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Stratified Approximation Results in Singular Spaces

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Abstract

Determining conditions under which a given map is close to a homeomorphism has been an important problem in geometric topology. One of the major results related to the problem is the α-Approximation Theorem of Chapman and Ferry, which asserts that a small homotopy equivalence between manifolds is small homotopic to a homeomorphism. In this context, the smallness condition on a homotopy means that the size of the track covered by each point during the homotopy is small when measured by an open cover of the target space. In proving such a theorem, besides the original approach of Chapman-Ferry which uses some results from topological surgery theory, there is another more geometric approach that is more suitable to establish a similar theorem for classes of spaces more general than manifolds. This second approach, due to Chapman himself, is to use controlled topological engulfing to prove a geometric result on approximate fibrations called the Sucking Principle. The α-Approximation Theorem then follows from an application of this principle together with the Cell-Like Approximation Theorem of Siebenmann. In this thesis, based on previous work of B. Hughes, we develop various tools that address the above approximation questions in a stratified setting of possibly singular spaces. In particular, we establish the Stratified Radial Engulfing Theorem, the Stratified Wrapping Up Theorem, the Stratified Handle Theorem, and the Stratified γ-Sucking Theorem. As a consequence we obtain a Stratified Sucking Theorem with unstratified polyhedral target space.

Zusammenfassung

Die Bestimmung von Bedingungen, unter denen eine bestimmte Abbildung einem Homöomorphismus nahe kommt, ist ein wichtiges Problem der geometrischen Topologie. Eines der Hauptergebnisse in diesem Zusammenhang ist der α-Approximationssatz von Chapman und Ferry, der besagt, dass eine kleine Homotopieäquivalenz zwischen Mannigfaltigkeiten eine kleine Homotopie zu einem Homöomorphismus besitzt. Hierbei bedeutet die Kleinheitsbedingung an die Homotopie, dass die Spur, die von jedem Punkt während der Homotopie abgedeckt wird, in den Mengen einer offenen Überdeckung des Zielraums enthalten ist. Um den Satz zu beweisen, gibt es neben dem ursprünglichen Ansatz von Chapman-Ferry, der einige Ergebnisse aus der topologischen Chirurgie Theorie verwendet, einen anderen geometrischeren Ansatz, der besser geeignet ist, die Erweiterung des Satzes für Klassen von Räumen zu beweisen, die allgemeiner als Mannigfaltigkeiten sind. Dieser zweite Ansatz, der Chapman selbst zu verdanken ist, besteht darin, ein kontrolliertes topologisches „Umfangen” zu verwenden, um ein geometrisches Ergebnis, genannt „Ansaugeprinzip” zu beweisen. Der α-Approximationssatz folgt dann aus einer Anwendung dieses Prinzips zusammen mit dem zellförmigen Approximationssatz von L. Siebenmann. In dieser Arbeit entwickeln wir, ausgehend von Ansätzen von B. Hughes, etliche Werkzeuge, die obige Approximationsfragen in einem stratifizierten Kontext möglicherweise singulärer Räume adressieren. Insbesondere etablieren wir den stratifizierten radialen Umfangungssatz, den Satz über stratifiziertes Aufrollen, den stratifizierten Henkelsatz, und den stratifizierten γ-Ansaugesatz. Als Folgerung erhalten wir einen stratifizierten Ansaugesatz, in dem der Zielraum ein unstratifizierter Polyeder ist.

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Contents

Chapter 1

Introduction

When is a map close to a homeomorphism?

The homeomorphism type of a closed manifold of dimensions 0, 1, and 2 is fully determined by its homotopy type. In higher dimensions, homotopy equivalences between closed manifolds are much harder to understand, particularly in the case of non-trivial fundamental groups. Indeed, there are closed manifolds that are homotopy equivalent but not homeomorphic. Historically, the first such examples were the 3-dimensional lens spaces, classified by the Reidemeister torsion [Coh73]. These phenomena motivated further investigation on the relationship between homotopy equivalences and homeomorphisms. One of the typical questions then was when is a homotopy equivalence homotopic to a homeomorphism.

The α-Approximation Theorem of T. A. Chapman and Steve Ferry asserts that a small homotopy equivalence between manifolds is homotopic to a homeomorphism via a small homotopy. In this context, the smallness condition on a homotopy means that the size of the track covered by each point during the homotopy is small when measured by an open cover of the target space. Such a homotopy is then called an α-homotopy, where the term α denotes the open cover. This also explains the terminology of the theorem. More precisely, the theorem states that if M^n and N^n are manifolds then for any open cover α of N^n there is another open cover β of N^n such that if $f : M^n \to N^n$ is a β-homotopy equivalence which is already a homeomorphism from ∂M to ∂N, then f is α-homotopic to a homomorphism which agrees with the homeomorphism between boundaries.

The α-Approximation Theorem is true for manifolds of all dimensions. It was first proved for infinite-dimensional Hilbert-cube manifolds by Steve Ferry using global mapping cylinder constructions which do not seem to have analogs in finite-dimensions [Fer77]. Chapman-Ferry proved the theorem for $n \geq 5$ by using a handle decomposition of the target space to analyze the situation on each handle and then get a series of handle problems [CF79]. The handle problems are then solved by using the torus geometry in the form of Kirby's torus trick and Siebenmann's inversion trick. Some results from topological surgery in the form of a splitting theorem and a uniqueness theorem of homotopy tori are also needed. Then, the conclusion of the theorem is achieved by patching all solutions of the handle problems using the deformation theorem of Edwards and Kirby [EK71]. In low dimensions, W. Jakobsche proved the case $n = 2, 3$ by proving the splitting theorem in these dimensions and then following the arguments of Chapman-Ferry [Jak83] [Jak88]. The $n = 4$ case follows from the $n = 5$ case and the 5-dimensional Quinn's Thin h-Cobordism Theorem [Qui82] [FW91]. In Chapter 2, we are presenting the techniques and results that are used in the Chapman-Ferry's proof as well as describing the main idea of the proof itself.

The α-Approximation Theorem is also related to another classical problem in geometric topology, namely determining conditions under which a map is close to a homeomorphism. The first hint to solve this problem is the fact that the point inverses of a homeomorphism are all precisely points. Therefore a map that is close to a homeomorphism must have point inverses that are close to points in some suitable sense. One of the first notions of a set being close to a point is the concept of the cellular subset of a manifold which was firstly introduced by M. Brown in [Bro60]. A compact subset of a manifold is cellular if it has arbitrarily small open neighbourhoods that are homeomorphic to open cells. A map between manifolds with cellular point inverses is called a cellular map. A cellular subset X of a manifold M is close to a point in the sense that the quotient space M/X is a manifold and the quotient projection map $\pi : M \to M/X$, which is a cellular map, is a uniform limit of homeomorphisms. On the other hand, R. Finney observed that any map which is a limit of homeomorphisms is necessarily cellular [Fin68]. This motivated one to make a conjecture that cellular maps are precisely limits of homeomorphisms

Nevertheless, cellularity is clearly not an intrinsic concept. Whether the image of an embedding $\phi : X \to M$ is cellular on M or not, does depend on the embedding rather than being a property of the embedded space X. Any finite-dimensional cellular subset of a manifold, obviously except a point, can be embedded as a non-cellular subset in Euclidean space with a dimension greater than twice of its dimension [Edw78]. This motivated R. Lacher to consider embeddability as a cellular subset of some manifold rather than cellularity itself [Lac77]. A space is cell-like if it can be embedded as a cellular subset in a manifold. A map between ANRs with cell-like point inverses is then called a cell-like map. A proper cell-like map is called a CE map. Furthermore, W. E. Haver also showed that a cell-like map between separable ANRs is an α-homotopy equivalence for every α [Hav75]. Therefore, for a map between separable ANRs, being CE is an example of being α-homotopy equivalence.

Responding to the conjecture that cellular maps are precisely limits of homeomorphisms, Siebenmann proved his CE Approximation Theorem [Sie72]. The theorem states that a given CE map $f : M^n \to N^n$ between metric n-manifolds with $n \neq 4$ can be approximated arbitrarily closely by a homeomorphism provided that if $n = 5$ then the CE map f is already a homeomorphism between boundaries and if $n = 3$ then each point inverse of f has an open neighborhood that is prime for connected sum. An important consequence of this theorem that more closely resembles the conjecture is that the set of cell-like maps between closed n-manifolds is precisely the closure of the set of homeomorphisms in the space of all maps [Lac77]. In particular, we obtain that such a cell-like map is homotopic to a homeomorphism. Hence, this theorem is a special case of the α-Approximation Theorem.

There is a second approach to proving the α-Approximation Theorem for $n \geq 5$ which directly uses the CE Approximation Theorem. This approach, which is due to Chapman himself, is more geometric because it does not need any facts from surgery [Cha81] [Wei94]. Instead, it uses a result from topological engulfing called the Sucking Principle of manifold approximate fibrations. An α-homotopy equivalence is an example of a weaker version of fibration called α-fibration. A map is called an approximate fibration if it is an α-fibration for each α. Loosely speaking, the Sucking Principle says that for a given α-fibration there exists an approximate fibration that is small homotopic to it. Hence, the given α-homotopy equivalence is small homotopic to an approximate fibration. Furthermore, such an approximate fibration is CE and therefore Siebenmann's theorem can be used to achieve the conclusion of the theorem.

The idea of Chapman's proof of the Sucking Theorem in [Cha81] is by constructing a sequence of α_i-fibrations so that it starts from the given α-fibration and its limit is the required approximate fibration. The existence of such a sequence is guaranteed by a result called the Handle Theorem which is proved by solving some handle problems. This handle problem is solved by a torus argument that comes from wrapping up constructions, this is the stage where the engulfing comes to play. The type of engulfing that is needed in such constructions is a controlled topological version of Stallings' engulfing due to Siebenmann, Guillou, and Hähl [SGH74].

The non-manifold case

The geometric nature of Chapman's approach makes it more applicable in strategies towards obtaining versions of the α-Approximation Theorem for spaces that are more general than manifolds. M. Steinberger and J. West have used this approach to prove the Equivariant α-Approximation Theorem for the orbit spaces of locally linear group actions that satisfy some gap condition of codimensions [SW87]. In this thesis, based on earlier works of T. Chapman [Cha81] and B. Hughes [Hug04], we develop stratified tools that we hope may be helpful in establishing the α-Approximation Theorem for stratified spaces. In doing so, we need to use the stratified adaption of various notions of maps used in Chapman's approach; we will work with classes of maps such as stratum-preserving homeomorphisms, stratum-preserving α-equivalences, stratified α-fibrations, etc. We are recalling the definitions of such classes in Chapter 3.

We will mainly work in a class of spaces that is suitable for topologically stratified situations, namely the class of Quinn's Homotopically Stratified Spaces. F. Quinn introduced his class of stratified spaces in [Qui88] to provide "a setting for the study of purely topological stratified phenomena, particularly group actions on manifolds." The conditions on how the strata should fit together used to define such stratified spaces are homotopic rather than geometric conditions. Hence, this class includes the classes of geometrically stratified spaces of Whitney, Thom-Mather, and Browder-Quinn, as well as the class of topologically stratified spaces of Siebenmann. In Chapter 3, we are also recalling the definitions of those various classes of stratified spaces and then stating some of the important results that will be of importance to our work. Namely, the Stratified Isotopy Extension Theorem of Quinn and the Approximate Tubular Neighborhood Theorem of Hughes. For terminology, a homotopically stratified space with manifold strata will be called a manifold homotopically stratified space or an MHSS.

In light of the above manifold results, a natural approach towards a possible stratified version of the α-Approximation Theorem and the Sucking Principle is to provide the required stratified engulfing techniques. In [Hug04], B. Hughes indicated an idea to extend Chapman's engulfings to the stratified settings. The idea is as follows, Chapman's engulfing steps produce a series of self-homeomorphisms that are then realized by small ambient isotopies of the source space. In the stratified setting, we can regard these ambient isotopies as isotopies on the strata of the source space, and then using Quinn's Stratified Isotopy Extension Theorem, we can construct stratum-preserving small ambient isotopies of the whole space. This idea will be realized in Chapter 3 and, among other things, yields a detailed proof of the following result suggested by B. Hughes:

Stratified Radial Engulfing Theorem. *Let B be a compact polyhedron and X be an MHSS such that the bottom stratum has dim ≥ 5. For every $\epsilon > 0$ there exists a $\delta > 0$, such that if $f : X \to B \times \mathbb{R}$ is a stratified δ-fibration over $B \times [-4, 4]$, then there is a stratum-preserving homeomorphism $h : X \to X$ such that*

(1) $f^{-1}(B \times (-\infty, 1]) \subset hf^{-1}(B \times (-\infty, 0))$

(2) *h may be chosen so that there is a stratum-preserving $(p_B f)^{-1}(\epsilon)$-isotopy $h_t : id_X \simeq h$ which is supported on $f^{-1}(B \times [-3, 3])$.*

Next, the stratified radial engulfing will be used to do wrapping up constructions for stratified α-fibrations. In the following statements, B continues to be a compact polyhedron.

Stratified Wrapping Up Theorem I. *Let X be an MHSS without boundary such that the bottom stratum has $dim \geq 5$. For every $\epsilon > 0$ there exists a $\delta > 0$, such that if $f : X \to B \times \mathbb{R}$ is a stratified δ-fibration over $B \times [-4, 4]$, then there exists an MHSS without boundary $\tilde{X}$, a stratified ϵ-fibration $\tilde{f} : \tilde{X} \to B \times S^1$, and a stratum-preserving open embedding*

$\phi : f^{-1}(B \times (-1,1)) \to \tilde{X}$ *such that the following diagram commutes:*

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\tilde{f}"] & & B \times S^1 \\ f^{-1}(B \times (-1,1)) \arrow[u, hook, "\phi"] \arrow[rr, "f|"'] & & B \times (-1,1). \arrow[u, "id \times e"'] \end{tikzcd}$$

The Stratified Wrapping Up Theorem will then be used to solve stratified handle problems in the form of the following theorem. Note that $c_t(B)$ and $\mathring{c}_t(B)$ denote the closed t-subcone and the open t-subcone of the cone on some compact polyhedron B, respectively.

Stratified Handle Theorem. *Let X be an MHSS without boundary such that the bottom stratum has $dim \geq 5$. Given $\epsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ if $f : X \to \mathring{c}(B) \times \mathbb{R}^n$ is a stratified δ-fibration over $\mathring{c}_3(B) \times D^n_3$ and a stratified ν-fibration over $(c_3(B) - \mathring{c}_{1/3}(B)) \times D^n_3$, then there exists a stratified μ-fibration $\tilde{f} : X \to \mathring{c}(B) \times \mathbb{R}^n$ over $c_1(B) \times D^n_1$ which is ϵ-homotopic to f $rel(X - f^{-1}(\mathring{c}_{2/3}(B) \times \mathring{D}^n_3))$.*

The Stratified Handle Theorem is used to solve the handle problems that arise in proving the Stratified γ-Sucking Theorem of Chapter 4:

Stratified γ-Sucking Theorem. *Let X be an MHSS without boundary such that the bottom stratum has $dim \geq 5$ and let B be a compact polyhedron. For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $f : X \to B$ is a stratified δ-fibration then for every $\gamma > 0$ there exists a stratified γ-fibration $f' : X \to B$ which is ϵ-close to f.*

This theorem has several applications, for example, we use it to prove the following:

Stratified Sucking Theorem. *Let X be an MHSS without boundary such that the bottom stratum has dim ≥ 5 and let B be a compact polyhedron. Then for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $f : X \to B$ is a proper stratified δ-fibration, then f is ϵ-near to a stratified approximate fibration.*

The proof is by using the Stratified γ-Sucking Theorem to construct sequences $\{\delta_i\}_{i=1,2,\ldots}$ and a sequence of stratified δ_i-fibrations starting from the given stratified δ-fibration so that the limit is the required stratified approximate fibration. Based on arguments of Chapman, B. Hughes sketched a proof of a similar theorem but with manifold target space [Hug04]. The next step in this program would be to develop a version of the Stratified Sucking Theorem in which the target is allowed to be stratified. A possible idea to do this is to use Hughes' Approximate Tubular Neighborhood Theorem.

Chapter 2

Geometric Topology of Manifolds

The purpose of this chapter is to describe key tools in geometric topology of manifold. We discuss the Torus Trick, the Wrapping Up Construction, and the Handle Problem. We also present some related results on topological manifolds such as the Generalized Schoenflies Theorem, Local Contractibility Theorem of the Homeomorphisms Group of Manifolds, and the Deformation Theorem of Embeddings.

2.1 Kirby's Torus Trick

At the 1963 Conference on Differential and Algebraic Topology in Seattle, Washington, John Milnor put forward the following list of the seven most important problems in geometric topology:

(1) (The Double Suspension Problem) Let M^3 be a homology sphere with non-trivial fundamental group, is the double suspension of M^3 homeomorphic to S^5?

(2) (Topological Invariance of Whitehead Torsion) Is simple homotopy type a topological invariant?

(3) (Topological Invariance of Rational Pontrjagin classes) Can rational Pontryagin classes be defined as topological invariants?

(4) (The Hauptvermutung) If two PL-manifolds are homeomorphic, does it follow that they are PL-homeomorphic?

(5) (The Triangulation Problem) Can topological manifolds be triangulated?

(6) (Low Dimensional Poincaré Conjecture) Is the Poincaré conjecture true in dimensions 3 and 4?

(7) (The Annulus Conjecture) Is the region bounded by two locally flat n-sphere in R^{n+1} necessarily homoeomorphic to $S^n \times [0,1]$

Much progress has been made on these problems, in fact, only the 4-dimensional PL/DIFF Poincaré conjecture remains open. Problems $1, 2, 3$, and 7 were solved affirmatively by Edwards-Cannon [Edw77] [Can79], Chapman [Cha74], Novikov [Nov66], and Kirby [Kir69], respectively. Whereas problems 4 and 5 were solved negatively by Kirby-Siebenmann [KS69]. M. Freedman solved the 4-dimensional TOP Poincaré conjecture [Fre82] and, relatively recently, G. Perelman solved the 3-dimensional Poincaré conjecture for all categories. For an excellent geometric topology reference towards the solutions of the above problems, see [Fer77].

This section is devoted to the Torus Trick which was first introduced in 1969 by R. Kirby in his solution of the Annulus Conjecture. This trick is so powerful in studying the geometric topology of topological manifolds. In fact, the trick has then inspired in solving other Milnor problems, namely problems 2, 3, 4, and 5. It also motivated the proof of several important theorems in geometric topology including the Deformation Theorem of Embeddings and the α-Approximation Theorem. We will describe the torus trick by considering the simplest case in which the trick is used. We will describe the sketch of the proof of the local contractibility theorem of the homeomorphisms group of $\mathbb{R}^n$. This proof came alongside the solution of the Annulus Conjecture in [Kir69].

In this context, a space X is locally contractible if each of its points has a contractible basis neighborhood. Let $\mathcal{H}(\mathbb{R}^n)$ denotes the space of self-homeomorphisms between $\mathbb{R}^n$ with the compact-open topology.

Theorem 2.1 (Kirby)**.** $\mathcal{H}(\mathbb{R}^n)$ *is locally contractible.*

It is a well-known fact that $\mathcal{H}(\mathbb{R}^n)$ is a topological group. Hence, to prove the local contractibility theorem, it is sufficient to only consider the basis neighborhood of the identity map. For any $\epsilon > 0$ and compact subset K in $\mathbb{R}^n$, such a basis neighborhood is given by

$$N_{K,\epsilon}(id) := \{h : \mathbb{R}^n \to \mathbb{R}^n \mid d(x, h(x)) < \epsilon, \text{ for } x \in K\}.$$

The idea of the proof is to show that any $h \in N_{K,\epsilon}(id)$ is isotopic to the identity by using the fact that a homeomorphism between torus can be lifted to a periodic homeomorphism via universal covering. Such a periodic homeomorphism is bounded and hence by Alexander's trick is isotopic to the identity.

