal{D}_I)$ of rank 1.*

The proof of the Theorem is similar to [35], (2.6)(comp [14], Section 2)

Proof. Let $\mathcal{A}b$ denote the category of abelian groups. Let $\mathcal{F}$ be a locally free $\mathcal{O}_S$-module of rank m together with an isomorphism $\varphi : Frob_S^*\mathcal{F} \longrightarrow \mathcal{F}$. Consider the functor G from Sch_S to $\mathcal{A}b$ given by $T \mapsto H^0(T, \mathcal{F}_T)^{\varphi=id}$ where $\mathcal{F}_T$ denote the sheaf $\mathcal{F}$ after base change to T. Locally on S, we have $\mathcal{F} \simeq \mathcal{O}_S^m$ and denote by $\Phi = (a_{ij}) \in GL_m(H^0(S, \mathcal{O}_S))$ the inverse of the matrix representing the isomorphism φ. Then, the functor G is represented by a closed subscheme of $\mathbb{G}_{a,S}^m$ group scheme, denote it by $\mathbb{G}$, given by the system of equations

$$x_j^q = \sum_{i=1}^{m} a_{ij} \cdot x_i \tag{1}$$

It is finite étale over S because Φ is invertible. Hence the functor $G : Sch_S \longrightarrow \mathcal{A}b$ defined by $T \mapsto H^0(T, \mathcal{F}_T)^{\varphi=id}$ is represented by a finite étale commutative group scheme $\mathbb{G}$ of order q^n that is given by the system of equations (1)(See also [12]).

Now, if we consider the direct image of $\mathcal{E}|_{I\times S}$ under the projection map $X \times S \longrightarrow S$, we get a locally free sheaf over S with a $Frob_S$-linear isomorphism. In Drinfeld case, there is an $H^0(I, \mathcal{O}_I)$ action on the direct image. But additional to Drinfeld we have a $\mathcal{D}$-action on $\mathcal{E}_{I\times S}$. So, $pr_*(\mathcal{E}_{I\times S})$ is a $H^0(I, \mathcal{D})$-module.

Hence, E_I is represented by a finite étale group scheme over S in $H^0(I, \mathcal{D}_I)$-modules of rank $|H^0(I, \mathcal{D}_I)|$.

To show that it is free of rank 1 over $H^0(I, \mathcal{D}_I)$-module, we follow [35]. We may assume that the support of I is x. Then, $\mathcal{O}_I$ is a quotient of $\mathcal{O}_x$. Define $E_x := \varprojlim_{I'}(E_{I'} \otimes_{\mathcal{O}_I} \mathcal{O}_{I'})$ where the limit is taken over all finite closed subschemes $I' \subset X$ such that $\operatorname{supp} I = \operatorname{supp} I'$. Then, E_x is a $\mathcal{D}_x = \mathbb{M}_d(\mathcal{O}_x)$-module. Now, by [15], Proposition 2.2 and since the order of E_I is $|H^0(I, \mathcal{D}_I)|$, it follows that E_x is a free $\mathcal{O}_x$ module of order d^2. So, E_x is a $\mathbb{M}_d(\mathcal{O}_x)$-module and free $\mathcal{O}_x$-module of rank d^2. Then, E_x is free over $\mathbb{M}_d(\mathcal{O}_x)$ of rank 1 by Nakayama's lemma. More precisely, let m_x be the maximal ideal of $\mathcal{O}_x$ and let $\kappa(x)$ denote the residue field. By the classification of simple $\mathbb{M}_d(\kappa(x))$-modules, we have

$$E_x \otimes \mathcal{O}_x/m_x \simeq \mathbb{M}_d(\kappa(x)).$$

Let $e \in E_x$ be an element that maps to a generator of $E_x \otimes \mathcal{O}_x/m_x$. Then,

$$\mathbb{M}_d(\mathcal{O}_x) \longrightarrow E_x, a \mapsto e$$

is surjective by Nakayama's lemma. Since $\mathbb{M}_d(\mathcal{O}_x)$ and E_x are both free over $\mathcal{O}_x$ of the same rank, they are isomorphic (cf [10], 26.24 (iii)).

□

Remark 2.23. Note that $(\mathcal{D}_I \boxtimes \mathcal{O}_S)^{t=id} \boxtimes \mathcal{O}_S \xrightarrow{\sim} \mathcal{D}_I \boxtimes \mathcal{O}_S$

Lemma 2.24. *Let $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ be a generalized $\mathcal{D}$-elliptic sheaf over a connected S. Then, the set of level I-structures on $\underline{\mathcal{E}}$ are in 1-1 correspondence with the set of isomorphisms of $H^0(I, \mathcal{D}_I)$-modules*

$$E_I(S) \xrightarrow{\sim} H^0(I, \mathcal{D}_I)$$

Proof. Suppose that S is connected. Then taking t-invariants thus gives an isomorphism $\mathcal{D}_I \xrightarrow{\sim} \mathcal{D}_I$ of $\mathcal{D}_I$-modules. And by the previous remark the latter isomorphism uniquely determines the former one.

$\square$

We will use this lemma to define adelic level structures in Section 7.

Remark 2.25. By the previous lemma, we see that for connected S, the set of I-level structures is a torsor over the unit group $D_I^\times$. More precisely, Let S be an $\mathbb{F}_q$-scheme. Then, the morphism

$$r_{I',I}(S) : \mathcal{GE}\ell\ell_{X,\mathcal{D},I'}(S) \longrightarrow \mathcal{GE}\ell\ell_{X,\mathcal{D},I}(S)$$

which associates a level I'-structure to its restriction gives us a $G_{I',I}$-torsor over $X \setminus I'$, i.e, the finite group $G_{I',I}$ acts on the set of level I'-structures transitively and freely.

Remark 2.26. The multiplicative group of the algebra $H^0(I, \mathcal{D}_I) = H^0(I, \mathcal{D})$ acts on the set of level structures via the composition

$$\mathcal{D}_{I\times S} \xrightarrow[g]{} \mathcal{D}_{I\times S} \xrightarrow[\varphi]{\simeq} \mathcal{E}_{I\times S}$$

Proof. (of Remark 2.25)

Let φ_1 and φ_2 be two level I'-structures on a generalized $\mathcal{D}$-elliptic sheaf $(\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ over an S-scheme T such that $\varphi_1|_I \simeq \varphi_2|_I$. By Remark 2.23, instead of the diagram

$$\begin{tikzcd}
\mathcal{D}_{I'} \boxtimes \mathcal{O}_T \arrow[r, "\varphi_1", "\simeq"'] \arrow[dr, dotted, "g'"'] & \mathcal{E}_{I'\times T} \\
& \mathcal{D}_{I'} \boxtimes \mathcal{O}_T \arrow[u, "\simeq", "\varphi_2"']
\end{tikzcd}$$

we can consider the following

$$\begin{tikzcd}
(\mathcal{D}_{I'} \boxtimes \mathcal{O}_T)^{t=id} \arrow[r, "\psi_1", "\simeq"'] \arrow[dr, dotted, "g"'] & (\mathcal{E}_{I'\times T})^{t=id} \\
& (\mathcal{D}_{I'} \boxtimes \mathcal{O}_T)^{t=id} \arrow[u, "\simeq", "\psi_2"']
\end{tikzcd}$$

where $g \in Isom((\mathcal{D}_{I'} \boxtimes \mathcal{O}_T)^{t=id}) = Isom((\mathcal{E}|_{I\times S})^{t=id})$. Then, by Theorem 2.22, we know that $g \in H^0(I', \mathcal{D}_{I'})$. Since $\psi_1|_I \simeq \psi_2|_I$, we have

$$g \in Ker\Big(GL_1\big(H^0(I', \mathcal{D}_{I'})\big) \longrightarrow GL_1\big(H^0(I, \mathcal{D}_I)\big)\Big).$$

So, the $G_{I',I}$ action is transitive.

On the other hand,

$$\varphi_1 \circ g = \varphi_1 \iff \varphi_1^{-1} \circ \varphi_1 \circ g = id,$$

hence the action is free. Therefore, for every S, the morphism $r_{I',I}(S)$ is a $G_{I',I}$-torsor.

$\square$

Remark 2.27. We want to remark that the functor E_I is defined for a fixed generalized $\mathcal{D}$-elliptic sheaves $\underline{\mathcal{E}}$. If we want to consider the functor for two different generalized $\mathcal{D}$-elliptic sheaves $\underline{\mathcal{E}}$ and $\underline{\mathcal{F}}$, we will write $E_I(\underline{\mathcal{E}})$ and $E_I(\underline{\mathcal{F}})$(e.g Construction 16.3).

3 Isogenies

In this section we will define and give some examples of (quasi-)isogenies of generalized $\mathcal{D}$-elliptic sheaves. First, we want to recall the following definition [25], page 109:

Definition 3.1. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be an $\mathcal{O}_X$-module.

1. We say $\mathcal{F}$ is *free* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$.
2. We say $\mathcal{F}$ is *locally free* if X can be covered by open sets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module. In that case the *rank* of $\mathcal{F}$ is the number of copies of the structure sheaf needed(finite or infinite).

Remark 3.2. If X is connected, the rank of a locally free sheaf is same everywhere.

Definition 3.3. 1. An *isogeny* between two generalized $\mathcal{D}$-elliptic sheaves $\underline{\mathcal{E}}$ and $\underline{\mathcal{E}}'$ is a morphism $f : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}'$ such that for all $\underline{i} \in \mathbb{Z}^t$

   (a) $f_{\underline{i}} : \mathcal{E}_{\underline{i}} \longrightarrow \mathcal{E}'_{\underline{i}}$ is injective,

   (b) locally on S, there is an effective divisor $\mathsf{D} \subset X$ such that $\operatorname{coker} f_i$ is supported on $\mathsf{D} \times S$,

   (c) $\operatorname{coker} f_i$ is locally free of finite rank as an $\mathcal{O}_S$-module.

2. A *quasi-isogeny* between two generalized $\mathcal{D}$-elliptic sheaves $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ and $\underline{\mathcal{E}}' = (\mathcal{E}'_{\underline{i}}, t'_{\underline{i}})$ is, locally on S, a pair (f, D) where $f : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}'$ is an isogeny for an effective divisor $\mathsf{D} \subset X$.

Example 3.4. Let $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ be a generalized $\mathcal{D}$-elliptic sheaf over S and let $a \in F^\times$ be an arbitrary element. Then, multiplication by a is a quasi-isogeny on $\underline{\mathcal{E}}$ since multiplying by a sends $\mathcal{E}_{\underline{i}}$ into $\mathcal{E}_{\underline{i}}((a)_\infty)$ where $(a)_\infty$ denotes the pole divisor of a.

Example 3.5. 1. Let $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, j_{\underline{i},\underline{i}'}, t_{\underline{i}})$ be a generalized $\mathcal{D}$-elliptic sheaf. The morphism $f : \underline{\mathcal{E}}[1] \longrightarrow \underline{\mathcal{E}}$ which given by the sequence of maps $(j_{\underline{i}-\underline{1},\underline{i}} : \underline{\mathcal{E}}[1] = \underline{\mathcal{E}}_{\underline{i}-\underline{1}} \longrightarrow \underline{\mathcal{E}}_{\underline{i}})$ is an isogeny. Indeed by condition ii, we know that $\operatorname{coker} j_{\underline{i}-\underline{1},\underline{i}}$ is locally free of finite rank over $\mathcal{O}_S$ and $\operatorname{coker} j_{\underline{i}-\underline{1},\underline{i}}$ is supported on $\mathbb{T} \times S$ for the effective divisor $\mathbb{T}$ of X.

2. Let $(\underline{\mathcal{E}}, \psi)$ be a generalized $\mathcal{D}$-elliptic sheaf such that the subscheme S' of $X \times S$ via ψ is supported on $T \times S$ then the sequence $(t_{\underline{i}})$ defines an isogeny $\sigma^*\underline{\mathcal{E}}[1] \longrightarrow \underline{\mathcal{E}}$. This follows since we know that $\operatorname{coker} t_{\underline{i}}$ is locally free of finite rank over $\mathcal{O}_S$ and $\operatorname{coker} t_{\underline{i}}$ is supported on $\mathbb{T} \times S$.

3. Let $(\mathcal{E}, \psi)$ is a generalized $\mathcal{D}$-elliptic sheaf such that the subscheme S' of $X \times S$ is supported on $p \times S$ for p in $X \smallsetminus \mathbb{T}$, then by item (ii) the $(t_{\underline{i}})$ define an isogeny $\sigma^*\mathcal{E}[1] \longrightarrow \mathcal{E}$ where the effective divisor of X in the definition 3.3 is $\{p\}$.

4 Categories of generalized $\mathcal{D}$-elliptic sheaves

In the following we will use the fact that a scheme S is identified by its functor of points h_S. Let B denote the image of $\underline{Bad}$ under the finite covering map $\pi : X \longrightarrow Y$ of degree t.

Generalized $\mathcal{D}$-elliptic sheaves relative to X/Y

Recall that the Hilbert scheme $\mathbf{Hilb}_t(U)$ for any open subset $U \subset X$ is the representable functor $Sch_{\mathbb{F}_q} \longrightarrow Sets$ defined by

$$S \mapsto \{\text{closed subschemes } S' \subset U \times_{\mathbb{F}_q} S \mid pr_2|_{S'} : S' \longrightarrow S \text{ is finite of degree } t\}.$$

Now, the functor that assigns to a generalized $\mathcal{D}$-elliptic sheaf $(\underline{\mathcal{E}}, \psi)$ to the graph of its characteristic morphism $\psi : S' \longrightarrow (X \setminus \pi^{-1}(\mathsf{B})) \times_k S$ defines a morphism

$$\begin{aligned} \mathcal{GE}\ell\ell_{X,\mathcal{D}} &\longrightarrow \mathbf{Hilb}_t(X \setminus \pi^{-1}(\mathsf{B})) \\ (\underline{\mathcal{E}}, \psi) &\longmapsto \Gamma\psi \end{aligned}$$

We have a morphism

$$can : h_{Y \setminus \mathsf{B}} \longrightarrow \mathbf{Hilb}_t(X \setminus \pi^{-1}(\mathsf{B}))$$

defined by $(T \longrightarrow Y \setminus \mathsf{B}) \mapsto \Big((X \setminus \pi^{-1}(\mathsf{B})) \times_{Y \setminus \mathsf{B}} T \xrightarrow{f} (X \setminus \pi^{-1}(\mathsf{B})) \times_{\mathbb{F}_q} T\Big)$ for any $Y \setminus \mathsf{B}$-scheme T. Note that $can(\infty) = \{\infty_1, \cdots, \infty_t\}$. For simplicity assume $X \setminus \pi^{-1}(\mathsf{B}) = \operatorname{Spec} M$, $Y \setminus \mathsf{B} = \operatorname{Spec} R$ and $T = \operatorname{Spec} S$. Then, $f : M \otimes_{\mathbb{F}_q} S \longrightarrow M \otimes_R S$ is defined by $\alpha \otimes_{\mathbb{F}_q} \beta \mapsto \alpha \otimes_R \beta$. Since R is a $\mathbb{F}_q$-module, f is surjective, i.e, $f : X \setminus \pi^{-1}(\mathsf{B}) \times_{Y \setminus \mathsf{B}} T \longrightarrow X \setminus \pi^{-1}(\mathsf{B}) \times_{\mathbb{F}_q} T$ is a closed immersion. Moreover

$$pr_2 \circ f : X \setminus \pi^{-1}(\mathsf{B}) \times_{Y \setminus \mathsf{B}} T \longrightarrow X \setminus \pi^{-1}(\mathsf{B}) \times_{\mathbb{F}_q} T \longrightarrow T$$

is of degree t. Indeed, since M is an S-module of degree t, we have:

$$M \otimes_R S \simeq R^t \otimes_R S \simeq S^t$$

which is of rank t over S.

Using these natural morphisms we will define a new stack:

Definition 4.1. The algebraic stack $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D}}$ is defined as the pullback

$$\begin{tikzcd} & \mathcal{GE}\ell\ell_{X,\mathcal{D}} \arrow[d] \\ h_{Y \setminus \mathsf{B}} \arrow[r, "can"'] & \mathbf{Hilb}_t(X \setminus \pi^{-1}(\mathsf{B})) \end{tikzcd}$$

It is a stack over $Y \smallsetminus \mathsf{B}$. The objects of $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D}}(T)$ for a $Y \smallsetminus \mathsf{B}$-scheme T is called *generalized $\mathcal{D}$-elliptic sheaf relative to X/Y*. More precisely, for a given $Y \smallsetminus \mathsf{B}$-scheme $\xi : T \longrightarrow Y$, such a generalized $\mathcal{D}$-elliptic sheaf is given by

$$(\underline{\mathcal{E}}, can(\xi) : (X \smallsetminus \pi^{-1}(\mathsf{B})) \times_{Y \smallsetminus \mathsf{B}} T \longrightarrow (X \smallsetminus \pi^{-1}(\mathsf{B})) \times_{\mathbb{F}_q} T)$$

Remark 4.2. Note that this puts a condition on the characteristic of a generalized $\mathcal{D}$-elliptic sheaf, namely we want ψ to be the map $(X \smallsetminus \pi^{-1}(\mathsf{B})) \times_{Y \smallsetminus \mathsf{B}} T \longrightarrow (X \smallsetminus \pi^{-1}(\mathsf{B})) \times_{\mathbb{F}_q} T$ where ψ is the characteristic of generalized $\mathcal{D}$-elliptic sheaf $(\underline{\mathcal{E}}, \psi)$.

Generalized $\mathcal{D}$-elliptic sheaves relative to X/Y with level structure

By Proposition 2.19, we know that the category of generalized $\mathcal{D}$-elliptic sheaves with level I-structure forms a stack where I is a closed subscheme of $X \smallsetminus Bad$ that is disjoint from $\mathbb{T} \cup pr_2(\Gamma\psi)$.

Let $J = \pi(I)$ and assume that $J \cup \mathsf{B} = \emptyset$. We will define the relative objects for schemes over $Y \smallsetminus (J \cup \mathsf{B})$.

As before, we have two morphism:

$$\begin{aligned} \mathcal{GE}\ell\ell_{X,\mathcal{D},I} &\longrightarrow \mathbf{Hilb}_t(X \smallsetminus \big(\pi^{-1}(\mathsf{B}) \cup \mathrm{supp}(I)\big)) \\ (\underline{\mathcal{E}}, \psi, \iota) &\longmapsto \psi \end{aligned}$$

where $\iota : \mathcal{D}_I \boxtimes \mathcal{O}_S \xrightarrow{\sim} \mathcal{E}_{\underline{j}}|_{I \times_{\mathbb{F}_q} S}$ is the level I-structure and

$$\begin{aligned} Y \smallsetminus (J \cup \mathsf{B}) &\longrightarrow \mathbf{Hilb}_t\big(X \smallsetminus (\pi^{-1}(\mathsf{B}) \cup \mathrm{supp}(I))\big) \\ (T \longrightarrow Y \smallsetminus (J \cup \mathsf{B})) &\mapsto ((X \smallsetminus (\underline{Bad} \cup \mathrm{supp}(I))) \times_{Y \smallsetminus (J \cup \mathsf{B})} T \longrightarrow (X \smallsetminus (\underline{Bad} \cup \mathrm{supp}(I))) \times_{\mathbb{F}_q} T) \end{aligned}$$

Definition 4.3. The algebraic stack $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$ is defined as the pullback

$$\begin{tikzcd} & \mathcal{GE}\ell\ell_{X,\mathcal{D},I} \arrow[d] \\ Y \smallsetminus (J \cup \mathsf{B}) \arrow[r, "can"'] & \mathbf{Hilb}_t(X \smallsetminus (\pi^{-1}(\mathsf{B}) \cup \mathrm{supp}(I))) \end{tikzcd}$$

It is a stack over $Y \smallsetminus (J \cup \mathsf{B})$. The objects of $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}(T)$ for a $Y \smallsetminus (J \cup \mathsf{B})$-scheme T are called *generalized $\mathcal{D}$-elliptic sheaves relative to X/Y with level I-structure.* More precisely, let $\zeta : T \longrightarrow Y \smallsetminus (J \cup \mathsf{B})$ be a $Y \smallsetminus (J \cup \mathsf{B})$-scheme. Then, the generalized $\mathcal{D}$-elliptic sheaf is given by

$$\Big(\underline{\mathcal{E}}, can(\zeta) : (X \smallsetminus (\pi^{-1}(\mathsf{B}) \cup \mathrm{supp}(I))) \times_{Y \smallsetminus (J \cup \mathsf{B})} T \longrightarrow (X \smallsetminus (\pi^{-1}(\mathsf{B}) \cup \mathrm{supp}(I))) \times_{\mathbb{F}_q} T, \iota\Big)$$

Remark 4.4. 1. In the next section we will prove that it is a Deligne-Mumford stack and moreover if I is nontrivial it is a disjoint union of quasi-projective schemes.

2. By considering relative objects, we put a restriction on the characteristic of the generalized $\mathcal{D}$-elliptic sheaves with level I-structures.

5 The Deligne-Mumford stack of generalized $\mathcal{D}$-elliptic sheaves

In this section, we will prove that the stack of generalized $\mathcal{D}$-elliptic sheaves relative to X/Y with a level I-structure is a smooth Deligne-Mumford stack. Moreover, for nontrivial I, it will be represented by a scheme. Throughout the next two sections, we will write only generalized $\mathcal{D}$-elliptic sheaves instead of generalized $\mathcal{D}$-elliptic sheaves relative to X/Y.

Definition 5.1. For any $n \in \mathbb{Z}$ define the substack $\mathcal{GE}\ell\ell^n_{X/Y,\mathcal{D},I}$ of $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$ consisting of generalized $\mathcal{D}$-elliptic sheaves $(\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ with fixed degree $\deg \mathcal{E}_{\underline{0}} = n$.

It follows from the definition that, we have

$$\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I} = \coprod_n \mathcal{GE}\ell\ell^n_{X/Y,\mathcal{D},I} \tag{2}$$

Theorem 5.2. *The stack $\mathcal{GE}\ell\ell^n_{X/Y,\mathcal{D},I}$ is of finite type which is smooth over $Y \smallsetminus (J \cup \mathsf{B} \cup \infty)$. Moreover if $I \neq \emptyset$, it is actually a quasi-projective scheme.*

This theorem together with the decomposition in (2) will give us the following theorem:

Theorem 5.3. *The stack $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$ is an algebraic stack in the sense of Deligne-Mumford [11] which is smooth over $Y \smallsetminus (J \cup \mathsf{B} \cup \infty)$. Moreover, if $I \neq \emptyset$, it is a scheme that is a disjoint union of quasi-projective schemes.*

To prove the Theorem 5.2, we will follow the steps of [35], Section 4. The smoothness follows from the following lemma:

Lemma 5.4. *([33], Lecture 2)*
Consider the diagram of stacks over $\mathbb{F}_q$

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d, "(\mathrm{Frob}_{\mathcal{U}}, id)"] \\ \mathcal{V} \arrow[r, "(\alpha,\beta)"] \arrow[d, "\pi"] & \mathcal{U} \times \mathcal{U} \\ Y & \end{tikzcd}$$

where Y is a scheme, $\mathcal{U}$ is algebraic and locally of finite type over $\mathbb{F}_q$, $\mathcal{V}$ is algebraic and locally of finite type over Y, the morphism $(\pi, \alpha) : \mathcal{V} \longrightarrow Y \times \mathcal{U}$ is representable and the square is 2-cartesian.

Then, $\mathcal{W}$ is algebraic and locally of finite type over Y and the diagonal morphism $\mathcal{W} \longrightarrow \mathcal{W} \times \mathcal{W}$ (which is representable, separated and of finite type) is everywhere unramified and therefore quasi-finite.

Moreover, if we assume that $\mathcal{U}$ is smooth over $\mathbb{F}_q$ and that $(\pi, \alpha) : \mathcal{V} \longrightarrow Y \times \mathcal{U}$ is smooth of pure relative dimension n, the algebraic stack $\mathcal{W}$ is smooth of pure relative dimension n over Y.

To prove theorem 5.2, we need to introduce new stacks.

Definition 5.5. 1. Let $\mathcal{E}$ be a locally free sheaf on $X \times S$ of rank r with a *level I-structure*, i.e, with an isomorphism $\eta : \mathcal{O}^r_{I\times S} \xrightarrow{\sim} \mathcal{E}|_{I\times S}$.

2. We say $(\mathcal{E}, \eta)$ as in (1) is *I-stable* if for any geometric point $s \longrightarrow S$ of S and for any locally free sheaves $\mathcal{F}$ on $X \times S$ which is properly contained in $\mathcal{E}_s$, we have

$$\frac{\deg \mathcal{F} - \deg I}{\operatorname{rank}(\mathcal{F})} < \frac{\deg(\mathcal{E}_s) - \deg I}{\operatorname{rank}(\mathcal{E}_s)}$$

(cf [45], 4.I, Définition 2)

Definition 5.6. (i) We denote by $\mathcal{GE}\ell\ell^{st}_{X/Y,\mathcal{D},I}$ the open substack of the stack of generalized $\mathcal{D}$-elliptic sheaves consisting of $(\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ such that $\mathcal{E}_{\underline{0}}$ is stable as a vector bundle.

(ii) Define the substack $\mathcal{GE}\ell\ell^{st,n}_{X/Y,\mathcal{D},I}$ of $\mathcal{GE}\ell\ell^{st}_{X/Y,\mathcal{D},I}$ via the pullback diagram

$$\begin{array}{ccc} & & \mathcal{GE}\ell\ell^{st}_{X/Y,\mathcal{D},I} \\ & & \big\downarrow \\ \mathcal{GE}\ell\ell^{n}_{X/Y,\mathcal{D},I} & \longrightarrow & \mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I} \end{array}$$

Theorem 5.7. *The stacks $\mathcal{GE}\ell\ell^{st,n}_{X/Y,\mathcal{D},I}$ are representable by quasi projective schemes when $I \neq \emptyset$. In particular, they are of finite type.*

To prove this theorem we will define more stacks.

Definition 5.8. 1. Let $Vec_{X,I}$ (and $Vec^n_{X,I}$) denote the stack over $Sch_{\mathbb{F}_q}$ of vector bundles over X (i.e. over $X \times S$ for any $\mathbb{F}_q$-scheme S) of rank d^2 with level I-structure (and degree n).

2. Denote the substack $Vec^{*,st}_{X,I}$ of $Vec^{*}_{X,I}$ where $* \in \{\emptyset, n\}$ classifying the I-stable vector bundles (of degree n if $* = n$) in the sense of Definition 5.5

Definition 5.9. 1. Let $Vec_{X,\mathcal{D},I}$ denote the stack of vector bundles over X with level I-structure and $\mathcal{D}$-action. It carries a natural morphism $\pi : Vec_{X,\mathcal{D},I} \longrightarrow Vec_{X,I}$.

2. Let $Vec^{*}_{X,\mathcal{D},I}$ denote the inverse image of $Vec^{*}_{X,I}$ where $*$ is n, st or $\{n, st\}$ under π.

Proposition 5.10. ***(Seshadri), ([31], Theorem 1.4.1)***
The open substack of I-stable vector bundles with fixed rank r and degree d

$$Vect^{r,d,st}_{X,I} \subset Vect^{r}_{X,I}$$

is a smooth quasiprojective scheme with dimension $r^2(g-1+\deg I)$. In particular, it is of finite type.

Lemma 5.11. *The morphism*

$$Vec_{X,\mathcal{D},I} \longrightarrow Vec_{X,I}$$

is relatively representable and affine.

Proof. [35], Lemma 4.4. □

Remark 5.12. The stack $Vec^{st}_{X,\mathcal{D},I}$ is representable by a disjoint union of quasi-projective schemes if $\deg I > 0$ and $Vec_{X,\mathcal{D},I}$ is smooth over $\mathbb{F}_q$([35], Lemma 4.5).

Definition 5.13. 1. Consider the sequence

$$\cdots \hookrightarrow \mathcal{E}_{\underline{i}} \overset{j_{\underline{i},\underline{s}}}{\hookrightarrow} \mathcal{E}_{\underline{s}} \overset{j_{\underline{s},\underline{k}}}{\hookrightarrow} \mathcal{E}_{\underline{k}} \hookrightarrow \cdots$$

for $\underline{i} \leq \underline{s} \leq \underline{k}$ where $\mathcal{E}_{\underline{i}}$ is a locally free sheaf over $X \times S$ with $\mathcal{D}$-action and the morphisms $j_{\underline{i},\underline{s}}$ which are compatible with the $\mathcal{D}$-action satisfy the conditions (i) and (ii) in the Definition 2.2.

2. Let $I \subset X \smallsetminus \underline{Bad}$ be a finite closed subscheme different than $\mathbb{T} \cup \psi(S')$. Then the restrictions $\mathcal{E}_{\underline{i}}|_{I \times \mathbb{T}}$ are all isomorphism via the morphisms $j_{\underline{i},\underline{i}'}$. So, we will write $\mathcal{E}|_{I\times S}$ for the restriction $\mathcal{E}_{\underline{i}}|_{I\times S}$ A level I-structure for such a sequence is collection of level I-structures $\iota_{\underline{i}} : \mathcal{D}_I \boxtimes \mathcal{O}_S \longrightarrow \mathcal{E}|_{I\times S}$ that are compatible with $j_{\underline{i}}$.

3. Let $t\text{-}Seq_{X,\mathcal{D},I}$ denote the stack classifying the sequences as above with level I-structure. And let $t\text{-}Seq^{st}_{X,\mathcal{D},I}$ denote the open substack of $t\text{-}Seq_{X,\mathcal{D},I}$ such that $\mathcal{E}_{\underline{0}}$ is stable with level I-structure.

Proposition 5.14. *The morphism $t\text{-}Seq_{X,\mathcal{D},I} \longrightarrow Vec_{X,\mathcal{D},I}$ sending an element*

$$\cdots \hookrightarrow \mathcal{E}_{\underline{i}} \overset{j_{\underline{i},\underline{s}}}{\hookrightarrow} \mathcal{E}_{\underline{s}} \overset{j_{\underline{s},\underline{k}}}{\hookrightarrow} \cdots$$

to $\mathcal{E}_{\underline{0}}$ is relatively representable by a product of flag varieties, in particular it is smooth.

Proof. Let $\mathcal{E}_{\underline{0}}$ on $X \times S$ be given. Then, the chain $(\mathcal{E}_{\underline{i}})_{i\in\mathbb{Z}^t}$ corresponds to a flag subsheaves via periodicity

$$\{0\} \subset \tilde{\mathcal{E}}_{\underline{1}} \subset \cdots \subset \tilde{\mathcal{E}}_{\underline{d}\deg\infty-\underline{1}} \subset \mathcal{E}_{\underline{0}}(\infty_1, \cdots, \infty_t)/\mathcal{E}_{\underline{0}}$$

such that the successive quotients are locally free over $\mathcal{O}_S$ and which are stable with respect to the $\mathcal{D}$-action, i.e under the action of $\mathcal{D} \otimes_{\mathcal{O}_X} \prod_{i=1}^t \kappa_{\infty_i} = \prod \mathcal{D} \otimes_{\mathcal{O}_X} \kappa(\infty_i)$. Now the resulting flag for each i is exactly the type of flag considered in the proof of [35], Lemma 4.6. Morevoer by [35], Lemma 4.6, for each i this flag is relatively representable by a flag variety. Hence, the morphism $t\text{-}Seq_{X,\mathcal{D},I} \longrightarrow Vec_{X,\mathcal{D},I}$ is relatively representable by the t-fold product of the flag varieties from [35]. □

Remark 5.15. Note that our definitions of objects are slightly different than [35]. Our indices are elements of $\mathbb{Z}^t$ eventhough indices in [35] are elements of $\mathbb{Z}$. We will indicate that in the future also by putting t before objects, e.g, we don't have $Seq_{X,\mathcal{D},I}$ as in [35] but $t\text{-}Seq_{X,\mathcal{D},I}$.

Definition 5.16. Let S be an $\mathbb{F}_q$-scheme and $\psi : S' \longrightarrow (X \smallsetminus \underline{Bad}) \times_k S$ such that $pr_2 \circ \psi : S' \longrightarrow S$ is finite locally free. Let I be a closed subscheme of X such that $I \cap \psi(S') = \emptyset$. We define $t\text{-}Hecke_{X,\mathcal{D},I}$ to be the stack classifying pairs consisting of a map ψ as above and a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, hook] & \mathcal{E}_{\underline{i}} \arrow[r, hook, "j_{\underline{i},\underline{s}}"] & \mathcal{E}_{\underline{s}} \arrow[r, hook, "j_{\underline{s},\underline{k}}"] & \mathcal{E}_{\underline{k}} \arrow[r, hook] & \cdots \\
\cdots \arrow[r, hook] & \mathcal{E}'_{\underline{i}-\underline{1}} \arrow[u, "t_{\underline{i}-\underline{1}}"] \arrow[r, hook, "j'_{\underline{i},\underline{s}}"] & \mathcal{E}'_{\underline{s}-\underline{1}} \arrow[u, "t_{\underline{s}-\underline{1}}"] \arrow[r, hook, "j'_{\underline{s},\underline{k}}"] & \mathcal{E}'_{\underline{k}-\underline{1}} \arrow[u, "t_{\underline{k}-\underline{1}}"] \arrow[r, hook] & \cdots
\end{tikzcd}$$

where the first and second row are elements of $t\text{-}Seq_{X,\mathcal{D},I}$ and the morphisms $t_{\underline{i}}$ satisfy the condition (iii) in the Definition 2.2.

Mapping a pair (ψ, commutative diagram) as above to ψ defines a morphism of stacks

$$\pi : t\text{-}Hecke_{X,\mathcal{D},I} \longrightarrow \mathbf{Hilb}_t(X \smallsetminus \pi^{-1}(\mathsf{B}))$$

Since $\operatorname{coker} t_{\underline{i}}$ are supported on the graph $\Gamma\psi$ of the morphism $\psi : S' \longrightarrow X \times S$, we can define a morphism of stacks $t\text{-}\pi : Hecke_{X,\mathcal{D},I} \longrightarrow \mathbf{Hilb}_t(X \smallsetminus \pi^{-1}(\mathsf{B}))$ by assigning a diagram to S'.

The stacks defined above form a 2-cartesian diagram:

$$
\begin{tikzcd}
\mathcal{GE}\ell\ell_{X,\mathcal{D},I} \arrow[r] \arrow[d] & t\text{-}Seq_{X,\mathcal{D},I} \arrow[d, "(\mathrm{Frob},id)"] \\
t\text{-}Hecke_{X,\mathcal{D},I} \arrow[r, "(\alpha{,}\beta)"] \arrow[d, "\pi"] & t\text{-}Seq_{X,\mathcal{D},I} \times t\text{-}Seq_{X,\mathcal{D},I} \\
\mathbf{Hilb}_t(X \smallsetminus \pi^{-1}(\mathsf{B})) &
\end{tikzcd}
\tag{3}
$$

where (α, β) is given by (1st row, 2nd row) of a diagram in $t\text{-}Hecke_{X,\mathcal{D},I}$.

Definition 5.17. Recall the morphism $Y \smallsetminus \mathsf{B} \longrightarrow \mathbf{Hilb}_t(X \smallsetminus \pi^{-1}(\mathsf{B}))$ defined in Section 4. We define $t\text{-}Hecke_{X/Y,\mathcal{D},I}$ via the pullback diagram

$$
\begin{tikzcd}
 & t\text{-}Hecke_{X,\mathcal{D},I} \arrow[d] \\
Y \smallsetminus \mathsf{B} \arrow[r] & \mathbf{Hilb}_t(X \smallsetminus \pi^{-1}(\mathsf{B}))
\end{tikzcd}
$$

Similarly as diagram 3, we have the following:

$$
\begin{tikzcd}
\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I} \arrow[r] \arrow[d] & \mathcal{GE}\ell\ell_{X,\mathcal{D},I} \arrow[r] \arrow[d] & t\text{-}Seq_{X,\mathcal{D},I} \arrow[d, "(\mathrm{Frob},id)"] \\
t\text{-}Hecke_{X/Y,\mathcal{D},I} \arrow[r] \arrow[d, "\pi"] & t\text{-}Hecke_{X,\mathcal{D},I} \arrow[r, "(\alpha{,}\beta)"] \arrow[d] & t\text{-}Seq_{X,\mathcal{D},I} \times t\text{-}Seq_{X,\mathcal{D},I} \\
Y \smallsetminus \mathsf{B} \arrow[r] & \mathbf{Hilb}_t(X \smallsetminus \pi^{-1}(\mathsf{B})) &
\end{tikzcd}
$$

To proceed we need to define another stack as in [23], Section 3. The Hilbert scheme $\mathbf{Hilb}_t(X/\mathbb{F}_q)$ parametrizes the closed subschemes of X. The *Quot scheme* is defined as the functor that parametrizes the quotients of locally free sheaves on X. For more details on Quot schemes one can see [51].

Definition 5.18. Let X be a scheme of finite type over S. Let $\mathcal{E}$ be a locally free sheaf over X. Define the functor $\mathcal{Q}uot_{\mathcal{E}/X/S}$ from the category of S-schemes by sending an S-scheme T to equivalence class $(\mathcal{F}, q)/\sim$ where $\mathcal{F}$ is a coherent sheaf over X_T which is flat over T and $q : \mathcal{E}_T \longrightarrow \mathcal{F}$ is an epimorphism together with an equivalence relation defined as follows:

$$(\mathcal{F}, q) \sim (\mathcal{F}', q') :\iff \ker(q) = \ker(q')$$

Let the base stack be $\mathcal{T} := \mathbf{Hilb}_t(X \smallsetminus \pi^{-1}(\mathsf{B})) \times t\text{-}Seq_{X,\mathcal{D},I}$

Over $\mathcal{T}$ we have a universal vector bundle $\mathcal{E}_{\underline{0}}$ of rank d^2 over $X \times \mathcal{T}$ coming from the $\underline{0}$-th term of $t\text{-}Seq_{X,\mathcal{D},I}$. The datum in $\mathbf{Hilb}_t(X \smallsetminus \pi^{-1}(\mathsf{B}))$ over $\mathcal{T}$ defines a closed substack $\mathcal{T}'$ of $X \times \mathcal{T}$, say given a by a closed immersion Ψ. By the defining property of $\mathbf{Hilb}_t(X \smallsetminus \pi^{-1}(\mathsf{B}))$, the induced morphism $\mathcal{T}' \to \mathcal{T}$ is finite locally free of degree t.

