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Periodic Bounce Orbits in Magnetic Billiard Systems

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Math is like ice cream, with more flavours than you can imagine.

Denise Gaskins

Abstract

This is an invitation to play magnetic billiards. We consider a billiard table that is an n-dimensional compact Riemannian manifold with smooth boundary. This is a generalization of the classical billiard game. In particular, we study periodic orbits on a prescribed energy level in the magnetic setup. We show that for sufficiently high energy values above the Mañé critical value, there exists a *periodic magnetic bounce orbit* with bounded period.

Résumé

C'est une invitation à jouer au billard magnétique. Nous consid'erons une table de billard qui est une variété riemannienne avec un bord lisse. C'est une généralisation du jeu de billard classique. En particulier, nous étudions les orbits périodiques dans des niveaux d'énergie prescrits. Nous montrons qu'il existe une *orbite périodique magneétique de collision* pour des valeurs d'énergie hautes, plus hautes que la valeur critique de Mañé.

Zusammenfassung

Dies ist eine Einladung, magnetisches Billard zu spielen. Als Billardtisch betrachten wir eine n-dimensionale kompakte Riemannsche Mannigfaltigkeit mit glattem Rand. Dies ist eine Verallgemeinerung des klassischen Billardspiels. Insbesondere studieren wir periodische Orbits auf einem vorgeschriebenen Energielevel. Wir zeigen, dass ein *periodischer magnetischer Anstoßorbit* für genügend große Energiewerte über dem Mañé kritischen Wert existiert.

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Contents

Chapter 1

Introduction

Billiards are a fascinating and active area of research. There are many different settings to focus on. This thesis contributes in a small way to the wide range of interesting results on the theory of billiards. Examples of billiard settings include polygonal billiards, smooth billiards, symplectic billiards or dual billiards. We guide the reader to further discussions in the literature. To start exploring the richness of billiards, there is a fantastic written book by Tabachnikov [Tab05]. More interesting references include [Kat05], [KH95], [MZ05], [Maz21] and [Roz19]. Albers–Tabachnikov introduce symplectic billiards in [AT18]. In this work, we concentrate on smooth billiards.

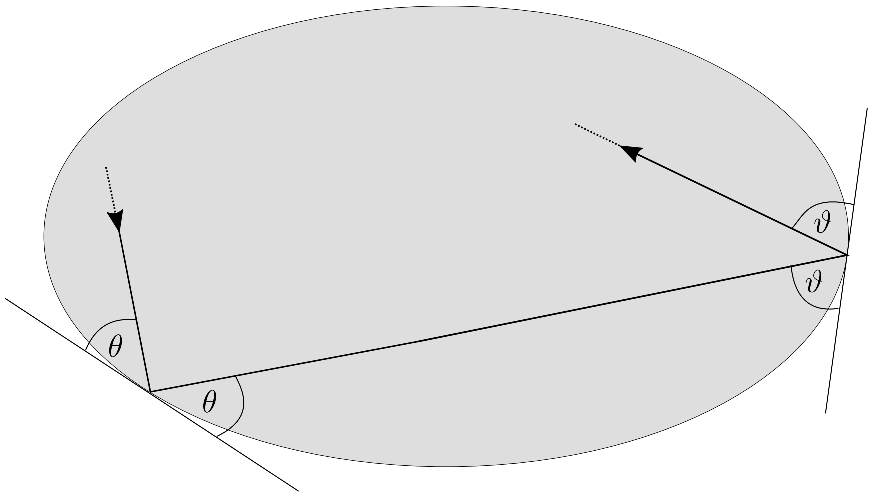

Figure 1.1: A classical billiard orbit on an ellipse in $\mathbb{R}^n$.

Smooth billiards describe the motion of a point mass, the *billiard ball*, inside a bounded n-dimensional region with smooth boundary, the *billiard table*, without application of any force except the first kick. In classical smooth billiards, the billiard ball moves along a straight line with constant speed until it hits the boundary. At boundary points, the ball is reflected according to the law of reflection, which means that the angle of incidence equals the angle of reflection. Afterwards, the ball moves on in a new direction with the same energy until it hits the boundary again. This procedure forms a *classical billiard orbit*, see Figure 1.1 for an exemplary illustration.

Of particular interest for us is studying the existence of periodic billiard trajectories on a given smooth billiard table. Historically, one of the first to study these was Birkhoff. He considered trajectories on a 2-dimensional *convex* billiard table, see [Bir79]. By applying the celebrated Poincaré–Birkhoff fixed point theorem, he proved the existence of infinitely many distinct periodic orbits on a strictly convex table. In the proof, he studied fixed points of an area-preserving twist map on an annulus.

Non-convex billiard tables $\Omega \subset \mathbb{R}^n$ have also been studied. In this more general situation, Benci–Giannoni proved the existence of a periodic bounce orbit with prescribed period and with at most $\dim(\Omega) + 1$ bounce points, compare [BG89]. The bound on the number of bounce points in dimension two is sharp in general: There are domains $\Omega \subset \mathbb{R}^2$ where every billiard trajectory has at least $2 + 1 = 3$ bounce points, see Figure 1.2 and [Tab05, Figure 6.6].

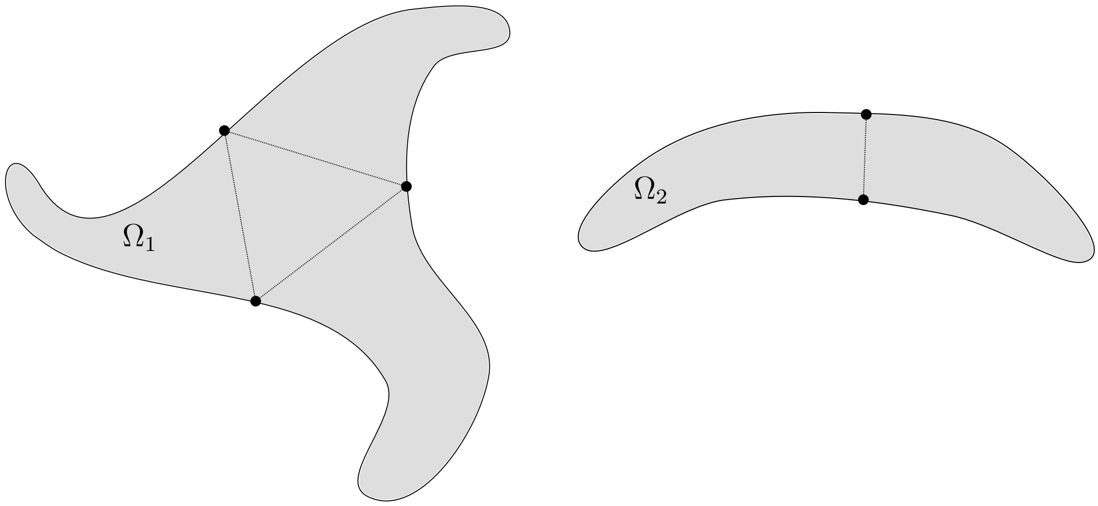

Figure 1.2: On the left hand side: a billiard table $\Omega_1 \subset \mathbb{R}^2$ without 2-periodic trajectories, but with a periodic billiard trajectory with 3 bounce points; On the right hand side: a billiard table $\Omega_2 \subset \mathbb{R}^2$ without 3-periodic trajectories but periodic trajectories with 2 bounces.

Benci-Giannoni developed a brilliant approximation scheme using the so-called fixed-period action functional. They approximated the bounce trajectories with regular solutions of a Lagrangian system. These modified Lagrangians have an additional potential term that explodes to infinity towards the boundary $\partial\Omega$ and vanishes at any point far from $\partial\Omega$. They showed that approximate solutions actually converge to periodic bounce orbits. In [AM11], Albers–Mazzucchelli adjusted this situation to the case of free period and prescribed energy. With this improved approximation scheme, they proved the existence of periodic bounce orbits of prescribed energy rather than prescribed period. They overcame new difficulties using techniques from symplectic geometry rather than variational methods which were used by Benci–Giannoni. A new difficulty in the approximation scheme is to obtain bounds on the periods for approximate solutions independent of the approximation parameter. We point out that this is necessary in order to pass to the limit. The advantage is that one obtains explicit bounds on the period of the bounce orbits in the limit.

In this thesis, we follow the approach given by Albers–Mazzucchelli and treat two generalizations that they considered in [AM11, Remark 1.6]. The first one is to study Riemannian manifolds instead of $\mathbb{R}^n$ and the second is to add a magnetic field. It is quite clear what is meant by studying Riemannian instead of Euclidean billiards. There are several interesting results in the realm of magnetic billiards, see for example [BMS20a], [BMS20b], [RB85]. In classical billiards one considers a free particle. Now instead, for magnetic billiards we consider a charged particle, which will be deflected by a magnetic field. Note that we recover the classical situation, if the electromagnetic field is zero.

The standard example is a billiard table $\Omega \subset \mathbb{R}^2$. In this case, the strength of a magnetic field, that is perpendicular to the plane, is given by a function B on the plane, compare [Tab05, Chapter 1]. The charged ball is acted upon by the Lorentz force which is proportional to B and to the speed of the ball. Note that the Lorentz force acts perpendicularly to the direction of motion making the ball move along arcs of circles whose curvature at every point is prescribed by B. If the magnetic field B is constant, then the trajectories are circles of Larmor radius. At a bounce point, the ball gets reflected according to the law of reflection. This means, that the magnetic field does not change the law of reflection, see [BMS20b]. We highlight one difference of billiards with or without magnetic field. A system of magnetic billiards is not reversible, i.e., the ball moves backwards on a different orbit.

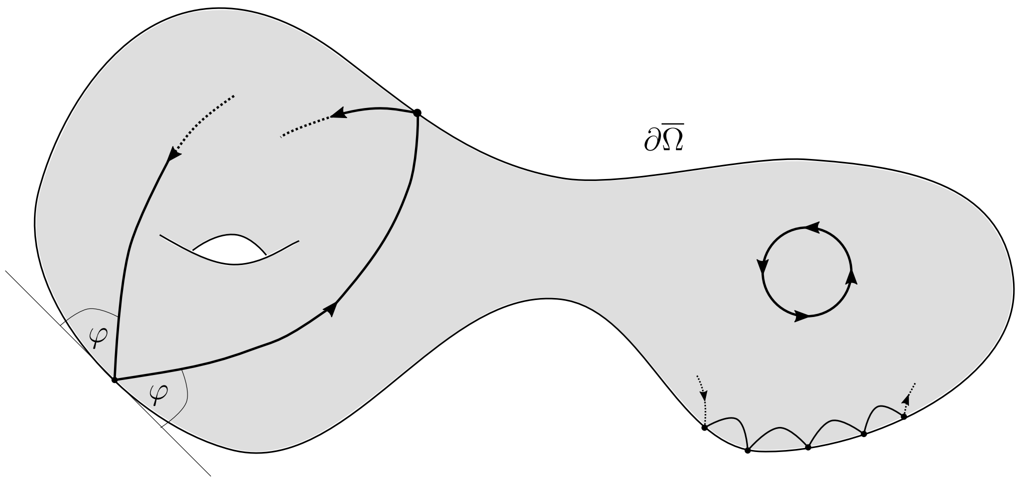

Figure 1.3: Magnetic billiard trajectories.

In Figure 1.3, we illustrate three possible scenari in magnetic billiards on a table $\Omega \subset \mathbb{R}^3$: a billiard ball, moving on an arc of a circle, that hits the boundary and gets reflected according to the law of reflection; second a billiard trajectory that does not touch the boundary at all and moves along a circle; and, lastly, a billiard trajectory that immediately turns back towards the boundary when being reflected, a so called *creeping orbit*. In this magnetic billiard game, we study periodic orbits on a prescribed energy level. The main result of this thesis, given below as Theorem 3.3, ensures the existence of a *periodic magnetic bounce orbit*, that we will define rigorously in Section 3.1, with bounded period when prescribing sufficiently high energy.

We present our guideline to achieve this result below. In Chapter 2, we prepare the playground. For this purpose, we recall the concepts of Tonelli Lagrangian and Hamiltonian systems. We introduce Mañé critical values which mark important changes in the dynamics of a mechanical system. After this, we proceed with the magnetic part of the billiard story in Chapter 3. We start in Chapter 3.1 by explaining the rigorous electromagnetic billiard setting, move on in Chapter 3.2 to the approximation scheme and finally prove the main theorem of this work in Chapter 3.4, see Theorem 3.3 below. There exists a more general framework called Tonelli systems. In Chapter 4, we derive some generalizations of parts of the results studied in Chapter 3 to the Tonelli situation. We point out that electromagnetic systems are the main examples of Tonelli systems. Finally, we collect several ideas for further studies in Section 5.

Enjoy the billiard game!

Chapter 2

Tonelli's Theory

This chapter recalls the language of classical mechanics based on [Abb13, Chapter 3], [CI99, Chapter 1], [Fat08, Chapter 3], [Maz12, Chapter 1], [Maz21, Chapter 1] and [Sor15, Chapter 1]. Classical Mechanics describes the motion of mechanical systems. We review the concepts of Lagrangian and Hamiltonian systems. All the results stated can be found in the above mentioned literature.

2.1 Lagrangian point of view

Let M be a compact n-dimensional manifold with smooth boundary endowed with a Riemannian metric $g = \langle \cdot, \cdot \rangle$. Denote by $\mathrm{T}M$ its tangent bundle. A point of $\mathrm{T}M$ will be denoted by (q, v), where $q \in M$, $v \in \mathrm{T}_q M$.

Definition 2.1. A function $L \in C^\infty(\mathrm{T}M)$ is called a *Tonelli Lagrangian* if

- L is *fiberwise uniformly convex*, i.e.

$$\frac{\partial^2 L}{\partial v^2}(q, v) > 0$$

  for every $(q, v) \in \mathrm{T}M$, where $\partial^2 L / \partial v^2$ denotes the fiberwise second differential of L and

- L has *superlinear growth on each fiber*, i.e.

$$\lim_{|v|_g \to \infty} \frac{L(q, v)}{|v|_g} = \infty,$$

  where we denote by $|\cdot|_g = g(\cdot, \cdot)$ the norm induced on $\mathrm{T}_q M$ by the Riemannian metric g. This condition is equivalent to asking whether for each $A \in \mathbb{R}$ there exists $B(A) \in \mathbb{R}$, such that for all $(q, v) \in \mathrm{T}M$

$$L(q, v) \geq A|v|_g - B(A).$$

We say that the *Tonelli assumption* holds if we consider such a Tonelli Lagrangian L.

Remark. The convexity assumption ensures that a line tangent to the Lagrangian L at a given point is always below the respective Lagrangian L. Moreover, the superlinearity condition guarantees that L grows faster than linear.

Remark. Some examples of Tonelli Lagrangians are:

- Riemannian Lagrangians. The *Riemannian Lagrangian* on (M, g) is given by the kinetic energy
$$L(q, v) = \tfrac{1}{2}|v|_g^2.$$
Analogous to the Riemannian Lagrangian, there is the *Finsler Lagrangian.* This Lagrangian is also given by the same formula, but where $|\cdot|$ is Finsler, i.e. $|\cdot|$ is a (non necessarily symmetric) norm on $\mathrm{T}_q M$ which varies smoothly on $q \in M$.

- Mechanical Lagrangians. These Lagrangians are widely studied in classical mechanics and given by the sum of the kinetic energy and a potential $V \colon M \to \mathbb{R}$:
$$L(q, v) = \tfrac{1}{2}|v|_g^2 - V(q).$$

- Electromagnetic Lagrangians. Let α be a smooth 1-form (the *magnetic potential*) and V a smooth function (the *electric potential*). The *electromagnetic Lagrangian* is defined by
$$L(q, v) = \tfrac{1}{2}|v|_g^2 + \alpha_q(v) - V(q).$$
They are the main objects in this work. We will study them in further details in Chapter 3 in a billiard context.

Associated with each Lagrangian is the *Euler–Lagrange flow* on the tangent bundle $\mathrm{T}M$, which is defined as follows. Consider the action functional on the space of continuous piecewise smooth curves $\gamma \colon [a, b] \to M$ (where $a \leq b$) given by
$$\int_a^b L(\gamma(t), \gamma'(t))\mathrm{d}t.$$
Extremizers of this functional among all curves with the same endpoints are solutions of the *Euler–Lagrange equation*, locally given by
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v}(\gamma(t), \gamma'(t)) - \frac{\partial L}{\partial q}L(\gamma(t), \gamma'(t)\right) = 0, \tag{2.1}$$
where $\frac{\partial L}{\partial v}$ denotes the fibrewise differential of L and $\frac{\partial L}{\partial q}$ the differential of L in the horizontal direction.

This is equivalent to
$$\frac{\partial^2 L}{\partial v^2}(\gamma(t), \gamma'(t))\gamma''(t) = \frac{\partial L}{\partial q}(\gamma(t), \gamma'(t)) - \frac{\partial^2 L}{\partial v \partial q}(\gamma(t), \gamma'(t))\gamma'(t).$$
Thus, the Tonelli assumption implies that the Euler–Lagrange equation is well-posed and allows to define a vector field X_L on $\mathrm{T}M$, such that the solutions of
$$\gamma''(t) = X_l(\gamma(t), \gamma'(t))$$
are exactly the curves satisfying the Euler-Lagrange equation. Its flow is called the *Euler–Lagrange flow* associated with L and we denote it by ϕ_L^t.

Remark. The Euler–Lagrange equation associated with the Riemannian Lagrangian is the geodesic equation of g and its Euler–Lagrange flow coincides with the geodesic flow. In Chapter 3, we examine the Euler–Lagrange equation that corresponds to the electromagnetic Lagrangian.

Definition 2.2. The *energy function* $\mathcal{E}\colon \mathrm{T}M \longrightarrow \mathbb{R}$ associated with the Tonelli Lagrangian L is defined by

$$\mathcal{E}(q,v) := \frac{\partial L}{\partial v}(q,v)[v] - L(q,v).$$

The Euler–Lagrange flow ϕ_L^t preserves the energy. Indeed, we observe that, if γ is a solution of the Euler–Lagrange equation (2.1), then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\gamma(t),\gamma'(t)) = 0.$$

The energy function of a Tonelli Lagrangian satisfies the following properties:

- $\mathcal{E}$ itself is a Tonelli Lagrangian.
- For any $q \in M$, the restriction of $\mathcal{E}$ to $\mathrm{T}_q M$ achieves its minimum at $q = 0$.

Let $E \in \mathbb{R}$ be an energy value. In our work, we are interested in proving the existence of periodic orbits on the energy level $\mathcal{E}^{-1}(E)$. Since such energy levels are compact, up to modifying the Lagrangian L far away from the energy level, we may assume that L is electromagnetic for $|v|_g$ large. Under this assumption, in particular for some numbers $L_0 > 0$ and $L_1 \in \mathbb{R}$ the following inequalities are true for $(q,v) \in \mathrm{T}M$ and $u \in \mathrm{T}_q M$:

$$\begin{aligned} L(q,v) &\geq L_0 |v|_g^2 - L_1 \\ \frac{\partial^2 L}{\partial v^2}(q,v)[u,u] &\geq 2L_0 |u|_g^2. \end{aligned} \tag{2.2}$$

These two features are important for future discussions (see Chapter 4.1, 4.2). This also ensures that E has the form of an electromagnetic energy for $|v|_g$ large.

In order to view the Lagrangian dynamics from a Hamiltonian point of view, we introduce the *Legendre transform*.

Definition 2.3. The *Legendre transform* associated with a Lagrangian L is defined as

$$\begin{aligned} \mathrm{Leg}\colon \mathrm{T}M &\longrightarrow \mathrm{T}^*M \\ (q,v) &\longmapsto \left(q, \frac{\partial L}{\partial v}(q,v)\right). \end{aligned}$$

We note the following important property of the Legendre transform.

Proposition 2.4. *The Legendre transform* $\mathrm{Leg}\colon \mathrm{T}M \to \mathrm{T}^*M$ *is a diffeomorphism if and only if* L *is Tonelli.*

A proof of this result can be found in [Fat08, Proposition 3.4.2] or in [Maz12, Chapter 1.2].

2.2 Hamiltonian point of view

Proposition 2.4 asserts that there is a second point of view from which one can discuss the motion of orbits on a certain energy level. This second perspective is given by the Hamiltonian approach on the cotangent bundle T^*M. We denote a point of T^*M by (q,p), where $q \in M$ and $p \in \mathrm{T}_q^*M$ is a linear form on T_qM. When considering the dynamics on the cotangent bundle T^*M, it makes sense to define a *Tonelli Hamiltonian* $H \in \mathcal{C}^\infty(\mathrm{T}^*M)$ associated with the Lagrangian L. We describe this Hamiltonian system and point out its relation to the Tonelli Lagrangian L.

Definition 2.5. We call a Hamiltonian $H\colon \mathrm{T}^*M \to \mathbb{R}$ *Tonelli* if it satisfies the following two conditions:

- H is fiberwise uniformly convex, i.e. $\frac{\partial^2 H}{\partial p^2}(q,p) > 0$ for every $(q,p) \in \mathrm{T}^*M$, where $\frac{\partial^2 H}{\partial p^2}$ denotes the fiberwise second differential of H, and
- H has superlinear growth on each fibre, i.e.

$$\lim_{|p|_{g^*}\to\infty} \frac{H(q,p)}{|p|_{g^*}} = \infty,$$

  where $|\cdot|_{g^*}$ denotes the norm on T_q^*M induced by the dual Riemannian metric g^* on T^*M, see Appendix A for further details on g^*.

In particular, we are interested in Hamiltonians $H\colon \mathrm{T}^*M \to \mathbb{R}$ that are obtained via the Legendre transform of a Lagrangian L:

$$H \circ \mathrm{Leg}(q,v) = H\left(q, \frac{\partial L}{\partial v}(q,v)\right) = \frac{\partial L}{\partial v}(q,v)[v] - L(q,v).$$

Then, we note that $H = \mathcal{E} \circ \mathrm{Leg}^{-1}$, if H is Tonelli.

The Tonelli Hamiltonian functions are precisely those Hamiltonians that are dual to the Tonelli Lagrangians, see [Maz12, Proposition 1.2.2]. This means that, Legendre duality guarantees a one-to-one correspondence between Tonelli Lagrangians $L\colon \mathrm{T}M \to \mathbb{R}$ and Tonelli Hamiltonians $H\colon \mathrm{T}^*M \to \mathbb{R}$. Therefore, Leg conjugates the Euler–Lagrange flow ϕ_L^t and the Hamiltonian flow ϕ_H^t of a system of motion, compare the diagram:

$$\begin{array}{ccc} \mathrm{T}M & \xrightarrow{\phi_L^t} & \mathrm{T}M \\ \mathrm{Leg}\big\downarrow & & \big\downarrow\mathrm{Leg} \\ \mathrm{T}^*M & \xrightarrow[\phi_H^t]{} & \mathrm{T}^*M. \end{array}$$

Remark. The Tonelli Hamiltonians corresponding to the examples of Tonelli Lagrangians, considered in Section 2.1, are the following:

- Riemannian Hamiltonians. The Legendre dual to the Riemannian Lagrangian is

$$H(q,p) = \tfrac{1}{2}|p|_{g^*}^2.$$

- Mechanical Hamiltonian. The associated Hamiltonian to the mechanical Lagrangian is
$$H(q,p) = \tfrac{1}{2}|p|^2_{g^*} + V(q).$$

- Electromagnetic Hamiltonian. The Hamiltonian that is Legendre dual to the electromagnetic Lagrangian is
$$H(q,p) = \tfrac{1}{2}|p-\alpha_q|^2_{g^*} + V(q).$$

To summarize this discussion, we assert that both approaches, the Lagrangian and the Hamiltonian points of view, provide useful tools and advantages. On the one hand, the tangent bundle is the natural setting to deal with classical calculus of variations, as we will do in Section 3.2. On the other hand, the cotangent bundle, equipped with a canonical symplectic structure, allows for the use of several symplectic tools, as we will see in Section 3.3. Working with both approaches results in a fruitful garden of discoveries.

2.3 Mañé critical values

When studying the existence of a periodic orbit on a certain energy level, there are some levels that mark important dynamical and geometric changes for the respective Euler–Lagrange flow ϕ^t_L induced by the Tonelli Lagrangian L. We recall some of these numbers, that are called Mañé critical values, see [Abb13, Chapter 4] and [CI99, Chapter 2] for references.

Definition 2.6. We define several *Mañé critical values* as follows:

$$c_0(L) := -\inf\left\{ \tfrac{1}{\tau}\int_0^\tau L(\gamma(t),\gamma'(t))\mathrm{d}t \mid \gamma\in\mathcal{C}^\infty(\mathbb{R}/\tau\mathbb{Z},M) \text{ homologous to zero, } \tau>0\right\},$$

$$c_u(L) := -\inf\left\{ \tfrac{1}{\tau}\int_0^\tau L(\gamma(t),\gamma'(t))\mathrm{d}t \mid \gamma\in\mathcal{C}^\infty(\mathbb{R}/\tau\mathbb{Z},M) \text{ contractible, } \tau>0\right\},$$

$$e_0(L) := \max_{q\in M}\mathcal{E}(q,0) = \max\{\mathcal{E}(q,v) \mid (q,v)\in \mathrm{Crit}\mathcal{E}\}.$$

The number $c_0(L)$ is called the *strict Mañé critical value* and $c_u(L)$ is called the *lowest Mañé critical value.*

In the case of a closed manifold M, the *strict Mañé critical value* has the following important characterization (see [CIP98] and [Fat97]):

$$c_0(L) = \inf\left\{ \max_{q\in M} H(q,\theta(q)) \mid \theta \text{ smooth closed 1-form on } M\right\},$$

where $H\colon \mathrm{T}^*M \to \mathbb{R}$ is the Hamiltonian associated to the Tonelli Lagrangian L via Legendre duality.

Remark. The three energy values are ordered as follows:

$$\min \mathcal{E} \leq e_0(L) \leq c_u(L) \leq c_0(L). \tag{2.3}$$

The second inequality in (2.3) is obtained from the following observation: let $q \in M$ be a point such that $\mathcal{E}(q,0) = e_0(L)$ and consider a constant curve $\gamma = q$, then

$$-\tfrac{1}{\tau}\int_0^\tau L(\gamma(t),\gamma'(t))\mathrm{d}t = -\tfrac{1}{\tau}\tau L(q,0) = \mathcal{E}(q,0).$$

The last inequality in (2.3) is true since contractible curves are nullhomologous.

These energy values vanish for a Riemannian Lagrangian $L(q,v) = \frac{1}{2}|v|_g^2$. In the case of mechanic Lagrangians $L(q,v) = \frac{1}{2}|v|^2 + V(q)$, $\min \mathcal{E} = \min V$ and $\min \mathcal{E} < e_0(L) = c_u(L) = c_0(L) = \max V$. For electromagnetic Lagrangians with non-vanishing magnetic potential α, in general $e_0(L)$ and $c_u(L)$ do not coincide, see [CIP98].

For the sake of completeness, we include an overview of results on the existence of periodic orbits on certain energy levels within the different ranges of energy values on a closed manifold M. The following theorem was first proved by Contreras in [Con06]. We cite the formulation given in [Abb13].

Theorem 2.7. *Let L be a Tonelli Lagrangian on the tangent bundle $\mathrm{T}M$ of the closed manifold M. Then the following holds:*

1. *If $E > c_u(L)$ and M is simply connected, then the energy level $\mathcal{E}^{-1}(E)$ has a periodic orbit.*
2. *For almost every $E \in (\min \mathcal{E}, c_u(L))$, the energy level $\mathcal{E}^{-1}(E)$ has a periodic orbit.*
3. *If the energy level $\mathcal{E}^{-1}(E)$ is stable then $\mathcal{E}^{-1}(E)$ has a periodic orbit.*
4. *If $E > c_0(L)$, then $H^{-1}(E)$ is of restricted contact type.*

Remark. In Section 3.3.2, we explain rigorously what is meant by *restricted contact type.*

In our work, we are particularly interested in the situation above the strict Mañé critical value. For some further discussion on the situation around the Mañé critical value, we guide the interested reader to Chapter 5.

Chapter 3

A Magnetic story

In this chapter, we play magnetic billiards on a billiard table that is an n-dimensional compact Riemannian manifold with smooth boundary. We prove, that for sufficiently high energy values that are above the strict Mañé critical value, there exists a *periodic magnetic bounce orbit* with bounded period.

3.1 The setting

Let $\overline{\Omega}$ be a compact n-dimensional manifold with smooth boundary endowed with a Riemannian metric $g = \langle \cdot, \cdot \rangle$. We denote its interior by Ω. Moreover, we fix a smooth function $V \in \mathcal{C}^\infty\left(\overline{\Omega}\right)$ (the *electric potential*) and a differential 1-form α (the *magnetic potential*) on $\overline{\Omega}$. We study the *electromagnetic Lagrangian system*

$$\begin{aligned} L \colon \mathrm{T}\overline{\Omega} &\longrightarrow \mathbb{R}, \\ (q, v) &\longmapsto \tfrac{1}{2}|v|_g^2 + \alpha_q(v) - V(q), \end{aligned} \tag{3.1}$$

where $|v|_g^2 := g(v, v)$. Consider the *Lorentz force* of the magnetic system that is the anti-symmetric bundle endomorphism $Y \colon \mathrm{T}\overline{\Omega} \to \mathrm{T}\overline{\Omega}$ uniquely defined by $g(Y_q(v), w) = \sigma_q(v, w)$ for all $q \in \overline{\Omega}$ and all $v, w \in \mathrm{T}_q\overline{\Omega}$, where $\sigma := \mathrm{d}\alpha$ is the *exact magnetic form*.