To be more precise, we are given a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $d(x, h(x)) < \epsilon$ for $\epsilon > 0$ and $x \in K$. We construct the following commutative diagram of spaces and maps in which the given homeomorphism appears at the bottom of the diagram.

1. For $r \geq 0$, let rD^n denotes the n-dimensional disk of radius r. The map $\alpha : T^n - D^n \to \mathbb{R}^n$ is an immersion such that $\alpha(T^n - D^n)$ is contained in K. The existence of such

immersions is guaranteed by the immersion theorem of Smale-Hirsch. Steve Ferry also has explicitly constructed such immersion.

2. Let D^n and $2D^n$ be concentric so that $\bar{h} : T^n - 2D^n \to T^n - D^n$ is an inclusion. Since both the source and the target spaces are contained in K, we can choose $\epsilon > 0$ small enough so that we have that $\bar{h}$ is a homeomorphism. Hence we have that the bottom rectangle in the diagram is commutative.

3. The vertical maps in the middle rectangle are inclusions. In the source space, we just coning off the boundary of $T^n - 2D^n$. Whereas on the target space we need to use an adaption of the generalized Schoenflies theorem. This theorem says that the image of any locally flat embedding of an n-sphere into an $(n+1)$-sphere bounds two $(n+1)$-disks. In fact, $\bar{h}(\partial 2D^n)$ is a locally flat submanifold of $T^n - D^n$, hence it bounds an $(n+1)$-disk and we can just cone off the boundary as before. Finally, we radially extend $\bar{h}$ along these disks to get a self-homeomorphism of torus $\hat{h}$.

$$
\begin{tikzcd}
 & \mathbb{R}^n \arrow[r, "\tilde{h}"] \arrow[d, "e^n"] & \mathbb{R}^n \arrow[d, "e^n"'] & \\
 & T^n \arrow[r, "\hat{h}"] & T^n & \\
B^n \arrow[ur, hook] \arrow[uur, hook] \arrow[dr, hook] \arrow[ddr, hook, "i"'] & & & B^n \arrow[ul, hook'] \arrow[uul, hook'] \arrow[dl, hook] \arrow[ddl, hook] \\
 & T^n - 2D^n \arrow[r, "\bar{h}"'] \arrow[uu, hook] \arrow[d, hook, "\alpha|"] & T^n - D^n \arrow[uu, hook] \arrow[d, hook, "\alpha"'] & \\
 & \mathbb{R}^n \arrow[r, "h"'] & \mathbb{R}^n. &
\end{tikzcd}
$$

4. Now we use the universal covering maps e^n to lift $\bar{h}$ and get a periodic homeomorphism $\tilde{h}$. This periodic homeomorphism is then bounded and we can use the result of Connell that a bounded homeomorphism is isotopic to the identity.

5. We can assume that D^n is in the top handle of T^n, so we just modify this handle. Let B^n be the 0-handle. Hence all the triangles in the diagram are commutative and we have that $h = \tilde{h}$ on B^n. Therefore, h is isotopic to the identity on B^n. By composing with Alexander's isotopy, we conclude that h is isotopic to the identity on the whole $\mathbb{R}^n$.

Note that the above construction can be made more canonical in the sense that if h varies continuously, then all the lifts will also vary continuously. Therefore, we can conclude that any neighborhood basis of the identity is contractible and hence $\mathcal{H}(\mathbb{R}^n)$ is locally contractible. To do this, we need the following canonical version of the generalized Schoenflies theorem.

The Canonical Generalized Schoenflies Theorem. *There exists an $\epsilon > 0$ such that if $f : S^{n-1} \times [-1,1] \to \mathbb{R}^n$ is an embedding within ϵ of the identity map, then $f|_{S^{n-1}\times\{0\}}$ extends canonically to an embedding $\bar{f} : D^n \to \mathbb{R}^n$. The embedding $\bar{f}$ is canonical in the sense that $\bar{f}$ depends continuously on f and if $f = id$, then $\bar{f} = id$.*

Moreover, we can regard $\mathbb{R}^n$ in the above construction as the 0-handle and we can cross all spaces in the above diagram with D^k for some integer $k > 0$ to get a similar construction for the k-handle. Hence, by the fact that compact manifolds admit handlebody decompositions, R. Edwards and R. Kirby proved the local contractibility of the homeomorphisms group of compact manifolds. This result is a consequence of their Deformation Theorem of Embeddings [EK71].

Deformation Theorem of Imbeddings (Edwards-Kirby)**.** *Let M^n be a topological manifold. If C is a compact subset of M which is contained in an open subset U, then for every $\epsilon > 0$ there is a $\delta > 0$ so that if $h : U \to M$ is an open embedding with $d(h(x), x) < \delta$, then there is a homeomorphism $\bar{h} : M \to M$ so that $\bar{h}|_C = h|_C$, $\bar{h}|_{(M-U)} = id$, and $d(\bar{h}(x), x) < \epsilon$.*

There is a variation of the Torus Trick in which, to produce a homeomorphism between torus, one replaces the use of punctured torus immersion with a construction called wrapping up, see Section 8 of [EK71].

Wrapping Up Lemma. *Let $h : 4B^n \to \mathbb{R}^n$ be a proper embedding that is sufficiently close to the identity map. Then, there exists a homeomorphism $\hat{h} : T^n \to T^n$ such that*

(1) $\hat{h}|_{2B^n} = h|_{2B^n}$

(2) *$\hat{h}$ depends continuously on h and if $h = id$, then $\hat{h} = id$.*

The idea of the proof is to regard $(-6, 6) \times T^{n-1}$ as an open subset of $\mathbb{R}^n$ and since h is sufficiently small it restricts to a homeomorphism $h| : (-6, 6) \times T^{n-1} \to (-6, 6) \times T^{n-1}$. Then, by using the universal covering $e : \mathbb{R} \to S^1$ we lift $h|$ to get homeomorphism $\hat{h} : T^n \to T^n$

$$\begin{tikzcd}
T^n \arrow[r, "\hat{h}"] & T^n \\
(-6,6) \times T^{n-1} \arrow[u, "e\times id_{T^{n-1}}"] \arrow[r, "h|"'] & (-6,6) \times T^{n-1}. \arrow[u, "e\times id_{T^{n-1}}"']
\end{tikzcd}$$

Note that the construction can be done carefully so that we get the conclusion of the lemma, that is by choosing so that $(-6, 6) \times T^{n-1} \subset 4B^n$ and $e \times id_{T^{n-1}}|_{B^n} = id_{B^n}$.

This variation is more applicable to consider the non-manifold case. For instance in [Sie71], Siebenmann used this to prove the local contractibility theorem of the homeomorphisms group of CS spaces. In this thesis, we will use a stratified wrapping up construction to solve our stratified handle problems.

2.2 Chapman-Ferry Handle Problems

In geometric topology, many theorems about manifolds can be solved by firstly solving handle problems. Suppose we are given a homeomorphism $f : M \to N$ between manifolds in which N admits a handle decomposition and we want to isotop f to a special map. By special map, we mean categories of maps such as CAT embeddings (CAT=TOP, DIFF or PL), cell-like maps, small-homotopy equivalences, or approximate fibrations. It suffices to analyse the situation in which the homeomorphism is to the handles of N and then to get series of handle problems that usually can be solved by using the torus trick.

To be more precise, we can regard $B^k \times \mathbb{R}^n$ as a model of an open k-handle with core $B^k \times 0$. A handle problem means a topological embedding $h : M^{n+k} \to B^k \times \mathbb{R}^n$ from a manifold M^{n+k} which is a special map over a neighborhood of $\partial B^k \times \mathbb{R}^n$. The problem h is said to be solved if there exists an isotopy $h_t : M^{n+k} \to B^k \times \mathbb{R}^n$ for $t \in [0, 1]$, such that:

(1) $h_0 = h$

(2) h_1 is a special map near $B^k \times B^n$

(3) $h_t = h$ over the complement of a compact set and over a neighborhood of $\partial B^k \times \mathbb{R}^n$ for all $t \in [0, 1]$.

In general, the torus trick takes a piece of the given embedding and extends it in such a way that it has nice properties near infinity. For example, in the proof of the local contractibility of $\mathcal{H}(\mathbb{R}^n)$ above, the torus trick is used to solve a 0-handle problem of TOP embeddings.

This section is devoted to describing the handle problems that occur in the Chapman-Ferry's proof of the α-Approximation Theorem for manifolds. First we recall the notions of small-homotopy and small-homotopy equivalence with respect to an open cover of the target space. Let α be an open cover of a space Y, a homotopy $h : X \times I \to Y$ from a space X is said to be an α-homotopy if for each $x \in X$ there is $U \in \alpha$ such that $F(\{x\} \times I) \subset U$. A map $f : X \to Y$ is said to be α-homotopic to another map $g : X \to Y$ if there is an α-homotopy connecting both maps. Let $A \subset Y$, a map $f : X \to Y$ is an α-homotopy equivalence (or α-equivalence) over A provided that there is a map $g : A \to X$, called an α-inverse of f over A, such that the composition fg is α-homotopic to id_Y and the composition $gf|_{f^{-1}(A)}$ is $f^{-1}(\alpha)$-homotopic to id_X. Note that if Y has a specified metric, for an $\epsilon > 0$ we can use an open cover of Y

which contains open balls of radius ϵ. Hence, by ϵ-homotopy and ϵ-equivalence we mean a small-homotopy and small-equivalence with respect to this open cover.

Theorem 2.2 (The Manifold α-Approximation Theorem). *Let N^n be a closed topological manifold with a fixed topological metric. For every $\epsilon > 0$ there exists a $\delta > 0$ such that if M^n is a manifold and $f : M \to N$ is a δ-equivalence, then f is ϵ-homotopic to a homeomorphism.*

Chapman-Ferry proved the theorem by using a handle decomposition of N to get some handle problems that then be solved by a torus argument. The handle problem is solved by proving the following Handle Lemma. The Handle Lemma is then used to prove the following Handle Theorem. Note that this proof is modelled on Siebenmann's proof of the CE Approximation Theorem in [Sie71].

For notation of the following Manifold Handle Lemma and Manifold Handle Theorem, let V^n be a manifold with a fixed topological metric in which $n = k + m \neq 4$. Let $f : V^n \to B^k \times \mathbb{R}^n$ be a proper map such that $f^{-1}(\partial B^k \times \mathbb{R}^m) = \partial V^n$ and f is a homeomorphism over a neighborhood of $\partial B^k \times \mathbb{R}^n$ such as $(B^k - \frac{1}{2}\mathring{B}^k) \times \mathbb{R}^n$.

Manifold Handle Lemma. *For every $\epsilon > 0$ there exists a $\delta > 0$ so that if f is a δ-equivalence over a neighborhood of 0 say $B^k \times 3B^m$, for $m \geq 1$, then:*

(1) *there exists an ϵ-equivalence $F : B^k \times \mathbb{R}^m \to B^k \times \mathbb{R}^m$ such that $F = id$ over a neighborhood of ∞ for instance $[(B^k - \frac{5}{6}\mathring{B}^k) \times \mathbb{R}^m] \cup [B^k \times (\mathbb{R}^m - 4\mathring{B}^m)]$, and*

(2) *there exist a homeomorphism $\varphi : f^{-1}(U) \to F^{-1}(U)$ such that $F \circ \varphi = f|_{f^{-1}(U)}$ where $U = (B^k - \frac{5}{6}\mathring{B}^k) \times \mathbb{R}^m \cup B^k \times 2B^m$.*

Manifold Handle Theorem. *For every $\epsilon > 0$ there exists a $\delta > 0$ so that if f is a δ-equivalence over a neighborhood of 0 such as $B^k \times 3B^m$, then there exists a proper map $\tilde{f} : V \to B^k \times \mathbb{R}$ which satisfies:*

(1) *$\tilde{f}$ is an ϵ-equivalence over a smaller neighborhood of 0 such as $B^k \times 2.5B^m$*

(2) *$\tilde{f} = f$ over a neighborhood of ∞ namely $[(B^k - \frac{5}{6}\mathring{B}^k) \times \mathbb{R}^n \cup B^k \times (\mathbb{R}^m - 2\mathring{B}^m)]$*

(3) *$\tilde{f}$ is a homeomorphism over another smaller neighborhood of 0 say $B^k \times B^m$.*

The proof of the Handle Lemma is by constructing a commutative diagram of spaces and maps consisting of the following:

1. We are given a δ-equivalence f over $B^k \times 3B^m$. As above, α is an immersion of punctured torus. The bottom rectangle comes from a pullback construction. Since $id_B \times \alpha$ is an

immersion, α' is also an immersion and hence W_0 is an n-manifold. For sufficiently small δ, the lift f_0 is a δ_0-equivalence where the size of δ_0 depends on the size of δ.

2. Since f is a homeomorphism near the boundary, we can add a copy of $(B^k - \frac{2}{3}B^k) \times \{x_0\}$ to get a manifold W_1 containing W_0 and a proper map f_1 which is an extension of f_0. Note that f_1 is a δ_1-equivalence where the size of δ_1 depends on the size of δ_0.

$$
\begin{tikzcd}
B^k \times \mathbb{R}^m \arrow[rr, "F"] & & B^k \times \mathbb{R}^m \\
B^k \times \mathbb{R}^m \arrow[u, "j"] \arrow[rr, "F'"] \arrow[d, "id_B \times e^m"'] & & B^k \times \mathbb{R}^m \arrow[u, "j"'] \arrow[d, "id_B \times e^m"] \\
B^k \times T^m & W_3 \arrow[l, "h"'] \arrow[r, "f_3"] & B^k \times T^m \\
 & W_2 \arrow[u, hook] \arrow[r, "f_2"] \arrow[d, hook] & (B^k \times T^m) - \mathring{D}^n \arrow[u, hook] \arrow[d, hook] \\
 & W_1 \arrow[r, "f_1"'] & (B^k \times T^m) - (\frac{2}{3}B^k \times \{x_0\}) \\
 & W_0 \arrow[u, hook] \arrow[r, "f_0"'] \arrow[d, "\alpha'"'] & B^k \times (T^m - \{x_0\}) \arrow[u, hook] \arrow[d, "id_B \times \alpha"] \\
 & V \arrow[r, "f"'] & B^k \times \mathbb{R}^m.
\end{tikzcd}
$$

3. The end of $(B^k \times T^m) - (\frac{2}{3}B^k \times \{x_0\})$ can be parametrized as $S^{n-1} \times \mathbb{R}$. Then, if δ_1 is sufficiently small, f_1 restricts to proper map $f_1| : f^{-1}(S^{n-1} \times \mathbb{R}) \to S^{n-1} \times \mathbb{R}$ which is a δ_1-equivalence over $S^{n-1} \times [-2, 2]$. By the Splitting Theorem, there exists a bicollared $(n-1)$-sphere $S \subset f_1^{-1}(S^{n-1} \times [-1, 1])$ such that $f_1|_S : S \to S^{n-1} \times \mathbb{R}$ is a homotopy equivalence. Choose D^n to be a disk in $B^k \times T^m$ containing $\frac{2}{3}B^k \times \{x_0\}$ and let W_2 to be the component of W_1 that does not contain D^n. Define $f_2 = f_1|_{W_2}$. Note that at this stage we have lost some control.

4. By attaching to W_2 a cone over S we can get W_3. Note that f_2 extends to a proper map f_3'. We regain the lost control by stretching out a collar on a disk $2D^n$ which contains D^n and squeezing D^n to be small. The result is a δ_2-equivalence f_3.

5. We choose h to be a homeomorphism which agrees with f_3 over $(B^k - \frac{5}{6}\mathring{B}^k) \times T^m$ and which is homotopic to f_3. For $n \geq 5$, the existence of h is a consequence of topological surgery, namely from the result that any homotopy $B^k \times T^m$ is a real $B^k \times T^m$ rel ∂

[Sie71]. For $n = 4$, the existence of h is a consequence of Waldhausen's Irreducibility Theorem [Jak88].

6. The map e^m is the universal covering. $F' : B^k \times \mathbb{R}^m \to B^k \times \mathbb{R}^m$ is the lift of $f_3 \circ h^{-1}$ which is the identity on $(B^k - \frac{5}{6}\mathring{B}^k) \times T^m$ and homotopic to the identity. Hence, from the elementary covering-space theory, F' is bounded. If δ_3 is sufficiently small, F' is an ϵ-equivalence in which the size of ϵ depends on the size of δ_3.

7. We define $J : \mathbb{R}^n \to 4\mathring{B}^k \times 4\mathring{B}^m$ to be the radial homeomorphism which is fixed on $2B^k \times 2B^m$. Then $j : B^k \times \mathbb{R}^m \to B^k \times \mathbb{R}^m$ is defined by restricting J. It is clear that j is an open embedding.

8. Next, $F : B^k \times \mathbb{R}^m \to B^k \times \mathbb{R}^m$ is defined to be the conjugation of F' by j on $B^k \times \mathbb{R}^m$ and to be the identity elsewhere.

9. Finally, from the above diagram we can construct the following diagram:

$$\begin{tikzcd}
F^{-1}(B^k \times 2B^m) \arrow[r, "F"] \arrow[d, "h(id\times e^m)j^{-1}"'] & B^k \times 2B^m \arrow[d, "id\times e^m"] \\
f_0^{-1}(id \times e^m)(B^k \times 2B^m) \arrow[r, "f_0|"] \arrow[d, "\alpha'"'] & (id \times e^m)(B^k \times 2B^m) \arrow[d, "id\times\alpha"] \\
f^{-1}(B^k \times 2B^m) \arrow[r, "f|"'] & B^k \times 2B^m.
\end{tikzcd}$$

Note that all vertical arrows are homeomorphisms. Thus, we can get a homeomorphism $\psi : f^{-1}(B^k \times 2B^m) \to F^{-1}(B^k \times 2B^m)$ by composing the inverse of the vertical maps on the left. We obviously have that $F\psi = f|_{f^{-1}(B^k\times 2B^m)}$ and $\psi = f$ on $(B^k - \frac{5}{6}\mathring{B}^k) \times 2B^m$. Hence, ψ extends to the desired homeomorphism φ.

Now, we turn to the proof of the Handle theorem. The idea is to switch the roles of 0 and ∞ by using a method called Siebenmann's Inversion Trick [Sie71]. The case when $m = 0$ is easy and it follows from the TOP generalized Poincaré Conjecture and coning. For $m > 0$, the inversion trick is roughly as follows, by applying the Handle Lemma we obtain a homeomorphism $F : B^k \times \mathbb{R}^m \to B^k \times \mathbb{R}^m$ which is equal to the identity on the neighborhood of ∞. We identify $R^m \cup \{\infty\}$ with S^m. Then compactify the homeomorphism by adding the point ∞ and extending by the identity to obtain a homeomorphism between $B^k \times S^m$. Consider the restriction $F| : B^k \times \mathbb{R}^m - F^{-1}(B^k \times \{0\}) \to B^k \times (\mathbb{R}^m - \{0\})$, this map will extend to a map $F_1 : V_1 \to B^k \times (S^m - \{0\})$ in which V_1 is a manifold. The compactification can be carried out so that F_1 is a δ_1-equivalence over a compactum in $B^k \times (S^m - \{0\})$ and a homeomorphism over $(B^k - \frac{5}{6}\mathring{B}^k) \times (S^m - \{0\})$. Note that in this stage we have switched 0 and ∞. Then, we apply the Handle Lemma to F_1 and obtain the desired $\tilde{f}$. Note that in this stage we also need to use the Splitting Theorem and the Generalized Schoenflies Theorem. The Handle Theorem is then used to prove the following.