Pulling back $\mathcal{E}_{\underline{0}}$ along $\mathcal{T}' \longrightarrow X \times \mathcal{T}$ gives us a vector bundle $\mathcal{F}_{\underline{0}}$ over $\mathcal{T}'$ of rank d. The bundle $\mathcal{F}_{\underline{0}}$ is finite flat over $\mathcal{T}$ of degree d^2t. Over $\mathcal{T}'$ the bundle $\mathcal{F}_{\underline{0}}$ carries a D-action.

We will consider the Quot scheme $\mathcal{Q}uot^{dt}_{\mathcal{F}_{\underline{0}}/\mathcal{T}'/\mathcal{T}}$. The quotients of $\mathcal{F}_{\underline{0}}$ are supported on $\mathcal{T}'$. Those where are locally free over $\mathcal{T}$ of rank dt are classified by $\mathcal{Q}uot^{dt}\,\mathcal{F}_{\underline{0}}/\mathcal{T}'/\mathcal{T}$. For a diagram in $t\text{-}Hecke_{X,\mathcal{D},I}$ over $\mathcal{T}$, the $\operatorname{coker} t_{-\underline{1}}$ over $\mathcal{T}$ defines an element in $\mathcal{Q}uot^{dt}\,\mathcal{F}_{\underline{0}}/\mathcal{T}'/\mathcal{T}$. It also carries a $\mathcal{D}$-action not parameterized by Quot-scheme. We shall deal with this in Proposition 5.19.

Now, $\mathcal{Q}uot_{\mathcal{F}_{\underline{0}}/\mathcal{T}'/\mathcal{T}}$ is a stack over $\mathcal{T}$. We know that $t\text{-}Seq_{X,\mathcal{D},I}$ is a stack over $\mathbb{F}_q$. Then we have the following pullback diagram

$$\begin{tikzcd} & \mathcal{T} \arrow[d] \\ t\text{-}Seq_{X,\mathcal{D},I} \arrow[r] & \mathbb{F}_q \end{tikzcd}$$

We will denote the pullback object $t\text{-}Seq_{X,\mathcal{D},I} \times \mathcal{T}$ by $t\text{-}Seq_{X,\mathcal{D},I/\mathcal{T}}$ and regard it as a stack over $\mathcal{T}$. Let $Vec_{\mathcal{T}}$ denote the stack of vector bundles over $\mathcal{T}$. By the morphism in Lemma 5.14, we have a map $t\text{-}Seq_{X,\mathcal{D},I/\mathcal{T}} \longrightarrow Vec_{\mathcal{T}}$. So we can form the fiber product over $Vec_{\mathcal{T}}$

$$\begin{tikzcd} & t\text{-}Seq_{X,\mathcal{D},I/\mathcal{T}} \arrow[d] \\ \mathcal{Q}uot^{dt}_{\mathcal{F}_{\underline{0}}/\mathcal{T}'/\mathcal{T}} \arrow[r] & Vec_{\mathcal{T}} \end{tikzcd}$$

which is a stack over $\mathcal{T}$.

Proposition 5.19. *(cf. [23], Lemma 3.8.) The morphism*

$$t\text{-}Hecke_{X,\mathcal{D},I} \longrightarrow \mathcal{Q}uot^{dt}_{\mathcal{F}_{\underline{0}}/\mathcal{T}'/\mathcal{T}} \times_{Vec_{\mathcal{T}}} t\text{-}Seq_{X,\mathcal{D},I/\mathcal{T}}$$

defined by assigning a diagram in $t\text{-}Hecke_{X,\mathcal{D},I}$ to $(\operatorname{coker} t_{-1}, 2^{nd}$ row$)$ is represented by a closed immersion.

Proof. Note that first line of $t\text{-}Hecke_{X,\mathcal{D},I}$ is already in the stack $\mathcal{T}$ and so on the stack $\mathcal{Q}uot^{dt}_{\mathcal{F}_{\underline{0}}/\mathcal{T}'/\mathcal{T}} \times_{Vec_{\mathcal{T}}} t\text{-}Seq_{X,\mathcal{D},I/\mathcal{T}}$ one has both lines and the universal quotient. So, to get a diagram in $t\text{-}Hecke_{X,\mathcal{D},I}$ we need to define the morphism $t_{\underline{i}}$ and check that the necessary conditions are satisfied. One can define $t\text{-}Hecke_{X,\mathcal{D},I}$ by the following conditions:

1. $\mathcal{E}'_{-\underline{1}}$ equals the kernel of the morphism from $\mathcal{E}_{\underline{0}}$ to the universal quotient,

2. for each $-\ell \le \underline{i} \le -\underline{2}$ the sheaf $\mathcal{E}'_{\underline{i}}$ is contained in the intersection of $\mathcal{E}_{\underline{i+1}}$ and $\mathcal{E}'_{-\underline{1}}$ which we view as subsheaves of $\mathcal{E}_{\underline{0}}$ via $j_{\underline{i+1},\underline{0}}$ and $t_{-\underline{1}}$

3. if we let $t_{\underline{i}}$ be the inclusion $\mathcal{E}_{\underline{i'}} \subset \mathcal{E}_{\underline{i+1}}$ then $\operatorname{coker} t_{\underline{i}}$ is supported on $Im\Psi$ for $\psi : S' \longrightarrow X \smallsetminus \pi^{-1}(\mathsf{B}) \times S$, and it is locally free of rank d over S'

4. $t_{-\underline{1}}$ is induces an isomorphism of level I-structures on $\mathcal{E}'_{-\underline{1}}$ and on $\mathcal{E}_{\underline{0}}$

5. $t_{-\underline{1}}$ is $\mathcal{D}$-equivariant.

Let us define $t_{-1} : \mathcal{E}'_{-\underline{1}} \longrightarrow \mathcal{E}_{\underline{0}}$ as the isomorphism of $\mathcal{E}'_{-\underline{1}}$ with the kernel of the universal quotient map. So, for $\underline{j}, \underline{s} \ge 1$, we have

$$\begin{tikzcd}
\cdots \arrow[r, hook] & \mathcal{E}_{-\underline{i}} \arrow[r, hook] & \mathcal{E}_{\underline{0}} \arrow[r, hook] & \mathcal{E}_{\underline{s}} \arrow[r, hook] & \mathcal{E}_{\underline{k}} \arrow[r, hook] & \cdots \\
\cdots \arrow[r, hook] & \mathcal{E}'_{-\underline{i}-\underline{1}} \arrow[r, hook] & \mathcal{E}'_{-\underline{1}} \arrow[u, "t_{-\underline{1}}"] \arrow[r, hook] & \mathcal{E}'_{\underline{s}-\underline{1}} \arrow[r, hook] & \mathcal{E}'_{\underline{k}-\underline{1}} \arrow[r, hook] & \cdots
\end{tikzcd}$$

For simplicity assume $\underline{i} = -\underline{2}$. Condition (ii) tells us that if $\operatorname{coker}(j_{-\underline{1},\underline{0}}) \circ t_{-\underline{1}} \circ j'_{-\underline{2},-\underline{1}} = 0$ then we can define $t_{-2} : \mathcal{E}'_{-\underline{2}} \longrightarrow \mathcal{E}_{-\underline{1}}$. More precisely, assume $\operatorname{coker}(j_{-\underline{1},\underline{0}}) \circ t_{-\underline{1}} \circ j'_{-\underline{2},-\underline{1}} = 0$. We have $(j'_{-\underline{2},-\underline{1}} \circ j_{-\underline{1},\underline{0}})(c) \in Im(j_{-\underline{1},\underline{0}})$, and hence we can define a map $\mathcal{E}'_{-\underline{2}} \longrightarrow \mathcal{E}_{-\underline{1}}$. Define $t_{-\underline{2}}$ to be this map. Then, the following diagram

$$\begin{tikzcd}
\mathcal{E}_{-\underline{1}} \arrow[r, hook, "j_{-\underline{1},\underline{0}}"] & \mathcal{E}_{\underline{0}} \\
\mathcal{E}'_{-\underline{2}} \arrow[u, hook, "t_{-\underline{2}}"] \arrow[r, hook, "j'_{-\underline{2},-\underline{1}}"] & \mathcal{E}'_{-\underline{1}} \arrow[u, hook, "t_{-\underline{1}}"']
\end{tikzcd}$$

is commutative. Consider the following short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker} t_{-\underline{2}} \arrow[r] & \operatorname{coker}(j_{-\underline{1},\underline{0}} \circ t_{-\underline{2}}) \arrow[r] \arrow[d, equal] & \operatorname{coker} j_{-\underline{1},\underline{0}} \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{coker} j'_{-\underline{2},-\underline{1}} \arrow[r] & \operatorname{coker}(t_{-\underline{1}} \circ j'_{-\underline{2},-\underline{1}}) \arrow[r] & \operatorname{coker} t_{-\underline{1}} \arrow[r] & 0
\end{tikzcd}$$

We know that $\big(\operatorname{coker}(j_{\underline{i},\underline{i'}})\big)|_{\infty_j \times S}$ is locally free of rank d over $\mathcal{O}_S$. Then, via diagram we see that $\operatorname{coker} t_{-\underline{2}}$ is locally free of rank dt over $\mathcal{O}_S$.

Therefore t-$Hecke_{X,\mathcal{D},I}$ is a substack of $\mathcal{Q}uot^{dt}_{\mathcal{F}_{\underline{0}}/\mathcal{T}'/\mathcal{T}} \times_{Vec_{\mathcal{T}}} t\text{-}Seq_{X,\mathcal{D},I/\mathcal{T}}$. And since each condition can be expressed in terms of algebraic relations, t-$Hecke_{X,\mathcal{D},I}$ is a closed substack.

□

Theorem 5.20. *([35], Theorem 5.2)*

There exists a constant $c > 0$ with the following property: Let I be any closed subscheme of $X \smallsetminus Bad$ of degree $> c$. Then for any generalized $\mathcal{D}$-elliptic sheaf with level I-structure $(\underline{\mathcal{E}}, \psi, \iota)$ over $\operatorname{Spec} L$, where L is an algebraically closed field, the vector bundle $\mathcal{E}_{\underline{0}}$ is I-stable.

The proof goes similarly as in [35], Theorem 5.2. Below, we will give the idea. The proof uses Harder-Narasimhan filtration of a vector bundle and some related properties. A Harder-Narasimhan filtration of a vector bundle used to determine the unstability-stability of a vector bundle in general. For more details, please see [35], Section 5.

Let $\mathcal{E}$ be a vector bundle over X. The *slope* of $\mathcal{E}$ is defined as

$$\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\operatorname{rank} \mathcal{E}}.$$

Let $\mathcal{F}$ be a subbundle of $\mathcal{E}$. Define

$$jump_{\mathcal{E}} = \mu_{min}(\mathcal{F}) - \mu_{max}(\mathcal{E}/\mathcal{F})$$

where μ_{min} is the minimal slope of the nontrivial quotients of $\mathcal{F}$ and μ_{max} is the maximal slope of the nonzero vector subbundles of $\mathcal{E}/\mathcal{F}$. We will denote by $\mathcal{E}^{(i)}$ the i-th term of the Harder-Narasimhan filtration of $\mathcal{E}$.

Proof. Let $\alpha_1, \cdots, \alpha_r \in \Gamma(X \smallsetminus \mathbb{T}, \mathcal{D})$ be generators of $\Gamma(X \smallsetminus T, \mathcal{D})$ as a $\Gamma(X \smallsetminus T, \mathcal{O}_X)$. Since the orders of the poles are bounded we find a constant n with $\alpha_j \in \Gamma(X, \mathcal{D}(n \cdot \infty))$ for $j = 1, \cdots, r$.

Now let $(\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ be a generalized $\mathcal{D}$-elliptic sheaf over $SpecL$. Then, via periodicity, we have $\alpha_j \cdot \mathcal{E}_{\underline{i}} \subset \mathcal{E}_{\underline{i}+\underline{n}d\cdot\deg\infty}$. Then, for each $\underline{i} \in \mathbb{Z}$ one gets an upper bound $jump_{\mathcal{E}_{\underline{i}}} \leq P(n, d, \deg\infty)$ for every non-zero proper subbundle $\mathcal{F} \subset \mathcal{E}_{\underline{i}}$ (cf [35], 5.3). Here $P(t, d, \deg\infty)$ stands for an expression in t, d and $\deg\infty$.

As the constant in the theorem, take $c := (d^2-1)^2 \cdot d^2 \cdot P(n, d, \deg\infty)$. By definition of the slope and I-stability of a vector bundle and Harder-Narasimhan filtration it is enough to show that

$$\mu(\mathcal{E}_{\underline{0}}^{(1)}) - \mu(\mathcal{E}_{\underline{0}}) \leq (d^2-1)P(n, d, \deg\infty)$$

where $\mathcal{E}_{\underline{0}}^{(1)}$ denotes the first nonzero element in the Harder-Narasimhan filtration of $\mathcal{E}_{\underline{0}}$. We have

$$\begin{aligned}
\mu(\mathcal{E}_{\underline{0}}^{(1)}) - \mu(\mathcal{E}_{\underline{0}}) &\leq \mu(\mathcal{E}_{\underline{0}}^{(1)}) - \mu(\mathcal{E}_{\underline{0}}/\mathcal{E}_0^{r-1}) \\
&= \sum_{j=1}^{r-1} (\mu(\mathcal{E}_{\underline{0}}^{(j)}/\mathcal{E}_{\underline{0}}^{(j-1)}) - \mu(\mathcal{E}_{\underline{0}}^{j+1}/\mathcal{E}_{\underline{0}}^{(j)})) \\
&= \sum_{j=1}^{r-1} jump_{\mathcal{E}_{\underline{0}}}(\mathcal{E}_{\underline{0}}^{(j)}) \\
&\leq (d^2-1)P(n, d, \deg\infty)
\end{aligned} \tag{4}$$

where the first inequality comes from the Harder-Narasimhan filtration of $\mathcal{E}_{\underline{0}}$. For the other (in)-equalities, use the fact that $r \leq d^2 = rank(\mathcal{E}_{\underline{0}})$ and the upper bound $P(n, d, \deg \infty)$ for the $jump_{\mathcal{E}_{\underline{0}}}$.

$\square$

Proof. (of Theorem 5.7)
We have morphism of stacks

$$\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I} \longrightarrow t\text{-}Hecke_{X,\mathcal{D},I} \longrightarrow \mathcal{Q}uot \times t\text{-}Seq_{X,\mathcal{D},I} \longrightarrow t\text{-}Seq_{X,\mathcal{D},I} \longrightarrow Vec_{X,\mathcal{D},I}$$

where the first two morphisms are closed immersions, the last one is representable by a product of flag varieties and the other is represented by a quasi-projective morphism.

Now, by Theorem 5.10, we know that $Vec^{n,st}_{X,\mathcal{D},I}$ is a quasi-projective scheme if $\deg I > 0$. Consider the inverse image of $Vec^{n,st}_{X,\mathcal{D},I}$ under the composition of maps above. We get the substack $\mathcal{GE}\ell\ell^{n,st}_{X/Y,\mathcal{D},I}$ of $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$(cf. [31], Proposition 1.4.6). $\square$

Now, we are ready to prove our first theorem:

Proof. (of Theorem 5.2) Let $G_{I',I} := Ker\Big(GL_1\big(H^0(I', \mathcal{D}_{I'})\big) \longrightarrow GL_1\big(H^0(I, \mathcal{D}_I)\big)\Big)$. Note that $\mathcal{GE}\ell\ell^{st,n}_{X/Y,\mathcal{D},I'}$ is stable with respect to $G_{I',I}$-action. By Remark 2.25, the quotient $\mathcal{GE}\ell\ell^{st,n}_{X/Y,\mathcal{D},I'}/G_{I',I}$ is quasi-projective scheme if $I \neq \emptyset$ and in particular of finite type.

Since any vector bundle becomes stable for some I' with degree big enough (cf. Theorem 5.20), the quotients

$$\mathcal{GE}\ell\ell^{st,n}_{X/Y,\mathcal{D},I'}/G_{I',I}$$

cover $\mathcal{GE}\ell\ell^{n}_{X/Y,\mathcal{D},I}$ as I' vary and so $\mathcal{GE}\ell\ell^{n}_{X/Y,\mathcal{D},I}$ is of finite type. Moreover, if $I \neq \emptyset$, it is a quasi-projective scheme.

$\square$

Corollary 5.21. *The stack $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$ is union of algebraic stacks of finite type.*

Proof. Since we can cover $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$ by $\bigcup_n \mathcal{GE}\ell\ell^{n}_{X/Y,\mathcal{D},I}$, the proof follows from the previous theorem. $\square$

Corollary 5.22. *The stack $\mathcal{GE}\ell\ell^{st}_{X/Y,\mathcal{D},I}$ is a disjoint union of quasi-projective schemes.*

Proof. Since one can write $\mathcal{GE}\ell\ell^{st}_{X/Y,\mathcal{D},I} = \coprod_n \mathcal{GE}\ell\ell^{st,n}_{X/Y,\mathcal{D},I}$, the proof follows from the Theorem 5.7. $\square$

Proof. (of Theorem 5.3)

Let $I' \subset X \smallsetminus \{\mathbb{T} \cup \psi(S')\}$ be a finite closed subscheme with $\deg I' > 0$ such that $I \subset I' \subset X$. By Remark 2.25, the morphism

$$r_{I',I}; \mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I'} \longrightarrow \mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$$

is a torsor over $X \smallsetminus I'$ under the finite group $G_{I',I} = \mathrm{Ker}\Big(GL_1\big(H^0(I', \mathcal{D}_{I'})\big) \longrightarrow GL_1\big(H^0(I, \mathcal{D}_I)\big)\Big)$.

Note that $r_{I',I}^{-1}(\mathcal{GE}\ell\ell^{st}_{X/Y,\mathcal{D},I}) \subset \mathcal{GE}\ell\ell^{st}_{X/Y,\mathcal{D},I'}$ and the open substack $\mathcal{GE}\ell\ell^{st}_{X/Y,\mathcal{D},I'}$ is stable under the finite group $G_{I',I}$. So, we can define the quotients

$$\mathcal{GE}\ell\ell^{st}_{X/Y,\mathcal{D},I'}/G_{I',I}$$

which are stacks in the sense of Deligne-Mumford since $G_{I',I}$ is finite étale ([20], Section 2.3). Since any vector bundle becomes stable for some I' with degree big enough (cf. Theorem 5.20), these quotients cover $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$ as I' vary. And so, $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I'}$ is itself a Deligne-Mumford stack.

□

6 Properness

The main aim of this section is to prove the following:

Theorem 6.1. *The morphism*

$$\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}/\underline{1}\mathbb{Z} \longrightarrow Y \smallsetminus \mathsf{B}$$

is proper.

Remark 6.2. We want to point out that our properness is stronger than the one in [35]. Namely, the image of our morphism can meet with the poles whereas in [35] it can't.

We will prove this by checking the valuative criteria of properness.

Theorem 6.3. *(Valuative Criterion of Properness) Let $f : X \longrightarrow Y$ be a morphism of schemes of finite type and quasi-separated. Then, f is proper iff the following condition holds: Let R be a valuation ring with quotient field K, let $T = SpecR$, $U = SpecK$ and let $i : U \longrightarrow T$ be the morphism induced by $R \subset K$. For every morphism $U \longrightarrow X$ and $T \longrightarrow Y$ forming a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

there exists a unique morphism $T \longrightarrow X$ making the whole diagram commutative.

Proof. [21], Theorem 15.9. □

Lemma 6.4. *The characteristic morphism $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}/\underline{1}\mathbb{Z} \longrightarrow Y \smallsetminus \mathsf{B}$ is of finite type and quasi-separated.*

Proof. By the action of $\mathbb{Z}^t$, we can write

$$\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I} = \coprod_{0<n<d\deg\infty} \mathcal{GE}\ell\ell^{n}_{X/Y,\mathcal{D},I}$$

where $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D}}$ is the stack of generalized $\mathcal{D}$-elliptic sheaves $(\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ with $\deg \mathcal{E}_{\underline{0}} = n$ is fixed. Then, by Theorem 5.2, the stack $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}/\underline{1}\mathbb{Z}$ is a stack of finite type. So, the characteristic morphism $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D}}/\underline{1}\mathbb{Z} \longrightarrow Y \smallsetminus \mathsf{B}$ is of finite type.

By the previous section, the stack $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$ is a Deligne-Mumford stack, which means the diagonal morphism $\Delta : \mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I} \times \mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I} \longrightarrow \mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$ is quasi-compact and separated ([20], Definition 2.20), equivalently Δ is quasi-separated (Tag 04YW). So, the morphism $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}/\underline{1}\mathbb{Z} \longrightarrow Y \smallsetminus \mathsf{B}$ is quasi-separated.

□

Before we start the proof of Theorem 6.1, we want to introduce some notations. Let $R \supset \mathbb{F}_q$ be a complete discrete valuation ring with quotient field K, perfect residue field κ and uniformizing element ϖ. Since R is a discrete valuation ring, $SpecR$ has only two points, $\operatorname{Spec} R = \{\eta, s\}$ where η is the generic point and s is the special point. Consider $X \times_{\mathbb{F}_q} \operatorname{Spec} R$. Let η' be generic point of $X \otimes_{\mathbb{F}_q} \kappa$. Note that via the composition

$$\eta' \longrightarrow X \otimes \kappa \longrightarrow X \otimes R$$

one can see η' as a point of $X \otimes R$. Denote by R' the local ring $\mathcal{O}_{X \otimes R, \eta'}$ at η'. Let K' be the fraction field of R' and κ' be the residue field of R'. Then, R' is a discrete valuation ring (cf. Lemma 00PD) with uniformizing element ϖ, quotient field K' and residue field κ' (Note that $\kappa' = Frac(F \otimes \kappa) = FF(X \otimes \kappa)$, i.e the function field of $X \otimes \kappa$ and $K' = Frac(F \otimes K) = FF(X \otimes K)$, i.e the function field of $X \otimes K$).

Theorem 6.5. *([13], Proposition 3.1) The category of locally free sheaves $\widetilde{\mathcal{F}}$ over $X \otimes R$ is equivalent to the category of pairs $(\mathcal{F}, N)$ where $\mathcal{F}$ is locally free sheaf over $X \otimes K$ and N is R'-lattice in $\mathcal{F}_{K'} = \mathcal{F}_{\eta'}$*

Proof. We will sketch the proof here. Let $\eta' = \operatorname{Spec}(K')$ be the generic point of $X \otimes K$. Now let us consider the diagram

$$\begin{tikzcd} X \otimes R & X \otimes K \arrow[l, "f"'] \\ \operatorname{Spec} R' \arrow[u, "g"] & \operatorname{Spec} K' \arrow[l, "f'"'] \arrow[u, "g'"'] \end{tikzcd}$$

Let $\widetilde{F}$ be a locally free sheaf over $X \otimes R$. Then, $\mathcal{F} := \widetilde{\mathcal{F}}|_{X \otimes K}$ and $N := H^0(Spec(R'), g^*\widetilde{\mathcal{F}})$ such that we have $\alpha : (f')^*(N) \xrightarrow{\sim} (g')^*(\mathcal{F})$.

Now, let $\mathcal{F}$ be a locally free sheaf over $X \otimes K$ and let N be a R'-lattice in $\mathcal{F}_{K'}$ such that they are isomorphic over $SpecK'$. Then, $\widetilde{\mathcal{F}} := g_*(N) \cap f_*(\mathcal{F})$.

□

Remark 6.6. For the details of the proof one can also check [18], Corollary 2.9.

Now we will start our proof. Let $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ be a generalized $\mathcal{D}$-elliptic sheaf over K. We want to apply valuative criteria, i.e, we want to extend $\underline{\mathcal{E}}$ to $X \times \operatorname{Spec} R$. Put $V := \mathcal{E}_{\underline{i}, \eta'}$ the stalk of $\mathcal{E}_{\underline{i}}$ at the generic point η' of $X \otimes K$. By definition of stalk this is a module over $\mathcal{O}_{X \times \operatorname{Spec} K, \eta'} = \mathrm{FF}(X \otimes K) = K'$ of finite rank, i.e a finite dimensional K'-vector space, so $\mathcal{E}_{\underline{i}, \eta'}$ is independent of $\underline{i}$. Define $\varphi := t_{\underline{i}, \eta'}$ as the stalk of $t_{\underline{i}}$ at η'. Then, φ is $id_F \otimes Frob_K$-linear endomorphism of V. The $\mathcal{D}$-action on $\mathcal{E}_{\underline{i}}$'s induces a $\mathcal{D}$-action on V and φ is D-linear. We will construct R'-lattices in V using φ so that we can use the equivalence of categories to get a locally free sheaves over $X \otimes R$.

Let L be an R'-lattice in V. We say L is *admissible* if the induced map

$$\bar{\varphi} : L/\varpi L \longrightarrow L/\varpi L$$

is not nilpotent, i.e, $\varphi^{\dim V} L \subsetneq \varpi L$. Now, in his paper [13], Drinfeld defined admissible lattices and proved the following:

Proposition 6.7. *([13], Proposition 3.2)*

1. *There exists a φ-invariant lattice M_0 in V containing all other φ-invariant lattices. If M_0 is not admissible then there are no admissible lattices in V.*

2. *After replacing K by a finite extension K_1, which means changing V with $V \otimes K_1$ and φ with $\varphi \otimes Frob_{K_1}$, there exists admissible lattices in V.*

Remark 6.8. In [35], Laumon, Rapoport and Stuhler remarked after Proposition 6.6 that eventhough Drinfeld proved Proposition 6.7 in the rank 2 case only, it is valid in general. Differently from [35], we have a finite covering $X \longrightarrow Y$ and multi-t-indices.

After passing to a finite extension K_1 of K if necessary, we may assume that the maximal φ-invariant R'-lattice M_0 is admissible, i.e, the induced map

$$\bar{\varphi} : M_0/\varpi M_0 \longrightarrow M_0/\varpi M_0$$

is not nilpotent, i.e, $\varphi^{\dim V} M_0 \not\subset \varpi M_0$. Note that M_0 is D-stable since φ is D-linear and M_0 is maximal. So, we have an R'-lattice in a K'-vector space V together with locally free sheaves $\mathcal{E}_{\underline{i}}$ over K. Then, by the equivalence of categories in Theorem 6.5, we have a ladder of locally free sheaves over $Y \otimes R$:

$$\begin{tikzcd}
 & \widetilde{\mathcal{E}}_{\underline{i}+\underline{1}} \arrow[r, hook, "\widetilde{j}_{\underline{i},\underline{i}'}"] & \widetilde{\mathcal{E}}_{\underline{i}'+\underline{1}} \\
\sigma^*\widetilde{\mathcal{E}}_{\underline{i}} \arrow[ur, "\widetilde{t}_{\underline{i}}"] \arrow[r, hook, "\sigma^*\widetilde{j}_{\underline{i},\underline{i}'}"'] & \sigma^*\widetilde{\mathcal{E}}_{\underline{i}'} \arrow[ur, "\widetilde{t}_{\underline{i}'}"'] &
\end{tikzcd}$$

Now, via the $\mathcal{D}$-action on $\mathcal{E}_{\underline{i}}$ and categorical equivalence in Theorem 6.5, there is a $\mathcal{D}$-action on $\widetilde{\mathcal{E}}_{\underline{i}}$'s and all morphisms are $\mathcal{D}$-linear. Moreover, we have $\widetilde{\mathcal{E}}_{\underline{i}+\underline{\ell}} \simeq \widetilde{\mathcal{E}}_{\underline{i}}(\infty_1, \ldots, \infty_t)$ where $\ell = d \cdot \deg \infty$.

Let $\underline{i}' = \underline{i} + (\delta_1, \ldots, \delta_t)$ where each $\delta_j = 0$ or 1. Since

$$\mathrm{supp}(\widetilde{\mathcal{E}}_{\underline{i}'}/\widetilde{\mathcal{E}}_{\underline{i}}) \subset \mathrm{supp}(\widetilde{\mathcal{E}}_{\underline{i}}(\infty_1, \ldots, \infty_t)/\widetilde{\mathcal{E}}_{\underline{i}}) = \mathbb{T} \times R,$$

the support of $\widetilde{\mathcal{E}}_{\underline{i}'}/\widetilde{\mathcal{E}}_{\underline{i}}$ is contained in $\mathbb{T} \times R$. We have to prove that $\widetilde{\mathcal{E}}_{\underline{i}'}/\widetilde{\mathcal{E}}_{\underline{i}}$ is locally free of rank d over $\mathrm{Spec}\, R$ around each ∞_j with $\delta_j = 1$. Since $(\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ is a generalized $\mathcal{D}$-elliptic sheaf over $X \otimes K$, $\mathrm{coker}\, \widetilde{j}_{\underline{i},\underline{i}'}$'s are locally free of rank $d\delta_j$ over $\infty_j \times K$ for

$\underline{i}' = \underline{i} + (\delta_1, \cdots, \delta_t)$ with $\delta_j = 0$ or 1. By categorical equivalence from Theorem 6.5, we also know that $\widetilde{\mathcal{E}}_{\underline{i}}|_{X \otimes K} = \mathcal{E}_{\underline{i}}$. Then, by Nakayama's lemma, we have

$$d\delta_j \leq \dim_\kappa \left((\widetilde{\mathcal{E}}_{\underline{i}'}/\widetilde{\mathcal{E}}_{\underline{i}})|_{\infty_j \times R} \otimes \kappa \right)$$

since we have a $\mathcal{D}$-action on $\widetilde{\mathcal{E}}_{\underline{i}'}/\widetilde{\mathcal{E}}_{\underline{i}}$, we see that $\dim_\kappa \left(\widetilde{\mathcal{E}}_{\underline{i}'}/\widetilde{\mathcal{E}}_{\underline{i}} \right) = d\delta_j$

We have $\psi : X \smallsetminus \pi^{-1}(\mathsf{B}) \times_{Y \smallsetminus \mathsf{B}} \operatorname{Spec} K \longrightarrow (X \smallsetminus \pi^{-1}(\mathsf{B})) \times_{\mathbb{F}_q} \operatorname{Spec} K$. Then, by the natural morphism $\operatorname{Spec} K \longrightarrow \operatorname{Spec} R$, we get the morphism $\widetilde{\psi} : (X \smallsetminus \pi^{-1}(\mathsf{B})) \times_{Y \smallsetminus \mathsf{B}} \operatorname{Spec} R \longrightarrow X \smallsetminus \pi^{-1}(\mathsf{B}) \times_{\mathbb{F}_q} \operatorname{Spec} R$.

Assume that we have a morphism $\widetilde{\psi} : S' \longrightarrow (X \smallsetminus \underline{Bad}) \times_k SpecR$. We want to show that $(\widetilde{\mathcal{E}}_{\underline{i}}, \widetilde{t}_{\underline{i}}, \widetilde{\psi})$ is a generalized $\mathcal{D}$-elliptic sheaf over $X \otimes R$. There are two cases:

(i) $R' \cdot \varphi(M_0) = M_0$
(ii) $R' \cdot \varphi(M_0) \subsetneq M_0$.

We will show that in the first case the cokernel of $\tilde{t}_{\underline{i}}$ is supported on the image of $\widetilde{\psi}$ and locally free of rank dt over R and if D is a division algebra second case cannot occur.

Lemma 6.9. *In the first case, the triple* $(\widetilde{\mathcal{E}}_{\underline{i}}, \widetilde{t}_{\underline{i}}, \widetilde{\psi})$ *is a generalized* $\mathcal{D}$*-elliptic sheaf over* $\operatorname{Spec} R$.

Proof. The only thing remaining to show is that the $\operatorname{coker} t_{\underline{i}}$ has support on $Im\widetilde{\psi}$ and locally free of rank dt over R.

Consider the stalk of $\operatorname{coker} \widetilde{t}_{\underline{i}}$ at the generic point $\operatorname{Spec} \kappa'$ of $X \otimes \kappa$. Over $\operatorname{Spec} \kappa'$, the $\operatorname{coker} \widetilde{t}_{\underline{i}}$ is same as the $\operatorname{coker} \bar{\varphi}$ where $\bar{\varphi} : M_0/\varpi M_0 \longrightarrow M_0/\varpi M_0$. Since we are in the first case and κ is perfect, $\bar{\varphi}$ is surjective. So, $\operatorname{Spec} \kappa' \notin Supp(\operatorname{coker} \widetilde{t}_{\underline{i}})$, so $\overline{\operatorname{Spec} \kappa'} \subsetneq Supp(\operatorname{coker} \widetilde{t}_{\underline{i}})$. Hence $\operatorname{coker} \widetilde{t}_{\underline{i}}$ has support on $\operatorname{Im} \widetilde{\psi}$. We need to show that $\operatorname{coker} \widetilde{t}_{\underline{i}}$ is locally free of rank $d \cdot t$ over R. To show this we will use:

Lemma 6.10. *Let* $f : X \longrightarrow Y$ *be a proper morphism of locally Noetherian schemes,* $\mathcal{F}$ *be a coherent sheaf on* X *which is flat over* Y*. Let* $X_y := X \times_Y Speck(y)$ *denote the fiber over* y *of* f *and* $\mathcal{F}_y$ *denote the sheaf* $\mathcal{F} \otimes_{\mathcal{O}_Y} k(y)$ *on* X_y *where* $k(y)$ *is the residue field of* y*. Then the function* $Y \longrightarrow \mathbb{Z}$ *defined by*

$$y \mapsto \chi(\mathcal{F}_y) = \sum (-1)^p \dim_{k(y) H^p(X_y, \mathcal{F}_y)}$$

is locally constant on Y.

Proof. [38], Chapter II, Section 5, Corollary 1. □

Now, consider the function $\operatorname{Spec} R \longrightarrow \mathbb{Z}$ defined as in the previous lemma. Since $SpecR = \{\eta, s\}$, the open sets containing s is $\emptyset, \{\eta\}, \operatorname{Spec} R$. Then, as the function χ is locally constant via previous lemma, we have that $\chi(\widetilde{\mathcal{E}}_{\underline{i}} \otimes \kappa) = \chi(\widetilde{\mathcal{E}})$. Then,

$$\dim_\kappa \widetilde{t}_{\underline{i}} \otimes \kappa = \chi(\widetilde{\mathcal{E}}_{\underline{i}+\underline{1}}) - \chi(\widetilde{\mathcal{E}}_{\underline{i}}) = \chi(\mathcal{E}_{\underline{i}+\underline{1}}) - \chi(\mathcal{E}_{\underline{i}}) = \dim_K \operatorname{coker} t_{\underline{i}}$$

Hence, $\operatorname{coker} \widetilde{t}_{\underline{i}}$ is locally free of rank dt over R. Therefore $(\widetilde{\mathcal{E}}_{\underline{i}}, \widetilde{t}_{\underline{i}}, \widetilde{\psi})$ is a generalized $\mathcal{D}$-elliptic sheaf over $\operatorname{Spec} R$.

□

Lemma 6.11. *If D is a division algebra, the second case cannot occur.*

Proof. The proof goes similarly as in [35]. Let $M_0, \overline{M}_0, \varphi, \bar{\varphi}$ be as before. In the second case, $\bar{\varphi}$ is neither surjective nor nilpotent. Let us consider the flag of κ'-vector spaces:

$$\overline{M}_0 \supsetneq \operatorname{Im} \bar{\varphi} \supsetneq \operatorname{Im} \bar{\varphi}^2 \supsetneq \cdots \supsetneq \operatorname{Im} \bar{\varphi}^n = \operatorname{Im} \bar{\varphi}^{n+1} = \cdots \supsetneq$$

which becomes stationary.

On the other hand $\operatorname{Im} \bar{\varphi}^i / \operatorname{Im} \bar{\varphi}^{i+1}$ is a $D \otimes \kappa'$-module, so its dimension over κ' is divisible by d. Therefore, $n \leq d-1$. Moreover, if we put $\overline{N} = \operatorname{Im} \bar{\varphi}^n$ then $\dim N = rd$ where $0 < r < d$.

Let $\overline{\mathcal{E}}_{\underline{i}} := \widetilde{\mathcal{E}}_{\underline{i}}|_{X \otimes \kappa}$. The stalk of $\overline{\mathcal{E}}_{\underline{i}}$ at the generic point of $X \otimes \kappa$ is $\overline{M}_0$. Define $\overline{\mathcal{F}}_{\underline{i}} \subset \overline{\mathcal{E}}_{\underline{i}}$ to be the locally free $\mathcal{O}_{X \otimes \kappa}$-submodule generated by $\overline{N}$, i.e, the maximal locally free $\mathcal{O}_{X \otimes \kappa}$-submodule of $\overline{\mathcal{E}}_{\underline{i}}$ with stalk at the generic point $\overline{N} \subset \overline{M}_0$. Now, by the maximality, $\overline{\mathcal{F}}_{\underline{i}}$ is a $\mathcal{D} \otimes \kappa$-submodule of $\overline{\mathcal{E}}_{\underline{i}}$ and we have the following diagram

$$\begin{tikzcd} \overline{\mathcal{E}}_{\underline{i}} \arrow[r, hook] & \overline{\mathcal{E}}_{\underline{i}'} \\ \overline{\mathcal{F}}_{\underline{i}} \arrow[r, hook] \arrow[u] & \overline{\mathcal{F}}_{\underline{i}'} \arrow[u] \end{tikzcd}$$

and $\overline{\mathcal{F}}_{\underline{i}+\underline{d}\deg\infty} = \overline{\mathcal{F}}_{\underline{i}}(\infty_1, \cdots, \infty_t)$. Now, the quotients $\big(\overline{\mathcal{F}}_{\underline{i}'}/\overline{\mathcal{F}}_{\underline{i}}\big)|_{(\infty_j)\times\kappa}$ are $(\mathcal{D} \otimes k_{\infty_j}) \otimes \kappa$-modules. Note that

$rd\deg(\infty) = \dim(\overline{\mathcal{F}}_{\underline{i}}/\overline{\mathcal{F}}_{\underline{i}}(-\infty_1, \cdots, -\infty_t) < dim(\overline{\mathcal{E}}_{\underline{i}}/\overline{\mathcal{E}}_{\underline{i}}(-\infty_1, \cdots, \infty_t) = d^2 deg(\infty)$
Hence there exists $\underline{i}_j \in \{\underline{i}, \cdots, \underline{i} + \underline{d}\deg\infty - 1\}$ such that

$$\big(\overline{\mathcal{F}}_{\underline{i}_j+\underline{1}}/\overline{\mathcal{F}}_{\underline{i}_j}\big)|_{\infty_j \otimes \kappa} = 0. \tag{5}$$

By using this observation we will prove there exists $\underline{i}$ for all $j \in \{1, \cdots, t\}$ such that $\big(\overline{\mathcal{F}}_{\underline{i}'}/\overline{\mathcal{F}}_{\underline{i}}\big)|_{\infty_j \otimes \kappa} = 0$, i.e, $\overline{\mathcal{F}}_{\underline{i}} = \overline{\mathcal{F}}_{\underline{i}+\underline{1}}$.

