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The Horofunction Compactification of Finite-Dimensional Normed Vector Spaces and of Symmetric Spaces

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Abstract

This work examines the horofunction compactification of finite-dimensional normed vector spaces with applications to the theory of symmetric spaces and toric varieties.

For any proper metric space X the horofunction compactification can be defined as the closure of an embedding of the space into the space of continuous real valued functions vanishing at a given basepoint. A point in the boundary is called a horofunction. This characterization though lacks an explicit characterization of the boundary points. The first part of this thesis is concerned with such an explicit description of the horofunctions in the setting of finite-dimensional normed vector spaces. Here the compactification strongly depends on the shape of the unit and the dual unit ball of the norm. We restrict ourselves to cases where at least one of the following holds true:

I) The unit and the dual unit ball are polyhedral.

II) The unit and the dual unit ball have smooth boundaries.

III) The metric space X is two-dimensional.

Based on a result of Walsh [Wal07] we provide a criterion for the convergence of sequences in the horofunction compactification in these cases to determine the topology. Additionally we show that then the compactification is homeomorphic to the dual unit ball. Later we give an explicit example, where our criterion for convergence fails in the general case and make a conjecture about the rate of convergence of some spacial sets in the boundary of the dual unit ball. Assuming the conjecture holds, we generalize the convergence criterion to any norm with the property that all horofunctions in the boundary are limits of almost-geodesics (so-called Busemann points). This part of the thesis ends with a construction of how to extend our previous results to a new class of norms using Minkowski sums:

IV) The dual unit ball is the Minkowski sum of a polyhedral and a smooth dual unit ball.

The second part of the thesis applies the results of part one to two different settings: first to symmetric spaces of non-compact type and then to projective toric varieties. For a symmetric space $X = G/K$ of non-compact type with a G-invariant Finsler metric we prove that the horofunction compactification of X is determined by the horofunction compactification of a maximal flat in X. With this result we show how to realize any Satake or Martin compactification of X as an appropriate horofunction compactification. Finally, as an application to projective toric varieties, we give a geometric 1-1 correspondence between projective toric varieties of dimension n and horofunction compactifications of $\mathbb{R}^n$ with respect to rational polyhedral norms.

Zusammenfassung

Diese Arbeit befasst sich mit der Horofunktions-Kompaktifizierung endlichdimensionaler normierter Vektorräume und Anwendungen derselben auf symmetrische Räume und torische Varietäten.

Die Horofunktions-Kompaktifizierung kann für jeden eigentlichen metrischen Raum X definiert werden als der Abschluss einer bestimmten Einbettung des Raumes in den Raum der stetigen reellwertigen Funktionen auf X, die an einem Basispunkt verschwinden. Die Punkte im Rand der Kompaktifizierung sind heißen Horofunktionen. Bei dieser Definition fehlt allerdings eine explizite Beschreibung der Randpunkte. Im ersten Teil dieser Arbeit geht es um eine solche explizite Charakterisierung der Horofunktionen für endlichdimensionale Vektorräume. Hierbei hängt die Kompaktifizierung des Raumes stark von der Form des Einheits- und des dualen Einheitsballes der Norm ab. Wir beschränken uns dabei auf Bälle, die mindestens eine der folgenden Bedingungen erfüllen:

I) Der Einheitsball und sein dualer Ball sind polyedrisch.

II) Der Einheitsball und sein dualer Ball haben einen glatten Rand.

III) Der metrische Raum ist zweidimensional.

Ausgehend von einem Resultat von Walsh [Wal07] geben wir für diese Fälle ein Kriterium für die Konvergenz von Folgen in der Horofunktions-Kompaktifizierung an, um die Topologie zu bestimmen. Außerdem zeigen wir, dass in diesen Fällen die Kompaktifizierung homöomorph zum dualen Einheitsball ist. Anschließend betrachten wir ein explizites Beispiel das zeigt, dass das Konvergenzkriterium im allgemeinen Fall nicht gilt und formulieren darauf aufbauend eine Vermutung über die Konvergenzrate spezieller Folgen im dualen Einheitsball. Unter der Voraussetzung, dass die Vermutung stimmt, verallgemeinern wir das Konvergenzkriterium für alle Normen, deren Horofunktionen Limiten von Fastgeodäten sind (sogenannte Busemann Punkte). Zum Abschluss dieses Teils der Arbeit erweitern wir unsere bisherigen Resultate mit Hilfe der Minkowski-Summe um alle Normen, die die folgende Bedingung erfüllen:

IV) Der duale Einheitsball ist die Minkowski-Summe eines polyedrischen und eines glatten Einheitsballs.

Im zweiten Teil der Arbeit werden die Ergebnisse des ersten Teils auf zwei Situationen angewendet, nämlich auf symmetrische Räume von nicht-kompaktem Typ und auf projektive torische Varietäten. Für einen symmetrischen Raum $X = G/K$ von nicht-kompaktem Typ, der mit einer G-invarianten Finslermetrik ausgestattet ist, zeigen wir, dass die Horofunktions-Kompaktifizierung des Raumes bestimmt ist durch die Kompaktifizierung eines maximalen Flachs in X bestimmt ist. Damit zeigen wir, wie jede Satake- und Martin-Kompaktifizierung von X als Horofunktions-Kompaktifizierung bezüglich einer geeigneten Norm realisiert werden kann. Als Anwendung auf torische Varietäten geben wir schließlich eine geometrische Bijektion zwischen n-dimensionalen projektiven torischen Varietäten und der Horofunktions-Kompaktifizierung von $\mathbb{R}^n$ bezüglich einer polyedrischen Norm an.

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Contents

1 | Introduction

The simplest compactification of a topological space is the Alexandroff compactification introduced by Alexandroff in 1924 [Ale24]. It is also called *one-point compactification* by its way of construction: Given a non-compact locally compact Hausdorff space X, construct a compact space $X_\omega := X \cup \{\omega\}$ by adding an additional point $\omega \notin X$ called the *point at infinity*. A topology on X_ω is given by the open sets as $\mathcal{T} = \{U \subseteq X \mid U \text{ open }\} \cup \{X_\omega \backslash C \mid C \subseteq X \text{ closed compact}\}$. Then X_ω is compact and contains X as a dense open subset. Moreover, the space X_ω is unique up to homeomorphism.

The one-point compactification of the real line $\mathbb{R}$ can be imagined as first shrinking the real line to the open interval $(-1, 1)$ and then bending it such that the ends -1 and 1 almost touch each other at the top. There we add the additional point ω. The result is a circle and the one-point compactification of $\mathbb{R}$ is indeed homeomorphic to $\mathbb{S}^1$. This concept generalizes to higher dimensions: for the Euclidean space $\mathbb{R}^n$ the one-point compactification $\mathbb{R}^n_\omega$ is homeomorphic to the n-Sphere $\mathbb{S}^n$.

There are many more compactifications of $\mathbb{R}^n$ when we allow more points to create the compact space. For example by retracting any point along a straight line centered at the origin, the space $\mathbb{R}^n$ is diffeomorphic to the interior of the unit sphere. Adding then the unit sphere gives another compactification of $\mathbb{R}^n$. Instead of shrinking the space homeomorphically into the interior of a compact set, we can also add the *sphere at infinity* $X(\infty)$ to the space. The sphere at infinity is defined as the set of equivalence classes of asymptotic geodesic rays, where two rays are equivalent, if they remain within bounded distance from each other as they go to infinity. This compactification is called the *geodesic compactification* and can not only be obtained for $\mathbb{R}^n$ but for every simply connected non-positively curved Riemannian manifold. The picture of the space shrunk into the interior of the sphere remains valid [BJ06, Prop. I.2.3]: the sphere at infinity $X(\infty)$ can be identified with the unit sphere in the tangent space $T_{p_0}X$ for any basepoint $p_0 \in X$.

We want to keep this picture in mind when talking about the horofunction compactification, which is the compactification we are most interested in.

The Horofunction Compactification

The horofunction compactification was introduced by Gromov [Gro81, §1.2] in 1981 as a general method to construct compactifications of metric spaces. As horofunction compactifications only require a proper metric space to be defined, they arise in many contexts. Alessandrini, Liu, Papadopoulos and Su for example show that the Thurston compactification of the Teichmüller space and its horofunction compactification with respect to the arc metric are homeomorphic. The horofunction compactification of a complete simply connected non-positively curved manifold was identified with the geodesic compactification in [BGS85, §3] by Ballmann, Gromov and Schröder. In this thesis we will focus on finite-dimensional normed spaces. Karlsson, Metz and Noskov [KMN06] describe horoballs for finite-dimensional normed spaces with polyhedral norm and for Hilbert metrics on simplices. In their recent paper [CKS20], Ciobotaru, Kramer and Schwer describe the horofunction compactification of finite-dimensional vector spaces with asymmetric polyhedral norms. They use the ultrapower of X with respect to a free ultrafilter and plan to use their techniques and results to obtain compactifications of buildings in a follow-up paper (in preparation). Related results of buildings were first obtained in [Bri06].

For the construction of the horofunction compactification of a proper metric space (X, d) with possibly asymmetric metric one embeds X into the space of continuous functions $\widetilde{C}(X)$ on X which vanish at a fixed base point p_0. The embedding is given in terms of the metric d:

$$\begin{aligned} X &\longrightarrow C_{p_0}(X) \\ z &\longmapsto \psi_z = d(\cdot, z) - d(p_0, z). \end{aligned}$$

The closure of the image is the horofunction compactification $\overline{X}_d^{hor}$ of (X, d), where the structure of the compactification crucially depends on the metric d. Though the embedding depends on the basepoint p_0, the compactifications with respect to different base points are homeomorphic. Within the boundary $\partial_{hor}(X)$ of horofunctions there are special elements called *Busemann points* which are given as the limits of almost geodesic sequences in X. It is often difficult to determine the horofunction compactification of a given space. One simplification sometimes is not to consider all horofunctions but to determine the set of Busemann points.

So let us go to the setting of a finite-dimensional normed vector space $(X, ||\cdot||)$. There Walsh [Wal07] explicitly described all horofunctions and showed that a necessary and sufficient condition for all horofunctions to be Busemann points is a relatively low condition on the shape of the dual unit ball. Given a norm on a finite-dimensional vector space X, its unit ball B determines a *dual unit ball* in the dual space given as the polar of the convex set B:

$$B^\circ := \{y \in X^* \mid \langle y|x\rangle \geq -1 \ \forall x \in B\}.$$

As B is closed, compact and convex, the same holds for its dual B° and both sets contain the origin in their interior. An *extreme* set of B° is a convex subset of the boundary ∂B° not containing any line-segment in B° without its endpoints (Definition 2.3.3). When B° is polyhedral, the extreme sets of B° are exactly its faces and all of them arise as the intersection of B° with some affine hyperplane. But in the more general setting, this is not true any more. Any boundary point of B° belongs to some extreme set and different extreme sets have disjoint relative interior.

Much work (like in [CKS20, KMN06]) has been done on the horofunction compactification of finite-dimensional normed spaces applying many different techniques. But as far as we know, only polyhedral norms have been considered up to now. We will generalize the theory to norms of the following classes:

I) The unit ball is polyhedral.

II) The unit and the dual unit ball have smooth boundaries.

III) The space X is two-dimensional.

The second case can equivalently be described as B° only having smooth extreme points as extreme sets. In all three cases the set of extreme sets of B° is closed and we can use the results by Walsh.

In order to explicitly describe the set of horofunctions, we follow [Wal07] and introduce a set of maps $h_{E,p} : X \to \mathbb{R}$ depending on a convex set $E \subseteq X^*$ and a point $p \in X$ by

$$h_{E,p}(x) := -\inf_{e\in E} \langle e|p - x\rangle + \inf_{e\in E} \langle e|p\rangle .$$

This then yields the following description:

Corollary 3.1.10 The set of horofunctions of X is given as

$$\partial_{hor}(X) = \{h_{E,p} \mid E \subsetneq B^\circ \text{ is a proper extreme set}, \ p \in T(E)^*\},$$

where $T(E)^* \subseteq X$ is a certain subspace of X of the same dimension as E. ◦

The above result determines the set of horofunctions in the boundary but gives no statement about its topology. We will define a topology in terms of the convergence of sequences.

An important result in this thesis is the description of the convergence behavior of sequences in the horofunction compactification to obtain a topology on $\overline{X}^{hor}$. To each extreme set F of the unit ball B we can assign a unique *exposed dual* extreme set F° of B° which is maximal among all those extreme sets of B° that minimize the dual pairing with F. For an unbounded sequence $(z_m)_{m\in\mathbb{N}}$ and a point $x \in X$ we consider its sequence of directions $\left(\frac{z_m - x}{\|z_m - x\|}\right)_{m\in\mathbb{N}}$, which is a sequence of points on the boundary of the unit ball B. Each such point lies in the relative interior of an extreme set $F_m(x)$ of B. Taking the associated exposed duals $D_m(x)$ of $F_m(x)$ gives a sequence $(D_m(x))_{m\in\mathbb{N}}$ of extreme sets of B°. As the set of extreme sets of B° is closed, all accumulation points of this sequence are extreme sets. By $D(x)$ we denote the set of accumulation points of $(D_m(x))_m$. So for each $x \in X$ the set $D(x)$ is a set of extreme sets of B°. They play an important role when describing convergent sequences and their limits.

Theorem 3.2.6 *Let $B \subseteq X$ be a unit ball and $B^\circ \subseteq X^*$ its dual such that they belong to one of the three cases I) − III) defined above. Let $(z_m)_{m\in\mathbb{N}}$ be an unbounded sequence in X.*
Then the sequence $(\psi_{z_m})_{m\in\mathbb{N}}$ converges to a horofunction $h_{E',p}$ associated to an extreme set $E' \subseteq B^\circ$ and a point $p \in T(E')^*$ if and only if the following conditions are satisfied:

1) *$E := \operatorname{aff}\{D(x) \mid x \in X\} \cap B^\circ$ is extreme.*

2) *The projection $(z_{m,E})_{m\in\mathbb{N}}$ of $(z_m)_{m\in\mathbb{N}}$ to $T(E)^*$ converges.*

If $(\psi_{z_m})_m$ converges, then $E' = E$ and $p = \lim_{m\to\infty} z_{m,E}$. ◦

We have to restrict ourselves to the three cases mentioned before because there the structure of the extreme sets of the dual unit ball is sufficiently nice. Otherwise the statement is not true anymore as will be seen by an explicit counterexample (more details in Section 3.2.6): Consider $\mathbb{R}^3$ equipped with a norm that has a cylinder as unit ball parallel to the z-Axis. Its dual is a rotated rhombus with peaks on the z-axis, as shown in Figure 1.1.

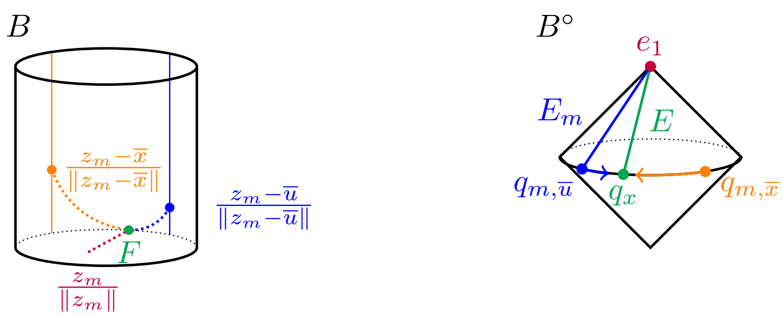


Figure 1.1: The cylindric unit ball (LEFT) and its dual (RIGHT).

For the sequence $z_m = (-m^2 + a, m + b, -m^2 + c)$ with real parameters satisfying $a - c > \frac{1}{2}$ we can explicitly compute the limiting horofunction and it is the map $h_{E,p}$ associated to the extreme set $E = \operatorname{conv}\{(0,0,1),\ (1,0,0)\} \subseteq B^\circ$ with parameter $p = \frac{1}{2}(a - c - \frac{1}{2}) \cdot (1,0,-1) \in T(E)^*$. But applying the theorem yields a horofunction $h_{E,\widetilde{p}}$ associated to the same extreme set but with a different parameter: $\widetilde{p} = p + \frac{1}{4} \cdot (1,0,-1) \in T(E)^*$. We see that the parameter which we obtained as the limit of the sequence $(z_m)_m$ projected to the subspace $T(E)^*$ is not the right one by an additive constant. It actually turns out that we get the correct p by projecting not to $T(E)^*$ but by projecting each element z_m to a different subspace $T(E_m)^*$, where $(E_m)_{m\in\mathbb{N}}$ is a sequence of subsets of B° of the same dimension as E and converging to E. Now the crucial point is that we can not take any such sequence of subsets (otherwise E would already do it) but the convergence has to happen with the correct rate depending on $(z_m)_m$: for every $x \in X$ the sequence $(q_{m,x})_m \subseteq \partial B^\circ$ satisfying $\langle q_{m,x} | z_m - x\rangle = -\|z_m - x\|$, approaches the set E_m faster than z_m goes to infinity.

Inspired from the above example we make the following conjecture for the general case:

Conjecture 3.2.12 *Let $u_1, \dots, u_k \in X$ be points and for each $j = 1, \dots, k$, let $(q_{m,u_j})_m \subseteq \partial B^\circ$ be a sequence of points satisfying $\left\langle q_{m,u_j} \middle| z_m - u_j \right\rangle = -\|z_m - u_j\|$ for all $m \in \mathbb{N}$, such that the set*

$$E_m := \operatorname{aff}\{q_{m,u_1}, \dots, q_{m,u_k}\} \cap B^\circ$$

converges to E with $\dim(E_m) = \dim(E)$ for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ and a point $x \in X$ let $q_{m,x} \in \partial B^\circ$ be an extreme point dual to $\frac{z_m - x}{\|z_m - x\|} \in \partial B$ and let $e_{m,x} \in E_m$ a point such that the sequences $(q_{m,x})_m$ and $(e_{m,x})_m$ (or subsequences, if necessary) both converge to the same point $q_x \in E$. Then it holds:

$$\langle e_{m,x} - q_{m,x} | z_m \rangle \longrightarrow 0 \quad \forall x \in X. \qquad \circ$$

We know that the left side of the pairing goes to 0 whereas the right side is unbounded and the pairing can only be bounded, if the left side converges faster than the right one. That is, the points $e_{m,x}$ and $q_{m,x}$ approach each other fast enough. Under the assumption that the conjecture holds, we can generalize our result for the convergence of sequences in the horofunction compactification to the general setting:

Theorem 3.2.14 *Assume Conjecture 3.2.12 holds. Let $B \subsetneq X$ be a unit ball and B° its dual such that the set of extreme sets of B° is closed. Let $(z_m)_{m\in\mathbb{N}}$ be an unbounded sequence in X and $E := \operatorname{aff}\{D(x) \mid x \in X\} \cap B^\circ$.*
Let $u_1, \dots, u_k \in X$ be points with $k = \dim(E) + 1$ and for each $j = 1, \dots, k$, let $(q_{m,u_j})_m \subseteq \partial B^\circ$ be a sequence of points satisfying $\left\langle q_{m,u_j} \middle| z_m - u_j \right\rangle = -\|z_m - u_j\|$ for all $m \in \mathbb{N}$, such that with

$$E_m := \operatorname{aff}\{q_{m,u_1}, \dots, q_{m,u_k}\} \cap B^\circ$$

there holds

(A) $\dim(E_m) = \dim(E)$ *and*

(B) $E_m \longrightarrow E$ *as* $m \to \infty$.

Then the sequence $(\psi_{z_m})_m$ converges to a horofunction $h_{E',p}$ for an extreme set $E' \subseteq B^\circ$ and a point $p' \in T(E')^*$ if and only if the following conditions are satisfied:

1) E as defined above is extreme.

2) The projection $(z_{m,E_m})_m$ of $(z_m)_m$ to $T(E_m)^*$ converges.

If $(\psi_{z_m})_m$ converges, then $E' = E$ and $p = \lim_{m\to\infty} z_{m,E_m}$. $\circ$

Earlier we said that we want to keep the picture in mind that we obtain the compactification by adding boundary components to the space. For the horofunction compactification we saw that the boundary points, i.e. the horofunctions, are indexed by the extreme sets E of the dual unit ball and parameters p in a subspace of the same dimension as E. We can keep this picture in mind by the following consideration: imagine the space X to be mapped into the interior of the dual unit ball B° and a horofunction $h_{E,p}$ from the boundary mapped into E in a way depending on $p \in T(E)^*$. This picture is maid rigorous in another main theorem of this thesis:

Theorem 3.3.10 *Let $(X, \|\cdot\|)$ be a finite-dimensional normed space with unit ball $B \subseteq X$ and dual unit ball B° belonging to one of the three cases $I) - III)$ above and such that B° has only finitely many connected components of extreme points.*
Then the horofunction compactification $\overline{X}^{hor}$ is homeomorphic to B°. ◦

For the polyhedral setting, analog results using different techniques were also obtained in [KL18], [CKS20] and [JS16].

The map that defines the homeomorphism has to respect the convergence behavior of sequences. It is built out of several "smaller" maps $m^C : \mathbb{R}^n \longrightarrow \operatorname{int}(C)$ that map $\mathbb{R}^n$ into the interior of a compact convex set C. Hereby the map m^C generalizes the idea of a convex combination and is inspired by the moment map from the theory of toric varieties. When C is a polytope, it can be described as the convex hull of its extreme points $c_1, \dots, c_k$ and the map m^C is given as a real convex combination over the extreme points of C:

$$m^C(x) = \sum_{i=1}^{k} \frac{e^{-\langle c_i | x\rangle}}{\sum_{j=1}^{k} e^{-\langle c_j | x\rangle}} c_i.$$

In the general setting where C has infinitely many extreme points, we can not take weighted sums any more. Nevertheless, the same concept will be used then: we consider the weighted sum over the finitely many isolated extreme points c_i plus the sum over (the finitely many) integrals over connected components A_j of extreme points:

$$m^C(x) = \frac{\sum_{i=1}^{k} e^{-\langle c_i | x\rangle} c_i + \sum_{j=1}^{l} \int_{A_j} e^{-\langle v | x\rangle} v dv}{\sum_{i=1}^{k} e^{-\langle c_i | x\rangle} + \sum_{j=1}^{l} \int_{A_j} e^{-\langle w | x\rangle} dw}.$$

The homeomorphism between the horofunction compactification and the dual unit ball B° is built out of maps of the kind of m^C in the following way, consistent with the picture we have in mind:

$$m : \overline{X}^{hor} \longrightarrow B^\circ, \qquad \begin{cases} x \in X & \longmapsto \quad m^{B^\circ}(x) \ \in \operatorname{int}(B^\circ), \\ h_{E,p} \in \partial_{hor} X & \longmapsto \quad m^E(p) \ \in \operatorname{int}(E). \end{cases}$$

So far we had to restrict ourselves to settings where the unit ball of the norm satisfies some conditions about its structure of extreme sets. There is a way to extend the class of allowed norms by taking a norm that is the sum of two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ that belong to the previously allowed cases $I), II)$ and $III)$:

$$\|\cdot\|_3 := \|\cdot\|_1 + \|\cdot\|_2.$$

Then the dual unit ball of our new norm $\|\cdot\|_3$ is the Minkowski sum of the dual unit ball of the other two norms. The only really new extension to the previous cases arises when we take one of the norms to be polyhedral and the other one to be smooth. We want to follow the way we went before and characterize convergent sequences with respect to the new norm and show that the resulting horofunction compactification is homeomorphic to the dual unit ball. The first statement follows immediately by the construction of the new norm as a sum of two norms for which we already know the convergence behavior: a sequence $(z_m)_{m\in\mathbb{N}}$ converges to a horofunction $h_{E,p}$ with respect to $\|\cdot\|_3$ if and only if $(z_m)_m$ converges with respect to both $\|\cdot\|_1$ and $\|\cdot\|_2$ to horofunctions h_{E_1,p_1} and h_{E_2,p_2}, respectively. The limiting function $h_{E,p}$ then is associated to the extreme set E obtained as the Minkowski sum of the extreme sets E_1 and E_2. Given a point p in the Minkowski sum $M = A + D$ of two convex sets A and D, we can decompose the point $p = a + d$ into summands, such that $a \in A$ and $d \in D$, but if p is not an extreme point of M, this decomposition is not unique. This is the reason why we can not sum up the maps m^{E_1} and m^{E_2} to get the map m^E for the homeomorphism. Indeed, by doing so, we loose injectivity. But nothing prevents us from defining m^E and m in the same way as we did before. Doing so we can use the theory developed before for norms belonging to the three cases to get the final result:

Theorem 3.4.7 *Let X be a finite-dimensional normed space equipped with the norm*

$$\|\cdot\|_3 = \|\cdot\|_1 + \|\cdot\|_2,$$

where $\|\cdot\|_1$ is a polyhedral norm and $\|\cdot\|_2$ is smooth. Denote by B_3° the dual unit ball of $\|\cdot\|_3$. Then the horofunction compactification of X with respect to $\|\cdot\|_3$ is homeomorphic to B_3°:

$$\overline{X}^{hor}_{\|\cdot\|_3} \simeq B_3^\circ. \qquad \circ$$

So we have gained a new class of allowed norms:

IV) The dual unit ball arises as the Minkowski sum of a polyhedral unit ball and a smooth dual unit ball with only extreme points.

Let us now explain what we actually did geometrically in terms of compactifications, a picture is given in Figure 1.2. As B_3° is the Minkowski sum of a polyhedral and a smooth set, the extreme sets of B_3° are either extreme points obtained as the sum of each an extreme point of B_1° and B_2°, or polyhedral extreme sets, obtained as the Minkowski sum of a face of the polytope B_1° and an extreme points of B_2°. By collapsing the connected components A_j of extreme sets of B_j to a point, we get a homeomorphism from B_3° to the polytope B_1°. On the other hand, shrinking every higher dimensional extreme set of B_3° to a points yields a homeomorphism from B_3° to the smooth dual unit ball B_2°.

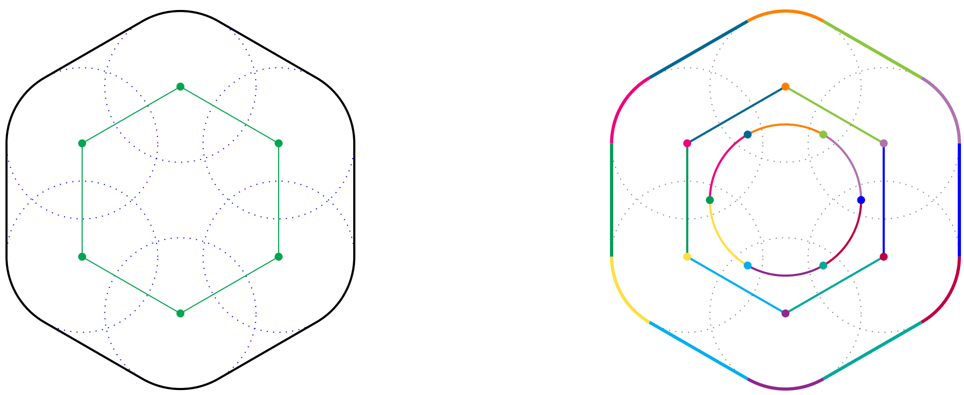


Figure 1.2: The Minkowski sum of a hexagon and a circle (LEFT) and the decomposition of faces (RIGHT).

These identifications of extreme sets of B_1° and B_2° with those of B_3° causes the identity map on X to extend to continuous maps $\overline{X}^{hor}_{\|\cdot\|_3} \longrightarrow \overline{X}^{hor}_{\|\cdot\|_1}$ and $\overline{X}^{hor}_{\|\cdot\|_3} \longrightarrow \overline{X}^{hor}_{\|\cdot\|_2}$. In other words, we constructed by taking the sum of two given norms the *least common refinement*:

Theorem 3.4.13 $\overline{X}^{hor}_{\|\cdot\|_3}$ *is the least common refinement of* $\overline{X}^{hor}_{\|\cdot\|_1}$ *and* $\overline{X}^{hor}_{\|\cdot\|_2}$. ◦

Constructing a new compactification out of given ones or identifying a given (new) compactification with other well-known compactifications can reveal new aspects of the compactifications. We will do so when dealing with the compactifications of symmetric spaces and compare the horofunction compactification of the symmetric space with its Satake and its Martin compactifications.

Symmetric Spaces

Symmetric spaces arise in many areas of mathematics and physics and are an important class of Riemannian or Finsler manifolds. They stand out due to their close relation to Lie groups and Lie algebras. The class of irreducible symmetric spaces can be divided into three types - Euclidean type, compact type and non-compact type. Symmetric spaces of Euclidean type are flat, i.e. have sectional curvature equal to 0 and arise as the Riemannian product of some Euclidean $\mathbb{R}^k$ and a flat $(n-k)$-torus. A symmetric space of compact type is indeed compact as it has constant positive Ricci curvature. A basic example is the sphere $\mathbb{S}^n$ in the Euclidean space $\mathbb{R}^{n+1}$. We are interested in symmetric spaces of non-compact type, which have non-positive sectional curvature, and are diffeomorphic to $\mathbb{R}^n$. One of the simplest examples of a symmetric space of non-compact type is the hyperbolic plane $\mathbb{H}^2$.

Before we go on, let us have a closer look at the example of the hyperbolic plane $\mathbb{H}^2$. The group $\mathrm{SL}(2,\mathbb{R})$ is a semisimple Lie group that acts isometrically and transitively on the upper half plane by fractional linear transformations, that is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+b}{cz+d}$$

for any $z \in \mathbb{H}^2$. The stabilizer of the imaginary unit i is given by the subgroup $\mathrm{SO}(2)$. As the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}^2$ is transitive, this yields an identification of $\mathbb{H}^2$ with the quotient $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$.

The occurrence of such an identification is not specific to $\mathbb{H}^2$, but instead is characteristic for symmetric spaces of non-compact type: Any symmetric space X of non-compact type can be identified with a quotient G/K, where G, the isometry group of X, is a semisimple Lie group and $K = G_{p_0}$ is the stabilizer of a base point $p_0 \in X$, which is a compact subgroup of G. Therefore symmetric spaces can be studied in terms of Lie groups and Lie algebras. Therefore denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively, then there is a maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p}$, where $\mathfrak{p}$ is the Killing-orthogonal of $\mathfrak{k}$ in $\mathfrak{g}$. As a Lie algebra, $\mathfrak{a}$ carries the structure of a finite-dimensional vector space and we can apply our results from Chapter 3 to investigate its horofunction compactification. In order to do that, it remains to determine a norm on $\mathfrak{a}$.

Symmetric spaces carry a Riemannian metric but they also carry a Finsler metric, which is much more interesting to us. A Finsler metric is a generalization of a Riemannian metric as it is a continuous family of (possibly asymmetric) norms on the tangent spaces, which are not necessarily induced by an inner product. Any G-invariant Finsler metric on X induces a Weyl group invariant norm on $\mathfrak{a}$ and also on a maximal flat $F = \exp(\mathfrak{a}).p_0 \cong \mathbb{R}^k$. The flat F can both be seen as a metric space of its own or as a part of the symmetric space X. A key result then is that both view points lead to the same result:

Theorem 4.2.18 *Let $X = G/K$ be a symmetric space of non-compact type. Consider a G-invariant Finsler metric on X such that the dual unit ball belongs to one of the cases I) - IV) and such that its set of extreme sets is closed. Let $\overline{X}^{hor}$ be the horofunction compactification of X with respect to this Finsler metric. Then the closure of a maximal flat F in $\overline{X}^{hor}$ is isomorphic to the horofunction compactification of F with respect to the induced metric.* ◦

Comparison with other compactifications The above theorem allows us to compare the horofunction compactification of X with other well-known compactifications of by studying the compactifications of the flat or of $\mathfrak{a}$. The first compactification we consider is the Satake compactification.

Specific Satake compactifications have been realized as horofunction compactifications of Finsler metrics before. Kapovich and Leeb [KL18] studied the polyhedral horofunction compactification of finite-dimensional vector spaces in order to understand the Satake compactifications of symmetric spaces of non-compact type. Satake compactifications have polyhedral unit balls in $\mathfrak{a}$ and we will come to the same conclusion in Chapter 4 but using different techniques. This is also shown in [HSWW18] with respect to generalized Satake compactifications as introduced in [GKW15]. Friedland and Freitas [FF04a, FF04b] described the horofunction compactification for Finsler p-metrics on $GL(n, \mathbb{C})/U_n$ for $p \in [1, \infty]$, which they showed to agree with the visual compactification for $p > 1$. Additionally they proved that the horofunction compactification of the Siegel upper half plane of rank n for the 1-metric agrees with the bounded symmetric domain compactification, a minimal Satake compactification. The two books [GJT98] and [BJ06] explain many more compactifications of symmetric spaces and show how they are related to each other. They additionally give a unifying approach of how to construct them by adding boundary components.

Before stating the result more precisely let us shortly describe the basic construction of Satake compactifications $\overline{X}^S_\tau$ as given by Satake [Sat60] in 1960. The index τ signifies that they are associated to irreducible faithful representations $\tau : G \to \mathrm{PSL}(n, \mathbb{C})$, which give rise to embeddings $X = G/K \to \mathbb{P}(\mathcal{H}(\mathbb{C}^n))$, $gK \mapsto [\tau(g)^*\tau(g)]$, into the space of positive definite Hermitian matrices. There are only finitely many isomorphism classes of Satake compactifications, determined by subsets of the set of simple roots. The closure of a flat in a Satake compactification with respect to the representation τ is homeomorphic to the convex hull $\mathrm{conv}(\mathcal{W}(2\mu_\tau))$ of the Weyl group orbit of the highest weight μ_τ of the representation as shown by Ji [Ji97]. Now we state

Theorem 4.3.22 *Let $X = G/K$ be a symmetric space of non-compact type. Let τ be a faithful irreducible projective representation of G, $\mu_1, \ldots, \mu_n$ the weights and μ_τ the highest weight of τ. With the Weyl group $\mathcal{W}$ let $D := \mathrm{conv}(\mu_1, \ldots, \mu_n) = \mathrm{conv}(\mathcal{W}(\mu_\tau))$ be the $\mathcal{W}$-orbit of the highest weight. Let $B = -D^\circ$ define a unit ball in the maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{g}$. Then the Satake compactification $\overline{X}^S_\tau$ is G-equivariantly isomorphic to the horofunction compactification of X with respect to the Finsler metric defined by B.* ◦

The other compactification we want to examine more closely is the Martin compactification. It is constructed using the spectrum of the Laplace-Beltrami operator and has no direct geometric meaning. When λ_0 denotes the bottom of the spectrum, then there is an associated Martin compactification $X \cup \partial_\lambda(X)$ for each $\lambda \leq \lambda_0$. A well know geometric interpretation of the Martin compactification is in terms of the maximal Satake and the geodesic compactification of X: $X \cup \partial_{\lambda_0}(X)$ is homeomorphic to the maximal Satake compactification $\overline{X}^S_\tau$ of X and $X \cup \partial_\lambda(X)$ is the least common refinement of the maximal Satake and the geodesic compactification of X. Our previous results from Section 3.4 then allow us immediately to realize any Martin compactification as a horofunction compactification:

Theorem 4.4.2 *Let $X = G/K$ be a symmetric space of non-compact type. Let τ be a faithful irreducible projective representation of G with generic highest weight $\mu_\tau \in \mathfrak{a}^+$. With the Weyl group $\mathcal{W}$ let $D := \mathrm{conv}(\mathcal{W}(\mu_\tau))$ be the $\mathcal{W}$-orbit of the highest weight. Denote the norm on $\mathfrak{a}$ defined by the unit ball $B_S := -D^\circ$ by $\|\cdot\|_S$. Let $\|\cdot\|_E$ be the Euclidean norm $\|\cdot\|_E$ on $\mathfrak{a}$.*

Then for $\lambda = \lambda_0$, the Martin compactification $X \cup \partial_{\lambda_0}(X)$ is homeomorphic to the horofunction compactification of X with respect the Finsler norm given by $\|\cdot\|_S$ on $\mathfrak{a}$.

For $\lambda < \lambda_0(X)$, the Martin compactification $X \cup \partial_\lambda(X)$ is homeomorphic to the horofunction compactification of X with respect to the Finsler norm given by the sum $\|\cdot\| = \|\cdot\|_S + \|\cdot\|_E$ on $\mathfrak{a}$. ◦

Toric Varieties

Our results about the horofunction compactification of a finite-dimensional normed spaces can also be applied in a different setting, namely for toric varieties. Toric varieties provide a basic class of algebraic varieties which are relatively simple. They are irreducible varieties over $\mathbb{C}$ that contain the complex torus as a Zariski open subset such that the action of the torus on itself extends to an algebraic action of the torus on the variety. A subclass of toric varieties are *normal* toric varieties which can be described as an abstract toric variety X_Σ constructed by a fan $\Sigma \subseteq \mathbb{R}^n$. A *fan* is a collection of strongly convex rational polyhedral cones satisfying the same building conditions as simplicial complexes: given a cone $\sigma \in \Sigma$, then also every face of σ belongs to Σ and the intersection of two cones is a common face of both. A polytope P in $\mathbb{R}^n$ defines a fan Σ_P by taking cones over its faces. But the converse is not true, there are fans that do not come from polytopes. Therefore we restrict ourselves to projective toric varieties where it is known that the underlying fan is induced by a rational convex polytope. It is well-known that many algebro-geometric and cohomology properties of toric varieties X_Σ are determined by combinatorial and convex properties of their fans. We use this correspondence to combine projective toric varieties and horofunction compactifications of $\mathbb{R}^n$. Every toric variety X_Σ has a nonnegative part $X_{\Sigma,\geq 0}$ and we show that this nonnegative part can be identified with a suitable horofunction compactification:

Theorem 5.3.8 *Let $X = X_{\Sigma_P}$ be a projective toric variety of dimension n. Then the following are homeomorphic:*

1) *the nonnegative part $X_{\geq 0}$ of the toric variety X*
2) *the image of the moment map of the toric variety X*
3) *the horofunction compactification $\overline{\mathbb{R}^n}^{hor}$ of $\mathbb{R}^n$ with respect to the norm $\|\cdot\|_P$*

These homeomorphisms give a bijective correspondence between projective toric varieties X of dimension n and rational polyhedral norms $\|\cdot\|$ on $\mathbb{R}^n$ up to scaling in every dimension $n \geq 1$. ◦

The homeomorphism here is given by the (toric) moment map μ which provides a homeomorphism between the nonnegative part $X_{\Sigma_P,\geq 0}$ of the toric variety and the dual of the polytope P that defines the fan Σ_P. This map is well-known in the context of toric varieties and can be found in standard literature (like [CLS11, Prop. 12.2.5] or [Ful93, §4.2]), but with slightly different notations. A similar convexity result about the image of the (symplectic) moment map is also well know in symplectic geometry and goes back to Atiyah [Ati82] and Guillemin-Sternberg [GS82]. The map that we use for the homeomorphism between the horofunction compactification and the dual unit ball is inspired by the moment map. Apart from the toric or symplectic setting, the moment map does not seem to be widespread, especially not in the context of convex sets where we used it.

To show Theorem 5.3.8 we define a *topological model* $\overline{T}_\Sigma$ of the variety where we explicitly describe the convergence behavior of sequences. The topological model is constructed as the complex torus T to which we attach some boundary components. A key point then is to show that the topological model $\overline{T}_\Sigma$ and the usual construction of X_Σ as the variety obtained from a fan are homeomorphic as T-topological spaces.

Structure of the thesis

We start in Chapter 2 with preliminaries about some concepts needed later, especially in the third chapter. We first unify notations abouts subspaces in Section 2.1 and then go over to convex sets and (asymmetric) norms in Section 2.2. Important for us is the structure of extreme sets and the notion of dual convex sets and faces, this will be treated in Sections 2.3 and 2.4. Further we will show some basics about the Minkowski sum in Section 2.5 and especially determine how faces of convex sets behave under the sum. The last part in the preliminary chapter (Section 2.6) introduces the maps $h_{E,p}$ that will later give the horofunctions.

Chapter 3 deals with the horofunction compactification, mainly in the setting of a finite-dimensional normed vector space. After a general introduction into horofunctions and especially horofunctions on a finite-dimensional normed vector space (Sections 3.1), we state the main result about the convergence behavior of sequences in the horofunction compactification in Section 3.2. The homeomorphism $\overline{X}^{hor} \simeq B^\circ$ is deduced in Section 3.3 and generalizes a result of joint work with Lizhen Ji that was published on the arXiv as [JS16]. The extension of the previous results to the case where the norm is the sum of two other norms is the content of Section 3.4.

Chapter 4 starts with an introduction to the theory of symmetric spaces and of Lie groups and Lie algebras in Section 4.1. The justification for determining the horofunction compactification of a symmetric space of non-compact type by compactifying a maximal flat is given in Section 4.2. We then compare the horofunction compactification with the Satake (Section 4.3) and the Martin compactification (Section 4.4). The results on the horofunction compactification and the Satake compactification of symmetric spaces are joint work with Thomas Heattel, Cormac Walsh and Anna Wienhard and published on the arXiv as [HSWW18].

Chapter 5 treats the theory of toric varieties. The basics on toric varieties and their construction from fans is given in Section 5.1. After defining the topological model of a toric variety in Section 5.2, we show the homeomorphism of the nonnegative part and the dual polytope in Section 5.3. The results of this chapter were deduced in joint work with Lizhen Ji and a shorter version containing all results was published as [JS17].

In Chapter 6 we state some open problems and questions for future research work.

2 | Preliminaries

In this section we introduce some basic definitions, notations and concepts used later, especially in Chapter 3. Throughout this section, let $(X, \|\cdot\|)$ be an n-dimensional normed real vector space. By $\langle\cdot|\cdot\rangle$ we denote the dual pairing of the dual space X^* and X. For a subset $A \subseteq X$ denote by $A^* \subseteq X^*$ its associated subspace by the identification of X and X^*. By identifying X with $\mathbb{R}^n$ we get an inner product on X from the Euclidean inner product on $\mathbb{R}^n$, which we denote by $\langle\cdot|\cdot\rangle_X$. The definitions and statements are inspired by [Sol15], though the notations and names there sometimes differ. Other references especially for convex sets and duality are [Bee93], [Mar77] and [Roc97].

2.1 Subspaces and Affine Spaces

We start by collecting some basic definitions and results about subspaces and affine spaces of $\mathbb{R}^n$ to unify notations.

Definition 2.1.1 Let $S \subseteq X$ be a non-empty set. The smallest subspace of X containing S is called the *subspace generated by* S and denoted by $V(S)$. ○

Definition 2.1.2 A non-empty subset $K \subseteq X$ is a *cone with apex* a, where $a \in X$, if for all $x \in K$ and $\lambda \geq 0$ it holds

$$a + \lambda(x - a) \in K.$$

If K is convex[1], we call it a *convex cone*. ○

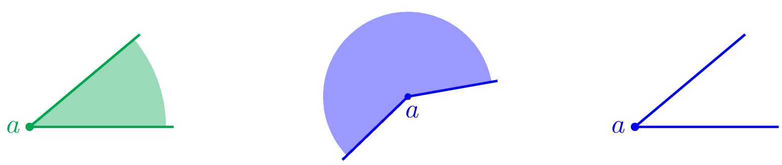


Figure 2.1: A convex (GREEN) and two non-convex cones (BLUE) in $\mathbb{R}^2$.

Remark 2.1.3 When the apex $a \in X$ of a cone is the origin, then we get the common condition $\lambda \cdot x \in K$ for all $\lambda \geq 0$ and $x \in K$ to define a cone and will then just call it a *cone*. ○

For any subset $S \subseteq X$ we can get at cone over S by taking all non-negative multiples of points in S:

Definition 2.1.4 Let $S \subseteq X$ be a set. The *cone* $K(S)$ *over* S is the smallest cone (with apex at the origin) in X containing S, that is,

$$K(S) := \{x \in X \mid x = \lambda \cdot s \text{ for some } \lambda \geq 0, s \in S\}. \qquad ○$$

[1] see Def. 2.2.1 on page 14

Definition 2.1.5 Let $V \subseteq \mathbb{R}^n$ be a subspace. Then the *orthogonal complement* $V^\perp$ of V is the subspace orthogonal to all elements of V:

$$V^\perp := \{y \in X \mid \langle y|x \rangle_X = 0 \ \forall x \in V\}. \qquad \circ$$

Remark 2.1.6 We could also have taken the quotient $X/V(F)$ instead of $F^\perp$, but since the orthogonal complement is more geometric, we use the complement $V(F)^\perp$. ◦

A subspace V and its orthogonal complement $V^\perp$ are complementary, that is, $V \oplus V^\perp = X$. Therefore every element $x \in X$ can uniquely be written as

$$x = x_V + x^V,$$

where $x_V \in V$ denotes the orthogonal projection to the subspace V and $x^V \in V^\perp$ the orthogonal projection to $V^\perp$.

Definition 2.1.7 An *affine subspace* $A \subseteq X$ is the translate of a subspace $V \subseteq X$, that means there is a point $a \in X$ such that

$$A = a + V = \{x = a + v \in X \mid v \in V\}.$$

The empty set is also considered as an affine subspace. If $\emptyset \neq A \neq X$, then the affine subspace A is called *proper*. ◦

Definition 2.1.8 Let $S \subseteq X$ be a non-empty set. The *affine hull* $\operatorname{aff}(S)$ over S is the smallest affine subspace containing S. ◦

The affine hull $\operatorname{aff}(S)$ of a set S can equivalently be defined as the intersection of all affine spaces containing S.

Just as we do for subspaces, we speak of affine (in)dependency of a set of points and we can make basic operations on affine spaces like sums and intersections. The exact definitions and results are given now.

Definition 2.1.9 A set $\{x_1, \dots, x_k\} \subseteq X$ is called *affinely independent*, if for $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ the combinations

$$\sum_{i=1}^k \lambda_i x_i = 0 \qquad \text{and} \qquad \sum_{i=1}^k \lambda_i = 0$$

are only possible for $\lambda_1 = \ldots = \lambda_k = 0$. Otherwise the set of points is called *affinely dependent*. The empty set $\emptyset$ and every set $\{x\}$ are considered to be affinely independent. ◦

Remark 2.1.10 For linear independency we do not require that $\sum_i \lambda_i = 0$. So a set can be linearly dependent but affinely independent but not the other way round. ◦

Example 2.1.11 In $\mathbb{R}^2$ consider the points

$$x_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x_4 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Then the set $\{x_1, x_2, x_3\} \subseteq \mathbb{R}^2$ is linearly dependent but affinely independent. The set $\{x_1, x_2, x_4\}$ is both linearly and affinely dependent. ◦

The following characterization helps to find affinely independent sets.

Proposition 2.1.12 ([Sol15, Thm 1.60]) *The set* $\{x_1, \dots, x_k\} \subseteq X$ *is affinely independent, if and only if the plane* $\operatorname{aff}(x_1, \dots, x_k)$ *has dimension* $k - 1$. ◦

Proposition 2.1.13 ([Sol15, Thm. 1.4]) *Intersections and finite weighted sums of affine spaces are again affine spaces.* ◦

Proposition 2.1.14 ([Sol15, Thm 1.2]) *Let $A \subseteq X$ be an affine subspace. Then A is the translate of a unique subspace $V \subseteq X$ given by*

$$V = A - A = \{x - y \mid x, y \in A\}.$$ ◦

Definition 2.1.15 The unique subspace of which the affine subspace A is a translate, is called the *space of translations* of A and denoted by $T(A)$.
For a subset $S \subseteq X$ we denote by $T(S)$ the space of translations of $\mathrm{aff}(S)$:

$$T(S) := T(\mathrm{aff}(S)) = \mathrm{aff}(S) - \mathrm{aff}(S).$$ ◦

To obtain the space of translations of A it is actually enough to consider A translated by a non-empty subset L of A. This holds especially when L only consists of a single point.

Corollary 2.1.16 ([Sol15, Thm. 1.2]) Let $A \subseteq X$ be an affine subspace and $L \subseteq A$ a non-empty subset. Then

$$T(A) = A - L.$$ ◦

Now that we assigned a subspace to every affine space A we can speak about the dimension of A. It is defined as the dimension its space of translations:

$$\dim(A) := \dim(T(A)).$$

Similarly, for any non-empty set $S \subseteq X$ we define ([Sol15, Def. 1.75]) the dimension of S by

$$\dim(S) := \dim(\mathrm{aff}(S)).$$

Then $dim(S)$ is the maximal number of affinely independent points in S minus 1.

Let the subset L in the corollary above be a single point, then we see that the space of translations $T(A)$ is a translate of A and we intuitively would say that A and $T(A)$ are "parallel". This statement is also consistent with the definition of parallelism:

Definition 2.1.17 Two non-empty affine subspaces A_1, A_2 are called *parallel*, if one of them contains a translate of the other. ◦

Note that we do not require the affine subspaces to have the same dimension for being parallel. Therefore parallelism is not an equivalence relation. An equivalent characterization of parallelism of A_1 and A_2 is to require that one of their space of translations contains the other one:

Proposition 2.1.18 ([Sol15, Thm. 1.13]) *Let $A_1, A_2 \subseteq X$ be non-empty affine subspaces with $\dim(A_1) \leq \dim(A_2)$. Then A_1 and A_2 are parallel if and only if $T(A_1) \subseteq T(A_2)$.* ◦

Corollary 2.1.19 In the situation of the proposition, if $\dim(A_1) = \dim(A_2)$, then A_1 and A_2 are parallel if and only if $T(A_1) = T(A_2)$, that is, A_1 and A_2 are translates of each other. ◦

Soltan [Sol15] calls affine subspaces *planes*. We keep the notation of affine subspaces, to avoid confusion with the common definition of a hyperplane, that is, an affine subspace of dimension $n - 1$.

Important for us will be the following characterization of a hyperplane as the level set of the dual pairing with some fixed vector:

Proposition and Definition 2.1.20 ([Sol15, Thm 1.17]) *An affine subspace $H \subseteq X$ is a hyperplane if and only if there is a point $g \neq 0 \in X^*$ and a scalar $\lambda \in \mathbb{R}$ such that*

$$H = H^g_\lambda := \{x \in X \mid \langle g|x\rangle = \lambda\}. \tag{2.1}$$

The above representation is unique up to a common non-zero scalar multiple of g and λ:

$$H^{kg}_{k\lambda} = H^g_\lambda \qquad \forall k \neq 0.$$ ◦

We will mainly consider the hyperplanes for $\lambda = -1$ and $\lambda = 0$.

Corollary 2.1.21 ([Sol15, Cor 1.18]) Two hyperplanes $H = H^h_{-1}$ and $G = G^g_{-1}$ in X are parallel if and only if they are defined by the same point up to a scalar multiple: there is an $\alpha \in \mathbb{R}$ such that

$$h = \alpha g.$$

Therefore their (common) space of translations is given by

$$T(H) = T(G) = \{x \in X \mid \langle h \mid x\rangle = 0\}.$$ ◦

With the notations introduced above in Equation (2.1), we see immediately that the space of translations of H is the orthogonal complement to the subspace generated by h:

$$T(H^h_{-1}) = \left(V(\{h\})^\perp\right)^* = H^h_0.$$

Every hyperplane H divides X into two closed half-spaces:

Definition 2.1.22 The affine half-spaces defined by a hyperplane H^h_λ will be denoted by:

$$\begin{aligned} V^h_{\geq\lambda} &:= \{x \in X \mid \langle h|x\rangle \geq \lambda\} \\ V^h_{\leq\lambda} &:= \{x \in X \mid \langle h|x\rangle \leq \lambda\}, \end{aligned}$$

and similarly for $V^h_{>\lambda}$ and $V^h_{<\lambda}$. ◦

2.2 Convex Sets and Norms

In this thesis we will often deal with compact convex sets as they can be seen as unit balls of a norm they define. Therefore we will now state some basic results about general (i.e. not necessarily compact) convex sets and the norm they define.

Definition 2.2.1 A non-empty subset $C \subseteq X$ is called *convex*, if for any two points $x, y \in C$ the interval between them is contained in C, that is, for all $0 \leq \lambda \leq 1$ it holds:

$$\lambda x + (1 - \lambda)y \in C.$$

Also the empty set is defined to be convex. ◦

Generalizing the sum above to more than two points, we get a convex combination:

Definition 2.2.2 A *convex combination* of points $x_1, \dots x_k \in X$ is a sum $\sum_{i=1}^k \lambda_i x_i$ where $\lambda_i \geq 0$ for all i and $\sum_{i=1}^k \lambda_i = 1$. When all scalars λ_i are positive, we call it a *positive convex combination*. ◦

The union of all convex combinations of two points gives the (closed) interval between them, where the union of all positive convex combinations yields the open interval. Similarly all (positive) convex combinations of three points not in a line produce an (open) closed triangle. This idea leads to the following proposition:

Proposition 2.2.3 ([Sol15, Thm. 2.3]) *Let $C \subseteq X$ be a non-empty set. Then C is convex if and only if it contains all convex combinations of elements in C.* ∘

Also on the set of convex sets basic operations are allowed:

Proposition 2.2.4 ([Sol15, Thm. 2.8/2.9]) *Intersections and finite weighted sums of convex sets are again convex.* ∘

Definition 2.2.5 Let $S \subseteq X$ be a non-empty set. The *convex hull* $\operatorname{conv}(S)$ over S is the intersection of all convex sets containing S. ∘

In Figure 2.2 we show some examples of convex hulls (green) over different sets (red). Note that different sets can define the same convex hull. In the second and third example we have points in the red set that are redundant for defining the convex hull, while in the example on the left, the convex hull changes if we take away one of the red points.

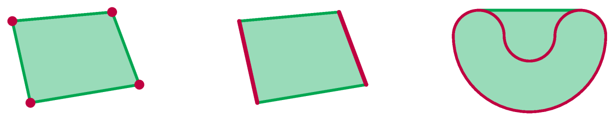


Figure 2.2: Some examples of convex hulls (green) over different sets of points (red).

Whenever a convex set C is given as the convex hull of a set of points, $C = \operatorname{conv}(S)$, we often want this set of points to be minimal, that is, we require $\operatorname{conv}(S) \neq \operatorname{conv}(S \setminus \{s\})$ for all $s \in S$. This means that each point s is a proper extreme point[2] of C.