Definition 3.1. A continuous and piecewise smooth map $\gamma \colon \mathbb{R}/\tau\mathbb{Z} \to \overline{\Omega}$, $\tau > 0$, is called *periodic magnetic bounce orbit* of the Lagrangian system (3.1) if there exists a (possibly empty) finite subset $\mathcal{B} \subset \mathbb{R}/\tau\mathbb{Z}$, such that

(1) for all $t \notin \mathcal{B}$ there exists $\gamma'(t)$ and γ solves the Euler–Lagrange equation $\forall t \notin \mathcal{B}$, i.e.

$$\nabla_{\gamma'}\gamma'(t) + Y_{\gamma(t)}(\gamma'(t)) + \nabla V(\gamma(t)) = 0, \tag{3.2}$$

where $\nabla_{\gamma'}$ denotes the Levi-Civita covariant derivative in direction $\gamma' := \partial\gamma/\partial t$ and ∇V denotes the gradient of V, both with respect to g,

(2) For every $t \in \mathcal{B}$ we have $\gamma(t) \in \partial\overline{\Omega}$, and there exist left and right derivatives of γ at t, i.e.

$$\gamma'\left(t^\pm\right) := \lim_{s \to t^\pm} \gamma'(s) = \begin{cases} \lim_{s \searrow t} \gamma'(s) & \text{for } t^+ \\ \lim_{s \nearrow t} \gamma'(s) & \text{for } t^- \end{cases}, \tag{3.3}$$

such that γ satisfies the law of reflection

$$\begin{aligned} \langle \gamma'(t^+), \nu(\gamma(t)) \rangle &= -\langle \gamma'(t^-), \nu(\gamma(t)) \rangle \neq 0, \\ \gamma'(t^+) - \langle \gamma'(t^+), \nu(\gamma(t)) \rangle \nu(\gamma(t)) &= \gamma'(t^-) - \langle \gamma'(t^-), \nu(\gamma(t)) \rangle \nu(\gamma(t)), \end{aligned} \tag{3.4}$$

where ν is the outer normal to $\partial\overline{\Omega}$ with respect to g.

We call the times $t \in \mathcal{B}$ *bounce times* and for $t \in \mathcal{B}$ the points $\gamma(t)$ are called *bounce points*.

Remark. Note that there exist $t \in \mathbb{R}/\tau\mathbb{Z}$ with $\gamma(t) \in \partial\overline{\Omega}$ that are not in $\mathcal{B}$: It may happen that an orbit γ coming from Ω hits the boundary $\partial\overline{\Omega}$ tangentially, these orbits still satisfy the Euler–Lagrange equation (3.2). We call these orbits *glancing orbits* and we will ignore them in the following.

Remark. In geometric terms, the law of reflection (3.4) means that the angle of incidence equals the angle of reflection. We point out that the magnetic field does not affect the law of reflection.

Remark. Note that Definition 3.1 includes the case of $\mathcal{B} = \emptyset$. We point out that we still call them *bounce orbits* although the orbits might not bounce at all. In case $\mathcal{B} = \emptyset$, we talk about *smooth* periodic orbits.

Remark. The *energy* of a magnetic bounce orbit is given by

$$E(\gamma(t)) := \tfrac{1}{2}|\gamma'(t)|_g^2 + V(\gamma(t)) \tag{3.5}$$

and independent of t. Note in particular that the energy E is independent of the magnetic influence given by α. This is a consequence of the fact that the Lorentz force Y is anti-symmetric.

Remark. As a special case of the electromagnetic system we can recover classical billiards in $\overline{\Omega}$ by setting $V = 0$ and $\alpha = 0$.

Remark. In this thesis, we restrict our attention to the case of an *exact magnetic field* $\sigma = \mathrm{d}\alpha$. One can also treat non-exact magnetic fields by using the Hamiltonian formalism described in Section 3.3. Note that, in this more general setup, energy hypersurfaces may not have periodic orbits, see e.g. [CMP04, Section 1] for an overview on magnetic flows. For further references and results compare Section 5.2.

In our main theorem, we use the following variant of the Mañé critical value.

Definition 3.2. (see [Abb13], [Mn96]) A variant of the *strict Mañé critical value* c_0 is defined by

$$c_0 := \inf \left\{ \max_{q \in \overline{\Omega}} H(q, \tau_q) \mid \tau \text{ smooth closed 1-form on } \overline{\Omega} \text{ vanishing near } \partial\overline{\Omega} \right\}$$

where $H(q,p) := \frac{1}{2}|p - \alpha_q|_{g^*}^2 + V(q) \colon \mathrm{T}^*\overline{\Omega} \to \mathbb{R}$ is the Legendre dual to the Lagrangian $L \colon \mathrm{T}^*\overline{\Omega} \to \mathbb{R}$.

Remark. As explained in [Abb13], this number should be interpreted as an energy level that marks important dynamical and geometric changes for the Euler–Lagrange flow corresponding to L.

Remark. In the case of a closed manifold $\overline{\Omega}$ there is an important characterization of the strict Mañé critical value c_0, see [Fat97] and [CIP98]. With this in mind, we recall the definition of the strict Mañé critical value c_0 in the case of a closed manifold:

$$c_0(L) := -\inf\left\{\frac{1}{\tau}\int_0^\tau L(\gamma(t),\gamma'(t))\ \mathrm{d}t \mid \gamma \in C^\infty(\mathbb{R}/\tau\mathbb{Z},\overline{\Omega}) \text{ homologous to } 0, \tau > 0\right\},$$

where L is the electromagnetic Lagrangian given in Equation (3.1). This value can be characterized by

$$c_0(L) = \inf\left\{\max_{q\in\overline{\Omega}} H(q,\tau_q) \mid \tau \text{ smooth closed 1-form on } \overline{\Omega}\right\}, \tag{3.6}$$

where $H\colon \mathrm{T}^*\overline{\Omega} \to \mathbb{R}$ is the Hamiltonian associated with the Lagrangian L (3.1) via Legendre duality. Equation (3.6) motivates our definition of the strict Mañé critical value c_0 in Definition 3.2. To the author's knowledge, it is not clear whether an analogous characterization is valid in the case of a manifold with boundary.

In this thesis we prove the following theorem.

Theorem 3.3. *There exists an energy value $E_0 \in \mathbb{R}$ such that for all $E \geq E_0$, there exists a periodic magnetic bounce orbit $\gamma\colon \mathbb{R}/\tau\mathbb{Z} \to \overline{\Omega}$, $\tau > 0$, with energy $E(\gamma) = E$ and at most* $\dim(\Omega)+1$ *bounce points. In addition, there exists an explicit upper bound on the period $\tau < \infty$.*

Remark. The implicit bound on E_0 can be found in Proposition 3.13 and the explicit bound on the period τ is given in the proof of Theorem 3.3 in (3.58).

Remark. On general Riemannian manifolds $\overline{\Omega}$, it is not possible to show the existence of periodic magnetic bounce orbits with at least one *bounce point* for prescribed energy values $E > c_0$. Indeed, if the Riemannian metric g admits a closed geodesic in the domain Ω (see [Kli78]), and the potential V, as well as the magnetic field α, vanish near such a geodesic, then for every energy value E this closed geodesic leads to a *smooth* periodic magnetic bounce orbit, i.e. $\mathcal{B} = \emptyset$.

If we restrict our considerations to the Euclidean situation and look for periodic magnetic bounce orbits *with* bounce points, we can show the following.

Theorem 3.4. *Consider the domain Ω endowed with the flat Euclidean metric $|\cdot|_{\mathrm{Eucl}}$. Let E_0 be the energy value as in Theorem 3.3. Then the set of bounce times $\mathcal{B}$ of a periodic magnetic bounce orbit is not empty under the conditions that $E \in \mathbb{R}$ with $E > E_0$ and*

$$E > \operatorname{diam}(\overline{\Omega}) \max_{\overline{\Omega}} |\nabla V| + \max_{\overline{\Omega}} V + C,$$

where C is a non-negative constant depending on Ω, α and V. The explicit formulation of the constant C can be found in the proof of Theorem 3.4 in Equation (3.62).

Remark. If $\alpha = 0$, then $C = 0$. Thus, the lower bound on E agrees with the result in the non-magnetic case (see [AM11, Corollary 1.4]).

Organization of this chapter. In Section 3.2 we introduce the approximation scheme for the free-time action functional and prove that approximate solutions converge to actual periodic magnetic bounce orbits, assuming that the Morse index of the corresponding free-time action functional is bounded. After that, we consider in Section 3.3 the Hamiltonian point of view of our setting using Legendre duality. In particular, we prove the existence of approximate solutions. Finally, we combine all the collected ingredients to prove our main theorem 3.3. The proofs of Theorem 3.3 and Theorem 3.4 are explained in Section 3.4.

3.2 An approximation scheme

We start by introducing an approximation scheme. Originally, this scheme was used by Benci–Giannoni who developed it for the fixed period case. In the article [BG89], Benci–Giannoni proved the existence of a periodic bounce orbit with prescribed period and with at most $\dim(\Omega) + 1$ bounce points. To that end, they developed a fantastic approximation scheme and studied the fixed-time action functional. In this work, we follow the approach by Albers–Mazzucchelli in [AM11, Chapter 2] who adjusted the approximation scheme to the case of free period and prescribed energy. They studied the free-time action functional instead and proved the existence of periodic bounce orbits of prescribed energy. They overcame new difficulties, that arise in showing the existence of a bounce orbit, using techniques coming from symplectic geometry rather than variational methods. In particular, they obtained explicit bounds on the period of the bounce orbits.

To obtain a Lagrangian system approximating the original one, we modify the Lagrangian system given in Equation (3.1) by adding an additional term, the so-called *penalty term*. Assume now that there exist solutions for the modified system. By passing to the limit, as the penalty term tends to zero, we show that under suitable assumptions these approximate solutions actually converge to magnetic bounce orbits. We begin with the variational setup.

3.2.1 Variational Setup

In this section, we define a Lagrangian system approximating the original Lagrangian that was given in Equation (3.1). We call a solution $\widehat{\gamma}$ of the Euler–Lagrange equation corresponding to this modified Lagrangian system an *approximate solution*. One can think of $\widehat{\gamma}$ as a curve which does not quite reach the boundary and instead has a sharp turn, see Figure 3.1. In the limit, the sharp turn becomes sharper and sharper and $\widehat{\gamma}$ converges to an actual magnetic bounce orbit. In the following, we explain this more formally.

We consider a compact Riemannian manifold $(\overline{\Omega}, g = \langle \cdot, \cdot \rangle)$ with smooth boundary and denote its interior by Ω. Let $l(c)$ denote the length of a curve c in $\overline{\Omega}$ (see [GHL04, Chapter 2.6]). Fix a sufficiently small $d_0 \in \left(0, \frac{1}{2}\right)$ such that the distance function

$$\mathrm{dist}_{\partial\Omega}(q) := \inf \left\{ l(c) \mid c \text{ is a piecewise } \mathcal{C}^1\text{-curve and } c \text{ joins } q \text{ to } \tilde{q} \in \partial\Omega \right\} \tag{3.7}$$

is smooth at all $q \in \Omega$ with $\mathrm{dist}_{\partial\Omega}(q) \leq 2d_0 < 1$. Moreover, we choose a smooth function $k\colon [0, \infty) \to [0, 2d_0]$ (compare Figure 3.2) with $0 \leq k' \leq 1$, $k(x) = x$ for

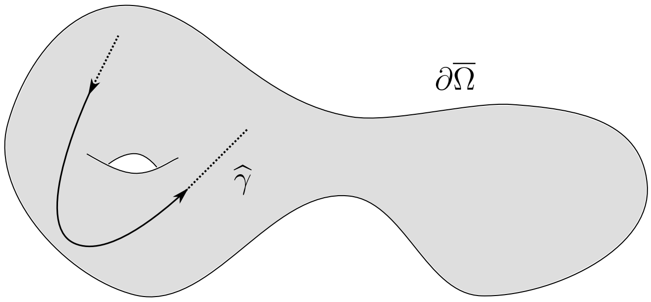

Figure 3.1: Idea of an approximate solution $\widehat{\gamma}$.

$x \leq d_0$ and $k(x) = const$ for $x \geq 2d_0$. Define $h \in \mathcal{C}^\infty\left(\overline{\Omega}\right)$ (see Figure 3.3) via

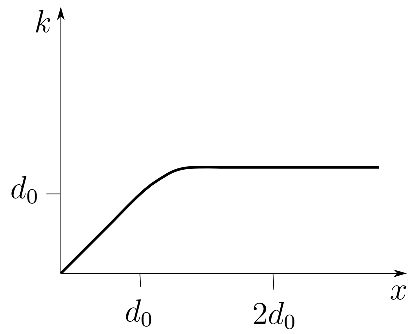

Figure 3.2: Example of a function k.

$$\begin{aligned} h\colon \overline{\Omega} &\longrightarrow [0, 2d_0], \\ q &\longmapsto k(\operatorname{dist}_{\partial\Omega}(q)). \end{aligned} \tag{3.8}$$

Then, h satisfies

- $h(q) = \operatorname{dist}_{\partial\Omega}(q)$ for all $q \in \overline{\Omega}$ with $\operatorname{dist}_{\partial\Omega}(q) \leq d_0$,
- $h(q) > d_0$ if $\operatorname{dist}_{\partial\overline{\Omega}}(q) > d_0$,
- $0 \leq h \leq 2d_0 < 1$,
- $h(q) = const$ if $\operatorname{dist}_{\partial\overline{\Omega}}(q) \geq 2d_0$,
- $|\nabla h|_g \leq 1$ and
- $|\nabla \operatorname{dist}_{\partial\overline{\Omega}}|_g = 1$.

Finally, we define the penalty term $U \in \mathcal{C}^\infty(\Omega)$ as

$$U(q) := \frac{1}{h^2(q)}. \tag{3.9}$$

Then U has the following properties (see Figure 3.3):

- $U = \frac{1}{\operatorname{dist}^2_{\partial\Omega}}$ near $\partial\overline{\Omega}$,

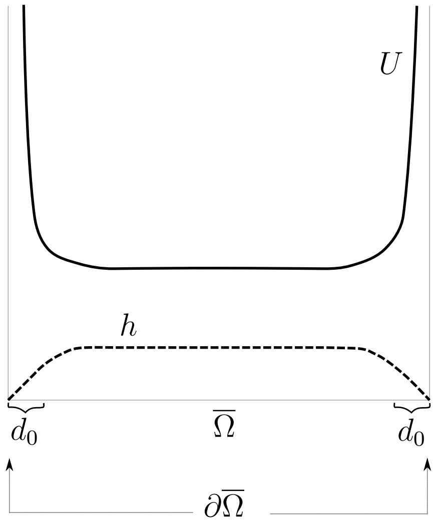

Figure 3.3: The functions h and U.

- U is constant in $\{\mathrm{dist}_{\partial\overline{\Omega}}(q) \geq 2d_0\}$.

The modified Lagrangian system. To modify the Lagrangian L (3.1), we use the penalty term as follows. For $\varepsilon > 0$ we define the modified Lagrangian L_ε as

$$\begin{aligned} L_\varepsilon \colon \mathrm{T}\Omega &\longrightarrow \mathbb{R} \\ (q, v) &\longmapsto \tfrac{1}{2}|v|_g^2 + \alpha_q(v) - V(q) - \varepsilon U(q) = L(q, v) - \varepsilon U(q). \end{aligned} \tag{3.10}$$

Remark. We will both write α as a function $\alpha \colon \mathrm{T}\overline{\Omega} \to \mathbb{R}$ with $\alpha_q = \alpha|_{\mathrm{T}_q\overline{\Omega}} \colon \mathrm{T}_q\overline{\Omega} \to \mathbb{R}$, $\alpha_q = \sum_j \alpha_j(q)\mathrm{d}q_j$ and as $\alpha \colon \overline{\Omega} \to \mathrm{T}\overline{\Omega}$ in the duality pairing given by $\alpha_q(v) = \langle \alpha(q), v \rangle$, changing notation to improve readability.

A solution γ_ε of the Euler–Lagrange equation corresponding to the modified Lagrangian system (3.10) is called an *approximate solution.* The energy of an approximate solution γ_ε is given by

$$E_\varepsilon(\gamma_\varepsilon) := \tfrac{1}{2}|\gamma_\varepsilon'|_g^2 + V(\gamma_\varepsilon) + \varepsilon U(\gamma_\varepsilon).$$

As we consider a magnetic system with prescribed energy, an approximate solution cannot reach the boundary of $\overline{\Omega}$ because the penalty term U explodes to ∞ near the boundary $\partial\Omega$.

In this section we will show that, under natural assumptions, sequences of approximate solutions converge to a magnetic bounce orbit as $\varepsilon \to 0$. To prove this convergence we introduce more background in variational methods.

3.2.2 The free-time action functional

We study the free-time Lagrangian action functional on the space of closed curves of arbitrary period. The latter space can be given a differentiable structure by reparametrizing each curve on $S^1 = \mathbb{R}/\mathbb{Z}$ and by keeping track of its period as a second variable (see [Abb13, Chapter 2]).

A Hilbert manifold of Sobolev loops. Let $\gamma\colon \mathbb{R}/\tau\mathbb{Z} \to \overline{\Omega}$, $\tau \in \mathbb{R}_{>0}$, be an absolutely continuous τ-periodic curve and define $\Gamma\colon \mathbb{R}/\mathbb{Z} \to \overline{\Omega}$ as $\Gamma(t/\tau) := \gamma(t)$. To streamline notation we often abbreviate $\gamma := \gamma(t)$ and $\Gamma := \Gamma(t)$. We identify the closed curve γ with the pair (Γ, τ). The action of γ on the time interval $[0, \tau] \subset \mathbb{R}$ is the number

$$\int_0^\tau L_\varepsilon\left(\gamma(t), \gamma'(t)\right) \mathrm{d}t = \tau \int_0^1 L_\varepsilon\left(\Gamma(t), \tfrac{1}{\tau}\Gamma'(t)\right) \mathrm{d}t.$$

For our setting, we fix an energy value $E \in \mathbb{R}$ and consider the *free-time action functional*

$$\begin{aligned} \mathscr{L}_\varepsilon^E(\Gamma, \tau) : &= \tau \int_0^1 \left[L_\varepsilon\left(\Gamma(t), \tfrac{1}{\tau}\Gamma'(t)\right) + E\right] \mathrm{d}t \\ &= \int_0^\tau \left[L_\varepsilon(\gamma(t), \gamma'(t)) + E\right] \mathrm{d}t. \end{aligned}$$

As the modified Lagrangian L_ε has fiberwise quadratic growth, the natural functional analytic setting to study the functional is the Hilbert manifold $H^1(S^1, \overline{\Omega})$ of Sobolev loops (see [AS09, Chapter 3], [Abb13, Chapter 3]) with $S^1 = \mathbb{R}/\mathbb{Z}$. For that we consider $\mathscr{L}_\varepsilon^E$ as a map

$$\mathscr{L}_\varepsilon^E\colon H^1(S^1, \overline{\Omega}) \times \mathbb{R}_{>0} \to \mathbb{R}.$$

The smooth structure on $H^1(S^1, \overline{\Omega})$. The Hilbert space $H^1(S^1, \overline{\Omega})$ has a natural differentiable structure, see [AS09, p.10] and [Kli78, Thm.1.2.9] for more details. The model space of $H^1(S^1, \overline{\Omega})$ is

$$H^1(\gamma^*\mathrm{T}\overline{\Omega}),$$

where $\gamma\colon S^1 \to \overline{\Omega}$ denotes a smooth closed curve. Let U be a sufficiently small tubular neighborhood of the zero-section in $\gamma^*\mathrm{T}\overline{\Omega}$. A chart is given by

$$\begin{aligned} \exp_\gamma\colon H^1(\gamma^*(U)) &\to \mathcal{U}(\gamma) = \exp_\gamma(H^1(\gamma^*(U))) \subset H^1(S^1, \overline{\Omega}) \\ (\exp_\gamma \xi)(t) &= \exp_{\gamma(t)} \xi(t). \end{aligned}$$

So a natural atlas can be given as

$$(\exp_\gamma^{-1}, \mathcal{U}(\gamma)),$$

where $\gamma \in \mathcal{C}^\infty(S^1, \overline{\Omega})$. The H^1-setup guarantees that $\mathscr{L}_\varepsilon^E$ has a well-defined gradient flow on $H^1(S^1, \overline{\Omega})$.

Differential of the free-time action functional. The regularity properties of the free-time action functional $\mathscr{L}_\varepsilon^E$ are proven in [AS09, Prop.3.1] as well as in [Con06, Chapter 2]. In particular, they show that $\mathscr{L}_\varepsilon^E$ is continuously differentiable on $H^1(S^1, \overline{\Omega}) \times \mathbb{R}_{>0}$. Now, we derive explicit formulas for the directional derivatives of the action functional $\mathscr{L}_\varepsilon^E$.
Let $(\Psi, \rho) \in \mathrm{T}_{(\Gamma,\tau)}\left(H^1(S^1, \overline{\Omega}) \times \mathbb{R}_{>0}\right)$ for some $(\Gamma, \tau) \in H^1(S^1, \overline{\Omega}) \times \mathbb{R}_{>0}$. To streamline notation we write $G_s(t) := \exp_{\Gamma(t)}\left(s\Psi(t)\right)$. The derivative in the direction of (Ψ, ρ) in (Γ, τ) can be computed as follows

$$\mathrm{d}\mathscr{L}_\varepsilon^E(\Gamma, \tau)(\Psi, \rho) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \tau \int_0^1 \left[L_\varepsilon\left(G_s, \tfrac{1}{\tau}G_s'\right) + E\right] \mathrm{d}t$$

$$+ \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} (\tau + s\rho) \int_0^1 \left[L_\varepsilon \left(\Gamma, \tfrac{1}{\tau+s\rho}\Gamma' \right) + E \right] \mathrm{d}t.$$

First we calculate $\rho = 0$ and obtain

$$\begin{aligned} \mathrm{d}\mathscr{L}_\varepsilon^E(\Gamma,\tau)(\Psi,0) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \tau \int_0^1 \left[L_\varepsilon \left(G_s, \tfrac{1}{\tau} G_s' \right) + E \right] \mathrm{d}t \\ &= \tau \cdot \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_0^1 \left[\tfrac{1}{2} \left| \tfrac{1}{\tau} G_s' \right|_g^2 + \alpha_{G_s} \left(\tfrac{1}{\tau} G_s' \right) - V(G_s) - \varepsilon U(G_s) + E \right] \mathrm{d}t. \end{aligned} \tag{3.11}$$

The first summand in Equation (3.11) is the first variation of the Riemannian energy functional (see [GHL04, Section 3.B.2]). With that, we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_0^1 \tfrac{1}{2} \left| \tfrac{1}{\tau} G_s' \right|_g^2 \, \mathrm{d}t = \frac{1}{\tau^2} \int_0^1 \langle \Gamma', \nabla_{\Gamma'} \Psi \rangle \, \mathrm{d}t,$$

where $\nabla_{\Gamma'}$ denotes the Levi-Civita connection in the direction of $\Gamma' := \partial\Gamma/\partial t$. Let $\mathcal{L}$ denote the *Lie derivative*. We continue with the next term in Equation (3.11) and derive

$$\begin{aligned} \int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left(\frac{1}{\tau} \alpha_{G_s(t)} \left(G_s'(t) \right) \right) \mathrm{d}t &= \frac{1}{\tau} \int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} G_s^* \alpha \\ &= \frac{1}{\tau} \int_0^1 G_0^* \left[\mathcal{L}_{\Psi(t)} \alpha \right] \\ &= \frac{1}{\tau} \int_0^1 \Gamma^* \left[\mathrm{d}\iota_{\Psi(t)} \alpha + \iota_{\Psi(t)} \mathrm{d}\alpha \right], \end{aligned}$$

where we used Cartan's magic formula in the last step. In the next step, we use that $\sigma = \mathrm{d}\alpha$ and that the integral of $\Gamma^* \mathrm{d}\iota_{\Psi(t)}\alpha$ vanishes, since Γ is a closed curve. Thus, we obtain

$$\begin{aligned} \frac{1}{\tau} \int_0^1 \Gamma^* \left[\mathrm{d}\iota_{\Psi(t)} \alpha + \iota_{\Psi(t)} \mathrm{d}\alpha \right] &= \frac{1}{\tau} \int_0^1 \sigma_{\Gamma(t)} \left(\Gamma'(t), \Psi(t) \right) \mathrm{d}t \\ &= -\frac{1}{\tau} \int_0^1 \left\langle Y_{\Gamma(t)} \left(\Gamma'(t) \right), \Psi(t) \right\rangle \mathrm{d}t. \end{aligned}$$

All in all, the differential of the free-time action functional $\mathscr{L}_\varepsilon^E$ with respect to the first variable is

$$\begin{aligned} \mathrm{d}\mathscr{L}_\varepsilon^E(\Gamma,\tau)(\Psi,0) = \frac{1}{\tau} \int_0^1 \langle \Gamma', \nabla_{\Gamma'} \Psi \rangle \, \mathrm{d}t &- \int_0^1 \langle Y_\Gamma(\Gamma'), \Psi \rangle \, \mathrm{d}t \\ - \tau \int_0^1 \langle \nabla V(\Gamma), \Psi \rangle \, \mathrm{d}t &- \tau \int_0^1 \varepsilon \langle \nabla U(\Gamma), \Psi \rangle \, \mathrm{d}t. \end{aligned}$$

where we denote by ∇V, ∇U the Riemannian gradients of the corresponding functions V, U.

We continue with the computation of the derivative with respect to the second variable and calculate

$$\mathrm{d}\mathscr{L}_\varepsilon^E(\Gamma,\tau)(0,\rho) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} (\tau + s\rho) \int_0^1 \left[L_\varepsilon \left(\Gamma, \tfrac{1}{\tau+s\rho}\Gamma' \right) + E \right] \mathrm{d}t$$

$$= \rho \int_0^1 \left[L_\varepsilon\left(\Gamma, \tfrac{1}{\tau}\Gamma'\right) + E\right] \mathrm{d}t + \tau \int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left[L_\varepsilon\left(\Gamma, \tfrac{1}{\tau+s\rho}\Gamma'\right) + E\right] \mathrm{d}t. \tag{3.12}$$

Computing the second summand in Equation (3.12) results in

$$\begin{aligned}
&\tau \int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left[L_\varepsilon\left(\Gamma, \tfrac{1}{\tau+s\rho}\Gamma'\right) + E\right] \mathrm{d}t \\
&= \tau \int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left[\frac{1}{2(\tau+s\rho)^2}|\Gamma'|_g^2 + \frac{1}{\tau+s\rho}\alpha_\Gamma(\Gamma') - V(\Gamma) - \varepsilon U(\Gamma) + E\right] \mathrm{d}t \\
&= \tau \int_0^1 \left[-\frac{1}{\tau^3}\rho|\Gamma'|_g^2 - \frac{1}{\tau^2}\rho \cdot \alpha_\Gamma(\Gamma')\right] \mathrm{d}t \\
&= \rho \int_0^1 \left[-\frac{1}{\tau^2}|\Gamma'|_g^2 - \frac{1}{\tau}\alpha_\Gamma(\Gamma')\right] \mathrm{d}t.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathrm{d}\mathscr{L}_\varepsilon^E(\Gamma,\tau)(0,\rho) &= \rho \int_0^1 \left[L_\varepsilon\left(\Gamma, \tfrac{1}{\tau}\Gamma'\right) + E\right] \mathrm{d}t + \rho \int_0^1 \left[-\frac{1}{\tau^2}|\Gamma'|_g^2 - \frac{1}{\tau}\alpha_\Gamma(\Gamma')\right] \mathrm{d}t \\
&= \rho \int_0^1 \left[\frac{1}{2\tau^2}|\Gamma'|_g^2 + \frac{1}{\tau}\alpha_\Gamma(\Gamma') - V(\Gamma) - \varepsilon U(\Gamma) + E - \frac{1}{\tau^2}|\Gamma'|_g^2 - \frac{1}{\tau}\alpha_\Gamma(\Gamma')\right] \mathrm{d}t \\
&= \rho \int_0^1 \left[-\frac{1}{2\tau^2}|\Gamma'|_g^2 - V(\Gamma) - \varepsilon U(\Gamma) + E\right] \mathrm{d}t.
\end{aligned}$$

Altogether, the differential of the free-time action functional $\mathscr{L}_\varepsilon^E$ is given by

$$\begin{aligned}
\mathrm{d}\mathscr{L}_\varepsilon^E(\Gamma,\tau)\left[(\Psi,\rho)\right] = \frac{1}{\tau}&\int_0^1 \langle \Gamma', \nabla_{\Gamma'}\Psi\rangle \;\mathrm{d}t - \int_0^1 \langle Y_\Gamma\left(\Gamma'\right), \Psi\rangle \;\mathrm{d}t \\
&- \tau \int_0^1 \langle \nabla V(\Gamma), \Psi\rangle \;\mathrm{d}t - \tau \int_0^1 \varepsilon\langle \nabla U(\Gamma), \Psi\rangle \;\mathrm{d}t \\
&+ \rho \int_0^1 \left[-\frac{1}{2\tau^2}|\Gamma'|_g^2 - V(\Gamma) - \varepsilon U(\Gamma) + E\right] \mathrm{d}t.
\end{aligned} \tag{3.13}$$

Expressing this differential in terms of $\gamma(t) = \Gamma(t/\tau)$ leads to

$$\begin{aligned}
\mathrm{d}\mathscr{L}_\varepsilon^E(\gamma)\left[\psi\right] = &- \int_0^\tau \langle \nabla_{\gamma'}\gamma' + Y_\gamma(\gamma') + \nabla V(\gamma) + \varepsilon \nabla U(\gamma), \psi\rangle \,\mathrm{d}t \\
&+ \frac{\rho}{\tau}\int_0^\tau \left(E - V(\gamma) - \varepsilon U(\gamma) - \tfrac{1}{2}|\gamma'|_g^2\right) \mathrm{d}t,
\end{aligned}$$

where $\psi \in \mathrm{T}_\gamma H^1(\mathbb{R}/\rho\mathbb{Z}, \overline{\Omega})$.

In the following lemma, we characterize the critical points of the free-time action functional $\mathscr{L}_\varepsilon^E$. This helps to describe the problem of finding magnetic bounce orbits in a variational manner.

