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Embedding topological quantum field theories functorially in the UV

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Funktorielle Einbettungen topologischer Quantenfeldtheorien in den UV

Der Renormierungsgruppenfluss (RG-Fluss) von Quantenfeldtheorien beschreibt die Veränderung von hohen Ultraviolett-Energien (UV) hin zu niedrigen Infrarot-Energien (IR). In dieser Thesis wird eine neue Methode entwickelt, die die Berechnung von topologischen IR-Korrelatoren direkt im UV ermöglicht. Diese Methode basiert auf Domain-Wall-Projektionsdefekten der Kodimension Eins und erlaubt eine vollständige Representation der topologischen IR-Theorie in der UV-Theorie. Der RG-Fluss einer Theorie kann auf diese Weise als Fluss des Identitätsdefekts in der fixierten UV-Theorie verstanden werden. Die vorgestellte Methode ist allerdings nicht beschränkt auf RG-Fluss, sondern verallgemeinert zu topologischen Quantenfeldtheorien, die über Projektionsdefekte auf Untertheorien projiziert werden.

Für Projektionsdefekte in triangulierten Defektkategorien, z.B. in topologisch getwisteten 2d $\mathcal{N} = (2,2)$-Theorien, wird gezeigt, dass diese immer einen komplementären Projektionsdefekt besitzen. Die ursprüngliche Theorie zerfällt dann automatisch in die beiden den Projektoren zugeordneten projizierten Theorien.

Die neue Methode wird angewendet auf Phasen von geeichten linearen Sigma-Modellen und den RG-Fluss zwischen 2d supersymmetrischen Landau-Ginzburg-Orbifold-Modellen. Die entsprechenden Defekte werden durch Matrixfaktorisierungen beschrieben.

Embedding topological quantum field theories functorially in the UV

Renormalization group (RG) flow describes how the behavior of quantum field theories changes from high ultra-violet (UV) energies to low infra-red (IR) energies. In this thesis, I describe a new method which allows the calculation of topological IR correlators directly within the UV. This method is based on codimension-one domain-wall defects and provides a complete representation of the topological IR theory in terms of the UV theory. From this perspective, RG flow of bulk theories can be regarded as RG flow of the codimension-one identity defect in the fixed UV bulk theory. The procedure is not restricted to RG flow but generalizes to topological quantum field theories projected to subtheories by projection defects.

It is furthermore shown that projection defects in triangulated defect categories (such as defects in 2d topologically twisted $\mathcal{N} = (2,2)$ theories) always come with complementary projection defects. The unprojected theory then decomposes into the theories associated to the two projection defects.

The new method is applied to phases of gauged linear sigma models and RG flows between 2d supersymmetric Landau-Ginzburg orbifold models, for which the respective defects can be described in terms of matrix factorizations.

to my wife and family

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Chapter 1

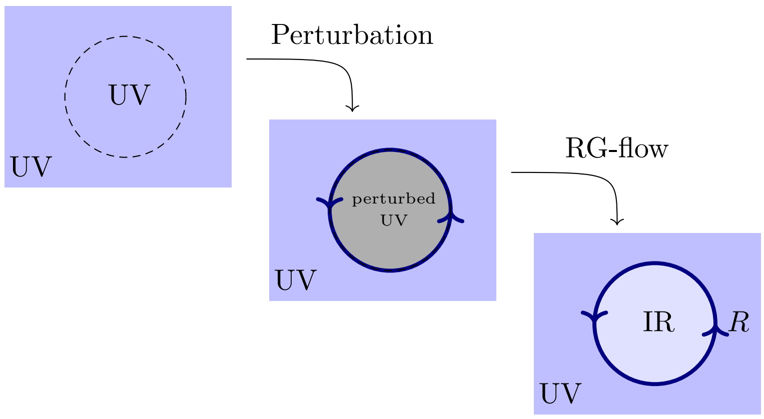
Introduction

Quantum field theories crucially depend on the energy scale. When this scale shifts from ultraviolet energies (UV) to infrared energies (IR), renormalization group (RG) flow describes how the behavior of the theory changes. For example, heavy fields might be integrated out and new interactions arise. In this thesis, I describe a new method which allows the calculation of IR correlators directly within the UV. This method is based on codimension-one domain-wall defects and provides a complete representation of the IR theory in terms of the UV theory. In fact, the procedure is not restricted to RG flow but generalizes to subtheories realized within a host theory.

The following introduction is mainly an adapted version of the introductions in [1, 2]. For more details on the connection between these papers and this thesis, see the end of this introduction.

While the presented approach might also be useful in more general situations, I focus on two-dimensional topological quantum field theories. In such theories, the quantum fields depend on the two coordinates of a two-dimensional manifold called worldsheet and all correlators are required to be independent of variations of the worldsheet metric.

Starting point of the construction are RG defects as introduced in [3]. These are domain walls between UV and IR theories obtained in the following way. Consider a perturbation of a scale invariant quantum field theory by a relevant local operator. The RG flow drives the theory from the original theory in the UV to some other theory in the IR. If the perturbation is restricted to a finite region of spacetime, the RG flow drives the theory to the IR on the domain of the perturbation, while leaving it at the UV on the rest of spacetime. Along the way, it creates a domain wall R on the boundary of the perturbation domain, separating the IR theory from the UV theory:

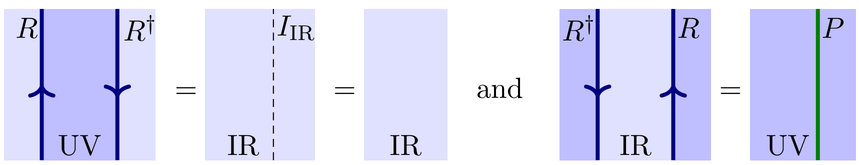


(1.1)

The *RG defects* R obtained in this way capture the entire relation between UV and IR theories. They project UV degrees of freedom onto the IR theory and embed IR degrees of freedom into the UV theory.

In order to get a good handle on defect lines, in particular the behavior of correlation functions under changes of their position, we now pass to the topologically twisted theory. Compatibility of the RG flow with the topological twist assures that the respective RG defect descends to a defect between the topologically twisted IR and UV theories. We still refer to this defect as RG defect and to the topologically twisted theories as IR and UV theories. (The notion of RG defects as defined in [3] does not require a topological twist. In fact, examples of RG defects are known in full CFTs [4], see also [5, 6]. We expect the ideas presented below to also be applicable in this more general context, albeit in a more intricate way.)

Fusion[1] of RG defects R with their downward oriented versions $R^\dagger$ gives rise to the trivial identity defect in the IR theory, $R \otimes R^\dagger \cong I_{\mathrm{IR}}$, while fusion in the opposite order yields non-trivial defects $P = R^\dagger \otimes R$ in the UV theory:



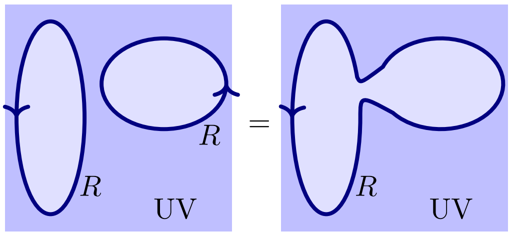
(1.2)

Intuitively, this can be understood as follows. Reading the first diagram from right to left, $R^\dagger$ embeds the IR degrees of freedom into the UV und R projects back down onto the IR. From a purely IR point of view, the resulting defect is invisible. Because the IR carries less information than

[1] In a TQFT one can move parallel defects infinitely close together resulting in a new, fused defect. We will denote fusion of defects by '$\otimes$'.

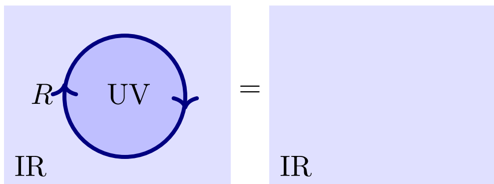
the UV, opposite fusion depicted in the second equation is more interesting: When projecting from the UV onto the IR and lifting the remaining degrees of freedom back to the UV, information is lost and we obtain a non-trivial defect in the UV.

The first equation of (1.2) is a central property of RG defects, which ensures locality in the sense that islands of IR theories trivially connect:



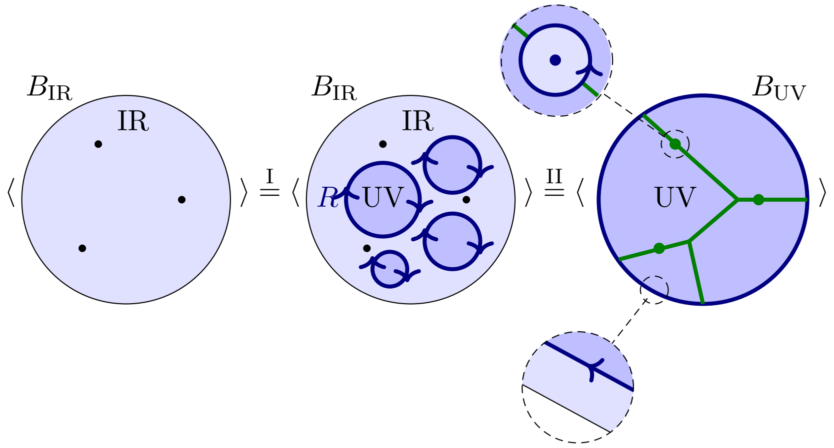
It also implies that the defects P of the second equation are projection defects, i.e. they are idempotent under fusion, $P \otimes P \cong P$. They project onto IR degrees of freedom in the UV theory. Next to idempotency, the defects P have a second defining property: They are (co)unital, which is discussed later in this introduction.

Another consequence of (1.2) is that right R-loops evaluate to the identity:



(1.3)

(Since the IR carries less information than the UV, the above loop-condition does not hold for left R-loops.) This can be used to express correlation functions of the IR theory in terms of UV correlators by the following trick familiar from the discussion of dualities and generalized orbifolds [7, 8, 9, 10]: Because of equation (1.3), a given IR correlator is not changed upon insertion of right R loops, c.f. step I in the example (1.4) below. Since we are dealing with a topological quantum field theory, the UV islands created in this way can be expanded without changing the correlation function until they cover the entire spacetime surface, see step II in (1.4). The result is a correlation function in the UV theory with a network of the projection defect P inserted. For instance, a disk correlator in the IR with boundary condition B_{IR} can be represented as a UV correlator in the following way.



(1.4)

Of course, steps I and II involve many choices leading to representations of one and the same IR correlation function by possibly different P-networks in the UV. The latter can be related by sequences of local transformations, which are generated by identities satisfied by the defects P and their junctions.

Carrying out this procedure on the level of correlators immediately reveals how objects of the IR theory are represented in the UV. For instance, IR bulk fields appear as field insertions on the defect P. Right[2] boundary conditions B_{IR} are mapped to boundary conditions $B_{\mathrm{UV}} = R^\dagger \otimes B_{\mathrm{IR}}$ in the UV. Similarly, IR defects D_{IR} are mapped to defects $D_{\mathrm{UV}} = R^\dagger \otimes D_{\mathrm{IR}} \otimes R$ in the UV. This in particular applies to the defects associated to symmetries of the IR theory. These symmetry defects encode the action of the symmetry group on all objects of the theory, and they fuse according to the multiplication in the symmetry group. Lifting IR symmetry defects to the UV one obtains UV defects whose fusion is still governed by the IR symmetry group. This yields a realization of the IR symmetry group in the UV, which however is not an honest representation. After all, the identity defect in the IR corresponding to the neutral element in the IR symmetry group is lifted to the projection defect P in the UV. Thus, the lifted symmetries are only invertible on the IR degrees of freedom.

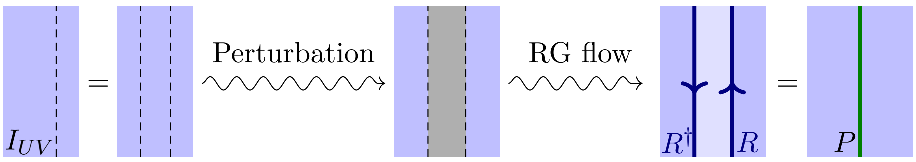
In fact, given the projection defect P, the objects in the UV theory representing IR objects can be characterized completely within the UV theory without any reference to R: IR bulk fields are represented by defect fields on P, right IR boundary conditions are represented by right UV boundary conditions B_{UV} which are invariant under fusion with P, $P \otimes B_{\mathrm{UV}} \cong B_{\mathrm{UV}}$. IR defects are represented by defects D_{UV} in the UV, which are invariant

[2] As discussed in chapter 2.1, worldsheet diagrams are typically read from right to left. A right boundary of a TQFT T is then considered to map the trivial theory to T. A left boundary (which might refer to the same boundary condition) however maps T to the trivial theory.

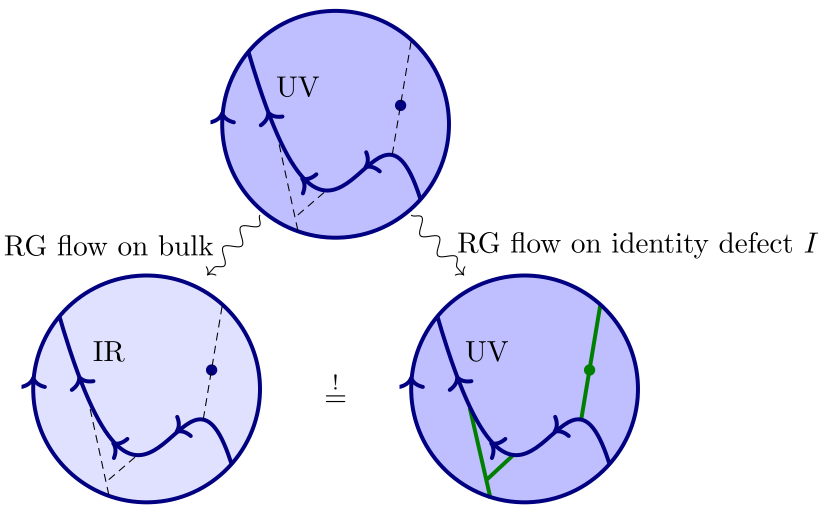
under fusion with P from both sides, $D_{\mathrm{UV}} \otimes P \cong D_{\mathrm{UV}} \cong P \otimes D_{\mathrm{UV}}$, etc. Given the projection defect P, one can therefore completely describe the IR theory in the framework of the UV theory.

Through perturbations by different relevant operators, a UV theory might permit many different RG flows leading to possibly different IR theories at various engery scales. All of these theories with all their symmetries etc. can be described by projection defects in one and the same UV theory. This applies in particular if the theory is asympotically free in the UV, in which case all possible IR theories can be realized by means of projection defects in a free theory.

Remarkably, the description of IR correlators in terms of UV correlators containing networks of the projection defect $P = R^\dagger \otimes R$ provides a radically new view on bulk perturbations: instead of perturbing the theory on the entire spacetime, we can restrict the perturbation on a network of thin strips. These strips can even be made infinitely thin, effectively reducing the bulk perturbation to a (one-dimensional) perturbation of the identity defect in the UV theory. RG flow leaves the bulk theory at the UV, but drives the identity defect to some projection defect P in the IR.



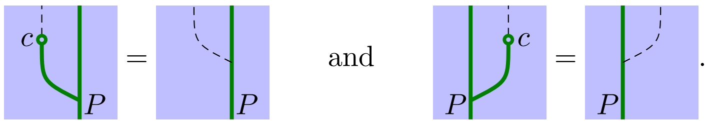
Concretely, one obtains the correlation function of the IR theory from the one in the UV by first inserting an (invisible) network of the identity defect, which in particular passes through all bulk insertions and runs parallel to every boundary and also on both sides of any defect. The IR correlation function can then be obtained by a defect RG flow on this network.



(1.5)

Under the flow UV boundary conditions and defects flow to their respective fusion with P (which has not been drawn above for ease of readability). In this way, bulk RG flows can be entirely studied in the fixed UV bulk theory by means of perturbations of the UV identity defect.

The fact that one can describe IR theories in the UV without reference to the RG defects R by using the respective ((co)unital) projection defects $P = R^\dagger \otimes R$ suggests applying this procedure to general such special defects P, which do not a priori arise from RG flows. In this way, new 'P-projected' theories can be constructed from *any* (co)unital projection defect P in a given TQFT. Spelled out, P is required to be idempotent with respect to fusion, $P \otimes P \cong P$, and it must have a counit or unit. A counit of P is a morphism (defect changing field) $c : P \to I$ from P to the identity (or invisible) defect I of the theory which satisfies

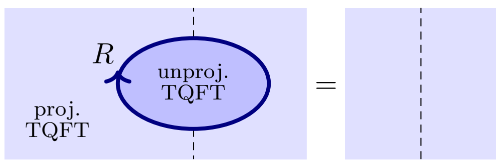


(1.6)

In these diagrams, which we read from bottom to top, dashed lines always represent the identity defect. The identity defect exists in any theory and does not affect correlation functions upon insertion. Moreover, there are natural junction fields by which it can end (respectively begin) on any other defect. The trivalent junctions of P are given by the isomorphisms $P \otimes P \cong P$. Similarly, a unit of P is a morphism $u : I \to P$ from the identity defect to P, satisfying the corresponding relations obtained by vertical reflection of the diagrams above.

Given such a (co)unital projection P in a TQFT, one can define the P-projected theory. Its bulk fields are the defect fields on P, its boundaries are the unprojected boundaries B satisfying $P \otimes B \cong B$ and its defects are the unprojected defects D which obey $P \otimes D \cong D \otimes P \cong D$. Correlation functions of the projected theory can then be obtained as correlation functions of the unprojected theory with a network of the defect P inserted.

Interestingly, counitality (unitality) of P ensures that P splits as $P \cong {}^\dagger R \otimes R$ ($P \cong R^\dagger \otimes R$), where R is a defect between the unprojected and projected theory and ${}^\dagger R$ ($R^\dagger$) is an adjoint, i.e. an oppositely oriented version of it. Moreover, R has invertible right quantum dimension



which is just (1.3) in a different terminology. Hence, we have come full circle and find that the existence of an RG defect R with invertible quantum dimension is equivalent to the existence of a (co)unital projection defect P.

In all examples considered in this thesis the boundary and defect spectra are described by triangulated categories. This triangulation has an important effect on the projection defects: Any counital projection defect P in this set-up comes with a complementary unital projection defect $\bar{P}$, and vice-versa, and the unprojected theory decomposes into the P-projected and $\bar{P}$-projected theories. In particular, the direct sum of the two defects P and $\bar{P}$ can be deformed to the invisible defect of the unprojected theory.

Thesis organization. The thesis is organized as follows. In chapter 2 the general method is presented in detail. The chapter starts with the RG defect point of view and the representation of IR degrees of freedom in the UV. Furthermore, the relation of RG flow to defect flow is discussed. Then, the opposite point of view starting with a (co)unital projection defect in any TQFT is presented. Also, the generalized orbifold procedure is reviewed and its relation to the presented method discussed. Proofs of various important identities are relegated to appendix A.

Chapter 3 contains a review of 2d quantum field theories, $\mathcal{N} = (2, 2)$-supersymmetry and the topological B-twist. The corresponding B-type defects and their mathematical description as matrix factorizations are discussed. The defect and boundary categories are furthermore embedded in the formalism of triangulated categories. Finally, orbifold models are discussed, which are the building blocks for all examples considered in this thesis.

In chapter 4 a major example is discussed in depth. Namely, the $\mathbb{Z}_d$-orbifold of the Landau-Ginzburg model with chiral superfield X and superpotential $W(X) = X^d$, which we denote by $\mathcal{M}_d$, admits RG flows to Landau-Ginzburg orbifolds $\mathcal{M}_{d'}$ for all $d' < d$. Applying the procedure described above to these flows yields a realization of all models $\mathcal{M}_{d'}$ in terms of projection defects in $\mathcal{M}_d$ for $d' < d$. In particular, taking $d \to \infty$ one obtains a representation of all models $\mathcal{M}_{d'}$ in the theory of a free twisted chiral field. The realization of the discussed models in terms of the generalized orbifold procedure can be found in appendix B. Multiple calculations for RG defects are relegated to appendix C.

Next, in chapter 5 the general method is applied to gauged linear sigma models (GLSMs). Such models admit different phases, and the procedure of this thesis allows to represent them directly in the linear sigma model. For the latter, a formulation for the invisible defect I is proposed. Furthermore, the results allow to describe D-brane transport between the different phases.

In chapter 6, (co)unital projection defects in theories with a triangulated defect category are considered. It is proven that every counital projection comes with a unital projection and vice-versa and that the direct sum of these two defects can be perturbed to the invisible defect of the unprojected theory. This finding is then applied to the example of chapter 4.

Finally, the most important findings of this thesis are summarized concisely in chapter 7 and a few directions of possible future research are discussed.

The method presented was developed with Prof. Dr. Daniel Roggenkamp [1] and extended with him [2] and Prof. Dr. Ilka Brunner [11]. This thesis is based on these three papers. In particular, chapters 2 and 4 are an adapted version of [1], chapter 5 of [11] and chapter 6 of [2]. When putting together publications [1, 2, 11] for this thesis, I have refrained from replacing the word "we" by "I" in order to honor the contributions of my coauthors.

Chapter 2

RG-networks in two dimensions

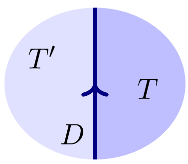
In this chapter, the basic mechanism of realizing TQFTs within a host theory is presented. First, the stage is set by a review of one-dimensional topological defects (section 2.1). Next up in section 2.2, I discuss RG defects and how they give rise to projection defects. These projections can be used to represent an IR TQFT within a UV TQFT (section 2.3) and indicate that bulk RG flow can be understood as defect flow on the invisible defect (section 2.4).

Conversely, given a (co)unital projection defect in some TQFT, a new (IR) TQFT can be described within the host theory (section 2.5). In fact, every such projector splits into RG defects (section 2.6). Finally, in section 2.7 the generalized orbifold procedure is reviewed and its relation to the method of this thesis discussed (section 2.8).

This chapter closely follows the paper [1].

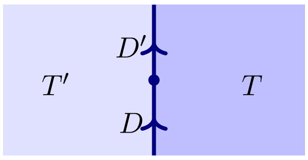
2.1 Topological defect lines

Of particular interest in this thesis are topological defect lines (and boundaries). Such lines can put constraints on fields along the defect contour (e.g. t'Hooft defects) or add additional one-dimensional terms to the lagrangian (e.g. Wilson lines). Because defect lines have codimension one, they can also separate different 2d TQFTs on the same spacetime surface. Locally, a neighborhood around a point on a defect $D : T \to T'$ separating two TQFTs T and T' can be depicted as



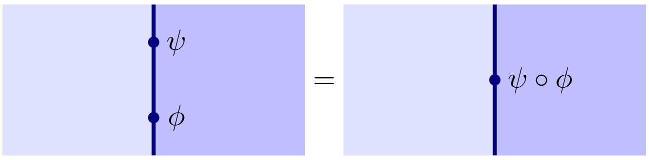
Such diagrams are typically read from right to left and D is said to be a defect from theory T to theory T'. Sometimes defect lines are displayed with arrows to emphasise this orientation. In the remainder of this section, we review properties of defect lines following [12, 13, 14, 15].

Defect lines carry local degrees of freedom, called defect fields, which can be inserted at points on defects. Defect fields can also separate different defects or glue together defects at junctions. We denote the space of defect changing fields between two defects $D, D' : T \to T'$ by $\mathrm{Hom}(D, D')$. Every defect carries the identity field $1_D \in \mathrm{Hom}(D, D)$.

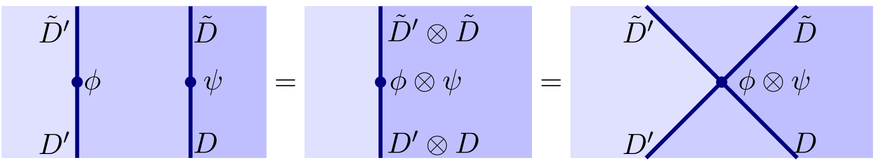


In every 2d TQFT T there is a special invisible or identity defect I_T, whose insertion does not change correlation functions, and which can be connected to any other defect. The defect fields on this defect are just the bulk fields of the underlying 2d TQFT, $\mathrm{Hom}(I_T, I_T) \cong \mathcal{H}_T$.

Due to topological invariance, defects and field insertions can be moved on the spacetime surface without changing correlation functions, as long as field insertions or defects do not cross. This in particular implies a vertical composition of defect fields.

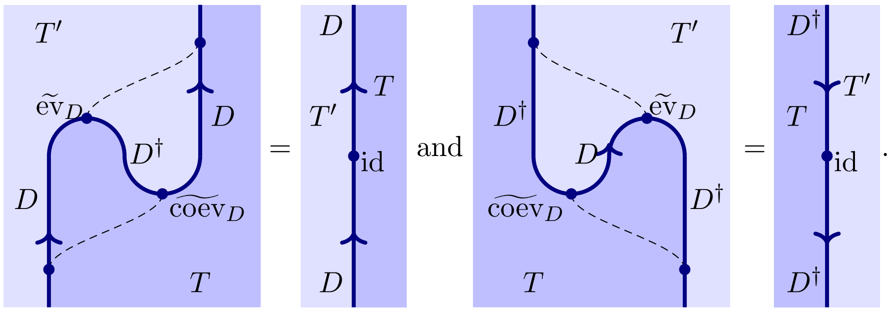


Similarly, when parallel defect lines $D' : T' \to T''$ and $D : T \to T'$ are brought close together, they fuse to the defect $D' \otimes D : T \to T''$ and analogously for field insertions:



Sometimes a field insertion in such diagrams is omitted. Then, there is an implicit insertion of the identity field. Furthermore, the invisible or identity defect I is the neutral object with respect to fusion, i.e. $D \otimes I \cong D \cong I \otimes D$ for all defects D. Due to topological invariance vertical composition and horizontal fusion commute.

Topological invariance also implies that one can bend a defect (to the right or left) without changing correlators. This is described by the following two Zorro move identities (relations like this hold locally when inserted in any correlator):

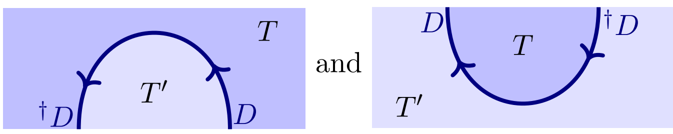


(2.1)

These diagrams involve additional structure: First of all, bending D to the right results in a downwards oriented version $D^\dagger$ of D, its right-adjoint. Secondly, dotted lines depict the (invisible) identity defect I, which connects to the defects D and $D^\dagger$ in defect (junction) fields

$$\begin{aligned}\widetilde{\mathrm{ev}}_D &: D \otimes D^\dagger \to I_{T'}\\ \widetilde{\mathrm{coev}}_D &: I_T \to D^\dagger \otimes D,\end{aligned} \tag{2.2}$$

called evaluation and coevaluation maps, respectively. Of course, one can equally well bend the defect D to the left

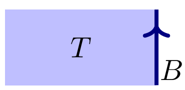


giving rise to the left-adjoint ${}^\dagger D$ of D. Topological invariance implies analogous Zorro move identities involving D and ${}^\dagger D$ and the respective (co-)evaluation maps

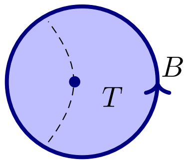
$$\begin{aligned}\mathrm{ev}_D &: {}^\dagger D \otimes D \to I_T\\ \mathrm{coev}_D &: I_{T'} \to D \otimes {}^\dagger D.\end{aligned}$$

Of course for all defects D, $\left({}^\dagger D\right)^\dagger \cong D \cong {}^\dagger\left(D^\dagger\right)$. For more details on adjunctions of defects, see [15].

Boundaries $B : 0 \to T$ are a special kind of defect from the trivial theory 0 to the TQFT T.



They too carry local degrees of freedom (open-string states) which can separate two different boundary conditions. In a well-defined TQFT, boundaries give rise to disk correlators



where the invisible defect I is used to insert bulk fields.

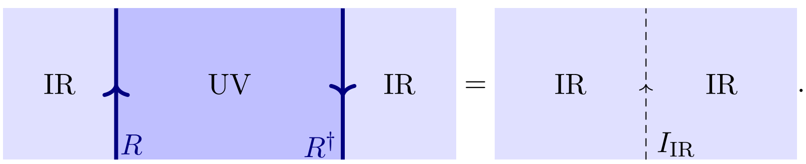
Mathematically, 2d TQFTs can be expressed in categorial language. Namely, theories (including the zero-theory) assigned to parts of a worldsheet are called *objects*, defects and boundaries separating two TQFTs are called *1-morphisms* and field insertions separating two 1-morphisms are called *2-morphisms*.

2.2 Projections from RG defects

Starting point of our construction are RG defects as defined in [3]. These defects arise when 2d field theories are perturbed by local operators only on part of the spacetime surface. The RG flow drives the theory to the IR on the perturbation domain, while leaving it at the UV on the rest, thus creating a defect on the boundary of the perturbation domain separating the IR from the UV theory as in (1.1).

This RG defect encodes all aspects of the relationship between UV and IR theories.[1] In the following we will consider such RG defects in the context of 2d topological quantum field theories.

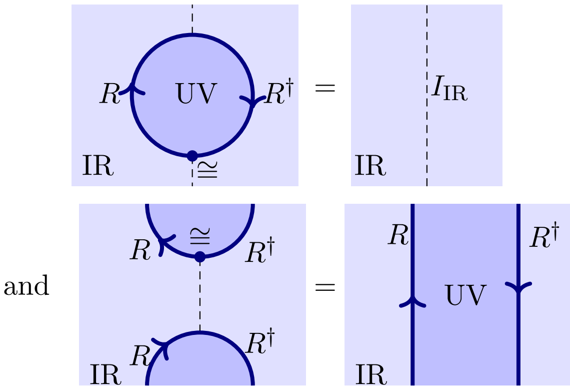
Arising from local perturbations, RG defects have rather special properties. Locality postulates that perturbation on two adjacent domains is nothing but the perturbation on the union of the domains. This implies that fusion of an RG defect R with its opposite defect $R^\dagger$ in the UV theory yields the identity defect I_{IR} in the IR:



In other words,the evaluation map $R \otimes R^\dagger \to I$ has an inverse such that the right quantum dimension of R is invertible and UV bubbles in the IR locally

[1] For instance, in [3] RG-defects are used to describe how boundary conditions behave under perturbations of the bulk theory.

connect:

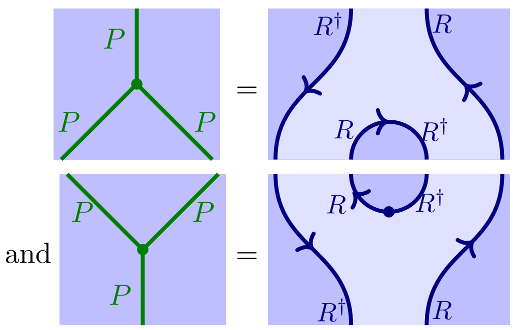


(2.3)

Since I_{IR} is self-adjoint, $R \otimes R^\dagger \cong I_{\mathrm{IR}}$ is equivalent to $R \otimes {}^\dagger R \cong I_{\mathrm{IR}}$. By the Zorro-moves, the latter isomorphism is given by the coevaluation map of R.

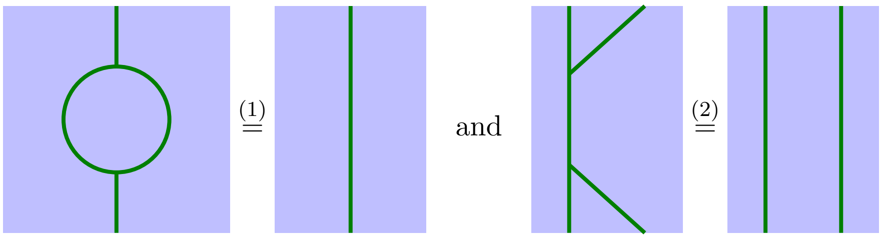
Utilizing that right R loops evaluate to the identity, it is possible to express correlation functions of the IR theory in terms of correlation functions in the UV by the following trick: Given a correlation function of the IR theory, one can insert right R loops without changing it. Expanding these islands of UV theory until they cover the entire surface, the IR correlation function is transformed into a correlation function of the UV theory with a network of defects as in equation (1.4).

The network is built out of the defect $P := R^\dagger \otimes R$ (in the following represented by green lines which we take as upwards oriented if an orientation is not specified) and its junctions



which we call multiplication and comultiplication, respectively.

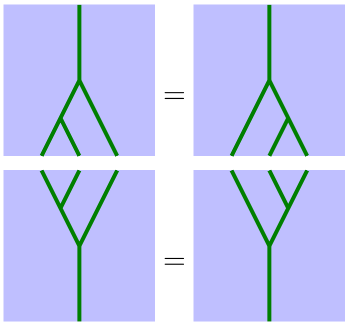
The defect P together with its junctions has some rather special features, which easily follow from the properties (2.3) of R. In particular, $P \otimes P \cong P$, and the following relations hold:



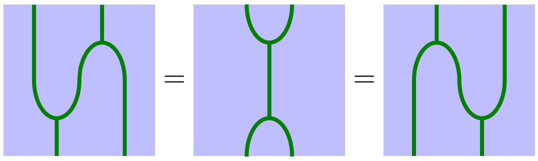
We call the first one *loop-omission property* (or *separability*) and the second one *projection property*. Beyond these, P also obeys the following identities:

associativity:

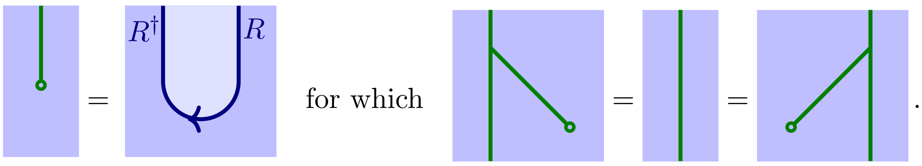
coassociativity:



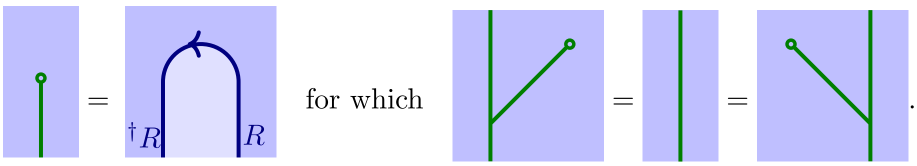
and the Frobenius identities:



Moreover, P comes with a unit



Indeed, instead of $P = R^\dagger \otimes R$ we could just as well have chosen $P' = {}^\dagger P = {}^\dagger R \otimes R$ as building block of the network above. The latter defect equally satisfies the relations above with the only difference that instead of a unit, it has a counit

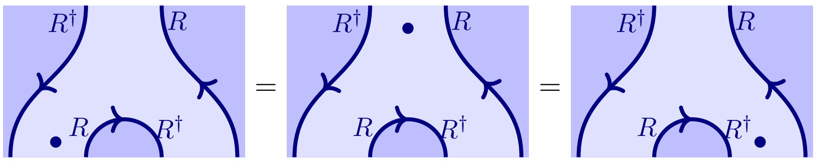


In summary, any correlation function of the IR theory can be written as a correlator in the UV with a P-network inserted. The correlation function is invariant under local changes of the P-network generated by loop-omission and projection properties, the associativity and coassociativity relations and the Frobenius identities. This in particular reflects the fact that the resulting correlation function does not depend on how exactly the UV islands are inserted into the IR correlators and how they are expanded.

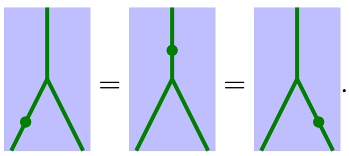
2.3 Representing the IR in the UV

Having expressed the IR correlators in terms of UV correlators in the last section, we now discuss how the defining structures of IR correlators such as bulk fields, boundaries, defects and symmetries are represented in the UV theory. The results can be summarized as follows: If one characterizes the respective IR object by its relation to the IR identity defect I_{IR}, then its UV realization is obtained by replacing the IR identity defect by the defect P of the UV theory, c.f. table 2.1. For simplicity we will restrict the discussion to the case of unital projection defect $P = R^\dagger \otimes R$. The results are the same for the counital case, and the argument is similar.

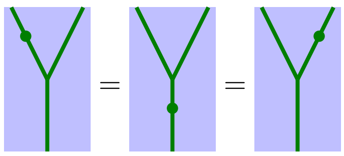
IR bulk fields. Let us first discuss bulk fields of the IR theory. Upon expanding the UV islands in the IR, bulk fields become defect fields on P, i.e. elements in $\mathrm{Hom}(P, P)$ (represented in diagrams by dots on defects). Due to topological invariance, they have to be compatible with the multiplication on P. Namely,



implying



Considering the algebra P as P-bimodule, the IR bulk fields become P-bimodule morphisms of P in the UV. By the same argument these morphisms also respect the P-comodule structure on P:



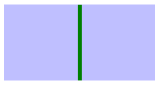
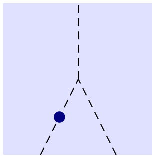
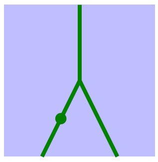
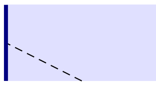
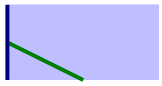
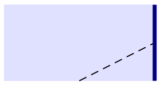
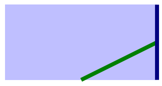
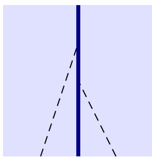
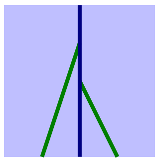
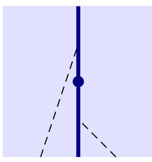
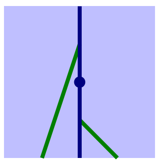
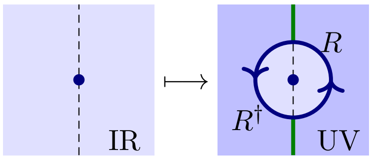
IR object	**IR realization**	**UV realization**
Identity defect	(separable Frobenius) algebra I_{IR}	(Co-)unital projection defect P 
IR bulk fields	I_{IR}-bimodule morphisms of I_{IR} 	P-bimodule morphisms of P 
Left boundary conditions	Right I_{IR}-modules 	Right P-modules 
Right boundary conditions	Left I_{IR}-modules 	Left P-modules 
Defects	I_{IR}-bimodules 	P-bimodules 
Defect changing fields	I_{IR}-bimodule morphisms 	P-bimodule morphisms 
Defect fusion	fusion in the IR $D \otimes D'$	fusion of UV lifted defects $D_{\mathrm{UV}} \otimes D'_{\mathrm{UV}}$
Adjunction	$D^{\dagger}$ ${}^{\dagger}D$	$D_{\mathrm{UV}}^{\dagger_P} = P \otimes D_{\mathrm{UV}}^{\dagger} \otimes P$ ${}^{\dagger_P}D_{\mathrm{UV}} = P \otimes {}^{\dagger}D_{\mathrm{UV}} \otimes P$

Table 2.1: Dictionary of IR structures lifted into the UV.

Now, not only are IR bulk fields lifted to P-bimodule morphisms of P in the UV, the Hilbert space of bulk fields of the IR theory is in fact isomorphic to the space of P-bimodule morphisms of P. More precisely, the map

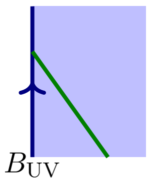


sending IR bulk fields to P-bimodule morphisms of P is an isomorphism. This is spelled out in appendix A.1.

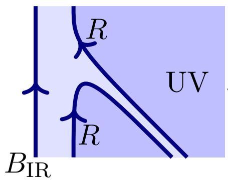
In fact, due to the special properties of P, all morphisms of P, i.e. all defect fields on P are automatically P-bimodule morphisms of P and at the same time also P-bicomodule morphisms of P, see appendix A.2. Thus, the IR bulk Hilbert space is isomorphic to the space of defect fields on P.

IR boundary conditions and defects. Next, let us discuss left IR boundary conditions. Upon inserting and expanding UV islands in the IR theory, a left IR boundary condition B_{IR} is lifted to the UV boundary condition $B_{\mathrm{UV}} := B_{\mathrm{IR}} \otimes R$. The latter comes equipped with a map

$B_{UV} \otimes P \to B_{UV}$

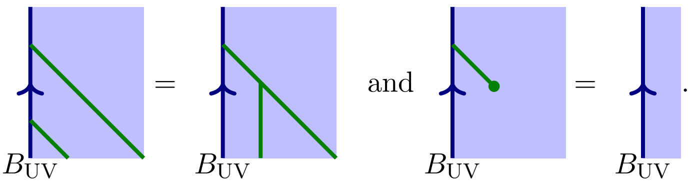


arising from

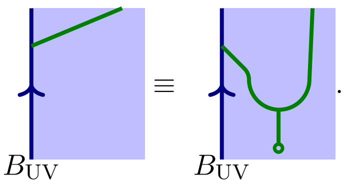


.

It satisfies the identities



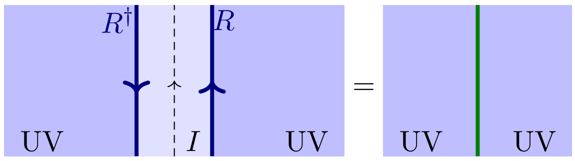
In other words, B_{UV} is a right P-module. In fact, the unit of P induces a P-comodule structure on any P-module, hence also on B_{UV}:



Therefore, left IR boundary conditions lift to right P-modules in the UV, which automatically are also P-comodules.

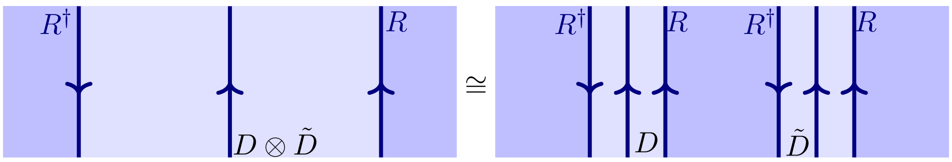
Conversely, all right P-modules arise in this way from IR boundary conditions. To see this, note that due to the special properties of the defect P, a left UV boundary condition B is a right P-module iff $B \cong B \otimes P$ as shown in appendix A.3. Hence, given a right P-module B, the IR boundary condition $B_{\mathrm{IR}} = B \otimes R^\dagger$ satisfies $B_{\mathrm{IR}} \otimes R = B \otimes R^\dagger \otimes R = B \otimes P \cong B$. Thus, left IR boundary conditions are in one-to-one correspondence with right P-modules in the UV.[2]

Analogously one finds that right IR boundary conditions B_{IR} lift to left P-modules $B_{\mathrm{UV}} = R^\dagger \otimes B_{\mathrm{IR}}$ in the UV, and defects D_{IR} of the IR theory lift to P-bimodules $D_{UV} = R^\dagger \otimes D_{\mathrm{IR}} \otimes R$. Importantly, P itself is the UV lift of the IR identity defect:



A straightforward generalization of the discussion of IR bulk fields shows that IR defect fields are lifted to bimodule morphisms of the respective UV lifted defects, which again due to the special properties of P are nothing but the defect fields of the UV lifts.

Fusion of IR defects. Because of $R \otimes R^\dagger \cong I_{\mathrm{IR}}$, the lift of fused IR defects is the fusion of the lifted defects:



This is a rather special property closely tied to the projection property of P.

Adjunction of IR defects. While fusion of defects in the IR lifts to fusion in the UV, adjunction is not compatible with the lift from IR to UV. If for instance, an IR defect D_{IR} is lifted to a defect $D_{\mathrm{UV}} = R^\dagger \otimes D_{\mathrm{IR}} \otimes R$ in the IR, then the right adjoint of the latter in the UV theory is given by $D_{\mathrm{UV}}^\dagger = R^\dagger \otimes D_{\mathrm{IR}}^\dagger \otimes R^{\dagger\dagger}$, which in general does not coincide with the lift $R^\dagger \otimes D_{\mathrm{IR}}^\dagger \otimes R$ of the right adjoint of D_{IR} to the UV theory. However, the two are related: Selfadjointness of the IR identity defect yields $R^{\dagger\dagger} \otimes R^\dagger \cong I_{\mathrm{IR}}$, and hence the UV lift of the adjoint can be expressed as $R^\dagger \otimes D_{\mathrm{IR}}^\dagger \otimes R \cong$

[2]Indeed, this also holds if one chooses to construct the network using $P' = {}^\dagger R \otimes R$ instead of $P = R^\dagger \otimes R$. In that case however $B_{UV} = B_{IR} \otimes {}^\dagger R$ inherits a natural P'-comodule structure, which by means of the counit on P' also induces a P'-module structure on B_{UV}.

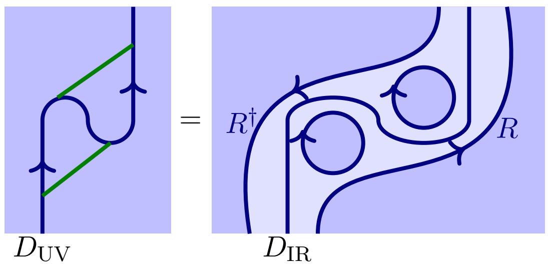
$D^\dagger_{\mathrm{UV}} \otimes P \cong P \otimes D^\dagger_{\mathrm{UV}} \otimes P$, leading to the notion of IR adjunction in the UV theory, which we denote by

$$D^{\dagger_P}_{\mathrm{UV}} := P \otimes D^\dagger_{\mathrm{UV}} \otimes P\,.$$

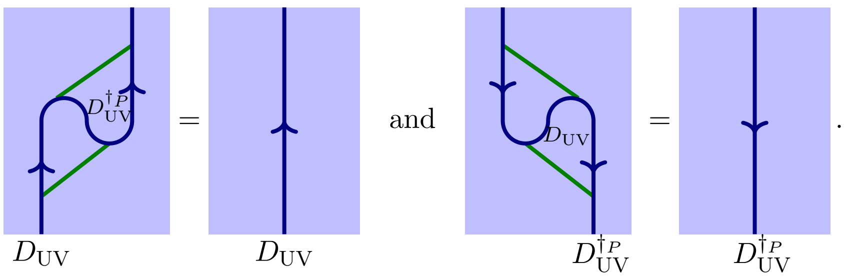
Similarly, the UV lift of a left adjoint defect is given by[3]

$${}^{\dagger_P}D_{\mathrm{UV}} = P \otimes {}^\dagger D_{\mathrm{UV}} \otimes P\,.$$

These formulas are very natural. After all, the defining relation of adjoints are the Zorro move identities (2.1), which involve the identity defect. Lifting these identities from the IR theory to the UV replaces the identity defect with the defect P:



For instance, lifting the IR Zorro move identities for the right adjoint to the UV results in the relations



It is easy to see that fusing the UV adjoint from both sides with P yields a defect which satisfies the P-Zorro move identities, c.f. appendix A.4.

A special case is P itself: Since it is the UV lift of the IR identity defect, which is selfadjoint, P is selfadjoint with respect to P-adjunction: $P \cong P^{\dagger_P} = P \otimes P^\dagger \otimes P$.

IR symmetries. Also symmetries of the IR theory can be easily described in the UV. As noted in [7] (see also [16]), symmetries of 2d field theories can be described by symmetry defects ${}_gI$ which describe the action of an element g of the symmetry group on any object in the field theory.

[3]The same formulas for left and right IR adjunction hold if one chooses to construct the network using the counital $P' = {}^\dagger R \otimes R$ instead of $P = R^\dagger \otimes R$.

These symmetry defects fuse according to the multiplication in the symmetry group:

$$ {}_gI \otimes {}_hI = {}_{g\cdot h}I $$

Now, IR symmetry defects lift to the UV as any other defect: ${}_gI \mapsto {}_gI_{\mathrm{UV}} = R^\dagger \otimes {}_gI \otimes R$. Since IR fusion lifts to UV fusion, the fusion of the lifted symmetry defects still respects the multiplication in the symmetry group, ${}_gI_{\mathrm{UV}} \otimes {}_hI_{\mathrm{UV}} = {}_{g\cdot h}I_{\mathrm{UV}}$. In that sense, the IR symmetry group is already present in the UV theory. However it is not realized as a symmetry group in the UV, since the lift of the IR identity defect, which is the symmetry defect associated to the neutral element of the symmetry group, does not lift to the identity defect, but rather to P. So the lifted symmetry defects are in general not invertible defects in the UV, but instead satisfy ${}_gI_{\mathrm{UV}} \otimes {}_{g^{-1}}I_{\mathrm{UV}} = P$.