Lemma 2.1. *Let $f : M^n \to N^n$ be a proper map in which $\partial M = \emptyset$ and N is an open subset of $\mathbb{R}^n$. Choose a compactum C in N which is contained in another compactum $\tilde{C}$. Then, for every $\epsilon > 0$ there exists a $\delta > 0$ so that if f is a δ-equivalence over $\tilde{C}$, then there exists a proper map $g : M^n \to N^n$ which is ϵ-close to f and which is a homeomorphism over C.*

With Lemma 2.1, we are ready to prove Theorem 2.2. Write N as an infinite union of openly embeddable subset N_i of $\mathbb{R}^n$ in which $\{N_i\}$ is a star-finite cover of N. For each i, let C_i be a compactum in N_i which is contained in the interior of another compactum $\tilde{C}_i$ so that $\cup C_i = N$. Hence by Lemma 2.1, there is a proper map $g_i : f_i^{-1}(N_i) \to N_i$ approximating $f|_{f_i^{-1}(N_i)}$ which is a homeomorphism over $\tilde{C}_i$. We then glue all the embeddings $g_i|_{C_i}$ together by using the Deformation Theorem of Edwards-Kirby to obtain the desired homeomorphism.

Note that in both the Handle Lemma and Handle Theorem, we need the Splitting Theorem. In fact, the Theorem is the most essential and difficult part of Chapman-Ferry's approach.

The Splitting Theorem. *Let W^n be a manifold without boundary such that $n \neq 4$ and $f : W \to S^{n-1} \times \mathbb{R}$ be a proper map which is a $p_R^{-1}(\epsilon)$-equivalence over $S^{n-1} \times [-2, 2]$ where $p_R : S^{n-1} \times \mathbb{R} \to \mathbb{R}$ is the projection map. If ϵ is sufficiently small, then there exists an $(n-1)$-sphere $S \subset (p_R f)^{-1}(-1, 1)$ such that $f| : S \to S^{n-1} \times \mathbb{R}$ is a homotopy equivalence, S is bicollared, and S separates the component of W^n containing $f^{-1}(S^{n-1} \times [-1, 1])$ into two components, one containing $f^{-1}(S^{n-1} \times \{-1\})$ and one containing $f^{-1}(S^{n-1} \times \{1\})$.*

In [CF79], Chapman-Ferry only gave the proof of the Splitting Theorem (and hence the proof of the α-Approximation Theorem) for dimensions $n \geq 5$. The proof is modeled on the Splitting Theorem for Siebenmann's Boundary Theorem which uses topological surgery theory. In [Jak83], W. Jakobsche proved the theorem for $n = 2, 3$ using PL techniques that exist in these dimensions, and then, in [Jak88], he proved the Manifold α-Approximation Theorem for these dimensions. Hence to sum up, the Manifold α-Approximation Theorem is true for all dimensions. The infinite-dimensional case was proved first by Steve Ferry himself using global mapping cylinder constructions which do not seem to have analogs in finite-dimensions. The case $n = 0$ is trivial and the case $n = 1$ is obvious. The $n = 4$ case follows from the $n = 5$ case and the 5-dimensional Quinn's Thin h-Cobordism Theorem [Qui82] [FW91].

Chapter 3

Topologically Stratified Spaces

The purpose of this chapter is twofold. Firstly, to present the definition of a class of stratified spaces called manifold homotopically stratified spaces. This notion of stratified spaces has been first introduced by Frank Quinn in [Qui88]. It includes geometrically stratified spaces of Whitney, Thom-Mather, and Browder-Quinn, as well as topologically stratified spaces of Siebenmann. Secondly, to develop stratified version of small-equivalences and approximate fibrations. We also show that some results concerning these categories of maps on manifold settings can be generalized to stratified settings.

3.1 Geometrically Stratified Spaces

The section is devoted to discussing some categories of geometrically stratified spaces. Those are stratified spaces in which there are geometric conditions on how the strata should fit together. Such conditions are also related to the existence of bundle neighborhoods of strata. The presentation of this section will closely follow the treatments in [HW00], [Ban07] and [Fri20].

We start by defining general stratified spaces or spaces with a stratification.

Definition 3.1. A *stratification* of a space X indexed by an index set $\mathcal{I}$ is a locally finite partition $\{X_i\}_{i\in\mathcal{I}}$ of locally closed subspaces of X. For $i \in \mathcal{I}$, the subset X_i is called the i*-stratum* of X, and the closed subset $X^i = \cup\{X_k \mid X_k \cap \mathrm{cl}(X_i) \neq \emptyset\}$ is called the i*-skeleton* of X. In the case when $\mathcal{I} = \mathbb{N}$, the *depth* of the stratification is defined to be $d = \sup\{i - j \mid X_i \neq \emptyset \neq X_j\}$. A space X is *a stratified space with a stratification* $\{X_i\}_{i\in\mathcal{I}}$ if it admits a stratification $\{X_i\}_{i\in\mathcal{I}}$.

However, we need to introduce a condition which eliminate some pathologies in how the strata of a stratification can fit together.

Definition 3.2. A stratification $\{X_i\}_{i\in\mathcal{I}}$ is said to satisfy *the frontier condition* if for every $i, j \in \mathcal{I}$, the condition $X_i \cap \mathrm{cl}(X_j) \neq \emptyset$ implies $X_i \subseteq \mathrm{cl}(X_j)$.

For a space X with stratification $\{X_i\}_{i\in\mathcal{I}}$, an ordering relation $\leq$ can be defined on $\mathcal{I}$ by setting $i \leq j$ if and only if $X_i \subseteq \mathrm{cl}(X_j)$. If the stratification satisfies the frontier condition, then $\leq$ is a partial ordering of $\mathcal{I}$ and for each $i \in \mathcal{I}$ we have $X^i = \mathrm{cl}(X_i)$. For the proofs of these facts, see [Hug99].

To motivate the notions of geometric stratifications, we consider a real algebraic set $V \subset \mathbb{R}^n$ which is a common locus of finitely many real polynomials. The singular set ΣV of all points where V fails to be a smooth variety is also an algebraic set. Hence, there is a finite filtration $V = V^m \supseteq V^{m-1} \supseteq \cdots \supseteq V^0 \supseteq V^{-1} = \emptyset$ where $V^{i-1} = \Sigma V^i$. We obtain a stratification of V by setting the i-stratum as $V_i = V^i - V^{i-1}$. However, the strata need not have well-behaved neighborhoods, the local topological type need not be locally constant along strata. To illustrate this situation, consider the locus of $x^2 = zy^2$ which is an algebraic set in $\mathbb{R}^3$ known as the Whitney umbrella. The singular set ΣV is the z-axis which is clearly a smooth manifold, hence we obtain just two strata namely $V - \Sigma V$ and ΣV. However, there is a drastic change in the neighborhood of ΣV in V when we pass through the origin, for negative z a small neighborhood of ΣV only meets ΣV and this is not the case for positive z. This motivated Whitney to introduce his conditions A and B.

Definition 3.3 (Whitney Stratified Spaces). Let M be a smooth manifold and $Z \subset M$ be a closed subset. A stratification $\{S_i\}_{i\in\mathcal{I}}$ of Z which satisfies the frontier condition is called *a Whitney stratification* provided that it fulfills the following.

(1) Each stratum S_i is a submanifold of M.

(2) For $i < j$, the strata S_i and S_j satisfy the following Whitney's conditions. Suppose:

- $(x_i) \subset S_j$ is a sequence of points converging to a point $y \in S_i$ such that the tangent spaces $T_{x_i}S_j$ converge to a limiting tangent space τ.
- $(y_i) \subset S_i$ is a sequence of points which is also converging to $y \in S_i$ such that the secant lines $l_i = \overline{x_i y_j}$ converge to a limiting line l.

Then, (Condition A) $T_y S_i \subset \tau$ and (Condition B) $l \subset \tau$.

A space with a Whitney stratification is called a *Whitney stratified space*.

Note the definition turns out to be somewhat redundant, as it was shown by J. Mather that Condition B implies Condition A. The Whitney umbrella with the above stratification is not a Whitney stratified space. However, we can choose another stratification in which the origin is regarded as a stratum to get a Whitney stratification. A similar construction also works for a class of spaces that is more general than algebraic sets called semi-algebraic sets. It is a finite union of sets that are the locus of finitely many polynomial equations and inequalities. For example real algebraic sets and polyhedra. Moreover, the class of Whitney stratified spaces also include real and complex analytic sets.

Next, we discuss other more general categories of geometric stratified spaces in which the geometric conditions are encoded by the existence of some neighborhoods of strata.

Definition 3.4 (Thom-Mather Stratified Spaces). Let k be an integer. A C^k*-Thom-Mather* stratified space is a triple $(X, \mathbf{S}, \mathbb{T})$ such that:

(1) $\mathbf{S} = \{X_i\}_{i\in\mathcal{I}}$ is a stratification of X such that each stratum X_i is a C^k-manifold.

(2) $\mathbb{T} = (T_i, \pi_i, \rho_i)$ is called a *tube system* and consists of open neighborhoods T_i of X_i which is called *tubular neighborhoods*, retractions $\pi_i : T_i \to X_i$ which is called the *local retractions of* T_i and maps $\rho_i : T_i \to [0, \infty)$ such that $X_i = \rho^{-1}(0)$.

(3) For each pair $X_i, X_j \in \mathbf{S}$, we have a C^k-submersion $(\pi_{ij}, \rho_{ij}) : T_{ij} \to X_i \times [0, \infty)$ where $T_{ij} = T_i \cap X_j$, $\pi_{ij} = \pi_i|_{T_{ij}}$ and $\rho_{ij} = \rho_i|_{T_{ij}}$.

(4) For each triple $X_i, X_j, X_k \in \mathbf{S}$ and $x \in T_{jk} \cap T_{ik} \cap \pi_{jk}^{-1}(T_{ij})$, we have the following *compatibility conditions*: $\pi_{ij} \circ \pi_{jk}(x) = \pi_{ki}(x)$ and $\rho_{ij} \circ \rho_{jk}(x) = \rho_{ki}(x)$.

J. Mather has proved that every Whitney stratified space admits a C^∞-Thom-Mather stratification. Moreover, by using the Thom Isotopy Lemmas, Mather has also proved that each stratum X_i in a Thom-Mather stratified space has a mapping cylinder neighborhood in which the projection map is a fiber bundle projection over that stratum. Furthermore, the existence of this mapping cylinder neighborhood is abstracted by W. Browder and F. Quinn.

Definition 3.5 (Browder-Quinn Stratified Spaces). Let X be a space with a stratification $\{X_i\}_{i\in\mathcal{I}}$ satisfying the frontier condition. The stratification $\{X_i\}_{i\in\mathcal{I}}$ is said to be a C^k*-Browder-Quinn stratification of* X provided that for every i, there is a closed neighborhood N_i of $\Sigma X_i = X^i - X_i$ in X^i and a map $\nu_i : \partial N_i \to \Sigma X_i$, such that:

(1) each stratum X_i is a C^k-manifold,

(2) ∂N_i is a codimension 0 submanifold of X_i,

(3) N_i is the mapping cylinder of ν_i,

(4) If $i < j$ and $W_i = X_i - \text{int}N_i$, then $\nu_j| : \nu_j^{-1}(W_i) \to W_i$ is a C^k-submersion.

Note that the definitions of both Thom-Mather space and Browder-Quinn space incorporate the topological case by taking $k = 0$.

3.2 Quinn's Manifold Homotopically Stratified Spaces

The notion of homotopically stratified spaces has been first introduced to provide a setting for the study of purely topological stratified phenomena [Qui88]. The conditions used to define homotopically stratified spaces are homotopic rather than geometric conditions. Hence, this class of stratified spaces is very general. It includes the topological cases of various classes of geometrically stratified spaces that have been discussed in the previous chapter.

To define the homotopically stratified spaces, we some notions of maps that are compatible with a given stratification.

Definition 3.6. A map between spaces with stratifications is *stratum-preserving* if it takes strata into strata. If X is a space with stratification $\{X_i\}_{i\in\mathcal{I}}$ then a map $F : Z \times A \to X$ is said to be *stratum-preserving along* A with respect to the stratification if for each $z \in Z$ there is some $i \in \mathcal{I}$ such that $F(\{z\} \times A) \subset X_i$. Particularly, for the unit interval I, a map $F : Z \times I \to X$ is a *stratum-preserving homotopy* if it is stratum-preserving along I and is a *nearly stratum-preserving homotopy* if its restriction to $Z \times [0, 1)$ is stratum-preserving along $[0, 1)$.

Definition 3.7. Let X and Y be spaces with stratifications $\{X_i\}_{i\in\mathcal{I}}$ and $\{Y_j\}_{j\in\mathcal{J}}$, respectively. Let Z be a space, a map $f : Z \to Y$ is said to be *stratum-preserving homotopic* to another map $g : Z \to Y$ if there exist a stratum-preserving homotopy $F : Z \times I \to Y$ such that $F_0 = f$ and $F_1 = g$. A map $f : X \to Y$ is said to be a *stratum-preserving homotopy equivalence* provided that there exists another map $g : Y \to X$ such that $f \circ g : Y \to Y$ is stratum-preserving homotopic to id_Y with respect to $\{Y_j\}_{j\in\mathcal{J}}$ and $g \circ f : X \to X$ is stratum-preserving homotopic to id_X with respect to $\{X_i\}_{i\in\mathcal{I}}$.

Next, we discuss a homotopy condition of subspaces that will be of importance in defining homotopically stratified spaces.

Definition 3.8. Let X be a space with a stratification $\{X_i\}_{i\in\mathcal{I}}$ and $Y \subset X$. Then Y is said to be *forward tame* in X if there exists a neighborhood U of Y in X and a homotopy $h : U \times I \to X$ such that $h_t|_Y = \text{inc.} : Y \hookrightarrow X$ for each $t \in I$, $h_0 = \text{inc.} : U \hookrightarrow X$, $h_1(U) = Y$, and $h((U \setminus Y) \times [0, 1)) \subseteq X \setminus Y$.

Intuitively, the definition says that points of $U \setminus Y$ remain in the same stratum until the last moment of the homotopy when everything gets pushed into Y.

Another notion that will be important in defining homotopically stratified spaces is the notion of the homotopy link. It is a path space that will be a homotopical model of neighborhoods of strata. The idea to use path spaces as neighborhoods originally goes back to E. Fadell in which he used it as the total spaces of topological normal bundles in order to construct a topological version of Stiefel-Whitney classes [Fad65]. It should be noted that all mapping space will be assumed to be given the compact-open topology. We denote by Y^X the space of continuous maps from X to Y.

Definition 3.9. Let X be a space with a stratification $\{X_i\}_{i\in\mathcal{I}}$ and $Y \subset X$.

(1) The *homotopy link of Y in X* is the space of paths that start in Y but leave it instantly:

$$\text{holink}(X, Y) = \{\omega \in X^I \mid \omega(t) \in Y \text{ if and only if } t = 0\}.$$

(2) The *stratified homotopy link of Y in X* is defined by

$$\text{holink}_s(X, Y) = \{\omega \in \text{holink}(X, y) \mid \omega(t) \in X_i \text{ for some } i \text{ and for all } t \in (0, 1]\}.$$

(3) Let $x_0 \in X_i \subseteq X$. The *local homotopy link at x_0* is defined by

$$\text{holink}(X, x_0) = \{\omega \in \text{holink}_s(X, X_i) \mid \omega(0) = x_0\}.$$

(4) Evaluation at 0 defines a map $q : \text{holink}(X, Y) \to Y$ called *holink evaluation map*.

The stratified homotopy link $\text{holink}_s(X, Y)$ has a natural stratification induced by the stratification of X in which the i-stratum is defined by $\text{holink}_s(X, Y)_i = \{\omega \in \text{holink}_s(X, Y) \mid \omega(1) \in X_i\}$. The local holink at x_0 inherits a natural stratification from $\text{holink}_s(X, Y)$.

Definition 3.10. A space X with a stratification $\{X_i\}_{i\in\mathcal{I}}$ satisfying the frontier condition is a *manifold homotopically stratified space (MHSS)* if the following conditions are fulfilled:

(1) *Manifold strata property.* X is a locally compact, separable metric space and each stratum X_i is a topological manifold without boundary.

(2) *Forward tameness.* For each $k > i$, X_i is forward tame in $X_i \cup X_k$.

(3) *Normal fibration.* For each $k > i$, the holink evaluation map $q : \text{holink}(X_i \cup X_k, X_i) \to X_i$ is a fibration.

(4) *Compactly dominated local holinks.* For each $x_0 \in X$ there exists a compact subset $C \subset \text{holink}(X, x_0)$ and a stratum-preserving homotopy

$$h : \text{holink}(X, x_0) \times I \to \text{holink}(X, x_0)$$

such that $h_0 = id$ and $h_1(\text{holink}(X, x_0)) \subseteq C$.

If X only satisfies the condition (2) and (3), then X is called a homotopically stratified space.

One of the important classes of examples of MHSS is the class of Siebenmann CS sets that was originally considered as a suitable setting for topologically stratified phenomena. L. Siebenmann introduced this class of stratified space in his work proving the local contractibility of the homeomorphisms group of compact polyhedra. To define Siebenmann's stratified spaces we need the notion of a cone over a space. Let B be a compact space, then the *cone over* B is defined to be the quotient $c(B) = B \times [0, \infty]/\sim$ in which the equivalence relation $\sim$ is generated by $(b, 0) \sim (b', 0)$ for all $b, b' \in B$. Similarly, *the open cone over* B is defined by $\mathring{c}(B) = B \times [0, \infty)/\sim$.

Definition 3.11. Let X be a space with stratification $\{X_i\}_{i\in\mathcal{I}}$. Then X is *locally cone-like* if for all $i \in \mathcal{I}$ and for each $x \in X_i$ there is a neighborhood U of x in X_i, a neighborhood N of x in X which is called a *distinguished neighborhood*, a compact space with stratification L which is called a *link*, and a homeomorphism $h : U \times \mathring{c}(L) \to N$ such that $h(U \times \mathring{c}(L^k)) = X^{i+k+1} \cap N$. A locally cone-like space is called *a CS space* if each stratum is a manifold.

Note that in defining the locally cone-like spaces as above, Siebenmann did not require the Frontier Condition, however it can be shown that the condition follows from the definition. See [Fri20] for a proof of this fact. It is immediate from the definition that a distinguished neighborhood of a CS space is of the form $D^n \times \mathring{c}(L)$. This induces a natural stratified handle decomposition in which a k-handle is defined as $D^k \times \mathring{c}(L)$, see [Sie71].

Example 3.1. Let B be a polyhedron with a specified PL triangulation. This triangulation induces a stratification in which, for each i, the open i-simplices of B can be regarded as the i-strata of this stratification. This stratification can be shown to satisfy the frontier condition. The cone-like structure is constructed as follows. For each barycenter $b \in B$, let C_b be the closed star of b in the second barycentric subdivision of B. It is clearly PL homeomorphic to $c_1(X_b) \times D^k$, where X_b is a compact polyhedron and D^k is a neighborhood of b in the corresponding simplex.

The rest of this section is devoted to stating two fundamental results in [Qui88], namely the Quinn's Isotopy Extension Theorem and the Whole Tameness Theorem.

Theorem 3.1 (Stratified Isotopy Extension). *Let X be a MHSS such that if there exist indices $j < k$ such that $j \leq i$ then $\dim(X_k) \geq 5$,*

(1) *If U is a neighborhood of the i-skeleton X^i in X and $h : X^i \times I \to X^i \times I$ is a stratum-preserving isotopy, then there exists an extension of h to a stratum-preserving isotopy $\tilde{h} : X \times I \to X \times I$ supported on U.*

(2) *If $C \subseteq V \subseteq X$ such that $C \subset X$ is closed in X, V is open and $h_t|_{V\cap X^i}$ is the inclusion for each $t \in I$, then $\tilde{h}$ may be chosen such that $\tilde{h}_t|_C$ is the inclusion for each $t \in I$.*

The defining property of MHSS says that each stratum is forward tame in its higher strata. The whole tameness says that such property can be generalized to the whole space. See also [Hug99]. To state the theorem, we need the notion of stratified fibration.