Lemma 3.5. *A pair $(\Gamma,\tau) \in H^1(S^1, \overline{\Omega}) \times \mathbb{R}_{>0}$ is a critical point of the free-time action functional $\mathscr{L}_\varepsilon^E$ if and only if the corresponding τ-periodic curve γ is a solution of the Euler–Lagrange equation*

$$\nabla_{\gamma'}\gamma' + Y_\gamma(\gamma') + \nabla V(\gamma) + \varepsilon \nabla U(\gamma) = 0 \tag{3.14}$$

and the energy of γ *satisfies*

$$E_\varepsilon(\gamma) := \tfrac{1}{2}|\gamma'|_g^2 + V(\gamma) + \varepsilon U(\gamma) = E.$$

Proof. The point (Γ, τ) is a critical point of the free-time action functional $\mathscr{L}_\varepsilon^E$ if and only if $\mathrm{d}\mathscr{L}_\varepsilon^E \equiv 0$. We note that we can analyse the two summands of the differential separately. To begin with, we examine the first summand of the differential of the free-time action functional $\mathscr{L}_\varepsilon^E$ in Equation (3.13) for arbitrary $\psi \in H^1(S^1, \gamma^*\mathrm{T}\overline{\Omega})$:

$$\int_0^\tau \left(\langle -\nabla_{\gamma'}\gamma', \psi\rangle - \langle Y_\gamma(\gamma') + \nabla V(\gamma) + \varepsilon\nabla U(\gamma), \psi\rangle\right) \mathrm{d}t = 0.$$

Applying the *Lemma of Du Bois–Raymond* (see [Kli78, Thm.1.3.11]) leads to

$$\nabla_{\gamma'}\gamma' + Y_\gamma(\gamma') + \nabla V(\gamma) + \varepsilon\nabla U(\gamma) = 0,$$

i.e. the curve γ is a solution of the Euler–Lagrange equation as claimed. The second summand of the differential of $\mathscr{L}_\varepsilon^E$ shows that

$$\int_0^\tau \left(E - \tfrac{1}{2}|\gamma'|_g^2 - V(\gamma) - \varepsilon U(\gamma)\right) \mathrm{d}t = 0$$

and

$$\nabla_{\gamma'}\gamma' + Y_\gamma(\gamma') + \nabla V(\gamma) + \varepsilon\nabla U(\gamma) = 0.$$

The Euler–Lagrange equation implies that $E_\varepsilon(\gamma(t))$ is independent of t, as the following computation proves:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon(\gamma(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}|\gamma'(t)|_g^2 + V(\gamma(t)) + \varepsilon U(\gamma(t))\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\langle\gamma'(t), \gamma'(t)\rangle + V(\gamma(t)) + \varepsilon U(\gamma(t))\right) \\
&= \tfrac{1}{2}\langle\nabla_{\gamma'}\gamma'(t), \gamma'(t)\rangle + \tfrac{1}{2}\langle\gamma'(t), \nabla_{\gamma'}\gamma'(t)\rangle \\
&\quad + \langle\nabla V(\gamma(t)), \gamma'(t)\rangle + \varepsilon\langle\nabla U(\gamma(t)), \gamma'(t)\rangle \\
&= \langle\nabla_{\gamma'}\gamma'(t), \gamma'(t)\rangle + \langle\nabla V(\gamma(t)), \gamma'(t)\rangle + \varepsilon\langle\nabla U(\gamma(t)), \gamma'(t)\rangle \\
&= \langle\nabla_{\gamma'}\gamma'(t) + \nabla V(\gamma(t)) + \varepsilon\nabla U(\gamma(t)), \gamma'(t)\rangle \\
&= -\langle Y_\gamma(\gamma'), \gamma'\rangle = -\sigma_\gamma(\gamma', \gamma') = -\mathrm{d}\alpha_\gamma(\gamma', \gamma') \\
&= 0.
\end{aligned}$$

Therefore

$$E_\varepsilon(\gamma) = \tfrac{1}{2}|\gamma'|_g^2 + V(\gamma) + \varepsilon U(\gamma) = E.$$

□

3.2.3 Convergence of approximate solutions

In this section, we prove that a sequence of approximate solutions converges to a periodic magnetic bounce orbit in H^1 under suitable assumptions. This is a magnetic analogue of [AM11, Prop.2.1].

Proposition 3.6. *Let $K > 0$ and let $T_2 > T_1 > 0$. For each $\varepsilon > 0$, let $(\Gamma_\varepsilon, \tau_\varepsilon)$ be a critical point of the free-time action functional $\mathscr{L}_\varepsilon^E$ with period $T_1 \le \tau_\varepsilon \le T_2$ and energy $E_\varepsilon \le K$. Then, up to choosing a subsequence, $(\Gamma_\varepsilon, \tau_\varepsilon)$ converges in $H^1(S^1, \overline{\Omega}) \times \mathbb{R}_{>0}$ to (Γ, τ) as $\varepsilon \to 0$. Moreover, there exists a finite Borel measure μ on $\mathcal{C} := \{t \in \mathbb{R}/\tau\mathbb{Z} \mid \gamma(t) \in \partial\Omega\}$ for $\gamma(t) := \Gamma(t/\tau)$ such that*

1. *for all $\psi \in H^1(S^1, \gamma^*\mathrm{T}\overline{\Omega})$*

$$\int_0^\tau [\langle \gamma', \nabla_{\gamma'}\psi\rangle - \langle Y_\gamma(\gamma') + \nabla V(\gamma), \psi\rangle]\,\mathrm{d}t = \int_{\mathcal{C}} \langle \nu(\gamma), \psi\rangle \mathrm{d}\mu, \tag{3.15}$$

   where ν is the outer normal with respect to $\partial\overline{\Omega}$,

2. *outside* $\mathrm{supp}(\mu)$ *the curve γ is a smooth solution of the Euler–Lagrange equation* (3.2) *corresponding to L with energy $E(\gamma) = \lim_{\varepsilon\to 0} E_\varepsilon(\gamma_\varepsilon)$ and*

3. *γ has left and right derivatives that are left and right continuous on $\mathbb{R}/\tau\mathbb{Z}$, respectively. Moreover, γ satisfies the law of reflection given in Equation* (3.4) *at each time $t \in \mathcal{C}$ which is an isolated point of* $\mathrm{supp}(\mu)$.

In particular, if $\mathrm{supp}(\mu)$ *is a finite set, then γ is a periodic magnetic bounce orbit of the Lagrangian system given in Equation* (3.1) *and $\mathcal{B} :=$* $\mathrm{supp}(\mu)$ *is its set of bounce times.*

Proof. Let $(\Gamma_\varepsilon, \tau_\varepsilon)$ be a sequence as above. The sequences (τ_ε) and (E_ε) are bounded since $T_1 \le \tau_\varepsilon \le T_2$ and $0 \le E_\varepsilon \le K$. Up to the choice of a subsequence we have $\tau_\varepsilon \to \tau$ and $E_\varepsilon \to E$ as $\varepsilon \to 0$ with $T_1 \le \tau \le T_2$ and $E \le K$. Define $\gamma_\varepsilon(t) = \Gamma_\varepsilon(t/\tau_\varepsilon)$ as the periodic orbit associated to the sequence $(\Gamma_\varepsilon, \tau_\varepsilon)$. The energy of γ_ε is given by

$$\begin{aligned} E_\varepsilon \equiv E_\varepsilon(\gamma_\varepsilon) &= \tfrac{1}{2}|\gamma_\varepsilon'|_g^2 + V(\gamma_\varepsilon) + \varepsilon U(\gamma_\varepsilon) \\ &= \tfrac{1}{2\tau_\varepsilon}|\Gamma_\varepsilon'|_g^2 + V(\Gamma_\varepsilon) + \varepsilon U(\Gamma_\varepsilon). \end{aligned} \tag{3.16}$$

$\varepsilon\nabla\mathbf{U}(\gamma_\varepsilon)$ is uniformly bounded in $\mathbf{L^1}$. Let us first consider a tangent vector of the form $(\Psi, 0) \in H^1(S^1, \Gamma_\varepsilon^*(\mathrm{T}\overline{\Omega})) \times \mathbb{R}_{>0}$ at the critical point $(\Gamma_\varepsilon, \tau_\varepsilon)$. An integration by parts of Equation (3.13) leads to

$$\begin{aligned} 0 &= \mathrm{d}\mathscr{L}_\varepsilon^E(\Gamma_\varepsilon, \tau_\varepsilon)(\Psi, 0) \\ &= \tau_\varepsilon \int_0^1 \left[\tau_\varepsilon^{-2}\langle \Gamma_\varepsilon', \nabla_{\Gamma_\varepsilon'}\Psi\rangle - \langle \tau_\varepsilon^{-1} \cdot Y_{\Gamma_\varepsilon}(\Gamma_\varepsilon') + \nabla V(\Gamma_\varepsilon) + \varepsilon\nabla U(\Gamma_\varepsilon), \Psi\rangle\right] \mathrm{d}t. \end{aligned}$$

In particular, we know

$$\int_0^1 \left[\tau_\varepsilon^{-2}\langle \Gamma_\varepsilon', \nabla_{\Gamma_\varepsilon'}\Psi\rangle - \langle \tau_\varepsilon^{-1} \cdot Y_{\Gamma_\varepsilon}(\Gamma_\varepsilon') + \nabla V(\Gamma_\varepsilon), \Psi\rangle\right] \mathrm{d}t = \int_0^1 \langle \varepsilon\nabla U(\Gamma_\varepsilon), \Psi\rangle\ \mathrm{d}t \tag{3.17}$$

for all $\Psi \in H^1(S^1, \Gamma^*(\mathrm{T}\overline{\Omega}))$. To prove that $\varepsilon \nabla U(\gamma_\varepsilon)$ is uniformly bounded in L^1, we first show that the term in the integral of the left hand side of Equation (3.17) is uniformly bounded in L^∞ for a good choice of Ψ. For that, we fix $\Psi := \Psi_\varepsilon = -\nabla h(\Gamma_\varepsilon)$. Then $\nabla_{\Gamma'}\Psi_\varepsilon = -\nabla_{\Gamma'_\varepsilon}\nabla h(\Gamma_\varepsilon)$ is uniformly bounded in L^∞ since Γ'_ε is uniformly bounded in L^∞ and h is a smooth function on $\overline{\Omega}$. In a next step, we bound the second integrand in Equation (3.17). We start by recalling the operator norm of the *Lorentz force* $Y\colon \mathrm{T}\overline{\Omega} \to \mathrm{T}\overline{\Omega}$ as

$$|Y_q|_{\mathrm{op}} := \max\left\{\langle Y_q(v), w\rangle \mid v, w \in \mathrm{T}_q\overline{\Omega}, |v|_g, |w|_g \leq 1\right\}.$$

Set $|Y|_{\max} := \max_{q\in\overline{\Omega}} |Y_q|_{\mathrm{op}}$. The maximum $|Y|_{\max}$ exists because $\overline{\Omega}$ is compact. By definition, we obtain

$$|\langle Y_{\Gamma_\varepsilon}(\Gamma'_\varepsilon), \Psi_\varepsilon\rangle|_{L^\infty} \leq |Y|_{\max} \cdot |\Gamma'_\varepsilon|_{L^\infty} \cdot |\Psi_\varepsilon|_{L^\infty}.$$

Recall that $\tau_\varepsilon \in \mathbb{R}_{>0}$ is uniformly bounded and $V \in \mathcal{C}^\infty(\overline{\Omega})$, so it follows that

$$\begin{aligned}
&\int_0^1 \left[\tau_\varepsilon^{-2}\langle\Gamma'_\varepsilon, \nabla_{\Gamma'_\varepsilon}\Psi_\varepsilon\rangle - \langle\tau_\varepsilon^{-1}\cdot Y_{\Gamma_\varepsilon}(\Gamma'_\varepsilon) + \nabla V(\Gamma_\varepsilon), \Psi_\varepsilon\rangle\right] \mathrm{d}t \\
&\leq \tau_\varepsilon^{-2}\cdot|\Gamma'_\varepsilon|_{L^\infty}\cdot|\nabla_{\Gamma'_\varepsilon}\Psi_\varepsilon|_{L^\infty} + \tau_\varepsilon^{-1}\cdot|Y|_{\max}\cdot|\Gamma'_\varepsilon|_{L^\infty}\cdot|\Psi_\varepsilon|_{L^\infty} + |\nabla V(\Gamma_\varepsilon)|_{L^\infty}\cdot|\Psi_\varepsilon|_{L^\infty}
\end{aligned}$$

is uniformly bounded in ε, since Γ'_ε is uniformly bounded in L^∞. Therefore, the left hand side of Equation (3.17) is uniformly bounded in ε and hence so is the right hand side:

$$\int_0^1 \langle\varepsilon\nabla U(\Gamma_\varepsilon), \Psi\rangle \;\mathrm{d}t \leq C, \tag{3.18}$$

where C is a constant, independent of ε. Recall that we want to show that $\varepsilon\nabla U(\gamma_\varepsilon)$ is uniformly bounded in L^1. Inserting $\nabla U(\Gamma_\varepsilon) = -2h^{-3}(\Gamma_\varepsilon)\nabla h(\Gamma_\varepsilon)$ in Equation (3.18), we get the estimate

$$\begin{aligned}
C &\geq \int_0^1 \langle\varepsilon\nabla U(\Gamma_\varepsilon), \Psi\rangle \;\mathrm{d}t \\
&= \int_0^1 \left\langle 2\varepsilon h^{-3}(\Gamma_\varepsilon)\nabla h(\Gamma_\varepsilon), \nabla h(\Gamma_\varepsilon)\right\rangle \;\mathrm{d}t \\
&= \int_0^1 2\varepsilon h^{-3}(\Gamma_\varepsilon)\,|\nabla h(\Gamma_\varepsilon)|_g^2 \;\mathrm{d}t.
\end{aligned} \tag{3.19}$$

Next, let $\Omega' \subset \overline{\Omega}$ be the compact neighborhood of $\partial\overline{\Omega}$ given by

$$\Omega' := \{q \in \overline{\Omega} \mid h(q) \leq d_0\},$$

compare Equation (3.8) and Figure 3.3 for the definition of h. By definition, $h(q) = \mathrm{dist}_{\partial\overline{\Omega}}(q)$ for $q \in \Omega'$ and thus, $|\nabla h|_g = 1$ on Ω'. Moreover, on $\overline{\Omega}\backslash\Omega'$ we have $h > d_0$ and so $|\nabla h|_g \leq 1$. Then, we conclude

$$\int_0^1 2\varepsilon h^{-3}(\Gamma_\varepsilon)\;\mathrm{d}t = \int\limits_{\{t|\Gamma_\varepsilon(t)\in\Omega'\}} 2\varepsilon h^{-3}(\Gamma_\varepsilon)\;\mathrm{d}t + \int\limits_{\{t|\Gamma_\varepsilon(t)\notin\Omega'\}} 2\varepsilon h^{-3}(\Gamma_\varepsilon)\;\mathrm{d}t$$

$$\leq \int_{\{t|\Gamma_\varepsilon(t)\in\Omega'\}} 2\varepsilon h^{-3}(\Gamma_\varepsilon)|\nabla h(\Gamma_\varepsilon)|^2 \, \mathrm{d}t + \int_{\{t|\Gamma_\varepsilon(t)\notin\Omega'\}} \frac{2\varepsilon}{d_0^3} \, \mathrm{d}t$$
$$\leq \int_0^1 2\varepsilon h^{-3}(\Gamma_\varepsilon)|\nabla h(\Gamma_\varepsilon)|^2 \, \mathrm{d}t + \frac{2\varepsilon}{d_0^3}.$$

Combining this with the previous estimate (3.19) leads to

$$\int_0^1 2\varepsilon h^{-3}(\Gamma_\varepsilon) \, \mathrm{d}t \leq C + \frac{2\varepsilon}{d_0^3}. \tag{3.20}$$

Now, we are able to prove that $\varepsilon\nabla U(\Gamma_\varepsilon)$ is uniformly bounded in L^1. Recall that $\nabla U(\Gamma_\varepsilon) = -2h^{-3}(\Gamma_\varepsilon)\nabla h(\Gamma_\varepsilon)$ and $|\nabla h|_g \leq 1$ to derive from Equation (3.20)

$$\begin{aligned}\int_0^1 \varepsilon \left|\nabla U(\Gamma_\varepsilon)\right|_g \mathrm{d}t &= \int_0^1 2\varepsilon h^{-3}(\Gamma_\varepsilon) \left|\nabla h\left(\Gamma_\varepsilon\right)\right|_g \mathrm{d}t \\ &\leq \int_0^1 2\varepsilon h^{-3}\left(\Gamma_\varepsilon\right) \, \mathrm{d}t < C + \frac{2\varepsilon}{d_0^3}.\end{aligned} \tag{3.21}$$

Γ_ε converges to Γ in $\mathbf{H^1}$. Lemma 3.5 implies that γ_ε is a solution of the Euler–Lagrange equation corresponding to the modified Lagrangian system given in Equation (3.10):

$$\begin{aligned} 0 &= \nabla_{\gamma_\varepsilon'}\gamma_\varepsilon' + Y_{\gamma_\varepsilon}(\gamma_\varepsilon') + \nabla V(\gamma_\varepsilon) + \varepsilon\nabla U(\gamma_\varepsilon) \\ &= \tfrac{1}{\tau_\varepsilon^2}\nabla_{\Gamma_\varepsilon'}\Gamma_\varepsilon' + \tfrac{1}{\tau_\varepsilon} Y_{\Gamma_\varepsilon}(\Gamma_\varepsilon') + \nabla V(\Gamma_\varepsilon) + \varepsilon\nabla U(\Gamma_\varepsilon)\end{aligned} \tag{3.22}$$

Since $Y_{\Gamma_\varepsilon}(\Gamma_\varepsilon')$ and $\nabla V(\Gamma_\varepsilon)$ are uniformly bounded in L^∞ and $\varepsilon\nabla U(\Gamma_\varepsilon)$ is uniformly bounded in L^1, we conclude that $\nabla_{\Gamma_\varepsilon'}\Gamma_\varepsilon'$ is uniformly bounded in L^1, that means Γ_ε is uniformly bounded in $W^{2,1}$. The Sobolev embedding theorem asserts that the embedding

$$W^{2,1}(S^1, \Gamma_\varepsilon^*(\mathrm{T}\overline{\Omega})) \hookrightarrow W^{1,2}(S^1, \Gamma_\varepsilon^*(\mathrm{T}\overline{\Omega})) = H^1(S^1, \Gamma_\varepsilon^*(\mathrm{T}\overline{\Omega})) \tag{3.23}$$

is compact. Therefore, Γ_ε converges in H^1 to $\Gamma \in H^1(S^1, \Gamma_\varepsilon^*(\mathrm{T}\overline{\Omega}))$ as $\varepsilon \to 0$, after choosing a subsequence.

Defining the Borel measure $\boldsymbol{\mu}$. Equation (3.20) shows that the sequence of functions $\widetilde{\mu_\varepsilon} := 2\varepsilon h^{-3}(\Gamma_\varepsilon)$ is uniformly bounded in L^1. Therefore, $\widetilde{\mu_\varepsilon}$ converges in the weak-$*$ topology to a measure $\widetilde{\mu}$, up to subsequence. The Riesz representation theorem (see [Rud87, Chapter 2.2]) shows that $\widetilde{\mu}$ is a finite, positive Borel measure.

Consider $\mathcal{C}' := \{t \in \mathbb{R}/\mathbb{Z} \mid \Gamma(t) \in \partial\Omega\}$. By definition of h (see Equation (3.8)) the functions $\widetilde{\mu_\varepsilon}$ clearly converges uniformly to 0 in a neighborhood of any $t \notin \mathcal{C}'$. Thus, $\mathrm{supp}(\mu)$ is contained in $\mathcal{C}'$. For $t \in \mathcal{C}'$, we know that $\nabla h\left(\Gamma_\varepsilon(t)\right) \longrightarrow -\nu(\Gamma(t))$ as $\varepsilon \to 0$, where ν is the outer normal to $\partial\overline{\Omega}$. Thus, taking the limit $\varepsilon \to 0$ in Equation (3.17) we obtain

$$\tau^{-2}\int_0^1 \langle \Gamma', \nabla_{\Gamma'}\Psi\rangle \mathrm{d}t - \int_0^1 \langle \tau^{-1} \cdot Y_\Gamma(\Gamma') + \nabla V(\Gamma), \Psi\rangle \mathrm{d}t = \int_{\mathcal{C}'} \langle \nu(\Gamma), \Psi\rangle \mathrm{d}\widetilde{\mu} \tag{3.24}$$

for all $\Psi \in H^1(S^1, \Gamma^*(\mathrm{T}\overline{\Omega}))$. Then, we define μ as the pull-back of $\widetilde{\mu}$ by the reparametrisation $\mathbb{R}/\tau\mathbb{Z} \to S^1$, $t \to t/\tau$. In particular, we conclude that $\mathrm{supp}(\mu) \subset \mathcal{C} := \{t \in \mathbb{R}/\tau\mathbb{Z} \mid \gamma(t) \in \partial\Omega\}$.

Euler–Lagrange equation. For $t \notin \mathrm{supp}(\mu)$ we choose $\widehat{\varepsilon} > 0$ such that $[t-\widehat{\varepsilon}, t+\widehat{\varepsilon}] \cap \mathrm{supp}(\mu) = \emptyset$. Equation (3.24) shows that

$$\int_{t-\widehat{\varepsilon}}^{t+\widehat{\varepsilon}} [\langle \gamma', \nabla_{\gamma'}\psi\rangle - \langle Y_\gamma(\gamma') + \nabla V(\gamma), \psi\rangle]\mathrm{d}t = \int_{\mathcal{C}} \langle \nu(\gamma), \psi\rangle \mathrm{d}\mu = 0$$

for all $\psi \in H^1(\mathbb{R}/\tau\mathbb{Z}, \gamma^*\mathrm{T}\overline{\Omega})$ with support in $[t-\widehat{\varepsilon}, t+\widehat{\varepsilon}]$. Note that due to the *Lemma of Du Bois–Raymond* (see [Kli78, Thm.1.3.11]) we know that $\nabla_{\gamma'}\gamma'$ exists. Thus, we obtain

$$\int_{t-\widehat{\varepsilon}}^{t+\widehat{\varepsilon}} \langle Y_\gamma(\gamma') + \nabla V(\gamma), \psi\rangle \mathrm{d}t = \int_{t-\widehat{\varepsilon}}^{t+\widehat{\varepsilon}} \langle \gamma', \nabla_{\gamma'}\psi\rangle \mathrm{d}t = -\int_{t-\widehat{\varepsilon}}^{t+\widehat{\varepsilon}} \langle \nabla_{\gamma'}\gamma', \psi\rangle \mathrm{d}t$$

for all $\psi \in H^1(\mathbb{R}/\tau\mathbb{Z}, \gamma^*\mathrm{T}\overline{\Omega})$ with support in $[t-\widehat{\varepsilon}, t+\widehat{\varepsilon}]$. We conclude that the equation

$$\nabla_{\gamma'}\gamma' = -Y_\gamma(\gamma') - \nabla V(\gamma)$$

holds outside $\mathrm{supp}(\mu)$. A bootstrap argument shows that $\gamma \in H^k$ for all $k \in \mathbb{N}$. By the Sobolev embedding theorem we obtain a compact embedding for a special $m = m(k) \in \mathbb{N}$:

$$H^k = W^{k,2} \hookrightarrow C^m.$$

Therefore, $\gamma \in C^m$ for all $m \in \mathbb{N}$ and for $m(k) \to \infty$ as $k \to \infty$, we get $\gamma \in C^\infty$. Thus, γ is a smooth solution of the Euler–Lagrange equation outside $\mathrm{supp}(\mu)$ with energy

$$E(\gamma) := \lim_{\widehat{\varepsilon}\to 0} E_{\widehat{\varepsilon}}(\gamma(t)) = \tfrac{1}{2}|\gamma'(t)|_g^2 + V(\gamma(t))$$

for any $t \notin \mathrm{supp}(\mu)$.

Law of reflection. In order to establish the law of reflection, we first prove that the equality

$$E(\gamma) = \tfrac{1}{2}|\gamma'(t)|_g^2 + V(\gamma(t)) \tag{3.25}$$

holds actually almost everywhere. For that, we recall the inequality

$$E(\gamma_\varepsilon) := \tfrac{1}{2}|\gamma_\varepsilon'|_g^2 + V(\gamma_\varepsilon) + \varepsilon U(\gamma_\varepsilon) \leq K.$$

This implies that there exists $u \in L^\infty(\mathbb{R}/\tau\mathbb{Z}, \overline{\Omega})$ such that (after choosing a subsequence)

$$\lim_{\varepsilon\to 0} \varepsilon U(\gamma_\varepsilon(t)) = u(t) \;\text{ almost everywhere.}$$

Now, assume that $u(t) \neq 0$ for all $t \in I$ for some set I. Then

$$0 \neq u(t) = \lim_{\varepsilon\to 0} \varepsilon U(\gamma_\varepsilon(t)) = \lim_{\varepsilon\to 0} \frac{\varepsilon}{h^2(\gamma_\varepsilon(t))}$$

implies that $\lim_{\varepsilon\to 0} h(\gamma_\varepsilon(t)) = 0$. The properties of h show that

$$\lim_{\varepsilon\to 0} \nabla h(\gamma_\varepsilon(t)) = 1.$$

Thus, we can conclude from

$$\varepsilon\nabla U\left(\gamma_\varepsilon(t)\right) = -\frac{2\varepsilon}{h^3(\gamma_\varepsilon(t))}\nabla h(\gamma_\varepsilon(t)) = -2\varepsilon U(\gamma_\varepsilon(t))\frac{\nabla h\left(\gamma_\varepsilon(t)\right)}{h(\gamma_\varepsilon(t))},$$

that $\lim_{\varepsilon\to 0} |\varepsilon\nabla U\left(\gamma_\varepsilon(t)\right)| = +\infty$ for all $t \in I$. If we assume that I is a set of positive Lebesgue measure, then the lemma of Fatou shows that

$$+\infty = \int_I \liminf_{\varepsilon\to 0} |\varepsilon\nabla U(\gamma_\varepsilon(t))|\mathrm{d}t \le \liminf_{\varepsilon\to 0} \int_I |\varepsilon\nabla U(\gamma_\varepsilon(t))|\mathrm{d}t, \tag{3.26}$$

which is a contradiction to the uniform boundedness of $\varepsilon\nabla U(\gamma_\varepsilon)$ in L^1 (see Equation (3.20)). Therefore, we proved that

$$\lim_{\varepsilon\to 0} \varepsilon U(\gamma_\varepsilon(t)) = 0 \quad \text{almost everywhere}$$

and thus, the definition of $E(\gamma)$ as $\lim_{\varepsilon\to 0} E_\varepsilon(\gamma(t))$ immediately implies

$$E(\gamma) = \tfrac{1}{2}|\gamma'(t)|_g^2 + V(\gamma(t))$$

almost everywhere. Next, we recall that $\gamma \in H^1(\mathbb{R}/\tau\mathbb{Z}, \overline{\Omega})$. This ensures that γ' is a curve of bounded variation. Thus, left respectively right derivatives, $\gamma'(t^-)$ resp. $\gamma'(t^+)$, as defined in Equation (3.3), exist at every point and are left and right continuous. In particular, we obtain for all $t \in \mathbb{R}/\tau\mathbb{Z}$

$$\tfrac{1}{2}|\gamma'(t^\pm)|_g^2 + V(\gamma(t)) = E. \tag{3.27}$$

Now, we can prove the law of reflection at an isolated point $t \in \operatorname{supp}(\mu)$. For that, we choose $\psi \in H^1(\mathbb{R}/\tau\mathbb{Z}, \gamma^*\mathrm{T}\overline{\Omega})$ with support in the interval $[t-\widehat{\varepsilon}, t+\widehat{\varepsilon}]$, where $\widehat{\varepsilon} > 0$ is chosen such that $[t-\widehat{\varepsilon}, t+\widehat{\varepsilon}] \cap \operatorname{supp}(\mu) = \{t\}$. Then Equation (3.24) reduces to

$$\int\limits_{[t-\widehat{\varepsilon},t+\widehat{\varepsilon}]\setminus\{t\}} \left[\langle\gamma', \nabla_{\gamma'}\psi\rangle - \langle Y_\gamma(\gamma') + \nabla V(\gamma), \psi\rangle\right] \mathrm{d}t = \langle \nu\left(\gamma(t)\right), \psi(t)\rangle\, \mu\left(\{t\}\right).$$

After integration by parts, we obtain

$$\left\langle\gamma'(t^-) - \gamma'(t^+), \psi(t)\right\rangle - \int\limits_{[t-\widehat{\varepsilon},t+\widehat{\varepsilon}]\setminus\{t\}} \left\langle \nabla_{\gamma'}\gamma'(t) + Y_{\gamma(t)}(\gamma'(t)) + \nabla V(\gamma(t)), \psi(t)\right\rangle \mathrm{d}t$$

$$= \langle \nu\left(\gamma(t)\right), \psi(t)\rangle\, \mu\left(\{t\}\right).$$

Above we proved that γ is a solution of the Euler–Lagrange equation corresponding to L for $t \notin \operatorname{supp}(\mu)$, i.e. on $[t-\widehat{\varepsilon}, t+\widehat{\varepsilon}\,]\setminus\{t\}$ we have $\nabla_{\gamma'}\gamma' + Y_\gamma(\gamma') + \nabla V(\gamma) = 0$. Thus, the integrand vanishes and we conclude that

$$\left\langle\gamma'(t^-) - \gamma'(t^+), v\right\rangle = \left\langle\nu\big(\gamma(t)\big), v\right\rangle \mu\left(\{t\}\right) \tag{3.28}$$

for all $v \in H^1(\mathbb{R}/\tau\mathbb{Z}, \gamma^*\mathrm{T}\overline{\Omega})$. By choosing an arbitrary vector v tangent to $\partial\overline{\Omega}$, i.e.

$$\langle \nu(\gamma(t)), v \rangle = 0,$$

Equation (3.28) shows that the components of $\gamma'(t^-)$ and $\gamma'(t^+)$ tangent to $\partial\overline{\Omega}$ are identical. In other words, the orthogonal projection of $\gamma'(t^-)$ and $\gamma'(t^+)$ along $\nu(\gamma(t))$ agree:

$$\gamma'(t^-) - \left\langle \gamma'(t^-), \nu(\gamma(t)) \right\rangle \ \nu(\gamma(t)) = \gamma'(t^+) - \left\langle \gamma'(t^+), \nu(\gamma(t)) \right\rangle \ \nu(\gamma(t)). \qquad (3.29)$$

Equation (3.27) implies that $|\gamma'(t^-)| = |\gamma'(t^+)|$ and we conclude from Equation (3.29) the following equality

$$\left|\left\langle \gamma'(t^+), \nu(\gamma(t)) \right\rangle\right| = \left|\left\langle \gamma'(t^-), \nu(\gamma(t)) \right\rangle\right|.$$

In case $\langle \gamma'(t^+), \nu(\gamma(t)) \rangle = \langle \gamma'(t^-), \nu(\gamma(t)) \rangle$, Equation (3.29) implies $\gamma'(t^+) = \gamma'(t^-)$ and we obtain from (3.28), by choosing $v = \nu(\gamma(t))$, that $\mu(\{t\}) = 0$. This directly contradicts the assumption $t \in \mathrm{supp}(\mu)$ as μ ia a positive Borel measure. Thus, we have to have

$$\left\langle \gamma'(t^+), \nu(\gamma(t)) \right\rangle = -\left\langle \gamma'(t^-), \nu(\gamma(t)) \right\rangle.$$

In this case Equation (3.28) with $v = \mu(\gamma(t))$ implies

$$2\big\langle \gamma'(t^-), \nu\big(\gamma(t)\big) \big\rangle = \big\langle \gamma'(t^-) - \gamma'(t^+), \nu\big(\gamma(t)\big) \big\rangle = \mu(\{t\}) \neq 0.$$

All in all, we obtain

$$\begin{gathered} \left\langle \gamma'(t^+), \nu(\gamma(t)) \right\rangle = -\left\langle \gamma'(t^-), \nu(\gamma(t)) \right\rangle \neq 0 \\ \gamma'(t^+) - \left\langle \gamma'(t^+), \nu(\gamma(t)) \right\rangle \ \nu(\gamma(t)) = \gamma'(t^-) - \left\langle \gamma'(t^-), \nu(\gamma(t)) \right\rangle \ \nu(\gamma(t)), \end{gathered}$$

which is Equation (3.4), as claimed.