We want to remark that if we fix the j^{th} entry of $\underline{i}_j$ and change the other entries of $\underline{i}_j$, the equation 5 still works. So, we may assume that all entries of $\underline{i}_j$ is zero except i_j. Let $\underline{i}_1 = (i_{11}, 0, \cdots, 0)$, $\underline{i}_2 = (0, i_{22}, \cdots, 0), \cdots, \underline{i}_t = (0, \cdots, 0, i_{tt})$. We define $\underline{i} = (i_{11}, \cdots, i_{tt})$ and we claim that $\big(\overline{\mathcal{F}}_{\underline{i}+\underline{1}}/\overline{\mathcal{F}}_{\underline{i}}\big)|_{\infty_j \otimes \kappa} = 0$.

We want to recall that $\widetilde{\mathcal{E}}_{\underline{i}}$ is independent of $\underline{i}$ on the affine part. Then, we have:

$$
\begin{tikzcd}
\widetilde{\mathcal{E}}_{\underline{i}_j}|_{((X\smallsetminus\mathbb{T})\cup\infty_j)\otimes\kappa} \arrow[r, equal] \arrow[d, equal] & \widetilde{\mathcal{E}}_{\underline{i}}|_{((X\smallsetminus\mathbb{T})\cup\infty_j)\otimes\kappa} \arrow[d, equal] \\
\overline{\mathcal{E}}_{\underline{i}_j}|_{((X\smallsetminus\mathbb{T})\cup\infty_j)\otimes\kappa} \arrow[r, equal] & \overline{\mathcal{E}}_{\underline{i}}|_{((X\smallsetminus\mathbb{T})\cup\infty_j)\otimes\kappa} \\
\overline{\mathcal{F}}_{\underline{i}_j}|_{((X\smallsetminus\mathbb{T})\cup\infty_j)\otimes\kappa} \arrow[r, equal] \arrow[u, phantom, sloped, "\subseteq"] & \overline{\mathcal{F}}_{\underline{i}}|_{((X\smallsetminus\mathbb{T})\cup\infty_j)\otimes\kappa} \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd} \tag{6}
$$

So, $\overline{\mathcal{F}}_{\underline{i}_j} = \overline{\mathcal{F}}_{\underline{i}}$ around ∞_j. Now, consider the following diagram

$$
\begin{tikzcd}
 & \overline{\mathcal{E}}_{\underline{i}_j+\underline{1}} \arrow[r, hook] & \overline{\mathcal{E}}_{\underline{i}+\underline{1}} \\
\overline{\mathcal{E}}_{\underline{i}_j} \arrow[r, hook] \arrow[ur] & \overline{\mathcal{E}}_{\underline{i}} \arrow[ur] &
\end{tikzcd}
$$

Since $\overline{\mathcal{F}}_{\underline{i}} \subset \overline{\mathcal{E}}_{\underline{i}}$ for each $\underline{i} \in \mathbb{Z}^t$, we have a similar diagram for $\overline{\mathcal{F}}_{\underline{i}}$'s:

$$
\begin{tikzcd}
 & \overline{\mathcal{F}}_{\underline{i}_j+\underline{1}} \arrow[r, hook] & \overline{\mathcal{F}}_{\underline{i}+\underline{1}} \\
\overline{\mathcal{F}}_{\underline{i}_j} \arrow[r, hook] \arrow[ur] & \overline{\mathcal{F}}_{\underline{i}} \arrow[ur] &
\end{tikzcd} \tag{7}
$$

We know by the Equation 5 that $\overline{\mathcal{F}}_{\underline{i}_j+\underline{1}}/\overline{\mathcal{F}}_{\underline{i}} = 0$ around ∞_j. By Diagram 6 we have $\overline{\mathcal{F}}_{\underline{i}_j} = \overline{\mathcal{F}}_{\underline{i}}$ around ∞_j. Similarly, we have $\overline{\mathcal{F}}_{\underline{i}_j+\underline{1}} = \overline{\mathcal{F}}_{\underline{i}+\underline{1}}$. Then, in the previous diagram around ∞_j we have

$$
\begin{tikzcd}
 & \overline{\mathcal{F}}_{\underline{i}_j+\underline{1}} \arrow[r, equal] & \overline{\mathcal{F}}_{\underline{i}+\underline{1}} \\
\overline{\mathcal{F}}_{\underline{i}_j} \arrow[r, equal] \arrow[ur, equal] & \overline{\mathcal{F}}_{\underline{i}} \arrow[ur] &
\end{tikzcd}
$$

which implies that $\left(\overline{\mathcal{F}}_{\underline{i}+\underline{1}}/\overline{\mathcal{F}}_{\underline{i}}\right)|_{\infty_j\otimes\kappa} = 0$ for any $j = 1, \cdots, t$.

Consider the following diagram

$$
\begin{tikzcd}
 & \overline{\mathcal{E}}_{\underline{i}} \arrow[rr] & & \overline{\mathcal{E}}_{\underline{i}'} \\
\sigma^*\overline{\mathcal{E}}_{\underline{i}-\underline{1}} \arrow[ur, "\bar{t}_{\underline{i}-\underline{1}}"] \arrow[rr] & & \sigma^*\overline{\mathcal{E}}_{\underline{i}'-\underline{1}} \arrow[ur] & \\
 & \overline{\mathcal{F}}_{\underline{i}} \arrow[uu] \arrow[rr, equal] & & \overline{\mathcal{F}}_{\underline{i}'} \arrow[uu] \\
\sigma^*\overline{\mathcal{F}}_{\underline{i}-\underline{1}} \arrow[uu] \arrow[ur] \arrow[rr] & & \sigma^*\overline{\mathcal{F}}_{\underline{i}'-\underline{1}} \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

The dotted arrows in the diagram are defined via the maximality of $\overline{\mathcal{F}}_{\underline{i}}$'s. So, we have a morphism $t : \sigma^*\overline{\mathcal{F}}_{\underline{i}} \longrightarrow \overline{\mathcal{F}}_{\underline{i}}$ whose stalk at the generic point is equal to $\bar{\varphi}|_{\overline{N}} : \overline{N} \longrightarrow \overline{N}$. Since $\bar{\varphi}|_{\overline{N}}$ is bijective, t is injective. As $\deg \overline{\mathcal{F}}_{\underline{i}} = \deg \sigma^*\overline{\mathcal{F}}_{\underline{i}}$, we conclude that t is an isomorphism. By Galois descent data, we conclude that $\overline{\mathcal{F}}_{\underline{i}}$ is of the form

$$\overline{\mathcal{F}}_{\underline{i}} = \mathcal{F}' \otimes \kappa$$

where $\mathcal{F}'$ is a locally free sheaf over X. Moreover, $\mathcal{F}'$ is a $\mathcal{D}$-module and its rank over $\mathcal{O}_X$ is equal to $rd = \dim \overline{N}$. Then the generic stalk $\mathcal{F}'_F$ is a $\mathcal{D}$-module of dimension $rd < d^2$ over F. If D is a division algebra such a module cannot exist.

$\square$

7 Adelic Level structures

We have defined level I-structures on a generalized $\mathcal{D}$-elliptic sheaf for closed subschemes I of $X \smallsetminus \underline{Bad}$ which is disjoint from $\mathbb{T} \cup \operatorname{im}\psi$ in Definition 2.15. Now, we will define level structures in the adelic point of view. The main tool for this section is Lemma 2.24.

In Section 2, we have defined the t-invariant functor E_I and in Theorem 2.22 we have shown that E_I is a free $H^0(I, \mathcal{D}_I)$-module of rank 1. Recall that $B = \Gamma(X \smallsetminus \mathbb{T}, \mathcal{O}_X)$ and define $\widehat{B} := \prod_{x \in X \smallsetminus (\mathbb{T} \cup \underline{Bad})} B_x$. Define the functor

$$E_{\widehat{B}} : Sch_S \longrightarrow \mathcal{D}(\widehat{B})\text{-modules}$$

by $T \mapsto \varprojlim_{I'} E_{I'}(T)$ where the limit is taken over all closed subschemes I' of $X \smallsetminus \underline{Bad}$ that are disjoint from $\mathbb{T} \cup pr_2(\Gamma\psi)$.

Remark 7.1. Note that

$$E_{\widehat{B}}(S) \simeq \varprojlim_{I'} E_{I'}(S) \simeq \varprojlim_{I'} \mathbb{M}_d(B_{I'}/I') = \mathcal{D}(\widehat{B})$$

where $B_{I'} := H^0(I', \mathcal{O}_{I'})$.

Suppose S is connected. Let $\iota : s \longrightarrow S$ be a geometric point. Since each E_I is representable by an étale scheme by Theorem 2.22 and $E_{\widehat{B}}$ is an $\mathcal{D}(\widehat{B})$-module, we may see $E_{\widehat{B}}$ as an $\mathcal{D}(\widehat{B})[\pi_1(S,s)]$-module $\iota^* E_{\widehat{B}}(s)$. Consider the set $\operatorname{Isom}(E_{\widehat{B}}, \mathcal{D}(\widehat{B})) := \operatorname{Isom}_{\mathcal{D}(\widehat{B})}(\iota^* E_{\widehat{B}}(s), \mathcal{D}(\widehat{B}))$ of isomorphism of $\mathcal{D}(\widehat{B})$-modules. By definition, there is a right action of $\mathcal{D}(\widehat{B})$ and a $\pi_1(S,s)$-action from the left on $\operatorname{Isom}_{\widehat{B}}(E_{\widehat{B}}, \mathcal{D}(\widehat{B}))$.

Definition 7.2. Let $H \subset \mathcal{D}(\widehat{B})$ be a compact open subgroup. An *H-level structure* on a generalized $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{E}}$ is an H-orbit in $\operatorname{Isom}(E_{\widehat{B}}, \mathcal{D}(\widehat{B}))$ which is fixed by $\pi_1(S,s)$.

Remark 7.3.

1. The condition to be fixed by $\pi_1(S,s)$ tells us that the level structure is independent of the choice of the base point.
2. If $H = \ker(\mathcal{D}(\widehat{B}) \longrightarrow \mathcal{D}(I))$ then an H-level structure is a level I-structure (Definition 2.15).

Now, we will modify the definition of an H-level structure. Let $\mathbb{A}_f$ denote the finite adeles of the function field F of X. For a generalized $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{E}}$, define the functor

$$E_{\mathbb{A}_f} : Sch_S \longrightarrow \mathcal{D}(\mathbb{A}_f)\text{-modules}$$

by $T \mapsto \mathcal{D}(\mathbb{A}_f) \otimes_{\mathcal{D}(\widehat{B})} E_{\widehat{B}}$.

Assume S is connected and let $\iota : s \longrightarrow S$ be an algebraically closed point. As before, we may see $E_{\mathbb{A}_f}$ as an $\mathcal{D}(\mathbb{A}_f)[\pi_1(S,s)]$-module $\iota^* E_{\mathbb{A}_f}(s)$. Consider $\mathrm{Isom}(E_{\mathbb{A}_f}, \mathcal{D}(\mathbb{A}_f)) := \mathrm{Isom}(\iota^* E_{\mathbb{A}_f}(s), \mathcal{D}(\mathbb{A}_f))$. Once more by definition, there is a right action of $\mathcal{D}(\mathbb{A}_f)$ and a left action of $\pi_1(S,s)$ on this set.

Definition 7.4. Let $H \subset \mathcal{D}(\mathbb{A}_f)$ be a compact open subgroup. A *rational H-level structure* on a generalized $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{E}}$ over S is an H-orbit in $\mathrm{Isom}(E_{\mathbb{A}_f}, \mathcal{D}(\mathbb{A}_f))$ which is fixed by $\pi_1(S,s)$

Remark 7.5.

1. Again the condition to be fixed by $\pi_1(S,s)$ implies that the level structure is independent from the choice of the base point.
2. In Section 16, we will see that a quasi-isogeny will give us an H-level structure.

Part II Divisible groups

The work on p-divisible groups is useful to study p-adic structure of abelian varieties and their local study. In this chapter we will give an analogy of some of the main objects such as p-divisible groups, Dieudonné modules and isocrystals that was used by Rapoport-Zink [43], stating first the analogues definitions in abelian sheaf case [23]. In the theory of Rapoport-Zink there are three main theorems and we will give analogues of the first two:

1. Rigidity of quasi-isogenies of p-divisible groups
2. Serre-Tate theorem
3. Grothendieck-Messing theorem

In this part in each section, we will first consider the case when $t = 1$ and so $X = Y$. So we will be looking at the $\mathcal{D}$-elliptic sheaf case. Then, by using these objects, we will define the objects for generalized $\mathcal{D}$-elliptic sheaf case. We will consider certain analogues of p-divisible groups with $\mathcal{D}_\infty$-action (resp, Dieudonné modules) which we call *z-divisible $\mathcal{D}_\infty$-module* (resp, *Dieudonné $\mathcal{D}_\infty$-modules*) generalizing the work of Hartl in [23], and define the corresponding moduli functors. By using z-divisible $\mathcal{D}_\infty$-module , we will define *$\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-modules.*

8 Generalized z-divisible $\mathcal{D}_\infty$-modules

First we will consider the case when $t = 1$ and $X = Y$. So we have only one $\infty \in |X|$. In the classical case p-divisible groups are sequence of finite flat group schemes with certain conditions. Similarly, we will use "balanced group schemes" defined by Poguntke in [42].

8.1 Balanced Group Schemes

This part is a summary of the first 5 sections of [42].

Let $S = SpecR$ be an affine scheme over $\mathbb{F}_q$. Denote by Gr_S the category whose objects are finite flat affine commutative group schemes over S locally of finite presentation.

An $\mathbb{F}_q$-action on a group scheme $G \in \mathrm{Gr}_S$ is a ring homomorphism:

$$\mathbb{F}_q \longrightarrow \mathrm{End}_{Gr_S} G$$

Assume locally on S there exists an $\mathbb{F}_q$-equivariant closed embedding $G \hookrightarrow \mathbb{G}_a^N$ for some finite set N. Such group schemes are called *of $\mathbb{F}_q$-additive type.*

Let $\mathcal{C}$ denote the category of objects $G \in \mathrm{Gr}_S$ that are of $\mathbb{F}_q$-additive type and that in addition carry an $\mathbb{F}_q$-action.

Take any $G \in \mathcal{C}$. One can write $G = \operatorname{Spec} A$. By using the group structure on G, we have the following maps on A:

$m : A \longrightarrow A \otimes A$ the co-multiplication map

$\eta : A \longrightarrow R$ the augmentation(or co-unit) map

$\iota : A \longrightarrow A$ the co-inverse map

making A a Hopf algebra.

Definition 8.1. Let $G = \operatorname{Spec} A$ be an affine group scheme over $S = \operatorname{Spec} R$ with an $\mathbb{F}_q$-action: $\mathbb{F}_q \longrightarrow \operatorname{End}_{Gr_S} G$. Let $I := \ker(\eta)$ be the augmentation ideal. Now, we get an induced $\mathbb{F}_q^*$-action on I. The corresponding eigenspaces are:

$$I_j = \{x \in I | \alpha.x = \alpha^j x, \forall \alpha \in \mathbb{F}_q^*\}$$

for $0 < j < q$ where the multiplication on the LHS is the $\mathbb{F}_q^*$-action.

Definition 8.2.

1. Let $G = \operatorname{Spec} A$ be an affine group scheme over $S = \operatorname{Spec} R$. Define the space of primitive elements of A as

$$\operatorname{Prim}(A) = \{x \in I | m(x) = x \otimes 1 + 1 \otimes x\}$$

2. Define $\operatorname{Prim}_j(A) := \operatorname{Prim}(A) \cap I_j$

Remark 8.3.

1. One can write

$$I = \oplus_{j=1}^{q-1} I_j.$$

By [48], Lemma 2, one can find orthogonal idempotents $e_1, \ldots, e_{q-1}$ of $\operatorname{End}(I)$ such that $I_j = e_j I$.

2. Similar to (1), if $PrimB_G$ is flat, one can write $\operatorname{Prim} A = \oplus_{j=1}^{q-1} \operatorname{Prim}_j A$.

Definition 8.4. For $s \in \mathbb{N}$, define the *p-Frobenius map*

$$f_s : \operatorname{Prim}_{p^s} A \longrightarrow \operatorname{Prim}_{p^{s+1}}(A) \text{ as } x \mapsto x^p$$

Proposition 8.5. *Let $G = \operatorname{Spec} A \in \mathcal{C}$. The R-module $\operatorname{Prim} A$ is locally free and* $\operatorname{ord} G = p^{\mathrm{rk(PrimA)}}$.

Proof. [42], Proposition3.6 □

Proposition 8.6. $\operatorname{Prim}_j A \neq 0$ *only if* $j = p^s$ *for some* $s \in \mathbb{N}$

Proof. [42], Theorem5.10 □

Definition/Theorem. Let $G = \operatorname{Spec} A \in \mathcal{C}$. We say G is a *balanced group scheme* if one of the following equivalent conditions hold:

1. For $0 \leq s < r-1$, the maps f_s are bijective
2. The map defined by composition of p-Frobenius maps

$$f' : \mathrm{Prim}_1(A) \longrightarrow \mathrm{Prim}_{p^{r-1}}(A)$$
$$x \mapsto x^{p^{r-1}}$$

   is injective.

3. The rank of $\mathrm{Prim}_{p^s}(A)$ as an R-module is same for all $0 \leq s \leq r-1$
4. The order of G is equal to $q^{\mathrm{rank}_R(\mathrm{Prim}_1(A))}$

Proof. [42], Lemma 5.12 □

Remark 8.7. The balanced group schemes are categorically equivalent to *finite locally free strict* $\mathbb{F}_q$*-module schemes*" as defined in [24]. A finite locally free $\mathbb{F}_q$-module scheme G is a *strict* $\mathbb{F}_q$*-module scheme* if it has a deformation carrying a strict $\mathbb{F}_q$-action which lifts the $\mathbb{F}_q$-action on G. It is proved in [1] that finite locally free strict $\mathbb{F}_q$-module schemes are categorically equivalent to the category of "*finite* $\mathbb{F}_q$*-shtukas*" (cf. Remark 9.3 below).

Example 8.8. Recall that $q = p^r$. The group scheme $\alpha_{p^s} := \mathrm{Spec}(A[x]/(x^{p^s}))$ for $s \in \mathbb{N}$ with the usual $\mathbb{F}_q$-action is balanced iff $r|s$.

Remark 8.9. Note that the additive group scheme is not in the category $\mathcal{C}$ since it is not finite, and so not a balanced group scheme.

Lemma 8.10. *Let $G, H \in \mathcal{C}$. If two of G, H and $G \times H$ are balanced, then so is the third.*

Proof. [42], Lemma 5.19

□

8.2 z-divisible $\mathcal{D}_\infty$-module

Before we define $\mathcal{D}_\infty$-groups, we need to introduce some notation:

Notation: Let $q_\infty := q^{\deg \infty}$. Recall that z is a uniformizer of $\mathcal{O}_{X,\infty}$. Identify its completion, $\mathcal{O}_\infty$, with $k_\infty[\![z]\!]$ and F_∞ with $k_\infty(\!(z)\!)$. Let ζ be an indeterminant over k_∞ and $k_\infty[\![\zeta]\!]$ be the ring of formal power series. From now on, all base schemes S will be schemes over $\mathrm{Spec}\, k_\infty[\![\zeta]\!]$. Relate $k_\infty[\![z]\!]$ with $k_\infty[\![\zeta]\!]$ by fixing the characteristic map $f : \mathrm{Spec}\, k_\infty[\![\zeta]\!] \longrightarrow X$ such that $f^*(z) = \zeta$. We will use the notation z as a uniformizer of $\mathcal{O}_{X,\infty}$ and ζ as an element of $\mathcal{O}_S$.

We denote by $\mathcal{N}ilp_{k_\infty[\![\zeta]\!]}$ the category of schemes over $\mathrm{Spf}\, k_\infty[\![\zeta]\!]$, viz., the category of schemes over $\mathrm{Spec}\, k_\infty[\![\zeta]\!]$ on which ζ is locally nilpotent.

Denote by $k_\infty^{(d)}$ the field extension of k_∞ such that $[k_\infty^{(d)} : k_\infty] = d$. Let Δ be the central $\mathcal{O}_\infty$-algebra $k_\infty^{(d)}[\![\Pi]\!]$ where

$$\Pi^l = z, \Pi.a^{q_\infty} = a.\Pi \ \ \forall a \in k_\infty^{(d)}$$

Let $S \in \mathcal{N}ilp_{k_\infty[\![\zeta]\!]}$.

Definition 8.11.

1. Let R be a ring. An *R-module scheme* over S is a flat commutative S-group scheme E with a ring homomorphism

$$R \longrightarrow \mathrm{End}_S(E).$$

2. An R-module scheme E is *finite of order r* if $E \longrightarrow S$ is finite flat of degree r.

3. A *morphism* of R-module schemes is a morphism between underlying S-group schemes which is compatible with the R-action.

Definition 8.12. Let G be a commutative group scheme over S and let $\varepsilon : S \longrightarrow G$ be its unit section. Then $\omega_G := \varepsilon^*\Omega^1_{G/S}$ is its *co-Lie module.*

Definition 8.13. ([24], Definition 7.1) Let $h \in \mathbb{Z}^{>0}$. A *z-divisible group of height h* is an inductive system of finite $\mathcal{O}_\infty$-module schemes over S

$$(E_1 \xrightarrow{i_1} E_2 \xrightarrow{i_2} E_3 \xrightarrow{i_3} \ldots)$$

such that for each integer $n \geq 1$

1. $E_n \simeq E_{n+1}[z^n]$ where $E_{n+1}[z^n] := \ker(z^n : E_{n+1} \longrightarrow E_{n+1})$

2. The underlying group scheme of E_n is a balanced group scheme(cf. Appendix 8.1), denote it by G_n

3. the order of E_n is q_∞^{hn},

4. locally on S, there exists $e \in \mathbb{Z}^{>0}$ such that $(z - \zeta)^e = 0$ on $\omega_E := \varprojlim \omega_{E_n}$.

A *morphism of z-divisible groups over S* is a morphism of inductive systems of $\mathcal{O}_\infty$-module schemes.

Definition/Remark: By Lemma 8.2 and Theorem 10.7 in [24], ω_E is a locally free $\mathcal{O}_S$-module and the rank of ω_E is locally constant on S. We define the *dimension of E* as $\mathrm{rk}(\omega_E)$

Now, we will consider z-divisible groups with a $\mathcal{D}_\infty$-action.

Definition 8.14. A *z-divisible $\mathcal{D}_\infty$-module of height h and dimension e* is a z-divisible group of height dh and dimension de with an $\mathcal{O}_\infty$-algebra homomorphism $\mathcal{D}_\infty \longrightarrow \mathrm{End}_{\mathcal{O}_\infty}(E)$ extending the action of $\mathcal{O}_\infty$.

Remark 8.15. Assume $\deg \infty = 1$ and $\mathcal{D}_\infty = \mathcal{O}_\infty$. Then, one can identify $\mathcal{O}_\infty$ with $\mathbb{F}_q[\![z]\!]$. In this case, the z-divisible $\mathcal{O}_\infty$-module is same as a z-divisible group in Definition 8.13

Remark 8.16. Our definition of z-divisible $\mathcal{D}_\infty$-modules are similar to Tate and called in the classical case *Barsotti-Tate groups.* In [37], Messing defined p-divisible groups in the classical case in a different, but equivalent, way than Tate in [50]. In [24], Hartl and Singh, defines z-divisible groups following [37] so that it is an fppf sheaf of $\mathbb{F}_q[\![z]\!]$-modules. Then, they show that it is equivalent to the Definition 8.13. Following [24], one can give a definition of z-divisible $\mathcal{D}_\infty$-modules as in [37] and then state that the two definitions are equivalent. Since it is technical and very similar to the case [24], we will give the idea briefly:

Let $E = (E_n, i_n)$ be a z-divisible $\mathcal{D}_\infty$-module. Then, $G := \varinjlim_{n\in\mathbb{N}} E_n$ defines us a commutative fppf sheaf of groups. Now, G is z-divisible, i.e, the morphism $z : G \longrightarrow G$ is an epimorphism. By condition (1) And G is z-torsion, i.e, $G = \varinjlim_{n\in\mathbb{N}} G_n$ where $G_n = \ker(z^n : G \longrightarrow G)$. Also, by the condition (2) in the definition of z-divisible group, each G_n is representable by a balanced group scheme.

Remark 8.17. Let $\underline{E}$ be a z-divisible $\mathcal{D}_\infty$-module over S. Then, pulling back $\underline{E}$ under the morphism of schemes $S' \longrightarrow S$ gives us a z-divisible $\mathcal{D}_\infty$-module over S'. We will use this in Proposition 9.17.

Morita equivalence for z-divisible $\mathcal{D}_\infty$-modules

Proposition 8.18. *The category of z-divisible $\mathcal{O}_\infty$-modules of height h and dimension d and z-divisible $\mathcal{D}_\infty$-modules of height hd and dimension de are Morita equivalent.*

Proof. Let $E = (E_n, i_n)$ be an z-divisible $\mathcal{O}_\infty$-module over S. Consider the functor

$$E \mapsto E' = (E'_n, i'_n) := (E_n \otimes_{\mathcal{O}_\infty} N, i_n \otimes_{\mathcal{O}_\infty} N)$$

where N is the $\mathcal{O}_\infty$-$\mathbb{M}_d(\mathcal{O}_\infty)$- bimodule $\mathcal{O}_\infty^{1\times d}$. Note that $E_n \otimes N$ is same as taking d copies $E_n \times \cdots \times E_n$ of E_n, so $E_n \otimes N$ is finite $\mathcal{O}_\infty$-module scheme and the group scheme underlying $E_n \otimes N$ is balanced(cf. Lemma 8.10). Therefore, we have an inductive system

$$(E'_1 \xrightarrow{i'_1} E'_2 \xrightarrow{i'_2} E'_3 \xrightarrow{i'_3} \ldots)$$

of finite $\mathcal{O}_\infty$-module schemes. Note that z acts on each $E_n \otimes_{\mathcal{O}_\infty} N$ through only first factor. And, there is an $\mathcal{D}_\infty \simeq \mathbb{M}_d(\mathcal{O}_\infty)$-action on E' extending the action of $\mathcal{O}_\infty$.

Now, let us consider the order of E'_n. For simplicity, assume $E_n := SpecA_n$ where A_n is a $\mathcal{O}_S$-module of rank q^{hn} since $\operatorname{ord} E_n = q^{hn}$. Now, $\mathcal{O}_{E'_n} = \mathcal{O}_{E_n \times \cdots \times E_n} = A_n \otimes \cdots \otimes A_n$ where all operations are taken with d copies, the order of E'_n is q^{dhn}.

Since $\omega_{E'} = \omega_{E \oplus \cdots \oplus E} \simeq \omega_E \oplus \cdots \omega_E$, it follows that E' has dimension de.

Therefore, $E' = (E'_n, i'_n)$ is a z-divisible $\mathcal{D}_\infty$-module.

□

A morphism of z-divisible $\mathcal{D}_\infty$-modules is a morphism of z-divisible groups which is compatible with the $\mathcal{D}_\infty$-action. Let $E = (E_n, i_n)$ and $E' = (E'_n, i'_n)$ be two z-divisible $\mathcal{D}_\infty$-modules. Denote by $\operatorname{Hom}_{\mathcal{D}_\infty}(E, E')$ the set of morphisms $E \longrightarrow E'$ of z-divisible $\mathcal{D}_\infty$-modules and $\operatorname{End}_{\mathcal{D}_\infty}(E) := \operatorname{Hom}_{\mathcal{D}_\infty}(E, E)$.

Isogenies of z-divisible $\mathcal{D}_\infty$-modules

Definition 8.19. Let S be an $\mathbb{F}_q$-scheme and X, Y be two $\mathcal{O}_S$-module schemes. Define the sheaf $\underline{\operatorname{Hom}}_S$ as $U \mapsto \operatorname{Hom}_S(X(U), Y(U))$ on the Zariski site over S. This sheaf is called *sheaf of germs of morphisms on S.*

Recall that we can see z-divisible $\mathcal{D}_\infty$-modules as fppf-sheaves Remark 8.16. Now, we will define isogenies of z-divisible $\mathcal{D}_\infty$-modules by seeing them as fppf sheaves.

Definition 8.20. A morphism between two z-divisible $\mathcal{D}_\infty$-modules E and E' is an *isogeny* :⇔ it is an fppf-epimorphism between E and E' whose kernel is represented by a finite locally free group scheme.

Example 8.21. The multiplication by z on a z-divisible $\mathcal{D}_\infty$-module E is an isogeny. We will denote this isogeny by $[z]$.

Remark 8.22. Note that the composition of two isogeny is again an isogeny.

Proposition 8.23. *Let E and E' be two z-divisible $\mathcal{D}_\infty$-modules over S. The group of morphisms* $\operatorname{Hom}_{\mathcal{D}_\infty}(E, E')$ *is torsion free $k_\infty[\![z]\!]$-module.*

Proof. Let $\Phi : E \longrightarrow E'$ be a morphism of z-divisible $\mathcal{D}_\infty$-modules, i.e, Φ is an inductive system of morphisms $\Phi_n : E_n \longrightarrow E'_n$. Assume that

$$[z]^n \Phi = 0 \text{ for some } n. \tag{8}$$

Consider the diagram

$$\begin{tikzcd} E_{n+m} \arrow[r, "{[z^n]}"] \arrow[d, "\Phi_{m+n}"] & E_m \arrow[d, "\Phi_m"] \\ E'_{n+m} \arrow[r, "{[z^n]}"] & E'_m \end{tikzcd}$$

By assumption, we have $[z^n] \circ \Phi_{n+m} = 0$ and hence $\Phi_m \circ [z^n] = 0$. The latter means $\mathrm{im}[z^n] \subset \ker \Phi_m$. Since $[z^n]$ is surjective, we see that $\Phi_m = 0$ for all $m \in \mathbb{N}$. Hence, $\Phi = 0$.

$\square$

Now, we will define quasi-isogenies for z-divisible $\mathcal{D}_\infty$-modules.

Definition 8.24. A *quasi-isogeny* ρ between two z-divisible $\mathcal{D}_\infty$-modules E, E' is a global section of the sheaf $\underline{\mathrm{Hom}}_S(E, E') \otimes_{k_\infty[\![z]\!]} k_\infty(\!(z)\!)$ of $k_\infty(\!(z)\!)$-modules on S such that locally on S there exists an $n \in \mathbb{Z}$ for which $z^n\rho$ is an isogeny. Denote by $\mathrm{QIsog}_S(E, E')$ the set of quasi-isogenies between E and E'.

Definition 8.25. The category $\mathcal{C}$ of z-divisible $\mathcal{D}_\infty$-modules up to isogeny has z-divisible $\mathcal{D}_\infty$-modules as objects and all global sections of the sheaf $\underline{\mathrm{Hom}}_S(E, E') \otimes_{k_\infty[\![z]\!]} k_\infty(\!(z)\!)$ as morphisms.

Remark 8.26. Let E, E' be two z-divisible $\mathcal{D}_\infty$-modules over S. Then, E and E' are isomorphic in $\mathcal{C}$ iff they are isogeneous. More precisely, let $\rho : E \longrightarrow E'$ be an isogeny. By definition we have an exact sequence

$$0 \longrightarrow H \xrightarrow{\iota} E \xrightarrow{f} E' \longrightarrow 0$$

where H denotes the kernel of f. Note that $f \in \mathrm{Hom}_S(E, E')$. We claim that $f \otimes 1 \in \underline{\mathrm{Hom}}_S(E, E') \otimes_{k_\infty[\![z]\!]} k_\infty(\!(z)\!)$ is an isomorphism of objects in $\mathcal{C}$. The latter holds iff $\iota \otimes 1 = 0$. We know that $\mathrm{ord}\, H$ is finite, say n. Then, $n\iota = 0$. But then, we have

$$\iota \otimes 1 = \iota \otimes \frac{n}{n} = n\iota \otimes \frac{1}{n} = 0 \otimes \frac{1}{n} = 0.$$

and hence $f \otimes 1$ is an isomorphism.

Definition 8.27.
1. Let $\rho : E \longrightarrow E'$ be an isogeny between two z-divisible $\mathcal{D}_\infty$-modules over S. The rank of the kernel of ρ is a power of q. If the rank is constant, say q^h, we call *h the height of the isogeny ρ.*

2. Let $\rho : E \longrightarrow E'$ be a quasi-isogeny between E, E'. Then, by definition locally on S there is $n \in \mathbb{Z}$ such that $z^n\rho$ is an isogeny. Let h be the smallest of such n's. We define h to be *the height of the quasi-isogeny ρ.*

Remark 8.28.
1. Note that the number h in item 2 need not to exist.

2. We want to remark that any isogeny or quasi-isogeny between two z-divisible $\mathcal{D}_\infty$-modules is, by definition, compatible with $\mathcal{D}_\infty$-action.

Let $\rho : E \longrightarrow E'$ be a quasi-isogeny of z-divisible $\mathcal{D}_\infty$-modules. By definition, locally on S, there exists $n \in \mathbb{Z}$ such that $z^n\rho$ is an isogeny. The question is: Is there a characterization that will tell us when ρ is an isogeny itself to begin with? The answer is given by the following lemma:

Lemma 8.29. *Let $\underline{E} = (E_n, i_n)$ and $\underline{E'} = (E'_n, i'_n)$ be two z-divisible $\mathcal{D}_\infty$-modules over S. Let $\rho : E \longrightarrow E'$ be a quasi-isogeny of z-divisible $\mathcal{D}_\infty$-modules. Let $n \in \mathbb{Z}$ such that $z^n\rho$ is an isogeny. Then*

$$\rho \textit{ is an isogeny } \iff z^n\rho : E[n] \longrightarrow E'[n] \textit{ is the zero morphism}$$

where $E[n]$(resp, $E'[n]$) denotes the kernel of multiplication by z^n on E (resp. E').

Remark 8.30.

1. Note that in the lemma, the n that satisfies the considition $z^n\rho$ to be an isogeny and the n in the condition $z^n\rho : E[n] \longrightarrow E'[n]$ is the zero morphism are same.

2. Let $f : E \longrightarrow E'$ be a morphism of z-divisible $\mathcal{D}_\infty$-modules. Note that since $E[n] \subset E$ and $E'[n] \subset E'$ we can restrict the morphism f on these subsets. We denote both the restriction and the morphism on E and E' by f. So, one can define $z^n f$ as the composition

$$E[n] \xrightarrow{f} E'[n] \xrightarrow{z^n} E'$$

Since f is a morphism of z-divisible $\mathcal{D}_\infty$-modules, it sends $E[n]$ to $E'[n]$ and the composition $z^n f$ is the zero morphism. The key point to say that the composition is zero is the fact that $E[n]$ mapped to $E'[n]$, i.e, f respects the group structure.

Proof. (of Lemma 8.29) Let ρ be an isogeny. Then, by definition $z^n\rho$ is the zero morphism on E_n.

Conversely, let us assume that

$$E[n] \xrightarrow{\rho} E'[n] \xrightarrow{z^n} E'[n]$$

is the zero morphism.

Assume for now that ρ is a morphism. Since $[z^n]$ is an isogeny, $\ker[z^n] = E[n]$ is finite locally free group scheme. Since $z^n\rho$ is the zero morphism, we have $\ker \rho \subset \ker(z^n\rho) = E[n]$ and so $\ker \rho$ is also finite locally free group scheme. So, we only need to prove that $\rho : E \longrightarrow E'$ is a morphism. We denote n-shift of E' by $E'(n)$, i.e, $E'(n) = (E'_{j-n}, i_{j-n})$. The image of $z^n : E \longrightarrow E$ lies in $E(n)$. Since $z^n\rho : E[n] \longrightarrow E'[n]$ is the zero morphism, we get an isogeny $z^n\rho : E \longrightarrow E'(n)$. And $z^{-n} : E'(n) \longrightarrow E'$ is a morphism. Then, $\rho = z^{-n} \circ z^n\rho$ is composition of morphisms, hence a morphism.

□

Proposition 8.31. *Let $\alpha : E \longrightarrow E'$ be a quasi isogeny of z-divisible $\mathcal{D}_\infty$-modules over S. The functor defined on $\mathcal{N}ilp_{k_\infty[\![\zeta]\!]}$ by*

$$T \mapsto \{f \in \mathrm{Hom}(T, S) \mid f^*\alpha \textit{ is an isogeny } \}$$

is representable by a closed subscheme of S.

The proof of this proposition will follow from the following lemma:

Lemma 8.32. *Let $\alpha : \mathcal{E} \longrightarrow \mathcal{F}$ be a morphism of $\mathcal{O}_S$-modules on a scheme S. Assume $\mathcal{F}$ is finite locally free. The functor defined on Sch_S by*

$$T \mapsto \{f \in \mathrm{Hom}(T, S) \mid f^*\alpha = 0\}$$

is representable by a closed subscheme of S.

Proof. [43], Lemma 2.10. □

Proof. (of Proposition 8.31) Let $\alpha : E \longrightarrow E'$ be a quasi-isogeny of z-divisible $\mathcal{D}_\infty$-modules over S. Then, locally on S, there is an n such that $z^n\rho$ is an isogeny. Let $f : T \longrightarrow S$ be a morphism of schemes such that $f^*\alpha$ is an isogeny. By Lemma 8.29, it is equivalent to say that $z^n(f^*\alpha)$ is zero. Note that $z^n f^*\alpha = f^*(z^n\alpha)$. But then by Proposition 8.32, the functor is representable by a closed subscheme of S. □

Definition 8.33. ([23], Definition 6.6) A *z-divisible $\mathcal{O}_\Delta$-module over S* is a z-divisible group E over S with an action $\mathcal{O}_\Delta \to \mathrm{End}_{\mathcal{O}_\infty} E$ of $\mathcal{O}_\Delta$, which prolongs the natural action of $\mathcal{O}_\infty$. A morphism of z-divisible $\mathcal{O}_\Delta$-modules which is an isogeny of z-divisible groups is called an *isogeny*.

Remark 8.34. The height and dimension of a z-divisible $\mathcal{O}_\Delta$-module is the height and the dimension of the underlying z-divisible group.