Similarly to Theorem 2.2.3 one can describe the convex hull *conv*(S) over a set S as the set of all convex combinations of elements of S, see [Sol15, Thm. 3.3] for a proof. It is also easy to see that $\operatorname{conv}(S)$ is convex, contains S and if $R \subseteq S$ are two non-empty sets then $\operatorname{conv}(R) \subseteq \operatorname{conv}(S)$.

Consider a (filled) square in $\mathbb{R}^3$ lying in the xy-plane. Then by the common definition[3] it has empty interior. Nevertheless, the square is not empty, it is filled and considering the same definition not in $\mathbb{R}^3$ but in the subspace of the xy-plane, we get the "filling" as the interior. This is the basic idea of the definition of the *relative* interior of a set, see also Figure 2.3 for some examples.

Definition 2.2.6 Let $S \subseteq X$ be a non-empty set. The *relative interior* $\operatorname{relint}(S)$ of S is the interior of S within the affine subspace $\operatorname{aff}(S)$. In other words, a point $c \in X$ lies in $\operatorname{relint}(S)$ if and only if there is a $\rho > 0$, such that $B_\rho(c) \cap \operatorname{aff}(S) \subseteq S$.
The relative interior of the empty set is the empty set and the relative interior of a point is the point itself.
Similarly, we define the *relative boundary* of A as $\partial_{\mathrm{rel}}(S) := (\operatorname{cl} S) \setminus (\operatorname{relint}(S))$ where $\operatorname{cl}(S)$ denotes the closure of S. ∘

We now state some facts about the behavior of the relative interior, the relative boundary and convex sets and hulls.

[2] Extreme sets will be defined in Definition 2.3.3 on page 19. A point x in the boundary of a convex set C is extreme, if for all closed intervals $I \subseteq C$ with $x \in I$ the point x has to be an endpoint of the interval.

[3] Let $S \subseteq X$ be a set. Then a point $c \in X$ is *interior* of S, if there is a $\rho > 0$, such that $B_\rho(c) \subseteq S$. Additionally $\operatorname{int}(\emptyset) = \emptyset$. The union of all interior points of S is called the interior of S and denoted by $\operatorname{int}(S)$.

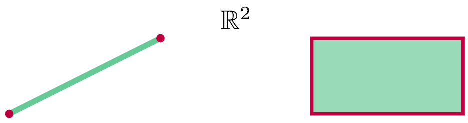


Figure 2.3: The relative interior of a line segment in $\mathbb{R}^2$ is the segment between the two endpoints. For a square in $\mathbb{R}^2$ the relative and the normal interior coincide.

Lemma 2.2.7 ([Sol15, Cor. 2.22]) *The relative interior* $\operatorname{relint}(C)$ *of a convex set* $C \subseteq X$ *is convex.* ◦

Proposition 2.2.8 ([Sol15, Thm. 3.2/3.17/3.19]) *Let* $S \subseteq X$ *be a non-empty set. Then it holds:*

1) $\operatorname{aff}(\operatorname{conv} S) = \operatorname{aff}(S)$.

2) $\operatorname{conv}(\operatorname{cl} S) \subseteq \operatorname{cl}(\operatorname{conv} S)$ *with equality if* S *is bounded.*

3) $\operatorname{conv}(\operatorname{relint} S) \subseteq \operatorname{relint}(\operatorname{conv} S)$. ◦

As a consequence of the latter proposition, if S is compact, then so is its convex hull $\operatorname{conv}(S)$.

Definition 2.2.9 A set $S \subseteq C$ is *compact* if every open cover has a finite subcover. That is, for every family of open sets $(U_\alpha)_\alpha$ such that $S \subseteq \bigcup_\alpha U_\alpha$, there exists a finite subfamily $(U_{\alpha_i})_{i=1,\dots,k}$ such that $S \subseteq \bigcup_{i=1,\dots,k} U_{\alpha_i}$. ◦

Proposition 2.2.10 (Heine-Borel) *Let* X *be a finite-dimensional normed vector space and let* $S \subseteq X$ *be a subset. Then* S *is compact if and only if* S *is closed and bounded.* ◦

A convex set can not only be described as the set of all convex combinations (which is an intrinsic description) but also as the intersection of half-spaces. To obtain this, we first have to get to know supporting hyperplanes.

Definition 2.2.11 Let $A \subseteq X$ be an affine subspace and $C \subseteq X$ a non-empty convex set. Then A is said to *properly support* C if A intersects exactly the relative boundary of C: $A \cap C \neq \emptyset$ and $A \cap \operatorname{relint}(C) = \emptyset$. If either $C \subseteq A$ or A properly supports C then we just say that A *supports* C. ◦

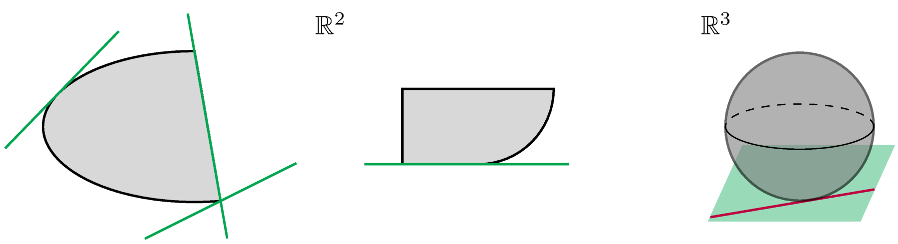


Figure 2.4: Some examples of supporting hyperplanes (green) in $\mathbb{R}^2$ (LEFT, MIDDLE) and in $\mathbb{R}^3$ (RIGHT). The red line supports properly, but is not a hyperplane in $\mathbb{R}^3$.

The third picture in the example above shows that by the convexity of C, any affine subspace properly supporting C can be enlarged to a properly supporting hyperplane. This is also the content of the following proposition:

Proposition 2.2.12 ([Sol15, Thm. 6.8]) *Let* $C \subseteq X$ *be a convex set and* $A \subseteq X$ *an affine subspace properly supporting* C*. Then there is a hyperplane* $H \subseteq X$ *containing* A *and properly supporting* C*.* ◦

The affine hull of a convex set $\emptyset \neq D \subseteq \partial_{\mathrm{rel}} C$ in the relative boundary of a closed compact convex set $C \subseteq X$ is an affine subspace properly supporting C. Therefore D lies in a properly supporting hyperplane to C. See [Sol15, Cor 6.9] for a proof.

Corollary 2.2.13 ([Sol15, Cor.6.10]) Let $C \subseteq X$ be a proper convex set. Then a hyperplane $H \subseteq X$ properly supports C if and only if $H \cap \mathrm{cl}\, C \neq \emptyset$ and C is contained in one of the closed half-spaces determined by H. ∘

The following observation will help us later when dealing with the dual unit ball of a norm.

Corollary 2.2.14 Let $C \subseteq X$ be a compact convex set and $H = H^h_\lambda$ ba a hyperplane properly supporting C. Then

$$\begin{aligned} \text{either} \quad & \langle h|c\rangle \geq \lambda \qquad \forall c \in C \\ \text{or} \quad & \langle h|c\rangle \leq \lambda \qquad \forall c \in C. \end{aligned}$$ ∘

As compact sets are bounded, it holds:

Corollary 2.2.15 (see also Prop. 2.3.1) Every closed compact convex set $C \subseteq X$ has a supporting hyperplane ∘

Hyperplanes are also used to separate convex sets:

Definition 2.2.16 Let $S_1, S_2 \subseteq X$ be non-empty set. Then a hyperplane $H \subseteq X$ *separates* S_1 and S_2 if S_1 and S_2 lie in the opposite closed half-spaces determined by H. The hyperplane H *properly separates* S_1 and S_2 if $S_1 \cup S_2 \nsubseteq H$. If there is a scalar $\rho > 0$ such that H separates the ρ-neighborhoods $B_\rho(S_1)$ and $B_\rho(S_2)$, then H *strongly separates*. ∘

Proposition 2.2.17 ([Roc97, Thm. 11.1]) *Let S_1 and S_2 be non-empty sets in X*

1) *S_1 and S_2 are properly separated by a hyperplane, if and only if there exists a point $h \in X^*$ such that*

   a) $\inf\{\langle h|s\rangle \mid s \in S_1\} \geq \sup\{\langle h|t\rangle \mid t \in S_2\}$

   b) $\sup\{\langle h|s\rangle \mid s \in S_1\} > \inf\{\langle h|t\rangle \mid t \in S_2\}$.

2) *S_1 and S_2 are strongly separated by a hyperplane, if and only if there exists a point $h \in X^*$ such that*
$$\inf\{\langle h|s\rangle \mid s \in S_1\} > \sup\{\langle h|t\rangle \mid t \in S_2\}.$$ ∘

For convex sets, we even have:

Proposition 2.2.18 ([Sol15, Thm. 6.30/6.32]) *Let $C_1, C_2 \subseteq X$ be two non-empty convex sets.*

1) *C_1 and C_2 are properly separated if and only if* $\mathrm{relint}(C_1) \cap \mathrm{relint}(C_2) = \emptyset$.

2) *C_1 and C_2 are strongly separated if and only if* $\inf\{\|x_1 - x_2\| \mid x_1 \in C_1, x_2 \in C_2\} > 0$. ∘

As mentioned before, compact convex sets in X are closely related to norms. We now show this relation and start with the definition of an asymmetric norm:

Definition 2.2.19 An *asymmetric norm* $\|\cdot\|$ on X is a function $\|\cdot\| : X \to \mathbb{R}_{\geq 0}$ satisfying:

1) For any $x \in X$, if $\|x\| = 0$, then $x = 0$.

2) For any $\alpha \geq 0$ and $x \in X$, $\|\alpha x\| = \alpha\|x\|$.

3) For any two vectors $x, y \in X$, $\|x + y\| \leq \|x\| + \|y\|$. ∘

In particular, $||x||$ and $||-x||$ may not be equal to each other. If the second condition is replaced by the stronger condition: $||\alpha x|| = |\alpha|||x||$ for all $\alpha \in \mathbb{R}$, then $||\cdot||$ is symmetric and is a usual norm on X.

Remark 2.2.20 It is rather confusing in the beginning to consider asymmetric norms and to get used to the fact that possibly $||x|| \neq ||-x||$. But in the relation with compact convex sets, this asymmetric definition is much more natural than the symmetric one. More about asymmetric norms can be found for example in [Cob13]. ∘

Given an asymmetric norm $||\cdot||$ on X, the unit ball $B_{||\cdot||}$ of the norm is given by

$$B_{||\cdot||} = \{x \in X \mid ||x|| \leq 1\}.$$

It is a compact convex subset of X which contains the origin as an interior point. Conversely, given any convex compact subset C of X which contains the origin as an interior point, this defines an asymmetric norm $||\cdot||_C$ on X by

$$||x||_C := \inf\{\lambda > 0 \mid x \in \lambda C\}. \tag{2.2}$$

If C is symmetric with respect to the origin then $||\cdot||_C$ is a norm on X.

It is easy to see that the unit ball of $||\cdot||_C$ is equal to C. Since any asymmetric norm $||\cdot||$ on X is uniquely determined by its unit ball, it is of the form $||\cdot||_C$ for some closed convex domain C in X containing the origin in its interior.

Definition 2.2.21 When P is a polytope, the asymmetric norm $||\cdot||_P$ is called a *polyhedral norm*. If $P \subseteq \mathbb{R}^n$ is a rational polytope with respect to the integral structure $\mathbb{Z}^n \subseteq X$, the norm $||\cdot||_P$ is also called a *rational polyhedral norm*. ∘

Remark 2.2.22 (Connections to the Minkowski and Hilbert geometry) Finite-dimensional normed vector spaces are sometimes also called Minkowski geometry or Minkowski spaces as in [Tho96] or [FK05]. This interplay between convex subsets of $\mathbb{R}^n$ and norms on $\mathbb{R}^n$ plays a foundational role in the convex analysis of Minkowski geometry, see for example [Gru07] and [Tho96].

There is another metric space associated with a convex domain Ω of $\mathbb{R}^n$. It is the domain Ω itself equipped with the Hilbert metric defined on it. When Ω is the unit ball of $\mathbb{R}^2$, this is the Klein's model of the hyperbolic plane. In general, the Hilbert metric is a complete metric on Ω defined through the cross-ratio, see [dlH93] for details. Since Ω is diffeomorphic to $\mathbb{R}^n$, the Hilbert metric induces a metric on $\mathbb{R}^n$.

The polyhedral Hilbert metric associated with a polytope P is isometric to a normed vector space if and only if the polytope P is the simplex [FK05, Theorem 2].

These discussions show that polyhedral norms on $\mathbb{R}^n$, in particular rational polyhedral norms, are very special in the context of the Minkowski geometry [Tho96] and the Hilbert geometry [dlH93]. ∘

2.3 Extremal Structure of Convex Sets

In this section we will examine more closely the structure of the (convex) sets in the (relative) boundary of a convex set. This will lead us to the notion of extreme sets and points, which play a very important role in the description of horofunctions in the next section.

Proposition 2.3.1 (Minkowski; [Gal08, Thm 3.17]) *Let $C \subseteq X$ be a non-empty, closed subset. If C is convex, then there is a supporting hyperplane to C through c for every boundary point $c \in \partial C$.* ◦

When we require C to have non-empty interior, then the converse of the proposition is also true ([Gal08, Thm 3.18]). We distinguish between boundary points with exactly one or more supporting hyperplanes to it.

Definition 2.3.2 Let $A \subseteq X$ be a d-dimensional affine subspace and $\emptyset \neq C \subseteq A$ a closed convex subset. A boundary point $C \in \partial C$ is called *smooth*, if the supporting hyperplane to c is unique. ◦

Definition 2.3.3 Let $C \subseteq X$ be a convex set. A non-empty convex subset $F \subseteq C$ is called an *extreme set* or *extreme face* of C, if for any $x \in F$ and every closed interval $I \subseteq C$ with $x \in \operatorname{relint}(I)$ there holds $I \subseteq F$. The empty set is defined to be extreme. F is called *proper extreme* if $\emptyset \neq F \neq C$. An extreme face F with $\dim F = 0$ is called an *extreme point* and if it has dimension $n-1$ then it is called a *facet*. ◦

Note that we require the set F to be convex to be an extreme face.

Definition 2.3.4 Let C be a compact convex set. The set of extreme points of C will be denoted by $\mathcal{E}_C$. ◦

Figure 2.5 below shows a convex set C in $\mathbb{R}^2$ and its extreme sets. Every orange point is an extreme point of C. Every green line is a proper extreme facet. Note that there are infinitely many extreme points where the boundary is smoothly curved.

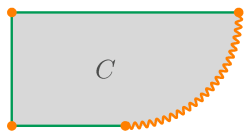


Figure 2.5: A convex set with its proper extreme sets: green facets and orange extreme points.

Definition 2.3.5 An extreme point $x \in \partial C$ of a compact convex set C is called *isolated*, if there is an $\varepsilon > 0$ such that x is the only extreme point of C in the ε-ball $B_\varepsilon(x)$ around x. Otherwise x lies in the closure of a smooth part of ∂C. An isolated extreme point is also called a *vertex* of C. ◦

A proper extreme set always lies in the relative boundary of C. Indeed, assume an extreme set F meets the relative interior of C, that is, $F \cap \operatorname{relint} C \neq \emptyset$. Let $x \in F \cap \operatorname{relint}(C)$ be a point in this intersection and $c \in C$ be any point in C. Denote by g the line-segment starting at c and going through x until the boundary of C. Then x is an interior point of the line-segment g lying in F. By extremality of F, c also lies in F. As $c \in C$ was arbitrary $F = C$. So as a proper extreme set, F lies in the relative boundary of C and $\dim F \leq \dim C - 1$. ([Sol15, Thm.7.4]).

Extreme sets are always defined relatively to the set they are subsets of. Therefore there are several results about the behavior of extreme sets depending on the structure of C:

Proposition 2.3.6 ([Sol15, Thm. 7.2/7.3/7.4]) *Let $C \subseteq X$ be a convex set, then it holds:*

1) *Arbitrary intersections of extreme sets of C are again extreme sets of C.*
2) *Let $F \subseteq C$ be an extreme set. If C is closed, then so is F.*
3) *A convex set F is an extreme set of C if and only if the set $C \setminus F$ is convex and $C \cap \operatorname{aff}(F) = F$.*
4) *Let $F \subseteq C$ be an extreme set and $G \subseteq F$ also be extreme. Then G is an extreme set of C.*
5) *Distinct extreme faces of C have disjoint relative interior.* ◦

The extreme points are special extreme sets as they are the smallest proper ones. Any compact convex set of X has some extreme point in its relative boundary. The following proposition (see for example [Sol15, Thm. 7.17/7.18]) shows that we get any compact convex set as a convex hull of all its extreme points:

Proposition 2.3.7 (**Krein-Milman Theorem**) *Let $C \subseteq X$ be a compact convex set and $\mathcal{E}_C$ the set of all extreme points of C. Then for any subset $S \subseteq C$ we have $C = \operatorname{conv}(S)$ if and only if $\mathcal{E}_C \subseteq S$. In particular it holds*

$$C = \operatorname{conv}(\mathcal{E}_C). \qquad \circ$$

So every point of C can be written as a convex combination of its extreme points. Even more, this convex combination is not only finite but the number of extreme points needed is bounded above by $\dim(C) + 1$, see [Sol15, Cor. 7.19].

We know that an extreme set F of a convex set C lies in the relative boundary of C intersected with the affine hull over F. In other words, $\operatorname{aff}(F)$ is a supporting affine subspace to C with $F = \operatorname{aff}(F) \cap C$. This holds for any extreme set. If F is not only the intersection of C with an supporting affine subspace of lower dimension but with a supporting hyperplane, we call this set *exposed*:

Definition 2.3.8 Let $C \subseteq X$ be a convex set. A subset $F \subseteq C$ is called an *exposed face* of C, if either there is a supporting hyperplane $H \subseteq X$ to C such that $F = H \cap C$ or $F = \emptyset$ or $F = C$. In the first case, F is called a *proper exposed face*. $\circ$

Any hyperplane H satisfying $F = H \cap C$ for an exposed face F of C is actually supporting C and it is properly supporting it, if F is a proper exposed face (see [Sol15, Cor. 8.4]).

Proposition 2.3.9 ([Sol15, Thm. 8.2]) *Every exposed face $F \subseteq C$ of a convex set $C \subseteq X$ is also extreme.* $\circ$

The proposition follows from the convexity of C and because F lies in a supporting hyperplane to it. Therefore every line with interior point in F also lies in the hyperplane and then has its endpoints in F.

The opposite of the statement is not true as the following example shows.

Example 2.3.10 Consider the two convex sets given in Figure 2.6. The orange points are extreme and exposed. The pink points are extreme but not exposed, as any intersection with a hyperplane containing one of them also contains the neighboring one-dimensional extreme face. The green lines are again both extreme and exposed. In the convex set on the right the green facets contain the pink points as relative boundary points. $\circ$

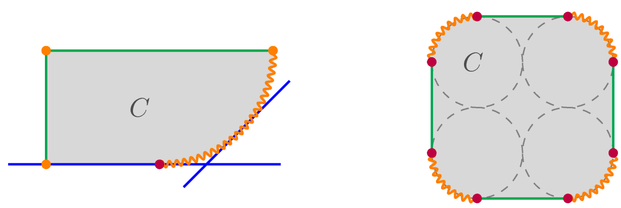


Figure 2.6: Two convex sets and their extreme set. The orange points are extreme and exposed whereas the pink extreme points are not exposed. In both cases the green lines are facets.

With the above proposition, some properties of general extreme sets stated previously can be taken over to exposed faces directly.

Proposition 2.3.11 ([Sol15, Cor. 8.3/Thm. 8.9]) *Let $C \subseteq X$ be a convex set and $F \subseteq C$ be an exposed face. Then it holds:*

1) *The intersection of exposed faces of C is again an exposed face of C.*

2) *If $F \cap \mathrm{relint}(C) \neq \emptyset$, then $F = C$.*

3) *Then $F \subseteq \partial_{rel}(C)$ and $\dim(F) \leq \dim(C) - 1$.*

4) *If C is closed, then so is F.*

5) *Distinct exposed faces of C have disjoint relative interior.* ○

Remark 2.3.12 The property of being an extreme set was transitive as seen in 2.3.6[(4)]. This is not true any more for exposed faces as can be seen by the convex set on the right in Example 2.3.10, where the pink points are exposed faces of the green lines but not of C. Nevertheless, we can find a sequence of sets such that each one is an exposed face of the next bigger one. ○

Proposition 2.3.13 ([Sol15, Thm. 8.5]) *Let $C \subseteq X$ be a convex set and $F \subseteq C$ an extreme set. Then we can find a sequence S_i of sets such that*

$$F = S_k \subseteq S_{k-1} \subseteq \cdots \subseteq S_1 \subseteq S_0 = C$$

where $S_j \subseteq S_{j-1}$ is a proper exposed face for all $1 \leq j \leq k-1$. ○

By the transitivity of extremality, each S_j in the proposition above is an extreme set of C.

Corollary 2.3.14 ([Sol15, Cor. 8.6]) Let $C \subseteq X$ be a convex set of dimension $m > 0$. Then every extreme face $F \subseteq C$ of dimension $m - 1$ is exposed. ○

Although not every exposed point is extreme, the set of exposed points lies dense in the set of extreme points:

Proposition 2.3.15 ([Sol15, Cor. 8.20]) *Let $C \subseteq X$ be a compact convex set, $\mathcal{E}_C$ the set of extreme points of C and $\mathcal{EP}_C$ those of exposed points. Then*

$$\mathcal{EP}_C \subseteq \mathcal{E}_C \subseteq \mathrm{cl}(\mathcal{EP}_C).$$ ○

2.4 Duality of Convex Sets and Faces

From now on (if not stated otherwise) we denote by $B \subseteq X$ a compact convex set of dimension n containing the origin 0 in its interior. As mentioned before, such a set B defines a norm of which it is the unit ball. This is why we will also name B a *unit ball* from now on. We will call extreme faces just *faces* of B and state individually whether they are exposed or not. To any unit ball B in X we can assign a set B° in the dual space X^* which is also the unit ball of a norm. It this section we want to explore the structure of this dual unit ball and of its faces and give a correspondence between the faces of B and those of B°. The understanding of this extremal structure and correspondence is crucial for the proof of Theorem 3.2.6.

Definition 2.4.1 Let $B \subseteq X$ be a non-empty compact convex set. Then its *dual* B° is a subset of the dual space X^* and defined as the polar of B:

$$B^\circ := \{y \in X^* \mid \langle y|x \rangle \geq -1 \ \forall x \in B\}.$$ ○

Remark 2.4.2 Some authors define the polar and thereby the dual of a compact convex set by the condition $\langle y|x\rangle \leq 1 \;\; \forall x \in B$. As long as B is symmetric, this makes as a set no difference. If B is not symmetric, we get the same result with our definition by replacing x with $-x$ and then acting carefully with the signs. ○

In Figure 2.7 there are given two examples of unit balls B in $\mathbb{R}^n$ and their duals B°. The colors indicate the pair of extreme sets that have dual pairing -1 with each other.

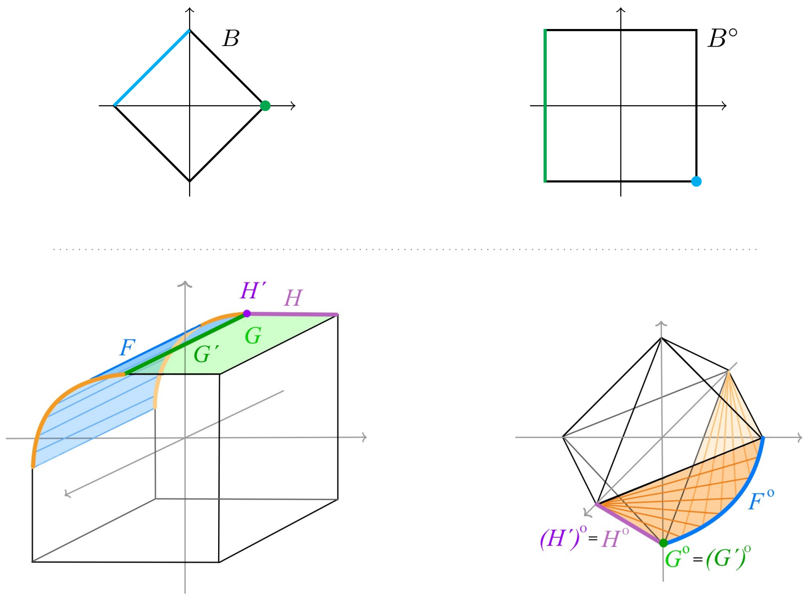


Figure 2.7: Two examples of a compact convex set B (LEFT) and its dual B° (RIGHT) in $\mathbb{R}^2$ and $\mathbb{R}^3$. The color of faces indicate the pair of faces that have pairing equal to -1 with each other.

Coming from $B \subseteq X$ we can define its dual B°, but what happens if we take the dual again, so how can we describe $B^{\circ\circ} = (B^\circ)^\circ$?. As for the origin $\langle 0|x\rangle = 0 > -1$ for any $x \in X$ it is clear that $0 \in B^\circ$ and therefore also $0 \in B^{\circ\circ}$. By definition it is also obvious that $B \subseteq B^{\circ\circ}$. A precise description is the following (from [Bee93, Thm. 1.4.6]):

Proposition 2.4.3 (Bipolar Theorem) *Let $S \subseteq X$ be a non-empty set. Then*

$$S^{\circ\circ} = \mathrm{cl\,conv}(S \cup \{0\}).$$ ○

Corollary 2.4.4 Let $B \subseteq X$ be a compact convex set with the origin $\{0\}$ in its interior. Then

$$B^{\circ\circ} = B.$$ ○

To determine the dual of a given convex set C can sometimes be difficult. In such a case it might be helpful to see C as the intersection or union of other convex sets, whose duals are already known. The following relations then help to get C°:

Lemma 2.4.5 *Let $A, B \subseteq X$ be compact convex sets containing the origin in their interior. Then*

1) *If $A \subseteq B$ then $B^\circ \subseteq A^\circ$.*

2) $(A \cup B)^\circ = A^\circ \cap B^\circ$.

3) $(A \cap B)^\circ = \mathrm{conv}(A^\circ \cup B^\circ)$. ○

Proof. The proof of the statements is a direct calculation using the definition of duality:

1) Let $A \subseteq B$ and $x \in B^\circ$. Then $\langle x|b\rangle \geq -1$ for all $b \in B$. As B contains A we obviously have $x \in A^\circ$.
2) Let $x \in (A \cup B)^\circ$, i.e $\langle x|a\rangle \geq -1$ and $\langle x|b\rangle \geq -1$ for all $a \in A$ and $b \in B$. This is equivalent to $x \in A^\circ \cap B^\circ$ which shows equality.
3) We use the previous result and the Bipolar Theorem 2.4.3 to compute:

$$\begin{aligned}(A \cap B)^\circ &= (A^{\circ\circ} \cap B^{\circ\circ})^\circ \\ &= [(A^\circ)^\circ \cap (B^\circ)^\circ]^\circ \\ &= [(A^\circ \cup B^\circ)^\circ]^\circ \\ &= (A^\circ \cup B^\circ)^{\circ\circ} \\ &= \operatorname{conv}(A^\circ \cup B^\circ)\end{aligned}$$ □

We will now investigate how the extremal structure of B determines B°. By the previous corollary all these results can be applied to B° to get back the structure of B.

Take an extreme point $p \in \mathcal{E}_B$. It defines us a hyperplane

$$H^p_{-1} = \{y \in X^* \mid \langle y|p\rangle = -1\}$$

and out of this two closed affine half-spaces $V^p_{\geq -1}$ and $V^p_{\leq -1}$ in X^* (see Definition 2.1.22 on page 14). As the dual set B° consists of all points with dual pairing ≥ -1 with all elements of B, it is clear that $B^\circ \subseteq V^p_{\geq -1}$ and H^p_{-1} supports B°. This is the basic idea of the following characterization of B°:

Lemma 2.4.6 *Let $B = \operatorname{conv}(\mathcal{E}_B) \subseteq X$ be compact convex with the origin in its interior. Then*

$$B^\circ = \bigcap_{p \in \mathcal{E}_B} V^p_{\geq -1}.$$ ○

Proof. By the previous discussion the inclusion $\subseteq$ is clear. For the other direction take a point $x \in \bigcap_{p \in \mathcal{E}_B} V^p_{\geq -1}$ and $b \in B$. As x and b are arbitrary, we have to show that $\langle x|b\rangle \geq -1$ to finish the proof. By the Krein-Milman Theorem (Thm. 2.3.7) and the following comment on page 20 there are finitely many extreme points $p_i \in \mathcal{E}_B$ and coefficients $\lambda_i \geq 0$ satisfying $\sum_i \lambda_i = 1$ such that

$$b = \sum_i \lambda_i p_i.$$

Therefore, as $\langle x|p\rangle \geq -1$ for any extreme point $p \in \mathcal{E}_B$:

$$\langle x|b\rangle = \sum_i \lambda_i \langle x|p_i\rangle \geq -\sum_i \lambda_i = -1,$$

as it was to show. □

Example 2.4.7 We calculate the unit balls of two standard norms in $\mathbb{R}^n$.

1) Let $\{v_1, \ldots, v_n\}$ be a system of orthogonal vectors in $\mathbb{R}^n$ and $D = \operatorname{conv}\{v_1, -v_1, v_2, -v_2, \ldots, -v_n\}$ be our unit ball. Denote by $\{v_1^*, \ldots, v_n^*\}$ the dual basis in $(\mathbb{R}^n)^*$. For each $j \in \{1, \ldots, n\}$ we have

$$H^{\pm v_j}_{-1} = \left\{ \sum_{i=1}^n a_i v_i^* \,\middle|\, a_i \in \mathbb{R},\ a_j = \mp \frac{1}{\left\langle v_j^* | v_j \right\rangle} \right\}$$

by the orthogonality. So by the previous lemma,

$$D^\circ = \bigcap_{j=1}^{n} V_{\geq -1}^{v_j} \cap \bigcap_{j=1}^{n} V_{\leq -1}^{-v_j}.$$

This dual set is a rotated $n-$dimensional cube with facets orthogonal to the basis vectors. If we choose the standard basis vectors as the v_j, then D is the unit ball of the L^1-norm on $\mathbb{R}^n$ and D° is the unit ball of the L^∞-norm.

2) Let $B := B_r(0) \subseteq \mathbb{R}^n$ be the Euclidean unit ball of radius $r > 0$. We claim that

$$(B_r(0))^\circ = B_{\frac{1}{r}}(0). \tag{2.3}$$

This is easy to see with the previous calculation, Lemma 2.4.5 and the Bipolar Theorem: Let

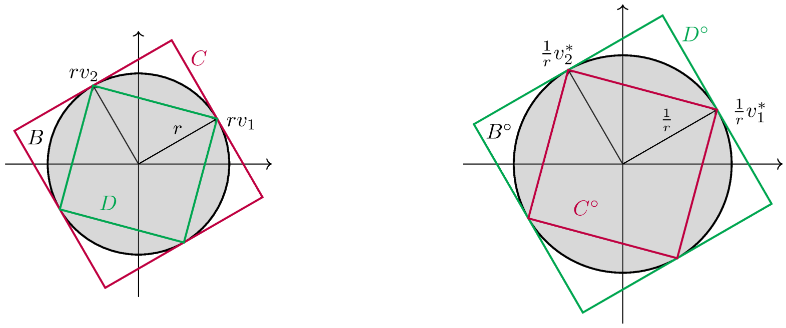


Figure 2.8: LEFT: The Euclidean unit ball of radius r contains the (green) diamond D and is itself contained in the (red) square C° RIGHT: The dual Euclidean unit ball has radius $\frac{1}{r}$. It contains the (red) diamond C° and is itself contained in the (green) square D°.

$D = \operatorname{conv}\{v_1, -v_1, v_2, -v_2, \ldots, -v_n\}$ be as in the previous example with respect to a system $\{v_j\}_{j=1,\ldots,n}$ of unit orthogonal vectors. In Figure 2.8 the two-dimensional case is shown.

Then $B = B_r(0)$ contains the set D and therefore D° contains B°. On the other hand, B is contained in a cube C with facets of distance r from the origin, so C° is contained in B°. As C° has vertices $\pm\frac{1}{r}v_j^*$ and the facets of D° are orthogonal to $\pm\frac{1}{r}v_j^*$ for all $j = 1, \ldots, n$ and this holds for any system of orthogonal vectors, claim follows. ◦

The above characterization of B° as the intersection of closed half-spaces leads to the following result.

Lemma 2.4.8 *Let B be a compact convex set with the origin in its interior. Then B° is also compact and convex and $\{0\} \in \operatorname{relint}(B^\circ)$. If P is polyhedral, then so in P°.* ◦

Proof. It is obvious that B° is closed and contains the origin in its interior. Let $r > 0$ be maximal such that $B_r(0) \subseteq B$. Then $B^\circ \subseteq B_{\frac{1}{r}}(0)$ by Equation 2.3 and we see that B° is bounded. For convexity of B° let $x, y \in B^\circ$ and $\lambda \in [0, 1]$. Then for the point $m = \lambda x + (1 - \lambda)y$ and any point $b \in B$ it holds

$$\begin{aligned} \langle m|b\rangle &= \lambda \langle x|b\rangle + (1-\lambda) \langle y|b\rangle \\ &\geq -\lambda - (1-\lambda) = -1, \end{aligned}$$

which means that $m \in B^\circ$.

For a polyhedral P, the set of extreme points is finite and by the previous lemma, P° is the intersection of finitely many closed half-spaces containing the origin in their interior and therefore also polyhedral. □

Our compact convex set $B \subseteq X$ can not only be described as the convex hull of its extreme points but also as the intersection of half-spaces $V^g_{\geq -1}$ defined by hyperplanes H^g_{-1} supporting B, where $g \in X^*$. Actually, as $B \subseteq V^g_{\geq -1}$ because $\{0\} \in B$, we have $g \in B^\circ$. This gives us another way to describe the dual B°, which is the complementary description to the previous one:

Lemma 2.4.9 *Let $B \subseteq X$ be compact convex with the origin in its interior. Then*

$$B^\circ = \operatorname{conv}\{g \in X^* \mid H^g_{-1} \text{ supports } B\}. \qquad \circ$$

Proof. Let $B = \bigcap_{g \in G} V^g_{\geq -1}$ be given as the intersection of half-spaces for a set $G \subseteq X^*$. Then each hyperplane H^g_{-1} with $g \in G$ supports B. The proof follows immediately by the third point of Lemma 2.4.5: With $B = \bigcap V^g_{\geq -1}$ we get

$$\begin{aligned} B^\circ &= \left(\bigcap V^g_{\geq -1}\right)^\circ \\ &= \operatorname{conv}\left\{\left(V^g_{\geq -1}\right)^\circ \;\middle|\; H^g_{-1} \text{ supports } B\right\} \\ &= \operatorname{conv}\{g \;| g \in G\}, \end{aligned}$$

because $\left(V^g_{\geq -1}\right)^\circ = \operatorname{conv}(g, \{0\})$. $\square$

Any boundary point v of B has at least one supporting hyperplane. When we intersect all closed affine half-spaces containing the origin that are defined by the hyperplanes supporting B at v, then we obtain an cone with apex v containing B. Figure 2.9 shows an example for a smooth boundary point v_1 with a unique supporting hyperplane and for a boundary point v_2 with non-unique hyperplane. Such a cone is minimal in the sense that we can not find a hyperplane passing through v in the interior of the cone that does not intersect the interior of B. So the cone is tangent to B at v and it is actually enough to consider the the hyperplanes tangent to ∂B at v to define the cone.

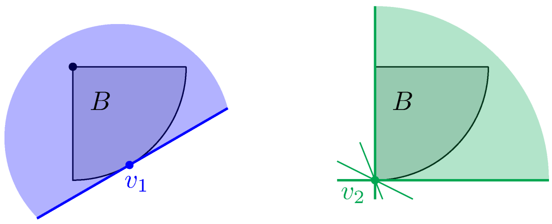


Figure 2.9: A convex set B with two tangent cones. LEFT: At a smooth boundary point v_1 the hyperplane is unique and the cone is a half-space; RIGHT: A point v_2 with infinitely many supporting hyperplanes.

In the last section we talked about extreme and exposed faces of compact convex sets. As the dual of a compact convex set is again compact and convex we can ask ourselves how faces behave under duality. For this we first have to define duals of faces. Recall that for (extreme) faces we have to distinguish whether they are exposed or not.

Definition 2.4.10 Let $F \subseteq C$ be a proper face. The *exposed dual* F° of F is defined as the set

$$F^\circ := \{y \in C^\circ \mid \langle y|f\rangle = -1 \;\forall f \in F\}. \qquad \circ$$

Definition 2.4.11 Let $F \subseteq C$ be a proper face. A proper extreme set $E \subseteq C^\circ$ is called *dual to* F if either $E = F^\circ$ or $E \subseteq \partial_{\text{rel}} F$ is not an exposed face of C.
By $\mathcal{D}(F)$ we denote the set of all duals of F. $\circ$

Example 2.4.12 Two examples of compact convex sets B and their duals B° are given in Figure 2.10. The colors indicate the duality between the faces. In the polyhedral case (a) every face has exactly one dual face.

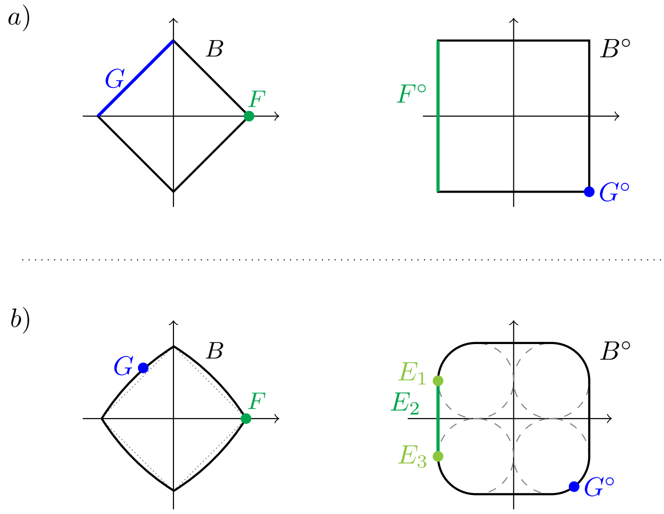


Figure 2.10: Two convex sets and their duals with dual faces. In the polyhedral case (a), every face has exactly one dual face. In the general case (b), a face can have several dual faces.

In example (b), the face F has three faces dual to it: E_1, E_2, E_3. The face E_2 is the exposed dual face and the other two are in its relative boundary and not exposed faces of B°. Such a duality relation holds for any of the four points where the boundary is not smooth. In the smooth part of the boundary of B, as for the extreme point G, we again have only one dual face. ○

Lemma 2.4.13 *Let $F \subseteq C$ be a proper face. Then $F^\circ \subseteq C^\circ$ is an exposed face given by*

$$F^\circ = \bigcap_{f \in F} H^f_{-1} \cap C^\circ. \qquad \circ$$

Proof. The characterization of the exposed dual face as the intersection of all hyperplanes follows directly by definition:

$$\begin{aligned} F^\circ &= \{y \in C^\circ \mid \langle y \mid f \rangle = -1 \ \forall f \in F\} \\ &= \bigcap_{f \in F} H^f_{-1} \cap C^\circ. \end{aligned}$$

To see that F° is exposed we first show that the intersection is not empty. As an extreme set of C, the set F is contained in some hyperplane H_F supporting C. So there is a point $d \in X^*$ such that $H_F = H^d_{-1}$ and therefore especially $\langle d | f \rangle = -1$ for all $f \in F$. For the intersection not to be trivial it remains to show that $d \in C^\circ$. As H_F supports C and $0 \in C$ we have $C \setminus (H_F \cap C) \subseteq V^d_{\geq}$, that is, $\langle d | x \rangle > -1$ for all $x \in C \setminus (H_F \cap C)$. Together with $\langle d \mid x \rangle = -1$ for all $x \in H_F \cap C$ we have $d \in C^\circ$. The next step is to show that F° is extreme. Recall that $F^\circ \subseteq C^\circ$ is an extreme set, if some interior point of a line in C° lies in F°, then also both endpoints of the line. Therefore let $y \in F^\circ$ be a point and $y_1, y_2 \in C^\circ$ such that $y = \lambda y_1 + (1-\lambda) y_2 \in F^\circ$ for some $\lambda \in (0,1)$. For any $x \in F$ we have

$$-1 = \langle y | x \rangle = \lambda \langle y_1 | x \rangle + (1-\lambda) \langle y_2 | x \rangle \geq -1,$$

as both $y_1, y_2 \in B^\circ$. Equality holds if and only if $\langle y_1 | x \rangle = \langle y_2 | x \rangle = -1$ and therefore $y_1, y_2 \in F^\circ$. As F° contains all points that have dual pairing -1 with every element of F, it is exposed. Otherwise there would be an exposed face $G \subseteq C^\circ$ in whose relative boundary F° lies and also $\langle G \mid F \rangle = -1$ would hold in contradiction to the definition. □

Remark 2.4.14 In Example 2.4.12 above we saw that for a face F there might be several faces of B° dual to F. Actually, a face can have infinitely many faces dual to it. But they all are in the relative boundary of a unique exposed dual face F°. As for a polytope every extreme set is exposed, a face F of a polytope has a unique dual face and then $\mathcal{D}(F) = \{F^\circ\}$. $\circ$

We described F° as the intersection of B° with the hyperplanes H^f_{-1} for all $f \in F$. Actually it is also enough to either intersect all hyperplanes defined by the extremal points of F or to take just one hyperplane associated to a relative interior point of F:

Lemma 2.4.15 *Let $F \subseteq B$ be a face and $\mathcal{E}_F$ the set of extreme points of F. Then there are the following two descriptions of the exposed dual of F:*

1) $F^\circ = \bigcap_{e\in\mathcal{E}_F} H^e_{-1} \cap B^\circ$.
2) $F^\circ = H^g_{-1} \cap B^\circ$ *for any* $g \in \operatorname{relint}(F)$. $\circ$

Proof. (1) As every extreme point of F lies in the relative boundary of F, the inclusion $\subseteq$ is clear. For the other way round let $y \in \bigcap_{e\in\mathcal{E}_F} H^e_{-1} \cap B^\circ$, then $\langle y \mid e\rangle = -1$ for all $e \in \mathcal{E}_F$ and $\langle y \mid x\rangle \geq -1$ for all $x \in B$. We have to show that $\langle y \mid f\rangle = -1$ for all $f \in F$ and not only for the extreme points. By the discussion after Proposition 2.3.7 there are finitely many $\lambda_i \geq 0$ and $e_i \in \mathcal{E}_F$ such that

$$f = \sum_i \lambda_i e_i \quad \text{and} \quad \sum_i \lambda_i = 1.$$

Then

$$\langle y \mid f\rangle = \sum_i \lambda_i \langle y \mid e_i\rangle = -\sum_i \lambda_i = -1.$$

(2) In the case where F is a single point, the statement in obviously true. So let's assume F is not a point. The inclusion $\subseteq$ is again trivial and it remains to show that $H^g_{-1} \cap B^\circ \subseteq F^\circ = \bigcap_{f\in F} H^f_{-1}$ for any $g \in \operatorname{relint}(F)$. Let $y \in H^g_{-1} \cap B^\circ$, that is, $\langle y|g\rangle = -1$ where $g \in \operatorname{relint}(F)$ is fixed, and $\langle y|f\rangle \geq -1$ for every $f \in F$ by the duality of B and B°. We have to show that $\langle y|f\rangle = -1$ for all $f \in F$. Assume there is an $f \in F$ such that $\langle y|f\rangle > -1$ and let $\varepsilon > 0$ be small enough such that $g + \varepsilon(g - f)$ is a point in F. Such an ε can always be found because F is convex and $g \in \operatorname{relint}(F)$ and $f \in F$. Then

$$\langle y|g + \varepsilon(g - f)\rangle = -1 - \varepsilon - \varepsilon\,\langle y|f\rangle < -1 - \varepsilon + \varepsilon = -1,$$

which means that $g + \varepsilon(g - f) \notin B$. As ε was arbitrary, this contradicts that g lies in the relative interior of F, so we conclude $\langle y|f\rangle = -1$ for all $f \in F$. $\square$

We conclude this section about the duality of faces by a look at the subspaces that are defined by dual faces.

Proposition 2.4.16 *Let $F \subseteq B$ be a proper face and $E \in \mathcal{D}(F)$ a face of $B^\circ \subseteq X^*$ dual to F. Then*

$$T(E)^* \subseteq T(F^\circ)^* \subseteq V(F)^\perp$$

and

$$\dim(F) + \dim(E) \leq n - 1.$$

If B is polyhedral, then $T(E)^* = V(F)^\perp$ and $\dim(F) + \dim(E) = n - 1$. $\circ$

Proof. Let $E \in \mathcal{D}(F)$ be an extreme set, $E \subseteq F^\circ \subseteq B^\circ$. The first inclusion is clear. To see that $T(F^\circ)^* \subseteq V(F)^\perp$ we remember the definition of the space of translations (Def. 2.1.15): $T(F^\circ) = F^\circ - F^\circ = \{x - y \mid x, y \in F^\circ\}$. As every element of F° has dual pairing equal to -1 with any element of F we have $\langle t|f\rangle = 0$ for any $t \in T(F^\circ)$ and $f \in F$, so $T(F^\circ)^* \subseteq V(F)^\perp$.

For the statement about the dimensions we use that $\dim(V(F)) = \dim(F) + 1$ because $0 \notin F$ and thereby $\dim(V(F)^\perp) = n - \dim(F) - 1$. As $\dim(E) \leq \dim(F^\circ) \leq \dim V(F)^\perp$ for any $E \in \mathcal{D}(F)$ we calculate

$$\dim(F) + \dim(E) \leq \dim(F) + n - \dim(F) - 1 = n - 1.$$

If B is polyhedral, then $\mathcal{D}(F) = \{F^\circ\}$ and it remains to show that $V(F)^\perp \subseteq T(F^\circ)^*$. By Corollary 2.1.21 we fix some $z \in F^\circ$ such that $T(F^\circ) = \mathrm{aff}(F^\circ) - z$. Let $x \in (V(F)^\perp)^*$ and $\varepsilon > 0$, then

$$\langle z + \varepsilon x|f\rangle = \langle z|f\rangle + \varepsilon \langle x|f\rangle = -1 \quad \forall f \in F.$$

As a polytope has only finitely many vertices, the pairing of z with any extreme point not in F can be uniformly bounded away from -1 by some positive constant. So there is a $\delta > 0$ such that $\langle z|e\rangle > -1 + \delta$ for all $e \in \mathcal{E}_B \setminus \mathcal{E}_F$. By $e_1, \ldots, e_k$ let us denote the extreme points of B that are not in F and by $f_1, \ldots f_l$ those contained in F. Set $\alpha := \min_i \langle x|e_i\rangle$. Let $y \in B \setminus F$ and $\lambda_i, \mu_j \in [0, 1]$ such that $\sum_i \lambda_i + \sum_j \mu_j = 1$ and

$$y = \sum_{i=1}^{k} \lambda_i e_i + \sum_{j=1}^{l} \mu_j f_j.$$

Note that the coefficients λ_i and μ_j in the convex combination are not necessarily unique. As $y \notin F$ it is $k \geq 1$ and we can arrange the e_i in a way such that $\lambda_1 \neq 0$. Then

$$\begin{aligned}
\langle z + \varepsilon x|y\rangle &= \sum_i \lambda_i \langle z|e_i\rangle + \sum_j \mu_j \left\langle z|f_j\right\rangle + \varepsilon \sum_i \langle x|e_i\rangle + \varepsilon \sum_j \mu_j \left\langle x|f_j\right\rangle \\
&> -\sum_i \lambda_i + \delta \sum_i \lambda_i - \sum_j \mu_j + \varepsilon \sum_i \lambda_i \langle x|e_i\rangle \\
&\geq -1 + \sum_i \lambda_i(\delta + \varepsilon\alpha).
\end{aligned}$$

As δ and α are independent of y, we can choose ε small enough such that $\langle z + \varepsilon x|y\rangle > -1$ for all $y \in B \setminus F$. This implies that $z + \varepsilon x \in F^\circ$ or equivalently $x \in \frac{1}{\varepsilon}(\mathrm{aff}(F^\circ) - z) = T(F^\circ)$. Therefore $T(F^\circ)^* = V(F)^\perp$ and the formula for the dimension follows in the same way as above. □

For more details on polars and polyhedral convex sets see for example [Bee93] or [Roc97, §19].

In Theorem 3.2.6 we want to characterize sequences in X that converge to horofunctions in the horofunction compactification $\overline{X}^{hor}$. The structure of the compactification strongly depends on the face structure of the dual unit ball B° of the norm we are considering. We will have to split up the sequence into parts lying in different subspaces depending on the faces $F \subseteq B$ and $E \in \mathcal{D}(F) \subseteq B^\circ$.

Notation Let $A \subseteq X$ be a subset and $x \in X$. We fix the following notations for projections of x onto the subspaces $T(A)$ and $T(A)^\perp$:

$$\begin{aligned}
x_A &:= x_{T(A)} := \mathrm{proj}_{T(A)^*}(x) \\
x^A &:= x^{T(A)} := \mathrm{proj}_{(T(A)^\perp)^*}(x).
\end{aligned}$$

We use the same notation for projections with respect to subspaces of X^* Let $S \subseteq X^*$ be a subset and $x \in X$. We fix the following notations for projections of x onto the subspaces dual to $T(S)$ and $T(S)^\perp$:

$$x_S := x_{T(S)} := \mathrm{proj}_{T(S)^*}(x)$$
$$x^S := x^{T(S)} := \mathrm{proj}_{(T(S)^\perp)^*}(x).$$

Note that $(T(S)^\perp)^* = (T(S)^*)^\perp$. $\circ$

According to the inclusions given in the previous proposition we have the following splitting up of an element $x \in X$:

$$x = x_{V(F)} + x_E + (x_{F^\circ})^E + \left[x^{V(F)}\right]^{F^\circ}.$$

As $T(F^\circ)^* \subseteq V(F)^\perp$, the expression $\left(x^{V(F)}\right)^{F^\circ}$ means that we project to the orthogonal complement of $T(F^\circ)^*$ within the subspace $V(F)^\perp$. By Proposition 2.4.16 we have the identities

$$\left(\left(x^{V(F)}\right)_{F^\circ}\right)_E = x_E, \qquad \left(\left(x^{V(F)}\right)_{F^\circ}\right)^E = (x_{F^\circ})^E, \qquad \left(x^{V(F)}\right)_{F^\circ} = x_{F^\circ}.$$

Figure 2.11 shows schematically how to obtain this splitting step-by-step.

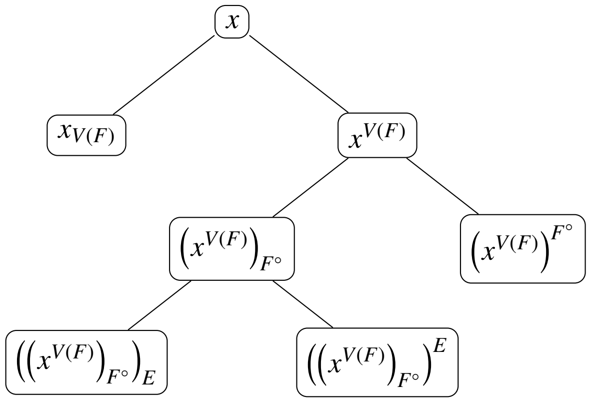


Figure 2.11: Splitting up an element $x \in X$ into the various subspaces depending on a face $F \subseteq B$ and a face $E \in \mathcal{D}(F) \subseteq B^\circ$ dual to it.

Additionally we get

$$\begin{aligned} x^E &= x_{V(F)} + (x_{F^\circ})^E + \left[x^{V(F)}\right]^{F^\circ}, \\ x^{V(F)} &= x_E + (x_{F^\circ})^E + \left[x^{V(F)}\right]^{F^\circ}, \\ x_{F^\circ} &= x_E + (x_{F^\circ})^E, \\ x^{F^\circ} &= x_{V(F)} + \left[x^{V(F)}\right]^{F^\circ}. \end{aligned}$$

2.5 Basics about the Minkowski Sum

A nice way to obtain new unit balls out of given ones is to take the Minkowski sum of the unit balls. This gives us a new unit ball (that is a compact convex set). In this section we first want to collect some basic properties of the Minkowski sum and then examine in more detail the face structure of the new unit ball and how it is related to the faces of the summands. In the following chapter we will also see that the Minkowski sum behaves well with the horofunction compactification.