To conclude the proof, we observe that, if $\mathcal{B} := \mathrm{supp}(\mu)$ is a finite set, then γ is a periodic magnetic bounce orbit of the Lagrangian system given in Equation (3.1) by the following reason. Since $\gamma \in H^1$, the Sobolev embedding theorem guarantees that this curve is continuous. Moreover, γ is piecewise smooth, namely smooth on $(\mathbb{R}/\tau\mathbb{Z})\backslash\mathcal{B}$, where γ satisfies the Euler–Lagrange equation. For $t \in \mathcal{B}$, left and right derivatives exist and γ satisfies the law of reflection. □

3.2.4 Morse index of the free-time action functional

The following proposition gives an upper bound on the number of bounce points in terms of the Morse index of the free-time action functional $\mathscr{L}^E_\varepsilon$. This proposition is an analogue of [AM11, Prop.2.2]. We include some details of their proof for the reader's convenience.

Proposition 3.7. *In the situation of Proposition 3.6, let $(\Gamma_\varepsilon, \tau_\varepsilon)$ be the subsequence converging to (Γ, τ). Then, the cardinality $|\mathrm{supp}(\mu)|$ (up to taking a subsequence of $(\Gamma_\varepsilon, \tau_\varepsilon)$) is bounded from above by the Morse index μ_{Morse} of the restricted action functional $\mathscr{L}^E_\varepsilon\big|_{H^1\times\{\tau_\varepsilon\}}$ at Γ_ε for $\varepsilon > 0$ sufficiently small, that is,*

$$|\mathrm{supp}(\mu)| \leq \liminf_{\varepsilon\to 0} \mu_{\mathrm{Morse}}\left(\Gamma_\varepsilon; \mathscr{L}^E_\varepsilon\big|_{H^1\times\{\tau_\varepsilon\}}\right).$$

Proof. We keep the notation of Proposition 3.6. According to that the measure μ is the pullback of the measure $\widetilde{\mu}$ on $S^1 = \mathbb{R}/\mathbb{Z}$ by the reparametrisation $\iota\colon \mathbb{R}/\tau\mathbb{Z} \to S^1$, $\iota(t) = t/\tau$. In particular, we have

$$\iota(\operatorname{supp}(\mu)) = \operatorname{supp}(\widetilde{\mu}) \quad \text{und} \quad |\operatorname{supp}(\widetilde{\mu})| = |\operatorname{supp}(\mu)|.$$

We claim that it is sufficient to show that for each point $t \in \operatorname{supp}(\widetilde{\mu})$ and for every sufficiently small $\varepsilon > 0$ there exists a vector field $\Psi_\varepsilon \in H^1(S^1, \Gamma_\varepsilon^*(\mathrm{T}\overline{\Omega}))$ with support in a sufficiently small neighborhood of t satisfying

$$\mathrm{d}^2\mathscr{L}_\varepsilon^E(\Gamma_\varepsilon, \tau_\varepsilon)[(\Psi_\varepsilon, 0), (\Psi_\varepsilon, 0)] < 0. \tag{3.30}$$

Indeed, assume this is shown. Then for sufficiently small $\varepsilon > 0$ and k different points $t_1, \ldots, t_k \in \operatorname{supp}(\widetilde{\mu})$ we are able to find k vector fields $\Psi_{\varepsilon,1}, \ldots, \Psi_{\varepsilon,k}$ such that each vector field is supported in a sufficiently small neighborhood of t_j, $j = 1, \ldots, k$, and satisfies Equation (3.30). In particular, we may assume that the support of the vector fields $\Psi_{\varepsilon,j}$ is pairwise disjoint. Hence the vector fields span a k-dimensional vector subspace of $H^1(S^1, \Gamma_\varepsilon^*(\mathrm{T}M))$ on which the Hessian $\mathrm{d}^2\mathscr{L}_\varepsilon^E|_{H^1\times\{\tau_\varepsilon\}}(\Gamma_\varepsilon)$ of the restricted action functional is negative definite. For $\Psi,\ \Xi \in H^1(S^1, \Gamma_\varepsilon^*(\mathrm{T}\overline{\Omega}))$ we know that

$$\mathrm{d}^2\mathscr{L}_\varepsilon^E|_{H^1\times\{\tau_\varepsilon\}}(\Gamma_\varepsilon)[\Psi, \Xi] = \mathrm{d}^2\mathscr{L}_\varepsilon^E(\Gamma_\varepsilon, \tau_\varepsilon)\big[(\Psi, 0), (\Xi, 0)\big].$$

This then implies

$$\mu_{\text{Morse}}\left(\Gamma_\varepsilon; \mathscr{L}_\varepsilon^E\big|_{H^1\times\{\tau_\varepsilon\}}\right) \geq k$$

for all $\varepsilon > 0$ sufficiently small and in particular proves the proposition under the assumption above which we will verify next.

Computation of the Hessian. Let $t \in \operatorname{supp}(\widetilde{\mu})$ and fix some $\delta > 0$. For $\varepsilon > 0$ and $\delta > \delta' > 0$ we choose a smooth cut off function $f\colon S^1 \to [0,1]$ with $\operatorname{supp}(f) \subset [t-\delta, t+\delta]$ and $f \equiv 1$ on $[t - \delta', t + \delta']$. We define the vector field $\Psi_\varepsilon \in H^1(S^1, \Gamma_\varepsilon^*(\mathrm{T}\overline{\Omega}))$ by

$$\Psi_\varepsilon(s) := -f(s)\nabla h(\Gamma_\varepsilon(s)). \tag{3.31}$$

Using $L_\varepsilon = L - \varepsilon U$, the Hessian of the free-time action functional $\mathscr{L}_\varepsilon^E$ is given by (see [AS09, Proposition 3.1])

$$\begin{aligned}
&\mathrm{d}^2\mathscr{L}_\varepsilon^E(\Gamma_\varepsilon, \tau_\varepsilon)\left[(\Psi_\varepsilon, 0), (\Psi_\varepsilon, 0)\right] \\
&\quad = \tau_\varepsilon \int_0^1 \bigg[\frac{\partial^2 L_\varepsilon}{\partial q^2}(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')[\Psi_\varepsilon, \Psi_\varepsilon] + 2\frac{\partial^2 L_\varepsilon}{\partial q \partial v}(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')\tfrac{1}{\tau_\varepsilon}[\Psi_\varepsilon', \Psi_\varepsilon] \\
&\qquad\qquad + \frac{\partial^2 L_\varepsilon}{\partial v^2}(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')\tfrac{1}{\tau_\varepsilon}[\Psi_\varepsilon', \tfrac{1}{\tau_\varepsilon}\Psi_\varepsilon']\bigg]\, \mathrm{d}t \\
&\quad = \tau_\varepsilon \int_0^1 \frac{\partial^2 L}{\partial q^2}(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')[\Psi_\varepsilon, \Psi_\varepsilon]\, \mathrm{d}t + 2\tau_\varepsilon \int_0^1 \frac{\partial^2 L}{\partial q \partial v}(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')[\tfrac{1}{\tau_\varepsilon}\Psi_\varepsilon', \Psi_\varepsilon]\, \mathrm{d}t \\
&\qquad + \tau_\varepsilon \int_0^1 \frac{\partial^2 L}{\partial v^2}(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')[\tfrac{1}{\tau_\varepsilon}\Psi_\varepsilon', \tfrac{1}{\tau_\varepsilon}\Psi_\varepsilon']\, \mathrm{d}t - \tau_\varepsilon \int_0^1 \varepsilon\frac{\partial^2 U}{\partial q^2}(\Gamma_\varepsilon)[\Psi_\varepsilon, \Psi_\varepsilon]\, \mathrm{d}t.
\end{aligned}$$

We know that

$$\mathrm{d}U(\Gamma_\varepsilon)[\Psi_\varepsilon] = -2h^{-3}(\Gamma_\varepsilon)\mathrm{d}h(\Gamma_\varepsilon)\Psi_\varepsilon = -2h^{-3}(\Gamma_\varepsilon)\langle \nabla h(\Gamma_\varepsilon), \Psi_\varepsilon\rangle$$

and therefore

$$\frac{\partial^2 U}{\partial q^2}(\Gamma_\varepsilon)[\Psi_\varepsilon, \Psi_\varepsilon] = 6h^{-4}(\Gamma_\varepsilon)\langle \nabla h(\Gamma_\varepsilon), \Psi_\varepsilon\rangle^2 - 2h^{-3}(\Gamma_\varepsilon)\ \frac{\partial^2 h}{\partial q^2}(\Gamma_\varepsilon)[\Psi_\varepsilon, \Psi_\varepsilon].$$

These observations allow us to express

$$\mathrm{d}^2\mathscr{L}_\varepsilon^{E_\varepsilon}(\Gamma_\varepsilon, \tau_\varepsilon)\left[(\Psi_\varepsilon, 0), (\Psi_\varepsilon, 0)\right] = A_\varepsilon - B_\varepsilon,$$

where

$$\begin{aligned} A_\varepsilon :=\ & \tau_\varepsilon \int_0^1 \frac{\partial^2 L}{\partial q^2}(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')[\Psi_\varepsilon, \Psi_\varepsilon]\ \mathrm{d}t + 2\tau_\varepsilon \int_0^1 \frac{\partial^2 L}{\partial q \partial v}(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')[\tfrac{1}{\tau_\varepsilon}\Psi_\varepsilon', \Psi_\varepsilon]\ \mathrm{d}t \\ & + \tau_\varepsilon \int_0^1 \frac{\partial^2 L}{\partial v^2}(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')[\tfrac{1}{\tau_\varepsilon}\Psi_\varepsilon', \tfrac{1}{\tau_\varepsilon}\Psi_\varepsilon']\ \mathrm{d}t \\ & + \tau_\varepsilon \int_0^1 2\varepsilon h^{-3}(\Gamma_\varepsilon)\ \frac{\partial^2 h}{\partial q^2}(\Gamma_\varepsilon)[\Psi_\varepsilon, \Psi_\varepsilon]\ \mathrm{d}t \end{aligned}$$

and

$$B_\varepsilon := -\tau_\varepsilon \int_0^1 6\varepsilon h^{-4}(\Gamma_\varepsilon)\langle \nabla h(\Gamma_\varepsilon), \Psi_\varepsilon\rangle^2\ \mathrm{d}t.$$

$\mathbf{A}_\varepsilon$ is bounded. First we recall that the considered Lagrangian L is electromagnetic, see Equation (3.1). Thus, there exists some number $l \geq 0$ (see [Abb13, Eq.(5.2)], [AS09, Eq.(3.2)]) such that

$$\begin{aligned} &\frac{\partial^2 L}{\partial q^2}(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')[\Psi_\varepsilon, \Psi_\varepsilon] \leq l\left(1 + |\tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon'|^2\right)\ |\Psi_\varepsilon|\ |\Psi_\varepsilon|, \\ &\frac{\partial^2 L}{\partial q \partial v}(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')[\Psi_\varepsilon', \Psi_\varepsilon] \leq l(1 + |\tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon'|)\ |\Psi_\varepsilon'|\ |\Psi_\varepsilon|, \\ &\frac{\partial^2 L}{\partial v^2}(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')[\Psi_\varepsilon', \Psi_\varepsilon'] \leq l|\Psi_\varepsilon'|^2. \end{aligned}$$

Furthermore, we know that h is a smooth function on $\overline{\Omega}$, so that we may assume in addition

$$\frac{\partial^2 h}{\partial q^2}(\Gamma_\varepsilon)[\Psi_\varepsilon, \Psi_\varepsilon] \leq l|\Psi_\varepsilon|_g^2.$$

Proposition 3.6 asserts that Γ_ε converges in H^1 up to subsequence. Therefore the vector field Ψ_ε given in Equation (3.31) is uniformly bounded in H^1. Using the properties of h, we see from Equation (3.19) that $2\varepsilon h^{-3}(\Gamma_\varepsilon)$ is bounded in L^1. Therefore $|A_\varepsilon|$ is uniformly bounded in ε.

$\mathbf{B}_\varepsilon$ is unbounded . For B_ε we will actually show that $B_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Proposition 3.6 shows that $\Gamma_\varepsilon \to \Gamma$ converges in H^1 as $\varepsilon \to 0$. The Sobolev embedding theorem implies that Γ_ε also converges in C^0 (see [Ada75, Chapter 5]). Thus, the assumption $t \in \operatorname{supp}(\widetilde{\mu})$, in particular $\Gamma(t) \in \partial\overline{\Omega}$, and $|\nabla h|_g = 1$ on $\partial\overline{\Omega}$ imply that we can find a $\delta'' \in (0, \delta]$ such that

$$|\nabla h(\Gamma_\varepsilon(s))|_g^4 \geq \frac{1}{2}$$

for $s \in [t - \delta'', t + \delta'']$ and $\varepsilon > 0$ sufficiently small. By definition in Equation (3.31) we have $\Psi_\varepsilon(s) = -f(s)\nabla h(\Gamma_\varepsilon(s))$. Since $f \equiv 1$ on $[t - \delta', t + \delta']$ and $\delta'' \in (0, \delta]$, we conclude

$$B_\varepsilon = 6\tau_\varepsilon \varepsilon \int_{t-\delta''}^{t+\delta''} \frac{|\nabla h(\Gamma_\varepsilon)|_g^4}{h^4(\Gamma_\varepsilon)} \mathrm{d}s.$$

Using the assumption $\tau_\varepsilon \geq T_1$ from Proposition 3.6 and Hölder's inequality we estimate for sufficiently small ε

$$\begin{aligned}
B_\varepsilon &\geq 6T_1\varepsilon \int_{t-\delta''}^{t+\delta''} \frac{|\nabla h(\Gamma_\varepsilon)|^4}{h^4(\Gamma_\varepsilon)} \mathrm{d}s \\
&\geq \frac{6}{2} T_1 \varepsilon \int_{t-\delta''}^{t+\delta''} \frac{1}{h^4(\Gamma_\varepsilon)} \mathrm{d}s \\
&\geq \frac{3T_1\varepsilon}{(2\delta'')^{\frac{1}{3}}} \left(\int_{t-\delta''}^{t+\delta''} \frac{1}{h^3(\Gamma_\varepsilon)} \mathrm{d}s \right)^{\frac{4}{3}} \\
&= \frac{3T_1}{(2\varepsilon\delta'')^{\frac{1}{3}}} \Bigg(\underbrace{\int_{t-\delta''}^{t+\delta''} \frac{\varepsilon}{h^3(\Gamma_\varepsilon)} \mathrm{d}s}_{=:B'_\varepsilon} \Bigg)^{\frac{1}{3}}.
\end{aligned}$$

In the proof of Proposition (3.6) we established that the function $2\varepsilon h^{-3}(\Gamma_\varepsilon)$ converges to the measure $\tilde{\mu}$ in the weak-$*$ topology. In particular,

$$\lim_{\varepsilon \to 0} B'_\varepsilon \geq \frac{1}{2}\tilde{\mu}(\{t\}) > 0,$$

where $t \in \operatorname{supp}(\widetilde{\mu})$. This implies that $B_\varepsilon \to +\infty$. □

3.2.5 What happens if the periods tend to zero?

In Section 3.3.2 we use methods from symplectic geometry to find approximate solutions as required by Proposition 3.6, except that we cannot guarantee that the periods τ_ε of these critical points $(\Gamma_\varepsilon, \tau_\varepsilon)$ of $\mathscr{L}_\varepsilon^E$ are bounded away from zero. Therefore we still need to consider the case where the periods τ_ε tend to zero. The following proposition is a magnetic analogue of [AM11, Proposition 2.3]. The arguments are included for the reader's convenience.

Proposition 3.8. *Let $K > 0$ and let $(\Gamma_\varepsilon, \tau_\varepsilon)$ be a sequence of critical points of $\mathscr{L}_\varepsilon^{E_\varepsilon}$, where $E_\varepsilon \leq K$ and $\tau_\varepsilon \to 0$ as $\varepsilon \to 0$. Then, up to a subsequence, Γ_ε converges in $\mathcal{C}^0$ to a constant curve $\Gamma \equiv q \in \overline{\Omega}$ for $\varepsilon \to 0$. Moreover, one of the following holds.*

(i) q is a critical point of V or

(ii) q lies in $\partial\overline{\Omega}$ and there exists $a > 0$ such that $\nabla V(q) = -a\nu(q)$, where ν is the outer normal to $\partial\overline{\Omega}$.

Remark. Case (ii) of Proposition 3.8 can be interpreted in the following way. The stationary curve $\Gamma(t) \equiv q \in \overline{\Omega}$ describes a particle confined by a potential wall, see Figure 3.4.

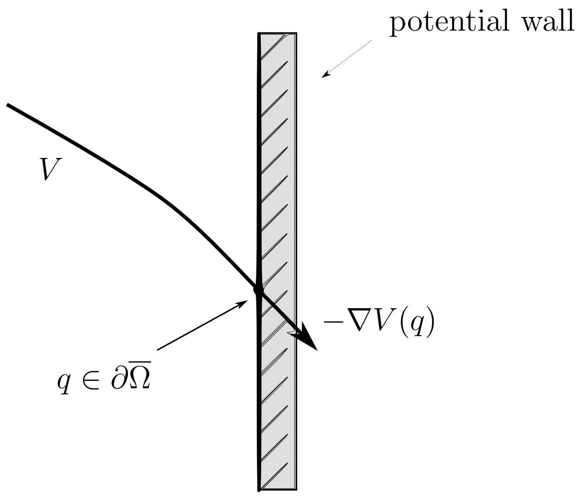

Figure 3.4: A particle q confined by the potential V.

Proof. To begin with, we prove that a sequence of (Γ_ε) converges to a constant curve. For that, we choose a sequence of positive integers (κ_ε) such that $T_1 < \kappa_\varepsilon\tau_\varepsilon < T_2$ for suitable $T_2 > T_1 > 0$ and we define $(\Theta_\varepsilon, \sigma_\varepsilon) \in H^1(S^1; \overline{\Omega}) \times \mathbb{R}_{>0}$ as $\Theta_\varepsilon(t) := \Gamma_\varepsilon(\kappa_\varepsilon t)$ and $\sigma_\varepsilon := \kappa_\varepsilon\tau_\varepsilon$. Then $(\Theta_\varepsilon, \sigma_\varepsilon)$ is also a critical point of the functional $\mathscr{L}_\varepsilon^{E_\varepsilon}$. Now, Proposition 3.6 implies that $(\Theta_\varepsilon, \sigma_\varepsilon)$ converges to (Θ, σ) in $H^1(S^1; \overline{\Omega}) \times \mathbb{R}_{>0}$ as $\varepsilon \to 0$, up to subsequence. In particular, we have $\Theta_\varepsilon \to \Theta$ in C^0. We claim that Θ is a constant curve. To prove it we assume by contradiction that there exist $0 \leq t_1 < t_2 < 1$ such that

$$\Theta(t_1) \neq \Theta(t_2). \tag{3.32}$$

Since every Θ_ε is κ_ε^{-1}-periodic, we know that

$$\Theta_\varepsilon(t_2) = \Theta_\varepsilon(t_2 - j\kappa_\varepsilon^{-1})$$

for all $j \in \mathbb{N}$. As $\kappa_\varepsilon \to \infty$, we find a sequence (j_ε) of positive integers such that

$$j_\varepsilon\kappa_\varepsilon^{-1} \to t_2 - t_1.$$

Together with the C^0 convergence of $\Theta_\varepsilon \to \Theta$, this implies

$$\Theta(t_1) = \lim_{\varepsilon\to 0} \Theta_\varepsilon(t_1) = \lim_{\varepsilon\to 0} \Theta_\varepsilon(t_2 - j_\varepsilon \kappa_\varepsilon^{-1}) = \lim_{\varepsilon\to 0} \Theta_\varepsilon(t_2) = \Theta(t_2).$$

This is a contradiction to our assumption that $\Theta(t_1) \neq \Theta(t_2)$. Thus, Θ is constant. Since every curve Θ_ε is an iteration of Γ_ε, the curve Γ_ε needs to converge in C^0 to the same constant curve $\Gamma = \Theta \equiv q \in \overline{\Omega}$. Therefore, Equation (3.15) in Proposition 3.6 reduces to

$$-\int_0^\sigma \langle \nabla V(q), \psi \rangle \mathrm{d}t = \int_{\mathcal{C}} \langle \nu(q), \psi \rangle \mathrm{d}\mu \tag{3.33}$$

for all $\psi \in C^\infty(\mathbb{R}/\sigma\mathbb{Z}; \gamma^*\mathrm{T}\overline{\Omega})$. If $q \in \Omega$, then $\mathcal{C} = \emptyset$ and so we have

$$-\int_0^\sigma \langle \nabla V(q), \psi \rangle \mathrm{d}t = 0 \tag{3.34}$$

for all ψ. As a result, we see that $\nabla V(q) = 0$, i.e. q is a critical point of V. If $q \in \partial\overline{\Omega}$, then $\mathcal{C} = \mathbb{R}/\sigma\mathbb{Z}$ and there exists an $a > 0$ such that $\nabla V(q) = -a\nu(q)$. □

3.3 Existence of magnetic bounce orbits

In Section 3.2 we explained, that a sequence of approximate solutions converges under suitable assumptions in H^1 to a periodic magnetic bounce orbit. Next, we use methods from symplectic geometry to construct suitable approximate solutions.

3.3.1 Hamiltonian setup

Recall that $(\overline{\Omega}, g = \langle \cdot, \cdot \rangle)$ is a compact Riemannian manifold with boundary and $\Omega \subset \overline{\Omega}$ its interior. In this section, we will consider the electromagnetic Lagrangian system given in (3.1) from a Hamiltonian point of view.

Before that, we recall that for a general Tonelli Lagrangian system L on $\mathrm{T}\overline{\Omega}$, Legendre duality (see e.g. [Maz12, Chapter 1]) provides an associated Hamiltonian system H on $\mathrm{T}^*\overline{\Omega}$ with the following property. There is a one-to-one correspondence between τ-periodic solutions $\gamma\colon \mathbb{R}/\tau\mathbb{Z} \to \overline{\Omega}$ of the Euler–Lagrange equation for L with energy $E(\gamma) = E$ and τ-periodic orbits $v\colon \mathbb{R}/\tau\mathbb{Z} \to \mathrm{T}^*\overline{\Omega}$ of H with $H(v) = E$. This correspondence is via $\pi(v) = \gamma$, where $\pi\colon \mathrm{T}^*\overline{\Omega} \to \overline{\Omega}$ is the canonical projection. Here, a *periodic orbit of H* is shorthand for a periodic orbit of the associated Hamiltonian vector field X_H. Our sign convention is $\omega_0(X_H, \cdot) = -\mathrm{d}H$, and ω_0 is the canonical symplectic form on a cotangent bundle. We point out, that electromagnetic Lagrangians are special cases of Tonelli Lagrangians, see for instance [Abb13, Section 3].

Next, we compute the associated Hamiltonian system in our electromagnetic setting. We denote the induced Riemannian metric on $\mathrm{T}^*\overline{\Omega}$ by g^*.

Lemma 3.9. *Let $L_\varepsilon(q, v) = \frac{1}{2}|v|_g^2 + \alpha_q(v) - V(q) - \varepsilon U(q)$ be the modified Lagrangian, see Equation (3.10). Then the corresponding Hamiltonian $H_\varepsilon\colon \mathrm{T}^*\Omega \to \mathbb{R}$, Legendre dual to L_ε, is given by*

$$H_\varepsilon(q, p) = \tfrac{1}{2}|p - \alpha_q|_{g^*}^2 + V(q) + \varepsilon U(q),$$

where we denote $|\cdot|_{g^*}^2 = g^*(\cdot, \cdot)$.

Proof. By definition of Legendre duality for Tonelli Lagrangian systems, the Hamiltonian H_ε is defined by

$$H_\varepsilon\left(q, \frac{\partial L_\varepsilon}{\partial v}(q,v)\right) = \frac{\partial L_\varepsilon}{\partial v}(q,v)[v] - L_\varepsilon(q,v).$$

In our setting, we calculate

$$\frac{\partial L_\varepsilon}{\partial v}(q,v)[\cdot] = \langle v, \cdot\rangle + \alpha_q(\cdot) = \langle v + \alpha(q), \cdot\rangle.$$

This implies

$$\begin{aligned} H_\varepsilon\big(q, \langle v+\alpha(q),\cdot\rangle\big) &= H_\varepsilon\left(q, \frac{\partial L_\varepsilon}{\partial v}(q,v)\right) \\ &= |v|_g^2 + \alpha_q(v) - \tfrac{1}{2}|v|_g^2 - \alpha_q(v) + V(q) + \varepsilon U(q) \\ &= \tfrac{1}{2}|v|_g^2 + V(q) + \varepsilon U(q) \end{aligned}$$

and therefore

$$H_\varepsilon(q,p) = \tfrac{1}{2}|p - \alpha_q|_{g^*}^2 + V(q) + \varepsilon U(q)$$

is the Legendre dual to L_ε. □

To find approximate solutions, as required by Proposition 3.6, we want to apply a theorem from Reeb dynamics originally due to Felix Schlenk, see Theorem 3.18 below, in the Hamiltonian setting. For this we need to adjust our setting. In particular, we prefer to work on a cotangent bundle T^*M over a closed manifold M.

For that, we enlarge $\overline{\Omega}$ by attaching to $\partial\overline{\Omega}$ a collar neighborhood, see e.g. [Mil65, Corollary 3.5], and let M be the double of this enlarged compact manifold with boundary. Then M is a closed manifold. We still denote by $\overline{\Omega} \subset M$ one of the copies of $\overline{\Omega}$ inside M.

Next, we will modify $H_\varepsilon \colon \mathrm{T}^*\Omega \to \mathbb{R}$ to a Hamiltonian $K_\varepsilon \colon \mathrm{T}^*M \to \mathbb{R}$ without changing the relevant dynamics. By assumption, g, α and V are smooth on $\overline{\Omega}$ and therefore they admit smooth extensions to M, which we will denote by the same letters. To modify εU, we need to fix an energy value $E \in \mathbb{R}$, which we will suppress in the following notation. We note that the energy hypersurface $\{H_\varepsilon = E\}$ projects onto the compact set $\Omega_\varepsilon := \{\varepsilon U \leq E\} \subset \overline{\Omega}$. This allows us to modify the potential εU outside of Ω_ε without changing the energy hypersurface $\{H_\varepsilon = E\}$. For this modification, we choose functions $U_\varepsilon \in C^\infty(M)$, such that $U_\varepsilon = \varepsilon U$ on $\Omega_{\varepsilon/2}$, $U_\varepsilon > E$ outside of $\Omega_{\varepsilon/2}$ and $U_\varepsilon \equiv E' > E$ outside of Ω, see Figure 3.5. We set

$$\begin{aligned} K_\varepsilon \colon \mathrm{T}^*M &\longrightarrow \mathbb{R} \\ (q,p) &\longmapsto K_\varepsilon(q,p) := \tfrac{1}{2}|p-\alpha_q|_{g^*}^2 + V(q) + U_\varepsilon(q). \end{aligned} \tag{3.35}$$

By construction, we see that $\{H_\varepsilon = E\} = \{K_\varepsilon = E\}$. Moreover, the Hamiltonian vector fields X_{H_ε} and X_{K_ε} agree on this energy hypersurface. In particular, periodic orbits of H_ε and K_ε of energy E are the same and correspond to periodic solutions of the Euler–Lagrange equation for L_ε.

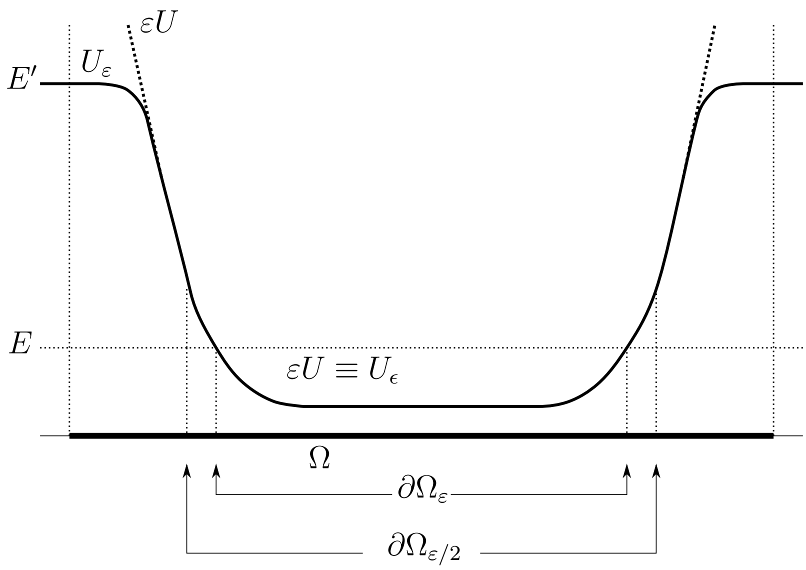

Figure 3.5: The potential εU and its modification U_ε.

We recall our aim of proving existence of periodic solutions of the Euler–Lagrange equation for L_ε of energy E, i.e. of periodic orbits of X_{K_ε} on $\{K_\varepsilon = E\}$. However, it is known, that for electromagnetic Lagrangian systems, in general, not all energy levels actually admit periodic orbits but they do for energies above the strict Mañé critical value, see [Abb13]. In our situation, we slightly modify the definition of the strict Mañé critical value and define

$$c_0 := \inf \left\{ \max_{q \in \overline{\Omega}} K(q, \tau_q) \mid \tau \text{ smooth closed 1-form on } \overline{\Omega} \text{ vanishing near } \partial\overline{\Omega} \right\}$$

with $K(q,p) := \frac{1}{2}|p - \alpha_q|^2_{g^*} + V(q) : \mathrm{T}^*M \to \mathbb{R}$. We will apply Schlenk's theorem, given here as Theorem 3.18, to the Hamiltonian vector field X_{K_ε}, in order to find periodic orbits with energy E. This requires further preparation.