Definition 3.12. Let X and Y be spaces with stratifications $\{X\}_{i\in\mathcal{I}}$ and $\{Y\}_{j\in\mathcal{J}}$, respectively, and $p : X \to Y$ be a map. Then,

(1) a *stratified homotopy lifting problem* (SHLP) of p consists of a space Z, a map $f : Z \to X$ and a stratum-preserving homotopy $F : Z \times I \to Y$ such that $F(0, z) = pf(z)$ for each $z \in Z$, i.e. the following diagram is commutative

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, hook, "\times 0"'] & X \arrow[d, "p"] \\ Z \times I \arrow[r, "F"'] & Y. \end{tikzcd}$$

(2) p is a *stratified fibration* provided that given any SHLP as above, there exists a *stratified solution*, i.e. a stratum-preserving homotopy $\tilde{F} : Z \times I \to X$ such that $\tilde{F}(z, 0) = f(z)$ for each $z \in Z$ and $p\tilde{F} = F$

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, hook, "\times 0"'] & X \arrow[d, "p"] \\ Z \times I \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}"'] & Y. \end{tikzcd}$$

Example 3.2. If X is a space with a stratification $\{X_i\}_{i\in\mathcal{I}}$ and A is a space then we can regard $X \times A$ as a space with a stratification $\{X_i \times A\}_{i\in\mathcal{I}}$ and the projection onto the second factor $pr_2 : X \times A \to A$ is a stratified fibration. For suppose we are given an SHLP as in the following diagram:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, hook, "\times 0"'] & X \times A \arrow[r, "pr_1"] \arrow[d, "pr_2"] & X \\ Z \times I \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}"'] & A, & \end{tikzcd}$$

we can construct a stratified solution $\tilde{F} : Z \times I \to X \times A$ of pr_2 by $\tilde{F}(z, t) = (pr_1 f(z), F(z, t))$ for each $(z, t) \in Z \times I$. Note that $\tilde{F}$ is clearly stratum-preserving since $pr_1 f(z) \in X_i$ for all $z \in Z$ and some i.

Example 3.3. Let X be a space with a stratification $\{X_i\}_{i\in\mathcal{I}}$ and $P_{sp}(X)$ be the space of stratum-preserving paths in X, i.e. $P_{sp}(X) = \{\omega \in X^I \mid \omega(I) \subset X_i \text{ for some } i \in \mathcal{I}\}$. Then the evaluation map $q : P_{sp}(X) \to X$ is a stratified fibration. This fact follows from the standard proof that the evaluation map $q : X^I \to X$ is a fibration.

Theorem 3.2 (Whole Tameness). *Let X be a MHSS and $Y \subseteq X$ be a closed union of strata of X. Then*

(1) *Y is stratified forward tame in X*

(2) *the holink evaluation map $q : \mathrm{holink}_s(X, Y) \to Y$ is a stratified fibration.*

3.3 Approximate Tubular Neighborhood

There is an important difference between smooth and topological manifolds concerning the existence of mapping cylinder neighborhoods of submanifolds. By the Tubular Neighborhood Theorem, every smooth submanifold of a smooth manifold has a neighborhood which is the mapping cylinder of a smooth fiber bundle. On the other hand, by the examples of Rourke and Sanderson , neighborhoods of submanifolds in a topological manifold which are the mapping cylinders of topological fiber bundles may not exist [RS67]. However, R. D. Edwards proved that for topological manifolds of $\dim \geq 5$, locally flat submanifolds do have mapping cylinder neighborhoods in which the projection maps are approximate fibrations [Edw].

Similarly, such difference also occurs in stratified spaces. In smoothly stratified spaces, skeleta have mapping cylinder neighborhoods whose projection maps are stratified systems of fiber bundles. On the other hand, the work of B. Hughes in [Hug02a] showed that, under a certain dimensional assumption, skeleta in an MHSS do have a neighborhood in which the projection map is a stratified approximate fibration, but the topology of the neighborhood is not necessarily the usual quotient topology of mapping cylinders.

Definition 3.13. Let X and Y be spaces. Given a map $p : X \to Y \times \mathbb{R}$, *the teardrop of* p is the space denoted by $X \cup_p Y$ whose underlying set is the disjoint union $X \sqcup Y$ with the minimal topology such that

(1) $X \subset X \cup_p Y$ is an open embedding,

(2) the mapping $c : X \cup_p Y \to Y \times (-\infty, \infty]$ defined by

$$c(x) = \begin{cases} p(x) & \text{for } x \in X \\ (x, \infty) & \text{for } x \in Y \end{cases}$$

is continuous.

The teardrop is a generalization of the open mapping cylinder construction. If $g : X \to Y$ then the teardrop $(X \times \mathbb{R}) \cup_{g \times id} Y$ is the open mapping cylinder $\overset{\circ}{cyl}(g)$ with the teardrop topology. Note that if $f : X \to Y$, the teardrop topology of the mapping cylinder is the topology on $X \times (0, 1) \sqcup Y$ generated by the open subsets of $X \times (0, 1)$ and sets of the form

$U \cup (f^{-1}(U) \times (0, \epsilon))$, where U is open in Y and $\epsilon > 0$. If f is a proper map between locally compact and Hausdorff spaces X and Y, then this topology coincides with the usual quotient topology. Not all teardrops are open mapping cylinders because not all maps to $Y \times \mathbb{R}$ split as products. See [HTWW00] for more about teardrop construction and [Hug02a] for its relation with mapping cylinder construction.

Definition 3.14. Let X and Y be spaces with stratifications $\{X\}_{i\in\mathcal{I}}$ and $\{Y\}_{j\in\mathcal{J}}$, respectively, and $p : X \to Y$ be a map. Then,

(1) for an open cover α of Y, p is a *stratified α-fibration* provided that given any SHLP as in the following diagram, there exists a *stratified α-solution*, i.e. a stratum-preserving homotopy $\tilde{F} : Z \times I \to X$ such that $\tilde{F}(z, 0) = f(z)$ for each $z \in Z$ and $p\tilde{F}$ is α-close to F. The latter statement means that given any $(z, t) \in Z \times I$ there is a $U \in \alpha$ containing both $f\tilde{F}(z, t)$ and $F(z, t)$.

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, hook, "\times 0"'] & X \arrow[d, "p"] \\ Z \times I \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}"] & Y. \end{tikzcd}$$

(2) p is said to be a *stratified approximate fibration* if it is a stratified α-fibration for every open cover α of Y.

The notion of stratified approximate fibration was introduced by D. Coram and P. Duvall in [CD77] as a generalization of both Hurewicz fibrations and CE maps.

Example 3.4. It is clear from the definitions that any stratified fibration is a stratified approximate fibration. Proposition 3.2 of the next section give us an example of stratified α-fibrations.

Definition 3.15. Let X be a space with a stratification. A subset $A \subseteq X$ is said to have an *approximate tubular neighborhood in X* if there exists an open neighborhood U of A and a stratified approximate fibration $p : U \setminus A \to A \times \mathbb{R}$ such that the natural map $(U \setminus A) \cup_p A \to U$ is a homeomorphism.

Remark 3.1. The above condition is equivalent to that the natural extension $\tilde{p} : U \to A \times (-\infty, \infty]$ of p is continuous. It will be more convenient for us to replace $\mathbb{R}$ by $(0, \infty)$ and to switch the role of $\{\infty\} \in (-\infty, \infty]$ to $\{0\} \in [0, \infty)$. Hence, by this convention, $A \subset X$ has the approximate tubular neighborhood U in X if and only if there is a stratified approximate fibration $\varphi : U \to A \times [0, \infty)$ where $\varphi^{-1}(A \times \{0\}) = A$ and $\varphi| : A \to A \times \{0\}$ is the identity.

By *the singular set* of a MHSS we mean the union of all non-maximal strata.

Theorem 3.3 (Approximate Tubular Neighborhood). *Let X be an MHSS with compact singular set Σ_{sing} such that all non-minimal strata of X have dimension greater than 4. If $Y \subset \Sigma_{sing}$ is a union of strata, then Y has an approximate tubular neighorhood in X.*

Prior to [Hug02b], the work of Hughes, Taylor, Weinberger and William has proved the theorem for the two strata cases [HTWW00]. The book of Hughes and Ranicki [HR96] contains the proof of the two strata cases in which the bottom strata are points.

3.4 Stratum-Preserving Small-Homotopies and CE maps

In this section we define stratified versions of small-homotopies and small-equivalences. We prove some results that relate these classes of maps with the previously defined ones. We also prove the stratified version of some preparatory results that are needed in this thesis.

Definition 3.16. Let X and Y be spaces with stratifications $\{X_i\}_{i\in\mathcal{I}}$ and $\{Y_j\}_{j\in\mathcal{J}}$, respectively, and α be an open cover of Y. Let Z be a space, a stratum-preserving homotopy $F : Z \times I \to Y$ is a *stratum-preserving α-homotopy* if for every $z \in Z$ there exists a $U \in \alpha$ such that the track $\{z\} \times I$ is contained in U. A map $f : Z \to Y$ is said to be *stratum-preserving α-homotopic* to another map $g : Z \to Y$ if there exists a stratum-preserving α-homotopy $F : Z \times I \to Y$ such that $F_0 = f$ and $F_1 = g$. A map $f : X \to Y$ is said to be a *stratum-preserving α-equivalence* provided there exists another map $g : Y \to X$, called *stratified α-inverse*, such that $f \circ g : Y \to Y$ is stratum-preserving α-homotopic to id_Y and $g \circ f : X \to X$ is stratum-preserving $f^{-1}(\alpha)$-homotopic to id_X where $f^{-1}(\alpha) := \{f^{-1}(U) \mid U \in \alpha\}$.

Recall that a subset A of a space X is a neighborhood retract of X if there exists a neighborhood of A in X which retracts to A. A space X is called an *Absolute Neigborhood Retract* (abbreviated ANR) provided that it is a neighborhood retract of every space Y containing it as a closed subset. For excellent references on the ANR theory, see [vM89] and [Hu65]. An important result on ANR that will be used several times in this thesis is the fact that, for maps to ANR, the notion of small-nearness is equivalent to the notion of small-homotopic.

Proposition 3.1. *Let Y be an ANR. For every open cover α of Y there exists an open cover β of Y, which is a refinement of α, such that if X is a metrizable space and $f, g : X \to Y$ are maps which are β-near, then f and g are α-homotopic.*

For a detailed proof, see Chapter 4 of [Hu65].

The first result of this section is a stratum-preserving version of the Estimated Homotopy Extension Theorem of Chapman-Ferry. The proof is a direct modification of the proof of the unstratified case in [CF79].

Proposition 3.2 (Stratum-Preserving Estimated Homotopy Extension). *Let X be an ANR and A be a closed ANR subspace of X. If Y is a space with a stratification and $F : A \times I \to Y$ is a stratum-preserving α-homotopy such that $F_0 : A \to Y$ extends to $\tilde{F}_0 : X \to Y$, then there exists a stratum-preserving α-homotopy $\tilde{F} : X \times I \to Y$ extending F.*

Proof. From the fact that both X and A are ANRs, it is clear that $(X \times \{0\}) \cup (A \times I)$ is also an ANR (in $X \times I$). Thus, there exists a neighborhood N of A in X which retracts to A so that we can construct a retraction $r_1 : (X \times \{0\}) \cup (N \times I) \to (X \times \{0\}) \cup (A \times I)$. Note that if N is close to A, the retraction r_1 does not move points very far. Next, we define another retraction $r_2 : X \times I \to (X \times \{0\}) \cup (N \times I)$ by $r_1(x, t) = (x, t\varphi(x))$, where $\varphi : X \to I$ is a map which is 0 on $X - N$ and 1 on A. By using the given maps, we define $H : (X \times \{0\}) \cup (A \times I) \to Y$ by $H(x, 0) = \tilde{F}_0(x)$ and $H(x, t) = F(x, t)$. Note that since F is stratum-preserving, H is also stratum-preserving. We define the required extension as $\tilde{F} = Hr_2r_1 : X \times I \to Y$ which is obviously stratum-preserving. To check the α-smallness condition, note that by the definition of φ each track $\tilde{F}(\{x\} \times I)$ is a single point for $x \notin N$. For $x \in N$, we can choose N sufficiently close to A so that the track $\tilde{F}(\{x\} \times I)$ will be close to some track $F(\{x'\} \times I)$, where $x' \in A$. Hence, $\tilde{F}$ is an α-homotopy because it is close to an α-homotopy F. □

The following result says that a stratum-preserving α-equivalence is an example of stratified α-fibration. For example, this will be important in a program towards a stratified version of the α-Approximation Theorem via Chapman's approach.

Proposition 3.3. *Let X and Y be spaces with stratifications $\{X_i\}_{i \in \mathcal{I}}$ and $\{Y_j\}_{j \in \mathcal{J}}$, respectively, and α be an open cover of Y. If $p : X \to Y$ is a stratum-preserving α-equivalence, then p is a stratified α-fibration.*

Proof. Given an SHLP of p as shown by the following commutative diagram:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, hook, "\times 0"'] & X \arrow[d, "p"] \\ Z \times I \arrow[r, "F"'] & Y. \end{tikzcd}$$

We need to find a stratum-preserving homotopy $\tilde{F} : Z \times [0, 1] \to X$ such that $\tilde{F}(z, 0) = f(z)$ for each $z \in Z$ and $p\tilde{F}$ is α-close to F. By hypothesis, there exists a stratified α-inverse $q : Y \to X$ of p such that qp is stratum-preserving $p^{-1}(\alpha)$-homotopic to id_X. It is clear that $qpf : Z \to X$ is stratum-preserving $p^{-1}(\alpha)$-homotopic to $f : Z \to X$. Thus, there is a stratum-preserving $p^{-1}(\alpha)$-homotopy $H : Z \times I \to X$ such that $H(z, 0) = f(z)$ and $H(z, 1) = qpf$. We set $\tilde{F} = H$. The required condition when $t = 0$ is clearly satisfied. From the commutativity we have that $pf(z) = F(z, 0)$ and hence $pf(z)$ is stratum-preserving homotopic to $F(z, t)$. On the other hand, from the relation $f(z) = H(z, 0)$ and the fact that H is a stratum-preserving $p^{-1}(\alpha)$-homotopy we have that $pf(z)$ is stratum-preserving α-homotopic to $pH(z,t) = p\tilde{F}(z,t)$. Hence, we have that $p\tilde{F}(z,t)$ is stratum-preserving α-homotopic to $F(z,t)$ and finally, by Lemma 3.1, the conclusion of the proposition follows. $\square$

In order to prove the stratified version of the Radial Engulfing Lemma in Chapter 3, we will need to restrict a stratified α-fibration to a stratum. Hence, we prove the following:

Lemma 3.1. *Let X and Y be spaces with stratifications $\{X_i\}_{i\in\mathcal{I}}$ and $\{Y_j\}_{j\in\mathcal{J}}$, respectively, and α be an open cover of Y. If $p : X \to Y$ is a stratified α-fibration, then its restriction to a stratum X_i of X is also a stratified α-fibration.*

Proof. For a given SHLP of $p| : X_i \to Y$, we construct the following diagram:

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, hook, "\times 0"'] & X_i \arrow[r, hook] \arrow[d, "p|"] & X \arrow[d, "p"] \\
Z \times I \arrow[r, "F"'] \arrow[ur, dashed, "?"] \arrow[urr, dashed, "\tilde{F}"] & Y \arrow[r, "id"'] & Y.
\end{tikzcd}$$

Note that the compositions with the inclusions give us an SHLP of $p : X \to Y$. Since p is a stratified α-fibration, there is a stratified α-solution $\tilde{F} : Z \times I \to X$. The required conclusion will follow by showing that this stratified α-solution factors to X_i and setting this factor to be the required stratified α-solution of $p|$. From the definition of a stratified α-solution, for all $z \in Z$, we have that $\tilde{F}(z,0) = f(z) \in X_i$. Since $\tilde{F}$ is stratum-preserving, we have that $F(\{z\} \times I) \subset X_i$ for all $z \in Z$. Hence, $\tilde{F}$ factors to X_i. $\square$

Next, we consider the pullback construction of stratified fibrations. We will prove a result asserting that the pullback of a stratified fibration is also a stratified fibration. For instance, this will be needed in Chapter 4 when we are trying to solve some stratified handle problems for stratified α-fibrations. The setting is as follows, let $p : X \to Y$ be a fibration between unstratified spaces and $f : Y' \to Y$ be a map in which Y' is a stratified space with a stratification $\{Y'\}_{i\in\mathcal{I}}$. Let $X' = \{(y', x) \in Y' \times X \mid f(y') = p(x)\}$. We can stratify $Y' \times X$ with the product stratification and stratify X' as a subset of $Y' \times X$, i.e. by setting $X'_i = \{(y', x) \in X' \mid y' \in Y'_i\}$. We obtain maps $f' : X' \to X$ and $p' : X' \to Y'$ induced by the projections so that the following diagram is commutative:

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "p'"'] & X_i \arrow[d, "p"] \\
Y' \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Note that p' is stratum-preserving. Assuming this notation, we prove the following important facts:

Lemma 3.2 (Universal Property). *If $G : Z \times I \to X$ is a homotopy and $G' : Z \times I \to Y'$ is a stratum-preserving homotopy such that $pG = fG'$, then there exists a unique stratum-preserving-$H : Z \times I \to X'$ satisfying $f'H = G$ and $p'H = G'$.*

$$\begin{tikzcd}
Z \times I \arrow[dr, dashed, "H"] \arrow[drr, "G"] \arrow[ddr, "G'"'] & & \\
& X' \arrow[r, "f'"'] \arrow[d, "p'"] & X \arrow[d, "p"] \\
& Y' \arrow[r, "f"'] & Y
\end{tikzcd}$$

Proof. Consider the homotopy $(G', G) : Z \times I \to Y' \times X$ which is obviously stratum-preserving with respect to the product stratification on $Y' \times X$. From the condition $pG = fG'$, we have that (G', G) factors to X'. We set this factor as H. It is clear that the required commutativity relations hold. Uniqueness follow from the fact that the projections f' and p' define the image point uniquely in X', but these are fixed by the hypothesis. $\square$

Proposition 3.4. *The induced map $p' : X' \to Y'$ is a stratified fibration.*

Proof. Given an SHLP of p', we can construct a commutative diagram as follows:

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, hook, "\times 0"'] & X' \arrow[r, "f'"] \arrow[d, "p'"] & X \arrow[d, "p"] \\
Z \times I \arrow[r, "G"'] \arrow[ur, dashed, "\bar{G}"] \arrow[urr, dashed, "\tilde{G}"] & Y' \arrow[r, "f"'] & Y.
\end{tikzcd}$$

It is obvious that the compositions $f'g : Z \times 0 \to X$ and $fG : Z \times I \to Y$ clearly form an SHLP of p. Since p is a fibration, there exists a solution $\tilde{G} : Z \times I \to X$. Note that the stratum-preserving homotopy G and the homotopy $\tilde{G}$ satisfy the conditions of Lemma 3.2. Hence, there is a unique stratum-preserving homotopy $\bar{G} : Z \times I \to X'$. Note that this $\bar{G}$ satisfies all requirements to be a stratified solution of p'. $\square$

Note that in [Fri03], G. Friedman has considered a similar case in which X is stratified and both Y, Y' is unstratified. The proofs above are inspired by his proofs of the related case.

We now expand the definitions of stratum-preserving α-equivalence and stratified α-fibration. Let X and Y be spaces with stratifications, α be an open cover of Y, and $A \subset Y$. A map $f : X \to Y$ is a *stratum-preserving α-equivalence over A* if it has an α-inverse that is only defined on A, that is there exists a map $g : A \to X$ such that $fg : A \hookrightarrow Y$ is stratum-preserving α-homotopic to id_A and $gf|_{f^{-1}(A)} : f^{-1}(A) \hookrightarrow X$ is stratum-preserving $f^{-1}(A)$-homotopic to $id_{f^{-1}(A)}$. A map $p : X \to Y$ is said to be a *stratified α-fibration over A* if for any given SHLP over A, i.e. a map $f : Z \to X$ and a stratum-preserving homotopy $F : Z \times I \to A$ for which $pf(z) = F(z, 0)$ for all $z \in Z$, there exists a stratum-preserving homotopy $\tilde{F} : Z \times I \to X$ such that $\tilde{F}(z, 0) = f(z)$ and $p\tilde{F}$ is α-close to F.