IR projectors and subsequent flows. Projection defects

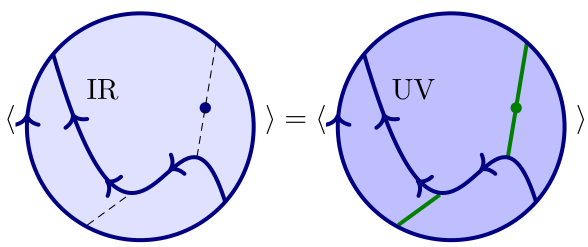
$$ P_2 = (R_2)^\dagger \otimes R_2 $$

in the IR theory associated to some RG flow from the IR theory to some theory IR_2 can also be lifted to the UV. The corresponding defects in the UV theory are given by

$$ \widetilde{P} = R^\dagger \otimes P_2 \otimes R = R^\dagger \otimes R_2^\dagger \otimes R_2 \otimes R = (R_2 \otimes R)^\dagger \otimes (R_2 \otimes R)\,. $$

These are precisely the projection defects built out of the RG defect $R_2 \otimes R$ associated to the concatenation of RG flows from the UV via IR to IR_2.

IR correlation functions. Having described how to realize IR objects inside the UV theory, it is straightforward to represent IR correlators in the UV theory: First, prepare the IR correlator by placing identity defects through all field insertions, in particular bulk fields and at defect cusps. Then replace all IR objects by the respective UV objects as described above. Importantly, this includes the IR identity defect, which has to be replaced by the UV projection defect P. The resulting UV correlator coincides with the original IR correlator.



2.4 Bulk RG flow as defect flow

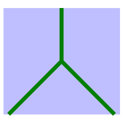
The previous discussion suggests a radically new view on bulk RG flow. Namely, that bulk perturbations of a 2d theory can be understood as a perturbation of a defect network in the fixed UV bulk theory. More precisely, insertion and expansion of UV islands in the perturbed theory confines the perturbation on ever smaller domains, which eventually become one-dimensional. Hence, perturbed correlation functions are nothing but UV correlation functions with networks of perturbed identity defects inserted. RG flow then does not change the bulk UV theory, but drives the identity defect in the UV to some projection defect P and defects (boundaries) to their IR images, c.f. (1.5).

The two-dimensional RG flow in the bulk can hence be reduced to a one-dimensional RG flow on the identity defect I_{UV}. Such defect flows are of course much easier to handle, because the underlying bulk theory does not change. For instance, UV bulk fields (I_{UV}-endomorphisms) and boundaries (I_{UV}-modules) flow to bulk fields and boundaries in the UV theory, which are compatible with P, i.e. to P-bimodule morphisms of P and P-modules, respectively.

Thus, if one can get a handle on perturbations of the identity defect in a given TQFT, the structures (bulk space, boundaries, correlators, etc.) associated to the corresponding perturbed bulk theory can be easily extracted.

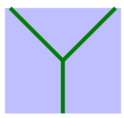
2.5 IR theories from projections

In the previous discussion, we represented correlation functions of a perturbed 2d TQFT as correlation functions of the unperturbed UV theory with a defect network inserted. While the starting point of the construction were RG defects R, the correlation functions of the perturbed theory only depended on the projection defect $P = R^\dagger \otimes R$. This suggests applying this method to arbitrary unital or counital projection defects P, which have the same properties as the defects associated to RG flows discussed in section 2.2: The projection property, $P \otimes P \cong P$ means that there are two junctions [4]



multiplication

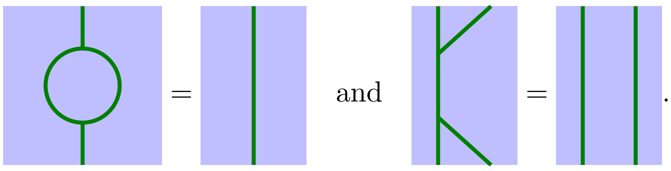
and



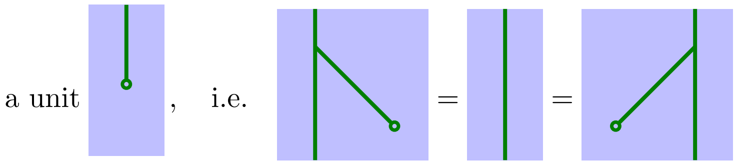
comultiplication

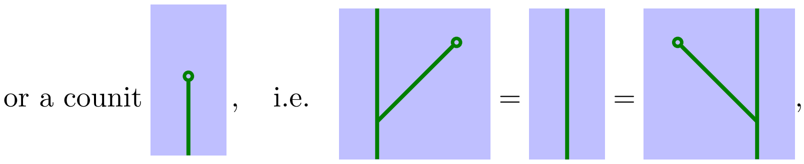
[4] As before, P is depicted in green, oriented from bottom to top.

satisfying the loop-omission (separability) and projection properties:



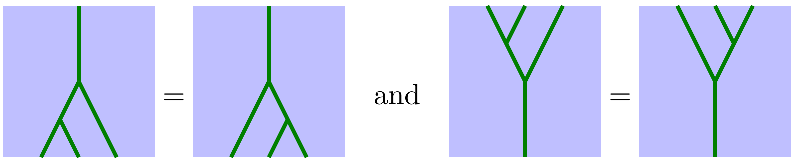
The junctions turn P into an algebra as well as a coalgebra. We require P to either have[5]

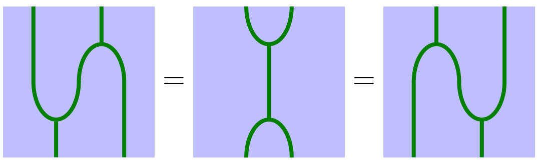




see also (1.6).

As is shown in appendix A.6, the existence of a unit for a projection defect implies coassociativity, while the existence of a counit implies associativity. In fact, for projection defects, associativity, coassociativity and the Frobenius identities





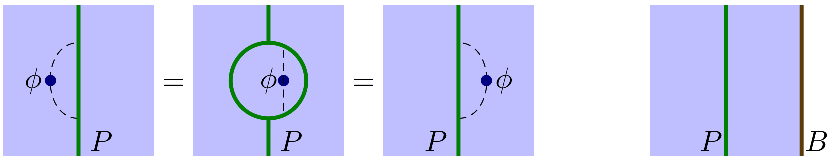
are all equivalent to one another, c.f. appendix A.6. Thus, unital or counital projection defects satsify all of them.

As in the context of RG defects discussed in section 2.3, replacing the identity defect I in a 2d TQFT by an arbitrary projection defect P, and inserting P-networks into the correlation functions one obtains correlation functions of a new, P-projected 2d TQFT. The relation between the projected and unprojected theories is exactly the same as the relation between IR and UV theories discussed in section 2.3.

[5]The special case in which P has a unit as well as a counit is discussed in appendix A.5.

Namely, given such a projection defect P, the P-projected theory can be described in terms of the unprojected theory as follows. While bulk fields of the unprojected theory can be regarded as endomorphisms (i.e. defect fields) of the identity defect I, bulk fields of the projected theory can be realized as endomorphisms of P in the unprojected theory. Boundary conditions of the projected theory can be represented as boundary conditions B of the unprojected theory, which are invariant under fusion with P, i.e. $B \cong P \otimes B$, and defect lines in the projected theory correspond to defect lines D in the unprojected theory, which are invariant under fusion with P from both sides, $D \otimes P \cong D \cong P \otimes D$.

To every object (such as fields, boundary conditions, defects) in the unprojected theory, one can associate a respective object of the projected theory by surrounding it with the defect P. For instance, bulk fields ϕ of the unprojected theory can be mapped to bulk fields of the projected theory by encircling their insertions with P, and fusion with P maps any boundary condition B of the unprojected theory to a P-invariant one representing a boundary condition of the projected theory:



(2.4)

Correlation functions of the projected theory can then be obtained as correlation functions of the unprojected theory with a network of the defect P inserted. As will be shown in the next chapter, counitality (unitality) of P ensures that P splits as $P \cong {}^{\dagger}R \otimes R$ ($P \cong R^{\dagger} \otimes R$), where R is a defect between the unprojected and projected theory and ${}^{\dagger}R$ ($R^{\dagger}$) is an adjoint, i.e. an opposite oriented version of it. Moreover, R has invertible right quantum dimension, which allows to apply the trick of diagram (1.4).

2.6 Factorization of projection defects

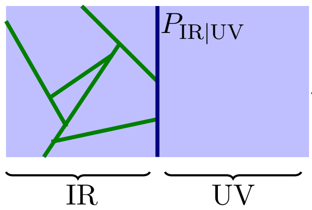
We now come full circle by showing that any (unital or counital) projection defect P factorizes as

$$\begin{aligned} P &= R^{\dagger} \otimes R \qquad \text{in case } P \text{ is unital} \\ P &= {}^{\dagger}R \otimes R \qquad \text{in case } P \text{ is counital}\,, \end{aligned}$$

where R is an RG type[6] defect between the P-projected theory on one side and the original unprojected theory on the other. By analogy to the case of RG flows, we call the original, unprojected theory UV and the P-projected theory IR.

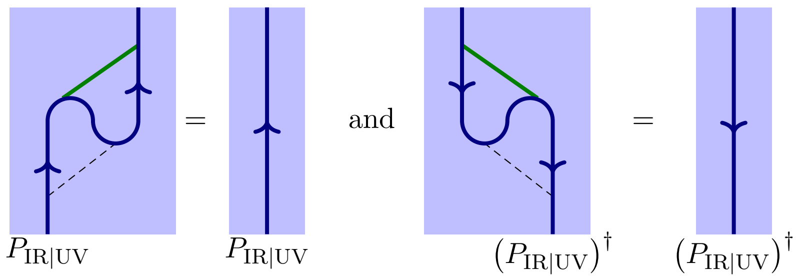
[6] $R \otimes R^{\dagger}$ is isomorphic to the identity defect

The basic idea is the following. As P is a (co)algebra, it can be viewed as a left and/or right (co)module over itself. Thus, the defect P can be regarded as a defect in the original (UV) theory (the defect P itself), a defect in the P-projected (IR) theory (the identity defect), or a defect separating one of those from the other. To indicate which of the interpretations we are referring to, we denote the respective defects as $P_{\mathrm{UV}|\mathrm{UV}}$, $P_{\mathrm{IR}|\mathrm{IR}}$, $P_{\mathrm{IR}|\mathrm{UV}}$ or $P_{\mathrm{UV}|\mathrm{IR}}$, respectively. For instance, viewed as a left P-(co)module and a right I_{UV}-(co)module P represents the defect $P_{\mathrm{IR}|\mathrm{UV}}$ between the P-projected (IR) theory and the original (UV) theory

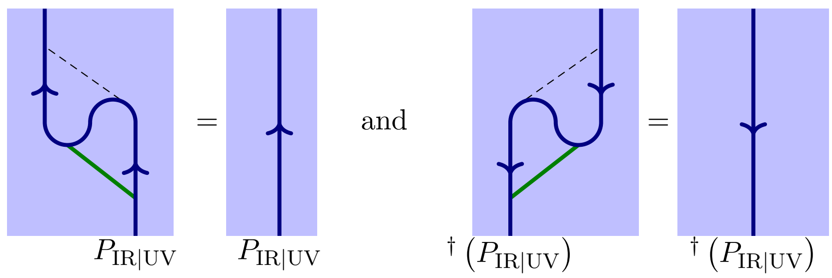


This defect plays the role of the RG defect R.

To show that it is indeed of RG type, we first need to determine its adjoints. We will restrict our discussion to the case that P is unital. (There is an analogous argument for the case of counital P.) Since $P_{\mathrm{IR}|\mathrm{UV}}$ is a defect between IR and UV theory, the adjoints have to satisfy mixed Zorro identities:



for the right adjoint and



for the left adjoint. Here, the defect P plays the role of the identity defect on the IR side of the defect. For unital P, comultiplication induces a coevaluation map $I_{\mathrm{UV}} \to P \to P \otimes P$, and, as is shown in appendix A.7

$$\begin{aligned} \left(P_{\mathrm{IR}|\mathrm{UV}}\right)^{\dagger} &= P_{\mathrm{UV}|\mathrm{IR}} \\ {}^{\dagger}\left(P_{\mathrm{IR}|\mathrm{UV}}\right) &= \left({}^{\dagger}P\right)_{\mathrm{UV}|\mathrm{IR}} \,. \end{aligned}$$

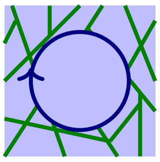
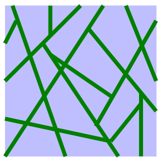
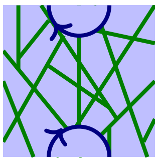
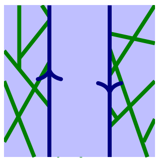
(${}^{\dagger}P$ denotes the left adjoint of P in the UV theory.) Now, since fusion over the IR theory is the same as fusion in the UV, it follows from the projection property of P that

$$P_{\mathrm{UV}|\mathrm{UV}} = P \cong P \otimes P = P_{\mathrm{UV}|\mathrm{IR}} \otimes P_{\mathrm{IR}|\mathrm{UV}} = (P_{\mathrm{IR}|\mathrm{UV}})^{\dagger} \otimes P_{\mathrm{IR}|\mathrm{UV}} \,.$$

Moreover, the identity defect in the IR theory is represented by P in the UV theory, and hence

$$I_{\mathrm{IR}} = P_{\mathrm{IR}|\mathrm{IR}} = P \cong P \otimes P = P_{\mathrm{IR}|\mathrm{UV}} \otimes P_{\mathrm{UV}|\mathrm{IR}} = P_{\mathrm{IR}|\mathrm{UV}} \otimes (P_{\mathrm{IR}|\mathrm{UV}})^{\dagger} \,.$$

Thus, any unital projection defect P factorizes as $P = R^{\dagger} \otimes R$, where $R = P_{\mathrm{IR}|\mathrm{UV}}$ has the property that $R \otimes R^{\dagger} \cong I_{\mathrm{IR}}$. Note that all the defects R, $R^{\dagger}$ and I_{IR} are represented by P in the UV theory, and the isomorphism $R \otimes R^{\dagger} \to I_{\mathrm{IR}}$ and its inverse are just given by the multiplication and comultiplication of P, respectively. The loop-omission and projection property of P then imply

 $=$  and  $=$ 

Similar considerations lead to an analogous factorization of counital projection defects P. The role of the RG defect is again played by $R = P_{\mathrm{IR}|\mathrm{UV}}$. But the adjoints differ from the unital case:

$$\begin{aligned} \left(P_{\mathrm{IR}|\mathrm{UV}}\right)^{\dagger} &= \left(P^{\dagger}\right)_{\mathrm{UV}|\mathrm{IR}} \\ {}^{\dagger}\left(P_{\mathrm{IR}|\mathrm{UV}}\right) &= P_{\mathrm{UV}|\mathrm{IR}}, \end{aligned}$$

which leads to slightly different factorizations

$$P_{\mathrm{UV}|\mathrm{UV}} = P \cong P \otimes P = P_{\mathrm{UV}|\mathrm{IR}} \otimes P_{\mathrm{IR}|\mathrm{UV}} = {}^{\dagger}(P_{\mathrm{IR}|\mathrm{UV}}) \otimes P_{\mathrm{IR}|\mathrm{UV}} \,,$$

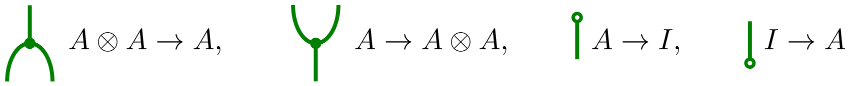
and

$$I_{\mathrm{IR}} = P_{\mathrm{IR}|\mathrm{IR}} = P \cong P \otimes P = P_{\mathrm{IR}|\mathrm{UV}} \otimes P_{\mathrm{UV}|\mathrm{IR}} = P_{\mathrm{IR}|\mathrm{UV}} \otimes {}^{\dagger}(P_{\mathrm{IR}|\mathrm{UV}}) \,.$$

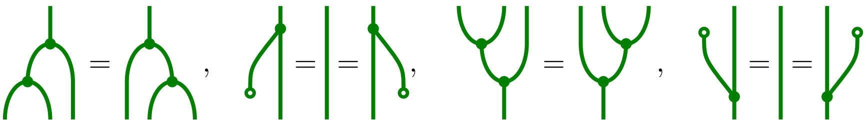
If P comes with both, a unit and a counit, it is self-adjoint ($P^{\dagger} \cong P \cong {}^{\dagger}P$, see appendix A.5), and the left and right adjoint of the induced RG defect R are isomorphic, $R^{\dagger} \cong {}^{\dagger}R$.

2.7 Review: Generalized orbifold theories

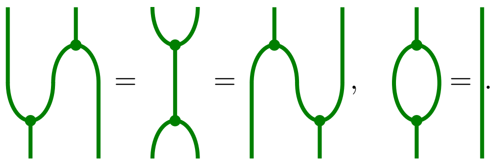
The generalized orbifold procedure [8, 17, 18, 19, 9, 20] is a method to produce a new theory out of a given 2d TQFT T by inserting networks of an endo-defect $A : T \to T$ into its correlation functions. These modified correlation functions are well-defined if the defect A satisfies the following special properties. It has to come with (co)multiplication and (co)unit fields



which turn A into a separable Frobenius algebra, i.e. it obeys the (co)associativity and (co)unit conditions

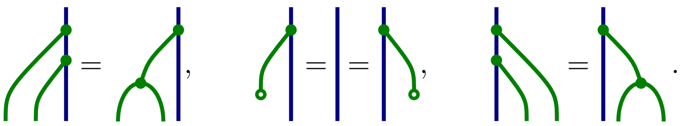


as well as the Frobenius and loop-omission properties:

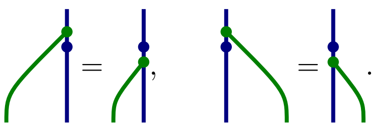


The respective orbifold theory is denoted by (T, A). An obvious example for a defect satsifying the above conditions is the identity defect $A = I$ in any TQFT. Orbifolding by I of course just gives back the original theory, $(T, I) \cong T$. In the following we will briefly outline how objects in the orbifold (T, A) are defined in terms of objects in A.

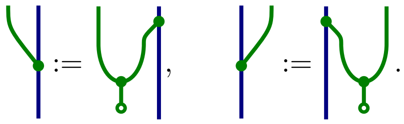
For any two TQFTs T and T' with defects A and A' as above an A-A'-bimodule D is a defect $D : T' \to T$ with junctions $A \otimes D \to D$, $D \otimes A' \to D$ such that



For two such bimodules D and $\tilde{D}$, $\mathrm{Hom}_{A,A'}(D, \tilde{D})$ denotes the space of all defect changing fields $D \to \tilde{D}$ commuting with the bimodule structure, i.e.



Via the unit, such modules are automatically also comodules, c.f. [9, eqn. (4.1)]:



With these notations at hand, one can now represent objects of the generalized orbifold theory (T, A) in terms of objects of T as follows:

i) Its invisible defect is A.

ii) Its bulk Hilbert space is $\mathrm{Hom}_{A,A}(A, A)$, the space of $A - A$-bimodule endomorphisms of A.

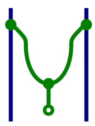
iii) Boundary conditions B of (T, A) are those boundary conditions B of T carrying an appropriate A-module structure.

iv) The space of boundary condition changing fields between boundary conditions B and $\tilde{B}$ is given by $\mathrm{Hom}_A(B, \tilde{B})$, the space of A-module morphisms from B to $\tilde{B}$.

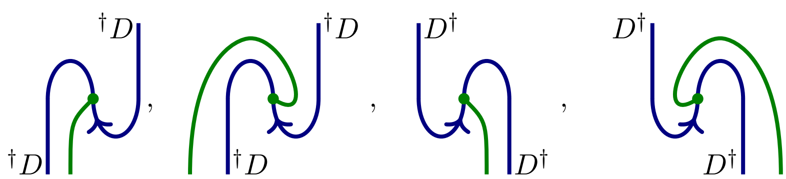
v) Defects D from (T', A') to (T, A) are A-A'-bimodules.

vi) The space of defect changing fields from defects D to $\tilde{D}$ is given by $\mathrm{Hom}_{A,A'}(D, \tilde{D})$, the space of $A - A'$-bimodule morphisms from D to $\tilde{D}$.

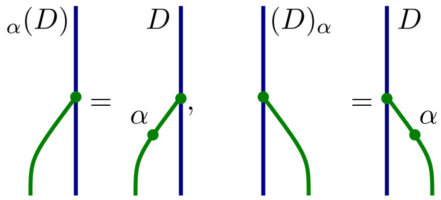
vii) The fusion product $D \otimes_A \tilde{D}$ in the orbifold theory (T, A) of two defects D and $\tilde{D}$ is given by the image of the fusion $D \otimes \tilde{D}$ in the unorbifolded theory T under



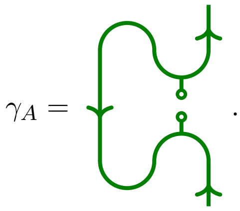
viii) The adjoints of defects D in the orbifold theory are defined in the following way in terms of the adjoints in the unorbifolded theory. The A-actions on any defect D in (T, A) induce actions on its non-orbifold adjoints ${}^\dagger D$ and $D^\dagger$:



The action of an algebra A on any module can be twisted by an algebra automorphism $\alpha : A \to A$. So for any defect D in the orbifold theory, one can define twisted defects ${}_\alpha(D)$ and $(D)_\alpha$ by twisting the left, respectively right A-action:



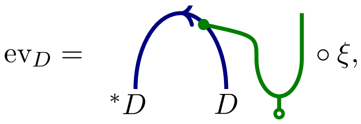
Left and right adjoints in the orbifold theory can be obtained by twisting the respective adjoints in the unorbifolded theory by the Nakayama automorphism

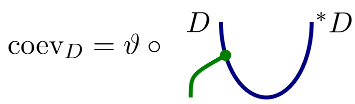


More precisely, the left and right adjoints in the orbifold theory[7] are given by [9, Prop. 4.7]

$$ {}^*D = {}_{\gamma_A^{-1}}\left({}^\dagger D\right), \qquad D^* = \left(D^\dagger\right)_{\gamma_{A'}}. \tag{2.5}$$

For ${}^*D$, the (co)evaluation maps are given by





with the inclusion and projections maps $\xi : {}^*D \otimes_A D \to {}^*D \otimes D$ and $\vartheta : D \otimes {}^*D \to D \otimes_A {}^*D$. There are similar formulas for the (co)evaluation maps for D^*.

In its mostly used form, the generalized orbifold procedure allows to describe defect and boundary spectra of orbifold theories in terms of the original non-orbifold version. In this thesis, this is done for the example of chapter 4 in appendix B. However, as shown in the next section this version of the generalized orbifold is inherently non-local and only the additional projection property allows to describe RG flow and the projection to subtheories.

[7] In this section we denote adjunction in the orbifold by '$*$' to distinguish it from the adjunction '$\dagger$' in the unorbidolded theory.

2.8 Relation to the generalized orbifold procedure

The method described in section 2.5 above to construct a new 2d TQFT by replacing the identity defect by a projection defect P is very close to and in fact inspired by the generalized orbifold procedure summarized in the previous section. The difference to our construction is the requirements imposed on A.

In the generalized orbifold construction the defect A has to be a separable Frobenius algebra[8]. This condition is very similar to the properties of projection defects with two differences: On the one hand the defect A does not have to satisfy the projection property, but is on the other hand required to have both, a unit and a counit, which we do not demand of projection defects. Moreover, it is often assumed in the generalized orbifold procedure that left and right adjoints of any defect D are isomorphic, i.e. $D^\dagger \cong {}^\dagger D$, so that further conditions such as pivotality and symmetry can be demanded (see e.g. [9]). We do not require such a condition, and in fact it is not met in the example of chapter 4.

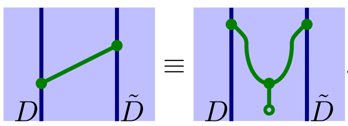
A projection defect P has both a unit and counit if and only if left and right adjoints of the respective RG defects are isomorphic

$$\left(P_{\mathrm{IR|UV}}\right)^\dagger \cong P_{\mathrm{UV|IR}} \cong {}^\dagger\left(P_{\mathrm{IR|UV}}\right) ,$$

c.f. appendix A.5. In that case P is a separable Frobenius algebra, and the construction described in section 2.6 is a special case of the generalized orbifold construction.

The projection property of P brings about interesting new phenomena in the generalized orbifold construction, which we will spell out in the remainder of this section.

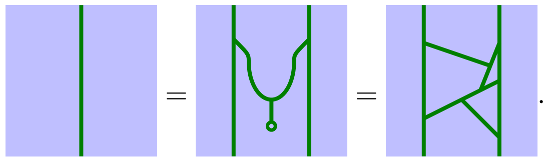
Let A be a separable Frobenius algebra in a given 2d theory. We will represent it by green line segments in diagrams. Defects in the generalized orbifold theory defined by A are given by defects in the underlying 2d theory, which are A-(bi)modules. Let D and $\tilde{D}$ be two such (bi)modules. Their fusion in the generalized orbifold theory is given by their tensor product $D \otimes_A \tilde{D}$ over the algebra A, pictorially



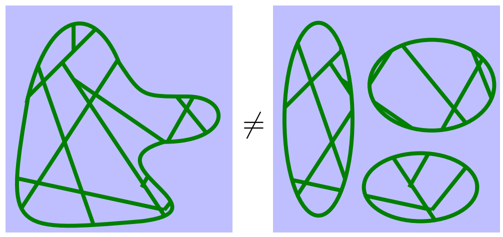
In general, it is different from the fusion $D \otimes \tilde{D}$ in the underlying unorbifolded theory.

[8]a unital, counital, associative, coassociative algebra and coalgebra satisfying loop-omission and Frobenius properties

Indeed, similarly to projection defects, also separable Frobenius algebras always factorize into defects between the orbifold and the underlying unorbifolded theory and their adjoints. Namely, considered as a left A- and right I-module[9], A represents a defect R between orbifolded and unorbifolded theory. Considered as right A- and left I-module it represents the adjoint defect $R^\dagger \cong {}^\dagger R$. Now, for any separable Frobenius algebra we have $A \otimes_A A \cong A$, or pictorially

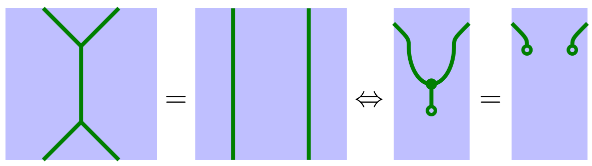


Hence, A as a defect in the unorbifolded theory factorizes as $A \cong R^\dagger \otimes_A R$. However, for generic A the defect R is not of RG type, i.e. $R \otimes R^\dagger$ is not isomorphic to the identity defect in the orbifold theory. Hence, bubbles of a generalized orbifold theory inserted in the unorbifolded theory do not in general connect trivially:

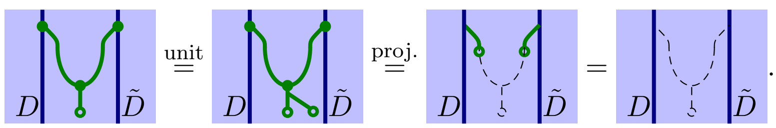


Instead, pushing two bubbles of the generalized orbifold against each other creates a non-trivial defect at the interface of the two bubbles. Thus, the generalized orbifold cannot be obtained by a local perturbation of the original theory. This is only true if A additionally satisfies the projection property.

In that case, fusion in the generalized orbifold simplifies dramatically – it reduces to fusion in the unorbifolded theory. Namely, for a separable Frobenius algebra the projection property can be rephrased as



leading to the following simplification for defect fusion in the orbifold theory:



[9] I is the identity defect of the underlying unorbifolded theory.

Our construction can be viewed as a generalization of orbifold equivalence [21, 22]. Two TQFTs are said to be orbifold equivalent if they are separated by a defect whose left and right quantum dimensions are both invertible.

Chapter 3

TQFTs from supersymmetric Landau-Ginzburg theories

In order to apply the previously developed theory, this chapter reviews a large class of two-dimensional topological field theories. First, 2d $\mathcal{N} = (2, 2)$ supersymmetric LG models are summarized concisely. Second, it is reviewed how these models can be twisted to obtain a topological theory. Next, supersymmetric boundaries and defects surviving the topological twist are introduced. Finally, LG orbifold models and their defect spectra are discussed. The following two review chapters follow [23] unless indicated otherwise.

3.1 2d $\mathcal{N} = (2, 2)$ supersymmetry

Consider a two-dimensional plane parametrized by time x^0 and space x^1 with metric $\eta_{00} = -1, \eta_{11} = 1$. The Poincaré algebra is then generated by the Hamiltonian H corresponding to time translations ∂_0, momentum P corresponding to space translations ∂_1 and Lorentz boost M corresponding to $x^0\partial_1 + x^1\partial_0$. Here, $\partial_i := \partial/\partial x^i$. The Poincaré relations are given by

$$\begin{aligned}
[H, P] &= 0 \\
i[M, H] &= -\eta_{00} P \\
i[M, P] &= -\eta_{11} H
\end{aligned}$$

An $\mathcal{N} = (2, 2)$ supersymmetric extension of this algebra by complex fermionic generators $Q_\pm$, $\bar{Q}_\pm = (Q_\pm)^*$ can be constructed as follows. The (anti-)commutators are given by

$$\begin{aligned}
Q_+^2 = Q_-^2 = \bar{Q}_+^2 = \bar{Q}_-^2 &= 0 \\
\{Q_\pm, \bar{Q}_\pm\} &= H \pm P \\
[iM, Q_\pm] = \mp Q_\pm, \quad [iM, \bar{Q}_\pm] &= \mp \bar{Q}_\pm
\end{aligned} \tag{3.1}$$

with all others vanishing. There shall also be an axial R-symmetry $U(1)_A$ and a vector R-symmetry $U(1)_V$ whose respective generators F_A and F_V obey the commutation relations[1]

$$\begin{aligned}[iF_V, Q_\pm] = -iQ_\pm, [iF_V, \bar{Q}_\pm] = i\bar{Q}_\pm \\ [iF_A, Q_\pm] = \mp iQ_\pm, [iF_A, \bar{Q}_\pm] = \pm i\bar{Q}_\pm\end{aligned} \tag{3.2}$$

The corresponding representations of the $\mathcal{N} = (2,2)$ supersymmetry algebra are called supermultiplets and can conveniently be described by formally adding fermionic anti-commuting variables $\theta^\pm$, $\bar{\theta}^\pm := (\theta^\pm)^*$ to the bosonic spacetime variables $x^\pm := x^0 \pm x^1$. In terms of this *superspace*, the supercharges become

$$\begin{aligned}Q_\pm &= \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm\partial_\pm \\ \bar{Q}_\pm &= -\frac{\partial}{\partial\bar{\theta}^\pm} - i\theta^\pm\partial_\pm\end{aligned}$$

where $\partial_\pm = \partial_0 \pm \partial_1$. Supermultiplets can then be written as functions of superspace. Important superfields are the *chiral* superfield X (also called (c,c)) and the *twisted* chiral superfield U (also called (a,c)). They admit the component decompositions

$$\begin{aligned}X(x^\mu, \theta^\pm, \bar{\theta}^\pm) = \phi(y^\pm) + \theta^+\psi_+(y^\pm) + \theta^-\psi_-(y^\pm) + \theta^+\theta^- F(y^\pm) + ... \\ U(x^\mu, \theta^\pm, \bar{\theta}^\pm) = v(\tilde{y}^\pm) + \theta^+\bar{\chi}_+(\tilde{y}^\pm) + \bar{\theta}^-\chi_-(\tilde{y}^\pm) + \theta^+\bar{\theta}^- E(\tilde{y}^\pm) + ...\end{aligned} \tag{3.3}$$

where $y^\pm = x^\pm - i\theta^\pm\bar{\theta}^\pm$ and $\tilde{y}^\pm = x^\pm \mp i\theta^\pm\bar{\theta}^\pm$.[2] Here, F and E are auxiliary fields and ... involves only derivatives in ϕ, ψ and v, χ, respectively. The complex conjugates $\bar{X}$ and $\bar{U}$ are called anti-chiral (or (a,a)) and twisted anti-chiral (or (c,a)), respectively. Sums and products of (anti-)chiral superfields are again (anti-)chiral superfields.

Axial and vector R-rotations act on a superfield with axial R-charge q_A and vector R-charge q_V as

$$\begin{aligned}e^{i\alpha F_V} &: \mathcal{F}(x^\mu, \theta^\pm, \bar{\theta}^\pm) \mapsto e^{i\alpha q_V}\mathcal{F}(x^\mu, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar{\theta}^\pm) \\ e^{i\beta F_A} &: \mathcal{F}(x^\mu, \theta^\pm, \bar{\theta}^\pm) \mapsto e^{i\beta q_A}\mathcal{F}(x^\mu, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\bar{\theta}^\pm).\end{aligned}$$

Actions invariant under the supersymmetry variation

$$\delta_{\epsilon,\bar{\epsilon}} = \epsilon_+ Q_- - \epsilon_- Q_+ - \bar{\epsilon}_+\bar{Q}_- + \bar{\epsilon}_-\bar{Q}_+$$

[1] If one or both of these R-symmetries is broken, the $\mathcal{N} = (2,2)$ supersymmetry algebra can be extended by central charges Z, $\tilde{Z}$. However, most examples discussed in this thesis have unbroken R-symmetry.

[2] In the literature, these superfields are usually defined to be general superfields which vanish upon action with differential operators $D_\pm$ or $\overline{D}_\pm$ anti-commuting with the supercharges. This allows to easily prove the invariance of action functionals under supersymmetry variations.

can now be built out of the invariant D-term, F-term and twisted F-term

$$
\begin{aligned}
S_D &= \int \mathrm{d}^2x\mathrm{d}^4\theta K(\mathcal{F}_i, \bar{\mathcal{F}}_i) \\
S_W &= \int \mathrm{d}^2x\mathrm{d}\theta^-\mathrm{d}\theta^+ W(X_j)\big|_{\bar\theta^\pm=0} + \int \mathrm{d}^2x\mathrm{d}\bar\theta^+\mathrm{d}\bar\theta^- \overline{W}(\overline{X}_j)\big|_{\theta^\pm=0} \\
S_{\widetilde{W}} &= \int \mathrm{d}^2x\mathrm{d}\bar\theta^-\mathrm{d}\theta^+ \widetilde{W}(U_k)\Big|_{\bar\theta^+=\theta^-=0} + \int \mathrm{d}^2x\mathrm{d}\bar\theta^+\mathrm{d}\theta^- \overline{\widetilde{W}}(\overline{U}_k)\Big|_{\theta^+=\bar\theta^-=0}
\end{aligned}
$$

where K is any differentiable function of superfields $\mathcal{F}_i$, W is a holomorphic function of chiral superfields X_j and $\widetilde{W}$ is a holomorphic function of twisted chiral superfields U_k.

As a first concrete model, consider a Lagrangian of pure D-type with chiral superfields $X_1, ..., X_n$ [23, 2.22]:[3]

$$
\begin{aligned}
\mathcal{L}_{\text{kin}} &= \int \mathrm{d}^4\theta K\left(X_k, \bar{X}_k\right) \\
&= -g_{i\bar{j}}\partial^\mu\phi_i\partial_\mu\bar\phi_{\bar{j}} + ig_{i\bar{j}}\bar\psi_-^{\bar{j}}\left(D_0 + D_1\right)\psi_-^i \\
&\quad + ig_{i\bar{j}}\bar\psi_+^{\bar{j}}\left(D_0 - D_1\right)\psi_+^i + R_{i\bar{k}j\bar{l}}\psi_+^i\psi_-^j\bar\psi_-^{\bar{k}}\bar\psi_+^{\bar{l}}
\end{aligned}
$$

Here, the final equality holds up to total derivatives in x^μ, the auxiliary fields have been eliminated and

$$
\begin{aligned}
g_{i\bar{j}} &:= \partial_i\partial_{\bar{j}}K\left(X_k, \overline{X}_k\right), \\
D_\mu\psi_\pm^k &:= \partial_\mu\psi_\pm^i + \partial_\mu\psi^j\Gamma^i_{jk}\psi_\pm^k
\end{aligned}
$$

in the component decomposition of the Lagrangian and Γ^i_{jk} and $R_{i\bar{j}k\bar{l}}$ are expressions in $g_{i\bar{j}}$.

If we assume that $g_{i\bar{j}}$ is a positive definite matrix, then $\{\phi_i\}$ describe the local coordinates of an n-dimensional Kähler manifold M with Kähler metric induced by $g_{i\bar{j}}$, Levi-Civita connection Γ^i_{jk} and Riemannian curvature $R_{i\bar{j}k\bar{l}}$. Because the action is invariant under Kähler transformations

$$
K\left(X_k, \bar{X}_k\right) \to K\left(X_k, \bar{X}_k\right) + f\left(X_k\right) + \bar{f}\left(\bar{X}_k\right)
$$

for holomorphic functions $f\left(X_k\right)$, this patch-wise construction extends to an action for a map from the worldsheet to any Kähler manifold M of arbitrary topology:

$$
\phi : \Sigma \to M
$$

This model is called *supersymmetric non-linear sigma model (NLSM) on the Kähler manifold M with metric g*. M is called *Calabi-Yau* (CY) if its first Chern class vanishes $c_1(M) = 0$.

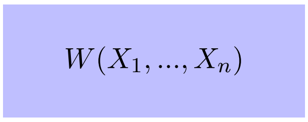
[3] For ease of readibility, the fermionic components $\psi_\pm^k$ of the superfield X_k carries a superscript.

Building on top of this model, one can add an F-term S_W for a holomorphic function W on M. This model is called *Landau-Ginzburg (LG) model on M with superpotential* W. If W is *quasi-homogeneous*, i.e.

$$W(\lambda^{q_i} X_i) = \lambda W(X_i)$$

for the scaling dimensions q_i of X_i, then the F-term is invariant under changes of scale $z \to \lambda z, \mathrm{d}\theta \to \lambda^{-\frac{1}{2}}\mathrm{d}\theta$. In fact, one can prove [24, p. 331ff.] that supersymmetry protects such superpotentials from receiving any renormalization terms upon scale variations. One can also show that for quasi-homogeneous superpotentials W both R-charges $U(1)_A$ and $U(1)_V$ are preserved classically and at the quantum level by assigning vector R-charge $2q_i$ to X_i and axial R-charge 0 to all chiral fields [23]. The D-term contains only irrelevant operators and all relevant operators are included in the F-term. Hence, the fixed point of the renormalization group flow is dictated only by the superpotential [25].

After this section, the models considered will be conformal fixed points of LG (orbifold) models with quasi-homogeneous superpotentials, which will also be referred to as LG models by abuse of language. Because they are completely specified by their chiral superfields $X_1, \ldots, X_n$ and their superpotential $W(X_1, \ldots, X_n)$, they will be depicted in spacetime diagrams as



For a proper treatment of the full $\mathcal{N} = 2$ superconformal algebra see [26, 27].

Superpotentials W which are not quasi-homogeneous exhibit *effective flow* [24, p. 336f.]. This means that W changes upon a redefinition of the fields. An important example in this thesis is the theory with a single chiral field X and superpotential

$$W(X) = X^d + X^k$$

with $d > k$. Under $W \to \lambda W$ one can redefine $X \to \lambda^{-\frac{1}{d}} X$ such that scale changes leave the higher dimension operator X^n invariant

$$W(X) \to X^n + \lambda^{\frac{n-k}{n}} X^k.$$

Thus, the superpotential effectively flows to $W = X^k$ in the IR ($\lambda \to 0$) and to $W = X^n$ in the UV ($\lambda \to \infty$). In other words, if one perturbes the CFT described by the superpotential X^d by adding the relevant term X^k to the action, the conformal invariance is broken and the theory flows to the CFT described by the LG model with superpotential X^k.

3.2 Topological twist

Instead of calculating in the full field theory, one can alter the theory slightly and restrict to a topological subsector which contains important information about the full theory.

Chiral ring. Setting

$$Q_B := \bar{Q}_+ + \bar{Q}_-,$$

a *chiral operator* is defined to be an operator $\mathcal{O}$ commuting with Q_B

$$[Q_B, \mathcal{O}] = 0.$$

A particular example of such an operator is the lowest component ϕ of a chiral superfield X (3.3). One can show that the spacetime translation of a chiral operator is Q_B-exact [24, p. 398]

$$\frac{i}{2}\partial_\pm \mathcal{O} = \{Q_B, [Q_\pm, \mathcal{O}]\}$$

and hence the Q_B cohomology class of such an operator is invariant under worldsheet translations. In particular, two chiral operators can be moved to space-like separation and hence they commute up to Q_B-exact terms. Because the product of two chiral operators is again a chiral operator, the full set of such operators forms a commutative ring called *chiral ring*. Analogous statements hold for the operator $Q_A := \bar{Q}_+ + Q_-$, which gives rise to the *twisted chiral ring* [28].

One can show [24, p. 306] that the supersymmetric ground states are in one-to-one correspondence with the elements of the chiral ring and for LG models also with the critical points of the superpotential. In particular, for a Landau-Ginzburg model with chiral fields $X_1, ..., X_n$ and superpotential W, the chiral ring is given by [25, 29, 30]

$$\mathbb{C}[X_1, ..., X_n] / \langle \partial_{X_1} W, ..., \partial_{X_n} W \rangle .$$

Topological B-twist. Supersymmetric Landau-Ginzburg models with unbroken R-symmetry can be turned into a topological theory. To see this, one first Wick rotates the time coordinate $x^0 = -ix^2$ such that the worldsheet metric becomes Euclidean. In particular, the Lorentz group $\mathrm{SO}(1,1)$ turns into the Euclidean rotation group $SO(2)_E = U(1)_E$. The supersymmetry algebra (3.1), (3.2) remains largely unaffected, one simply replaces M by $-iM_E$.

Twisting the theory means to regard the diagonal subgroup of $U(1)_E \times U(1)_A$ as the Euclidean rotation group. In other words, the generator $M_E + F_A$ is seen as the rotation generator. This of course alters the matter content's spin. For example, the scalar component ϕ of a chiral multiplet (3.3) with axial R-charge 0 is charged trivially under $U(1)_E$ and therefore remains a scalar in the topological theory. The following table indicates the transformation behavior of the component fields of a (twisted) chiral superfield and the supercharges before and after the twist.[4]

	ϕ	ψ_+	ψ_-	$\bar{\psi}_+$	$\bar{\psi}_-$	Q_+	Q_-	$\bar{Q}_+$	$\bar{Q}_-$
$U(1)_E$	0	-1	1	-1	+1	-1	+1	-1	+1
$U(1)_A$	0	-1	1	+1	-1	-1	+1	+1	-1
$U(1)_{E'}$	0	-2	2	0	0	-2	+2	0	0

In particular, the supercharge $Q_B = \bar{Q}_+ + \bar{Q}_-$ is a scalar in the B-twisted theory. Note that twisting only makes sense when the axial R-charges are integral.

At the same time, the energy-momentum tensor also changes. For all models of interest in this thesis, it is Q_B-exact, i.e.

$$T_{\mu\nu}^{\text{twisted}} = \{Q_B, G_{\mu\nu}\}$$

for some fermionic symmetric tensor $G_{\mu\nu}$ [24]. This implies that correlation functions of Q_B-closed operators are independent of variations of the worldsheet metric h:

$$\delta_h \langle \mathcal{O}_1 ... \mathcal{O}_s \rangle = \langle \frac{1}{4\pi} \int \sqrt{h} \mathrm{d}^2 x \delta h^{\mu\nu} T_{\mu\nu}^{\text{twisted}} \mathcal{O}_1 ... \mathcal{O}_s \rangle = 0$$

Therefore, one restricts the operator content of the *B-twisted topological Landau-Ginzburg model* to Q_B-cohomology elements, i.e. the ground states of the original theory. Then, the twisted theory is a topological theory [24, p. 404].

Mirror symmetry. The supersymmetry algebra (3.1), (3.2) is invariant under the outer automorphism

$$Q_- \longleftrightarrow \overline{Q}_- \qquad F_V \longleftrightarrow F_A.$$

Mirror symmetry describes the equivalence of two $\mathcal{N} = (2,2)$ supersymmetric field theories whose symmetry generators are exchanged in this way. This symmetry extends to theories with broken R-symmetries.

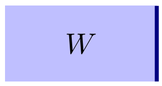
Instead of the axial R-symmetry $U(1)_A$ one can also use the vector R-symmetry $U(1)_V$ to twist to a topological *A-twisted* theory. Mirror symmetry then relates the B-twist of one $\mathcal{N} = (2,2)$ supersymmetric theory to the A-twist of a second $\mathcal{N} = (2,2)$ supersymmetric theory.

[4] This table is an adapted version of the tables in [24, p. 401f.].

3.3 B-type boundaries and defects

In order to apply the theory of chapter 2 to topologically B-twisted LG models, a description for defects in these models is needed. In the following, supersymmetric defects and boundaries in LG models which become topologial defects in the twisted theory are reviewed. The mathematical formulation in terms of matrix factorizations was first suggested by Kontsevic and discussed in [31, 32, 33, 16, 34]. The first part of this chapter closely follows [16] and the review [35].

Boundaries. On a worldsheet with boundary



the LG bulk action is a priori not invariant under B-type supersymmetry variations. The breaking of this symmetry is due to the *Warner term* [36] coming from

$$Q_B \cdot \int_\Sigma \mathrm{d}^2x \mathrm{d}^2\theta \, W(X_i) = \int_\Sigma \mathrm{d}^2x \mathrm{d}\theta^+ \mathrm{d}\theta^- \, (\theta^+\partial_+ + \theta^-\partial_-) W(X_i)$$
$$= \int_{\partial\Sigma} \mathrm{d}x \mathrm{d}\theta \, W\left(X_i|_{\partial\Sigma}\right).$$

This variation can be cancelled by introducing $d = 1, \mathcal{N} = 2$ boundary superfields $\Pi_a = \pi_a + \theta l_a$ consisting of auxiliary fields l_a and fermions π_a satisfing a Clifford algebra $\{\pi_a, \overline{\pi}_b\} = \delta_{ab}$. These superfields obey[5]

$$\overline{D}\Pi_a = E_a\left(X_i|_{\partial\Sigma}\right)$$

for some polynomials E_a. One can now add a boundary superpotential

$$S_{\partial\Sigma} = \int_{\partial\Sigma} \mathrm{d}x \mathrm{d}\theta \sum_a \Pi_a J_a \left(X_i|_{\partial\Sigma}\right)$$

whose variation exactly cancels the Warner term if

$$\sum_a J_a E_a = W. \tag{3.4}$$

Letting $a = 1, ..., k$, the fermions can be written as generalized $2^k \times 2^k$-dimensional Pauli matrices and the boundary BRST operator takes the form

$$Q = \sum_{a=1}^{k} (\pi_a J_a + \overline{\pi}_a E_a) = \begin{pmatrix} 0 & \mathrm{d}_{B1} \\ \mathrm{d}_{B0} & 0 \end{pmatrix}.$$

[5] The differential operator $\overline{D}$ is a one-dimensional analogue of the differential operators in two dimensions which appeared in the footnote around equation (3.3).

In particular, condition (3.4) can be recast into $Q^2 = \mathrm{d}_{B0} \cdot \mathrm{d}_{B1} = \mathrm{d}_{B1} \cdot \mathrm{d}_{B0} = W \cdot \mathbb{1}$. Any such decomposition of the superpotential in terms of square matrices $\mathrm{d}_{B0}, \mathrm{d}_{B1} \in \mathrm{Mat}\,(n, n; \mathbb{C}[X_i])$ gives a boundary condition for the LG model. In the derivation just reviewed n must be a power of 2 for the matrices d_{Bi} to describe a spinor representation. However, by allowing more general boundary contributions it can be shown [37] that any $n \in \mathbb{N}_{>0}$ leads to a valid boundary condition.