Lemma 3.10. *An energy value $E > \max_{\overline{\Omega}} V$ is a regular value of K_ε for $\varepsilon > 0$ sufficiently small.*

Remark. We point out that $c_0 \geq \max_{\overline{\Omega}} V$ since $K(q, \tau_q) \geq V(q)$ for all q.

Remark. This lemma follows from [AM11, Lemma 3.1], since K_ε is a translation by α of the classical Hamiltonian $G_\varepsilon(q,p) := \frac{1}{2}|p|^2_{g^*} + V(q) + U_\varepsilon(q)$.

Proof. We will show that the map $\mathrm{d}K_\varepsilon(q,p)\colon \mathrm{T}_{(q,p)}\mathrm{T}^*M \to \mathbb{R}$ is surjective for all $(q,p) \in K_\varepsilon^{-1}(E)$. Let $(v,w) \in \mathrm{T}_{(q,p)}\mathrm{T}^*M \cong \mathrm{T}_qM \oplus \mathrm{T}^*_qM$, where the splitting is given by the Levi-Civita connection. To compute the differential of K_ε, we consider $v \in \mathrm{T}_qM$ and $w \in \mathrm{T}^*_qM$ separately. For $v = 0$ we obtain

$$\begin{aligned} \mathrm{d}K_\varepsilon(q,p)[0,w] &= (0,w)[K_\varepsilon(q,p)] \\ &= (0,w)\left[\tfrac{1}{2}|p - \alpha_q|^2_{g^*} + V(q) + U_\varepsilon(q)\right] \\ &= g^*(w, p - \alpha_q) \end{aligned}$$

and for $w = 0$ we have

$$\begin{aligned} \mathrm{d}K_\varepsilon(q,p)[v,0] &= (v,0)[K_\varepsilon(q,p)] \\ &= (v,0)\left[\tfrac{1}{2}|p-\alpha_q|^2_{g^*} + V(q) + U_\varepsilon(q)\right] \\ &= g^*(-\nabla_v\alpha_q, p-\alpha_q) + \mathrm{d}V[v] + \mathrm{d}U_\varepsilon[v]. \end{aligned}$$

Therefore, the differential of K_ε is given by

$$\mathrm{d}K_\varepsilon(q,p)[v,w] = g^*(w - \nabla_v\alpha_q, p-\alpha_q) + \mathrm{d}V[v] + \mathrm{d}U_\varepsilon[v]. \tag{3.36}$$

We know that $\mathrm{d}K_\varepsilon(q,p)$ is not surjective if and only if $\mathrm{d}K_\varepsilon(q,p) = 0$. For that, we consider two cases. First, let $|p-\alpha_q|_{g^*} \neq 0$, then $\mathrm{d}K_\varepsilon(q,p)[0,w] \neq 0$ for at least one $0 \neq w \in \mathrm{T}^*_qM$. Second, let $|p-\alpha_q|_{g^*} = 0$, then $\mathrm{d}K_\varepsilon(q,p)[0,w] = 0$, $g^*(-\nabla_v\alpha_q, p-\alpha_q) = 0$ for any $v \in \mathrm{T}_qM$ and $\mathrm{d}K_\varepsilon(q,p)[v,0] = \mathrm{d}V[v] + \mathrm{d}U_\varepsilon[v]$. Therefore, in this case it remains to prove that $\mathrm{d}V[v] + \varepsilon\mathrm{d}U[v] \neq 0$. Then this lemma is proven.
In order to prove that $\mathrm{d}V[v] + \varepsilon\mathrm{d}U[v] \neq 0$, we show that $|\nabla V(q) + \varepsilon\nabla U(q)|_g$ can be bounded from below. We consider

$$\partial\pi\left(\{|p-\alpha_q|_{g^*} = 0\} \cap \{K_\varepsilon = E\}\right) = \{q \in M \mid V(q) + U_\varepsilon(q) = E\} =: \Upsilon_\varepsilon,$$

where $\pi\colon \mathrm{T}^*M \to M$. For $q \in \Upsilon_\varepsilon$, we have

$$h^2(q) = \frac{\varepsilon}{E - V(q)}$$

and we know by definition of U that

$$|\nabla U(q)|_g = \frac{2}{h^3(q)}|\nabla h(q)|_g.$$

Therefore, we estimate

$$\begin{aligned} |\nabla V(q) + \varepsilon\nabla U(q)|_g &\geq |\varepsilon\nabla U(q)|_g - |\nabla V(q)|_g \\ &\geq \frac{2\varepsilon}{h^3(q)}|\nabla h(q)|_g - |\nabla V(q)|_g \\ &= 2\varepsilon\cdot\varepsilon^{-3/2}(E - V(q))^{3/2}|\nabla h(q)|_g - |\nabla V(q)|_g \\ &\geq 2\varepsilon\cdot\varepsilon^{-3/2}\left(E - \max_M V\right)^{3/2}|\nabla h(q)|_g - |\nabla V(q)|_g. \end{aligned}$$

We claim, that for $q \in \Upsilon_\varepsilon$, we have $|\nabla h|_g = 1$ for $\varepsilon > 0$ sufficiently small. For that, we consider

$$0 \leq h^2(q) = \frac{\varepsilon}{E - V(q)} \leq \frac{\varepsilon}{E - \max_M V}.$$

Thus, the restricted function $h\big|_{\Upsilon_\varepsilon}$ tends uniformly to 0 as $\varepsilon \to 0$. That means, for sufficiently small $\varepsilon > 0$ we have

$$h\big|_{\Upsilon_\varepsilon} = \mathrm{dist}_{\partial\overline{\Omega}}\big|_{\Upsilon_\varepsilon}$$

and $|\nabla h|_g = 1$. As a result, we obtain for all $q \in \Upsilon_\varepsilon$

$$|\nabla V(q) + \varepsilon\nabla U(q)|_g \geq \varepsilon^{-1/2}(E - \max_{\overline{\Omega}} V)^{3/2} - |\nabla V(q)|_g.$$

Finally, we conclude that an energy value $E > \max_{\overline{\Omega}} V$ is a regular value of K_ε for $\varepsilon > 0$ sufficiently small. □

From now on we consider $\varepsilon > 0$ sufficiently small, such that Lemma 3.10 holds. Therefore, the energy hypersurface

$$\Sigma_\varepsilon := \left\{(q,p) \in \mathrm{T}^*M \mid K_\varepsilon(q,p) = \tfrac{1}{2}|p-\alpha_q|^2_{g^*} + V(q) + U_\varepsilon(q) = E\right\}$$

is a nonempty and smooth closed manifold. Next, we compute the Hamiltonian vector field X_{K_ε}, that is needed to apply Theorem 3.18.

Lemma 3.11. *The Hamiltonian vector field X_{K_ε} associated to the Hamiltonian system K_ε is in local coordinates given by*

$$\begin{aligned} X_{K_\varepsilon(q,p)} &= \sum_k \left(g^{ik}(q)(p_i-\alpha_i(q))\right)\frac{\partial}{\partial q_k} \\ &\quad - \sum_{k,i,j}\left(\tfrac{1}{2}\frac{\partial g^{ij}}{\partial q_k}(q)(p_i-\alpha_i(q))(p_j-\alpha_j(q)) - g^{ij}(q)\frac{\partial \alpha_i}{\partial q_k}(q)(p_j-\alpha_j(q))\right. \\ &\qquad\qquad \left. + \frac{\partial V}{\partial q_k}(q) + \frac{\partial U_\varepsilon}{\partial q_k}(q)\right)\frac{\partial}{\partial p_k}. \end{aligned}$$

Our sign convention is $\omega_0(X_{K_\varepsilon},\cdot) = -\mathrm{d}K_\varepsilon$.

Proof. The differential of the Hamiltonian K_ε can be computed in local coordinates:

$$\begin{aligned} \mathrm{d}K_\varepsilon &= \mathrm{d}\left(\tfrac{1}{2}|p-\alpha_q|^2_{g^*} + V(q) + U_\varepsilon(q)\right) \\ &= \mathrm{d}\left(\tfrac{1}{2}\sum_{i,j} g^{ij}(q)(p_i-\alpha_i(q))(p_j-\alpha_j(q))\right) + \mathrm{d}V + \mathrm{d}U_\varepsilon \\ &= \tfrac{1}{2}\sum_{k,i,j}\frac{\partial g^{ij}}{\partial q_k}(q)(p_i-\alpha_i(q))(p_j-\alpha_j(q))\mathrm{d}q_k \\ &\quad - \sum_{k,i,j} g^{ij}(q)\frac{\partial \alpha_i}{\partial q_k}(q)(p_j-\alpha_j(q))\mathrm{d}q_k \\ &\quad + \sum_k \frac{\partial V}{\partial q_k}(q)\mathrm{d}q_k + \sum_k \frac{\partial U_\varepsilon}{\partial q_k}(q)\mathrm{d}q_k \\ &\quad + \sum_{i,j} g^{ij}(q)(p_i-\alpha_i(q))\mathrm{d}p_j. \end{aligned}$$

We write the Hamiltonian vector field X_{K_ε} of K_ε as

$$X_{K_\varepsilon} = \sum_k \left(X_{q_k}\frac{\partial}{\partial q_k} + X_{p_k}\frac{\partial}{\partial p_k}\right).$$

Thus, with our sign convention, we have

$$\mathrm{d}K_\varepsilon = -\omega_0(X_{K_\varepsilon},\cdot) = \sum_k -X_{p_k}\mathrm{d}q_k + X_{q_k}\mathrm{d}p_k.$$

By comparison of coefficients, we derive

$$X_{q_k} = \sum_i g^{ik}(q)(p_i-\alpha_i(q))$$

and

$$X_{p_k} = -\sum_{i,j}\left(\tfrac{1}{2}\frac{\partial g^{ij}}{\partial q_k}(q)(p_i-\alpha_i(q))(p_j-\alpha_j(q)) - g^{ij}(q)\frac{\partial \alpha_i}{\partial q_k}(q)(p_j-\alpha_j(q))\right) \\ - \frac{\partial V}{\partial q_k}(q) - \frac{\partial U_\varepsilon}{\partial q_k}(q).$$

□

3.3.2 Restricted contact type

We recall our goal of proving existence of periodic orbits of the Hamiltonian vector field X_{K_ε} on the energy hypersurface $\Sigma_\varepsilon := \{K_\varepsilon = E\}$. In order to apply Schlenk's theorem, see Theorem 3.18 below, two properties of Σ_ε are required. Namely, the energy hypersurface Σ_ε has to be of restricted contact type and Hamiltonianly displaceable. This section is dedicated to the proof that Σ_ε is of restricted contact type.

Definition 3.12. An energy hypersurface $\Sigma_\varepsilon \subset \mathrm{T}^*M$ is of *restricted contact type*, if there exists a primitive λ_ε of the canonical symplectic form ω_0 on T^*M such that $\lambda_\varepsilon|_{\Sigma_\varepsilon}$ is a contact form.

Proposition 3.13. *Let $\Lambda > 0$ be arbitrarily small and let E_0 be the unique real solution of the equation*

$$E_0 - E_0^{2/3}(6\Lambda + 3c_0)^{1/3} = \max_M V + \tfrac{1}{2}\left(2\sqrt{2(c_0 - \min_M V)} + 2\sqrt{\Lambda}\right)^2. \tag{3.37}$$

Then, for all $E \in \mathbb{R}$, $E > E_0$ and for sufficiently small ε, there exists a 1-form λ_ε on T^*M with $\mathrm{d}\lambda_\varepsilon = \omega_0$ which restricts to a contact form on the energy hypersurface $\Sigma_\varepsilon = \{K_\varepsilon = E\}$. Moreover, we have the estimate

$$\lambda_\varepsilon\left(X_{K_\varepsilon}\right)\big|_{\Sigma_\varepsilon} \geq \Lambda > 0.$$

In particular, the energy hypersurface Σ_ε is of restricted contact type.

Before diving into the long proof of Proposition 3.13, we highlight several crucial observations in the subsequent remarks.

Remark. In (3.51) and (3.52) below, we explain that Equation (3.37) has a unique real solution.

Remark. We point out that the assumption $E > E_0$, where E_0 is the unique real solution of the following equation

$$E_0 - E_0^{2/3}(6\Lambda + 3c_0)^{1/3} = \max_M V + \tfrac{1}{2}\left(2\sqrt{2(c_0 - \min_M V)} + 2\sqrt{\Lambda}\right)^2,$$

guarantees that $E > c_0$ (this is not obvious but needs a calculation) and that

$$E > \max_M V + \tfrac{1}{2}\left(2\sqrt{2(c_0 - \min_M V)} + 2\sqrt{\Lambda}\right)^2,$$

since $E_0^{2/3}(6\Lambda + 3c_0)^{1/3} > 0$ and $\max_M V + \frac{1}{2}\left(2\sqrt{2(c_0 - \min_M V)} + 2\sqrt{\Lambda}\right)^2 > 0$. Thus, in particular we have $E > \max_M V$.

Remark. Proposition 3.13 ensures that we can choose λ_ε such that $\lambda_\varepsilon(X_{K_\varepsilon})$ is bounded away from zero *uniformly* in ε. This property will be needed in (3.54).

Proof. We start by explaining that the energy hypersurface Σ_ε is of restricted contact type. A primitive λ_ε of ω_0 restricts to a contact form on Σ_ε if and only if the associated Liouville vector field P_ε, defined by $\omega_0(P_\varepsilon, \cdot) = \lambda_\varepsilon(\cdot)$, is transverse to Σ_ε. This is equivalent to showing $\lambda_\varepsilon(X_{K_\varepsilon}) \neq 0$, where X_{K_ε} denotes the Hamiltonian vector field associated to the Hamiltonian function K_ε. Indeed,

$$\lambda_\varepsilon(X_{K_\varepsilon}) = \omega_0(P_\varepsilon, X_{K_\varepsilon}) = -\omega_0(X_{K_\varepsilon}, P_\varepsilon) = \mathrm{d}K_\varepsilon(P_\varepsilon)$$

and $\mathrm{d}K_\varepsilon(P_\varepsilon) \neq 0$ is equivalent to P_ε being transverse to Σ_ε. It remains to show that $\lambda_\varepsilon(X_{K_\varepsilon}) \neq 0$ on Σ_ε, such that Σ_ε is of restricted contact type.

In this proof we show a stronger property. We prove that Σ_ε is actually of uniform restricted contact type, independent of ε. In particular, this guarantees that $\lambda_\varepsilon(X_{K_\varepsilon}) > 0$ on Σ_ε and Σ_ε is of restricted contact type.

Since $E > c_0$, there exists a smooth closed 1-form θ such that

$$c_0 \geq \max_{q \in M} \left(\tfrac{1}{2} |\theta_q - \alpha_q|^2_{g^*} + V(q) \right) \geq \tfrac{1}{2} |\theta_q - \alpha_q|^2_{g^*} + V(q). \tag{3.38}$$

Then, we define the 1-form $\lambda := (p - \theta_q)\mathrm{d}q$ and remark that λ is a primitive of ω_0, since θ is closed. The Hamiltonian vector field X_{K_ε}, see Lemma 3.11, is in local coordinates given by

$$\begin{aligned} X_{K_\varepsilon(q,p)} = & \sum_k \left(\sum_{i,k} g^{ik}(p_i - \alpha_i(q)) \right) \frac{\partial}{\partial q_k} \\ & - \sum_{i,j,k} \Bigg(\tfrac{1}{2} \frac{\partial g^{ij}}{\partial q_k}(q)(p_i - \alpha_i(q))(p_j - \alpha_j(q)) - g^{ij}(q) \frac{\partial \alpha_i}{\partial q_k}(q)(p_j - \alpha_j(q)) \\ & \qquad + \frac{\partial V}{\partial q_k}(q) + \frac{\partial U_\varepsilon}{\partial q_k}(q) \Bigg) \frac{\partial}{\partial p_k}. \end{aligned}$$

We compute that

$$\begin{aligned} \lambda(X_{K_\varepsilon}) &= \sum_k (p_k - \theta_k(q)) \mathrm{d}q_k(X_{K_\varepsilon}) \\ &= \sum_{i,k} (p_k - \theta_k(q)) g^{ik}(p_i - \alpha_i(q)) \\ &= g^*(p - \theta_q, p - \alpha_q). \end{aligned}$$

For a vanishing magnetic term, i.e. $\alpha = 0$, we know that $\theta = 0$ and thus

$$\lambda(X_{K_\varepsilon}) = g^*(p - \theta_q, p - \alpha_q) = |p|^2_{g^*} \geq 0.$$

In general, $g^*(p - \theta_q, p - \alpha_q) \geq 0$ is not true.

Now, we perturb the 1-form λ on T^*M by an exact term as follows. Consider the function $u_\varepsilon \colon \mathrm{T}^*M \to \mathbb{R}$ defined by

$$u_\varepsilon(q,p) = (p - \alpha_q)(\nabla U_\varepsilon).$$

For $\varepsilon > 0$, we modify λ to

$$\lambda_\varepsilon := \lambda - C\varepsilon \mathrm{d}u_\varepsilon,$$

where $C > 0$ is a constant independent of ε, that we will fix later in the proof. We remark that λ_ε is still a primitive of ω_0. The gradient ∇U_ε of U_ε is locally expressed by (see Appendix A, in particular (A.3) for further details)

$$(\nabla U_\varepsilon)_i = \sum_j g^{ij} \frac{\partial U_\varepsilon}{\partial q_j},$$

where we write $(g^{ij}) := (g_{ij})^{-1}$. Thus, we have

$$u_\varepsilon(q,p) = (p - \alpha_q)(\nabla U_\varepsilon(q)) = \sum_{i,j} (p_i - \alpha_i(q)) g^{ij}(q) \frac{\partial U_\varepsilon}{\partial q_j}.$$

Locally, the differential of u_ε can be computed as follows:

$$\begin{aligned}
\mathrm{d}u_\varepsilon &= \mathrm{d}\left(\sum_{i,j} g^{ij}(q)(p_i - \alpha_i(q)) \frac{\partial U_\varepsilon}{\partial q_j}(q) \right) \\
&= \sum_k \frac{\partial}{\partial q_k} \left(\sum_{i,j} g^{ij}(q)(p_i - \alpha_i(q)) \frac{\partial U_\varepsilon}{\partial q_j}(q) \right) \mathrm{d}q_k \\
&\quad + \sum_k \frac{\partial}{\partial p_k} \left(\sum_{i,j} g^{ij}(q)(p_i - \alpha_i(q)) \frac{\partial U_\varepsilon}{\partial q_j}(q) \right) \mathrm{d}p_k \\
&= \sum_{i,j,k} \left(\frac{\partial g^{ij}}{\partial q_k}(q)(p_i - \alpha_i(q)) \frac{\partial U_\varepsilon}{\partial q_j}(q) - g^{ij}(q) \frac{\partial \alpha_i}{\partial q_k}(q) \frac{\partial U_\varepsilon}{\partial q_j}(q) \right) \mathrm{d}q_k \\
&\quad + \sum_{i,j,k} \left(g^{ij}(q)(p_i - \alpha_i(q)) \frac{\partial^2 U_\varepsilon}{\partial q_k \partial q_j}(q) \right) \mathrm{d}q_k + \sum_{i,j} g^{ij}(q) \frac{\partial U_\varepsilon}{\partial q_j}(q) \mathrm{d}p_i.
\end{aligned}$$

Thus, for $\varepsilon > 0$, the modified 1-form λ_ε is given by

$$\begin{aligned}
\lambda_\varepsilon = (p - \theta_q)\mathrm{d}q - C\varepsilon \Bigg[&\sum_{i,j,k} \left(\frac{\partial g^{ij}}{\partial q_k}(q)(p_i - \alpha_i(q)) \frac{\partial U_\varepsilon}{\partial q_j}(q) - g^{ij}(q) \frac{\partial \alpha_i}{\partial q_k}(q) \frac{\partial U_\varepsilon}{\partial q_j}(q) \right) \mathrm{d}q_k \\
&+ \sum_{i,j,k} \left(g^{ij}(q)(p_i - \alpha_i(q)) \frac{\partial^2 U_\varepsilon}{\partial q_k \partial q_j}(q) \right) \mathrm{d}q_k + \sum_{i,j} g^{ij}(q) \frac{\partial U_\varepsilon}{\partial q_j}(q) \mathrm{d}p_i \Bigg].
\end{aligned}$$

After inserting the Hamiltonian vector field X_{K_ε} into the modified Liouville form λ_ε, we obtain

$$\begin{aligned}
\lambda_\varepsilon(X_{K_\varepsilon}) &= \lambda(X_{K_\varepsilon}) - C\varepsilon \mathrm{d}u_\varepsilon(X_{K_\varepsilon}) \\
&= g^*(p - \theta_q, p - \alpha_q)
\end{aligned}$$

$$
\begin{aligned}
&- C\varepsilon \sum_{i,j,k,l} \bigg(\frac{\partial g^{ij}}{\partial q_k}(q)(p_i - \alpha_i(q)) \frac{\partial U_\varepsilon}{\partial q_j}(q) - g^{ij}(q) \frac{\partial \alpha_i}{\partial q_k}(q) \frac{\partial U_\varepsilon}{\partial q_j}(q) \\
&\qquad\qquad + g^{ij}(q)(p_i - \alpha_i(q)) \frac{\partial^2 U_\varepsilon}{\partial q_k \partial q_j}(q) \bigg) g^{lk}(q)(p_l - \alpha_l(q)) \\
&+ C\varepsilon \sum_{i,j,l,m} g^{ij}(q) \frac{\partial U_\varepsilon}{\partial q_j}(q) \tfrac{1}{2} \frac{\partial g^{lm}}{\partial q_i}(q)(p_l - \alpha_l(q))(p_m - \alpha_m(q)) \qquad (3.39) \\
&- C\varepsilon \sum_{i,j,l,m} g^{ij}(q) \frac{\partial U_\varepsilon}{\partial q_j}(q) g^{lm}(q) \frac{\partial \alpha_l}{\partial q_i}(q)(p_m - \alpha_m(q)) \\
&+ C\varepsilon \sum_{i,j} g^{ij}(q) \frac{\partial U_\varepsilon}{\partial q_j}(q) \frac{\partial V}{\partial q_i}(q) \\
&+ C\varepsilon \sum_{i,j} g^{ij}(q) \frac{\partial U_\varepsilon}{\partial q_j}(q) \frac{\partial U_\varepsilon}{\partial q_i}(q).
\end{aligned}
$$

By definition of $U = 1/h^2$ and the fact that $U_\varepsilon = \varepsilon U$ on Σ_ε, we compute

$$\nabla U_\varepsilon = -2\varepsilon h^{-3} \nabla h \qquad (3.40)$$

and

$$\frac{\partial^2 U_\varepsilon}{\partial q_k \partial q_j}(q) = -2\varepsilon h^{-3}(q) \frac{\partial^2 h}{\partial q_k \partial q_j}(q) + 6\varepsilon h^{-4}(q) \frac{\partial h}{\partial q_k}(q) \frac{\partial h}{\partial q_j}(q). \qquad (3.41)$$

With Equations (3.40) and (3.41), we have in Equation (3.39)

$$
\begin{aligned}
&\lambda_\varepsilon(X_{K_\varepsilon}) = g^*(p - \theta_q, p - \alpha_q) + \\
&\quad + 2C\varepsilon^2 h^{-3}(q) \sum_{i,j,k,l} \frac{\partial g^{ij}}{\partial q_k}(q)(p_i - \alpha_i(q)) \frac{\partial h}{\partial q_j}(q) g^{lk}(q)(p_l - \alpha_l(q)) \\
&\quad - 2C\varepsilon^2 h^{-3}(q) \sum_{i,j,k,l} \frac{\partial \alpha_i}{\partial q_k}(q) g^{ij}(q) \frac{\partial h}{\partial q_j}(q) g^{lk}(q)(p_l - \alpha_l(q)) \\
&\quad + 2C\varepsilon^2 h^{-3}(q) \sum_{i,j,k,l} g^{ij}(q)(p_i - \alpha_i(q)) \frac{\partial^2 h}{\partial q_k \partial q_j}(q) g^{lk}(q)(p_l - \alpha_l(q)) \\
&\quad - 6C\varepsilon^2 h^{-4}(q) \sum_{i,j,k,l} g^{ij}(q)(p_i - \alpha_i(q)) \frac{\partial h}{\partial q_j}(q) \frac{\partial h}{\partial q_k}(q) g^{lk}(q)(p_l - \alpha_l(q)) \qquad (3.42) \\
&\quad - 2C\varepsilon^2 h^{-3}(q) \sum_{i,j,l,m} \tfrac{1}{2} g^{ij}(q) \frac{\partial h}{\partial q_j}(q) \frac{\partial g^{lm}}{\partial q_i}(q)(p_l - \alpha_l(q))(p_m - \alpha_m(q)) \\
&\quad + 2C\varepsilon^2 h^{-3}(q) \sum_{i,j,l,m} g^{ij}(q) \frac{\partial h}{\partial q_j}(q) g^{lm}(q) \frac{\partial \alpha_l}{\partial q_i}(q)(p_m - \alpha_m(q))
\end{aligned}
$$

$$
\begin{aligned}
&- 2C\varepsilon^2 h^{-3}(q) \sum_{i,j} g^{ij}(q) \frac{\partial h}{\partial q_j}(q) \frac{\partial V}{\partial q_i}(q) \\
&+ 4C\varepsilon^3 h^{-6}(q) \sum_{i,j} g^{ij}(q) \frac{\partial h}{\partial q_j}(q) \frac{\partial h}{\partial q_i}(q).
\end{aligned}
$$

In the next step, we use the local descriptions of the Riemannian metric (see Appendix A, in particular Equations (A.3), (A.4), (A.5), (A.6) for details) and derive further from Equation (3.42):

$$
\begin{aligned}
\lambda_\varepsilon(X_{K_\varepsilon}) = {} & g^*(p - \theta_q, p - \alpha_q) \\
& + 2C\varepsilon^2 h^{-3}(q) \sum_{i,j,k,l} \frac{\partial g^{ij}}{\partial q_k}(q)(p_i - \alpha_i(q)) \frac{\partial h}{\partial q_j}(q)((p - \alpha_q)^\sharp)_k \\
& - 2C\varepsilon^2 h^{-3}(q) \sum_{i,j,k} \frac{\partial \alpha_i}{\partial q_k}(q)(\nabla h)_i ((p - \alpha_q)^\sharp)_k \\
& + 2C\varepsilon^2 h^{-3}(q) \sum_{i,j,k,l} \frac{\partial^2 h}{\partial q_k \partial q_j}(q)((p - \alpha_q)^\sharp)_j ((p - \alpha_q)^\sharp)_k \\
& - 6C\varepsilon^2 h^{-4}(q)\ g((p - \alpha_q)^\sharp, \nabla h)^2 \qquad (3.43) \\
& - 2C\varepsilon^2 h^{-3}(q) \sum_{i,j,l,m} \tfrac{1}{2} (\nabla h)_i \frac{\partial g^{lm}}{\partial q_i}(q)(p_l - \alpha_l(q))(p_m - \alpha_m(q)) \\
& + 2C\varepsilon^2 h^{-3}(q) \sum_{i,j,l,m} (\nabla h)_i \frac{\partial \alpha_l}{\partial q_i}(q)((p - \alpha_q)^\sharp)_l \\
& - 2C\varepsilon^2 h^{-3}(q)\ g(\nabla h, \nabla V) \\
& + 4C\varepsilon^3 h^{-6}(q)\ g(\nabla h, \nabla h).
\end{aligned}
$$

On the energy hypersurface Σ_ε, we know that $K_\varepsilon = E$ and thus, we observe for $(q, p) \in \Sigma_\varepsilon$ that

$$
h^2(q) = \frac{\varepsilon}{E - V(q) - \frac{1}{2}|p - \alpha_q|^2_{g^*}}.
$$

We obtain the following equations:

$$
\begin{aligned}
\varepsilon^2 h^{-3}(q) &= \varepsilon^{1/2} \left(E - V(q) - \tfrac{1}{2}|p - \alpha_q|^2_{g^*}\right)^{3/2} \\
\varepsilon^2 h^{-4}(q) &= \left(E - V(q) - \tfrac{1}{2}|p - \alpha_q|^2_{g^*}\right)^2 \\
\varepsilon^3 h^{-6}(q) &= \left(E - V(q) - \tfrac{1}{2}|p - \alpha_q|^2_{g^*}\right)^3 .
\end{aligned}
$$

Using bilinearity, the Cauchy–Schwarz inequality and the properties of h, we further estimate Equation (3.43) for constants $\kappa_i \in \mathbb{R}_{>0}$, $i = 1, \dots, 3$, that are independent of q and ε (see Appendix A, in particular Equations (A.7), (A.8), (A.9), (A.10) for further details). We write $|\nabla\alpha|_{\max} := \max_{q\in M}\left|\frac{\partial\alpha_i}{\partial q_k}(q)\right|_{g^*}$. Then, we have the estimate

$$\begin{aligned}\lambda_\varepsilon\left(X_{K_\varepsilon}\right)\big|_{\Sigma_\varepsilon} \geq\, & g^*(p-\alpha_q, p-\theta_q) \\ & - 2C\varepsilon^{1/2}\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^{3/2}\kappa_1|p-\alpha_q|^2_{g^*} \\ & - 2C\varepsilon^{1/2}\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^{3/2}|\nabla\alpha|_{\max}|p-\alpha_q|_{g^*} \\ & - 2C\varepsilon^{1/2}\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^{3/2}\kappa_2|p-\alpha_q|^2_{g^*} \\ & - 6C\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^{2}|p-\alpha_q|^2_{g^*} \\ & - C\varepsilon^{1/2}\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^{3/2}\kappa_3|p-\alpha_q|^2_{g^*} \\ & - 2C\varepsilon^{1/2}\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^{3/2}|\nabla\alpha|_{\max}|p-\alpha_q|_{g^*} \\ & - 2C\varepsilon^{1/2}\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^{3/2}|\nabla V|_g \\ & + 4C\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^{3}|\nabla h|^2_g \\ \geq\, & g^*(p-\alpha_q, p-\theta_q) - 6C\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^{2}|p-\alpha_q|^2_{g^*} \\ & + 4C\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^{3}|\nabla h|^2_g - c_\varepsilon,\end{aligned} \tag{3.44}$$

where we define

$$\begin{aligned}c_\varepsilon := 2C\varepsilon^{1/2}\Big(E - \min_M V - \tfrac{1}{2}|p-\alpha_q|^2_g\Big)^{3/2}\big(&\kappa_1|p-\alpha_q|^2_{g^*} + 2|\nabla\alpha|_{\max}|p-\alpha_q|_{g^*} \\ &+ \kappa_2|p-\alpha_q|^2_{g^*} + \kappa_3|p-\alpha_q|^2_{g^*} + |\nabla V|_g\big).\end{aligned}$$