Definition 2.5.1 Let A, D be two sets[4] in X. Then the Minkowski sum M of A and D is defined as

$$M := A + D = \{a + d \,|a \in A, d \in D\}. \qquad \circ$$

Remark 2.5.2 The Minkowski sum is associative and can therefore be calculated for arbitrarily many sets. For simplifying notations we restrict ourselves to the sum of two sets. Nevertheless, all results presented here are also true for more summands, the proofs go through the same. ∘

Lemma 2.5.3 *The Minkowski sum commutes with taking the convex hull:*

$$\mathrm{conv}(A + D) = \mathrm{conv}(A) + \mathrm{conv}(D),$$

for sets $A, D \subseteq X$. ∘

Proof. We first show the inclusion $\subseteq$. Let $m \in \mathrm{conv}(A + D)$ be an arbitrary element. Then m can be written as a convex combination of elements $m_1, \ldots, m_k \in A + D$:

$$m = \sum_{i=1}^{k} \lambda_i m_i,$$

with $0 \le \lambda_1, \ldots, \lambda_k \le 1$ satisfying $\sum_i \lambda_i = 1$. Moreover, each m_i can be split into $m_i = a_i + d_i$, with $a_i \in A$ and $d_i \in D$ for all $i \in \{1, \ldots, k\}$. Together we obtain:

$$m = \sum_{i=1}^{k} \lambda_i (a_i + d_i) = \sum_i \lambda_i a_i + \sum_i \lambda_i d_i \in \mathrm{conv}(A) + \mathrm{conv}(D).$$

For the other inclusion $\supseteq$ let $m \in \mathrm{conv}(A) + \mathrm{conv}(D)$ be arbitrary with $a \in \mathrm{conv}(A)$ and $d \in \mathrm{conv}(D)$ such that $m = a + d$. Then we express a and d as convex combinations of elements $a_1, \ldots, a_k \in A$ and $d_1, \ldots, d_l \in D$:

$$a = \sum_{i=1}^{k} \lambda_i a_i, \qquad d = \sum_{j=1}^{l} \mu_j d_j,$$

with coefficients $0 \le \lambda_1, \ldots, \lambda_k \le 1$ and $0 \le \mu_1, \ldots, \mu_l \le 1$ satisfying $\sum_i \lambda_i = 1$ as well as $\sum_j \mu_j = 1$. These partitions of 1 are used in the next calculation, where we replace a and d step by step by the above sums to obtain

$$\begin{aligned} m = a + d &= \sum_{i=1}^{k} \lambda_i a_i + \sum_{i=1}^{k} \lambda_i d \\ &= \sum_{i=1}^{k} \lambda_i \,(a_i + d) \\ &= \sum_{i=1}^{k} \lambda_i \left(\sum_{j=1}^{l} \mu_j a_i + \sum_{j=1}^{l} \mu_j d_j \right) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{l} \lambda_i \mu_j (a_i + d_j). \end{aligned}$$

As both the sums over λ_i and μ_i are equal to 1, the coefficients in the last sum also add up to 1 and we conclude that m is a convex combination of elements in A and D and therefore $m \in \mathrm{conv}(A + D)$. □

[4]To minimize confusion, we avoid letters B and C for general (non necessarily convex) set in X.

Lemma 2.5.4 *Let C_1, C_2 be convex sets and $\mathcal{E}_1, \mathcal{E}_2$ their sets of extreme points, respectively. Then their Minkowski sum M is also convex and given by the convex hull of their pairwise sum of extreme points:*

$$\begin{aligned} M := C_1 + C_2 &= \operatorname{conv}(\mathcal{E}_1 + \mathcal{E}_2) \\ &= \operatorname{conv}(\{e_1 + e_2 \mid e_1 \in \mathcal{E}_1, e_2 \in \mathcal{E}_2\}). \end{aligned} \qquad \circ$$

Proof. By the previous lemma and the Krein-Milman Theorem (2.3.7) we get immediately:

$$\begin{aligned} M = C_1 + C_2 &= \operatorname{conv}(\mathcal{E}_1) + \operatorname{conv}(\mathcal{E}_2) \\ &= \operatorname{conv}(\mathcal{E}_1 + \mathcal{E}_2) \\ &= \operatorname{conv}(\{e_1 + e_2 \mid e_1 \in \mathcal{E}_1, e_2 \in \mathcal{E}_2\}). \end{aligned} \qquad \square$$

Note that not all sums of points $p_i + q_j$ in the Minkowski sum M are vertices of the polytope M. Some of them will give interior or non-extreme boundary points.

Corollary 2.5.5 Let $P = \operatorname{conv}(p_1, \ldots, p_k)$ and $Q = \operatorname{conv}(q_1, \ldots, q_l)$ be two polytopes. Then their Minkowski sum M is a polytope whose vertices are the sum of each a vertex of P and Q:

$$M := P + Q = \operatorname{conv}(\{p_i + q_j \mid i = 1, \ldots, k,\ j = 1, \ldots, l\}). \qquad \circ$$

Example 2.5.6 Let P be the cube with vertices $(\pm 1, \pm 1, \pm 1)$ and let

$$Q = \operatorname{conv}((1, -1, -1), (1, -1, 1), (-1, -1, 1), (-1, -1, -1), (1, 2, -1)).$$

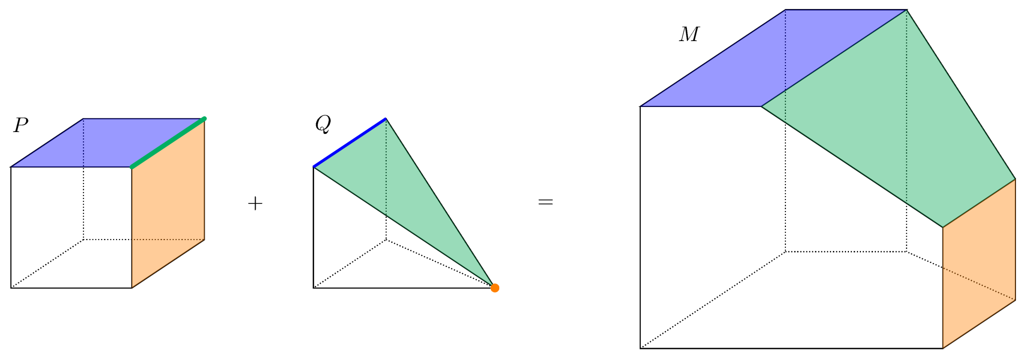


Figure 2.12: The polytopes P and Q and their Minkowski sum M as in Example 2.5.6. The colors indicate which faces of P and Q sum up to the corresponding face in M.

By the previous lemma we immediately get their Minkowski sum as

$$\begin{aligned} M := P + Q = \operatorname{conv}\big(&(1, -1, -1), (1, 1, -1), (1, 1, 1), (1, -1, 1), (-1, -1, -1), \\ &(-1, 1, -1), (-1, 1, 1), (-1, -1, 1)\big). \end{aligned} \qquad \circ$$

The Minkowski sum of the two convex sets $C_1 = \operatorname{conv}(\mathcal{E}_1)$ and $C_2 = \operatorname{conv}(\mathcal{E}_2)$ can be constructed as the convex hull of the (shifted) convex sets $C_{1,e_2} := C_1 + e_2$ for each $e_2 \in \mathcal{E}_2$. Therefore the dimension of $C_1 + C_2$ is at least the maximal dimension of C_1 and C_2 and can not be bigger than the sum of their dimensions. This consideration motivates the following lemma (see [Wei07, p. 16] for polytopes):

Lemma 2.5.7 *Let C_1, C_2 be closed convex sets in X and $M = C_1 + C_2$ their Minkowski sum. Then*

$$\max\{\dim(C_1), \dim(C_2)\} \le \dim(M) \le \dim(C_1) + \dim(C_2). \qquad \circ$$

Given a point m in the Minkowski sum M, there might be several pairs of points $(c_1, c_2) \in C_1 \times C_2$ such that their sum is equal to m. But for exposed faces of M the decomposition is unique (see also [Wei07, Thm. 3.1.2] for polytopes and [Sol15, Thm 8.10 (2)] for general convex sets), as the following proposition shows:

Proposition 2.5.8 *Let $C_1, C_2 subsetX$ be compact convex sets in X and $M = C_1 + C_2$ their Minkowski sum. Let $F \subseteq M$ be an exposed face of M. Then there are unique exposed faces $F_1 \subseteq C_1, F_2 \subseteq C_2$ such that $F = F_1 + F_2$.* ∘

Proof. To not get confused by the notation, see Figure 2.13 for a visualization. Without loss of generality, we assume that all sets contain the origin in their relative interior. As an exposed face,

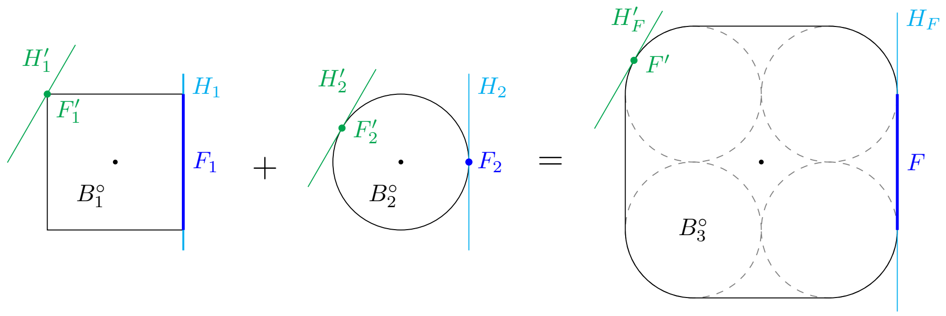


Figure 2.13: The Minkowski sum of a square and a circle gives a bigger square with rounded corners. The blue and green faces $F \subseteq M$ are exposed and uniquely the sum of exposed faces $F_1 \subseteq C_1$ and $F_2 \subseteq F_2$.

F is the intersection of M with a supporting hyperplane H_F, $F = M \cap H_F$. So there is a unique $h \in X^*$ such that $H_F = H^h_{-1} = \{x \in X \mid \langle h|x\rangle = -1\}$. Then h is a point in the boundary of $M^\circ \subseteq X^*$ but not necessarily an extreme set of it. For each $j \in \{1, 2\}$ there are two hyperplanes supporting C_j that are parallel to H_F. Let H_j be the one that has smaller dual pairing with h. Then as C_j lies entirely in one of the two closed half-spaces defined by H_j, the hyperplane can also be characterized as

$$H_j = \left\{ x \in X \;\middle|\; \langle h|x\rangle = \inf_{q_j \in C_j} \left\langle h \middle| q_j \right\rangle \right\}.$$

The sets C_j are compact, so $\inf_{q_j \in C_j} \left\langle h \middle| q_j \right\rangle$ is a scaler, the hyperplanes H, H_1 and H_2 are parallel, as required. Set $F_j := C_j \cap H_j$, that is,

$$F_j = \left\{ x \in C_j \;\middle|\; \langle h|x\rangle = \inf_{q_j \in C_j} \left\langle h \middle| q_j \right\rangle \right\}.$$

Then $F_j \subseteq C_j$ are exposed faces and their sum gives back F:

$$\begin{aligned}
F &= \{x \in M \mid \langle h|x\rangle = -1\} \\
&= \left\{ x \in M \mid \langle h|x\rangle = \inf_{q \in M} \langle h|q\rangle \right\} \\
&= \left\{ c_1 + c_2 \in M = C_1 + C_2 \;\middle|\; \langle h|c_1 + c_2\rangle = \inf_{q_j \in C_j} \langle h\,|q_1 + q_2\rangle \right\} \\
&= \left\{ c_1 + c_2 \in M = C_1 + C_2 \;\middle|\; \langle h|c_1\rangle + \langle h|c_2\rangle = \inf_{q_1 \in C_1} \langle h\,|q_1\rangle + \inf_{q_2 \in C_2} \langle h\,|q_2\rangle \right\} \\
&= F_1 + F_2.
\end{aligned}$$

The last line in the calculation above enforces the uniqueness of the decomposition, as all the points in $C_j \setminus F_j$ have dual pairing bigger than $\inf_{q_j \in C_j} \left\langle h \middle| q_j \right\rangle$ with h. □

Corollary 2.5.9 Exposed points of M are the sum of exposed points of C_1 and C_2. ∘

Recall that every exposed face is also extreme but not vice versa. Nevertheless, the proposition above is also true for extreme sets (see [Sol15, Thm. 7.15 (2)] for an alternative proof):

Proposition 2.5.10 *Let C_1, C_2 be compact convex sets in X and $M = C_1 + C_2$ their Minkowski sum. Let $F \subseteq M$ be an extreme set of M. Then there are unique extreme sets $F_1 \subseteq C_1, F_2 \subseteq C_2$ such that $F = F_1 + F_2$.* ∘

Proof. For a examples of the notation see Figure 2.14. If F is exposed, the statement is the same

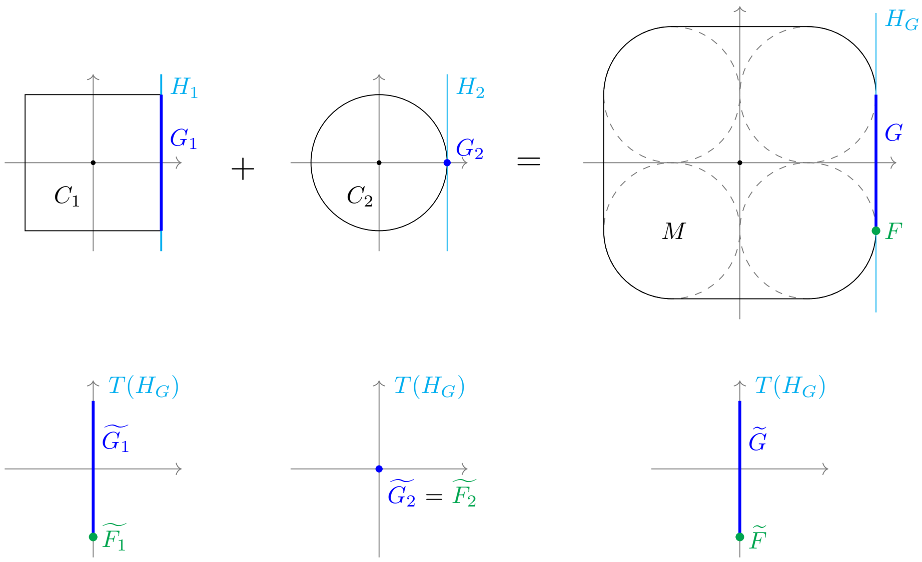


Figure 2.14: To get the decomposition of the orange extreme point F which is not exposed, we consider the decomposition of the exposed blue face G.

as in Proposition 2.5.8 before. Therefore assume that F is extreme but not exposed. Then F is an extreme set of an exposed face $G \subseteq M$ and lies in its relative boundary. By the previous proposition, G can uniquely be decomposed into two exposed faces $G_j \subseteq C_j$ (with $j \in \{1, 2\}$):

$$G = G_1 + G_2.$$

All supporting hyperplanes H_G and H_j at $G \subseteq C$ and $G_j \subseteq C_j$, respectively, that were constructed in the proof of Prop. 2.5.8, were defined as supporting hyperplanes minimizing the dual pairing with a fixed $h \in X^*$. Therefore all of them are parallel, that is, they all have the same set of translations:

$$T(H_G) = T(H_1) = T(H_2) =: T(H).$$

Now we consider orthogonal projections $\widetilde{G}, \widetilde{G_j} \subseteq T(H)$ to this subspace, denoted by a tilde over the set. Then there are $s, s_j \in T(H)^\perp$ such that

$$\begin{aligned} G &= \widetilde{G} + s \\ G_j &= \widetilde{G_j} + s_j, \qquad (j \in \{1, 2\}). \end{aligned}$$

Together with $G = G_1 + G_2 = \widetilde{G_1} + s_1 + \widetilde{G_2} + s_2$ we get

$$\widetilde{G} = \widetilde{G_1} + \widetilde{G_2}.$$

The orthogonal projection $\widetilde{F}$ of F to $T(H)$ is given such that $F = \widetilde{F} + s$ with the same $s \in T(H)^{\perp}$ as before and therefore $\widetilde{F}$ is an extreme set of $\widetilde{G}$ so that we now have the following situation in $T(H)$:

- $\widetilde{G} = \widetilde{G_1} + \widetilde{G_2}$ is the Minkowski sum of two compact convex sets,
- $\widetilde{F} \subseteq \widetilde{G}$ is an extreme set.

If $\widetilde{F}$ is an exposed face of $\widetilde{G}$ in $T(H)$, then we use Proposition 2.5.8 to decompose $\widetilde{F} = \widetilde{F_1} + \widetilde{F_2}$ and then set $F_j = \widetilde{F_j} + s_j$ to be done. If $\widetilde{F}$ is not exposed, we conclude by induction that F can be decomposed as the unique sum of two extreme sets. As $\dim(T(H)) = n - 1 < n = \dim(X)$ and since every extreme point in the boundary of a line segment is exposed, the statement follows. □

Every extreme set of a polytope P, that is not a facet, lies in the relative boundary of some higher dimensional proper face or facet. This is not true for general compact convex sets, as the example of the Euclidean unit ball shows. But if an extreme set is lying in the relative boundary of another one, this structure is compatible with the Minkowski sum (see [Wei07, Cor. 3.1.5] for polytopes):

Lemma 2.5.11 *Let $M = C_1 + C_2$ be the Minkowski sum of two compact convex sets. Let $F \subseteq G$ be extreme sets of M with unique decompositions $F = F_1 + F_1$ and $G = G_1 + G_1$. Then $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$ are extreme.* ○

Proof. The main idea of the proof is that extreme sets of a Minkowski sum can uniquely be decomposed into extreme sets of the summands.
We first consider the case where G is exposed. As in the proof of Proposition 2.5.10 we project to the subspace $T(H)$, where H denotes a supporting hyperplane to M such that $G = H \cap M$. Then in $T(H)$ we have

$$\widetilde{G} = \widetilde{G_1} + \widetilde{G_2}$$
$$\widetilde{F} = \widetilde{F_1} + \widetilde{F_2},$$

where $\widetilde{F_j}$ is an extreme set of $\widetilde{G_j}$, respectively. Therefore $F_j \subseteq G_j$ are extreme sets. In the case when G is not exposed, we project to lower dimensional subspaces until the projection of G is exposed and conclude as before. As the relation of being an extreme set of another is remained under orthogonal projections, the proof follows. □

We now show a proposition that explicitly constructs the faces F_1 and F_2 in the decomposition of F by considering the decomposition of the extreme points of F.

Proposition 2.5.12 *Let $F = F_1 + F_2$ be the decomposition of an extreme set in the Minkowski sum $M = C_1 + C_2$ of two compact convex sets C_1, C_2. For each extreme point $e \in \mathcal{E}_F$ let $e_1 \subseteq F_1$ and $e_2 \subseteq F_2$ be the unique extreme points such that $e = e_1 + e_2$. Denote by $\mathcal{E}_j \subseteq F_j$, $j = 1, 2$, the set of all these summands. Then the faces F_1 and F_2 are the convex hulls of the points $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively:*

$$F_j = \operatorname{conv}(\mathcal{E}_j).$$ ○

Proof. By Lemma 2.5.11 we know that for $j = 1, 2$, the points e_j are extreme points of F_j, respectively. Therefore all sums of the form $e_1 + e_2$ give elements of F but not necessarily extreme points of it.

Using this and Lemma 2.5.3 we get:

$$\begin{aligned} F_1 + F_2 = F &= \operatorname{conv}(\mathcal{E}_F) \\ &= \operatorname{conv}\{e = e_1 + e_2 \mid e \in \mathcal{E}_F, e_j \in \mathcal{E}_j \text{ for } j \in \{1,2\}\} \\ &= \operatorname{conv}(\mathcal{E}_1 + \mathcal{E}_2) \\ &= \operatorname{conv}(\mathcal{E}_1) + \operatorname{conv}(\mathcal{E}_2). \end{aligned}$$

By the unique decomposition of faces we immediately get $F_1 = \operatorname{conv}(\mathcal{E}_1)$ and $F_2 = \operatorname{conv}(\mathcal{E}_2)$. □

Example 2.5.13 We look again at the polytopes given in Example 2.5.6, the colors illustrate the unique decomposition of faces of M as given in Proposition 2.5.8. Label the vertices as given in Figure 2.15.

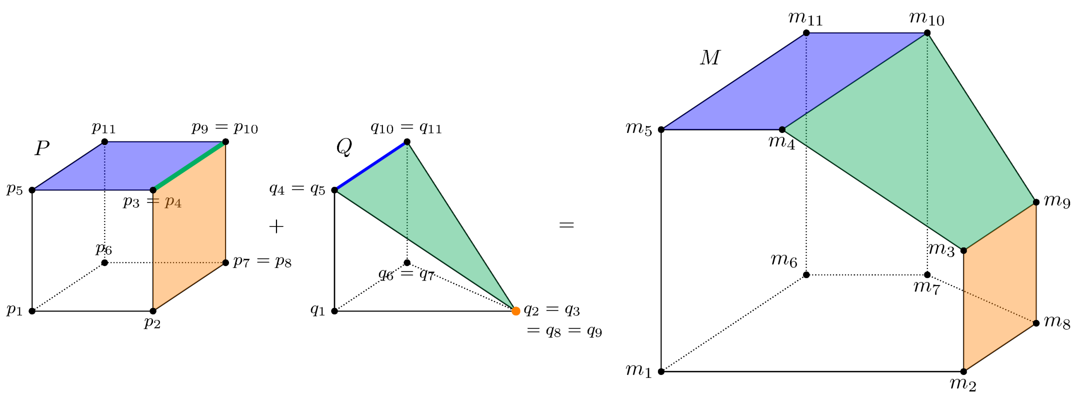


Figure 2.15: The polytopes P and Q and their Minkowski sum M. The colors indicate which faces of P and Q sum up to the corresponding face in M, the labels show the decomposition of vertices.

This labeling shows the decomposition of vertices: $m_j = p_j + q_j$.
The blue face of M for instance is the convex hull of m_4, m_5, m_{10} and m_{11}. In P the blue face has vertices p_4, p_5, p_{10} and p_{11} as expected. In Q the vertices m_4 and m_5 as well as m_{10} and m_{11} yield to the same vertices $q_4 = q_5$ and $q_{10} = q_{11}$ and the blue face is just the line with these two endpoints. This illustrates Proposition 2.5.12. ○

2.6 A Pseudo-Norm and the Functions $h_{E,p}$

In this preliminary part we want to introduce some notations and a "pseudo-norm" that will be used in the end of this section to define some functions that will later turn out to be the horofunctions. We start with a basic observation:

Lemma 2.6.1 *Let $F \subseteq B \subseteq X$ be a face and $F^\circ \subseteq B^\circ \subseteq X^*$ its exposed dual. Then there is an $s \in X^*$ such that*

$$\langle e|f\rangle = \langle s|f\rangle$$

for all $e \in F^\circ$ and $f \in V(F)$. ○

Proof. We know by Lemma 2.4.16 that the subspace $T(F^\circ)$ parallel to F° is orthogonal to the dual subspace generated by F. Therefore there is an $s \in V(F)^*$ such that

$$\operatorname{aff}(F^\circ) = T(F^\circ) + s.$$

So every $e \in F^\circ$ can be written as $e = e_t + s$ for some $e_t \in T(F^\circ) \subseteq (V(F)^\perp)^*$. Then for any $f \in V(F)$ it holds

$$\langle e|f\rangle = \langle e_t + s|f\rangle = \langle s|f\rangle$$

as it was to show. □

The lemma shows that all elements of F° have the same dual pairing with an element of F. We will sometimes write $\langle F^\circ|f\rangle$ in such a situation when $f \subseteq T(F^\circ)^\perp$.

Remark 2.6.2 As all duals of F that are not exposed are in the relative boundary of the exposed dual, the lemma above holds for every dual $E \in \mathcal{D}(F)$ of F. ◦

In the proof of Proposition 2.5.8 on page 32 we defined a hyperplane (for $j = 1, 2$) by

$$H_j = \left\{ x \in X \,\middle|\, \langle h|x\rangle = \inf_{q_j \in C_j} \left\langle h \middle| q_j \right\rangle \right\},$$

where C_j are two compact convex sets. The infimum on the right is attained at a point $q \in C_j$ that is "the furthest away" from h. More geometrically, take the orthogonal hyperplane H_0^h and shift it in the direction away from h, such that the pairing of the hyperplane with h gets smaller and smaller. Then q will be a point in the intersection of C_j with the latest shifted parallel hyperplane that has non-empty intersection with C_j. So it is then is a supporting hyperplane.

This is a useful concept when dealing with duality and the basic idea of the pseudo-norm, see also [Wal07, p.5]:

Definition 2.6.3 Let $C \subseteq X^*$ be a convex set. For every $x \in X$ define

$$|x|_C := -\inf_{q \in C} \langle q|x\rangle .$$ ◦

In general, this is not a norm. But by the polarity of the unit balls B and B°, $|\cdot|_{B^\circ}$ is a norm, since

$$|\cdot|_{B^\circ} = -\inf_{q \in B^\circ} \langle q|\cdot\rangle = \|\cdot\|.$$

Therefore we call $|x|_C$ the *pseudo-norm* of x with respect to C.

The following technical lemma will be used later in the proof of Theorem 3.2.6

Lemma 2.6.4 *Let C be a compact convex set and $\mathcal{E}_C$ be the set of its extreme points. Then the pseudo-norm over C is the infimum of the dual pairing with the extreme points of C:*

$$|x|_C = -\inf_{e \in \mathcal{E}_C} \langle e|x\rangle .$$ ◦

Proof. Define a function $f : C \longrightarrow \mathbb{R}$ via $f(q) = \langle q|x\rangle$. As C is compact and f is continuous and affine, f takes its minimum and its maximum on the boundary of C. Indeed, if the extrema would only lie in the interior of C, the derivative would be 0 there. As f is affine, it would be constant in contradiction to the assumption that it takes its extrema not on the boundary. As the boundary of C is the union of compact convex sets, we can conclude in the same way that f takes its minimum and maximum on the extreme points $\mathcal{E}_C$ of C. □

Corollary 2.6.5 If $P = \mathrm{conv}\{p_1, \ldots, p_k\}$ is a convex polytope, then

$$|x|_P = -\inf_{i=1,\ldots,k} \langle p_i|x\rangle.$$ ◦

We now have all ingredients to introduce real-valued functions on X which will later turn out to be the horofunctions of X with respect to our norm $\|\cdot\|$ (see [Wal07, p.5] for a different notation):

Definition 2.6.6 For every proper face $E \subseteq B^\circ \subseteq X^*$ and every $p \in T(E)^* \subseteq X$ we define the function

$$\begin{aligned} h_{E,p} : X &\longrightarrow \mathbb{R}, \\ y &\longmapsto |p-y|_E - |p|_E. \end{aligned} \qquad \circ$$

The above definition of $h_{E,p}$ would also be well-defined for any $p \in X$. We restrict ourselves to $p \in T(E)^*$ to gain uniqueness of functions as the following lemmas show.

Lemma 2.6.7 *Let $E \subseteq B^\circ$ be an extreme set. Then for all $p, y \in X$ there holds*

$$h_{E,p}(y) = |p-y|_E - |p|_E = |p_E - y|_E - |p_E|_E = h_{E,p_E}(y),$$

where as usual p_E denotes the projection of p to $T(E)^*$. $\circ$

Proof. Let $\mathcal{E}_E$ be the set of extreme points of E. Then as $p^E \in (T(E)^*)^\perp$ and by Lemma 2.6.1 we obtain

$$\begin{aligned} h_{E,p}(y) = |p-y|_E - |p|_E &= -\inf_{e\in\mathcal{E}_E} \langle e|p-y\rangle + \inf_{e\in\mathcal{E}_E} \langle e|p\rangle \\ &= -\inf_{e\in\mathcal{E}_E} \left[\langle e|p_E - y\rangle + \langle e|p^E\rangle\right] + \inf_{e\in\mathcal{E}_E} \left[\langle e|p_E\rangle + \langle e|p^E\rangle\right] \\ &= |p_E - y|_E - |p_E|_E. \end{aligned} \qquad \square$$

As indicated in the proof above, shifting the extreme set E behaves well with the functions $h_{E,p}$ in the sense, that the shift can be added separately:

Lemma 2.6.8 *Let $E \subseteq B^\circ$ be an extreme set and $s \in X^*$ a parameter to shift E. Then for any $p \in T(E)^*$ and $y \in X$, we have*

$$h_{E+s,p}(y) = h_{E,p}(y) + \langle s|y\rangle. \qquad \circ$$

Proof. Let $\mathcal{E}_E$ be the set of extreme points of E. Then the extreme points $\mathcal{E}_{E+s}$ of the shifted set $E+s$ are given by $\mathcal{E}_{E+s} = \{e+s | e \in \mathcal{E}_E\}$, and thereby

$$\begin{aligned} h_{E+s,p}(y) &= -\inf_{q\in\mathcal{E}_{E+s}} \langle q|p-y\rangle + \inf_{q\in\mathcal{E}_{E+s}} \langle q|p\rangle \\ &= -\inf_{e\in\mathcal{E}_E} \left[\langle e|p-y\rangle + \langle s|p-y\rangle\right] + \inf_{e\in\mathcal{E}_E} \left[\langle e|p\rangle + \langle s|p\rangle\right] \\ &= -\inf_{e\in\mathcal{E}_E} \langle e|p-y\rangle + \inf_{e\in\mathcal{E}_E} \langle e|p\rangle - \langle s|p-y\rangle + \langle s|p\rangle \\ &= h_{E,p}(y) + \langle s|y\rangle. \end{aligned} \qquad \square$$

With the restriction $p \in T(E)^*$, the definition of the function $h_{E,p}$ is unambiguous: Two function $h_{E,p}$ and $h_{F,q}$ are the same, if and only if $E = F$ and $p = q \in T(E)^*$. In other words:

Lemma 2.6.9 *Let $E_1 \neq E_2$ be two faces of B° with dual faces $F_1, F_2 \subseteq B$, respectively. Then there are no points $p_1, p_2 \in X$ such that $h_{E_1,p_1} = h_{E_2,p_2}$.* $\circ$

Proof. Without loss of generality let $\dim E_1 \geq \dim E_2$. Assume there are $p_1 \in T(E_1)^*$ and $p_2 \in T(E_2)^*$ such that $h_{E_1,p_1} = h_{E_2,p_2}$. Then

$$-|p_1|_{E_1} = h_{E_1,p_1}(p_1) = h_{E_2,p_2}(p_1) = |p_2 - p_1|_{E_2} - |p_2|_{E_2}. \tag{2.4}$$

Now consider the cones $K(F_j)$ over the exposed duals $F_j = E_j^\circ \subseteq B$ for $j \in \{1, 2\}$. If the two cones are not the same, choose $y \in X$ such that $p_1 - y \in K(F_1)$ but $p_1 - y \notin K(F_2)$. If the two cones coincide, this means that $E_2 \subsetneq E_1$ is an extreme set that is not an exposed set of B°. Then we can choose $y \in X$ such that the infimum $\inf_{q \in E_1} \langle q|p_1 - y\rangle$ is not taken in E_2 but at some other extreme point of E_1. In both cases we have $\langle e \mid p_1 - y\rangle > \inf_{q \in E_1}\langle q \mid p_1 - y\rangle$ for all $e \in E_2 \subseteq F_2^\circ$. As E_1 and E_2 are compact convex, their infimum is attained in their boundary and we get

$$\inf_{q \in E_2} \langle q|p_1 - y\rangle > \inf_{q \in E_1} \langle q|p_1 - y\rangle \,.$$

Using this and Equation (2.4) we compute

$$\begin{aligned}
h_{E_2,p_2}(y) &= -\inf_{q \in E_2} \langle q|p_2 - y\rangle - |p_2|_{E_2} \\
&= -\inf_{q \in E_2} \{\langle q|p_2 - p_1\rangle + \langle q|p_1 - y\rangle\} - |p_2|_{E_2} \\
&< \ |p_2 - p_1|_{E_2} + \inf_{q \in E_1} \langle q|p_1 - y\rangle - |p_2|_{E_2} \\
&= -|p_1|_{E_1} + \inf_{q \in E_1} \langle q|p_1 - y\rangle = h_{E_1,p_1}(y).
\end{aligned}$$

This contradicts the assumption that $h_{E_1,p_1} = h_{E_2,p_2}$, as we found a point on which they do not coincide. □

3 | Horofunction Compactification

The horofunction compactification was introduced by Gromov [Gro81, §1.2] in 1981 as a general method to construct compactifications of metric spaces. Walsh described the horofunction compactification of finite-dimensional normed vector spaces in [Wal07]. In the case when the convex unit ball B is a finite sided polytope, the horofunction compactification has been described in detail in [JS16], see also [KMN06] for a description of horoballs.

This section is structured as follows: After a short introduction following [Wal14a] we will concentrate on the compactification of a finite-dimensional normed vector space. We explicitly describe (Theorem 3.2.6) the topology of the compactification using the convergence behavior of sequences. Hereby we extend the results for polyhedral norms in [JS16] to all norms in a two-dimensional space and to smooth norms in any dimension. Based on an example (Section 3.2.6) we make a conjecture (Conjecture 3.2.12) for the general case with the only restriction that the set of extreme sets of the dual unit ball is closed and that it only has finitely many connected components of extreme points. Theorem 3.3.10 provides a homeomorphism between the compactification and the dual unit ball B°. At the end of the chapter (Section 3.4) we generalize the previous results (namely Theorem 3.2.6 and Theorem 3.3.10) to normed spaces where the dual unit ball is the Minkowski sum of a polyhedral and a smooth norm.

Throughout the section, we will use the notation introduced in the preliminary chapter. If there is no danger of confusion, we sometimes write $(z_m)_m$ instead of $(z_m)_{m\in\mathbb{N}}$ for sequences in X.

3.1 Introduction to Horofunctions

We start with short introduction to the horofunction compactification of a metric space and then focus on finite-dimensional normed spaces. Finally we prove a convexity result (Convexity Lemma 3.1.16) that will be used in Section 4.2.

3.1.1 General Introduction to Horofunctions

For this general introduction let (X, d) be a locally compact not necessarily symmetric metric space, that is, $d(x, y) \neq d(y, x)$ for $x, y \in X$ is possible. Assume the topology to be induced by the symmetrized distance

$$d_{sym}(x, y) := d(x, y) + d(y, x)$$

for all $x, y \in X$. Let $C(X)$ be endowed with the topology of uniform convergence on bounded sets with respect to d_{sym}. Fix a basepoint $p_0 \in X$ and let $C_{p_0}(X)$ be the set of continuous functions on

X which vanish at p_0. This space is homeomorphic to the quotient of $C(X)$ by constant functions, $\widetilde{C}(X) := C(X)\,/\,const$. The compactification is obtained by embedding X into $\widetilde{C}(X)$ via the map

$$\begin{aligned} \psi : X &\longrightarrow \widetilde{C}(X) \\ z &\longmapsto \psi_z, \end{aligned} \tag{3.1}$$

where for all $x \in X$

$$\psi_z(x) := d(x,z) - d(p_0,z). \tag{3.2}$$

By using the triangle inequality it can be shown that this map is injective and continuous. If (X,d) is nice enough, it is also an embedding with compact image:

Lemma 3.1.1 ([Wal14a, Prop. 2.2])

1) *If d_{sym} is proper, i.e. every closed ball is compact, then the closure of the set $\{\psi_z \mid z \in X\}$ in $\widetilde{C}(X)$ is compact.*

2) *Let additionally X be geodesic, i.e., every two points are connected by a geodesic, and let d be symmetric with respect to convergence, that is, for a sequence $(x_m)_{m\in\mathbb{N}}$ in X and some $x \in X$ the following condition holds:*

$$d(x_m,x) \longrightarrow 0 \ \text{ iff } \ d(x,x_m) \longrightarrow 0.$$

Then ψ is an embedding of X into $\widetilde{C}(X)$. ◦

Definition 3.1.2 The *horofunction boundary* $\partial_{hor}(X)$ of X in $\widetilde{C}(X)$ is defined as

$$\partial_{hor}(X) := \big(\operatorname{cl}\psi(X)\big) \setminus \psi(X).$$

Its elements are called *horofunctions*. If $\operatorname{cl}\psi(X)$ is compact, then the set

$$\overline{X}^{hor} := \operatorname{cl}\psi(X) = X \cup \partial_{hor}X$$

is called the *horofunction compactification* of X. ◦

Remark 3.1.3

1) The definition of ψ_z and therefore also those of ψ and $\partial_{hor}(X)$ depend on the choice of the basepoint p_0. One can show by a short calculation (see also [Wal14a, p.4]) that if we choose an alternative basepoint, the corresponding boundaries are homeomorphic.

2) All elements of $\operatorname{cl}\psi(X)$ are 1-Lipschitz with respect to d_{sym}. Indeed, by the triangle inequality, Equation (3.2) immediately turns to $\psi_z(x) \leq d(x,y) + \psi_z(y)$ for all $z \in X$. Similarly, for horofunctions $\eta \in \partial_{hor}(X)$ it holds $\eta(x) \leq d(x,y) + \eta(y)$ for all $x,y \in X$. ◦

From now on we assume all conditions from Lemma 3.1.1 to be satisfied such that ψ is an embedding with compact image and identify X with $\psi(X)$. Then a sequence $(z_m)_m \subseteq X$ converges to a horofunction $\eta \in \partial_{hor}(X)$ if the sequence of associated maps converges uniformly over compact subsets. We will use the following notation: $\psi_{z_m} \longrightarrow \eta$.

Rieffel [Rie02, Thm. 4.5] showed that there are special sequences that always converge to a horofunction $\eta \in \partial_{hor}X$, namely those along so-called almost-geodesics.

Definition 3.1.4 An *almost geodesic* in a metric space (X,d) with base point p_0 is a sequence $(x_m)_{m\in\mathbb{N}}$ in X such that $d(p_0,x_m)$ is unbounded and for all $\varepsilon > 0$ and

$$d(p_0,x_m) + d(x_m,x_n) < d(p_0,x_n) + \varepsilon, \tag{3.3}$$

for m and n large enough, with $m \leq n$. ◦

Note that this is a slight variation of the original definition by Rieffel [Rie02, Def. 4.3]. The main difference is that his almost geodesics were parameterized to have approximately unit speed. The equivalence to our definition can be found in [Rie02] as Lemma 4.4 .

Definition 3.1.5 A *Busemann point* is a horofunction in $\partial_{hor}(X)$ that is the limit of some almost geodesic sequence in X. ◦

Note that not all horofunctions have to be Busemann points.

3.1.2 Horofunctions of Normed Vector Spaces

From now on we consider $(X, \|\cdot\|)$ to be a finite-dimensional normed vector space where the norm is not required to be symmetric. As basepoint we choose the origin. Then the map ψ for defining horofunctions can be written as

$$\psi_z(x) = \|z - x\| - \|z\|.$$

Note that we have to distinguish between $\|z - x\|$ and $\|x - z\|$ as the norm might not be symmetric. In this setting, Walsh obtains a very nice criterion to answer the question when all horofunctions are Busemann points:

Proposition 3.1.6 ([Wal07, Thm. 1.2]) *Consider any finite-dimensional normed vector space. Then every horofunction is a Busemann point if and only if the set of extreme sets of the dual unit ball is closed.* ◦

We assume from now on that the set of extreme sets of the dual unit ball of our space is closed.

The topology used in the proposition is the Chabauty topology on the space of all closed subspaces of the dual unit ball: If X is any locally compact topological space, then the space $\mathrm{Sub}(X)$ of all closed subspaces of F is endowed with a natural compact topology called the Chabauty topology (see [Bou63] for details). When X is metrizable, then $\mathrm{Sub}(X)$ is also metrizable, and a sequence of closed subspaces $(F_n)_{n\in\mathbb{N}}$ converges to F in $\mathrm{Sub}(X)$ if:

- for any $x \in F$, for any $n \in \mathbb{N}$, there exists $x_n \in F_n$ such that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x, and
- for any sequence $(x_n)_{n\in\mathbb{N}}$ in X such that for any $n \in \mathbb{N}$ we have $x_n \in F_n$, every accumulation point of $(x_n)_n$ belongs to F.

As our space is locally compact and Hausdorff, the Chabauty topology coincides with the Painlevé-Kuratowski topology used by Walsh. More details about the different topologies can be found in [Bee93] or [Pat21, Prop. 2.11].

In the very same paper [Wal07] Walsh also gives a rather explicit description of the set of all Busemann points. He describes them as the Legendre-Fenchel-transforms $f^*_{E,p}$ of certain functions depending on proper faces $E \subsetneq B^\circ$ and points $p \in X$:

$$\begin{aligned} f_{E,p} : X^* &\longrightarrow [0, \infty], \\ q &\longmapsto f_{E,p}(q) := I_E(q) + \langle q|p\rangle - \inf_{y\in E} \langle y|p\rangle, \end{aligned} \tag{3.4}$$

where the indicator function $I_E(q)$ is 0 for $q \in E$ and ∞ elsewhere. The Legendre-Fenchel-transform f^* of a function $f : X \to \mathbb{R} \cup \{\infty\}$ is given by

$$\begin{aligned} f^* : X^* &\longrightarrow \mathbb{R} \cup \{\infty\}, \\ w &\longmapsto \sup_{x\in X} (\langle w|x\rangle - f(x)). \end{aligned}$$

More about it can be found for example in [Bee93, §7.2]. The result of Walsh can be stated as follows.

Proposition 3.1.7 ([Wal07, Thm. 1.1.]) *Let $(X, \|\cdot\|)$ be a finite-dimensional normed vector space and the notations be as above. Then the set of Busemann points is the set*

$$\{f^*_{E,p} \mid E \subseteq B^\circ \text{ is a (proper) extreme face, } p \in X\}. \qquad \circ$$

We show now that our previously defined maps $h_{E,p}$ (see Definition 2.6.6 on page 37) are exactly these Busemann points.

Lemma 3.1.8 ([Wal07, p.5]) *Let E be a face of B° and $p \in X$. Then*

$$f^*_{E,p}(\cdot) = h_{E,p}(\cdot) = |p - \cdot|_E - |p|_E. \qquad \circ$$

Proof. By definition, we obtain for all $y \in X$:

$$\begin{aligned} f^*_{E,p}(y) &= \sup_{x \in X^*} \Big(\langle x|y \rangle - f_{E,p}(x) \Big) \\ &= \sup_{x \in X^*} \left(\langle x|y \rangle - I_E(x) - \langle x|p \rangle + \inf_{q \in E} \langle q|p \rangle \right) \\ &= \sup_{x \in E} \left(\langle x|y - p \rangle \right) + \inf_{q \in E} \langle q|p \rangle \\ &= -\inf_{x \in E} \left(\langle x|p - y \rangle \right) + \inf_{q \in E} \langle q|p \rangle \\ &= |p - y|_E - |p|_E. \end{aligned} \qquad \square$$

Corollary 3.1.9 Let p_E be the projection of p to the subspace $T(E)^* \subseteq X$. Then it holds

$$f^*_{E,p} = f^*_{E,p_E}. \qquad \circ$$

Proof. The statement follows directly by Lemma 2.6.7. □

Corollary 3.1.10 In summary (because we assume that the set of extreme sets of B° is closed) we can describe the set of horofunctions easily as

$$\partial_{hor} X = \{h_{E,p} \mid E \subseteq B^\circ \text{ is a (proper) face, } p \in T(E)^*\}. \qquad \circ$$

Proof. The statement is a direct consequence of Proposition 3.1.7 and Lemma 3.1.8. □

To describe the topology of $\overline{X}^{hor}$, we characterize converging sequences in Section 3.2.

Remark 3.1.11 For a normed vector space $(X, \|\cdot\|)$, the map ψ to define the horofunction was given as $\psi_z(x) = \|z - x\| - \|z\|$ for all $x, z \in X$. When B denotes the unit ball of $\|\cdot\|$, it holds $\|\cdot\| = |\cdot|_{B^\circ}$ and we can rewrite this expression as

$$\psi_z(x) = |z - x|_{B^\circ} - |z|_{B^\circ} = h_{B^\circ, z}.$$

So it is reasonable to expect the limit of the sequence $(\psi_{z_m})_m$ be related to a function $h_{E,p}$ where E and p are related to the sequence $(z_m)_m$. ∘

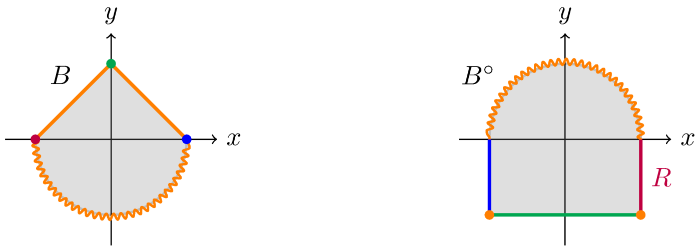


Figure 3.1: The unit ball B and its dual B°

Example 3.1.12 As an example let us determine some horofunctions explicitly. Consider $\mathbb{R}^2$ equipped with a norm that is the 1-norm in the upper half-plane and the Euclidean norm in the lower half-plane. Its dual is a half-circle in the upper half-plane and half of a square in the lower half-plane. See also Figure 3.1 below for a picture.

Then B° has three one-dimensional extreme faces (red, blue and green in the picture on the right) and infinitely many extreme points (orange in the picture) out of which two are exceptional, as they are isolated.

Let $E = \{e\} \subseteq B^\circ$ be one of the extreme points. Then the associated horofunction is given as

$$\begin{aligned} h_{E,p}(x) &= |p-x|_E - |p|_E \\ &= -\inf_{q\in E} \langle q|p-x\rangle + \inf_{q\in E} \langle q|p\rangle \\ &= -\langle e|p-x\rangle + \langle e|p\rangle \\ &= \langle e|x\rangle \,. \end{aligned}$$

Note that here $h_{E,p}$ is independent of p which is reasonable as $T(E)^* = \{0\}$.

Now we have a look at an one-dimensional face, as an example we take the red one called R. It is the convex hull of $(1,0)$ and $(1,-1)$. Then as the infimum of a pairing over a set is taken at the boundary of the set, we get

$$\begin{aligned} h_{R,p}(x) &= |p-x|_R - |p|_R \\ &= -\inf_{q\in R} \langle q|p-x\rangle + \inf_{q\in R} \langle q|p\rangle \\ &= -\min\{\langle (-1,0)|\, p-x\rangle\,,\ \langle (-1,-1)|\, p-x\rangle\} + \min\{\langle (-1,0)|\, p\rangle\,,\ \langle (-1,-1)|\, p\rangle\} \\ &= -\min\{-p_1+x_1,\ -p_1+x_1-p_2+x_2\} + \min\{-p_1,\ -p_1-p_2\} \\ &= p_1 - x_1 - \min\{0,\ -p_2+x_2\} - p_1 + \min\{0,\ -p_2\} \\ &= -x_1 + \max\{p_2-x_2,\ 0\} + \max\{p_2,\ 0\}\,, \end{aligned}$$

where $p = (0, p_2)$ and $x = (x_1, x_2) \in \mathbb{R}^2$. Since $T(R)^*$ is the y-axis in $\mathbb{R}^2$, we have $p \in T(R)^*$ as we wanted it to be. ○

Remark 3.1.13 It is a general result that if $E = \{e\} \subseteq B^\circ$ is a vertex, then $h_{E,p} = \langle e|\cdot\rangle$ is independent of the point p. ○

3.1.3 Convexity Lemma

In this section, we will prove a technical convexity result, which will be used later in Section 4.2 to determine the compactification of a flat.

Lemma 3.1.14 *Let $(x_m)_{m\in\mathbb{N}}$ be an almost geodesic in a metric space (X, d). Then, for any $\varepsilon > 0$,*

$$d(x_i, x_j) + d(x_j, x_k) < d(x_i, x_k) + \varepsilon$$

for i, j, and k large enough, with $i \leq j \leq k$, ◦

Proof. Applying Equation (3.3) to both summands on the left hand side and using the triangle inequality we get, for i, j, and k large enough, with $i \leq j \leq k$,

$$\begin{aligned} d(x_i, x_j) + d(x_j, x_k) &< d(p_0, x_k) - d(p_0, x_i) + 2\delta \\ &\leq d(p_0, x_i) + d(x_i, x_k) - d(p_0, x_i) + 2\delta \\ &= d(x_i, x_k) + 2\delta, \end{aligned}$$

where $p_0 \in X$ denotes the basepoint. The conclusion follows with $\varepsilon = \frac{1}{2}\delta$. □

If $(x_m)_m$ is an almost geodesic converging to a Busemann point ξ, then

$$\xi(x) = \lim_{m\to\infty} \left(d(x, x_m) + \xi(x_m)\right) \qquad \text{for all } x \in X, \tag{3.5}$$

where $\xi(x_m) = \psi_{x_m}(x_m) = -d(p_0, x_m)$.

Lemma 3.1.15 *Let $(x_m)_m$ and $(y_m)_m$ be almost geodesics in a metric space (X, d) converging to the same Busemann point ξ. Then there exists an almost geodesic $(z_m)_m$ that has infinitely many points in common with $(x_m)_m$ and also infinitely many points in common with $(y_m)_m$.* ◦

Proof. Choose a sequence $(\varepsilon_i)_i$ of positive real numbers such that $\sum_{i=0}^{\infty} \varepsilon_i$ is finite. Define the sequence $(z_i)_i$ inductively in the following way. Start with $z_0 := p_0$. Given z_i with i even, use (3.5) to define $z_{i+1} := x_j$, where $j \geq i$ is large enough such that $\xi(z_i) > d(z_i, z_{i+1}) + \xi(z_{i+1}) - \varepsilon_i$. Given z_i with i odd, do the same but this time using the sequence $(y_m)_m$.

Observe that by Equation (3.5) we know that the sequence $(d(p_0, z_i)+\xi(z_i))_i$ converges to $\xi(p_0) = 0$ as i tends to infinity.

Since horofunctions are 1-Lipschitz, it holds $\xi(x) - \xi(y) \leq d(x, y)$ for all $x, y \in X$ and thereby especially $-\xi(y) \leq d(p_0, y)$ as $\xi(p_0) = 0$. So for all $m, n \in \mathbb{N}$, with $m \leq n$ it holds

$$\begin{aligned} d(z_m, z_n) &\leq \sum_{i=m}^{n-1} d(z_i, z_{i+1}) \\ &< \sum_{i=m}^{n-1} (\xi(z_i) - \xi(z_{i+1}) + \varepsilon_i) \\ &= \xi(z_m) - \xi(z_n) + \sum_{i=m}^{n-1} \varepsilon_i \\ &\leq \xi(z_m) + d(p_0, z_n) + \sum_{i=m}^{n-1} \varepsilon_i. \end{aligned}$$

Adding $d(p_0, z_m)$ to both sides, we see that $(z_i)_i$ is an almost geodesic because the error term $\sum_{i=m}^{n-1} \varepsilon_i$ becomes arbitrarily small as m and n become large and $d(p_0, z_m) + \xi(z_m) \longrightarrow 0$. □

We will now prove a convexity result for a pair of almost geodesics converging to the same Busemann point.

Lemma 3.1.16 (Convexity Lemma) *Let $(x_n)_n$ and $(y_n)_n$ be almost geodesics in a finite-dimensional normed space $(X, \|\cdot\|)$ converging to the same Busemann point ξ. Let $(\lambda_n)_n$ be a sequence of coefficients in $[0,1]$, and write $m_n := (1-\lambda_n)x_n + \lambda_n y_n$, for all $n \in \mathbb{N}$. Then $(m_n)_n$ converges to ξ and has an almost geodesic subsequence.* ∘

Proof. Since the horofunction compactification is compact and metrizable, to show that $(m_n)_n$ converges to ξ it is enough to show that every limit point η of $(m_n)_n$ is equal to ξ. By taking a subsequence if necessary, we may assume that $(m_n)_n$ converges to a horofunction η.

By Lemma 3.1.15, there exists an almost geodesic sequence $(z_n)_n$ that has infinitely many points in common with both $(x_n)_n$ and $(y_n)_n$. Since almost geodesics always converge to a horofunction, $(z_n)_n$ has a limit, which must necessarily be ξ. By taking subsequences if necessary, we may assume that $z_n = x_n$ when n is even, and $z_n = y_n$ when n is odd.

Define the sequence $(w_n)_n$ by

$$w_n := \begin{cases} x_n, & \text{if } n \text{ is even;} \\ m_n, & \text{if } n \text{ is odd.} \end{cases} \tag{3.6}$$

The construction of the sequences $(z_n)_n$ and $(w_n)_n$ are schematically shown in Figure 3.2. We will

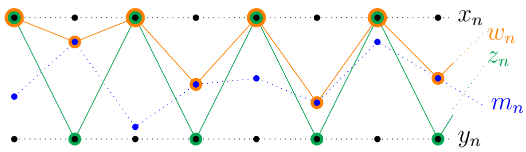


Figure 3.2: A schematic description how the sequences $(z_n)_n$ and $(w_n)_n$ in the proof of Lemma 3.1.16 are constructed out of $(x_n)_n$, $(y_n)_n$ and $(m_n)_n$.

show that $(w_n)_n$ is an almost geodesic.

We first claim that, given any $\varepsilon > 0$, if $i, j, k \in \mathbb{N}$ with $i < j < k$ are large enough and such that i and k are even, and j is odd, then

$$d(w_i, w_j) + d(w_j, w_k) < d(w_i, w_k) + \varepsilon. \tag{3.7}$$

Here, $d(x, y) := \|y - x\|$ is the distance function associated to the norm.

Indeed, note that the distance function $d(\cdot, \cdot)$ is convex in each of its arguments. This implies that

$$\begin{aligned} d(w_i, w_j) &= d(x_i, m_j) \le (1-\lambda_j) d(x_i, x_j) + \lambda_j d(x_i, y_j) \quad \text{and} \\ d(w_j, w_k) &= d(m_j, x_k) \le (1-\lambda_j) d(x_j, x_k) + \lambda_j d(y_j, x_k). \end{aligned}$$

Adding the two equations and applying Lemma 3.1.14 to the almost geodesics $(x_n)_n$ and $(z_n)_n$, we get

$$d(w_i, w_j) + d(w_j, w_k) < d(x_i, x_k) + \varepsilon,$$

for i, j, and k large enough. This establishes the claim as $w_n = x_n$ for n even.

Let k and n be natural numbers satisfying $k < n$. We now want to show than $(w_n)_n$ is an almost geodesic. There are four cases, depending on whether k and n are even or odd. We consider only the case where both are odd; the other cases are similar but less complicated. Using the triangle inequality, the claim just established and the fact that $(x_i)_i$ is an almost geodesic, we have for any $\varepsilon > 0$:

$$\begin{aligned} d(p_0, w_k) + d(w_k, w_n) &\leq d(p_0, w_{k-1}) + d(w_{k-1}, w_k) + d(w_k, w_{k+1}) + d(w_{k+1}, w_n) \\ &< d(p_0, w_{k-1}) + d(w_{k-1}, w_{k+1}) + d(w_{k+1}, w_{n+1}) - d(w_n, w_{n+1}) + 2\varepsilon \\ &\leq d(p_0, w_{n+1}) - d(w_n, w_{n+1}) + 4\varepsilon \\ &\leq d(p_0, w_n) + 4\varepsilon, \end{aligned}$$

if k and n are large enough. The same inequality can be proven in the other cases. We conclude that $(w_n)_n$ is an almost geodesic.