The following result will be of importance throughout the thesis, it says that for a sufficiently fine open cover we can restrict small-homotopies and small-fibrations. For a space Y with an open cover of α and $A \subset Y$, recall that the star of A in α is defined as $St(A, \alpha) = \bigcup\{A \cup U \mid A \cap U \neq \emptyset \text{ and } U \in \alpha\}$.

Proposition 3.5. *Let X and Y be spaces with stratifications, $U \subset Y$ is open, $A \subset U$, and α is an open cover of Y that is chosen so that no element of $St(A, \alpha)$ meets $Y - U$. Then,*

(1) *if $f : X \to Y$ is a stratum-preserving α-equivalence, then the restriction $f| : f^{-1}(U) \to U$ is a stratum-preserving $(\alpha \cap U)$-equivalence over A.*

(2) *Similarly, if $p : X \to Y$ is a stratified α-fibration, then $p| : p^{-1}(U) \to U$ is a stratified $(\alpha \cap U)$-fibration over A.*

Remark 3.2. If $A \subset Y$ is closed, the open cover α can always be chosen fine enough so that the hypothesis of Proposition 3.5 is satisfied, see [CF79].

The following result says that we can sew together stratified α-fibrations. Its proof which uses the Stratum-Preserving Estimated Homotopy Extension Theorem 3.2, is a modification of the proof of the unstratified case in [Cha80].

Lemma 3.3. *Let B be an ANR with chosen closed subsets A_1 and A_2. Let $\tilde{A}_1$ and $\tilde{A}_2$ be closed neighborhoods of A_1 and A_2, respectively. For every open cover α of B there exists an open cover β of B such that if X is a space with a stratification and $f : X \to B$ is a proper map which is a stratified β-fibration over $\tilde{A}_1$ and $\tilde{A}_2$, then f is a stratified α-fibration over $A_1 \cup A_2$.*

Proof. Suppose we have an SHLP of f over $A_1 \cup A_1$ as in the following diagram:

$$\begin{tikzcd} Z \arrow[r, "\tilde{F}_0"] \arrow[d, hook, "\times 0"'] & X \arrow[d, "f"] \\ Z \times I \arrow[r, "F"'] & A_1 \cup A_2. \end{tikzcd}$$

Choose open sets $N_i \subset B$ so that $A_i \subset N_i$ and $\bar{N}_i \subset \tilde{A}_i$. Let α' be an open cover of B.

Assertion. *The open cover β can be choosen so that if for each $z \in Z$, $F(\{z\} \times I)$ lies in either N_1 or N_2, then there is a stratum-preserving homotopy $G : Z \times I \to X$ extending $\tilde{F}_0$ and for which fG is α'-close to F.*

Proof. Let $Z_i := \{z \in Z \mid F(\{z\} \times I) \subset N_i\}$. Consider the following SHLP which is a restriction of the previous one:

$$\begin{tikzcd} Z_1 \arrow[r, "\tilde{F}_0|"] \arrow[d, hook, "\times 0"'] & X \arrow[d, "f"] \\ Z_1 \times I \arrow[r, "F|"'] \arrow[ur, dashed, "G'"] & \tilde{A}_1. \end{tikzcd}$$

By using the hypothesis that f is a stratified β-fibration over $\tilde{A}_1$, there exists a stratum-preserving homotopy $G' : Z_1 \times I \to X$ extending $\tilde{F}_0|$ such that fG is β-close to $F|$. By the Estimated Homotopy Extension Lemma 3.2 we can adjust F slightly rel $Z \times \{0\}$ so that $F| = fG'$. Note that we also have to cut down the domain of G' a little bit.

Since Z_1 and Z_2 are open, we can find compacta $C_i \subset Z_i$ so that $Z = C_1 \cup C_2$. Let $R : Z_2 \times I \times [0,1] \to Z_2 \times I$ be a homotopy which affects only the I-coordinate and satisfies the following properties:

(1) $R_0(Z_2 \times I) \subset (Z_2 \times \{0\}) \cup ((Z_1 \cap Z_2) \times I)$

(2) $R_t = id$ on $(Z_2 \times \{0\}) \cap ((C_1 \cap Z_2) \times I)$, for all t

(3) $R_1 = id$.

We define $g : (Z_2 \times \{0\}) \cup ((Z_1 \cap Z_2) \times I) \to X$ by $g = G'$ on $(Z_1 \cap Z_2) \times I$ and $g = \tilde{F}_0$. Consider the following diagram:

$$\begin{tikzcd} Z_2 \times I \arrow[r, hook, "R_0"] \arrow[d, hook, "\times 0"] & (Z_2 \times \{0\}) \cup ((Z_1 \cap Z_2) \times I) \arrow[r] & X \arrow[d, "f"] \\ Z_2 \times I \times [0,1] \arrow[r, "R"'] \arrow[urr, dashed, "H"] & Z_2 \times I \arrow[r, "F|"'] & \tilde{A}_2. \end{tikzcd}$$

It is clear that the diagram is an SHLP of f and since f is a stratified β-fibration over $\tilde{A}_2$, there is a stratum-preserving homotopy $H : Z_2 \times I \times [0,1] \to X$ so that $H_0 = gR_0$ and fH is β-close to FR. Note that by using the $[0,1]$-coordinate and the β-closeness, the restriction of $H_1 : Z_2 \times I \to X$ to $(Z_2 \times \{0\}) \cup ((C_1 \cap Z_2) \times I)$ is stratum preseving $f^{-1}(\beta)$-homotopic to the restriction of g to $(Z_2 \times \{0\}) \cup ((C_1 \cap Z_2) \times I)$. Hence, by Proposition 3.2 there exists a stratum-preserving $f^{-1}(\beta)$-of H_1 to a stratum-preserving homotopy $G^2 : Z_2 \times I \to X$ so that

$G^2_0 = \tilde{F}|_{Z_2}$ and $G^2 = G^1$ on $C_1 \cap Z_2$. Thus, we can piece G^1 and G^2 together to get the desired stratum-preserving homotopy $G : Z \times I \to X$. □

Returning to the proof of the lemma, we choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of I so that for each $z \in Z$ and i we have that $F(\{z\} \times [t_{i-1}, t_i])$ is close to A_1 or A_2. We will construct the desired small solution inductively. For each i, we will construct a map G^i : $Z \times [0, t_i] \to X$ so that

(1) $G^i_0 = \tilde{F}_0$

(2) $fG^i|_{Z\times\{i\}}$ is α-close to $F|_{Z\times\{i\}}$

(3) fG^i is α-close to $F|_{Z\times[0,t_i]}$

and then $G^n : Z \times I \to X$ will be the desired extension of $\tilde{F}_0$. The case $i = 0$ is trivial. Let $G^i : Z \times [0, t_i] \to X$ be given with the properties above, we need to construct $G^{i+1} : Z \times [0, t_{i+1}] \to X$. We only need to consider the definition of G^{i+1} on $Z \times [t_i, t_{i+1}]$ since G^{i+1} is an extension of G^i to this region. As in the proof of the assertion, we use Proposition 3.2 to adjust $F|_{Z\times[t_i,t_{i+1}]}$ to obtain a homotopy $F' : Z \times [t_i, t_{i+1}] \to B$ so that $F'_{t_i} = fG^i_{t_i}$ and $F'_{t+1} = F_{t_{i+1}}$. The purpose of doing this is so that we still have that $F'(\{z\} \times [t_i, t_{i+1}])$ is close to A_1 or A_2 for all $z \in Z$ and therefore the assertion can be applied. By the assertion, there is a map $G^{i+1} : Z \times [t_i, t_{i+1}] \to X$ for which $G^{i+1}_{t_{i_1}} = G^i_{t_{i_1}}$ and fG^{i+1} is α'-close to F'. If α' is sufficiently fine, fG^{i+1} is α-close to $F|_{[t_i,t_{i+1}]}$. The induction is now complete. □

Finally, we come to the main result of this section. This result is very useful when we are in the situation that we have to check whether a given map is a stratified α-fibration. Such a situation occurs, for instance, in stratified wrapping up constructions in Chapter 4.

Proposition 3.6. *Let B be a locally compact ANR and γ be an open cover of B. For every open cover α of B there exists an open cover β of B so that if X is a space with a stratification and $f : X \to B$ is a stratified β-fibration over the closure of each element of γ, then f is a stratified α-fibration.*

Proof. *The case when B is compact.* We may assume that $\gamma = \{U_i\}_{i=1}^n$ is a finite cover of B. Choose a sequence of $\{\gamma_k\}_{k=1}^{n-1}$ of open covers of B in which $\gamma_k = \{U_{k,i}\}_{i=1}^n$ so that elements of it satisfy $\bar{U}_{1,i} \subset U_i$ and $\bar{U}_{k+1,i} \subset U_{k+i}$. We are given that f is a stratified β-fibration over each $\bar{U}_i$. By Lemma 3.3, f is a stratified β_1-fibration over the closure of $U_{1,1} \cup U_{1,2}$ and $\bar{U}_{1,i}$ for $3 \leq i \leq n$, where β_1 can be made as fine as we want corresponding to a fine choice of β. Again, by Proposition 3.3, f is a stratified β_2-fibration over the closure of $U_{2,1} \cup U_{2,2} \cup U_{2,3}$ and $\bar{U_{2,i}}$ for $4 \leq i \leq n$ where β_1 can also be made as fine as we want. By continuing this process, we eventually will get that f is a stratified α-fibration over the closure of $\bigcup_{i=1}^n U_{n-1,i} = B$.

The case when B is locally compact. We can write $B = \bigcup_{i=1}^{\infty} B_i$ where each B_i is compact and $B_i \cap B_j = \emptyset$ for $|i-j| \geq 2$. Let $\tilde{B}_i$ be a closed neighborhood of B_i so that $\tilde{B}_i \cap \tilde{B}_j = \emptyset$ for $|i-j| \geq 2$. From the compact case above, we have that f is a stratified α'-fibration over each $\tilde{B}_i$ where α' can be made as fine as we want corresponding to a fine choice of β. Obviously f is also a stratified α'-fibration over $B' = \bigcup\{\tilde{B}_i \mid i \text{ odd}\}$ and $B'' = \bigcup\{\tilde{B}_i \mid i \text{ even}\}$. Hence, by Proposition 3.3 f is a stratified α-fibration over $B' \cup B'' = B$. □

Note that in the case when the target space Y has a specified metric and $\epsilon > 0$ is given, we will also use ϵ to denote the open cover of Y by open balls of diameter ϵ. This convention means that we can also define the notions of stratum-preserving ϵ-homotopy, stratum-preserving ϵ-equivalence, and stratified ϵ-fibration. Note that all the above results are also true in the metric case.

The rest of this section is devoted to recalling some facts about CE maps. The term CE is the abbreviation of "cellular-equivalent". This notion is introduced by R. Lacher in [Lac77]. We adapt the definitions and results to the stratified setting.

Definition 3.17. A subset A of an ANR X is said to have *property* UV^∞ in x provided that for each neighborhood U of A in X there exists a neighborhood $V \subset U$ of A such that the inclusion $V \hookrightarrow U$ is homotopic to a constant map.

Definition 3.18. Let X and Y be stratified spaces in which the strata are ANRs. A stratum-preserving map $f : X \to Y$ is said to be *cell-like* provided that each point inverse $f^{-1}(y)$ has property UV^∞ in X. A proper onto stratum-preserving cell-like map is called a *stratum-preserving CE map*.

One of the fundamental results in the theory of CE maps is the following homotopy characterization. Note that, R. Lacher proved the finite dimensional case in [Lac77] and then W. Haver proved the general case in [Hav75].

Theorem 3.4 (Homotopy Characterization of CE Maps). *Let $f : X \to Y$ be a proper onto mapping between locally compact ANRs, then the following are equivalent:*

(1) *f is cell-like,*

(2) *for each contractible open subset $U \subset Y$, $f^{-1}(U)$ is contractible,*

(3) *f is a hereditary proper homotopy equivalence, in the sense that for every open set $V \subset Y$, the restriction $f|_{f^{-1}(V)}$ is a proper homotopy equivalence.*

The following is a useful homotopy property of point inverses of an approximate fibration that can be used to show the relation between approximate fibration and CE map.

Theorem 3.5 (Theorem 2.4 of [CD77])**.** *Let X and Y be locally compact ANRs. If $p : X \to Y$ is an approximate fibration and $y \in Y$, then given any neighborhood U of $p^{-1}(y)$ in X there is a neighborhood V of $p^{-1}(y)$ in U such that for any neighborhood W of $p^{-1}(y)$ in V there is a neighborhood W_0 of $p^{-1}(y)$ in W and a homotopy $H : X \times I \to X$ satisfying*

(1) $H_0 = id_X$,

(2) $H_t|cl(X - U) \cup W_0 = id$ *for all* $t \in I$,

(3) $H_t(V) \subset U$ *for all* t,

(4) $H_1(V) \subset W$.

Remark 3.3. The condition that the point inverses of a CE map have the property UV^∞ as given in Definition 3.18 is equivalent to the condition that the point inverses have trivial shape in the sense of Borsuk. It is also equivalent with our description in the Introduction, see [Lac77]. The third condition of Theorem 3.4 equivalently says that a CE map is a α-equivalence for each α. Moreover, it is well-known that an approximate fibration which is also a homotopy equivalence is CE. For example [Cha81], [Wei94] and [FL18]. contain observations of this fact.

Chapter 4

Stratified Engulfing Results

In this chapter, we are going to develop stratified topological engulfing results that will for example be applied to solve the handle problems of Chapter 4. Engulfing is a process by which an open subset of a manifold is adjusted via an ambient isotopy to absorb a predetermined polyhedron. In [Hug04], B. Hughes indicated that by using Quinn's Stratified Isotopy Extension Theorem, one may extend the engulfing methods to the stratified settings.

4.1 Controlled Topological Engulfing

The method of engulfing has been one of the most useful discoveries in geometric topology. Nevertheless, there are several distinct versions of engulfing, but all of them are connected by a common thread. The first versions, developed in the PL category, are the Zeeman's and the Stallings' engulfing that firstly appeared in the early Sixties. Among other things, Zeeman used his engulfing method to solve the weak high dimensional Poincaré conjecture and Stallings used his method to prove the Hauptvermutung for Euclidean spaces. In 1969, M. H. A. Newman developed a topological engulfing method. For a complete treatment of PL topology one can consult the book of J. P. Hudson [Hud69] and the book of Rushing [Rus73] for various engulfing methods and their applications.

The version of engulfing that will be used in this thesis is the controlled topological engulfing of Siebenmann, Guillou and Hähl [SGH74] [Cha81] which is modelled on the engulfing theorem of Stallings. Hence, we start by recalling the statement of this engulfing theorem. Roughly speaking, Stallings' engulfing says that an open subset of a PL-manifold can expand like an amoeba to engulf any given subpolyhedron, provided that certain dimensional and connectivity conditions are met. To be more precise, recall that a pair (M, U) is *p-connected* if $\pi_i(M, U) = 0$ for all $i \leq p$. The following is first appear in [Sta62].

Theorem 4.1 (Stallings' Engulfing). *Let M^n be a PL-manifold without boundary, U be an open subset of M^n and P be a p-dimensional polyhedron in M^n. If (M, U) is p-connected,*

$P \cap (M - U)$ is compact and $p \leq 3$, then there is a compact subset $E \subset M$ and an ambient PL-isotopy $e_t : M \to M$ which is the identity outside E and connecting id_M to a PL homeomorphism $h : M \to M$ such that $P \subset h(U)$.

The idea of the proof is to let U acts as an amoeba which sends out feelers to engulf the vertices of $P \cap (M - U)$ one at a time, all the while keeping the part of P already in U covered. After all the vertices of $P \cap (M - U)$ are covered, the engulfing is extended to the 1-simplices of $P \cap (M - U)$, one at a time. It is not so much like sending out feelers anymore, but more like sliding the new U sideways along a singular disk bounded by the 1-simplex to be engulfed and an arc in the extended U joining the ends of the simplex. This process is then extended to all simplices of $P \cap (M - U)$.

Now we discuss the controlled topological engulfing that will be used in this thesis. The controlled term means that we give controls on the open subset that is going to engulf the given polyhedron. It is a topological engulfing so we need the notion of a subset being locally polyhedral in a manifold.

Definition 4.1. Let M be a topological manifold and $P \subset M$ be a polyhedron. For $x \in M$, the polyhedron P is said to be *locally polyhedral at point x* if there exists an open neighborhood U of x in M and a triangulation of U as a PL-manifold such that $U \cap P$ is a subpolyhedron of this triangulation. The polyhedron P is said to be *locally polyhedral in M* if it is locally polyhedral at all points of M.

Theorem 4.2 (Controlled Topological Engulfing). *Let M be a connected n-manifold without boundary. Let p be an integer ≥ 0, and*

(1) *let P be a closed and locally polyhedral polyhedron in M, possibly non-compact, with $\dim P \leq n - 3$ and $P_0 \subset P$ be a subpolyhedron such that $Q = \mathrm{cl}(P - P_0)$ is compact with $\dim Q \leq p$,*

(2) *let $U_0 \subset U_1 \subset \cdots \subset U_p$ and $M_0 \subset M_1 \subset \cdots \subset M_p = M$ be sequences of non-empty open sets of M such that $P \subset M_0$, $P_0 \subset U_0$, $U_i \subset M_i$ such that for $0 \leq i \leq p-1$, all maps $(K, L) \to (M_i, U_i)$ from a finite simplicial pair of $\dim \leq p-1$ are homotopic in (M_{i+1}, U_i) to a map $(K, L) \to (U_{i+1}, U_i)$.*

Then, there exists a compactly supported isotopy of id_M to a homeomorphism $h : M \to M$ for which $P \subset h(U_p)$.

For completeness we also include the treatment of T. Chapman in [Cha81], that is to apply the theorem to the Euclidean plane to get useful engulfing results. For notation, let $u, v : [0, \infty) \to (-2, 2)$ so that $v(s) \leq u(s)$, for all $s \geq 2$. Consider the areas $\Gamma(u)$ and $\Gamma(v)$ under the graphs of u and v as subsets of $[0,\infty)\times\mathbb{R}$. That are

$$\Gamma(u) := \{(s,t) \mid -\infty < t \le u(s)\}$$
$$\Gamma(v) := \{(s,t) \mid -\infty < t \le v(s)\}.$$

Lemma 4.1. *For an m-manifold without boundary M there exists an $\epsilon > 0$ such that if*

(1) $f : M \to [0,\infty)\times\mathbb{R}$ *is an ϵ-fibration over* $[0,4]\times[-4,4]$*,*

(2) $P \subset M$ *is a closed polyhedron which is locally polyhedral in M and* $\dim(P) \le m-3$.

Then there exists an isotopy, which is supported on $f^{-1}([0,3]\times[-3,3])$, of id_M to a homeomorphism $h : M \to M$ satisfying $P \cap f^{-1}(\Gamma(v)) \subset hf^{-1}(\Gamma(u))$.

Proof. Without loss of generality we can assume that $P \subset f^{-1}(\Gamma(v))$. Choose a closed subpolyhedron $P_0 \subset P$ such that $P_0 \subset f^{-1}([2,\infty)\times\mathbb{R})$ and $P \cap f^{-1}([2.5,\infty)\times\mathbb{R}) \subset P_0$ and functions $u_i, v_i : [0,\infty) \to (-2,2)$, for $0 \le i \le m$, that satisfy

(1) $u_0 < u_1 < \cdots < u_m = u$,

(2) $v < v_0 < \cdots < v_m$,

(3) $u_i < v_i$, for all i,

(4) $v(s) < u_0(s) < \cdots < u_m(s) < v_m(s)$, for all $s \ge 2$.