We will show that $c_\varepsilon \to 0$ as $\varepsilon \to 0$. Indeed, on the energy hypersurface Σ_ε it holds

$$\tfrac{1}{2}|p-\alpha_q|^2_{g^*} = E - V(q) - \varepsilon U(q) \leq E - \min_M V,$$

since V is a smooth function defined on the compact manifold M and $\varepsilon U(q) \geq 0$ for all $q \in M$. Thus, we can bound $|p-\alpha_q|_{g^*} \leq \sqrt{2(E - \min_M V)}$. Since α is a 1-form defined on the compact manifold $\overline{\Omega}$, α and all its derivatives are bounded. Therefore, we conclude that $c_\varepsilon \to 0$ as $\varepsilon \to 0$. We proceed with the estimate in (3.44). Using the reversed triangle inequality, we obtain

$$\begin{aligned}g^*(p-\alpha_q, p-\theta_q) &= |p-\alpha_q|_{g^*}|p-\theta_q|_{g^*} = |p-\alpha_q|_{g^*}|p-\alpha_q+\alpha_q-\theta_q|_{g^*} \\ &\geq |p-\alpha_q|_{g^*}\left(|p-\alpha_q|_{g^*} - |\alpha_q-\theta_q|_{g^*}\right) \\ &= |p-\alpha_q|^2_{g^*} - |p-\alpha_q|_{g^*}|\alpha_q-\theta_q|_{g^*}.\end{aligned}$$

Thus, we continue in (3.44) as follows:

$$\begin{aligned}\lambda_\varepsilon\left(X_{K_\varepsilon}\right)\big|_{\Sigma_\varepsilon} \geq\, & |p-\alpha_q|^2_{g^*} - |p-\alpha_q|_{g^*}|\alpha_q-\theta_q|_{g^*} \\ & - 6C\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^{2}|p-\alpha_q|^2_{g^*}\end{aligned}$$

$$
\begin{aligned}
&+ 4C\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^3 |\nabla h|^2_g - c_\varepsilon \\
&= |p-\alpha_q|^2_{g^*}\left(1 - 6C\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^2\right) - |p-\alpha_q|_{g^*}|\alpha_q - \theta_q|_{g^*} \\
&+ 4C\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^3 |\nabla h|^2_g - c_\varepsilon \\
&\geq |p-\alpha_q|_{g^*}\left(\left(1 - 6C\left(E - \min_M V\right)^2\right)|p-\alpha_q|_{g^*} - |\alpha_q-\theta_q|_{g^*}\right) \\
&+ 4C\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^3 |\nabla h|^2_g - c_\varepsilon.
\end{aligned} \tag{3.45}
$$

Set

$$C := \frac{1}{12\left(E - \min_M V\right)^2} > 0.$$

Thus, $C = C(E,V) > 0$ satisfies

$$1 > 6C\left(E - \min_M V\right)^2$$

and we have $1 - 6C\left(E - \min_M V\right)^2 = \frac{1}{2}$. We proceed in our estimate in (3.45):

$$
\begin{aligned}
\lambda_\varepsilon\left(X_{K_\varepsilon}\right)\big|_{\Sigma_\varepsilon} &\geq |p-\alpha_q|_{g^*}\left(\tfrac{1}{2}|p-\alpha_q|_{g^*} - |\alpha_q-\theta_q|_{g^*}\right) \\
&+ \frac{1}{3\left(E-\min_M V\right)^2}\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^3 |\nabla h|^2_g - c_\varepsilon.
\end{aligned} \tag{3.46}
$$

Distinction of cases. We start the distinction of cases with exploring the region

$$\Sigma_{\varepsilon,1} := \Sigma_\varepsilon \cap \left\{|p-\alpha_q|_{g^*} \geq 2|\alpha_q-\theta_q|_{g^*} + 2\sqrt{\Lambda}\right\}.$$

For $(q,p) \in \Sigma_{\varepsilon,1}$ we continue with the estimate in (3.46) and obtain for sufficiently small ε:

$$
\begin{aligned}
\lambda_\varepsilon\left(X_{K_\varepsilon}\right)\big|_{\Sigma_{\varepsilon,1}} &\geq |p-\alpha_q|_{g^*}\underbrace{\left(\tfrac{1}{2}|p-\alpha_q|_{g^*} - |\alpha_q-\theta_q|_{g^*}\right)}_{\geq\sqrt{\Lambda}>0} \\
&\quad + \frac{1}{3\left(E-\min_M V\right)^2}\underbrace{\left(E - V(q) - \tfrac{1}{2}|p-\alpha_q|^2_{g^*}\right)^3}_{=(\varepsilon U(q))^3\geq 0} |\nabla h|^2_g - c_\varepsilon \\
&\geq \left(2|\alpha_q-\theta_q|_{g^*} + 2\sqrt{\Lambda}\right)\sqrt{\Lambda} - c_\varepsilon \\
&\geq 2\sqrt{\Lambda}|\alpha_q-\theta_q|_{g^*} + 2\Lambda - c_\varepsilon \\
&\geq 2\Lambda - c_\varepsilon \\
&\geq \Lambda.
\end{aligned}
$$

Second, we consider the region

$$\Sigma_{\varepsilon,2} := \Sigma_\varepsilon \cap \left\{|p-\alpha_q|_{g^*} \leq 2|\alpha_q-\theta_q|_{g^*} + 2\sqrt{\Lambda}\right\}.$$

Continuing with (3.46), we estimate further. Recall that (3.38) guarantees that $|\alpha_q - \theta_q|_{g^*}$ is uniformly bounded by

$$\tfrac{1}{2}|\alpha_q - \theta_q|^2_{g^*} \leq c_0 - \min_M V.$$

For $\varepsilon > 0$ small enough, we show that the definition of h (see Figure 3.3) and (3.38) imply that $|\nabla h|_g = 1$ in the region $\Sigma_{\varepsilon,2}$. Since by assumption

$$E > \max_M V + \tfrac{1}{2}\left(2\sqrt{2(c_0 - \min_M V)} + 2\sqrt{\Lambda}\right)^2,$$

we have for $(q,p) \in \Sigma_{\varepsilon,2}$

$$\begin{aligned} 0 \leq h^2(q) &= \frac{\varepsilon}{E - V(q) - \frac{1}{2}|p - \alpha_q|^2_{g^*}} \\ &\leq \frac{\varepsilon}{E - V(q) - \frac{1}{2}\left(2|\alpha_q - \theta_q|_{g^*} + 2\sqrt{\Lambda}\right)^2} \\ &\leq \frac{\varepsilon}{E - \max_M V - \frac{1}{2}\left(2\sqrt{2(c_0 - \min_M V} + 2\sqrt{\Lambda}\right)^2}. \end{aligned} \tag{3.47}$$

For $\varepsilon \to 0$ the right hand side in (3.47) tends to zero, because the denominator in (3.47) is independent of ε. Thus, we have for ε sufficiently small

$$0 \leq h^2(q) \leq \frac{\varepsilon}{E - \max_M V - \frac{1}{2}\left(2\sqrt{2(c_0 - \min_M V} + 2\sqrt{\Lambda}\right)^2} < (d_0)^2,$$

where d_0 is given as in Figure 3.3, and for that, we obtain $|\nabla h|_g = 1$ on $\Sigma_{\varepsilon,2}$. We continue with the estimate in (3.46) and compute the minimum of the first summand in (3.46):

$$\begin{aligned} \lambda_\varepsilon\left(X_{K_\varepsilon}\right)\big|_{\Sigma_{\varepsilon,2}} &\geq -\frac{|\alpha_q - \theta_q|^2_{g^*}}{2} + \frac{\left(E - \max_M V - \frac{1}{2}|p - \alpha_q|^2_{g^*}\right)^3}{3\left(E - \min_M V\right)^2} - c_\varepsilon \\ &\geq -\frac{|\alpha_q - \theta_q|^2_{g^*}}{2} + \frac{\left(E - \max_M V - \frac{1}{2}\left(2|\alpha_q - \theta_q|_{g^*} + 2\sqrt{\Lambda}\right)^2\right)^3}{3\left(E - \min_M V\right)^2} - c_\varepsilon. \end{aligned} \tag{3.48}$$

We use again (3.38) and estimate further in (3.48):

$$\begin{aligned} \lambda_\varepsilon\left(X_{K_\varepsilon}\right)\big|_{\Sigma_{\varepsilon,2}} \geq &-\left(c_0 - \min_M V\right) - c_\varepsilon \\ &+ \frac{\left(E - \max_M V - \frac{1}{2}\left(2\sqrt{2(c_0 - \min_M V)} + 2\sqrt{\Lambda}\right)^2\right)^3}{3\left(E - \min_M V\right)^2}. \end{aligned} \tag{3.49}$$

For ease of computation, we set $\min_M V = 0$. We remark that this assumption does not change our considered dynamics on the energy hypersurface Σ_ε. Thus, we obtain from (3.49) the following:

$$\lambda_\varepsilon\left(X_{K_\varepsilon}\right)\Big|_{\Sigma_{\varepsilon,2}} \geq \underbrace{\frac{-3c_0E^2 + \left(E - \max_M V - \frac{1}{2}\left(2\sqrt{2c_0} + 2\sqrt{\Lambda}\right)^2\right)^3}{3E^2}}_{=:A} - c_\varepsilon. \tag{3.50}$$

We point out that the summand A does not have a unique minimum, as a short calculation, that we omit here, shows. For sufficiently small $\varepsilon > 0$, we obtain that

$$\begin{aligned}\lambda_\varepsilon\left(X_{K_\varepsilon}\right)\Big|_{\Sigma_{\varepsilon,2}} &\geq A - c_\varepsilon \\ &\geq \underbrace{\frac{-3c_0E^2 + \left(E - \max_M V - \frac{1}{2}\left(2\sqrt{2c_0} + 2\sqrt{\Lambda}\right)^2\right)^3}{6E^2}}_{=:B}.\end{aligned}$$

In a next step, we show that $B \geq \Lambda$. For that, we consider first that

$$\begin{aligned} 6\Lambda E^2 &= -3c_0E^2 + \left(E - \max_M V - \tfrac{1}{2}\left(2\sqrt{2c_0} + 2\sqrt{\Lambda}\right)^2\right)^3 \\ \Leftrightarrow (6\Lambda + 3c_0)E^2 &= \left(E - \max_M V - \tfrac{1}{2}\left(2\sqrt{2c_0} + 2\sqrt{\Lambda}\right)^2\right)^3 \\ \Leftrightarrow E - E^{2/3}(6\Lambda + 3c_0)^{1/3} &= \underbrace{\max_M V + \tfrac{1}{2}\left(2\sqrt{2c_0} + 2\sqrt{\Lambda}\right)^2}_{=:R}. \end{aligned} \tag{3.51}$$

We set $F := E^{1/3}$ and obtain from (3.51) the following cubic equation

$$F^3 - (6\Lambda + 3c_0)^{1/3}F^2 - R = 0. \tag{3.52}$$

We show that Equation (3.52) has one real solution in the following. For that, we point out that $c_0, \Lambda, R > 0$. Write

$$P(F) := F^3 - (6\Lambda + 3c_0)^{1/3}F^2 - R.$$

The first derivative of P is $P'(F) = 3F^2 - 2(6\Lambda + 3c_0)^{1/3}F = F(3F - 2(6\Lambda + 3c_0)^{1/3})$ and the second derivative is $P''(F) = 6F - 2(6\Lambda + 3c_0)^{1/3}$. Then, critical points of P are $F_0 = 0$ and $F_1 = \frac{2}{3}(6\Lambda + 3c_0)^{1/3}$. Thus, we check for local extrema and obtain

$$P''(F_0) = -2(6\Lambda + 3c_0)^{1/3} < 0$$

and

$$P''(F_1) = 2(6\Lambda + 3c_0)^{1/3} > 0.$$

Therefore, we have the following schematic picture for the function P, see Figure 3.6. We obtain that

$$P(F_1) = -\tfrac{2}{9}(6\Lambda + 3c_0) - R < -R = P(0) < 0.$$

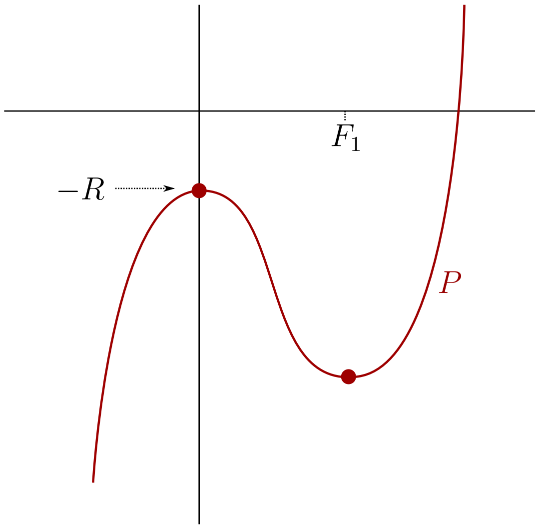

Figure 3.6: Schematic plot of P.

Thus, we have exactly *one real solution* of Equation (3.52). Therefore, there exists exactly one energy value $E_0 \in \mathbb{R}$, that is uniquely defined by the following equation

$$E_0 - E_0^{2/3}(6\Lambda + 3c_0)^{1/3} = R. \tag{3.53}$$

Therefore, we know that

$$B \geq \frac{-3c_0E_0^2 + \left(E_0 - \max_M V - \frac{1}{2}\left(2\sqrt{2c_0} + 2\sqrt{\Lambda}\right)^2\right)^3}{6E_0^2} = \Lambda.$$

Since $c_\varepsilon \to 0$ as $\varepsilon \to 0$, the above considerations in the two regions $\Sigma_{\varepsilon,1}$ and $\Sigma_{\varepsilon,2}$ show that we have for sufficiently small ε (keep in mind that $\min_M V = 0$):

$$\lambda_\varepsilon\left(X_{K_\varepsilon}\right)\big|_{\Sigma_\varepsilon} \geq \Lambda > 0.$$

□

Associated to the contact form λ_ε on Σ_ε, there is a preferred vector field R_ε, called the *Reeb vector field*. It is uniquely defined by the conditions $\mathrm{d}\lambda_\varepsilon\big|_{\Sigma_\varepsilon}(R_\varepsilon, \cdot) = 0$ and $\lambda_\varepsilon\big|_{\Sigma_\varepsilon}(R_\varepsilon) = 1$. Let P_ε be the Liouville vector field which is transverse to Σ_ε and let $\lambda_\varepsilon\big|_{\Sigma_\varepsilon}$ be the induced contact form. Then we see

$$\mathrm{d}\lambda_\varepsilon\big|_{\Sigma_\varepsilon}(X_{K_\varepsilon}, \cdot) = \omega_0\big|_{\Sigma_\varepsilon}(X_{K_\varepsilon}, \cdot) = -\mathrm{d}K_\varepsilon(\cdot)\big|_{\Sigma_\varepsilon} = 0$$

and

$$\lambda_\varepsilon\big|_{\Sigma_\varepsilon}(X_{K_\varepsilon}) = \omega_0(P_\varepsilon, X_{K_\varepsilon}) = -\omega_0(X_{K_\varepsilon}, P_\varepsilon) = \mathrm{d}K_\varepsilon(P_\varepsilon) \neq 0.$$

Thus, the Hamiltonian vector field X_{K_ε} is a non-vanishing multiple of the Reeb vector field R_ε, i.e. the Hamiltonian vector field X_{K_ε} and R_ε differ only by reparametrisation.

In particular, their orbits also coincide up to reparametrisation. Hence, there exists a function $r_\varepsilon\colon \Sigma_\varepsilon \to \mathbb{R}_{>0}$ such that $X_{K_\varepsilon} = r_\varepsilon R_\varepsilon$ and

$$\lambda_\varepsilon\big|_{\Sigma_\varepsilon}(X_{K_\varepsilon}) = r_\varepsilon \cdot \underbrace{\lambda_\varepsilon\big|_{\Sigma_\varepsilon}(R_\varepsilon)}_{=1}.$$

Moreover, Propositon 3.13 ensures that

$$r_\varepsilon = \lambda_\varepsilon\big|_{\Sigma_\varepsilon}(X_{K_\varepsilon}) \geq \Lambda > 0. \tag{3.54}$$

In (3.54), it is necessary that $\lambda_\varepsilon(X_{K_\varepsilon})\big|_{\Sigma_\varepsilon} > 0$ is *uniformly* positive, because the Reeb vector field R_ε is only proportional (depending on ε) to the Hamiltonian vector field X_{K_ε}. Let v be a Reeb orbit of period T and γ_ε a Hamiltonian orbit of period τ_ε. Then, we have

$$\tau_\varepsilon \Lambda \leq T.$$

In general, the periods τ_ε of a Hamiltonian orbit of X_{K_ε} are not bounded away from zero. For that, we considered the case where the periods τ_ε of the critical points of the action functional $\mathscr{L}_\varepsilon^E$ tend to zero separately in Section 3.2.5.

3.3.3 Hamiltonian Displaceability

The remaining ingredient to apply Schlenk's theorem, stated below as Theorem 3.18, is Hamiltonian displaceability. For that, we start by explaining this concept.

Definition 3.14. The energy hypersurface Σ_ε is called *Hamiltonianly displaceable*, if there exists a Hamiltonian diffeomorphism $\Phi_G^t \in \mathrm{Ham}_c(\mathrm{T}^*M)$ generated by a time-dependent Hamiltonian function $G\colon [0,1] \times \mathrm{T}^*M \to \mathbb{R}$ with compact support, such that

$$\phi_G^1(\Sigma_\varepsilon) \cap \Sigma_\varepsilon = \emptyset.$$

Then the *displacement energy* $e(\Sigma_\varepsilon)$ of Σ_ε is given by

$$e(\Sigma_\varepsilon) := \inf\left\{ \int_0^1 \left[\max_{\mathrm{T}^*M} G(t,\cdot) - \min_{\mathrm{T}^*M} G(t,\cdot)\right] \mathrm{d}t \;\middle|\; \phi_G^1(\Sigma_\varepsilon) \cap \Sigma_\varepsilon = \emptyset \right\}.$$

More information on the displacement energy and its definition can be found in [Pol01]. In this section, we derive an upper bound on the displacement energy. For that, we need further preparation. We start by considering a lemma from Morse theory.

Lemma 3.15 (Contreras)**.** *Let M be a closed Riemannian manifold. Given an open non-empty subset $W \subset M$, there exists a smooth function $f\colon M \to \mathbb{R}$ whose critical points are all in the open set W.*

An appropriate scaling of the function f and its differential $\mathrm{d}f$ imply an immediate corollary of Lemma 3.15.

Corollary 3.16. *Let M be a closed Riemannian manifold. For all non-empty open subsets $W \subset M$ there exists a function $f\colon M \to \mathbb{R}_{\geq 0}$ with $\mathrm{Crit}(f) \subset W$ and $|\mathrm{d}_q f|_{g^*} \geq 1$ for all $q \notin W$.*

Now, we prove that Σ_ε is Hamiltonianly displaceable. The proof contains ideas from [Con06, Proposition 8.2].

Proposition 3.17. *Let M be a closed Riemannian manifold. Then, the energy hypersurface Σ_ε is Hamiltonianly displaceable with displacement energy*

$$e(\Sigma_\varepsilon) \leq 2\Big(2(E - \min_M V)\Big)^{1/2}\Big(\max_{\overline{\Omega}} f - \min_{\overline{\Omega}} f\Big),$$

where $f\colon M \to \mathbb{R}_{\geq 0}$ is a smooth function with $\mathrm{Crit}(f) \subset M\backslash\overline{\Omega}$ and $|\mathrm{d}_q f|_{g^*} \geq 1$ for all $q \in \overline{\Omega}$.

Proof. Consider the non-empty open subset $W := M\backslash\overline{\Omega}$ of M. Thus, by Corollary 3.16 there exists a function $f\colon M \to \mathbb{R}_{\geq 0}$ with $\mathrm{Crit}(f) \subset W$ and $|\mathrm{d}_q f|_{g^*} \geq 1$ for all $q \in \overline{\Omega}$. Define the compact set $\mathcal{A}$ in T^*M by

$$\mathcal{A} := \left\{(q,p) \in \mathrm{T}^*M \;\middle|\; q \in \overline{\Omega}, |p - \alpha_q|_{g^*} \leq \Big(2(E - \min_M V)\Big)^{1/2}\right\}.$$

We claim that $\Sigma_\varepsilon \subset \mathcal{A}$ and that $\mathcal{A}$ is displaceable. When the claim is shown, we are able to conclude that

$$e(\Sigma_\varepsilon) \leq e(\mathcal{A}) < \infty.$$

Since $U_\varepsilon(q) \geq 0$ for all $q \in M$, we have

$$\Sigma_\varepsilon \subset \{(q,p) \in \mathrm{T}^*M \mid q \in \overline{\Omega}, \tfrac{1}{2}|p - \alpha_q|^2_{g^*} + V(q) \leq E\} = \mathcal{A}.$$

$\mathcal{A}$ is displaceable. Consider the Hamiltonian function

$$\begin{aligned} G\colon \mathrm{T}^*M &\longrightarrow \mathbb{R} \\ (q,p) &\longmapsto -f(q). \end{aligned}$$

Its Hamiltonian equations are

$$\begin{cases} \dot{q} &= \quad 0 \\ \dot{p} &= \mathrm{d}_q f \end{cases}$$

and its Hamiltonian diffeomorphism is given by

$$\phi^t_G(q,p) = (q, p + t\mathrm{d}_q f).$$

Next, we determine a time $T > 0$, such that

$$\phi^T_G(\mathcal{A}) \cap \mathcal{A} = \emptyset.$$

We claim that for time

$$T > 2\Big(2(E - \min_M V)\Big)^{1/2},$$

we have $\phi_G^T(\mathcal{A}) \notin \mathcal{A}$. In order to prove this claim, we assume by contradiction the following:

$$\phi_G^T(q,p) = (q, p + T\mathrm{d}_q f) \in \mathcal{A}$$

for $(q,p) \in \mathcal{A}$. Thus, we have

$$|p + T\mathrm{d}_q f - \alpha_q|_{g^*} \leq \left(2(E - \min_M V)\right)^{1/2}. \tag{3.55}$$

Applying the reversed triangle inequality and Corollary 3.16, we obtain for $(q,p) \in \mathcal{A}$:

$$\begin{aligned}|p + T\mathrm{d}_q f - \alpha_q|_{g^*} = |(p - \alpha_q) + T\mathrm{d}_q f|_{g^*} &\geq T|\mathrm{d}_q f|_{g^*} - |p - \alpha_q|_{g^*} \\ &\geq T - \left(2(E - \min_M V)\right)^{1/2} \\ &> 2\left(2(E - \min_M V)\right)^{1/2} - \left(2(E - \min_M V)\right)^{1/2} \\ &= \left(2(E - \min_M V)\right)^{1/2}.\end{aligned}$$

This is a contradiction to (3.55). Therefore, we have

$$\phi_G^T(\mathcal{A}) \cap \mathcal{A} = \emptyset$$

for time $T > 2\left(2(E - \min_M V)\right)^{1/2} > 0$.

Displacement energy $\mathbf{e(\Sigma_\varepsilon)}$. Consider the compact set $K := \bigcup_{t \in [0,T]} \phi_G^t(\mathcal{A})$. Let $\lambda \colon \mathrm{T}^*M \to \mathbb{R}_{\geq 0}$ be a smooth function such that $\lambda\big|_K \equiv T$ and $\lambda = 0$ outside of K. We define a Cut-Off function F of the Hamiltonian G outside a small neighborhood of K as follows:

$$F := \lambda G \colon \mathrm{T}^*M \longrightarrow \mathbb{R}.$$

Then, F has compact support and its Hamiltonian flow satisfies

$$\phi_F^s(q,p) = \phi_G^{sT}(q,p),$$

when $(q,p) \in \mathcal{A}$ and $s \in [0,1]$. Thus, we have

$$\phi_F^1(\mathcal{A}) \cap \mathcal{A} = \emptyset,$$

and hence $\mathcal{A}$ is Hamiltonianly displaceable with displacement energy

$$e(\mathcal{A}) \leq \int_0^1 \left[\max_{\mathrm{T}^*M} F - \min_{\mathrm{T}^*M} F\right] \mathrm{d}t < \infty. \tag{3.56}$$

Therefore, we obtain for the displacement energy $e(\Sigma_\varepsilon)$ of the energy hypersurface Σ_ε:

$$\begin{aligned}e(\Sigma_\varepsilon) &\leq e(\mathcal{A}) \\ &\leq \int_0^1 \left[\max_{\mathrm{T}^*M} F - \min_{\mathrm{T}^*M} F\right] \mathrm{d}t\end{aligned}$$

$$\leq \int_0^1 \left[\max_K \lambda G - \min_K \lambda G\right] \mathrm{d}t$$

$$= T\left(\max_K G - \min_K G\right)$$

$$= T\left(\max_{\overline{\Omega}} f - \min_{\overline{\Omega}} f\right),$$

where the last equality holds, since $\pi(K) \subset \overline{\Omega}$.

All in all, we estimate the displacement energy $e(\Sigma_\varepsilon)$ of the energy hypersurface Σ_ε as follows:

$$e(\Sigma_\varepsilon) \leq 2\left(2(E - \min_M V)\right)^{1/2}\left(\max_{\overline{\Omega}} f - \min_{\overline{\Omega}} f\right).$$

□

The following theorem was proved by F. Schlenk in [Sch06] and can also be found in [CFP10, Theorem 4.9].

Theorem 3.18 (Schlenk). *Let Σ be a displaceable, contact type hypersurface in a symplectically aspherical, geometrically bounded, symplectic manifold (V, ω). Then Σ carries a Reeb orbit $v\colon \mathbb{R}/T\mathbb{Z} \to \Sigma$ with period T bounded by the displacement energy $e(\Sigma)$ of Σ, i.e.*

$$T \leq e(\Sigma).$$

We already know that

$$\tau_\varepsilon \Lambda \leq T.$$

Together with Theorem 3.18, this immediately implies the following lemma.

Lemma 3.19. *Let M be a closed Riemannian manifold. Let $f\colon M \to \mathbb{R}_{\geq 0}$ be a smooth function with $\mathrm{Crit}(f) \subset M\backslash\overline{\Omega}$ and $|\mathrm{d}_q f|_{g^*} \geq 1$ for all $q \in \overline{\Omega}$. Then, the Hamiltonian vector field X_{K_ε} associated to the Hamiltonian K_ε on Σ_ε has a periodic orbit of period τ_ε satisfying*

$$\tau_\varepsilon \Lambda \leq e(\Sigma_\varepsilon) \leq 2\left(2E - \min_M V\right)^{1/2}\left(\max_{\overline{\Omega}} f - \min_{\overline{\Omega}} f\right)$$

and thus

$$\tau_\varepsilon \leq \frac{e(\Sigma_\varepsilon)}{\Lambda} \leq \frac{2\left(2E - \min_M V\right)^{1/2}\left(\max_{\overline{\Omega}} f - \min_{\overline{\Omega}} f\right)}{\Lambda}.$$

Therefore, we proved the existence of a periodic orbit of X_{K_ε} on $\Sigma_\varepsilon = \{K_\varepsilon = E\}$ of period τ_ε. Hence, the Euler–Lagrange equation corresponding to the modified Lagrangian L_ε, see (3.10), has a solution γ_ε of energy $E_\varepsilon(\gamma_\varepsilon) = E$ and period τ_ε.

To achieve an upper bound on the number of bounce times, it remains to bound the Morse index μ_{Morse} of γ_ε. This works analogously to the proof in [AM11, Proposition 3.7]. We outline the result in the following proposition.

Proposition 3.20. *Let M be a closed Riemannian manifold. Let $f\colon M \to \mathbb{R}_{\geq 0}$ be a smooth function with* $\mathrm{Crit}(f) \subset M \backslash \overline{\Omega}$ *and* $|\mathrm{d}_q f|_{g^*} \geq 1$ *for all* $q \in \overline{\Omega}$*. Let* $\Lambda > 0$ *be arbitrarily small and let* E_0 *be the unique solution of*

$$E_0 - 3c_0 E_0^{2/3} = \max_M V + \tfrac{1}{2}\left(2\sqrt{2(c_0 - \min_M V)} + 2\sqrt{\Lambda}\right)^2. \tag{3.57}$$

Then, for all $E \in \mathbb{R}$, $E > E_0$ *and any* $\varepsilon > 0$, *there exists a critical point* $(\Gamma_\varepsilon, \tau_\varepsilon)$ *of the free-time action functional* $\mathscr{L}_\varepsilon^E$ *with*

$$\tau_\varepsilon \leq \frac{e(\Sigma_\varepsilon)}{\Lambda} \leq \frac{2\left(2E - \min_M V\right)^{1/2}\left(\max_{\overline{\Omega}} f - \min_{\overline{\Omega}} f\right)}{\Lambda},$$

$$E_\varepsilon\left(\Gamma_\varepsilon\left(t/\tau_\varepsilon\right)\right) = E,$$

$$\mu_{\mathrm{Morse}}\left(\Gamma_\varepsilon; \mathscr{L}_\varepsilon^E\big|_{H^1 \times \{\tau_\varepsilon\}}\right) \leq n + 1.$$

Proof. ([AM11, Proposition 3.7]) For the reader's convenience, we present the proof in the present setting.

We first show that the Morse index μ_{Morse} of γ_ε is bounded by $n+1$. Legendre duality between the Lagrangian L_ε and the Hamiltonian K_ε guarantees that the Morse index μ_{Morse} and the Conley-Zehnder index μ_{CZ} agree. Indeed, if the critical point $(\Gamma, \tau) \in \mathrm{Crit}\left(\mathscr{L}_\varepsilon^E\right)$ and the Reeb orbit v correspond to each other, then

$$\mu_{\mathrm{Morse}}\left(\Gamma; \mathscr{L}_\varepsilon^E\big|_{H^1 \times \{\tau\}}\right) = \mu_{\mathrm{CZ}}(v).$$

Proofs can be found in [Vit90] by C. Viterbo who extends a result by Duistermat [Dui76] (see also [Abb03] for a functional analytic proof).