Observe that both ξ and η are limit points of $(w_n)_n$. Since this sequence is an almost geodesic, it has a unique limit. Hence, ξ and η are equal. □

3.2 Characterization of Horofunctions via Converging Sequences

The main theorem of this section (Theorem 3.2.6) characterizes all sequences converging to a horofunction depending on the structure of the unit ball B and its dual B° in X. It shows the strong dependence of the horofunctions on the shape of the dual unit ball, which is the underlying principle of the homeomorphism in Theorem 3.3.10. This result is also used in Section 5.3 to establish a geometric 1-1 correspondence between the nonnegative part of n-dimensional projective toric varieties and horofunction compactifications of $\mathbb{R}^n$ with respect to rational polyhedral norms. If not stated otherwise, we assume that at least one of the following holds true:

I) The unit ball is polyhedral.

II) The unit and the dual unit ball have smooth boundaries.

III) The space X is two-dimensional.

The second case can equivalently be described as B° only having smooth extreme points as extreme sets.

In all three cases the set of extreme sets of B° is closed and so every horofunction is a Busemann point by Proposition 3.1.6. Therefore we can use Corollary 3.1.10 to determine all horofunctions.

This subsection is structured as follows: We will start with some notational conventions and then specify the special properties of the unit ball B and its dual B° in the three cases above. The proof of Theorem 3.2.6 will be based on Lemma 3.2.4, which shows that we can always find subsequences that satisfy the conditions we need for characterizing convergent sequences. The proof of both the lemma and the theorem will be split up in three parts according the three cases for B. After some examples to illustrate the theorem, we will explain in detail an example in Section 3.2.6 where the unit ball does not belong to one of the three cases above and where the statement of Theorem 3.2.6 is not true. This will lead us to a conjecture for the convergence behavior in the general setting in Section 3.2.7.

3.2.1 Dual Sequences of Directions

From now on (unless stated otherwise) let $B \subseteq X$ be the unit ball of a norm and $B^\circ \subseteq X^*$ its dual. Let $(z_m)_{m\in\mathbb{N}}$ be an unbounded sequence. For some $x \in X$, the normed sequence of directions $\left(\frac{z_m-x}{\|z_m-x\|}\right)_{m\in\mathbb{N}}$ is a sequence of points in the boundary of B. So each point of this sequence lies is the relative interior of some extreme set $F_m(x)$ of B. Let $D_m(x) \subseteq B^\circ$ denote the exposed dual of $F_m(x)$. Then $(D_m(x))_{m\in\mathbb{N}} \subseteq B^\circ$ is a sequence of extreme sets of B° and by duality it holds $\langle q_m|z_m - x\rangle = -\|z_m - x\|$ for all $q_m \in D_m(x)$. As the set of extreme sets of B° is closed, all accumulation points of this sequence are extreme sets. To stress that these accumulation points can be extreme points, but also higher dimensional extreme sets, we call them *accumulation sets*. Denote by $D(x)$ the set of all accumulation sets of the sequence $(D_m(x))_m$.

Doing this construction for every point $x \in X$, we consider the following set:

$$E := \operatorname{aff}\{D(x) \mid x \in X\} \cap B^\circ.$$

Though $D(x)$ is a set of faces of B°, E is not necessarily extreme or a subset of the boundary of B°, but may also contain interior points. Note that E strongly depends on the sequence $(z_m)_m$.

Definition 3.2.1 For a sequence $(x_m)_{m\in\mathbb{N}}$ and a set S we denote by $x_m \overset{\subseteq}{\longrightarrow} S$ that $(x_m)_m$ has all its accumulation points in $\operatorname{relint}(S)$. ∘

Assume for our sequence $(z_m)_m \subseteq X$ that there is an extreme face $F \subseteq \partial B$ with $\frac{z_m}{\|z_m\|} \overset{\subseteq}{\longrightarrow} F$. Then also $\frac{z_m-x}{\|z_m-x\|} \overset{\subseteq}{\longrightarrow} F$ for any $x \in X$ and the sequence $(D_m(x))_m$ of duals will converge to an extreme set in the relative boundary of the exposed dual F° or to F° itself . Now it is an interesting question whether we only obtain subsets in the relative boundary as limits or whether we get the whole exposed dual. It can be answered partially:

Lemma 3.2.2 *Let $(z_m)_m \subseteq X$ be an unbounded sequence and B the unit ball. Let $F \subseteq \partial B$ be an extreme face such that $\frac{z_m}{\|z_m\|} \overset{\subseteq}{\longrightarrow} F$ and $F^\circ \subseteq B^\circ$ its exposed dual. If the projected sequence $(z_{m,F^\circ})_m$ is bounded, then $E = F^\circ$.* ∘

Proof. Let $d = \dim(F^\circ) \leq n - \dim(F) - 1$ be the dimension of the exposed dual. As $E \subseteq F^\circ$ we already know that $\dim(E) \leq d$ and we want to show that equality holds. We will do this by showing that we can find $d + 1$ points $y_1, \dots, y_{d+1} \in X$ such that $D(y_j)$ is an extreme point of B° for all $j = 1, \dots d + 1$ and their affine hull $\operatorname{aff}\{D(y_1), \dots, D(y_{d+1})\}$ has dimension d, that is, $D(y_1), \dots, D(y_{d+1})$ are affinely independent.

We first consider the case where B and B° are polyhedral. Then B has only finitely many facets and the idea of the proof is to shift the sequence such that it remains in the cone over a facet As then $\dim(F^\circ) = d$, the face F° has at least $d+1$ vertices e_j, $j \in \{1, \dots, d+1\}$, whose affine hull has dimension d. So there are at least $d + 1$ facets F_j of B that have F in their relative boundary. As $(z_{m,F^\circ})_m$ is bounded, it has a converging subsequence $(y_m)_m$ with $y_m \longrightarrow y$ for some $y \in T(F^\circ)^*$. Choose now $y_1, \dots y_{d+1} \in X$ such that for each $j \in \{1, \dots, d + 1\}$ and m big enough the sequence $(y_m - y_j)_m$ lies in the interior of the cone $K(F_j)$. Then $\frac{y_m-y_j}{\|y_m-y_j\|} \in \operatorname{relint}(F_j)$ and so $D_m(y_j) = e_j$ for m big enough. Therefore all dual sequences are disjoint and the statement is shown.

Now we look at the case where B is not polyhedral. Then it has infinitely many extreme points and we can not conclude as before. But since it is enough to find $d + 1$ points such that their dual accumulation sets are affinely independent, we construct a polytope $P^\circ \subseteq B^\circ$ indescribed in the dual unit ball around the origin such that there holds:

- P° and B° coincide at every non-smooth extreme point of B°;
- if F° is polyhedral (i.e. has only finitely many extreme points), then P° and B° coincide at F° and the round parts of B° are cut off to obtain P°;
- if F° is not polyhedral, let P° and B° coincide at the polyhedral parts of F° and cut off the round parts of F° such that B° and P° coincide at at least $2(d+1)$ extreme points;

See Figure 3.3 for an example.

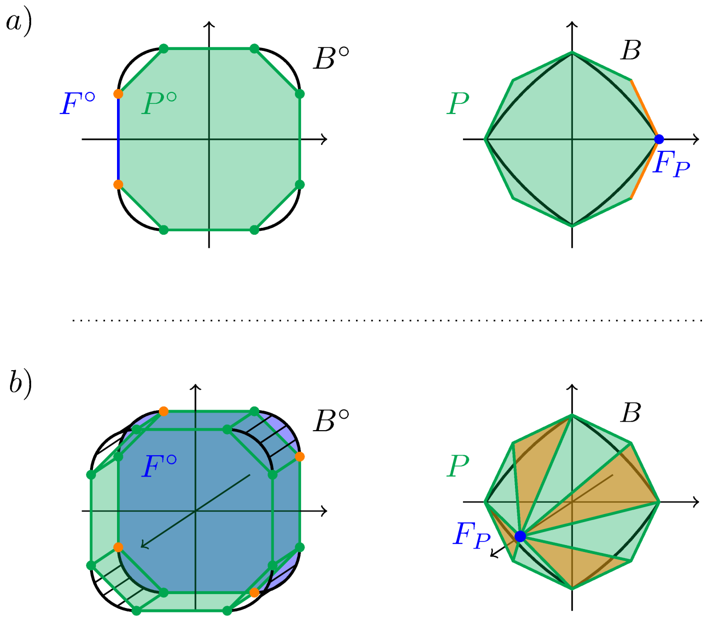


Figure 3.3: Two examples of a non-polyhedral dual unit ball B° and an inscribed polytope P° (LEFT) and their duals (RIGHT) as constructed in the proof of Lemma 3.2.2. In example *a*), the face F° is polyhedral, so B° and P° coincide there. The round parts of B° are cut off. In example *b*), the face F° is not polyhedral. By construction, B° and P° are the same on the polyhedral parts.

Then P° has a d-dimensional face $F^\circ_{P^\circ}$ that is contained in F°. By Lemma 2.4.5 the dual polytope $P = (P^\circ)^\circ$ contains B and the dual face $F_P = (F^\circ_{P^\circ})^\circ$ in P contains $F \subseteq \partial B$. Note that the dimension of F_P and F do not have to be the same. Any extreme point of $F^\circ_{P^\circ}$ corresponds to a facet of P containing F_P in its relative boundary. Just as in the polyhedral case, we can find a converging subsequence $(y_m)_m$ of $(z_{m,F^\circ})_m$ and now we choose the points $y_1, \dots, y_{d+1}$ such that each sequence $(y_m - y_j)_m$ lies in the cone of a facet around F_P. As $F \subseteq F_P$, the duals of these shifted sequences with respect to P converge to $d+1$ extreme points of $F^\circ_{P^\circ}$. With respect to B the sequences may converge to other extreme points. In case that two limit points that were different with respect to P do now coincide, add more extreme points in the corresponding area when constructing P°. Then the dual sequences converge to $d+1$ different points in $\partial_{\mathrm{rel}} F^\circ$ as required. They are affinely independent and so $E = F^\circ$. □

In the situation of Lemma 3.2.2, the converse is not true, as the following examples shows:

Example 3.2.3 Let $\mathbb{R}^2$ be equipped with the 1-norm, its dual is the L^∞-norm with the unit square as unit ball, see also Figure 3.4. Consider the sequence $(z_m)_m$ given by

$$z_m = \begin{pmatrix} m^2 \\ (-1)^m m \end{pmatrix}.$$

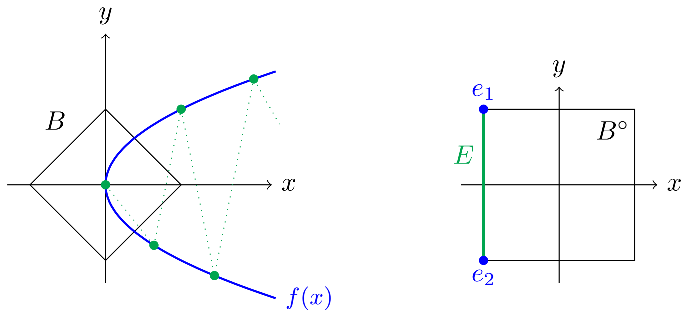


Figure 3.4: The converse of Lemma 3.2.2 is not true: also an unbounded projection can yield to the whole exposed dual.

Then $\frac{z_m}{\|z_m\|} \xrightarrow{\subseteq} F = \{(1,0)\}$ and $F^\circ = \text{conv}\{(-1,-1),(-1,1)\} \subseteq B^\circ$. The projection to $T(F^\circ)^*$ is

$$z_{m,F^\circ} = \begin{pmatrix} 0 \\ (-1)^m m \end{pmatrix},$$

which is unbounded. For $x \in \mathbb{R}^2$ the sequence $(D_m(x))_m$ of duals is given by

$$D_m(x) = \left\{ \begin{pmatrix} -1 \\ (-1)^m + 1 \end{pmatrix} \right\},$$

and has the two accumulation sets $e_1 := \{(-1,1)\}$ and $\mathrm{e}_2 := \{(-1,-1)\}$. Therefore $E = F^\circ$. ◦

3.2.2 Specific Properties of the Three Cases I), II) and III)

We briefly discuss the properties of B and B° in the three cases given in the beginning of this section.

I) Polyhedral norms Let B be polyhedral. Then also B° is polyhedral, that is, both can be described as the convex hull of finitely many distinct points or, equivalently, as the intersection of finitely many half-spaces. All extreme faces of B and B° are exposed, every extreme face of B has exactly one (exposed) dual face in B° and their dimensions sum up to $n-1$. Any extreme face that is not a facet lies in the relative boundary of a facet and the union of all closed facets covers the whole boundary.

II) B and B° are smooth Consider a unit ball $B \subseteq X$ such that every extreme face of both B and B° is an extreme point. Then the boundaries of B and B° are smooth and every extreme face is also exposed. So also here we have a 1-1 correspondence between the extreme points of B and those of B° and for an extreme face $F \subseteq B$ it holds

$$(F^\circ)^\circ = F. \tag{3.8}$$

An example for such a unit ball would be the Euclidean unit ball, not necessarily centered but such that the origin is still contained in its interior. The dual unit ball then is an ellipsoid, probably shifted and rotated, but with the origin in the interior.

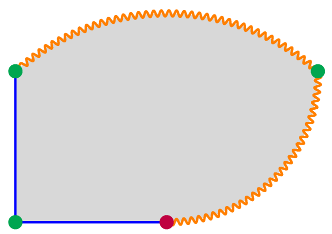


Figure 3.5: The different types of faces of B in X: facets (blue), corner points (green), smoothly exposed points (orange) and an extreme point that is not exposed (purple).

III) dim(X) = 2 and B is arbitrary Let us now consider a two-dimensional normed space equipped with a unit ball B. Then it can have four kinds of extreme faces, see also Figure 3.5:

- facets, i.e. one-dimensional extreme faces. They are always exposed and have a unique supporting hyperplane (blue lines in the picture).
- corner points: exposed points that have more than one supporting hyperplane. They can be isolated but do not have to be. On each side, they can be either in the relative boundary of a facet or of a smoothly curved part (green points in the picture).
- smoothly exposed points: extreme points that lie in a smooth part of the boundary with extreme points on both sides. They are exposed and have exactly one supporting hyperplane (every point in the orange part of the boundary in the picture).
- extreme points that are not exposed: points of this type are always in the relative boundary of a facet on one side and of a smoothly curved part on the other side, where the transition from the curved part to the facet is smooth. They lie in the unique supporting hyperplane which defines the facet (purple point in the picture).

Considering the dual unit ball B°, it can have the same types of faces as described above. There is a strong relation between the faces of B and those of B° and shown in Tabular 3.1 below and in Figure 3.6:

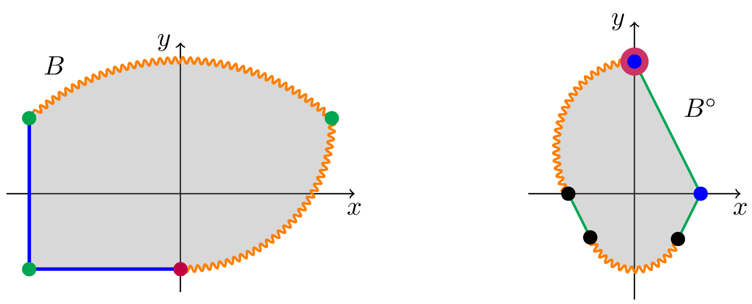


Figure 3.6: A unit ball B (LEFT) and the dual unit ball B° (RIGHT) with faces corresponding to those of B. The black points are extreme but not exposed and correspond to those green extreme points of B that are in the relative boundary of a smooth orange part.

The slope of the two lines tangent to a corner point fixes the length of the dual facet by determining the position of the endpoints in the relative boundary of the facet. Such an endpoint is the unique point having dual pairing -1 with all points of the hyperplane. Note that the dimensions of a face and its dual do not have to sum up to 1 and that a face of B can have more than one face of B° dual to it and vice versa.

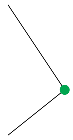
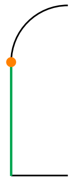
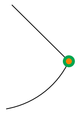
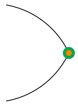
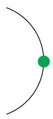
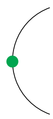
faces of B			**faces of B°**
facets			corner points
without non-exposed point			between two facets
with one non-exposed point			between a facet and a smooth part
with two non-exposed points			between two smooth parts
smoothly exposed points			smoothly exposed points
smoothly exposed point			smoothly exposed point

Table 3.1: An overview over the types of faces and their duals in two dimensions.

3.2.3 A Useful Lemma

Before we state the theorem to characterize converging sequences, we first show a lemma which already contains the main idea of the characterization.

Lemma 3.2.4 *Let $B \subseteq X$ be a unit ball and B° its dual such that they belong to one of the three cases I) - III) defined above.*
Then every unbounded sequence $(z_m)_{m\in\mathbb{N}}$ has an unbounded subsequence satisfying the following two conditions:

1) $E := \operatorname{aff}\{D(x) \mid x \in X\} \cap B^\circ$ is an extreme set of B°.

2) The projection of $(z_m)_m$ to $T(E)^*$ converges to a point $p := \lim_{m\to\infty} z_{m,E}$. ∘

Proof. As mentioned before, we will split up the proof into the three cases and prove them separately. Recall that E depends on the sequence $(z_m)_{m\in\mathbb{N}}$ via $D(x)$, which is the accumulation set of the sequence $(D_m(x))_m$, where for each $x \in X$ and $m \in \mathbb{N}$, the set $D_m(x) \subseteq B^\circ$ is the face dual to that face of B the contains $\frac{z_m - x}{\|z_m - x\|}$ in its relative interior.

I): B is polyhedral When B is polyhedral, it only has finitely many faces and especially only finitely many facets. Therefore we can find a facet G and a subsequence also called $(z_m)_m$ such that $(z_m)_m$ lies in the (closed) cone $K(G)$ generated by G. Now consider the distance of $(z_m)_m$ to the relative boundary of $K(G)$.

If $(z_m)_m$ has no subsequence with bounded distance to the relative boundary of $K(G)$, then for any $x \in X$, also the sequence $(z_m - x)_m$ lies in the interior of $K(G)$ for m big enough. Thereby for every $x \in X$ with m big enough

$$D_m(x) = D(x) = G^\circ = \{g\} \subseteq B^\circ$$

where g is an extreme point of B°. So $E = G^\circ$ is extreme and the projection $(z_{m,E})_m$ is trivial and obviously converges.

If $(z_m)_m$ has bounded distance to the relative boundary of G, the idea is to split up the faces in those with bounded and those with unbounded distance to a subsequence. So take a subsequence $(z_{m_k})_{k\in\mathbb{N}}$ such that the cones in the relative boundary of $K(G)$ can be split up in those to which $(z_{m_k})_k$ has bounded distance and those to which the distance goes to infinity. That is, if G_j for $j \in \{1, \dots, s\}$ are the faces in the relative boundary of G, we can order the labeling such that

$$\partial_{\mathrm{rel}} K(G) = \bigcup_{j\le l} K(G_j) \cup \bigcup_{j \ge l+1} K(G_j),$$

where

$$\operatorname{dist}(z_{m_k}, K(G_j)) = \begin{cases} < \infty & j \in \{1, \dots, l\} \\ \to \infty & j \in \{l+1, \dots, s\}, \end{cases}$$

as $m_k \longrightarrow \infty$. Note that $l \ge 1$ as by assumption there is a face to which $(z_{m_k})_k$ has bounded distance and that $l < s$ because the sequence is unbounded. We define

$$F := \bigcap_{j\in\{1,\dots,l\}} \operatorname{cl}(G_j) \tag{3.9}$$

to be the intersection of all (closed) faces in the relative boundary of G to which $(z_{m_k})_k$ has bounded distance. As the relative boundary of the cone $K(G)$ is the union of parts of subspaces all intersecting in the origin, their distance from each other outside of any compactum is unbounded if they do not have a common subspace. Therefore the intersection in Equation (3.9) is not trivial and F is a face of B of dimension smaller than $\dim(G)$. Figure 3.7 shows an example of the above construction in $\mathbb{R}^3$ equipped with the 1-norm.

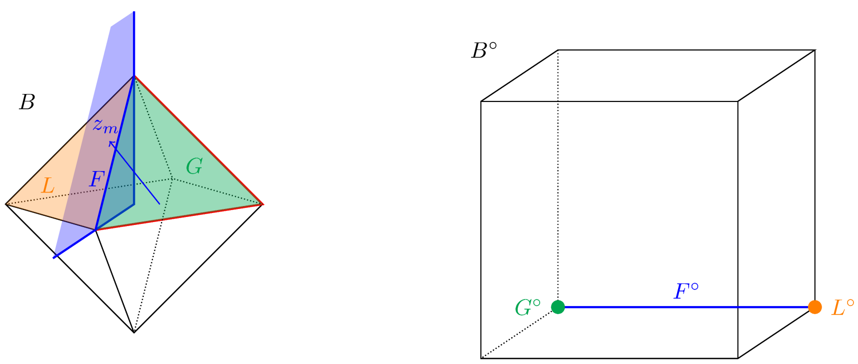


Figure 3.7: An example in $\mathbb{R}^3$ with the 1-norm and a sequence $z_m = (m, k, am)$ with $a, k > 0$. The notations are as in the proof.

It remains to show that the dual face F° actually is $E = \operatorname{aff}\{D(x) \mid x \in X\}$. Let $L^\circ = \{l\}^\circ$ be an extreme point of F° (i.e. an extreme point of B° in the relative boundary of F°) and $L \subsetneq B$ its dual facet. Then the cone $K(L)$ has $K(F)$ in its boundary and as the distance of $(z_{m_k})_k$ to $K(F)$ is bounded, there is an $x \in X$ such that $z_{m_k} - x \in K(L)$ for k big enough. This yields

$$D_{m_k}(x) = \left(\frac{z_{m_k} - x}{\|z_{m_k} - x\|}\right)^\circ \in L^\circ,$$

for k big enough. Such an $x \in X$ can be found for every extreme point of F° which shows that $F^\circ = \operatorname{aff}\{D(x) \mid x \in X\} \cap B^\circ = E$. For the lemma to be proven in the polyhedral case we still have to show that the projection of $(z_{m_k})_k$ to $T(E)^*$ converges. By Proposition 2.4.16, $T(E)^* = V(F)^\perp$ and as $(z_{m_k})_k$ has bounded distance to $K(F)$, the projection $(z_{m_k,E})_k$ is bounded. Therefore $(z_{m_k,E})_h$ has a converging subsequence and the lemma is proven.

II): B and B° only have extreme points In this case, B and B° are smooth and only have extreme points in their boundaries. Let $E = \operatorname{aff}\{D(x) \mid x \in X\} \cap B^\circ$ be given as above. We have to show that E is an extreme point in the boundary of B°. Equivalently, all sequences $(D_m(x))_m$ of extreme points have to converge to one point $e \in \partial_{\mathrm{rel}} B^\circ$. By the definition of $D_m(x)$ and Equation (3.8) we get the condition

$$D_m(x) = \left(\frac{z_m - x}{\|z_m - x\|}\right)^\circ \overset{!}{\longrightarrow} e,$$

or, on the side of B:

$$\frac{z_m - x}{\|z_m - x\|} \longrightarrow f, \quad \forall x \in X$$

where $F = \{f\} = E^\circ$. Since the sequence of directions of a shifted sequence converges if and only if the sequence of directions of the unshifted sequence converges, we obtain for our smooth unit ball the condition

$$\exists F = \{f\} \subseteq \partial B : \quad \frac{z_{m_k}}{\|z_{m_k}\|} \longrightarrow f,$$

for some subsequence $(z_{m_k})_k$ of $(z_m)_m$. As ∂B is compact, we can always find such a subsequence. Therefore $E = \{e\}$ is an extreme point.

As $T(E) = 0$, the projection onto this subspace is trivial and the second condition is redundant in this case.

III): $\dim(X) = 2$ and B is arbitrary In the two-dimensional case there are only two kinds of extreme sets in the boundary of B and B°: one-dimensional facets and zero-dimensional extreme points. The sequence of directions $\left(\frac{z_m - x}{\|z_m - x\|}\right)_{m \in \mathbb{N}}$ lives on the boundary ∂B which is a compact subset of X. Therefore the sequence has a convergent subsequence and without loss of generality we call this subsequence again $\left(\frac{z_m - x}{\|z_m - x\|}\right)_m$.

If the sequence $\left(\frac{z_m}{\|z_m\|}\right)_m$ converges within a facet F, then we have the same situation as in the first part of the polyhedral case: for any $x \in X$ also the sequence $\left(\frac{z_m - x}{\|z_m - x\|}\right)_m$ converges within F and $E = F^\circ$ is an extreme point. The subspace $T(E)^*$ is trivial and the projection of $(z_m)_m$ to it converges obviously.

Now assume $\frac{z_m}{\|z_m\|} \overset{\subseteq}{\longrightarrow} F$ where $F = \{f\}$ is an extreme point. Then also for any shifted sequence it holds $\frac{z_m - x}{\|z_m - x\|} \overset{\subseteq}{\longrightarrow} F$ and $E \subseteq F^\circ$, where F° is the exposed dual of F. If $E = F^\circ$ we are done with the first condition. Otherwise recall that the sequence $(D_m(x))_m$ was a sequence of extreme sets of

B° converging to the relative boundary of F°. As F° only has two extreme points in its relative boundary and we assume $E \neq F^\circ$, E has to be one of these two extreme points and is therefore also an extreme set.

To see that the second condition of the lemma is fulfilled, we assume $\dim(E) = 1$, otherwise the statement is trivial. Then its dual face F is a corner point (because $\dim(F) + \dim(E) = 1$) and $V(F) \perp T(E)^*$. Since the set $E = F^\circ$ is the whole exposed dual (and not only a relative boundary point of it), we know that for both sides of the relative boundary of B around F there is an $x \in X$ such that the sequence of directions (or a subsequence) remains on that side. This is equivalent to $(z_{m,E})_m$ being bounded. Therefore we can find a subsequence such that the projection $(z_{m,E})_m$ converges. □

3.2.4 Characterization of Converging Sequences

In Corollary 3.1.10 we described the set of Busemann points of X with respect ot a given unit ball as the set of functions $h_{E,p}$, where $E \subseteq B^\circ$ is a proper extreme set and $p \in T(E)^*$ is a point. Our goal now is to give the topology of this set by determining the limit of unbounded sequences converging in the compactification. Recall that we only consider unit balls whose set of extreme sets is closed. So the set of Busemann points $h_{E,p}$ is the set of all horofunctions of the compactification by Proposition 3.1.6.

Remark 3.2.5 The approach to describe the topology by convergent sequences is justified by the following discussion in [BJ06, §I.8.9]: A *convergence class of sequences* C is a class of pairs $((y_m)_{m\in\mathbb{N}}, y_\infty)$ consisting of an unbounded sequence $(y_m)_{m\in\mathbb{N}}$ and a limit point $y_\infty \in X$ satisfying several convergence conditions. Elements of the class are called C-convergent and denoted by $y_m \xrightarrow{C} y_\infty$. To a subset $A \subseteq X$ assign the set $\overline{A}$, the set of all points y in X such that there is a sequence in A that C-converges to y. Then a subset $A \subseteq X$ is called *closed* provided $A = \overline{A}$. The convergence class of sequences C uniquely defines a topology on X such that a sequence $(y_m)_{m\in\mathbb{N}}$ converges to a point $y_\infty \in X$ with respect to this topology, if and only if $((y_m)_{m\in\mathbb{N}}, y_\infty) \in C$. The obtained topological space is a compact Hausdorff space if and only if the limit of every convergent sequence is unique and if every sequence in X has a convergent subsequence. ○

Theorem 3.2.6 *Let $B \subseteq X$ be a unit ball and $B^\circ \subseteq X^*$ its dual such that they belong to one of the following cases:*

I) The unit ball is polyhedral.

II) The unit and the dual unit ball have smooth boundaries.

III) The space X is two-dimensional.

Let $(z_m)_{m\in\mathbb{N}}$ be an unbounded sequence in X. Then the sequence $(\psi_{z_m})_{m\in\mathbb{N}}$ converges to a horofunction $h_{E',p}$ associated to an extreme set $E' \subseteq B^\circ$ and a point $p \in T(E^*$ if and only if the following conditions are satisfied:

1) $E := \operatorname{aff}\{D(x) \mid x \in X\} \cap B^\circ$ is extreme.

2) The projection $(z_{m,E})_{m\in\mathbb{N}}$ of $(z_m)_{m\in\mathbb{N}}$ to $T(E)^*$ converges.

If $(\psi_{z_m})_m$ converges, then $E' = E$ and $p = \lim_{m\to\infty} z_{m,E}$. ○

Proof. We have to show two directions for the proof. We start with a sequence $(z_m)_{m\in\mathbb{N}}$ fulfilling both conditions with respect to some extreme face $E \subseteq B^\circ$ and $p := \lim_{m\to\infty} z_{m,E} \in T(E)^*$ and show, that it converges to the associated horofunction $h_{E,p}$. Just as in the previous lemma, the proof will be split up in three parts.

I): B is polyhedral Let B be polyhedral and assume both conditions are satisfied for a sequence $(z_m)_{m\in\mathbb{N}}$. Then by the first condition, all accumulation sets $D(x)$ of the sequences $(D_m(x))_m$ dual to the shifted sequences $\left(\frac{z_m-x}{\|z_m-x\|}\right)_m$ for any $x \in X$ are either in the relative boundary of the extreme set E or they are E itself. As B and B° are polyhedral, they only have finitely many faces and therefore for any $x \in X$ the sets $D_m(x)$ are contained in $D(x)$ for m big enough: there is an $M \in \mathbb{N}$ for each $x \in X$ such that

$$D_m(x) = D(x) \quad \forall m \geq M.$$

Let $\{E_j\}_{j\in\{1,\ldots,k\}}$ be the set of extreme sets in the relative boundary of E. Their dual faces $F_j := E_j^\circ$ all have the face $F := E^\circ$ in their relative boundary. For every $x \in X$ and $m \in \mathbb{N}$ big enough there is a $j \in \{1,\ldots,k\}$ such that $\frac{z_m-x}{\|z_m-x\|} \in F_j$, that is, it lies in one of the faces around F. Then for any point $e_j \in E_j$ it holds

$$-1 = \left\langle e_j \;\middle|\; \frac{z_m - x}{\|z_m - x\|} \right\rangle = \inf_{q\in B^\circ} \left\langle q \;\middle|\; \frac{z_m - x}{\|z_m - x\|} \right\rangle.$$

This means that (for m big enough) the infimum over B° is attained at some extreme point of E.

Using Lemma 2.6.1 with $z_{m,E} \in K(F) = (T(E)^*)^\perp$ we compute for m big enough:

$$\begin{aligned}
\psi_{z_m}(x) &= \|z_m - x\| - \|z_m\| \\
&= -\inf_{q\in B^\circ} \langle q \,|z_m - x\rangle + \inf_{q\in B^\circ} \langle q \,|z_m\rangle \\
&= -\inf_{q\in E} \langle q \,|z_m - x\rangle + \inf_{q\in E} \langle q \,|z_m\rangle \\
&\overset{2.6.1}{=} -\left\langle E \,\middle|z_m^E\right\rangle - \inf_{q\in E} \left\langle q \,\middle|z_{m,E} - x\right\rangle + \left\langle E \,\middle|z_m^E\right\rangle + \inf_{q\in E} \left\langle q \,\middle|z_{m,E}\right\rangle \\
&\to -\inf_{q\in E} \langle q \,|p - x\rangle + \inf_{q\in E} \langle q \,|p\rangle = h_{E,p}(x).
\end{aligned}$$

II): B and B° only have extreme points The idea of the proof in the smooth case is to define (for every $x \in X \setminus \{0\}$ because $x = 0$ is trivially true) a sequence of new unit balls $B_m(x)$, where each of them coincides with B at the points $\frac{z_m}{\|z_m\|}$ and $\frac{z_m-x}{\|z_m-x\|}$ such that B_m behaves locally polyhedral. See Figure 3.8 for a picture.

To do so, let $x \in X \setminus \{0\}$ be a point and let

$$u_m := \frac{z_m}{\|z_m\|} \quad \text{and} \quad v_m := \frac{z_m - x}{\|z_m - x\|}$$

be extreme points in ∂B. We first take only those $m \in \mathbb{N}$ into account such that $u_m \neq v_m$. If $\dim(X) = n \geq 3$ let $y_1, \ldots, y_{n-2} \in X$ be points, such that for all $m \in \mathbb{N}$ with $w_m^j := \frac{z_m-y_j}{\|z_m-y_j\|} \in \partial B$ the affine plane

$$H_m := \operatorname{aff}\{u_m, v_m, w_m^1, \ldots, w_m^{n-2}\}$$

is a well-defined hyperplane in X not containing the origin. Denote by V_m^+ the half-space defined by H_m containing the origin. We set

$$B_m := B \cap V_m^+.$$

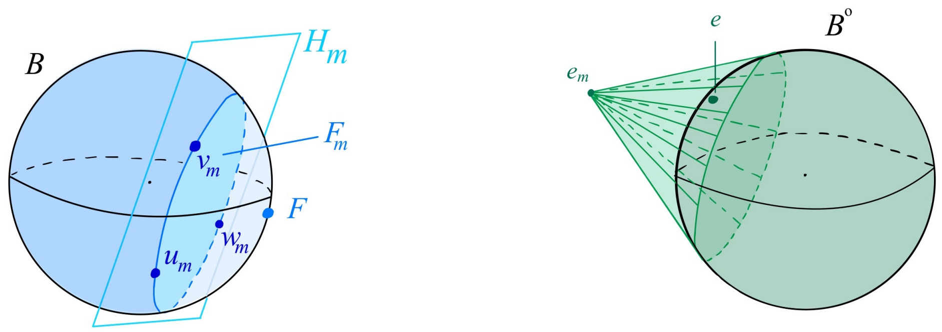


Figure 3.8: Schematic pictures to illustrate the notations and the idea of the proof.

B_m is then for each $m \in \mathbb{N}$ a closed convex ball containing the origin in its interior and defines a norm $\|\cdot\|_m$. As B was convex, so is $B_m(x)$ and $F_m := H_m \cap B_m$ is a facet of B_m with H_m as supporting hyperplane. Therefore there is exactly one extreme point $E_m = \{e_m\} \subseteq B_m^\circ$ dual to F_m.

Such a unique extreme point e_m exists for every m and we claim that the sequence $(e_m)_{m\in\mathbb{N}} \subseteq X^*$ converges to a point $e \in \partial B^\circ$.
As $E = \operatorname{aff}\{D(x) \mid x \in X\} \cap B^\circ$ is an extreme point, we know by the proof of Lemma 3.2.4 on page 53 that $\frac{z_m}{\|z_m\|} \xrightarrow{\subseteq} f$, where $\{f\} = F = E^\circ$ is the extreme point dual to E. Then also $\frac{z_m - z}{\|z_m - z\|} \xrightarrow{\subseteq} f$ for any $z \in X$. Note that the facet F_m and therefore also the point e_m strongly depends on the choice of the points x and $y_1, \dots, y_{n-1}$. Nevertheless the point e is unique. Indeed, the sequence $(H_m)_{m\in\mathbb{N}}$ of hyperplanes (which define the point e_m) converges to a hyperplane H supporting B at f. By the smoothness of ∂B, the hyperplane H is unique. The point e then is the point defined by H via $\langle e|h\rangle = -1$ for any $h \in H$. In other words, $E = \{e\} = F^\circ$.

As $B_m \subseteq B$ it holds $\|\cdot\|_m \geq \|\cdot\|$ and especially since B_m and B coincide at u_m and v_m by construction we have

$$\|z_m\|_m = \|z_m\| \quad \text{and} \quad \|z_m - x\|_m = \|z_m - x\|.$$

As E_m is dual to the facet F_m it also holds

$$\begin{aligned}
\|z_m\|_m &= -\inf_{q\in B_m} \langle q|z_m\rangle = -\inf_{q\in E_m} \langle q|z_m\rangle = -\langle e_m|z_m\rangle\,, \\
\|z_m - x\|_m &= -\inf_{q\in B_m} \langle q|z_m - x\rangle = -\inf_{q\in E_m} \langle q|z_m - x\rangle = -\langle e_m|z_m - x\rangle\,.
\end{aligned}$$

Therefore we calculate:

$$\begin{aligned}
\psi_{z_m}(x) &= \|z_m - x\| - \|z_m\| \\
&= \|z_m - x\|_m - \|z_m\|_m \\
&= -\langle e_m|z_m - x\rangle + \langle e_m|z_m\rangle = \langle e_m|x\rangle \\
&\longrightarrow \langle e|x\rangle = h_{E,p}(x).
\end{aligned}$$

In the case where $u_m = v_m$ for a subsequence $(z_{m_k})_k \subseteq (z_m)_m$, let $\{d_{m_k}\} = \{u_{m_k}\}^\circ \subseteq \partial B_m^\circ$ be the sequence of dual points. Then

$$\begin{aligned}
\psi_{z_{m_k}}(x) &= \|z_{m_k} - x\| - \|z_{m_k}\| \\
&= -\langle d_{m_k}|z_{m_k} - x\rangle + \langle d_{m_k}|z_{m_k}\rangle = \langle d_{m_k}|x\rangle \\
&\longrightarrow \langle e|x\rangle = h_{E,p}(x).
\end{aligned}$$

III): $\dim(X) = 2$ and B is arbitrary We now look at the two-dimensional case. Figure 3.9 shows a schematic picture of the idea of the proof with the notations we use. The basic idea of the proof is the following: We first replace the norm by the dual pairing with an appropriate point. Then we have to treat the cases $\dim(E) = 0$ and $\dim(E) = 1$ separately. If E is an extreme point, we construct a sequence $(e_m)_m$ of points for each $x \in X$, such that for each $m \in \mathbb{N}$ it holds $\langle e_m | z_m \rangle = -\|z_m\|$ and $\langle e_m | z_m - x \rangle = -\|z_m - x\|$. Using this new sequence, the calculation with be simplified. For $\dim(E) = 1$ and for each $x \in X$ we construct two new sequences $(h_{m,0})_m$ and $(h_{m,x})_m$ contained in $\mathrm{aff}(E)$. In the final calculation we will use that $h_{m,x} - h_{m,0} \in T(E)$ and that $(z_{m,E})_m$ is bounded.

For some $x \in X \setminus \{0\}$ denote by $q_{m,x}, q_{m,0} \in \partial B^\circ$ extreme points such that

$$\langle q_{m,0} | z_m \rangle = -\|z_m\|$$
$$\langle q_{m,x} | z_m - x \rangle = -\|z_m - x\|.$$

In other words, $q_{m,0} \in D_m(0)$ and $q_{m,x} \in D_m(x)$. These points are not uniquely determined, but if there is more than one possibility, they both lie in the relative boundary of a common facet.

Let us first assume that $\dim(E) = 0$, that is, $E = \{e\}$ is an extreme point and $q_{m,x}, q_{m,0} \longrightarrow e$. We set the following notations:

$$u_m := \frac{z_m}{\|z_m\|} \quad \text{and} \quad v_m := \frac{z_m - x}{\|z_m - x\|}.$$

Let $H_{-1}^{u_m}, H_{-1}^{v_m} \subseteq X^*$ be the hyperplanes defined as in Definition 2.1.20, supporting B° at $q_{m,0}$ and $q_{m,x}$, respectively. Let $e_m \in H_{-1}^{u_m} \cap H_{-1}^{v_m}$ be a point in the intersection. As $H_{-1}^{u_m}$ and $H_{-1}^{v_m}$ are parallel if and only if $u_m = \pm v_m$, the intersection is not empty for m big enough. Then it holds

$$\langle e_m | z_m \rangle = -\|z_m\| \quad \text{and} \quad \langle e_m | z_m - x \rangle = -\|z_m - x\|.$$

As our extreme set $E = \{e\}$ was only a point we have $e_m \longrightarrow e$ as $m \longrightarrow \infty$ and therefore

$$\begin{aligned}\psi_{z_m}(x) &= \|z_m - x\| - \|z_m\| \\ &= -\langle e_m | z_m - x \rangle + \langle e_m | z_m \rangle = \langle e_m | x \rangle \\ &\longrightarrow \langle e | x \rangle = h_{E,p}(x).\end{aligned}$$

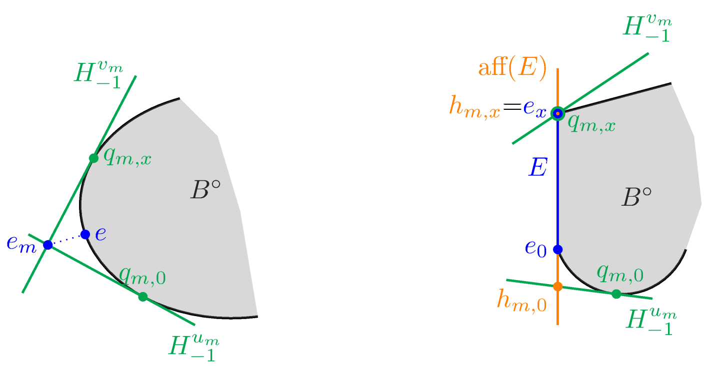


Figure 3.9: Schematic pictures to illustrate the notations used in the proof for $\dim(E) = 0$ (LEFT) and $\dim(E) = 1$ (RIGHT).

Now assume E is a facet, i.e. $\dim(E) = 1$. Then there are exactly two points $e_1, e_2 \in \partial B^\circ$ such that $E = \mathrm{conv}(e_1, e_2)$. Let $e_x, e_0 \in \{e_1, e_2\}$ be such that $q_{m,x} \to e_x$ and $q_{m,0} \to e_0$. Just as in the previous case, we look at the hyperplanes $H_{-1}^{v_m}, H_{-1}^{u_m}$ supporting B° at $q_{m,x}, q_{m,0}$, respectively. But now we consider points in the intersection of $\operatorname{aff}(E)$ with these two hyperplanes:

$$\begin{aligned} h_{m,x} &\in H_{-1}^{v_m} \cap \operatorname{aff}(E) \\ h_{m,0} &\in H_{-1}^{u_m} \cap \operatorname{aff}(E). \end{aligned} \tag{3.10}$$

As all hyperplanes are supporting and $q_{m,x} \to e_x$ as well as $q_{m,0} \to e_0$, the intersections are non-empty for m big enough and we have

$$\begin{aligned} h_{m,x} &\longrightarrow e_x \\ h_{m,0} &\longrightarrow e_0. \end{aligned} \tag{3.11}$$

Recall from page 36 and page 42 our description of horofunctions for any $x \in X$:

$$h_{E,p}(x) = |p - x|_E - |p|_E = -\inf_{q \in E} \langle q|p - x\rangle + \inf_{q \in E} \langle q|p\rangle .$$

Now we claim that $\inf_{q\in E} \langle q|p - x\rangle = \langle e_x|p - x\rangle$. We already know that the infimum over E is attained in one of its two relative boundary points (or both, if $p - x \in V(E)^\perp$). Without loss of generality, let us assume $\inf_{q\in E} \langle q|p - x\rangle = \langle e_1|p - x\rangle$. Then

$$\langle e_1 - e_2|p - x\rangle < 0 \quad \text{ if } p - x \notin (V(E)^\perp)^*. \tag{3.12}$$

As $q_{m,x} \in \partial B^\circ$ was the point in the boundary minimizing the dual pairing with $z_m - x$, we also have $\langle e_1 - q_{m,x}|z_m - x\rangle \geq 0$. As $\left\langle e_1 - q_{m,x} \;\middle|\; z_m^E \right\rangle \leq 0$ because $\left\langle E \;\middle|\; \frac{z_m^E}{\|z_m^E\|} \right\rangle = -1$ is minimal, we see that

$$\langle e_1 - q_{m,x}|z_{m,E} - x\rangle \geq 0.$$

Since $z_{m,E} \to p$, the convergence $q_{m,x} \to e_2$ would lead to a contradiction with Equation (3.12) and thereby

$$e_x = e_1$$

as we wanted to show.
If $p - x \in (V(E)^\perp)^*$, then $\langle e_1|p - x\rangle = \langle e_2|p - x\rangle$ and the claim is trivially true.

Similarly we get $\inf_{q\in E} \langle q|p\rangle = \langle e_0|p\rangle$. Together we obtain

$$\begin{aligned} h_{E,p}(x) &= |p - x|_E - |p|_E \\ &= -\inf_{q\in E} \langle q|p - x\rangle + \inf_{q\in E} \langle q|p\rangle \\ &= -\langle e_x|p - x\rangle + \langle e_0|p\rangle . \end{aligned} \tag{3.13}$$

Finally we compute

$$\begin{aligned} \psi_{z_m}(x) &= \|z_m - x\| - \|z_m\| \\ &= -\langle q_{m,x}|z_m - x\rangle + \langle q_{m,0}|z_m\rangle \\ &\overset{(3.10)}{=} -\langle h_{m,x}|z_m - x\rangle + \langle h_{m,0}|z_m\rangle \\ &= \langle h_{m,0} - h_{m,x}|z_m\rangle + \langle h_{m,x}|x\rangle \\ &\overset{(3.10)}{=} \langle h_{m,0} - h_{m,x}|z_{m,E}\rangle + \langle h_{m,x}|x\rangle \\ &\xrightarrow{(3.11)} \langle e_0 - e_x|p\rangle + \langle e_x|x\rangle \\ &= -\langle e_x|p - x\rangle + \langle e_0|p\rangle \overset{(3.13)}{=} h_{E,p}(x). \end{aligned}$$

If $x = 0$, then $\psi_{z_m}(0) = 0 = h_{E,p}(0)$, which completes the first part of the proof in this case.

The other direction This part of the proof is the same for all three cases and based on Lemma 3.2.4. We assume the sequence $(\psi_{z_m})_m$ to converges to $h_{E,p}$ with $E \subseteq B^\circ$ an extreme set and $p \in T(E)^*$. We have to show that $(z_m)_{m\in\mathbb{N}}$ fulfills both conditions above. By Lemma 3.2.4, $(z_m)_m$ has a subsequence fulfilling both conditions with respect to some extreme set $E' \subseteq B^\circ$ and a point $p' \in T(E')^*$. By the first part of the proof, this subsequence converges to some horofunction $h_{E',p'}$. As two horofunctions only coincide if their associated extreme sets and points coincide (recall Lemma 2.6.9 on page 37), it has to be $E = E'$ and $p = p'$. This follows for any (sub-)subsequence of $(z_m)_m$ for which reason both conditions of the theorem are fulfilled for E and p. □

Remark 3.2.7 In [JS16] it is shown that if the norm is polyhedral, then a sequence $(z_m)_{m\in\mathbb{N}} \subseteq X$ converges to a horofunction, if and only if the following conditions are satisfied:

1) The sequence is unbounded: $\|z_m\| \longrightarrow \infty$.
2) The projection $z_{m,V(F)}$ of z_m to $V(F)$ lies in the cone $K(F)$ for m big enough.
3) The distance of the projection to the relative boundary of the cone is unbounded: $d(z_{m,V(F)}, \partial_{\text{rel}}K(F)) \longrightarrow \infty$ as $m \to \infty$.
4) The orthogonal projection of z_m to $V(F)^\perp$ is bounded and converges to p: $\|z_m^{V(F)} - p\| \longrightarrow 0$ as $m \to \infty$.

This "old" criterion is actually equivalent to the more general one we have shown here. Some arguments used in the proof of Lemma 3.2.4 and Theorem 3.2.6 for the polyhedral case are based on this equivalence. As we are dealing with polytopes, there is a 1-1-correspondence between the faces of B and those of B° (see Remark 2.4.14) and their dimensions sum up to $n-1$. Therefore, when $E \subseteq F$ denotes the dual face of $F \subseteq B$, we have $T(E) = V(F)^\perp$, which gives us immediately the equivalence between the two respectively last items of the two criteria. So it remains to show that the following two statements are equivalent:

(a) z_m^E lies in the cone $K(F)$ and has unbounded distance to its relative boundary.

(b) $E = \operatorname{aff}\{D(x) \mid x \in X\} \cap B^\circ$ is extreme.

Assume condition (a) holds. Then there are finitely many vertices $e_1, \ldots, e_k \in \partial B^\circ$ such that $E = \operatorname{conv}\{e_1, \ldots, e_k\}$ and for each of them there is an $x_i \in X$ with

$$D(x_i) = e_i. \tag{3.14}$$

On the other hand, $D(x) \in \{e_1, \ldots, e_k\}$ for all $x \in X$, otherwise E would have an additional extreme point. Let $F_i := \{e_i\}^\circ \subseteq B$ be the corresponding facets of B. Then as E is a face, their intersection

$$F := \bigcap_{i=1}^{k} F_i$$

is non-empty and F is the face of B that is dual to E. As F is a common face of all F_i, for all $x \in X$ there holds $z_m - x \in K(F_i)$ for some $i \in \{1, \ldots, k\}$ and m big enough. The relative boundary of F consists of faces of B that are the intersection of some facets F_i with some facets that do not belong to $\{F_1, \ldots, F_k\}$. If the distance of the sequence $\left(z_{m,V(F)}\right)_m$ to $\partial_{\text{rel}}K(F)$ would be bounded, that is, has bounded distance to at least one boundary face, we could find an $x \in X$ such that for m big enough $z_{m,V(F)} - x$ lies in a facet that is not dual to an extreme points of E. As $\left(z_m^{V(F)}\right)_m$ is bounded, also $z_m - x$ lies for m big enough in the (probably closed) cone over a facet not belonging to $\{F_1, \ldots, F_k\}$, so E had to have another extreme point. (See the proof of Lemma 3.9 in [JS16] for more details.) By the boundedness of $\left(z_m^{V(F)}\right)_m$ and as $z_m - x \in K(F_i)$ for all $x \in X$ and m big enough, also the projection $z_{m,V(F)} \in K(F)$ for m big enough. Otherwise $(z_{m,V(F)})_m$ would stay within bounded distance to $\partial_{\mathrm{rel}} K(F)$.

Now assume (b) holds. Then the equivalence follows from the proof of Lemma 3.2.4 on page 51: there we constructed E as the dual face of F, where F was the intersection of all faces to which $(z_m)_m$ has bounded distance. ◦

3.2.5 Examples

Before we go on with a discussion why we had to restrict B to some special cases, we want to give some examples to illustrate the conditions of Theorem 3.2.6 and to give the reader some intuition how sequences converge. In all examples below we consider $\mathbb{R}^2$ but equipped with different norms and sequences of the form $z_m = (n, f(n)) \in \mathbb{R}^2$ following a function $f : \mathbb{R} \to \mathbb{R}$.

Example 3.2.8 We start with $\mathbb{R}^2$ equipped with the 1-norm. Its dual is the ∞-norm as seen in Example 2.4.12 before. The unit ball B and its dual B° as well as the notation of faces are shown in Figure 3.10, the functions we consider are shown in Figure 3.11.

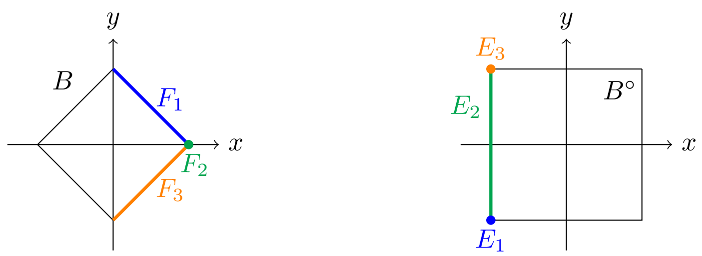


Figure 3.10: The unit ball B and its dual B° with some of their faces colored: faces that are dual to each other have the same color.

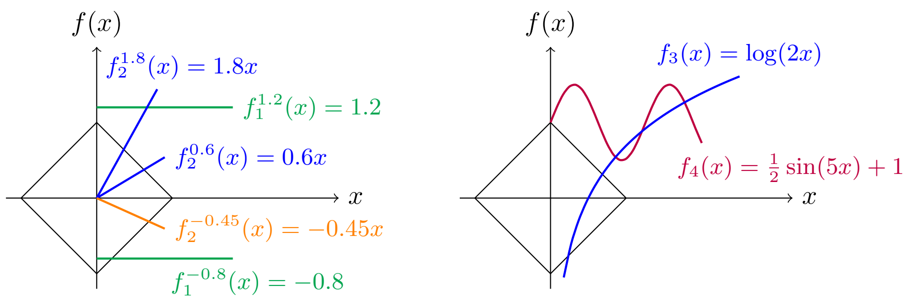


Figure 3.11: The functions the sequences in this example follow. The colors correspond to those in Figure 3.10 and show the extreme set of B° that defines the horofunction the sequence is converging to. The purple function f_4 defines a sequence that does not converge.

1) For some constant $c \in \mathbb{R}$ and $m \in \mathbb{N}$ consider the constant function $f_1^c(x) = c$. This gives us the sequence $\left(z_m^{(1)}\right)_m$ with

$$z_m^{(1)} = \begin{pmatrix} m \\ c \end{pmatrix} \in \mathbb{R}^2.$$

The sequence runs along a line parallel to the x-axis shifted by c. Then for any $x \in \mathbb{R}^2$ the sequence $\left(z_m^{(1)} - x\right)_m$ goes again along a line parallel to the x-axis but now with a different y-component. Therefore the sequence of directions lies either in F_1 (if $c - x_2 > 0$), in F_2 (if

$c - x_2 = 0$) or in F_3 (if $c - x_2 < 0$) and always converges to F_2. This gives us $E = E_1 \cup E_2 \cup E_3$ and as E_1, E_3 are extreme points in the relative boundary of E_2 we have

$$E = E_2.$$

The projection of $z_m^{(1)}$ to E is its second component, therefore we conclude

$$\psi_{z_m^{(1)}} \longrightarrow h_{E_2,p}$$

with $p = (0, c) \in T(E)^*$.

2) Next we consider sequences $\left(z_m^{(2)}\right)_{m \in \mathbb{N}}$ of the form

$$z_m^{(2)} = \begin{pmatrix} m \\ sm \end{pmatrix} \in \mathbb{R}^2$$

with $s \neq 0$. For any $s > 0$ the direction of the sequence lies in F_1. Shifting the sequence by some $x \in \mathbb{R}^2$ may lead to a direction through a different face for some small m, but at some point, the sequence of directions will come back to F_1 and remain there. So no matter how we chose the slope s of the sequence, all sequences of this type converge to the same horofunction $h_{E_1,p}$ where $p = 0$. If $s < 0$ then $\psi_{z_m^{(2)}} \to h_{E_3,p}$ ($p = 0$) by the same argument.