Note that the choice of u_i, v_i depends only on u, v and the ϵ must be calculated in term of this choice. Set $f^{-1}(\Gamma(u_i))$ and $f^{-1}(\Gamma(v_i))$ to be the U_i and M_i of Theorem 4.2, respectively. If ϵ is sufficiently small, the first assumption implies that for each i there is a homotopy of the identity on $(f^{-1}(\Gamma(v_i)), f^{-1}(\Gamma(u_i)))$ to a mapping into $(f^{-1}(\Gamma(u_{i+1})), f^{-1}(\Gamma(u_i)))$ which is supported on $(f^{-1}(\Gamma(v_{i+1})), f^{-1}(\Gamma(u_i)))$. We also have $P_0 \subset f^{-1}(\Gamma(u_0))$. The required isotopy and homeomorphism follow from Theorem 4.2. Hence, we have $P \subset h(f^{-1}(\Gamma(u)))$. □

The next step is to engulf the whole of $f^{-1}(\Gamma(v))$. This can be done by patching the isotopy that engulfs a polyhedron P with the one that engulfs a certain dual polyhedron Q of P in the complement of P. The existence of a such dual polyhedron is guaranteed by the following theorem of R. D. Edwards [SGH74]. From now on, superscripts will denote dimensions.

Theorem 4.3 (Topological Dual Skeleta). *Let M^n be a manifold without boundary with $n \ge 5$ and let $2 \le k \le n-3$. Then for any $\epsilon > 0$, there exists a pair of disjoint closed polyhedra (P^k, Q^{m-k-1}) which are locally polyhedral in M such that for any compact subset $C \subset M - Q$ and any neighbourhood E of P in M there exists an ϵ-isotopy h_t of id_M such that $C \subset h_1(E)$. Moreover, one may require that $h_t = id_M$ outside a neighbourhood of C for all $t \in I$.*

Definition 4.2. In the situation of the above theorem, the pair (P, Q) is said to be *topological dual skeleta* in M.

Lemma 4.2. *For a manifold without boundary M^n with $n \geq 5$, there exists an $\epsilon > 0$ such that if $f : M \to [0, \infty) \times \mathbb{R}$ is an ϵ-fibration over $[0, 4] \times [-4, 4]$ then there is an isotopy, which is supported on $f^{-1}([0, 3] \times [-3, 3])$, of id_M to a homeomorphism $h : M \to M$ satisfying $f^{-1}(\Gamma(v)) \subset h(f^{-1}(\Gamma(u)))$.*

Proof. Choose maps $v_1, v_2 : [0, \infty) \to (-2, 2)$ such that $v < v_1 < v_2$ and $v_2(s) < u(s)$ for all $s \geq 2$. If $P^{n-3} \subset M$ is a closed locally polyhedral polyhedron, then by Lemma 4.1 there exists a homeomorphism $h_1 : M \to M$ which is supported on $f^{-1}([0, 2] \times [-2, 2])$ and satisfies $P \cap f^{-1}(\Gamma(v_2)) \subset h_1(f^{-1}(\Gamma(u)))$. If $Q^2 \subset M - P$ is also a closed and locally polyhedral polyhedron then also by Lemma 4.1 and the relation $v_1 < v_2$, there exists a homeomorphism $h_2^{-1} : M \to M$ supported on $f^{-1}(\Gamma(v_1) \cap ([0, 3] \times [-3, 3]))$ and $h_2(Q \cap f^{-1}([0, 2.1] \times [-2.1, \infty]))$ lies in the complement of $f^{-1}(\Gamma(v))$.

By Theorem 4.3, for each $\epsilon > 0$ we can choose P and Q so that (P, Q) is topological dual skeleta in M by setting the closure of $h_1(M - f^{-1}(\Gamma(u))) \cap f^{-1}(\Gamma(v_2))$ to be the compact subset C. Let $h_3 : M \to M$ to be the corresponding homeomorphism supported on $f^{-1}([0, 3] \times [-3, 3])$ which is ϵ-close to id_M and takes C close to Q. If ϵ is small enough, it follows that the composition $h_2 h_3 h_1$ fulfills our requirement. $\square$

4.2 Stratified Radial Engulfings

In this section, we are going to prove our stratified version of Radial Engulfings. For the manifold settings, the theorem was firstly proved for the infinite dimensional cases (i.e. on Hilbert-cube manifolds) in [Cha80] by using the unknotting theorem of Z-set embeddings. See [vM89] or [Cha76] for a detailed account on Hilbert-cube manifolds containing the Z-set Unknotting Theorem. The finite-dimensional case was then proved in [Cha81] based on the controlled topological engulfing results of the previous section. We will prove the theorem for finite-dimensional MHSSs by following the idea of B. Hughes indicated in [Hug04] to use the Quinn's Stratified Isotopy Extension Theorem as a tool.

For notation, let B be a compact polyhedron having metric topology determined by a fixed triangulation and $p_B : B \times \mathbb{R} \to B$ is the projection to the first factor.

Theorem 4.4 (Stratified Radial Engulfing I). *Let B be a compact polyhedron and X is an MHSS such that the bottom stratum has dim ≥ 5. For every $\epsilon > 0$ there exists a $\delta > 0$, such that if $f : X \to B \times \mathbb{R}$ is a stratified δ-fibration over $B \times [-4, 4]$, then there is a stratum-preserving homeomorphism $h : X \to X$ such that*

(1) $f^{-1}(B \times (-\infty, 1]) \subset hf^{-1}(B \times (-\infty, 0))$

(2) *h may be chosen so that there is a stratum-preserving $(p_B f)^{-1}(\epsilon)$-isotopy $h_t : id_X \simeq h$ which is supported on $f^{-1}(B \times [-3, 3])$.*

Proof. We start by choosing a fine triangulation of B such that its mesh depends on ϵ and then the δ is calculated in term of this choice. The idea is to do a double inductive arguments. We start by following [Cha81] to do an induction up through the skeleta of B to get a small ambient isotopy on a strata of X. Then, we do an induction up through the skeleta of X by using the Stratified Isotopy Extension Theorem 3.1 to extend the ambient isotopy to a stratum-preserving ambient isotopy on the whole X.

For each vertex $v \in B$, let C_v be a small closed neighborhood of v. Let $\tilde{C}_v$ be an open set containing C_v such that the collection $\{\tilde{C}_v\}_{v \in B}$ is pairwise disjoint. For instance, we can choose $\tilde{C}_v$ as the open star of v in the second barycentric subdivision of B. We are given a stratified δ-fibration $f : X \to B \times \mathbb{R}$ over $B \times [-1, 1]$. For each i-stratum X_i of X, we will apply Lemma 3.2 to the composition

$$(q \times id_{\mathbb{R}})f| : f^{-1}(\tilde{C}_v \times \mathbb{R}) \cap X_i \hookrightarrow f^{-1}(\tilde{C}_v \times \mathbb{R}) \to \tilde{C}_v \times \mathbb{R} \to [0, \infty) \times \mathbb{R}$$

where $q : \tilde{C}_v \to [0, \infty)$ is a proper retraction, for instance, a cone parameter of the open star. If δ is small enough, by Proposition 3.5, $f| : f^{-1}(\tilde{C}_v \times \mathbb{R}) \to \tilde{C}_v \times \mathbb{R}$ is a stratified δ-fibration over $q^{-1}([0, 4] \times [-4, 4])$. Moreover, the restriction $f| : f^{-1}(\tilde{C}_v \times \mathbb{R}) \cap X_i \to \tilde{C}_v \times \mathbb{R}$ is also a stratified δ-fibration by Lemma 3.4. Nevertheles, $(q \times id_{\mathbb{R}})f| : f^{-1}(\tilde{C}_v \times \mathbb{R}) \cap X_i \to [0, \infty) \times \mathbb{R}$ need not necessarily be a stratified δ-fibration over $[0, 4] \times [-4, 4]$ due to the arbitrariness of q. However, we can still apply Lemma 4.2 because on its proof we only needed the ϵ-lifting property for homotopies which move only in the $\mathbb{R}$-direction and this property is also true for the map $(q \times id)f$. By Lemma 4.2, we obtain a homeomorphism $h_v : X_i \to X_i$ supported on $f^{-1}(\tilde{C}_v \times [-2, 2])$ such that $f^{-1}(C_v \times (-\infty, 1.5]) \subset h_v f^{-1}(\tilde{C}_v \times (-\infty, 0))$. Then, by composing all of h_v, we get a homeomorphism $h^0 : X_i \to X_i$ such that

$$f^{-1}(\cup_{v \in B} C_v \times (-\infty, 1.5]) \subset h^0 f^{-1}(B \times (-\infty, 0)).$$

By our choice of triangulation of B, for each simplex Δ of B we have that $\operatorname{diam}(\Delta) << \epsilon$. Hence, the homeomorphism h^0 can be realized by a small ambient isotopy in the sense that there is a $(p_B f)^{-1}(\epsilon)$-isotopy $h_t^0 : id \simeq h$. By the absolute part of the Isotopy Extension Theorem 3.1, we can extend h_t^0 to a stratum-preserving $(p_B f)^{-1}(\epsilon)$-isotopy of X which is supported on a neighborhood of X^i. That is how we deal with the 0-skeleton.

For the 1-skeleton, let $\sigma =< v_1, v_2 >$ be a 1-simplex of B. Let C_σ be a closed subset containing the closure of $\sigma - (C_{v_1} \cup C_{v_2})$ in its interior. As above, let $\tilde{C}_\sigma$ be an open subset containing C_σ such that $\{\tilde{C}_\sigma\}_{\sigma \in B}$ is a pairwise disjoint collection. Hence, by Lemma 4.2 there is a homeomorphism $h_\sigma : X_i \to X_i$ supported on $f^{-1}(\tilde{C}_\sigma \times [-2.5, 2.5])$ for which $f^{-1}((C_\sigma \cup [\tilde{C}_\sigma \cap (C'_{v_1} \cup C'_{v_2})]) \times (-\infty, 1.4])$ lies in $h_\sigma f^{-1}[(\tilde{C}_\sigma \times (-\infty, -2)) \cup (\tilde{C}_\sigma \cap (C_{v_1} \cup C_{v_2})) \times (-\infty, 1.5))]$ where C'_{v_1} and C'_{v_2} are slightly smaller neighborhood of v_1 and v_2. By doing this for all 1-simplices σ of B and composing all the yielded homeomorphisms, we obtain a homeomorphism $h^1 : X_i \to X_i$. By a similar argument as in the previous paragraph, this homeomorphism is realized by a small ambient isotopy that can be extended to a stratum-preserving small isotopy of the whole X. Note that in this step we use the relative part of Theorem 3.1 such that this isotopy agrees with the one in the pervious paragraph. The composition $h^1 h^0 f^{-1}(B \times (-\infty, 0))$ contains $f^{-1}(B^1 \times (-\infty, 1.4])$ where B^1 is the 1-skeleton of B. Hence, we are done for the 1-skeleton. We do the similar step until all the skeleta of B are exhausted. □

In the above Stratified Radial Engulfing Theorem, we get ambient isotopies on X which are small when measured in B. In the following version of the theorem, we also impose a control on the $\mathbb{R}$-factor so that the isotopies are also small when measured in $\mathbb{R}$. These following radial engulfings are the versions that will be used to solve the handle problems.

Theorem 4.5. *Let $\theta : \mathbb{R} \to \mathbb{R}$ be a homeomorphism which is supported on $[-1, 1]$. Suppose X is an MHSS without boundary such that the bottom stratum has dim ≥ 5. For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $f : X \to B \times \mathbb{R}$ is a stratified δ-fibration over $B \times [-4, 4]$, then there is a stratum-preserving homeomorphism $\tilde{\theta} : X \to X$ supported on $f^{-1}(B \times [-3, 3])$ such that $d(f\tilde{\theta}, (id_B \times \theta) f) < p_B^{-1}(\epsilon)$. Moreover, $\tilde{\theta}$ may be chosen so that there is a stratum-preserving $(p_B f)^{-1}(\epsilon)$-isotopy $\tilde{\theta}_t : id_X \simeq \tilde{\theta}$ which is supported on $f^{-1}(B \times [-3, 3])$.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\tilde{\theta}"'] \arrow[dr, phantom, "p_B^{-1}(\epsilon)"] & B \times \mathbb{R} \arrow[d, "id_B \times \theta"] & \\
X \arrow[r, "f"'] & B \times \mathbb{R} \arrow[r, "p_B"'] & B
\end{tikzcd}$$

Proof. The idea is to choose a finite partition $-1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ of $[-1, 1]$ and then, by using Theorem 4.4, to construct the required stratum-preserving homeomorphism $\tilde{\theta} : X \to X$ as a 'stacking' on this partition. The latter means that $\tilde{\theta}$ is supported on $f^{-1}(B \times [-2, 2])$, satisfies $d(p_B f\tilde{\theta}, p_B f) < \epsilon/2$ and also satisfies

$$f^{-1}(B \times (-\infty, \theta(x_{i-1})]) \subset \tilde{\theta} f^{-1}(B \times (-\infty, x_i]) \subset f^{-1}(B \times (-\infty, \theta(x_i)])$$

for $1 \leq i \leq n-1$. If the partition $\{x_i\}$ is chosen sufficiently fine and the support of $\tilde{\theta}$ is sufficiently close to $f^{-1}(B \times [-1, 1])$, then the condition $d(f\tilde{\theta}, (id_B \times \theta) f) < p_B^{-1}(\epsilon)$ will be clearly satisfied. The stratum-preserving stacking homeomorphism $\tilde{\theta}$ will be constructed as a composition $\tilde{\theta} = \tilde{\theta}_{n-1} \circ \cdots \circ \tilde{\theta}_1$, in which each stratum-preserving homeomorphism $\tilde{\theta}_i$ comes from an application of Theorem 4.4. The required stratum-preserving $(p_B f)^{-1}(\epsilon)$-isotopy will be clear from the construction.

First, we construct the stratum-preserving homeomorphism $\tilde{\theta}_1$. We are given a stratified δ-fibration $f : X \to B \times \mathbb{R}$ over $B \times [-4, 4]$, from Theorem 4.4, there is a stratum-preserving homeomorphism $h : X \to X$ and a stratum-preserving $(p_B f)^{-1}(\epsilon)$-isotopy $h_t : id_X \simeq h$ which is supported on $f^{-1}(B \times [-1, 1])$. By setting $\tilde{\theta}_1 = h^{-1}$, we obviously have that $\tilde{\theta}_1$ satisfies $d(p_B f \tilde{\theta}_1, p_B f) < \epsilon$ and

$$f^{-1}(B \times (-\infty, -1]) \subset \tilde{\theta}_1 f^{-1}(B \times (-\infty, x_1]) \subset f^{-1}(B \times (-\infty, \theta(x_1)]) \quad (*)$$

in which the first inclusion follows from the facts that $\tilde{\theta}_1$ is supported on $f^{-1}(B \times [-1, 1])$ and is small-homotopic to id_X. We also use the relation $-1 < x_1$. The second inclusion follows from Theorem 4.4.

Next, we construct the stratum-preserving homeomorphism $\tilde{\theta}_2$ as a composition $\tilde{\theta}_2 = \theta_2'' \circ \theta_2'$. It must be supported on $\tilde{\theta}_1 f^{-1}(B \times [x_1, 1])$ so that the inclusions in (*) will be automatically preserved. By similar reasoning as in the previous step, θ_2' comes from Theorem 4.4. It is supported on $\tilde{\theta}_1 f^{-1}(B \times [x_1, 1])$ and satisfies

$$f^{-1}(B \times (-\infty, \theta(x_1)]) \subset \theta_2' \tilde{\theta}_1 f^{-1}(B \times (-\infty, x_2]). \quad (**)$$

The θ_2'' also comes from Theorem 4.4, but we directly use the resulted stratum-preserving homeomorphism instead of its inverse. It is supported on $f^{-1}(B \times (\theta(x_1), 1])$ so that the inclusion in (**) will be preserved and it also satisfies

$$\theta_2'' \theta_2' \tilde{\theta}_1 f^{-1}(B \times (-\infty, x_2]) \subset f^{-1}(B \times (-\infty, \theta(x_2)]). \quad (***)$$

From (**) and (***), we have that

$$\begin{aligned} f^{-1}(B \times (-\infty, \theta(x_1)]) \subset \theta_2'' f^{-1}(B \times (-\infty, \theta(x_1)]) &\subset \theta_2'' \theta_2' \tilde{\theta}_1 f^{-1}(B \times (-\infty, x_2]) \\ &\subset f^{-1}(B \times (-\infty, \theta(x_2)]) \end{aligned}$$

and then by defining $\tilde{\theta}_2 = \theta_2'' \circ \theta_2'$, we have that

$$f^{-1}(B \times (-\infty, \theta(x_1)]) \subset \tilde{\theta}_2 \tilde{\theta}_1 f^{-1}(B \times (-\infty, x_2]) \subset f^{-1}(B \times (-\infty, \theta(x_2)]).$$

Finally, we are actually done if we inductively continue this process for all $1 \leq i \leq n-1$. $\square$

The proof of the above lemmas can be adapted so that the engulfing moves can also be along B. Let $\phi_B : B \to [0, \infty)$ be a proper map and define $B_t = \phi_B^{-1}([0,\ t])$ for each $t \in [0, \infty)$. Let $\theta_s : \mathbb{R} \to \mathbb{R}$ be an isotopy supported on $[-1, 1]$ in which $\theta_s = id$ for $s \geq 1$. It induces a homeomorphism $\theta : B \times \mathbb{R} \to B \times \mathbb{R}$ defined by $\theta(b, r) = (b, \theta_{\phi_{B}(b)}(r))$. It is obvious that θ is supported on $B_1 \times [-1, 1]$. The following are adaption of Theorem 4.4 and Theorem 4.5, respectively.

Theorem 4.6 (Stratified Radial Engulfing II). *If X is an MHSS without boundary such that the bottom stratum has dim ≥ 5. Then for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $f : X \to B \times \mathbb{R}$ is a stratified δ-fibration over $B_4 \times [-4, 4]$, then there is a stratum-preserving homeomorphism $h : X \to X$ such that*

(1) $f^{-1}(B \times (-\infty, 1]) \subset hf^{-1}(B \times (-\infty, 0))$

(2) *if $p_B : B \times \mathbb{R} \to B$ is the projection then h may be chosen so that there is a stratum-preserving $(p_B f)^{-1}(\epsilon)$-isotopy $h_t : id_X \simeq h$, supported on $f^{-1}(B_3 \times [-3, 3])$.*

Theorem 4.7. *Let X be an MHSS without boundary such that the bottom stratum has dim ≥ 5.*

(1) *For every given $\epsilon > 0$ there exists a $\delta > 0$ such that if $f : X \to B \times \mathbb{R}$ is a stratified δ-fibration over $B_3 \times [-3, 3]$, then there is a stratum-preserving homeomorphism $\tilde{\theta} : X \to X$ which is supported on $f^{-1}(B_2 \times [-2, 2])$ and which satisfies $d(f\tilde{\theta}, \theta f) < \epsilon$. Furthermore, $\tilde{\theta}$ may be chosen so that there is a stratum-preserving $(p_B f)^{-1}(\epsilon)$-isotopy of $\tilde{\theta}$ to id_X which is supported on $f^{-1}(B_2 \times [-2, 2])$.*

(2) *Moreover, for every $\mu > 0$ there is a $\nu > 0$ such that if f is additionally given to be a stratified ν-fibration over $(B_3 - \mathring{B}_{1/3}) \times [-3, 3]$, then the stratum-preserving homeomorphism $\tilde{\theta} : X \to X$ additionally satisfies $d(f\tilde{\theta}, \theta f) < \mu$ over $(B - \mathring{B}_{2/3}) \times \mathbb{R}$. Also, the $\tilde{\theta}$ may be chosen so that additionally the stratum-preserving $(p_B f)^{-1}(\mu)$-isotopy is also over $(B - \mathring{B}_{2/3}) \times \mathbb{R}$.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\tilde{\theta}"'] \arrow[dr, phantom, "p_B^{-1}(\epsilon)"] & B \times \mathbb{R} \arrow[d, "\theta"] & \\
X \arrow[r, "f"'] & B \times \mathbb{R} \arrow[r, "p_B"'] & B
\end{tikzcd}$$

Proof. This is similar the proof of Theorem 4.5, but using Lemma 4.6 instead of Theorem 4.4. The idea is to choose a finite partition $-1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ of $[-1, 1]$ and then, by using Lemma 4.6, to construct a stratum-preserving stacking homeomorphism $\tilde{\theta}$ satisfying

$$f^{-1}(B_{x_{i-1}} \times (-\infty, \theta_{x_{i-1}}(x_{i-1})]) \subset \tilde{\theta} f^{-1}(B_{x_i} \times (-\infty, x_i]) \subset f^{-1}(B_{x_i} \times (-\infty, \theta_{x_i}(x_i)]).$$

The second part is achieved by using the ν-control over $(B_3 - \mathring{B}_{1/3}) \times [-3, 3]$ to further refine the stacking so that we have $d(f\tilde{\theta}, \theta f) < \mu$ over $(B - \mathring{B}_{2/3}) \times \mathbb{R}$. $\square$

Chapter 5

Stratified Handle Problems

In this chapter, we are going to prove the following:

Stratified γ-Sucking Theorem. *Let X be an MHSS without boundary such that the bottom stratum has $dim \geq 5$ and let B be a compact polyhedron. For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $f : X \to B$ is a stratified δ-fibration then for every $\gamma \geq 0$ there exists a stratified γ-fibration $f' : X \to B$ which is ϵ-close to f.*

The idea of the proof is to start by using a handle decomposition of the target space and analyze the situation on each handle. From this step, we get handle problems that will be solved by a stratified wrapping-unwrapping technique. We will be working backward starting from the wrapping up construction in which the stratified engulfing results will be used.