Definition 8.35. ([23], Definition 6.7) A z-divisible $\mathcal{O}_\Delta$-module E which as a z-divisible $\mathcal{O}_\infty$-module is of height $r\ell$ and dimension $d\ell$ over $S \in \mathcal{N}ilp_{\mathrm{Spf}\, k_\infty^{(\ell)}[\![\zeta]\!]}$ is called *special* if the action of $\mathcal{O}_\Delta$ induced on ω_E, makes ω_E into a locally free $k_\infty^{(\ell)} \otimes \mathcal{O}_S$-module of rank d.

Definition 8.36. (a)A z-divisible $\mathcal{D}_\infty \otimes \mathcal{O}_\Delta$-module over S of height $h\ell$ and dimension $e\ell$ is a z-divisible $\mathcal{O}_\Delta$-module E of height $dh\ell$ and dimension $de\ell$ together with an action $\mathcal{D}_\infty \otimes \mathcal{O}_\Delta \longrightarrow End_S(E)$

(b) A z-divisible $\mathcal{D}_\infty \otimes \mathcal{O}_\Delta$-module E over $S \in \mathcal{N}ilp_{Spf k_\infty^{(d)}[\![\zeta]\!]}$ of height hl and dimension el is called *special* if the action of $\mathcal{D}_\infty \otimes \mathcal{O}_\Delta$ makes ω_E a locally free $k_\infty^{(d)} \otimes \mathcal{O}_S$-module of rank e.

Definition 8.37.
1. A morphism of z-divisible $\mathcal{D}_\infty \otimes \mathcal{O}_\Delta$-modules is a morphism of z-divisible $\mathcal{D}_\infty$-modules that respects the $\mathcal{D}_\infty \otimes \mathcal{O}_\Delta$-action.
2. A morphism of z-divisible $\mathcal{D}_\infty \otimes \mathcal{O}_\Delta$-modules which is an isogeny of z-divisible $\mathcal{D}_\infty$-module is called an *isogeny*

8.3 Generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-modules

We will extend the definition of z-divisible groups in the setting of generalized $\mathcal{D}$-elliptic sheaves. Denote by E_j the z_j-divisible $\mathcal{D}_{\infty_j}$-module at ∞_j for $j = 1, \ldots, t$.

Put $\mathcal{D}_{\underline{\infty}} = \prod_{j=1}^{t} \mathcal{D}_{\infty_j}$ and $\underline{z} = (z_1, \ldots, z_t)$. Let $k_{\infty_j}^{(d)}$ denote the field extension of k_{∞_j} of degree d and put $q_{\infty_j} := q^{\deg \infty_j}$. For each $j = 1, \ldots, t$, define the k_{∞_j}-algebra $\Delta_j := k_{\infty_j}^{(d)}((\Pi_j))$ where

$$\Pi_j^d = z_i, \Pi_j \cdot a^{q_{\infty_j}} = a \cdot \Pi_j \text{ for all } a \in k_{\infty_j}^{(d)}$$

Put $\underline{\Delta} := \prod \Delta_j$. Note that if $t = 1$ then $\underline{\Delta}$ is same as in the section of z-divisible $\mathcal{D}_\infty$-module (Section 8). Let $S \in \mathcal{N}ilp_{k_\infty[\![\zeta_1, \cdots, \zeta_t]\!]}$

Definition 8.38. A generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-module over S is t-tuple

$$\underline{E} = (E_1, \cdots, E_t)$$

where each E_j is a z_j-divisible $\mathcal{D}_{\infty_j}$-module. Note that $\mathcal{D}_{\underline{\infty}}$ acts on $\underline{E}$ by acting on j-component via the projection onto j-factor.

Definition 8.39. Let $\underline{E} = (E_1, \cdots, E_t)$ and $\underline{E}' = (E_1', \cdots, E_t')$ be two generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-modules. A *morphism* $f : \underline{E} \longrightarrow \underline{E}'$ is a t-tuple $(f_1, \cdots, f_t)$ of morphisms of z_j-divisible $\mathcal{D}_{\infty_j}$-modules $f_j : E_j \longrightarrow E_j'$.

Definition 8.40. Let $\underline{E} = (E_1, \cdots, E_t)$ and $\underline{E}' = (E_1', \cdots, E_t')$ be two generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-modules over S.

1. An *(quasi-)isogeny* $f : \underline{E} \longrightarrow \underline{E}'$ between $\underline{E}$ and $\underline{E}'$ is a t-tuple $(f_1, \cdots, f_t)$ where each $f_j : E_j \longrightarrow E_j'$ is (quasi-)isogeny of z_j-divisible $\mathcal{D}_{\infty_j}$-modules E_j and E_j'. We say $\underline{E}$ and $\underline{E}'$ are *(quasi-)isogeneous* if there is an (quasi-)isogeny between them.

2. We denote the $\mathcal{D}_{\underline{\infty}}$-module of isogenies between $\underline{E}$ and $\underline{E}'$ by $\mathrm{Isog}_S(\underline{E}, \underline{E}')$ and by $\mathrm{QIsog}_S(\underline{E}, \underline{E}')$ the $\mathcal{D}_{\underline{\infty}}$-module of quasi-isogenies.

Remark 8.41. The composition of two isogenies is again an isogeny.

Definition 8.42. The category of generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-modules up to isogeny has $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-modules as objects and all global sections of the sheaf $\underline{\mathrm{Hom}}_S(\underline{E}, \underline{E}') \otimes_{k(T)[\![\underline{z}]\!]} k(T)((\underline{z}))$ as morphisms.

As in the z-divisible $\mathcal{D}_\infty$-module case, in this case also, we have the following:

Morita equivalence

Recall that each ∞_i is split, i.e, $\mathcal{D}_{\infty_i} \simeq \mathbb{M}_d(\mathcal{O}_{\infty_i})$. Now, let $\underline{E} = (E_1, \cdots, E_t)$ be a generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-module over S. By Morita equivalence of z-divisible $\mathcal{D}_{\infty}$-modules (Proposition 8.18), we have that each z_i-divisible $\mathcal{D}_{\infty_i}$-module E_i is Morita equivalent to $E_i' \otimes_{\mathcal{D}_{\infty_i}} \mathcal{O}_{\infty_i}^{d\times 1}$ where E_i' is an z_i-divisible $\mathcal{O}_{\infty_i}$-module. So, we have

$$\underline{E} = \big((E_1' \otimes_{\mathcal{D}_{\infty_1}} \mathcal{O}_{\infty_1}^{d\times 1}), \cdots, (E_t' \otimes_{\mathcal{D}_{\infty_t}} \mathcal{O}_{\infty_t}^{d\times 1})\big) \simeq (E_1', \cdots, E_t') \otimes_{\mathcal{D}_{\underline{\infty}}} (\prod \mathcal{O}_{\infty_i}^{d\times 1})$$

where $(E_1', \cdots, E_t')$ is $\underline{z}$-divisible $\mathcal{O}_{\underline{\infty}}$-module.

Definition 8.43.

1. We say a generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}}$-module $\underline{E} = (E_1, \cdots, E_t)$ is a *generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}} \otimes \mathcal{O}_{\underline{\Delta}}$-module* over S if each E_i is a z_i-divisible $\mathcal{D}_{\infty_i} \otimes \Delta_i$-module.

2. A generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}} \otimes \mathcal{O}_{\underline{\Delta}}$-module $\underline{E} = (E_1, \cdots, E_t)$ over S is called *special* if each z_i-divisible $\mathcal{D}_{\infty_i} \otimes \Delta_i$-module E_i is special.

Definition 8.44. Let $\underline{E} = (E_1, \cdots, E_t)$ and $\underline{E}' = (E_1', \cdots, E_t')$ be two generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}} \otimes \mathcal{O}_{\underline{\Delta}}$-modules over S. A *morphism (resp., isogeny, quasi-isogeny)* between $\underline{E}$ and $\underline{E}'$ is a t-tuple of morphisms (resp., isogeny, quasi-isogeny) between z_i-divisible $\mathcal{D}_{\infty_i} \otimes \mathcal{O}_{\Delta_i}$-modules E_i and E_i'.

9 Generalized Dieudonné $\mathcal{D}_\infty$-modules

9.1 Dieudonné $\mathcal{D}_\infty$-modules

First we will define *Dieudonné $\mathcal{D}_\infty$-modules* for $t = 1$ case. Denote the structure sheaf of the completion of $X \times S$ along $\infty \times V(\zeta)$ by $\mathcal{O}_\infty \widehat{\otimes} \mathcal{O}_S$. Let $S \in \mathcal{N}ilp_{k_\infty[\![\zeta]\!]}$. We extend the map σ^* on S to $\mathcal{O}_\infty \widehat{\otimes} \mathcal{O}_S$ as follows:

$$\sigma^*(x) = x^q \text{ for } x \in \mathcal{O}_S \text{ and } \sigma^* = id \text{ otherwise.}$$

Now, define $\sigma^* M = M \otimes_{\mathcal{O}_\infty \widehat{\otimes} \mathcal{O}_S, \sigma^*} (\mathcal{O}_\infty \widehat{\otimes} \mathcal{O}_S)$.

Definition 9.1. 1. ([24], Definition 2.4) A Dieudonné $\mathcal{O}_\infty$-module over S of dimension e and rank r is a pair (M, F) where

   (i) M is a locally free sheaf of $\mathcal{O}_\infty \widehat{\otimes} \mathcal{O}_S$-modules of rank r and

   (ii) $F : \sigma^* M \longrightarrow M$ is an $\mathcal{O}_\infty \widehat{\otimes} \mathcal{O}_S$-module homomorphism where $\operatorname{coker} F$ is locally free of rank e as an $\mathcal{O}_S$-module.

2. A Dieudonné $\mathcal{D}_\infty$-module of rank r and dimension e over S is a Dieudonné $\mathcal{O}_\infty$-module $\underline{M} = (M, F)$ of rank rd and dimension ed over S together with an $\mathcal{O}_\infty$-algebra homomorphism $\mathcal{D}_\infty \longrightarrow \operatorname{End}(M)$ extending the action of $\mathcal{O}_\infty$ and which F is compatible with $\mathcal{D}_\infty$-action.

A *morphism* of Dieudonné modules is a morphism of locally free $\mathcal{O}_\infty \widehat{\otimes} \mathcal{O}_S$-modules which is compatible with F and the $\mathcal{D}_\infty$-action.

Remark 9.2. If $\deg \infty = 1$ and $\mathcal{D}_\infty = \mathcal{O}_\infty$, a Dieudonné $\mathcal{D}_\infty$-module is nothing but Dieudonné $\mathcal{O}_\infty$-module.

Remark 9.3. ([24], ex 2.8) For every Dieudonné $\mathcal{O}_\infty$-module $\underline{M} = (M, F)$ over S of rank r, we have $\underline{M} = \varprojlim (M/z^n M, F \bmod z^n M)$ where $M/z^n M$ considered as locally free $\mathcal{O}_S$-module of rank rn and $F \bmod z^n M$ is an $\mathcal{O}_S$-module homomorphism for every $n \in \mathbb{N}$.

Morita equivalence for Dieudonné Modules

Let $\underline{M} = (M, F)$ be a dieudonné $\mathcal{O}_\infty$-module of rank r and dimension e. Define $\underline{M}' = (M', F')$ as follows:

$$M' := M \otimes_{\mathcal{O}_\infty} N$$

and

$$F' := F \otimes_{\mathcal{O}_\infty} N$$

where $N = \mathcal{O}_\infty^{1\times d}$ is an $\mathcal{O}_\infty$-$\mathbb{M}_d(\mathcal{O}_\infty)$-bimodule. Since ∞ is unramified, we have $\mathcal{D}_\infty \simeq \mathbb{M}_d(\mathcal{O}_\infty)$. So, M' is a locally free $(\mathcal{O}_\infty\widehat{\otimes}\mathcal{O}_S)$-module of rank $r\cdot d$ with $\mathcal{D}_\infty$-action and F' is a $(\mathcal{O}_\infty\widehat{\otimes}\mathcal{O}_S)$-module homomorphism that respects the $\mathcal{D}_\infty$-action with $\operatorname{coker} F' \simeq \operatorname{coker} F^{\oplus d}$. So, we have:

Proposition 9.4. *The category of Dieudonné $\mathcal{O}_\infty$ -modules of rank r and dimension e is equivalent to the category of Dieudonné $\mathcal{D}_\infty$-modules of rank $r\cdot d$ and dimension $e\cdot d$.*

Theorem 9.5. *([24], Theorem 8.3)*

The functor $E\mapsto (M_E, F_E)$ gives an anti-equivalence between categories of $\mathcal{O}_\infty$-z-divisible groups of height h and dimension d over S and Dieudonné $\mathcal{O}_\infty$-modules of rank h and dimension d over S. Furthermore, the $\mathcal{O}_S[\![z]\!]$-modules ω_E and $\operatorname{coker} F_E$ are isomorphic.

Proof. Let $S\in \mathcal{N}ilp_{k_\infty[\![\zeta]\!]}$ and let $\mathbb{G}_a = \operatorname{Spec}\mathcal{O}_S[\![\xi]\!]$ be the additive group scheme over S. There is a Frobenius endomorphism on $\mathbb{G}_a$ defined by $Frob_q^*\xi = \xi^q$. Let $\underline{E} = (E_n, i_n)$ be an $\mathcal{O}_\infty$-z-divisible group over S. Define the sheaf on S

$$M_E := \varprojlim \mathcal{H}om_S(G_n, \mathbb{G}_a)$$

where G_n is the balanced group scheme underlying E_n for each n. We make M_E a sheaf of $\mathcal{O}_S[\![z]\!]$-modules by the multiplication by z on $\underline{E}$, which is locally free by [24], Lemma 8.1. Moreover, composition with σ defines a map

$$F_{M_E} : \sigma^* M_E \longrightarrow M_E,\ \ f\mapsto \sigma\circ f \tag{9}$$

So, (M_E, F_E) is a Dieudonné $\mathcal{O}_\infty$-module.

Let (M,F) be a Dieudonné $\mathcal{O}_\infty$ module. By Remark 9.3, $M = \varprojlim(M/z^nM)$. Put $M_n := M/z^nM$

Claim: $E_M = \varinjlim \operatorname{Spec}(\operatorname{Sym}(M_n)/\mathfrak{f}_n)$ is a z-divisible group where $\operatorname{Sym} M_n$ denotes the symmetric algebra of M_n and $\mathfrak{f}_n$ is the ideal $(x^{\otimes q} - F_n x | x\in M_n)$

Proof: One can easily see that the kernel of the map $z^n : M_{n+1} \longrightarrow M_{n+1}$ is M_n. We need to show that for each $n\geq 1$, $\operatorname{Spec}(\operatorname{Sym}(M_n)/\mathfrak{f}_\mathfrak{n})$ is a balanced group scheme. The comultiplication and the $\mathbb{F}_q$-action are given by

$$\Delta(x) = x\otimes 1 + 1\otimes x \ \text{ and } \ \alpha.x = \alpha x$$

Via the surjective map $\operatorname{Sym} M_n \twoheadrightarrow \operatorname{Sym} M_n/\mathfrak{f}_n$, we get an embedding

$$\operatorname{Spec}(\operatorname{Sym} M_n/\mathfrak{f}_n) \hookrightarrow \mathbb{G}_a^N$$

for some set N. Moreover, the products $\{\prod x_i^{m_i}\}$ where $x_i \in M_n$ and $m_i \in \{0,\dots,q-1\}$ form a basis of $\operatorname{Sym} M_n/\mathfrak{f}_n$. Therefore, $\operatorname{ord}(\operatorname{Sym}(M_n)/\mathfrak{f}_n) = q^{\operatorname{rank} M_n}$, hence $\operatorname{Spec}(\operatorname{Sym} M_n/\mathfrak{f}_n)$ is a balanced group scheme over S by Definition 8.1, (iv).

This also gives us the third condition since $rank_{\mathcal{O}_S} M_n = hn$. The last condition follows from [24], Theorem 8.3(e).

$\square$

Corollary 9.6. *The functor $E \mapsto (M_E, F_E)$ gives an anti-equivalence between categories of z-divisible $\mathcal{D}_\infty$-modules of height h and dimension d over S and Dieudonné $\mathcal{D}_\infty$-modules of rank h and dimension d over S. Furthermore, the $\mathcal{O}_\infty \widehat{\otimes} \mathcal{O}_S$-modules ω_E and $coker F_E$ are isomorphic.*

Proof. Let $E = (E_n, i_n)$ be a z-divisible $\mathcal{D}_\infty$-module over S. Then by Morita equivalence, $(E_n, i_n) = (E'_n \otimes_{\mathcal{O}_\infty} N, i'_n \otimes id)$ where (E'_n, i'_n) is an $\mathcal{O}_\infty$-z-divisible group and $N = \mathcal{O}_\infty^{d\times 1}$. By Theorem 9.5, there is a Dieudonné- $\mathcal{O}_\infty$-module (M, F). So, by Morita equivalence we get the Dieudonné $\mathcal{D}_\infty$-module $(M \otimes_{\mathcal{O}_\infty} \mathcal{O}_{1\times d}, F \otimes_{\mathcal{O}_\infty} \mathcal{O}_{1\times d})$ corresponding to the z-divisible $\mathcal{D}_\infty$-module (E_n, i_n).

Conversely, let $\underline{M} = (M, F)$ be a Dieudonné $\mathcal{D}_\infty$-module. We can express $\underline{M}$ as $(M' \otimes_{\mathcal{O}_\infty} N, F' \otimes_{\mathcal{O}_\infty} N)$ where (M', F') is a Dieudonné $\mathcal{O}_\infty$-module and $N = \mathcal{O}_\infty^{d\times 1}$. Then, by Theorem 9.5, there exists a $\mathcal{O}_\infty$-z-divisible group $\underline{E} = (E_n, i_n)$. By Morita equivalence there we get a z-divisible $\mathcal{D}_\infty$-modules $(E_n \otimes_{\mathcal{O}_\infty} \mathcal{O}_\infty^{d\times 1}, i_n \otimes_{\mathcal{O}_\infty} \mathcal{O}_\infty^{d\times 1})$.

$\square$

To get the classification of p-divisible groups in the classical case, one works with isocrystals. In [23], the isocrystals are defined in the case of abelian sheaves:

Definition 9.7. A Dieudonné $\mathbb{F}_q((z))$-module over S is a tuple (M, F) where M is finite locally free $\mathcal{O}_S[\![z]\!][1/z]$-module and $F : \sigma^* M \longrightarrow M$ is an isomorphism

Example 9.8. Let (M, F) be a Dieudonné $\mathbb{F}_q[\![z]\!]$-module over S. Its corresponding Dieudonné $\mathbb{F}_q((z))$-module is :

$$\underline{M}[1/z] := \big(M \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S[\![z]\!][1/z], F \otimes id\big)$$

We define the quasi-isogeny between two Dieudonné $\mathbb{F}_q[\![z]\!]$-modules by using their isocrystals:

Definition 9.9. A *quasi-isogeny* between Dieudonné $\mathbb{F}_q[\![z]\!]$-modules $f : (M, F) \longrightarrow (M', F')$ is an isomorphism of the corresponding Dieudonné $\mathbb{F}_q((z))$-modules

$$f : M \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S[\![z]\!][1/z] \xrightarrow{\sim} M' \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S[\![z]\!][1/z]$$

such that $f \circ F = F' \circ \sigma^*(f)$.

By using Morita equivalence we can also define a quasi-isogeny between Dieudonné $\mathcal{D}_\infty$-modules $\underline{M}$ and $\underline{M}'$ by using the isocrystals of the corresponding Dieudonné $\mathcal{O}_\infty$-modules:

Definition 9.10. 1. Let $\underline{N}, \underline{N}'$ be two Dieudonné $\mathcal{D}_\infty$-modules. Then, by Morita equivalence $\underline{N} = \underline{M} \oplus \cdots \oplus \underline{M}$ and $\underline{N}' = \underline{M}' \oplus \cdots \oplus \underline{M}'$ for some Dieudonné $\mathcal{O}_\infty$-modules $\underline{M}, \underline{M}'$. A *quasi-isogeny* $f : \underline{N} \longrightarrow \underline{N}'$ is an isomorphism $(\underline{M} \oplus \cdots \oplus \underline{M})[1/z] \longrightarrow (\underline{M}' \oplus \cdots \oplus \underline{M}')[1/z]$ which is compatible with the $\mathcal{D}_\infty$-action.

2. We will denote the group of quasi-isogenies between two Dieudonné $\mathcal{D}_\infty$-modules $\underline{M}$ and $\underline{M}'$ by $\mathrm{QIsog}_{\mathcal{D}_\infty}(\underline{M}, \underline{M}')$.

Remark 9.11. Note that the map $E \mapsto (M_E, F_E)$ in Theorem 9.6 sends isogenies to isogenies and quasi-isogenies to quasi-isogenies.

Now, we can define the *Newton polygon* of a Dieudonné $\mathbb{F}_q[\![z]\!]$-module. For this we need the *slope decomposition* as in the classical case.

Let m/n be a rational number written in lowest term with $n \geq 0$. Define the Dieudonné $\mathbb{F}_q[\![z]\!]$-module $\mathcal{V}(m/n) = (\mathcal{V}, F)$ over $S = \operatorname{Spec} \mathbb{F}_q$ as follows:

$$\mathcal{V} = (\mathbb{F}_q(\!(z)\!))^n \,, \qquad F = \begin{pmatrix} 0 & \dots & & z^m \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \cdot \sigma^* \colon\; \sigma^*\mathcal{V} \to \mathcal{V} \,.$$

As an analogue to the classical case as in [43], we have the following

Theorem 9.12. *([23], Theorem 7.6) Let K be an algebraically closed field with $SpecK \in Nilp_{\mathbb{F}_q[\![\zeta]\!]}$. Then, every Dieudonné $\mathbb{F}_q[\![z]\!]$-module (M, F) over $SpecK$ is isomorphic to a decomposition*

$$\oplus_i\, \mathcal{V}(m_i/n_i) \otimes K(\!(z)\!) \tag{10}$$

where $m_1/n_1 \leq m_2/n_2 \leq \dots$ are determined uniquely. The Dieudonné $\mathbb{F}_q[\![z]\!]$-modules $\mathcal{V}(m_i/n_i)$ are called the component of slope m_i/n_i.

Let $\underline{M} = (M, F)$ be a Dieudonné $\mathbb{F}_q[\![z]\!]$-module over $\operatorname{Spec} K$ where $\operatorname{Spec} K \in \mathcal{N}ilp_{\mathbb{F}_q[\![z]\!]}$. By Theorem 9.12, we know that over an algebraically closed extension of K, the module M decomposes as (10). Then, the Newton polygon of $\underline{M}$ is the polygon passes through the points

$$(n_1 + \cdots + n_i, m_1 + \cdots + m_i)$$

for all i and extend linearly between these points.

Remark 9.13. The Newton polygon begins at $(0,0)$ and ends at $(\mathrm{rk}(\underline{M}), \dim(\underline{M}))$.

Definition 9.14. 1. Let $\underline{M} = (M, F)$ be a Dieudonné $\mathbb{F}_q[\![z]\!]$-module over $S = \operatorname{Spec} K$ where K is a field. We say $\underline{M}$ is *isoclinic* if $\underline{M}$ has only one slope appearing in the slope decomposition.

2. Let $\underline{M} = (M, F)$ be a Dieudonné $\mathcal{D}_\infty$-module. then, by Morita equivalence there is a Dieudonné $\mathcal{O}_\infty$-module $\underline{M}'$ such that $\underline{M} = \underline{M}' \otimes_{\mathcal{D}_\infty} \mathcal{O}_\infty^{d\times 1}$. We say that $\underline{M}$ is *isoclinic* if its corresponding Dieudonné $\mathcal{O}_\infty$-module is isoclinic.

Similarly, one can define a *Hodge polygon* of $\underline{M}$ over $S = \operatorname{Spec} K \in \mathcal{N}ilp_{\mathbb{F}_q[\![z]\!]}$. The Hodge polygon of a Dieudonné $\mathbb{F}_q[\![z]\!]$-module is defined by the elementary divisors of the $K[\![z]\!]$-module $\operatorname{coker} F$. The Hodge polygon has same initial and terminal point as the Newton polygon. An analogue of a theorem of Grothendieck-Katz ([30]) and Katz's constancy theorem([27], Theorem 2.7.1) in the function field world is stated in [23].

Let $\underline{M}$ be a Dieudonné $\mathcal{D}_\infty$-modules over S. For a morphism $f : S' \longrightarrow S$ in $\mathcal{N}ilp_{k_\infty[\![\zeta]\!]}$ we can define the pullback of $\underline{M} = (M, F)$ to the Dieudonné $\mathcal{D}_\infty$-module

$$(M \otimes_{\mathcal{O}_\infty \widehat{\otimes} \mathcal{O}_S} (\mathcal{O}_\infty \widehat{\otimes} \mathcal{O}_{S'}), F \otimes \mathrm{id})$$

over S'. We have the following rigidity theorem for Dieudonné $\mathcal{D}_\infty$-modules:

Proposition 9.15. *Let $\underline{M}$ and $\underline{M}'$ be two Dieudonné $\mathcal{O}_\infty$-modules over S. We denote the group of quasi-isogenies between $\underline{M}$ and $\underline{M}'$ by $\mathrm{QIsog}_S(\underline{M}, \underline{M}')$. Let $\iota : \bar{S} \hookrightarrow S$ be a closed scheme defined by a sheaf of ideals $\mathcal{I}$ that is locally nilpotent. Then,*

$$\mathrm{QIsog}_S(\underline{M}, \underline{M}') \longrightarrow \mathrm{QIsog}_{\bar{S}}(\iota^* \underline{M}, \iota^* \underline{M}')$$

is a bijection.

Proof. We may assume that $\mathcal{I}^q = (0)$. Then, the Frobenius morphism $\mathrm{Frob}_S : S \longrightarrow S$ factors as

$$S \xrightarrow{j} \bar{S} \xrightarrow{\iota} S$$

where $j : S \longrightarrow \bar{S}$ is the identity map between the underlying topological spaces $|\bar{S}| = |S|$. On the structure sheaves, we have

$$\mathcal{O}_S \xrightarrow{\iota^*} \mathcal{O}_{\bar{S}} \xrightarrow{j*} \mathcal{O}_S$$

defined by $x \mapsto x \bmod \mathcal{I} \mapsto x^q$. So, we have $\sigma^* f = j^*(\iota^* f)$ for $f \in \mathrm{QIsog}_S(\underline{M}, \underline{M}')$. We have the diagram

$$\begin{tikzcd} M \arrow[r, "f"] & M' \\ \sigma^* M \arrow[u, "F"] \arrow[r, "j^*\iota^* f"] & \sigma^* M' \arrow[u, "F'"'] \end{tikzcd}$$

Let $f, g \in \mathrm{QIsog}_S(\underline{M}, \underline{M}')$ such that $\iota^* f = \iota^* g$. Via a similar diagram as above we have

$$f \circ (F \otimes \mathrm{id}) = (F \otimes \mathrm{id}) \circ j^* \iota^* f = (F \otimes \mathrm{id}) \circ j^* \iota^* g = g \circ (F \otimes id)$$

where the last equality comes from considering a diagram as above for the quasi-isogeny g. Hence, we have $f = g$, which shows the injectivity. The surjectivity also follows directly from the diagram above and the fact that $\sigma^* f = j^* \iota^* f$ for a quasi-isogeny f.

$\square$

Proposition 9.16. *Let $\underline{N}, \underline{N}'$ be two Dieudonné $\mathcal{D}_\infty$-modules over S. Let $\iota : \bar{S} \hookrightarrow S$ be a closed subscheme defined by locally nilpotent sheaf of ideals. Then,*

$$\mathrm{QIsog}\,\mathcal{D}_\infty(\underline{N}, \underline{N}') \longrightarrow \mathrm{QIsog}_{\mathcal{D}_\infty}(\iota^*\underline{N}, \iota^*\underline{N}')$$

Proof. Let $f \in \mathrm{QIsog}_{\mathcal{D}_\infty}(\underline{N}, \underline{N}')$ be any. by definition f gives us a quasi-isogeny $g : \underline{M} \longrightarrow \underline{M}'$ of Dieudonné $\mathcal{O}_\infty$-modules such that $f\mathcal{D}_\infty = \mathcal{D}_\infty f$. By Proposition 9.15, we have $\bar{g} \in \mathrm{QIsog}_{\bar{S}}(\iota^*\underline{M}, \iota^*\underline{M}')$ with $\bar{g}\mathcal{D}_\infty = \mathcal{D}_\infty \bar{g}$. By using $\bar{g}$ we get an $\bar{f} \in \mathrm{QIsog}_{\mathcal{D}_\infty}(\iota^*\underline{M}, \iota^*\underline{M}')$. $\square$

By Remark 9.11, we can rewrite the rigidity theorem as follows:

Proposition 9.17. *Let E and E' be two z-divisible $\mathcal{D}_\infty$-modules over S. Let $\iota : \bar{S} \longrightarrow S$ be a closed subscheme of S defined by a sheaf of ideals that is locally nilpotent. Then,*

$$\mathrm{QIsog}_S(E, E') \longrightarrow \mathrm{QIsog}_{\bar{S}}(\iota^*E, \iota^*E')$$

is a bijection.

By using this theorem, we can prove the rigidity for generalized $\mathcal{D}_{\underline{\infty}}$-$\underline{z}$-divisible groups:

Theorem 9.18 (Rigidity theorem)**.** *Let $\iota : S' \longrightarrow S$ be a closed subscheme defined by a locally nilpotent sheaf of ideals. Let $\underline{E}$ and $\underline{E}'$ be two generalized $\mathcal{D}_{\underline{\infty}}$-$\underline{z}$-divisible groups. Then, every (quasi-)isogeny $\rho' : \iota^*\underline{E} \longrightarrow \iota^*\underline{E}'$ lifts uniquely to a (quasi-)isogeny $\rho : \underline{E} \longrightarrow \underline{E}'$.*

Proof. Let $\rho' : \iota^*\underline{E} \longrightarrow \iota^*\underline{E}'$ be a quasi-isogeny between generalized $\mathcal{D}_{\underline{\infty}}$-$z$-divisible groups $\underline{E} = (E_1, \cdots, E_t)$ and $\underline{E}' = (E_1', \cdots, E_t')$ where each E_i and E_i' is a $\mathcal{D}_{\infty_i}$-z-divisible groups. Then, by definition we have quasi-isogenies $\rho_i' : \iota^*E_i \longrightarrow \iota^*E_i'$. By Theorem 9.17, ρ_i' lifts uniquely to a quasi-isogeny $\rho_i : E_i \longrightarrow E_i'$. Hence, ρ' liftly uniquely to a quasi-isogeny $\rho := (\rho_1, \cdots, \rho_t)$.

$\square$

9.2 Generalized Dieudonné $\mathcal{D}_{\underline{\infty}}$-modules

Let S be as before. Denote the formal completion of $X \times S$ along the closed subscheme $\mathbb{T} \times S$ by $\widehat{X \times S}^{\mathbb{T}\times S}$. Then,

$$\widehat{X \times S}^{\mathbb{T}\times S} = \coprod Spf(\mathcal{O}_{\infty_i}\hat{\otimes}\mathcal{O}_S)$$

Let us denote the structure sheaf of the completion of $X \times S$ along the closed subscheme $\{\infty_j\} \times V(\zeta_j)$ by $\mathcal{O}_{\infty_j}\widehat{\otimes}\mathcal{O}_S$ where $j \in \{1, \ldots, t\}$.

Definition 9.19. A *generalized Dieudonné $\mathcal{D}_{\underline{\infty}}$-module of rank r and dimension e* $\underline{M}$ over S is t-tuple $(\underline{M}_1, \cdots, \underline{M}_t)$ of Dieudonné $\mathcal{D}_{\infty_j}$-modules $\underline{M}_j = (M_j, F_j)$ of rank r and dimension e for $j = 1, \cdots, t$.

Definition 9.20. Let $\underline{M} = (M, F) = \big((M_1, F_1), \cdots, (M_t, F_t)\big)$ and $\underline{M}' = (M', F') = \big((M'_1, F'_1), \cdots, (M'_t, F'_t)\big)$ be two generalized Dieudonné $\mathcal{D}_{\underline{\infty}}$-modules over S.

1. An *(quasi-)isogeny* between two generalized Dieudonné $\mathcal{D}_{\underline{\infty}}$-modules $\underline{M}$ and $\underline{M}'$ is a t-tuple of (quasi-)isogenies between Dieudonné $\mathcal{D}_{\infty_j}$-modules.

2. We denote the set of quasi-isogenies between two generalized Dieudonné $\mathcal{D}_{\underline{\infty}}$-modules $\underline{M}$ and $\underline{M}'$ by $\mathrm{QIsog}(\underline{M}, \underline{M}')$.

Theorem 9.21. *There is a categorical equivalence between the category of generalized $\mathcal{D}_{\underline{\infty}}$-$\underline{z}$-divisible groups and the category of generalized Dieudonné $\mathcal{D}_{\underline{\infty}}$-modules.*

Proof. The proof follows from the fact that a generalized $\mathcal{D}_{\underline{\infty}}$-$\underline{z}$-divisible group (resp. a generalized Dieudonné $\mathcal{D}_{\underline{\infty}}$-module) is tuples of $\mathcal{D}_{\infty_j}$-z_j-divisible groups (resp. Dieudonné $\mathcal{D}_{\infty_j}$-modules) and Theorem 9.5. □

Remark 9.22. Similar in the $\mathcal{D}$-elliptic sheaf case, Theorem 9.21 sends quasi-isogenies to quasi-isogenies.

Morita Equivalence

Let $(M, F) = \big((M_1, F_1), \cdots, (M_t, F_t)\big)$ be a generalized Dieudonné $\mathcal{D}_{\underline{\infty}}$-module. We know that each (M_i, F_i) is Morita equivalent to a Dieudonné $\mathcal{O}_{\infty_i}$-module (M'_i, F'_i). Hence, (M, F) is Morita equivalent to $\big((M'_1, F'_1), \cdots, (M'_t, F'_t)\big)$.

10 Generalized Formal $\mathcal{D}_\infty$-elliptic sheaves

10.1 Formal $\mathcal{D}_\infty$-elliptic sheaves

We are back to the case when $X = Y$. Let $S \in \mathcal{N}ilp_{k_\infty^{(d)}[\![\zeta]\!]}$. Recall that we denote the structure sheaf of the completion of $X \times S$ along $\infty \times V(\zeta)$ by $\mathcal{O}_\infty \widehat{\otimes} \mathcal{O}_S$. Denote the sheaf $\mathcal{O}_\Delta \otimes_{\mathcal{O}_\infty} (\mathcal{O}_\infty \widehat{\otimes} \mathcal{O}_S)$ on S by $\mathcal{O}_\Delta \widehat{\otimes} \mathcal{O}_S$.

Definition 10.1. (a) ([23], Definition 7.11)Let $S \in \mathcal{N}ilp_{k_\infty^{(d)}[\![z]\!]}$. A formal abelian sheaf of rank r and dimension e over S is a sheaf $\widehat{\mathcal{F}}$ of $\mathcal{O}_\Delta \widehat{\otimes} \mathcal{O}_S$-modules on S together with an isomorphism of $\mathcal{O}_\Delta \widehat{\otimes} \mathcal{O}_S$-modules $F : \sigma^* \widehat{\mathcal{F}} \longrightarrow \widehat{\mathcal{F}}$ such that $(\widehat{\mathcal{F}}, F)$ is a Dieudonné $\mathcal{O}_\infty$-module over S of rank $r\ell$ and $\operatorname{coker} F$ is locally free of rank e as an $k_\infty^{(\ell)} \otimes \mathcal{O}_S$-module.

(b) A morphism of formal abelian sheaves is a morphism of the corresponding Dieudonne $\mathcal{O}_\infty$-modules.

Definition 10.2. Let $S \in \mathcal{N}ilp_{k_\infty^{(d)}[\![\zeta]\!]}$. A *formal $\mathcal{D}_\infty$-abelian sheaf* over S of rank r and dimension e is a formal abelian sheaf of rank rd and dimension ed with an $\mathcal{O}_\infty$-algebra homomorphism $\mathcal{D}_\infty \longrightarrow \operatorname{End}(\underline{\widehat{\mathcal{F}}})$.

Remark 10.3. 1. Let $\widehat{\underline{\mathcal{F}}} = (\widehat{\mathcal{F}}, F)$ be a formal $\mathcal{D}_\infty$-abelian sheaf. Then, $\underline{\mathcal{F}}$ has an $(\mathcal{D}_\infty \otimes_{\mathcal{O}_\infty} \mathcal{O}_\Delta) \widehat{\otimes} \mathcal{O}_S$-action and so F respects the $(\mathcal{D}_\infty \otimes_{\mathcal{O}_\infty} \mathcal{O}_\Delta) \widehat{\otimes} \mathcal{O}_S$-action.

2. We want to note that a formal $\mathcal{D}_\infty$-abelian sheaf is in particular a Dieudonné $\mathcal{D}_\infty$-module.

Definition 10.4. A morphism of formal $\mathcal{D}_\infty$-abelian sheaves is a morphism of Dieudonné $\mathcal{D}_\infty$-modules that is compatible with the $(\mathcal{D}_\infty \otimes_{\mathcal{O}_\infty} \mathcal{O}_\Delta) \widehat{\otimes} \mathcal{O}_S$-module action and the $\mathcal{D}_\infty$-action.

We can extend Theorem 9.6 to special z-divisible $\mathcal{D}_\infty \otimes_{\mathcal{O}_\infty} \mathcal{O}_\Delta$-modules and formal $\mathcal{D}_\infty$-elliptic sheaves:

Theorem 10.5. *The functor $E \mapsto (M_E, F_E)$ is an anti-equivalence of categories between the category of special z-divisible $\mathcal{D}_\infty \otimes \mathcal{O}_\Delta$ of height rl and dimension el and the category of formal $\mathcal{D}_\infty$-abelian sheaf of rank r and dimension e.*

Proof. We know by Theorem 9.6 that the category of z-divisible $\mathcal{D}_\infty$-modules and the category of Dieudonné $\mathcal{D}_\infty$-modules are anti-equivalent and $\operatorname{coker} F$ and ω_E are isomorphic. The condition on ranks of $\operatorname{coker} F$ and ω_E as $k_\infty^{(\ell)} \otimes \mathcal{O}_S$-modules follows immediately.