3) One might think that an easier condition for finding the appropriate face F is to look at the limit of the sequence $\left(\frac{z_m - x}{\|z_m - x\|}\right)_m$ of directions and then take the dual face instead of first taking the sequence of dual faces and then their limit. The following example shows that is does not work: take $f_3(x) = \log(2x)$ for $x > 0$ and

$$z_m^{(3)} = \begin{pmatrix} m \\ \log(2m) \end{pmatrix} \in \mathbb{R}^2.$$

Then the sequence of directions converges to F_2. But the second component of each $z_m^{(3)}$ is unbounded and therefore $\left(z_{m,E_2}^{(3)}\right)_m$ does not converge. This happens because $\log(2m)$ grows slower than m.

We do not have convergence with respect to E_2, but maybe this was just the wrong extreme set to look at. A closer look to the sequence of directions shows that for any $x \in \mathbb{R}^2$, the direction $\frac{z_m - x}{\|z_m - x\|}$ lies in F_1 for $m \in \mathbb{N}$ big enough. Therefore $D(x) = E_1$ and with $E = E_1$ the second requirement of the convergence of the projection is trivial. So we get with $p = 0$

$$\psi_{z_m^{(3)}} \longrightarrow h_{E_1,p}.$$

4) Now let $f_4(x) = \frac{1}{2}\sin(5x) + 1$ be the function defining the sequence

$$z_m^{(4)} = \begin{pmatrix} m \\ \frac{1}{2}\sin(5m) + 1 \end{pmatrix} \in \mathbb{R}^2.$$

Similar as in the first example, shifting $\left(z_m^{(4)}\right)_m$ by some $x \in \mathbb{R}^2$ yields no relevant difference in the first component and only in the second one. Again all directions can lie in F_1, F_2 or F_3 and we get again

$$E = E_2.$$

But now the projection of $\left(z_m^{(4)}\right)_m$ to $T(E)^*$ is $z_{m,E}^{(4)} = (0, f_3(m))$, which does not converge. The second condition of the theorem is not satisfied and we conclude that in this case $\psi_{z_m^{(4)}}$ does not converge at all. (This also turns out when doing the calculation directly.) ◦

Example 3.2.9 Wo now consider the same sequences as before but with respect to a different norm, namely a norm that can be seen as a blown-up version of the 1-norm. We already have seen this unit ball and its dual in Example 2.4.12. Figure 3.12 shows B and B° with the notation of faces, and the sequences we consider are again shown in Figure 3.13, where the colors indicate the extreme set associated to the limiting horofunction.

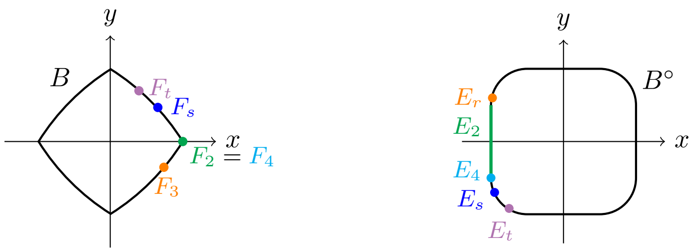


Figure 3.12: The unit ball B of Example 3.2.9 can be seen as a blown-up of the 1-norm. Its dual is the convex hull of four small circles. The colors in the picture show again some faces of B and their duals in B°, according to the sequences in Figure 3.13.

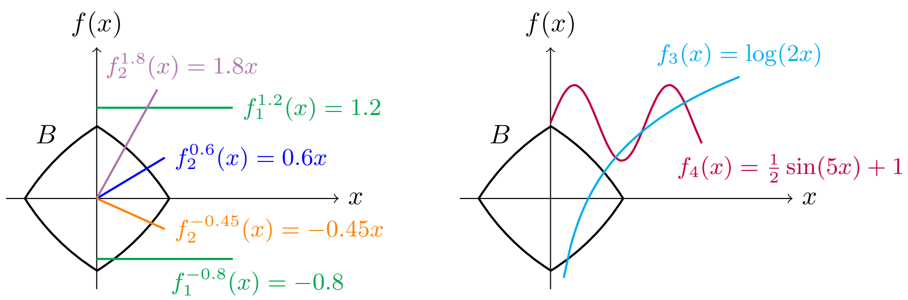


Figure 3.13: In Example 3.2.9 we consider the same sequences as before but now with a curved norm. The colors correspond to those in Figure 3.12.

1) The sequence $\left(z_m^{(1)}\right)_m$ with

$$z_m^{(1)} = \begin{pmatrix} m \\ c \end{pmatrix} \in \mathbb{R}^2$$

shows the same converging behavior as in the first example above because the boundary point $(1,0) \in \partial B$ is still extreme and not smooth. Therefore $\left(\psi_{z_m^{(1)}}\right)_m$ converges to $h_{E_2,p}$ with $p = (0,c) \in T(E_2)^*$.

2) In the polyhedral case, all sequences of the form

$$z_m^{(2)} = \begin{pmatrix} m \\ sm \end{pmatrix} \in \mathbb{R}^2$$

converged to the same horofunction associated to the vertex in the third quadrant. In the blown-up 1-norm, the dual unit ball has infinitely many smooth extreme points there between the facets. If $s \neq t \neq 0$, then the two sequences $z_{m,s}^{(2)} = (m, sm)$ and $z_{m,t}^{(2)} = (m, tm)$ will converge to different horofunctions $\psi_{E_s,p}, \psi_{E_t,p}$, respectively, where $E_s \neq E_t$ are exposed points of B° in the second (if $s,t < 0$) or third (if $s,t > 0$) quadrant. As there holds $T(E_s)^* = T(E_t)^* = \{0\}$, we have $p = (0,0)$ in both cases.

3) For the function f_3 that gives us

$$z_m^{(3)} = \begin{pmatrix} m \\ \log(2m) \end{pmatrix} \in \mathbb{R}^2,$$

the sequence of directions still converges to $F_2 = \{(1,0)\}$. Also all sequences of the shifted directions converge to F_2. For their dual sequence it holds $D_m(x) \to E_4$ for all $x \in \mathbb{R}^2$, because there is always an $M \in \mathbb{N}$ such that $z_m - x$ lies in the first quadrant for all $m \geq M$. E_4 is the lower extreme point of E_2. Therefore, with $p = 0$,

$$\psi_{z_m^{(3)}} \longrightarrow h_{E_4,p}.$$

4) The sequence $\left(z_m^{(4)}\right)_m$ with

$$z_m^{(4)} = \begin{pmatrix} m \\ \frac{1}{2}\sin(5m) + 1 \end{pmatrix} \in \mathbb{R}^2$$

still does not converge by the same reason as above: the affine hull $\operatorname{aff}\{D(x) \mid x \in \mathbb{R}^2\}$ is the extreme set E_2, but the projection $(z_{m,E_2})_m$ is not convergent. ∘

Example 3.2.10 Next we consider $\mathbb{R}^2$ equipped with the Euclidean norm which has the unit circle as unit and dual unit ball. For the notations Figure 3.14 and Figure 3.15.

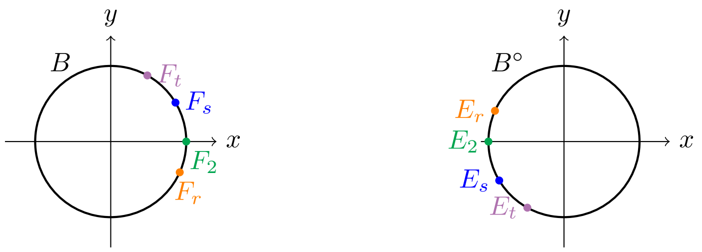


Figure 3.14: The unit ball B of Example 3.2.10 is the unit circle. Its dual is also a unit circle and for a face $F = \{f\} \in \partial B$ the dual face is $F^\circ = \{-f\}^*$. The colors correspond to the sequences in Figure 3.15.

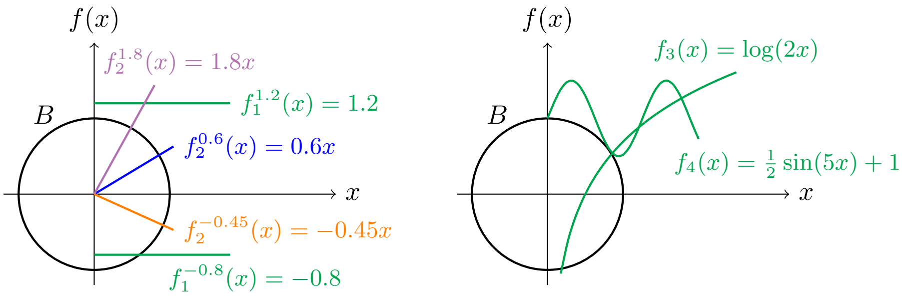


Figure 3.15: In Example 3.2.10 we take the same sequences as in the two examples before but now with respect to the Euclidean norm. The colors are in accordance with Figure 3.14.

Here every extreme set of B° is an extreme point and the second condition of the theorem is redundant. Now all sequences we considered so far converge and those with directions converging to F all converge to the same horofunction associated to the extreme point $E_1 = \{(-1,0)\} \in B^\circ$ with $p = (0,0)$:

$$\psi_{z_m^{(1)}}, \psi_{z_m^{(3)}}, \psi_{z_m^{(4)}} \longrightarrow h_{E_2,p}.$$

For the sequences of the second type $z_{m,s}^{(2)} = (m, sm)$ the limit again depends on the parameter s. With

$$E_s := \begin{pmatrix} -\frac{1}{\sqrt{1+s^2}} \\ -\frac{s}{\sqrt{1+s^2}} \end{pmatrix} \in \partial B^\circ$$

we have with $p = (0, 0)$

$$\psi_{z^{(2)}_{m,s}} \longrightarrow h_{E_s,p}.$$

This is because of the special geometry of the Euclidean unit ball, where $v^\circ = -v^*$ for any extreme point $v \in \partial B$. Two sequence with different slope s and t converge do different horofunctions. ◦

We have seen in the examples that it is not enough to consider the direction of a sequence to determine the right face associated to the horofunction. But the direction gives the extreme face in whose relative boundary the actual limiting extreme set will be.

Remark 3.2.11 The easiest examples to consider are sequences following straight lines and they are important enough to show the general behavior of convergence. All sequences in a regular direction, that is, within the interior of a facet, collapse and converge to the horofunction associated to the dual vertex, independent of any translation or direction. For a sequence in a singular direction associated to a lower dimensional face F, we have the same collapsing behavior for the $z_{m,F}$-part and a blowing-up in the orthogonal direction $V(F)^\perp$, which is encoded by the point $p \in T(E)^* = V(F)^\perp$ in the definition of $h_{E,p}$. ◦

3.2.6 Counterexample to Theorem 3.2.6 in $\mathbb{R}^3$: the Cylinder

If B is not one of the three cases considered above, the conclusion in Theorem 3.2.6 does not hold. To see what can go wrong, we look at the example of a cylinder in $\mathbb{R}^3$, see also Figure 3.16.

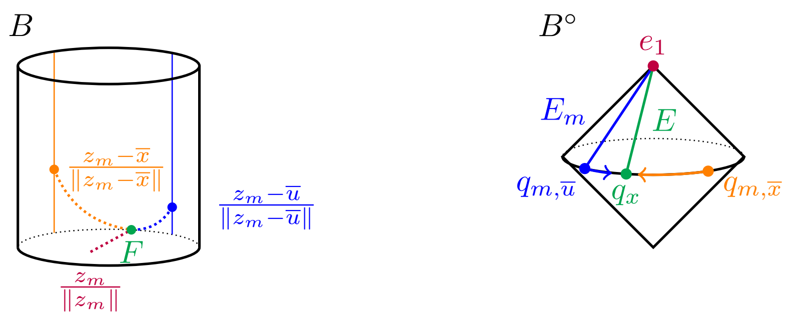


Figure 3.16: The cylindric unit ball and its dual.

On $\mathbb{R}^3$ we consider the norm

$$\|(x,\ y,\ z)\| := \max\left(\sqrt{x^2 + y^2}, |z|\right),$$

whose unit ball is a symmetric cylinder along the z-axis with radius and height 1. It can also be obtained by rotating the unit ball of the ∞-norm in two dimensions around the z-axis. Its dual is the rotated unit ball of the 1-norm, namely the convex hull of the points $e_1 := (0, 0, 1)$ and $e_2 := (0, 0, -1)$ and the unit circle in the xy-plane.

For some parameters $a, c \in \mathbb{R}$ satisfying

$$a - c - \frac{1}{2} > 0 \tag{3.15}$$

and some $b \in \mathbb{R}$ we will investigate the behavior of the sequence $(z_m)_{m\in\mathbb{N}}$ given by

$$z_m := \begin{pmatrix} -m^2 + a \\ m + b \\ -m^2 + c \end{pmatrix}$$

with norm $\|z_m\| = m^2 - c$ for m big enough by Condition (3.15). The sequence $\left(\frac{z_m}{\|z_m\|}\right)_m$ of directions stays in the cylinder bottom and converges to $F := \{(0,0,-1)\} \in \partial B$.

Now take a point $\bar{x} = (x,y,z) \in \mathbb{R}^3$ with

$$\|z_m - \bar{x}\| = \sqrt{(m^2 - a + x)^2 + (m + b - y)^2} =: \sqrt{W(\bar{x})}.$$

In this example we will denote points in $\mathbb{R}^3$ over-lined and components of such vectors without. Note that $\sqrt{W(\bar{x})}$ is also of order 2 with respect to m. Then with

$$q_{m,\bar{x}} := \frac{1}{\sqrt{W(\bar{x})}} \begin{pmatrix} m^2 - a + x \\ -m - b + y \\ 0 \end{pmatrix} \in \partial B^\circ$$

it holds

$$\langle q_{m,\bar{x}} | z_m - \bar{x} \rangle = -\|z_m - \bar{x}\|.$$

This means that $q_{m,\bar{x}} \in D_m(\bar{x})$ is dual to $\frac{z_m - \bar{x}}{\|z_m - \bar{x}\|}$. If we choose instead a different point $\bar{x}' \in \mathbb{R}^3$ and m big enough such that $\|z_m - \bar{x}'\| = |-m^2 + c - z'| = m^2 - c + z'$, then

$$\langle e_1 | z_m - \bar{x}' \rangle = -\|z_m - \bar{x}'\|,$$

that is, the point e_1 is dual to $\frac{z_m - \bar{x}'}{\|z_m - \bar{x}'\|}$. So for every $\bar{x} = (x,y,z) \in \mathbb{R}^3$ and m big enough, the dual $D_m(\bar{x})$ of the sequence of directions belongs to one of the following cases:

$$D_m(\bar{x}) = \begin{cases} q_{m,\bar{x}} & \text{if} \quad \sqrt{W(\bar{x})} > |m^2 - c + z|, \\ e_1 & \text{if} \quad \sqrt{W(\bar{x})} < |m^2 - c + z|, \\ \operatorname{conv}\{q_{m,\bar{x}}, e_1\} & \text{if} \quad \sqrt{W(\bar{x})} = |m^2 - c + z|. \end{cases} \tag{3.16}$$

The last case occurs exactly when $\frac{z_m}{\|z_m\|}$ lies in the circular intersection of the cylinder barrel and the bottom. To determine $E = \operatorname{aff}\{D(\bar{x}) \mid \bar{x} \in \mathbb{R}^3\} \cap B^\circ$ we have to know where $q_{m,\bar{x}}$ converges to. So we calculate:

$$\begin{aligned} q_{m,\bar{x}} &= \frac{1}{\sqrt{W(\bar{x})}} \begin{pmatrix} m^2 - a + x \\ -m - b + y \\ 0 \end{pmatrix} \\ &= \left[\sqrt{m^4 + (1 - 2(a-x))m^2 + 2(b-y)m + (a-x)^2 + (b-y)^2} \right]^{-1} \begin{pmatrix} m^2 - a + x \\ -m - b + y \\ 0 \end{pmatrix} \\ &= \left[\sqrt{1 + \frac{1 - 2(a-x)}{m^2} + \frac{1}{m^4}[2(b-y)m + (a-x)^2 + (b-y)^2]} \right]^{-1} \begin{pmatrix} 1 - \frac{a-x}{m^2} \\ -\frac{1}{m} - \frac{b-y}{m^2} \\ 0 \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

By Equation (3.16) we then have

$$E = \operatorname{conv}\{\underbrace{(0,0,1)}_{e_1}, (1,0,0)\}.$$

As a next step, the criterion in Theorem 3.2.6 on page 54 tells us to compute the limit of the projected sequence $(z_{m,E})_m$ to get the parameter p. Doing this we obtain for all $m \in \mathbb{N}$:

$$z_{m,E} = \frac{a-c}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix} =: \widetilde{p}. \tag{3.17}$$

Following the theorem, we would now conclude $\psi_{z_m} \longrightarrow h_{E,\widetilde{p}}$ as $m \to \infty$.

But when we do the calculation explicitly, we get a different result. To see this, consider the Taylor expansion of the square root in $\mathbb{R}$ around $s = 0$,

$$\sqrt{1+s} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{(1-2n)(n!)^2 4^n} s^n = 1 + \frac{1}{2}s - \frac{1}{8}s^2 + \frac{1}{16}s^3 - \dots, \tag{3.18}$$

which converges for $|s| < 1$. This gives us

$$\begin{aligned}
\sqrt{W(\bar{x})} &= \sqrt{m^4 + (1-2(a-x))m^2 + 2(b-y)m + (a-x)^2 + (b-y)^2} \\
&= m^2\left(1 + \frac{1-2(a-x)}{2m^2} + \frac{1}{2m^4}\left[2(b-y)m + (a-x)^2 + (b-y)^2\right] + O(m^{-4})\right) \\
&= m^2 + \frac{1}{2} - a + x + O(m^{-1})
\end{aligned} \tag{3.19}$$

and similarly for $\bar{x} = 0$:

$$\sqrt{W(0)} = m^2 + \frac{1}{2} - a + O(m^{-1}).$$

Using this we compute for a general $\bar{x} \in \mathbb{R}^3$:

$$\begin{aligned}
\psi_{z_m}(\bar{x}) &= \|z_m - \bar{x}\| - \|z_m\| \\
&= \max\left(m^2 + \frac{1}{2} - a + x + O(m^{-1}),\ m^2 - c + z\right) - \max\left(m^2 + \frac{1}{2} - a + O(m^{-1}),\ m^2 - c\right) \\
&\longrightarrow \max\left(x + \frac{1}{2} - a,\ z - c\right) - \max\left(\frac{1}{2} - a,\ -c\right),
\end{aligned}$$

because in the limit, the m^2 annihilate each other. This expression can be simplified as

$$\begin{aligned}
\max&\left(x + \frac{1}{2} - a,\ z - c\right) - \max\left(\frac{1}{2} - a, -c\right) = \frac{1}{2}\left[\max(2x + 1 - 2a,\ 2z - 2c) - \max(1 - 2a,\ -2c)\right] \\
&= \frac{1}{2}\left[\max\left(2x + 1 - 2a - \frac{1}{2} + a + c,\ 2z - 2c - \frac{1}{2} + a + c\right) + \frac{1}{2} - a - c + \min(2a - 1, 2c)\right] \\
&= \frac{1}{2}\left[\max\left(2x + \frac{1}{2} - a + c,\ 2z - \frac{1}{2} + a - c\right) + \min\left(a - c - \frac{1}{2},\ -a + c + \frac{1}{2}\right)\right] \\
&= -\min\left(\frac{a-c-\frac{1}{2}}{2} - x,\ -\frac{a-c-\frac{1}{2}}{2} - z\right) + \min\left(\frac{a-c-\frac{1}{2}}{2},\ -\frac{a-c-\frac{1}{2}}{2}\right) \\
&= -\min\left\{\left\langle\begin{pmatrix}1\\0\\0\end{pmatrix}\middle|\ \frac{a-c-\frac{1}{2}}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix} - \begin{pmatrix}x\\y\\z\end{pmatrix}\right\rangle,\ \left\langle\begin{pmatrix}0\\0\\1\end{pmatrix}\middle|\ \frac{a-c-\frac{1}{2}}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix} - \begin{pmatrix}x\\y\\z\end{pmatrix}\right\rangle\right\} \\
&\quad + \min\left\{\left\langle\begin{pmatrix}1\\0\\0\end{pmatrix}\middle|\ \frac{a-c-\frac{1}{2}}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\rangle,\ \left\langle\begin{pmatrix}0\\0\\1\end{pmatrix}\middle|\ \frac{a-c-\frac{1}{2}}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\rangle\right\} = h_{E,p}(\bar{x}),
\end{aligned}$$

where

$$p := \frac{a - c - \frac{1}{2}}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \widetilde{p} - \frac{1}{4} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

The additional summand $\frac{1}{4}$ comes from the m in the second component of z_m, in particular from the relation of the first and the second component because of the Taylor expansion of the root (Equation (3.18)).

To see the dependence even better, we consider the more general sequence

$$z_m^\lambda = \begin{pmatrix} -m^2 + a \\ \lambda m + b \\ -m^2 + c \end{pmatrix}$$

for any $\lambda \in \mathbb{R}$. When we do the calculation (so without using the theorem), we get convergence to h_{E,p_λ} with

$$p_\lambda := \widetilde{p} - \frac{\lambda^2}{4} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Therefore if and only if $\lambda = 0$, the calculated parameter and the parameter obtained by using the theorem coincide.

So the extreme set E we determined following Theorem 3.2.6 is the right one, only the parameter $\widetilde{p}$ we obtained by projecting $(z_m)_m$ to $T(E)^*$ is not correct and differs from the correct p by an additive constant.

The question now is how to obtain the right parameter p. We will find it again by projecting to a subspace, but this time not to $T(E)^*$ but to some subspace associated to the dual point $q_{m,\bar{u}}$ of $\frac{z_m - \bar{u}}{\|z_m - \bar{u}\|}$ for some $\bar{u} \in \mathbb{R}^3$.

To construct E_m, let $\bar{u} = (u, v, w) \in \mathbb{R}^3$ be a fixed point such that $\|z_m - \bar{u}\| = \sqrt{W(\bar{u})}$ is given as above:

$$\begin{aligned} \sqrt{W(\bar{u})} &= \sqrt{(m^2 - a + u)^2 + (m + b - v)^2} \\ &= \sqrt{m^4 + (1 - 2(a - u))m^2 + 2(b - v)m + (a - u)^2 + (b - v)^2}. \end{aligned}$$

According to the notations used before we set

$$q_{m,\bar{u}} := \frac{1}{\sqrt{W(\bar{u})}} \begin{pmatrix} m^2 - a + u \\ -m - b + v \\ 0 \end{pmatrix} \in \partial B^\circ$$

and define the sequence $(E_m)_m \subseteq B^\circ \subseteq X^*$ by

$$E_m := \operatorname{conv}\{e_1, q_{m,\bar{u}}\}.$$

Note that here E_m is a one-dimensional extreme set of B° and as $q_{m,\bar{u}} \longrightarrow (1, 0, 0)$, the sequence $(E_m)_m$ satisfies

$$E_m \longrightarrow E.$$

For each $m \in \mathbb{N}$, the subspace $T(E_m)^*$ is spanned by $t_m := q_{m,\bar{u}} - e_1$. It holds

$$\langle t_m | t_m \rangle = \frac{1}{W(\bar{u})} \left\{ (m^2 - a + u)^2 + (m + b - v)^2 + W(\bar{u}) \right\} = 2.$$

For the projection of $(z_m)_m$ to the subspace $T(E_m)^*$ we obtain (again by using the Taylor expansion of the square root):

$$\begin{aligned} z_{m,E_m} &= \frac{\langle t_m | z_m \rangle}{\langle t_m | t_m \rangle} t_m \\ &= \frac{1}{2\sqrt{W(\bar{u})}} \left[-m^4 + m^2(2a - u) - m^2 + m^4 + m^2 \left(-a + u + \frac{1}{2} \right) - cm^2 + O(m) \right] t_m \\ &= \frac{1}{2\sqrt{W(\bar{u})}} \left[m^2 \left(a - \frac{1}{2} - c \right) + O(m) \right] t_m \\ &\longrightarrow \frac{a - c - \frac{1}{2}}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = p. \end{aligned}$$

So projecting to E_m seems - at least in this example - to be the right way. Note that the limit of the projection is independent of the point $\bar{u} \in \mathbb{R}^3$. Therefore we can project z_m to any subspace parallel to an affine space of the form $\mathrm{conv}\{q_{m,\bar{u}'}, e_1\}$ for some $\bar{u}' \in \mathbb{R}^3$ with $\|z_m - \bar{u}'\| = \sqrt{W(\bar{u}')}$.

For each $m \in \mathbb{N}$ big enough, the point $q_{m,\bar{u}} \in \partial B^\circ$ is an extreme point of B° lying on the circle that is the intersection of B° with the xy-plane. Now one could think of other sequences $(G_m)_m$ of extreme sets of the form

$$G_m = \mathrm{conv}\{g_m, e_1\}$$

where $(g_m)_m \in \partial B^\circ \cap (xy)$-plane is a sequence with $g_m \longrightarrow (1, 0, 0)$ and then compute the projection of $(z_m)_m$ to $T(G_m)^*$. To see what happens, we consider the following sequences:

$$g_m^{(1)} = \begin{pmatrix} \sqrt{\frac{m^4-1}{m^4}} \\ -1/m^2 \\ 0 \end{pmatrix}, \quad g_m^{(3)} = \begin{pmatrix} \cos(1/m) \\ -\sin\frac{1}{m} \\ 0 \end{pmatrix}, \quad g_m^{(5)} = \begin{pmatrix} \sqrt{1 - \frac{1}{m^{1.9}}} \\ 1/m^{0.95} \\ 0 \end{pmatrix},$$

$$g_m^{(2)} = \begin{pmatrix} \sqrt{1 - \frac{1}{m^{2.1}}} \\ \frac{-1}{m^{1.05}} \\ 0 \end{pmatrix}, \quad g_m^{(4)} = \begin{pmatrix} \sqrt{\frac{m^2-1}{m^2}} \\ -1/m \\ 0 \end{pmatrix}, \quad g_m^{(6)} = \begin{pmatrix} \sqrt{\frac{m-1}{m}} \\ -1/\sqrt{m} \\ 0 \end{pmatrix}.$$

We take the projection of $(z_m)_m$ to the subspaces $T(G_m^j)^*$ for $j = 1, \ldots, 6$, with

$$G_m^j := \mathrm{conv}\left\{ g_m^{(j)}, e_1 \right\}$$

and then compute the limits

$$p_j := \lim_{m \to \infty} z_{m, G_m^j}.$$

As results we get

$$p_1 = p_2 = \frac{a - c}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \widetilde{p}, \qquad p_3 = p_4 = \frac{a - c - \frac{1}{2}}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = p,$$

and the projections to the subspaces $T(G_m^5)^*$ and $T(G_m^6)^*$ diverge. Why do we get such a different behavior? The six sequences $\left(g_m^{(j)}\right)_m$ above all converge to the point $(1, 0, 0)$, but they distinguish each other by how fast they do so. Only when the ratio of the second to the first component roughly goes by $\frac{1}{m}$ (as for $\left(g_m^{(3)}\right)_m$ and $\left(g_m^{(4)}\right)_m$), we get the limit p. If the rate of convergence is faster, the projection converges to $\widetilde{p}$. This happens for $\left(g_m^{(1)}\right)_m$, $\left(g_m^{(2)}\right)_m$ and also when we project directly to E. When the convergence of $\left(G_m^j\right)_m$ to E is too slow, then the projection of $(z_m)_m$ to $T(G_m^j)^*$ diverges, this can be observed for $\left(g_m^{(5)}\right)_m$ and $\left(g_m^{(6)}\right)_m$. So whether or not we obtain the right p depends on the velocity of $(g_m)_m$ converging to $(1,0,0)$.

To see even more explicitly how the limit of $(z_{m,G_m})_m$ depends on the rate of the convergence $g_m^{(j)} \to (1,0,0)$, we calculate another example. For $\gamma > 0$ let

$$g_m^{(7)} = \begin{pmatrix} \sqrt{1-\frac{1}{m^{2\gamma}}} \\ -1/m^\gamma \\ 0 \end{pmatrix}$$

and set $G_m^7 = \operatorname{conv}\{g_m^{(7)}, e_1\}$ to be the extreme set whose space of translates $T(G_m^7)^*$ is spanned by the vector $t_{m,7} := g_m^{(7)} - e_1$. Then there holds $\langle t_{m,7} | t_{m,7}\rangle = 2$ and with the Taylor expansion $\sqrt{1-\frac{1}{m^{2\gamma}}} = 1 - \frac{1}{2m^{2\gamma}} + \frac{1}{8m^{4\gamma}} + O(m^{-6\gamma})$ we get

$$\begin{aligned}
z_{m,G_m^7} &= \frac{\langle t_{m,7} | z_m\rangle}{\langle t_{m,7} | t_{m,7}\rangle} t_{m,7} \\
&= \frac{1}{2}\left[\left(1 - \frac{1}{2m^{2\gamma}} + \frac{1}{8m^{4\gamma}} + O(m^{-6\gamma})\right)(-m^2+a) - \frac{1}{m^\gamma}(m+b) + m^2 - c\right] t_{m,7} \\
&= \frac{1}{2}\left[-m^2 + \frac{1}{2}m^{2-2\gamma} - \frac{1}{8}m^{2-4\gamma} + O(m^{2-6\gamma}) + a + O(m^{-2\gamma}) - m^{1-\gamma} - bm^{-\gamma} + m^2 - c\right] t_{m,7} \\
&= \frac{1}{2}\left[a - c + \frac{1}{2}m^{2-2\gamma} - m^{1-\gamma} + O(m^{2-4\gamma})\right] t_{m,7} \\
&\longrightarrow \begin{cases} \widetilde{p} & \text{if} \quad \gamma > 1, \\ p & \text{if} \quad \gamma = 1, \\ \infty & \text{if} \quad \gamma < 1 \end{cases}
\end{aligned}$$

Let us summarize these observations: For an extreme set $G_m = \operatorname{conv}\{g_m, e_1\}$ where $g_m \to (1,0,0)$ is a sequence of extreme points with first components $[g_m]_1 \in O(m^2)$ and second components $[g_m]_2$ we have for $\varepsilon > 0$

$$z_{m,G_m} \longrightarrow \begin{cases} \widetilde{p} & \text{if} \quad [g_m]_2 \in O(m^{1+\varepsilon}), \\ p & \text{if} \quad [g_m]_2 \in O(m), \\ \pm\infty & \text{if} \quad [g_m]_2 \in O(m^{1-\varepsilon}). \end{cases}$$

Here is another point of view: To have the right rate for the sequence to converge does also mean, that $q_{m,\bar{u}} \in E_m$ and $q_{m,\bar{x}} \in D_m(\bar{x})$ approach each other in the right speed, which depends on $(z_m)_m$. To see this we compute:

$$\begin{aligned}
\langle q_{m,\bar{x}} - q_{m,\bar{u}} | z_m\rangle &= \left\langle \frac{1}{\sqrt{W(\bar{x})}}\begin{pmatrix} m^2-a+x \\ -m-b+y \\ 0\end{pmatrix} - \frac{1}{\sqrt{W(\bar{u})}}\begin{pmatrix} m^2-a+u \\ -m-b+v \\ 0\end{pmatrix} \middle| \begin{pmatrix} -m^2+a \\ m+b \\ -m^2+c\end{pmatrix}\right\rangle \\
&= \frac{1}{\sqrt{W(\bar{x})}}\left[-m^4 + m^2(2a-x-1) + O(m)\right] + \frac{1}{W(\bar{u})}\left[m^4 + m^2(1-2a+u) + O(m)\right] \\
&= \frac{1}{m^2-a+x-\frac{1}{2}+O(m^{-1})}\left[-m^4 + m^2(2a-1-x) + O(m)\right] \qquad (3.20)
\end{aligned}$$

$$+\frac{1}{m^2-a+u-\frac{1}{2}+O(m^{-1})}\left[m^4+m^2(1-2a+u)+O(m)\right]$$
$$=\frac{-m^6+m^4(2a-1-x)+m^4(a-u+\frac{1}{2})+m^6+m^4(1-2a+u)+m^4(-a+x-\frac{1}{2})+O(m^3)}{m^4+O(m^2)}$$
$$=\frac{O(m^3)}{m^4+O(m^3)}$$
$$\longrightarrow 0.$$

So all terms of order m^4 annihilate each other. If we had not taken $q_{m,\bar{u}}$ but one of the sequences $g_m^{(j)}$ with a different rate of convergence, we would not have had such a nice canceling that gives us the convergence to 0.

3.2.7 A Conjecture for the General Case

Inspired by the example in the previous section, we now want to reformulate Theorem 3.2.6 such that it holds for any norm on X whose set of extreme sets of the dual unit ball is closed. Although we are convinced that the statement is true, its proof relies on a conjecture about the convergence behavior.

We already know how to determine the extreme set $E \subseteq B^\circ$ and that for finding the parameter $p \in T(E)^*$ we have to determine the limit of the projection of the sequence $(z_m)_m$ to an appropriate subspace. Now the important question is how to characterize this subspace. It should be the space of translates of a subset $E_m \subseteq B^\circ$ (not necessarily extreme), that has the same dimension as E and converges to it. But we know that "convergence of E_m to E" alone is not enough, the example above shows that the crucial point is the rate of convergence. In Equation (3.20) we saw that $(E_m)_m$ has the right speed if for every point $x \in X$, the points $q_{m,u} \in E_m$ and $q_{m,x} \in B^\circ$ approach each other faster than $(z_m)_m$ goes to infinity. Note that from now on we write again $x \in X$ instead of $\bar{x} \in \mathbb{R}^3$ because we will not make calculations using the components of a point.

The correct sequence E_m in the example was defined as the convex hull of the two points e_1 and $q_{m,u}$, that were both of the form $D_m(y)$ for some $y \in X$. So we guess that E_m in general should be defined as the intersection of B° with the affine hull of several points of the form $D_m(u_j)$ with $u_j \in X$. Defined like this, Conjecture 3.2.12 states that E_m actually has the right rate of convergence, that is, an analog of Equation (3.20) holds.

As usual denote by $B \subseteq X$ a unit ball and by $B^\circ \subseteq X^*$ its dual. Let $(z_m)_{m\in\mathbb{N}}$ be an unbounded sequence in X and $E := \operatorname{aff}\{D(x) \mid x \in X\} \cap B^\circ$ an extreme set. See also Figure 3.17 for the following notations: Let $u_1, \dots, u_k \in X$ be points with $k = \dim(E) + 1$ and for each $j = 1, \dots, k$, let $(q_{m,u_j})_m \subseteq D_m(u_j) \subseteq \partial B^\circ$ be a sequence of points satisfying $\left\langle q_{m,u_j} \middle| z_m - u_j \right\rangle = -\|z_m - u_j\|$ for all $m \in \mathbb{N}$ such that for

$$E_m := \operatorname{aff}\{q_{m,u_1}, \dots, q_{m,u_k}\} \cap B^\circ \tag{3.21}$$

the following two conditions hold:

(A) $\dim(E_m) = \dim(E)$;

(B) $E_m \longrightarrow E$ as $m \to \infty$;

For a point $x \in X$ we denote by $q_{m,x} \in \partial B^\circ$ a point dual to $\frac{z_m - x}{\|z_m - x\|} \in \partial B$. Then by the definition of E we know that any converging subsequence of $(q_{m,x})_{m\in\mathbb{N}}$ converges to a point in the relative

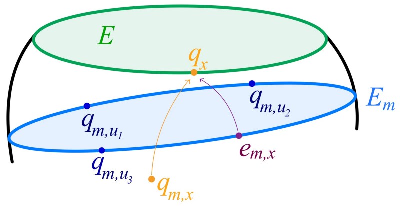


Figure 3.17: The notation used in the conjecture.

boundary of E. Let q_x be such a limit point of a subsequence, also denoted by $(q_{m,x})_m$. As E_m converges to E, there is a sequence of points $e_{m,x} \in E_m$ such that

$$e_{m,x} \longrightarrow q_x.$$

Note that $e_{m,x} \in E_m$ can be of the form $e_{m,x} = q_{m,u_j}$ for some $j \in \{1, \ldots, k\}$ but doesn't have to be. Let the sequence $(z_m)_m$ projected to $T(E_m)^*$ be convergent:

$$z_{m,E_m} \longrightarrow p.$$

Conjecture 3.2.12 *With the notations introduced above it holds:*

$$\langle e_{m,x} - q_{m,x} | z_m \rangle \longrightarrow 0 \quad \forall x \in X. \qquad \circ$$

An equivalent reformulation of the conjecture is the statement that the projection of $(z_m)_m$ to any subspace of the form (3.21) have the same limit:

Lemma 3.2.13 *Let $u_1, \ldots, u_k$ and $s_1, \ldots, s_k \in X$ be points and for each $j = 1, \ldots, k$, let $(q_{m,u_j})_m \subseteq \partial B^\circ$ and $(q_{m,s_j})_m \subseteq \partial B^\circ$ be two sequences of points dual to $\frac{z_m - u_j}{\|z_m - u_j\|}$, $\frac{z_m - s_j}{\|z_m - s_j\|}$, respectively, such that*

$$\begin{aligned} E_m^1 &:= \operatorname{aff}\{q_{m,u_1}, \ldots, q_{m,u_k}\} \cap B^\circ \\ E_m^2 &:= \operatorname{aff}\{q_{m,s_1}, \ldots, q_{m,s_k}\} \cap B^\circ \end{aligned} \tag{3.22}$$

are two sequences of sets satisfying conditions (A) and (B) above. Let $\lim_m z_{m,E_m^1} = p_1 \in T(E)$ and $\lim_m z_{m,E_m^2} = p_2 \in T(E)$ denote the limits. Then Conjecture 3.2.12 holds if and only if

$$p_1 = p_2. \qquad \circ$$

Proof. We use the notations introduced before with a superscript $j \in \{1, 2\}$ associated to the sequences, respectively. Assume we have $\left\langle e_{m,x}^j - q_{m,x} \middle| z_m \right\rangle \longrightarrow 0$ for any $x \in X$ and for $j = 1, 2$. Recall that by our notations it holds

$$\begin{aligned} e_{m,x}^j,\ q_{m,x} &\longrightarrow q_x \in \operatorname{aff}(E), \\ e_{m,0}^j,\ q_{m,0} &\longrightarrow q_0 \in \operatorname{aff}(E). \end{aligned}$$

Then by assumption we know that

$$\underbrace{\left\langle e_{m,x}^1 - q_{m,x} \middle| z_m - x \right\rangle - \left\langle e_{m,x}^2 - q_{m,x} \middle| z_m - x \right\rangle - \left\langle e_{m,0}^1 - q_{m,0} \middle| z_m \right\rangle + \left\langle e_{m,0}^2 - q_{m,0} \middle| z_m \right\rangle}_{:=\mu} \longrightarrow 0,$$

because each summand goes to 0.

Computing the expression yields

$$\begin{aligned}\mu &= \left\langle e^1_{m,x} - e^1_{m,0} | z_m \right\rangle - \left\langle e^1_{m,x} | x \right\rangle - \left\langle e^2_{m,x} - e^2_{m,0} | z_m \right\rangle + \left\langle e^2_{m,x} | x \right\rangle \\ &= \left\langle e^1_{m,x} - e^1_{m,0} \middle| z_{m,E^1_m} \right\rangle - \left\langle e^1_{m,x} \middle| x \right\rangle - \left\langle e^2_{m,x} - e^2_{m,0} \middle| z_{m,E^2_m} \right\rangle + \left\langle e^2_{m,x} \middle| x \right\rangle \\ &\longrightarrow \langle q_x - q_0 | p_1 \rangle - \langle q_x | x \rangle - \langle q_x - q_0 | p_2 \rangle + \langle q_x | x \rangle \\ &= \langle q_x - q_0 | p_1 - p_2 \rangle .\end{aligned}$$

So it must hold $\langle q_x - q_0 | p_1 - p_2 \rangle = 0$ for all $x \in X$. Note that the points $p_1, p_2 \in T(E)^*$ are independent of x. Since E is spanned by $k + 1$ elements of the form q_x, the condition can only be satisfied if $p_1 = p_2$.

To conclude in the other direction we first consider the case where $\dim(E) = 0$. Then given two sequences $\left(E^1_m\right)_m$ and $\left(E^2_m\right)_m$ as in the lemma, the projections are trivial. So we have to show that the conjecture holds for any sequence $(E_m)_m$ of extreme points of the form $e_{m,x} = q_{m,u} \in \left(\frac{z_m - u}{\|z_m - u\|}\right)^\circ$ for some $u \in X$ with $E_m \longrightarrow E$. Let $x \in X$ be some point. Then $(q_{m,x})_m$ and $(q_{m,u})_m$ are sequences of extreme points of B° converging to $E =: \{e\}$. We assume that we have subsequences (also denoted with the index m) such that $q_{m,u} \neq q_{m,x}$, otherwise the statement is trivial. Pick a sequence $(r_m)_{m \in \mathbb{N}} \subseteq X^*$ of points satisfying $\langle q_{m,u} | z_m - u \rangle = \langle r_m | z_m - u \rangle$ and $\langle q_{m,x} | z_m - x \rangle = \langle r_m | z_m - x \rangle$ and such that $r_m \to e$. Such a sequence can be found by looking at the supporting hyperplanes at $q_{m,u}$ and $q_{m,x}$ which are not parallel, because $q_{m,x} \neq q_{m,u}$. Then as $q_{m,u}, q_{m,x} \to e$, we have

$$\begin{aligned}\langle q_{m,u} - q_{m,x} | z_m \rangle &= \langle q_{m,u} | z_m - u \rangle + \langle q_{m,u} | u \rangle - \langle q_{m,x} | z_m - x \rangle - \langle q_{m,x} | x \rangle \\ &= \langle r_m | z_m - u \rangle + \langle q_{m,u} | u \rangle - \langle r_m | z_m - x \rangle - \langle q_{m,x} | x \rangle \\ &= \langle q_{m,u} - r_m | u \rangle - \langle q_{m,x} - r_m | x \rangle \longrightarrow 0.\end{aligned}$$

When $\dim(E) \geq 1$, we define a new set by

$$E^x_m := \operatorname{aff}\{q_{m,x}, q_{m,s_2}, \ldots, q_{m,s_k}\}$$

with some points $s_2, \ldots, s_k \in X$ such that $\{s_2, \ldots s_k\} \cap \{u_1, \ldots u_k\} \neq \emptyset$ and such that (A) and (B) hold for E^x_m. As $\{q_{m,x}\} \in D_m(x)$ is a point dual to $\frac{z_m - x}{\|z_m - x\|}$, the set E^x_m is as in Equation (3.22) and we know that

$$p_1 = \lim_m z_{m,E^1_m} = \lim_m z_{m,E^x_m}.$$

We want to show that the conjecture holds with respect to a sequence $(e^1_{m,x})_m \subseteq \left(E^1_m\right)_m$ and $(q_{m,x})_m$. As E^x_m and E^1_m both converge to E, there is a point $q_x \in E$ and a sequence $(e^1_{m,x})_m$ with $e^1_{m,x} \in E^1_m$ such that $q_{m,x}, e^1_{m,x} \to q_x \in E$. Note that by the choice of $s_2, \ldots, s_k$, the sets E^x_m and E^1_m have for each $m \in \mathbb{N}$ at least one point in common, which we call y_m. Then we calculate

$$\begin{aligned}\left\langle e^1_{m,x} - q_{m,x} \middle| z_m \right\rangle &= \left\langle e^1_{m,x} - y_m \middle| z_m \right\rangle + \left\langle y_m - q_{m,x} \middle| z_m \right\rangle \\ &= \left\langle e^1_{m,x} - y_m \middle| z_{m,E^1_m} \right\rangle + \left\langle y_m - q_{m,x} \middle| z_{m,E^x_m} \right\rangle \\ &= \left\langle e^1_{m,x} - y_m \middle| z_{m,E^1_m} \right\rangle + \left\langle y_m - q_{m,x} \middle| z_{m,E^1_m} + z_{m,E^x_m} - z_{m,E^1_m} \right\rangle \\ &= \left\langle e^1_{m,x} - q_{m,x} \middle| z_{m,E^1_m} \right\rangle + \left\langle y_m - q_{m,x} \middle| z_{m,E^x_m} - z_{m,E^1_m} \right\rangle \\ &\longrightarrow \langle q_x - q_x | \, p_1 \rangle = 0.\end{aligned}$$

As $\left(E^1_m\right)_m \subseteq B^\circ$ was an arbitrary sequence of sets fulfilling the conditions given previous to the conjecture, we have shown that the conjecture holds also if $\dim(E) \geq 1$. □

Under the assumption that the conjecture holds, we can now reformulate Theorem 3.2.6 for X equipped with any norm (see again Figure 3.17 for the notations):

Theorem 3.2.14 *Assume Conjecture 3.2.12 holds. Let $B \subseteq X$ be a unit ball and B° its dual such that the set of extreme sets of B° is closed. Let $(z_m)_{m\in\mathbb{N}}$ be an unbounded sequence in X and $E := \operatorname{aff}\{D(x) \mid x \in X\} \cap B^\circ$. Let $u_1, \dots, u_k \in X$ be points with $k = \dim(E) + 1$ and for each $j = 1, \dots, k$, let $(q_{m,u_j})_m \subseteq \partial B^\circ$ be a sequence of points dual to $\frac{z_m - u_j}{\|z_m - u_j\|}$, such that with*

$$E_m := \operatorname{aff}\{q_{m,u_1}, \dots, q_{m,u_k}\} \cap B^\circ$$

there holds

(A) $\dim(E_m) = \dim(E)$ *and*

(B) $E_m \longrightarrow E$ *as* $m \to \infty$.

Then the sequence $(\psi_{z_m})_m$ converges to a horofunction $h_{E',p}$ for an extreme set $E' \subseteq B^\circ$ and a point $p' \in T(E')^*$ if and only if the following conditions are satisfied:

1) E as defined above is extreme.

2) The projection $(z_{m,E_m})_m$ of $(z_m)_m$ to $T(E_m)^*$ converges.

If $(\psi_{z_m})_m$ converges, then $E' = E$ and $p = \lim_{m\to\infty} z_{m,E_m}$. ◦

Proof. We want to show that the sequence $(\psi_{z_m})_m$ converges to the horofunction $h_{E,p}$ where E and p are as given in the conjecture. Let $x \in X$ be a point and let $q_{m,0}, q_{m,x} \in \partial B^\circ$ be points dual to $\frac{z_m}{\|z_m\|}$ and $\frac{z_m - x}{\|z_m - x\|}$, respectively. Then by the definition of E we know that any converging subsequence of $(q_{m,x})_m$ converges to a point in the relative boundary of E. Let q_x be such a limit point of a subsequence, which we also denote by $(q_{m,x})_m$. Then as $E_m \longrightarrow E$ there is a sequence $(e_{m,x})_m$ of points $e_{m,x} \in E_m$ such that

$$e_{m,x} \longrightarrow q_x.$$

Similarly for $q_{m,0} \longrightarrow q_0$ we get a sequence $(e_{m,0})_m \subseteq E_m$ with

$$e_{m,0} \longrightarrow q_0.$$

So using the conjecture we calculate:

$$\begin{aligned}
\psi_{z_m}(x) &= \|z_m - x\| - \|z_m\| \\
&= -\langle q_{m,x} | z_m - x\rangle + \langle q_{m,0} | z_m\rangle \\
&= -\langle e_{m,x} | z_m - x\rangle + \langle e_{m,x} - q_{m,x} | z_m - x\rangle + \langle e_{m,0} | z_m\rangle + \langle q_{m,0} - e_{m,0} | z_m\rangle \\
&= \langle e_{m,0} - e_{m,x} | z_m\rangle + \langle e_{m,x} | x\rangle + \langle e_{m,x} - q_{m,x} | z_m\rangle - \langle e_{m,x} - q_{m,x} | x\rangle + \langle q_{m,0} - e_{m,0} | z_m\rangle \\
&= \langle e_{m,0} - e_{m,x} | z_{m,E_m}\rangle + \langle e_{m,x} | x\rangle - \langle e_{m,x} - q_{m,x} | x\rangle + \langle e_{m,x} - q_{m,x} | z_m\rangle + \langle q_{m,0} - e_{m,0} | z_m\rangle \\
&= -\langle e_{m,x} | z_{m,E_m} - x\rangle + \langle e_{m,0} | z_{m,E_m}\rangle - \langle e_{m,x} - q_{m,x} | x\rangle + \underbrace{\langle e_{m,x} - q_{m,x} | z_m\rangle}_{\longrightarrow 0} + \underbrace{\langle q_{m,0} - e_{m,0} | z_m\rangle}_{\longrightarrow 0} \\
&\longrightarrow -\langle q_x | p - x\rangle + \langle q_0 | p\rangle = h_{E,p}(x).
\end{aligned}$$

The fact than $h_{E,p}$ can be written as the sum of pairings as in the last equation, follows by the same argument as in the two-dimensional case in the proof of Theorem 3.2.6 on page 58.

For the other direction we have to show that Lemma 3.2.4 on page 51 is satisfied also for an arbitrary norm. The proof goes as in the two-dimensional case (see page 57) combined with an induction over the dimension. Assume $\frac{z_m}{\|z_m\|} \overset{\subseteq}{\longrightarrow} F$ where $F \subseteq B$ is an extreme set. If F is a facet or $(z_{m,F^\circ})_m$ is bounded, we are done. Otherwise we consider the subspace $T(F^\circ)^* \subseteq X$ equipped with a norm B' which is the dual of $\widetilde{F^\circ} \subseteq T(F^\circ)$, where $\widetilde{F^\circ}$ is obtained by projecting and shifting F° to $T(F^\circ)$ such that it contains the origin in its interior. In this subspace we now have an unbounded sequence which has a subsequence $(z'_m)_m$ with

$$\frac{z'_m}{\|z'_m\|_{B'}} \xrightarrow{\subseteq} F' \subseteq B'.$$

By induction and as the two-dimensional case is already shown, it follows that Lemma 3.2.4 is also true for an arbitrary norm. The rest of the proof is the same as the proof of Theorem 3.2.6. □

3.3 The Homeomorphism between the Compactification and B°

In the last part of this section we construct a homeomorphism m between the horofunction compactification $\overline{X}^{hor}$ and the dual unit ball B°. This will be done in Theorem 3.3.10. The homeomorphism will be put together by a map m^{B° from X into the interior of B° and maps m^E from $\partial_{hor}X$ into the interior of each extreme set E in the boundary of B°. To do so, we first define a more general map m^C, which maps a finite-dimensional vector space to the interior of a compact convex set C of the same dimension. Hereby we restrict ourselves to the cases where C is polyhedral, smooth or two-dimensional, so to the same cases as B° was restricted to in Theorem 3.2.6. The structure of the map is motivated by the moment map known from the theory of toric varieties. See for example [Ful93, §4.2] for a description. Up to some signs which come from the definition of the dual unit ball, the same result as Proposition 3.3.9 for a polytope C can be found in [Ful93, p. 82] but with a different proof. The moment map was also used to realize the closure of a flat in the Stake compactifications as bounded polytopes in [Ji97]. In this section, we will use a lot of calculus, which is justified by the identification $X \simeq \mathbb{R}^n$. More about it can be found in [Col12].

3.3.1 Definition and Properties of the Map m^C

Let $C \subseteq X^*$ be an m-dimensional closed compact convex set belonging to one of the following three cases:

I) C is polyhedral.

II) Every extreme set of C is an extreme point and all of them are smooth.

III) $m = \dim(C) = 2$.

Additionally we make the following constraint:

Constraint: We only consider convex sets C that have finitely many connected components of extreme points.

The set $\mathcal{E}_C$ consists of extreme points of C, there are isolated extreme points and extreme points in a smooth part (recall Definition 2.3.5 on page 19). The set $\mathcal{E}_C$ can now be split up in its connected components: Let $c_i \in \partial C$ (for $i \in \{1, \dots, k\}$) denote the isolated extreme points and $A_j \subseteq \partial C$ (for $j \in \{1, \dots, l\}$) the connected components of smooth parts of extreme points. Then

$$\mathcal{E}_C = \dot{\bigcup} c_i \sqcup \dot{\bigcup} A_j.$$

The basic idea for the map m^C is to define it as a convex combination of the extreme points in $\mathcal{E}_C$. If all extreme points of C are isolated, that is, C is polyhedral, then we really have a convex combination. If there is a smooth part in the boundary, we would have a sum over uncountably many extreme points. Instead we will integrate over smooth parts using Dirac functions.

For a simplified notation, we define for a bounded function $f : X^* \to \mathbb{R}$ or $f : X^* \to X^*$:

$$\widetilde{\int_{\partial C}} f(v)dv := \sum_{i=1}^{k} f(c_i) + \sum_{j=1}^{l} \int_{A_j} f(v)dv$$

where we use the (component wise) Lebesgue measure for the integrals. Note that non-extreme points of the boundary ∂C are not considered by this notation as they are obtained as convex combinations of extreme boundary points. For each connected component

$$D \in \{A_j, \{c_i\} \mid j = 1, \dots, l;\ i = 1, \dots, k\} \subseteq \partial C$$

and a point $x \in X$ define a map $\varphi_x^D : X^* \longrightarrow \mathbb{R}$ by

$$\varphi_x^D(v) := \frac{e^{-\langle v|x\rangle}}{\widetilde{\int_{\partial C}} e^{-\langle w|x\rangle}dw} \chi_D(v),$$

where χ_D is the indicator function of the set D. Summing them all up for all $D \in \{A_j, \{c_i\}\}$ gives us the following function $\varphi_x : X^* \to \mathbb{R}$:

$$\varphi_x = \sum_{i=1}^{k} \varphi_x^{c_i} + \sum_{j=1}^{l} \varphi_x^{A_j}.$$

As $\varphi_x(v) \geq 0$ for all $v \in X^*$ and as integration of φ_x over X^* gives 1 because of the indicator functions, φ is a probability measure on X^*. Thus integrating over the boundary of C as defined above gives an element in the interior of C, see also the proof of the main theorem in [RW58]. Using this we define the map m^C from X into the interior of C in such a way that will later turn out to be compatible with the convergence to horofunctions.