Let us first assume that the functional $\mathscr{L}_\varepsilon^E$ is Morse-Bott. By Legendre duality, this translates to the Hamiltonian formulation that Σ_ε is non-degenerate, i.e. all Reeb orbits are isolated and non-degenerate, that is, the linearized Poincaré return map along any Reeb orbit has only one eigenvalue equal to 1. Due to the autonomous character of the Reeb flow this one exists necessarily. By the proof of Theorem 4.9 in [CFP10], the Conley-Zehnder index of the Reeb orbit v_ε satisfies

$$\mu_{\mathrm{CZ}}(v_\varepsilon) \in \{n, n+1\}.$$

The proof of [CFP10, Theorem 4.9] is based on a homotopy stretching argument for a time dependent perturbation of the Rabinowitz action functional, where the perturbation is given by the Hamiltonian that displaces, see [CFP10, Proof of Theorem 4.9]. Then, it turns out that the perturbed Rabinowith action functional does not have any critical points anymore. Assuming that Σ_ε is non-degenerate, the proof, see [CFP10], shows that a gradient flow line (in the sense of Floer) of the Rabinowitz action functional connecting the orbit v_ε and a maximum of an auxiliary Morse function on Σ_ε has to exist. Using the index formula [CF09, Proposition 4.1] and the μ-grading for Morse-Bott homology [CF09, Appendix A] this translates to "index difference equals 1 between two critical points on a connecting gradient flow line":

$$1 = \mu(v_\varepsilon) - \mu(\max)$$

$$
\begin{aligned}
&= \mu_{\mathrm{CZ}}(v_\varepsilon) + \mathrm{ind}_f^{\mathrm{Morse}}(v_\varepsilon) + \tfrac{1}{2}\left(\mathrm{dim}\Sigma_\varepsilon - \mathrm{dim}(C_{v_\varepsilon})\right) \\
&\quad - \mu_{\mathrm{CZ}}(\max) - \mathrm{ind}_f^{\mathrm{Morse}}(\max) - \tfrac{1}{2}\left(\mathrm{dim}\Sigma_\varepsilon - \mathrm{dim}(C_{\max})\right) \\
&= \mu_{\mathrm{CZ}}(v_\varepsilon) + \eta(v_\varepsilon) - \tfrac{1}{2} - \mu_{\mathrm{CZ}}(\max) - \tfrac{1}{2}(2n-1),
\end{aligned}
$$

where f is a Morse function, C_{v_ε}, $C_{\max}$ are critical manifolds and $\eta(v_\varepsilon) \in \{0,1\}$. The summand $\eta(v_\varepsilon)$ is there due to the fact that a critical point on the critical manifold represented by the periodic orbit v_ε has Morse index 0 or 1.
Therefore, we conclude

$$\mu_{\mathrm{CZ}}(v_\varepsilon) \in \{n, n+1\}.$$

Thus, $\gamma_\varepsilon = \pi(v_\varepsilon)$ has Morse index n or $n+1$ under the assumption that $\mathscr{L}_\varepsilon^E$ is Morse-Bott.

If the action functional $\mathscr{L}_\varepsilon^E$ is degenerate, we choose a sequence of compactly supported perturbations $f_n \colon \mathrm{T}^*\overline{\Omega} \to \mathbb{R}$ with $f_n \to 0$ in $\mathcal{C}^\infty$, such that the action functional $\mathscr{L}_\varepsilon^{E,f_n}$ corresponding to the Lagrangian $L_\varepsilon + f_n + E$ is Morse-Bott. By our previous discussion, we find a sequence (v_ε^n) of critical points of $\mathscr{L}_\varepsilon^{E,f_n}$, such that all orbits v_ε^n have period uniformly bounded from above by $e(\Sigma_\varepsilon) + \delta$ for some small $\delta > 0$, energy E and Morse index n or $n+1$. Since $f_n \to 0$ in C^∞ and the period of (v_ε^n) is uniformly bounded, see Lemma 3.19, a subsequence of (v_ε^n) converges. Thus, we obtain a critical point $\gamma_\varepsilon \colon \mathbb{R}/\tau_\varepsilon\mathbb{Z} \to \Omega$ of $\mathscr{L}_\varepsilon^E$ with

$$\Lambda\tau_\varepsilon \le e(\Sigma_\varepsilon) + \delta, \quad E_\varepsilon(\gamma_\varepsilon) = E, \quad \mu_{\mathrm{Morse}}(\gamma_\varepsilon) \le n+1.$$

Moreover, we can choose δ as small as we like and thus we conclude

$$\Lambda\tau_\varepsilon \le e(\Sigma_\varepsilon).$$

□

3.4 Proofs

Finally, we have collected all ingredients to prove Theorem 3.3. The proof is a magnetic analogue to [AM11, p.17-18].

Proof of Theorem 3.3. Fix an energy value E with the properties given in Proposition 3.13 and consider the sequence $(\Gamma_\varepsilon, \tau_\varepsilon)$ given in Proposition 3.20. Proposition 3.8 implies that (τ_ε) is uniformly bounded from below by some constant $T_1 > 0$, as follows. By contradiction, we assume that up to taking a subsequence $\tau_\varepsilon \to 0$ for $\varepsilon \to 0$. Up to further subsequence, Proposition 3.8 shows that Γ_ε converges uniformly to a constant curve $\Gamma \equiv q \in \overline{\Omega}$ with $E(\Gamma) = V(\Gamma) = E$. Then q is either (see Propositon 3.8 (i)) a critical point of V or (see Proposition 3.8 (ii)) there exists $a > 0$ such that $\nabla V(q) = -a\nu(q)$, where ν is the outer normal to $\partial\overline{\Omega}$. This contradicts $E > \max_{\overline{\Omega}} V$.

Thus, we obtain

$$0 < T_1 \le \tau_\varepsilon \le T_2 := \frac{e(\Sigma_\varepsilon)}{\Lambda} \le \frac{2\left(2E - \min_M V\right)^{1/2}\left(\max_{\overline{\Omega}} f - \min_{\overline{\Omega}} f\right)}{\Lambda}. \tag{3.58}$$

Proposition 3.6 implies that $(\Gamma_\varepsilon, \tau_\varepsilon) \longrightarrow (\Gamma, \tau)$ in $H^1(S^1, \Gamma^*(\mathrm{T}\overline{\Omega})) \times \mathbb{R}_{>0}$ as $\varepsilon \to 0$, where $T_1 \leq \tau \leq T_2$. Let μ be the Borel measure given in Proposition 3.6. Proposition 3.7 together with Proposition 3.20 ensure that $|\mathrm{supp}(\mu)| \leq n+1$. Finally, Proposition 3.6 shows that the τ periodic curve $\gamma(t) := \Gamma(t/\tau)$ is a τ periodic orbit of the Lagrangian system (3.1) with energy $E(\gamma) = E$ and at most $n+1$ bounce points. □

Now we restrict our considerations to an open bounded domain Ω that is endowed with the Euclidean metric $|\cdot| := |\cdot|_{\mathrm{Eucl}}$. This allows to prove Theorem 3.4. The idea of the proof goes back to [AM11, p.17-18].

Proof of Theorem 3.4. Assume there exists a closed curve γ of energy E without bounce point. We will derive an upper limit on the energy E from the obvious diameter bound:

$$\mathrm{diam}\left(\overline{\Omega}\right) \geq |\gamma(t) - \gamma(0)|. \tag{3.59}$$

If this inequality is not satisfied, then the periodic magnetic bounce orbit has at least one bounce point.

Since the magnetic term σ is exact, we know that

$$\begin{aligned} \sigma_q(v,w) = (\mathrm{d}\alpha_q)(v,w) &= \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial \alpha_i}{\partial q_j}(q) \mathrm{d}q_j \right) \wedge \mathrm{d}q_i(v,w) \\ &= \sum_{j<i} \left(\frac{\partial \alpha_i}{\partial q_j}(q) - \frac{\partial \alpha_j}{\partial q_i}(q) \right) \mathrm{d}q_j \wedge \mathrm{d}q_i(v,w). \end{aligned}$$

We define

$$A_{ij}(q) := \frac{\partial \alpha_i}{\partial q_j}(q) - \frac{\partial \alpha_j}{\partial q_i}(q)$$

and in matrix notation

$$A := A(q) := (A_{ij}(q))_{i,j=1,\dots,n}\,.$$

The matrix A is skew-symmetric, i.e. $A^T = -A$. This allows to rewrite the Euler–Lagrange equation (3.2) for all $t \notin \mathcal{B}$, where $\mathcal{B}$ is defined in Definition 3.1, as

$$\gamma''(t) + A(\gamma(t))\gamma'(t) + \nabla V(\gamma(t)) = 0. \tag{3.60}$$

At first, we consider the case $\max_{\overline{\Omega}} |A| \neq 0$ or $\max_{\overline{\Omega}} |\nabla V| \neq 0$. Inserting the reformulated Euler–Lagrange equation (3.60) and the energy equation (3.5) in (3.59) leads to:

$$\begin{aligned} \mathrm{diam}\left(\overline{\Omega}\right) &\geq |\gamma(t) - \gamma(0)| \\ &= \left| \int_0^t \gamma'(s) \,\mathrm{d}s \right| \\ &\geq |\gamma'(0)| \cdot |t| - \left| \int_0^t \int_0^s \gamma''(r) \,\mathrm{d}r\mathrm{d}s \right| \end{aligned}$$

$$
\begin{aligned}
&\geq |\gamma'(0)| \cdot |t| - \int_0^t \int_0^s |A(\gamma(r))| \cdot |\gamma'(r)| \; \mathrm{d}r\mathrm{d}s \\
&\quad - \int_0^t \int_0^s |\nabla V(\gamma(r))| \; \mathrm{d}r\mathrm{d}s \\
&\geq |\gamma'(0)| \cdot |t| - \int_0^t \int_0^s |A(\gamma(r))| \cdot \left(2\left(E - V\left(\gamma(r)\right)\right)\right)^{1/2} \; \mathrm{d}r\mathrm{d}s \\
&\quad - \int_0^t \int_0^s \max_{\overline{\Omega}} |\nabla V| \; \mathrm{d}r\mathrm{d}s \\
&\geq |\gamma'(0)| \cdot |t| - \int_0^t \int_0^s |A(\gamma(r))| \cdot \left(2\left(E - \min_{\overline{\Omega}} V\right)\right)^{1/2} \; \mathrm{d}r\mathrm{d}s \\
&\quad - \tfrac{1}{2}t^2 \max_{\overline{\Omega}} |\nabla V|.
\end{aligned}
$$

Since α is smooth on the compact manifold $\overline{\Omega}$, we have $\max_{\overline{\Omega}} |A|_2 < \infty$. Without loss of generality we assume that $t \geq 0$, such that

$$
\begin{aligned}
\operatorname{diam}\left(\overline{\Omega}\right) \geq \left(2\left(E - \max_{\overline{\Omega}} V\right)\right)^{1/2} t - \max_{\overline{\Omega}} |A| \tfrac{1}{2}t^2 \left(2\left(E - \min_{\overline{\Omega}} V\right)\right)^{1/2} \\
- \tfrac{1}{2}t^2 \max_{\overline{\Omega}} |\nabla V|
\end{aligned}
\tag{3.61}
$$

Now when considering the right hand side as a function depending on t, we can search for critical points. Thus, we compute the derivative with respect to t:

$$
\begin{aligned}
0 &= \left(2\left(E - \max_{\overline{\Omega}} V\right)\right)^{1/2} - t \max_{\overline{\Omega}} |A| \left(2\left(E - \min_{\overline{\Omega}} V\right)\right)^{1/2} - t \max_{\overline{\Omega}} |\nabla V| \\
&= \left(2\left(E - \max_{\overline{\Omega}} V\right)\right)^{1/2} - t \left(\max_{\overline{\Omega}} |A| \left(2\left(E - \min_{\overline{\Omega}} V\right)\right)^{1/2} + \max_{\overline{\Omega}} |\nabla V|\right).
\end{aligned}
$$

This is equivalent to

$$
t = \frac{(2(E - \max_{\overline{\Omega}} V))^{1/2}}{\max_{\overline{\Omega}} |A| (2\left(E - \min_{\overline{\Omega}} V\right))^{1/2} + \max_{\overline{\Omega}} |\nabla V|}.
$$

Inserting t in Equation (3.61) shows:

$$
\operatorname{diam}(\overline{\Omega}) \geq \frac{E - \max_{\overline{\Omega}} V}{\max_{\overline{\Omega}} |A| (2\left(E - \min_{\overline{\Omega}} V\right))^{1/2} + \max_{\overline{\Omega}} |\nabla V|}
$$

Since we assumed that $\max_{\overline{\Omega}} |A| \neq 0$ or $\max_{\overline{\Omega}} |\nabla V| \neq 0$ this is possible only if

$$
\left(E - \max_{\overline{\Omega}} V\right) - \operatorname{diam}\left(\overline{\Omega}\right) \left(\left(2\left(E - \min_{\overline{\Omega}} V\right)\right)^{1/2} \max_{\overline{\Omega}} |A|_2 + \max_{\overline{\Omega}} |\nabla V|_2\right) \leq 0.
$$

A lengthy computation shows that this is equivalent to

$$\begin{aligned} E \leq\ & \operatorname{diam}(\overline{\Omega}) \max_{\overline{\Omega}} |\nabla V| + \max_{\overline{\Omega}} V + \operatorname{diam}(\overline{\Omega})^2 \left(\max_{\overline{\Omega}} |A| \right)^2 \\ & + \operatorname{diam}(\overline{\Omega}) \max_{\overline{\Omega}} |A| \\ & \quad \left(\operatorname{diam}(\overline{\Omega})^2 \left(\max_{\overline{\Omega}} |A| \right)^2 + 2\operatorname{diam}(\overline{\Omega}) \max_{\overline{\Omega}} |\nabla V| + 2 \left(\max_{\overline{\Omega}} V - \min_{\overline{\Omega}} V \right) \right)^{1/2}. \end{aligned}$$

So for all energy values E with

$$E > \operatorname{diam}(\overline{\Omega}) \max_{\overline{\Omega}} |\nabla V| + \max_{\overline{\Omega}} V + C,$$

where we write

$$\begin{aligned} C :=\ & \operatorname{diam}(\overline{\Omega})^2 \left(\max_{\overline{\Omega}} |A| \right)^2 \\ & + \operatorname{diam}(\overline{\Omega}) \max_{\overline{\Omega}} |A| \\ & \quad \cdot \sqrt{\operatorname{diam}(\overline{\Omega})^2 \left(\max_{\overline{\Omega}} |A| \right)^2 + 2\operatorname{diam}(\overline{\Omega}) \max_{\overline{\Omega}} |\nabla V| + 2 \left(\max_{\overline{\Omega}} V - \min_{\overline{\Omega}} V \right)}, \end{aligned} \tag{3.62}$$

all periodic magnetic bounce orbits have bounce points. Together with Theorem 3.3, there exists a periodic magnetic bounce orbit with bounce point for high energy values, where the energy value is as described above.

It remains to consider the case where both, $\max_{\overline{\Omega}} |A|$ and $\max_{\overline{\Omega}} |\nabla V|$, are zero. We know that $\max_{\overline{\Omega}} |A| = 0$ is equivalent to $|A| = 0$ and this is equivalent to $A = 0$. In particular, then the Euler–Lagrange equation reduces to

$$\gamma''(t) + \nabla V(\gamma(t)) = 0. \tag{3.63}$$

Moreover, we know that $\max_{\overline{\Omega}} |\nabla V| = 0$. So the potential V is constant. Therefore the solutions of the Euler–Lagrange equation (3.63) of the Lagrangian L with energy $E > \max_{\overline{\Omega}} V = \textit{constant}$ are straight curves with constant positive velocity, because the energy equation (3.5) shows that

$$\tfrac{1}{2}|\gamma'(t)|^2 = E - \max_{\overline{\Omega}} V = E - \textit{constant} > 0.$$

□

Chapter 4

A Tonelli approximation scheme

In this chapter, we develop a more general approach to the billiard game than the one with electromagnetic Lagrangians. For that, we check whether it is possible to generalize our electromagnetic ideas developed in Chapter 3 to the Tonelli context. In particular, we will see that the approximation scheme derived in Section 3.2 still holds true for Tonelli Lagrangians. Analogous to the magnetic chapter 3, we follow the approach by Albers–Mazzucchelli in [AM11, Section 2]. For that, we modify the Tonelli Lagrangian system by adding a penalty term and we assume that there exist periodic solutions for the modified system. We prove that these approximate solutions actually converge to *Tonelli periodic bounce orbits*, assuming the Morse index of the corresponding free-time action functional is bounded. These Tonelli bounce orbits also satisfy a law of reflection. Currently, we do not see a nice geometric interpretation in this general context. We work in a variational environment analogous to the one in Chapter 3.2.1 and set up the notation accordingly.

4.1 Variational setting

Let $\overline{\Omega}$ be a compact n-dimensional manifold with smooth boundary endowed with a Riemannian metric $g = \langle\cdot,\cdot\rangle$ and let Ω be its interior. We study the *Tonelli Lagrangian system*

$$\begin{aligned} L\colon \mathrm{T}\overline{\Omega} &\longrightarrow \mathbb{R} \\ (q,v) &\longmapsto L(q,v). \end{aligned} \tag{4.1}$$

Recall from Definition 2.1 that L is fiberwise uniformly convex and superlinear. Thus, for some numbers $L_0 > 0$ and $L_1 \in \mathbb{R}$ the Lagrangian L satisfies the following inequalities for $(q,v) \in \mathrm{T}M$ and $u \in \mathrm{T}_qM$ (see (2.2)):

$$\begin{aligned} L(q,v) &\geq L_0|v|^2 - L_1 \\ \mathrm{d}^2_{vv}L(q,v)[u,u] &\geq 2L_0|u|^2. \end{aligned} \tag{4.2}$$

Analogously to the periodic magnetic bounce orbit, we define a Tonelli version.

Definition 4.1. A continuous and piecewise smooth map $\gamma\colon \mathbb{R}/\tau\mathbb{Z} \to \overline{\Omega}$, $\tau > 0$, is called *periodic Tonelli bounce orbit* of the Lagrangian system (4.1) if there exists a (possibly empty) finite subset $\mathcal{B} \subset \mathbb{R}/\tau\mathbb{Z}$ such that

(1) for all $t \notin \mathcal{B}$ there exists $\gamma'(t)$ and γ solves the Euler–Lagrange equation for all $t \notin \mathcal{B}$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\partial_v L(\gamma(t), \gamma'(t))\right) = \partial_q L(\gamma(t), \gamma'(t)), \tag{4.3}$$

where ∂_v denotes the derivative in direction of the fiber and ∂_q the derivative in direction of q.

(2) For every $t \in \mathcal{B}$ we have $\gamma(t) \in \partial\overline{\Omega}$, and there exist left and right derivatives of γ at t, i.e.

$$\gamma'\left(t^{\pm}\right) := \lim_{s\to t^{\pm}} \gamma'(s) = \begin{cases} \lim_{s\searrow t} \gamma'(s) & \text{for } t^{+} \\ \lim_{s\nearrow t} \gamma'(s) & \text{for } t^{-} \end{cases},$$

and there exists a measure μ on $\mathcal{B}$, such that γ satisfies

$$\Big(\mathrm{d}_v L(\gamma(t), \gamma'(t^-)) - \mathrm{d}_v L(\gamma(t), \gamma'(t^+))\Big)[w] = \langle \nu(\gamma(t)), w\rangle\ \mu(\{t\}), \tag{4.4}$$

where $w \in H^1(\mathbb{R}/\tau\mathbb{Z}, \gamma^*\mathrm{T}\overline{\Omega})$ and the measure μ is as in Proposition 4.4 below.

As in Definition 3.1, we call $t \in \mathcal{B}$ *bounce times* and for $t \in \mathcal{B}$ the points $\gamma(t)$ are called *bounce points.*

Remark. In future work, we plan to derive a proper law of reflection from Equation (4.4) in order to see the geometric meaning.

Approximate Lagrangian system. We use the penalty term U, see (3.9), to modify the Tonelli Lagrangian (4.1) as follows

$$\begin{aligned} L_\varepsilon\colon \mathrm{T}\Omega &\longrightarrow \mathbb{R} \\ (q, v) &\longmapsto L(q, v) - \varepsilon U(q). \end{aligned} \tag{4.5}$$

Free-time action functional. With the modified Lagrangian (4.5), we are able to define the free-time action functional as

$$\begin{aligned} \mathscr{L}_\varepsilon^E\colon H^1(S^1, \Omega) \times \mathbb{R}_{>0} &\longrightarrow \mathbb{R} \\ (\Gamma, \tau) &\longmapsto \tau \int_0^1 \left[L_\varepsilon(\Gamma(t), \Gamma'(t)/\tau) + E\right] \mathrm{d}t = \int_0^\tau \left[L_\varepsilon(\gamma, \gamma') + E\right] \mathrm{d}t, \end{aligned}$$

where $\gamma(t) := \Gamma(t/\tau)$. We cite the regularity properties of the free-time action functional $\mathscr{L}_\varepsilon^E$ in the subsequent lemma. Those are proven in [AS09, Proposition 3.1] as well as in [Con06, Section 2].

Lemma 4.2. *The free-time action functional $\mathscr{L}_\varepsilon^E$ satisfies the following:*

- $\mathscr{L}_\varepsilon^E$ *is in* $\mathcal{C}^{1,1}\left(H^1(S^1,\Omega)\times\mathbb{R}_{>0}\right)$ *and twice Gateaux differentiable at every point.*
- $\mathscr{L}_\varepsilon^E$ *is twice Fréchet differentiable at every point if and only if* L_ε *is electromagnetic on the whole* $\mathrm{T}\overline{\Omega}$*. In this case,* $\mathscr{L}_\varepsilon^E$ *is actually smooth on* $H^1(S^1,\Omega)\times\mathbb{R}_{>0}$*.*

Recall the energy function $E\colon \mathrm{T}\overline{\Omega}\longrightarrow\mathbb{R}$ from (2.2), defined by

$$(x,v)\longmapsto \mathrm{d}_vL(x,v)[v]-L(x,v).$$

Then, the differential of $\mathscr{L}_\varepsilon^E$ can be given as follows (see [AS09] and [Con06]):

$$\begin{aligned}\mathrm{d}\mathscr{L}_\varepsilon^E(\Gamma,\tau)[(\Psi,\rho)] = &\int_0^\tau \left(\mathrm{d}_qL_\varepsilon(\gamma,\gamma')[\Psi]+\mathrm{d}_vL_\varepsilon(\gamma,\gamma')[\nabla_t\Psi]\right)\mathrm{d}t\\ &+\frac{\rho}{\tau}\int_0^\tau \left(E-\mathcal{E}(\gamma,\gamma')-\varepsilon U(\gamma)\right)\mathrm{d}t,\end{aligned}\tag{4.6}$$

where $(\Psi,\rho)\in\mathrm{T}_{(\Gamma,\tau)}\left(H^1(S^1,\overline{\Omega})\times\mathbb{R}_{>0}\right)$ for some $(\Gamma,\tau)\in H^1(S^1,\overline{\Omega})\times\mathbb{R}_{>0}$.

Lemma 4.3. *The point* $(\Gamma,\tau)\in H^1(S^1,\overline{\Omega})\times\mathbb{R}_{>0}$ *is a critical point of* $\mathscr{L}_\varepsilon^E$ *if and only if* γ *is a solution of the Euler–Lagrange equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\partial_vL(\gamma,\gamma')\right)=\partial_qL(\gamma,\gamma')+\varepsilon\nabla U(\gamma)\tag{4.7}$$

and the energy of γ *is*

$$E_\varepsilon(\gamma):=\mathrm{d}_vL(\gamma,\gamma')[\gamma']-L(\gamma,\gamma')+\varepsilon U(\gamma)=E.\tag{4.8}$$

The proof of this lemma works in the same way as the magnetic proof of Lemma 3.5.

Remark. We mention that using the chain rule, the Euler–Lagrange equation (4.7) is equivalent to the following equation, see [Sor15, Chapter 1]:

$$\partial_{vv}L(\gamma,\gamma')\nabla_{\gamma'}\gamma'=\partial_qL(\gamma,\gamma')-\partial_{qv}L(\gamma,\gamma')\gamma'+\varepsilon\nabla U(\gamma).$$

Note that $\partial_{vv}L(\gamma,\gamma')$ is invertible, since L is fiberwise convex. Thus, we can solve this equation for $\nabla_{\gamma'}\gamma'$ and obtain

$$\nabla_{\gamma'}\gamma'=\left(\partial_{vv}L(\gamma,\gamma')\right)^{-1}\Big(\partial_qL(\gamma,\gamma')-\partial_{qv}L(\gamma,\gamma')\gamma'+\varepsilon\nabla U(\gamma)\Big).\tag{4.9}$$

4.2 Convergence of approximate solutions

In this section, we prove that under suitable assumptions a sequence of approximate solutions of (4.5) converges to a Tonelli periodic bounce orbit in H^1. This is a Tonelli analogue of Proposition 3.6. For this generalization, we stay close to the notation in Proposition 3.6.

Proposition 4.4. *Let $K > 0$ and let $T_2 > T_1 > 0$. For each $\varepsilon > 0$, let $(\Gamma_\varepsilon, \tau_\varepsilon)$ be a critical point of the free-time action functional $\mathscr{L}_\varepsilon^E$ with period $T_1 \leq \tau_\varepsilon \leq T_2$ and energy $E_\varepsilon \leq K$. Then, up to choosing a subsequence, $(\Gamma_\varepsilon, \tau_\varepsilon)$ converges in $H^1(S^1, \overline{\Omega} \times \mathbb{R}_{>0})$ to (Γ, τ) as $\varepsilon \to 0$. Moreover, there exists a finite Borel measure μ on $\mathcal{C} := \{t \in \mathbb{R}/\tau\mathbb{Z} \mid \gamma(t) \in \partial\overline{\Omega}\}$ for $\gamma(t) := \Gamma(t/\tau)$ such that*

1. *for all $\psi \in H^1(S^1, \gamma^*\mathrm{T}\overline{\Omega})$*

$$\int_0^\tau \left(\mathrm{d}_q L(\gamma, \gamma')[\psi] + \mathrm{d}_v L(\gamma, \gamma')[\nabla_t \psi]\right) \mathrm{d}t = \int_{\mathcal{C}} \langle \nu(\gamma), \psi \rangle \mathrm{d}\mu, \tag{4.10}$$

   where ν is the outer normal with respect to $\partial\overline{\Omega}$.

2. *outside* $\operatorname{supp}(\mu)$ *the curve γ is a smooth solution of the Euler–Lagrange equation* (3.2) *corresponding to L with energy $E(\gamma) = \lim_{\varepsilon\to 0} E_\varepsilon(\gamma_\varepsilon)$ and*

3. *γ has left and right derivatives that are left and right continuous on $\mathbb{R}/\tau\mathbb{Z}$, respectively. Moreover, γ satisfies the law of reflection given in* (4.4) *at each time $t \in \mathcal{C}$ which is an isolated point of* $\operatorname{supp}(\mu)$.

In particular, if $\operatorname{supp}(\mu)$ *is a finite set, then γ is a periodic Tonelli bounce orbit of the Lagrangian system given in Equation* (4.1) *and $\mathcal{B} :=$* $\operatorname{supp}(\mu)$ *is its set of bounce times.*

Remark. We point out that the law of reflection coincides with the well-known law of reflection, i.e. the angle of incidence equals the angle of reflection, for electromagnetic Lagrangians, compare Equations (3.4).

Proof. In this proof, we explain the necessary changes to derive Proposition 3.6 in the Tonelli context. For that, we follow an analogous recipe as in the proof of Proposition 3.6.

Let $(\Gamma_\varepsilon, \tau_\varepsilon)$ be a sequence as above. As L is a Tonelli Lagrangian, we know that L is electromagnetic for $|v|$ large, compare (2.2). In particular, the estimates in (2.2) deliver a lower bound on L. An analogous upper bound can be given by

$$L(q, v) \leq L_2|v|^2 + L_3$$

for numbers $L_2, L_3 > 0$. We recall that the energy function $E\colon \mathrm{T}\overline{\Omega} \to \mathbb{R}$ is also Tonelli and thus also satisfies (2.2). Therefore, we can estimate using [CI99, Lemma 1-4.4] that

$$\begin{aligned} \mathrm{d}_v L(\gamma_\varepsilon, \gamma_\varepsilon')[\gamma_\varepsilon'] &\geq -E(\gamma_\varepsilon, \gamma_\varepsilon') - L(\gamma_\varepsilon, \gamma_\varepsilon') \\ &\geq L_4|\gamma_\varepsilon'|^2 + L_5, \end{aligned}$$

where L_4, L_5 are suitable positive numbers. Therefore, the energy equation (4.8)

$$E_\varepsilon \equiv E_\varepsilon(\gamma_\varepsilon) = \mathrm{d}_v L(\gamma_\varepsilon, \gamma_\varepsilon')[\gamma_\varepsilon'] - L(\gamma_\varepsilon, \gamma_\varepsilon') + \varepsilon U(\gamma_\varepsilon)$$

shows that $|\gamma_\varepsilon'|$ is uniformly bounded in L^∞, since $E_\varepsilon \leq K$ and $U \geq 0$.