$\square$

Definition 10.6. An *(quasi-)isogeny* of formal $\mathcal{D}_\infty$-abelian sheaves is an (quasi-)isogeny between the corresponding Dieudonné $\mathcal{D}_\infty$-modules that are compatible with the $(\mathcal{D}_\infty \otimes \mathcal{O}_\Delta)\widehat{\otimes}\mathcal{O}_S$-module action and the $\mathcal{D}_\infty$-action.

In the next example we will construct a formal $\mathcal{D}_\infty$-abelian sheaf associated to a $\mathcal{D}$-elliptic sheaf. We will call this formal $\mathcal{D}_\infty$-abelian sheaf associated to a $\mathcal{D}$-elliptic sheaf *formal $\mathcal{D}_\infty$-elliptic sheaf.* In the classical case, one can obtain a p-divisible group associated to an abelian variety. By this construction, we will get the z-divisible $\mathcal{D}_\infty$-module associated to a $\mathcal{D}$-elliptic sheaf.

Construction 10.7. Let $S \in \mathcal{N}ilp_{k_\infty[\![\zeta]\!]}$ and $\underline{\mathcal{E}} = (\mathcal{E}_i, j_i, t_i)$ be a $\mathcal{D}$-elliptic sheaf over S. Assume $\deg(\infty) = m$, so $k_\infty = \mathbb{F}_{q^m}$. We will denote by $\sigma^\infty = \sigma|_{\mathbb{F}_{q^m}}$ the relative Frobenius with respect to $\mathbb{F}_{q^m}$ and $\sigma = \mathrm{id}_X \times \sigma^\infty$.

Since $\deg\infty = m$ the periodicity of the $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{E}}$ is dm. Assume there is a morphism $\beta : S \longrightarrow \operatorname{Spec} k_\infty^{(d)}[\![\zeta]\!]$. We will define the formal $\mathcal{D}_\infty$-elliptic sheaf corresponding to the $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{E}}$. Now, we define

$$\mathcal{E}_i^\infty := \mathcal{E}_i \otimes_{\mathcal{O}_{X\times S}} (\mathcal{O}_\infty \widehat{\otimes}_{\mathbb{F}_q} \mathcal{O}_S)$$

We want to note that $\mathcal{E}_i^\infty$'s have periodicity dm since $\mathcal{E}_i$'s have periodicity dm. Note that there is a $\mathcal{D}_\infty = \mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{O}_\infty$-action on each $\mathcal{E}_i^\infty$ as $\mathcal{D}$ and $\mathcal{O}_\infty$ acts on $\mathcal{E}_i$ and $\mathcal{O}_\infty\widehat{\otimes}\mathcal{O}_S$, respectively. We can define the maps j_i^∞ and t_i^∞ via the morphisms j_i and t_i's of $\underline{\mathcal{E}}$:

$$j_i^\infty : \mathcal{E}_i^\infty \longrightarrow \mathcal{E}_{i+1}^\infty$$

$$t_i^\infty : \sigma^*\mathcal{E}_i^\infty \longrightarrow \mathcal{E}_{i+1}^\infty$$

We put

$$\mathcal{E}^\infty = (\mathcal{E}_i^\infty, j_i^\infty, t_i^\infty)$$

Define $\widetilde{\mathcal{E}}^\infty := (\widetilde{\mathcal{E}}_i^\infty, \widetilde{j}_i^\infty : \widetilde{\mathcal{E}}_i^\infty \longrightarrow \widetilde{\mathcal{E}}_{i+1}^\infty, \widetilde{t}_i^\infty : \sigma^*\widetilde{\mathcal{E}}_i^\infty \longrightarrow \widetilde{\mathcal{E}}_{i+1})$ as follows:

$$\widetilde{\mathcal{E}}_i^\infty := \mathcal{E}_{mi}^\infty,$$

$$\widetilde{j}_i^\infty := j_{mi+m-1}^\infty \circ \cdots \circ j_{mi}^\infty : \mathcal{E}_{mi}^\infty \longrightarrow \mathcal{E}_{mi+m}^\infty \tag{11}$$

and

$$\widetilde{t}_i^\infty : (\sigma^0)^* t_{mi+m-1}^\infty \circ \cdots \circ (\sigma^{m-1})^* t_{mi} : \sigma^*\mathcal{E}_{mi}^\infty \longrightarrow \mathcal{E}_{mi+m}^\infty \tag{12}$$

We want to note that $\widetilde{\mathcal{E}}^\infty$ has periodicity d by definition. Now, define

$$\widehat{\mathcal{E}} := \oplus_{i=0}^{d-1} \widetilde{\mathcal{E}}_i^\infty$$

We let $\lambda \in k_\infty^d$ act on $\widetilde{\mathcal{E}}_i^\infty$ as the scalar $\beta^*\lambda^{q^i}$. Define $F : \sigma^*\widehat{\mathcal{E}} \longrightarrow \widehat{\mathcal{E}}$ by 12. We can express Π, λ and F as block matrices:

$$\Pi = \begin{pmatrix} 0 & \dots & & z\widetilde{j}_{d-1}^\infty \\ \widetilde{j}_0^\infty & \ddots & & \vdots \\ & \ddots & \ddots & \\ & & \widetilde{j}_{d-2}^\infty & 0 \end{pmatrix}, \quad \lambda = \begin{pmatrix} \beta^*\lambda\,\mathrm{Id}_d & & & \\ & \beta^*\lambda^q\,\mathrm{Id}_d & & \\ & & \ddots & \\ & & & \beta^*\lambda^{q^{d-1}}\,\mathrm{Id}_d \end{pmatrix} \text{ and}$$

$$F = \begin{pmatrix} 0 & \dots & & z\widetilde{t}_{d-1}^\infty \\ \widetilde{t}_0^\infty & \ddots & & \vdots \\ & \ddots & \ddots & \\ & & \widetilde{t}_{d-2}^\infty & 0 \end{pmatrix}.$$

It can be easily seen that $F\Pi = \Pi F$ and $F\Lambda = \Lambda F$. Then, $\widehat{\mathcal{E}}$ is a sheaf of $\mathcal{O}_\Delta \otimes \mathcal{O}_S$-modules. Moreover we have:

1) $\widehat{\mathcal{E}}$ has rank d^3 over $\mathcal{O}_\infty \widehat{\otimes} \mathcal{O}_S$,

2) $\operatorname{coker} F$ has rank d^2 over $\mathcal{O}_S$,

3) $\operatorname{coker} F$ has rank d over $k_\infty^{(d)} \otimes \mathcal{O}_S$.

Now, via 1) and 2), $(\widehat{\mathcal{E}}, F)$ is a Dieudonné $\mathcal{D}_\infty$-module of rank d^3 and dimension d^2. And, via 3) we conclude that $(\widehat{\mathcal{E}}, F)$ is a formal $\mathcal{D}_\infty$-elliptic sheaf.

Remark 10.8. Any quasi-isogeny of a $\mathcal{D}$-elliptic sheaf induces a quasi-isogeny of a formal $\mathcal{D}_\infty$-elliptic sheaf.

Remark 10.9. We can reconstruct $\mathcal{E}^\infty$ from a given $\widetilde{\mathcal{E}}^\infty$. For this we will use the fact that there is a $\mathcal{D}_\infty \widehat{\otimes} \mathcal{O}_S$-action on $\widetilde{\mathcal{E}}^\infty$. Note that we have

$$\mathcal{D}_\infty \widehat{\otimes}_{\mathbb{F}_q} \mathcal{O}_S = (\mathcal{D} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}) \widehat{\otimes}_{\mathbb{F}_{q^m}} \mathcal{O}_S = (\mathcal{D}_\infty e_1 \oplus \cdots \oplus \mathcal{D}_\infty e_m) \widehat{\otimes} \mathcal{O}_S$$

Now let $\widetilde{\mathcal{E}}^\infty = (\widetilde{\mathcal{E}}^\infty, \widetilde{j}^\infty, \widetilde{t}_i^\infty)$ be as in the Construction 10.7. More precisely, recall that $\widetilde{\mathcal{E}}_i^\infty = \mathcal{F}_{mi}^\infty$. Define for $j \in \{0, \cdots, m-1\}$

$$\mathcal{E}_{mi+j}^\infty = \oplus_{j'=0}^{j-1} (\sigma^{j'})^* \widetilde{\mathcal{E}}_{i+1}^\infty e_{j'} \oplus \oplus_{j'=j}^{m-1} (\sigma^{j'})^* \widetilde{\mathcal{E}}_i^\infty e_{j'}.$$

Then, we have $\mathcal{E}_{mi+j}^\infty \subset \mathcal{E}_{mi+j+1}^\infty$ because $(\sigma^j)^* \widetilde{\mathcal{E}}_i^\infty \subset (\sigma^j)^* \widetilde{\mathcal{E}}_{i+1}^\infty$ and we define j_i^∞ as the inclusion $\mathcal{E}_{mi+j}^\infty \subset \mathcal{E}_{mi+j+1}^\infty$. Now, we will define t_i^∞'s. First observe that

$$\sigma^*(\mathcal{E}_{mi+j}^\infty) = \left(\oplus_{j'=1}^{j} (\sigma^j)^* \widetilde{\mathcal{E}}_{i+1}^\infty e_{j'} \right) \oplus \left(\oplus_{j'=j+1}^{m-1} (\sigma^{j'})^* \widetilde{\mathcal{E}}_i^\infty e_{j'} \right) \oplus (\sigma^m)^* \widetilde{\mathcal{E}}_i^\infty e_0$$

Now, note that $\widetilde{t}_i^\infty$ is defined from the last compontent $(\sigma^m)^*\widetilde{\mathcal{E}}_i^\infty e_0$ of $(\sigma^m)^*\widetilde{\mathcal{E}}_i^\infty$ to $\widetilde{\mathcal{E}}_{i+1}^\infty$. Hence, we can define $t_i^\infty : \sigma^*\mathcal{E}_i^\infty \longrightarrow \mathcal{E}_{i+1}^\infty$ as the identity on the first $m-1$ components and as $\widetilde{t}_i^\infty$ on the last component. Therefore we have $(\mathcal{E}_i^\infty, j_i^\infty, t_i^\infty)$.

Remark 10.10. 1. A formal $\mathcal{D}_\infty$ abelian sheaf $\underline{\widehat{\mathcal{E}}} = (\widehat{\mathcal{E}}, F)$ carries an action of $\mathcal{O}_\Delta$. The latter contains $k_\infty^{(\ell)}$ as a subfield. Denote this action by φ. We can therefore decompose $\widehat{\mathcal{E}}$ into eigenspaces for the $k_\infty^{(\ell)}$-action as follows:

$$\widetilde{\mathcal{E}}_i := \{\alpha \in \widehat{\mathcal{E}} \mid \varphi(\lambda)\cdot\alpha = \beta^*\lambda^{q^i}\alpha, \forall\lambda \in k_\infty^{(\ell)}\}$$

Then, $\widehat{\mathcal{E}} = \oplus_{i=1}^{\ell}\widetilde{\mathcal{E}}_i$. Note that if $\widehat{\mathcal{E}}$ is constructed from a $\mathcal{D}$-elliptic sheaf $\mathcal{E}$ and $d=\ell$, these eigenspaces recover precisely $\widetilde{\mathcal{E}}^\infty$ in the Construction 10.7.

2. Let $\underline{\widehat{\mathcal{E}}} = (\widehat{\mathcal{E}}, F)$ and $\underline{\widehat{\mathcal{E}}}' = (\widehat{\mathcal{E}}', F')$ be two formal $\mathcal{D}_\infty$-elliptic sheaf. Let $\widehat{f} : \underline{\widehat{\mathcal{E}}} \longrightarrow \underline{\widehat{\mathcal{E}}}'$ be a quasi-isogeny. By (1), we can write $\widehat{\mathcal{E}} = \oplus_{i=1}^{\ell}\widetilde{\mathcal{E}}_i$ and $\widehat{\mathcal{E}}' = \oplus_{i=1}^{\ell}\widetilde{\mathcal{E}}_i'$. By definition, $\widehat{f}$ commutes with the $\mathcal{O}_\Delta$-action, in particular with the $k_\infty^{(d)}$-action. Therefore, $\widehat{f}$ sends $\widetilde{\mathcal{E}}_i$ to $\widetilde{\mathcal{E}}_i'$. This means that quasi-isogenies between formal $\mathcal{D}_\infty$-elliptic sheaves is componentwise.

Proposition 10.11. *The category of formal $\mathcal{O}_\infty$-elliptic sheaves of rank r and dimension e is Morita equivalent to the category of formal $\mathcal{D}_\infty$-elliptic sheaves of rank r and dimension e.*

Proof. Let $(\widehat{\mathcal{E}}, F)$ be a formal $\mathcal{O}_\infty$-elliptic sheaf of rank r and dimension e. Then, $(\widehat{\mathcal{E}}, F)$ is a Dieudonné $\mathcal{O}_\infty$-module of rank $r\ell$ and dimension $e\ell$. Similar to the previous sections, define

$$(\widehat{\mathcal{E}}', F') = (\widehat{\mathcal{E}} \otimes \mathcal{O}_\infty^{1\times d}, F \otimes id).$$

By Morita equivalence for Dieudonné $\mathcal{D}_\infty$-modules, $(\widehat{\mathcal{E}}', F')$ is a Dieudonné $\mathcal{D}_\infty$-module of rank $r\ell \cdot d$ and dimension $e\ell$. Now, the fact that $\operatorname{coker} F'$ is locally free of rank ed over $k_\infty^{(\ell)} \otimes \mathcal{O}_S$ follows. □

Definition 10.12. We say a formal $\mathcal{D}_\infty$-elliptic sheaf is *isoclinic* if it is isoclinic as a Diedonné $\mathcal{D}_\infty$-module.

Theorem 10.13. *Let $\underline{\widehat{\mathcal{E}}} = (\widehat{\mathcal{E}}, F)$ be a formal $\mathcal{D}_\infty$-elliptic sheaf over S. Then, $\underline{\widehat{\mathcal{E}}}$ is isoclinic.*

Proof. By Morita equivalence of $\mathcal{D}_\infty$-elliptic sheaves we can write $\underline{\widehat{\mathcal{E}}} \simeq \underline{\widehat{\mathcal{F}}} \otimes \mathcal{O}_\infty^{d\times 1}$ where $\underline{\widehat{\mathcal{F}}}$ is a formal $\mathcal{O}_\infty$-elliptic sheaf. Since $\dim \underline{\widehat{\mathcal{F}}} = 1$ and $rk\underline{\widehat{\mathcal{F}}} = d$, there is only one choice of slope in the decomposition. Therefore, $\underline{\widehat{\mathcal{E}}}$ has only one slope in its slope decomposition, hence isoclinic. □

10.2 Generalized Formal $\mathcal{D}_{\underline{\infty}}$ elliptic sheaves

Now, we are in the situation that $\pi : X \longrightarrow Y$ is of degree $t \geq 1$.
Let $S \in \mathcal{N}ilp_{k_\infty^{(d)}[\![\zeta_1,\cdots,\zeta_t]\!]}$.

Definition 10.14. A generalized formal $\mathcal{D}_{\underline{\infty}}$-abelian sheaf $\widehat{\underline{\mathcal{E}}}$ of rank r and dimension e is t-tuple $(\widehat{\underline{\mathcal{E}}}_1, \cdots, \widehat{\underline{\mathcal{E}}}_t)$ of formal $\mathcal{D}_{\infty_j}$-abelian sheaves of rank r and dimension e for $j = 1, \cdots, t$.

Definition 10.15. A *morphism*(resp, *(quasi)-isogeny*) $f : \widehat{\underline{\mathcal{E}}} \longrightarrow \widehat{\underline{\mathcal{F}}}$ between two generalized formal $\mathcal{D}_{\underline{\infty}}$-abelian sheaves $\widehat{\underline{\mathcal{E}}} = (\widehat{\underline{\mathcal{E}}}_1, \cdots, \widehat{\underline{\mathcal{E}}}_t)$ and $\widehat{\underline{\mathcal{F}}} = (\widehat{\underline{\mathcal{F}}}_1, \cdots, \widehat{\underline{\mathcal{F}}}_t)$ is t-tuple $(f_1, \cdots, f_t)$ where each $f_j : \widehat{\underline{\mathcal{E}}}_j \longrightarrow \widehat{\underline{\mathcal{F}}}_j$ is a morphism (resp, (quasi)-isogeny) of formal $\mathcal{D}_{\infty_j}$-abelian sheaves.

Remark 10.16. (Morita equivalence) Let $S \in \mathcal{N}ilp_{k_\infty^{(d)}[\![\zeta_1,\cdots,\zeta_t]\!]}$ and let $\widehat{\underline{\mathcal{E}}} = (\widehat{\mathcal{E}}, F) = \big((\widehat{\mathcal{E}}_1, F_1), \cdots, (\widehat{\mathcal{E}}_t, F_t)\big)$ be a generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf over S. By Proposition 10.11, we know that each $(\widehat{\mathcal{E}}_j, F_j)$ is Morita equivalent to a formal $\mathcal{O}_{\infty_j}$-elliptic sheaf $(\widehat{\mathcal{E}}'_j, F'_j)$. So, $\widehat{\underline{\mathcal{E}}}$ is Morita equivalent to $((\widehat{\mathcal{E}}'_1, F'_1), \cdots, (\widehat{\mathcal{E}}'_t, F'_t))$.

Theorem 10.17. *The category of generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaves is anti-equivalent to the category of generalized special $\underline{z}$-divisible $\mathcal{D}_\infty \otimes \mathcal{O}_{\underline{\Delta}}$-modules.*

Proof. Let $\underline{E} = (\underline{E}^{(1)}, \cdots, \underline{E}^{(t)})$ be a generalized special $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}} \otimes \mathcal{O}_\Delta$-module, i.e, each $\underline{E}^{(j)}$ is a special z_j-divisible $\mathcal{D}_{\infty_j} \otimes \Delta_j$-module. By Theorem 10.5, we know that via the functor $E \mapsto (M_E, F_E)$ the category of special z-divisible $\mathcal{D}_\infty \otimes \mathcal{O}_\Delta$-modules is anti-equivalent to the category of formal $\mathcal{D}_\infty$-elliptic sheaves. By using this functor at each component we see that the category of generalized $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}} \otimes \mathcal{O}_{\underline{\Delta}}$-modules is anti-equivalent to the category of generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaves. $\square$

Construction 10.18. : Let $S \in \mathcal{N}ilp_{k_\infty[\![\zeta_1,\cdots,\zeta_t]\!]}$ and let $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ be a generalized $\mathcal{D}$-elliptic sheaf over S. We will denote the structure sheaf of the formal completion of $X \times S$ along the closed subscheme $\infty_j \times V(\zeta_j)$ by $\mathcal{O}_{\infty_j} \widehat{\otimes} \mathcal{O}_S$ for each $j = 1, \cdots, t$.

Assume $\deg(\infty_j) = m$, so $k_{\infty_j} = \mathbb{F}_{q^m}$. We will denote by $\sigma^{\infty_j} = \sigma|_{\mathbb{F}_{q^m}}$ the relative Frobenius with respect to $\mathbb{F}_{q^m}$ and $\sigma = \mathrm{id}_X \times \sigma^\infty$. Since $\deg \infty = m$ the periodicity of the $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{E}}$ is dm. Assume there is a morphism $\beta : S \longrightarrow \operatorname{Spec} k_\infty^{(d)}[\![\zeta_1, \cdots, \zeta_t]\!]$. Now, for each $j = 1, \cdots, t$ by proceeding as in the Construction 10.7 we get formal $\mathcal{D}_{\infty_j}$-elliptic sheaves $\widehat{\underline{\mathcal{E}}}^{(j)}$. Hence

$$\widehat{\underline{\mathcal{E}}} = (\widehat{\underline{\mathcal{E}}}^{(1)}, \cdots, \widehat{\underline{\mathcal{E}}}^{(t)})$$

is the generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf corresponding to the generalized $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{E}}$.

Remark 10.19. Let $\underline{\widehat{\mathcal{E}}} = (\underline{\widehat{\mathcal{E}}}^{(1)}, \cdots, \underline{\widehat{\mathcal{E}}}^{(t)})$ be a generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf. We know by Theorem 10.13 that each $\underline{\widehat{\mathcal{E}}}^{(j)}$ has only one slope, namely $\frac{1}{d}$ in its slope decomposition. So, the generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf has $(\frac{1}{d}, \cdots, \frac{1}{d})$ as its slope decomposition.

11 Moduli space of formal $\mathcal{D}_\infty$-elliptic sheaves

11.1 Main Example

In this section we will consider the case when $t = 1$ and hence $X = Y$. We will construct an example of $\mathcal{D}$-elliptic sheaf. First we will assume that $\deg \infty = 1$ and then $\deg \infty > 1$.

We will define the objects around ∞ and on the affine part $U := X \smallsetminus \infty$, and by glueing we will get $\mathcal{D}$-elliptic sheaves (see Section 20.2). Recall that for a $\mathcal{D}$-elliptic sheaf $(\mathcal{E}_i, j_i, t_i)$, we had $\mathcal{E}_i \simeq \mathcal{E}_{i+1}$ on U. In the following $\lfloor a \rfloor$ denotes the integer part of the given rational number a.

CASE 1($\deg \infty = 1$):

Let X be a smooth projective geometrically irreducible curve with constant field $\mathbb{F}_q$ and let $S = \operatorname{Spec} \mathbb{F}_q$. Assume $deg \infty = 1$. We will define $\mathcal{D}$-elliptic sheaf over $X \times \mathbb{F}_q$ with $\mathcal{E}_0 = \mathcal{D} \otimes \mathbb{F}_q$. Since $\mathcal{E}_i|_{X \smallsetminus \infty \times S}$ is the same for all i, the construction is by gluing and by exploring the $\mathcal{D}$-elliptic sheaf near ∞.

Define the decreasing chain of $\mathcal{O}_\infty$-lattices in F_∞^d:

$$L_i := \oplus_{j=1}^{d} \mathcal{O}_\infty z^{\lfloor (d-j-i)/d \rfloor}.$$

Define $\mathcal{E}_{i,\infty} := \operatorname{Hom}(L_0, L_i)$. Then, we have $\mathcal{E}_{0,\infty} = \mathbb{M}_d(\mathcal{O}_\infty) \simeq \mathcal{D}_\infty$. Clearly, $\operatorname{Hom}(L_0, L_i) \subseteq \operatorname{Hom}(L_0, L_{i+1})$, i.e, $\mathcal{E}_{i,\infty} \subset \mathcal{E}_{i+1,\infty}$. Note that each $\mathcal{E}_{i,\infty}$ is an $\mathbb{M}_d(\mathcal{O}_\infty)$-module. Now, define

$$j_i : \mathcal{E}_i \longrightarrow \mathcal{E}_{i+1}$$

by the natural inclusion $\operatorname{Hom}(L_0, L_i) \subseteq \operatorname{Hom}(L_0, L_{i+1})$. And directly by definition of L_i's, one can see that

$$L_d \simeq L_0(\infty).$$

which means we have the periodicity $\ell = d \deg \infty = d$. And, since $\deg \infty = 1$, we have $\sigma^* = id$. So, one can define

$$t_i : \sigma^* \mathcal{E}_i \longrightarrow \mathcal{E}_{i+1}$$

as j_i.

Now, we glue $\mathcal{E}_{i,\infty}$'s with $M := \Gamma(X \smallsetminus \{\infty\} \otimes \mathbb{F}_q, \mathcal{D} \otimes \mathbb{F}_q)$ and get locally free sheaves $\mathcal{E}_i$ over $X \times S$.

Then, $\underline{\mathcal{E}} = (\mathcal{E}_i, j_i, t_i)_{i \in \mathbb{Z}}$ is a $\mathcal{D}$-elliptic sheaf.

CASE 2($\deg \infty > 1$)

Let X be a smooth projective geometrically irreducible curve over $\mathbb{F}_q$ and let $S = \operatorname{Spec} \mathbb{F}_{q^m}$ where $m = \deg \infty$. Note that $\mathcal{O}_\infty \otimes \mathcal{O}_S = \oplus_{i=0}^{m-1} \mathcal{O}_{\infty_i}$, where $\mathcal{O}_{\infty_i} \simeq \mathcal{O}_\infty$ for each $i \in \{0, \ldots, m-1\}$.

Similarly in the $\deg\infty = 1$ case, we will define $\mathcal{D}$-elliptic sheaf over $X \times \mathbb{F}_{q^m}$ with $\mathcal{E}_0 = \mathcal{D} \otimes \mathbb{F}_{q^m}$. Since $\mathcal{E}_i|_{X \smallsetminus \infty \times S}$ is same for all i, the construction is by gluing and investigating the $\mathcal{D}$-elliptic sheaves around ∞.

We define

$$L_{s,i} = \oplus_{j=0}^{d-1} z^{\lfloor \frac{i-1-s-jm}{md} + 1 \rfloor} e_j \mathcal{O}_{\infty_s}$$

where e_j's are the bases at ∞_s

Now, put $\mathcal{E}_{i,\infty} := \oplus_{s=0}^{m-1} \operatorname{Hom}(L_{s,i}, L_{s,0})$. We glue $\mathcal{E}_{i,\infty}$ with $M = \Gamma(X \smallsetminus \{\infty\} \otimes \mathbb{F}_{q^m}, \mathcal{D} \otimes \mathbb{F}_{q^m})$ and get locally free sheaves $\mathcal{E}_i$ over $X \times \mathbb{F}_{q^m}$.

Define the morphisms $j_i : \mathcal{E}_i \longrightarrow \mathcal{E}_{i+1}$ via the inclusions $\operatorname{Hom}(L_{s,i}, L_{s,0}) \subseteq \operatorname{Hom}(L_{s,i+1}, L_{s,0})$. One can easily see that we have a periodicity $\mathcal{E}_{\ell,\infty} = \mathcal{E}_{d\cdot\deg\infty,\infty} = \mathcal{E}_0(\infty)$.

Now, we need to define t_i's. Note that σ interchange ∞_i's, say $\sigma(\infty_i) = \infty_{i+1}$ for $i = 0, \cdots, m-2$ and $\sigma(\infty_{m-1}) = \infty_0$. Let us consider $\mathcal{E}_{0,\infty}$. For each $s = 0, \cdots, m-1$, we have

$$\operatorname{Hom}(L_{s,0}, L_{0,0}) \simeq \mathbb{M}_d(\mathcal{O}_{\infty_s})$$

And σ sends $\mathbb{M}_d(\mathcal{O}_{\infty_i})$ to $\mathbb{M}_d(\mathcal{O}_{\infty_{i+1}})$ for $i = 1, \cdots, md-1$ and $\mathbb{M}_d(\mathcal{O}_{\infty_t})$ to $\mathbb{M}_d(\mathcal{O}_{\infty_1})$. Therefore, $\sigma^*\mathcal{E}_0 \simeq \mathcal{E}_0$. So define t_0 by using this isomorphism : $t_0 : \sigma^*\mathcal{E}_0 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{E}_1$. Now we need to define t_i for $i = 1, \ldots, md$. Note that

$$\begin{aligned}
\sigma^*\mathcal{E}_{i,\infty} &= \oplus_{s=0}^{m-1} \operatorname{Hom}(\sigma^* L_{s,i}, \sigma^* L_{s,0}) \\
&= \operatorname{Hom}\left(\sigma^*(\oplus_{j=0}^{d-1} z^{\lfloor \frac{i-1-s-jm}{md} + 1 \rfloor} e_j \mathcal{O}_{\infty_s}), \sigma^*(\oplus_{j=0}^{d-1} z^{\lfloor \frac{-1-s-jm}{dm} + 1 \rfloor} e_j \mathcal{O}_{\infty_s})\right) \\
&= \oplus_{s=0}^{m-2} \operatorname{Hom}\left(\oplus_{j=0}^{d-1} z^{\lfloor \frac{i-1-s-jm}{dm} + 1 \rfloor} e_j \mathcal{O}_{\infty_{s+1}}, \oplus_{j=0}^{d-1} z^{\lfloor \frac{-1-s-jm}{dm} + 1 \rfloor} e_j \mathcal{O}_{\infty_{s+1}}\right) \\
&\oplus \operatorname{Hom}\left(\oplus_{j=0}^{d-1} z^{\lfloor \frac{i-m-jm}{dm} + 1 \rfloor} e_j \mathcal{O}_{\infty_0}, \oplus_{j=0}^{d-1} z^{\lfloor \frac{-m-jm}{dm} + 1 \rfloor} e_j \mathcal{O}_{\infty_0}\right)
\end{aligned}$$

And

$$\begin{aligned}
\mathcal{E}_{i+1,\infty} &= \oplus_{s=0}^{m-1} \operatorname{Hom}(L_{s,i+1}, L_{s,0}) \\
&\operatorname{Hom}(\oplus_{j=0}^{d-1} z^{\lfloor \frac{i-jm}{dm} + 1 \rfloor} e_j \mathcal{O}_{\infty_0}, \oplus_{j=0}^{d-1} z^{\lfloor \frac{1-jm}{dm} + 1 \rfloor} e_j \mathcal{O}_{\infty_0}) \\
&\oplus \oplus_{s=1}^{m-1} \operatorname{Hom}\left(\oplus_{j=0}^{d-1} z^{\lfloor \frac{i-s-jm}{dm} + 1 \rfloor} e_j \mathcal{O}_{\infty_s}, \oplus_{j=0}^{d-1} z^{\lfloor \frac{-s-jm}{dm} + 1 \rfloor} e_j \mathcal{O}_{\infty_s}\right)
\end{aligned}$$

Now, it is easy to see that one gets the components of each $\mathcal{O}_{\infty_s}$'s change by a multiple of $z^{1/md}$. Hence, we define $t_i : \sigma^*\mathcal{E}_i \longrightarrow \mathcal{E}_{i+1}$. We also want to note that if $i \neq \ell m$ or a multiple of ℓm, then we have $\mathcal{E}_{i+1} = \mathcal{E}_i + t_i(\sigma^*\mathcal{E}_i)$.

Therefore we have a $\mathcal{D}$-elliptic sheaf, again denote it by $\underline{\boldsymbol{\mathcal{E}}}$. And, as in the Construction 10.7, we can compute the formal $\mathcal{D}_\infty$-elliptic sheaf corresponding to

$\underline{\boldsymbol{\mathcal{E}}}$. Denote this formal $\mathcal{D}_\infty$-elliptic sheaf by $\underline{\widehat{\boldsymbol{\mathcal{E}}}} = (\widehat{\boldsymbol{\mathcal{E}}}, F)$. Now, by Theorem 10.5, there exists a z-divisible $\mathcal{D}_\infty \otimes \mathcal{O}_\Delta$, which we will denote by $\underline{\mathbb{E}}$.

11.2 The Moduli functor

Now, we will define a moduli functor for the formal $\mathcal{D}_\infty$-elliptic sheaves which are quasi-isogeneous to $\underline{\widehat{\boldsymbol{\mathcal{E}}}}$ in the example in section 11.1 similar to [23], in which the author refers to [12] and [43]. As in previous cases, the solution of our moduli problem will be a formal scheme over $\operatorname{Spf} k_\infty[\![\zeta]\!]$.

Let $S \in \mathcal{N}ilp_{k_\infty[\![\zeta]\!]}$. Denote by $\bar{S}$ the closed subscheme of S given by the sheaf of ideals $\zeta\mathcal{O}_S$. If $\beta : S \longrightarrow \operatorname{Spf} k_\infty^{(\ell)}[\![\zeta]\!]$ is a morphism of formal schemes, let $\bar{\beta} : \bar{S} \longrightarrow \operatorname{Spec} k_\infty^{(\ell)}$ denote the restriction of β to the special fibers.

Definition 11.1. (Moduli Problem) Define the functor $G : \mathcal{N}ilp_{k_\infty[\![\zeta]\!]} \longrightarrow Sets$ as

$$S \longmapsto \Big\{ \text{ Isomorphism classes of triples } (\beta, \underline{\widehat{\mathcal{F}}}, \widehat{\alpha}) \text{ where}$$

- $\beta : S \to \operatorname{Spf} k_\infty^{(d)}[\![\zeta]\!]$ is a morphism of formal schemes,
- $\underline{\widehat{\mathcal{F}}}$ is a formal $\mathcal{D}_\infty$-elliptic sheaf of rank d and dimension 1 over S,
- $\widehat{\alpha} : \underline{\widehat{\mathcal{F}}}_{\bar{S}} \to \bar{\beta}^* \underline{\widehat{\boldsymbol{\mathcal{E}}}}$ is a quasi-isogeny of formal $\mathcal{D}_\infty$-elliptic sheaves. $\Big\}$

Two triples $(\beta_1, \underline{\widehat{\mathcal{F}}}_1, \widehat{\alpha}_1)$ and $(\beta_2, \underline{\widehat{\mathcal{F}}}_2, \widehat{\alpha}_2)$ are isomorphic if $\beta_1 = \beta_2$ and there is an isomorphism between $\underline{\widehat{\mathcal{F}}}_1$ and $\underline{\widehat{\mathcal{F}}}_2$ over S that is compatible with $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$.

By Theorem 10.5, we can reformulate the moduli problem as

Proposition 11.2. *The functor G from Definition 11.1 is equivalent to the functor $\mathcal{N}ilp_{k_\infty[\![\zeta]\!]} \longrightarrow Sets$ such that*

$$S \longmapsto \Big\{ \textit{ Isomorphism classes of triples } (\beta, E, \gamma) \textit{ where}$$

- $\beta : S \to \operatorname{Spf} k_\infty^{(d)}[\![\zeta]\!]$ *is a morphism of formal schemes,*
- E *is a special z-divisible $\mathcal{D}_\infty \otimes \mathcal{O}_\Delta$-module of height d^2 and dimension d over S,*
- $\gamma : \bar{\beta}^* \underline{\mathbb{E}} \to E_{\bar{S}}$ *is a quasi-isogeny of special z-divisible $\mathcal{D}_\infty \otimes \mathcal{O}_\Delta$.* $\Big\}$

Lemma 11.3. *Let $\widetilde{G}$ be the functor $\mathcal{N}ilp_{k_\infty[\![\zeta]\!]} \longrightarrow Sets$ defined by*

$$S \longmapsto \Big\{ \textit{ Isomorphism classes of triples } (\beta, E', \rho') \textit{ where}$$

- $\beta : S \to \operatorname{Spf} k_\infty^{(d)}[\![\zeta]\!]$ *is a morphism of formal schemes,*
- E' *is a special z-divisible $\mathcal{O}_\Delta$-module of height d^2 and dimension d over S,*
- $\rho : \bar{\beta}^* \underline{\mathbb{E}}' \to E'_{\bar{S}}$ *is a quasi-isogeny of $\mathcal{O}_\infty$-z-divisible $\mathcal{O}_\Delta$-modules.* $\Big\}$

Then, the functor G is Morita equivalent to $\widetilde{G}$.

Proof. We define the natural transformation $\eta : \widetilde{G} \longrightarrow G$. Define $\eta_S : \widetilde{G}(S) \longrightarrow G(S)$ as follows:

$$(\beta', E', \rho') \mapsto (\beta, E' \otimes_{\mathcal{O}_\infty} \mathcal{O}_\infty^{1\times d}, \rho)$$

where $\rho : \overline{\beta}^*(\underline{\mathbb{E}}) \longrightarrow (E' \otimes_{\mathcal{O}_\infty} \mathcal{O}_\infty^{1\times d})_{\bar{S}}$. Note that the height of $E' \otimes_{\mathcal{O}_\infty} \mathcal{O}_\infty^{1\times d} \simeq E'_n \oplus \cdots \oplus E'_n$ is d^3 and the dimension of $E' \otimes_{\mathcal{O}_\infty} \mathcal{O}_\infty^{1\times d}$ is d^2.

$$\begin{tikzcd}
\widetilde{G}(S) \arrow[r, "\widetilde{G}(f)"'] \arrow[d, "\eta_S"'] & \widetilde{G}(S') \arrow[d, "\eta'_S"'] \\
G(S) \arrow[r, "G(f)"'] & G(S')
\end{tikzcd}$$

Note that by Proposition 8.18, for every $S \in \mathcal{N}ilp_{k_\infty[\![\zeta]\!]}$, the component η_S is an isomorphism. Therefore, we have an isomorphism of functors.

□

12 Moduli space of generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaves

12.1 Main Example

Now, we are back to the case where $\pi : X \longrightarrow Y$ is a finite morphism of degree t. We will construct example of generalized $\mathcal{D}$-elliptic sheaves using gluing Section 20.

Example 12.1. ($\deg \infty = 1$ case:)

Recall that for each $j = 1, \cdots, t$, we have $k_{\infty_j} \simeq k_{\infty}$. Assume $\deg \infty = 1$. So, $k(T) = \prod k_{\infty_i} = k_{\infty}^{\oplus t} \simeq \mathbb{F}_q^t$. Let $S = \operatorname{Spec} \mathbb{F}_q^t$. Let $\mathcal{D}$ be an Azumaya algebra of dimension d^2 over $\mathcal{O}_X$. We will construct generalized $\mathcal{D}$-elliptic sheaf over $X \times \mathbb{F}_q^t$ with $\mathcal{E}_{\underline{0}} = \mathcal{D} \otimes \mathbb{F}_q^t$. As in the $\mathcal{D}$-elliptic sheaf case since for a generalized $\mathcal{D}$-elliptic sheaf $\mathcal{E}_{\underline{i}}|_{X \smallsetminus \mathbb{T} \times S}$ is same for all $\underline{i}$, our construction will be base on gluing and investigating the generalized $\mathcal{D}$-elliptic sheaf around ∞_j's.

Before we continue, let us make a remark. Let $\underline{i} = (i_1, \cdots, i_t) \in \mathbb{Z}^t$ be any. Each i_j stands for each ∞_j, i.e, every change in one i_j means to modify around that ∞_j. Keeping this in mind we will define lattices. Let $\underline{i} = (i_1, \cdots, i_t)$ and $s \in \{1, \cdots, t\}$. Define

$$L_{\underline{i}, \infty_s} := \oplus_{j=1}^{d} \mathcal{O}_{\infty_s} z_s^{\lfloor (d-j-i_s)/d \rfloor}$$

and

$$L_{\underline{i}, \mathbb{T}} := \oplus_{s=1}^{t} L_{\underline{i}, \infty_s}.$$

Define $\mathcal{E}_{\underline{i}, \infty_s} := \operatorname{Hom}(L_{\underline{0}, \infty_s}, L_{\underline{i}, \infty_s})$ and let

$$\mathcal{E}_{\underline{i}, \mathbb{T}} := \oplus_{s=1}^{t} \mathcal{E}_{\underline{i}, \infty_s}.$$

We glue $\mathcal{E}_{\underline{i}, \mathbb{T}}$ with $\Gamma(X \smallsetminus \mathbb{T} \times \mathbb{F}_q^t, \mathcal{D} \otimes \mathbb{F}_q^t)$ to get locally free sheaves $\mathcal{E}_{\underline{i}}$ over $X \times S$.