Definition 3.3.1 Let $C \subseteq X^*$ be an m-dimensional closed compact convex set. We define

$$\begin{aligned} m^C : X &\longrightarrow \operatorname{int}(C) \\ x &\longmapsto m^C(x) = \widetilde{\int_{\partial C}} \varphi_x(v)v dv. \end{aligned}$$ ○

Writing out all short notations we get due to the indicator functions in the definition of φ_x^D:

$$m^C(x) = \frac{\sum_{i=1}^k e^{-\langle c_i|x\rangle} c_i + \sum_{j=1}^l \int_{A_j} e^{-\langle v|x\rangle} v dv}{\sum_{i=1}^k e^{-\langle c_i|x\rangle} + \sum_{j=1}^l \int_{A_j} e^{-\langle w|x\rangle} dw},$$

where $c_1, \dots, c_k$ are the isolated extreme points of C and $A_1, \dots, A_l$ are the connected components of smooth parts of $\mathcal{E}_C$. Since the dual pairing is continuous, it is clear that m^C is continuous.

Remark 3.3.2 If C is polyhedral, then all of its extreme points are isolated and the map m^C simplifies to

$$m^C(x) = \sum_{i=1}^{k} \frac{e^{-\langle c_i|x\rangle}}{\sum_{j=1}^k e^{-\langle c_j|x\rangle}} c_i. \tag{3.23}$$

If there is only one connected component in $\mathcal{E}_C$ then we have

$$m^C(x) = \frac{\int_{\partial C} e^{-\langle v|x\rangle} v dv}{\int_{\partial C} e^{-\langle w|x\rangle} dw}. \qquad \circ$$

Example 3.3.3 Let us consider $\mathbb{R}^2$ with the convex set

$$C = \operatorname{conv}\left\{\begin{pmatrix}0.8\\0.7\end{pmatrix}, \begin{pmatrix}-1.3\\1\end{pmatrix}, \begin{pmatrix}-0.8\\-0.7\end{pmatrix}, \begin{pmatrix}1.3\\-1\end{pmatrix}\right\}.$$

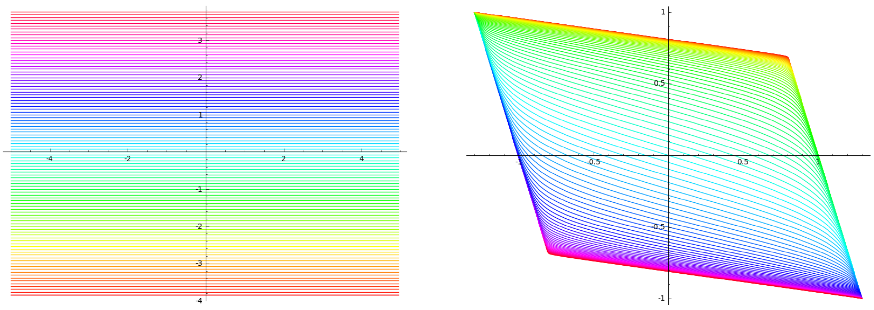


Figure 3.18: The horizontal lines in $\mathbb{R}^2$ (LEFT) are mapped into the interior of C (RIGHT). The picture was done with Sage.

Indicated by colors, Figure 3.18 shows how $\mathbb{R}^2$ is mapped into C by the map m^C. $\circ$

We do not require C to have the origin as an interior point. But we later want to take the dual of C as a unit ball, therefore we have to consider how m^C behaves under shifting. The following lemma shows that m^C behaves as desired:

Lemma 3.3.4 *Let $C_s = C + s$ be the convex set obtained by shifting C by an element $s \in X^*$. Then for all $x \in X$*

$$m^{C_s}(x) = m^C(x) + s. \qquad \circ$$

Proof. By $c_{s,i}, A_{s,j}$ ($i \in \{1, \dots, k\}, j \in \{1, \dots, l\}$) we denote the isolated and smooth connected components of $\mathcal{E}_{C_s}$. Then we label the indices such that for all i, j it holds

$$c_{s,i} = c_i + s \quad \text{and} \quad A_{s,j} = A_j + s.$$

Let $x \in X$ be arbitrary, then we have

$$\begin{aligned}
m^{C_s}(x) &= \frac{\sum_{i=1}^k e^{-\langle c_{s,i}|x\rangle} c_{s,i} + \sum_{j=1}^l \int_{A_{s,j}} e^{-\langle v|x\rangle} v dv}{\sum_{i=1}^k e^{-\langle c_{s,i}|x\rangle} + \sum_{j=1}^l \int_{A_{s,j}} e^{-\langle v|x\rangle} dv} \\
&= \frac{\sum_i e^{-\langle c_i+s|x\rangle}(c_i + s) + \sum_j \int_{A_j+s} e^{-\langle v|x\rangle} v dv}{\sum_i e^{-\langle c_i+s|x\rangle} + \sum_j \int_{A_j+s} e^{-\langle v|x\rangle} dv} \\
&= \frac{\sum_i e^{-\langle c_i|x\rangle} e^{-\langle s|x\rangle}(c_i + s) + \sum_j \int_{A_j} e^{-\langle v|x\rangle} e^{-\langle s|x\rangle}(v + s) dv}{\sum_i e^{-\langle c_i|x\rangle} e^{-\langle s|x\rangle} + \sum_j \int_{A_j} e^{-\langle v|x\rangle} e^{-\langle s|x\rangle} dv} \\
&= \frac{\sum_i e^{-\langle c_i|x\rangle} c_i + \sum_j \int_{A_j} e^{-\langle v|x\rangle} v dv}{\sum_i e^{-\langle c_i|x\rangle} + \sum_j \int_{A_j} e^{-\langle v|x\rangle} dv} + \frac{\sum_i e^{-\langle c_i|x\rangle} s + \sum_j \int_{A_j} e^{-\langle v|x\rangle} s dv}{\sum_i e^{-\langle c_i|x\rangle} + \sum_j \int_{A_j} e^{-\langle v|x\rangle} dv} \\
&= m^C(x) + s.
\end{aligned}$$

$\square$

Remark 3.3.5 When $\dim(C) = m < n$, we can write any $x \in X$ as $x = x_C + x^C$, where $x_C \in T(C)^*$ and $x^C \in (T(C)^\perp)^*$. Then as the pairing of any element of C with x^C is the same (see Lemma 2.6.1 on page 35) we get

$$m^C(x) = m^C(x_C). \qquad \circ$$

Therefore we will from now on assume C to have full dimension, that is, $m = n$.

In the end we want to have a map m built out of several maps like m^C that is compatible with the convergence of sequences to horofunctions. To achieve that, we first examine the behavior of sequences and extreme points under dual pairing:

Notation From now on let $C \subseteq X^*$ be an n-dimensional compact convex set with the origin in its interior belonging to one of the three cases I), II) III) stated at the beginning of the section. Denote by $C^\circ \subseteq X$ its dual set. Let $(z_m)_{m\in\mathbb{N}} \subseteq X$ be an unbounded sequence such that

$$\frac{z_m}{\|z_m\|} \xrightarrow{\subseteq} F$$

for some extreme set $F \subseteq C^\circ$. As in the previous section let $E = \mathrm{aff}\{D(x) | x \in X\} \cap C \subseteq F^\circ$. $\circ$

Lemma 3.3.6 *For all extreme points $e, e_1, e_2 \in \mathcal{E}_E$ of E and an extreme point $v \in \mathcal{E}_C \setminus \mathcal{E}_E$ it holds:*

a) $\left\langle e_1 - e_2 \middle| z_m^E \right\rangle = 0$

b) $\langle e - v | z_m \rangle \longrightarrow -\infty$ *and* $\left\langle e - v \middle| z_m^E \right\rangle \longrightarrow -\infty.$ $\circ$

Proof.

a) Let $e_1, e_2 \in \partial_{\mathrm{rel}} E$ be extreme points, then their difference is an element of $T(E)$. As the notation z_m^E denotes the projection of z_m to the space orthogonal to $T(E)$, the statement follows.

b) Now let $e \subseteq \mathcal{E}_E$ be an extreme point of E and $v \in \mathcal{E}_C \setminus \mathcal{E}_E$ extreme but not in the relative boundary of E. Since $E \subseteq F^\circ$ is a subset, we have to distinguish between $x \in \mathcal{E}_{F^\circ}$ and $v \in \mathcal{E}_C \backslash \mathcal{E}_{F^\circ}$. We first assume v to be an extreme point of B° not in the relative boundary of F°. Let $\gamma_m = \gamma_m(v)$ and $\delta_m = \delta_m(e) \in \mathbb{R}$ be defined by

$$\left\langle v \middle| \frac{z_m}{\|z_m\|} \right\rangle = -1 + \gamma_m,$$
$$\left\langle e \middle| \frac{z_m}{\|z_m\|} \right\rangle = -1 + \delta_m.$$

By the duality of C and C° we know that $\gamma_m, \delta_m \geq 0$. As $E \subseteq F^\circ$, the point v is not an extreme point of E, so the dual exposed set $\{v\}^\circ \subseteq C^\circ$ is an extreme set of C° having empty intersection with F. As $\frac{z_m}{\|z_m\|} \xrightarrow{\subseteq} F$, we conclude that there is some $\gamma = \gamma(v) > 0$ such that

$$\gamma_m > \gamma > 0$$

whereas

$$\delta_m \longrightarrow 0 \quad \text{as} \quad m \to \infty.$$

Note that $\gamma = \gamma(v)$ is a positive constant only depending on v that bounds $\gamma_m(v)$ away from 0. Additionally we know that $\frac{z_{m,E}}{\|z_m\|} \longrightarrow 0$ as $E \subseteq V(F)^\perp$. Therefore there is an $M \in \mathbb{N}$ such that $\delta_m - \left\langle e - v \,\middle|\, \frac{z_{m,E}}{\|z_m\|} \right\rangle < \frac{1}{2}\gamma$ for all $m \geq M$.

Together we compute

$$\begin{aligned}
\langle e - v | z_m \rangle &= \|z_m\| \left\langle e - v \,\middle|\, \frac{z_m}{\|z_m\|} \right\rangle \\
&= \|z_m\|(-1 + \delta_m + 1 - \gamma_m) \\
&< -\|z_m\|(\gamma - \delta_m) \longrightarrow -\infty
\end{aligned}$$

and

$$\begin{aligned}
\left\langle e - v \,\middle|\, z_m^E \right\rangle &= \langle e - v | z_m - z_{m,E} \rangle \\
&= \|z_m\| \left(\delta_m - \gamma_m - \left\langle e - v \,\middle|\, \frac{z_{m,E}}{\|z_m\|} \right\rangle \right) \\
&< -\frac{1}{2}\|z_m\|\gamma \longrightarrow -\infty.
\end{aligned}$$

It remains the case where $v \in \partial_{\text{rel}}(F^\circ)$ (and still $v \notin E$). Such an extreme point only exists if $E \subsetneq F^\circ$ which we thereby assume. So $(z_{m,F^\circ})_m$ has to be unbounded, otherwise $E = F^\circ$ by Lemma 3.2.2. As $e - v \in T(F^\circ)$, because $e \in \partial_{\text{rel}}(E) \subseteq \partial_{\text{rel}}(F^\circ)$, the dual pairings simplify to

$$\begin{aligned}
\langle e - v \,| z_m \rangle &= \left\langle e - v \,\middle|\, z_{m,F^\circ} \right\rangle \\
\left\langle e - v \,\middle|\, z_m^E \right\rangle &= \left\langle e - v \,\middle|\, (z_{m,F^\circ})^E \right\rangle,
\end{aligned}$$

where the orthogonal complement is taken within $T(F^\circ)$. We will show the result by iteratively constructing new unit balls and their duals in lower dimensional subspaces. The idea of the first step (and analogous for the following ones) is to take F° as new dual unit ball in $T(F^\circ)$ having E and v in its relative boundary. As F° is a face of C and therefore does not contain the origin as an relative interior point, we will have to shift F° with all its extreme sets to $T(F^\circ)$. See Figure 3.19 for a sketch.

The details go as follows. First let $s_1 \in X^*$ be a shifting parameter such that

$$C_1 := F^\circ + s_1 \subseteq T(F^\circ)$$

is a convex compact set having the origin in its relative interior. Then taking its dual within $T(F^\circ)$ and $T(F^\circ)^*$ gives us a new compact convex set $C_1^\circ \subseteq T(F^\circ)^*$ around the origin. Remember the definition

$$E = \text{aff}\,\{D(x) | x \in X\}$$

where $D(x) = \lim_{m\to\infty} D_m(x)$ is the set of accumulation sets $D_m(x) = \left(\frac{z_m - x}{\|z_m - x\|}\right)^\circ$. The $z_{m,V(F)}$-part is dominant and guarantees that $\frac{z_m}{\|z_m\|} \overset{\subseteq}{\longrightarrow} F$, whereas the z_{m,F°-part determines the behavior within aff(F°) and thereby to which extreme set of F° the sequence converges. Note that we assumed $E \neq F^\circ$ and therefore $(z_{m,F^\circ})_m$ is unbounded by Lemma 3.2.2. When only considering the part in $T(F^\circ)$, the sequence of duals converges to the extreme set

$$E_1 := E + s_1 \subseteq C_1.$$

This means that when we only consider the subspace $T(F^\circ)^*$ we have the following situation: we have an unbounded sequence $(z_{m,F^\circ})_m$ with $\frac{z_{m,F^\circ}}{\|z_{m,F^\circ}\|} \overset{\subseteq}{\longrightarrow} F_1$ for some extreme set

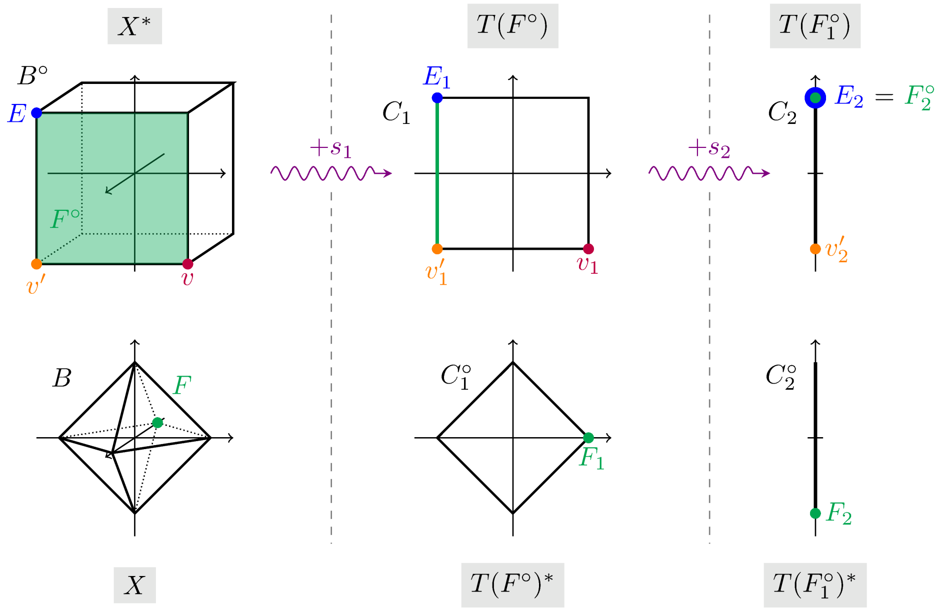


Figure 3.19: Sketch of the idea of constructing new unit balls.

$F_1 \subseteq C_1^\circ$ such that $E_1 \subseteq (F_1)^\circ \subseteq C_1$. Note that the dual of F_1 now is taken only in the subspace $T(F^\circ)^*$ of lower dimension. Let $e_1 := e + s_1 \in \partial_{\text{rel}} F_1^\circ$ and $v_1 := v + s_1 \in \partial_{\text{rel}} C_1$ be the shifted extreme point. If $v_1 \notin \partial_{\text{rel}} F_1^\circ$ we conclude by the first part of the proof:

$$\langle e - v | z_m \rangle = \langle e_1 - v_1 | z_{m,F^\circ} \rangle \longrightarrow -\infty.$$

If $v_1 \in \partial_{\text{rel}} F_1^\circ$ (as v_1' in the picture) we go on in the same way by projecting to $T(F_1^\circ)$ and shifting with some $s_2 \in T(F^\circ)$ such that $C_2 := F_1^\circ + s_2 \subseteq T(F_1^\circ)$ contains the origin as an interior point. Continuing this procedure, we get a sequence $C^\circ, C_1^\circ, C_2^\circ \ldots$ of compact convex sets $C_j^\circ \subseteq T(F_{j-1}^\circ)^*$ (set $F_0^\circ := F^\circ$) and their duals $C_j \subseteq T(F_{j-1}^\circ)$. They are obtained as $C_j := F_{j-1}^\circ + s_j$ for some $s_j \in T(F_{j-2}^\circ)$ (with $j \geq 2$) where $F_j \subseteq C_j^\circ$ is extreme such that

$$\frac{z_{m,F_{j-1}^\circ}}{\|z_{m,F_{j-1}^\circ}\|_{C_j^\circ}} \xrightarrow{\subseteq} F_j \subseteq C_j^\circ.$$

The extreme sets in the relative boundary then are given as

$$\begin{aligned} e_j &:= e_{j-1} + s_j, \\ v_j &:= v_{j-1} + s_j, \\ E_j &:= E_{j-1} + s_j. \end{aligned}$$

We can only break out of this iterating process, when $v_k \notin \partial_{\text{rel}} F_k^\circ$ for some $k \geq 1$. As the dimension of the exposed dual F° decreases in each step, that is, $\dim(F_j^\circ) < \dim(F_{j-1}^\circ)$, we finally come to the point where $\dim(F_k^\circ) = 0$ (if $v_j \in \partial_{\text{rel}} F_j^\circ$ all the time). Then we have $E_k = F_k^\circ \in T(F_{k-1}^\circ)$ and, as we required $v \notin \partial_{\text{rel}} E$, it is $v_k \notin \partial_{\text{rel}} E_k$ and the iterating process finally terminates. As $e - v = e_k - v_k \in T(F_{k-1}^\circ)$ we have

$$\langle e - v | z_m \rangle = \left\langle e_k - v_k \middle| z_{m,F_{k-1}^\circ} \right\rangle \longrightarrow -\infty,$$

which is the desired convergence behavior. □

Lemma 3.3.7 *With the notations as on page 77 we require $p = \lim_{m\to\infty} z_{m,E}$ to exist. Additionally let $e_0 \in \mathcal{E}_E$ be some extreme point of E. Then it holds:*

a) *If $c \in \mathcal{E}_C$ is a vertex of C, then*

$$e^{\langle e_0 - c|z_m\rangle} \longrightarrow \begin{cases} e^{\langle e_0 - c|p\rangle} & \text{if} \quad c \in E \\ 0 & \text{if} \quad c \notin E. \end{cases}$$

b) *If $f : X^* \to \mathbb{R}$ or $f : X^* \to X^*$ is bounded on A_j and $A_j \cap E = \emptyset$, then*

$$\int_{A_j} e^{\langle e_0 - v|z_m\rangle} f(v)dv \longrightarrow 0.$$

c) *If $f : X^* \to \mathbb{R}$ or $f : X^* \to X^*$ is bounded on A_j, $\{e_0\} = A_j \cap E$ and $\{e_0\}$ and A_j do not both intersect the boundary of a common facet at a point different to e_0, then*

$$\int_{A_j} \frac{e^{-\langle v|z_m\rangle}}{\int_{A_j} e^{-\langle w|z_m\rangle}dw} f(v)dv \longrightarrow f(e_0). \qquad \circ$$

Proof.

a) When both e_0 and c are extreme points of E ($c = e_0$ is also possible), then $e_0 - c \in T(E)$ and therefore $\langle e_0 - c|z_m\rangle = \langle e_0 - c|z_{m,E}\rangle \longrightarrow \langle e_0 - c|p\rangle$. If $c \notin E$ is a vertex of C, then the convergence to 0 follows directly by Lemma 3.3.6.

b) Let A_j be a connected component of the set $\mathcal{E}_C$ of extreme points of C having empty intersection with E. This case only occurs when $\dim(C) = 2$ since there is only one connected component of $\mathcal{E}_C$ in the smooth case and none in the polyhedral one. Then for any point $v \in A_j$ we have $e^{\langle e_0 - v|z_m\rangle} \longrightarrow 0$ by Lemma 3.3.6. We now show that the convergence is uniform, as we then also have convergence of the integral over A_j.

As A_j is a connected component of extreme points not intersecting E, it is strongly separated from E.

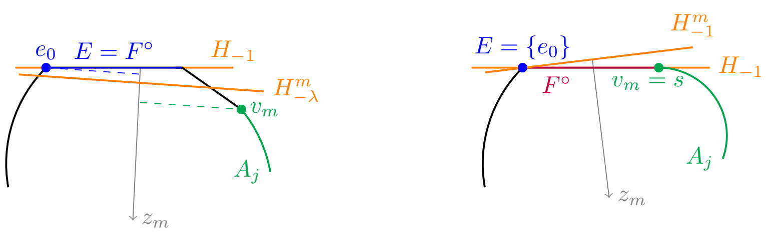


Figure 3.20: LEFT: A_j is strongly separated from E and $H^m_{-\lambda}$ lies in between for m big enough. RIGHT: $E = \{e_0\}$ lies in the boundary of the facet F° and H^m_{-1} converges to F°, so there is no strong separation between the limit H_{-1} and A_j.

Let $H^m_{-1} := H^{\frac{z_m}{\|z_m\|}}_{-1}$ be the the hyperplane supporting C at a point dual to $\frac{z_m}{\|z_m\|} \in \partial C^\circ$. Let H_{-1} denote the limit of this sequence of hyperplanes for $m \to \infty$. As $\frac{z_m}{\|z_m\|} \overset{\subseteq}{\longrightarrow} F$ this limit actually exists and is a hyperplane supporting C at F°. There are two cases to distinguish now, see Figure 3.20 for a picture: either $H_{-1} \cap A_j = \emptyset$ or there is an $s \in A_j$ such that $H_{-1} \cap A_j = \{s\}$.

- In the first case, we can find a $0 < \lambda < 1$ such that for m big enough $H^m_{-\lambda}$ and also the limiting hyperplane $H_{-\lambda}$ strongly separate E and A_j. Let $v_m \in A_j$ be a point for each $m \in \mathbb{N}$ with minimal distance to H_m. By the strong separation, there is a $\delta > 0$ and an $M \in \mathbb{N}$ such that $\left\langle e_0 - v_m \middle| \frac{z_m}{\|z_m\|} \right\rangle < -\delta < 0$ for all $m \geq M$. Then

$$\langle e_0 - v|z_m\rangle \leq \langle e_0 - v_m|z_m\rangle < -\delta\|z_m\| \quad \forall v \in A_j,\ m \geq M.$$

- In the second case where $H_{-1} \cap A_j = \{s\}$, the extreme point $E = \{e_0\}$ lies in the relative boundary of the facet F°, and s is the other extreme point of F°, like shown in Figure 3.20 on the right. In this case $\left\langle s \,\middle|\, \frac{z_m}{\|z_m\|} \right\rangle \to -1$ just as for the pairing with e_0 and we do not have a strong separation. We assume that s is the point of A_j that minimizes the pairing $\left|\left\langle e_0 - s \,\middle|\, \frac{z_{m,F^\circ}}{\|z_m\|} \right\rangle\right|$, otherwise divide A_j into two parts, where the part not containing s belongs to the first case above. Then as $\left(z_m^{V(F)}\right)_m = (z_{m,F^\circ})_m$ is unbounded, we can find an $\delta > 0$ such that

$$\left\langle e_0 - v \,\middle|\, \frac{z_{m,F^\circ}}{\|z_m\|} \right\rangle \leq \left\langle e_0 - s \,\middle|\, \frac{z_{m,F^\circ}}{\|z_m\|} \right\rangle < -\delta < 0$$

for all $v \in A_j$. Since $\left\langle e_0 - v \,\middle|\, \frac{z_m^{F^\circ}}{\|z_m\|} \right\rangle \leq 0$ for m big enough, we have for all $v \in A_j$:

$$e^{\langle e_0 - v | z_m \rangle} = e^{\left\langle e_0 - v | \frac{z_{m,F^\circ}}{\|z_m\|} \right\rangle} e^{\left\langle e_0 - v | \frac{z_{m,F^\circ}}{\|z_m\|} \right\rangle} < e^{-\delta \|z_m\|}.$$

By the compactness of A_j, let $\tilde{v} \in A_j$ be a point maximizing $|f(v)|$ over A_j. Then for any $\varepsilon > 0$ there is an $N > M \in \mathbb{N}$ such that for all $m \geq N$ it holds

$$\begin{aligned} |e^{\langle e_0 - v | z_m \rangle} f(v)| &= e^{\langle e_0 - v | z_m \rangle} |f(v)| \\ &\leq e^{\langle e_0 - v_m | z_m \rangle} |f(\tilde{v})| \\ &< e^{-\delta \|z_m\|} |f(\tilde{v})| < \varepsilon. \end{aligned}$$

Therefore we have uniform convergence and the integral over A_j goes to 0 as it was to show.

c) Now let the intersection $\{e_0\} := A_j \cap E$ be non-empty. Then for all $v \in A_j \setminus \{e_0\}$ we still have $e^{\langle e_0 - v | z_m \rangle} \longrightarrow 0$, but now there is no strong separation between E and $A_j \backslash \{e_0\}$ as in the case before. Let $\gamma : I \longrightarrow A_j$ be a parametrization of A_j with a closed set $I \subseteq \mathbb{R}^n$ containing the origin such that $\gamma(0) = e_0$, and extend γ by 0 to $\mathbb{R}^n$. We now want to show that

$$\delta_m(v) := \frac{e^{-\langle v | z_m \rangle}}{\int_{A_j} e^{-\langle w | z_m \rangle} dw} \chi_{A_j}(v)$$

is a Dirac-sequence around e_0. This then gives us the convergence we have to show. It is obvious that $\delta_m(v) \geq 0$ for all $v \in A_j$ and that $\int_{A_j} \delta_m(v) dv = 1$. As last condition for $(\delta_m)_m$ to be a Dirac-sequence, we have to show that for any $\varepsilon > 0$ it holds $\int_{A_j \backslash B_\varepsilon(e_0)} \delta_m(v) dv \longrightarrow 0$, where $B_\varepsilon(e_0)$ is the ball with radius ε around e_0.

When for all points in $A_j \backslash B_\varepsilon(e_0)$ the numerator of δ_m uniformly goes to 0 while the denominator remains positive, we are done. $\delta_m(v)$ remains the same if we expand the fraction with $e^{\langle b | z_m \rangle}$ for some $b \in X^*$. So our goal now is to find a point $b \in A_j$ such that $\langle b - v | z_m \rangle \longrightarrow -\infty$ for all $v \in A_j \setminus B_\varepsilon(e_0)$ and $\langle b - w | z_m \rangle > 0$ for all points w in some subset of $A_j \cap B_\varepsilon(e_0)$ of measure greater than 0.

Let $\varepsilon > 0$ be given. Let $E^\circ \subseteq C^\circ \subseteq X$ be the face dual to E and define the cones

$$\begin{aligned} K_{e_0}^+ &:= \bigcap_{g \in E^\circ} V_{\geq -1}^g \\ K_{e_0}^- &:= \bigcap_{g \in E^\circ} V_{\leq -1}^g, \end{aligned}$$

where (recall Definition 2.1.22) $V^g_{\geq -1} = \{y \in X^* \mid \langle y|g\rangle \geq -1\}$ is the affine half-space defined by the hyperplane H^g_{-1} which contains the origin. Similarly $V^g_{\leq -1}$ is the half-space not containing the origin. So $K^+_{e_0}$ and $K^-_{e_0}$ are two opposite cones with apex e_0 and $C \subseteq K^+_{e_0}$ is contained in the positive cone. To find the point b, we will shift the cones $K^+_{e_0}$ and $K^-_{e_0}$ around to have different apexes. Therefore we introduce the following notation: for some $y \in X^*$ we set

$$K^+_y := (y - e_0) + K^+_{e_0}$$
$$K^-_y := (y - e_0) + K^-_{e_0}$$

to be the shifted cones with apex y and boundaries parallel to H^g_{-1} for all extreme points $g \in \mathcal{E}_{E^\circ}$. Note that the hyperplanes H^g_{-1} are tangent to ∂C for all $g \in \mathcal{E}_{E^\circ}$.

The point $e_0 \in A_j \subseteq \partial C$ can either be a corner point (only in two dimensions) or the boundary is smooth around it. Let us look at the two cases separately, see also Figure 3.21 for a picture.

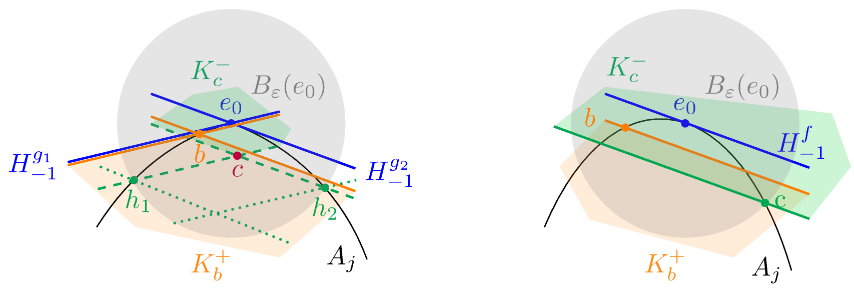


Figure 3.21: The configuration of the hyperplanes and cones as in the proof. LEFT: The point e_0 is a corner point and has infinitely many supporting hyperplanes out of which two are tangent to ∂C. RIGHT: If e_0 is a smooth boundary point, it only has one supporting hyperplane and all cones are half-spaces.

- If e_0 is a corner point, we are in the case $\dim(C) = 2$ and $E^\circ = \mathrm{conv}\{g_1, g_2\}$ is a facet. The two extreme points g_1, g_2 determine the cone $K^+_{e_0} = V^{g_1}_{\geq -1} \cap V^{g_2}_{\geq -1}$ and the corresponding hyperplanes $H^{g_1}_{-1}$ and $H^{g_2}_{-1}$ are tangent to C at the two sides of e_0. Let h_1, h_2 be the (two) intersection points of $B_\varepsilon(e_0)$ with ∂C (we assume that ε is small enough such that they exist and at least one of them is in A_j). Consider the cone

$$K^-_c := K^-_{h_1} \cap K^-_{h_2}.$$

  If $e_0 \in \partial_{\mathrm{rel}} A_j$ is a relative boundary point of A_j with a facet on its other side containing one of the points h_i, say h_1, we set $c = h_2 \in A_j$. Then K^-_c again is a cone with apex $c \in C$ and bounding hyperplanes parallel to $H^{g_1}_{-1}$ and $H^{g_2}_{-1}$. By K^+_c we denote the cone opposite of K^-_c with apex c.

- If ∂C is smooth at e_0, then $G = F = \{f\}$ is an extreme point and there is a unique supporting hyperplane H^f_{-1} tangent to C at e_0. So in this case the cones $K^+_{e_0} = V^f_{\geq -1}$ and $K^-_{e_0} = V^f_{\leq -1}$ are affine half-spaces. For unifying notations with the non-smooth case we keep the notation as a cone rather than a half-space. Let $c \in \partial B_\varepsilon(e_0) \cap A_j$ be a point in the intersection of the boundary of $B_\varepsilon(e_0)$ with A_j with minimal distance to H^f_{-1}. Hereby $\varepsilon > 0$ is assumed to be small enough such that c exists. Then the half-space $K^-_c = (c - e_0) + K^-_{e_0}$ contains e_0.

In both cases (where e_0 is a corner point or ∂C is smooth at e_0), we found a point c and a pair of cones with apex c such that $e_0 \in K_c^-$. Now we can choose a point $b \neq e_0 \in A_j \cap \mathrm{int}(K_c^-)$ and consider the (shifted) cones K_b^- and K_b^+ with apex b. Their bounding hyperplanes are again parallel to those of $K_{e_0}^-$ and $K_{e_0}^+$. By construction it holds:

$$A_j \setminus B_\varepsilon(e_0) \subseteq K_b^+, \qquad \text{and} \qquad e_0 \in K_b^-. \tag{3.24}$$

If there is a part of A_j not contained in K_b^+, take b closer to e_0. As e_0 is not in the relative boundary of a facet that also touches A_j on the other side, we can always choose $b \neq e_0$. Then by Equation (3.24) we have

$$\begin{aligned} b - v &\in K_0^- \qquad && \forall\, v \in A_j \setminus B_\varepsilon(e_0), \\ b - w &\in K_0^+ \qquad && \forall\, w \in K_b^-. \end{aligned}$$

As $\frac{z_m}{\|z_m\|} \xrightarrow{\subseteq} F$, we know that (for m big enough) the hyperplanes $H_0^{z_m}$ orthogonal to z_m and passing through the origin either converge to the bounding hyperplanes of K_0^- and K_0^+ or (only for $\dim(C) = 2$) intersect K_0^- only at the origin. This means that $b - v \in V_{\leq 0}^{z_m}$ for all m big enough. So for any $v \in A_j \setminus B_\varepsilon(e_0)$ it is $\left\langle b - v \,\middle|\, \frac{z_m}{\|z_m\|} \right\rangle < \delta < 0$ for m big enough and for some $\delta > 0$ by strong separation. Note that we chose b to be in the interior of K_c^-, so even if v is close to $\partial B_\varepsilon(e_0) \cap A_j$, the vector $b - v$ points into K_0^-, and the pairing is strictly negative and bounded away from 0.

Similarly $\left\langle b - w \,\middle|\, \frac{z_m}{\|z_m\|} \right\rangle > 0$ for all $w \in K_b^- \cap A_j$ and m big enough. By convexity of C, the set $K_b^- \cap A_j \subseteq B_\varepsilon(e_0)$ is connected and contains more than one point, so it has measure greater than 0.

Therefore we found the point $b \in A_j$ satisfying

$$\begin{aligned} \langle b - v \,|z_m \rangle = \|z_m\| \left\langle b - v \,\middle|\, \frac{z_m}{\|z_m\|} \right\rangle &\xrightarrow{unif.} -\infty \qquad && \forall v \in A_j \setminus B_\varepsilon(e_0), \\ \langle b - w|z_m \rangle &> 0 \qquad && \forall w \in K_b^- \cap A_j. \end{aligned}$$

Now we can compute the integral of $\delta_m(v)$ over $A_j \setminus B_\varepsilon(e_0)$ and get:

$$\begin{aligned} \int_{A_j \setminus B_\varepsilon(e_0)} \delta_m(v) dv &= \int_{A_j \setminus B_\varepsilon(e_0)} \frac{e^{-\langle v|z_m \rangle}}{\int_{A_j} e^{-\langle w|z_m \rangle} dw} dv \\ &= \int_{A_j \setminus B_\varepsilon(e_0)} \frac{e^{\langle b - v|z_m \rangle}}{\int_{A_j \setminus B_\varepsilon(e_0)} e^{\langle b - w|z_m \rangle} dw + \int_{B_\varepsilon(e_0)} e^{\langle b - w|z_m \rangle} dw} dv \\ &\longrightarrow 0, \end{aligned}$$

as the second term in the denominator is not vanishing whereas the numerator is. Therefore δ_m is a Dirac-sequence.

Using the Dirac-sequence δ_m around e_0 we compute:

$$\int_{A_j} \frac{e^{-\langle v|z_m \rangle}}{\int_{A_j} e^{-\langle w|z_m \rangle} dw} f(v) dv = \int_{A_j} \delta_m(v) f(v) \longrightarrow f(e_0). \qquad \square$$

Let us come back to the map m^C. The following lemma contains the most important feature of the map: it guarantees its surjectivity and will later be the continuity of the map m, which will be constructed as several maps m^C put together according to the combinatorics of B°.

Lemma 3.3.8 *With the notations as on page 77, assume that $p = \lim_{m\to\infty} z_{m,E}$ exists. Then*

$$m^C(z_m) \longrightarrow m^E(p). \qquad \circ$$

Proof. This proof is based on the previous lemma and we have to distinguish several cases depending on the shape of C.

I) C is polyhedral C being polyhedral means that there are only isolated extreme points, therefore m^C has the simplified expression (3.23) without integrals. By the first part of the previous lemma, we get for some $e_0 \in \mathcal{E}_E$:

$$\begin{aligned} m^C(z_m) &= \sum_{i=1}^{k} \frac{e^{-\langle c_i|z_m\rangle}}{\sum_{j=1}^k e^{-\langle c_j|z_m\rangle}} c_i \;=\; \sum_i \frac{e^{\langle e_0 - c_i|z_m\rangle} c_i}{\sum_j e^{\langle e_0 - c_j|z_m\rangle}} \\ &\longrightarrow \sum_{c_i'\in\mathcal{E}_E} \frac{e^{\langle e_0 - c_i'|p\rangle} c_i'}{\sum_{c_j'\in\mathcal{E}_E} e^{\langle e_0 - c_j'|p\rangle}} = m^E(p). \end{aligned}$$

II) C and C° are smooth C being smooth is the contrary case to C being polyhedral, as we now only have one (smooth) connected component of $\mathcal{E}_C$. Then $E = \{e_0\}$ consists of a single point and $m^E(p) = e_0$. The convergence now follows immediately with the third part of the previous lemma ,where now $f(v)fv$:

$$m^C(z_m) = \int_{\partial C} \frac{e^{-\langle v|z_m\rangle}}{\int_{\partial C} e^{-\langle w|z_m\rangle} dw} v dv \longrightarrow e_0 = m^E(p).$$

III) $\dim(C) = 2$ and C is arbitrary In this case we have to consider the full expression for m^C:

$$m^C(z_m) = \frac{\sum_{i=1}^k e^{-\langle c_i|z_m\rangle} c_i + \sum_{j=1}^l \int_{A_j} e^{-\langle v|z_m\rangle} v dv}{\sum_{i=1}^k e^{-\langle c_i|z_m\rangle} + \sum_{j=1}^l \int_{A_j} e^{-\langle w|z_m\rangle} dw}.$$

For the extreme set E there are several cases to consider, refer also to Table 3.1 on page 51 and the discussion before.

- If E is isolated, that is, all of its (one or two) extreme points are isolated, then the integrals over all smooth parts go to 0 and we basically have the same calculation as in the polyhedral case.

- Next we consider the case where $E = \mathrm{conv}\{e_1, e_2\}$ is a non-isolated facet, that is, at least one of e_1 and e_2 lies in the relative boundary of some smooth connected component. Then we always have one of the following two situations (see also Figure 3.22):

  (A) E is surrounded by two smooth parts, say $e_1 \in A_1$ and $e_2 \in A_2$. Let the enumeration be such that A_1 is the component whose second endpoint $a_1 \neq e_1 \in \partial_{\mathrm{rel}} A_1$ is closer to $\mathrm{aff}(E)$ than the second endpoint $a_2 \neq e_2$ of A_2.

  (B) E has only one vertex in the relative boundary of a smooth part, let e_1 be this one. The other vertex e_2 is isolated and, by relabeling the extreme points, we assume $e_2 = c_2$.

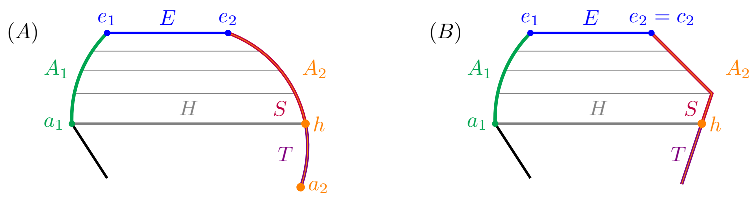


Figure 3.22: The decomposition of A_2 to construct the translating homeomorphism. LEFT: The case (A) where E is surrounded by smooth parts. RIGHT: In case (B) the extreme point e_1 lies in the boundary of the smooth part A_1 and e_2 is a vertex.

In both cases, we next want to identify a part S of A_2 with A_1 by a (bijective) translation φ parallel to E. Using this bijection we will later replace the integral over S by an integral over A_1 when computing the convergence of m^C. The identification is illustrated in Figure 3.22. We now explain it rigorously:

Let H be a hyperplane passing through a_1 parallel to E, that is, with $T(H) = T(E)$. Let $a_1 \neq h \in H \cap \partial C$ be the other intersection point of H with the boundary of C. In case (A) we have $h \in A_2$ and h splits up A_2 in two parts S and T where $e_2 \in S$. In case (B) we define $S \subseteq \partial C$ to be the part between e_2 and h without E. In both cases, for each point $a \in A_1$ there is a corresponding point $s \in S$ such that their difference $t_a := s - a$ lies in $T(E)$, that is, there is a bijection

$$\begin{aligned} \varphi : A_1 &\longrightarrow S \\ a &\longmapsto a + t_a \end{aligned} \tag{3.25}$$

with $t_a \in T(E)$. Especially $e_1 + t_{e_1} = e_2$.

We first consider case (A), where E is surrounded by two smooth parts A_1 and A_2 and the two vertices e_1, e_2 of E are not isolated. Then

$$\begin{aligned} m^C(z_m) &= \frac{\sum_{i=1}^k e^{-\langle c_i | z_m\rangle} c_i + \sum_{j=1}^l \int_{A_j} e^{-\langle v|z_m\rangle} v dv}{\sum_{i=1}^k e^{-\langle c_i|z_m\rangle} + \sum_{j=1}^l \int_{A_j} e^{-\langle v|z_m\rangle} dv} \\ &= \frac{\sum_i e^{\langle a_1 - c_i|z_m\rangle} c_i + \sum_{j\geq 3} \int_{A_j} e^{\langle a_1 - v|z_m\rangle} v dv}{\sum_i e^{\langle a_1 - c_i|z_m\rangle} + \sum_{j\geq 3}\int_{A_j} e^{\langle a_1-v|z_m\rangle}dv + \int_{A_1} e^{\langle a_1 - v|z_m\rangle}dv + \int_{A_2} e^{\langle a_1-v|z_m\rangle}dv} \\ &\quad + \frac{\int_{A_1} \frac{e^{-\langle v|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw} v dv + \int_{A_2} \frac{e^{-\langle v|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw} v dv}{\sum_i \frac{e^{\langle a_1 - c_i|z_m\rangle}}{\int_{A_1} e^{\langle a_1 - w|z_m\rangle}dw} + \sum_{j\geq 3}\int_{A_j} \frac{e^{\langle a_1 - v|z_m\rangle}}{\int_{A_1} e^{\langle a_1-w|z_m\rangle}dw}dv + 1 + \int_{A_2}\frac{e^{-\langle v|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw}dv}. \end{aligned} \tag{3.26}$$

The extreme point $a_1 \in \partial_{\text{rel}} A_1$ has the following property: any point of A_1 and S has less or equal distance to $\text{aff}(E)$ than a_1 and any other extreme point (both isolated or in a smooth part) has bigger distance to $\text{aff}(E)$. Therefore

$$\begin{aligned} \langle a_1 - c_i | z_m\rangle &\longrightarrow -\infty && \forall i \in \{1, \dots, k\} \\ \langle a_1 - v|z_m\rangle &\longrightarrow -\infty && \forall v \in A_j \cup T,\ j \geq 3 \\ \langle a_1 - v|z_m\rangle &\geq 0 && \forall v \in A_1 \cup S. \end{aligned}$$

This shows that the first fraction in Equation (3.26) goes to 0. To get the convergence of the second fraction we first consider the integral over A_2 in the numerator. In the computation we will replace the integral over $S \subseteq A_2$ by an integral over A_1 using the translation map φ:

$$\begin{aligned}
\int_{A_2} \frac{e^{-\langle v|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw} v dv &= \int_S \frac{e^{-\langle v|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw} v dv + \int_T \frac{e^{-\langle v|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw} v dv \\
&= \int_{A_1} \frac{e^{-\langle a+t_a|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw}(a+t_a)da + \int_T \frac{e^{-\langle v|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw} v dv \\
&= \int_{A_1} \frac{e^{-\langle a|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw} e^{-\left\langle t_a|z_{m,E}\right\rangle}(a+t_a)da + \int_T \frac{e^{\langle a_1-v|z_m\rangle}}{\int_{A_1} e^{\langle a_1-w|z_m\rangle}dw} v dv \\
&= \int_{A_1} \delta_m(a) e^{-\left\langle t_a|z_{m,E}\right\rangle}(a+t_a)da + \int_T \frac{e^{\langle a_1-v|z_m\rangle}}{\int_{A_1} e^{\langle a_1-w|z_m\rangle}dw} v dv \\
&\longrightarrow e^{-\left\langle t_{e_1}|p\right\rangle}(e_1+t_{e_1}) + 0 = e^{-\langle e_2-e_1|p\rangle} e_2.
\end{aligned} \tag{3.27}$$

For the convergence we used the Dirac-sequence $\delta_m(a)$ and the fact that $\langle a_1 - v|z_m\rangle \to -\infty$ for all $v \in T$. For the integral over A_2 in the denominator of the second fraction in Equation (3.26) we obtain similarly

$$\int_{A_2} \frac{e^{-\langle v|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw} dv \longrightarrow e^{-\left\langle t_{e_1}|p\right\rangle} + 0 = e^{-\langle e_2-e_1|p\rangle}.$$

All together we get from Equation (3.26):

$$\begin{aligned}
m^C(z_m) &= \frac{\sum_i e^{\langle a_1-c_i|z_m\rangle} c_i + \sum_{j\geq 3} \int_{A_j} e^{\langle a_1-v|z_m\rangle} v dv}{\sum_i e^{\langle a_1-c_i|z_m\rangle} + \sum_{j\geq 3} \int_{A_j} e^{\langle a_1-v|z_m\rangle} dv + \int_{A_1} e^{\langle a_1-v|z_m\rangle} dv + \int_{A_2} e^{\langle a_1-v|z_m\rangle} dv} \\
&\quad + \frac{\int_{A_1} \frac{e^{-\langle v|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw} v dv + \int_{A_2} \frac{e^{-\langle v|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw} v dv}{\sum_i \frac{e^{\langle a_1-c_i|z_m\rangle}}{\int_{A_1} e^{\langle a_1-w|z_m\rangle}dw} + \sum_{j\geq 3} \int_{A_j} \frac{e^{\langle a_1-v|z_m\rangle}}{\int_{A_1} e^{\langle a_1-w|z_m\rangle}dw} dv + 1 + \int_{A_2} \frac{e^{-\langle v|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw} dv} \\
&\longrightarrow \frac{e_1 + e^{\langle e_1-e_2|p\rangle} e_2}{1 + e^{\langle e_1-e_2|p\rangle}} = \frac{e^{-\langle e_1|p\rangle} e_1 + e^{-\langle e_2|p\rangle} e_2}{e^{-\langle e_1|p\rangle} + e^{-\langle e_2|p\rangle}} = m^E(p).
\end{aligned}$$

In case (B) where E has an isolated vertex $e_2 = c_2$, the computation goes similar. But instead of summing over the isolated vertex e_2 we replace it by an integral over S combined with a Delta distribution and get (by using the translation φ and the transformation rule):

$$\begin{aligned}
e^{-\langle e_2|z_m\rangle} e_2 &= \int_S e^{-\langle v|z_m\rangle} v \delta(v - e_2) dv \\
&= \int_{A_1} e^{-\langle a|z_m\rangle} e^{-\left\langle t_a|z_{m,E}\right\rangle}(a+t_a)\delta(a+t_a-e_2)da.
\end{aligned}$$

So dividing by an integral over A_1 yields:

$$\begin{aligned}
\frac{e^{-\langle e_2|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw} e_2 &= \int_{A_1} \frac{e^{-\langle a|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw} e^{-\left\langle t_a|z_{m,E}\right\rangle}(a+t_a)\delta(a+t_a-e_2)da \\
&\longrightarrow e^{-\left\langle t_{e_1}|p\right\rangle}(e_1+t_{e_1})\delta(e_1+t_{e_1}-e_2) = e^{-\langle e_2-e_1|p\rangle} e_2,
\end{aligned}$$

and similarly

$$\frac{e^{-\langle e_2|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw} \longrightarrow e^{-\langle e_2-e_1|p\rangle}.$$

If there is an isolated vertex c_i ($i \neq 2$) or a smooth connected component A_j ($j \neq 1$) that is closer to $\mathrm{aff}(E)$ than a_1, we take a point $\tilde{a} \in \mathrm{relint}(A_1)$ which is closer to $\mathrm{aff}(E)$ than a_1, to extend the fractions. Then $\int_{A_1} e^{\langle \tilde{a}-w|z_m\rangle}dw > 0$ is positive even in the limit and the calculations go through the same. In total we then get:

$$
\begin{aligned}
m^C(z_m) &= \frac{\sum_{i=1}^k e^{-\langle c_i|z_m\rangle}c_i + \sum_{j=1}^l \int_{A_j} e^{-\langle v|z_m\rangle}v\,dv}{\sum_{i=1}^k e^{-\langle c_i|z_m\rangle} + \sum_{j=1}^l \int_{A_j} e^{-\langle v|z_m\rangle}dv} \\
&= \frac{\sum_{i\neq 2} e^{\langle a_1-c_i|z_m\rangle}c_i + \sum_{j\geq 2}\int_{A_j} e^{\langle a_1-v|z_m\rangle}v\,dv}{\sum_i e^{\langle a_1-c_i|z_m\rangle} + \sum_{j\geq 2}\int_{A_j} e^{\langle a_1-v|z_m\rangle}dv + \int_{A_1} e^{\langle a_1-v|z_m\rangle}dv} \\
&\quad + \frac{\frac{e^{-\langle e_2|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw}e_2 + \int_{A_1}\frac{e^{-\langle v|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw}v\,dv}{\sum_{i\neq 2}\frac{e^{\langle a_1-c_i|z_m\rangle}}{\int_{A_1} e^{\langle a_1-w|z_m\rangle}dw} + \sum_{j\geq 2}\int_{A_j}\frac{e^{\langle a_1-v|z_m\rangle}}{\int_{A_1} e^{\langle a_1-w|z_m\rangle}dw}dv + 1 + \frac{e^{-\langle e_2|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw}} \\
&\longrightarrow 0 + \frac{e^{-\langle e_2-e_1|p\rangle}e_2 + e_1}{1 + e^{-\langle e_2-e_1|p\rangle}} = \frac{e^{-\langle e_1|p\rangle}e_1 + e^{-\langle e_2|p\rangle}e_2}{e^{-\langle e_1|p\rangle} + e^{-\langle e_2|p\rangle}} = m^E(p).
\end{aligned}
$$

In special cases where for example there is exactly one connected component A_1 connecting both (non-isolated) endpoints e_1, e_2 of E, we can split up A_1 into three parts and then conclude as in case (A).

- When $E = \{e_0\}$ is not isolated and consists of a single point not in the relative boundary of a facet, then we enumerate the connected components A_j such that $A_1 \cap E = \{e_0\}$ but $A_j \cap E = \emptyset$ for all $j \geq 2$. The set A_1 is strongly separated from the isolated extreme points c_i (for all $i \in \{1, \dots, k\}$) and the other connected components A_j of $\mathcal{E}_C$ (for all $j \in \{2, \dots, l\}$), as can be seen in Figure 3.23.

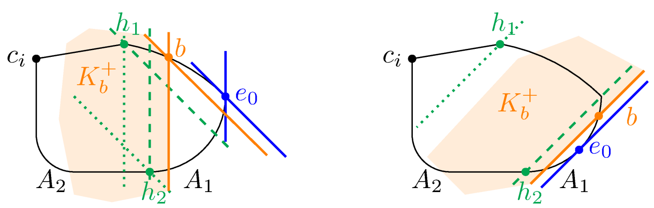


Figure 3.23: When $E = \{e_0\}$ is not isolated and lies in the relative interior of A_1, there is a point $b \in A_1$ such that all other isolated extreme points and connected components are contained in the cone K_b^+. LEFT: e_0 is a corner point with two tangent supporting hyperplanes defining the cones. RIGHT: e_0 is a smooth extreme point with only one supporting hyperplane.

Now we conclude in the same way as we did in the proof of Lemma 3.3.7(c), where $A_j \setminus B_\varepsilon(e_0)$ was separated from a subset of $B_\varepsilon(e_0)$. Here we take the two endpoints of A_1 as the points h_1, h_2 we constructed in the proof above. Let $b \neq e_0 \in A_1 \cap K_{h_1}^- \cap K_{h_2}^-$ be a point. Then all c_i and $x \in A_j$ (for $j \geq 2$ lie in the interior of the cone K_b^+ with apex b and bounding hyperplanes parallel to the hyperplanes tangent to ∂C at e_0 because e_0 is exposed. So for m big enough and a non-vanishing subset $S \subseteq A_1$ we have positive pairing

$$\langle b - x|z_m\rangle > 0 \qquad \forall x \in S,$$

whereas

$$
\begin{aligned}
\langle b - v|z_m\rangle &\xrightarrow{unif.} -\infty && \forall v \in A_j,\ j \geq 2 \\
\langle b - c_i|z_m\rangle &\longrightarrow -\infty && \forall c_i \text{ isolated extreme point.}
\end{aligned}
$$

Using this we compute:

$$
\begin{aligned}
m^C(z_m) &= \frac{\sum_{i=1}^k e^{-\langle c_i|z_m\rangle} c_i + \sum_{j=1}^l \int_{A_j} e^{-\langle v|z_m\rangle} v dv}{\sum_{i=1}^k e^{-\langle c_i|z_m\rangle} + \sum_{j=1}^l \int_{A_j} e^{-\langle v|z_m\rangle} dv} \\
&= \frac{\int_{A_1} \frac{e^{-\langle v|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle} dw} v dv}{\sum_i \frac{e^{\langle b-c_i|z_m\rangle}}{\int_{A_1} e^{\langle b-x|z_m\rangle} dx} + 1 + \sum_{j\geq 2} \frac{\int_{A_j} e^{\langle b-v|z_m\rangle} dv}{\int_{A_1} e^{\langle b-x|z_m\rangle} dx}} + \frac{\sum_i e^{\langle e_0-c_i|z_m\rangle} c_i + \sum_{j\geq 2} \int_{A_j} e^{\langle e_0-v|z_m\rangle} v dv}{\sum_i e^{\langle e_0-c_i|z_m\rangle} + \sum_j \int_{A_j} e^{\langle e_0-w|z_m\rangle} dw} \\
&\xrightarrow{Lem. 3.3.7} e_0 + 0 = m^E(p).
\end{aligned}
$$

Recall that in the first fraction we have the integral over a Dirac sequence in the numerator and both fractions in the denominator vanish by the choice of the point b.