5.1 Stratified Wrapping Up

The use of the wrapping up technique as an alternative to Kirby's immersion of punctured torus in solving handle problems firstly appeared in section 8 of [EK71] to prove the local contractibility of the homeomorphism group of manifolds. The power of this alternative technique becomes obvious when we are dealing with stratified spaces in which the Torus Trick cannot be applied. In the stratified settings, Siebenmann used such a technique to prove the local contractibility of the homeomorphism group of CS spaces [Sie71]. Moreover, in [AH80] Anderson and Hsiang used the wrapping up construction to study the extension problems of PL structures on CS spaces.

In Chapter 1, we have described the wrapping up construction for a topological embedding onto $\mathbb{R}^n$. The idea is by regarding the embedding to have an $\mathbb{R}$-factor and then wrapping this factor up using the universal covering. This section is devoted to the wrapping up construction of some stratified δ-fibrations. We will use Lemma 4.4 and Lemma 4.6 to wrap up a stratified δ-fibration having an $\mathbb{R}$-factor around S^1. For notation, let $e : \mathbb{R} \to S^1$ be the universal covering projection defined by $e(x) = \exp(\pi i x/4)$ and B be a compact polyhedron.

Theorem 5.1 (Stratified Wrapping Up I). *Let X be an MHSS without boundary such that the bottom stratum has $dim \geq 5$. For every $\epsilon > 0$ there exists a $\delta > 0$, such that if $f : X \to B \times \mathbb{R}$ is a stratified δ-fibration over $B \times [-4, 4]$, then there exists an MHSS without boundary $\tilde{X}$, a stratified ϵ-fibration $\tilde{f} : \tilde{X} \to B \times S^1$, and a stratum-preserving open embedding $\phi : f^{-1}(B \times (-1, 1)) \to \tilde{X}$ such that the following diagram commutes:*

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\tilde{f}"] & & B \times S^1 \\ f^{-1}(B \times (-1,1)) \arrow[u, hook, "\phi"] \arrow[rr, "f|"'] & & B \times (-1,1). \arrow[u, "id \times e"'] \end{tikzcd}$$

Proof. We are given a stratified δ-fibration $f : X \to B \times \mathbb{R}$ over $B \times [-4, 4]$. By Theorem 4.5, for some $\mu > 0$ there exist a stratum-preserving homeomorphism $\tilde{\theta}$ and a stratum-preserving $(p_B f)^{-1}(\mu)$-isotopy $\tilde{\theta}_t : id_X \simeq \tilde{\theta}$ which is supported on $f^{-1}(B \times [-3, 3])$. Hence it is clear that, by applying Lemma 3.1, $p_B f$ is ν-close to $p_B f \tilde{\theta}_t$ for some $\nu > 0$, and that, by choosing an appropriate homeomorphism θ in Theorem 4.5, $\tilde{\theta}$ can be conditioned so that we have a controlled condition that $p_R f \tilde{\theta} f^{-1}(B \times \{-t\})$ is close to $\{-t + 4\}$ for $1.8 \leq t \leq 2.2$.

Construction of $\tilde{X}$. We consider

$$Y = \tilde{\theta} f^{-1}(B \times (-\infty, -2]) - f^{-1}(B \times (-\infty, -2))$$

which is a compact subset of X and define $\tilde{X} = Y/\sim$ where the equivalence relation $\sim$ is generated by $x \sim \tilde{\theta}(x)$ for all $x \in f^{-1}(B \times \{-2\})$. Since $\tilde{\theta}$ is stratum-preserving, the identified points x and $\tilde{\theta}(x)$ lie in the same stratum of X and hence the identification only modifies that stratum. The modified stratum is clearly a manifold and thus $\tilde{X}$ is an MHSS with the same stratification of X except for the modified stratum.

Construction of $\tilde{f} : \tilde{X} \to B \times S^1$. By representing S^1 as $[-2, 2]/\sim$ where $\sim$ is generated by $-2 \sim 2$, we can naturally identify $f^{-1}(B \times (-1, 1))$ as an open subset of $\tilde{X}$ and $(-1, 1)$ as an open subset of S^1. Therefore, we have inclusion maps $\phi : f^{-1}(B \times (-1, 1)) \hookrightarrow \tilde{X}$ and $e| : (-1, 1) \hookrightarrow S^1$, in which the former is stratum-preserving. We define a map $g : Y \to B \times [-2, 2]$ such that the $\mathbb{R}$-component satisfies:

$$p_R g(x) = \begin{cases} -2, \text{ for } x \in f^{-1}(B \times \{-2\}) \\ 2, \text{ for } x \in \tilde{\theta} f^{-1}(B \times \{-2\}) \\ \text{close to } p_R f|_Y, \text{ otherwise.} \end{cases}$$

Note that the controlled condition implies that $p_R f(x)$ is close to 2 for $x \in \tilde{\theta} f^{-1}(B \times \{-2\})$ and hence $p_R g$ is well-defined. To define the B-component of g we will use the stratum-preserving inverse isotopy $\tilde{\theta}_t^{-1} := \tilde{\theta}_{1-t} \tilde{\theta}^{-1} : id \simeq \tilde{\theta}^{-1}$. Let $\mu : Y \to [0, 1]$ be a Urysohn function which is 0 on $f^{-1}(B \times \{-2\})$ and 1 on $\tilde{\theta} f^{-1}(B \times \{-2\})$. We define $p_B g(x) = p_B f \tilde{\theta}^{-1}_{\mu(x)}(x)$. Note that, for $x \in f^{-1}(B \times \{-2\})$, we have $p_B g(x) = p_B f(x)$ and $p_R g(x) = -2 = p_R f(x)$. Moreover, for $x \in \tilde{\theta} f^{-1}(B \times \{-2\})$, we have $p_B g(x) = p_B f \tilde{\theta}^{-1}(x)$ and $p_R g(x) = 2 = p_R f \tilde{\theta}^{-1}(x)$. Therefore $g = f$ on $f^{-1}(B \times \{-2\})$ and $g = f \tilde{\theta}^{-1}(x)$ on $\tilde{\theta} f^{-1}(B \times \{-2\})$. Hence, g factors to the quotient and yields a map $\tilde{f} : \tilde{X} \to B \times S^1$ making the diagram above commute.

Proof that $\tilde{f} : \tilde{X} \to B \times S^1$ is a stratified ϵ-fibration. From the above construction, we have that g is close to $f|_Y$ and hence by Lemma 3.1, there is a $\nu > 0$ such that g is ν-homotopic to $f|_Y$. Thus, by Proposition 3.6, it suffices to find a neighborhood of $B \times \{-2\}$ in the quotient over which $\tilde{f}$ is a stratified ϵ_1-fibration for smaller ϵ_1. Let

$$U = \tilde{\theta} f^{-1}(B \times (-\infty, -1.8)) - f^{-1}(B \times (-\infty, 1.8])$$

and define $g' : U \to B \times (1.8, 2.2)$ by

$$p_B g'(x) = \begin{cases} p_B g(x), & \text{for } x \in U \cap Y \\ p_B g \tilde{\theta}^{-1}(x), & \text{for } x \in U - Y \end{cases}$$

and

$$p_R g'(x) = \begin{cases} p_R g(x), & \text{for } x \in U \cap Y \\ p_R g \tilde{\theta}^{-1}(x) + 4, & \text{for } x \in U - Y. \end{cases}$$

It is clear that U can be regarded as an open subset of $\tilde{X}$ and $B \times (1.8, 2.2)$ as an open subset of $B \times S^1$ such that $\tilde{f}|_U = g'$. Thus if we can prove that g' is a stratified ϵ_1-fibration over $B \times [1.9, 2.1]$, then $\tilde{f}$ will be a stratified ϵ_1-fibration over a neighborhood of $B \times \{-2\}$ as desired.

To show that g' is a stratified ϵ_1-fibration over $B \times [1.9, 2.1]$, we prove that g' is small homotopic to the stratified ϵ-fibration $f|_U$. By construction $p_R g'$ is close to $p_R f|_U$, so by Lemma 3.1 there is a small homotopy $p_R g' \simeq p_R f|_U$. To treat the B-factor, we define $u : U \to B$ by

$$u(x) = \begin{cases} p_B g(x), & \text{for } x \in U \cap Y \\ p_B f \tilde{\theta}^{-1}(x), & \text{for } x \in U - Y \end{cases}$$

and we will show that this is small homotopic to $p_B g'$. From the definition of $p_B g$ we have a small homotopy $e_t : p_B g \simeq p_B f \mathrm{rel} f^{-1}(B \times \{-2\})$ and hence we get a small homotopy $p_B g' \simeq u$. By applying the homotopy e_t again to $U \cap Y$ and by extending it over U using Proposition 3.2, we get a homotopy $u \simeq p_B f|_U$. Therefore, by putting all the homotopies together, we have that g' is γ-homotopic to $f|_U$ for a $\gamma > 0$. Thus g' is a stratified ϵ_1-fibration for an $\epsilon > 0$ as desired. □

The proof of the next Theorem is by doing similar things as in the proof of Theorem 5.1, but using Theorem 4.7 instead of Theorem 4.5. Let B be a compact polyhedron. Recall that $\phi : B \to [0, \infty)$ is a proper map and $B_t = \phi^{-1}([0, t])$.

Theorem 5.2 (Stratified Wrapping Up II). *Let X be an MHSS without boundary such that the bottom stratum has $dim \geq 5$. For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $f : X \to B \times \mathbb{R}$ is a stratified δ-fibration over $B_3 \times [-3,3]$, then there exist an MHSS $\tilde{X}$, a stratified δ-fibration $\tilde{f} : \tilde{X} \to \mathring{B}_{2.5} \times S^1$ over $B_2 \times S^1$ and a stratum-preserving embedding $\phi : f^{-1}(\mathring{B}_1 \times (-1,1)) \to \tilde{X}$ such that the following diagram commutes:*

$$\begin{tikzcd}
\tilde{X} \arrow[rr, "\tilde{f}"] & & \mathring{B}_{2.5} \times S^1 \\
f^{-1}(\mathring{B}_1 \times (-1,1)) \arrow[u, hook, "\phi"] \arrow[rr, "f|"'] & & \mathring{B}_1 \times (-1,1). \arrow[u, "id \times e"']
\end{tikzcd}$$

Moreover, for every $\mu > 0$ there exists a $\nu > 0$ such that if f is additionally given to be a ν-fibration over $(B_3 - \mathring{B}_{1/3}) \times [-3,3]$, then the map $\tilde{f}$ is additionally a stratified μ-fibration over $(B_3 - \mathring{B}_{2/3}) \times S^1$. □

5.2 Stratified Handle Results

In this section, we will prove the main result of this chapter, the Stratified γ-Sucking Theorem 5.4. The proof is by using a handle decomposition of the target space and then solving some related handle problems. The solutions of the handle problems will be presented as Theorem 5.1 and Theorem 5.3 in which they are proved by some variants of the torus trick. We will apply the stratified radial engulfing theorems and the stratified wrapping up theorems in these torus arguments.

The first theorem contains the solution of top-dimensional handle problems. For notation, let D^n_k denotes the n-dimensional disk of radius k and let $e^n = e \times \cdots \times e : \mathbb{R}^n \to T^n$ be the product of n universal covering projections.

Lemma 5.1 (Stratified Handle: Top-Dimensional Handles). *Let X be an MHSS without boundary such that the bottom stratum has $dim \geq 5$. Given $\epsilon > 0$ there exists a $\delta > 0$, such that for every $\mu > 0$, if $f : X \to \mathbb{R}^n$ is a stratified δ-fibration over D^n_3, then there exists a stratified μ-fibration $\tilde{f} : X \to \mathbb{R}^n$ over D^n_1 which is ϵ-homotopic to f $rel(X - f^{-1}(\mathring{D}^n_3))$.*

Proof. We will work through the following commutative diagram of spaces and maps, the required $\tilde{f} : X \to \mathbb{R}$ will be obtained from a modification of the map in the top of the diagram. We choose the projection $\varepsilon^n : \mathbb{R}^n \to T^n$ such that $\varepsilon^n|_{D^n_3} : D^n_3 \to T^n$ is an open embedding and regard $T^{n-1} \times \mathbb{R}$ as an open subset of $\mathring{D}^n_3$ in such a way that $\varepsilon^{n-1} \times id : D^n_2 \to D^n_2$ is the identity. The latter can be done using the argument in Section 8 of [EK71], see also Section 1.1 of this thesis. Consider the restriction $f| : f^{-1}(T^{n-1} \times \mathbb{R}) \to T^{n-1} \times \mathbb{R}$. By Proposition 3.5, the restriction $f|$ is a stratified δ-fibration over $T^{n-1} \times [-3,3]$ provided that δ is sufficiently small.

From Lemma 5.1 there is an MHSS X', a stratified δ'-fibration f' and a stratum-preserving open embedding ϕ so that the lowest rectangle in the above diagram commutes.

$$\begin{tikzcd}
X'' \arrow[r, "\bar{f}"] & \mathbb{R}^n & \\
X'' \arrow[u, "id"] \arrow[r, "f''"] \arrow[d, "p"'] & \mathbb{R}^n \arrow[u, "\gamma"'] \arrow[d, "e^n"] & D_2^n \arrow[ul, hook'] \arrow[l, hook'] \arrow[dl, hook] \arrow[ddl, hook] \\
X' \arrow[r, "f'"'] & T^n & \\
f^{-1}(T^{n-1} \times (-1,1)) \arrow[u, hook, "\phi"] \arrow[r, "f|"'] & T^{n-1} \times (-1,1) \arrow[u, "id \times e"'] &
\end{tikzcd}$$

Next, we unwrap all of this by using the pullback construction of f' and the universal covering $e^n : \mathbb{R}^n \to T^n$. This step is indicated by the middle rectangle in the diagram above. Since e^n is a local homeomorphism, $p : X'' \to X'$ is a stratum-preserving local homeomorphism by the facts that p is stratum-preserving and the pullback of a local homeomorphism is a local homeomorphism. Hence, by the fact that the property of being homotopically stratified is local one, X'' is an MHSS. Since we chose that $(e^{n-1} \times id)|_{D_2^n} = id_{D_2^n}$, it follows that $p| : (f'')^{-1}(\mathring{D}_2^n) \to f^{-1}(\mathring{D}_2^n)$ is a stratum-preserving homeomorphism. Thus we can choose that $f'' = f$ over $\mathring{D}_2^n$. To prove that f'' is a stratified δ''-fibration, for some $\delta'' > 0$, let $F : Z \times I \to \mathbb{R}^n$ and $\tilde{F}_0 : Z \to X''$ be maps whose form an SHLP of f''. In order to solve the SHLP, we construct the following diagram:

$$\begin{tikzcd}
Z \arrow[r, "\tilde{F}_0"] \arrow[d, hook, "\times 0"'] & X'' \arrow[r, "p"] \arrow[d, "f''"] & X' \arrow[d, "f'"] \\
Z \times I \arrow[r, "F"'] \arrow[ur, dashed, "G"] \arrow[urr, dashed, "H"'] & \mathbb{R}^n \arrow[r, "e^n"'] & T^n
\end{tikzcd}$$

It is clear that $e^n F : Z \times I \to T^n$ and $pf : Z \to X'$ yield an SHLP of f'. Since f' is a stratified δ'-fibration, there is a stratum-preserving homotopy $H : Z \times I \to X'$ such that $f'H$ is δ'-close to $e^n F$ and $H(z, 0) = p\tilde{F}_0(z)$ for all $z \in Z$. Since e^n and f' clearly satisfy the hypothesis of Proposition 3.4, we have that p is a stratified fibration. Note that H and $\tilde{F}_0$ can be regarded as inputs of an SHLP of p, therefore there is a stratum-preserving homotopy $G : Z \times I \to X''$ such that $G_0 = \tilde{F}_0$ and $pG = H$. We claim that G is the required stratified δ''-solution of f''. From the facts that $f'H$ is δ'-close to $e^n F$, that $pG = H$ and that the right rectangle in the above diagram is commutative, we have that $f'pG = e^n f'' G$ is δ'-close to $e^n F$. Hence, since e^n is a local homeomorphism, $f''G$ is δ''-close to F for some $\delta'' > 0$.

Now, we modify f'' using a radial homeomorphism $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ which compresses D_K^n

to D^n_3 for large $K > 0$ and keeps D^n_2 pointwisely fixed. For a detailed construction of such a radial homeomorphism, we refer to page 18 of [KS77]. We define $\bar{f} = \gamma f''$. It is clear from the construction that $\bar{f} = f$ over $\mathring{D}^n_2$, $\bar{f}$ is a stratified δ'''-fibration, and $\bar{f}$ is a stratified μ-fibration over $\mathbb{R}^n - \mathring{D}^n_3$ where the size of μ depends on the size of $1/K$ and the size of δ''' depends on the size of δ''.

The final step is to modify $\bar{f}$. Let $\theta : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism supported on $\mathring{D}^n_8$ and defined to be $\theta(x_1, x_2, \cdots, x_n) = (x_1 + 5, x_2, \cdots, x_n)$ on $\mathring{D}^n_2$ such that it only affects the first coordinate of $\mathbb{R}^n$. It is clear that $\theta(\mathring{D}^n_2) \subset \mathbb{R}^n - \mathring{D}^n_3$. By the first part of Lemma 4.7, there is a stratum-preserving homeomorphism $\tilde{\theta} : X'' \to X'$ such that $\bar{f}\tilde{\theta}$ is ϵ''-close to $\theta\bar{f}$ in which the size of ϵ'' depends on the size of δ'''. We define $\hat{f} = \theta^{-1}\bar{f}\tilde{\theta} : X'' \to \mathbb{R}^n$ which is a stratified μ-fibration over $D^n_{1.5}$. Moreover, there is a small homotopy from $\hat{f}$ to $\bar{f}$ where the size depends on ϵ''. Finally, we define $f : X \to \mathbb{R}^n$ to be $\hat{f}$ over $D^n_{1.5}$, to be f over $\mathbb{R}^n - \mathring{D}^n_2$, and over $D^n_2 - \mathring{D}^n_{1.5}$ to be defined by the homotopy $\hat{f} \simeq \bar{f}$. It is clear that $\tilde{f}$ is as desired. □

Recall that for a compact space B, the *cone over* B is defined to be the quotient $c(B) = B \times [0, \infty]/\sim$ in which the equivalence relation $\sim$ is generated by $(b, 0) \sim (b', 0)$ for all $b, b' \in B$. Similarly, *the open cone over* B is defined by $\mathring{c}(B) = B \times [0, \infty)/\sim$. For any $t \in [0, \infty)$, define the *t-subcone* by $c_t(B) = B \times [0, t]/\sim$ and the *open t-subcone* by $\mathring{c}_t(B) = B \times [0, t)/\sim$.