Rephrased mathematically[6], every B-type boundary is determined by a $\mathbb{Z}_2$-graded free module $B = B_0 \oplus B_1$ over the polynomial ring $S := \mathbb{C}[X_1, ..., X_n]$, with an odd endomorphism $d_B : B \to B$, which squares to W times the identity map, i.e. $d_B^2 = W\,\mathrm{id}_B$, and B_0 and B_1 must be of the same rank. One often unfolds such matrix factorizations into 2-periodic complexes

$$B : B_1 \underset{\mathrm{d}_{B0}}{\overset{\mathrm{d}_{B1}}{\rightleftarrows}} B_0\,, \qquad \mathrm{d}_B = \begin{pmatrix} 0 & \mathrm{d}_{B1} \\ \mathrm{d}_{B0} & 0 \end{pmatrix}$$

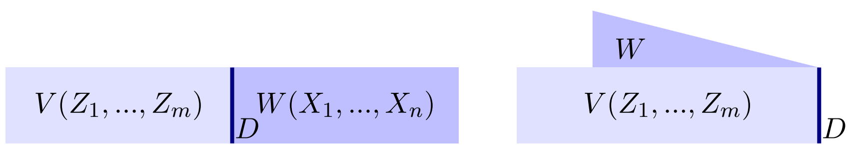
twisted by W: $\mathrm{d}_{B1} \circ \mathrm{d}_{B0} = W \cdot \mathrm{id}_{B_0}$ and $\mathrm{d}_{B0} \circ \mathrm{d}_{B1} = W \cdot \mathrm{id}_{B_1}$.

The space of boundary condition-changing fields $\mathrm{Hom}(B, B')$ between two boundaries represented by matrix factorizations B, B' of W is given by the homology of the induced $\mathbb{Z}_2$-graded complex on the space of homomorphisms $\mathrm{Hom}_S(B, B')$ of the respective S-modules[7]. More precisely,

$$\begin{aligned} &\mathrm{Hom}(B, B') = H^*_{\mathrm{d}}(\mathrm{Hom}_S(B, B'))\,, \\ \text{with differential} \quad &\mathrm{d}\phi = \mathrm{d}_{B'} \circ \phi - (-1)^{\deg}\phi \circ \mathrm{d}_B\,, \end{aligned} \tag{3.5}$$

for $\phi \in \mathrm{Hom}_S(B, B')$. Here deg denotes the $\mathbb{Z}_2$-degree. The space of boundary-changing fields is $\mathbb{Z}_2$-graded with even and odd elements corresponding to bosons and fermions, respectively. The operator product of boundary-changing fields is just the composition of homomorphisms.

Defects. The above construction can be applied to defects by considering them as boundaries of the product theory (folding trick). Namely, such a defect D



between two LG models specified by superpotentials $V \in \mathbb{C}[Z_1, ..., Z_m]$ and $W \in \mathbb{C}[X_1, ..., X_n]$ is described by a matrix factorization of $V - W$ over the ring $\mathbb{C}[Z_1, ..., Z_m, X_1, ..., X_n]$.

[6]The remainder of this paragraph is an adapted version of the review in [1].

[7]Note that the Hom-complex is untwisted!

After twisting, two defects can be brought together closely giving a new defect. Such defect fusion is described by the tensor product of matrix factorizations [16]. Namely, let $U \in \mathbb{C}[X_1, \ldots, X_m]$, $V \in \mathbb{C}[Y_1, \ldots, Y_n]$, $W \in \mathbb{C}[Z_1, \ldots, Z_o]$ and $D : W \to V$ and $D' : V \to U$ be matrix factorizations of $V - W$ and $U - V$, respectively. Then the fused defect is given by the tensor product $D' \otimes D$ of matrix factorizations. This is the matrix factorization built on the $\mathbb{Z}_2$-graded $\mathbb{C}[X_1, \ldots, X_m, Z_1, \ldots, Z_o]$-module $D' \otimes_{\mathbb{C}[Y_1,\ldots,Y_n]} D$ with homomorphism

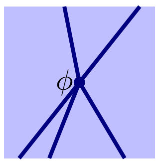
$$\mathrm{d}_{D'\otimes D} = \mathrm{d}_{D'} \otimes \mathrm{id}_D + \mathrm{id}_{D'} \otimes \mathrm{d}_D \,. \tag{3.6}$$

This differential is to be understood with Koszul signs

$$(\mathrm{id}_{D'} \otimes \mathrm{d}_D)(\nu \otimes \omega) = (-1)^{\deg}(\nu) \otimes \mathrm{d}_D(\omega).$$

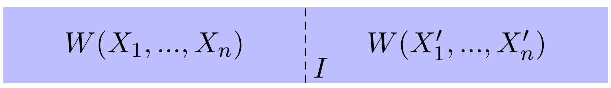
Since the factorized polynomials add upon taking the tensor product, this is indeed a matrix factorization of $(U - V) + (V - W) = U - W$, i.e. $D' \otimes D : W \to U$.[8]

Mathematically, defect-changing fields take the same form as their boundary equivalent. However, multiple defects can be joint together at a junction, i.e. a defect-changing field $\phi \in \mathrm{Hom}\,(D_1 \otimes ... \otimes D_l, D_{l+1} \otimes ... \otimes D_{l+N})$ between tensor products of defects.



Note that boundary conditions can be considered a special case of defects, namely those with a trivial theory on one side. The trivial LG theory is the theory with no chiral fields and zero superpotential. Right (left) B-type boundary conditions of a Landau-Ginzburg theory with superpotential W can therefore be described by matrix factorizations of W $(-W)$.

Identity defect. The invisible or identity defect I has been determined in [15], see also [39]. If the variables to the left of I are called X_i and the ones to the right carry an additional prime,



[8] A priori, such tensor product matrix factorizations are of infinite rank. It can be shown however, that tensor products of finite-rank matrix factorizations are isomorphic to finite-rank matrix factorizations [16]. See also [38].

then the module of I is given by the exterior algebra

$$\Lambda\left(\bigoplus_{i=1}^{n}\mathbb{C}[X_1,...,X_n,X_1',...,X_n']\cdot\theta_i\right)$$

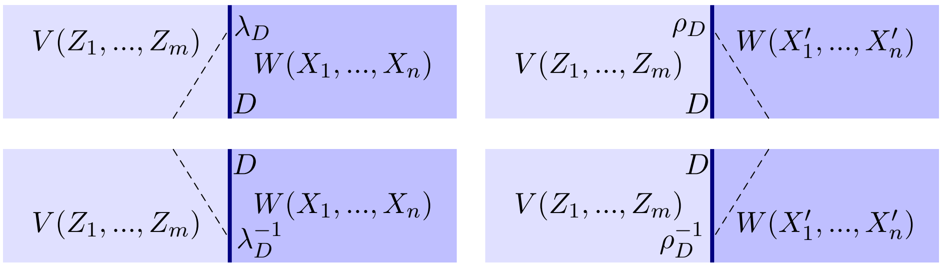
based on n anticommuting variables θ_i. Defining the polynomial

$$\partial_i^{X,X'}W := \frac{W(X_1',...,X_{i-1}',X_i,...,X_n)-W(X_1',...,X_i',X_{i+1},...,X_n)}{X_i-X_i'},$$

the twisted differential on I is

$$\mathrm{d}_{I_W}=\sum_{i=1}^{n}\left[(X_i-X_i')\cdot\theta_i^*+\partial_i^{X,X'}W\cdot\theta_i\right].$$

In [15, 39] also the left and right actions of I on a defect D have been determined.



Namely, λ_D first projects I_V to θ-degree zero and subsequently identifies $Z_i'=Z_i$ and similarly for ρ_D. Their inverses are given by

$$\lambda_D^{-1}(e_i)=\sum_{l\geq 0}\sum_{a_1<...<a_l}\sum_j\theta_{a_1}...\theta_{a_l}\left\{\partial_{a_l}^{Z,Z'}\mathrm{d}_D...\partial_{a_1}^{Z,Z'}\mathrm{d}_D\right\}_{ji}\otimes e_j$$

$$\rho_D^{-1}(e_i)=\sum_{l\geq 0}\sum_{a_1<...<a_l}\sum_j(-1)^{\binom{l}{2}+l|e_i|}e_j\otimes\left\{\partial_{a_1}^{X,X'}\mathrm{d}_D...\partial_{a_l}^{X,X'}\mathrm{d}_D\right\}_{ji}\theta_{a_1}...\theta_{a_l}$$

where e_i are the generators of $D_0\oplus D_1$.

Associated modules. The computation of defect fusion for non-trivial superpotentials can be simplified by using that matrix factorizations of W over a polynomial ring S are related to finitely generated modules over $\hat{S}:=S/(W)$ as explained in [16, 3]. Namely, if such a matrix factorization is given by the matrices $(\mathrm{d}_1,\mathrm{d}_0)$, one associates to it the $\hat{S}$-module

$$M_D=\mathrm{coker}(\mathrm{d}_1: D_1\otimes_S\hat{S}\to D_0\otimes_S\hat{S}).$$

Reversely, free resolutions of finitely generated modules over $\hat{S}:=S/(W)$ necessarily turn two-periodic after finitely many steps [40]. This two-periodic part gives a matrix factorization of W over S. For example, M_D has the following free two-periodic resolution.

$$\ldots \xrightarrow{\mathrm{d}_0} D_1 \otimes_S \hat{S} \xrightarrow{\mathrm{d}_1} D_0 \otimes_S \hat{S} \xrightarrow{\mathrm{d}_0} D_1 \otimes_S \hat{S} \xrightarrow{\mathrm{d}_1} D_0 \otimes_S \hat{S} \longrightarrow M_D \to 0 \, .$$

Isomorphisms between modules associated to matrix factorizations of the same polynomial W give rise to isomorphisms of the respective matrix factorizations. This is relevant for defect fusion, because the two-periodic part of the free resolution of $\mathrm{coker}(\mathrm{d}_{D'\otimes D})_1$ in (3.6) equals the two-periodic part of the free resolution of

$$\mathrm{coker}\left((\mathrm{d}_{D'})_1 \otimes \mathrm{id}_{D0}, \mathrm{id}_{D'0} \otimes (\mathrm{d}_D)_1\right).$$

This equation simplifies defect fusion considerably. For example, the invisible defect I can this way be regarded to be a tensor product over the complex numbers of the one-dimensional matrix factorizations

$$\mathbb{C}[X_1, ..., X_n, X'_1, ..., X'_n] \underset{\partial_i^{X,X'} W}{\overset{(X_i - X'_i)}{\rightleftarrows}} \mathbb{C}[X_1, ..., X_n, X'_1, ..., X'_n].$$

Indeed, the module corresponding to a fused defect $D \otimes I$ is then given by

$$\mathrm{coker}(\mathrm{d}_{D1}, (X_1 - X'_1), ..., (X_n - X'_n)) \cong \mathrm{coker}(\mathrm{d}_{D1})$$

where the latter differential d_{D1} has shifted variables $X_i \mapsto X'_i$. In other words, $D \otimes I \cong D$.

Adjunctions. Adjoints, i.e. left and right bent versions satisfying the Zorro moves of chapter 2.1, of B-type defects and boundaries in LG models have been studied in [41, 15] (see [39] for a review). They are given by

$$D^\dagger \cong D^\vee[n] \, , \quad {}^\dagger D \cong D^\vee[m] \, , \tag{3.7}$$

where $D^\vee$ is the dual of a matrix factorization D, consisting of the dual modules $(D^\vee)_i = (D_i)^\vee$, and the maps

$$\mathrm{d}_{D^\vee} = \begin{pmatrix} 0 & \mathrm{d}_{D0}^\vee \\ -\mathrm{d}_{D1}^\vee & 0 \end{pmatrix} .$$

Moreover, $(\cdot)[m]$ denotes the shift of $\mathbb{Z}_2$-degree by m:

$$D[m] : D_{1+m \bmod 2} \underset{(-1)^m \mathrm{d}_{m \bmod 2}}{\overset{(-1)^n \mathrm{d}_{1+m \bmod 2}}{\rightleftarrows}} D_{m \bmod 2}$$

While the first isomorphism in (3.7) is given by the identity map, the second isomorphism contains a crucial minus sign [15, remark 5.1]

$$\begin{aligned} {}^{\dagger}D &\to D^{\vee}[m] \\ \nu &\mapsto (-1)^{m|\nu|}\nu \end{aligned}$$

for $\nu \in D_0$ or D_1. The corresponding (co)evaluation maps (2.2) are described by

$$\begin{aligned} \widetilde{\mathrm{ev}}_D(e_j \otimes e_i^*) = \sum_{l\geq 0} \sum_{a_1<...<a_l} & (-1)^{l+(n+1)|e_j|}\theta_{a_1}...\theta_{a_l} \cdot \\ & \cdot \operatorname{Res}\left[\frac{\left\{\partial_{a_l}^{Z,Z'}\mathrm{d}_D...\partial_{a_1}^{Z,Z'}\mathrm{d}_D \cdot \Lambda^X \mathrm{d}X\right\}_{ij}}{\partial_{X_1}W,...,\partial_{X_n}W}\right] \end{aligned}$$

$$\begin{aligned} \mathrm{ev}_D(e_i^* \otimes e_j) = \sum_{l\geq 0} \sum_{a_1<...<a_l} & (-1)^{\binom{l}{2}+l|e_j|}\theta_{a_1}...\theta_{a_l} \cdot \\ & \cdot \operatorname{Res}\left[\frac{\left\{\Lambda^Z \cdot \partial_{a_l}^{X,X'}\mathrm{d}_D...\partial_{a_1}^{X,X'}\mathrm{d}_D \mathrm{d}Z\right\}_{ij}}{\partial_{Z_1}W,...,\partial_{Z_m}W}\right] \\ \widetilde{\mathrm{coev}}_D(\bar{\gamma}) = \sum_{i,j} & (-1)^{(\bar{r}+1)|e_j|+s_n}\left\{\partial_{\bar{b}_{\bar{r}}}^{X,X'}\mathrm{d}_D...\partial_{\bar{b}_1}^{X,X'}\mathrm{d}_D\right\}_{ji} e_i^* \otimes e_j \\ \mathrm{coev}_D(\gamma) = \sum_{i,j} & (-1)^{\binom{r+1}{2}+mr+s_m}\left\{\partial_{b_1}^{Z,Z'}\mathrm{d}_D...\partial_{b_r}^{Z,Z'}\mathrm{d}_D\right\}_{ij} e_i \otimes e_j^* \end{aligned}$$

where $\Lambda^X := (-1)^n \partial_{X_1}\mathrm{d}_D...\partial_{X_n}\mathrm{d}_D$ and $\Lambda^Z := \partial_{Z_1}\mathrm{d}_D...\partial_{Z_m}\mathrm{d}_D$. $b_i, \bar{b}_{\bar{j}}$ and $s_n, s_m \in \mathbb{Z}_2$ are the unique numbers with $b_1 < ... < b_r$, $\bar{b}_1 < ... < \bar{b}_{\bar{r}}$ and $\bar{\gamma}\theta_{\bar{b}_1}...\theta_{\bar{b}_{\bar{r}}} = (-1)^{s_n}\theta_1...\theta_n$, $\gamma\theta_{b_1}...\theta_{b_r} = (-1)^{s_m}\theta_1...\theta_m$.

The categorial structure. Defects between Landau-Ginzburg models give rise to the structure of a 2-category. Objects are the Landau-Ginzburg models (potentials), 1-morphisms are the defects between different Landau-Ginzburg models (matrix factorizations of the difference of potentials) and 2-morphisms are the defect changing fields (morphisms between matrix factorizations).

In the case of B-type LG models, the defect and boundary categories carry even more structure – they are triangulated. As discussed in the next section, such a triangulation requires the existence of exact triangles, which - in the case at hand - arises from the cone-construction [42]. The cone of a morphism $\phi : D \to E$ between two matrix factorizations $D, E \in \mathrm{hmf}(W)$ is given by

$$\mathrm{cone}(\phi : D \to E) : E_1 \oplus D_0 \quad \underset{\begin{pmatrix} \mathrm{d}_{E0} & \phi_1 \\ & -\mathrm{d}_{D1} \end{pmatrix}}{\overset{\begin{pmatrix} \mathrm{d}_{E1} & \phi_0 \\ & -\mathrm{d}_{D0} \end{pmatrix}}{\rightleftarrows}} \quad E_0 \oplus D_1. \tag{3.8}$$

Physically, this corresponds to a deformation (or perturbation) of the sum $E \oplus D[-1]$. The exact triangles, which can be shown to satisfy all the necessary properties of the next section [42], are those isomorphic to triangles of the form

$$D \xrightarrow{\phi} E \xrightarrow{\begin{pmatrix} \mathrm{id}_E \\ 0 \end{pmatrix}} \mathrm{cone}(\phi : D \to E) \xrightarrow{\left(0, -\mathrm{id}_{D[1]}\right)} D[1].$$

A straightforward computation reveals that fusion $\otimes$, shift $[\cdot]$ and cone construction all pairwise commute. In other words, shift and fusion take exact triangles to exact triangles. Hence, the defect categories in Landau-Ginzburg models satisfies all the requirements of a tensor triangulated category as spelled out in the next section. Furthermore, the boundary categories are triangulated and fusion between defects and boundaries respects the triangulated structure of both defect and boundary categories.

3.4 Triangulated defect categories

In this section, which appeared in [2], some basic properties of triangulated tensor categories are reviewed. For the full set of axioms see e.g. [42]. A tensor triangulated category T comes with automorphisms $[n]$ called shift functors for all $n \in \mathbb{Z}$. Moreover, it comes with a collection of exact (or distinguished) triangles

$$C \xrightarrow{\phi} D \xrightarrow{\psi} E \xrightarrow{\chi} C[1] \tag{3.9}$$

satisfying a list of axioms. Here $C, D, E \in \mathrm{obj}(T)$. For instance, for every morphism $\phi : C \to D$ between objects $C, D \in \mathrm{obj}(T)$ there is an object $\mathrm{cone}(\phi) \in \mathrm{obj}(T)$ which fits into an exact triangle

$$C \xrightarrow{\phi} D \longrightarrow \mathrm{cone}(\phi) \longrightarrow C[1].$$

The cone of the identity morphism id_C of any object C is trivial, $\mathrm{cone}(\mathrm{id}_D) = 0$. Moreover, the shift functor takes exact triangles to exact triangles, and the triangle (3.9) is exact if and only if the rotated triangle

$$D \xrightarrow{\psi} E \xrightarrow{\chi} C[1] \xrightarrow{-\phi[1]} D[1]$$

is exact.

A morphism of triangles consists of morphisms, c, d, e such that all squares in

$$\begin{tikzcd} C' \arrow[r, "\phi'"] & D' \arrow[r, "\psi'"] & E' \arrow[r, "\chi'"] & C'[1] \\ C \arrow[r, "\phi"] \arrow[u, "c"] \arrow[ur, phantom, "\square"] & D \arrow[r, "\psi"] \arrow[u, "d"] \arrow[ur, phantom, "\square"] & E \arrow[r, "\chi"] \arrow[u, "e"] \arrow[ur, phantom, "\square"] & C[1] \arrow[u, "c[1]"'] \end{tikzcd}$$

commute. If c, d, e are isomorphisms, they define an isomorphism of triangles. Another important property of triangulated categories is that any triangle isomorphic to an exact triangle is itself exact. Moreover, for exact triangles, the existence of the first two (and hence also the fourth) vertical morphisms in

$$\begin{tikzcd} C' \arrow[r, "\phi'"] & D' \arrow[r, "\psi'"] & E' \arrow[r, "\chi'"] & C'[1] \\ C \arrow[r, "\phi"] \arrow[u] \arrow[ur, phantom, "\square"] & D \arrow[r, "\psi"] \arrow[u] \arrow[ur, phantom, "\square"] & E \arrow[r, "\chi"] \arrow[u, dashed] \arrow[ur, phantom, "\square"] & C[1] \arrow[u] \end{tikzcd}$$

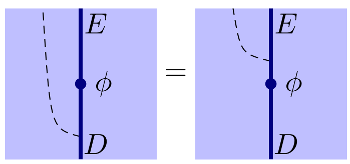
implies the existence of the dashed morphism with all squares commuting. The morphism induced in this way by two isomorphisms is itself an isomorphism.

Since our aim is to describe defects, we require T to be a tensor category, in particular it comes with a product $\otimes$ on objects and morphisms with neutral element $I \in \text{obj}(T)$. Moreover, this product is compatible with the triangulated structure. More precisely, it commutes with the shift functor $[n]$ and cone construction of T, i.e.

$$\begin{aligned} &(D \otimes E)[1] \cong D[1] \otimes E \cong D \otimes E[1] \\ &\text{cone}(\phi : D \to E) \otimes F \cong \text{cone}(\phi \otimes \text{id}_F : D \otimes F \to E \otimes F) \end{aligned} \tag{3.10}$$

To put it differently, the induced functors $\cdot \otimes D$ and $D \otimes \cdot$ for any $D \in \text{obj}(T)$ are triangulated, i.e. compatible with the triangulated structure. Furthermore, morphisms are taken to commute with the isomorphisms $D \cong I \otimes D$ and $D \cong D \otimes I$. This is expressed by the existence of four commutative diagrams of the type

$$\begin{tikzcd} I \otimes D \arrow[r, "\text{id} \otimes \phi"] & I \otimes E \\ D \arrow[r, "\phi"'] \arrow[u, "\sim"] \arrow[ur, phantom, "\square"] & E \arrow[u, "\sim"'] \end{tikzcd}$$



On the right hand side we have provided the relation in string diagram notation. The other three diagrams can be obtained by mirroring the given defect diagrams.

3.5 LG orbifolds and equivariant defects

An ordinary topological LG model specified by a superpotential W can be orbifolded by a finite group G giving rise to the LG *orbifold* model denoted W/G [43, 18]. The B-twists of these models constitute the primary examples in this thesis.

In detail, consider a finite group G acting linearly on the polynomial ring $\mathbb{C}[X_1, ..., X_n]$ of chiral fields such that the superpotential $W \in \mathbb{C}[X_1, ..., X_n]$ is invariant,

$$W(g \cdot X_1, ..., g \cdot X_n) = W(X_1, ..., X_n)$$

for all $g \in G$. In the orbifold construction, one first allows new representations constituting the g-twisted sectors $\mathcal{H}^g$. The untwisted sector $\mathcal{H}^0$ is the original Hilbert space. In the CFT operator picture on the cylinder, a g-twisted field obeys $\phi(e^{2\pi i}\sigma) = g\phi(\sigma)$. Second, one keeps only the g-invariant states in the g-twisted sector. In other words, the Hilbert space of the LG orbifold model is

$$\mathcal{H}_{\text{orbifold}} = \bigoplus_{\substack{\text{conj. class } \{g\} \\ \text{of } G}} P_g \mathcal{H}^g$$

where P_g implements the projection onto g-invariant states and the sum is only over conjugacy classes because $\mathcal{H}^g \cong \mathcal{H}^{hgh^{-1}}$.

Equivariant defects. The description of B-type defects in Landau-Ginzburg models by means of matrix factorizations extends in a straightforward manner to the context of Landau-Ginzburg orbifolds. (For more details on defects in Landau-Ginzburg orbifolds see [3].) Let $V \in \mathbb{C}[X_1, \ldots, X_n]$ and $W \in \mathbb{C}[Y_1, \ldots, Y_m]$ be two superpotentials and G_V and G_W be orbifold groups leaving the respecting superpotential invariant. Then, B-type defects between the respective LG orbifolds can be described by $G = G_V \times G_W$-equivariant matrix factorizations of $V - W$ [3, 44, 45]. These are matrix factorizations $D : W \to V$ as before, which are additionally equipped with a representation ρ_D of G. The latter has to be compatible with the module structure on D and has to commute with d_D. Denoting by ρ the representation of $G = G_V \times G_W$ on the combined polynomial ring $S = \mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ this means that for all $g \in G$

$$\begin{aligned} &\rho_D(g)(s \cdot p) = \rho(g)(s) \cdot \rho_D(g)(p), \qquad \forall s \in S, p \in D = D_0 \oplus D_1, \\ &\rho_D(g) \circ \mathrm{d}_D = \mathrm{d}_D \circ \rho_D(g)\,. \end{aligned}$$

Given two equivariant matrix factorizations $D, D' : W \to V$, the complex $\mathrm{Hom}_S(D, D')$ carries an action of $G = G_V \times G_W$ which commutes with the differential d of (3.5), inducing a representation on the homology $H^*_{\mathrm{d}}(\mathrm{Hom}_S(D, D'))$. The space of defect-changing fields in the orbifold theory is then given by the G-invariant part $\mathrm{Hom}^G(D, D') = (H^*_{\mathrm{d}}(\mathrm{Hom}_S(D, D')))^G$.

The operator product of defect-changing fields is again just composition of homomorphisms.

Defect fusion carries over from the unorbifolded LG models by taking invariant parts. More precisely, let $U \in \mathbb{C}[X_1, \dots, X_m]$, $V \in \mathbb{C}[Y_1, \dots, Y_n]$ and $W \in \mathbb{C}[Z_1, \dots, Z_o]$ be polynomials invariant under actions of groups G_U, G_V, G_W on the respective polynomial rings. And let $D : W \to V$ and $D' : V \to U$ be $G_V \times G_W$-, respectively $G_U \times G_V$-equivariant matrix factorizations. Then the tensor product $D' \otimes D$ is a $G_U \times G_V \times G_W$-equivariant matrix factorization of $U - W$. Fusion of the defects in the orbifold theory is then given by the G_V-invariant part $D' \otimes_{G_V} D := (D' \otimes D)^{G_V}$ of $D' \otimes D$, which is of course $G_U \times G_W$-equivariant.

Adjunction of defects D in the orbifold theory is given by adjunction (3.7) in the underlying unorbifolded theory, where however the G-action on the adjoints is twisted. This can be seen in a systematic way in the generalized orbifold construction reviewed in section 2.7 which offers a completely general framework to describe orbifold theories using defects in the underlying unorbifolded theory. In the example of the next chapter, the generalized orbifold procedure is used to explicitely determine adjoints and (co)evaluations maps.

Importantly, categories of equivariant matrix factorizations inherit the tensor triangulated structure from the categories of matrix factorizations.

Identity defect. Given an abelian symmetry group G in the original non-orbifold theory, one can construct defects ${}_gI$ that implement the symmetry operation. The matrix factorizations of these symmetry defects are built on the same module as the identity matrix factorization, but the differential is twisted by the symmetry [16]

$$\mathrm{d}_{{}_gI} = \sum_{i=1}^{n} \left[(X_i - g(X'_i)) \cdot \theta_i^* + \partial_i^{X,X'} W(X, g(X')) \cdot \theta_i \right] .$$

The identity defect in Landau-Ginzburg orbifolds can now be constructed from the identity defect of the unorbifolded LG model by summing over all symmetry defects ${}_gI$ of the unorbifolded Landau-Ginzburg model associated to orbifold group elements [3]

$$I^{\mathrm{orb}} = \bigoplus_{g \in G} {}_gI \ . \tag{3.11}$$

This defect can be brought in the equivariant form discussed above. For $G = \mathbb{Z}_d$ this is executed explicitely in appendix B.6.

Chapter 4

Application: LG orbifolds with one chiral superfield

As a first example, the procedure of chapter 2 is applied to the (topologically B-twisted conformal fixed point of the) LG orbifold with a single chiral field X, superpotential X^d and orbifold group $\mathbb{Z}_d$. An element $a \in \mathbb{Z}_d$ of the orbifold group acts on the chiral field X by $X \mapsto e^{\frac{2\pi i a}{d}} X$. This section is an excerpt from [1].

LG orbifolds and their defects have been described in section 3.5. Explicit formulae for adjunctions and (co)evaluation maps are obtained by applying the generalized orbifold procedure of section 2.7 to the non-orbifold formulae of section 3.3, see appendix B.

4.1 Defects and adjoints

A defect $D : X^d/\mathbb{Z}_d \to Z^{d'}/\mathbb{Z}_{d'}$ is given by a $G = \mathbb{Z}_{d'} \times \mathbb{Z}_d$-equivariant matrix factorization of $Z^{d'} - X^d$. Since G is commutative, its representations on D can be specified by G-gradings or -charges of the generators of the free $S = \mathbb{C}[Z, X]$-module $D = D_0 \oplus D_1$. We will indicate them in square brackets and specify a G-equivariant matrix factorization as

$$D : S^M \begin{pmatrix} [l_M, r_M] \\ [l_{M+1}, r_{M+1}] \\ \vdots \\ [l_{2M-1}, r_{2M-1}] \end{pmatrix} \underset{\mathrm{d}_{D0}}{\overset{\mathrm{d}_{D1}}{\rightleftarrows}} S^M \begin{pmatrix} [l_0, r_0] \\ [l_1, r_1] \\ \vdots \\ [l_{M-1}, r_{M-1}] \end{pmatrix} ,$$

where d_{D1} and d_{D0} are $M \times M$-square matrices.

According to the generalized orbifold procedure adjoints then take the form, c.f. appendix B.4,

$$
\begin{aligned}
D^\dagger : S^M \begin{pmatrix} [-r_0+1,-l_0] \\ [-r_1+1,-l_1] \\ \vdots \\ [-r_{M-1}+1,-l_{M-1}] \end{pmatrix} & \underset{-\mathrm{d}_{D0}^T}{\overset{\mathrm{d}_{D1}^T}{\rightleftarrows}} S^M \begin{pmatrix} [-r_M+1,-l_M] \\ [-r_{M+1}+1,-l_{M+1}] \\ \vdots \\ [-r_{2M-1}+1,-l_{2M-1}] \end{pmatrix} \\
{}^\dagger D : S^M \begin{pmatrix} [-r_0,-l_0+1] \\ [-r_1,-l_1+1] \\ \vdots \\ [-r_{M-1},-l_{M-1}+1] \end{pmatrix} & \underset{-\mathrm{d}_{D0}^T}{\overset{\mathrm{d}_{D1}^T}{\rightleftarrows}} S^M \begin{pmatrix} [-r_M,-l_M+1] \\ [-r_{M+1},-l_{M+1}+1] \\ \vdots \\ [-r_{2M-1},-l_{2M-1}+1] \end{pmatrix}
\end{aligned} \tag{4.1}
$$

Note that left adjoints differ from right adjoints by a shift in G-charges by $[-1,1]$. We write ${}^\dagger D = D^\dagger\{[-1,1]\}$.

An important example is the identity defect $I_d : X^d/\mathbb{Z}_d \to Z^d/\mathbb{Z}_d$ which is represented by the following $\mathbb{Z}_d \times \mathbb{Z}_d$-equivariant matrix factorization (c.f. appendix B.6)

$$
I_d : S^d \begin{pmatrix} [1,0] \\ [2,-1] \\ [3,-2] \\ \vdots \end{pmatrix} \underset{\mathrm{d}_{I_d0}}{\overset{\begin{pmatrix} Z & 0 & \dots & 0 & -X \\ -X & Z & & & \\ 0 & -X & Z & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & -X & Z \end{pmatrix}}{\rightleftarrows}} S^d \begin{pmatrix} [0,0] \\ [1,-1] \\ [2,-2] \\ \vdots \end{pmatrix}.
$$

One easily reads off that this defect is self-adjoint, i.e. $I_d^\dagger \cong I_d \cong {}^\dagger I_d$.

4.2 RG defects

For $d > 2$, the models $X^{d'}/\mathbb{Z}_{d'}$ exhibit relevant perturbations by twisted chiral fields which trigger renormalization group flows to Landau-Ginzburg orbifolds $X^{d'}/\mathbb{Z}_{d'}$ of the same type but with $d' < d$.[1] The associated RG defects have been constructed in [3]. They preserve B-type supersymmetry and can therefore be described by $\mathbb{Z}_{d'} \times \mathbb{Z}_d$-equivariant matrix factorizations of $Z^{d'} - X^d$. Indeed, due to a singularity in the parameter space, there are different flows from $\mathcal{M}_d/\mathbb{Z}_d$ to $\mathcal{M}_{d'}/\mathbb{Z}_{d'}$. The corresponding RG defects $R = R(m, n_0, \dots, n_{d'-1})$ are specified by $m \in \mathbb{Z}_d$, and integers $n_0, \dots, n_{d'-1} \geq 1$,

[1] These are mirror to flows between the unorbifolded Landau-Ginzburg models with superpotentials $W = X^d$ and $W' = X^{d'}$ triggered by deformation of the superpotential W by lower degree polynomials.

such that $n_0 + ... + n_{d'-1} = d$. They are represented by matrix factorizations

$$R : S^{d'} \begin{pmatrix} {\scriptstyle [1,-m]} \\ {\scriptstyle [2,-m-n_1]} \\ {\scriptstyle [3,-m-n_1-n_2]} \\ \vdots \end{pmatrix} \underset{\mathrm{d}_{R0}}{\overset{\mathrm{d}_{R1}}{\rightleftarrows}} S^{d'} \begin{pmatrix} {\scriptstyle [0,-m]} \\ {\scriptstyle [1,-m-n_1]} \\ {\scriptstyle [2,-m-n_1-n_2]} \\ \vdots \end{pmatrix},$$

$$\mathrm{d}_{R1} = \begin{pmatrix} Z & 0 & \dots & 0 & -X^{n_0} \\ -X^{n_1} & Z & & & \\ 0 & -X^{n_2} & Z & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & -X^{n_{d'-1}} & Z \end{pmatrix} \tag{4.2}$$

where $S = \mathbb{C}[X, Z]$. For more details see [3]. In the following we will sometimes take the subscripts of the n_i to be elements in $\mathbb{Z}_{d'}$ by defining $n_{i+z\,d'} = n_i$ for all $z \in \mathbb{Z}$.

Using this concrete realization of RG defects, one can now explicitly carry out the construction outlined in section 2 and represent the LG orbifolds $\mathcal{M}_{d'}/\mathbb{Z}_{d'}$ in $\mathcal{M}_d/\mathbb{Z}_d$ for any $d' < d$. In order to construct the respective projection defects, we need right and left adjoints of the defects R, which can easily be read off from formula (4.1). They are given by

$$R^\dagger : S^{d'} \begin{pmatrix} {\scriptstyle [m+1,0]} \\ {\scriptstyle [m+1+n_1,-1]} \\ {\scriptstyle [m+1+n_1+n_2,-2]} \\ \vdots \end{pmatrix} \underset{\mathrm{d}_{R^\dagger 0} = \mathrm{d}_{R0}^T}{\overset{\mathrm{d}_{R^\dagger 1}}{\rightleftarrows}} S^{d'} \begin{pmatrix} {\scriptstyle [m+1,-1]} \\ {\scriptstyle [m+1+n_1,-2]} \\ {\scriptstyle [m+1+n_1+n_2,-3]} \\ \vdots \end{pmatrix}$$

and

$$^\dagger R : S^{d'} \begin{pmatrix} {\scriptstyle [m,1]} \\ {\scriptstyle [m+n_1,0]} \\ {\scriptstyle [m+n_1+n_2,-1]} \\ \vdots \end{pmatrix} \underset{\mathrm{d}_{^\dagger R0} = \mathrm{d}_{R0}^T}{\overset{\mathrm{d}_{^\dagger R1}}{\rightleftarrows}} S^{d'} \begin{pmatrix} {\scriptstyle [m,0]} \\ {\scriptstyle [m+n_1,-1]} \\ {\scriptstyle [m+n_1+n_2,-2]} \\ \vdots \end{pmatrix}$$

for

$$\mathrm{d}_{R^\dagger 1} = \mathrm{d}_{^\dagger R1} = \begin{pmatrix} Z & -X^{n_1} & & & \\ & Z & -X^{n_2} & & \\ & & \ddots & \ddots & \\ & & & Z & -X^{n_{d'-1}} \\ -X^{n_0} & & & & Z \end{pmatrix}$$

A straightforward calculation presented in appendix C.1 then shows that indeed

$$R \otimes_{\mathbb{Z}_d} R^\dagger \cong I_{d'}$$
$$R \otimes_{\mathbb{Z}_d} {}^\dagger R \cong I_{d'},$$

i.e. the defects R are indeed of RG type. Fusion in the opposite order yields the respective projection defects (see appendix C.2 for the explicit calculation). For the unital projection defect $P = R^\dagger \otimes_{\mathbb{Z}_{d'}} R$ one obtains

$$P : P_1 \underset{\mathrm{d}_{P0}}{\overset{\mathrm{d}_{P1}}{\rightleftarrows}} S^{d'} \begin{pmatrix} [m+1+\sum_{l=1}^{d'-1} n_l, -m] \\ [m+1, -m-n_1] \\ [m+1+n_1, -m-n_1-n_2] \\ \vdots \\ [m+1+\sum_{l=1}^{d'-2} n_l, -m-\sum_{l=1}^{d'-1} n_l] \end{pmatrix},$$

where

$$\mathrm{d}_{P1} = \begin{pmatrix} Z^{n_0} & 0 & \dots & 0 & -X^{n_0} \\ -X^{n_1} & Z^{n_1} & & & \\ 0 & -X^{n_2} & Z^{n_2} & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & -X^{n_{d'-1}} & Z^{n_{d'-1}} \end{pmatrix}. \tag{4.3}$$

and

$$P_1 = S^{d'} \begin{pmatrix} [m+1, -m] \\ [m+1+n_1, -m-n_1] \\ [m+1+n_1+n_2, -m-n_1-n_2] \\ \vdots \\ [m+1+\sum_{l=1}^{d'-1} n_l, -m-\sum_{l=1}^{d'-1} n_l] \end{pmatrix}$$

The counital projection defect $P' = {}^\dagger R \otimes_{\mathbb{Z}_{d'}} R$ is given by the left adjoint $P' = {}^\dagger P$ of P.

4.3 Representing $\mathcal{M}_{d'}/\mathbb{Z}_{d'}$ in $\mathcal{M}_d/\mathbb{Z}_d$ for $d' < d$

The projection defects constructed from RG defects in the previous section can now be used to represent Landau-Ginzburg orbifolds $\mathcal{M}_{d'}/\mathbb{Z}_{d'}$ in orbifolds $\mathcal{M}_d/\mathbb{Z}_d$ for $d' < d$.

Bulk Hilbert space. The orbifolds $\mathcal{M}_{d'}/\mathbb{Z}_{d'}$ only possess a single bulk chiral field, namely the identity field. Therefore, the bulk Hilbert space in the B-twisted model is trivial, it just contains the vaccuum. One easily checks, that this is also true for $\mathrm{Hom}(P, P)$. Hence, the bulk Hilbert space of $\mathcal{M}_{d'}/\mathbb{Z}_{d'}$ agrees with the space of defect fields on the projection defect in $\mathcal{M}_d/\mathbb{Z}_d$.[2]

[2] Since the bulk Hilbert spaces are trivial, this is not that interesting. However, there is a way to describe also the twisted chiral fields in the B-twisted LG orbifolds $\mathcal{M}_d/\mathbb{Z}_d$. Namely, being orbifold twist fields, they can be realized as defect changing fields between symmetry defects. This realization then lifts from IR to UV using projection defects, i.e. one can realize the twisted chiral fields in $\mathcal{M}_{d'}/\mathbb{Z}_{d'}$ by defect changing fields in $\mathcal{M}_d/\mathbb{Z}_d$.

Boundary conditions. Next, we demonstrate how to represent the boundary conditions of $\mathcal{M}_{d'}/\mathbb{Z}_{d'}$ as P-invariant boundary conditions in the models $\mathcal{M}_d/\mathbb{Z}_d$.

Elementary left boundary conditions in a theory $\mathcal{M}_d/\mathbb{Z}_d$ are represented by the $\mathbb{Z}_d$-equivariant matrix factorizations

$$B^d_{k,N} : \mathbb{C}[X]\big([N+k]\big) \underset{-X^{d-k}}{\overset{X^k}{\rightleftarrows}} \mathbb{C}[X]\big([N]\big)$$

of $-X^d$, where $k \in \{1, \dots, d-1\}$ and $N \in \mathbb{Z}_d$.

As is shown in appendix C.3, a UV boundary condition $B_{\mathrm{UV}} = B^d_{k,N}$ is invariant under fusion with P, i.e. $B_{\mathrm{UV}} \otimes P \cong B_{\mathrm{UV}}$ iff

$$\begin{aligned} k &= n_i + \dots + n_{i-l} \\ \text{and} \quad N &= \left[-m - \sum_{a=1}^{i} n_a\right] \end{aligned}$$

for an $i \in \mathbb{Z}_{d'}$ and an $l \in \{0, \dots, d'-2\}$. These are of course nothing but the lifts $B_{\mathrm{IR}} \otimes_{\mathbb{Z}_{d'}} R$ of IR boundary conditions to the UV. Namely, for $B_{\mathrm{IR}} = B^{d'}_{l,M}$ one finds [3]

$$B_{\mathrm{IR}} \otimes_{\mathbb{Z}_{d'}} R = B^d_{(n_{-M-l+1}+\dots+n_{-M}),(-m-\sum_{a=1}^{-M} n_a)} .$$

IR symmetries. The Landau-Ginzburg orbifold model $\mathcal{M}_{d'}/\mathbb{Z}_{d'}$ exhibits a $\mathbb{Z}_{d'}$-symmetry. The action of an element $a \in \mathbb{Z}_{d'}$ on the theory is described by the symmetry defect ${}_aI_{d'} = I_{d'}\{[a,0]\} \cong I_{d'}\{[0,-a]\}$ obtained by shifting the charges of the identity defect $I_{d'}$ by $[a,0]$ or equivalently by $[0,-a]$. These defects fuse according to the group multiplication in the symmetry group $\mathbb{Z}_{d'}$:

$${}_aI_{d'} \otimes_{\mathbb{Z}_{d'}} {}_bI_{d'} = {}_{a+b}I_{d'} , \quad \text{for} \quad a, b \in \mathbb{Z}_{d'}.$$

As any IR defects, they lift into the UV theory $\mathcal{M}_d/\mathbb{Z}_d$ by fusion with RG defects

$${}_aI_{d'} \longmapsto R^\dagger \otimes_{\mathbb{Z}_{d'}} {}_aI_{d'} \otimes_{\mathbb{Z}_{d'}} R =: {}_aP.$$

These lifted defects also fuse according to multiplication in the symmetry group, i.e. ${}_aP \otimes_{\mathbb{Z}_d} {}_bP = {}_{a+b}P$, and therefore give a realization of the IR symmetry in the UV. The neutral element of the group however lifts to the defect ${}_0P = P$ and not to the identity defect in the UV. The lifted IR symmetries are therefore not invertible in the full UV theory, and hence are not symmetries of the UV theory.

The explicit form of ${}_aP$ can be easily derived by means of a slight variation of the calculation of P as carried out in appendix C.4. The result is

$$
{}_aP : ({}_aP)_1 \underset{\mathrm{d}_{P0}}{\overset{\mathrm{d}_{P1}}{\rightleftarrows}} S^{d'} \begin{pmatrix} [m+1+\sum_{l=1}^{d'-1} n_l, -m-\sum_{j=1}^{-a} n_j] \\ [m+1, -m-\sum_{j=1}^{1-a} n_j] \\ [m+1+n_1, -m-\sum_{j=1}^{2-a} n_j] \\ \vdots \\ [m+1+\sum_{l=1}^{d'-2} n_l, -m-\sum_{l=1}^{d'-1-a} n_l] \end{pmatrix},
$$

where

$$
\mathrm{d}_{P1} = \begin{pmatrix} Z^{n_0} & 0 & \dots & 0 & -X^{n_{0-a}} \\ -X^{n_{1-a}} & Z^{n_1} & & & \\ 0 & -X^{n_{2-a}} & Z^{n_2} & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & -X^{n_{d'-1-a}} & Z^{n_{d'-1}} \end{pmatrix}.
$$

and

$$
({}_aP)_1 = S^{d'} \begin{pmatrix} [m+1, -m-\sum_{j=1}^{-a} n_j] \\ [m+1+n_1, -m-\sum_{j=1}^{1-a} n_j] \\ [m+1+n_1+n_2, -m-\sum_{j=1}^{2-a} n_j] \\ \vdots \\ [m+1+\sum_{l=1}^{d'-1} n_l, -m-\sum_{l=1}^{d'-1} n_l] \end{pmatrix}
$$

For $a = 0$, this is the matrix factorization describing P. The lifted IR symmetry defects ${}_aP$ are obtained from it by shifting the exponents of X by a steps while keeping left $\mathbb{Z}_d$-charges fixed and adapting the right ones accordingly.

4.4 The limit $d \to \infty$

As discussed in the beginning of this section, the RG flows between LG orbifolds $\mathcal{M}_d/\mathbb{Z}_d$ are nothing but the mirror versions of flows between LG models $\mathcal{M}_d$ generated by deformations of the superpotentials $W = X^d$ by lower degree polynomials. Indeed, all the models $\mathcal{M}_{d'}$ can be obtained as perturbations of the free chiral field theory ($W = 0$) by superpotential deformations. Thus, employing our procedure provides a representation of all the models $\mathcal{M}_{d'}$ inside the theory of a free chiral field, which can be thought of as the limit $\mathcal{M}_\infty = \lim_{d\to\infty} \mathcal{M}_d$ of the models $\mathcal{M}_d$.

In order to make this more explicit we again take the mirror perspective. The representation of the respective RG defects in terms of matrix factorizations then allows us to explicitly realize all the LG orbifolds $\mathcal{M}_{d'}/\mathbb{Z}_{d'}$ by means of projection defects in the theory of the free twisted chiral field. The latter can be described as the limit $\mathcal{M}_\infty/\mathbb{Z}_\infty = \lim_{d\to\infty} \mathcal{M}_d/\mathbb{Z}_d$ and can be thought of as a $U(1)$-equivariant version of the free chiral field.

RG defects between $\mathcal{M}_{d'}/\mathbb{Z}_{d'}$ and $\mathcal{M}_\infty/\mathbb{Z}_\infty$ can be obtained as limits of the RG defects (4.2) representing flows $\mathcal{M}_d/\mathbb{Z}_d \to \mathcal{M}_{d'}/\mathbb{Z}_{d'}$, where one n_i is sent to ∞ while the others are kept fixed. Since $d = \sum_i n_i$, then also $d \to \infty$. Indeed, we can choose $n_0 \to \infty$ and compensate for this choice by allowing a shift of the charges of R by $[k, 0]$, $k \in \mathbb{Z}_{d'}$. In the limit, entries X^{n_0} in the matrix factorization have to be replaced by 0, and the $\mathbb{Z}_d$-equivariance turns into a $U(1)$-equivariance. This way, one obtains the $\mathbb{Z}_{d'} \times U(1)$-equivariant matrix factorizations

$$R_\infty : S^{d'}\begin{pmatrix} [k+1,-m] \\ [k+2,-m-n_1] \\ [k+3,-m-n_1-n_2] \\ \vdots \end{pmatrix} \underset{\mathrm{d}_{R0}}{\overset{\mathrm{d}_{R1}}{\rightleftarrows}} S^{d'}\begin{pmatrix} [k,-m] \\ [k+1,-m-n_1] \\ [k+2,-m-n_1-n_2] \\ \vdots \end{pmatrix}$$

of $Z^{d'}$. They are specified by integers $m \in \mathbb{Z}$, $n_1, ..., n_{d'-1} \in \mathbb{N}$ and $k \in \mathbb{Z}_{d'}$. The maps are given by

$$\mathrm{d}_{R1} = \begin{pmatrix} Z & 0 & \dots & 0 & 0 \\ -X^{n_1} & Z & & & \\ 0 & -X^{n_2} & Z & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & -X^{n_{d'-1}} & Z \end{pmatrix}$$

$$\mathrm{d}_{R0} = \begin{pmatrix} Z^{d'-1} & 0 & 0 & \dots \\ Z^{d'-2}X^{n_1} & Z^{d'-1} & 0 & \dots \\ Z^{d'-3}X^{n_1+n_2} & Z^{d'-2}X^{n_2} & Z^{d'-1} & \ddots \\ Z^{d'-4}X^{n_1+n_2+n_3} & Z^{d'-3}X^{n_2+n_3} & Z^{d'-2}X^{n_3} & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

These matrix factorizations represent RG defects between $\mathcal{M}_{d'}/\mathbb{Z}_{d'}$ and $\mathcal{M}_\infty/\mathbb{Z}_\infty$. Indeed $R_\infty \otimes R_\infty^\dagger \cong I_{\mathrm{IR}}$ as is spelled out in appendix C.5.