- It remains the case where $E = \{e_0\}$ lies in the relative boundary of a facet $G \subsetneq \partial C$. We can not conclude as in the exposed case: let s denote the other extreme point of G. Then we can not find a point $b \neq e_0$ such that $s \in \operatorname{int} K_b^+$ as it was needed before.

  If $F^\circ = E = \{e_0\} \subsetneq G$, that is, the exposed dual F° of F is E, then the hyperplanes $H_{-1}^{\frac{z_m}{\|z_m\|}}$ will not converge to G and we conclude as is the previous case with shifted cones, but replace the hyperplane tangent to G by one that supports at e_0 but is not parallel to $T(G)$.

  If the exposed dual F° of F is G we will make a calculation similar to the one in the case were we had $E = \operatorname{conv}\{e_1, e_2\}$, but with the following difference: The sequence $(z_{m,F^\circ})_m$ is unbounded now and does not converge. Just as we did before (with F° playing the role of E, e_0 the one of e_1 and s the one of e_2), let S denote some part of the boundary ∂C on the other side of F° and define a translation map $\varphi : A_1 \to S$ by $a \mapsto a + t_a$ with $t_a \in T(F^\circ)$. See Figure 3.24 for a picture

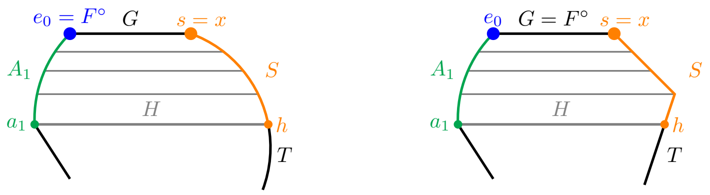


Figure 3.24: The decomposition of ∂C to construct the translating homeomorphism between A_1 and $S \subseteq \partial C$. LEFT: When $\{e_0\} = F^\circ$, we conclude as before by strong separation. RIGHT: e_0 nd $s \in S$ are both in the relative boundary of $F^\circ = G$.

Let $x \in S$ be the point closest to e_0 in the direction of F° and let S be small enough such that $\langle e_0 - x|z_{m,F^\circ}\rangle < -\delta < 0$ for some $\delta > 0$. Then we compute (like in case (A) above)

$$
\begin{aligned}
\left| \int_S \frac{e^{-\langle v|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle} dw} v dv \right| &= \left| \int_{A_1} \frac{e^{-\langle a+t_a|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle} dw} (a + t_a) da \right| \\
&= \left| \int_{A_1} \frac{e^{-\langle a|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle} dw} e^{-\langle t_a|z_{m,E}\rangle} (a + t_a) da \right| \\
&= \left| \int_{A_1} \delta_m(a) e^{-\langle t_a|z_{m,F^\circ}\rangle} (a + t_a) da \right| \\
&\leq e^{-\langle t_x|z_{m,F^\circ}\rangle} \left| \int_{A_1} \delta_m(a)(a + t_a) da \right| \longrightarrow 0,
\end{aligned}
$$

or similarly (like in case (B)):

$$\begin{aligned}\left|\frac{e^{-\langle e_2|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw}e_2\right| &= \left|\int_{A_1} \frac{e^{-\langle a|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw} e^{-\langle t_a|z_{m,E}\rangle}(a+t_a)\delta(a+t_a-e_2)da\right| \\ &\leq e^{-\langle t_x|z_{m,F^\circ}\rangle}\left|\int_{A_1} \frac{e^{-\langle a|z_m\rangle}}{\int_{A_1} e^{-\langle w|z_m\rangle}dw}(a+t_a)\delta(a+t_a-e_2)da\right| \\ &\longrightarrow 0.\end{aligned}$$

Therefore we get

$$m^C(z_m) \longrightarrow e_0 = m^E(p).$$

In total we have shown, that if our convex set C is two-dimensional, we always get the desired convergence $m^C(z_m) \longrightarrow m^E(p)$. □

The convergence result $m^C(z_m) \longrightarrow m^E(p)$ we proved in the previous lemma guarantees the surjectivity of the map m^C. This is shown in the next proposition. For notational reasons, we state the proposition for $\mathbb{R}^n$. As X is a finite-dimensional normed vector space, the same statement and proof also holds for X, see [Col12] for more details.

Proposition 3.3.9 *Let $C \subseteq (\mathbb{R}^n)^*$ be a compact convex set. Then the map $m^C : \mathbb{R}^n \longrightarrow \mathrm{int}(C)$ is bijective.* ∘

Proof. To show injectivity, define the function

$$\begin{aligned} f : \mathbb{R}^n &\longrightarrow \mathbb{R}, \\ x &\longmapsto \ln\left(\sum_{i=1}^{k} e^{-\langle c_i|x\rangle} + \sum_{j=1}^{l}\int_{A_j} e^{-\langle v|x\rangle}dv\right).\end{aligned}$$

Then $m^C = -\nabla f$ is the negative gradient of f. To prove injectivity of m^C on $\mathbb{R}^n$, we show that f is strictly convex and then use a description of strict convexity including the derivative. We define the function

$$\begin{aligned} g : \mathbb{R}^n &\longrightarrow \mathbb{R}, \\ x &\longmapsto \sum_{i=1}^{k} e^{-\langle c_i|x\rangle} + \sum_{j=1}^{l}\int_{A_j} e^{-\langle v|x\rangle}dv,\end{aligned}$$

such that $f(x) = -\ln(g(x))$. Recall Hölder's inequalities for any $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$:

$$\sum_{i=1}^{n}|a_i b_i| \leq \left(\sum_{i=1}^{n}|a_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}|b_i|^q\right)^{\frac{1}{q}} \tag{3.28}$$

and

$$\int_S |f(x)g(x)|dx \leq \left(\int_S |f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_S |g(x)|^q dx\right)^{\frac{1}{q}} \tag{3.29}$$

for all $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$, and $S \subseteq \mathbb{R}^n$ a measurable subset with f, g measurable functions on S. Take $p = \frac{1}{\lambda}$ and $q = \frac{1}{1-\lambda}$ for some $\lambda \in (0,1)$, then we have for all $x \neq y \in \mathbb{R}^n$:

$$\begin{aligned} g(\lambda x + (1-\lambda)y) &= \sum_{i=1}^{k} e^{-\langle c_i | \lambda x + (1-\lambda) y\rangle} + \sum_{j=1}^{l} \int_{A_j} e^{-\langle v | \lambda x + (1-\lambda) y\rangle} dv \\ &= \sum_i e^{-\lambda\langle c_i|x\rangle} e^{-(1-\lambda)\langle c_i|y\rangle} + \sum_j \int_{A_j} e^{-\lambda\langle v|x\rangle} e^{-(1-\lambda)\langle v|y\rangle} dv \\ &\overset{(3.29)}{\leq} \sum_i e^{-\lambda\langle c_i|x\rangle} e^{-(1-\lambda)\langle c_i|y\rangle} + \sum_j \left(\int_{A_j} e^{-\langle v|x\rangle} dv \right)^{\frac{1}{p}} \left(\int_{A_j} e^{-\langle v|y\rangle} dv \right)^{\frac{1}{q}}, \end{aligned}$$

by the choice of p and q. To apply Hölder's inequality again, we put the two sums together. We set

$$a_t := \begin{cases} e^{-\lambda\langle c_t|x\rangle} & \text{if } t = 1, \dots, k, \\ \left(\int_{A_{t-k}} e^{-\langle v|x\rangle} dv \right)^{\frac{1}{p}} & \text{if } t = k+1, \dots, k+l, \end{cases}$$

and

$$b_t := \begin{cases} e^{-(1-\lambda)\langle c_t|y\rangle} & \text{if } t = 1, \dots, k, \\ \left(\int_{A_{t-k}} e^{-\langle v|y\rangle} dv \right)^{\frac{1}{q}} & \text{if } t = k+1, \dots, k+l. \end{cases}$$

Then we get (using $p = \frac{1}{\lambda}$ and $q = \frac{1}{1-\lambda}$):

$$\begin{aligned} g(\lambda x + (1-\lambda)y) &\leq \sum_{t=1}^{k+l} |a_t b_t| \\ &\overset{(3.28)}{\leq} \left(\sum_t |a_t|^p \right)^{\frac{1}{p}} \left(\sum_t |b_t|^q \right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^{k} e^{-\langle c_i|x\rangle} + \sum_{j=1}^{l} \int_{A_j} e^{-\langle v|x\rangle} dv \right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} e^{-\langle c_i|y\rangle} + \sum_{j=1}^{l} \int_{A_j} e^{-\langle v|y\rangle} dv \right)^{\frac{1}{q}} \\ &= g(x)^{\lambda} g(y)^{1-\lambda}. \end{aligned}$$

Therefore

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \ln\left(g(\lambda x + (1-\lambda)y)\right) \\ &\leq \ln\left(g(x)^{\lambda} g(y)^{1-\lambda}\right) \\ &= \lambda f(x) + (1-\lambda) f(y) \end{aligned}$$

by the monotonicity of the logarithm. So f is convex. It is actually strictly convex as the following argument shows. In the discrete Hölder's inequality (3.28) it holds equality if and only if $a_i^p = \alpha b_i^q$ for all $i \in 1, \dots, n$ with $\alpha > 0$ as all our summands are positive. For the continuous inequality (3.29) it must be $|f|^p = \alpha |g|^q$ almost everywhere. Both conditions together yield to $e^{-\langle e|x\rangle} = \alpha e^{-\langle e|y\rangle}$ for all $e \in \mathcal{E}_C$ or, equivalently,

$$\langle e | y - x \rangle = \ln(\alpha) \quad \forall e \in \mathcal{E}_C.$$

As the extreme points $\mathcal{E}_C$ span an n-dimensional subset of $\mathbb{R}^n$, this can only be satisfied if $x = y$, which is a contradiction to our assumption. So we never have equality in Hölder's inequality which means that f is strictly convex.

For a function $s : D \longrightarrow \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ convex, strict convexity is equal to the generalized monotonicity condition[1]

$$s(x) > s(y) + \langle \nabla s(y) | \, x - y \rangle \tag{3.30}$$

for all $x, y \in D$ with $x \neq y$. So let $x, y \in \mathbb{R}^n$ with $x \neq y$, then applying Equation (3.30) twice we get

$$\begin{aligned} f(x) &> f(y) + \langle \nabla f(y) | \, x - y \rangle \\ &> f(x) + \langle \nabla f(x) | \, y - x \rangle + \langle \nabla f(y) | \, x - y \rangle . \end{aligned}$$

Consequently, as $\nabla f(x) = -m^C(x)$, we have $\langle m^C(x) - m^C(y) | x - y \rangle < 0$. Therefore $m^C(x) \neq m^C(y)$ for all $x \neq y \in \mathbb{R}^n$ and injectivity is shown.

We show that m^C is onto by showing that the derivative of m^C is a negative definite matrix and therefore invertible. Using the Inverse Mapping Theorem[2] we then prove surjectivity. Recall that

$$m^C(x) = \frac{\sum_{i=1}^k e^{-\langle c_i | x \rangle} c_i + \sum_{j=1}^l \int_{A_j} e^{-\langle v | x \rangle} v dv}{\sum_{i=1}^k e^{-\langle c_i | x \rangle} + \sum_{j=1}^l \int_{A_j} e^{-\langle w | x \rangle} dw}$$

for all $x \in \mathbb{R}^n$. We use the notation that an superscript denotes the corresponding component of the vector. Then with the abbreviation

$$M := \left(\sum_{i=1}^k e^{-\langle c_i | x \rangle} + \sum_{j=1}^l \int_{A_j} e^{-\langle w | x \rangle} dw \right)^2 > 0$$

and we compute

$$\begin{aligned} \frac{\partial (m^C)^\alpha}{\partial x^\beta}(x) &= \frac{1}{M} \Bigg[\left(\sum_{i=1}^k e^{-\langle c_i | x \rangle} + \sum_{\mu=1}^l \int_{A_\mu} e^{-\langle w | x \rangle} dw \right) \left(- \sum_{j=1}^k e^{-\langle c_j | x \rangle} c_j^\alpha c_j^\beta - \sum_{\nu=1}^l \int_{A_\nu} e^{-\langle v | x \rangle} v^\alpha v^\beta dv \right) \\ &\quad - \left(- \sum_{i=1}^k e^{-\langle c_i | x \rangle} c_i^\beta - \sum_{\mu=1}^l \int_{A_\mu} e^{-\langle w | x \rangle} w^\beta dw \right) \left(\sum_{j=1}^k e^{-\langle c_j | x \rangle} c_j^\alpha + \sum_{\nu=1}^l \int_{A_\nu} e^{-\langle v | x \rangle} v^\alpha dv \right) \Bigg] \\ &= \frac{-1}{M} \Bigg[\sum_{i,j=1}^k e^{-\langle c_i + c_j | x \rangle} (c_j^\alpha c_j^\beta - c_j^\alpha c_i^\beta) + \sum_{\mu,\nu=1}^l \int_{A_\mu} \int_{A_\nu} e^{-\langle v + w | x \rangle} (v^\alpha v^\beta - v^\alpha w^\beta) dv dw \\ &\quad + \sum_{i=1}^k \sum_{\mu=1}^l \int_{A_\mu} e^{-\langle c_i + v | x \rangle} (v^\alpha v^\beta + c_i^\alpha c_i^\beta - c_i^\alpha v^\beta - v^\alpha c_i^\beta) dv \Bigg] \\ &= \frac{-1}{M} \Bigg[\sum_{i<j} e^{-\langle c_i + c_j | x \rangle} (c_i^\alpha - c_j^\alpha)(c_i^\beta - c_j^\beta) + \sum_{\mu<\nu} \int_{A_\mu} \int_{A_\nu} e^{-\langle v + w | x \rangle} (v^\alpha - v^\beta)(w^\alpha - w^\beta) dv dw \\ &\quad + \sum_{i,\mu} \int_{A_\mu} e^{-\langle c_i + v | x \rangle} (v^\alpha - c_i^\alpha)(v^\beta - c_i^\beta) dv \Bigg]. \end{aligned}$$

Let $a = (a^1 \dots a^m) \in \mathbb{R}^n$ be some arbitrary vector. We want to show that the quadratic form defined by the derivative of m^C is negative definite:

$$a^T \frac{\partial m^C}{\partial x} a = (a^1, \dots, a^n) \begin{pmatrix} \frac{\partial (m^C)^1}{\partial x^1}(x) & \cdots & \frac{\partial (m^C)^1}{\partial x^n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial (m^C)^n}{\partial x^1}(x) & \cdots & \frac{\partial (m^C)^n}{\partial x^n}(x) \end{pmatrix} \begin{pmatrix} a^1 \\ \vdots \\ a^n \end{pmatrix} = \sum_{\alpha,\beta=1}^n a^\alpha a^\beta \frac{\partial (m^C)^\alpha}{\partial x^\beta}(x) \overset{!}{<} 0.$$

[1] See [Col12, Thm 7.5] for the general setting.

[2] See [Col12, Thm. 8.1] for a general normed space X.

As all three summands of the derivative have the same structure, we only show the calculation for the first summand. The other two go the same.

$$\begin{aligned}
\left[a^T \frac{\partial m^C}{\partial x} a\right]_{\text{1st summand}} &= \frac{-1}{M} \sum_{\alpha,\beta=1}^{n} a^\alpha a^\beta \left[\sum_{i<j} e^{-\langle c_i + c_j | x\rangle} (c_i^\alpha - c_j^\alpha)(c_i^\beta - c_j^\beta)\right] \\
&= \frac{-1}{M} \sum_{i<j} e^{-\langle c_i + c_j | x\rangle} \left[\sum_{\alpha,\beta} a^\alpha (c_i^\alpha - c_j^\alpha) a^\beta (c_i^\beta - c_j^\beta)\right] \\
&= \frac{-1}{M} \sum_{i<j} e^{-\langle c_i + c_j | x\rangle} \left(\sum_{\alpha} a^\alpha (c_i^\alpha - c_j^\alpha)\right)^2 \\
&= \frac{-1}{M} \sum_{i<j} e^{-\langle c_i + c_j | x\rangle} \left(\langle a | c_i - c_j\rangle\right)^2 < 0.
\end{aligned}$$

By the Inverse Mapping Theorem we know now that m^C is a local isomorphism and that its image is open in $\mathrm{int}(C)$. It remains to show that the image is also closed. Assume that the image is open but not closed, then there exists a point on the boundary of the image that lies in the interior of C, say $y \in \partial m^C(\mathbb{R}^n) \cap \mathrm{int}(C)$. So we can find a sequence $(y_m)_{m\in\mathbb{N}} \subseteq m^C(\mathbb{R}^n)$ converging to y. Let $(x_m)_{m\in\mathbb{N}} \subseteq \mathbb{R}^n$ be the corresponding sequence of preimages. Then there are two cases to distinguish. If $x_m \longrightarrow \infty$, we can find a subsequence also denoted by $(x_m)_m$, which fulfills all conditions of Lemma 3.2.4 with respect to C°. As we are only interested in limits, we can assume by Lemma 3.3.4 that C contains the origin. Let $F \subseteq C^\circ$ be the corresponding face and $E \subseteq C$ its dual. Then by Lemma 3.3.8 we know that $y_m = m^C(x_m) \longrightarrow m^E(p) \in \mathrm{relint}(E)$. As E is an extreme set in the boundary of C, this contradicts our assumption. It remains to prove the case where $(x_m)_{m\in\mathbb{N}}$ is contained inside a compactum. Then we can find a subsequence $(x_{m_k})_k$ converging to some point $x \in \mathbb{R}^n$. By continuity of m^C and uniqueness of limits we conclude $y = m^C(x)$ lies in the image of m^C and not in its boundary. As y was some arbitrary boundary point, the image of m^C is also closed in C and therefore the whole C. Overall we have shown that the map m^C is bijective. □

3.3.2 The Homeomorphism $\overline{X}^{hor} \simeq B^\circ$

The extreme sets of a dual unit ball are compact convex sets. So for each of them we have a homeomorphism from a (sub)space into the interior of the extreme set given by the map m^C we defined in the previous section (see Proposition 3.3.9). Putting all of them together respecting the combinatorics of B° will give us a homeomorphism from the horofunction compactification $\overline{X}^{hor}$ to the dual unit ball B°.

Theorem 3.3.10 *Let $(X, \|\cdot\|)$ be a finite-dimensional normed space with unit ball $B \subseteq X$ and dual unit ball B° belonging to one of the following three cases:*

I) B is polyhedral.

II) Every extreme set of B is an extreme point and all of them are smooth.

III) $\dim(B) = 2$ and $\mathcal{E}_{B^\circ}$ has finitely many connected components.

Then the horofunction compactification $\overline{X}^{hor}$ is homeomorphic to B° via the map

$$m : \overline{X}^{hor} \longrightarrow B^\circ, \qquad \begin{cases} x \in X & \longmapsto \quad m^{B^\circ}(x), \\ h_{E,p} \in \partial_{hor} X & \longmapsto \quad m^E(p). \end{cases}$$ ○

Proof. The proof is structured as follows. We first show that the map is well-defined and bijective, then we prove continuity. As both spaces involved are Hausdorff and compact, this is enough to conclude that the map is a homeomorphism.

The map m^C (recall Definition 3.3.1) maps an n-dimensional normed vector space X^n into the interior of a closed compact convex set $C \subseteq (X^n)^*$ of dimension n. We will use this map for with respect to the dual unit ball $C = B^\circ$ and X and also with respect to faces $C = E$ of B° and the space $T(E)^*$.

The map m is well-defined: Recall that in the previous discussion, we assumed the set C to have the same dimension as the surrounding space to obtain injectivity. A face $E \subseteq B^\circ$ lies not in the space $T(E)$ but in the affine space $\operatorname{aff}(E) \subseteq X^*$ of the same dimension. Nevertheless, with m^E as defined in Definition 3.3.1 on page 75, we have $m^E(p) \in \operatorname{int}(E) \subseteq \operatorname{aff}(E)$ by Lemma 3.3.4 and m^E is bijective.

So for each extreme set $E \subseteq B^\circ$ (including B° itself), $m^E(p)$ is a homeomorphism from $T(E)$ into the interior of E. Since the interiors of any two different extreme sets have empty intersection and $B^\circ = \dot{\bigcup}_{E \subseteq B^\circ \text{ extreme}} \operatorname{int}(E)$, we have bijectivity.

It remains to show that m is continuous on the boundary of the faces. For continuity from the interior of B° to the boundary, we first take a sequence $(z_m)_{m\in\mathbb{N}} \subseteq X$ that converges to a horofunction $h_{E,p}$. Then by Theorem 3.2.6, we know that $z_{m,E} \to p$ and $E = \operatorname{aff}\{D(x) | x \in X\} \cap B^\circ$ is an extreme set. With Lemma 3.3.8 we conclude that

$$m(z_m) = m^{B^\circ}(z_m) \longrightarrow m^E(p) = m(h_{E,p}).$$

as $m \longrightarrow \infty$.

For the continuity within the boundary, the argument is similar. The basic idea is to use the already shown continuity from the interior to the boundary on a lower dimensional subspace, where the unit ball is given by the dual of a projected and translated face of B° (similar as in the proof of Lemma 3.3.6). Let $h_{E_m,p_m} \longrightarrow h_{E,p}$ be a sequence of converging horofunctions. Now we have to consider the cases depending on the shape of B separately.

I) B is polyhedral If B and B° are polyhedral, then B° has only finitely many faces and we can go over to a subsequence $\left(h_{E_0,p_m}\right)_m$ with a fixed face E_0 of B° and it holds $h_{E_0,p_m} \longrightarrow h_{E,p}$. Let $\widetilde{E_0} := E_0 + t$ denote the projection of E_0 to $T(E)$ with $t \in X^*$ such that $\widetilde{E_0}$ contains the origin in its interior. By Lemma 2.6.8 we get

$$h_{\widetilde{E_0},p_m} = h_{E_0,p_m} + \langle t|\cdot\rangle \longrightarrow h_{E,p} + \langle t|\cdot\rangle = h_{E+t,p}.$$

The set $\widetilde{E_0} \subseteq T(E_0) \subseteq X^*$ is a polytope containing the origin as an interior point and can be taken as the dual unit ball for a norm on $T(E)^*$ with unit ball $B_{E_0} := (\widetilde{E_0})^\circ \subseteq T(E_0)^* \subseteq X$. So for some $y \in T(E_0)^*$ we have by Remark 3.1.11

$$\psi_{p_m}^{B_{E_0}}(y) = h_{\widetilde{E_0},p_m}(y) \longrightarrow h_{E+t,p}(y).$$

This tells us that $E + t$ is a face of $\widetilde{E_0} = E_0 + t$, so E is a face of E_0. By the convergence shown above we know that

$$m^{\widetilde{E_0}}(p_m) \longrightarrow m^{E+t}(p)$$

which is (by Lemma 3.3.4) equivalent to the convergence

$$m^{E_0}(p_m) \longrightarrow m^E(p),$$

as it was to show. As there are only finitely many subsequences each of them consisting of the horofunctions associated to a different face of B°, the convergence follows.

II) B **and** B° **are smooth** When B and B° are both smooth, then any extreme set E of B° is an extreme point $E = \{e\}$ with

$$h_{E,p}(x) = \langle e|x\rangle\,,$$
$$m^E(p) = e$$

for any $x \in X$ where $p = 0$. A sequence of horofunctions $(h_{E_m,p_m})_{m\in\mathbb{N}}$ converging to a horofunction $h_{E,p}$ now is nothing else than the convergence $\langle e_m|x\rangle \longrightarrow \langle e|x\rangle$ for any $x \in X$ where $E_m = \{e_m\}$. So clearly $m^{E_m}(p) \longrightarrow m^E(p)$ as desired.

III) $\dim(X) = 2$ **and** B **is arbitrary** For the case where $\dim(B^\circ) = \dim(B) = 2$ recall that we assume B° to have only finitely many connected components of $\mathcal{E}_{B^\circ}$. So we can consider a subsequence of $\left(h_{E_m,p_m}\right)_m$ converging to $h_{E,p}$ such that either $E_m = E_0$ is a fixed facet of B° or $E_m = \{e_m\} \in A_0$ is a sequence of extreme points in one fixed connected component of $\mathcal{E}_{B^\circ}$. Then we conclude as either in the polyhedral or smooth case to see that $m^{E_m}(p_m) \longrightarrow m^E(p)$. Note that we were only considering the subsequence so far. But as this holds for any of the finitely many subsequences, the statement is shown. □

3.4 Sum of Norms

In this section we want to generalize the results from the previous section to a new norm. So far we only considered polyhedral or smooth norms in X and the two-dimensional case. Now we want to add another case IV) to the three cases investigated before.

Given two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X that belong to the three cases I) - III) considered so far, we we can define a third norm on X by

$$\|\cdot\|_3 := \|\cdot\|_1 + \|\cdot\|_2.$$

For each $i \in \{1, 2, 3\}$ denote by X_i the space X with norm $\|\cdot\|_i$ and by B_i and B_i° the unit and dual unit ball associated to $\|\cdot\|_i$. We assume that the set of extreme sets of B_3° is closed.

3.4.1 Horofunctions and Sum of Norms

We first describe the horofunction compactification of X_3, that is, we want to generalize Theorem 3.2.6. By Corollary 3.1.10 we know that

$$\partial_{hor}X_3 = \{h_{E,p} \mid E \subseteq B_3^\circ \text{ proper extreme, } p \in T(E)^*\}.$$

So the first thing we have to know is how the unit and dual unit ball of $X_3 = (X, \|\cdot\|_3)$ look like:

Lemma 3.4.1 *There holds:*

1) $B_3^\circ = B_1^\circ + B_2^\circ$.

2) $B_3 = \left(B_1^\circ + B_2^\circ\right)^\circ$.

where we take the Minkowski sum of the sets. ◦

Proof.

1) We start with the inclusion $\subseteq$. Let $v \in B_3 \subseteq X$ be a point of B_3, that is, there holds $\|v\|_3 = \|v\|_1 + \|v\|_2 \leq 1$. Let $\lambda := \|v\|_1 \in [0,1]$. Then $\|v\|_2 \leq 1-\lambda$ and therefore $v \in \lambda B_1$ and $v \in (1-\lambda)B_2$ by the definition of a norm with respect to a compact convex set as given in Equation (2.2) on page 18. We have to show that the dual pairing of v with any element of $B_1^\circ + B_2^\circ$ is bigger or equal than -1. Let $y = a+b \in B_1^\circ + B_2^\circ \subseteq X^*$ be arbitrary. Then we conclude
$$\langle y|v\rangle = \langle a|v\rangle + \langle b|v\rangle \geq -\lambda - (1-\lambda) = -1.$$
As y was arbitrary, we have $v \in (B_1^\circ + B_2^\circ)^\circ$. For the other inclusion $\supset$, let $v \in (B_1^\circ + B_2^\circ)^\circ \subseteq X$. Then $\langle x|v\rangle \geq -1$ for all $x \in B_1^\circ + B_2^\circ \subseteq X^*$. Equivalently,
$$\langle a|v\rangle + \langle b|v\rangle \geq -1 \qquad \forall a \in B_1^\circ,\ b \in B_2^\circ. \tag{3.31}$$
We have to show that $v \in B_3$, i.e. $\|v\|_3 \leq 1$. Assume $\lambda := \|v\|_1 > 1$. Then $v \notin B_1$ but $v \in \lambda B_1$. Set $v' = \frac{1}{\lambda}v \in \partial B_1$ with $\|v'\|_1 = 1$ and let $a \in B_1^\circ$ be such that $\langle a|v'\rangle = -1$. Then $\langle a|v\rangle = \lambda\langle a|v'\rangle = -\lambda < -1$. So for some $b \in B_2^\circ$ with $\langle b|v\rangle < 0$ it would be $\langle a|v\rangle + \langle b|v\rangle < -1$, which is a contradiction to Equation (3.31). Such a b always exists because B_2 and therefore also B_2° is a convex set containing the origin in its interior. Therefore $\lambda = \|v\|_1 \leq 1$.
Assume $\delta := \|v\|_2 > 1-\lambda$. Then $v \in \delta B_2$ and we define $v'' = \frac{1}{\delta}v \in \partial B_2$. Let $b \in B_2^\circ$ with $\langle b|v''\rangle = -1$. Then we would obtain $\langle a|v\rangle + \langle b|v\rangle = \lambda\langle a|v'\rangle + \delta\langle b|v''\rangle = -\lambda - \delta < -1$ in contradiction to Equation (3.31). Therefore $\|v\|_2 \leq 1-\lambda$ and in total we obtain
$$\|v\|_3 = \|v\|_1 + \|v\|_2 \leq \lambda + 1 - \lambda = 1,$$
so $v \in B_3$.

2) For the statement about the unit ball B_3 note that all sets B_1, B_2, B_1° and B_2° are convex sets with the origin in their interior. Therefore by the Bipolar Theorem 2.4.3
$$B_3^\circ = ((B_1^\circ + B_2^\circ)^\circ)^\circ = B_1^\circ + B_2^\circ. \qquad \square$$

The dual unit ball B_3 is the Minkowski sum of the two dual unit balls B_1 and B_2. So let us shortly recall from Section 2.5 what we know about Minkowski sums:

Let E be an extreme set of $B_3^\circ = B_1^\circ + B_2^\circ$. Then there are unique proper extreme sets $E_1 \subseteq B_1^\circ$ and $E_2 \subseteq B_2^\circ$ such that
$$E = E_1 + E_2$$
and it holds for $j = 1, 2$:
$$\dim E_j \leq \dim E \leq \dim E_1 + \dim E_2$$

Any extreme point e of $E = \operatorname{conv}(\mathcal{E}_E)$ can be decomposed uniquely into two extreme points $e_1 \in E_1, e_2 \in E_2$ and it holds
$$E_1 = \operatorname{conv}(\mathcal{E}_1), \qquad \text{and} \qquad E_2 = \operatorname{conv}(\mathcal{E}_2),$$
where $\mathcal{E}_1, \mathcal{E}_2$ are the sets of all these summands for all extreme points e of E. For more details see Proposition 2.5.12 on page 34.

Now we want to see how horofunctions and Minkowski sums of the dual unit ball interact. For $p \in T(E)^*$ let
$$p_1 := \operatorname{proj}_{E_1}(p), \qquad \text{and} \qquad p_2 := \operatorname{proj}_{E_2}(p)$$

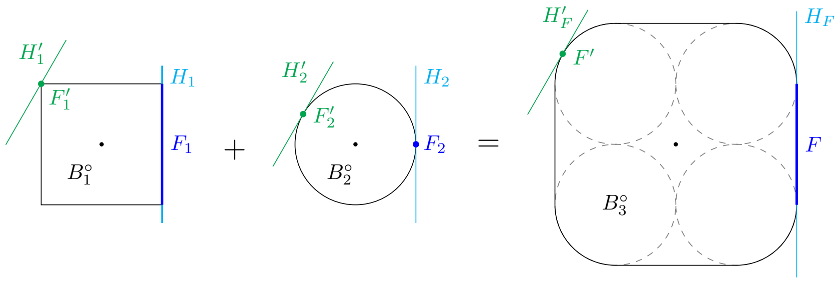


Figure 3.25: The Minkowski sum of a square and a circle gives a bigger square with rounded corners. The blue (green) face $F \subseteq B_3^\circ$ ($F' \subseteq B_3^\circ$ are exposed and uniquely the sum of exposed faces $F_1 \subseteq B_1^\circ$ ($F_1' \subseteq B_1^\circ$) and $F_2 \subseteq B_2^\circ$ ($F_2' \subseteq B_2^\circ$).

denote the projections of p to the subspaces $T(E_1^*)$ and $T(E_2)^*$, respectively.

As an example to illustrate the notation we give here again the same picture as before in the proof of Proposition 2.5.8:

Lemma 3.4.2 *It holds:*

$$\inf_{q\in E}\langle q|p-x\rangle = \inf_{r\in E_1}\langle r|p_1-x\rangle + \inf_{s\in E_2}\langle s|p_2-x\rangle. \qquad \circ$$

Proof. The statement is shown by a short calculation where we use that the extreme set E is a compact set, so that the infimum over E is attained at its extreme points $\mathcal{E}_E$ (see Lemma 2.6.4). In the second step we will use this fact in the other direction by taking the infimum not only over the extreme points but also over some interior points:

$$\begin{aligned}
\inf_{q\in E}\langle q|p-x\rangle &= \inf_{q\in\mathcal{E}_E}\langle q|p-x\rangle \qquad = \inf_{q\in\mathcal{E}_1+\mathcal{E}_2}\langle q|p-x\rangle \\
&= \inf_{\substack{r\in\mathcal{E}_1;\\ s\in\mathcal{E}_2}}\langle r+s|p-x\rangle \quad = \inf_{r\in\mathcal{E}_1}\langle r|p-x\rangle + \inf_{s\in\mathcal{E}_2}\langle s|p-x\rangle \\
&= \inf_{r\in E_1}\left\langle r\,\middle|\,p_{E_1}+p^{E_1}-x\right\rangle + \inf_{s\in E_2}\left\langle s\,\middle|\,p_{E_2}+p^{E_2}-x\right\rangle \\
&= \inf_{r\in E_1}\langle r|p_{E_1}-x\rangle + \inf_{s\in E_2}\langle s|p_{E_2}-x\rangle \\
&= \inf_{r\in E_1}\langle r|p_1-x\rangle + \inf_{s\in E_2}\langle s|p_2-x\rangle. \qquad \square
\end{aligned}$$

Corollary 3.4.3 With the notations above, we have for $h_{E,p} \in \partial_{hor}X_3$:

$$h_{E,p} = h_{E_1,p_1} + h_{E_2,p_2},$$

where $h_{E_1,p_1} \in \overline{X_1}^{hor}$ and $h_{E_2,p_2} \in \overline{X_2}^{hor}$ are the horofunctions associated to $E_1 = \mathrm{conv}(\mathcal{E}_1) \subseteq B_1^\circ$ and $E_2 = \mathrm{conv}(\mathcal{E}_2) \subseteq B_2^\circ$. $\circ$

Proof. The main calculation was already done in Lemma 3.4.2, it remains to put the results together:

$$\begin{aligned}
h_{E,p}(x) &= -\inf_{q\in E}\langle q|p-x\rangle + \inf_{q\in E}\langle q|p\rangle \\
&= -\inf_{r\in E_1}\langle r|p_1-x\rangle + \inf_{s\in E_2}\langle s|p_2-x\rangle + \inf_{r\in E_1}\langle r|p_1\rangle + \inf_{s\in E_2}\langle s|p_2\rangle \\
&= h_{E_1,p_1}(x) + h_{E_2,p_2}(x). \qquad \square
\end{aligned}$$

Remark 3.4.4 In the two lemmas above we do not need that the convex set $E = E_1 + E_2$ is a face of B_3°. The proof works the same for every convex sets $G_1 \subseteq B_1^\circ, G_2 \subseteq B_2^\circ$ with $G = G_1 + G_2$ and we get:

$$h_{G,q} = h_{G_1,q_1} + h_{G_2,q_2},$$

with $q \in T(G)^*$ and $q_1 = q_{G_1} \in T(G_1)^*$ as well as $q_2 = q_{G_2} \in T(G_2)^*$. $\circ$

As long as $E_1 + E_2$ is a face of B_3° and there is a point $p \in T(E)^*$ such that it holds for the projections $p_j = \mathrm{proj}_{E_j}(p) = p_{E_j} \in T(E_j)^*$ for $j = 1, 2$, we can construct a horofunction of X_3 as $h_{E_1,p_1} + h_{E_2,p_2}$. Therefore we have:

Corollary 3.4.5 The set of horofunctions of X_3 is determined by the horofunctions of X_1 and X_2:

$$\partial_{hor} X_3 = \left\{ h_{E_1,p_1} + h_{E_2,p_2} \;\middle|\; \begin{array}{ll} h_{E_j,p_j} \in \partial_{hor} X_j, & E_1 + E_2 \subseteq B_3^\circ \text{ proper face,} \\ p_j = p_{E_j} \in T(E_j)^* & \text{for some } p \in T(E)^* (j \in \{1,2\}) \end{array} \right\}. \qquad \circ$$

We now describe the convergence of sequences to horofunctions in the boundary of X equipped with $\|\cdot\|_3$, as we did in the last section for the horofunction compactification with a norm belonging to one of the three cases I) - III). For some $z \in X$ let ψ_z^j denote the map $\psi_z \in \widetilde{C}(X)$ (as defined in Equation (3.2) on page 40) with respect to the norm $\|\cdot\|_j$ (for $j \in \{1, 2, 3\}$). Then we compute for all $x \in X$:

$$\begin{aligned} \psi_z^3(x) &= \|z - x\|_3 - \|z\|_3 = \|z - x\|_1 + \|z - x\|_2 - \|z\|_1 - \|z\|_2 \\ &= \psi_z^1(x) + \psi_z^2(x) \end{aligned}$$

So also the converging behavior in X_3 is given by the convergence of sequences with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$. The following Lemma makes it even more obvious:

Lemma 3.4.6 *Let $(z_m)_{m\in\mathbb{N}} \subseteq X$ be an unbounded sequence. Then with $E = E_1 + E_2 \subseteq B_3^\circ$ extreme it holds:*

$$\psi_{z_m}^3 \longrightarrow h_{E,p} \in \partial_{hor} X_3 \quad \textit{if and only if} \quad \begin{cases} \psi_{z_m}^1 \longrightarrow h_{E,p} \in \partial_{hor} X_1 & \textit{and} \\ \psi_{z_m}^2 \longrightarrow h_{E,p} \in \partial_{hor} X_2. \end{cases} \qquad \circ$$

Proof. We start with a converging sequence $(z_m)_{m\in\mathbb{N}} \subseteq X$ such that $\psi_{z_m}^3 \longrightarrow h_{E,p} \in \overline{X_3}^{hor}$. As $E = E_1 + E_2$ with $p_1 = p_{E_1}$ and $p_2 = p_{E_2}$ we know by Corollary 3.4.3 that

$$h_{E,p} = h_{E_1,p_1} + h_{E_2,p_2}.$$

By assumption,

$$\psi_{z_m}^3 = \psi_{z_m}^1 + \psi_{z_m}^2 \longrightarrow h_{E,p} = h_{E_1,p_1} + h_{E_2,p_2}.$$

Let $h_{G_1,q_1} \in \overline{X_1}^{hor}$ be the limit of $\left(\psi_{z_m}^1\right)_m$ with respect to $\|\cdot\|_1$, that is, $G_1 \subseteq B_1^\circ$ is a proper extreme set and $q_1 \in T(G_1)^*$. Similarly denote by $h_{G_2,q_2} \in \overline{X_2}^{hor}$ the limit of $\left(\psi_{z_m}^2\right)_m$ with respect to $\|\cdot\|_2$. Then we obtain by Remark 3.4.4

$$\psi_{z_m}^3 \xrightarrow{\|\cdot\|_3} h_{G_1,q_1} + h_{G_2,q_2} = h_{G_1+G_2,q} \overset{!}{=} h_{E,p},$$

where $q \in T(G_1 + G_2)^*$ such that $q_1 = q_{G_1}$ and $q_2 = q_{G_2}$. Therefore again by Remark 3.4.4 $E = G_1 + G_2$ with $G_1 \subseteq B_1^\circ$ and $G_2 \subseteq B_2^\circ$ is an extreme set and $p = q \in T(E)^*$. As we also have $E = E_1 + E_2$ we conclude by the uniqueness of the face decomposition in the Minkowski sum:

$$E_1 = G_1 \text{ and } E_2 = G_2. \tag{3.32}$$

Using this we immediately get

$$p_1 = p_{E_1} = q_{G_1} = q_1 \qquad \text{and} \qquad p_2 = p_{E_2} = q_{G_2} = q_2.$$

Therefore we have shown that

$$\psi^1_{z_m} \longrightarrow h_{E_1,p_1} \qquad \text{and} \qquad \psi^2_{z_m} \longrightarrow h_{E_2,p_2}.$$

Now assume we have the convergence $\psi^1_{z_m} \longrightarrow h_{E,p} \in \partial_{hor}X_1$ and $\psi^2_{z_m} \longrightarrow h_{E,p} \in \partial_{hor}X_2$ such that $E = E_1 + E_2 \subseteq B^\circ_3$ is extreme. Then the convergence follows directly from Corollary 3.4.3. Note that we can always find a unique $p \in T(E)^*$ with projections $p_1 \in T(E_1)^*$ and $p_2 \in T(E_2)^*$ because $E = E_1 + E_2$. □

3.4.2 The Compactification is Homeomorphic to the Dual Unit Ball

Finally we want to show that also in case IV) the horofunction compactification is again homeomorphic to the dual unit ball. As the Minkowski sum of two polytopes is again a polytope and the Minkowski sum of two smooth convex sets is again smooth, the only new case to consider is the sum of two norms where one of them is polyhedral and the other one is smooth:

Theorem 3.4.7 *Let X be a finite-dimensional normed space equipped with the norm*

$$\|\cdot\|_3 = \|\cdot\|_1 + \|\cdot\|_2,$$

where $\|\cdot\|_1$ is a polyhedral norm and $\|\cdot\|_2$ is smooth. Denote by B°_3 the dual unit ball of $\|\cdot\|_3$. Then the horofunction compactification of X with respect to $\|\cdot\|_3$ is homeomorphic to B°_3:

$$\overline{X_3}^{hor} \simeq B^\circ_3. \qquad \circ$$

Before we give the proof, let us look at the face structure of B°_3. In Figure 3.26 the notation is illustrated for the 1−norm and the Euclidean norm in $\mathbb{R}^2$.

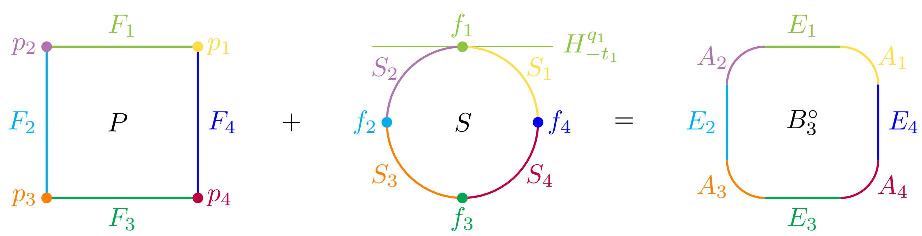


Figure 3.26: The decomposition of faces for the Minkowski sum of a square and a circle. The Minkowski sum on the right is scaled by $\frac{1}{2}$.

We know that B°_3 is the Minkowski sum of a polytope P and a smooth ball S. Let $p_1, \dots, p_l$ be the vertices of P and $F_1, \dots, F_k$ its facets. Then each facet $F_i \subseteq P \subseteq X^*$ determines a parallel hyperplane $H^{q_i}_{-1}$, where $q_i \in P^\circ \subseteq X$ is the vertex dual to F_i. Let $t_i := \max\{s > 0 \mid H^{q_i}_{-s} \cap S \neq \emptyset\}$ be the parameter such that the intersection of the hyperplane $H^{q_i}_{t_i}$ and S consists of a single point. Set

$$\{f_i\} := H^{q_j}_{-t_i} \cap S$$

to be this intersection point. So for each $i \in \{1, \ldots, k\}$ we obtain a point $f_i \in \partial S$ and these k points $f_1, \ldots, f_k$ determine a decomposition of the boundary in the following way: to each vertex p_j of P (for $j = 1, \ldots, l$) there corresponds a facet $G_j \subseteq P^\circ$. We set

$$S_j := \bigcup_{g \in \overline{G_j}} \left\{ s \in S \mid s = \inf_{r \in S} \langle g | r \rangle \right\}.$$

So each component S_j corresponds to a vertex p_j of P. By construction, if p_j is a vertex of the facets $F_{j_1}, \ldots, F_{j_r}$ of P, then S_j has corner points $f_{j_1}, \ldots, f_{j_r} \in S$. Two components S_i and S_j only intersect in their common boundary and the points in the boundary of each S_j are in relation to lower-dimensional faces of P: the point $s \in S_{i_1} \cap \ldots \cap S_{i_r}$ corresponds to the face spanned by $p_{i_1}, \ldots, p_{i_r}$. So especially if $F_i \subseteq P$ is a facet with vertices $p_{i_1}, \ldots, p_{i_s}$, then f_i is a corner point of all corresponding parts and

$$f_i = S_{i_1} \cap \cdots \cap S_{i_s}.$$

Coming from the decomposition just described, the facets E_i and smooth parts A_j of B_3° have the following form:

$$\begin{array}{rclll} E_i & = & F_i + f_i & \text{for } i = 1, \ldots, k & \text{facets of } B_3^\circ \\ A_j & = & p_j + S_j & \text{for } j = 1, \ldots, l & \text{smooth parts of } B_3^\circ. \end{array} \tag{3.33}$$

Note that B_3° can not have isolated extreme points: any extreme point of B_3° is uniquely the sum of two extreme points, one of P, which is isolated and one of S, which lies in a smooth part. Thereby also the extreme point of B_3° is at least in the boundary of a smooth part and therefore not isolated.

The homeomorphism between $\overline{X}_3^{hor}$ and B_3° will be given by the same map as before in Theorem 3.3.10:

$$m : \overline{X}_3^{hor} \longrightarrow B_3^\circ, \qquad \begin{cases} x \in X & \longmapsto & m^{B^\circ}(x), \\ h_{E,p} \in \partial_{hor} X & \longmapsto & m^E(p), \end{cases}$$

where for an n-dimensional compact convex set $C \subseteq X^*$ we have (as defined in Definition 3.3.1):

$$\begin{aligned} m^C : X &\longrightarrow \operatorname{int}(C) \\ x \longmapsto m^C(x) &= \widetilde{\int_{\partial C}} \varphi_x(v) v dv. \end{aligned}$$

As B_3° has no isolated extreme points, the expression simplifies to

$$m^C(x) = \frac{\sum_{j=1}^{l} \int_{A_j} e^{-\langle v|x \rangle} v dv}{\sum_{j=1}^{l} \int_{A_j} e^{-\langle w|x \rangle} dw}.$$

We have already shown the homeomorphism between the compactification and the dual unit ball for polyhedral and smooth unit and dual unit balls. This will help us now as we can often (but not always) use some of the results given in Section 3.3.

Let us state and proof the most important ingredients for the proof first. The lemmas we have to adapt are the convergence of fractions of integrals over smooth parts in the boundary (Lemma 3.3.7) and the continuity of m^C for a sequence in the interior (Lemma 3.3.8).

Let $(z_m)_{m \in \mathbb{N}} \subseteq X$ be an unbounded sequence such that

$$\frac{z_m}{\|z_m\|} \xrightarrow{\subseteq} F$$

for some extreme set $F \subseteq B_3$. Then the sequence of directions also converges with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$. As in the previous section let $E = \operatorname{aff}\{D(x)|x \in X_3\} \cap B_3^\circ \subseteq F^\circ$ be an extreme set (see Lemma 3.4.6) and assume that the projection converges to a point: $p := \lim_{m\to\infty} z_{m,E}$.

As we did in the proof of Lemma 3.3.8 we want to replace the integral over a smooth part $A_j \subseteq B_3^\circ$ using a translation φ (as defined in Equation (3.25)). According to the decomposition $B_3^\circ = P + S$ let $G \subseteq P$ be a face and $s_0 \in S$ be an extreme point such that

$$E = G + s_0. \tag{3.34}$$

Let $\mathcal{E}_G$ be the extreme points of G and $I_G \subseteq \{1, \dots, l\}$ be those indices belonging to G such that $G = \operatorname{conv}\{p_j \mid j \in I_G\}$. Without loss of generality let the parts S_j be enumerated in a way such that $1 \in I_G$. Then we find a small subset $U_1 \subseteq S_1$ with $s_0 \in U_1$ such that for each $j \in I_G$ we have a translating map $\varphi_j : U_1 \to\subseteq S_j$ given by $u \mapsto u + t_u$ for some $t_u \in T(G) = T(E)$. The subset $U_1 \subseteq S_1$ and the map φ_j can be found by the convexity of S and because the hyperplane tangent to S at s_0 is parallel to $T(E) = T(G)$. Translations in the directions of $T(E)$ are enough because the parts S_j with $j \in I_G$ correspond to the vertices of G. Note that $\varphi_j(s_0) = s_0$ which means that $u_{s_0} = 0$, because $s_0 \in \bigcap_{j\in I_G} S_j$.

Lemma 3.4.8 *It holds:*

$$\begin{aligned}
&\int_{S_j} \frac{e^{-\langle v|z_m\rangle}}{\int_{U_1} e^{-\langle w|z_m\rangle}dw} f(v)dv \longrightarrow f(s_0) && \forall j \in I_G,\\
&\int_{S_j} \frac{e^{-\langle v|z_m\rangle}}{\int_{U_1} e^{-\langle w|z_m\rangle}dw} f(v)dv \longrightarrow 0 && \forall j \notin I_G.
\end{aligned}$$

for all functions $f : X^* \to \mathbb{R}$ or $f : X^* \to X^*$ that are bounded on S. ◦

Proof. Let $V_1 := S_1 \backslash U_1$ be the complementary subset of U_1 in S_1 not containing s_0. For all $j \in I_G$ denote by $U_j = \varphi_j(U_1) \subseteq S_j$ the image of U_1 in S_j and $V_j := S_j \backslash U_j$ the corresponding complement. Note that the boundary of S is smooth and only contains extreme points, so we can use the result of Lemma 3.3.7(c) for U_j. A similar calculation as in Equation (3.27) in the poof of Lemma 3.3.8 yields for all $j \in I_G$:

$$\begin{aligned}
\int_{S_j} \frac{e^{-\langle v|z_m\rangle}}{\int_{U_1} e^{-\langle w|z_m\rangle}dw} f(v)dv &= \int_{U_j} \frac{e^{-\langle v|z_m\rangle}}{\int_{U_1} e^{-\langle w|z_m\rangle}dw} f(v)dv + \int_{V_j} \frac{e^{-\langle v|z_m\rangle}}{\int_{U_1} e^{-\langle w|z_m\rangle}dw} f(v)dv\\
&= \int_{U_1} \frac{e^{-\langle \varphi(u)|z_m\rangle}}{\int_{U_1} e^{-\langle w|z_m\rangle}dw} f(\varphi(u))du + \int_{V_j} \frac{e^{-\langle s_0 - v|z_m\rangle}}{\int_{U_1} e^{-\langle s_0 - w|z_m\rangle}dw} f(v)dv\\
&= \int_{U_1} \frac{e^{-\langle u|z_m\rangle}}{\int_{U_1} e^{-\langle w|z_m\rangle}dw} e^{-\langle t_u|z_m\rangle} f(\varphi(u))du + \int_{V_j} \frac{e^{-\langle s_0 - v|z_m\rangle}}{\int_{U_1} e^{-\langle s_0 - w|z_m\rangle}dw} f(v)dv\\
&= \int_{U_1} \delta_m(u) e^{-\left\langle t_u|z_{m,E}\right\rangle} f(\varphi(u))du + \int_{V_j} \frac{e^{-\langle s_0 - v|z_m\rangle}}{\int_{U_1} e^{-\langle s_0 - w|z_m\rangle}dw} f(v)dv\\
&\longrightarrow e^{-\left\langle u_{s_0}|p\right\rangle} f(s_0) = f(s_0).
\end{aligned}$$

The second equation we want to show has to be valid for all $j \notin I_G$. These are exactly those indices such that the corresponding smooth parts $S_j \subseteq \partial S$ do not contain the point s_0. Therefore we have a strict separation between s_0 and S_j and we can directly adapt the proof of Lemma 3.3.7 (b) and get the desired convergence. □

The next thing we have to show is the compatibility of m^C with the convergence of interior sequences to horofunctions, that is, the analogue of Lemma 3.3.8.