Theorem 5.3 (Stratified Handle: General Case). *Let X be an MHSS without boundary such that the bottom stratum has $dim \geq 5$. Given $\epsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ if $f : X \to \mathring{c}(B) \times \mathbb{R}^n$ is a stratified δ-fibration over $\mathring{c}_3(B) \times D^n_3$ and a stratified ν-fibration over $(c_3(B) - \mathring{c}_{1/3}(B)) \times D^n_3$, then there exists a stratified μ-fibration $\tilde{f} : X \to \mathring{c}(B) \times \mathbb{R}^n$ over $c_1(B) \times D^n_1$ which is ϵ-homotopic to f $rel(X - f^{-1}(\mathring{c}_{2/3}(B) \times \mathring{D}^n_3))$.*

Proof. For $n = 0$. We are given $\epsilon > 0$, $\mu > 0$ and a proper map $f : X \to \mathring{c}(B)$ which is a stratified δ-fibration over $\mathring{c}_3(B)$ and a stratified ν-fibration over $c_3(B) - \mathring{c}_{1/3}(B)$. Choose t close to 0 and let $\theta : \mathring{c}(B) \to \mathring{c}(B)$ be a homeomorphism which is supported on $c_{2/3}(B) - \mathring{c}_{t/2}(B)$ and takes $c_t(B)$ to $c_{1/2}(B)$ so that all moves occur along the $[0, \infty)$-direction in $\mathring{c}(B)$. By the first part of Lemma 4.7 there is a stratum-preserving homeomorphism $\tilde{\theta} : X \to X$ such that $f\tilde{\theta}^{-1}$ is $\tilde{\delta}$-close to $\theta^{-1}f$, where the size of $\tilde{\delta}$ depends on the size of δ. Then we define $f' = \theta^{-1}f\tilde{\theta} : X \to \mathring{c}(B)$. It is a stratified $\tilde{\mu}$-fibration over $c_3(B) - \mathring{c}_{1/3}(B)$ which is $\epsilon/2$-homotopic to f rel $(X - f^{-1}(\mathring{c}_{2/3}(B)))$, where the size of $\tilde{\mu}$ depends on the size of μ. Let $\gamma : \mathring{c}(B) \to \mathring{c}(B)$ be a homeomorphism which is supported on $c_{2/3}(B)$ and squeezes $c_t(B)$ close to the cone point. It can be seen that $\tilde{f} = \gamma f'$ fulfills the required conditions.

For $n \geq 1$. In this case the proof is similar to the proof of Lemma 5.1 with only two significant changes. The first occurs in the wrapping up procedure in which Lemma 5.2 is used in place of Lemma 5.1. The second one occurs in the definition of radial homeomorphism γ. In this case we define $\gamma : \mathring{c}(B) \times \mathbb{R}^n \to \mathring{c}(B) \times \mathbb{R}^n$ as the composition $\gamma = \gamma_2\gamma_1$ where γ_1 gives a radial squeezing on $\mathbb{R}^n$ toward the origin as the one in the proof of Lemma 5.1 and γ_2 gives a radial squeezing on $\mathring{c}(B)$ toward the cone point as the one in the case $n = 0$ above. □

Finally, we come to the main theorem of this chapter.

Theorem 5.4 (Stratified γ-Sucking). *Let X be an MHSS without boundary such that the bottom stratum has $dim \geq 5$ and let B be a compact polyhedron. For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $f : X \to B$ is a stratified δ-fibration then for every $\gamma > 0$ there exists a stratified γ-fibration $f' : X \to B$ which is ϵ-close to f.*

Proof. We use a stratified handle decomposition of B as in Example 2.1 and do an inductive argument from the handle neighborhood in the top stratum down. For top handles, we use Lemma 5.1 so we have that for δ small enough then there is a stratified ϵ_1-fibration $f_1 : X \to B$ which is close to f and which is a stratified γ_1-fibration over a neighborhood of $U_n = \cup N_b$ where ϵ_1 and γ_1 are the δ and the μ in the Lemma, respectively, and N_b is a distinguished neighorhood of the barycenter b of some n-simplex of B. The size of ϵ_1 depends on the size of ϵ and the size of γ_1 can be chosen as small as we please so that the following next steps can be done. Next, we similarly treat the open $(n-1)$-handles neighborhood $\mathring{c}(B_b) \times \mathbb{R}^{n-1}$ of barycenter b of some $(n-1)$-simplex. In this case, we use Lemma 5.3 to find a stratified ϵ_2-fibration over a neighborhood of $U_n \cup U_{n-1}$ where $U_{n-1} = \cup N_b$ for some barycenter b of the $(n-1)$-simplex where the size of ϵ_2 depends on the size of ϵ_1 and the size of γ_2 depends on the size of γ_1. We then continue doing this process until all handles in B are exhausted. □

As an application of the Stratified γ-Sucking Theorem, we prove our Stratified Sucking Theorem by constructing a sequence of stratified ϵ-fibrations. Loosely speaking, the theorem asserts that a map that is nearly a stratified approximate fibration can be sucked into the space of stratified approximate fibrations. This also explains the terminology.

Theorem 5.5 (Stratified Sucking). *Let X be an MHSS without boundary such that the bottom stratum has dim ≥ 5 and let B be a compact polyhedron. Then for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $f : X \to B$ is a proper stratified δ-fibration, then f is ϵ-near to a stratified approximate fibration.*

Proof. We are given a proper stratified δ-fibration $f : X \to B$, by Theorem 5.4, for some $\epsilon_1 > 0$ there is a proper stratified δ_1-fibration $f_1 : X \to B$ which is ϵ_1-close to f. Inductively, we can construct sequences $\{f_n\}$ and $\{\delta_n\}$ for $n = 1, 2, 3, \ldots$, such that f_n is a stratified δ_n-fibration, $\lim_{n\to\infty}\delta_n = 0$, each f_n is ϵ_n-close to f, the uniform limit $p := \lim_{n\to\infty} f_n$ is proper, and $\lim_{n\to\infty} \epsilon_n = \epsilon$. We will show that p is the desired stratified approximate fibration.

To show that p is a stratified approximate fibration, by Proposition 3.6, it suffices to show that for every compact subset $C \subset B$ and for every $\gamma > 0$, p is a stratified γ-fibration over C. Hence, suppose that we have an SHLP of p over C, i.e. a homotopy $F : Z \times I \to C$ and a map $\tilde{F}_0 : Z \to p^{-1}(C)$ such that $p\tilde{F}_0 = F_0$. We construct the following diagram:

$$\begin{tikzcd}
Z \times 0 \arrow[r, "\tilde{F}_0"] \arrow[d, hook, "\times 0"'] & p^{-1}(C) \arrow[r] \arrow[d, "p|"] & p^{-1}(C) \arrow[dl, "f_n|"] \\
Z \times [0,1] \arrow[r, "F"'] & C. &
\end{tikzcd}$$

By definition of p, we have that f_n is close to p. Then, by Proposition 3.1, we have a small-homotopy from $p|_{p^{-1}(C)}$ to $f_n|_{p^{-1}(C)}$. Since $F_0 = p\tilde{F}_0$, we have that F_0 is small homotopic to $f_n\tilde{F}_0 : Z \times 0 \to C$, say with a homotopy $H : (Z \times 0) \times I \to C$. Since $F_0 = H_0 : Z \times 0 \to C$ extends to $F : (Z \times [0,1]) \times 0 \to C$, by the Stratum-Preserving Estimated Homotopy Extension Theorem 3.2, there is a small-homotopy $\tilde{H} : (Z \times [0,1]) \times I \to C$ extending H so that $\tilde{H}_0 = F$ and $\tilde{H}_1 : Z \times [0,1] \times 1 \to C$ extends $f_n\tilde{F}_0 : Z \times 0 \to C$. Hence, we can regard both $\tilde{H}_1$ and $\tilde{F}_0$ as inputs of an SHLP of $f_n|$. We construct the following diagram:

$$\begin{tikzcd}
Z \arrow[r, "\tilde{F}_0"] \arrow[d, hook, "\times 0"'] & p^{-1}(C) \arrow[r] \arrow[d, "p|"] & p^{-1}(C) \arrow[dl, "f_n|"] \\
Z \times [0,1] \arrow[r, "\tilde{H}_1"'] \arrow[urr, dashed] & C. &
\end{tikzcd}$$

Since f_n is a stratified δ_n-fibration, there is a stratum-preserving homotopy $G : Z \times [0,1] \to p^{-1}(C)$ such that $G_0 = \tilde{F}_0$ and f_nG is δ_n-close to $\tilde{H}_1$. We claim that, for n large enough, G is a desired γ-solution of the original SHLP of $p|$. We have that pG is close to f_nG and $\tilde{H}_1$ is small-homotopic to F. Hence, from the fact that f_nG is δ_n-close to $\tilde{H}_1$ we have that pG is close to F. Note that all closeness and smallness relations above depend only on n, thus we can choose n large enough so that pG is γ-close to F. $\square$

Chapter 6

Outlook

A possible strategy towards a stratified version of the α-Approximation Theorem might be based on proving a more general stratified sucking theorem that allows for the target space to be stratified. In doing so one would have to overcome the problem that the stratum-preserving property of solutions in the limiting process might be lost due to collapsing of strata phenomena.

Suppose we knew the following version of the stratified sucking principle:

(*) Let X be an MHSS without boundary such that the bottom stratum has dim ≥ 5 and let Y be a compact stratified polyhedron. Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that if $f : X \to Y$ is a proper stratified δ-fibration, then f is ϵ-near to a stratified approximate fibration.

Then the strategy might be as follows. Given a stratum-preserving α-equivalence $f : X \to Y$ in which X is a CS set and Y is a compact stratified polyhedron in which both have dim ≥ 5 bottom strata. Then, by Proposition 3.3, f is a stratified α-fibration. From (*), f is β-near to a stratified approximate fibration $p : X \to Y$. Next, by Remark 3.3, we have that p is a stratum-preserving CE map. Finally, from the following stratum-Preserving CE approximation theorem of M. Handel, p is arbitrarily near to a stratum-preserving homeomorphism $h : X \to Y$. Hence, the given stratum-preserving α-equivalent f is β-near to a stratum-preserving homeomorphism $h : X \to Y$.

Theorem 6.1 (**Stratum-Preserving CE Approximation Theorem**). *Let X and Y be CS sets with only finitely many strata such that the bottom strata have dim ≥ 5. Let $\epsilon : X \to (0, \infty)$ be a continuous function. If $f : X \to Y$ is a stratum-preserving CE map, then there is a stratum-preserving homeomorphism $h : X \to Y$ such that $d(f(x), h(x)) < \epsilon(x)$ for all $x \in X$.*

This theorem is proved in [Han75]. It is a generalization of Siebenmann's CE Approximation Theorem for manifolds [Sie72].

Bibliography

[AH80] D. R. Anderson and W. Hsiang. Extending Combinatorial Piecewise Linear Structures on Stratified Spaces.I. *Transactions of the American Mathematical Society*, 260(1):223–253, 1980.

[Ban07] M. Banagl. *Topological Invariants of Stratified Spaces*. Springer Monographs in Mathematics. Springer Verlag Berlin-Heidelberg, 2007.

[Bro60] M. Brown. A Proof of Generalized Schönflies Theorem. *Bulletin of the American Mathematical Society*, 66:74–76, 1960.

[Can79] J. W. Cannon. Shrinking Cell-Like Decompositions of Manifolds. Codimension Three. *Annals of Mathematics*, 110(1):83–112, 1979.

[CD77] D. Coram and P. Duvall. Approximate Fibrations. *Rocky Mountain Journal of Mathematics*, 7(2):275–288, 1977.

[CF79] T. A. Chapman and S. Ferry. Approximating Homotopy Equivalences by Homeomorphisms. *American Journal of Mathematics*, 101(3):583–607, 1979.

[Cha74] T. A. Chapman. Topological Invariance of Whitehead Torsion. *American Journal of Mathematics*, 96(3):488–497, 1974.

[Cha76] T. A. Chapman. *Lectures on Hilbert Cube Manifolds*, volume 28 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, 1976.

[Cha80] T. A. Chapman. Approximation Results in Hilbert Cube Manifolds. *Transaction of the American Mathematical Society*, 262(2):303–334, 1980.

[Cha81] T. A. Chapman. Approximation Results in Topological Manifolds. *Memoirs of the American Mathematical Society*, 34(251), 1981.

[Coh73] M. M. Cohen. *A Course in Simple-Homotopy Theory*, volume 10 of *Graduate Text in Mathematics*. Springer-Verlag, New York Heidelberg Berlin, 1973.

[Edw] R. D. Edwards. Topological Regular Neighborhoods. *Unpublished Manuscript. arXiv:0904.4665 [math.GT]*, pages 1—33.

[Edw77] R. D. Edwards. Suspensions of Homology Spheres. *Unpublished Manuscript. arXiv:math/0610573 [math.GT]*, pages 1–84, 1977.

[Edw78] R. D. Edwards. The Topology of Manifolds and Cell-like Maps. *Proceedings of the International Congress of Mathematicians, Helsinski*, pages 111–127, 1978.

[EK71] R. D. Edwards and R. C. Kirby. Deformations of Spaces of Imbeddings. *Annals of Mathematics*, 93(1):63–88, 1971.

[Fad65] E. Fadell. Generalized Normal Bundles for Locally-Flat Imbeddings. *Transactions of the American Mathematical Society*, 114(2):488–513, 1965.

[Fer77] Steve Ferry. The Homeomorphism Group of a Compact Hilbert-Cube Manifold is an ANR. *Annals of Mathematics*, 106(1):101–119, 1977.

[Fin68] R. Finney. Uniform Limits of Compact Cell-like Maps. *Notices of the American Mathematical Society*, 15:942, 1968.

[FL18] T. Farrell and W. Lück. Approximately Fibering A Manifold over An Aspherical One. *Mathematische Annalen*, 370:669–726, 2018.

[Fre82] M. Freedman. The Topology of Four-Dimensional Manifolds,. *Journal of Differential Geometry*, 17(3):83–112, 1982.

[Fri03] G. Friedman. Stratified Fibrations and the Intersection Homology of the Regular Neighborhoods of Bottom Strata. *Topology and its Applications*, 134:69–109, 2003.

[Fri20] G. Friedman. *Singular Intersection Homology*. New Mathematical Monographs. Cambridge University Press, 2020.

[FW91] Steve Ferry and S. Weinberger. Curvature, Tangentiality, and Controlled topology. *Inventiones Mathematicae*, 105:401—414, 1991.

[Han75] M. Handel. *Approximating Stratum Preserving CE Maps Between CS sets by Stratum Preserving Homeomorphisms*, volume 438 of *Lecture Notes in Mathematics: Geometric Topology (L. C. Glaser and T. B. Rushing, eds.)*. Springer-Verlag New York, New York, 1975.

[Hav75] W. E. Haver. Mappings Between ANRs that are Fine Homotopy Equivalences. *Pacific Journal of Mathematics*, 58(2):457—461, 1975.

[HR96] B. Hughes and A. Ranicki. *Ends of Complexes*, volume 123 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[HTWW00] B. Hughes, L. Taylor, S. Weinberger, and B. Williams. Neighborhood in Stratified Spaces With Two Strata. *Topology*, 39(3):873–919, 2000.

[Hu65] S. T. Hu. *Theory of Retracts*. Wayne State University Press, Detroit, 1965.

[Hud69] J. P. Hudson. *Piecewise Linear Topology*. Mathematical Lecture Notes Series. W. A. Benjamin, New York, 1969.

[Hug99] B. Hughes. Stratified Path Spaces and Fibrations. *Proceedings of the Royal Society of Edinburgh Section A Mathematics*, 129(2):1–33, 1999.

[Hug02a] B. Hughes. Products and Adjunctions of Manifold Stratified Spaces. *Topology and its Applications*, 124(1):47–67, 2002.

[Hug02b] B. Hughes. The Approximate Tubular Neighborhood Theorem. *Annals of Mathematics*, 156(3):867–889, 2002.

[Hug04] B. Hughes. Neighborhoods of Strata in Manifold Stratified Spaces. *Glasgow Mathematical Journal*, 46(1):1–28, 2004.

[HW00] B. Hughes and S. Weinberger. *Surgery and Stratified Spaces*, volume 149 of *Surveys on Surgery Theory, Vol. 2 (S. Cappell, A. Ranicki, and J. Rosenberg, eds.)*. Princeton University Press, Princeton, 2000.

[Jak83] W. Jakobsche. Approximating Homotopy Equivalences of Surfaces by Homeomorphisms. *Fundamenta Mathematicae*, 118(1):1–9, 1983.

[Jak88] W. Jakobsche. Approximating Homotopy Equivalences of 3-Manifolds by Homeomorphisms. *Fundamenta Mathematicae*, 130(3):157–168, 1988.

[Kir69] R. C. Kirby. Stable Homeomorphisms and the Annulus Conjecture. *Annals of Mathematics*, 89(3):575–582, 1969.

[KS69] R. C. Kirby and L. C. Siebenmann. On the Triangulation of Manifolds and the Hauptvermutung. *Annals of Mathematics*, 75(4):742–749, 1969.

[KS77] R. C. Kirby and L.C. Siebenmann. *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*, volume 88 of *Annals of Mathematical Studies*. Princeton University Press, 1977.

[Lac77] R. C. Lacher. Cell-like Mappings and Generalizations. *Bulletin of the American Mathematical Society*, 83:495–552, 1977.

[Nov66] S. P. Novikov. On Manifolds with Free Abelian Fundamental Group and Their Applications (Pontrjagin Classes, Smooth Structures, High-Dimensional Knots) . *Izv. Akad. Nauk SSSR ser mat*, 30:208–246, 1966.

[Qui82] F. Quinn. Ends of Maps. III: Dimensions 4 and 5. *Journal of Differential Geometry*, 17:503—521, 1982.

[Qui88] F. Quinn. Homotopically Stratified Sets. *Journal of the American Mathematical Society*, 1(2):441–499, 1988.

[RS67] C. Rourke and B. Sanderson. An Embedding Without A Normal Microbundle. *Inventiones Mathematicae*, 3(2):293—299, 1967.

[Rus73] T. B. Rushing. *Topological Embeddings*, volume 52 of *Pure and Applied Mathematics*. Academic Press, New York and London, 1973.

[SGH74] L. Siebenmann, L. Guillou, and H. Hähl. Les Voisinages Ourvets Reguliers: Criteres Homotopiques d'Existence. *Annales Scientifiques de L'E.N.S*, 7:431–462, 1974.

[Sie71] L. C. Siebenmann. Deformations of Homeomorphisms on Stratified Sets. *Commentarii Mathematici Helvetici*, 47:123–162, 1971.

[Sie72] L. C. Siebenmann. Approximating Cellular Maps by Homeomorphisms. *Topology*, 11:271–294, 1972.

[Sta62] J. Stallings. The Piecewise-Linear Structure of Euclidean Space. *Mathematical Proceedings of the Cambridge Philosophical Society*, 58(3):481—488, 1962.

[SW87] M. Steinberger and J. West. Approximation by Equivariant Homeomorphism, I. *Transactions of the American Mathematical Society*, 302:297–317, 1987.

[vM89] J. van Mill. *Infinite-Dimensional Topology: Prerequisites and Introduction*, volume 43 of *North-Holland Mathematical Library*. Elsevier (North-Holland), 1989.

[Wei94] S. Weinberger. *The Topological Classification of Stratified Spaces*. Chicago Lectures in Mathematics Series. The University of Chicago Press, Chicago, 1994.