In explicit calculations, it is not difficult to see that fusion commutes with the limit $d \to \infty$, at least as long as the theory squeezed between the defects is kept fixed in the limit. In particular, the limit $d \to \infty$ of projection defects is the fusion $P_\infty = R_\infty^\dagger \otimes R_\infty$ of the limit of RG defects. The projection defect realizing $\mathcal{M}_{d'}/\mathbb{Z}_{d'}$ within the limit theory $\mathcal{M}_\infty/\mathbb{Z}_\infty$ takes the form

$$P_\infty : (P_\infty)_1 \underset{\mathrm{d}_{P0}}{\overset{\mathrm{d}_{P1}}{\rightleftarrows}} S^{d'}\begin{pmatrix} [m+1+\sum_{l=1}^{d'-1} n_l, -m] \\ [m+1, -m-n_1] \\ [m+1+n_1, -m-n_1-n_2] \\ \vdots \\ [m+1+\sum_{l=1}^{d'-2} n_l, -m-\sum_{l=1}^{d'-1} n_l] \end{pmatrix},$$

where $S = \mathbb{C}[Z, X]$,

$$\mathrm{d}_{P1} = \begin{pmatrix} 0 & & & & 0 \\ -X^{n_1} & Z^{n_1} & & & \\ & -X^{n_2} & Z^{n_2} & & \\ & & \ddots & \ddots & \\ & & & -X^{n_{d'-1}} & Z^{n_{d'-1}} \end{pmatrix}.$$

and

$$(P_\infty)_1 = S^{d'} \begin{pmatrix} [m+1, -m] \\ [m+1+n_1, -m-n_1] \\ [m+1+n_1+n_2, -m-n_1-n_2] \\ \vdots \\ [m+1+\sum_{l=1}^{d'-1} n_l, -m-\sum_{l=1}^{d'-1} n_l] \end{pmatrix}$$

Chapter 5

Application: Phases of gauged linear sigma models

The topic of this chapter, which was published in [11], are two-dimensional gauged linear sigma models (GLSMs) with $U(1)$ gauge groups. These are 2d $N = (2, 2)$ supersymmetric gauge theories coupled to chiral superfields carrying possibly different charges under the $U(1)$ gauge group, such that the respective superpotentials W are $U(1)$ invariant.

Gauged linear sigma models exhibit different phases for different ranges of the Fayet-Iliopoulos parameter r associated to the $U(1)$ gauge group [46]. For non-anomalous gauged linear sigma models, where axial and vector R-symmetries are preserved at the quantum level, the RG flow drives the GLSM to a (Kähler) moduli space of superconformal field theories parametrized by the complexified Fayet-Iliopoulos parameter t. The phases correspond to different domains of this moduli space. In contrast, in the anomalous case, the FI parameter is a running coupling constant, and the different phases correspond to fixed points under the RG flow.

The phases typically exhibit gauge symmetry breaking. For instance, in geometric phases, in which the theory can be effectively described by a non-linear sigma model, the gauge group is typically completely broken. On the other hand, in phases in which the theory can be described by Landau-Ginzburg models (Landau-Ginzburg phases), a finite subgroup of the gauge group remains unbroken and survives as an orbifold group.

The question addressed in this chapter is how the boundary sectors (i.e. the D-branes) behave under transitions between different phases of GLSMs. This transport of D-branes between different phases of abelian gauged linear sigma models has initially been studied in [47] for the non-anomalous "Calabi-Yau" case. Results on the anomalous "non-Calabi-Yau" case appeared more recently in [48, 49]. With a careful analysis, the authors of these papers obtain a prescription of the D-brane transport on the level of individual D-branes: Starting in one phase, a D-brane is first lifted to the gauged linear sigma model. This lift is a priori not unique, but requires certain choices. These choices correspond to the homotopy classes of paths along which the D-brane can be smoothly transported in parameter space ("grade restriction rule"). Having lifted a D-brane in such a way that it can be smoothly transported along the chosen path, one only has to push down the lift to the other phase.

In this chapter, the transport of D-branes is discussed from a defect point of view providing a uniform and functorial description of brane transport. In this setting, supersymmetry preserving defect lines (domain walls) connect different phases of a GLSM or a GLSM and one of its phases.

Supersymmetry preserving defects can be merged with boundaries (fusion) and in this way give rise to an action on D-branes.

This action is functorial, and hence any supersymmetric defect between two theories yields a functor between the respective D-brane categories.

In this chapter, the strategies of the previous chapters are employed in the context of abelian GLSMs to obtain transition defects between different phases of GLSMs as well as defects between GLSMs and their phases. In this way, we derive a novel method for brane transport and in particular recover the grade restriction rule of [47, 48, 49] from this point of view. While we do have compatible results, our derivations are rather different from those of [47, 48, 49]. In our discussion we decouple all gauge degrees of freedom and merely take into account the matter sector, the only remnant of the gauge symmetry being an equivariance condition. This subsector is under good control and still captures the physics of the (B-type) supersymmetry preserving sector, including perturbations, boundary conditions and defects. Our arguments mainly rely on the rigidity of defect constructions in this setting. The defects we construct on this level directly mediate between the different phases and do not exhibit an explicit t dependence.

In section 5.1 we outline the general ideas. In particular, we discuss the construction of defects connecting different phases of a GLSM and explain how they factorize into defects lifting phases to the GLSM and those pushing down the GLSM to phases. Furthermore, we introduce projection defects which realize the phases inside the GLSM. Their action on the D-brane category of the GLSM corresponds, in the language of [47, 49], to the projection of GLSM branes to "grade restricted" representatives.

The starting point for the construction of the transition defects is the identity defect of the GLSM theory which will be constructed explicitly in section 5.1.3. For this, one needs to generalize the constructions of orbifold identity defects of chapter 3.5, which will be revisited here thoroughly, to continuous (abelian) groups. We show that this can be achieved (in the context of equivariant matrix factorizations) by introducing new bosonic fields constrained to the defect. Our expectation is that this idea can be applied more generally in topological field theories.

In section 5.2, we illustrate the construction in an explicit class of examples, namely the $U(1)$-gauged linear sigma models with two chiral fields, X and P, and superpotential $W = P^{d'} X^d$, $d' < d$. These anomalous models have two different Landau-Ginzburg orbifold phases. The UV phase is described by the Landau-Ginzburg orbifold with superpotential $W = X^d$ and orbifold group $\mathbb{Z}_d$, and the IR phase by the Landau-Ginzburg orbifold with superpotential $W = P^{d'}$ and orbifold group $\mathbb{Z}_{d'}$. Along the RG flow, $d - d'$ vacua decouple to a Coulomb branch taking with them a set of D-branes. All of this is encoded in the transition defects we construct here. Moreover, in this example the phase transition between UV and IR phase of the GLSM corresponds the RG flow between the Landau-Ginzburg models describing the UV and IR phases [50]. The RG defects for these flows are the ones of chapter 4, and we indeed find that our transition defects between UV and IR phase agree with the respective RG defects.

This chapter is based on [11].

5.1 Phases of GLSMs and defects

5.1.1 Phases of GLSMs

We are considering two-dimensional $\mathcal{N} = (2,2)$ gauged linear sigma models with abelian gauge groups [46], which we review now. By X_i, $i = 1, \ldots, n$ we denote the chiral superfields of the theory. Their representation under the gauge group $U(1)^k$ is specified by the charge matrix Q_i^a, where $i = 1, \ldots, n$ and $a = 1, \ldots, k$. For each $U(1)$-factor of the gauge group the theory contains a field strength multiplet, a twisted chiral field Σ_a, $a = 1, \ldots, k$. We also allow for a superpotential W, which is a gauge invariant polynomial in the superfields X_i.

The classical bosonic potential for the scalar parts x_i of the chiral superfields X_i and σ_a of the twisted chiral fields Σ_a is given by

$$
\begin{aligned}
U = &\sum_{i=1}^{n} \left| \sum_{a=1}^{k} Q_i^a \sigma_a x_i \right|^2 + \frac{e^2}{2} \sum_{a=1}^{k} \left(\sum_{i=1}^{n} Q_i^a |x_i|^2 - r^a \right)^2 \\
&+ \sum_{i=1}^{n} \left| \frac{\partial W}{\partial x_i}(x_1, ..., x_n) \right|^2 .
\end{aligned}
\tag{5.1}
$$

Here, $r^a \in \mathbb{R}$ is the Fayet-Iliopoulos (FI) parameter of the ath $U(1)$ gauge factor. Together with the corresponding θ-angle θ^a it forms a complex parameter $t^a = r^a - i\theta^a$. (The gauge couplings, e of the $U(1)$-factors are assumed to be equal.)

The classical vacuum manifold is obtained as the space of solutions to the equation $U = 0$ modulo gauge transformations. Its nature depends crucially on the specific values of $(r^1, ..., r^k)$. The subspace parametrized by the expectation values of the matter fields is commonly referred to as the Higgs branch, whereas the scalars σ_a parametrize a Coulomb branch. Phases in which the gauge group is completely broken and all modes transverse to $\{U = 0\}$ are massive are called *geometric phases*. In these phases, the Higgs branch is effectively described by a non-linear sigma model with target space $\{U = 0\}/U(1)^k$. If on the other hand the space of vacua $\{U = 0\}/U(1)^k$ consists of a single point and all modes transverse to the orbit of the complexified gauge group remain massless, the Higgs branch is effectively described by a Landau-Ginzburg (orbifold) theory. Such phases are called *Landau-Ginzburg phases*. Besides these extreme ones, GLSMs can also exhibit various mixed phases. Furthermore, classically at $r = 0$, all fields can be 0. This means that some of the σ_a can be non-zero, and parametrize vacua on another branch, the Coulomb branch.

An important quantum effect is the renormalization of the Fayet-Iliopoulus parameters:

$$r^a(\mu) = r^a_{\text{UV}} + Q^a_{\text{tot}} \log \frac{\mu}{M_{\text{UV}}}. \tag{5.2}$$

Here M_{UV} denotes a UV energy scale, μ the scale under consideration, and $Q^a_{\text{tot}} = \sum_{i=1}^N Q^a_i$ is the total charge of the respective $U(1)$ factor. If $Q^a_{\text{tot}} = 0$ for all a, the axial R-symmetry of the theory is non-anomalous and the FI parameters do not run. The t^a are genuine parameters of the theory. This case is called the "Calabi-Yau case".

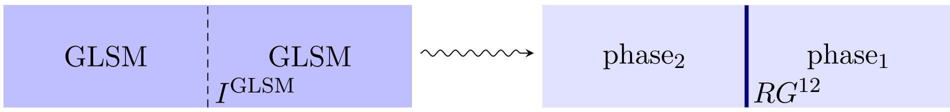
If one of the total charges is non-zero, the respective FI parameter does run under the RG flow. The direction of the running and with it the nature of the low energy IR phase is determined by the sign of the total charge.

In general, the low energy IR phase to which the system is driven by the RG flow consists of several branches. In the specific example considered in section 5.2, there is a Higgs branch described by a Landau-Ginzburg model as well as several massive vacua located on a Coulomb branch.

Note that also in the anomalous case the system can explore various different phases [49]. For this, one chooses r_{UV} such that at some intermediate energy scale the system is well described by the desired phase. Our main example in section 5.2 features, besides the IR phase, an additional phase corresponding to the UV fixed point. This UV phase is a Landau-Ginzburg phase as well, but in contrast to the IR phase, it is a pure Higgs phase, i.e. it does not have additional Coulomb vacua.

5.1.2 Phases of GLSMs and defects

We now want to obtain defects describing the transition between different phases of the same GLSM, much in line with the construction of RG defects in previous chapters. The general idea is to start with the identity or invisible defect I^{GLSM} in the GLSM, and to push the GLSM down to different phases on the two sides of it:



A priori this requires tuning the t^a on the two sides of the defect to different regimes. We avoid doing this explicitly by going to an extreme UV limit of the theory, in which the gauge coupling e becomes very small and the gauge sector decouples [47]. In this limit, the theory reduces to the matter sector, describing the Higgs branch of the original theory. The gauge group still acts on the matter fields, and physical observables must be gauge invariant. The defects are B-type supersymmetric, and depend on the parameters t^a only indirectly. In this setup, the transition to a phase restricts the allowed field configurations and breaks the (remnant of the) gauge symmetry to a subgroup. The details strongly depend on the respective phase.

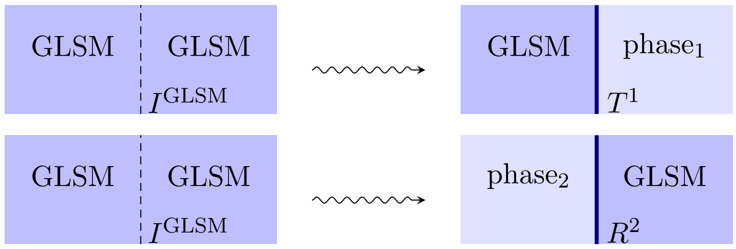
For instance, the class of examples we will discuss in section 5.2 features Landau-Ginzburg phases. In such phases, some of the fields obtain a vacuum expectation value, reducing the spectrum of massless excitations and breaking the gauge symmetry to a finite subgroup. Pushing down to such a phase then involves setting the respective fields to their vacuum expectation values and relaxing the invariance condition accordingly. The general strategy outlined here should be applicable to any phases of abelian GLSMs. Prof. Brunner and Prof. Roggenkamp will discuss transitions to geometric phases in a forthcoming paper [51].

Note that one obtains a possibly different defect for every homotopy class of paths connecting two given phases in the parameter space spanned by the t^a. Thus, in general there will not be one transition defect descending from the gauged linear sigma model, but many, and the choices of defects should correspond to choices of paths. On the other hand, there can be more RG flows and with it RG defects between different phases of a GLSM then the ones described within the GLSM.

Indeed, the transition defects RG^{12} between two phases of a GLSM factorize over the GLSM, i.e. RG^{12} can be obtained as the fusion of a defect T^1 from phase$_1$ to the GLSM and a defect R^2 from the GLSM to phase$_2$:

$$RG^{12} \cong R^2 \otimes T^1 \; .$$

The defects T^1 and R^2 are obtained from the GLSM identity defect, by pushing down only on one side. T^1 is obtained by pushing down I^{GLSM} on the right to phase$_1$ and R^2 by pushing down I^{GLSM} on the left to phase$_2$:



The R^i encode the push down from the GLSM to phase$_i$ and the T^i, the embedding of phase$_i$ into the GLSM. The functors associated to those defects describe the respective operation on D-brane categories.

Note that in the same way that there can be several transition defects RG^{12} between different phases we expect more than one possible defect T^i lifting the phase to the GLSM.[1] This will be discussed in more detail for the concrete example in section 5.2.

Since the transition between one and the same phase has to be trivial, the defects R^i and T^i have to satisfy the condition

$$R^i \otimes T^i \cong I^{\mathrm{phase}_i}, \tag{5.3}$$

where I^{phase_i} is the invisible defect of phase$_i$. This implies that the combination

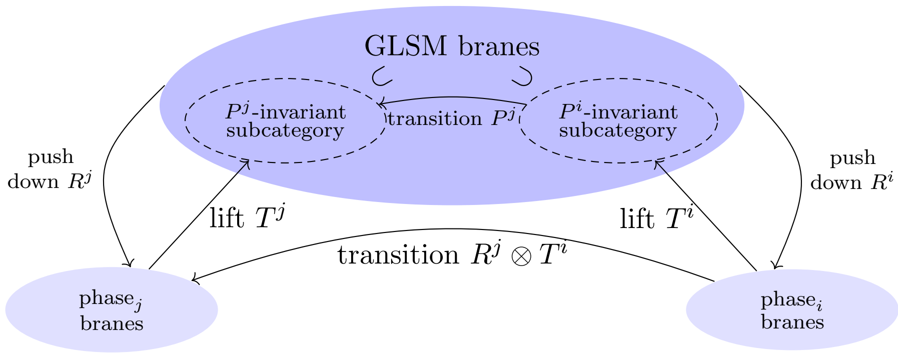
$$P^i = T^i \otimes R^i$$

is a projection defect from the GLSM to itself. That means P^i is an idempotent with respect to fusion, $P^i \otimes P^i \cong P^i$, and realizes phase$_i$ inside the GLSM in the sense of chapter 2. In particular, the corresponding functor projects the category of D-branes of the GLSM onto the image of the functor associated to T^i. Thus, the phase$_i$ branes are realized by P^i-invariant branes in the GLSM. Indeed, the latter precisely play the role of the branes called *grade restricted* in [47] and the action of the projection defects corresponds to the operation of associating to a GLSM brane a grade restricted representative. We will see this explicitly in the example discussed in section 5.2.

We have collected the various defects and their actions on D-branes in

[1] Indeed, one could also push the path dependence on the R^j.

the following diagram:



Note that along an RG flow, Higgs vacua can migrate to the Coulomb branch and become massive. Since we only include the Higgs branch in our discussion, we cannot see this directly. Instead, we observe that D-branes attached to those vacua decouple from the theory. This decoupling of D-branes is encoded in the defects introduced above. They can be constructed out of the identity defect of the respective GLSM, which will be introduced in the next section.

5.1.3 GLSM Identity Defects

Starting point of our construction are the identity defects of abelian GLSMs, which have not appeared in the literature so far. As discussed above, we will focus on the Higgs branch of the respective GLSM. In particular, we will decouple the gauge sector and only consider the $U(1)^k$-orbifold of the matter sector. The relevant defects and D-branes can then be described by means of $U(1)^k$-equivariant matrix factorizations.

Before discussing the identity defects in GLSMs, we reformulate the identity defects in Landau-Ginzburg (orbifold) models presented in sections 3.3 and 3.5.

A reformulation of the orbifold identity defect. For this chapter, we introduce the following notation.

$$\begin{aligned} S_{(X),(X')} &= \mathbb{C}[X_1, \ldots, X_n, X'_1, \ldots, X'_n] \\ C_{(X),(X')} &= S_{(X),(X')}/(W(X_1, \ldots, X_n) - W(X'_1, \ldots, X'_n)) \end{aligned}$$

Replacing X or X' by $-$ in the subscripts means setting the respective variables to zero. The module of the non-orbifold identity defect of chapter 3.3 is then given by the tensor product of the algebra of chiral fields $S_{(X)(X')}$ with the exterior algebra of a vector space $V = \text{span}_{\mathbb{C}}\{\theta_1, \ldots, \theta_n\}$ spanned by additional variables $\theta_1, \ldots, \theta_n$:

$$I^{\text{non-orb.}} = I_0 \oplus I_1 = S_{(X)(X')} \otimes \Lambda(V) \tag{5.4}$$

The identity defect in Landau-Ginzburg orbifolds is now constructed from the identity defect of the unorbifolded LG model using the method of images (i.e. summing images under the orbifold group and specifying a representation of the stabilizer subgroup). Let G be a finite abelian orbifold group. The identity defect of the orbifold model can be obtained by summing over all symmetry defects ${}_gI^{\text{non-orb}}$ of the unorbifolded Landau-Ginzburg model associated to orbifold group elements

$$I^{\text{orb}} = \bigoplus_{g\in G} {}_gI^{\text{non-orb}}\ . \tag{5.5}$$

Note here that one has to orbifold by $G\times G$, the product of the orbifold groups on the left and on the right of the defect, but that the diagonal subgroup acts as an isomorphism on the non-orbifolded identity defect. Hence only a non-diagonal copy of G, which we take to be the copy G_r acting trivially on the left of the defect contributes to the sum above.

The module on which the orbifolded identity matrix factorization is built is therefore a direct sum of $|G|$ copies of the module (5.4) associated to the identity defect in the unorbifolded Landau-Ginzburg model. We can regard it as a tensor product of the module $I^{\text{non-orb}}$ with the regular representation V_{reg} of the group G_r:

$$I^{\text{orb}} \cong I^{\text{non-orb}} \otimes V_{\text{reg}}.$$

The differential acts diagonally in the standard basis $g \in G_r$ of the regular representation, while the orbifold group acts in this basis by permuting the copies of the modules $I^{\text{non-orb}}$ according to the group law.

Since G_r is finite and abelian, we can diagonalize the group action on I^{orb}. This can be accomplished by decomposing the regular representation into irreducibles, which in the case of abelian G_r are all one-dimensional. In this way, we obtain a basis of I^{orb}, in which G_r acts diagonally. Any finite abelian group is isomorphic to a product $\mathbb{Z}_{d_1} \times \ldots \times \mathbb{Z}_{d_r}$. We will spell out the details for the case, in which it is isomorphic to a single factor $G_r \cong \mathbb{Z}_d$. The generalization to more factors is straightforward.

A basis of V_{reg} corresponding to the irreducible representations can be obtained by performing the following transformation:

$$e_j = \sum_{g\in\mathbb{Z}_d} \xi^{-gj} g\,, \quad 0 \le j < d\,, \tag{5.6}$$

where $\xi = \exp(2\pi i/d)$ is an elementary dth root of unity. $g \mapsto \xi^{jg}$ is the character of the irreducible representation ρ_j defined by $\rho_j([n]_d) = \xi^{jn}$. Hence, e_j is the basis vector of the irreducible representation $\rho_j([n]_d) = \xi^{jn}$, which is of course nothing but the j-fold tensor product of ρ_1: $\rho_j = \rho_1^{\otimes j}$. Thus, we can write $e_j = e_1^{\otimes j}$. Note that $\rho_1^{\otimes d} = \rho_0$. Writing $e_j = \alpha^{-j}$, the regular representation can be expressed as

$$V_{\text{reg}} \cong \mathbb{C}[\alpha]/(\alpha^d - 1)\,. \tag{5.7}$$

Note that this is not only a vector space, but also a ring, and that multiplication in this ring corresponds to taking the tensor product of representations. This allows us to rewrite the identity matrix factorization in the Landau-Ginzburg orbifold with orbifold group $G = \mathbb{Z}_d$ as

$$I^{\mathrm{orb}} = S_{(X)(X')} \otimes \Lambda(V) \otimes \mathbb{C}[\alpha]/(\alpha^d - 1) \tag{5.8}$$

with differential

$$\mathrm{d}_{I^{\mathrm{orb}}} = \sum_{i=1}^{n} \Big[(X_i - \alpha^{Q_i} X_i') \cdot \theta_i^* + \partial_i^{X,\alpha X'} W \cdot \theta_i\Big] \quad \text{for}$$

$$\partial_i^{X,\alpha X'} W = \frac{\begin{pmatrix} W(\alpha^{Q_1} X_1', ..., \alpha^{Q_{i-1}} X_{i-1}', X_i, ..., X_n) \\ \qquad - W(\alpha^{Q_1} X_1', ..., \alpha^{Q_i} X_i', X_{i+1}, ..., X_n) \end{pmatrix}}{X_i - \alpha^{Q_i} X_i'} \,.$$

Here, Q_i denote the charges of the chiral fields X_j under the orbifold group $\mathbb{Z}_d$, i.e. $[n] \in \mathbb{Z}_d$ acts on the chiral fields as $X_j \mapsto \xi^{Q_j n} X_j$. α can be regarded as a new bosonic defect field carrying charg $(1, -1)$ under the product $\mathbb{Z}_d \times \mathbb{Z}_d$ of the left and right orbifold groups.[2]

The representation on I^{orb} under $G \times G$ is now completely fixed by the choice of a one-dimensional representation of the diagonal subgroup, since the latter left the identity defect of the non-orbifold theory invariant. We choose it to be trivial to obtain the identity defect in the orbifold theory. (Other choices lead to defects implementing the quantum symmetry of the orbifold theory.) The representations of $G \times G$ on the module (5.8) is determined by the representation on the chiral fields X_i, the θ_i (which transform like the X_i) and the representation on α^i.

Let us give an explicit example which will be important later. Consider the Landau-Ginzburg model with a single chiral superfield X, superpotential $W(X) = X^d$ and orbifold group $G = \mathbb{Z}_d$. $[n]_d \in \mathbb{Z}_d$ acts on X by multiplication with a phase

$$X \mapsto e^{2\pi i n d'/d} X$$

which leaves $W(X)$ invariant. (X has charge d' under $\mathbb{Z}_d$.) Following the construction above, one obtains the identity matrix factorization

$$I^{\mathrm{orb}} : S^{\alpha}\{[1]_d, [0]_d\} \underset{i_0 = \prod_{i=1}^{d-1}(X - \xi^i \alpha^{d'} X')}{\overset{i_1 = (X - \alpha^{d'} X')}{\rightleftarrows}} S^{\alpha}\{[0]_d, [0]_d\} \,,$$

[2] The charge under the action of the right group is clear, because α represents the basis vector of the irreducible representation ρ_{-1} under the right group. That it has charge 1 under the left group follows because V_{reg} was chosen to be invariant under the left-right diagonal subgroup.

where $S^\alpha := \mathbb{C}[X, X', \alpha]/(\alpha^d - 1)$, and the $\{\cdot,\cdot\}$ denote a shift in $\mathbb{Z}_d \times \mathbb{Z}_d$-charges. The associated $C_{(X)(X')}$-module is given by

$$S^\alpha/(X - \alpha^{d'} X') \ . \tag{5.9}$$

One can unpack this, by replacing α by a cyclic shift matrix. This yields the equivalent representation

$$\begin{aligned} I^{\mathrm{orb}} : & S^d\begin{pmatrix} \{[1]_d,[0]_d\} \\ \{[2]_d,[-1]_d\} \\ \vdots \\ \{[d]_d,[-1+d]_d\} \end{pmatrix} \\ & \qquad \underset{\imath_0 = \prod_{i=1}^{d-1}(X\mathbb{I}_d - \xi^i \epsilon^{d'} X')}{\overset{\imath_1 = (X\mathbb{I}_d - \epsilon^{d'} X')}{\rightleftarrows}} S^d\begin{pmatrix} \{[0]_d,[0]_d\} \\ \{[1]_d,[-1]_d\} \\ \vdots \\ \{[d-1]_d,[-1+d]_d\} \end{pmatrix}, \end{aligned} \tag{5.10}$$

see also chapter 4.1. Here $S = \mathbb{C}[X, X']$, $\mathbb{I}_d$ is the $d \times d$-identity matrix, and

$$\epsilon_d = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \tag{5.11}$$

denotes the $d \times d$-shift matrix.

The identity defect in abelian gauged linear sigma models. For gauged linear sigma models, mainly boundaries were considered in the literature, see [47, 48, 52]. Defects can in principle be discussed along the same lines, for example using the folding trick, see [53, 54] or section 3.3. In [53] identity defects of Landau-Ginzburg phases of GLSMs are lifted to GLSMs. Using such constructions, one cannot obtain an identity defect of the GLSM itself, because the lifts are matrix factorizations of finite rank.

We now use the method presented in the previous section to construct an identity defect for a $U(1)$-orbifold of a Landau-Ginzburg model with chiral fields $X_1, \ldots, X_n$ and superpotential $W \in \mathbb{C}[X_1, \ldots, X_n]$. The action of $U(1)$ on the chiral fields is specified by their charges $(Q_1, \ldots, Q_n)$, where $\varphi \in U(1)$ acts on X_j by $X_j \mapsto e^{2\pi i Q_j \varphi} X_j$. (Generalizations to higher rank abelian gauge groups are straightforward.)

Since the orbifold group is infinite (and not even countable), the method of images cannot be applied in this situation. As it turns out, the formulation with the additional defect field α however can be adapted. The irreducible representations of $U(1)$ are countable, $\rho_j(\varphi) = e^{2\pi i j\varphi}$, $j \in \mathbb{Z}$, with $\rho_i \otimes \rho_j \cong \rho_{i+j}$. But in contrast to the case of representations of $\mathbb{Z}_d$, not all representations can be obtained as tensor products of a single representation

ρ_{-1}. One needs an additional representation ρ_1 to generate all representations by means of the tensor product. So in the $U(1)$-case, instead of one additional bosonic defect field α, one has to introduce two fields α, α^{-1} which are inverse to each other, i.e. $\alpha\alpha^{-1} = 1$. They carry $U(1) \times U(1)$-charges $(1, -1)$ and $(-1, 1)$, respectively.

With these additional fields one can construct the following defect in complete analogy to (5.8)

$$I = S_{(X),(X')} \otimes \Lambda(V) \otimes \mathbb{C}[\alpha, \alpha^{-1}]/(\alpha\alpha^{-1} - 1) \tag{5.12}$$

with differential

$$\mathrm{d}_I = \sum_{i=1}^{n} \left[(X_i - \alpha^{Q_i} X_i') \cdot \theta_i^* + \partial_i^{X,\alpha X'} W \cdot \theta_i \right] ,$$

$$\partial_i^{X,\alpha X'} W = \frac{\begin{pmatrix} W(\alpha^{Q_1} X_1', ..., \alpha^{Q_{i-1}} X_{i-1}', X_i, ..., X_n) \\ -W(\alpha^{Q_1} X_1', ..., \alpha^{Q_i} X_i', X_{i+1}, ..., X_n) \end{pmatrix}}{X_i - \alpha^{Q_i} X_i'} .$$

The $U(1) \times U(1)$-representation on this matrix factorization is completely determined by the transformation properties of the fields X_i, i.e. their $U(1)$-charges Q_i. The θ_i transform as the X_i.

Note that this matrix factorization is of infinite rank!

As in the case of finite orbifolds, there is the possibility of shifting the charges of the orbifold group on one side, relative to the one on the other. The defect with such a shift implements a quantum symmetry. Since we are interested in the identity defect, we set this shift to zero.

As alluded to above, the generalization to orbifold groups $U(1)^k$ $k > 1$ is straightforward. One just has to introduce a pair of additional fields (α, α^{-1}) for each $U(1)$-factor.

In the discussion of the identity defect of finite Landau-Ginzburg orbifolds in the last subsection, we just gave a different but equivalent representation of the known identity defect. In the case of the $U(1)$-orbifold, the defect (5.12) is new, and we have to show that it really is the identity defect, i.e. that it behaves as the unit with respect to fusion.

To do so, we use associated modules as explained in section 3.3 and introduce some additional notation:

$$\begin{aligned} S_{(X),(X')}^{(\alpha,\alpha^{-1})} &= S[\alpha, \alpha^{-1}]/(\alpha\alpha^{-1} - 1) \\ C_{(X),(X')}^{(\alpha,\alpha^{-1})} &= C_{(X),(X')}[\alpha, \alpha^{-1}]/(\alpha\alpha^{-1} - 1) . \end{aligned}$$

To the matrix factorization I constructed above we associate the $C_{(X),(X')}$-module

$$M_I = C_{(X),(X')}^{(\alpha,\alpha^{-1})} / \left(X_1 - \alpha^{Q_1} X_1', \ldots, X_n - \alpha^{Q_n} X_n' \right) \tag{5.13}$$

following the discussion of associated modules in chapter 3.3. Namely, this module has a free resolution, which after finitely many steps turns into the two-periodic complex induced by the matrix factorization I.

Let P' be a $U(1)$-equivariant matrix factorization of $W(X'_1, \ldots, X'_n)$. The analysis of the fusion $I \otimes P'$ now runs in complete analogy of the discussion of the fusion of the identity defect in the unorbifolded Landau-Ginzburg models, except for the fact that the fusion in the orbifold corresponds only to the part of the tensor product matrix factorization which is invariant under the gauge group associated to the model squeezed in between the two defects.

Let

$$M_{P'} = \mathrm{coker}(p'_1 : P'_1 \otimes_{S_{(-)(X')}} C_{(-)(X')} \to P'_0 \otimes_{S_{(-)(X')}} C_{(-)(X')})$$

be the module associated to P'. The matrix factorization given by the fusion of I and P' can be extracted from the $U(1)$-invariant part $M^{U(1)}$ of the $C_{(X),(-)}$-module

$$M = M_I \otimes_{C_{(-)(X')}} M_{P'} .$$

The relations in (5.13) can now be used to replace all X'_i by $\alpha^{-Q_i} X_i$. This eliminates all the variables X'_i. Let us next choose generators e_r of $M_{P'}$, on which $U(1)$ acts diagonally, with respective $U(1)$-charge q_r. Then M is generated by $\alpha^i \otimes e_r$, where $i \in \mathbb{Z}$. Most of these generators are not $U(1)$-invariant though. Only the $\tilde{e}_r := \alpha^{-q_r} \otimes e_r$ generate $M^{U(1)}$. Thus, for each generator e_r of $M_{P'}$ of $U(1)$-charge q_r there is exactly one generator of $M^{U(1)}$, which also has $U(1)$-charge q_r. (Recall that α has $U(1)\times U(1)$-charge $(1, -1)$.) The relations between the generators e_r in $M_{P'}$ become relations between the respective generators $\tilde{e}_r$, where all the X'_i are replaced by X_i. Thus, $M^{U(1)}$ is isomorphic to the module M_P associated to the matrix factorization P obtained from P' by replacing all X'_i by X_i. Therefore, fusion with I maps matrix factorizations to equivalent ones, and the matrix factorization I acts as the identity matrix factorization.

5.2 Example: superpotential $W(X, P) = X^d P^{d'}$

5.2.1 The model and its phases

In this section we exemplify our method in a concrete example of a gauged linear sigma model with two Landau-Ginzburg phases. The model has a single $U(1)$-gauge group and two chiral fields X and P of $U(1)$-charges $Q_x = d'$, respectively $Q_p = -d$. Its superpotential is given by $W = X^d P^{d'}$. We assume $d > d'$, and for simplicity we restrict to the case where d and d' are coprime integers.

For this model, the scalar potential (5.1) takes the form

$$U = |\sigma|^2 \left(Q_x^2|x|^2 + Q_p^2|p|^2\right) + \frac{e^2}{2}\left(Q_x|x|^2 + Q_p|p|^2 - r\right)^2 + |\partial_x W(p,x)|^2 + |\partial_p W(p,x)|^2 .$$

The total charge $Q_{tot} = d' - d < 0$ is negative, which means that the Fayet-Iliopoulos parameter (5.2) runs under the RG flow, from $r << 0$ in the UV to $r >> 0$ in the IR. The model exhibits two Landau-Ginzburg phases.

For $r < 0$, the D-term constraint coming from the second term above requires $p \neq 0$. This breaks the $U(1)$ gauge symmetry to $\mathbb{Z}_{|Q_p|} = \mathbb{Z}_d$ and σ must vanish according to the first term. Because of the first superpotential term, x also vanishes and hence $|p|^2 = \frac{r}{Q_p} = -\frac{r}{d}$. We obtain a Landau-Ginzburg orbifold model with one chiral field X, superpotential X^d and orbifold group $\mathbb{Z}_d$.

For $r > 0$, the roles of X and P are interchanged. The D-term constraint yields $x \neq 0$, which further implies that σ and p vanish. The $U(1)$ gauge group is broken to $\mathbb{Z}_{|Q_x|} = \mathbb{Z}_{d'}$, and $|x|^2 = \frac{r}{Q_x} = \frac{r}{d'}$. We arrive at a Landau-Ginzburg orbifold model with chiral field P, superpotential $P^{d'}$ and orbifold group $\mathbb{Z}_{d'}$.

Classically, there is a Coulomb branch emerging at $r = 0$, parametrized by σ. Due to a twisted superpotential, the values of σ will be restricted to a finite set of $d - d'$ massive vacua that appear in the IR phase.

In the following we will use our general strategy to construct defects describing the transitions between UV and IR phase of this model, defects embedding the two phases in the GLSM as well as defects projecting the GLSM to the phases.

Note that there is an effective description of the mirror of this GLSM in terms of an ordinary Landau-Ginzburg model [55], which we have already met at the end of section 3.1 - namely the Landau-Ginzburg model with one chiral field X and superpotential

$$W = X^d + e^{t/d} X^{d'} .$$

The deformation parameter $\lambda = e^{t/d}$ of the superpotential is related to the complexified Fayet-Iliopoulus parameter $t = r - i\theta$ of the GLSM. λ runs under the RG flow from $\lambda = 0$ in the UV to $\lambda = \infty$ in the IR. In the UV, the model is therefore described by a LG model with superpotential $W = X^d$ and in the IR by a LG model with superpotential $W = X^{d'}$.

Now, flows of Landau-Ginzburg models triggered by deformations of the superpotentials are relatively well under control, and at least some aspects of them can be studied very explicitly. For instance, it is not difficult to analyze what happens to the vacua of the model, which correspond to critical points of the superpotential. In the case at hand, some of these vacua ($d - d'$ many) move off to infinity under the RG flow, and decouple from the theory, taking with them some (A-type) D-branes attached to them.

This decoupling of D-branes is well described by RG defects associated to the flows. Indeed, all the RG defects corresponding to flows of Landau-Ginzburg models with a single chiral superfield but general deformations of the superpotential

$$W = X^d + \sum_{i=1}^{d-1} \lambda_i X^i \ , \tag{5.14}$$

have been constructed in [3].[3] Thus, the transition defects between UV and IR phases of the GLSM which we will obtain here can be checked against known results. We find complete agreement.

5.2.2 GLSM identity defect

The starting point of our analysis is the identity defect of the GLSM as constructed for the general abelian GLSMs in section 5.1.3. In this case, it is a $U(1) \times U(1)$-equivariant matrix factorization of the difference $W(X,P) - W(Y,Q) = X^d P^{d'} - Y^d Q^{d'}$ of the superpotentials of the gauged linear sigma models on either side of the defects. The two $U(1)$-factors correspond to the gauge groups of the models on the left and the right of the defect, respectively.

For clarity, we will repeat and spell out some details of the construction of the identity defect in this example. We first introduce new variables (corresponding to degrees of freedom on the identity defect) α and α^{-1} which satisfy $\alpha\alpha^{-1} = 1$ and which carry $U(1) \times U(1)$-charges $|\alpha| = (1,-1)$ and $|\alpha^{-1}| = (-1,1)$. Using these fields, we can write the difference of the superpotentials as follows

$$\begin{aligned} W(X,P) - W(Y,Q) &= X^d P^{d'} - Y^d Q^{d'} \\ &= X^d P^{d'} - \alpha^{dd'} Y^d P^{d'} + \alpha^{dd'} Y^d P^{d'} - Y^d Q^{d'} \\ &= \left(X^d - (\alpha^{d'} Y)^d\right) P^{d'} + Y^d \left((\alpha^d P)^{d'} - Q^{d'}\right) \\ &= P^{d'} \prod_{i=0}^{d-1} (X - \xi^i \alpha^{d'} Y) + Y^d \prod_{i=0}^{d'-1} (\alpha^d P - (\xi')^i Q) \,. \end{aligned}$$

Here $\xi = e^{2\pi i/d}$ and $\xi' = e^{2\pi i/d'}$ are elementary dth, respectively d'th roots of unity.

The matrix factorization associated to the identity defect is then given by the Koszul-type matrix factorization associated to $(X - \alpha^{d'} Y)$ and $(\alpha^d P - Q)$. More precisely, denoting the $\mathbb{C}[X,P,Y,Q]$-modules

$$\begin{aligned} S &= S_{(X,P)(Y,Q)} = \mathbb{C}[X,P,Y,Q] \\ \text{and} \quad \tilde{S} &= S^{(\alpha,\alpha^{-1})}_{(X,P)(Y,Q)} = S_{(X,P)(Y,Q)}[\alpha,\alpha^{-1}]/(\alpha\alpha^{-1} - 1) \,. \end{aligned}$$

[3]More precisely, they have been constructed in the mirror theories.

the identity matrix factorization can be written as

$$I : \tilde{S}^2 \begin{pmatrix} \{d',0\} \\ \{0,-d\} \end{pmatrix} \underset{i_0}{\overset{i_1}{\rightleftarrows}} \tilde{S}^2 \begin{pmatrix} \{0,0\} \\ \{d',-d\} \end{pmatrix} \tag{5.15}$$

Here $\{\cdot,\cdot\}$ indicates the $U(1)\times U(1)$-charge of the respective generator and

$$\begin{aligned} i_1 &= \begin{pmatrix} (X-\alpha^{d'}Y) & -(\alpha^d P - Q) \\ Y^d \prod_{i=1}^{d'-1}(\alpha^d P - (\xi')^i Q) & P^{d'} \prod_{i=1}^{d-1}(X - \xi^i \alpha^{d'} Y) \end{pmatrix} \\ i_0 &= \begin{pmatrix} P^{d'} \prod_{i=1}^{d-1}(X - \xi^i \alpha^{d'} Y) & (\alpha^d P - Q) \\ -Y^d \prod_{i=1}^{d'-1}(\alpha^d P - (\xi')^i Q) & (X-\alpha^{d'}Y)\ . \end{pmatrix} \end{aligned}$$

This is nothing but the GLSM identity matrix factorization (5.12) spelled out for the special case at hand. To it we associate the module

$$M_I = C^{(\alpha,\alpha^{-1})}_{(X,P)(Y,Q)}/((X-\alpha^{d'}Y),(\alpha^d P - Q)) \tag{5.16}$$

over the ring

$$C = C_{(X,P)(Y,Q)} = S_{(X,P)(Y,Q)}/(W(X,P) - W(Y,Q))\,,$$

where

$$C^{(\alpha,\alpha^{-1})}_{(X,P)(Y,Q)} = C_{(X,P)(Y,Q)}[\alpha,\alpha^{-1}]/(\alpha\alpha^{-1}-1)\,,$$

c.f. the general case (5.13). The Koszul resolution of M_I turns into the two-periodic complex induced by the identity matrix factorization I after two steps.

5.2.3 Pushing down the identity defect into phases

Going into the phases of the GLSM, one of the two chiral fields gets a vacuum expecation value (which we can take to be 1), and the gauge group is broken to the subgroup leaving this chiral field invariant. We can therefore push down any defect of the GLSM into a phase by setting the respective chiral field to 1 in the associated matrix factorization and considering it equivariant with respect to the residual gauge group. In fact, this can be done on either side of the defect yielding defects from the GLSM into the phases or from the phases to the GLSM. Moreover, one can push down to phases on both sides of the defect, possibly into different phases on the two sides, which gives rise to defects in the phases or from one phase to another.

In the next section, we will apply this push-down to the GLSM identity defect. Pushing down to the UV phase on the right side and the IR phase on the left, we will obtain a defect describing the transition from the UV phase to the IR phase. Indeed, we will reproduce RG defects of chapter 4, as expected.

Before we come to this, as a warm-up we first discuss the simpler case, where the GLSM identity defect is pushed down to the same phase on both sides, which we choose to be the UV phase. The push down to the IR can be dealt with in a similar way.

To push down the GLSM identity defect to the UV phase on both sides, we have to set P and Q to 1 in the matrix factorization (5.15) and consider it equivariant with respect to the residual gauge group $\mathbb{Z}_d \times \mathbb{Z}_d$. We can do this on the level of the associated module (5.16).

It will be useful to introduce some notation. Replacing the name of a variable with a '$\cdot$' in the subscripts of the rings $S_{(X,P)(Y,Q)}$, $C_{(X,P)(Y,Q)}$ or $C^{(\alpha,\alpha^{-1})}_{(X,P)(Y,Q)}$ just means setting the respective variable to one.[4] For instance

$$\begin{aligned} S_{(X,\cdot)(Y,Q)} &= \mathbb{C}[X,Y,Q] \\ C_{(X,\cdot)(Y,Q)} &= S_{(X,\cdot)(Y,Q)}/(W(X,1)-W(Y,Q)) \\ C^{(\alpha,\alpha^{-1})}_{(X,\cdot)(Y,Q)} &= C_{(X,\cdot)(Y,Q)}[\alpha,\alpha^{-1}]/(\alpha\alpha^{-1}-1)\,. \end{aligned}$$

Pushing down the GLSM identity defect to the UV phase on both sides yields the module

$$M_I^{\mathrm{UV\,UV}} = C^{(\alpha,\alpha^{-1})}_{(X,\cdot)(Y,\cdot)}/((X-\alpha^{d'}Y),(\alpha^d-1))\,.$$

Note that due to the relation α^d-1 this module is of finite rank, and in fact isomorphic to the module (5.9) associated to the identity matrix factorization of the LG model describing the UV phase. In fact, identifying[5]

$$C^{(\alpha,\alpha^{-1})}_{(X,\cdot)(Y,\cdot)}/(\alpha^d-1) \cong C^d_{(X,\cdot)(Y,\cdot)}\begin{pmatrix} \{[0]_d,[0]_d\} \\ \{[1]_d,[-1]_d\} \\ \vdots \\ \{[d-1]_d,[-d+1]_d\} \end{pmatrix},$$

where $\{\cdot,\cdot\}$ denotes the shift in $\mathbb{Z}_d\times\mathbb{Z}_d$-charge of the respective generators, one can write $M_I^{\mathrm{UV\,UV}}$ as cokernel of the map $\imath_1^{\mathrm{UV}} = (X\mathbb{1}_d - Y\epsilon_d^{d'})$,

$$\imath_1^{\mathrm{UV}} : C^d_{(X,\cdot)(Y,\cdot)}\begin{pmatrix} \{[d']_d,[0]_d\} \\ \{[d'+1]_d,[-1]_d\} \\ \vdots \\ \{[d'+d-1]_d,[-d+1]_d\} \end{pmatrix} \to C^d_{(X,\cdot)(Y,\cdot)}\begin{pmatrix} \{[0]_d,[0]_d\} \\ \{[1]_d,[-1]_d\} \\ \vdots \\ \{[d-1]_d,[-d+1]_d\} \end{pmatrix}.$$

Here, as before $\mathbb{1}_d$ denotes the $d\times d$-identity matrix and ϵ_d the $d\times d$-shift matrix (5.11).

Indeed, $\imath_1^{\mathrm{UV}}\imath_0^{\mathrm{UV}} = (X^d-Y^d)\mathbb{1}_d$ for

$$\imath_0^{\mathrm{UV}} = \prod_{i=1}^{d-1}(X\mathbb{1}_d - \xi^i Y\epsilon_d^{d'})\,.$$

[4]Or to put it differently, by diving the rings by the corresponding ideals.
[5]the generator α^i is sent to the generator e_i

Hence, $\imath_1^{\mathrm{UV}}$ is a factor of a matrix factorization of $W(X,1) - W(Y,1)$, namely

$$I^{\mathrm{UV}} : S^d_{(X,\cdot)(Y,\cdot)} \begin{pmatrix} \{[1]_d,[0]_d\} \\ \{[2]_d,[-1]_d\} \\ \vdots \\ \{[d]_d,[-d+1]_d\} \end{pmatrix} \underset{\imath_0^{\mathrm{UV}}}{\overset{\imath_1^{\mathrm{UV}}}{\rightleftarrows}} S^d_{(X,\cdot)(Y,\cdot)} \begin{pmatrix} \{[0]_d,[0]_d\} \\ \{[1]_d,[-1]_d\} \\ \vdots \\ \{[d-1]_d,[-d+1]_d\} \end{pmatrix} \tag{5.17}$$

This matrix factorization corresponds to the identity defect (5.10) in the UV phase. Thus, the module $M_I^{\mathrm{UV\,UV}}$ is associated to the identity matrix factorization in the UV phase. Pushing down the GLSM identity defect on both sides to the UV, one therefore produces the identity defect in the UV phase. Similarly, pushing down the GLSM identity defect to the IR phase on both sides yields the IR identity defect. This is of course what is to be expected.

5.2.4 RG defects from the GLSM identity

Next, we push down the GLSM identity defect to the UV on the right and to the IR on the left, to obtain a transition defect between UV and IR phase. Note that there is more than one (homotopy class of) paths from UV to IR. So there should be more than one such transition defects.

Implementing the push-down involves setting those variables to 1 in the GLSM identity matrix factorization, which correspond to fields acquiring a vacuum expectation value in the respective phases. These are X (IR phase on the left of defect) and Q (UV phase on the right). On the level of $C_{(\cdot,P)(Y,\cdot)}$-modules this yields

$$M_I^{\mathrm{UV\,IR}} = C^{(\alpha,\alpha^{-1})}_{(\cdot,P)(Y,\cdot)}/((Y-\alpha^{-d'}),(P-\alpha^{-d}))\,.$$

In contrast to the push down to the same phase on both sides, this module is of infinite rank, leading to a matrix factorization of infinite rank, which does not correspond to one of the RG defects between the respective Landau-Ginzburg models. We propose that under the push-down to the phases, the module (and the respective matrix factorization) has to be truncated to finite rank. The truncation is not unique, but it turns out, that the different choices of truncation exactly correspond to the different paths from UV to IR.[6] Concretely, we introduce an upper bound N on the α-exponents[7]

$$M_I^{\mathrm{UV\,IR}}(N) = \frac{\alpha^N C_{(\cdot,P)(Y,\cdot)}[\alpha^{-1}]}{((Y-\alpha^{-d'}),(P-\alpha^{-d}))\alpha^N C_{(\cdot,P)(Y,\cdot)}[\alpha^{-1}]}\,.$$

[6] We expect the truncation to be related to the gradability of the resulting matrix factorization with respect to R-symmetry. The latter ensures definite gluing conditions for the spectral flow operators of the respective SCFTs along the defect. This is needed to impose a stability condition in the sense of [56] on the level of the defect.

[7] Later we will show that N determines the charge window for the grade restriction rule which appears in [47].