By definition, we have

$$\mathcal{E}_{\underline{0}, \mathbb{T}} = \mathbb{M}_d(\mathcal{O}_{\infty_1}) \oplus \cdots \oplus \mathbb{M}_d(\mathcal{O}_{\infty_t}) \simeq \mathcal{D}_{\infty_1} \oplus \cdots \oplus \mathcal{D}_{\infty_t}$$

Let $\underline{i}' \in \mathbb{Z}^t$ be such that $\underline{i} \leq \underline{i}'$. By definition we have $\mathcal{E}_{\underline{i}, \infty_s} \subseteq \mathcal{E}_{\underline{i}', \infty_s}$ for each ∞_s. We define the morphisms $j_{\underline{i}, \underline{i}'} : \mathcal{E}_{\underline{i}} \longrightarrow \mathcal{E}_{\underline{i}'}$ via this natural inclusion. The periodicity

$$L_{\underline{i} + \underline{d} \cdot t, \mathbb{T}} \simeq L_{\underline{0}, \mathbb{T}}(\infty_1, \cdots, \infty_t)$$

also directly follows from definition. Now, we will define the morphisms $t_{\underline{i}} : \sigma^* \mathcal{E}_{\underline{i}} \longrightarrow \mathcal{E}_{\underline{i} + \underline{1}}$.

Now, let us define the maps $t_{\underline{i}}$ for $\underline{i} \in \mathbb{Z}^t$. Since $S = \operatorname{Spec} \mathbb{F}_q^t$, we have $\sigma^* = id$. Define $t_{\underline{i}} : \sigma^* \mathcal{E}_{\underline{i}} \longrightarrow \mathcal{E}_{\underline{i} + \underline{1}}$ to be the monomorphisms $j_{\underline{i}, \underline{i} + \underline{1}}$ that were defined above.

Therefore, we get a generalized $\mathcal{D}$-elliptic sheaf $\underline{\boldsymbol{\mathcal{E}}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})_{\underline{i} \in \mathbb{Z}^t}$. As before, one can construct corresponding generalized formal $\mathcal{D}_\infty$-elliptic sheaf, $\widehat{\underline{\boldsymbol{\mathcal{E}}}}$. By categorical equivalence, we get a generalized special z-divisible $\mathcal{D}_\infty \otimes \mathcal{O}_\Delta$-module, $\underline{\mathbb{E}}$.

Remark 12.2. We want to note that in the example above we have $S' = \mathbb{T} \times S$.

Example 12.3. ($\deg \infty > 1 case$):

Now, assume $\deg \infty = m > 1$. Let $S = \operatorname{Spec} \mathbb{F}_{q^m}^t$. Then we have

$$\mathcal{O}_{\infty_k} \otimes \mathcal{O}_S \simeq \oplus_{s=0}^{m-1} \mathcal{O}_{\infty_{k,s}}$$

for each $\infty_k \in \mathbb{T}$ where $\mathcal{O}_{\infty_{k,s}} \simeq \mathcal{O}_{\infty_k}$. Now, for each $\infty_k \in \mathbb{T}$ define

$$L_{s,\underline{i}}^k = \oplus_{j=0}^{d-1} z^{\lfloor \frac{i_s - 1 - s - jm}{md} + 1 \rfloor} e_j \mathcal{O}_{\infty_{k,s}}$$

where e_j's are the bases at $\infty_{k,s}$. Now put

$$\mathcal{E}_{\underline{i}, \infty_{s,k}} = \operatorname{Hom}(L_{s,\underline{i}}^k, L_{s,\underline{0}}^k)$$

Define

$$\mathcal{E}_{\underline{i}, \infty_k} := \oplus_{s=0}^{m-1} \mathcal{E}_{\underline{i}, \infty_{s,k}}$$

and

$$\mathcal{E}_{\underline{i}, \mathbb{T}} = \oplus_{k=1}^{t} \mathcal{E}_{\underline{i}, \infty_k}$$

Note that by definition we have

$$\mathcal{E}_{\underline{i}, \mathbb{T}} = \Big(\big[\mathbb{M}_d(\mathcal{O}_{\infty_{1,1}}) \oplus \cdots \oplus \mathbb{M}_d(\mathcal{O}_{1,m}) \big] \oplus \cdots \oplus \big[\mathbb{M}_d(\mathcal{O}_{\infty_{t,1}}) \oplus \cdots \oplus \mathbb{M}_d(\mathcal{O}_{\infty_{t,m}}) \big] \Big)$$

We glue $\mathcal{E}_{\underline{i}, \mathbb{T}}$ with $\Gamma(X \smallsetminus \mathbb{T} \otimes \mathbb{F}_{q^m}^t, \mathcal{D} \otimes \mathbb{F}_{q^m}^t)$ and get locally free sheaves $\mathcal{E}_{\underline{i}}$ over $X \times \mathbb{F}_{q^m}^t$.

Let $\underline{i}, \underline{i}' \in \mathbb{Z}^t$ with $\underline{i} \leq \underline{i}'$. Then, by definition we have $\mathcal{E}_{\underline{i}, \infty_k} \subset \mathcal{E}_{\underline{i}', \infty_k}$ for each ∞_k. We define the morphisms $j_{\underline{i}, \underline{i}'} : \mathcal{E}_{\underline{i}} \longrightarrow \mathcal{E}_{\underline{i}'}$ by this inclusion. We have the periodicity

$$\mathcal{E}_{\underline{i} + \underline{d} t \deg \infty} \simeq \mathcal{E}_{\underline{i}}(\infty_1, \cdots, \infty_t)$$

by definition. We only need to define the morphisms $t_{\underline{i}} : \sigma^* \mathcal{E}_{\underline{i}} \longrightarrow \mathcal{E}_{\underline{i} + \underline{1}}$. Note that

$$\begin{aligned} \sigma^* \mathcal{E}_{\underline{i}, \mathbb{T}} &= \oplus_{k=1}^t \sigma^* \mathcal{E}_{i, \infty_k} = \oplus_{k=1}^t \oplus_{s=0}^{m-1} \sigma^* \mathcal{E}_{\underline{i}, \infty_{s,k}} \\ &= \oplus_{k=1}^t \oplus_{s=0}^{m-1} \operatorname{Hom}(\sigma^* L_{s,\underline{i}}^k, \sigma^* L_{s,\underline{0}}^k) \end{aligned}$$

Similarly as in Example 11.1, Case 2, we define $t_{\underline{i}} : \sigma^* \mathcal{E}_i \longrightarrow \mathcal{E}_{\underline{i+1}}$.

Hence $(\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ is a generalized $\mathcal{D}$-elliptic sheaf. We will denote this also by $\underline{\boldsymbol{\mathcal{E}}}$.

12.1.1 Generalized Moduli Functor

Let $S \in \mathcal{N}ilp_{k_\infty[\![\zeta_1,\cdots,\zeta_t]\!]}$. We will consider the generalized special $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}} \otimes \mathcal{O}_\Delta$-modules which are isogeneous to $\mathbb{E}$ which is defined in Example 12.1 and then show that this is equivalent to product of the moduli schemes of the $\mathcal{D}_{\infty_j}$-z_j-divisible groups at each ∞_i, as in [46].

Definition 12.4. Define the functor G_{gen} from $\mathcal{N}ilp_{k_\infty[\![\zeta_1,\cdots,\zeta_t]\!]} \longrightarrow Sets$

$S \longmapsto \Big\{$ Isomorphism classes of triples $(\beta, \widehat{\underline{\mathcal{E}}}, \widehat{\alpha})$ where

- $\widehat{\beta} : S \to \operatorname{Spf} k_\infty^{(d)}[\![\zeta_1, \cdots, \zeta_t]\!]$ is a morphism of formal schemes,
- $\widehat{\underline{\mathcal{E}}}$ is a generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf over S,
- $\widehat{\alpha} : \widehat{\underline{\mathcal{E}}}_{\bar{S}} \to \bar{\beta}^* \widehat{\underline{\boldsymbol{\mathcal{E}}}}$ is a quasi-isogeny of generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaves. $\Big\}$

By Theorem 10.17, we can reformulate the moduli problem as follows:

Definition 12.5. The functor G_{gen} is equivalent to the functor from $\mathcal{N}ilp_{k_\infty[\![\zeta_1,\cdots,\zeta_t]\!]} \longrightarrow Sets$ defined by

$S \longmapsto \Big\{$ Isomorphism classes of triples $(\bar{\beta}, \underline{E}, \alpha)$ where

- $\bar{\beta} : S \to \operatorname{Spf} k_\infty^{(d)}[\![\zeta_1, \cdots, \zeta_t]\!]$ is a morphism of formal schemes,
- $\underline{E}$ is a generalized special $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}} \otimes \mathcal{O}_\Delta$-module over S,
- $\alpha : \underline{E}_{\bar{S}} \to \bar{\beta}^* \mathbb{E}$ is a quasi-isogeny of generalized $\mathcal{D}_{\underline{\infty}}$-$\underline{z}$-divisible groups. $\Big\}$

Proposition 12.6. *The moduli functor G_{gen} of generalized special $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}} \otimes \mathcal{O}_\Delta$-modules is*

$$G_{gen} \simeq G \times \cdots \times G$$

where G is the moduli functor of special z-divisible $\mathcal{D}_\infty \otimes \mathcal{O}_\Delta$-modules defined in Definition 11.2.

Proof. The proof follows by definition of a generalized special $\underline{z}$-divisible $\mathcal{D}_{\underline{\infty}} \otimes \mathcal{O}_\Delta$-modules. ∎

13 The Serre-Tate Theorem

In [23], Hartl gives an analogue of classical Serre-Tate theorem in the abelian sheaf case. In this section, we will adapt his result to the generalized $\mathcal{D}$-elliptic sheaf case. Namely, we will show that the deformation of a generalized $\mathcal{D}$-elliptic sheaf is same as the deformation of a generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf.

Let $S \in \mathcal{N}ilp_{k_\infty^{(d)}[\![\zeta_1,\cdots,\zeta_t]\!]}$ and $\iota : \bar{S} \hookrightarrow S$ be a closed subscheme of S that is defined by the sheaf of ideals $\mathcal{I}$ which is locally nilpotent.

Definition 13.1. 1. Let $\underline{\mathcal{E}}$ be a generalized $\mathcal{D}$-elliptic sheaf over $\bar{S}$. We say a generalized $\mathcal{D}$-elliptic sheaf $\underline{\widetilde{\mathcal{E}}}$ over S is a *deformation* of $\underline{\mathcal{E}}$ if there exists an isomorphism of generalized $\mathcal{D}$-elliptic sheaves $f : \iota^*\underline{\widetilde{\mathcal{E}}} \xrightarrow{\sim} \underline{\mathcal{E}}$

2. Two deformations $(\underline{\widetilde{\mathcal{E}}}, f)$ and $(\underline{\widetilde{\mathcal{F}}}, g)$ of a generalized $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{E}}$ are *isomorphic* if there exists an isomorphism of generalized $\mathcal{D}$-elliptic sheaves $\alpha : \underline{\widetilde{\mathcal{E}}} \longrightarrow \underline{\widetilde{\mathcal{F}}}$ such that the following diagram is commutative:

$$\begin{tikzcd}
\iota^*\underline{\widetilde{\mathcal{E}}} \arrow[r, "\iota^*\alpha"] \arrow[d, "f"] & \iota^*\underline{\widetilde{\mathcal{F}}} \arrow[d, "g"] \\
\underline{\mathcal{E}} \arrow[r] & \underline{\mathcal{E}}
\end{tikzcd}$$

Let $\underline{\mathcal{E}}$ be a generalized $\mathcal{D}$-elliptic sheaf over $\bar{S}$. The *category of deformations of* $\underline{\mathcal{E}}$ has

- objects: pairs of deformations $(\underline{\widetilde{\mathcal{E}}}, f)$
- morphisms: isomorphisms of deformations

Similarly, one can define the *category of deformations of a generalized formal* $\mathcal{D}_{\underline{\infty}}$*-elliptic sheaf*. Let $\underline{\widehat{\mathcal{E}}}$ denote the generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf corresponding $\underline{\mathcal{E}}$.

Definition 13.2. 1. We say a generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf $\underline{\widehat{\mathcal{F}}}$ is a deformation of $\underline{\widehat{\mathcal{E}}}$ if there exists an isomorphism of generalized formal $\mathcal{D}_{\underline{\infty}}$-sheaves $g : \iota^*\underline{\widehat{\mathcal{F}}} \xrightarrow{\sim} \underline{\widehat{\mathcal{E}}}$.

2. Two deformations $(\underline{\widehat{\mathcal{F}}}, g)$ and $(\underline{\widehat{\mathcal{G}}}, h)$ are *isomorphic* if there is an isomorphism of generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaves $\underline{\widehat{\mathcal{F}}} \xrightarrow{\sim} \underline{\widehat{\mathcal{G}}}$ that is compatible with g and h.

The *category of deformations of a generalized formal* $\mathcal{D}_{\underline{\infty}}$*-elliptic sheaf* $\underline{\widehat{\mathcal{E}}}$ has pairs $(\underline{\widehat{\mathcal{F}}}, g)$ as objects and isomorphisms of deformations as morphisms.

Theorem 13.3 (Serre-Tate Theorem)**.** *Let $S \in \mathcal{N}ilp_{k_\infty^{(d)}[\![\zeta_1,\cdots,\zeta_t]\!]}$ and let $\iota : \bar{S} \hookrightarrow S$ be as before. Then, the category of deformations of a generalized $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{E}}$ and the category of deformations of its corresponding generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf $\widehat{\underline{\mathcal{E}}}$ are equivalent.*

To prove this theorem, we need some preparations. The following proposition tells us, together with rigidity of generalized $\mathcal{D}_{\underline{\infty}}$-$\underline{z}$-divisible groups, that the set of morphisms of category of deformations is non-zero.

Proposition 13.4. *Let $\underline{\mathcal{E}}$ be a generalized $\mathcal{D}$-elliptic sheaf and $\widehat{\underline{\mathcal{E}}}$ be its generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf. Let $\widehat{\underline{\mathcal{G}}}$ be a generalized formal $\mathcal{D}_{\underline{\infty}}$ such that there exists a quasi-isogeny $\widehat{\alpha} : \widehat{\underline{\mathcal{G}}} \longrightarrow \widehat{\underline{\mathcal{E}}}$. Then, there exists a generalized $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{F}}$ and a quasi-isogeny $\gamma : \underline{\mathcal{F}} \longrightarrow \underline{\mathcal{E}}$ which is an isomorphism over $X' \times S$ where $X' = X \smallsetminus \mathbb{T}$ such that its completion is isomorphic to $\widehat{\alpha}$, i.e., there exists an isomorphism $\widehat{\mathcal{F}} \longrightarrow \widehat{\mathcal{G}}$ such that the following diagram commutes:*

$$\begin{tikzcd} \widehat{\mathcal{F}} \arrow[rr, "\simeq"] \arrow[dr, "\widehat{\gamma}"'] & & \widehat{\mathcal{G}} \arrow[dl, "\widehat{\alpha}"] \\ & \widehat{\underline{\mathcal{E}}} & \end{tikzcd}$$

Moreover, γ is unique up to isomorphism. So, we will denote the generalized abelian sheaf $\underline{\mathcal{F}}$ by $\widehat{\alpha}^*\underline{\mathcal{E}}$.

To prove this proposition, we will get the lattices around each ∞_j and vector bundle on the affine part so that we can glue them to get a generalized $\mathcal{D}$-elliptic sheaf on $X \times S$. For more on gluing lattices around ∞ and vector bundles on the affine part, please see Section 20.2 and Section 20. We want to recall that for a generalized $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$, the module $\mathcal{E}_{\underline{i}}|_{(X \smallsetminus \mathbb{T}) \times S}$ is independent of $\underline{i}$ (cf. Remark 2.5).

Proof. Let $\underline{\mathcal{E}}$ be a generalized $\mathcal{D}$-elliptic sheaf and $\widehat{\underline{\mathcal{E}}}$ be its generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf. Write $\widehat{\underline{\mathcal{E}}} = (\widehat{\underline{\mathcal{E}}}^{(1)}, \cdots, \widehat{\underline{\mathcal{E}}}^{(t)})$ and $\widehat{\underline{\mathcal{G}}} = (\widehat{\underline{\mathcal{G}}}^{(1)}, \cdots, \widehat{\underline{\mathcal{G}}}^{(t)})$. Recall that a quasi isogeny $\widehat{\alpha} : \widehat{\underline{\mathcal{G}}} \longrightarrow \widehat{\underline{\mathcal{E}}}$ is a t-tuple $(\widehat{\alpha}^{(1)}, \cdots, \widehat{\alpha}^{(t)})$ of quasi-isogenies $\widehat{\alpha}^{(j)} : \widehat{\underline{\mathcal{G}}}^{(j)} \longrightarrow \widehat{\underline{\mathcal{E}}}^{(j)}$ of formal $\mathcal{D}_{\infty_j}$-elliptic sheaves. By definition of quasi-isogenies of formal $\mathcal{D}_{\infty_j}$-elliptic sheaves, we have isomorphisms of the corresponding isocrystals

$$\widehat{\alpha}^{(j)}[1/z] : \widehat{\underline{\mathcal{G}}}^{(j)}[1/z] \xrightarrow{\simeq} \widehat{\underline{\mathcal{E}}}^{(j)}[1/z]$$

By Remark 10.10(i), we can write $\widehat{\underline{\mathcal{G}}}^{(j)} = \oplus_{i=1}^{m} \widetilde{\mathcal{G}}_i^j$. Then by Remark 10.10 (ii), we know that a quasi-isogeny between two formal $\mathcal{D}_\infty$-elliptic sheaves is a quasi-isogeny componentwise:

$$\widetilde{\alpha}^{(j)} : \widetilde{\mathcal{G}}^{\infty_j} \longrightarrow \widetilde{\mathcal{E}}^{\infty_j}$$

Now, by Remark 10.9, construct $\mathcal{G}^{\infty_j}$ from each $\widetilde{\mathcal{G}}^{\infty_j}$. Then, by the quasi-isogeny $\widehat{\alpha}^{(j)}$, we have an injection

$$\mathcal{G}_{\underline{i}}^{\infty_j} \hookrightarrow \widehat{\mathcal{E}}[1/z]$$

Therefore, we can glue $\mathcal{G}_{\underline{i}}^{\infty_j}, \cdots, \mathcal{G}_{\underline{i}}^{\infty_j}$ with $\mathcal{E}_{\underline{i}}|_{(X\smallsetminus\mathbb{T})\times S}$ to get locally free sheaves $\mathcal{F}_{\underline{i}}$ on $X\times S$. Since each $\widehat{\mathcal{E}}^{(j)}$ has a $\mathcal{D}_{\infty_j}$-action and $\underline{\mathcal{E}}$ has a $\mathcal{D}$-action, there is a $\mathcal{D}$-action on each $\mathcal{F}_{\underline{i}}$. By the morphisms Π' and F' of the generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf $\underline{\widehat{\mathcal{G}}}$ and by the morphisms $j_{\underline{i},\underline{i}'} : \mathcal{E}_{\underline{i}} \longrightarrow \mathcal{E}_{\underline{i}'}$ and $t_{\underline{i}} : \sigma^*\mathcal{E}_{\underline{i}} \longrightarrow \mathcal{E}_{\underline{i}+\underline{1}}$, we get morphisms

$$j'_{\underline{i}} : \mathcal{F}_{\underline{i}} \longrightarrow \mathcal{F}_{\underline{i}+\underline{1}} \text{ and } t'_{\underline{i}} : \sigma^*\mathcal{F}_{\underline{i}} \longrightarrow \mathcal{F}_{\underline{i}+\underline{1}}.$$

Therefore, we get a generalized $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{F}} = (\mathcal{F}_{\underline{i}}, j'_{\underline{i}}, t'_{\underline{i}})$. By construction, we see that the generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf $\underline{\widehat{\mathcal{F}}}$ corresponding to the generalized $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{F}}$ is isomorphic to $\underline{\widehat{\mathcal{G}}}$ and there is a quasi-isogeny $\rho : \underline{\mathcal{F}} \longrightarrow \underline{\mathcal{E}}$.

□

Proposition 13.5. *Let $\iota : \bar{S} \longrightarrow S$ be a closed subscheme of S defined by a sheaf of ideals that is locally nilpotent. Let $\underline{\mathcal{E}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ and $\underline{\mathcal{E}}' = (\mathcal{E}'_{\underline{i}}, t'_{\underline{i}})$ be two generalized $\mathcal{D}$-elliptic sheaves over S. Then, every quasi-isogeny $\iota^*\underline{\mathcal{E}} \longrightarrow \iota^*\underline{\mathcal{E}}'$ gives a quasi isogeny $\underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}'$ in a unique way.*

Proof. The proof goes similar to the rigidity of quasi-isogenies of Dieudonné $\mathcal{D}_\infty$-modules. We may assume by induction that $\mathcal{I}^q = (0)$. Then, Frobenius σ_S on S and Frobenius $\sigma_{\bar{S}}$ on $\bar{S}$ factors as

$$S \xrightarrow{j} \bar{S} \xrightarrow{\iota} S$$

and

$$\bar{S} \xrightarrow{\iota} S \xrightarrow{j} \bar{S}$$

where j is the identity between the underlying topological spaces $|\bar{S}| = |S|$.

Recall that by using the morphisms $t_{\underline{i}} : \sigma^*\mathcal{E}_{\underline{i}} \longrightarrow \mathcal{E}_{\underline{i}+\underline{1}}$ and $t'_{\underline{i}} : \sigma^*\mathcal{E}'_{\underline{i}} \longrightarrow \mathcal{E}'_{\underline{i}+\underline{1}}$, one can define isogenies $t : \sigma^*\underline{\mathcal{E}}[1] \longrightarrow \underline{\mathcal{E}}$ and $t' : \sigma^*\underline{\mathcal{E}}'[1] \longrightarrow \underline{\mathcal{E}}'$ (cf. Example 3.5). Here $\underline{\mathcal{E}}[1]$ denotes the shift by 1, i.e, $\underline{\mathcal{E}}[1] = (\mathcal{E}_{\underline{i}-\underline{1}}, j_{\underline{i}-\underline{1}}, t_{\underline{i}-\underline{1}})$.

Now, let $\bar{\rho} : \iota^*\underline{\mathcal{E}} \longrightarrow \iota^*\underline{\mathcal{E}}'$ be a quasi-isogeny. Consider the shift by 1 and pullback under j and we have

$$j^*\bar{\rho}[1] : j^*\iota^*\underline{\mathcal{E}}[1] \longrightarrow j^*\iota^*\underline{\mathcal{E}}'[1]$$

Hence, we get the following diagram

$$\begin{tikzcd} \underline{\mathcal{E}} \arrow[rr, "\rho"] & & \underline{\mathcal{E}}' \\ \sigma_S^*\underline{\mathcal{E}}[1] = j^*\iota^*\underline{\mathcal{E}}[1] \arrow[u, "t"] \arrow[rr, "j^*\bar{\rho}[1]"] & & \sigma_S^*\underline{\mathcal{E}}'[1] = j^*\iota^*\underline{\mathcal{E}}'[1] \arrow[u, "t'"'] \end{tikzcd}$$

which gives us a quasi-isogeny $\rho : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}'$. It follows from the diagram that $\iota^*\rho = \bar{\rho}$ and ρ is defined uniquely by $\bar{\rho}$.

□

Proof. (of Theorem 13.3)

Let $\underline{\mathcal{F}}$ be a deformation of $\underline{\mathcal{E}}$ with $f : \iota^*\underline{\mathcal{F}} \xrightarrow{\sim} \mathcal{E}$. Let $\widehat{\underline{\mathcal{E}}}$ and $\widehat{\underline{\mathcal{F}}}$ denote the generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf corresponding $\underline{\mathcal{E}}$ and $\underline{\mathcal{F}}$ respectively. Then, $\widehat{\underline{\mathcal{F}}}$ is a deformation of $\widehat{\underline{\mathcal{E}}}$. So, we have a functor sending a deformation $(\underline{\mathcal{F}}, f)$ of $\underline{\mathcal{E}}$ to the corresponding deformation $(\widehat{\underline{\mathcal{F}}}, \widehat{f})$ of $\widehat{\underline{\mathcal{E}}}$. Denote this functor by F.

Let $(\underline{\mathcal{F}}, f)$ and $(\underline{\mathcal{F}}', f')$ be two deformations of $\underline{\mathcal{E}}$. Then, the map

$$\mathrm{Hom}_{\mathcal{D}}(\underline{\mathcal{F}}, \underline{\mathcal{F}}') \longrightarrow \mathrm{Hom}_{\mathcal{D}_{\underline{\infty}}}(\widehat{\underline{\mathcal{F}}}, \widehat{\underline{\mathcal{F}}}')$$

is injective and surjective by Proposition 13.4 and 13.5. We will show that F is essentially surjective.

Let $(\widehat{\underline{\mathcal{F}}}, \widehat{f})$ be a deformation of $\widehat{\underline{\mathcal{E}}}$ where $\widehat{f} : \iota^*\widehat{\underline{\mathcal{F}}} \xrightarrow{\sim} \widehat{\underline{\mathcal{E}}}$ is an isomorphism of generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaves. As before, it is enough to consider the case when the ideal sheaf $\mathcal{I}$ of $\bar{S}$ satisfies $\mathcal{I}^q = (0)$. Then, the Frobenius σ_S on S factors as

$$S \xrightarrow{j} \bar{S} \xrightarrow{\iota} S$$

Consider the sheaf $\underline{\mathcal{E}}' := j^*\underline{\mathcal{E}}[1] = (j^*\mathcal{E}_{\underline{i-1}}, j^*t_{\underline{i-1}})$. By using the morphisms $t_{\underline{i}}$'s of $\underline{\mathcal{E}}$, we get a quasi-isogeny $t : \iota^*(\sigma * \underline{\mathcal{E}}[1]) \longrightarrow \underline{\mathcal{E}}$ that is an isomorphism over X'(cf. Example 3.5). But $\sigma^*\underline{\mathcal{E}}[1] = \iota^*j^*\underline{\mathcal{E}}[1] = \iota^*\underline{\mathcal{E}}'$. So, we have a quasi-isogeny $t : \iota^*\underline{\mathcal{E}}' \longrightarrow \underline{\mathcal{E}}$. Consider the generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf $\widehat{\underline{\mathcal{E}}}'$ corresponding to the generalized $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{E}}'$. We have $\widehat{t} : \iota^*\widehat{\underline{\mathcal{E}}}' \longrightarrow \widehat{\underline{\mathcal{E}}}$ and we obtain a diagram

$$\begin{tikzcd}
\iota^*\widehat{\underline{\mathcal{F}}} \arrow[r, "\widehat{f}", "\simeq"'] \arrow[dr, dotted, "\widehat{t}^{-1}\circ\widehat{f}"'] & \widehat{\underline{\mathcal{E}}} \\
& \iota^*\widehat{\underline{\mathcal{E}}}' \arrow[u, "\widehat{t}"']
\end{tikzcd}$$

where $\widehat{\alpha}' := \widehat{t}^{-1} \circ \widehat{f}$ is a quasi-isogeny of generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaves. By Theorem 9.17, the quasi-isogeny $\widehat{\alpha}'$ extends uniquely $\widehat{\alpha} : \widehat{\underline{\mathcal{F}}} \longrightarrow \widehat{\underline{\mathcal{E}}}'$. Then, by Proposition 13.4, there is a generalized $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{F}}$ that is quasi-isogeneous to $\underline{\mathcal{E}}'$ via $\alpha : \underline{\mathcal{F}} \longrightarrow \underline{\mathcal{E}}'$ so that the corresponding generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaf of $\underline{\mathcal{F}}$ is $\widehat{\underline{\mathcal{F}}}$. By the diagram

$$\begin{tikzcd}
\underline{\mathcal{F}} \arrow[r, "\alpha"] \arrow[dr, dotted, "t\circ\alpha"'] & j^*\underline{\mathcal{E}}[1] \arrow[d, "t"] \\
& \underline{\mathcal{E}}
\end{tikzcd}$$

the tuple $(\mathcal{F}, t \circ \alpha)$ is a deformation of $\underline{\mathcal{E}}$.

∎

Part III Uniformization

14 Algebraization

Assume $t = 1$ and so $X = Y$. Recall that in this case a generalized $\mathcal{D}$-elliptic sheaf is called $\mathcal{D}$-elliptic sheaf (cf. Remark 2.5 (ii)). We will give an interpretation of the moduli space G that was defined in Section in 11.1. Let $S \in \mathcal{N}ilp_{k_\infty[\![\zeta]\!]}$ and let $\bar{S}$ be the closed subscheme of S defined by $\zeta = 0$. One can define a morphism $G \longrightarrow \operatorname{Spf} k_\infty^{(d)}[\![\zeta]\!]$ by $(\beta, \underline{\widehat{\mathcal{F}}}, \alpha) \mapsto \beta$. We will define an action of $\operatorname{Gal}(k_\infty^{(d)}/k_\infty)$ on G. Note that $\operatorname{Gal}(k_\infty^{(d)}/k_\infty) = \operatorname{Gal}(k_\infty^{(d)}[\![\zeta]\!]/k_\infty[\![\zeta]\!])$. Take any $\gamma \in \operatorname{Gal}(k_\infty^{(d)}/k_\infty)$. We define $\gamma\beta$ via the diagram

$$\begin{tikzcd} S^\gamma \arrow[r, "\gamma_S"] \arrow[d, "\gamma\beta"] & S \arrow[d, "\beta"] \\ \operatorname{Spf} k_\infty^{(d)}[\![\zeta]\!] \arrow[r, "\gamma"] & \operatorname{Spf} k_\infty^{(d)}[\![\zeta]\!] \end{tikzcd} \tag{1}$$

where γ_S is the composition $S \longrightarrow \operatorname{Spf} k_\infty^{(d)}[\![\zeta]\!] \xrightarrow{\gamma} \operatorname{Spf} k_\infty^{(d)}[\![\zeta]\!]$. We write S^γ for S considered as a $\operatorname{Spf} k_\infty^{(d)}[\![\zeta]\!]$-scheme via γ_S. Then, we define the action of γ on $G(S)$ by

$$\gamma \cdot (\beta, \underline{\widehat{\mathcal{F}}}, \widehat{\alpha}) := (\gamma\beta, \gamma_S\underline{\widehat{\mathcal{F}}}, \gamma_S\widehat{\alpha}) \in G(S^\gamma)$$

that is compatible with $\mathcal{D}_\infty$-action.

Definition 14.1. Define the functor $G' : \mathcal{N}ilp_{k_\infty[\![\zeta]\!]} \longrightarrow Sets$ as

$$S \longmapsto \left\{ \begin{array}{l} \text{Isomorphism classes of pairs } (\underline{\mathcal{F}}, \alpha) \text{ where} \\ \quad \bullet\ \underline{\mathcal{F}} \text{ is a } \mathcal{D}\text{-elliptic sheaf over } S\,, \\ \quad \bullet\ \alpha : \underline{\mathcal{F}}_{\bar{S}} \to \underline{\boldsymbol{\mathcal{E}}}_{\bar{S}} \text{ is a quasi-isogeny of } \mathcal{D}\text{-elliptic sheaves.} \end{array} \right\}$$

Two such pairs $(\underline{\mathcal{F}}, \alpha)$ and $(\underline{\mathcal{F}}', \alpha')$ are *isomorphic* if there is an isomorphism of $\mathcal{D}$-elliptic sheaves between $\underline{\mathcal{F}}$ and $\underline{\mathcal{F}}'$ that is compatible with α and α'.

Theorem 14.2. *The functors G and $G' \times_{\operatorname{Spf} k_\infty[\![\zeta]\!]} \operatorname{Spf} k_\infty^{(d)}[\![\zeta]\!]$ are isomorphic as $\operatorname{Gal}(k_\infty^{(d)}/k_\infty)$-modules where $\operatorname{Gal}(k_\infty^{(d)}/k_\infty)$ acts trivially on G'*

Proof. The proof goes similarly as in [23], Theorem 10.2. Let $(\underline{\mathcal{F}}, \alpha) \in G'(S)$ and $\beta : S \longrightarrow \operatorname{Spf} k_\infty^{(d)}[\![\zeta]\!]$. By Construction 10.7, we get a formal $\mathcal{D}_\infty$-elliptic sheaf $\underline{\widehat{\mathcal{F}}}$ and the quasi-isogeny α of $\mathcal{D}$-elliptic sheaves gives a quasi isogeny of formal $\mathcal{D}_\infty$-elliptic sheaves $\widehat{\alpha} : \underline{\widehat{\mathcal{F}}}_{\bar{S}} \longrightarrow \bar{\beta}^*\underline{\widehat{\boldsymbol{\mathcal{E}}}}$. So, we get a triple $(\beta, \underline{\widehat{\mathcal{F}}}, \widehat{\alpha}) \in G(S)$. Since

$\mathrm{Gal}(k_\infty^{(d)}/k_\infty)$ acts trivially on G' and by definition of the Galois action, the map $G' \times \mathrm{Spf}\, k_\infty^{(d)}[\![\zeta]\!] \longrightarrow G$ is $Gal(k_\infty^{(d)}/k_\infty)$-equivariant.

Conversely, let $(\beta, \underline{\widehat{\mathcal{F}}}, \widehat{\alpha}) \in G(S)$. By Proposition 9.17, there is a unique lift of $\widehat{\alpha}$ to $\widehat{\rho} : \underline{\widehat{\mathcal{F}}} \longrightarrow \underline{\widehat{\mathcal{E}}}$ over S. Then, by Proposition 13.4, we get a $\mathcal{D}$-elliptic sheaf, say $\underline{\mathcal{F}}$ whose corresponding formal $\mathcal{D}_\infty$-elliptic sheaf is $\underline{\widehat{\mathcal{F}}}$ and a quasi-isogeny $\rho : \underline{\mathcal{F}} \longrightarrow \underline{\boldsymbol{\mathcal{E}}}$. So, $(\underline{\mathcal{F}}, \rho_{\bar{S}}) \in G'(S)$. Therefore, we get a map $G \longrightarrow G' \times \mathrm{Spf}\, k_\infty^{(d)}$. The fact that the map is $Gal(k_\infty^{(d)}/k_\infty)$-equivariant follows by definition. Now, one can easily see that the two maps are mutually inverse.

$\square$

Definition 14.3. The tuple $(\underline{\mathcal{F}}, \alpha) \in G'(S)$ associated to a $(\beta, \underline{\widehat{\mathcal{F}}}, \widehat{\alpha})$ is called *algebraization of* $(\beta, \underline{\widehat{\mathcal{E}}'}, \widehat{\alpha})$.

14.1 Algebraization of generalized moduli functor

Let $S \in \mathcal{N}ilp_{k_\infty[\![\zeta_1,\cdots,\zeta_t]\!]}$ and $\overline{S}$ be its closed subscheme of S. Define the functor $G'_{gen} : \mathcal{N}ilp_{k_\infty[\![\zeta_1,\cdots,\zeta_t]\!]} \longrightarrow Sets$ as

$$S \longmapsto \left\{ \begin{array}{l} \text{Isomorphism classes of pairs } (\underline{\mathcal{F}}, \alpha) \text{ where} \\ \bullet\ \underline{\mathcal{F}} \text{ is a } \mathcal{D}\text{-elliptic sheaf over } S\,, \\ \bullet\ \alpha : \underline{\mathcal{F}}_{\bar{S}} \to \underline{\boldsymbol{\mathcal{E}}}_{\bar{S}} \text{ is a quasi-isogeny of } \mathcal{D}\text{-elliptic sheaves.} \end{array} \right\}$$

Recall that G_{gen} be the moduli functor of generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaves that was defined in Definition 12.4. We will write $\widehat{\otimes} k_\infty[\![\zeta[\![$ for

$$k_\infty[\![\zeta]\!] \widehat{\otimes}_{k_\infty} \cdots \widehat{\otimes}_{k_\infty} k_\infty[\![\zeta]\!]$$

where completed tensor product is taken for t-copies and write $\widehat{\otimes} k_\infty^{(d)}[\![\zeta]\!]$ for

$$k_\infty^{(d)}[\![\zeta]\!] \widehat{\otimes}_{k_\infty[\![\zeta]\!]} \cdots \widehat{\otimes}_{k_\infty[\![\zeta]\!]} k_\infty^{(d)}[\![\zeta]\!].$$

where in both cases, we have t factors in the tensor product.

Theorem 14.4. *The functors G_{gen} and $G'_{gen} \times_{\widehat{\otimes} k_\infty[\![\zeta]\!]} \widehat{\otimes} k_\infty^{(d)}[\![\zeta]\!]$ are isomorphic as* $\mathrm{Gal}(k_\infty^{(d)}/k_\infty)^t$*-modules.*

Proof. The proof goes similarly as in Theorem 14.2. Let $\beta_j : S \longrightarrow k_\infty^{(d)}[\![\zeta[\![$ be a $k_\infty[\![\zeta]\!]$-morphism for $j = 1, \cdots, t$. Put

$$\beta := \prod_j \beta_j : S \longrightarrow k_\infty^{(d)}[\![\zeta]\!] \widehat{\otimes}_{k_\infty[\![\zeta]\!]} \cdots \widehat{\otimes}_{k_\infty[\![\zeta]\!]} k_\infty^{(d)}[\![\zeta]\!].$$

Then we have an action of $\mathrm{Gal}(k_\infty^{(d)}/k_\infty)^t$ on $k_\infty^{(d)}[\![\zeta]\!]\widehat{\otimes}_{k_\infty[\![\zeta]\!]}\cdots\widehat{\otimes}_{k_\infty[\![\zeta]\!]}k_\infty^{(d)}[\![\zeta]\!]$ where each $\mathrm{Gal}(k_\infty^{(d)}/k_\infty)$ act on each β_j as in the diagram 1.

Now, the functor $G'_{gen} \longrightarrow G_{gen}$ is defined by the Construction 10.18, and the functor $G_{gen} \longrightarrow G'_{gen}$ is defined by using Proposition 13.4 as in Theorem 14.2. □

The following theorem gives us another interpretation of G_{gen}:

Theorem 14.5. *We have*

$$G'_{gen} \simeq (G' \times \cdots \times G') \times_{\widehat{\otimes} k_{\infty_i}[\![\zeta_i]\!]} \widehat{\otimes} k_{\infty_i}^{(d)}[\![\zeta_i]\!]$$

where G' is the functor defined in Definition 14.1.