$\varepsilon\nabla\mathbf{U}(\mathbf{\Gamma}_\varepsilon)$ is uniformly bounded in $\mathbf{L^1}$. Let us start of by considering a tangent vector of the form $(\Psi, 0) \in H^1(S^1, \Gamma_\varepsilon^*(\mathrm{T}\overline{\Omega})) \times \mathbb{R}$ at the critical point $(\Gamma_\varepsilon, \tau_\varepsilon)$. Then Equation (4.6) gives

$$\begin{aligned} 0 &= \mathrm{d}\mathscr{L}_\varepsilon^E(\Gamma_\varepsilon, \tau_\varepsilon)[(\Psi, 0)] \\ &= \tau_\varepsilon \int_0^1 \Big[\mathrm{d}_q L(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')[\Psi] + \mathrm{d}_v L(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')[\nabla_{\Gamma_\varepsilon'} \tfrac{\Psi}{\tau_\varepsilon}] - \varepsilon\nabla U(\Gamma_\varepsilon)[\Psi]\Big]\, \mathrm{d}t. \end{aligned}$$

Thus, in particular we know for all $\Psi \in H^1(S^1, \Gamma_\varepsilon^*(\mathrm{T}\overline{\Omega})$ that

$$\int_0^1 \Big[\mathrm{d}_q L(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')[\Psi] + \mathrm{d}_v L(\Gamma_\varepsilon, \tfrac{1}{\tau_\varepsilon}\Gamma_\varepsilon')[\nabla_{\Gamma_\varepsilon'} \tfrac{\Psi}{\tau_\varepsilon}]\Big]\, \mathrm{d}t = \int_0^1 \varepsilon\nabla U(\Gamma_\varepsilon)[\Psi]\mathrm{d}t. \tag{4.11}$$

We can show that the integrand on the left hand side of Equation (4.11) is uniformly bounded in L^∞ for $\Psi := -\nabla h(\Gamma_\varepsilon)$. We obtain $\nabla_{\Gamma_\varepsilon'}\Psi = -\nabla_{\Gamma_\varepsilon'}\nabla h(\Gamma_\varepsilon)$. Since Γ_ε' is uniformly bounded in L^∞ and h is smooth, $\nabla_{\Gamma_\varepsilon'}\Psi$ is uniformly bounded in L^∞. Since the Tonelli Lagrangian $L \in \mathcal{C}^\infty(\mathrm{T}\overline{\Omega})$ is electromagnetic for $|v|_g$ large, its horizontal and vertical differentials are uniformly bounded. Therefore, the left hand side of (4.11) is bounded and we obtain a bound for the right hand side:

$$\int_0^1 \varepsilon\nabla U(\Gamma_\varepsilon)[\Psi]\mathrm{d}t \leq C,$$

where C is a constant, that is independent of ε. Analogously to the proof of Proposition 3.6, we can continue with the same argumentation and obtain that $\varepsilon\nabla U(\Gamma_\varepsilon)$ is uniformly bounded in L^1. The details can be found in (3.19) and (3.20). Thus, we obtain the same bound on $\int_0^1 \varepsilon|\nabla U(\Gamma_\varepsilon)|_g\mathrm{d}t$ as in (3.21):

$$\int_0^1 \varepsilon|\nabla U(\Gamma_\varepsilon)|_g\mathrm{d}t < C + \frac{2\varepsilon}{d_0^3},$$

where d_0 is given in Figure 3.3. Therefore, $\varepsilon\nabla U(\Gamma_\varepsilon)$ is uniformly bounded in L^1.

γ_ε converges to γ in $\mathbf{H^1}$. Using the Euler–Lagrange equation (4.7) derived in Lemma 4.3 and the equivalent equation (4.9), we obtain

$$\nabla_{\gamma_\varepsilon'}\gamma_\varepsilon' = \left(\partial_{vv}L(\gamma_\varepsilon, \gamma_\varepsilon')\right)^{-1}\left(\partial_q L(\gamma_\varepsilon, \gamma_\varepsilon') - \partial_{qv}L(\gamma_\varepsilon, \gamma_\varepsilon')\gamma_\varepsilon' + \varepsilon\nabla U(\gamma_\varepsilon)\right).$$

Since L is defined on a compact domain $\overline{\Omega}$, all its derivatives are bounded. Thus $\partial_q L(\gamma_\varepsilon, \gamma_\varepsilon')$ and $\partial_{qv}L(\gamma_\varepsilon, \gamma_\varepsilon')$ are uniformly bounded. We know that the boundedness condition, see [CI99, 1-1(c)], is immediately true for compact manifolds $\overline{\Omega}$. Thus we can deduce that $\left(\partial_{vv}L(\gamma_\varepsilon, \gamma_\varepsilon')\right)^{-1}$ is bounded as a continuous linear operator. Moreover, γ_ε' is uniformly bounded in L^∞ and $\varepsilon\nabla U(\gamma_\varepsilon)$ is uniformly bounded in L^1. We conclude that $\nabla_{\gamma_\varepsilon'}\gamma_\varepsilon'$ is uniformly bounded in L^1, i.e. γ_ε is uniformly bounded in $W^{2,1}$. As in the magnetic proof, see (3.23), the Sobolev embedding theorem ensures that γ_ε converges in H^1 to $\gamma \in H^1(\mathbb{R}/\tau\mathbb{Z}, \gamma^*(\mathrm{T}\overline{\Omega}))$ as $\varepsilon \to 0$, after choosing a subsequence.

Defining the Borel measure μ. To define the Borel measure μ, there are no changes necessary for the generalization to Tonelli Lagrangians. For that, we just shortly sum up the magnetic discussion. As in (3.20) and the subsequent paragraphs, one can show in exactly the same manner that $\widetilde{\mu}_\varepsilon := 2\varepsilon h^{-3}(\Gamma_\varepsilon)$ is uniformly bounded in L^1 and thus $\widetilde{\mu}_\varepsilon$ converges in the weak-$*$ topology to a measure $\widetilde{\mu}$. Applying the Riesz representation theorem shows that $\widetilde{\mu}$ is a finite, positive Borel measure. By reparametrisation, we define μ as the pull-back of $\widetilde{\mu}$. In particular, we conclude that

$$\operatorname{supp}(\mu) \subset \mathcal{C} = \{t \in \mathbb{R}/\tau\mathbb{Z} \mid \gamma(t) \in \partial\overline{\Omega}\}.$$

Euler–Lagrange equation. In this paragraph, there are some adjustements necessary. For that, here is a detailed consideration: For $t \notin \operatorname{supp}(\mu)$ we choose $\widehat{\varepsilon} > 0$ such that $[t-\widehat{\varepsilon}, t+\widehat{\varepsilon}] \cap \operatorname{supp}(\mu) = \emptyset$. For all $\psi \in H^1(\mathbb{R}/\tau\mathbb{Z}, \gamma^*\mathrm{T}\overline{\Omega})$ with support in $[t-\widehat{\varepsilon}, t+\widehat{\varepsilon}]$ we have by taking the limit $\varepsilon \to 0$ in Equation (4.11)

$$\int_{t-\widehat{\varepsilon}}^{t+\widehat{\varepsilon}} [\mathrm{d}_q L(\gamma,\gamma')[\psi] + \mathrm{d}_v L(\gamma,\gamma')[\nabla_{\gamma'}\psi]]\, \mathrm{d}t = \int_{\mathcal{C}} \langle \nu(\gamma), \psi\rangle \mathrm{d}\mu = 0.$$

With the help of the *Lemma of Du Bois–Raymond*, see [Kli78, Thm.1.3.11], we obtain for all $\psi \in H^1(\mathbb{R}/\tau\mathbb{Z}, \gamma^*\mathrm{T}\overline{\Omega})$ with support in $[t-\widehat{\varepsilon}, t+\widehat{\varepsilon}]$

$$\begin{aligned} 0 &= \int_{t-\widehat{\varepsilon}}^{t+\widehat{\varepsilon}} (\mathrm{d}_q L(\gamma,\gamma')[\psi] + \mathrm{d}_v L(\gamma,\gamma')[\nabla_{\gamma'}\psi])\, \mathrm{d}t \\ &= \int_{t-\widehat{\varepsilon}}^{t+\widehat{\varepsilon}} (\mathrm{d}_q L(\gamma,\gamma')[\psi] - \nabla_{\gamma'}\mathrm{d}_v L(\gamma,\gamma')[\psi])\, \mathrm{d}t. \end{aligned}$$

We conclude that

$$\mathrm{d}_q L(\gamma,\gamma') - \nabla_{\gamma'}\mathrm{d}_v L(\gamma,\gamma') = 0$$

holds outside $\operatorname{supp}(\mu)$ and recall the equivalent formulation (4.9):

$$\nabla_{\gamma'}\gamma' = (\partial_{vv}L(\gamma,\gamma'))^{-1}\left(\partial_q L(\gamma,\gamma') - \partial_{qv}L(\gamma,\gamma')\gamma' + \varepsilon\nabla U(\gamma)\right).$$

Bootstrapping then shows that γ is smooth and solves the Euler–Lagrange equation outside $\operatorname{supp}(\mu)$. We define the energy $E(\gamma)$ to be

$$E(\gamma) := \lim_{\varepsilon\to 0} E_\varepsilon(\gamma(t)) = \mathrm{d}_v L(\gamma,\gamma')[\gamma'] - L(\gamma,\gamma')$$

for any $t \notin \operatorname{supp}(\mu)$.

Law of reflection. Lastly, we prove a *Tonelli* law of reflection. For that, we follow the same steps as in the magnetic proof, see (3.25). First, we prove that

$$E(\gamma) = \mathrm{d}_v L(\gamma,\gamma')[\gamma'] - L(\gamma,\gamma')$$

holds actually almost everywhere. We recall the inequality

$$E_\varepsilon(\gamma_\varepsilon) := \mathrm{d}_v L(\gamma_\varepsilon,\gamma_\varepsilon')[\gamma_\varepsilon'] - L(\gamma_\varepsilon,\gamma_\varepsilon') + \varepsilon U(\gamma_\varepsilon) \leq K.$$

This implies that there exists $u \in L^\infty(\mathbb{R}/\tau\mathbb{Z}, \overline{\Omega})$, such that, after choosing a subsequence,

$$\lim_{\varepsilon\to 0} \varepsilon U(\gamma_\varepsilon(t)) = u(t) \text{ almost everywhere.}$$

The same argument as in the magnetic situation, see (3.26), shows by applying the lemma of Fatou, that

$$\lim_{\varepsilon\to 0}\varepsilon U(\gamma_\varepsilon(t)) = 0 \text{ almost everywhere.}$$

Thus, the definition of $E(\gamma) := \lim_{\varepsilon\to 0} E_\varepsilon(\gamma(t))$ immediately implies that

$$E(\gamma) = \mathrm{d}_v L(\gamma(t),\gamma'(t))[\gamma'(t)] - L(\gamma(t),\gamma'(t)) \text{ almost everywhere.}$$

We continue as in the magnetic proof (3.27) and obtain by the same arguments for all $t\in\mathbb{R}/\tau\mathbb{Z}$ that

$$E = \mathrm{d}_v L(\gamma(t),\gamma'(t^\pm))[\gamma'(t^\pm)] - L(\gamma(t),\gamma'(t^\pm)).$$

Now, we are ready to prove the Tonelli law of reflection at an isolated point $t\in\operatorname{supp}(\mu)$. Consider a vector field $\psi\in H^1(\mathbb{R}/\tau\mathbb{Z},\gamma^*\mathrm{T}\overline{\Omega})$ with $\operatorname{supp}(\psi)\subset[t-\widehat{\varepsilon},t+\widehat{\varepsilon}]$, where $\widehat{\varepsilon}>0$ is such that $[t-\widehat{\varepsilon},t+\widehat{\varepsilon}]\cap\operatorname{supp}(\mu)=\{t\}$. Then, Equation (4.11) (and taking the limit $\varepsilon\to 0$ in (4.11)) reduces to

$$\int\limits_{[t-\widehat{\varepsilon},t+\widehat{\varepsilon}]\setminus\{t\}} \left(\mathrm{d}_q L(\gamma,\gamma')[\psi] + \mathrm{d}_v L(\gamma,\gamma')[\nabla_{\gamma'}\psi]\right)\mathrm{d}t = \langle\nu(\gamma(t)),\psi(t)\rangle\ \mu(\{t\}).$$

After integration by parts and applying the Euler–Lagrange equation (4.3), we obtain

$$\Big(\mathrm{d}_v L(\gamma(t),\gamma'(t^-)) - \mathrm{d}_v L(\gamma(t),\gamma'(t^+))\Big)[w] = \langle\nu(\gamma(t)),w\rangle\ \mu(\{t\})$$

for all $w\in H^1(\mathbb{R}/\tau\mathbb{Z},\gamma^*\mathrm{T}\overline{\Omega})$, as we claimed in Equation (4.4).

As in the magnetic situation, we conclude the proof by observing, that γ is a periodic Tonelli bounce orbit, if $\mathcal{B} := \operatorname{supp}(\mu)$ is a finite set, by the same reasoning. We shortly recap the reasoning here for the reader's convenience. Since $\gamma\in H^1$, it is continuous. Moreover, γ is piecewise smooth on $(\mathbb{R}/\tau\mathbb{Z})\setminus\mathcal{B}$, where γ satisfies the Euler–Lagrange equation, compare (4.3). For $t\in\mathcal{B}$, left and right derivatives exist and γ satisfies Equation (4.4). □

4.3 Morse index of the Tonelli free-time action functional

In this section, we recall that Proposition 3.7 proved in Section 3.2.4 holds true for Tonelli Lagrangians L. In fact, we already proved this proposition for Tonelli Lagrangians in Section 3.2.4. We only used that the considered Lagrangrian is electromagnetic for $|v|$ large and that is true for a Tonelli Lagrangian. For completeness, we also put the statement in here. Recall that the proposition gives an upper bound on the number of bounce points in terms of the Morse index of the free-time action functional $\mathscr{L}_\varepsilon^E$.

Proposition 4.5. *In the situation of Proposition 4.4, let $(\Gamma_\varepsilon,\tau_\varepsilon)$ be the subsequence converging to (Γ,τ). Then, the cardinality $|\operatorname{supp}(\mu)|$ (up to taking a subsequence of*

$(\Gamma_\varepsilon, \tau_\varepsilon)$*) is bounded from above by the Morse index* μ_{Morse} *of the restricted action functional* $\mathscr{L}_\varepsilon^E\big|_{H^1\times\{\tau_\varepsilon\}}$ *at* Γ_ε *for* $\varepsilon > 0$ *sufficiently small, that is,*

$$|\mathrm{supp}(\mu)| \leq \liminf_{\varepsilon\to 0} \mu_{\text{Morse}}\left(\Gamma_\varepsilon; \mathscr{L}_\varepsilon^E\big|_{H^1\times\{\tau_\varepsilon\}}\right).$$

4.4 Tonelli Wrap-up

In the previous two sections, we explained a generalization of the approximation scheme for electromagnetic Lagrangians, derived in Chapter 3.2, to Tonelli Lagrangians. Thus, a sequence of approximate solutions converges under suitable assumptions in H^1 to a periodic Tonelli bounce orbit. In order to prove the existence of Tonelli periodic bounce orbits as we did it in the magnetic case in Theorem 3.3, one crucial step is missing. We did not construct suitable approximate solutions to the Euler–Lagrange equation, compare Section 3.3 for the magnetic construction.

In Section 2.2, we highlighted that Tonelli systems satisfy the Legendre duality. Let us take a look at the Tonelli Hamiltonian system corresponding to the modified Tonelli Lagrangian system (4.5). Let H be the Tonelli Hamiltonian, that is Legendre dual to the given Tonelli Lagrangian L (4.1). Then, the Hamiltonian H_ε corresponding to the modified Lagrangian L_ε, considered in 4.5, is given by

$$\begin{aligned} H_\varepsilon\colon (\mathrm{T}^*\overline{\Omega}, \omega_0) &\longrightarrow \mathbb{R} \\ (q,p) &\mapsto H(q,p) + \varepsilon U(q) \end{aligned}$$

with respect to the canonical symplectic form $\omega_0 = \mathrm{d}p \wedge \mathrm{d}q$ on the cotangent bundle $\mathrm{T}^*\overline{\Omega}$.

If we wanted to apply the same ideas as in Section 3.3, several difficulties arise. For example, an explicit description of the Hamiltonian H is lacking. Therefore, different ideas to prove the existence of Tonelli bounce orbits are needed. This can be done in future work. In the next chapter, we describe some more ideas for further studies.

Chapter 5

Keep on bouncing!

There are several directions for further studies related to the topic of magnetic billiards. We assume familiarity with the magnetic situation of Chapter 3. Recall our main theorem, where we showed that for energy values above the strict Mañé critical value c_0, there exists a *periodic magnetic bounce orbit* with bounded period, see Theorem 3.3. This outroduction does not aim for completeness. It describes the author's personal taste of possible future projects.

5.1 Magnetic situation on lower energy levels

As a natural next step, one could explore energy levels below c_0. Recall from Chapter 2.3, that there are four important energy values:

$$\min_{\mathrm{T}^*M} E \leq e_0(L) \leq c_u(L) \leq c_0(L).$$

Observe that c_0 is the highest value among these. When the magnetic term α vanishes, then $e_0(L) = c_u(L) = c_0(L) = \max_{\overline{\Omega}} V$. In the situation of $\alpha = 0$, Albers–Mazzucchelli proved the existence of periodic bounce orbits of prescribed energy on an open bounded domain in $\mathbb{R}^n$, see [AM11]. In general, the values $e_0(L)$ and $c_u(L)$ are distinct when α does not vanish. Moreover, note that $c_u(L)$ and $c_0(L)$ are distinct only if the fundamental group of $\overline{\Omega}$ is non-abelian. Therefore, it makes sense to explore the dynamics of an electromagnetic billiard system for different energy levels.

We give a short recap of known results for Tonelli systems from highest to lowest energy level. These results serve as an overview for the reader. For energy values $E > c_0$, Contreras–Iuturriaga–Paternain–Paternain showed in [CIP98] that the magnetic flow of a Tonelli Lagrangian on a closed manifold can be seen as a reparametrization of the geodesic flow of a suitable Finsler metric. In this region (c_0, ∞), we proved the existence of magnetic bounce orbits in Theorem 3.3.

If we consider energy values $E \geq c_u(L)$, then the action functional for the electromagnetic Lagrangian L is bounded from below . For the modified Lagrangian L_ε, this is however not guaranteed. If $E < c_u(L)$, then even the action functional corresponding to L is not bounded from below on every connected component of its domain, see [Abb13, Lemma 4.1]. Thus, one cannot find periodic orbits minimizing the action functional. Moreover, the Palais-Smale condition is not satisfied in this region and thus compactness for a minimax argument fails. Therefore, these regions could be interesting to explore with different methods that could lead to similar results as in Theorem 2.7.

In dimension two, some interesting work for exact magnetic flows was performed in [AMMP17] by Abbondandolo–Macarini–Mazzucchelli–Paternain. They consider an exact magnetic flow on a closed surface. In particular, they prove that for almost every energy level $E < c_u$, there are infinitely many periodic orbits with energy E. In [AMMP17, Chapter 1], they refer to it as "mysterious range of energies". In order to prove this theorem, they used variational techniques and studied the free-time Lagrangian action functional. We already pointed out the main difficulties in this range of energies: The action functional is unbounded from below on every connected component of its domain and does not satisfy the Palais-Smale condition. We shortly recall some history concerning this interval $(0, c_u)$. For $E \in (0, c_u)$, $\mathcal{E}^{-1}(E)$ has always at least one periodic orbit. Originally, this was proven by Taĭmanov in [Tai92a], [Tai92b], [Tai91]. In the context of Mañé critical values, this was reproved by Contreras–Macarini–Paternain in [CMP04] using geometric measure theory. To the author's knowledge, no more results on the existence of periodic orbits for magnetic flows on *all* energy levels in $(0, c_u)$ are known. There are several results for *almost every* $E \in (0, c_u)$. The existence of at least three periodic orbits is known for almost every $E \in (0, c_u)$. This result is due to Contreras in [Con06] and Abbondandolo–Macarini–Paternain in [AMP15]. Contreras even proved the existence of the second periodic orbit for any Tonelli Lagrangian on manifolds of arbitrary dimension. In [AMP15], Abbondandolo–Macarini–Paternain already proved the existence of infinitely many periodic orbits on $\mathcal{E}^{-1}(E)$ under some non-degeneracy condition.

Open questions: What happens for energy values in the intervals $(e_0(L), c_u(L))$ and $(c_u(L), c_0(L))$? When is the action functional corresponding to the modified Lagrangian L_ε bounded from below? Is it possible to show more for Lagrangians defined on surfaces? Do there exist periodic bounce orbits in this setup?

5.2 Non-exact situation

Another direction is the discussion of the non-exact magnetic situation. There arise some difficulties, as Albers–Mazzucchelli already pointed to in [AM11, Remark 1.6]. The Lagrangian action functional is not available in the non-exact case because the magnetic form σ does not have a primitive. Thus, the approximation scheme, as it is for the exact case, does not exist. Another problem concerns the approximating energy hypersurfaces, which may not be of contact type and thus might not contain periodic orbits.

We point out that the transformation law of Hamiltonian vector fields, see [HZ94, Chapter 1], uses that the magnetic 2-form σ is exact. In the exact situation, there are two equivalent Hamiltonian descriptions available. On the one hand, there is the electromagnetic Hamiltonian given by

$$\begin{aligned} H\colon (\mathrm{T}^*\overline{\Omega}, \omega_0) &\longrightarrow \mathbb{R} \\ (q,p) &\longmapsto \tfrac{1}{2}|p-\alpha_q|^2 + V(q), \end{aligned}$$

compare Section 2.2, with respect to the standard symplectic structure $\omega_0 = \mathrm{d}p \wedge \mathrm{d}q$ on the cotangent bundle $\mathrm{T}^*\overline{\Omega}$. On the other hand, a symplectomorphic description is given by the mechanical Hamiltonian

$$\begin{aligned} H\colon (\mathrm{T}^*\overline{\Omega}, \omega_\sigma) &\longrightarrow \mathbb{R} \\ (q,p) &\longmapsto \tfrac{1}{2}|p|^2_{g^*} + V(q) \end{aligned}$$

with respect to the twisted symplectic form $\omega_\sigma := \mathrm{d}p \wedge \mathrm{d}q + \pi^*\sigma$. In the non-exact case, only the second formulation is available.

Nevertheless, there are interesting results available for weakly exact magnetic fields on closed Riemannian manifolds that overcome these issues. For example, Merry shows in [Mer10] that for weakly exact magnetic fields, locally there is a Legendre duality in terms of the differential of the action functional. He proves that for a weakly exact magnetic system on a closed connected Riemannian manifold, almost all energy levels contain a closed orbit. Independently, Asselle–Benedetti proved the same result in [AB15].

When M is a surface, the existence of periodic orbits of magnetic flows has been studied a lot. Interesting references are [Gin96], [Ker00] and [CMP04]. There exist mainly two quite different approaches to prove the existence of magnetic orbits. On the one hand, there is Morse–Novikov theory, developed by Novikov and Taimanov. References for this study can be found in [Gin96] and [CMP04]. And on the other hand, there is an approach using symplectic methods that has been introduced by Arnold and developed by Ginzburg. One difficulty that arises in this approach is the following. The considered energy levels may fail to be of contact type. In this outroduction, we just mention 3 interesting examples: S^2, T^2 with flat metric and hyperbolic surfaces of constant curvature -1. For $M = S^2$ with the standard metric, there is at least one closed characteristic on every energy level, see [Gin96, Example 3.8]. In the case $M = T^2$ equipped with a flat metric and magnetic form equal to the area form, there exist at least 3 periodic orbits on all energy levels, compare [Ker00, Chapter 1] and [Gin96, Example 2.2]. Lastly, we consider a hyperbolic surface with constant curvature -1 and magnetic field given by the standard area form. In this example, magnetic geodesics are the curves with constant curvature, [CMP04]. On energy levels at the Mañé critical value the magnetic flow coincides with the horocycle flow and hence is minimal. In particular, no trajectory is closed. Thus, for a non-exact magnetic form, one cannot expect to find periodic orbits in all energy levels. This is different to the geodesic flow. From this small discussion on surfaces, we may deduce that it could be worthwhile to study the non-exact magnetic billiard problem first in dimension two because also for billiard systems more is known for surfaces than in higher dimensions.

Open questions: What does Legendre duality look like for the billiard situation? What can be said for higher energy values in the non-exact case? How to develop a non-exact billiard game? Do there exist periodic orbits in the billiard setup?

Appendix A

Coda

In this chapter, we explain the local calculations used in the proof of Proposition 3.13. For that, we recall some facts on Riemannian geometry, see for example [Lee18, Chapter 3].

A.1 Musical considerations for Riemannian metrics

Let (M, g) be an n-dimensional Riemannian manifold and let (U, q), $q = (q_1, \ldots, q_n)$, be a chart of M. Then, $\frac{\partial}{\partial q_i}$ form a coordinate frame and $\mathrm{d}q_1, \ldots, \mathrm{d}q_n$ its coframe. The Riemannian metric in U is given by

$$g = \sum_{i,j} g_{ij} \mathrm{d}q_i \otimes \mathrm{d}q_j,$$

where $g_{ij} = g\left(\frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}\right)$.

Riemannian manifolds allow us to convert vectors to covectors and vice versa by *raising and lowering indices*. Define a map $\flat\colon \mathrm{T}M \to \mathrm{T}^*M$ by sending a vector v to a covector $v^\flat$ defined by

$$v^\flat(w) := g(v, w), \qquad w \in \mathrm{T}M.$$

In coordinates, this reads as $v^\flat = g(v_i \frac{\partial}{\partial q_i}, \cdot) = g_{ij} v^i \mathrm{d}q_j$. The covector $v^\flat$ is commonly written as $v^\flat = \sum_j (v^\flat)_j \mathrm{d}q_j$, where

$$(v^\flat)_j := \sum_i g_{ij} v^i. \tag{A.1}$$

Thus, $v^\flat$ is obtained from v by lowering an index. The matrix of the flat operator $\flat$ in terms of a coordinate basis is therefore the matrix of g itself. Since the matrix of g is invertible, the flat operator $\flat$ is also invertible. The inverse $\sharp\colon \mathrm{T}^*M \to \mathrm{T}M$ is defined by

$$\eta \mapsto \eta^\sharp.$$

In coordinates, the vector $\eta^\sharp$ is given by $\eta^\sharp = \sum_{i,j} g^{ij}\eta_j \mathrm{d}q_i$, where g^{ij} are the components of the inverse matrix $(g_{ij})^{-1}$. The vector $\eta^\sharp$ is usually written as $\eta^\sharp = \sum_i (\eta^\sharp)^i \mathrm{d}q_i$, where

$$(\eta^\sharp)^i = \sum_j g^{ij}\eta_j. \tag{A.2}$$

Thus, $\eta^\sharp$ is obtained by raising an index. We point out that $\flat$ and $\sharp$ are isometries:

$$(\mathrm{T}M, g) \simeq (\mathrm{T}^*M, g^*).$$

By application of the sharp tensor, one can extend the classical gradient operator to Riemannian manifolds. Let $h\colon M \to \mathbb{R}$ be a smooth function. We define the *gradient* ∇h of h by

$$\nabla h := (\mathrm{d}h)^\sharp.$$

Thus, the gradient can be characterized by

$$g(\nabla h, \cdot) = \mathrm{d}h\cdot$$

and in coordinates by

$$(\nabla h)_i = \sum_j g^{ij}\frac{\partial h}{\partial q_j}. \tag{A.3}$$

In the proof of Proposition 3.13, we use the following calculations. We return Equation (3.42) to our mind and start computing. First, we consider

$$g^{ki}(p_k - \alpha_k(q)) = ((p-\alpha_q)^\sharp)_i, \tag{A.4}$$

where we used Equation (A.2). Next, we compute

$$\begin{aligned}\sum_{i,j} g^{ij}(p_j - \alpha_j(q))\frac{\partial h}{\partial q_i} &= \sum_{i,j,k,l} g_{ij}g^{ki}(p_k - \alpha_k(q))g^{lj}\frac{\partial h}{\partial q_l} \\ &= \sum_{i,j} g_{ij}((p-\alpha_q)^\sharp)_i(\nabla h)_j \\ &= g((p-\alpha_q)^\sharp, \nabla h).\end{aligned} \tag{A.5}$$

Using the definition (A.3), we have

$$g(\nabla h, \nabla h) = \sum_{i,j} g_{ij}(\nabla h)_i(\nabla h)_j = \sum_{i,j,r,s} g_{ij}g^{ir}\frac{\partial h}{\partial q_r}g^{js}\frac{\partial h}{\partial q_s} = \sum_{j,s} g^{js}\frac{\partial h}{\partial q_j}\frac{\partial h}{\partial q_s}. \tag{A.6}$$

Thus, we can derive Equation (3.43) from Equation (3.42).

Now, we explain the details to obtain Equation (3.44) from Equation (3.43). Note that

$$|p^\sharp|_g^2 = g(p^\sharp, p^\sharp) = \sum_{ij} g_{ij}(p^\sharp)_i(p^\sharp)_j = \sum_{i,j,k,l} g_{ij}g^{ik}p_k g^{jl}p_l = \sum_{j,l} g^{jl}p_j p_l = g^*(p,p) = |p|_{g^*}^2.$$

Using bilinearity and the Cauchy–Schwarz inequality, we obtain the following estimates for constants $\kappa_i \in \mathbb{R}_{>0}$, $i = 1, 2, 3$, independent of q and ε:

$$\sum_{i,j,k,l} \frac{\partial g^{ij}}{\partial q_k}(q)(p_i - \alpha_i(q)) \frac{\partial h}{\partial q_j}(q)((p-\alpha_q)^\sharp)_k \geq -\kappa_1 |(p-\alpha_q)^\sharp|_g^2 = -\kappa_1 |p-\alpha_q|_{g^*}^2, \tag{A.7}$$

$$\sum_{j,k} \frac{\partial^2 h}{\partial q_k \partial q_j}(q)((p-\alpha_q)^\sharp)_j((p-\alpha_q)^\sharp)_k \geq -\kappa_2 |p-\alpha_q|_{g^*}^2, \tag{A.8}$$

$$\sum_{i,l,m} (\nabla h)_i \frac{\partial g^{lm}}{\partial q_i}(q)(p_l - \alpha_l(q))(p_m - \alpha_m(q)) \geq -\kappa_3 |\nabla h|_g |p-\alpha_q|_{g^*}^2. \tag{A.9}$$

We define $|\nabla\alpha|_{\max} := \max_{q\in M} \left|\frac{\partial \alpha_i}{\partial q_k}(q)\right|_{g^*}$ and estimate as follows:

$$\begin{aligned} \sum_{i,k} \frac{\partial \alpha_i}{\partial q_k}(q)(\nabla h)_i((p-\alpha_q)^\sharp)_k &\leq |\nabla\alpha|_{\max} |\nabla h|_g |p-\alpha_q|_{g^*}, \\ \sum_{i,l} (\nabla h)_i \frac{\partial \alpha_l}{\partial q_i}(q)((p-\alpha_q)^\sharp)_l &\leq |\nabla\alpha|_{\max} |\nabla h|_g |p-\alpha_q|_{g^*}. \end{aligned} \tag{A.10}$$

Fine

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