This module is now of finite rank. It has generators $e_i := \alpha^{N-i}$ for $0 \leq i < d'$ of $\mathbb{Z}_{d'} \times \mathbb{Z}_d$-charges $([N-i]_{d'}, -[N-i]_d)$. To write down the relations in a convenient way we define $a, b \in \mathbb{N}$ with $b < d'$ such that

$$d = ad' + b\,. \tag{5.18}$$

Then the generators satisfy relations

$$\begin{aligned} Pe_i &= P\alpha^{N-i} = \alpha^{N-d-i} = Y^a \alpha^{N-d-i+ad'} \\ &= \begin{cases} Y^a e_{i+b}\,, & i+b < d' \\ Y^{a+1} e_{i+b-d'}\,, & i+b \geq d' \end{cases} \end{aligned}$$

and the module $M_I^{\mathrm{UV\,IR}}(N)$ is isomorphic to the cokernel of a map

$$\begin{aligned} rg_1 :& C^{d'}_{(\cdot,P)(Y,\cdot)} \begin{pmatrix} {\scriptstyle \{[N-d]_{d'},[-N]_d\}} \\ {\scriptstyle \{[N-d-1]_{d'},[-N+1]_d\}} \\ \vdots \\ {\scriptstyle \{[N-d-d'+1]_{d'},[-N-1+d']_d\}} \end{pmatrix} \\ &\to C^{d'}_{(\cdot,P)(Y,\cdot)} \begin{pmatrix} {\scriptstyle \{[N]_{d'},[-N]_d\}} \\ {\scriptstyle \{[N-1]_{d'},[-N+1]_d\}} \\ \vdots \\ {\scriptstyle \{[N-d'+1]_{d'},[-N+d'-1]_d\}} \end{pmatrix}, \end{aligned}$$

which can be written as $rg_1 = (P\mathbb{1}_{d'} - \epsilon^b_{d'} I_Y)$, where I_Y denotes the diagonal $d' \times d'$-matrix with Y^a as its first $d'-b$ diagonal entries and Y^{a+1} as the last b diagonal entries. Explicitly

$$rg_1 = \begin{pmatrix} P & & & -Y^{a+1} & & \\ & \ddots & & & \ddots & \\ & & \ddots & & & -Y^{a+1} \\ -Y^a & & & \ddots & & \\ & \ddots & & & \ddots & \\ & & -Y^a & & & P \end{pmatrix}.$$

Now

$$\prod_{i=0}^{d'-1} (P\mathbb{1}_{d'} - (\xi')^i \epsilon^b_{d'} I_Y) = (P^{d'} - Y^d)\mathbb{1}_{d'}\,,$$

and hence, rg_1 together with $rg_0 = \prod_{i=1}^{d'-1}(P\mathbb{1}_{d'} - (\xi')^i\epsilon^b_{d'} I_Y)$ defines a matrix factorization RG_N:

$$\begin{aligned} &S^{d'}_{(\cdot,P)(Y,\cdot)} \begin{pmatrix} {\scriptstyle \{[N-d]_{d'},-[N]_d\}} \\ {\scriptstyle \{[N-1-d]_{d'},-[N-1]_d\}} \\ \vdots \\ {\scriptstyle \{[N-d'+1-d]_{d'},-[N-d'+1]_d\}} \end{pmatrix} \\ &\qquad \underset{rg_0}{\overset{rg_1}{\rightleftarrows}} S^{d'}_{(\cdot,P)(Y,\cdot)} \begin{pmatrix} {\scriptstyle \{[N]_{d'},-[N]_d\}} \\ {\scriptstyle \{[N-1]_{d'},-[N-1]_d\}} \\ \vdots \\ {\scriptstyle \{[N-d'+1]_{d'},-[N-d'+1]_d\}} \end{pmatrix} \end{aligned} \tag{5.19}$$

Pushing down the GLSM identity defect to the UV on the right and to the IR on the left with truncation N yields a defect between the Landau-Ginzburg models in the UV and in the IR, given by the matrix factorization RG_N. Note that N only appears in the grading of the matrix factorization RG_N, and that $RG_N = RG_{N+dd'}$. Furthermore, the shift in N corresponds to conjugation with the quantum symmetries of the respective Landau-Ginzburg phases, $RG_{N+1} = Q_{\mathrm{IR}}^{-1} \otimes R_N \otimes Q_{\mathrm{UV}}$.

Thus, we obtain dd' many different transitions defects between the two phases. These indeed correspond to particular renormalization group defects between Landau-Ginzburg orbifolds describing the UV and IR phases [3]. In fact RG defects between these Landau-Ginzburg orbifolds corresponding to general perturbations of type (5.14) would allow for a more generic distribution of powers of Y in the map rg_1, of the form

$$rg_1^{gen} = \begin{pmatrix} P & & & -Y^{n_b} & & \\ & \ddots & & & \ddots & \\ & & \ddots & & & -Y^{n_{d'}} \\ -Y^{n_1} & & & \ddots & & \\ & \ddots & & & \ddots & \\ & & -Y^{n_{b-1}} & & & P \end{pmatrix} .$$

where $\sum n_a = d$ and the grades appearing in (5.19) have to be modified accordingly, c.f. [3]. The transition defects we obtain from the gauged linear sigma model are special cases of these defects which exhibit a maximally homogeneous distribution of powers of Y in rg_1.

Summarizing, by pushing down to UV and IR on the right, respectively left side of the GLSM identity defect with an additional truncation we obtain an RG defect between the UV and IR phases, which is known to describe the transport between UV and IR Landau-Ginzburg models. Different choices of the truncation parameter N only shift the charges of the matrix factorization, in particular $RG_N \cong RG_0\{[N]_{d'}, -[N]_d\} \cong Q_{\mathrm{IR}}^N \otimes RG_0 \otimes Q_{\mathrm{UV}}^{-N}$. The charge shifts are quantum symmetries of the Landau-Ginzburg orbifolds in IR and UV. They can be obtained as monodromies of the GLSM upon encircling the Landau-Ginzburg points in the Kähler parameter space. Thus, up to winding around the limit points, we obtain one defect describing the flow between UV and IR phases of the GLSM.

5.2.5 Factorization of RG defects

Using the fact that the GLSM identity defect is an idempotent,

$$I \cong I \otimes I \,, \tag{5.20}$$

we can factorize the RG defects RG_N over the GLSM. More precisely

$$RG_N \cong R^{\mathrm{IR}} \otimes T_N^{\mathrm{UV}} ,$$

where R^{IR} is a defect from the GLSM to the IR phase obtained by pushing down the GLSM identity defect to the IR on the left side, and T_N^{UV} is a defect from the UV phase to the GLSM model obtained by pushing the GLSM identity defect to the UV on the right and truncating.[8] Let us discuss the factor defects in turn.

$\boldsymbol{T_N^{\mathrm{UV}}}$**.** The module associated to T_N^{UV} is obtained by setting $Q=1$ in (5.16) and then truncating the α-spectrum. This yields the $C_{(X,P)(Y,\cdot)}$-module

$$M_I^{\mathrm{UV\,GLSM}}(N) = \frac{\alpha^N C_{(X,P)(Y,\cdot)}[\alpha^{-1}]}{((Y-\alpha^{-d'}X),(P-\alpha^{-d}))\alpha^N C_{(X,P)(Y,\cdot)}[\alpha^{-1}]} .$$

It is finitely generated with generators $e_i = \alpha^{N-i}$, of $U(1)\times\mathbb{Z}_d$-charge $(N-i, -[N-i]_d)$, where $0\le i<d$. The generators satisfy relations

$$\begin{aligned} Ye_i &= Y\alpha^{N-i} = X\alpha^{N-i-d'} \\ &= \begin{cases} Xe_{i+d'}\,, & i+d'<d \\ PXe_{i+d'-d}\,, & i+d'\geq d \end{cases} , \end{aligned} \tag{5.21}$$

and $M_I^{\mathrm{UV\,GLSM}}(N)$ is isomorphic to the cokernel of the map

$$\begin{aligned} t_1 : C^d_{(X,P)(Y,\cdot)} &\begin{pmatrix} \{N,-[N-d']_d\} \\ \{N-1,-[N-1-d']_d\} \\ \vdots \\ \{N-d+1,-[N-d+1-d']_d\} \end{pmatrix} \\ &\to C^d_{(X,P)(Y,\cdot)} \begin{pmatrix} \{N,[-N]_d\} \\ \{N-1,-[N-1]_d\} \\ \vdots \\ \{N-d+1,-[N-d+1]_d\} \end{pmatrix} , \end{aligned}$$

with $t_1 = (\epsilon_d^{d'} I_P X - Y\mathbb{1}_d)$. Here I_P is the diagonal $d\times d$-matrix with 1 in the first $b=d-d'$ diagonal entries and P in the last d' diagonal entries. Explicitly,

$$t_1 = \begin{pmatrix} -Y & & & PX & & \\ & \ddots & & & \ddots & \\ & & \ddots & & & PX \\ X & & & \ddots & & \\ & \ddots & & & \ddots & \\ & & X & & & -Y \end{pmatrix} .$$

[8] Note that the truncation could also be implemented on R instead, or on both R and T. As it turns out, pushing the truncation on T and not on R leads to a nice interpretation of the factor defects.

Now,

$$\prod_{i=0}^{d-1}(\epsilon_d^{d'} I_P X - \xi^i Y \mathbb{1}_d) = P^{d'} X^d - Y^d \,,$$

and hence, together with the map $t_0 = \prod_{i=1}^{d-1}(\epsilon_d^{d'} I_P X - \xi^i Y \mathbb{1}_d)$, t_1 defines a matrix factorization of $W(X,P) - W(Y,1)$. Thus, the module $M_I^{\mathrm{UV\,GLSM}}(N)$ has a free two-periodic resolution induced by the matrix factorization

$$\begin{aligned} T_N^{\mathrm{UV}} : S^d_{(X,P)(Y,\cdot)} &\begin{pmatrix} \{N,-[N-d']_d\} \\ \{N-1,-[N-1-d']_d\} \\ \vdots \\ \{N-d+1,-[N-d+1-d']_d\} \end{pmatrix} \\ &\overset{t_1}{\underset{t_0}{\rightleftarrows}} S^d_{(X,P)(Y,\cdot)} \begin{pmatrix} \{N,-[N]_d\} \\ \{N-1,-[N-1]_d\} \\ \vdots \\ \{N-d+1,-[N-d+1]_d\} \end{pmatrix} . \end{aligned} \tag{5.22}$$

This matrix factorization represents the factor defect T_N^{UV}. Note that it is of rank d and exhibits $U(1)$-charge shifts only in the set $\{N-d+1, N-d+2, \ldots, N\}$ of d consecutive integers starting at $N-d+1$. Fusion with the defect T_N^{UV} therefore lifts D-branes from the UV phase to GLSM branes in the charge window $\{N-d+1, \ldots, N\}$ in the terminology of [47].

Indeed, there is another way to arrive at the defects T_N^{UV}. One can start with the identity defect in the UV phase and then lift on the left to the GLSM. Lifting in this case means inserting variables P into the rank-d $\mathbb{Z}_d \times \mathbb{Z}_d$-equivariant matrix factorization of $X^d - Y^d$ in such a way as to make it into a $U(1) \times \mathbb{Z}_d$-equivariant matrix factorization of $P^{d'} X^d - Y^d$. (Lifting D-branes in such a manner is an important ingredient in the discussion of D-brane transfer between LG and geometric phases of abelian GLSMs in [47].) In this way one obtains a defect from the UV phase to the GLSM. This defect is automatically of finite rank, so a truncation of the kind we had to impose when coming from the GLSM is not necessary. On the other hand, the lift involves many choices. One of the choices corresponds to the choice of N, the maximal $U(1)$-charge. When that is fixed there are still choices left, and only one of them leads to the defects T_N^{UV}. In fact, T_N^{UV} corresponds to the unique lift of the UV identity defect, which has maximal $U(1)$-charge N, and whose $U(1)$-charges populate $\{N-d+1, \ldots, N\}$. That means it is the only such lift, which upon fusion sends all UV branes to GLSM branes in the respective charge window of length d in the terminology of [47].

As a side remark, pushing down the defect T_N^{UV} on the left to the IR (setting $X = 1$), one obtains the RG defect RG_N. Pushing down on the left to the UV (setting $P = 1$), yields the identity defect in the UV phase.

Of course, defects T_N can be constructed for any phase. Pushing the GLSM identity defect to the IR on the right, in a similar fashion yields defects T_N^{IR} from the IR phase to the GLSM.

R^{IR}. R^{IR} is obtained by pushing down to the IR on the left of the GLSM identity defect. Pushing down on the level of modules yields the $C_{(\cdot,P)(Y,Q)}$-module

$$M_I^{\mathrm{GLSM\,IR}} = C_{(\cdot,P)(Y,Q)}^{(\alpha,\alpha^{-1})}/((Y-\alpha^{-d'}),(P-\alpha^{-d}Q))\,.$$

This module is of infinite rank. Note that Y is invertible in this module. One way to look at it is as a limit of truncated modules

$$M_I^{\mathrm{GLSM\,IR}} = \lim_{N\to\infty} M_I^{\mathrm{GLSM\,IR}}(N)$$

with

$$M_I^{\mathrm{GLSM\,IR}}(N) = \frac{\alpha^N C_{(\cdot,P)(Y,Q)}[\alpha^{-1}]}{((Y-\alpha^{-d'}),(P-\alpha^{-d}Q))\alpha^N C_{(\cdot,P)(Y,Q)}[\alpha^{-1}]}\,.$$

The truncated module is finitely generated with generators $e_i = \alpha^{N-i}$, $0 \le i < d'$ of $\mathbb{Z}_{d'} \times U(1)$-charges $([N-i]_{d'}, -N+i)$. They satisfy relations

$$\begin{aligned} Pe_i &= P\alpha^{N-i} = Q\alpha^{N-i-d} = Y^a Q\alpha^{N-i-b} \\ &= \begin{cases} Y^a Q e_{i+b}\,, & i+b<d' \\ Y^{a+1}Qe_{i+b-d'}\,, & i+b\ge d' \end{cases} \end{aligned}$$

Therefore, $M_I^{\mathrm{GLSM\,IR}}(N)$ is isomorphic to the cokernel of the map

$$\begin{aligned} r_1^{\mathrm{IR}} : C^{d'} &\begin{pmatrix} \{[N-d]_{d'},-N\} \\ \{[N-1-d]_{d'},-N+1\} \\ \vdots \\ \{[N-d'+1-d]_{d'},-N+d'-1\} \end{pmatrix} \to C^{d'} \begin{pmatrix} \{[N]_{d'},-N\} \\ \{[N-1]_{d'},-N+1\} \\ \vdots \\ \{[N-d'+1]_{d'},-N+d'-1\} \end{pmatrix} \\ C &:= C_{(\cdot,P)(Y,Q)} \end{aligned}$$

with $r_1^{\mathrm{IR}} = (P\mathbb{1}_{d'} - Q\epsilon_{d'}^b I_Y)$. Here I_Y is the $d' \times d'$-diagonal matrix with Y^a as the first $d'-b$ diagonal entries and Y^{a+1} as the last b diagonal entries. Explicitly,

$$r_1^{\mathrm{IR}} = \begin{pmatrix} P & & & -QY^{a+1} & & \\ & \ddots & & & \ddots & \\ & & \ddots & & & -QY^{a+1} \\ -QY^a & & & \ddots & & \\ & \ddots & & & \ddots & \\ & & -QY^a & & & P \end{pmatrix}.$$

Now, $\prod_{i=0}^{d'-1}(P\mathbb{1}_{d'} - (\xi')^i Q\epsilon_{d'}^b I_Y) = P^{d'} - Y^d Q^{d'} =: r_1^{\mathrm{IR}} r_0^{\mathrm{IR}}$, and therefore the truncated modules have two-periodic resolutions induced by the matrix factorizations

$$R_N^{\mathrm{IR}} : S^{d'}\begin{pmatrix} \{[N-d]_{d'},-N\} \\ \{[N-1-d]_{d'},-N+1\} \\ \vdots \\ \{[N-d'+1-d]_{d'},-N+d'-1\} \end{pmatrix} \underset{r_0^{\mathrm{IR}}}{\overset{r_1^{\mathrm{IR}}}{\rightleftarrows}} S^{d'}\begin{pmatrix} \{[N]_{d'},-N\} \\ \{[N-1]_{d'},-N+1\} \\ \vdots \\ \{[N-d'+1]_{d'},-N+d'-1\} \end{pmatrix}$$
$$S := S_{(X,P)(Y,\cdot)}$$

One can think of the matrix factorization associated to R^{IR} as the limit $\lim_{N\to\infty} R_N^{\mathrm{IR}}$. Note that N only shifts the charges of this matrix factorization!

As we will see later, left-fusion with the defect R^{IR} just sets X to 1 in the matrix factorization R^{IR} is fused with.

In the following we will also need the defects R^{UV} obtained by pushing the GLSM identity defect to the UV on the left. The associated module is given by

$$M_I^{\mathrm{GLSM\,UV}} = C_{(X,\cdot)(Y,Q)}^{(\alpha,\alpha^{-1})}/((X-\alpha^{d'}Y),(\alpha^d-Q))\,,$$

which can be obtained as a limit $N\to-\infty$ of the truncated modules[9]

$$M_I^{\mathrm{GLSM\,UV}}(N) = \alpha^N C_{(X,\cdot)(Y,Q)}[\alpha]/((X-\alpha^{d'}Y),(Q-\alpha^d))\alpha^N C_{(X,\cdot)(Y,Q)}[\alpha]\,.$$

An analysis analogous to the IR case yields an associated matrix factorization

$$R_N^{\mathrm{UV}} : S^{d}\begin{pmatrix} \{[N+d']_{d},-N\} \\ \{[N+1+d']_{d},-N-1\} \\ \vdots \\ \{[N+d-1+d']_{d},-N-d+1\} \end{pmatrix} \underset{r_0^{\mathrm{UV}}}{\overset{r_1^{\mathrm{UV}}}{\rightleftarrows}} S^{d}\begin{pmatrix} \{[N]_{d},-N\} \\ \{[N+1]_{d},-N-1\} \\ \vdots \\ \{[N+d-1]_{d},-N-d+1\} \end{pmatrix}$$
$$S := S_{(X,P)(Y,\cdot)}$$

with $r_1^{\mathrm{UV}} = (X\mathbb{I}_d - Y\epsilon_d^{d'}I_Q)$, $r_0^{\mathrm{UV}} = \prod_{i=1}^{d-1} = (X\mathbb{I}_d - \xi^i Y\epsilon_d^{d'}I_Q)$, where I_Q is the diagonal matrix with 1 in its first $d-d'$ diagonal entries and Q in its last d'. Explicitly,

$$r_1^{\mathrm{UV}} = \begin{pmatrix} X & & & -QY & & \\ & \ddots & & & \ddots & \\ & & \ddots & & & -QY \\ -Y & & & \ddots & & \\ & \ddots & & & \ddots & \\ & & -Y & & & X \end{pmatrix}.$$

The matrix factorization R^{UV} can then be thought of as the limit

$$\lim_{N\to-\infty} R_N^{\mathrm{UV}}.$$

Left-fusion with R^{UV} implements the push-down to the UV phase, i.e. setting P to 1.

[9]Note that compared to the IR case, truncation is implemented in the opposite direction.

5.2.6 Projection defects

In the previous section we have shown that the defects RG_N describing the transition from UV to IR phase factorize as $RG_N \cong R^{\mathrm{IR}} \otimes T_N^{\mathrm{UV}}$. Here T_N^{UV} is the defect lifting the UV phase into the GLSM, and R^{IR} is the defect from the GLSM to the IR phase implementing the push-down to the IR.

Indeed, we can also consider the fusion $R^{\mathrm{UV}} \otimes T_N^{\mathrm{UV}}$. This defect describes the lift of the UV to the GLSM and the subsequent push-down to the same phase. Since identity defects are idempotent, $I \otimes I \cong I$, this defect can be obtained by pushing down to the UV phase on both sides of the GLSM identity defect, combined with a truncation. The untruncated push-down was calculated in section 5.2.3. The result is the identity defect of the UV phase. Indeed, it is not difficult so see that the truncation essentially does not change the calculation, and that also the truncated push-down yields the identity defect of the UV phase

$$R^{\mathrm{UV}} \otimes T_N^{\mathrm{UV}} \cong I^{\mathrm{UV}} . \tag{5.23}$$

Another way to obtain this result is to use the fact (discussed below) that left-fusion with defects R^i just implements the push-down to phase$_i$, i.e. it just sets the variable to 1, which is associated to the field obtaining a non-trivial vacuum expectation value in phase$_i$. In the case of R^{UV}, this is the variable P. Setting $P = 1$ in the matrix factorization T_N^{UV} given in (5.22) indeed yields the UV identity matrix factorization I^{UV}, c.f. (5.17).

Of course, one can also straightforwardly calculate the fusion. As we already used in section 5.1.3, on the level of modules, fusion corresponds to the part of the tensor product which is invariant under the gauge group of the model in between the fused defects [3]. One obtains

$$\left(M_I^{\mathrm{GLSM\,UV}} \otimes M_I^{\mathrm{UV\,GLSM}}(N)\right)^{U(1)}$$

which is the same as

$$\left(\frac{C^{(\alpha,\alpha^{-1})}_{(X,\cdot)(Y,Q)}}{((X-\alpha^{d'}Y),(\alpha^d - Q))} \otimes_{\mathbb{C}[Y,Q]} \frac{\beta^N C_{(Y,Q)(Z,\cdot)}[\beta^{-1}]}{((Z-\beta^{-d'}Y),(Q-\beta^{-d}))\beta^N C_{(Y,Q)(Z,\cdot)}[\beta^{-1}]}\right)^{U(1)}$$

and turns into

$$\left(\frac{\beta^N C_{(X,\cdot)(Z,\cdot)}[\alpha,\alpha^{-1},\beta^{-1}]}{((\alpha\alpha^{-1}-1),(Z-(\alpha\beta)^{-d'}X),(\alpha^d-\beta^{-d}))\beta^N C_{(X,\cdot)(Z,\cdot)}[\alpha,\alpha^{-1},\beta^{-1}]}\right)^{U(1)}$$

$$\cong \frac{(\alpha\beta)^N C_{(X,\cdot)(Z,\cdot)}[(\alpha\beta)^{-1}]}{((Z-(\alpha\beta)^{-d'}X),(1-(\alpha\beta)^{-d}))(\alpha\beta)^N C_{(X,\cdot)(Z,\cdot)}[(\alpha\beta)^{-1}]} .$$

With the same arguments as in section 5.2.3 this can be seen to be a module associated to the identity matrix factorization I^{UV} of the Landau-Ginzburg orbifold in the UV.

Analogously one finds

$$R^{\mathrm{IR}} \otimes T_N^{\mathrm{IR}} \cong I^{\mathrm{IR}} .$$

Relation (5.23) implies that the defect

$$P_N^{\mathrm{UV}} = T_N^{\mathrm{UV}} \otimes R^{\mathrm{UV}}$$

is idempotent, i.e. $P_N^{\mathrm{UV}} \otimes P_N^{\mathrm{UV}} \cong P_N^{\mathrm{UV}}$. This defect realizes the UV phase inside the GLSM in the sense of chapter 2. In particular, the category of D-branes in the UV phase is equivalent to the subcategory of GLSM branes invariant under fusion with P_N^{UV}.

A module associated to P_N^{UV} can be obtained as

$$M_{P_N^{\mathrm{UV}}} = M_I^{\mathrm{UV\,GLSM}}(N) \otimes M_I^{\mathrm{GLSM\,UV}}$$

which is given by

$$\left(\frac{\alpha^N C_{(X,P)(Y,\cdot)}[\alpha^{-1}]}{((Y-\alpha^{-d'}X),(P-\alpha^{-d}))\alpha^N C_{(X,P)(Y,\cdot)}[\alpha^{-1}]} \otimes_{\mathbb{C}[Y]} \frac{C^{(\beta,\beta^{-1})}_{(Y,\cdot)(Z,R)}}{((Y-\beta^{d'}Z),(\beta^d - R))} \right)^{\mathbb{Z}_d} .$$

This is isomorphic to

$$\left(\frac{\alpha^N C_{(X,P)(Z,R)}[\alpha^{-1},\beta,\beta^{-1}]}{((\beta\beta^{-1}-1),(P-\alpha^{-d}),(\beta^d-R),(\alpha^{-d'}X-\beta^{d'}Z))\alpha^N C_{(X,P)(Z,R)}[\alpha^{-1},\beta,\beta^{-1}]} \right)^{\mathbb{Z}_d}$$

The $\mathbb{Z}_d$-invariant generators are given by $(\alpha\beta)^{N-i}(\beta^d)^m$ for $i \in \mathbb{N}_0$ and $m \in \mathbb{Z}$. They carry $U(1)\times U(1)$-charges $(N-i, -N+i-md)$. Defining $\gamma := \alpha\beta$ and $\delta := \beta^{-d}$ of $U(1)\times U(1)$ charges $(1,-1)$ and $(0,d)$ respectively, one can write

$$M_{P_N^{\mathrm{UV}}} \cong \frac{\gamma^N C_{(X,P)(Z,R)}[\gamma^{-1},\delta]}{((P-\gamma^{-d}R),(X\gamma^{-d'}-Z),(\delta R-1))\gamma^N C_{(X,P)(Z,R)}[\gamma^{-1},\delta]} .$$

Note that R is invertible in this module! It can be considered as a module over the ring

$$C^{\delta}_{(X,P)(Z,R)} := \frac{C_{(X,P)(Z,R)}[\delta]}{(\delta R - 1)C_{(X,P)(Z,R)}[\delta]}$$

in which R is invertible. Over this ring, $M_{P_N^{\mathrm{UV}}}$ is finitely generated with generators $e_i := \gamma^{N-d+1+i}$, $0 \leq i < d$ of $U(1) \times U(1)$-charges $(N-d+1+i, -N+d-1-i)$. They satisfy relations

$$\begin{aligned} Xe_i &= Ze_{i+d'}\,, & \qquad i+d' &< d \\ PXe_i &= RZe_{i+d'-d}\,, & \qquad i+d' &\geq d \end{aligned}$$

Thus, as module over the ring $C^{\delta}_{(X,P)(Z,R)}$, $M_{P_N^{\mathrm{UV}}}$ is isomorphic to the cokernel of the map

$$\begin{aligned} p_1 : C^d \begin{pmatrix} \{N-d+1+d', -N+d-1\} \\ \{N-d+2+d', -N+d-2\} \\ \vdots \\ \{N, -N+d'\} \\ \{N-d+1, -N+d'-1\} \\ \{N-d+2, -N+d'-2\} \\ \vdots \\ \{N-d+d', -N\} \end{pmatrix} &\to C^d \begin{pmatrix} \{N-d+1, -N+d-1\} \\ \{N-d+2, -N+d-2\} \\ \vdots \\ \{N, -N\} \end{pmatrix} \\ C := C^{\delta}_{(X,P)(Z,R)} & \end{aligned}$$

with $p_1 = (XI_P - Z\epsilon_d^{d'} I_R)$. Here I_P is the diagonal $d \times d$-matrix whose first $d-d'$ diagonal entries are 1 and whose last d' diagonal entries are P, and I_R is the diagonal $d \times d$-matrix whose first $d-d'$ entries are 1 and whose last d' entries are R. Concretely,

$$p_1 = \begin{pmatrix} X & & & -PZ & & & & \\ & \ddots & & & \ddots & & & \\ & & X & & & \ddots & & \\ & & & PX & & & \ddots & \\ & & & & \ddots & & & -PZ \\ -Z & & & & & \ddots & & \\ & \ddots & & & & & \ddots & \\ & & -Z & & & & & PX \end{pmatrix}$$

Note that[10]

$$\prod_{i=0}^{d-1} \left(X\epsilon_d^{-id'} I_P \epsilon_d^{id'} - \xi^i Z \epsilon_d^{d'} I_R \right) = X^d P^{d'} - Z^d R^{d'}\,.$$

Hence, p_1 together with $p_0 = \prod_{i=1}^{d-1} \left(X\epsilon_d^{-id'} I_P \epsilon_d^{id'} - \xi^i Z \epsilon_d^{d'} I_R \right)$ forms a matrix

[10] In fact, this is true for any choice of diagonal matrices $I_P = \mathrm{diag}(P^{n_1}, \ldots, P^{n_d})$ and $I_R = \mathrm{diag}(R^{m_1}, \ldots, R^{m_d})$ with $\sum n_i = d' = \sum m_i$.

factorization

$$P_N^{\mathrm{UV}} : S^d \begin{pmatrix} \{N{-}d{+}1{+}d',{-}N{+}d{-}1\} \\ \{N{-}d{+}2{+}d',{-}N{+}d{-}2\} \\ \vdots \\ \{N,{-}N{+}d'\} \\ \{N{-}d{+}1,{-}N{+}d'{-}1\} \\ \{N{-}d{+}2,{-}N{+}d'{-}2\} \\ \vdots \\ \{N{-}d{+}d',{-}N\} \end{pmatrix} \underset{p_0}{\overset{p_1}{\rightleftarrows}} S^d \begin{pmatrix} \{N{-}d{+}1,{-}N{+}d{-}1\} \\ \{N{-}d{+}2,{-}N{+}d{-}2\} \\ \vdots \\ \{N,{-}N\} \end{pmatrix}$$

$$S := S^{\delta}_{(X,P)(Z,R)}$$

of $W(X,P) - W(Y,Q)$ over the ring

$$S^{\delta}_{(X,P)(Z,R)} = \mathbb{C}[X,P,Z,R,\delta]/(\delta R - 1)\mathbb{C}[X,P,Z,R,\delta]$$

of chiral fields of the GLSM on the left and right of the defect, in which the field R is made invertible.

5.2.7 Action on D-branes

Here, we will discuss the action of the defects T^i_N, R^i and P^i_N on D-branes (boundary conditions).

$\boldsymbol{R^i}$**.** Fusion with a defect R^i acts on D-branes by pushing down the respective GLSM matrix factorizations to phase$_i$ by setting the variable obtaining a vacuum expectation value in the phase to 1. More precisely, let

$$P : P_1 = S^r_{(Y,Q)} \begin{Bmatrix} b_1 \\ \vdots \\ b_r \end{Bmatrix} \underset{p_0}{\overset{p_1}{\rightleftarrows}} S^r_{(Y,Q)} \begin{Bmatrix} a_1 \\ \vdots \\ a_r \end{Bmatrix} = P_0$$

be a $U(1)$-equivariant matrix factorization of $Y^d Q^{d'}$ representing a D-brane in the GLSM. Here, we use the following notation:

$$\begin{aligned} S_{(Y,Q)} &= \mathbb{C}[Y,Q] \\ C_{(Y,Q)} &= S_{(Y,Q)}/(Y^d Q^d)\,. \end{aligned}$$

As before, replacing one of the variables in the subscript with a '$\cdot$' means that we set the respective variable to 1. So, in particular $S_{(Y,\cdot)} = \mathbb{C}[Y]$ and $C_{(Y,\cdot)} = \mathbb{C}[Y]/(Y^d)$. To this matrix factorization we associate the $C_{(X,P)}$-module

$$M_P = \operatorname{coker}\left(P'_1 := C^r_{(Y,Q)} \begin{Bmatrix} b_1 \\ \vdots \\ b_r \end{Bmatrix} \xrightarrow{p_1} C^r_{(Y,Q)} \begin{Bmatrix} a_1 \\ \vdots \\ a_r \end{Bmatrix} =: P'_0 \right).$$

We can now calculate the fusion $\widehat{P} = R^{\mathrm{UV}} \otimes P$. On the level of modules we obtain an associated $C_{(X,\cdot)}$-module

$$\begin{aligned} M_{\widehat{P}} &= \left(M_I^{\mathrm{GLSM\,UV}} \otimes_{S_{(Y,Q)}} M_P\right)^{U(1)} \\ &= \left(\frac{C^{(\alpha,\alpha^{-1})}_{(X,\cdot),(Y,Q)}}{((Y-\alpha^{-d'}X),(Q-\alpha^d))} \otimes_{S_{(Y,Q)}} M_P\right)^{U(1)} \end{aligned}$$

The $U(1)$-invariant generators of this module are given $\hat{e}_i := \alpha^{a_i} e_i$, where the e_i, $1 \le i \le r$ are the generators the module P'_0 of $U(1)$-charge a_i. Note that α has $\mathbb{Z}_d \times U(1)$-charge $([1]_d, -1)$, and hence, the $\mathbb{Z}_d$-charge of $\hat{e}_i$ are just the induced $\mathbb{Z}_d$-charges $[a_i]_d$. The relations from the first tensor factor set the variable Y to $\alpha^{-d'}X$ and Q to α^d. The relations from the second tensor factor, coming from the matrix p_1 can then be written in terms of the matrix $\widehat{p}_1 = p_1(Y = X, Q = 1)$ obtained from p_1 by setting Y to X and Q to 1. One obtains

$$M_{\widehat{P}} \cong \operatorname{coker}\left(\widehat{P}'_1 := C^r_{(X,\cdot)}\left\{\begin{array}{c}[b_1]_d\\ \vdots \\ [b_r]_d\end{array}\right\} \xrightarrow{\widehat{p}_1} C^r_{(X,\cdot)}\left\{\begin{array}{c}[a_1]_d\\ \vdots \\ [a_r]_d\end{array}\right\} =: \widehat{P}'_0\right).$$

This module is associated to the $\mathbb{Z}_d$-equivariant matrix factorization

$$\widehat{P} : \widehat{P}_1 = S^r_{(X,\cdot)}\left\{\begin{array}{c}[b_1]_d\\ \vdots \\ [b_r]_d\end{array}\right\} \underset{\widehat{p}_0}{\overset{\widehat{p}_1}{\rightleftarrows}} ; S^r_{(X,\cdot)}\left\{\begin{array}{c}[a_1]_d\\ \vdots \\ [a_r]_d\end{array}\right\} = \widehat{P}_0$$

of X^d. The matrices $\widehat{p}_i$ are obtained from the respective p_i by setting Y to X and Q to 1. An analogous result holds for the action of R^{IR}.

Thus, R^i indeed fuses with GLSM branes by setting the variables acquiring a non-trivial vacuum expectation value in phase$_i$ to 1 in the respective matrix factorization, and breaking the gauge symmetry accordingly.

$\boldsymbol{T^i_N}$**.** Fusion with T^{UV}_N lifts $\mathbb{Z}_d$-equivariant matrix factorizations of X^d to $U(1)$-equivariant matrix factorizations of $P^{d'}X^d$. Since $R^{\mathrm{UV}} \otimes T^{\mathrm{UV}}_N \cong I^{\mathrm{UV}}$ and R^{UV} acts by setting $P = 1$, the lifted matrix factorization has to reduce to the original one upon setting $P = 1$. Thus, such lifts are obtained by inserting P's into the matrices of the original matrix factorizations in such a way that the $\mathbb{Z}_d$-representations on the matrix factorizations lift to $U(1)$-representations. In fact, for a given matrix factorization there are many possible lifts. As it turns out, fusion with T^{UV}_N produces lifts whose $U(1)$-representations have charges in $\{N-d+1, N-d+2, \ldots, N\}$.

Let us illustrate this in the example of $\mathbb{Z}_d$-equivariant linear rank-1 factorizations

$$L^{\mathrm{UV}}_{[a]_d} : S_{(Y,\cdot)}\{[a+d']_d\} \underset{Y^{d-1}}{\overset{Y}{\rightleftarrows}} S_{(Y,\cdot)}\{[a]_d\}$$

of X^d. These matrix factorizations generate the category of $\mathbb{Z}_d$-equivariant matrix factorizations of X^d, i.e. the category of UV D-branes.

Now, any of the $U(1)$-equivariant rank-1 matrix factorizations

$$L^{\mathrm{GLSM}}_{c,m} : S_{(X,P)}\{c + d' - md\} \underset{Y^{d-1}Q^{d'-m}}{\overset{YQ^m}{\rightleftarrows}} S_{(Y,Q)}\{c\}$$

of $X^d P^{d'}$ is a lift of $L^{\mathrm{UV}}_{[a]_d}$ for $c \in a + d\mathbb{Z}$ and $0 \leq m \leq d'$. Namely,

$$R^{\mathrm{UV}} \otimes L^{\mathrm{GLSM}}_{c,m} \cong L^{\mathrm{UV}}_{[a]_d}\,,$$

or to put it differently, setting $Y = X$ and $Q = 1$ in $L^{\mathrm{GLSM}}_{c,m}$ produces $L^{\mathrm{UV}}_{[a]_d}$.

Next, we will compute to which of the lifts $L^{\mathrm{GLSM}}_{c,m}$ a matrix factorization $L^{\mathrm{UV}}_{[a]_d}$ is mapped under fusion with T^{UV}_N. As before we will compute the fusion on the level of modules. To $L^{\mathrm{UV}}_{[a]_d}$ we associate the $C_{(Y,\cdot)}$-module

$$M_{L^{\mathrm{UV}}_{[a]_d}} = C_{(Y,\cdot)}\{[a]_d\}/(Y)\,.$$

The fusion $T^{\mathrm{UV}}_N \otimes L^{\mathrm{UV}}_{[a]_d}$ is then given by the matrix factorization associated to the $C_{(X,P)}$-module given by the $\mathbb{Z}_d$-invariant part of the tensor product

$$\begin{aligned}
&\left(M_I^{\mathrm{UV\,GLSM}} \otimes_{\mathbb{C}[Y]} M_{L_a^{\mathrm{UV}}}\right)^{\mathbb{Z}_d} \\
&= \left(\frac{\alpha^N C_{(X,P)(Y,\cdot)}[\alpha^{-1}]}{((Y - \alpha^{-d'}X), (P - \alpha^{-d}))\alpha^N C_{(X,P)(Y,\cdot)}[\alpha^{-1}]} \otimes_{\mathbb{C}[Y]} \frac{C_{(Y,\cdot)}\{[a]_d\}}{(Y)}\right)^{\mathbb{Z}_d} \\
&\cong \left(\frac{\alpha^N C_{(X,P)}[\alpha^{-1}]\{a\}}{((P - \alpha^{-d}), \alpha^{-d'}X)\alpha^N C_{(X,P)}[\alpha^{-1}]\{a\}}\right)^{\mathbb{Z}_d}.
\end{aligned}$$

There is just one $\mathbb{Z}_d$-equivariant generator of this module over $C_{(X,P)}$, namely $\alpha^{N-\{N-a\}_d}$ of $U(1)$-charge $N - \{N-a\}_d$. Here $\{\cdot\}_d$ denotes the representative of the rest class $[\cdot]_d$ modulo d in the range $\{0, \ldots, d-1\}$. There is one relation, namely

$$P^n X \alpha^{N-\{N-a\}_d} = 0\,, \quad \text{where} \quad n = \left\{\begin{array}{ll} 0\,, & d' - \{N-a\}_d \leq 0 \\ 1\,, & d' - \{N-a\}_d > 0 \end{array}\right. . \tag{5.24}$$

Hence,

$$\left(M_I^{\mathrm{UV\,GLSM}} \otimes_{\mathbb{C}[Y]} M_{L^{\mathrm{UV}}_{[a]_d}}\right)^{\mathbb{Z}_d} \cong C_{(X,P)}/P^n X C_{(X,P)} \,.$$

which is associated to the matrix factorization

$$S_{(X,P)}\{N-\{N-a\}_d+d'-nd\} \underset{X^{d-1}P^{d'-n}}{\overset{XP^n}{\rightleftarrows}} S_{(X,P)}\{N-\{N-a\}_d\}\ ,$$

This is nothing but $L^{\mathrm{GLSM}}_{N-\{N-d\}_d,n}$, where the value of n depends on a as stated in (5.24). Hence:

$$T_N^{\mathrm{UV}} \otimes L^{\mathrm{UV}}_{[a]_d} \cong L^{\mathrm{GLSM}}_{N-\{N-d\}_d,n} \,.$$

Note that due to the specific dependence of n on a, the $U(1)$-charges of the generators (of the module of) the matrix factorization lie in the set $\{N-d+1, N-d+2, \ldots, N\}$ of d consecutive integers $\leq N$.

Indeed, this is the way T_N^{UV} acts on any boundary condition[11]. It lifts the $\mathbb{Z}_d$-equivariant matrix factorization of X^d to a $U(1)$-equivariant matrix factorization of $P^{d'}X^d$ by inserting factors of P into the matrix factorization in such a way that the $\mathbb{Z}_d$-representation lifts to $U(1)$, and that furthermore the $U(1)$-charges of the lifted representation all lie in $\{N-d+1, N-d+2, \ldots, N\}$. More precisely, let

$$P: S^r_{(Y,\cdot)}\left\{\begin{array}{c}[b_1]_d\\ \vdots\\ [b_r]_d\end{array}\right\} \underset{p_0}{\overset{p_1}{\rightleftarrows}} S^r_{(Y,\cdot)}\left\{\begin{array}{c}[a_1]_d\\ \vdots\\ [a_r]_d\end{array}\right\}$$

be a rank-r $\mathbb{Z}_d$-equivariant matrix factorization of Y^d. Then one can show that $T_N^{\mathrm{UV}} \otimes P$ is given by the $U(1)$-equivariant matrix factorization

$$\widehat{P}: S^r_{(X,P)}\left\{\begin{array}{c}N-\{N-b_1\}_d\\ \vdots\\ N-\{N-b_r\}_d\end{array}\right\} \underset{\widehat{p}_0}{\overset{\widehat{p}_1}{\rightleftarrows}} S^r_{(X,P)}\left\{\begin{array}{c}N-\{N-a_1\}_d\\ \vdots\\ N-\{N-a_r\}_d\end{array}\right\}$$

of $X^dP^{d'}$, where the matrix $\widehat{p}_1$ is obtained from p_1 by replacing each monomial Y^r in the matrix entry $(p_1)_{ij}$ by P^nX^r, with

$$n = \max\{0, -\big(\{N-a_i\}_d - d'r\big) \text{ div } d\} \,.$$

'div' denotes the division with (non-negative) remainder. Similarly $\widehat{p}_0$ is obtained from p_0 by replacing monomials Y^r in $(p_0)_{ij}$ by P^nX^r with

$$n = \max\{0, -\big(\{N-b_i\}_d - d'r\big) \text{ div } d\} \,.$$

One arrives at a similar conclusion for the action of T_N^{IR}, where however the $U(1)$-charges of the lifted matrix factorization have to lie in the smaller set $\{N-d'+1, \ldots, N\}$ of d' consecutive integers $\leq N$.

[11] or more generally defects

$\boldsymbol{P^i_N}$. Since fusion is associative, the last two sections imply the following action of the projection defects $P_N^{\mathrm{UV}} \cong T_N^{\mathrm{UV}} \otimes R^{\mathrm{UV}}$. Let

$$P : S^r_{(Y,Q)} \left\{ \begin{array}{c} b_1 \\ \vdots \\ b_r \end{array} \right\} \underset{p_0}{\overset{p_1}{\rightleftarrows}} S^r_{(Y,Q)} \left\{ \begin{array}{c} a_1 \\ \vdots \\ a_r \end{array} \right\}$$

be a $U(1)$-equivariant matrix factorization of $Y^d Q^{d'}$. Then $P_N^{\mathrm{UV}} \otimes P$ is isomorphic to the $U(1)$-equivariant matrix factorization

$$\widehat{P} : S^r_{(X,P)} \left\{ \begin{array}{c} N - \{N - b_1\}_d \\ \vdots \\ N - \{N - b_r\}_d \end{array} \right\} \underset{\widehat{p}_0}{\overset{\widehat{p}_1}{\rightleftarrows}} S^r_{(X,P)} \left\{ \begin{array}{c} N - \{N - a_1\}_d \\ \vdots \\ N - \{N - a_r\}_d \end{array} \right\}$$

of $X^d P^{d'}$. Here the matrix $\widehat{p}_1$ is obtained from p_1 by replacing each monomial $Y^r Q^s$ in the matrix entry $(p_1)_{ij}$ by $X^r P^n$, with

$$n = \max\{0, -\big(\{N - a_i\}_d - d'r\big) \operatorname{div} d\}\,.$$

Similarly $\widehat{p}_0$ is obtained from p_0 by replacing monomials $Y^r Q^s$ in $(p_0)_{ij}$ by $X^r P^n$ with

$$n = \max\{0, -\big(\{N - b_i\}_d - d'r\big) \operatorname{div} d\}\,.$$

Thus, the matrix factorization $\widehat{P}$ is obtained from P by shifting all $U(1)$-charges into the range $\{N - d + 1, \ldots, N\}$ by adding integer multiples of d, setting all Q in the matrices to 1 and inserting factors of P in a way ensuring $U(1)$-equivariance of $\widehat{P}$.

One finds an analogous result for P^{IR}, where the charges are shifted by integer multiples of d' into the smaller set $\{N - d' + 1, \ldots, N\}$, Y is set to 1 and factors of X are inserted in a way ensuring $U(1)$-equivariance.

$\boldsymbol{RG_N}$. As alluded to above, the defects RG_N describing the transitions between UV and IR phase are special RG defects between the Landau-Ginzburg orbifolds in the UV and the IR. The action of general RG defects have been discussed at length in [3]. In particular, there is an instructive picture of the D-brane transport coming from the corresponding flow between unorbifolded Landau-Ginzburg models in the mirror theory. These flows are tiggered by lower order perturbations of the superpotential $W(X) = X^d + \sum_{i<d} \lambda_i X^i$. During the flows some vacua of the theory, corresponding to critical points of W move off to infinity and decouple, taking with them certain A-branes attached to them. (For more details see [3].)

The factorization $RG_N \cong R^{\mathrm{IR}} \otimes T_N^{\mathrm{UV}}$ together with the action of the R^{IR} and T_N^{UV} discussed in the previous sections now leads to a stepwise description of the action of RG_N. Start with a D-brane in the UV phase. For simplicity we only discuss D-branes described by a rank-1 matrix factorizations

$$P : S_{(Y,\cdot)}\{[a+rd']_d\} \underset{Y^{d-r}}{\overset{Y^r}{\rightleftarrows}} S_{(Y,\cdot)}\{[a]_d\}\,. \tag{5.25}$$

Under the action of T_N^{UV} P gets lifted to the $U(1)$-equivariant matrix factorization

$$P' : S_{(X,P)}\{N-\{N-a-rd'\}_d\}\} \underset{X^{d-r}P^{d'-n}}{\overset{X^rP^n}{\rightleftarrows}} S_{(X,P)}\{N-\{N-a\}_d\}\,,$$

of $X^dP^{d'}$, where

$$\begin{aligned} nd &= rd' + \{N-a-rd'\}_d - \{N-a\}_d \\ \Longrightarrow \quad n &= rd' \operatorname{div} d + \begin{cases} 0\,, & \{N-a\}_d \geq \{rd'\}_d \\ 1\,, & \{N-a\}_d < \{rd'\}_d \end{cases} \end{aligned} \tag{5.26}$$

R^{IR} then pushes down this matrix factorization to the IR Landau-Ginzburg model by setting $X = 1$, resulting in the $\mathbb{Z}_{d'}$-equivariant matrix factorization

$$P'' : S_{(\cdot,P)}\{[N-\{N-a-rd'\}_d]_{d'}\} \underset{P^{d'-n}}{\overset{P^n}{\rightleftarrows}} S_{(\cdot,P)}\{[N-\{N-a\}_d]_{d'}\}\,, \tag{5.27}$$

of $P^{d'}$. Thus, $RG_N \otimes P \cong P''$.

Note that in case $n = 0$ and $n = d'$ in (5.26), the matrix factorization P'' is trivial, and hence the D-brane corresponding to the matrix factorization P in (5.25) decouples under the RG flow. That is the case whenever $rd' < d$ and $\{N-a\}_d \geq \{rd'\}_d$ ($n = 0$) or $r \geq d-s$ with $sd' \leq d$ and $\{N-a\}_d < \{-sd'\}_d$ ($n = d'$).

So for instance, the degree-1 linear matrix factorizations P (i.e. those with $r = 1$) with $a = N-b$ for $d > b \geq d'$ decouple under the RG flow, whereas the ones for $d' > b \geq 0$ are mapped to degree-1 linear matrix factorizations in the IR.

In general, it follows from (5.26) that $n \leq r$, so the degree of the matrix factorization does not increase during the flow. Either it stays the same, or it decreases. A decrease means that the corresponding D-brane decays during the flow and at least one constituent decouples.