We will use the algebraization of the moduli functors G and G' to algebraize the moduli functor G_{gen}.

Proof. By definition of generalized formal $\mathcal{D}_{\underline{\infty}}$-elliptic sheaves, we have

$$G_{gen} \simeq G \times \cdots \times G$$

where G is the moduli functor of formal $\mathcal{D}_\infty$-elliptic sheaves that was defined in Theorem 11.1. Then, by algebraization in $\mathcal{D}$-elliptic sheaf case (Theorem 14.2), we have

$$G_{gen} \simeq (G' \otimes_{k_{\infty_1}[\![\zeta_1]\!]} k_{\infty_1}^{(d)}[\![\zeta_1]\!]) \times \cdots \times (G' \otimes_{k_{\infty_t}[\![\zeta]\!]} k_{\infty_t}^{(d)}[\![\zeta_t]\!]) \simeq (G' \times \cdots \times G') \times_{\widehat{\otimes} k_{\infty_i}[\![\zeta_i]\!]} \widehat{\otimes} k_{\infty_i}^{(d)}[\![\zeta_i]\!].$$

□

Corollary 14.6. *The functors $G'_{gen} \simeq G' \times \cdots \times G'$.*

15 Representability of the Moduli Functor

15.1 Representability of G'

In this Section we will prove that the functor G' that was defined in Definition 14.1 is representable by $\mathbb{Z} \times \widehat{\Omega}^{(d)}$. For the definition and some of the properties of $\widehat{\Omega}^{(d)}$ we refer to [4], Section 4.3 and [46], Section 4. Our main reference for this section is [19]. We want to recall the $\mathcal{D}$-elliptic sheaf $\underline{\boldsymbol{\mathcal{E}}}$ defined in Section 11.1. By using Construction 10.7, one can define the formal $\mathcal{D}_\infty$-elliptic sheaf associated to $\underline{\boldsymbol{\mathcal{E}}}$. We will denote this formal $\mathcal{D}_\infty$-elliptic sheaf by $\widehat{\underline{\boldsymbol{\mathcal{E}}}}$.

Definition 15.1. ([19], Chapter I, Definition 4.3.3) Define the functor $G_{\mathcal{O}}$ from $\mathcal{N}ilp_{k_\infty[\![\zeta]\!]}$ to $Sets$ by sending an $S \in \mathcal{N}ilp_{k_\infty[\![\zeta]\!]}$ to the isomorphism class of triples $\big((\mathcal{M}_i, \Pi_i, F_i)_{i\in\mathbb{Z}}, R_0\big)$ where

1. $\mathcal{M}_i$ is locally free $k_\infty[\![\zeta]\!]\widehat{\otimes}\mathcal{O}_S$-module of rank d
2. the morphisms
$$\Pi_i : \mathcal{M}_i \longrightarrow \mathcal{M}_{i+1}$$
$$F_i : \sigma^*\mathcal{M}_i \longrightarrow \mathcal{M}_{i+1}$$
are morphism of $k_\infty[\![\zeta]\!]\widehat{\otimes}\mathcal{O}_S$-modules such that the following conditions are satisfied for $i \in \mathbb{Z}$:
   (a) The diagram
$$\begin{tikzcd} \sigma^*\mathcal{M}_i \arrow[r, "F_i"] \arrow[d, "\sigma^*\Pi_i"'] & \mathcal{M}_{i+1} \arrow[d, "\Pi_{i+1}"] \\ \sigma^*\mathcal{M}_{i+1} \arrow[r, "F_{i+1}"] & \mathcal{M}_{i+2} \end{tikzcd}$$
   is commutative
   (b) We have $\mathcal{M}_{i+d} \simeq \mathcal{M}_i(\infty)$ by the composition
$$\mathcal{M}_i \xrightarrow{\Pi_i} \mathcal{M}_{i+1} \longrightarrow \cdots \longrightarrow \mathcal{M}_{i+d}$$
   (c) (cf. Definition 9.1 1, item 1ii)
   There exists a locally free B-module ω_i such that $\operatorname{coker} F_i = \mathit{\Gamma}_*\omega_i$ where
$$\mathit{\Gamma} : \mathcal{O}\widehat{\otimes}B \longrightarrow B$$
$$a\widehat{\otimes}b \mapsto \beta(a)b$$
   Here $\beta : \mathcal{O}_d \longrightarrow B$ gives B an $\mathcal{O}_d$-algebra structure where $\mathcal{O}_d$ is the maximal unramified extension $k_\infty^{(d)}[\![\pi]\!] \subset \mathcal{O}_\Delta = k_\infty^{(d)}[\![\Pi]\!]$.

(d) There exists $n \in \mathbb{N}$(cf. Construction 10.7) such that the composition

$$(\mathrm{Id}_{k_\infty[\![\zeta]\!]} \widehat{\otimes} Fr^n)^* \mathcal{M}_0 \xrightarrow{(\mathrm{Id}_{k_\infty[\![\zeta]\!]} \widehat{\otimes} Fr^{n-1})^* F_0} (\mathrm{Id}_{k_\infty[\![\zeta]\!]} \widehat{\otimes} Fr^{n-1})^* \mathcal{M}_1 \longrightarrow \cdots$$

$$\cdots \xrightarrow{F_{n-1}} \mathcal{M}_n/\mathcal{M}_n(\infty)$$

is the zero morphism.

3. the map

$$R_0 : \mathcal{M}_{i,\bar{S}} \otimes_{k_\infty[\![\zeta]\!]} k_\infty(\!(\zeta)\!) \xrightarrow{\simeq} \widehat{\boldsymbol{\mathcal{E}}}_{i,\bar{S}} \otimes_{k_\infty[\![\zeta]\!]} k_\infty(\!(\zeta)\!)$$

is an isomorphism that satisfies the following commutative diagram:
Put $\mathcal{N}_{i,\bar{S}} := \mathcal{M}_{i,\bar{S}} \otimes_{k_\infty[\![\zeta]\!]} k_\infty(\!(\zeta)\!)$ and $\boldsymbol{\mathcal{F}}_{i,\bar{S}} := \widehat{\boldsymbol{\mathcal{E}}}_{i,\bar{S}} \otimes_{k_\infty[\![\zeta]\!]} k_\infty(\!(\zeta)\!)$

$$\begin{tikzcd}
(\mathrm{Id}_{k_\infty} \widehat{\otimes} Fr)^* \mathcal{N}_{0,\bar{S}} \arrow[r, "F_{0,\bar{S}} \otimes \mathrm{Id}_{k_\infty}"] \arrow[d, "(\mathrm{Id}_{k_\infty} \otimes Fr)^* R_0"'] & \mathcal{N}_{1,\bar{S}} \arrow[d, "R_1"] \\
(\mathrm{Id}_{k_\infty} \otimes Fr)^* \boldsymbol{\mathcal{F}}_{0,B_0} \arrow[r, "F_{0,\bar{S}} \otimes \mathrm{Id}_{k_\infty}"] & \boldsymbol{\mathcal{F}}_{1,\bar{S}}
\end{tikzcd}$$

4. if $n \in \mathbb{N}$ such that

$$R_0(\mathcal{M}_{0,\bar{S}}) \subset \widehat{\boldsymbol{\mathcal{E}}}_{n,\bar{S}} \subset \widehat{\boldsymbol{\mathcal{E}}}_{0,\bar{S}} \otimes_{k_\infty[\![\zeta]\!]} k_\infty(\!(\zeta)\!)$$

then $\widehat{\boldsymbol{\mathcal{E}}}_{n,\bar{S}}/R_0(\mathcal{M}_{0,\bar{S}})$ is locally free over $\bar{S}$ of rank n.

As one can see from the definition, one can think of the triples $(\mathcal{M}_i, \Pi_i, F_i)$ as a ladder over $\mathcal{O}_\infty$.

Proposition 15.2. *The functors G_O and G' are naturally isomorphic.*

Proof. We define a functor $G' \longrightarrow G_O$ and leave the verification of details to the reader.

Let $(\underline{\mathcal{F}}, \rho : \underline{\mathcal{F}}_{\bar{S}} \longrightarrow \underline{\boldsymbol{\mathcal{E}}}_{\bar{S}}) \in G'(S)$. By taking completion along the fiber over ∞ as in the Construction 10.7, we get a ladder $\widetilde{\mathcal{F}}^\infty$ over $\mathcal{O}_\infty$ which almost satisfies the necessary conditions in Definition 15.1 since the $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{E}}$ satisfies similar conditions.

Let d_ρ denote the degree of the quasi isogeny $\rho : \underline{\mathcal{F}}_{\bar{S}} \longrightarrow \underline{\boldsymbol{\mathcal{E}}}_{\bar{S}}$. Define $(\widehat{\mathcal{F}}_i, \Pi_i, \tau_i)$ as $\mathcal{F}^\infty$ shifted by d_ρ

Then R_0 is the isomorphism

$$\widehat{\mathcal{F}}_{d_\rho}[1/z] \longrightarrow \widehat{\boldsymbol{\mathcal{E}}}_0[1/z]$$

□

Then, we have the following theorem:

Theorem 15.3. *The functor G' is representable by $\mathbb{Z} \times \widehat{\Omega}^{(d)}$ over $\mathcal{O}_\infty$.*

Proof. By [19], in Chapter III, Section 3(Theorem 3.1.1 together with Lemma 3.3.1) we know that G_O is representable by $\widehat{\Omega}^{(d)}$ over $\mathcal{O}_\infty$. By the Proposition 15.2, we conclude that G' is representable by $\mathbb{Z} \times \widehat{\Omega}^{(d)}$ over $\mathcal{O}_\infty$. $\square$

15.2 Representability of G'_{gen}

Theorem 15.4. *the functor G'_{gen} is representable by $\prod_i(\mathbb{Z} \times \widehat{\Omega}^{(d)})$*

Proof. The proof follows immediately by Corollary 14.6 and Theorem 15.3. $\square$

Part IV Uniformization

16 The Uniformization Theorem

16.1 The group of quasi-isogenies for $\mathcal{D}$-elliptic sheaf

Recall the definition of $\underline{\boldsymbol{\mathcal{E}}}$ in 11.1. In [5], it is proved that the space of quasi-morphisms between abelian sheaves is isomorphic to the space of morphisms between the generic fibres η of $X \times S$. Similar to the abelian sheaf case, we have the following

Proposition 16.1. *Let $\underline{\mathcal{E}} = (\mathcal{E}_i, j_i, t_i)$ and $\underline{\mathcal{E}}' = (\mathcal{E}'_i, j'_i, t'_i)$ be two $\mathcal{D}$-elliptic sheaves of the same rank and same characteristic over $S = \operatorname{Spec} L$ where L is a field. Then, Q-vector space $\operatorname{QIsog}_{\mathcal{D}}(\underline{\mathcal{E}}, \underline{\mathcal{E}}')$ is isomorphic to the group of units of the space of morphisms between the fibers at the generic point η of $X \times S$*

$$\{f_{0,\eta} : \mathcal{E}_{0,\eta} \longrightarrow \mathcal{E}'_{0,\eta} \mid f_{0,\eta} \circ j_{0,\eta}^{-1} \circ t_{0,\eta} = (j'_{0,\eta})^{-1} \circ t'_{0,\eta} \circ \sigma^*(f_{0,\eta})\}$$

We will use this theorem to compute the group of quasi-isogenies of $\underline{\boldsymbol{\mathcal{E}}}$.

Theorem 16.2. *The group of quasi-isogenies of $\underline{\boldsymbol{\mathcal{E}}} = (\mathcal{E}_i, j_i, t_i)$ over the algebraic closure k_∞^{alg} of k_∞ in the example 11.1 is equal to D^*, the invertible elements of the division algebra D.*

Proof. By Proposition 16.1 in [5], it is enough to consider the morphisms between the fibers at the generic point η :

$$\mathcal{E}_{0,\eta} \longrightarrow (\mathcal{E}_0(\infty))_\eta = (\mathcal{E}_0 \otimes \mathcal{O}_{X \times S}(\infty))_\eta \qquad (1)$$

Recall that by definition $\mathcal{E}_0 = \mathcal{D}$ and $\mathcal{D}_\eta = D$. Let $\rho : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}$ be a quasi-isogeny. By the Proposition 16.1, it is enough to consider $\rho_{0,\eta} : \mathcal{E}_{0,\eta} \longrightarrow (\mathcal{E}_0(\mathsf{B}))_\eta$ where B is an effective divisor of X, i.e, we are looking for the group of morphisms

$$f : D \otimes \mathbb{F}_{q^m} \longrightarrow D \otimes \mathbb{F}_{q^m}.$$

such that the following diagram commutes:

$$\begin{tikzcd}
D \otimes \mathbb{F}_{q^m} \arrow[r, "f"'] & D \otimes \mathbb{F}_{q^m} \\
D \otimes \mathbb{F}_{q^m} \arrow[u, "j_0^{-1}"'] & D \otimes \mathbb{F}_{q^m} \arrow[u, "j_0^{-1}"'] \\
D \otimes^\sigma \sigma^*\mathbb{F}_{q^m} \arrow[u, "\tau_0"'] \arrow[r, "\sigma^* f"'] & D \otimes^\sigma \sigma^*\mathbb{F}_{q^m} \arrow[u, "\tau_0"']
\end{tikzcd}$$

Note that $\sigma^*\mathbb{F}_{q^m} = \mathbb{F}_{q^m} \otimes_{\mathbb{F}_{q^m},\sigma} \mathbb{F}_{q^m}$ and consider the following diagram:

$$
\begin{tikzcd}
\mathbb{F}_{q^m} \arrow[r, "\varphi"'] & \mathbb{F}_{q^m} \\
\mathbb{F}_{q^m} \otimes^{\sigma}_{\mathbb{F}_{q^m}} \mathbb{F}_{q^m} \arrow[u, "g"] \arrow[r, "\sigma^*\varphi"'] & \mathbb{F}_{q^m} \otimes^{\sigma}_{\mathbb{F}_{q^m}} \mathbb{F}_{q^m} \arrow[u, "g"] \\
\mathbb{F}_{q^m} \arrow[u, "h", "\simeq"'] \arrow[r, "\varphi"'] & \mathbb{F}_{q^m} \arrow[u, "h", "\simeq"']
\end{tikzcd}
$$

where the maps $g : \mathbb{F}_{q^m} \otimes^{\sigma}_{\mathbb{F}_{q^m}} \mathbb{F}_{q^m} \longrightarrow \mathbb{F}_{q^m}$ is defined by $\alpha \otimes \beta \mapsto \alpha.\sigma(\beta)$ and $h : \mathbb{F}_{q^m} \longrightarrow \mathbb{F}_{q^m} \otimes^{\sigma}_{\mathbb{F}_{q^m}} \mathbb{F}_{q^m}$ is defined by $\gamma \mapsto 1 \otimes \gamma$. Hence the composition of vertical arrows in the diagram is σ. So, we have by the second diagram above, $\sigma \circ \varphi = \varphi \circ \sigma$. Therefore, we have $(D \otimes \mathbb{F}_{q^m})^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)} = D$, and so, the group of quasi isogenies of the $\mathcal{D}$-elliptic sheaf is D^*.

□

16.2 The Uniformization Theorem

Following the way of p-adic uniformization of Shimura curves and uniformization of stack of abelian sheaves, we will use the scheme $\widehat{\Omega}^{(d)}$ representing the moduli functor G'. We know that the moduli functor G'_{gen} is represented by $\prod \mathbb{Z} \times \widehat{\Omega}^{(d)}$. We will use this to uniformize generalized $\mathcal{D}$-elliptic sheaves. Before we continue we want to recall our conventions:

$$
\begin{tikzcd}
X \arrow[d, "\pi"] & \mathbb{T} = \{\infty_1, \ldots, \infty_t\} \arrow[d, dash] & B = \Gamma(X \smallsetminus \mathbb{T}, \mathcal{O}_X) & F = \mathbb{F}_q(X) \\
Y & \infty & A = \Gamma(Y \smallsetminus \{\infty\}, \mathcal{O}_Y) \arrow[u] & L = \mathbb{F}_q(Y) \arrow[u]
\end{tikzcd}
$$

We will use the symbol $D^{\times}$ for both the group of units of D and the algebraic group of units of the division algebra D defined by

$$D^{\times}(R) = (D \otimes R)^{\times}$$

where R is an F-algebra. Let

$$\mathbb{A}_f = \left\{ (a_x) \in \widehat{\prod}_{x \notin \mathbb{T}} F_x \mid a_x \in B_x \text{ for almost all } x \right\}$$

denote the finite adeles and define

$$D^{\times}(\mathbb{A}_f) = \prod_{x \notin \mathbb{T}} (D_x, \mathcal{D}_x) = \left\{ (a_x) \in \prod_{x \notin \mathbb{T}} D_x \mid a_x \in \mathcal{D}_x \text{ for almost all } x \right\}.$$

Let $(\mathcal{D}^{\mathbb{T}})^{\times} = \prod_{x \notin \mathbb{T}} \mathcal{D}_x^{\times} = (\prod_{x \notin \mathbb{T}} \mathcal{D}_x)^{\times}$. And $(\mathcal{D}^{\mathbb{T}})^{\times}_I$ denote the kernel of the group homomorphism

$$(\mathcal{D}^{\mathbb{T}})^{\times} \longrightarrow H^0(X, \mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{O}_I)^{\times}$$

where the morphism $(\mathcal{D}^{\mathbb{T}})^{\times} \longrightarrow H^0(X, \mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{O}_I)^{\times}$ defined as $(a \otimes b_x)_{x \in X'} \mapsto (a \otimes \bar{b}_x)_{x \in I}$. We want to construct an isomorphism over $\mathbb{A}_f$ for a generalized $\mathcal{D}$-elliptic sheaf that is quasi-isogeneous to the generalized $\mathcal{D}$-elliptic sheaf $\underline{\boldsymbol{\mathcal{E}}}$ that was constructed in Examle 12.1. We will use this isomorphism to define the action of $(\mathcal{D}^{\mathbb{T}})_I^{\times}$.

Construction 16.3. Let $S \in \mathcal{N}ilp_{k_\infty[\![\zeta_1, \cdots, \zeta_t]\!]}$ and let $\bar{S}$ be its closed subscheme defined by $\zeta_j = 0$ for all j. Recall the generalized $\mathcal{D}$-elliptic sheaf $\underline{\boldsymbol{\mathcal{E}}} = (\mathcal{E}_{\underline{i}}, t_{\underline{i}})$ over $\operatorname{Spec} \mathbb{F}_q^t$ that was defined in Example 12.1. Consider the pullback of $\underline{\boldsymbol{\mathcal{E}}}$ along $s : \bar{S} \longrightarrow \operatorname{Spec} \mathbb{F}_q^t$. Denote this pullback by $\underline{\boldsymbol{\mathcal{E}}}'$, i.e, $\underline{\boldsymbol{\mathcal{E}}}' = (\mathcal{E}'_{\underline{i}}, t'_{\underline{i}}) = (s^*\mathcal{E}_{\underline{i}}, s^* t_{\underline{i}})$. On $X' = X \smallsetminus \mathbb{T}$, all $\mathcal{E}'_{\underline{i}}|_{X' \times \bar{S}}$ are isomorphic via the morphisms $j'_{\underline{i}, \underline{i}'}$. So, we denote this sheaf by $\mathcal{E}|_{X' \times \bar{S}}$. Similar for the morphisms $t_{\underline{i}}$'s, we get a map $t'|_{X' \times \bar{S}} : \sigma^* \mathcal{E}|_{X' \times \bar{S}} \longrightarrow \mathcal{E}|_{X' \times \bar{S}}$. Recall the t-invariant functor $E_{\widehat{B}}$ defined in Section 2 and consider $E_{\widehat{B}}(S)$. Now,

$$E_{\widehat{B}}(S) = \varinjlim_{I' \subset X'} E_{I'}(S) \simeq \varinjlim_{I'} \mathbb{M}_d(B_{I'}/I') = \mathcal{D}(\widehat{B})$$

where $B_{I'} = H^0(I', \mathcal{O}_{I'})$. We denote this isomorphism by f. Clearly, $(\mathcal{D}^{(\infty)})_I^{\times}$ acts on $\mathcal{D}(\widehat{B})$.

The isomorphism f gives rise to an isomorphism

$$\psi : E_{\mathbb{A}_f} \xrightarrow{\sim} \mathcal{D}(\mathbb{A}_f)$$

which induces a rational H-level structure on $\underline{\boldsymbol{\mathcal{E}}}$.

Let $\underline{\mathcal{F}}$ be a generalized $\mathcal{D}$-elliptic sheaf over S which is quasi-isogeneous to $\underline{\boldsymbol{\mathcal{E}}}$ via $\alpha : \underline{\mathcal{F}} \longrightarrow \underline{\boldsymbol{\mathcal{E}}}$. As in the previous paragraph, we can look at $\varinjlim E_{I'}(\underline{\mathcal{F}})$. The composition $\alpha \circ \psi$ gives us an isomorphism

$$\psi \circ \alpha : \varinjlim E_{I'}(\underline{\mathcal{F}}) \otimes_{\mathcal{D}(\widehat{B})} \mathcal{D}(\mathbb{A}_f) \longrightarrow \mathcal{D}(\mathbb{A}_f)$$

So, we get a level structure on $\underline{\mathcal{F}}$.

Let $S \in \mathcal{N}ilp_{k_\infty[\![\zeta_1, \cdots, \zeta_t]\!]}$. Let $(\underline{\mathcal{F}}, \rho) \in G'_{gen}$ be any where $\rho : \underline{\mathcal{F}}_{\bar{S}} \longrightarrow \underline{\boldsymbol{\mathcal{E}}}$ is a quasi-isogeny. Then, by Construction 16.3, we get a level structure η on $\underline{\mathcal{F}}$. So, we can define

$$\begin{aligned} G'_{gen} \times D^{\times}(\mathbb{A}_f)/(\mathcal{D}^{\mathbb{T}})_I^{\times} &\longrightarrow \mathcal{GE}\ell\ell_{X/Y, \mathcal{D}, I} \times_{Sch_{\mathbb{F}_q}} \mathcal{N}ilp_{k_\infty[\![\zeta_1, \cdots, \zeta_t]\!]} \\ ((\underline{\mathcal{F}}, \alpha), a(\mathcal{D}^{\mathbb{T}})_I^{\times}) &\mapsto (\underline{\mathcal{F}}, a^{-1}\eta) \end{aligned} \tag{1}$$

Remark 16.4. Note that $(\mathcal{D}^{\mathbb{T}})_I^{\times}$ is a compact open subgroup of $D^{\times}(\mathbb{A}_f)$.

Now, let us define the $D^\times$-action on $G'_{gen} \times D^\times(\mathbb{A}_f)/(\mathcal{D}^{\mathbb{T}})^\times_I$. We want to recall that the group of quasi-isogenies of $\underline{\boldsymbol{\mathcal{E}}}$ is $D^\times$.

Let $g \in D^\times$ and let $\underline{\mathcal{F}}$ be a generalized $\mathcal{D}$-elliptic sheaf that is quasi-isogeneous to $\underline{\boldsymbol{\mathcal{E}}}$ via $\rho : \underline{\mathcal{F}}_{\bar{S}} \longrightarrow \underline{\boldsymbol{\mathcal{E}}}_{\bar{S}}$. Then the action of g on α is defined by

$$\underline{\mathcal{F}}_{\bar{S}} \xrightarrow{\rho} \underline{\boldsymbol{\mathcal{E}}}_{\bar{S}} \xrightarrow{g_{\bar{S}}} \underline{\boldsymbol{\mathcal{E}}}_{\bar{S}}(\mathsf{D})$$

where $g_{\bar{S}}$ denotes $\iota^* g$ with $\iota : \bar{S} \longrightarrow S$. By using this action we can define the action of $D^\times$ on G'_{gen} as follows:

$$(g \cdot (\underline{\mathcal{F}}, \rho)) := (\underline{\mathcal{F}}, g_{\bar{S}} \circ \rho)$$

which is compatible with the $\mathcal{D}$-action.

On the other hand, by the diagonal embedding $D^\times \hookrightarrow D^\times(\mathbb{A}_Q^{\mathbb{T}})$, we have an action of $D^\times$ on $G'_{gen} \times D^\times(\mathbb{A}_f)/(\mathcal{D}^{\mathbb{T}})^\times_I$ by

$$\big((\underline{\mathcal{F}}, \rho), \bar{a}\big) \mapsto \big((\underline{\mathcal{F}}, g_{\bar{S}} \circ \rho), g \cdot \bar{a}\big)$$

Let Z be the pullback defined by the diagram

$$\begin{tikzcd} \mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I} \arrow[r] & Y \setminus \mathsf{B} \\ Z \arrow[u] \arrow[r] & \infty \arrow[u] \end{tikzcd}$$

We denote the formal completion of $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$ along the closed subscheme Z by $\widehat{\mathcal{GE}\ell\ell}^Z_{X/Y,\mathcal{D},I}$. Before we continue, we want to recall that the stack of generalized $\mathcal{D}$-elliptic sheaves $\mathcal{GE}\ell\ell_{X/Y,\mathcal{D},I}$ with nontrivial level structures is actually a scheme(Theorem 5.3). So, when we take the formal completion $\widehat{\mathcal{GE}\ell\ell}^Z_{X/Y,\mathcal{D},I}$ we have a formal scheme. For an introduction to formal schemes we refer to [17].

Now, put

$$\mathcal{X} := D^\times \backslash G'_{gen} \times D^\times(\mathbb{A}_f)/(\mathcal{D}^{\mathbb{T}})^\times_I$$

$$\mathcal{Y} := \widehat{\mathcal{GE}\ell\ell}^Z_{X,\mathcal{D},I}$$

Before we continue, we need to understand the double coset space $D^\times \backslash G'_{gen} \times D^\times(\mathbb{A}_f)/(\mathcal{D}^{\mathbb{T}})^\times_I$ better. For convenience, put $H := (\mathcal{D}^{\mathbb{T}})^\times_I$.

Let $d_i H$ be representatives of the quotient $D^\times \backslash D^\times(\mathbb{A}_f)/H$. Now, $stab_{D^\times}(d_i H) := \{d \in D^\times \mid d d_i H = d_i H\}$. Let $d \in stab_{D^\times}(d_i H)$ be any. We have

$$d \in stab_{D^\times}(d_i H) \iff d d_i H = d_i H \iff d \in d_i H d_i^{-1}$$

Hence, $d \in d_i H d_i^{-1} \cap D^\times$. Let us denote the group $d_i H d_i^{-1} \cap D^\times$ by Γ_i. Note that Γ_i is a discrete subgroup of $D^\times$. Indeed, by the Strong Approximation Theorem, we know that $D^\times$ is discrete in $D^\times(\mathbb{A})$. This implies that $\mathcal{D}^*$ is discrete in $D^\times(\prod_i F_{\infty_i})$. Then, Γ_i is discrete since any subgroup of a discrete group is discrete.

Then we have:

Proposition 16.5. $D^\times \backslash G'_{gen} \times D^\times(\mathbb{A}_f)/H = \coprod_i G'_{gen}/\Gamma_i$.

Proof. Consider $D^\times \backslash D^\times(\mathbb{A}_f)/H$. We can write it as a finite disjoint union $\coprod D^\times d_i H$. Then,

$$G'_{gen} \times D^\times(\mathbb{A}_f) = \coprod_i G'_{gen} \times D^\times d_i H.$$

Therefore, we can write $D^\times \backslash D^\times(\mathbb{A}_f)/H$ as $\coprod_i \Gamma_i \backslash G'_{gen}$ which is nothing but just $\coprod_i \Gamma_i \backslash (\widehat{\Omega}^{(d)})^t$. □

Proposition 16.6. *Assume $t = 1$ and hence $X = Y$. In this case we are only considering $\mathcal{D}$-elliptic sheaves. Then, the double coset space $D^\times \backslash G' \times D^\times(\mathbb{A}_f)/H$ is a formal scheme if Γ_i is discrete and torsion free.*

Proof. By the previous proposition, we know that the double coset space is isomorphic to $\Gamma_i \backslash (\widehat{\Omega}^{(d)})$. We know that $\widehat{\Omega}^{(d)}$ is a formal scheme. By [40], Theorem 3.1 (or Section 3 in [26]), $\widehat{\Omega}^{(d)}/\Gamma_i$ is a formal scheme since Γ_i is discrete.

□

Proposition 16.7. *In $t > 1$ case, we can state a similar result. Namely, with the notation and assumptions as before, $(\widehat{\Omega}^{(d)})^t/\Gamma_i$ is a formal scheme.*

Proof. The proof is analogous to [39] and [40]. □

We have defined the action of $(\mathcal{D}^{\mathbb{T}})^\times_I$ on $G'_{gen} \times D^\times(\mathbb{A}_f)$ in Section 16.2. So, we have morphism

$$G'_{gen} \times D^\times(\mathbb{A}_f)/H \longrightarrow \mathcal{GE}\ell\ell_{X,\mathcal{D},I}.$$

We also defined the action of $D^\times$ on $G'_{gen} \times D^\times(\mathbb{A}_f)/H$ so that we get a morphism

$$D^\times \backslash G'_{gen} \times D^\times(\mathbb{A}_f)/(\mathcal{D}^{\mathbb{T}})^\times_I \longrightarrow \mathcal{GE}\ell\ell_{X,\mathcal{D},I}$$

Now, this morphism factors through $\widehat{\mathcal{GE}\ell\ell}^Z_{X,\mathcal{D},I}$. Indeed, on the left, we have a formal scheme defined by ζ-completion, so the left side is of the form $\varprojlim X/(\pi)^n$. And if $\mathcal{I}$ denotes the ideal sheaf of Z, we see that $X/(\pi^n)$ mapsto $Y/\mathcal{I}^n$, i.e, I^n mapsto to ζ^n. So, we can define

$$\theta : \mathcal{X} \longrightarrow \mathcal{Y}$$

Now we can state our main theorem:

Theorem 16.8. *One has an isomorphism of formal schemes*

$$\widehat{\mathcal{GE}\ell\ell}^Z_{X/Y,\mathcal{D},I} \simeq D^\times \backslash G'_{gen} \times D^\times(\mathbb{A}_f)/(\mathcal{D}^{\mathbb{T}})^*_I.$$

Remark 16.9. We want to remark that if the proof of the previous proposition is worked out in a details then, we can recover Stuhler's result on uniformization of Frobenius-Hecke sheaves in [46].

Recall the representability theorem in Section 15, Theorem 15.4.

Theorem 16.10. *We can reformulate the Theorem 16.8 as follows:*

$$\widehat{\mathcal{GE}\ell\ell}^{Z}_{X/Y,\mathcal{D},I} \simeq D^{\times}\backslash\big(\prod_{i=1}^{t}(\mathbb{Z}\times\widehat{\Omega}^{(d)})\big)\times D^{\times}(\mathbb{A}_f)/(\mathcal{D}^{\mathbb{T}})^{*}_{I}$$

Remark 16.11. Let us assume $t=1$ and so we have $\mathcal{D}$-elliptic sheaves. We want to note that in [4], Blum and Stuhler consider $\mathcal{D}$-elliptic sheaves with a normalization condition. In this case G' is representable by $\widehat{\Omega}^{(d)}$. Hence our theorem becomes:

$$\widehat{\mathcal{E}\ell\ell}_{X,\mathcal{D},I} \simeq D^{\times}\backslash\mathbb{Z}\times\widehat{\Omega}^{(d)}\times D^{\times}(\mathbb{A}_f)/(\mathcal{D}^{\infty})^{\times}_{I}.$$

where $\mathcal{E}\ell\ell_{X,\mathcal{D},I}$ denotes the stack of $\mathcal{D}$-elliptic sheaves with level I-structure. This theorem is stated in [4], Theorem 4.4.11 without a proof.

17 Proof of Uniformization Theorem

Let $S \in \mathcal{N}ilp_{k_\infty[\![\zeta_1,\cdots,\zeta_t]\!]}$. We will prove that the stack $\mathcal{X} \times_{\mathcal{Y}} S$ is a scheme and that we have an isomorphism of schemes

$$\mathcal{X} \times_{\mathcal{Y}} S \xrightarrow{\sim} S$$

Proposition 17.1. *The morphism $\theta_{red} : \mathcal{X}_{red} \longrightarrow \mathcal{Y}_{red}$ is bijective on K-points where K is a field, i.e, the morphism $\theta_{red}(K) : \mathcal{X}(K) \longrightarrow \mathcal{Y}(K)$ is an isomorphism.*

Proof. First we will show the surjectivity. Let $s \in \mathcal{Y}_{red}$ be a point. Since it is isoclinic (Theorem 10.13), there is a quasi-isogeny $\rho : \underline{\mathcal{E}}_s \longrightarrow \underline{\boldsymbol{\mathcal{E}}}$. We can assume, by multiplying with a quasi-isogeny of $\underline{\boldsymbol{\mathcal{E}}}$ if necessary, that ρ respects the level structure. The induced quasi-isogeny $\underline{\mathcal{F}}_s \longrightarrow \widehat{\rho}^* \underline{\boldsymbol{\mathcal{E}}}_s$ is also compatible with the level structures. Hence, s lies in the image.

Now, let $s_1 := \big((\underline{\mathcal{F}}_1, \alpha_1), h_1 H\big)$ and $s_2 := \big(\underline{\mathcal{F}}_2, \alpha_2), h_2 H\big)$ be two elements in $\mathcal{X}_{red}(S)$. Assume $\theta_{red}\Big(\big(\underline{\mathcal{F}}_1, \alpha_1), h_1 H\big)\Big) = \theta_{red}\Big(\big(\underline{\mathcal{F}}_2, \alpha_2), h_2 H\big)\Big)$. We will show that s_1 and s_2 lie in the same orbit wrt the $D^\times$-action.

Since $(\underline{\mathcal{F}}_1, h_1 H) = (\underline{\mathcal{F}}_2, h_2 H)$, we have a quasi-isogeny $\varphi : \underline{\mathcal{F}}_1 \longrightarrow \underline{\mathcal{F}}_2$. Consider the diagram

$$\begin{tikzcd}
(\underline{\mathcal{F}}_1)_{\bar{S}} \arrow[r, "\alpha_1"] \arrow[d, "\bar{\varphi}"] & \underline{\boldsymbol{\mathcal{E}}}_{\bar{S}} \arrow[d, dashed, "f"] \\
(\underline{\mathcal{F}}_2)_{\bar{S}} \arrow[r, "\alpha_2"] & \underline{\boldsymbol{\mathcal{E}}}_{\bar{S}}
\end{tikzcd}$$

where f is defined vis the diagram. We claim that $f = g_{\bar{S}}$ for some $g \in D^\times$. This will follow from the next lemma.

□

Lemma 17.2. *Let $S \in \mathcal{N}ilp_{k_\infty[\![\zeta_1,\cdots,\zeta_t]\!]}$ and let $\bar{S}$ be its special fiber. Assume $\bar{S} =$ $\operatorname{Spec} K$ where K is an algebraically closed field. The map*

$$D^\times \longrightarrow \mathrm{QIsog}_S(\underline{\boldsymbol{\mathcal{E}}}_S)$$

defined by $g \mapsto g_{\bar{S}}$ is surjective.

Proof. Recall the generalized $\mathcal{D}$-elliptic sheaf $\underline{\boldsymbol{\mathcal{E}}}$ defined in Example 2. We consider $\underline{\boldsymbol{\mathcal{E}}}$ over $\overline{S}$, denote it by $\underline{\boldsymbol{\mathcal{E}}}_K = \underline{\boldsymbol{\mathcal{E}}} \times K$. Let $f : \underline{\boldsymbol{\mathcal{E}}}_K \longrightarrow \underline{\boldsymbol{\mathcal{E}}}_K(\mathsf{D})$ be an isogeny for some effective divisor D of X which is an isomorphism over $(X \smallsetminus \mathbb{T}) \times \overline{S}$. Consider the diagram

$$\begin{tikzcd}
\underline{\boldsymbol{\mathcal{E}}}_K \arrow[r, "f"] & \underline{\boldsymbol{\mathcal{E}}}_K(\mathsf{D}) \\
\sigma^*(\underline{\boldsymbol{\mathcal{E}}}_K) \arrow[u, "t"] \arrow[r, "\sigma^* f"] & \sigma^*(\underline{\boldsymbol{\mathcal{E}}}_K(\mathsf{D})) \arrow[u, "t \otimes \mathrm{id}"']
\end{tikzcd}$$

Note that $\sigma^*(\underline{\boldsymbol{\mathcal{E}}}_K) = \sigma^*(\underline{\boldsymbol{\mathcal{E}}} \otimes K) = \underline{\boldsymbol{\mathcal{E}}} \otimes \sigma^* K$ by definition of $\underline{\boldsymbol{\mathcal{E}}}$.

Note that $\sigma^* K^n \simeq K^n$.

Note that $\sigma^*(\underline{\boldsymbol{\mathcal{E}}}_K) = \sigma^*(\underline{\boldsymbol{\mathcal{E}}} \otimes K) = \underline{\boldsymbol{\mathcal{E}}} \otimes \sigma^* K \simeq \underline{\boldsymbol{\mathcal{E}}} \otimes K^n$. Then the morphism $t : \sigma^*(\underline{\boldsymbol{\mathcal{E}}}_K) \longrightarrow \underline{\boldsymbol{\mathcal{E}}_K}$ is represented by a matrix T. Over $F = \mathbb{F}_q(X)$, the matrix T is invertible. Hence, we have $\sigma^* f = f$ and therefore lies in $\mathbb{F}_q$.

□

Remark 17.3. 1. Note that by the rigidity in Proposition 13.5, we can reformulate the previous lemma as

$$D^\times \longrightarrow \mathrm{QIsog}_{\bar{S}}(\boldsymbol{\mathcal{E}}_{\overline{S}}) \xrightarrow{\simeq} \mathrm{QIsog}_S(\boldsymbol{\mathcal{E}}_S)$$

2. We also want to point out that the morphism $D^\times \longrightarrow \mathrm{QIsog}_{\bar{S}}(\boldsymbol{\mathcal{E}}_{\overline{S}})$ defined in the lemma is in fact an isomorphism of groups.

Proposition 17.4. *The morphism θ_{red} is radicial.*

Proof. By [2], Chapter VI, Proposition 5.2, it is enough to show that for any field K, the map of K points $\theta_{red}(K) : \mathcal{X}_{red}(K) \longrightarrow \mathcal{Y}_{red}(K)$ is injective. This follows from the previous proposition. □

Proposition 17.5. *The morphism θ is formally étale.*

Proof. This follows from the fact that quasi-isogenies of z-divisible groups extend uniquely to deformations in the category $\mathcal{N}ilp_{k_{\infty q}[\![\zeta_1, \cdots, \zeta_t]\!]}$, which is satisfied by Serre-Tate theorem.