Let us illustrate this in a specific example, namely for $d = 8$ and $d' = 5$, i.e. we are considering a $U(1)$-GLSM with superpotential $W = X^8P^5$, where the $U(1)$-charges of X and P are 5 and -8, respectively. The transition defects RG_N describe a certain RG flows between the Landau-Ginzburg orbifolds $X^8/\mathbb{Z}_8$ and $P^5/\mathbb{Z}_5$. For simplicity we will discuss the action of RG_0, i.e. we set $N = 0$.[12] Let us first consider the action on linear rank-1 factorizations

$$P : S_{(Y,\cdot)}\{[-b+5]_8\} \underset{Y^7}{\overset{Y}{\rightleftarrows}} S_{(Y,\cdot)}\{[-b]_8\} \tag{5.28}$$

for $0 \leq b < 8$. Under the action of T_0 these are mapped to the matrix factorizations

$$P' : S_{(X,P)}\{-b+5\} \underset{X^7P^5}{\overset{X}{\rightleftarrows}} S_{(X,P)}\{-b\}\,, \tag{5.29}$$

for $5 \leq b < 8$, and to

$$P' : S_{(X,P)}\{-b-3\} \underset{X^7P^4}{\overset{XP}{\rightleftarrows}} S_{(X,P)}\{-b\}\,, \tag{5.30}$$

for $0 \leq b < 5$. These are the lifts of the $\mathbb{Z}_8$-equivariant matrix factorizations (5.28) of X^8 to $U(1)$-equivariant matrix factorizations of X^8P^5 whose charges are contained in $\{-7,-6,\ldots,0\}$. Acting with R^{IR} essentially sets $X = 1$ and breaks the $U(1)$ to $\mathbb{Z}_5$. In the first case, $5 \leq b < 8$, the matrix factorizations (5.29) are mapped to the trivial matrix factorizations

$$P'' : S_{(\cdot,P)}\{-[b]_5\} \underset{P^5}{\overset{1}{\rightleftarrows}} S_{(\cdot,P)}\{-[b]_5\}\,.$$

The D-branes corresponding to (5.28) for $5 \leq b < 8$ therefore decouple under the RG flow. For $0 \leq b < 5$, on the other hand, the matrix factorizations (5.30) are mapped to the linear factorizations

$$P'' : S_{(\cdot,P)}\{-[b+3]_5\} \underset{P^4}{\overset{P}{\rightleftarrows}} S_{(\cdot,P)}\{-[b]_5\}\,.$$

The corresponding D-branes do not decouple.

Next, let us discuss the action on quadratic matrix factorizations

$$P : S_{(Y,\cdot)}\{[-b+2]_8\} \underset{Y^7}{\overset{Y^2}{\rightleftarrows}} S_{(Y,\cdot)}\{[-b]_8\}\,. \tag{5.31}$$

[12] N can be shifted by a quantum symmetry. This is a charge shift which can be implemented by a charge shifted versions of the identity defect in the respective LG orbifold [3].

Acting on them with T_0, one obtains

$$P' : S_{(X,P)}\{-b+2\} \underset{X^6P^4}{\overset{X^2P}{\rightleftarrows}} S_{(X,P)}\{-b\}\,, \tag{5.32}$$

for $2 \leq b < 8$ and

$$P' : S_{(X,P)}\{-b-6\} \underset{X^6P^3}{\overset{X^2P^2}{\rightleftarrows}} S_{(X,P)}\{-b\}\,, \tag{5.33}$$

for $0 \leq b < 2$. Again, the matrix factorization P' is the lift of the matrix factorization P in (5.31) to the GLSM whose charges lie in $\{-7, \ldots, 0\}$. Acting with R^{IR} then yields the linear matrix factorizations

$$P'' : S_{(\cdot,P)}\{-[b+3]_5\} \underset{P^4}{\overset{P}{\rightleftarrows}} S_{(\cdot,P)}\{-[b]_5\}\,.$$

for the case $2 \leq b < 8$ and the quadratic matrix factorizations

$$P'' : S_{(\cdot,P)}\{-[b+1]_5\} \underset{P^3}{\overset{P^2}{\rightleftarrows}} S_{(\cdot,P)}\{-[b]_5\}\,.$$

for $0 \leq b < 2$. In the latter case, a quadratic matrix factorization is mapped to a quadratic matrix factorization under the action of RG_0. In the case $2 \leq b < 8$, the degree decreases from 2 to 1. Indeed, this can be completely understood in terms of the linear matrix factorizations. Namely, the quadratic matrix factorizations P in (5.31) can be written as a cone of two linear matrix factorizations as in (5.28), one specified by the same label b and one specified by $\{b-5\}_8$. In case both of those linear matrix factorizations survive the flow, i.e. for $0 \leq b < 2$ the quadratic matrix factorization P is again mapped to a quadratic matrix factorization under RG_0. For the other cases, $2 \leq b < 8$, however, one of the two linear matrix factorizations is mapped to the trivial one under RG_0. Under the RG flow, the quadratic matrix factorization decays into the two constituent linear factorizations and one of them decouples. Thus, the quadratic matrix factorization flows to a linear matrix factorization.

In this way, one can explain the action of RG_0 on any rank-1 matrix factorization

$$P : S_{(Y,\cdot)}\{[-b+5r]_8\} \underset{Y^{8-r}}{\overset{Y^r}{\rightleftarrows}} S_{(Y,\cdot)}\{[-b]_8\}\,. \tag{5.34}$$

The result can be read off from the general formulas above. We summarize it in the following table:

degree of $P : r$	charge shift of P specified by b	charges of lift $T_N \otimes P$	degree of $P'' : n$
1	$5 \le b < 8$	$-b, -b+5$	0
1	$0 \le b < 5$	$-b, -b-3$	1
2	$2 \le b < 8$	$-b, -b+2$	1
2	$0 \le b < 2$	$-b, -b-6$	2
3	$7 \le b < 8$	$-b, -b+7$	1
3	$0 \le b < 7$	$-b, -b-1$	2
4	$4 \le b < 8$	$-b, -b+4$	2
4	$0 \le b < 4$	$-b, -b-4$	3
5	$1 \le b < 8$	$-b, -b+1$	3
5	$0 \le b < 1$	$-b, -b-7$	4
6	$6 \le b < 8$	$-b, -b+6$	3
6	$0 \le b < 6$	$-b, -b-2$	4
7	$3 \le b < 8$	$-b, -b+3$	4
7	$0 \le b < 3$	$-b, -b-5$	5

5.2.8 Comparison with other approaches

The current section, which appeared in [11], was mainly written by Prof. Dr. Ilka Brunner and Prof. Dr. Daniel Roggenkamp and is included here for completeness.

D-Brane transport between phases of abelian gauged linear sigma models has been investigated before with very different methods. The non-anomalous "Calabi-Yau" case was studied in [47]. A discussion going beyond abelian gauge groups as well as an extension to anomalous models can be found in the more recent work [48, 49].

In [48, 49], hemisphere partition functions are computed in curved backgrounds with B-type boundary conditions on the equator by means of path integral localization. As a result of the curvature of the background, these precisely capture the dependence of B-type boundary conditions on the parameters appearing in the gauge sector. A thorough analysis of analytic and convergence properties of hemisphere partition functions, then allows to determine the brane transport between different phases. This as well as the arguments in [47] rely on a detailed analysis of the boundary conditions imposed in the gauge sector.

The approach taken in the present paper is very different. We decouple the gauge sector, and boundary conditions in this sector are not taken into account. Essentially[13], we only consider information accessible to the B-

[13] with the exception of the truncation, which we introduced to obtain the RG defects from the GLSM identity defect, and which presumably is related to stability

twisted model. That means that we cannot control any analycity or explicit dependence on t. Remarkably, our approach still yields many similar results that we highlight in the following.

A crucial ingredient in the discussion of D-brane transport in [47] as well as [48, 49] are so called "charge windows". A D-brane whose $U(1)$-charges all lie in this window can be transported smoothly from one phase to another. Partition functions of these grade restricted branes are well behaved in both phases involved. Any D-brane in the GLSM has a grade restricted representative, which can be obtained by binding D-branes which are trivial in the phase in which the transport starts. The charge window is determined by the choice of the homotopy class of paths in parameter space, along which the D-branes are transported.

In our approach, the defect RG_N automatically takes care that branes are transported through such windows. Indeed the defect T^i_N lifting a phase i to the GLSM automatically maps D-branes from phase$_i$ to grade restricted GLSM branes, where the exact window is determined by the truncation parameter N. The projection defect P^i_N realizing phase$_i$ in the GLSM projects the category of GLSM branes on the grade restricted subcategory, i.e. it maps every D-brane to the respective grade restricted representative.

Note that a in the treatment of [48, 49] a D-brane transport between two phases actually involves two charge windows, a "large window" which ensures smooth transport as alluded to above, and a "small window" lying in the large one [14]. D-branes, whose charges completely lie in the small window flow to the new conformal fixed points, while D-branes, whose charges lie in the large window, but not completely in the small one undergo some kind of decay. (In [48, 49] this is determined by analyzing the asympotics of the hemisphere partition functions.)

In our approach both these windows appear naturally and on the same footing. The large window is determined by the projection P^{UV}_N associated to the phase, in which the transport starts, and the small window comes from the projection P^{IR}_N associated to the phase, in which the transport ends. Indeed, on the level of the GLSM the transport from phase i to phase j can be described by the fusion $P^{\mathrm{IR}}_N \otimes P^{\mathrm{UV}}_N$ of the respective projection defects.

Transporting branes from one phase to another can involve monodromies. In [48, 49] these are naturally associated with shifts in the two windows, either the large window as a whole, or the small window inside the large window. In our case, the windows are determined by the truncation parameters N, which can be shifted by a quantum symmetry, which exactly realizes the monodromy around the fixpoint of the respective phase.

Transporting branes from the GLSM to a phase can be done using two different functors. The authors of [49] consider geometric phases and define two functors F_{flow} and F_{geom}. The first one corresponds to the actual flow

[14] The two windows coincide in the Calabi-Yau case.

from the GLSM to the phase, the second one to a restriction to field configuration allowed by the deleted sets of the toric geometry/GLSM description. In our case, we have two defects from the GLSM to a given phase$_i$, R^i_N and R^i, the truncated and the untruncated descent defects. R^i_N depends on the the truncation parameter, and hence a path in parameter space, whereas R^i merely sets certain fields to 1. So these are precisely the analogues of F_{flow} and F_{geom}. In the same way as in [49], where the two functors agree on grade restricted branes, we have $R^j_N \otimes P^i_N \cong R^j \otimes P^i$. This is also the reason, why R^{IR}_N does not feature more prominently in our discussion: The lifts T^{UV}_N directly lift the UV phase to grade restricted branes, and we chose to factorize $RG_N = R^{\text{IR}} \otimes T^{\text{UV}}_N$. We could have used the cutoff version of R^{IR} as well, writing equivalently $RG_N = R^{\text{IR}}_N \otimes T^{\text{UV}}_N$.

One reason, why our approach, which is essentially based on the B-twisted model, still captures all this information might be the fact that functoriality is a strong constraint. Functoriality is inherent in the defect approach, and B-type defects seem to be rather rigid. With the exception of the truncation, which we introduced in an ad-hoc fashion to obtain RG defects from the GLSM identity defect, and which probably has its origins in stability considerations, there were no choices involved in our construction. Furthermore, this choice exactly aligns with the choice of paths between the respective phases.

It would be very interesting to understand the relation of our approach to the ones in [47, 48, 49] even better. For one thing, in [48, 57] the D-brane central charge and concrete dependence on the twisted chiral moduli is investigated quite explicitly. In particular, in [57] the mathematics of central charges in Landau-Ginzburg orbifolds is studied in detail. By general arguments, we expect that RG (or deformation) defects act on these objects via fusion, and it should be possible to formulate this operation in a natural way. On the other hand, one could try to incorporate the functoriality constraint directly into the approach of [48, 49] by applying their analysis to the GLSM identity defect constructed in section 5.1.3.

5.3 Conclusions

In this chapter, we have constructed defects that concretely describe the behavior of D-branes under transitions between phases of abelian gauged linear sigma models. They act on objects and morphisms of the respective D-brane categories via fusion, and this action is automatically functorial. A key ingredient is the new construction of the identity defect in gauged linear sigma models presented in section 5.1.3. Our approach gives a novel perspective on earlier work [47, 48, 49] on D-brane transport in GLSMs. This chapter is concluded with a list of interesting points for future investigation.

- The starting point for the construction of our defects RG_N that implement the transition between a UV and IR Landau-Ginzburg phase of a $U(1)$-gauged linear sigma model is the identity defect of the GLSM. The bosonic defect fields that we use to construct it create an infinite dimensional Chan-Paton-like space. In other words, the modules on which the associated equivariant matrix factorization is built are of infinite rank. Introducing a finite cutoff N for these modules, we obtain defects RG_N in agreement with expectations and earlier results [3]. The choice of cutoff corresponds to a choice of homotopy class of paths in Kähler parameter space. While we formulate all defects and boundaries on the level of the B-twisted model, which decouples from Kähler parameters, a (mild) Kähler dependence sneaks back in via the cutoff. We expect the choice of cutoff to be related to stability, one of the indicators being that the cutoff is necessary to ensure consistent gluing conditions on a spectral flow operator of an IR conformal field theory. It would be very interesting to investigate further, whether stability conditions in phases can be discussed on the level of the GLSM, and how this relates to defects.

- It would be very interesting to combine our approach with the one of [47, 48, 49]. Applying their methods to the GLSM identity defect would at the same time explicitly incorporate the constraint of functoriality in their approach as well as elucidate the precise origin of the cutoff appearing in our construction.

- In section 5.2 we applied the general approach outlined in section 5.1 to a specific class of $U(1)$-gauged linear sigma models which only exhibit Landau-Ginzburg phases. It would be very interesting to apply it to more interesting models, in particular those featuring geometric or mixed phases. Indeed, a paper on this topic is already in preparation [51].

- The construction of the identity defect should also generalize to non-abelian gauged linear sigma models. It would be very interesting to spell this out and obtain transition and monodromy defects also for phases of non-abelian GLSMs.

- While in two dimensions our methods are particularly powerful, as the fusion of defects is well-controllable, our basic ideas are not limited to this and it would be quite interesting to discuss phase transitions and possibly dualities from this point of view also in higher dimensions.

Chapter 6

Complementary projections

We now consider topological quantum field theories whose defect categories are tensor triangulated as defined in section 3.4. This in particular holds for topologically twisted $\mathcal{N} = (2, 2)$ superconformal field theories [42], see the end of section 3.3. We show that in this setup any counital projection defect P comes with a complementary unital projection defect $\bar{P}$, and vice-versa, and that the unprojected theory decomposes into the P-projected and $\bar{P}$-projected theories. This is spelled out in section 6.1. In section 6.2 this finding is illustrated for the example of projection defects in B-twisted Landau-Ginzburg orbifold models.

This chapter is based on [2]. In contrast to the other chapters of this thesis, unital projection defects are denoted by $\bar{P}$ in this chapter and P is reserved for counital projections. In particular, P is given by $R^\dagger \otimes R$ in chapter 4 and by ${}^\dagger R \otimes R$ in the current chapter.

6.1 Complementary Projection Defects

From now on, we assume that the defect category is tensor triangulated as explained in section 3.4. We show that every counital projection defect P comes with a complementary unital projection defect $\bar{P}$ and vice-versa. Complementarity of a pair $(P, \bar{P})$ means that

$$\begin{aligned}&\text{mutual fusion vanishes } P \otimes \bar{P} \cong 0 \cong \bar{P} \otimes P\,, \text{ and}\\&\text{the identity defect is isomorphic to a cone } I \cong \mathrm{cone}(s : \bar{P}[-1] \to P)\,.\end{aligned} \tag{6.1}$$

In fact, we can also start with two defects P and $\bar{P}$ which are idempotent with respect to fusion and satisfy conditions (6.1). Then, P carries a unit and $\bar{P}$ a counit. Physically, the second condition in (6.1) means that the sum $P \oplus \bar{P}$ of the two projectors can be deformed (or perturbed) to the identity defect.

In the following we use the triangulated structure to show our claim in three steps. First, we prove that every counital projection defect comes with a complementary unital projection defect. Second, we argue that conversely, every unital projection defect comes with a complementary counital projection defect. And third, we show that for any pair $(P, \bar{P})$ of complementary idempotent defects, P is counital and $\bar{P}$ is unital. Afterwards we discuss how, given such a pair of complementary projection defects, the host theory decomposes into the projected theories associated to P and $\bar{P}$.

(i) Counital projections have unital complementary projections. Let P be a counital projection defect $P \in \mathrm{obj}(T)$. In other words, it satisfies $P \otimes P \cong P$ and there is a morphism $c : P \to I$ such that the following two squares commute:

$$
\begin{tikzcd}
P \otimes P \arrow[r, "c \otimes \mathrm{id}_P"] & I \otimes P \\
P \arrow[u, "\sim"] \arrow[r, "\mathrm{id}_P"'] \arrow[ur, phantom, "\square"] & P \arrow[u, "\sim"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P \otimes P \arrow[r, "\mathrm{id}_P \otimes c"] & P \otimes I \\
P \arrow[u, "\sim"] \arrow[r, "\mathrm{id}_P"'] \arrow[ur, phantom, "\square"] & P \arrow[u, "\sim"']
\end{tikzcd}
$$

These two relations are just the counit conditions (1.6), see also section 2.5. Defining $\bar{P} := \mathrm{cone}(c : P \to I)$, the exact triangle with respect to the counit becomes

$$P \xrightarrow{\mathrm{c}} I \xrightarrow{u} \bar{P} \xrightarrow{s[1]} P[1]. \tag{6.2}$$

The key idea is now the following: c obeys counit conditions, u obeys unit conditions and $\mathrm{cone}(s) \cong I$ is isomorphic to the identity defect. Given any of the three morphisms s, c or u satisfying the respective condition, the triangulated structure implies that the other morphisms exist and satisfy the respective conditions. Let us spell this out for the case at hand.

Applying $\cdot \otimes P$ to (6.2) one obtains the upper row of

$$
\begin{tikzcd}
P \otimes P \arrow[r, "c \otimes \mathrm{id}_P"] & I \otimes P \arrow[r] & \bar{P} \otimes P \arrow[r] & P[1] \otimes P \\
P \arrow[u, "\sim"] \arrow[r, "\mathrm{id}_P"] \arrow[ur, phantom, "\square"] & P \arrow[u, "\sim"'] \arrow[r] & 0 \arrow[u, dashed] \arrow[r] & P[1] \arrow[u, "\sim"']
\end{tikzcd}
$$

Since $\cdot \otimes P$ is a triangulated functor, it is an exact triangle. The counit conditions gives rise to the left commuting square. By the axioms of triangulated categories, the lower triangle is also exact, the dashed morphism exists and all squares commute. Also, the dashed morphism is an isomorphism. Hence, $\bar{P} \otimes P \cong 0$. Similar considerations lead to $P \otimes \bar{P} \cong 0$.

Next, applying the functor $\bar{P} \otimes \cdot$ to (6.2) yields

$$
\begin{tikzcd}
\bar{P} \otimes P \arrow[r, "\mathrm{id}_{\bar{P}} \otimes c"] & \bar{P} \otimes I \arrow[r, "\mathrm{id}_{\bar{P}} \otimes u"] & \bar{P} \otimes \bar{P} \arrow[r] & \bar{P} \otimes P[1] \\
0 \arrow[u, "\sim"] \arrow[r] \arrow[ur, phantom, "\square"] & \bar{P} \arrow[u, "\sim"'] \arrow[r, "\mathrm{id}_{\bar{P}}"] & \bar{P} \arrow[u, dashed] \arrow[r] & 0 \arrow[u, "\sim"']
\end{tikzcd}
.
$$

Again, both rows are exact triangles. Because morphisms from the zero object are unique, the first square commutes. Hence, the dashed morphism exists, makes all squares commute and is an isomorphism. Therefore, $\bar{P}$ is idempotent with respect to $\otimes$ and the first of the two unit conditions

$$\begin{tikzcd} \bar{P} \arrow[r, "\mathrm{id}_{\bar{P}}"] & \bar{P} \\ \bar{P}\otimes I \arrow[u, "\sim"] \arrow[r, "\mathrm{id}_{\bar{P}}\otimes u"'] \arrow[ur, phantom, "\square"] & \bar{P}\otimes\bar{P} \arrow[u, "\sim"] \end{tikzcd} \qquad \begin{tikzcd} \bar{P} \arrow[r, "\mathrm{id}_{\bar{P}}"] & \bar{P} \\ I\otimes\bar{P} \arrow[u, "\sim"] \arrow[r, "u\otimes\mathrm{id}_{\bar{P}}"'] \arrow[ur, phantom, "\square"] & \bar{P}\otimes\bar{P} \arrow[u, "\sim"] \end{tikzcd}$$

holds. In the same way, application of $\cdot\otimes\bar{P}$ to (6.2) implies the second unit condition. $\bar{P}$ is therefore a unital projection defect.

It remains to show that the identity defect I is isomorphic to a cone of a morphism $\bar{P}[-1]\to P$. This follows by rotating the exact triangle (6.2) to

$$\bar{P}[-1] \xrightarrow{s} P \xrightarrow{c} I \xrightarrow{u} \bar{P}.$$

(ii) Unital projections have counital complementary projections. The above arguments also work the other way around: the existence of an idempotent $\bar{P}\in\mathrm{obj}(T)$ with unit $u : I\to\bar{P}$ implies the existence of a complementary counital idempotent

$$P := \mathrm{cone}(u[-1] : I[-1]\to\bar{P}[-1]) \in \mathrm{obj}(T)\,.$$

Of course, applying the construction in (i) to this projection P gives back the original projection $\bar{P}$. Namely, the counit of P is the left morphism in (6.2) and automatically $\mathrm{cone}(c : P\to I)\cong\bar{P}$.

Vice-versa, starting with a unital projection defect P and first constructing the counital projection defect $\bar{P}$ as in (i) and then applying the construction in (ii) returns the original projection defect P.

(iii) Complementary projections are (co)unital. We now turn the above discussion around and start with two idempotents $P,\bar{P}\in\mathrm{obj}(T)$ satisfying

$$P\otimes P\cong P\,,\qquad \bar{P}\otimes\bar{P}\cong\bar{P}\,,\qquad P\otimes\bar{P}\cong 0\cong\bar{P}\otimes P\,.$$

Moreover, we assume that there is a morphism $s : \bar{P}[-1]\to P$ such that $\mathrm{cone}(s)\cong I$. In other words,

$$\bar{P}[-1]\xrightarrow{s} P\to I\to\bar{P}$$

is exact. Application of $P\otimes\cdot$ to this triangle gives rise to the first row of

$$\begin{tikzcd} P\otimes\bar{P}[-1] \arrow[r, "\mathrm{id}_P\otimes s"] & P\otimes P \arrow[r] & P\otimes I \arrow[r] & P\otimes\bar{P} \\ 0 \arrow[u, "\sim"] \arrow[r] \arrow[ur, phantom, "\square"] & P \arrow[u, "\sim"] \arrow[r, "\mathrm{id}_P"] & P \arrow[u, dashed, "\sim"] \arrow[r] & 0 \arrow[u, "\sim"] \end{tikzcd}\,.$$

The first square commutes because there is a unique morphism from the zero object into any object. Hence, the middle square also commutes and yields the second counit condition. Similarly, application of the exact functors $\cdot \otimes P$, $\bar{P} \otimes \cdot$ and $\cdot \otimes \bar{P}$ leads to the first counit condition on P and the unit conditions on $\bar{P}$. Hence, of two complementary projectors one is always unital and the other counital.

Next, we will show how, given a complementary pair $(P, \bar{P})$ of projection defects, the unprojected theory decomposes into the P-projected theory and the $\bar{P}$-projected theory. Let us start with the spectrum of boundary conditions.

Decomposition – boundary spectrum. The category of boundary conditions of the unprojected theory decomposes into the subcategories of boundary conditions of the two projected theories. Namely, every boundary condition B in the unprojected theory can be expressed as

$$\begin{aligned} B \cong I \otimes B &\cong \mathrm{cone}(s : \bar{P}[-1] \to P) \otimes B \\ &\cong \mathrm{cone}(s \otimes \mathrm{id}_B : \bar{P} \otimes B[-1] \to P \otimes B). \end{aligned} \tag{6.3}$$

Hence, every boundary condition in the unprojected theory is a cone of a morphism $s \otimes \mathrm{id}_B$ from a $\bar{P}$-invariant boundary condition to a P-invariant boundary condition. The category of boundary conditions in the unprojected theory is therefore generated by the subcategories of $\bar{P}$- and P-invariant boundary conditions. The latter correspond to the categories of boundary conditions in the $\bar{P}$- and P projected theories, respectively. By complementarity, the two subcategories are disjoint

$$\begin{aligned} P \otimes B \cong B \quad &\Rightarrow \quad \bar{P} \otimes B \cong \bar{P} \otimes P \otimes B \cong 0 \\ \bar{P} \otimes B \cong B \quad &\Rightarrow \quad P \otimes B \cong P \otimes \bar{P} \otimes B \cong 0 \end{aligned}$$

and due to (6.3) all boundary conditions in the kernel of $P \otimes \cdot$ are $\bar{P}$-invariant and vice-versa:

$$\begin{aligned} P \otimes B \cong 0 \quad &\Rightarrow \quad \bar{P} \otimes B \cong B \\ \bar{P} \otimes B \cong 0 \quad &\Rightarrow \quad P \otimes B \cong B \end{aligned}$$

Decomposition – bulk spectrum. Similarly, the bulk spectrum of the unprojected theory can be reconstructed from the bulk spectra of the two projected theories, once s is known.

First, every bulk field $\phi : I \to I$ induces an endomorphism of P by enclosing it with the appropriate projection defect.[1] Because of the counit

[1] There is also an equivalent point of view: Concatenation with the counit (unit) gives a morphism $P \xrightarrow{\phi \circ c} I$ ($I \xrightarrow{u \circ \phi} \bar{P}$) and the morphisms $P \to I$ ($I \to \bar{P}$) can be seen to be one-to-one with the endomorphisms of P ($\bar{P}$). In fact, one can regard the (co)unit as the one-dimensional equivalent of an RG defect mapping bulk fields of the unprojected (UV)

and projection properties of P, this endomorphism, which we call $\alpha(\phi)$ can be written in several ways, see (2.4). The first one is

$$\alpha(\phi): \; P \xrightarrow{\sim} I \otimes P \xrightarrow{\phi\otimes\mathrm{id}_P} I \otimes P \xrightarrow{\sim} P. \tag{6.4}$$

The same holds for the endomorphisms $\bar{\alpha}(\phi)$ induced on the complementary unital projector $\bar{P}$:

$$\bar{\alpha}(\phi): \; \bar{P} \xrightarrow{\sim} I \otimes \bar{P} \xrightarrow{\phi\otimes\mathrm{id}_{\bar{P}}} I \otimes \bar{P} \xrightarrow{\sim} \bar{P} \tag{6.5}$$

Thus, we have a map $\mathrm{End}(I) \to \mathrm{End}(P) \oplus \mathrm{End}(\bar{P})$, $\phi \mapsto (\alpha(\phi), \bar{\alpha}(\phi))$. In fact, the image of this map is not $\mathrm{End}(P) \oplus \mathrm{End}(\bar{P})$ but rather $\mathrm{End}(\bar{P}[-1] \xrightarrow{s} P)$, the pairs of morphisms $(\alpha, \bar{\alpha}) \in \mathrm{End}(P) \oplus \mathrm{End}(\bar{P})$ which are compatible with s, i.e. all those $(\alpha, \bar{\alpha})$ such that the following diagram commutes

$$\begin{tikzcd}
\bar{P}[-1] \arrow[r, "s"] & P \\
\bar{P}[-1] \arrow[r, "s"] \arrow[u, "{\bar{\alpha}[-1]}"] \arrow[ur, phantom, "\square"] & P \arrow[u, "\alpha"']
\end{tikzcd}$$

This can be read off from the first two columns of the following diagram

$$\begin{tikzcd}
\bar{P}[-1] \arrow[r, "s"] & P \arrow[r, "c"] & I \arrow[r, "u"] & \bar{P} \\
I \otimes \bar{P}[-1] \arrow[r, "\mathrm{id}_I\otimes s"] \arrow[u, "\sim"] \arrow[ur, phantom, "\square"] & I \otimes P \arrow[r, "\mathrm{id}_I\otimes c"] \arrow[u, "\sim"] & I \otimes I \arrow[r, "\mathrm{id}_I\otimes u"] \arrow[u, dashed, "\sim"] & I \otimes \bar{P} \arrow[u, "\sim"] \\
I \otimes \bar{P}[-1] \arrow[r, "\mathrm{id}_I\otimes s"] \arrow[u, "{\phi\otimes\mathrm{id}[-1]}"] \arrow[ur, phantom, "\square"] & I \otimes P \arrow[r, "\mathrm{id}_I\otimes c"] \arrow[u, "\phi\otimes\mathrm{id}"] & I \otimes I \arrow[r, "\mathrm{id}_I\otimes u"] \arrow[u, dashed, "\phi\otimes\mathrm{id}"] & I \otimes \bar{P} \arrow[u, "\phi\otimes\mathrm{id}"] \\
\bar{P}[-1] \arrow[r, "s"] \arrow[u, "\sim"] \arrow[ur, phantom, "\square"] & P \arrow[r, "c"] \arrow[u, "\sim"] & I \arrow[r, "u"] \arrow[u, dashed, "\sim"] & \bar{P} \arrow[u, "\sim"]
\end{tikzcd}$$

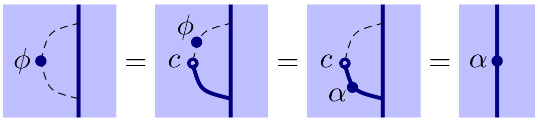
Note that all squares, and in particular the ones on the left commute. Composing the vertical maps, one arrives at the following diagram

$$\begin{tikzcd}
\bar{P}[-1] \arrow[r, "s"] & P \arrow[r, "c"] & I \arrow[r, "u"] & \bar{P} \\
\bar{P}[-1] \arrow[r, "s"] \arrow[u, "{\bar{\alpha}[-1]}"] \arrow[ur, phantom, "\square"] & P \arrow[r, "c"] \arrow[u, "\alpha"] & I \arrow[r, "u"] \arrow[u, dashed, "\phi"] & \bar{P} \arrow[u, "\bar{\alpha}"]
\end{tikzcd} \tag{6.6}$$

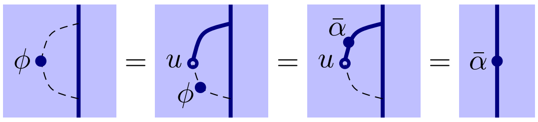
where $\alpha = \alpha(\phi)$ and $\bar{\alpha} = \bar{\alpha}(\phi)$ and of course all squares commute. This shows the claim that $(\alpha(\phi), \bar{\alpha}(\phi)) \in \mathrm{End}(\bar{P}[-1] \xrightarrow{s} P)$.

into the projected theory (IR). This connects to ideas in [58] and might be interesting for dimensions greater than two [59] where the RG defects of this chapter appear as a form of (co)unit.

On the other hand, since the rows in (6.6) are exact triangles, any endomorphism $(\alpha, \bar{\alpha}) \in \mathrm{End}(\bar{P}[-1] \xrightarrow{s} P)$ gives rise to an endomorphism $\phi \in \mathrm{End}(I)$. The latter in turn satisfies $\alpha(\phi) = \alpha$ and $\bar{\alpha}(\phi) = \bar{\alpha}$ which follows from the commutativity of squares in (6.6), namely



for the counital projection defect P and



for the unital projection defect $\bar{P}$.

Hence, $\mathrm{End}(I) \cong \mathrm{End}(\bar{P}[-1] \xrightarrow{s} P)$, and the algebra of bulk fields of the unprojected theory can be reconstructed from the ones of the two projected theories. This discussion naturally extends to the fermionic bulk spectrum $\mathrm{Hom}(I[-1], I)$.

6.2 Application to Landau-Ginzburg models

As explained at the end of section 3.3, the defect and boundary categories of B-twisted Landau-Ginzburg models and their orbifolds are tensor triangulated. Hence, the previous discussion of complementary projection defects applies to all these models. In the following, the construction is applied to the Landau-Ginzburg orbifolds of chapter 4.

Landau-Ginzburg orbifolds $X^d/\mathbb{Z}_d$. We again consider the Landau-Ginzburg orbifold with a single chiral field X, superpotential $W = X^d$, for some $d \in \mathbb{N}_{\geq 2}$, orbifolded by the group $\mathbb{Z}_d$. An element $[n] \in \mathbb{Z}_d$ acts on X by $X \mapsto \exp(2\pi i \frac{n}{d})X$, and hence leaves the superpotential W invariant.

RG defects describing the flow from the orbifold $X^d/\mathbb{Z}_d$ to $X^{d'}/\mathbb{Z}_{d'}$, $d > d'$, were constructed in [3] and discussed in chapter 4. They can be represented in terms of $\mathbb{Z}_d \times \mathbb{Z}_d$-equivariant matrix factorizations R of rank d' which are parametrized by a choice of $m \in \mathbb{Z}_d$ and $n_0, ..., n_{d'-1} \in \mathbb{N}_{>0}$ which sum up to $d = n_0 + ... + n_{d'-1}$. These RG defects give rise to counital[2] projection defects

[2]Similarly, $\bar{P} := R^\dagger \otimes R$ are unital projection defects realizing the IR theory within the theory at the UV. Note again the difference in notation to chapter 4: The unital projection P given by $R^\dagger \otimes R$ in that chapter is denoted by $\bar{P}$ in the current chapter. Here, P refers to the counital version ${}^\dagger R \otimes R$.

$P := {}^{\dagger}R \otimes R$ realizing the IR theory $X^{d'}/\mathbb{Z}_{d'}$ within the UV theory $X^{d}/\mathbb{Z}_{d}$. They are represented by the $\mathbb{Z}_d \times \mathbb{Z}_d$-equivariant matrix factorizations

$$P : P_1 \underset{\mathrm{d}_{P0}}{\overset{\begin{pmatrix} Z^{n_1} & 0 & \cdots & 0 & -X^{n_0} \\ -X^{n_1} & Z^{n_2} & & & \\ 0 & -X^{n_2} & \ddots & & \\ \vdots & & \ddots & Z^{n_{d'-1}} & \\ 0 & & & -X^{n_{d'-1}} & Z^{n_0} \end{pmatrix}}{\rightleftarrows}} P_0 = S^{d'} \begin{pmatrix} \{[r_0],[-r_0]\} \\ \{[r_1],[-r_1]\} \\ \{[r_2],[-r_2]\} \\ \vdots \\ \{[r_{d'-1}],[-r_{d'-1}]\} \end{pmatrix}$$

of $Z^d - X^d$. Here, $S = \mathbb{C}[X, Z]$, $\{\cdot,\cdot\}$ indicates the $\mathbb{Z}_d \times \mathbb{Z}_d$-charge of the respective S-module generator, and $[r_i] = [m + n_1 + ... + n_i] \in \mathbb{Z}_d$. The charges of P_1 can easily be inferred from the ones of P_0. For more details see chapter 4. To simpify notation, we will consider the indices i of r_i and n_i to be defined modulo d'.

The counit $c : P \to I$ is given by the evaluation map $P := {}^{\dagger}R \otimes R \to I$ which can be determined explicitely for the case at hand with the fomulae of appendix B.8 based on [15, 9]

$$(c_0)_{ji} = -\frac{d'-1}{d'}\delta_{j,r_i}, \qquad (c_1)_{kj} = -\frac{d'-1}{d'}\delta_{j,j_k} Z^{n_{j+1}-1-[k-r_j]} X^{[k-r_j]} \,.$$

Here, $[k - r_j]$ denotes the representative of $(k - r_j)$ mod d in $\{0, \ldots, d-1\}$, and j_k the unique $j \in \mathbb{Z}_{d'}$ that minimizes $[k - r_j]$.

Given the counit, a straightforward computation reveals that the complementary projection defect $\bar{P}$ of P is isomorphic to a direct sum

$$\bar{P} = \bigoplus_{\substack{i \in \mathbb{Z}_{d'} \\ \text{with } n_i > 1}} \bar{P}_i \tag{6.7}$$

of multiple unital projection defects $\bar{P}_i$, one for each $n_i > 1$. The summands $\bar{P}_i$ are given by the rank-n_i matrix factorizations

$$\bar{P}_i : S^{n_i} \underset{\mathrm{d}_{P0}}{\overset{\begin{pmatrix} -X & Z & & & \\ & -X & Z & & \\ & & \ddots & \ddots & \\ & & & -X & Z \\ Z^{d-n_i+1} & & & & -X^{d-n_i+1} \end{pmatrix}}{\rightleftarrows}} S^{n_i} \begin{pmatrix} \{[r_{i-1}+1],[-r_{i-1}-1]\} \\ \{[r_{i-1}+2],[-r_{i-1}-2]\} \\ \{[r_{i-1}+3],[-r_{i-1}-3]\} \\ \vdots \\ \{[r_i-1],[-r_i+1]\} \\ \{[r_i],[-r_{i-1}]\} \end{pmatrix}$$

One can check that each $\bar{P}_i$ is a projection defect with $\bar{P}_i \otimes \bar{P}_j \cong \delta_{i,j}\bar{P}_i$, and that it is unital. In fact, it factorizes as $\bar{P}_i \cong R_i^{\dagger} \otimes R_i$, where each R_i is an RG defect from $X^d/\mathbb{Z}_d$ to $X^{n_i}/\mathbb{Z}_{n_i}$. Therefore, $\bar{P}_i$ realizes the Landau-Ginzburg orbifold theory $X^{n_i}/\mathbb{Z}_{n_i}$ inside $X^d/\mathbb{Z}_d$.

We obtain the following picture. The counital projector $P = {}^{\dagger}R \otimes R$ associated to an RG defect from $X^d/\mathbb{Z}_d \to X^{d'}/\mathbb{Z}_{d'}$ specified by $(m, n_0, \ldots, n_{d'-1})$ projects the theory $X^d/\mathbb{Z}_d$ in the UV to the IR theory $X^{d'}/\mathbb{Z}_{d'}$. The respective complementary unital projection defect $\bar{P}$ projects to the stack[3]

$$\bigoplus_{\substack{i \in \mathbb{Z}_{d'} \\ \text{with } n_i > 1}} X^{n_i}/\mathbb{Z}_{n_i}$$

of Landau-Ginzburg orbifold theories.

$$\begin{tikzcd}
 & X^d/\mathbb{Z}_d \arrow[dl, "P"'] \arrow[dr, "\bar{P}"] & \\
X^{d'}/\mathbb{Z}_{d'} & & \bigoplus_{\substack{i \in \mathbb{Z}_{d'} \\ \text{with } n_i > 1}} X^{n_i}/\mathbb{Z}_{n_i}
\end{tikzcd}$$

In fact, the mirror of the RG flow described by the defect R is a flow from the unorbifolded Landau-Ginzburg model with superpotential $W = X^d$ to the one with superpotential $W' = X^{d'}$ triggered by a perturbation of the superpotential by lower order terms. The parameters n_i specify how the critical points of W behave under the flow. More precisely, the $(d-1)$-times degenerate cricial point of $W = X^d$ breaks up into a $(d'-1)$-times degenerate critical point which is associated to the IR theory, and $(n_i - 1)$-times degenerate cricial points for each $n_i > 1$. The latter run off to infinity and decouple from the theory. The complementary projection defect $\bar{P}$ projects onto all the decoupling parts of the theory.

We will conclude this example by discussing the decomposition of the category of boundary conditions of the UV theory with respect to the complementary pair $(P, \bar{P})$. B-type boundary conditions of the Landau-Ginzburg orbifold theory $X^d/\mathbb{Z}_d$ can be described by $\mathbb{Z}_d$-equivariant matrix factorizations of X^d. The category of these matrix factorizations is generated by the rank-1 linear matrix factorizations

$$B^1_{[c]} : \mathbb{C}[X]\{[c+1]\} \underset{X^{d-1}}{\overset{X^1}{\rightleftarrows}} \mathbb{C}[X]\{[c]\}.$$

Here $[c] \in \mathbb{Z}_d$ determines the $\mathbb{Z}_d$-action. All other $\mathbb{Z}_d$-equivariant matrix factorizations of X^d can be obtained from the $B^1_{[c]}$ via successive cone con-

[3]Such stacks are sometimes also referred to disjoint unions [60], because sigma models whose target spaces are disjoint unions are particular examples.

structions. For instance, any rank-1 $\mathbb{Z}_d$-equivariant matrix factorization

$$B^k_{[c]}:\mathbb{C}[X]\{[c+k]\} \underset{X^{d-k}}{\overset{X^k}{\rightleftarrows}} \mathbb{C}[X]\{[c]\}$$

of X^d can be expressed via (3.8) as cone of a morphism $\phi = (-1, X^{d-1})$: $B^1_{[c]}[1] \to B^{k-1}_{[c+1]}$,

$$B^k_{[c]} \cong \mathbb{C}[X]^2 \begin{pmatrix}\{[c+k]\}\\ \{[c+1]\}\end{pmatrix} \underset{\begin{pmatrix}X^{d-k+1} & X^{d-k}\\ & X^{d-1}\end{pmatrix}}{\overset{\begin{pmatrix}X^{k-1} & -1\\ & X\end{pmatrix}}{\rightleftarrows}} \mathbb{C}[X]^2 \begin{pmatrix}\{[c+1]\}\\ \{[c]\}\end{pmatrix}.$$

Thus, by induction on k, any rank-1 matrix factorization $B^k_{[c]}$ can be obtained as successive cone of linear matrix factorizations $B^1_{[c]}$.

Employing methods of [3, 16] to calculate fusion, we find that the action of P on the linear boundary conditions of the UV theory $X^d/\mathbb{Z}_d$ is given by

$$P \otimes B^1_{[c]} \cong \begin{cases} B^{n_i+1}_{[c]} & \text{if } [c]=[r_i] \text{ for some } i \\ 0 & \text{otherwise} \end{cases}. \tag{6.8}$$

Since $[r_i] \neq [r_j]$ for $i \neq j$, this implies that d' linear boundary conditions have a non-trivial image under fusion with P. Since fusion commutes with the cone construction, the images $B^{n_i+1}_{[r_i]}$ generate the category of P-invariant boundary conditions. The latter corresponds to the category of boundary condition of the P-projected theory, and is indeed isomorphic to the category of boundary conditions of the IR theory $X^{d'}/\mathbb{Z}_{d'}$.

On the other hand there are $(d-d')$ linear boundary conditions $B^1_{[c]}$, $[c] \notin \{[r_i]\,|\,i\}$ annihilated by the fusion with P. They generate the $\bar{P}$-invariant subcategory, i.e. the category of boundary conditions of the $\bar{P}$-projected theory. Indeed, the summands $\bar{P}_i$ of $\bar{P}$ fuse with linear boundary conditions according to

$$\bar{P}_i \otimes B^1_{[c]} \cong \begin{cases} B^1_{[c]} & \text{if } [c] \in \{[r_{i-1}+1], [r_{i-1}+2], ..., [r_i-1]\} \\ B^{d-n_i+1}_{[r_i]} & \text{if } [c]=[r_{i-1}] \\ 0 & \text{otherwise} \end{cases}. \tag{6.9}$$

We explicitly recover the decomposition of the category of boundary conditions into the P-invariant and $\bar{P}$-invariant subcategories (c.f. the discussion around (6.3)). The category of boundary conditions of $X^d/\mathbb{Z}_d$ is generated by the linear boundary conditions $B^1_{[c]}$. The ones with $[c] \notin \{[r_i] \,|\, i\}$ are annihilated by P and hence are $\bar{P}$-invariant. Moreover, for all i with $n_i = 1$ the linear boundary conditions $B^1_{[r_{i-1}]}$ are P-invariant: $P \otimes B^1_{[r_{i-1}]} \cong B^1_{[r_{i-1}]}$ and $\bar{P} \otimes B^1_{[r_{i-1}]} \cong 0$. The only linear boundary conditions which are neither P- nor $\bar{P}$-invariant are the $B^1_{[r_{i-1}]}$ with $n_i > 1$. For these, (6.8) and (6.9) yield

$$P \otimes B^1_{[r_{i-1}]} \cong B^{n_i}_{[r_{i-1}]} \quad \text{and} \quad \bar{P} \otimes B^1_{[r_{i-1}]} \cong B^{d-n_i+1}_{[r_i]} \,.$$

Hence, they can be represented as cones of morphisms

$$\bar{P} \otimes B^1_{[r_{i-1}]}[-1] \cong B^{d-n_i+1}_{[r_i]}[-1] \longrightarrow P \otimes B^1_{[r_{i-1}]} \cong B^{n_i}_{[r_{i-1}]}$$

between objects of the $\bar{P}$-invariant and the P-invariant subcategories, see (6.3). Explicitly,

$$B^1_{[r_{i-1}]} \cong \mathbb{C}[X]^2 \begin{pmatrix} \{[r_i]\} \\ \{[r_{i-1}+1]\} \end{pmatrix} \underset{\begin{pmatrix} X^{d-n_i} & -1 \\ & X^{n_i-1} \end{pmatrix}}{\overset{\begin{pmatrix} X^{n_i} & X \\ & X^{d-n_i+1} \end{pmatrix}}{\rightleftarrows}} \mathbb{C}[X]^2 \begin{pmatrix} \{[r_{i-1}]\} \\ \{[r_i]\} \end{pmatrix} \,.$$

The decomposition on the level of the generators determines the decomposition of the entire category of boundary conditions into P- and $\bar{P}$-invariant subcategories.

6.3 Conclusions

In this chapter it was shown that in theories whose defect categories are tensor triangulated, projection defects always come in complementary pairs $(P, \bar{P})$. These have the following properties. P is counital and $\bar{P}$ is unital, and the identity defect is isomorphic to a cone of a morphism $s : \bar{P}[-1] \to P$.

From chapter 2.6 we know that (co)unital projection defects always split, i.e. there are RG-type defects $\bar{R}$ (R) such that $\bar{P} \cong \bar{R}^\dagger \otimes \bar{R}$ ($P \cong {}^\dagger R \otimes R$), and the projection defects realize projected theories in the given host theory. The fact that in the triangulated setup, projection defects always come in complementary pairs means that whenever there is a projection defect in this context, the host theory decomposes into the projected theory associated to P and the complementary projected theory associated to $\bar{P}$.

In the explicit example we considered, the counital projection defect P splits as $P \cong {}^\dagger R \otimes R$, where R is the RG defect associated to a renormalization group flow between the Landau-Ginburg orbifolds $X^d/\mathbb{Z}_d$ in the UV and $X^{d'}/\mathbb{Z}_{d'}$ in the IR ($d > d'$). Hence, P realizes the IR theory inside the UV. The complementary projection defect $\bar{P}$ in this case collects all the parts of the theory which decouple during the RG flow; and the UV theory deomposes into two parts, the IR theory on the one hand and the decoupling parts on the other.

An insteresting open question is the physical significance of the fact that the identity defect in the host theory is isomorphic to the cone of some morphism $s : \bar{P}[-1] \to P$. Obviously this means that $P \oplus \bar{P}$ deforms to the identity defect. If $P \oplus \bar{P}$ was itself a (co)unital projection defect, with a well-defined projected theory associated to it, then this implies that the latter can be deformed (or perturbed) to the host theory. However, in general $P \oplus \bar{P}$ is neither unital nor counital. Hence, it is not clear to what extent $P \oplus \bar{P}$ describes an honest topological quantum field theory. It would be interesting to shed some light on the physical significance of the sum $P \oplus \bar{P}$ and the cone condition.

Chapter 7

Summary and outlook

In this thesis, I have presented a new method which allows for a complete representation of one topological quantum field theory within another (chapter 2 and appendix A). This procedure was found to have two equivalent points of view.

First, every (co)unital projection defect P gives rise to a new topological theory: Its spectrum equals the bulk spectrum of the new theory, boundaries and defects invariant under fusion with P constitute the boundary and defect spectra of the new theory and correlators of the new theory can be calculated in terms of the original theory by inserting networks of the defect P. The projection defect P also gives a functor from the original theory to the new one: Bulk fields encircled by P give a field on P and boundaries (defects) fused with P give P-invariant boundaries (defects).

Second and equivalently, every two such theories are related by an RG defect R - a defect whose loop around one theory collapses trivially. By fusion and encircling, R allows to functorially map objects from either theory to the other. What is more, this way the second theory is completely embedded into the first and all its properties (bulk/boundary/defect spectra, symmetries and correlators) are realized within the first. Conversely, R allows to project bulk, boundary and defect spectra of the first theory into the second. This process is not reversible - and tightly connected to RG flow. Namely, such RG defects arise naturally from the invisible defect I if the theory on one side of I is perturbed and flows to some IR fixed point.

The review chapter 3 then set the stage for concrete applications by introducing two-dimensional supersymmetric quantum field theories and their topological B-twist. It furthermore introduced (equivariant) matrix factorizations, which allow to describe the topological defect and boundary spectra of B-twisted Landau-Ginzburg (orbifold) theories. This formalism also provides a method to calculated defect fusion explicitely.

As a first application, the presented procedure was applied in depth to Landau-Ginzburg orbifolds with a single chiral field, superpotential $W = X^d$

and orbifold group $\mathbb{Z}_d$ (chapter 4 and appendices B, C). Taking the limit $d \to \infty$, it was possible to realize all these theories within the theory of a single free twisted chiral field.