□

Let $\mathcal{I}$ and $\mathcal{J}$ be ideal of definition of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Put $X_n := (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1})$ and $Y_n := \mathcal{O}_{\mathcal{Y}}/\mathcal{J}^{n+1}$. Then, X_n and Y_n are schemes locally of finite type over $S = \operatorname{Spec} k_\infty[\![\zeta_1, \cdots, \zeta_t]\!]$ for each n. Denote the restriction of θ to X_n by θ_n.

Proposition 17.6. *The morphism $X_n \longrightarrow Y_n$ is locally of finite type.*

Proof. This follows from Lemma 01T8.

□

Proof. (of the Uniformization Theorem)

By the previous lemmas, we have

$$\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_n \xrightarrow{\sim} \mathcal{Y}_n$$

where $\mathcal{Y}_n := \mathrm{Spec}(\mathcal{O}_{\mathcal{Y}}/\zeta^n)$(We want to note that ζ is an ideal of definition of $\mathcal{Y}$ since Z is the vanishing locus of ζ.)

Since the morphism θ is locally of finite type and formally étale, θ is étale. Since it is also radicial, it is an open immersion. So, θ is open immersion and bijection on points which means that it is an isomorphism. □

Appendix

18 Morita Equivalence

In this part, we will explain *Morita equivalence*, which later we will use it to show some categorical equivalences. For the proofs and more on Morita equivalence please see [6] and [28], Section 19.5.

Definition 18.1. Let R and S be two rings. We say R and S are Morita equivalence if $\operatorname{Mod}-R$ and $\operatorname{Mod}-S$ are equivalent where $\operatorname{Mod}-R$(respectively, $\operatorname{Mod}-S$) denotes the category of right R(resp, S)-modules .

Theorem 18.2. *(Eilenberg-Watts Theorem) If $F : \operatorname{Mod}-R \longrightarrow \operatorname{Mod}-S$ is an equivalence, then there exists an R-S-bimodule Q such that $F \simeq - \otimes_R Q$*

We can apply Theorem 18.2, to Morita equivalence. Let $F : \operatorname{Mod}-R \longrightarrow \operatorname{Mod}-S$ be an equivalence with inverse $G : \operatorname{Mod}-S \longrightarrow \operatorname{Mod}-R$. By Theorem 18.2, F is given by $- \otimes_R Q$ where Q is an R-S bimodule and G is given by $- \otimes_S P$ where P is an S-R bimodule. Hence,we have the following characterization of Morita equivalence:

Theorem 18.3. *The rings R and S are Morita equivalent iff there exists an R-S bimodule Q and S-R-bimodule P such that $P \otimes_R Q \simeq S$ (as S-S bimodules) and $Q \otimes_S P \simeq R$(as R-R bimodules).*

Remark 18.4. Let R and S be two Morita equivalent rings and Proj-R(respectively, Proj-S) denote the category of projective modules over R(resp, S). Then Proj-R and Proj-S are Morita equivalence. In general, any property defined categorically is preserved by Morita equivalence.

Example 18.5. Let $S = \mathbb{M}_d(R)$. Then, R and S are Morita equivalent with R-$\mathbb{M}_d(R)$ bimodule $Q = R^{1\times d}$, row vectors, and $\mathbb{M}_d(R)$-R bimodule $P = R^{d\times 1}$, column vectors. That is

$$F : \operatorname{Mod}-R \longrightarrow \operatorname{Mod}-\mathbb{M}_d(R)$$

is given by $A \longrightarrow A \otimes_R Q$ and

$$G : \operatorname{Mod}-\mathbb{M}_d(R) \longrightarrow \operatorname{Mod}-R$$

is given by $B \longrightarrow B \otimes_{\mathbb{M}_d(R)} P$

Remark 18.6. Note that an R-S-bimodule is just a left module over the ring $R \otimes_{\mathbb{Z}} S^{op}$ where S^{op} is the opposite ring of S.

One can define Morita equivalence for sheaves in a similar way, roughly speaking by simply replacing modules with *sheaf of modules*[see Example 9.1]. More generally, one has Morita equivalence for stacks, for more please see [28], Prop. 19.5.2. , which says that if we have an equivalence between $\mathcal{O}_X$ stacks then there exists a certain sheaf of modules such that equivalence functor is given by tensoring with that sheaf of modules.

Morita Equivalence for Sheaves and Stacks

In this section we will see that there is Morita equivalence for more general categories. We will start with Morita equivalence for $\mathcal{O}_X$-modules where $(X, \mathcal{O}_X)$ is a ringed space. The main reference is [21], Section 8.12.

Morita Equivalence for Sheaves

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{E}$ be an $\mathcal{O}_X$-module. One can associate the Hom-sheaf to $\mathcal{E}$ which is an $\mathcal{O}_X$-algebra $\mathcal{E}nd(\mathcal{E}) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$. Recall that $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$. For $U \subseteq X$ one can define the maps

$$\mathcal{E}nd(\mathcal{E}) \times \mathcal{E}(U) \longrightarrow \mathcal{E}(U) \text{ given by } (v, u) \mapsto v_U(s)$$

$$\mathcal{E}^\vee(U) \times \mathcal{E}nd(\mathcal{E})(U) \longrightarrow \mathcal{E}^\vee(U) \text{ given by } (\lambda, v) \mapsto \lambda \circ v$$

make $\mathcal{E}$ a left $\mathcal{E}nd(\mathcal{E})$-module and $\mathcal{E}^\vee$ a right $\mathcal{E}nd(\mathcal{E})$-module. So, we get functors

$$F : \mathcal{O}_X\text{- Mod} \longrightarrow \mathcal{E}nd(\mathcal{E})\text{- LeftMod}, \quad \mathcal{F} \mapsto \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$$
$$G : \mathcal{E}nd(\mathcal{E})\text{- LeftMod} \longrightarrow \mathcal{O}_X\text{- Mod}, \;\; \mathcal{H} \longrightarrow \mathcal{E}^\vee \otimes_{\mathcal{E}nd(\mathcal{E})} \mathcal{H}$$

Theorem 18.7. *([21], Proposition 8.26) Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module such that $\mathcal{E}_s \neq 0$ for all $x \in X$. Then, F and G are quasi-inverse to each other.*

Remark 18.8. If $\mathcal{E} = \mathcal{O}_X^n$ then $\mathcal{E}nd(\mathcal{E}) \simeq \mathbb{M}_n(\mathcal{O}_X)$. Hence, we obtain an equivalence between the category of $\mathcal{O}_X$-modules and the category of $\mathbb{M}_n(\mathcal{O}_X)$-modules.

Morita Equivalence for Stacks

So far, we had Morita equivalence for modules over a ring and for $\mathcal{O}_X$-modules. One has Morita equivalence for $\mathcal{O}_X$-stacks also.

Let $(X, \mathcal{O}_X)$ be a ringed site, i.e, a site X together with a sheaf of commutative rings $\mathcal{O}_X$ on X. Let $\mathcal{R}_i$ be a sheaf of $\mathcal{O}_X$-algebras on X.

Proposition 18.9. *([28], Proposition 19.5.2) Let P be an $\mathcal{R}_1 \otimes_{\mathcal{O}_X} \mathcal{R}_2^{op}$-module. the following are equivalent:*

(i) There is an $\mathcal{R}_2 \otimes_{\mathcal{O}_X} \mathcal{R}_1^{op}$-module Q such that $P \otimes_{\mathcal{R}_2} Q \simeq \mathcal{R}_1$ as an $\mathcal{R}_1 \otimes_{\mathcal{O}_X} \mathcal{R}_1^{op}$-modules and $Q \otimes_{\mathcal{R}_1} P \simeq \mathcal{R}_2$ as an $\mathcal{R}_2 \otimes_{\mathcal{O}_X} \mathcal{R}_2^{op}$-module.

(ii) $P \otimes_{\mathcal{R}_2} - : \mathrm{Mod}\,-\mathcal{R}_2 \longrightarrow \mathrm{Mod}\,-\mathcal{R}_1$ is an equivalence of $\mathcal{O}_X$-stacks

(iii) $\mathcal{H}om_{\mathcal{R}_1}(P, -) : \mathrm{Mod}\,-\mathcal{R}_1 \longrightarrow \mathrm{Mod}\,-\mathcal{R}_2$ is an equivalence of $\mathcal{O}_X$-stacks.

Moreover, under the condition of (i), Q is isomorphic to $\mathcal{H}om_{\mathcal{R}_1}(P, \mathcal{R}_1)$ and to $\mathcal{H}om_{\mathcal{R}_2^{op}}(P, \mathcal{R}_2)$ as an $\mathcal{R}_2 \otimes_{\mathcal{O}_X} \mathcal{R}_1^{op}$-module.

Remark 18.10. We stated only some parts of the proposition above to see the relation with the previous categories. To see the remaining items in the proposition please see [28].

Theorem 18.11. ***(Morita Equivalence)*** *Let $\Phi : \mathrm{Mod}\,-\mathcal{R}_2 \longrightarrow \mathrm{Mod}\,-\mathcal{R}_1$ be an equivalence of $\mathcal{O}_X$-stacks. Then, there exists an $\mathcal{R}_1 \otimes_{\mathcal{O}_X} \mathcal{R}_2^{op}$-module P satisfying one of the equivalent conditions in the previous proposition such that $P \otimes_{\mathcal{R}_2} - \simeq \Phi$ and $\mathcal{H}om_{\mathcal{R}_1}(P, -) \simeq \Phi^{-1}$*

Proof. [28], Theorem 19.5.4. □

19 Stacks

The main source for this part is [20], [47] and [52]. In this section we will give a summary of stacks.

Stacks can be thought of as a generalization of schemes in the following sense. In schemes the points are sets while in stacks the points are categories. So different than schemes in stacks each point comes with a set of automorphisms. A stack is a scheme iff the set of automorphisms of each point is trivial (Lemma 19.34). This fact plays a role in the representability by a scheme of the moduli problems. To have the representability by a scheme one usually puts extra conditions on automorphisms to satisfy. That is one of the reason that in the main body of this work we consider objects with level I-structures. In the presence of a non-trivial level structure, we prove that moduli functor is representable by a scheme. There are examples however where even after adding big level structures, the moduli functor still is not representable by a scheme(E.g [23], Section 3). In [23], Section 4, Hartl gives an example of a moduli functor which is not representable by a scheme.

Let S be a scheme. One can see S by its *functor of points.* One can define stacks as 2-*functor*, which emphasises that stacks are generalizations of schemes([20], Definition 2.10). Here, we will define stacks as categories. Note that these two definitions of stacks are equivalent. Then, we will put some condition on stacks so that we can see them as geometric objects.

Definition 19.1. A *category over* $\mathcal{C}$ is a category $\mathcal{F}$ together with a covariant functor $\rho_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{C}$. If $X \in \operatorname{Ob}\mathcal{F}$(resp. φ is a morphism) and $T \in \mathcal{C}$ such that $\rho_{\mathcal{F}}(X) = T$ (resp $\rho_{\mathcal{F}}(\varphi) = f$), we say that X *lies over* T (resp. φ *lies over* f).

Definition 19.2. Let $\mathcal{F}$ be a category over $\mathcal{C}$. A morphism $\varphi : X \longrightarrow X'$ is called *catresian* if for any other object $Y \in \operatorname{Ob}\mathcal{F}$ with a morphism $\psi : Y \longrightarrow X'$ and factorization

$$\rho(Y) \xrightarrow{h} \rho(X) \xrightarrow{\rho(\varphi)} \rho(X')$$

of $\rho(\psi)$, there exists unique morphism $\lambda : Y \longrightarrow X$ such that $\varphi \circ \lambda = \psi$ and $\rho(\lambda) = h$.

In a picture expression:

$$\begin{tikzcd}
Y \arrow[r, dashed, "\lambda"] \arrow[d] \arrow[rr, bend left, "\psi"] & X \arrow[r, "\varphi"] \arrow[d] & X' \arrow[d] \\
\rho(Y) \arrow[r, "h"] & \rho(X) \arrow[r, "\rho(\varphi)"] & \rho(X')
\end{tikzcd}$$

Definition 19.3. 1. A category $\mathcal{F}$ over $\mathcal{C}$ is called *fibered category* if given an object X of $\mathcal{F}$ and an arrow $f : T \longrightarrow \rho_{\mathcal{F}}(X)$ of $\mathcal{C}$, there exists a cartesian arrow $\varphi : X' \longrightarrow X$ of $\mathcal{F}$ over f, i.e, so that $\rho_{\mathcal{F}}(\varphi) = f$.

2. Let $\mathcal{F}$ be a fibered category and $T \in \operatorname{Ob}\mathcal{C}$. We define the *fiber of* $\mathcal{F}$ *over* T as the full subcategory of $\mathcal{F}$ whose objects lie over T and whose morphisms lie over id_T. We denote this fiber by $\mathcal{F}(T)$.

Definition 19.4. 1. A *groupoid* is a category in which every morphism is isomorphism.

2. We say a fibered category $\mathcal{F}$ is *fibered in groupoids* if all fibers are groupoids.

Proposition 19.5. *Let $\mathcal{F}$ be a category over $\mathcal{C}$. Then $\mathcal{F}$ is fibered in groupoids over $\mathcal{C}$ if and only if the following two conditions hold.*

1. *Every arrow in $\mathcal{F}$ is cartesian.*

2. *Given an object η of $\mathcal{F}$ and an arrow $f : U \longrightarrow \rho_{\mathcal{F}}(\eta)$ of $\mathcal{C}$, there exists an arrow $\varphi : \zeta \longrightarrow \eta$ of $\mathcal{F}$ with $\rho_{\mathcal{F}}(\varphi) = f$.*

Proof. [52], 3.22. $\square$

Remark 19.6. 1. the condition (1) implies that the morphism $\varphi : \zeta \longrightarrow \eta$ in the condition (2) is unique up to isomorphism: Assume there exists $\varphi_1 : \zeta_1 \longrightarrow \eta$ and $\varphi_2 : \zeta_2 \longrightarrow \eta$ as in the condition (2). Then by (1), there exists unique map $\lambda : \zeta_1 \longrightarrow \zeta_2$. By swaping ζ_1 and ζ_2, we see that λ is an isomorphism.

2. Condition (2) also implies that $\varphi : X \longrightarrow X'$ is an isomorphism $\iff$ $\rho_{\mathcal{F}}(\varphi)$ is an isomorphism in $\mathcal{C}$.

Convention: For each $X' \in \operatorname{Ob} \mathcal{F}$ and any $f : T \longrightarrow T'$ with X' over T' choose an X as in the condition one. Note that the map from $X' \longrightarrow X$ is unique by 18.6(1). We will denote this lift X by $f^* X'$(or by $X'|_T$). From now on we fix such choices for all f and X'. This kind of choice is called a *cleavage*.

Remark 19.7. From the previous remark, we see that any morphism in $\mathcal{F}(T)$ is an isomorphism.

Definition 19.8. 1. A *morphism* of fibered categories $\mathcal{F} \longrightarrow \mathcal{G}$ is a functor $f : \mathcal{F} \longrightarrow \mathcal{G}$ such that

   (i) $\rho_G \circ f = \rho_F$

   (ii) f sends cartesian morphisms in $\mathcal{F}$ to cartesian morphisms in $\mathcal{G}$.

2. Let $f, g : \mathcal{F} \longrightarrow \mathcal{G}$ be two morphisms of fibered categories. A *base preserving natural transformation* $\alpha : f \longrightarrow g$ is a natural transformation of functors such that for every $X \in \mathcal{F}$ the morphism $\alpha_X : f(X) \longrightarrow g(X)$ in $\mathcal{G}$ projects to the identity morphism in Sch_S.

Definition 19.9. Let $\mathcal{F} \longrightarrow \mathcal{C}$ be a fibered category. A *fibered subcategory* $\mathcal{G}$ of $\mathcal{F}$ is a subcategory of $\mathcal{F}$, such that the composite $\mathcal{G} \hookrightarrow \mathcal{F} \longrightarrow \mathcal{C}$ makes G into a fibered category over $\mathcal{C}$, and such that any cartesian arrow in $\mathcal{G}$ is also cartesian in $\mathcal{F}$.

Fibered category associated to a pseudo functor

Definition 19.10. ([52], Definition 3.10) Let $\mathcal{C}$ be a category. A *pseudo-functor* Φ on $\mathcal{C}$ consist of the following data

1. For each object U of $\mathcal{C}$ a category ΦU

2. For each morphism $f : U \longrightarrow V$ in $\mathcal{C}$ a functor $f^* : \Phi V \longrightarrow \Phi U$

3. For each object U of $\mathcal{C}$ an isomorphism $\varepsilon_U : id_U^* \simeq id_{\Phi U}$ of functors $\Phi U \longrightarrow \Phi U$

4. For each pair of morphisms $U \xrightarrow{f} V \xrightarrow{g} W$ an isomorphism

$$\alpha_{f,g} : f^* g^* \simeq (gf)^* : \Phi W \longrightarrow \Phi U$$

   of functors $\Phi W \longrightarrow \Phi U$

These data are required to satisfy the following conditions:

(a) If $f : U \longrightarrow V$ a morphism in $\mathcal{C}$ and η is an object of ΦV, we have

$$\alpha_{\mathrm{id}_U, f}(\eta) = \varepsilon_U(f^*\eta) : \mathrm{id}_U^* f^*\eta \longrightarrow f^*\eta$$

and

$$\alpha_{f,\mathrm{id}_U}(\eta) = f^*\varepsilon_V(\eta) : f^* \mathrm{id}_V \eta \longrightarrow f^*\eta$$

(b) Whenever we have morphisms $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} T$ and an object θ of ΦT the diagram

$$\begin{tikzcd} f^*g^*h^*\theta \arrow[r, "\alpha_{f,g}(h^*\theta)"] \arrow[d, "f^*\alpha_{g,h}(\theta)"] & (gf)^*h^*\theta \arrow[d, "\alpha_{gf,h}(\theta)"] \\ f(Y_1) \arrow[r, "\alpha_{f,gh}(\theta)"'] & g(Y_2) \end{tikzcd}$$

commutes.

By [52] Section 3.1.3, one can get a fibered category associated to a psedo-functor and vice versa. Moreover, these two procedures are inverse to each other(up to an isomorphism of fibered categories).

Example 19.11. Let Sch_S denote the category of schemes over a fixed base scheme S. For each scheme U we define $QCoh(U)$ to be the category of quasi-coherent sheaves on U. Given a morphism $f : U \longrightarrow V$, we have a functor $f^* : QCoh(V) \longrightarrow QCoh(U)$.

However, in general for $U \xrightarrow{f} V \xrightarrow{g} W$, $(g \circ f)^* \neq f^* \circ g^*$, so $U \longrightarrow QCoh(U)$ is not a functor. But $(g \circ f)^*$ and f^*g^* are canonically isomorphic since $(gf)_* = f_*g_*$ and f^* is left adjoin to f_*, Yoneda lemma induces the canonical isomorphism between functors $(g \circ f)^*$ and $f^* \circ g^*$. One can also check that the isomorphisms above satisfy the conditions in the definition. So we get a pseudo-functor, hence a fibered category $QCoh/S$. For details please see [52], 3.2.1.

From now on unless stated otherwise assume that $\mathcal{C} = Sch_S$ is equipped with the étale topology. Before we continue we want to recall the definition of a sheaf in a Grothendieck topology.

Definition 19.12. ([47], Section 2.2)

Let $\mathcal{C}$ be a category with Grothendick topology. A *presheaf* on $\mathcal{C}$ is a functor

$$F : \mathcal{C}^{op} \longrightarrow Sets$$

Definition 19.13. Let $\mathcal{C}$ be a category with Grothendieck topology. A presheaf F on $\mathcal{C}$ is a *sheaf* if for every $U \in \operatorname{Ob}\mathcal{C}$ and covering $\{U_i \longrightarrow U\}_{i\in I}$ the sequence

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j)$$

is exact, where the two maps on the right are induced by the two projections $U_i \times_U U_j \longrightarrow U_i$ and $U_i \times_U U_j \longrightarrow U_j$.

Remark 19.14. To say that the sequence above is exact means that the map $F(U) \longrightarrow \prod_{i\in I} F(U_i)$ identifies $F(U)$ with the equalizer of the two maps

$$\prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j)$$

Stacks

Let $\mathcal{F}$ be a fibered category over a $\mathcal{C}$. For any object S of C and any two objects ζ and η in $\mathcal{F}(S)$, define the presheaf on $(\mathcal{C}/S)$:

$$\underline{\mathrm{Hom}}_S(\zeta, \eta) : (\mathcal{C}/S)^{op} \longrightarrow Sets$$

Definition 19.15. We say a fibered category $\mathcal{F}$ is a *prestack* if for every choice of S, ζ, η the presheaf $\underline{\mathrm{Hom}}_S(\zeta, \eta)$ is a sheaf.

Definition 19.16. Let $\mathcal{C}$ be a site and $\mathcal{F}$ be a fibered category over $\mathcal{C}$. Let U be an object of $\mathcal{C}$. Given a covering $\mathcal{U} = \{U_i \longrightarrow U\}$. Set

$$U_{ij} = U_i \times_U U_j$$

and

$$U_{ijk} = U_i \times_U U_j \times_U U_k.$$

An *object with descent data* $\big(\{\zeta_i\}, \{\varphi_{ij}\}\big)$ on $\mathcal{U}$ is a collection of objects $\zeta_i \in \mathcal{F}(U_i)$ together with isomorphism

$$\varphi_{ij} : pr_2^*\zeta_j \xrightarrow{\sim} pr_1^*\zeta_i \text{ in } \mathcal{F}(U_i \times_U U_j)$$

such that the following cocycle condition is satisfied:

$$pr_{13}^*\varphi_{ik} = pr_{12}^*\varphi_{ij} \circ pr_{23}^*\varphi_{ik} : pr_3^*\zeta_k \longrightarrow pr_1^*\zeta_i$$

Definition 19.17. We say that an object with descent data $\big(\{\zeta_i\}, \{\varphi_{ij}\}\big)$ in $\mathcal{F}(\mathcal{U})$ is *effective* if there exists an object ζ of $\mathcal{F}(U)$ together with cartesian arrows $\zeta_i \longrightarrow \zeta$ over $f_i : U_i \longrightarrow U$ such that the following diagram

$$\begin{tikzcd}
pr_2^*\zeta_j \arrow[rr, "\varphi_{ij}"'] \arrow[d] & & pr_1^*\zeta_i \arrow[d] \\
\zeta_j \arrow[dr] & & \zeta_i \arrow[dl] \\
& \zeta &
\end{tikzcd}$$

commutes.

Definition 19.18. A *stack* is a prestack such that for every cover $(U_i \longrightarrow U)$ in the chosen Grothendieck topology, all descent data relative to $(U_i \longrightarrow U)$ are effective.

Example 19.19. Given a scheme S, we have constructed $(QCoh/S)$ of quasi-coherent sheaves, whose fiber of a scheme U over S is the category $QCoh(U)$ of quasi-coherent sheaves on U. The fibered category $(QCoh/S)$ over (Sch/S) is a stack with respect to the fpqc topology (cf. [52], Theorem 4.23).

Definition 19.20. Let $\mathcal{C}$ be a site and $\mathcal{F} \longrightarrow \mathcal{C}$ a stack. A *substack* of $\mathcal{F}$ is a fibered subcategory that is a stack.

Example 19.21. ([52], Example 4.19) Let $\mathcal{C}$ be a site, $\mathcal{F} \longrightarrow \mathcal{C}$ a stack, $\mathcal{G}$ a full subcategory of $\mathcal{F}$ satisfying the following two conditions.

1. Any cartesian arrow in $\mathcal{F}$ whose target is in $\mathcal{G}$ is also in $\mathcal{G}$.

2. Let $\{Ui \longrightarrow U\}$ be a covering in $\mathcal{C}$, η an object of $\mathcal{F}(U)$, η_i pullbacks of η to U_i. If η_i is in $\mathcal{G}$ for all i, then η is in $\mathcal{G}$.

Then $\mathcal{G}$ is a substack.

Example 19.22. The full subcategory of $(QCoh/S)$ consisting of locally free sheaves of finite rank satisfies the two conditions, hence it is a substack.

Stacks fibered in groupoids

Definition 19.23. A *stack in groupoids* is a category fibered in groupoid $\mathcal{F}$ such that the assignement

$$Sch_S \longrightarrow Set$$

given by $U \mapsto \mathcal{F}(U)$ is a *sheaf of groupoids*, i.e,

1. For all scheme T and pair of objects X, Y of $\mathcal{F}$ over T (i.e, pair of objects in $\mathcal{F}(T)$), the contravariant functor

$$\mathrm{Iso}_T(X, Y) : Sch_T \longrightarrow Sets$$

defined by $(f : T' \longrightarrow T) \mapsto \{\varphi : f^*X \xrightarrow{\simeq} f^*Y \text{ an isomorphism in } \mathcal{F}(T')\}$ is a sheaf (in the étale topology).

2. All descent data are effective.

Remark 19.24. We want to point out that the stack in Definition 19.23 is different than the one in the Definition 19.18. In Definition 19.23, we define " stacks in groupoids". We will usually supress the word "groupoid" in the "stack in groupoids".

Definition 19.25. 1. *Morphisms* (resp. *isomorphisms*) of stacks are defined to be morphisms (resp. isomorphisms) of fibered categories over Sch_S

2. We denote by $\mathrm{Hom}_S(\mathcal{F}, \mathcal{G})$ the category whose objects are morphisms of stacks and whose morphisms are base preserving natural transformations.

We can relate the two definitions of stack:

Definition 19.26. Let $\mathcal{F} \longrightarrow \mathcal{C}$ be a fibered category. The category fibered in groupoids associated with $\mathcal{F}$ is the subcategory $\mathcal{F}_{cart}$ of $\mathcal{F}$, whose objects are all the objects of $\mathcal{F}$, and whose arrows are the cartesian arrows of $\mathcal{F}$.

Remark 19.27. The stack $\mathcal{F}_{cart}$ is a groupoid stack.

Proposition 19.28. *Let $\mathcal{C}$ be a site, $\mathcal{F} \longrightarrow \mathcal{C}$ a fibered category. Let $\mathcal{F}_{cart}$ be the associated category fibered in groupoids.*

1. *If $\mathcal{F}$ is a stack, so is $\mathcal{F}_{cart}$.*

2. *If $\mathcal{F}$ is a prestack and $\mathcal{F}_{cart}$ is a stack, then $\mathcal{F}$ is also a stack.*

Proof. [52], Proposition 4.20. □

By using morphisms we can form a new stack, namely fiber product of stacks.

Definition 19.29. Let $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{G}$ be stacks. Given two morphisms $f : \mathcal{F}_1 \longrightarrow \mathcal{G}$ and $\mathcal{F}_2 \longrightarrow \mathcal{G}$, we define the *fiber product* $\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2$, as follows.

- The objects of $\mathcal{F}_1 \times \mathcal{F}_2$ are the triples (X_1, X_2, α) where $X_j \in \mathrm{Ob}\, \mathcal{F}_j$ lie over the same scheme U and $\alpha : f(X_1) \longrightarrow g(X_2)$ is an isomorphism in $\mathcal{G}$(in other words $\rho_{\mathcal{G}}(\alpha) = \mathrm{id}_U$).

- A *morphism* from (X_1, X_2, α) to (Y_1, Y_2, β) is a tuple (φ_1, φ_2) of morphisms $\varphi_j : X_j \longrightarrow Y_j$ that lie over the same morphisms of schemes $h : U \longrightarrow V$ such that the following diagram commutes

$$\begin{tikzcd} f(X_1) \arrow[r, "\alpha"] \arrow[d, "f(\varphi_1)"] & g(X_2) \arrow[d, "g(\varphi_2)"] \\ f(Y_1) \arrow[r, "\beta"'] & g(Y_2) \end{tikzcd}$$

The fiber product satisfy the universal property of fiber products.

Theorem 19.30. *The fiber products exist in the category of stacks stacks*

Proof. [22], Lemma 4.14. □

Example 19.31. Let T be a scheme over S. Consider Sch_T the category of schemes over T. Define the functor $Sch_T \longrightarrow Sch_S$ by the composition $V \longrightarrow T \longrightarrow S$ for any $V \in Sch_T$. Then, Sch_T becomes a stack. We will denote also this stack by T.

Definition 19.32.

1. We say a stack $\mathcal{F}$ is *representable by a scheme* T if it is isomorphic to the stack associated to T.

2. A morphism of stacks $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is called *representable* if for all $T \in \mathrm{Ob}\, Sch_S$ and morphisms $T \longrightarrow \mathcal{G}$ the fiber product of stacks $T \times_{\mathcal{G}} \mathcal{F}$ is representable by a scheme.

3. Let "P" be a property of morphisms of schemes which is local on the target and stable under base-change (e.g: smooth, étale, surjective, of finite type etc).We say that a representable morphism $f : \mathcal{F} \longrightarrow \mathcal{G}$ has "P" if for any $T \longrightarrow \mathcal{G}$ the induced morphism of schemes $T \times_{\mathcal{G}} \mathcal{F} \longrightarrow T$ has the property "P".

Remark 19.33. One can define the representability of a stack by algebraic spaces similarly as above. Since we will only use representability by a scheme, we won't give the definition of an algebraic space.

We have the following very useful lemma that shows us an obsticle to be representable.

Lemma 19.34. *If a stack has an object which has a nontrivial automorphism then the stack cannot be respresentable by a scheme.([20], Lemma 2.17)*

Lemma 19.35. *([20], Lemma 2.18) Let $\mathcal{F}$ be a stack and T a scheme. The functor*

$$u : \mathrm{Hom}_S(T, \mathcal{F}) \longrightarrow \mathcal{F}(T)$$

given by $(f : Sch_T \longrightarrow \mathcal{F}) \mapsto f(id_T)$ gives us an equivalence of categories.

Remark 19.36.

1. Note that the previous lemma tells us that an object of $\mathcal{F}$ that lies over T is equivalent to a morphism of stacks from $T \longrightarrow \mathcal{F}$.

2. We want to note this the previous lemma is the 2-Yoneda lemma(cf. [52], 3.6.2)

Let $\mathcal{F}$ be a stack and let $\Delta_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{F} \times \mathcal{F}$ be the obvious diagonal morphism. A morphism from a scheme T to $\mathcal{F} \times \mathcal{F}$ is equivalent to two objects $X_1, X_2 \in \mathcal{F}(T)$ by the previous lemma. By taking the fiber product we have

$$\begin{tikzcd} \mathrm{Isom}_T(X_1, X_2) \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "\Delta_{\mathcal{F}}"] \\ T \arrow[r, "(X_1, X_2)"'] & \mathcal{F} \times_S \mathcal{F} \end{tikzcd}$$

Hence the group of automorphisms of an object is encoded in the diagonal morphism.

Proposition 19.37. *Let $\mathcal{F}, X_1, X_2$ be as above. the following are equivalent:*

1. *The morphism $\Delta_{\mathcal{F}}$ is representable.*

2. *The stack* $\mathrm{Isom}_T(X_1, X_2)$ *is representable for all T, X_1 and X_2.*

3. *For any scheme T, every morphism $T \longrightarrow \mathcal{F}$ is representable.*

4. *For all schemes T_1, T_2 and morphisms $T_1 \longrightarrow \mathcal{F}$ and $T_2 \longrightarrow \mathcal{F}$, the fiber product $T_1 \times_{\mathcal{F}} T_2$ is representable.*

[32], Cor. 2.12. , [[52], Prop. 7.13] □

Definition 19.38. (Deligne-Mumford stack)

Let Sch_S be the category of S-schemes with étale topology. Let $\mathcal{F}$ be a stack and let $\Delta_{\mathcal{F}} : \mathcal{F} \times \mathcal{F}$ be the obvious diagonal morphism. Assume

1. The diagonal $\Delta_{\mathcal{F}}$ is representable.

2. There exists a scheme U (called *atlas*) and a surjective and étale morphism $u : U \longrightarrow \mathcal{F}$

   Then, we say that $\mathcal{F}$ is a *Deligne-Mumford stack.*

By the Proposition 19.37 and by the fact $\Delta_{\mathcal{F}}$ is representable, we see that the morphism $u : U \longrightarrow \mathcal{F}$ in (2) in the definition is representable. So, the notion of étale is well-defined for u.

Definition 19.39. Let Sch_S be the category of S-schemes with the fppf topology. Let $\mathcal{F}$ be a stack. Assume

1. The diagonal $\Delta_{\mathcal{F}}$ is representable.

2. There exists a scheme U and a smooth(hence locally of finite type) and surjective morphism $u : U \longrightarrow \mathcal{F}$.

   Then, we say that $\mathcal{F}$ is an *Artin stack.*

Example 19.40. Now, we will give the example "quotient stack" following [11], Example 4.8.

Let X be a scheme over S. Let G be a group scheme over S that is étale, separated and of finite type over S. Assume G acts on X. Define the stack $[X/G]$ over S as follows: Its category of sections over an S-scheme T is the category of principal homogeneous spaces over T under G_T. The principal homogeneous space $G \times X$ over X together with the morphism $G \times X \longrightarrow X$ is a section of $[X/G]$ over X. The corresponding morphism $X \longrightarrow [X/G]$ is étale and surjective, so $[X/G]$ is a Deligne-Mumford stack.

The stack $[X/G]$ is representable $\iff$ X is a principal homogeneous space over a scheme Y.

Remark 19.41. 1. Note that this example tells us that if G is étale over S, then the quotient stack $[X/G]$ is a Deligne-Mumford stack. We are using this fact in Section 5, Proof of Theorem 5.3.

2. For more details on the "moduli quotient", we refer to [20]. In the introduction he talks about "quotient scheme" vs "quotient stack" and in Section 3, he compare moduli scheme vs moduli stack of vector bundles.

There are some conditions on G to decide when the quotient of a group is a stack and when the quotient of a stack is a Deligne-Mumford stack. We collect some of them in the following proposition.

Proposition 19.42. *1. If G is smooth and affine then the groupoid $[X/G]$ is a stack.*

2. If the stabilizers of the geometric points of X are finite and reduced then $[X/G]$ is a Deligne-Mumford stack.

Proof. 1. [32], 2.4.2

2. [52], Example 7.17

□

Moduli stacks

For this section one can look at [47], Introduction.

Moduli spaces are spaces that answers the problem of classifying objects. These problems are called moduli problems. And the moduli problems are usually described by functors. We say that a functor F is representable by a scheme M if F is isomorphic to $\mathrm{Hom}(-, M)$ where $\mathrm{Hom}(-, M)$ is the functor of points. Then, the scheme M is called a *fine moduli space*. This means that there is a 1-1 correspondence between families of objects parametrized by B and the morphisms $B \longrightarrow M$. So, a fine moduli space has a universal family U corresponding to the identity morphism $\mathrm{id}_M \in \mathrm{Hom}(M, M)$ together with a morphism $U \longrightarrow M$. We say that two points on M are isomorphic if they correspond the isomorphic(or geometrically same) objects.

Often a fine moduli is desired but not obtained. Instead one obtains a *coarse moduli space*. A coarse moduli space is a scheme M with a morphism of functors $F \longrightarrow \mathrm{Hom}(-, M)$ that is universal for morphisms from F to representable functors and such that for any algebraically closed field k the induced map $F(\mathrm{Spec}(k)) \longrightarrow \mathrm{Hom}((Spec(k), M) = M(k)$ is a bijection. So, a coarse moduli space is a space that has the right information on points., i.e, if we only consider points not families, coarse moduli space has the right information.

Sometimes the moduli functor cannot be represented by a scheme and it has neither fine moduli space nor a coarse moduli space. The reason for that is the objects that are parametrized has nontrivial automorphisms(recall Lemma 19.34). One can solve this problem by considering isomorphisms. More precisely, consider the objects parametrized by B with only morphisms between them are isomorphisms. By remembering isomorphisms, we get a *moduli stack*.

20 Vector bundles over $X \times S$

Let $\infty_1, \cdots, \infty_n$ be closed places of X and let $A = \Gamma(X \smallsetminus \{\infty_1, \cdots, \infty_n\}, \mathcal{O}_X)$. We denote the function field of X by $F/\mathbb{F}_q$.

The main reference for this is [53], Section 2.

Theorem 20.1. *Let S be an $\mathbb{F}_q$-scheme. Given the data $(\mathcal{M}, \mathcal{E}_{\infty_j}, \iota_j)$ where*

- *$\mathcal{M}$ is vector bundle of rank d over* $\operatorname{Spec} A \times S$
- *$\mathcal{E}_{\infty_j}$ is vector bundle of rank d over $\mathcal{O}_{X,\infty} \times S$*
- *$\iota_j : \mathcal{M} \otimes_A F \simeq \mathcal{E}_{\infty_j} \otimes_{\mathcal{O}_{X,\infty_j}} F$ an isomorphism*

there exists (up to isomomorphism) a unique vector bundle $\mathcal{F}$ on $X \times S$ such that

$$(\mathcal{M}_{\mathcal{F}} = \mathcal{F}|_{\operatorname{Spec} A \times S}, \mathcal{E}_{\mathcal{F},\infty} = \mathcal{F} \otimes_{\mathcal{O}_{X \times S}} \mathcal{O}_{X,\infty_j} \times S, can_j : \mathcal{M}_{\mathcal{F}} \otimes_A F \cong \mathcal{E}_{\mathcal{F},\infty_j} \otimes F)$$

where can_j is the canonical isomorphism.

Proof. In [53], Proposition 2.69, the theorem is proved for the case $n = 1$. Now, the proof follows by induction on n. □

Therefore, one can think of a locally free space over $X \times S$ as in two parts: the affine part and the part around ∞. The isomorphism serves as gluing morphism.

Example 20.2. Let X be a smooth connected curve over K, and let $U \subset X$ be a non-empty affine open subset of X with $U = SpecR$. Denote by F the function field of X, i.e, $F = K(X) = Frac(R)$. Then, there is a bijection up to isomorphism between

1. rank n vector bundles on X
2. Data: $(M, (L_x)_{x \in X-U}, (i_x)_{x \in X-U})$ where M is a rank n projective R-module, for each $x \in X - U$, L_x is a rank n free $\mathcal{O}_{X,x}$-module and i_x is an isomorphism

$$i_x : M \otimes_R F \simeq L_x \otimes_{\mathcal{O}_{X,x}} F.$$

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