Next up, the method of chapter 2 was used to represent different phases of gauged linear sigma models within the GLSM itself (chapter 5). First, a clever rephrasing of the orbifold identity defect allowed a generalization to continuous orbifold groups, which gives a description of the identity defect of a topologically twisted gauge decoupled sector of the GLSM. Pushing down the two sides of this defect results in the sought-for RG defects which realize the two phases within the GLSM, describe the mapping of GLSM objects into both phases and allow to move D-branes between the two phases.

Finally, in chapter 6 an interesting observation was proved: If the defect spectra of the TQFTs in question are triangulated - as they are in our examples - then every (co)unital projection defect comes with a complementary projection defect. Hence, the host theory decomposes into the two projected theories associated to the two projectors. This finding was applied to the LG orbifold example of chapter 4.

Looking ahead, there are multiple avenues of possible investigation, some of which have already been discussed at the end of chapters 5 and 6. In the following, this list is extended along the lines of [1].

- It would be interesting to apply the construction outlined in this thesis to other examples of RG flows and to find representations of more elaborate 2d TQFTs in free theories by means of projectors. For example, the treatment of flows between LG orbifolds discussed in chapter 4 carries over to flows between orbifolds of free chiral field theories. The latter theories can be obtained from LG orbifolds by setting the superpotentials to zero. The respective RG defects can be described in terms of matrix factorizations and have been worked out in [61].

- In the example of LG orbifolds $\mathcal{M}_d/\mathbb{Z}_d$ one could explicitly compare the bulk flows studied in section 4.2 with the corresponding flows of the identity defects. While the identity defects can be described by means of matrix factorizations, the respective relevant perturbations are by twisted chiral fields. Such perturbations are not as easily treated in the matrix factorization framework as defect perturbations by chiral fields such as the ones discussed in [62]. In the case at hand however, the twisted chiral fields generating the perturbations are orbifold twist fields. As such, they do have a representation in the matrix factorization framework as defect changing fields. This possibly allows for an explicit analysis of the respective defect perturbations in this case. Hence, it might be possible to work out in this concrete example how the projection defects P associated to bulk flows arise as perturbations of identity defects.

- The LG orbifolds of chapter 4 were realized within the theory of a free twisted chiral field, which can be seen as a $U(1)$-equivariant version of the free chiral field. It would be interesting to write this down explicitely using the $U(1)$-equivariant identity defect of chapter 5.

- The construction described in this paper heavily relies on topological covariance. So an obvious question is whether it has any bearing on non-topological QFTs (beyond topologically twisted supersymmetric theories). Indeed, one can perturb the identity defect in non-topological QFTs and RG defects exist in more general theories, see e.g. [4]. However, fusion of non-topological defects is singular in general. Still, at least in some cases it is possible to define a reasonable notion of fusion [5, 63], so that defects P can be constructed from RG defects. The role of these defects is less clear, but it would be very interesting to study them in examples. Perhaps they are related to the line operators appearing in the context of integrable perturbations of conformal field theories [64, 65, 66, 67, 68].

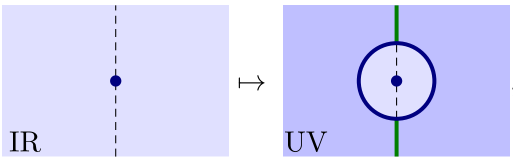
- While the discussion in this paper is restricted to 2d TQFTs, we expect that bulk perturbations of TQFTs can be described by means of codimension-one projection defects in any dimension. Indeed, the generalized orbifold construction has been extended from dimension two to higher dimensions in [59]. It would be very interesting to apply the methods described in this paper to higher-dimensional TQFTs.

Appendix A

Properties of projection defects

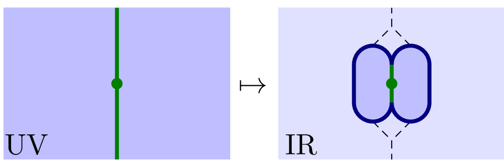
A.1 IR bulk fields in the UV

Here, we show that if a projection defect P in the UV factorizes as $P \cong R^\dagger \otimes R$ with $R \otimes R^\dagger \cong I_{\mathrm{IR}}$, then P-bimodule morphisms on P are one-to-one with IR bulk fields (I_{IR}-bimodule morphisms of I_{IR}). The latter are mapped into the UV by the homomorphism



, (A.1)

where the junctions are given by the isomorphisms $P \cong R^\dagger \otimes R$. (To keep the notation light, we refrain from putting arrows on RG defects. We mark UV and IR theory by dark, respectively light background.) By the projection property the right hand side is a P-bimodule morphism of P. We claim that the inverse to the homomorphism (A.1) is given by



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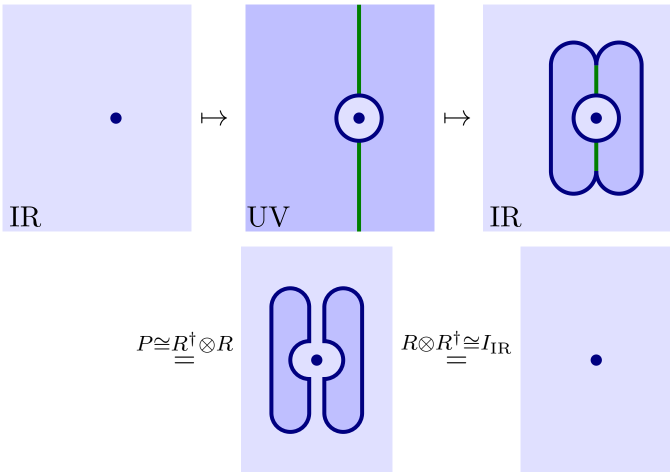
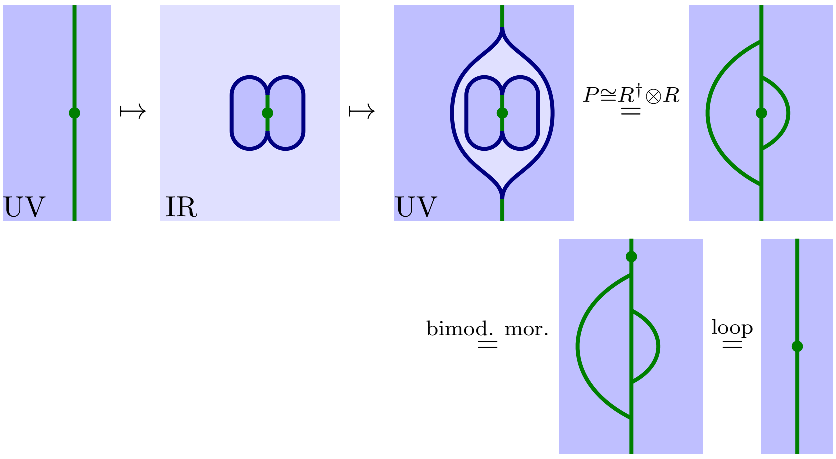
Let us first check that the composition IR → UV → IR evaluates to the identity on IR bulk fields:[1]



Similarly, the composition UV → IR → UV is the identity on P-bimodule morphisms of P:



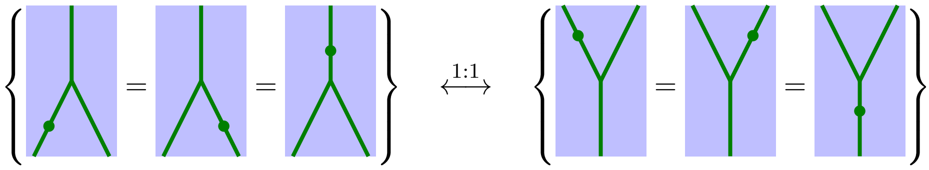
Therefore we have established that the map (A.1) is an isomorphism from the space of IR bulk fields to the space of P-bimodule morphisms of P.

A.2 Bimodule equal bicomodule morphisms

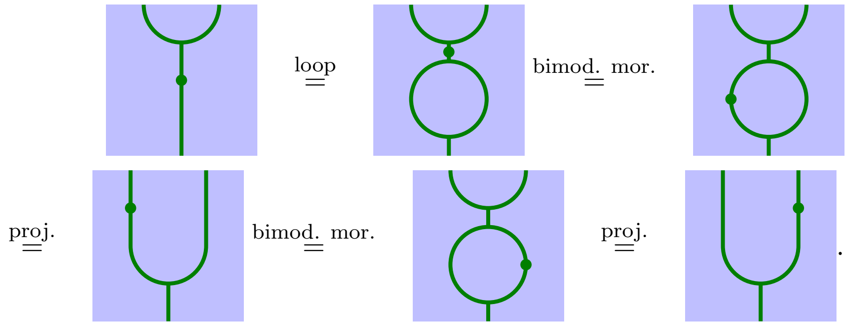
For any projection defect, bimodule morphisms over itself are automatically cobimodule morphisms and vice versa. In other words, the two types of

[1] For readability, I_{IR} has been omitted.

morphisms are one-to-one:



This is easy to see. A bimodule morphism for example (left-hand side above) automatically obeys



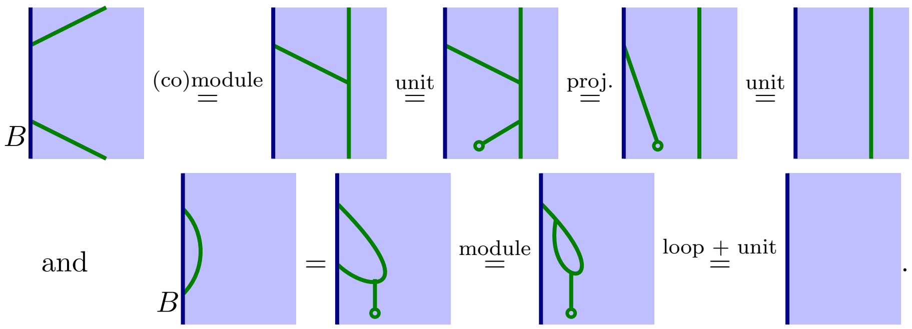
The argument that bicomodule morphisms also respect the bimodule structure follows by turning the diagrams above upside down.

Indeed, if the projection defect P comes with a unit then all morphisms of P automatically respect the bicomodule structure, and by the above also the bimodule structure on P. (This easily follows from the projection property.) Hence, in this case all morphisms of P are bimodule morphisms and bicomodule morphisms. The same is true if P has a counit.

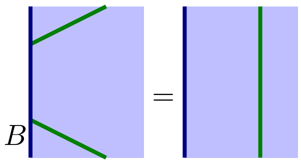
A.3 P-modules B and $B \otimes P \cong B$

In this appendix we show that for a projection defect P, any left boundary condition B is a right P-module if and only if $B \otimes P \cong B$. This in particular means that left IR boundary conditions can be represented by left boundary conditions B in the UV which are invariant under fusion with P, i.e. $B \otimes P \cong B$. This statement extends to right boundaries and defects.

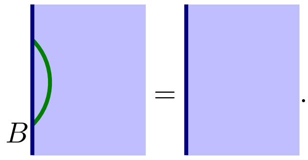
First, a left P-module (whose comodule structure is induced by the unit on P) obeys $B \otimes P \cong B$:



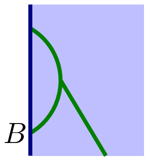
If on the other hand a left boundary condition B satisfies $B \otimes P \cong B$, B inherits the P-module structure of P itself. Namely, there are junctions $B \otimes P \to B$ and $B \to B \otimes P$ such that



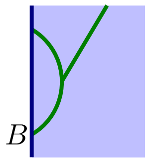
and



This implies that



and



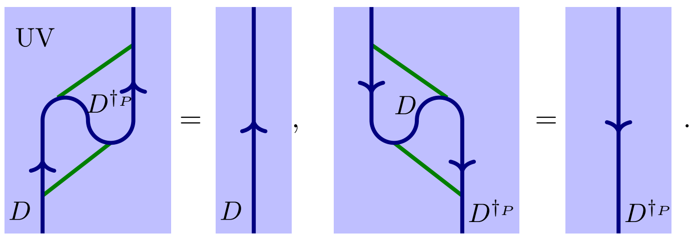
define P-module, respectively P-comodule structures on B, which are also inverse to each other and hence provide isomorphisms $B \otimes P \cong B$.

A.4 P-adjunction

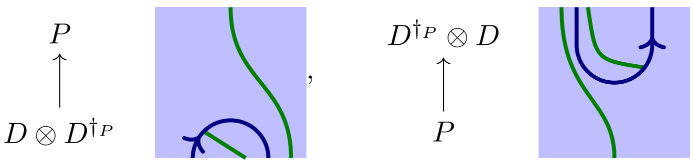
In this appendix we show that the adjunction of IR defects is lifted to the UV by the following formulas

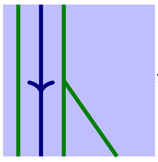
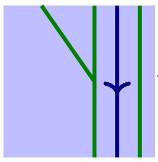
$$D^{\dagger_P} = P \otimes D^{\dagger} \otimes P$$
$$^{\dagger_P}D = P \otimes {}^{\dagger}D \otimes P$$

where P is the corresponding projection defect, and D is a defect in the UV theory representing an IR defect. We will only consider the first equation and will furthermore restrict to the case that P is unital. The arguments for the second equation and the counital case are similar. The IR right adjoints have to satisfy the following Zorro move identities

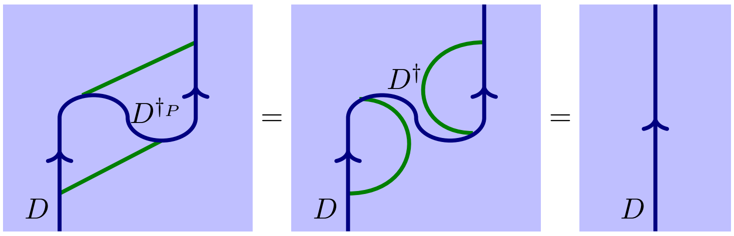


These are satisfied for $D^{\dagger_P}$ above when we choose the following natural (co-)evaluation maps

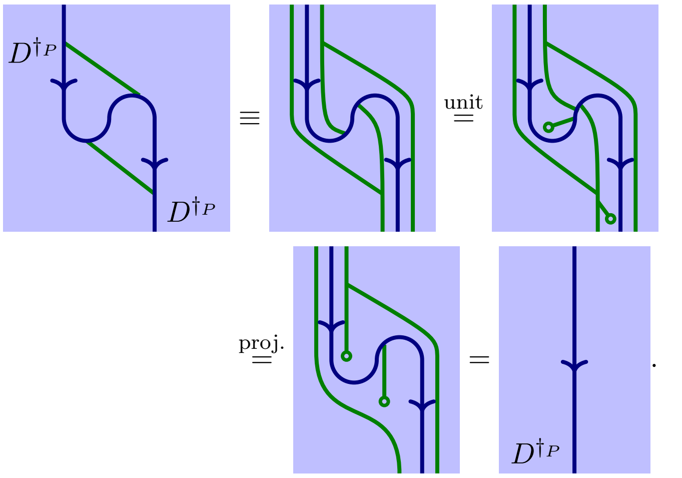


$$\begin{array}{c} D^{\dagger_P} \\ \uparrow \\ D^{\dagger_P} \otimes P \end{array}$$  , $$\begin{array}{c} P \otimes D^{\dagger_P} \\ \uparrow \\ D^{\dagger_P} \end{array}$$  .[2]

Namely,



and



using unit condition, Zorro-move and the definition of $D^{\dagger_P}$ in the last step. For counital P and left-adjoints the above diagrams have to be flipped appropriately.

The defect P is a P-module and a P-comodule, so it can be regarded as an IR defect. As such, it should be selfadjoint, and, using the above notion of IR adjunction one finds that this is indeed the case: $P^{\dagger_P} \cong P \cong {}^{\dagger_P}P$. If

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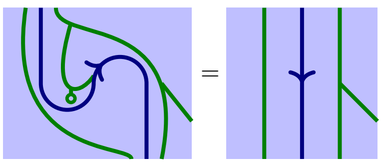
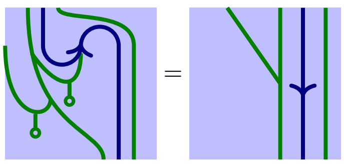
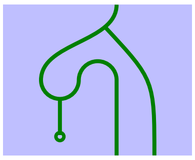
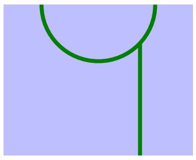
[2]The lower maps follow as natural generalizations from the generalized orbifold procedure [9, Prop. 4.7]. Namely,





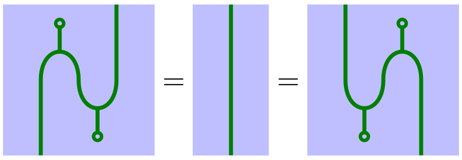
for instance P is unital, the two maps

$$\begin{array}{c} P \\ \uparrow \\ P^\dagger \otimes P \end{array}$$  and $$\begin{array}{c} P^\dagger \otimes P \\ \uparrow \\ P \end{array}$$ 

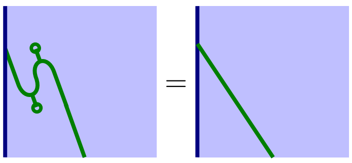
are inverse to each other and hence provide isomorphisms $P \cong P^\dagger \otimes P$. The projection property of P, $P \otimes P \cong P$ implies that $P^{\dagger_P} = P \otimes P^\dagger \otimes P \cong P$. The argument for counital P is analogous.

A.5 Projections with unit and counit

If a projection defect has a unit as well as a counit, it is automatically self-adjoint as there are natural (co)evaluation maps:



The equalities follow from the two Frobenius and (co)unit properties. If P has a unit, any P-module automatically carries a P-comodule structure. Vice versa, any P-comodule is automatically a P-module in case P has a counit. If P has both, a unit and a counit, these two constructions are compatible: Starting with a P-module, a P-comodule structure is induced which in turn induces a P-module structure. This P-module structure is identical to the original one:



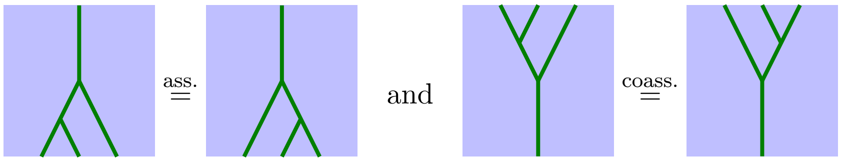
As discussed in section 2.6, all projection defects P factorize into RG type defects $P \cong R^\dagger \otimes R$. For example, unitary P factorize as

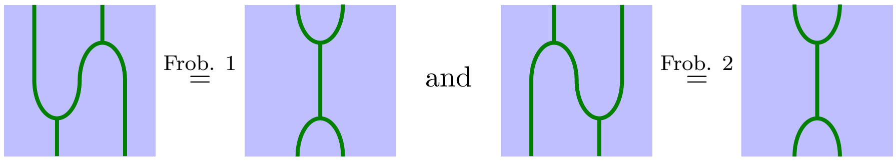
$$\begin{aligned} \left(P_{\mathrm{IR}|\mathrm{UV}}\right)^\dagger &= P_{\mathrm{UV}|\mathrm{IR}} \\ {}^\dagger\!\left(P_{\mathrm{IR}|\mathrm{UV}}\right) &= \left({}^\dagger P\right)_{\mathrm{UV}|\mathrm{IR}} . \end{aligned}$$

For selfadjoint projection defects P, the respective RG type defects R then satisfy ${}^\dagger R \cong R^\dagger$.

A.6 (Co)multiplication and Frobenius properties

Here, we show the equivalence of associativity, coassociativity and the two Frobenius properties for projection defects and how they follow from the existence of a (co)unit. The identities in question are:

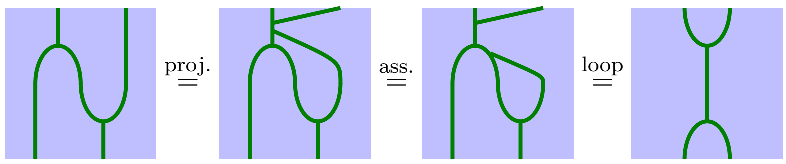




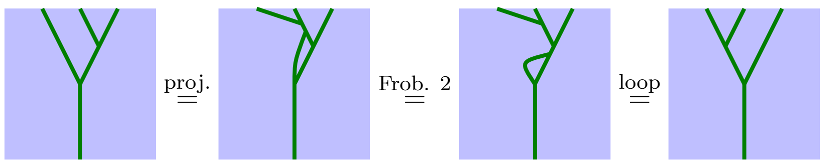
Equivalence is shown by the following chain of implications

associativity ⇒ Frob. 2 ⇒ coassociativity ⇒ Frob. 1 ⇒ associativity.

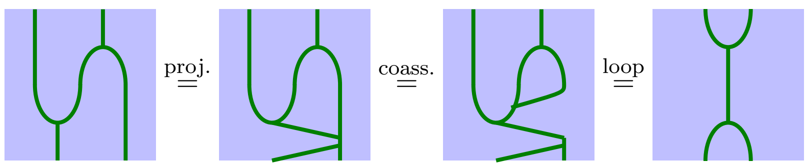
associativity ⇒ Frob. 2:



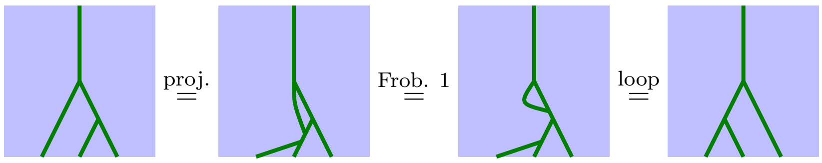
Frob. 2 ⇒ coassociativity:



coassociativity ⇒ Frob. 1:

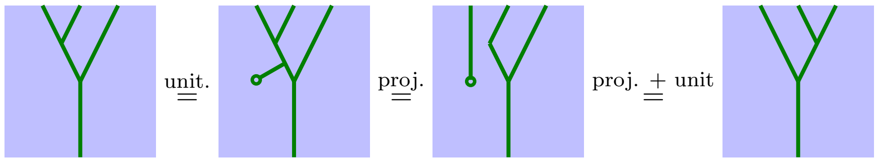


Frob. 1 $\Rightarrow$ associativity:



Next, we show how the existence of a unit for a projection defect implies coassociativity:

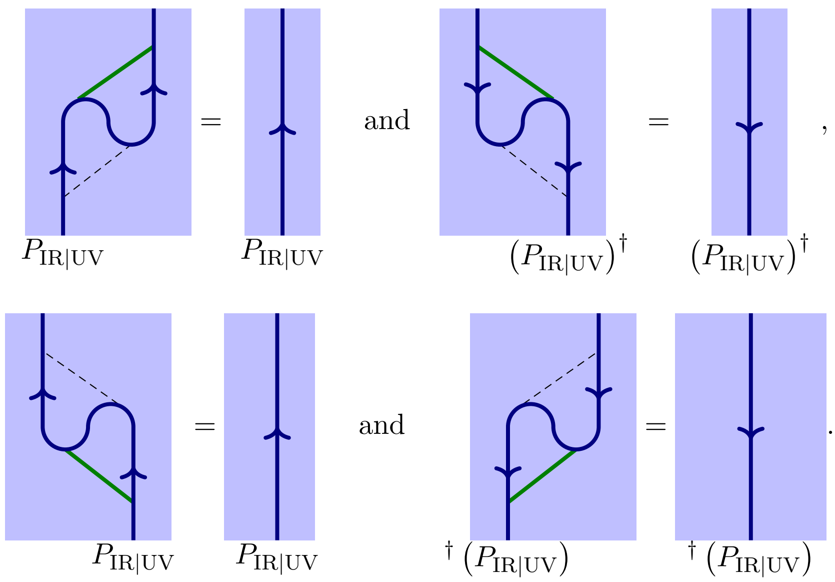
unit $\Rightarrow$ coassociativity:



In the last step we applied the projection property to the left and the lower defect. Turning these diagrams upside down shows how associativity follows from the existence of a counit.

A.7 Adjoints of induced RG defects

Right and left adjoints of the induced RG defect $P_{\mathrm{IR|UV}}$ must satisfy the Zorro move identities



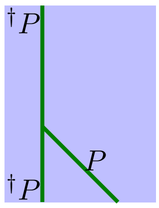
We will discuss the case that P has a unit. The counital case can be treated analogously. Indeed, it is easy to see that

$$(P_{\mathrm{IR|UV}})^\dagger = P_{\mathrm{UV|IR}},$$

i.e. P regarded as defect from IR to UV is the right adjoint of P regarded as a defect from UV to IR. The evaluation map is just given by the algebra $P \otimes P \to P$ and the coevaluation map is induced by the unit $I_{\mathrm{UV}} \to P \to P \otimes P$. (The Zorro identities immediately follow from associativity and the unit condition.) It is slightly more involved to see that the left adjoint is given by

$$^{\dagger}\left(P_{\mathrm{IR}|\mathrm{UV}}\right) = \left({}^{\dagger}P\right)_{\mathrm{UV}|\mathrm{IR}} ,$$

the left adjoint of P regarded as a defect from the IR to the UV theory. Evaluation and coevaluation are given by the maps



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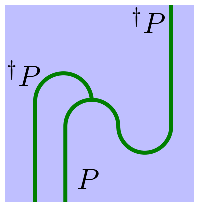

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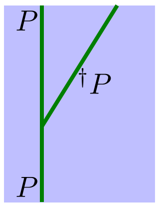

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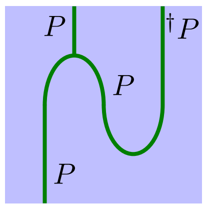

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which define the right P-module structure of ${}^{\dagger}P$ and the right ${}^{\dagger}P$-comodule structure of P, respectively. The first Zorro identity then follows from the UV Zorro move and loop omission, while the second one additionally requires associativity and the unit property.

Appendix B

Orbifold minimal models as generalized orbifolds

Here, we construct the Landau-Ginzburg orbifolds $\mathcal{M}_d/\mathbb{Z}_d$ as generalized orbifolds following [9, Chapter 7]. To distinguish objects in the orbifold from objects in the unorbifolded theory, we adopt the following notation, which is different from the one used in the main text: the identity defect in $\mathcal{M}_d$ is denoted by I, whereas the identity defect in the orbifold theory is represented by the orbifold defect A. Also, adjunction in the orbifold is denoted by '$*$' to distinguish it from adjunction '$\dagger$' in the unorbifolded theory. This notation is only used in this appendix. In chapter 4 of the main text, we do not explicitly refer to the orbifold construction and therefore do not need this distinction. There, I denotes the identity defect and '$\dagger$' the adjunction in the orbifold theory $\mathcal{M}_d/\mathbb{Z}_d$.

B.1 Orbifold identity defect

The models $\mathcal{M}_d/\mathbb{Z}_d$ are standard orbifolds. In this case the defect A is given by the direct sum of the defects implementing the respective actions of all the symmetries in the orbifold group: $A = \oplus_{g\in\mathbb{Z}_d}\,({}_gI)$. The symmetry defects ${}_gI$ can be represented by the rank-one matrix factorizations ($\eta = e^{2\pi i/d}$)

$$ {}_gI : \mathbb{C}[Z,X]e_{d+[g]} \underset{\frac{Z^d-X^d}{\eta^g Z-X}}{\overset{\eta^g Z - X}{\rightleftarrows}} \mathbb{C}[Z,X]e_{[g]}. \tag{B.1} $$

where $[g]$ denotes the representative of $g\in\mathbb{Z}_d$ in $\{0,\dots,d-1\}$, and the e_a, $a\in\{0,\dots,2d-1\}$ are the generators of the respective rank-one free modules $({}_gI)_{0,1}$. ${}_gI$ is the right twist of the identity defect I in $\mathcal{M}_d$ by $g\in\mathbb{Z}_d$.

Since A is the direct sum of the ${}_gI$, the modules A_0 and A_1 are rank-d free modules generated by $e_0,\dots,e_{d-1}$ and $e_d,\dots,e_{2d-1}$, respectively. In the basis (e_a), the differential of the matrix factorization A takes the form

$$(\mathrm{d}_A)_{ab} = \delta_{a,b-d}(\eta^a Z - X) + \delta_{a-d,b} \sum_{l=1}^{d} \eta^{-l\cdot a} Z^{d-l} X^{l-1}$$

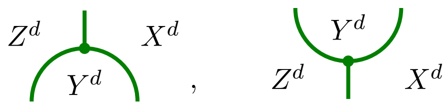
for $a, b = 0, ..., 2d-1$.

The following maps give A a separable Frobenius structure [9, Prop. 7.1]:

1. The unit $I = {}_0I \hookrightarrow A$ is given by the obvious inclusion while the counit is the projection multiplied by d

$$A \twoheadrightarrow I\,, \qquad e_i \mapsto d \cdot \begin{cases} e_i\,, & i \in \{0, d\} \\ 0\,, & \text{otherwise} \end{cases}$$

2. Multiplication and comultiplication



are given by

$$\begin{aligned} A \otimes A &\to A \\ e_{[g]} \otimes Y^q e_{[h]} &\mapsto (\eta^g Z)^q \, e_{[g+h]} \\ e_{[g]} \otimes Y^q e_{d+[h]} &\mapsto (\eta^g Z)^q \, e_{d+[g+h]} \\ e_{[g]+d} \otimes Y^q e_{[h]} &\mapsto 0 \\ e_{[g]+d} \otimes Y^q e_{[h]+d} &\mapsto 0 \end{aligned}$$

and

$$\begin{aligned} \Delta : A &\to A \otimes A \\ e_{[g]} &\mapsto \frac{1}{d} \sum_{h \in \mathbb{Z}_d} \Big[e_{[g-h]} \otimes e_{[h]} \\ &\qquad + e_{d+[g-h]} \otimes \left\{ \partial^{Z,Y} \frac{Z^d - X^d}{\eta^h Z - X} \right\}^{Z \to \eta^{g-h} Z} e_{d+[h]} \Big] \\ e_{d+[g]} &\mapsto \frac{1}{d} \sum_{h \in \mathbb{Z}_d} \left[e_{[g-h]} \otimes e_{d+[h]} + \eta^h e_{d+[g-h]} \otimes e_{[h]} \right], \end{aligned} \tag{B.2}$$

where $g, h \in \mathbb{Z}_d$ and $q \in \mathbb{N}$. Moreover,

$$\partial^{Z,Y} Z^i = \frac{Z^i - Y^i}{Z - Y}$$

and $\{\ldots\}^{Z \to \eta^{g-h} Z}$ means that all instances of Z within the brakets have to be replaced by $\eta^{g-h} Z$ after performing all operations. These formulas can be obtained from the natural junctions of symmetry defects ${}_hI$ with the identity defect $I = {}_0I$. The calculation for the comultiplication is sketched in appendix B.7 below.

In appendix B.6, we will reexpress the orbifold identity defect A using an equivariant basis.

B.2 Nakayama automorphism

The Nakayama automorphism (c.f. appendix 2.7) takes the form [18, Example 3.1]

$$\gamma_A = \sum_{g \in \mathbb{Z}_d} \det(g) \cdot 1_{gI}$$

where $\det(g)$ denotes the matrix representing the action of g on the chiral fields of the model to the right of A. In our case, g acts on X by multiplication with η^g and hence γ_A reduces to

$$\begin{aligned} \gamma_A : A &\to A \\ e_a &\mapsto \eta^a e_a. \end{aligned}$$

A Frobenius algebra B is symmetric iff $\gamma_B = \mathrm{id}_B$ [9, 69]. Since $\gamma_A \neq \mathrm{id}_A$, A is not symmetric. Therefore, left and right adjoints of defects in the orbifold theory (see equation (2.5)) generally differ

$$\left(D^\dagger\right)_{\gamma_{A'}} = D^* \ncong {}^*D = {}_{\gamma_A^{-1}}\left({}^\dagger D\right).$$

This means that we do not have a general prescription of how to close defect loops in $\mathcal{M}_d/\mathbb{Z}_d$. However, loops of RG defects can be closed with an explicit natural morphism.

B.3 Bulk space

The (c,c)-bulk space of the orbifold $\mathcal{M}_d/\mathbb{Z}_d$ contains only the identity $\mathrm{id} : A \to A$. However, in the unorbifolded theory the defect A carries additional fields – one for each $g \neq 0$:

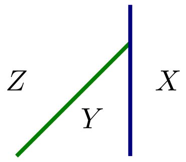
$$\begin{aligned} \psi_g : A &\to A \\ e_{[h]} &\mapsto \frac{Z^d - X^d}{(\eta^i Z - X)(\eta^{h+g} Z - X)} e_{d+[g+h]} \\ e_{d+[h]} &\mapsto e_{[g+h]} \end{aligned}$$

These correspond to the twist fields in the orbifold theory.

B.4 Defects and their adjoints

Consider a rank-M $\mathbb{Z}_{d'} \times \mathbb{Z}_d$-equivariant matrix factorization D of $Z^{d'} - X^d$ with equivariant generators $f_0, \ldots, f_{M-1}$ of D_0 and $f_M, \ldots, f_{2M-1}$. Let

$[l_k, r_k]$ be the $\mathbb{Z}_{d'} \times \mathbb{Z}_d$-charges of f_k. As discussed in the second part of B.7, these charges determine the A-action on D. Denoting the chiral fields as in



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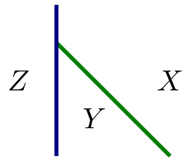

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one obtains

$$\begin{aligned} A \otimes D &\to D \\ e_a \otimes Y^p f_k &\mapsto \delta_{|e_a|,0} \cdot (\epsilon^a Z)^p \cdot \epsilon^{a \cdot l_k} \cdot f_k. \end{aligned}$$

and

$$\begin{aligned} D \otimes A &\to D \\ f_k \otimes Y^p e_a &\mapsto \delta_{|e_a|,0} \cdot f_k \cdot \eta^{-a \cdot r_k} (\eta^{-a} X)^p \end{aligned}$$

where $\epsilon = e^{\frac{2\pi i}{d'}}$ and $\eta = e^{\frac{2\pi i}{d}}$ are elementary d'th, respectively dth roots of unity.

In appendix B.6 below, we will define equivariant generators for A itself, and will reexpress the A-action on D in terms of these generators.

The adjoints in the orbifold theory are given by ${}^*D = {}_{\gamma_A^{-1}}\left({}^\dagger D\right)$ and $D^* = \left(D^\dagger\right)_{\gamma_{A'}}$, see equation (2.5). Here $D^\dagger \cong D^\vee[1] \cong {}^\dagger D$ denotes adjunction in the unorbifolded models, c.f. equation (3.7) in section 3.3. An explicit calculation carried out in the last part of appendix B.7 determines the induced A-action on $D^\dagger$, c.f. equation (B.10). From this, one can read off the $\mathbb{Z}_d \times \mathbb{Z}_{d'}$ charges of the equivariant generators $f_k^\dagger$ and ${}^\dagger f_k$ of $D^\dagger$ and ${}^\dagger D$ to be

$$[-r_{k+M} + 1, -l_{k+M} + 1].$$

Here, we have extended the range of indices of the charges r and l to $\mathbb{Z}$ by identification modulo $2M$, i.e. $r_{i+2Mz} = r_i$ and $l_{i+2Mz} = l_i$ for $i \in \{0, \ldots, 2M-1\}$ and $z \in \mathbb{Z}$.

Twisting by the Nakayama automorphism one then obtains the charges of the generators f_k^* and ${}^*f_k$ of the matrix factorizations describing the orbifold adjoints D^* and ${}^*D$. They are given by

$$\begin{aligned} &[-r_{k+M}, -l_{k+M} + 1] \\ \text{and} \quad &[-r_{k+M} + 1, -l_{k+M}], \end{aligned}$$

respectively. By construction, ${}^*D$ and D^* obey the Zorro moves whose building blocks are provided in B.8.

B.5 Left boundary conditions and their adjoints

As special case of defects, a left boundary condition in $\mathcal{M}_d/\mathbb{Z}_d$ is a $\mathbb{Z}_d$-equivariant matrix factorization B of $-X^d$. Using the same notation as in appendix B.4, we denote the generators of the modules B_0 and B_1 by f_k and their $\mathbb{Z}_d$-charges by $[r_k]$ which as in the general case determine the A-action on B. The induced charges on the right and left adjoint generators $f_k^\dagger$ of $B^\dagger$ and ${}^\dagger f_k$ of ${}^\dagger B$ are $[-r_{k+M}+1]$ and $[-r_k+1]$, respectively. Using $B^* = B^\dagger$ and ${}^*B = {}_{\gamma^{-1}}({}^\dagger B)$, the charges of the adjoint generators f_k^* and ${}^*f_k$ become

$$\begin{aligned} f_k^* &: [-r_{k+M}+1] \\ {}^*f_k &: [-r_k] \end{aligned}$$

The explicit expressions of the relevant (co-)evaluation maps for defects as well as boundaries are given in appendix B.8.

B.6 Equivariant generators of the orbifold identity defect

One can define equivariant generators of the orbifold identity matrix factorization A (c.f. section B.1) by

$$e'_b = \frac{1}{d}\sum_c \delta_{|e_b|,|e_c|}\eta^{-(b+|e_b|)c}e_c,$$

where the original generators e_c are expressed in terms of the equivariant ones as

$$e_c = \sum_b \delta_{|e_b|,|e_c|}\eta^{c(b+|e_c|)}e'_b$$

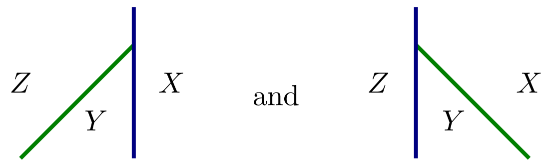
In this basis, the matrix factorization A takes the equivariant form

$$A : S^d \begin{pmatrix} [1,0] \\ [2,-1] \\ \vdots \end{pmatrix} \underset{\longleftarrow}{\overset{\begin{pmatrix} Z & 0 & \dots & 0 & -X \\ -X & \ddots & & & \\ 0 & \vdots & \ddots & & \\ \vdots & & \vdots & \vdots & \\ 0 & & & -X & Z \end{pmatrix}}{\longrightarrow}} S^d \begin{pmatrix} [0,0] \\ [1,-1] \\ \vdots \end{pmatrix}.$$

where $S := \mathbb{C}[Z, X]$. (This is the form used in [3].) The A-action on equivariant matrix factorizations determined in appendix B.7 and used in appendix B.4 simplifies in this basis.

Consider a $\mathbb{Z}_d \times \mathbb{Z}_d$-equivariant matrix factorization D of $Z^d - X^d$. Let $f_0, \dots, f_{M-1}$ and $f_M, \dots, f_{2M-1}$ be $\mathbb{Z}_d \times \mathbb{Z}_d$-equivariant generators of D_0

and D_1, respectively. Denote the $\mathbb{Z}_d \times \mathbb{Z}_d$-charges of f_k by $[l_k, r_k]$. In terms of the equivariant generators e'_i of A, the A-action



(B.3)

becomes

$$
\begin{aligned}
A \otimes D \to D, \qquad & e'_a \otimes Y^p f_k \mapsto \delta_{a,[p+l_k]}\, Z^p\, f_k \\
D \otimes A \to D, \qquad & f_k \otimes Y^p e'_a \mapsto \delta_{a,[-r_k-p]}\, f_k\, X^p.
\end{aligned}
$$

B.7 Important calculations

In this appendix we sketch some calculations used in the main text and the previous sections of this appendix.

Comultiplication of identity defect A

Following [9, 15], we define $\lambda^{-1}_{{}_hI} : {}_hI \to I \otimes {}_hI$ to be the natural junction of the identity defect with the symmetry defect ${}_hI$. It is given by

$$
\begin{aligned}
e_{[h]} &\mapsto 1 \otimes e_{[h]} + \theta \otimes \left\{ \partial^{Z,Y} \frac{Z^d - X^d}{\eta^h Z - X} \right\} e_{d+[h]} \\
e_{d+[h]} &\mapsto 1 \otimes e_{d+[h]} + \eta^h \theta \otimes e_{[h]}.
\end{aligned}
$$

Here X and Z are the chiral fields of the models to the right, respectively left of the defects, and Y is the chiral field of the model sandwiched between the defects I and ${}_hI$. Twisting by g from the left (i.e. fusion by ${}_gI$ from the left) one obtains junction fields $\Delta_{g,h} := {}_g\left(\lambda^{-1}_{{}_hI}\right) : {}_{g+h}I \to {}_gI \otimes {}_hI$:

$$
\begin{aligned}
e_{[g+h]} &\mapsto e_{[g]} \otimes e_{[h]} + e_{d+[g]} \otimes \left\{ \partial^{Z,Y} \frac{Z^d - X^d}{\eta^h Z - X} \right\}^{Z \to \eta^g Z} e_{d+[h]} \\
e_{d+[g+h]} &\mapsto e_{[g]} \otimes e_{d+[h]} + \eta^h e_{d+[g]} \otimes e_{[h]}.
\end{aligned}
$$

(Here, the notation $\{\dots\}^{Z \to \eta^g Z}$ means that that all instances of Z in the brackets have to be replaced by $\eta^g Z$ after performing all calculations.)

Summing up all the $\Delta_{g,h}$ yields the comultiplication

$$
\begin{aligned}
\Delta = \frac{1}{d} \sum_{g,h} \Delta_{g,h} : A &\to A \otimes A \\
e_{[g]} &\mapsto \frac{1}{d} \sum_{h \in \mathbb{Z}_d} \Delta_{g-h,h}(e_{[g]}) \\
e_{d+[g]} &\mapsto \frac{1}{d} \sum_{h \in \mathbb{Z}_d} \Delta_{g-h,h}(e_{d+[g]})
\end{aligned}
$$

of the identity defect A in the orbifold. It is spelled out completely in equation (B.2).

B.7.1 A-actions on equivariant defect

According to [9, Section 7.1], the data of a $G \times H$-equivariant defect is encoded in its left and right fusion with the symmetry defects A_G and A_H. Namely, it is described by a matrix factorization together with isomorphisms

- $\varphi_g : {}_gD \to D$ such that $\varphi_e = \mathrm{id}_D$ and $\varphi_{g_1} \circ {}_{g_1}(\varphi_{g_2}) = \varphi_{g_1+g_2}$ and
- $\phi_h : D_h \to D$ such that $\phi_e = \mathrm{id}_D$ and $\phi_{h_1} \circ (\phi_{h_2})_{h_1} = \phi_{h_1+h_2}$.

Here, one can think of ${}_gD$ as the matrix factorization where all variables Z_i to the left of D have been replaced by $g(Z_i)$, see for example ${}_gI$ in (B.1). Also, for some morphism $\alpha : D \to D'$ of matrix factorizations, ${}_g(\alpha)_h : {}_gD_h \to {}_gD'_h$ is the same morphism considered as a morphism between the respective twisted matrix factorizations. However, special attention has to be paid to morphisms including an identification of variables. For example, the left and right I-actions $\lambda_D : I \otimes D \to D$ and $\rho_D : D \otimes I \to D$ identify the middle variable with the one on the left or right, respectively. The identification of variables in the twisted versions ${}_g(\lambda_D)$ and $(\rho_D)_{-h}$ must respect the twist.

Following the proof of Thm. 7.2 in [9], the above data determine the left A_G-action on D:

$$\sum_{g \in G} \left(A_G \otimes D \twoheadrightarrow {}_gI \otimes D \xrightarrow{{}_g(\lambda_D)} {}_gD \xrightarrow{\varphi_g} D \right).$$

The right action includes the canonical isomorphism ${}_hI \to I_{-h}$ which we will comment on later:

$$\sum_{h \in H} \left(D \otimes A_H \twoheadrightarrow D \otimes {}_hI \to D \otimes I_{-h} \xrightarrow{(\rho_D)_{-h}} D_{-h} \xrightarrow{\phi_{-h}} D \right). \tag{B.4}$$

Turning to our example, set $G = \mathbb{Z}_{d'}$ and $H = \mathbb{Z}_d$ and consider a $G \times H$-equivariant defect D, i.e. a $\mathbb{Z}_{d'} \times \mathbb{Z}_d$-equivariant matrix factorization D of $Z^{d'} - X^d$. Let $f_0, \dots, f_{M-1}$ and $f_M, \dots, f_{2M-1}$ be equivariant generators of D_0, respectively D_1. Denote the $\mathbb{Z}_{d'} \times \mathbb{Z}_d$-charges of e_k by

$$[l_k, r_k].$$

In other words the action of $(g, h) \in \mathbb{Z}_{d'} \times \mathbb{Z}_d$ is given by

$$Z^p f_k X^q \;\mapsto\; (\epsilon^g Z)^p \cdot \epsilon^{g \cdot l_k} f_k \eta^{h \cdot r_k} \cdot (\eta^h X)^q \tag{B.5}$$

where $\epsilon = e^{\frac{2\pi i}{d'}}, \eta = e^{\frac{2\pi i}{d}}$.

We now reformulate this group action in terms of left and right A-actions. It is not hard to see that in our case the above isomorphisms are given by[1]

$$\begin{aligned} \varphi_g : {}_gD &\to D & \phi_h : D_h &\to D \\ f_k &\to \epsilon^{g\cdot l_k} f_k & f_k &\to f_k \eta^{h\cdot r_k}. \end{aligned}$$

The explicit form of the left A-action on D then turns out to be

$$\begin{aligned} A \otimes D &\to D \\ e_a \otimes Y^p f_k &\mapsto \delta_{|e_a|,0} \cdot (\epsilon^a Z)^p \cdot \epsilon^{a\cdot l_k} \cdot f_k \end{aligned}$$

for the same choice of variables as in (B.3) ($|\cdot|$ denotes the $\mathbb{Z}_2$-charge). This coincides with the expected action (B.5). The right A-action on the other hand takes the form

$$\begin{aligned} D \otimes A &\to D \\ f_k \otimes Y^p e_a &\mapsto \delta_{|e_a|,0} \cdot f_k \cdot \eta^{-a\cdot r_k} (\eta^{-a} X)^p \end{aligned}$$

where we emphasize the crucial minus sign for the right charges which differs from the expected (B.5). It originates from the fact that the symmetry defect A was defined as the direct sum of the *left* twisted identity morphisms which requires us to include the canonical isomorphism

$$\begin{aligned} {}_hI &\to I_{-h} \\ e_i &\mapsto \eta^{h\cdot|e_i|} e_i, \qquad i = 0, 1 \end{aligned}$$

in the construction (B.4).

B.7.2 Left A-action on right adjoint

As explained in appendix 2.7 item viii), adjoints of defects in the orbifold theory are induced by their non-orbifold counterpart. Here, given an equivarant matrix factorization D of $Z^{d'} - X^d$, we explicitely calculate the induced left A-action (i.e. the left charges, see previous calculation) on the non-orbifold adjoint $D^\dagger$. This leads to the charges of the right adjoint in the orbifold theory as $D^* \cong (D^\dagger)_\gamma$.

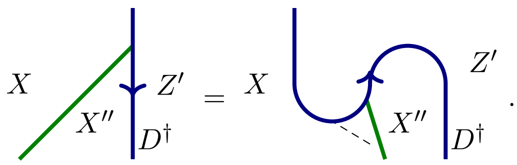
Let $f_0, \ldots, f_{M-1}$ be equivariant generators of D_0 and $f_M, \ldots, f_{2M-1}$ equivariant generators of D_1. We denote the $\mathbb{Z}_{d'} \times \mathbb{Z}_d$-charges of f_k by $[l_k, r_k]$. Then, $D_0^\dagger$ and $D_1^\dagger$ are generated by $f_0^\dagger, \ldots, f_{M-1}^\dagger$ and $f_M^\dagger, \ldots, f_{2M-1}^\dagger$ respectively, where $f_i^\dagger = f_{i+M}^\vee$ for $i < M$ and $f_i^\dagger = f_{i-M}^\vee$ for $i \geq M$. ('$\vee$' denotes the dual.)

[1] $\varphi_g \circ {}_g(\varphi_h) = \varphi_{g+h}$ is trivial and $\mathrm{d}_D \circ \varphi_g = \varphi_g \circ {}_g(\mathrm{d}_D)$ amounts to d_D being a degree zero map, i.e. e_k and $\mathrm{d}_D(e_k)$ carrying the same $\mathbb{Z}_{d'} \times \mathbb{Z}_d$ charges.

From the explicit expressions of the (co-)evaluation maps [15] we obtain the map $A \otimes D^\dagger \to D^\dagger$. Namely, generator $e_a \otimes f_i^\dagger$ is sent to $\delta_{|e_a|,0}$ times

$$
\begin{aligned}
&- \delta_{|f_i^\dagger|,0} \sum_{k=0}^{M-1} \operatorname{Res}\left[\frac{\begin{pmatrix} [\partial_X \mathrm{d}_0]^{X\to X''} \cdot \eta_{r,0}^{-a} \cdot \partial^{X,\eta^{-a}X''} \mathrm{d}_1 + \\ + \eta^a [\partial^{X,X''} \mathrm{d}_0]^{X \to \eta^a X} \cdot [\partial_X \mathrm{d}_1]^{X\to X''} \cdot \eta_{r,1}^{-a} \end{pmatrix}_{i,k}^{Z\to Z'} \mathrm{d}X''}{d \cdot X''^{d-1}} \right] f_k^\dagger \\
&- \delta_{|f_i^\dagger|,1} \sum_{k=M}^{2M-1} \operatorname{Res}\left[\frac{\begin{pmatrix} [\partial_X \mathrm{d}_1]^{X\to X''} \cdot \eta_{r,1}^{-a} \cdot \partial^{X,\eta^{-a}X''} \mathrm{d}_0 + \\ + \eta^a [\partial^{X,X''} \mathrm{d}_1]^{X \to \eta^a X} \cdot [\partial_X \mathrm{d}_0]^{X\to X''} \cdot \eta_{r,0}^{-a} \end{pmatrix}_{i-M,k-M}^{Z\to Z'} \mathrm{d}X''}{d \cdot X''^{d-1}} \right] f_k^\dagger
\end{aligned}
\tag{B.6}
$$

for the following choice of variables



Here $\eta = e^{2\pi i/d}$, and $\eta_{r,0}$ and $\eta_{r,1}$ are the diagonal matrices

$$
\begin{aligned}
\eta_{r,0} &= \mathrm{diag}(\eta^{r_0}, ..., \eta^{r_{M-1}}) \\
\eta_{r,1} &= \mathrm{diag}(\eta^{r_M}, ..., \eta^{r_{2M-1}}).
\end{aligned}
$$

Moreover, $\partial^{X,X''}$ is the divided difference operator which is defined as

$$
\partial^{X,X''} g(X, \ldots) = \frac{g(X, \ldots) - g(X'', \ldots)}{X - X''}
$$

on any polynomial g, and the residue $\operatorname{Res}\left[\frac{g \cdot \mathrm{d}X''}{X''^{d-1}}\right]$ picks out the prefactor of X''^{d-2} in the polynomial $g \in \mathbb{C}[Z', X, X'']$.

We now simplify expression (B.6) by calculating the X''^{d-2}-term in the numerator. We first derive a few identities which follow from the very definition of a graded matrix factorization.

From the basic property of matrix factorizations $\mathrm{d}_0 \mathrm{d}_1 = (Z^{d'} - X^d)\mathbb{1}$ one can deduce

$$
\begin{aligned}
\partial^{X,X''}(-d \cdot X^{d-1}) = {} & [\partial_X \mathrm{d}_0|^{X\to X''} \cdot \partial^{X,X''} \mathrm{d}_1 + \partial^{X,X''} \mathrm{d}_0 \cdot [\partial_X \mathrm{d}_1|^{X \to X''} \\
& + \left\{ \left(\partial^{X,X''} \partial_X \mathrm{d}_0\right) \cdot \mathrm{d}_1 + \mathrm{d}_0 \cdot \left(\partial^{X,X''} \partial_X \mathrm{d}_1\right) \right\}.
\end{aligned}
$$

Now, we will simplify the derivation by assuming that the matrices d_0 and d_1 do not contain terms X^n for $n \geq d$. This is certainly true for all the matrix factorizations relevant in this appendix and chapter 4, namely the ones associated to RG and projection defects, boundary conditions etc. Under this assumption, the curly bracket part of the last equation does not contain a term $\sim X''^{d-2}$, and hence

$$\left\{[\partial_X \mathrm{d}_0]^{X\to X''}\cdot\partial^{X,X''}\mathrm{d}_1+\partial^{X,X''}\mathrm{d}_0\cdot[\partial_X \mathrm{d}_1|^{X\to X''}\right\}_{ik}=-dX''^{d-2}\delta_{ik}+... \tag{B.7}$$

where ... contains only powers $(X'')^n$ with $n<d-2$. In order to make contact with equation (B.6) we replace $\partial^{X,X''}\mathrm{d}_0$ in (B.7) by $\partial^{X,X''}\mathrm{d}_0|^{X\to\eta^a X}$ which does not alter the leading X''-term:

$$\begin{aligned}&\left\{[\partial_X \mathrm{d}_0]^{X\to X''}\cdot\partial^{X,X''}\mathrm{d}_1+\partial^{X,X''}\mathrm{d}_0|^{X\to\eta^a X}\cdot[\partial_X \mathrm{d}_1|^{X\to X''}\right\}_{ik}\\ &=-\,dX''^{d-2}\delta_{ik}+...\end{aligned} \tag{B.8}$$

Also, since d_1 is of grade zero

$$\eta^{-a}_{r,0}\cdot[\mathrm{d}_1|^{X\to\eta^{-a}X}=\mathrm{d}_1\cdot\eta^{-a}_{r,1}$$

which together with the definition of the divided difference operator yields

$$\eta^{-a}_{r,0}\cdot\partial^{X,\eta^{-a}X''}\mathrm{d}_1\Big|_{\substack{\text{leading}\\ X''\text{-term}}}=\partial^{X,X''}\mathrm{d}_1\Big|_{\substack{\text{leading}\\ X''\text{-term}}}\cdot\eta^a\cdot\eta^{-a}_{r,1}. \tag{B.9}$$

Here, $\partial^{X,\eta^{-a}X''}\mathrm{d}_1\Big|_{\substack{\text{leading}\\ X''\text{-term}}}$ is the matrix d_1 with all entries X^{p+1} replaced by $(\eta^{-a}\cdot X'')^p$ and similarly for $\partial^{X,X''}\mathrm{d}_1\Big|_{\substack{\text{leading}\\ X''\text{-term}}}$. Finally, we evaluate the first summand of (B.6). It is given by $-\delta_{|e_a|,0}\delta_{|f_i^\dagger|,0}$ times

$$\sum_{k=0}^{M-1}\operatorname{Res}\left[\frac{\begin{pmatrix}[\partial_X \mathrm{d}_0]^{X\to X''}\cdot\eta^{-a}_{r,0}\cdot\partial^{X,\eta^{-a}X''}\mathrm{d}_1+\\ +\eta^a[\partial^{X,X''}\mathrm{d}_0]^{X\to\eta^a X}\cdot[\partial_X \mathrm{d}_1]^{X\to X''}\cdot\eta^{-a}_{r,1}\end{pmatrix}_{i,k}\mathrm{d}X}{d\cdot X''^{d-1}}\right]^{Z\to Z'} f_k^\dagger$$

For a given $k=0,...,M-1$, (B.9) turns the corresponding summand into

$$\operatorname{Res}\left[\frac{\begin{pmatrix}[\partial_X \mathrm{d}_0]^{X\to X''}\cdot\partial^{X,X''}\mathrm{d}_1\cdot\eta^a\cdot\eta^{-a}_{r,1}+...\\ +\eta^a[\partial^{X,X''}\mathrm{d}_0]^{X\to\eta^a X}\cdot[\partial_X \mathrm{d}_1]^{X\to X''}\cdot\eta^{-a}_{r,1}\end{pmatrix}_{i,k}\mathrm{d}X}{d\cdot X''^{d-1}}\right]^{Z\to Z'} f_k^\dagger$$

where '...' indicates that we omitted terms which do not contribute to the residue. This is the same as

$$\operatorname{Res}\left[\frac{\begin{pmatrix}[\partial_X \mathrm{d}_0]^{X\to X''}\cdot\partial^{X,X''}\mathrm{d}_1+\\ +[\partial^{X,X''}\mathrm{d}_0]^{X\to\eta^a X}\cdot[\partial_X \mathrm{d}_1]^{X\to X''}\end{pmatrix}_{i,k}\cdot\eta^{-a(r_{k+M}-1)}\mathrm{d}X}{d\cdot X''^{d-1}}\right]^{Z\to Z'} f_k^\dagger.$$

Finally, multiplication with $-\delta_{|e_a|,0}\delta_{|f_i^\dagger|,0}$, summation over k and application of (B.8) gives

$$
\begin{aligned}
&-\delta_{|e_a|,0}\delta_{|f_i^\dagger|,0}\sum_{k=0}^{M-1}\operatorname{Res}\left[\frac{(-dX''^{d-2}\delta_{ik}+...)\mathrm{d}X}{d\cdot X''^{d-1}}\right]^{Z\to Z'}\cdot\eta^{-a(r_{k+M}-1)}f_k^\dagger \\
&=\delta_{|e_a|,0}\delta_{|f_i^\dagger|,0}\eta^{a(-r_{i+M}+1)}f_i^\dagger .
\end{aligned}
$$

The second summand in equation (B.6) can be determined in a similar way and also takes a similar form. We find that the left charges of $D^\dagger$ are the negative right charges of D shifted by $+1$:

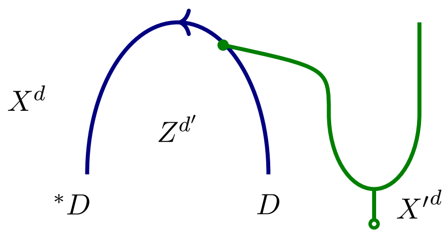
$$
\begin{aligned}
A\otimes D^\dagger &\to D^\dagger \\
e_a\otimes f_i^\dagger &\mapsto \delta_{|e_a|,0}\eta^{a(-r_{i+M}+1)}f_i^\dagger .
\end{aligned}
\tag{B.10}
$$

B.8 (Co)evaluation maps

Finally, we provide the explicit (co)evaluation maps used in calculations in the main text. They follow from the generalized orbifold construction, (c.f. appendix 2.7) and the expressions of [15]. Throughout, [...] denotes the representative in $\{0,\dots,d-1\}$ modulo d. Furthermore, $\partial^{Z,X}Z^i=\frac{Z^i-X^i}{Z-X}$, $\sigma=\begin{pmatrix}\mathbb{1}&0\\0&-\mathbb{1}\end{pmatrix}$, $\eta=e^{\frac{2\pi i}{d}}$, $\epsilon=e^{\frac{2\pi i}{d'}}$, $\eta_r=\mathrm{diag}(\eta^{r_0},\eta^{r_1},\eta^{r_2},...)$, $\epsilon_l=\mathrm{diag}(\epsilon^{l_0},\epsilon^{l_1},\epsilon^{l_2},...)$ and $\epsilon_{1+l}=\mathrm{diag}(\epsilon^{1+l_0},\epsilon^{1+l_1},\epsilon^{1+l_2},...)$.

Orbifold evaluation map (left)

$\mathrm{ev}_D=$

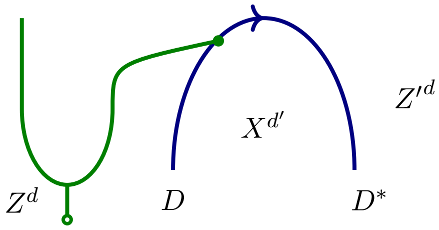


$\circ\,\xi,$

where $\xi: {}^*D\otimes_A D\to {}^*D\otimes D$ is the inclusion. $\mathrm{ev}_D: {}^*D\otimes_A D\to A$ sends ${}^*f_k\otimes Z^n f_i$ to

$$
\begin{aligned}
&\frac{1}{d}\sum_{h\in\mathbb{Z}_d}\operatorname{Res}\left[\frac{Z^n\left(\sigma\cdot\partial_Z\mathrm{d}_D\cdot\eta_r^h\right)_{(k+M),i}}{d'\cdot Z^{d'-1}}\right]e_{[h]} \\
&+\frac{1}{d}\sum_j\sum_{h\in\mathbb{Z}_d}\operatorname{Res}\left[\frac{Z^n\left(\sigma\cdot\partial_Z\mathrm{d}_D\cdot\eta_r^h\cdot\left[\partial^{X,X'}\mathrm{d}_D\right]^{X\to\eta^hX}\cdot\sigma\right)_{(k+M),i}}{d'\cdot Z^{d'-1}}\right]e_{d+[h]}
\end{aligned}
$$

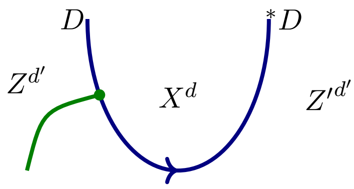
Orbifold evaluation map (right)

$\widetilde{\mathrm{ev}}_D =$  $\circ\, \xi$

where $\xi : D \otimes_A D^* \to D \otimes D^*$ is the inclusion. $\widetilde{\mathrm{ev}}_D : D \otimes_A D^* \to A$ sends $f_i \otimes X^n f_k^*$ to

$$
\begin{aligned}
&- \frac{1}{d'} \sum_{h \in \mathbb{Z}_{d'}} e_{[-h]} \mathrm{Res}\left[\frac{\left(\partial_X \mathrm{d}_D \cdot \epsilon_l^h\right)^{Z \to Z'}_{(k+M),i} X^n}{d \cdot X^{d-1}} \right] \\
&- \frac{1}{d'} \sum_{h \in \mathbb{Z}_{d'}} e_{d'+[-h]} \mathrm{Res}\left[\frac{\left(\partial_X \mathrm{d}_D^{Z \to Z'} \cdot \epsilon_{1+l}^h \cdot \partial^{Z,\epsilon^h Z'} \mathrm{d}_D\right)_{(k+M),i} X^n}{d \cdot X^{d-1}} \right]
\end{aligned}
$$

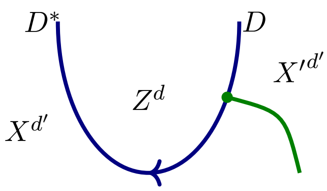
Orbifold coevaluation map (left)

$\mathrm{coev}_D = \vartheta \circ$ 

where $\vartheta : D \otimes {}^*D \to D \otimes_A {}^*D$ is the projection.

$$
\begin{aligned}
\mathrm{coev}_D &: A \to D \otimes_A {}^*D \\
&e_a \mapsto \delta_{|e_a|,0} \sum_{ij} (-1)^{|e_j|} \left(\left\{ \partial^{Z,Z'} \mathrm{d}_D \right\}^{Z \mapsto \epsilon^a Z} \right)_{ij} \epsilon^{a l_i} f_i \otimes f^*_{(j+M)} \\
&\quad + \delta_{|e_a|,1} \sum_i (-1)^{|e_i|} \epsilon^{a l_i} f_i \otimes f^*_{(i+M)}
\end{aligned}
$$

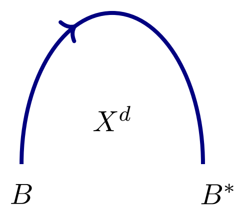
Orbifold coevaluation map (right)

$\widetilde{\mathrm{coev}}_D = \vartheta \circ$ 

where $\vartheta : D \otimes D^* \to D \otimes_A D^*$ is the projection.

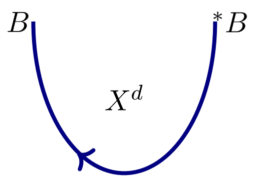
$$\begin{aligned}\widetilde{\mathrm{coev}}_D : A &\to D^* \otimes D \\ e_a \mapsto \delta_{|e_a|,0} \sum_{ij} &\left(\partial^{X,\eta^{-a}X'} \mathrm{d}_D\right)_{ji} f^*_{(i+M)} \otimes f_j \eta^{-ar_j} \\ &+ \delta_{|e_a|,1} \eta^a \sum_i (-1)^{|e_i|} f^*_{(i+M)} \otimes f_i \eta^{-ar_i}\end{aligned}$$

Orbifold evaluation map (right) for boundaries

$\widetilde{\mathrm{ev}}_B =$ 

$$\begin{aligned}\widetilde{\mathrm{ev}}_B : B \otimes_A B^* &\to \mathbb{C} \\ f_i \otimes X^p f_k^* &\mapsto -\mathrm{Res}\left[\frac{X^p \left(\partial_X \mathrm{d}_B\right)_{(k+M),i} \mathrm{d}X}{d \cdot X^{d-1}}\right]\end{aligned}$$

Orbifold coevaluation map (left) for boundaries

$\mathrm{coev}_B =$ 

$$\begin{aligned}\mathrm{coev}_B : \mathbb{C} &\to B \otimes_A {}^*B \\ 1 &\mapsto \sum_i f_i \otimes {}^*f_i\end{aligned}$$

Appendix C

Explicit calculations for RG defects in LG orbifolds

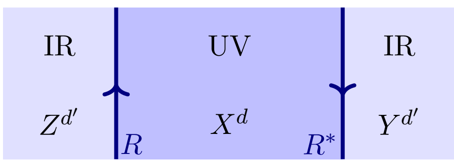
In this appendix we explicitly check that the RG defects R between LG orbifolds presented in section 4.2 satisfy the RG property that $R \otimes R^\dagger \cong I$ (appendix C.1) and determine the corresponding projection defects $P = R^\dagger \otimes R$ (appendix C.2). We show how IR boundary conditions and symmetries are realized in the UV (appendices C.3 and C.4) and we perform the calculation $R_\infty \otimes R_\infty^\dagger \cong I_{\mathrm{IR}}$ (appendix C.5). For the purpose of this appendix we again adopt the generalized orbifold notation of appendix B.

C.1 $R \otimes_A R^* \cong A$

Here, we show that $R \otimes_A R^* \cong A$. (In this appendix we adopt the following notation from appendix B: $\otimes_A$ denotes fusion in the generalized orbifold theory defined by A, while $\otimes$ denotes the fusion in the unorbifolded theory. Moreover, * denotes adjunction in the orbifold theory, while † refers to adjunction in the unorbifolded theory.) Fusion of B-type defects has been discussed in [16], for the orbifold version see [3].

As explained in those papers, matrix factorizations of W over a polynomal ring R are related to finitely generated modules over $\hat{R} := R/(W)$ as free resolutions of such modules always turn two-periodic after finitely many steps [40]. The two-periodic part then gives a matrix factorization of W.

In order to calculate $R \otimes_A R^*$, we fix the coordinates on all three parts of the worldsheet to be Z, X and Y:



The matrix factorization describing R is given by

$$\mathrm{d}_{R1} = \begin{pmatrix} Z & 0 & \dots & 0 & -X^{n_0} \\ -X^{n_1} & Z & & & \\ 0 & -X^{n_2} & Z & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & -X^{n_{d'-1}} & Z \end{pmatrix}$$

$$R : R_1 \underset{\mathrm{d}_{R0}}{\overset{\longrightarrow}{\longleftarrow}} R_0,$$

see section 4.2. The generators $f_{[i]}, i \in \mathbb{Z}_{d'}$, of R_0 carry $\mathbb{Z}_{d'} \times \mathbb{Z}_d$-charges $[i, -m - \sum_{l=1}^{i} n_l]$ while the generators $e_{d'+[i]}$ of R_1 have charges $[i+1, -m - \sum_{l=1}^{i} n_l]$.

According to section 4.1, the right adjoint R^* is given by the matrix factorization

$$\mathrm{d}_{R^*1} = \begin{pmatrix} Y & -X^{n_1} & 0 & \dots & 0 \\ 0 & Y & -X^{n_2} & & \\ \vdots & & Y & \ddots & \\ 0 & & & \ddots & -X^{n_{d'-1}} \\ -X^{n_0} & & & & Y \end{pmatrix}$$

$$R^* : R_1^* \underset{\mathrm{d}_{R^*0}}{\overset{\longrightarrow}{\longleftarrow}} R_0^*,$$

The generators $f_{[k]}^*$, $k \in \mathbb{Z}_{d'}$, of R_0^* carry $\mathbb{Z}_d \times \mathbb{Z}_{d'}$-charges $[+m + \sum_{l=1}^{k} n_l + 1, -k - 1]$, and the generators $f_{d'+[k]}^*$ of R_1^* carry charges $[+m + \sum_{l=1}^{k} n_l + 1, -k - 1]$.

Following the tensor product descriptions of sections 3.3 and 3.5, the matrix factorization describing $R \otimes_A R^*$ is the one associated to the $\mathbb{Z}_d$-invariant part of the $\mathbb{C}[Z,Y]/(Z^{d'} - Y^{d'})\mathbb{C}[Z,Y]$-module

$$M := \operatorname{coker} \begin{pmatrix} \mathrm{id}_{R0} \otimes \mathrm{d}_{R^*1} & \mathrm{d}_{R1} \otimes \mathrm{id}_{R^*0} \\ \mathrm{d}_{R0} \otimes \mathrm{id}_{R^*1} & -\mathrm{id}_{R1} \otimes \mathrm{d}_{R^*0} \end{pmatrix}.$$

The two-periodic resolution of M is isomorphic to the two-periodic part of the resolution of

$$M' := \operatorname{coker}\left(\mathrm{d}_{R1} \otimes \mathrm{id}_{R^*0}, \mathrm{id}_{R0} \otimes \mathrm{d}_{R^*1}\right).$$

The module M' is generated by $f_{[i],[k]}^l := f_{[i]} \otimes X^l f_{[k]}^*$. They satisfy the relations

$$Z f_{[i]} = X^{n_{i+1}} f_{[i+1]} \quad \text{and} \quad Y f_{[k]}^* = X^{n_{[k]}} f_{[k-1]}^*,$$

which allow to reduce the generators to $f^l_{[i],[k]}$ with $0 \leq l < \min(n_i, n_{k+1})$. These carry $\mathbb{Z}_{d'} \times \mathbb{Z}_d \times \mathbb{Z}_{d'}$-charges

$$[i, -m - \sum_{j=1}^{i} n_j + l + m + \sum_{j=1}^{k} n_j + 1, -k - 1].$$

The $\mathbb{Z}_d$-invariant part $(M')^{\mathbb{Z}_d}$ is generated by the $\mathbb{Z}_d$-invariant generators of M', which are given by $\hat{f}_{[i]} := f^{n_i-1}_{[i],[i-1]}$. They carry $\mathbb{Z}_{d'} \times \mathbb{Z}_{d'}$-charges $[i, -i]$ and satisfy the relations

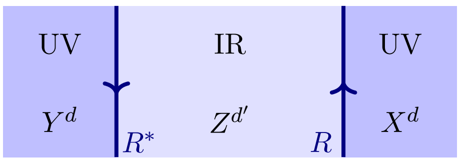
$$Z\hat{f}_{[i]} = Y\hat{f}_{[i+1]}.$$

Hence, $(M')^{\mathbb{Z}_d}$ is isomorphic to the module $\mathrm{coker}(\mathrm{d}_{A1})$, which implies that the matrix factorization $R \otimes_A R^*$ is isomorphic to the identity defect A in $\mathcal{M}_{d'}/\mathbb{Z}_{d'}$. Taking the left adjoint of this equation immediately yields $R \otimes_A {}^*R \cong A$ as well.

C.2 The projection defect P

Having shown $R \otimes_A R^* \cong A$ in the previous appendix, we are now in a position to determine the projection defect $P = R^* \otimes_A R$. The projection $P' = {}^*R \otimes_A R$ based on the left adjoint ${}^*R$ can then easily be obtained by left adjunction $P' = {}^*P$.

The calculation of P follows the same route as the calculation of $R \otimes_A R^*$ in appendix C.1 above. First, we fix the chiral fields on all three parts of the worldsheet to be Y, Z and X:



The matrix factorizations R and R^* are described in appendix C.1. As in the derivation of $R \otimes_A R^* \cong I$, the matrix factorization $R^* \otimes R$ is given by the two-periodic part of the free resolution of the $\mathbb{Z}_{d'}$-invariant part of the $\mathbb{C}[X,Y]/(Y^d - X^d)\mathbb{C}[X,Y]$-module

$$M' := \mathrm{coker}(d_{R^*1} \otimes \mathrm{id}_{R_0}, \mathrm{id}_{R^*0} \otimes d_{R1}).$$

The latter is generated by

$$f^l_{[k],[i]} := f^*_{[k]} Z^l \otimes f_{[i]}$$

subject to the relations

$$Y^{n_k} f^*_{[k-1]} = Z f^*_{[k]} \quad \text{and} \quad Z f_{[i]} = X^{n_{i+1}} f_{[i+1]}.$$

These relations allow to reduce the generators to the ones with $l = 0$. The remaining generators $f^0_{[k],[i]}$ carry $\mathbb{Z}_d \times \mathbb{Z}_{d'} \times \mathbb{Z}_d$-charges

$$\left[m + \sum_{j=1}^{[k]} n_j + 1, -k - 1 + i, -m - \sum_{j=1}^{[i]} n_j\right].$$

The $\mathbb{Z}_{d'}$-invariant part of M' is generated by the $\mathbb{Z}_{d'}$-invariant generators, i.e. those $f^0_{[k],[i]}$, for which $[-k-1+i] = 0$. These are the $\hat{f}_{[i]} := f^0_{[i-1],[i]}$, which are subject to the relations

$$Y^{n_i} \hat{f}_{[i]} = X^{n_{i+1}} \hat{f}_{[i+1]}.$$

They carry $\mathbb{Z}_d \times \mathbb{Z}_d$-charges

$$\left[m + \sum_{j=1}^{[i-1]} n_j + 1, -m - \sum_{j=1}^{[i]} n_j\right].$$

Comparing with the matrix factorization P given in equation (4.3), one finds that $(\widetilde{M'})^{\mathbb{Z}_{d'}} \cong \mathrm{coker}(p_1)$ (where Z has to be replaced by Y in p_1). Hence, $R^* \otimes R$ is isomorphic to the matrix factorization P given in section 4.2.

C.3 Boundary conditions satisfying $B \otimes_A P \cong B$

We now determine the boundary conditions, which are invariant under fusion with P. Elementary left boundary conditions in $\mathcal{M}_d/\mathbb{Z}_d$ are given by the $\mathbb{Z}_d$-equivariant matrix factorizations

$$B_{\mathrm{UV}} : \mathbb{C}[Z]\left([N+k]\right) \underset{-Z^{d-k}}{\overset{Z^k}{\rightleftarrows}} \mathbb{C}[Z]\left([N]\right).$$

of $-Z^d$, where $k, N \in \mathbb{Z}_d$, $k \neq 0$, c.f. section 4.3. The aim is to identify those boundary conditions, for which $B_{\mathrm{UV}} \otimes_A P \cong B_{\mathrm{UV}}$. To do so, we just calculate the fusion as is done in the previous appendices. We denote the generators of B_0 and B_1 by b_0 and b_1, respectively. They have $\mathbb{Z}_d$-charge $[N]$, respectively $[N+k]$. The generators $\hat{f}_{[i]}$ of P_0 have $\mathbb{Z}_d \times \mathbb{Z}_d$-charge $[m + 1 + \sum_{l=1}^{[i-1]} n_{[l]}, -m - \sum_{l=1}^{[i]} n_{[l]}]$, c.f. appendix C.2.

To determine the fusion $B_{\mathrm{UV}} \otimes P$, we again employ the method described in appendix C.1. For this, we determine generators and relations of the $\mathbb{Z}_d$-invariant part of the $\mathbb{C}[X]/X^d\mathbb{C}[X]$-module $M' := \mathrm{coker}(d_{B1} \otimes \mathrm{id}_{P0}, \mathrm{id}_{B0} \otimes d_{P1})$:

$$\begin{gathered} b_0 Z^k = 0 \\ Z^{n_i} f_{[i]} = X^{n_{[i+1]}} f_{[i+1]} \end{gathered} \tag{C.1}$$

For $B_{\mathrm{UV}} \otimes_A P \cong B_{\mathrm{UV}}$ to hold, out of all the generators $b_0 Z^q \otimes f_{[i]}$ of the fusion product exactly one generator may survive in $(M')^{\mathbb{Z}_d}$. It must

- be invariant under the left $\mathbb{Z}_d$-action, i.e.

$$\left[N + q + m + 1 + \sum_{l=1}^{[i-1]} n_l \right] = 0$$

- carry right $\mathbb{Z}_d$-charge $[N]$, i.e.

$$\left[-m - \sum_{l=1}^{[i]} n_l \right] = [N]$$

- has to be a generator with respect to $\mathbb{C}[X]$ and in particular cannot be eliminated by (C.1), i.e.

$$q < n_i \quad \text{and} \quad q < k$$

  and

- it has to satisfy $b_0 Z^q \otimes f_{[i]} X^k = 0$.

The first two conditions fix $N = -m - \sum_{j=1}^{[i]}$ and imply $q = n_i - 1$ which is consistent with $q < n_i$. The last condition becomes

$$k \in \{n_i, n_i + n_{i-1}, ..., n_i + ... + n_{i-d'-2}\} .$$

These conditions are equivalent to $B_{\mathrm{UV}} \otimes_A P \cong B_{\mathrm{UV}}$ and imply that B_{UV} must be of the form

$$B_{\mathrm{UV}} : \mathbb{C}[Z] \underset{-Z^{d-n_i-...-n_{i-I}}}{\overset{Z^{n_i+...+n_{i-I}}}{\rightleftarrows}} \mathbb{C}[Z] \left(-m - \textstyle\sum_{l=1}^{i} n_l\right)$$

for arbitrary $i \in \mathbb{Z}_{d'}$ and $I \in \{0, ..., d'-2\}$.

C.4 IR symmetry defects in the UV

Following section 4.3, the IR $\mathbb{Z}_{d'}$-symmetry is realized in the UV by means of the defects

$$R^* \otimes_A {}_aI_{d'} \otimes_A R =: {}_aP.$$

As ${}_aI_{d'} \otimes_A R$ is described by the same matrix factorization as R but with all left charges shifted by $+a$, we can employ the same set-up as in appendix C.2 and only shift charges by $+a$ where necessary. The corresponding module M' is generated by

$$f^l_{[k],[i]} := f^*_{[k]} Z^l \otimes f_{[i]}$$

with $\mathbb{Z}_d \times \mathbb{Z}_{d'} \times \mathbb{Z}_d$-charges

$$\left[m + \sum_{j=1}^{[k]} n_j + 1, -k - 1 + l + i + a, -m - \sum_{j=1}^{[i]} n_j\right]$$

subject to the relations

$$Y^{n_k} f^*_{[k-1]} = Z f^*_{[k]} \quad \text{and} \quad Z f_{[i]} = X^{n_{i+1}} f_{[i+1]}.$$

While the relations can be used to reduce generators to those with $l = 0$, $\mathbb{Z}_{d'}$-invariance gives the condition $[i + a - k - 1] = 0$. The remaining generators $\hat{f}_{[i]} := f^0_{[i-1],[i-a]}$ of $(M')^{\mathbb{Z}_{d'}}$ obey

$$Y^{n_i} \hat{f}[i] = X^{n_{i-a+1}} f_{[i+1]}$$

and carry $\mathbb{Z}_d \times \mathbb{Z}_d$-charges

$$\left[m + \sum_{j=1}^{[i-1]} n_j + 1, -m - \sum_{j=1}^{[i-a]} n_j\right].$$

One now easily reads off that $(M')^{\mathbb{Z}_{d'}}$ is isomorphic to the cokernel of the matrix p_1 of the matrix factorization ${}_aP$ given in section 4.2. Thus, the lifted symmetry defects are isomorphic to these matrix factorizations.

C.5 $R_\infty \otimes_{U(1)} R^*_\infty \cong I_{\mathbf{IR}}$

In this appendix we show that one can insert loops of the $U(1)$-equivariant Landau-Ginzburg theory with a single chiral superfield and zero superpotential into the Landau-Ginzburg orbifold models $\mathcal{M}_{d'}/\mathbb{Z}_{d'}$, $d' \geq 3$ without affecting correlators. The respective RG defects are described by the matrix factorizations R_∞ of $Z^{d'}$ presented in section 4.4:

$$R_\infty : S^{d'} \begin{pmatrix} {\scriptstyle [k+1,-m]} \\ {\scriptstyle [k+2,-m-n_1]} \\ {\scriptstyle [k+3,-m-n_1-n_2]} \\ \vdots \end{pmatrix} \underset{\mathrm{d}_{R0}}{\overset{\mathrm{d}_{R1}}{\rightleftarrows}} S^{d'} \begin{pmatrix} {\scriptstyle [k,-m]} \\ {\scriptstyle [k+1,-m-n_1]} \\ {\scriptstyle [k+2,-m-n_1-n_2]} \\ \vdots \end{pmatrix}.$$

Here $m \in \mathbb{Z}$, $k \in \mathbb{Z}_{d'}$ and $n_1, ..., n_{d'-1} \in \mathbb{N}$. Moreover, $S = \mathbb{C}[Z, X]$,

$$\mathrm{d}_{R1} = \begin{pmatrix} Z & 0 & \dots & 0 & 0 \\ -X^{n_1} & Z & & & \\ 0 & -X^{n_2} & Z & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & -X^{n_{d'-1}} & Z \end{pmatrix}$$

and

$$\mathrm{d}_{R0} = \begin{pmatrix} Z^{d'-1} & 0 & \dots & \dots & 0 \\ Z^{d'-2}X^{n_1} & Z^{d'-1} & 0 & \dots & 0 \\ Z^{d'-3}X^{n_1+n_2} & Z^{d'-2}X^{n_2} & Z^{d'-1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ X^{n_1+\dots+n_{d'-1}} & ZX^{n_2+\dots+n_{d'-1}} & \dots & Z^{d'-2}X^{n_{d'-1}} & Z^{d'-1} \end{pmatrix}.$$

The adjoint

$$R^*_\infty : \tilde{S}^{d'} \begin{pmatrix} {\scriptstyle [m+1,-k]} \\ {\scriptstyle [m+1+n_1,-k-1]} \\ {\scriptstyle [m+1+n_1+n_2,-k-2]} \\ \vdots \end{pmatrix} \underset{\mathrm{d}_{R^*0}}{\overset{\mathrm{d}_{R^*1}}{\rightleftarrows}} \tilde{S}^{d'} \begin{pmatrix} {\scriptstyle [m+1,-k-1]} \\ {\scriptstyle [m+1+n_1,-k-2]} \\ {\scriptstyle [m+1+n_1+n_2,-k-3]} \\ \vdots \end{pmatrix},$$

$$\mathrm{d}_{R^*1} = \begin{pmatrix} Y & -X^{n_1} & & & \\ & Y & -X^{n_2} & & \\ & & \ddots & \ddots & \\ & & & Y & -X^{n_{d'-1}} \\ & & & & Y \end{pmatrix}$$

can be obtained by taking the limit $d \to \infty$ of R^*. It is a matrix factorization of $-Y^{d'}$. $\tilde{S} = \mathbb{C}[X,Y]$ and d_{R^*0} is given by $-\mathrm{d}_{R0}^T$ with Z replaced by Y.

According to section 3.5, the fusion product $R_\infty \otimes_{U(1)} R^*_\infty$ is given by the $U(1)$-invariant part of the tensor product matrix factorization $R_\infty \otimes R^*_\infty$. The $U(1)$-invariant generators of the latter are

$$\begin{aligned} g_{(i,j)} &:= f_i \otimes X^{n_i+\dots+n_{j+1}-1} f^*_j & [k+i, -k-1-j] \\ g_{(d'+i,j)} &:= f_{d'+i} \otimes X^{n_i+\dots+n_{j+1}-1} f^*_j & [k+1+i, -k-1-j] \\ g_{(i,d'+j)} &:= f_i \otimes X^{n_i+\dots+n_{j+1}-1} f^*_{d'+j} & [k+i, -k-j] \\ g_{(d'+i,d'+j)} &:= f_{d'+i} \otimes X^{n_i+\dots+n_{j+1}-1} f^*_{d'+j} & [k+1+i, -k-j] \end{aligned}$$

for $1 \leq i \leq d'-1$ and $0 \leq j \leq i-1$. The $\mathbb{Z}_{d'} \times \mathbb{Z}_{d'}$-charges of the generators are specified in square brackets. Here, f_i and f^*_i label the generators of R_∞ and R^*_∞, respectively. The generators with $0 \leq i < d'$ are $\mathbb{Z}_2$-even and the ones with $d' \leq i < 2d'$ are $\mathbb{Z}_2$-odd. Setting

$$l = \frac{i(i+1)}{2} + j, \qquad 0 \leq l \leq M := \frac{(d'+1)(d'-2)}{2},$$

one can order the generators as follows

$$\begin{aligned} g_l &= g_{(i,j)} \\ g_{M+l} &= g_{(d'+i,d'+j)} \\ g_{2M+l} &= g_{(d'+i,j)} \\ g_{3M+l} &= g_{(i,d'+j)}. \end{aligned}$$

$\left(R_\infty \otimes_{U(1)} R^*_\infty\right)_0$ is then generated by the g_l and g_{M+l} for $0 \le l \le M$ and $\left(R_\infty \otimes_{U(1)} R^*_\infty\right)_1$ by the g_{2M+l} and g_{3M+l} for $0 \le l \le M$.

In terms of the generators, the $U(1)$-invariant tensor product matrix factorization

$$\mathrm{d} = \mathrm{d}_R \otimes_{U(1)} \mathbb{1} + \mathbb{1} \otimes_{U(1)} \mathrm{d}_{R^*} =: \begin{pmatrix} & \mathrm{d}_1 \\ \mathrm{d}_0 & \end{pmatrix}$$

takes the form

$$\begin{aligned}
(\mathrm{d}_1)_{(p,q),(d'+i,j)} &= \delta_{q,j}(Z\delta_{p,i} - \delta_{p,i+1}) \\
(\mathrm{d}_1)_{(d'+p,d'+q),(d'+i,j)} &= \delta_{p,i}\theta(j-q)Y^{d'-1-(j-q)} \\
(\mathrm{d}_1)_{(p,q),(i,d'+j)} &= \delta_{p,i}(Y\delta_{q,j} - \delta_{q+1,j}) \\
(\mathrm{d}_1)_{(d'+p,d'+q),(i,d'+j)} &= \delta_{q,j}\theta(p-i)Z^{d'-1-(p-i)} \\
(\mathrm{d}_0)_{(d'+i,j),(p,q)} &= \delta_{j,q}\theta(i-p)Z^{d'-1-(i-p)} \\
(\mathrm{d}_0)_{(i,d'+j),(p,q)} &= -\delta_{i,p}\theta(q-j)Y^{d'-1-(q-j)} \\
(\mathrm{d}_0)_{(d'+i,j),(d'+p,d'+q)} &= -\delta_{i,p}(Y\delta_{j,q} - \delta_{j+1,q}) \\
(\mathrm{d}_0)_{(i,d'+j),(d'+p,d'+q)} &= \delta_{j,q}(Z\delta_{i,p} - \delta_{i,p+1})
\end{aligned} \tag{C.2}$$

where $1 \le i, p \le d'$, $0 \le j < i$, $0 \le q < p$ and $\theta(x) = \begin{cases} 1, x \ge 0 \\ 0, x < 0 \end{cases}$.

For example, for $d' = 5$ one obtains

$$
\mathrm{d}_1 = \begin{pmatrix}
Z &&&&&&&&&& Y &&&&&&&&& \\
-1 & Z &&&&&&&&&& Y & -1 &&&&&&& \\
&& Z &&&&&&&&&& Y &&&&&&& \\
& -1 && Z &&&&&&&&&& Y & -1 &&&&& \\
&& -1 && Z &&&&&&&&&& Y & -1 &&&& \\
&&&&& Z &&&&&&&&&& Y &&&& \\
&&& -1 &&& Z &&&&&&&&&& Y & -1 && \\
&&&& -1 &&& Z &&&&&&&&&& Y & -1 & \\
&&&&& -1 &&& Z &&&&&&&&&& Y & -1 \\
&&&&&&&&& Z &&&&&&&&&& Y \\
Y^4 &&&&&&&&&& Z^4 &&&&&&&&& \\
& Y^4 & Y^3 &&&&&&&& Z^3 & Z^4 &&&&&&&& \\
&& Y^4 &&&&&&&&&& Z^4 &&&&&&& \\
&&& Y^4 & Y^3 & Y^2 &&&&& Z^2 & Z^3 && Z^4 &&&&&& \\
&&&& Y^4 & Y^3 &&&&&&& Z^3 && Z^4 &&&&& \\
&&&&& Y^4 &&&&&&&&&& Z^4 &&&& \\
&&&&&& Y^4 & Y^3 & Y^2 & Y & Z & Z^2 && Z^3 &&& Z^4 &&& \\
&&&&&&& Y^4 & Y^3 & Y^2 &&& Z^2 && Z^3 &&& Z^4 && \\
&&&&&&&& Y^4 & Y^3 &&&&&& Z^3 &&& Z^4 & \\
&&&&&&&&& Y^4 &&&&&&&&&& Z^4
\end{pmatrix}
$$

Stripping off trivial summands this matrix factorization reduces to the IR identity matrix factorization ($S' = \mathbb{C}[Z,Y]$)

$$I_{\mathrm{IR}} : S'^{d'}\begin{pmatrix} [1,0] \\ [2,-1] \\ [3,-2] \\ \vdots \\ [d',-d'+1] \end{pmatrix} \underset{\mathrm{d}_{I_{\mathrm{IR}}0}}{\overset{\begin{pmatrix} Z & 0 & \dots & 0 & -Y \\ -Y & Z & & & \\ 0 & -Y & Z & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & -Y & Z \end{pmatrix}}{\rightleftarrows}} S'^{d'}\begin{pmatrix} [0,0] \\ [1,-1] \\ [2,-2] \\ \vdots \\ [d'-1,-d'+1] \end{pmatrix}$$

In order to see this, we perform a change of basis on (C.2):

$$\mathrm{d}_1 = S \cdot \tilde{\mathrm{d}}_1 \cdot T^{-1}, \qquad \mathrm{d}_0 = T \cdot \tilde{\mathrm{d}}_0 \cdot S^{-1},$$

where S and T^{-1} are defined by

$$\begin{aligned}
(S)_{(p,q),(i,j)} &= \delta_{q,j}(\delta_{p,i} - Z\delta_{p+1,i}) \\
(S)_{(p,q),(d'+i,d'+j)} &= 0 \\
(S)_{(d'+p,d'+q),(i,j)} &= -\delta_{p+1,i}Y^{d'-1-(j-q)}\theta(j-q) \\
&\quad - \delta_{i,j+1}\Big[Z^{d'-p+i-1}Y^{q-i}\theta(q-i) \\
&\qquad\qquad + Z^{i-1-p}Y^{d'+q-i}\theta(i-2-p)\Big] \\
(S)_{(d'+p,d'+q),(d'+i,d'+j)} &= \delta_{q,j}(\delta_{p,i} + \theta(i-p-1)Z^{i-p}) \\
&\quad + \delta_{i,d'-1}\delta_{j,0}\theta(q-1)Z^{d'-1-p}Y^q
\end{aligned}$$

and

$$\begin{aligned}
(T^{-1})_{(d'+p,q),(d'+i,j)} &= \delta_{q,j}\delta_{p,i} \\
&\quad + \delta_{i,d'-1}\Big(-\delta_{q,j}Z^{d'-1-p}\theta(d'-1-p) \\
&\qquad + \delta_{p,d'-1}Y^{q-j}\theta(q-j-1)\Big) \\
(T^{-1})_{(p,d'+q),(d'+i,j)} &= -\delta_{i,d'-1}\delta_{q+1,p}\theta(q-j-1)Z^{d'-1-q}Y^{q-1-j} \\
(T^{-1})_{(d'+p,q),(i,d'+j)} &= -\delta_{q,j}Y\left(\delta_{p+1,i} + Z^{i-p-1}\theta(i-p-2)\right) \\
&\quad + \delta_{q+1,j}Z^{i-p-1}\theta(i-p-1) \\
(T^{-1})_{(p,d'+q),(i,d'+j)} &= \delta_{p,i}\delta_{q,j} + \delta_{q,j}\delta_{q+1,p}\theta(i-p-1)Z^{i-p}.
\end{aligned}$$

Here again $1 \le i,p \le d'$, $0 \le j < i$, $0 \le q < p$. Then

$$\begin{aligned}
(\tilde{\mathrm{d}}_1)_{(p,q),(d'+i,j)} &= -\delta_{q,j}\delta_{p,i+1} + Z\delta_{i,d'-1}\delta_{p,d'-1}\delta_{q,d'-2}\delta_{j,d'-2} \\
(\tilde{\mathrm{d}}_1)_{(d'+p,d'+q),(d'+i,j)} &= \delta_{p,d'-1}\delta_{i,d'-1}(W\delta_{q,j+1} - Y\delta_{q,0}\delta_{j,d'-2}) \\
(\tilde{\mathrm{d}}_1)_{(p,q),(i,d'+j)} &= -Y\delta_{p,i}\delta_{q,j}\delta_{i,j+1} + Z\delta_{p+1,i}\delta_{p,j}\delta_{p,q+1} \\
(\tilde{\mathrm{d}}_1)_{(d'+p,d'+q),(i,d'+j)} &= \delta_{i,p+1}\delta_{q,j}W + \delta_{i,1}\delta_{j,0}\delta_{p,d'-1}\delta_{q,0}Z
\end{aligned}$$

and

$$
\begin{aligned}
(\tilde{\mathrm{d}}_0)_{(d'+i,j),(p,q)} &= -\delta_{i+1,p}\delta_{j,q}W + \delta_{i,d'-1}\delta_{j,d'-2}\delta_{p,q+1}Z^pY^{d'-1-p} \\
(\tilde{\mathrm{d}}_0)_{(i,d'+j),(p,q)} &= -\delta_{i,j+1}\delta_{p,q+1}\Big(\theta(p-i)Y^{d'-1-p+i}Z^{p-i} \\
&\qquad +\theta(i-p-1)Z^{d'-i+p}Y^{i-p-1}\Big) \\
(\tilde{\mathrm{d}}_0)_{(d'+i,j),(d'+p,d'+q)} &= \delta_{p,d'-1}\delta_{i,d'-1}\Big(\delta_{j+1,q} - Y^{d'-1}\delta_{j,d'-2}\delta_{q,0}\Big) \\
(\tilde{\mathrm{d}}_0)_{(i,d'+j),(d'+p,d'+q)} &= -\delta_{i,p+1}\delta_{j,q} + \delta_{p,d'-1}\delta_{q,0}\delta_{i,j+1}Z^{d'-i}Y^j
\end{aligned}
$$

are matrix factorization of $W = Z^{d'} - Y^{d'}$ which reduce to the indentity matrix factorization.

In the example $d' = 5$ d_1 above turns into

$$\tilde{\mathrm{d}}_1 = \begin{pmatrix}
&&&&&&&&&& -Y && Z &&&&&&& \\
-1 &&&&&&&&&&&&&&&&&&& \\
&&&&&&&&&&&& -Y &&& Z &&&& \\
& -1 &&&&&&&&&&&&&&&&&& \\
&& -1 &&&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& -Y &&&& Z \\
&&& -1 &&&&&&&&&&&&&&&& \\
&&&& -1 &&&&&&&&&&&&&&& \\
&&&&& -1 &&&&&&&&&&&&&& \\
&&&&&&&&& Z &&&&&&&&&& -Y \\
&&&&&&&&& 0 & 0 & W &&&&&&&& \\
&&&&&&&&&&& 0 & 0 & W &&&&&& \\
&&&&&&&&&&&&&& W &&&&& \\
&&&&&&&&&&&&&& 0 & 0 & W &&& \\
&&&&&&&&&&&&&&&&& W && \\
&&&&&&&&&&&&&&&&&& W & \\
&&&&&&&&& -Y & Z &&&&&&&&& \\
&&&&&& W &&&&&&&&&&&&& \\
&&&&&&& W &&&&&&&&&&&& \\
&&&&&&&& W &&&&&&&&&&&
\end{pmatrix}$$

which is easily recognized as the matrix associated to a sum of the identity matrix factorization I_{IR} with a number of trivial rank-one matrix factorizations.

In the general case, the generators not belonging to trivial summands are the ones labelled by the restricted index sets

$$\begin{aligned}
\{(i,j)\,|i=j+1\} &\subset \big\{(i,j)\,\big|i=1,...,d'-1;\\
&\qquad j=0,...,i-1\}\\
\big\{(d'+i,d'+j)\,\big|i=d'-1,\ j=0\big\} &\subset \big\{(d'+i,d'+j)\,\big|i=1,...,d'-1;\\
&\qquad j=0,...,i-1\}\\
\big\{(d'+i,j)\,\big|i=d'-1,\ j=d'-2\big\} &\subset \big\{(d'+i,j)\,\big|i=1,...,d'-1;\\
&\qquad j=0,...,i-1\}\\
\big\{(i,d'+j)\,|i=j+1\big\} &\subset \big\{(i,d'+j)\,\big|i=1,...,d'-1;\\
&\qquad j=0,...,i-1\}
\end{aligned}$$

Restricting to these generators yields the IR identity defect I_{IR}.

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