Lemma 3.4.9 *With the notations given on page 100 it holds:*

$$m^C(z_m) \longrightarrow m^E(p). \qquad \circ$$

Proof. We will use the decomposition of connected components $A_j = p_j + S_j \subseteq P + S$ of extreme points as given in Equation (3.33) to compute:

$$\begin{aligned}
m^C(z_m) &= \frac{\sum_{j=1}^{l} \int_{A_j} e^{-\langle v|z_m\rangle} v dv}{\sum_{j=1}^{l} \int_{A_j} e^{-\langle w|z_m\rangle} dw} = \frac{\sum_j \int_{S_j} e^{-\langle v+p_j|z_m\rangle}(v+p_j)dv}{\sum_j \int_{S_j} e^{-\langle w+p_j|z_m\rangle} dw} \\
&= \frac{\sum_j e^{-\langle p_j|z_m\rangle} \int_{S_j} e^{-\langle v|z_m\rangle}(v+p_j)dv}{\sum_j e^{-\langle p_j|z_m\rangle} \int_{S_j} e^{-\langle w|z_m\rangle} dw} \\
&= \frac{\sum_{j\in I_G} e^{\langle p_1-p_j|z_m\rangle} \int_{S_j} e^{-\langle v|z_m\rangle}(v+p_j)dv}{\sum_{j\in I_G} e^{\langle p_1-p_j|z_m\rangle} \int_{S_j} e^{-\langle w|z_m\rangle} dw + \sum_{j\notin I_G} e^{\langle p_1-p_j|z_m\rangle} \int_{S_j} e^{-\langle w|z_m\rangle} dw} \qquad (3.35) \\
&+ \frac{\sum_{j\notin I_G} e^{\langle p_1-p_j|z_m\rangle} \int_{S_j} e^{\langle s_0-v|z_m\rangle}(v+p_j)dv}{\sum_{j\in I_G} e^{\langle p_1-p_j|z_m\rangle} \int_{S_j} e^{\langle s_0-w|z_m\rangle} dw + \sum_{j\notin I_G} e^{\langle p_1-p_j|z_m\rangle} \int_{S_j} e^{\langle s_0-w|z_m\rangle} dw}. \qquad (3.36)
\end{aligned}$$

By Lemma 3.4.8 the second fraction (3.36) goes to 0. For the first fraction we compute:

$$\begin{aligned}
(3.35) &= \frac{\sum_{j\in I_G} e^{\langle p_1-p_j|z_m\rangle} \int_{S_j} e^{-\langle v|z_m\rangle}(v+p_j)dv}{\sum_{j\in I_G} e^{\langle p_1-p_j|z_m\rangle} \int_{S_j} e^{-\langle w|z_m\rangle} dw + \sum_{j\notin I_G} e^{\langle p_1-p_j|z_m\rangle} \int_{S_j} e^{-\langle w|z_m\rangle} dw} \\
&= \frac{\int_{S_1} e^{-\langle v|z_m\rangle}(v+p_1)dv + \sum_{j\in I_G\setminus\{1\}} e^{\langle p_1-p_j|z_m\rangle} \int_{S_j} e^{-\langle v|z_m\rangle}(v+p_j)dv}{\int_{U_1} e^{-\langle v|z_m\rangle}dv + \int_{V_1} e^{-\langle v|z_m\rangle}dv + \sum_{j\in I_G\setminus\{1\}} e^{\langle p_1-p_j|z_m\rangle} \int_{S_j} e^{-\langle w|z_m\rangle} dw + \sum_{j\notin I_G} e^{\langle p_1-p_j|z_m\rangle} \int_{S_j} e^{-\langle w|z_m\rangle} dw} \\
&= \frac{\int_{S_1} \frac{e^{-\langle v|z_m\rangle}}{\int_{U_1} e^{-\langle w|z_m\rangle}dw}(v+p_1)dv + \sum_{j\in I_G\setminus\{1\}} e^{\langle p_1-p_j|z_{m,E}\rangle} \int_{S_j} \frac{e^{-\langle v|z_m\rangle}}{\int_{U_1} e^{-\langle w|z_m\rangle}dw}(v+p_j)dv}{1 + \int_{V_1} \frac{e^{-\langle v|z_m\rangle}}{\int_{U_1} e^{-\langle w|z_m\rangle}dw}dv + \sum_{j\in I_G\setminus\{1\}} e^{\langle p_1-p_j|z_{m,E}\rangle} \int_{S_j} \frac{e^{-\langle v|z_m\rangle}}{\int_{U_1} e^{-\langle w|z_m\rangle}dw}dv + \sum_{j\notin I_G} e^{\langle p_1-p_j|z_{m,E}\rangle} \int_{S_j} \frac{e^{\langle s_0-v|z_m\rangle}dv}{\int_{U_1} e^{\langle s_0-w|z_m\rangle}dw}} \\
&\longrightarrow \frac{s_0 + p_1 + \sum_{I\in I_G\setminus\{1\}} e^{\langle p_1-p_j|p\rangle}(s_0+p_j)}{1 + \sum_{I\in I_G\setminus\{1\}} e^{\langle p_1-p_j|p\rangle}} \\
&= \frac{\sum_{I\in I_G} e^{-\langle p_j+s_0|p\rangle}(p_j+s_0)}{\sum_{I\in I_G} e^{\langle p_j+s_0|p\rangle}} = m^E(p),
\end{aligned}$$

because the vertices of $E = G + s_0$ are given by $\mathcal{E}_E = \{p_j + s_0 \mid j \in I_G\}$. □

Let us now proof Theorem 3.4.7 using the two lemmas above.

Proof of Theorem 3.4.7. We want to show that the map m is a homeomorphism between $\overline{X}_3^{hor}$ and B_3°. Bijectivity follows directly from Lemma 3.3.9 because the proof given there was for a general compact convex set C independent of its shape. In the last part of the proof we have to use Lemma 3.4.9 to show that the image $m(\mathbb{R}^n) \subseteq \mathrm{int}(C)$ is closed.

For continuity from the interior we use again Lemma 3.4.9 to see that when $(z_m)_{m\in\mathbb{N}}$ is a sequence converging to a horofunction $h_{E,p}$, then

$$m(z_m) = m^{B_3^\circ}(z_m) \longrightarrow m^E(p) = m(h_{E,p}).$$

Now let $\left(h_{E_m,p_m}\right)_m \subseteq \partial_{hor}X_3$ be a sequence of horofunctions converging to a horofunction $h_{E,p} \in \partial_{hor}X_3$. The boundary of B_3° has infinitely many extreme points but we can decompose it into finitely many parts each consisting of extreme sets of the same type. To do so, denote by $A_i^r \subseteq \partial B_3^\circ$ for $i = 1, \dots l_r$ a connected component of r-dimensional extreme sets. In Figure 3.27 (page 105) on the right, each colored part is one connected component: A_i^0 is blue, A_i^1 is green and A_i^2 is orange. We already had $A_i^0 = A_j$ the connected components of extreme points before. For $r = n-1$ each connected component consists of a single facet, so $l_{n-1} = k$. Then

$$\partial B_3^\circ = \bigcup_{r=0}^{n-1} \bigcup_{i=1}^{l_r} A_i^r.$$

By the finiteness of the number of these components, $\left(h_{E_m,p_m}\right)_m$ has a subsequence such that the sequence $(E_m)_m$ of associated faces lies in one component A_i^r. If $r = 0$ or $r = n-1$ we know by Theorem 3.3.10 that $m(h_{E_m,p_m}) \to m(h_{E,p})$. So assume $0 < r < n-1$. Then each E_m is of the form $E_m = G + s_m$ where $G = \operatorname{conv}\{p_j \mid j \in I_G\} \subseteq P$ is an r-dimensional face and $s_m \in S$ are extreme points. Therefore we have

$$m(h_{E_m,p_m}) = \frac{\sum_{j\in I_G} e^{-\langle p_j+s_m|p_m\rangle}(p_j+s_m)}{\sum_{j\in I_G} e^{-\langle p_j+s_m|p_m\rangle}}.$$

Let $E = G' + s$ be the decomposition of E where $G' \subseteq G$ is a face and $s \in S$ with $s_m \to s$. If $G' = G$ then $p_m \in T(E_m)^* = T(E)^* = T(G)^*$ converges to $p \in T(E)^*$, and we get

$$m(h_{E_m,p_m}) = \frac{\sum_{j\in I_G} e^{-\langle p_j+s_m|p_m\rangle}(p_j+s_m)}{\sum_{j\in I_G} e^{-\langle p_j+s_m|p_m\rangle}} \longrightarrow \frac{\sum_{j\in I_G} e^{-\langle p_j+s|p\rangle}(p_j+s)}{\sum_{j\in I_G} e^{-\langle p_j+s|p\rangle}} = m(h_{E,p}).$$

If $G' \subsetneq G$ is a proper face then $p_m \in T(G)^*$ is unbounded by Lemma 3.2.2. As all sets E_m are parallel to G (and therefore of course also to each other) we conclude in the same way as in the polyhedral case of Theorem 3.3.10 by considering the unbounded sequence $(p_m)_m$ in the subspace $T(G)^*$. Thereby we get

$$m(h_{E_m,p_m}) \longrightarrow m(h_{E,p}).$$

So we have shown bijectivity and continuity of the map m. Since our spaces considered are Hausdorff, we conclude that m is a homeomorphism. □

Let us put all previous main results of this chapter in a common theorem:

Theorem 3.4.10 *Let X be a finite-dimensional normed vector space. Let $B \subseteq X$ be a unit ball and $B^\circ \subseteq X^*$ its dual such that they belong to one of the following cases:*

I) The unit ball is polyhedral.

II) The unit and the dual unit ball have smooth boundaries.

III) The space X is two-dimensional and $\mathcal{E}_{B^\circ}$ has finitely many connected components.

IV) The dual unit ball B° is the Minkowski sum of a polyhedral and a smooth dual unit ball with only extreme points.

Let $(z_m)_{m\in\mathbb{N}}$ be an unbounded sequence in X. Then the sequence $(\psi_{z_m})_{m\in\mathbb{N}}$ converges to a horofunction $h_{E',p}$ associated to an extreme set $E' \subseteq B^\circ$ and a point $p \in T(E')^*$ if and only if the following conditions are satisfied:

1) $E := \operatorname{aff}\{D(x) \mid x \in X\} \cap B^\circ$ is extreme.

2) The projection $(z_{m,E})_{m\in\mathbb{N}}$ of $(z_m)_{m\in\mathbb{N}}$ to $T(E)^*$ converges.

Further, $E' = E$ and $p = \lim_{m\to\infty} z_{m,E}$.

Additionally the horofunction compactification of X is homeomorphic to the dual unit ball B°:

$$\overline{X}^{hor} \simeq B^\circ. \qquad \circ$$

3.4.3 Refinement of Compactifications

Given two norms on a space X we constructed a third norm on X as the sum of the other two in the previous section. A question that comes up now is how to compare the associated compactifications of the same space equipped with different norms.

Definition 3.4.11 Let $\overline{X_1}, \overline{X_2}$ and $\overline{X_3}$ be three Hausdorff compactifications of a space X. Then $\overline{X_3}$ is called a *refinement* of $\overline{X_1}$ if the identity map on X extends to a continuous map $\overline{X_3} \longrightarrow \overline{X_1}$.
If $\overline{X_3}$ is a refinement of both $\overline{X_1}$ and $\overline{X_2}$, then it is called a *common refinement*. $\circ$

Lemma 3.4.12 ([BJ06, Prop. I.16.2]) *Given two compactifications $\overline{X_1}$ and $\overline{X_2}$, then they admit a unique least common refinement denoted by $\overline{X_1} \vee \overline{X_2}$.* $\circ$

Theorem 3.4.13 *Let two norms $\|\cdot\|_1, \|\cdot\|_2$ on a finite-dimensional normed vector space X belong to one of the three cases I) - III). Let a third norm on X be defined as*

$$\|\cdot\|_3 := \|\cdot\|_1 + \|\cdot\|_2.$$

For $j = 1, \ldots, 3$ set $X_j := (X, \|\cdot\|_j)$.
Then $\overline{X_3}^{hor}$ is the least common refinement of $\overline{X_1}^{hor}$ and $\overline{X_2}^{hor}$. $\circ$

Proof. To see that $\overline{X_3}^{hor}$ is a common refinement of $\overline{X_1}^{hor}$ and $\overline{X_2}^{hor}$ we have to show that the identity map on X extends to continuous maps $f_1 : \overline{X_3}^{hor} \longrightarrow \overline{X_1}^{hor}$ and $f_2 : \overline{X_3}^{hor} \longrightarrow \overline{X_2}^{hor}$ of the compactifications.
Using the previous notations $E = E_1 + E_2$, $p_1 = \operatorname{proj}_{E_1}(p)$ and $p_2 = \operatorname{proj}_{E_2}(p)$, we define the maps in the following way for j = 1,2:

$$f_j : \overline{X_3}^{hor} \longrightarrow \overline{X_j}^{hor}, \qquad \begin{cases} x \in X & \longmapsto \quad x \in X, \\ h_{E,p} \in \partial_{hor} X_3 & \longmapsto \quad h_{E_j,p_j} \in \partial_{hor} X_j. \end{cases}$$

As the Minkowski sum is symmetric, we will show everything only for f_1 and $\overline{X_1}^{hor}$. The proof for f_2 and $\overline{X_2}^{hor}$ goes the same.
By construction, $E_1 \subseteq B_1$ is an extreme set and $p_1 \in T(E_1)^*$, where $T(E_1)^*$ has the same dimension as E_1. Therefore the map f_1 is well-defined.

We have to show continuity both for a sequence $(z_m)_m \subseteq X$ of interior points and a sequence $(h_{E_m,p_m})_m \subseteq \partial_{hor} X_3$ of horofunctions. The convergence from the interior is exactly the content of Lemma 3.4.6.

For the convergence within the boundary, let $\left(h_{E_m,p_m}\right)_m \subseteq \partial_{hor} X_3$ be a sequence in the horofunction boundary with respect to $\|\cdot\|_3$ such that $h_{E_m,p_m} \longrightarrow h_{E,p} \in \partial_{hor} X_3$.

For each extreme set $E_m, E \subseteq B_3^\circ$ let $E_m = E_{m,1} + E_{m,2}$ and $E = E_1 + E_2$ be its unique decompositions with $E_{m,j}, E_j \subseteq B_j^\circ$ extreme sets for $j \in \{1,2\}$. By $p_{m,j} := (p_m)_{T(E_{m,j})} \in T(E_{m,j})$ and $p_j := p_{T(E_j)}$ we denote the projections of p_m and p for each $m \in \mathbb{N}$. Then

$$h_{E_{m,1},p_{m,1}} + h_{E_{m,2},p_{m,2}} = h_{E_m,p_m} \longrightarrow h_{E,p} = h_{E_1,p_1} + h_{E_2,p_2}.$$

Assume that $\left(h_{E_{m,1},p_{m,1}}\right)_m$ converges to some h_{G_1,q_1} with $G_1 \subseteq B_1^\circ$ and $q_1 \in T(G_1)^*$ and similarly $h_{E_{m,2},p_{m,2}} \longrightarrow h_{G_2,q_2}$ with $G_2 \subseteq B_2^\circ$ and $q_2 \in T(G_2)^*$. Again using the unique decomposition of the Minkowski sum as above (see Equation (3.32) and below), wo conclude

$$\begin{aligned} E &= G_1 + G_2 \\ p &= q_1 + q_2 \end{aligned}$$

and therefore

$$\begin{aligned} E_1 = G_1 &\text{ and } E_2 = G_2 \\ p_1 = q_1 &\text{ and } p_2 = q_2. \end{aligned}$$

Therefore f_1, f_2 are continuous extensions of the identity to the boundary and we have shown that $\overline{X_3}$ is a common refinement of $\overline{X_1}$ and $\overline{X_2}$. It is the least common refinement because there is a unique decomposition of faces of B_3° into those of B_1° and B_2° which determines the convergence behavior. Adding another compact convex set to B_3° still gives a refinement of $\overline{X_1}$ and $\overline{X_2}$, but not the least common refinement. □

Remark 3.4.14 Now that we know that $\overline{X}_3^{hor}$ is the least common refinement of $\overline{X}_1^{hor}$ and $\overline{X}_2^{hor}$, Lemma 3.4.6 is a special case of [GJT98, Lem. 8.23]. ○

3.4.4 Example

An example of a norm that is obtained as the sum of a polyhedral and a smooth norm, is the following:

Example 3.4.15 We consider $\mathbb{R}^3$ with the norm

$$\|(a,b,c)\| = |a| + |b| + |c| + \sqrt{a^2 + b^2 + c^2}$$

for all $(a,b,c) \in \mathbb{R}^3$. Then $\|\cdot\|$ is the sum of the 1-norm $\|\cdot\|_1$ and the Euclidean norm $\|\cdot\|_2$.

The dual unit ball B° of $\|\cdot\|$ is the Minkowski-sum of the dual unit ball B_1° of $\|\cdot\|_1$, which is a unit sphere, and the dual unit ball of $\|\cdot\|_2$, which is a cube. Figure 3.27 shows how to obtain B° by putting a sphere on each corner of the cube (left) and then taking the convex hull (middle). B° is shown on the right. The 8 blue parts A_j^0 are connected components of extreme points, the 12 green parts A_i^1 are one-dimensional extreme sets and the orange parts are the 6 facets. Dividing the sphere centered at the origin in its eight parts S_j each in one octant, then a blue part A_j^0 of B° is the sum of one the $S_j \subseteq \partial S$ and the vertex of the cube in the same octant. The green components A_i^1 arise as the sum of the boundary of two neighboring components $S_i \cap S_k$ of the unit sphere and an edge of the cube that connects the two corresponding octants. Finally, the faces are obtained as the sum of a point which is the intersection of four components of S and a facet of the cube.

The unit ball B corresponding to B° can be imagined as a blown up octahedron with vertices on the axes at distance $\frac{1}{2}$ from the origin. The vertices and the parts in the coordinate planes are not

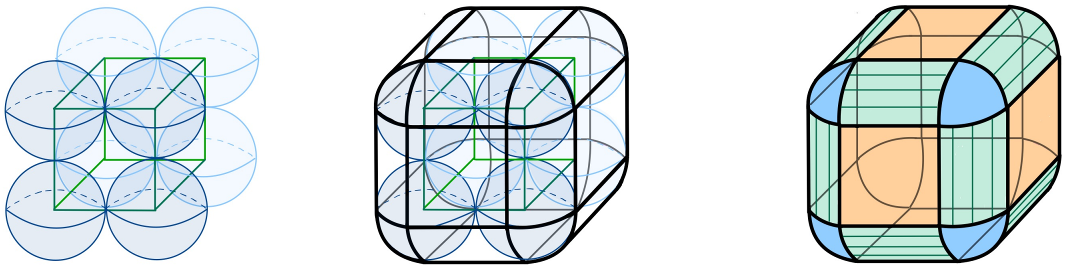


Figure 3.27: The dual unit ball as sum of a cube and a sphere.

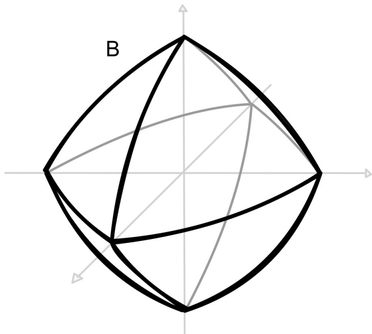


Figure 3.28: The corresponding unit ball B.

smooth, giving the one- and two-dimensional faces of B°. The parts of B within a quadrant are smooth, giving the blue parts of B°. A picture of B is drawn in Figure 3.28.

Let us now look at the converging behavior of some sequences to see how convergence with respect to $\|\cdot\|$ is related to convergence with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$.
Let the following sequence be given:

$$z_m = (-m, a, b)$$

for some $a, b \in \mathbb{R}$. Then with respect to $\|\cdot\|_1$, the sequence ψ_{z_m} converges to the horofunction h_{E_1,p_1} with face $E_1 = \{(1, y, z) \in \mathbb{R}^3 \mid |y|,\ |z| \leq 1\}$ and parameter $p_2 = (0, a, b) \in T(E_1)^*$. With respect to $\|\cdot\|_2$ the sequence converges to a horofunction associated to the extreme point $E_2 = \{(1, 0, 0)\}$. So we know that

$$\psi_{z_m} \longrightarrow h_{E,p}$$

with

$$E = \left\{(2, y, z) \in \mathbb{R}^3 \,\middle|\, |y|, |z| \leq 1\right\},$$
$$p = (0, a, b).$$

Let us look at a different sequence, now we take

$$y_m = (-am, -bm, c)$$

where $a, b > 0$. With respect to the two norms (for j = 1, 2) we have $\psi_{y_m}^{\|\cdot\|_j} \longrightarrow h_{E_j,p_j}$ with

$$E_1 = \{(1,1,z) \mid |z| \leq 1\}, \qquad p_1 = (0,0,c),$$
$$E_2 = \left\{\left(\tfrac{a}{\sqrt{a^2+b^2}}, \tfrac{b}{\sqrt{a^2+b^2}}, 0\right)\right\}, \quad p_2 = 0.$$

The extreme set to which $\left(\psi_{y_m}\right)_m$ converges to with respect to $\|\cdot\|$ is thereby given as

$$E = \left\{\left(1 + \frac{a}{\sqrt{a^2+b^2}},\ 1 + \frac{b}{\sqrt{a^2+b^2}},\ z\right) \middle| |z| \leq 1\right\}$$

and $p = (0,0,c)$.

In Lemma 3.4.6 we saw that the convergence with respect to $\|\cdot\|$ is uniquely determined by the convergence behavior with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$. To see this, let us look at a sequence that does not converge for B_1 but does for B_2. Let

$$x_m = (m, \sin(m), \cos(m)).$$

Then

$$\begin{aligned}\psi_{x_m}(x) &= \|x_m - x\| - \|x_m\| \\ &= \sqrt{(m-x)^2 + (\sin(m)-y)^2 + (\cos(m)-z)^2} + |m-x| + |\sin(m)-y| + |\cos(m)-z| \\ &\quad - \sqrt{m^2+1} - m - |\sin(m)| - |\cos(m)|\end{aligned}$$

does not converge. Looking more carefully, we see that the two summands with the roots together converge, they come from $\|\cdot\|_2$ and indeed, $\left(\psi_{x_m}^{\|\cdot\|_2}\right)_m$ converges. But the other parts do not converge because of the signs of sin and cos and also $\left(\psi_{x_m}^{\|\cdot\|_1}\right)_m$ does not converge. $\circ$

4 | Symmetric Spaces

Symmetric spaces arise in many areas of mathematics and physics and are an important class of Riemannian or Finsler manifolds. Especially helpful is their close relation to Lie groups and Lie algebras. In this chapter we examine the horofunction compactification of a symmetric space $X = G/K$ of non-compact type.

The symmetric space X carries a G-invariant Finsler metric and we could determine the horofunction compactification of X as a metric space, but this is rather difficult. Instead we apply our results on the horofunction compactification of a finite-dimensional normed space to a maximal abelian subalgebra $\mathfrak{a}$ contained in the Lie algebra $\mathfrak{g}$ of G whose norm will be obtained by the G-invariant Finsler metric on X. The corresponding flat $F = \exp(\mathfrak{a})$ in X, that is, a complete totally geodesic submanifold isometric to some $\mathbb{R}^k$, can be seen as a subspace of X and therefore one compactification of F is its closure in the compactification of X. It is also a metric space of its own by the connection to $\mathfrak{a}$ and thus has an intrinsic horofunction compactification. Our first main result (Theorem 4.2.18) will be that these two compactifications of F coincide. Afterwards we compare the horofunction compactification of X with two other well-known compactifications of X, namely the Satake and the Martin compactification. We give an explicit description of how to realize a Satake (Theorem 4.3.22) or a Martin compactification (Theorem 4.4.2) of X as a horofunction compactification respect to an appropriate G-invariant Finsler metric.

4.1 Preliminaries on Symmetric Spaces and Finsler Metrics

In this section we give some background about Lie theory, Symmetric Spaces and Finsler geometry. Basic references are [Hal15], [Hel78], [Kna02] and [Kir08] for Lie Groups and Symmetric Spaces and [Pla95], [Run59] or [BCS00] for Finsler Geometry.

4.1.1 Lie Groups and Lie Algebras

We first recall some basic properties of Lie groups and Lie algebras that we will need later to determine the horofunction compactification of a symmetric space of non-compact type.

Definition 4.1.1 A *Lie Group* is an algebraic group $(G, *)$ that is also a smooth manifold such that the following holds:

1) the group operation $G \times G \to G;\ (g, h) \mapsto g * h$ is smooth,

2) the inverse map $G \to G;\ g \mapsto g^{-1}$ is smooth. ◦

A Lie group homomorphism (respectively isomorphism) is a homomorphism (respectively isomorphism) of Lie groups that is a smooth map.

Example 4.1.2 We give some examples of Lie groups. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

- $(\mathbb{R}^n, +)$ and $(\mathbb{R}^n \setminus \{0\}, \cdot)$ are Lie groups
- $(\mathbb{S}^1, \cdot)$ is a Lie group
- $\mathrm{GL}(n, \mathbb{K}) = \{A \subseteq \mathrm{Mat}_n(\mathbb{K}) \mid \det(A) \neq 0\} \subseteq \mathbb{K}^{n^2}$ is a (matrix) Lie group. Some of its subgroups are Lie groups as well and called the *classical groups*:

$$\begin{aligned}
\mathrm{SL}(n, \mathbb{K}) &= \{A \in \mathrm{GL}(n, \mathbb{K}) \mid \det(A) = 1\} \\
\mathrm{O}(n) &= \{A \in \mathrm{GL}(n, \mathbb{R}) \mid AA^T = 1\} \\
\mathrm{SO}(n) &= \{A \in \mathrm{O}(n) \mid \det(A) = 1\} \\
\mathrm{U}(n) &= \{A \in \mathrm{GL}(n, \mathbb{C}) \mid A\overline{A}^T = 1\} \\
\mathrm{SU}(n) &= \{A \in U(n) \mid \det(A) = 1\}
\end{aligned}$$

- $\mathrm{GL}(V) = \mathrm{Aut}(V)$, the set of automorphisms, of a $\mathbb{K}$-vector space V is a Lie group with the composition as group structure. If V is finite-dimensional, $GL(V)$ and $GL(n, \mathbb{K})$ are isomorphic. ◦

Note that a Lie group does not have to be connected. Therefore we often consider the connected component of the identity.

Proposition 4.1.3 ([Kir08, Thm. 2.6]) *Let G be a real or complex Lie group and denote by G° the connected component of the identity. Then G° itself is a (real or complex) Lie group.* ◦

Definition 4.1.4 A *Lie Algebra* is a vector space V over a field $\mathbb{K}$ that carries a bilinear operation (called *bracket operation* or *Lie bracket*)

$$\begin{aligned}
[\cdot, \cdot] : V \times V &\longrightarrow V \\
(x, y) &\longmapsto [x, y],
\end{aligned}$$

that satisfies the following conditions:

1) $[x, x] = 0$ for all $x \in V$,

2) $[x,\ [y, z]] + [y,\ [z, x]] + [z,\ [x, y]] = 0$ for all $x, y, z \in V$ (Jacobi identity). ◦

We will consider Lie algebras over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. In these cases, where the field has characteristic zero, the first condition is equivalent to antisymmetry: $[x, y] = -[y, x]$.

A Lie algebra homomorphism (respectively isomorphism) $f : V_1 \to V_2$ is a linear homomorphism (respectively isomorphism) of Lie algebras V_1, V_2 such that

$$f([x, y]) = [f(x), f(y)]$$

for all $x, y \in V_1$.

Example 4.1.5 We give some examples of classical Lie algebras.

- $\mathbb{R}$ with the Lie bracket $[x, y] := xy - yx$ is a Lie algebra with trivial Lie bracket.

- $\mathfrak{gl}(n, \mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ is a Lie algebra with the commutator as Lie bracket containing all $n \times n$ matrices over $\mathbb{K}$. Some of its subalgebras are:

$$\begin{aligned} \mathfrak{sl}(n, \mathbb{K}) &= \{X \in \mathfrak{gl}(n, \mathbb{K}) \mid \operatorname{tr}(X) = 0\} \\ \mathfrak{so}(n) &= \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid X = -X^T\} \quad \text{(this implies } \operatorname{tr}(X) = 0) \\ \mathfrak{u}(n) &= \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X = -\overline{X}^T\} \\ \mathfrak{su}(n) &= \{X \in \mathfrak{u}(n) \mid \operatorname{tr}(X) = 0\} \end{aligned}$$

- $\mathfrak{gl}(V) = \operatorname{End}(V)$, the set of Endomorphisms, of a $\mathbb{K}$-vector space V is a Lie algebra with the commutator as Lie bracket. In the case where V is finite-dimensional, it is isomorphic to $\mathfrak{gl}(n, \mathbb{K})$. ◦

Connection between Lie group and Lie algebra

Let $(G, *)$ be a Lie group. We now want to assign a Lie algebra $\mathfrak{g}$ to G. Denote by $\mathcal{LV}(G)$ the set of all left-invariant vector fields on G and by $\pi : \mathcal{LV}(G) \to T_eG;\ X \mapsto X_e$ the evaluation at the identity $e \in G$. Then π is a linear isomorphism. We want to establish a Lie algebra structure on T_eG. As T_eG is the tangent space of a manifold, it is a vector space of the same dimension as the manifold G. So it remains to define a Lie bracket on T_eG. It will be induced by a Lie bracket on $\mathcal{LV}(G)$. Note that by the isomorphism π, the set $\mathcal{LV}(G)$ also carries the structure of a vector space and becomes a Lie algebra with the commutator as Lie bracket:

Lemma 4.1.6 ([Kna02, I.1, Ex.(5)]) *The bracket operation* $[X, Y] := X \circ Y - Y \circ X$ *for all* $X, Y \in \mathcal{LV}(G)$ *makes* $\mathcal{LV}(G)$ *into a Lie algebra.* ◦

Now that we have turned $\mathcal{LV}(G)$ into a Lie algebra, we want to use the isomorphism π to carry the Lie bracket over from $\mathcal{LV}(G)$ to T_eG:

Lemma 4.1.7 ([Kna02, I.10 (p.69)]) *Define a bracket operation on* T_eG *as the induced Lie bracket of* π*:*

$$[X, Y] := \pi([\pi^{-1}(X), \pi^{-1}(Y)]) \qquad \forall X, Y \in T_eG.$$

With this bracket operation, T_eG *is a Lie algebra and the map* π *is a Lie algebra isomorphism.* ◦

Definition 4.1.8 Let $(G, *)$ be a Lie group. The *Lie algebra* $\mathfrak{g}$ *of* G (also denoted by $\mathcal{L}(G)$) is the Lie algebra T_eG with the Lie bracket defined above. ◦

Lie's third Theorem states that given a Lie algebra $\mathfrak{g}$, we can find a Lie group G that has $\mathfrak{g}$ as its Lie algebra. But since $\mathcal{L}(G) = T_eG$ is a local property, G is not necessarily unique if G is not connected. But there is a unique connected simply-connected Lie group G with $\mathcal{L}(G) = \mathfrak{g}$ (see also [Kir08, Cor. 3.43]). Therefore when speaking about *the* Lie group G associated to a Lie algebra, we mean G°, the connected component of the identity.

The map between a Lie algebra $\mathfrak{g}$ to its Lie group G is given by the exponential map:

Definition 4.1.9 ([Hel78, Ch. II, Prop. 1.4, Cor. 1.5]) Let $X \in \mathfrak{g}$ and let $\gamma_X : (\mathbb{R}, +) \longrightarrow G$ be the unique geodesic such that $\dot{\gamma}_X(0) = X$. Then the exponential map $\exp : \mathfrak{g} \longrightarrow G$ is defined by

$$\exp(X) := \gamma_X(1). \qquad ◦$$

The map exp is not a global diffeomorphism but there are neighborhoods $0 \in U \subseteq \mathfrak{g}$ and $e \in V \subseteq G$ such that the restriction $\exp_{|U} : U \to V$ is a diffeomorphism.

Remark 4.1.10 ([Kir08, Ex. 3.3]) For matrix Lie groups $G \subseteq \mathrm{GL}(n, \mathbb{K})$ the exponential map exp as defined above coincides with the exponential of matrices: $e^A = \sum_k \frac{A^k}{k!}$ for $A \in \mathfrak{gl}(n, \mathbb{K})$. So the Lie algebra $\mathcal{L}(G)$ is given by

$$\mathcal{L}(G) = \{A \in \mathfrak{gl}(n, \mathbb{K}) \mid e^{tA} \in G \ \forall t \in \mathbb{R}\}.$$

Note that for $X \in \mathfrak{gl}(n, \mathbb{K})$ it holds

$$\det(e^X) = e^{\mathrm{tr}(X)}, \tag{4.1}$$

which can easily be seen for diagonal matrices but also holds in general by Jacobi's formula. ○

Example 4.1.11 Let us consider the matrix Lie groups and Lie algebras given in Example 4.1.2 and 4.1.5. The Lie group $\mathrm{GL}(n, \mathbb{K})$ consists of all invertible matrices. As for all $X \in \mathfrak{gl}(n, \mathbb{K})$ the exponential e^{tX} is invertible for all $t \in \mathbb{R}$, we have

$$\mathcal{L}(\mathrm{GL}(n, \mathbb{K})) = \mathfrak{gl}(n, \mathbb{K}).$$

The condition for a matrix to be in $\mathrm{SL}(n, \mathbb{K})$ was to have determinant equal to 1. By Equation (4.1), the matrix $X \in \mathfrak{gl}(n, \mathbb{K})$ lies in $\mathrm{SL}(n, \mathbb{K})$ if and only if $\mathrm{tr}(X) = 0$. This was exactly the defining condition for $\mathfrak{sl}(n, \mathbb{K})$ and we get

$$\mathcal{L}(\mathrm{SL}(n, \mathbb{K}) = \mathfrak{sl}(n, \mathbb{K}).$$

Now let us consider the Lie group $\mathrm{O}(n)$ which had $AA^T = 1$ as defining equation. This implies that A and A^T commute. Let $X \in \mathfrak{gl}(n, \mathbb{K})$ with $A = e^X$. Then $A^T = e^{X^T}$ and we get

$$1 = AA^T = e^X e^{X^T} = e^{X + X^T}$$

because X and X^T also commute. So we need $X + X^T = 0$ for X to be in $\mathcal{L}(\mathrm{O}(n))$. The skew-symmetry implies $\mathrm{tr}(X) = 0$ with was the equivalent condition for $\det(e^X) = 1$. Therefore

$$\mathcal{L}(\mathrm{O}(n)) = \mathcal{L}(\mathrm{SO}(n)) = \mathfrak{so}(n, \mathbb{K}).$$

A similar calculation shows that

$$\mathcal{L}(\mathrm{U}(n)) = \mathfrak{u}(n)$$
$$\mathcal{L}(\mathrm{SU}(n)) = \mathfrak{su}(n).$$

Here we get different Lie algebras associated to $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ because the condition $X = -\overline{X}^T$ does not imply $\mathrm{tr}(X) = 0$. ○

The strong connection between Lie groups and their algebras can also be seen in terms of homomorphisms. By taking the differential, Lie group homomorphisms (respectively isomorphisms) induce Lie algebra homomorphisms (respectively isomorphisms):

Lemma 4.1.12 ([Kna02, I. 10 (p.72)]) *Let G, H be Lie groups and let $\mathfrak{g}, \mathfrak{h}$ be their Lie algebras. If $f : G \to H$ is a Lie Group homomorphism (respectively isomorphism), then its differential $(\mathrm{d} f)_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism ((respectively isomorphism).* ○

We now want to recall the definition of representations of Lie groups and Lie algebras which will especially lead us to the adjoint representation that plays an important role in the theory of roots below.

Definition 4.1.13 Let G be a Lie group and V be a finite-dimensional real vector space. A *Lie group representation* is a homomorphism of G into the general linear group of V:

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

A group homomorphism $\rho : G \to \mathrm{PGL}(V)$ is called a *projective representation*. ◦

Definition 4.1.14 Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$ and let V be a $\mathbb{K}$-vector space. A *Lie algebra representation* is a Lie group homomorphism of $\mathfrak{g}$ into $\mathfrak{gl}(V)$:

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V). \qquad \circ$$

Remark 4.1.15 As $\mathfrak{gl}(V)$ is the Lie algebra of $GL(V)$, a Lie group representation $\rho : G \to \mathrm{GL}(V)$ induces a Lie algebra representation $(\mathrm{d}\rho)_e : \mathfrak{g} \to \mathfrak{gl}(V)$. ◦

For us the most important representation will be the adjoint representation which is associated to the Lie bracket. For some $g \in G$ let $c_g : G \to G$ denote the conjugation by g that sends $h \in G$ to ghg^{-1}. This is a Lie group isomorphism and its differential

$$\mathrm{Ad}(g) := (\mathrm{d}\, c_g)_e : \mathfrak{g} \to \mathfrak{g}$$

is an isomorphism of Lie algebras and especially it is a linear isomorphism. So for each $g \in G$ we associated an element in $\mathrm{GL}(\mathfrak{g})$ to it, in other words, we have a map

$$\begin{aligned} \mathrm{Ad} : G &\longrightarrow \mathrm{GL}(\mathfrak{g}) \\ g &\longmapsto \mathrm{Ad}(g). \end{aligned}$$

The map Ad is a Lie group homomorphism and called the *adjoint representation* of G. Its differential is a Lie algebra representation also called the *adjoint representation* of $\mathfrak{g}$, and is given by

$$\begin{aligned} \mathrm{ad} := (\mathrm{d}\,\mathrm{Ad})_e : &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ X &\longmapsto \mathrm{ad}(X), \end{aligned}$$

where $\mathrm{ad}(X)(Y) = [X, Y]$.

4.1.2 Symmetric Spaces

In this section we want to give a short overview over the theory of symmetric spaces of non-compact type and their connection to Lie groups and Lie algebras.

Definition 4.1.16 A *(globally) symmetric space* X is a connected Riemannian manifold (X, g) with an isometry $s_p : X \longrightarrow X$ for every point $p \in X$, such that p is a fixed point of the isometry, $s_p(p) = p$, and such that the differential at p is direction-reversing: $ds_{p|p} = -\mathrm{id}_{T_pX}$. ◦

Remark 4.1.17 Let γ be a geodesic on X with $\gamma(0) = p$, then $s_p(\gamma(t)) = \gamma(-t)$. Since additionally $s_p^2 = id_X$, the isometry s_p is also called a *geodesic symmetry*. ◦

From now on let X denote a symmetric space if not stated otherwise. Products of symmetric spaces are again symmetric spaces. Any irreducible symmetric space X allows a decomposition

$$X \cong \mathbb{E}^n \times X_+ \times X_-$$

where $\mathbb{E}^n$ is of Euclidean type, X_+ is of compact type and X_- is of non-compact type. Symmetric spaces of Euclidean type are flat and are isometric to some $\mathbb{R}^n$. A symmetric space of compact type has non-vanishing sectional curvature ≥ 0 and is compact. We are interested in symmetric spaces of non-compact type, who have non-vanishing sectional curvature ≤ 0 and are non-compact[1] The isometry group of a symmetric space of non-compact type is also non-compact and semisimple.

From now on let X be a symmetric space of non-compact type.

Equipped with the compact-open topology[2], the group $\mathrm{Isom}(X, g)$ of isometries becomes a locally compact topological group that acts continuously on X. As the topology on X comes from the distance associated to the metric g, the compact-open topology on $\mathrm{Isom}(X, g)$ coincides with the topology of uniform convergence on compact subsets. By the Myers-Steenrod Theorem, $\mathrm{Isom}(X, g)$ carries a smooth structure which is compatible with the group structure. Therefore

$$G := \mathrm{Isom}(X, g)$$

carries the structure of a Lie group (see also [Hel78, Ch. IV, Lem. 3.2]).

We fix some point $p_0 \in X$ and denote the stabilizer of p_0 in G by

$$K := G_{p_0} = \{f \in G \mid f(p_0) = p_0\}.$$

Then K is a compact subgroup of G ([Hel78, Ch.IV, Thm. 2.5]).

The group $G = \mathrm{Isom}(X, g)$ of isometries and also its connected component G° of the identity act transitively on X. So X is a homogeneous space and we can identity the symmetric space X with the space of left cosets G°/K:

Proposition 4.1.18 ([Hel78, Ch. IV, Thm. 3.3]) *Let (X, g) be a symmetric space with isometry group $G = \mathrm{Isom}^\circ(X, g)$ and $K = G_{p_0}$ for some $p_0 \in X$. Then K is a compact subgroup of G and*

$$G/K \cong X$$

by the analytic diffeomorphism $gK \mapsto gp_0$. ◦

Recall that we only consider symmetric spaces of non-compact type. In that case, the Lie group G is a semisimple Lie group with trivial center ([Ebe96, Prop. 2.1.1]).

So far we came from the geometric side of the story and assigned a pair of Lie groups G, K to a symmetric space X. This allows us to work with Lie groups and compact subgroups when talking about symmetric spaces.

We could also have defined a symmetric space algebraically via a (Riemannian) symmetric pair. This is a pair (G, H) of Lie groups satisfying an inclusion relation with respect to an involutive automorphism σ on G. Let

$$G^\sigma := \{g \in G \mid \sigma(g) = g\}$$

denote the set of fixed points of σ and $(G^\sigma)^\circ$ its connected component of the identity.

[1] Here is a precise definition: A symmetric space M is called a symmetric space of *non-compact type*, if M is of non-positive sectional curvature, simply connected and not the Riemannian product of an Euclidean space $\mathbb{R}^k$ $(k \geq 1)$ and another manifold N.

[2] This topology is generated by the open sets $W(U, C) := \{f \in \mathrm{Isom}(X, g) \mid f(C) \subseteq U\}$, where $U \subseteq X$ is open and $C \subseteq X$ is compact.

Proposition 4.1.19 ([Hel78, Ch. IV, Thm. 3.3]) *Let $X = G/K$ be a symmetric space with $G = \operatorname{Isom}^\circ(X, g)$ and $K = G_{p_0}$ for some $p_0 \in X$. Then the mapping*

$$\begin{aligned} \sigma_{p_0} : G &\longrightarrow G \\ h &\longmapsto s_{p_0}\, h\, s_{p_0} \end{aligned}$$

is an involutive automorphism of G such that

$$(G^{\sigma_{p_0}})^\circ \subseteq K \subseteq G^{\sigma_{p_0}}. \qquad \circ$$

Note that with the notations introduced on page 111 we have

$$\sigma_{p_0} = c_{s_{p_0}}.$$

The result above motivates the following definition that brings us to the algebraic way of constructing symmetric spaces.

Definition 4.1.20 Let G be a connected Lie group and $H \leq G$ a closed subgroup. We call (G, H) a *symmetric pair*, if there is an involutive automorphism $\sigma : G \longrightarrow G$ such that

$$(G^\sigma)^\circ \subseteq H \subseteq G^\sigma.$$

If $\mathrm{Ad}_H \subseteq \mathrm{GL}(\mathfrak{g})$ is compact, (G, H) is called a *Riemannian symmetric pair*. $\circ$

Proposition 4.1.21 ([Hel78, Ch. IV, Prop. 3.4]) *Let (G, H) be a symmetric pair with involution σ and $\pi : G \longrightarrow G/H$ the usual projection. Denote by $p_0 := \pi(e) = eH$ the image of the identity element of G. Then with any G-invariant Riemannian metric h on G/H, the manifold G/H is a symmetric space and the geodesic symmetry s_{p_0} is independent of the choice of h and fulfills*

$$s_{p_0} \circ \pi = \pi \circ \sigma. \qquad \circ$$

Example 4.1.22 We look at our favorite example from the algebraic side: let $G = \mathrm{SL}(n, \mathbb{R})$ and $K = SO(n)$. Then we define an involution on G with respect to the identity matrix $p_0 = I_n$ by

$$\begin{aligned} \sigma = \sigma_{I_n} : G &\longrightarrow G \\ g &\longmapsto (g^{-1})^T. \end{aligned}$$

Then

$$G^\sigma = \{g \in \mathrm{SL}(n, \mathbb{R}) \mid (g^{-1})^T = g\} = \mathrm{SO}(n) = K$$

For $n = 2$, the space $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2) \cong \mathbb{H}^2$ is a model of the hyperbolic plane and $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ can be seen as a generalization of it. $\circ$

For a symmetric space $X = G/K$, the groups G and K are Lie groups. Therefore it is reasonable to deal with X not only in terms of Lie groups but also via the associated Lie algebras to gain more structure. Let σ_{p_0} be the involutive automorphism as in Proposition 4.1.19 and let $\mathfrak{g}$ be the Lie algebra of G. By the identification $\mathfrak{g} = T_eG$ we obtain the involution $\theta_{p_0} : \mathfrak{g} \longrightarrow \mathfrak{g}$ given by the differential

$$\theta_{p_0} = (d\sigma_{p_0})_e.$$

The following connection using the exponential map between the involution σ on G and θ_{p_0} on $\mathfrak{g}$ holds true for all $X \in \mathfrak{g}$:

$$\sigma_{p_0}(e^{tX}) = e^{t\theta_{p_0}(X)}.$$

As θ_{p_0} is an involution, it is diagonalizable and the only possible eigenspaces are those to the eigenvalues 1 and -1. Then the positive eigenspace

$$\mathfrak{k} := \{X \in \mathfrak{g} \mid \theta_{p_0}(X) = X\}. \tag{4.2}$$

turns out to be the Lie algebra of K, see [Hel78, Ch. IV, Thm. 3.3]. For the negative eigenspace we set

$$\mathfrak{p} := \{X \in \mathfrak{g} \mid \theta_{p_0}(X) = -X\}. \tag{4.3}$$

We can write $\mathfrak{g}$ as the direct sum of vector spaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

This decomposition is called the *Cartan decomposition* of $\mathfrak{g}$ with *Cartan involution* θ_{p_0}.

As θ_{p_0} preserves the Lie bracket, that is $\theta_{p_0}[X, Y] = [\theta_{p_0}(X), \theta_{p_0}(Y)]$ for all $X, Y \in \mathfrak{g}$, it holds:

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

The usual projection $\pi : G \to G/K \cong X;\ g \mapsto g.p_0$ is a submersion and its differential $d(\pi)_e$ has kernel $\mathfrak{k}$. Therefore we get the isomorphism

$$\mathfrak{p} \cong T_{p_0}X.$$

Example 4.1.23 For the space $\mathrm{SL}(n, \mathbb{R})/SO(n)$ with $n \geq 2$ the group $G = \mathrm{SL}(n, \mathbb{R})$ is semisimple with Lie algebra

$$\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R}) = \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid \operatorname{tr}(X) = 0\}.$$

We want to determine the subspaces $\mathfrak{p}$ and $\mathfrak{k}$ as the eigenspaces of the involution $\theta_{p_0} = d(\sigma_{p_0})_e$. In Example 4.1.22 we defined the involution on G via $\sigma(g) = (g^{-1})^T$. Therefore $\theta_{p_0}(X) = -X^T$ for all $X \in \mathfrak{g}$. For the eigenspaces we get:

$$\begin{aligned}\mathfrak{p} &= \{X \in \mathfrak{g} \mid \theta_{p_0}(X) = -X\} = \{X \in \mathfrak{g} \mid X^T = X\} \\ &= \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid \operatorname{tr}(X) = 0,\ X = X^T\}\end{aligned}$$

is the space of traceless symmetric matrices and

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta_{p_0}(X) = X\} = \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid X = -X^T\} = \mathfrak{so}(n)$$

are the skew-symmetric matrices. ○

4.1.3 Root Spaces

We will now investigate more the structure of G and $\mathfrak{g}$ of symmetric spaces of non-compact type, that is, of semisimple Lie algebras. References for this section can be found in [Ebe96, p.71ff], [FH91, §14] or [Kna02].

Recall the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$ was the eigenspace of θ_{p_0} to -1 and $\mathfrak{p} \cong T_{p_0}X$. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subalgebra. Note that $\mathfrak{p}$ is not an algebra itself, so by an abelian subalgebra of $\mathfrak{p}$ we mean a subspace of $\mathfrak{p}$ that is an subalgebra of $\mathfrak{g}$. As $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{p} = \{0\}$, a subalgebra of $\mathfrak{p}$ is automatically abelian. There is not a unique maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p}$ but all of them are pretty similar to each other:

Lemma 4.1.24 ([Hel78, Ch. V, Lemma 6.3])

(i) All maximal abelian subalgebras of $\mathfrak{p}$ *are conjugate to each other over* K*, that is, for all* $\mathfrak{a}, \mathfrak{a}' \subseteq \mathfrak{p}$ *maximal abelian there is a* $k \in K$ *such that* $Ad(k)\mathfrak{a} = \mathfrak{a}'$.

(ii) Let $\mathfrak{a}$ *be a maximal abelian subalgebra of* $\mathfrak{p}$*. Then* $\mathfrak{p} = Ad(K)\mathfrak{a} = \bigcup_{k \in K} Ad(k)\mathfrak{a}$. ◦

The dimension of a maximal abelian subspace of $\mathfrak{p}$ is independent of our choice of $\mathfrak{a}$. So we will give it a name.

Definition 4.1.25 Let X be a symmetric space and $\mathfrak{p}$ as above. The *rank* of X is the dimension of some maximal abelian subspace of $\mathfrak{p}$. ◦

The rank of X can alternatively be defined as the maximal dimension of any flat totally geodesic subspace of X, a so-called flat:

Definition 4.1.26 A *k-flat* F in X is a complete, totally geodesic k-dimensional submanifold of X isometric to a Euclidean space $\mathbb{R}^k$. ◦

The two different ways of defining the rank of X are justified by the following lemma:

Lemma 4.1.27 ([Ji05, Prop. 4.70]) *Let* $\mathfrak{a}$ *be a maximal abelian subalgebra of* $\mathfrak{p}$ *and* $p_0 \in X$ *a chosen basepoint. Let* $A := \exp(\mathfrak{a})$ *be the corresponding subgroup of* G.

(i) The orbit $F := A.p_0$ *is a* k*-flat in* X.

(ii) Any k*-flat of* X *passing through the basepoint* p_0 *is of the form* $F = \exp(\mathfrak{a}).p_0$ *for some maximal abelian subalgebra* $\mathfrak{a} \subseteq \mathfrak{p}$. ◦

By the close relation between maximal abelian subalgebras and k-flats, it is not surprising that k-flats are also conjugate to each other:

Proposition 4.1.28 ([Ebe96, Prop. 2.10, p.85]) *Let* F_1 *and* F_2 *be* k*-flats in* X *and* $p_1 \in F_1, p_2 \in F_2$ *points. Then there is a* $g \in G$ *such that* $g(p_1) = p_2$ *and* $g(F_1) = F_2$. ◦

To determine the horofunction compactification of $\mathfrak{a}$ we need a bilinear form on it. We take the *Killing form* κ of $\mathfrak{g}$:

$$\begin{aligned} \kappa : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathbb{K}, \\ (X, Y) &\longmapsto \kappa(X, Y) := \operatorname{tr}(\operatorname{ad}(X) \circ \operatorname{ad}(Y)) \end{aligned}$$

As X is of non-compact type, the Killing form κ is a positive definite bilinear form on $\mathfrak{p}$. On the vector space $\mathfrak{k}$, κ is negative definite and a positive definite bilinear form on $\mathfrak{g}$ is given by

$$\phi_{p_0}(X, Y) := -\kappa(\theta_{p_0}(X), Y) \quad \text{for } X, Y \in \mathfrak{g},$$

where θ_{p_0} is the differential of the involution σ_{p_0} at $e \in G$. With respect to κ and ϕ_{p_0} the subspaces $\mathfrak{p}$ and $\mathfrak{k}$ are orthogonal ([Ebe96, 2.7.1]).

Example 4.1.29 Let us determine the Killing form κ on $\mathfrak{gl}(n, \mathbb{C})$. A basis of the space is given by the se of matrices $\{E^{ij}\}_{i,j}$, where E^{ij} is the matrix with an 1 at the (i, j)-entry and zeros anywhere else. For some $X, Y \in \mathfrak{gl}(n, \mathbb{C})$ we compute for the (a, b)-component of $(\operatorname{ad}(X) \circ \operatorname{ad}(Y))\,(E^{kl})$:

$$\begin{aligned} [X,\ [Y,\ E^{kl}]]_{ab} &= \sum_m \left(X_{am}[Y,\ E^{kl}]_{mb} - [Y,\ E^{kl}]_{am} X_{mb} \right) \\ &= \sum_m (X_{am} Y_{mk} \delta_{lb}) - X_{ak} Y_{lb} - X_{lb} Y_{ak} + \sum_m (X_{mb} Y_{lm} \delta_{ak}) \\ &= (XY)_{ak} \delta_{bl} + (YX)_{lb} \delta_{ak} - X_{ak} Y_{lb} - X_{lb} Y_{ak}. \end{aligned}$$

The entire matrix then is given by

$$\begin{aligned}
[X,\ [Y,\ E^{kl}]] &= \sum_{a,b}[X,\ [Y,\ E^{kl}]]_{ab}E^{ab} \\
&= \sum_{a,b} \left((XY)_{ak}\delta_{bl} + (YX)_{lb}\delta_{ak} - X_{ak}Y_{lb} - X_{lb}Y_{ak}\right) E^{ab} \\
&= \sum_{a}(XY)_{ak}E^{al} + \sum_{b}(YX)_{lb}E^{kb} - \sum_{a,b}\left(X_{ak}Y_{lb}E^{ab} + X_{lb}Y_{ak}E^{ab}\right).
\end{aligned}$$

The coefficient of this matrix with respect to the basis element E^{kl} is

$$\begin{aligned}
e_{kl} &= \sum_{a}\delta_{ak}(XY)_{ak} + \sum_{b}\delta_{bl}(YX)_{lb} - \sum_{a,b}\delta_{ak}\delta_{bl}(X_{ak}Y_{lb} + X_{lb}Y_{ak}) \\
&= (XY)_{kk} + (YX)_{ll} - (X_{kk}Y_{ll} + X_{ll}Y_{kk}).
\end{aligned}$$

To get the trace of the map $\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ we now have to sum over all these coefficients:

$$\begin{aligned}
\mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) &= \sum_{k,l} e_{kl} = n\sum_{k}(XY)_{kk} + n\sum_{l}(YX)_{ll} - 2\sum_{k,l}X_{kk}Y_{ll} \\
&= 2n\,\mathrm{tr}(XY) - 2\,\mathrm{tr}(X)\,\mathrm{tr}(Y).
\end{aligned}$$

On $\mathfrak{gl}(n, \mathbb{C})$, the Killing form κ is thereby given as

$$\kappa(A, B) = 2n\,\mathrm{tr}(AB) - 2\,\mathrm{tr}(A)\,\mathrm{tr}(B) \qquad \forall A, B \in \mathfrak{gl}(n, \mathbb{C}).$$

The Killing form κ on $\mathfrak{sl}(n, \mathbb{C})$ now can be obtained by restricting the Killing form of $\mathfrak{gl}(n, \mathbb{C})$ to $\mathfrak{sl}(n, \mathbb{C})$. Therefore we obtain

$$\kappa(A, B) = 2n\,\mathrm{tr}(AB) \qquad \forall A, B \in \mathfrak{sl}(n, \mathbb{C}).$$ ∘

The adjoint representation $\mathrm{ad}(X)$ for any $X \in \mathfrak{p}$ is symmetric with respect to the Killing form, that is,

$$\kappa(\mathrm{ad}(X)Y, Z) = \kappa(Y, \mathrm{ad}(X)Z) \quad \forall Y, Z \in \mathfrak{p}.$$

Now let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace. Then $[X, Y] = 0$ and by the Jacobi identity, the operators $\mathrm{ad}(X)$ and $\mathrm{ad}(Y)$ commute for all $X, Y \in \mathfrak{a}$. Therefore all maps $\mathrm{ad}(X)$ with $X \in \mathfrak{a}$ are simultaneously diagonalizable with a κ-orthogonal transformation. For each $\alpha \in \mathfrak{a}^* = \mathrm{Hom}(\mathfrak{a}, \mathbb{C})$ we thus define the following subset:

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid ad(H)X = \alpha(H)X \quad \forall H \in \mathfrak{a}\}.$$

Definition 4.1.30 A linear map $\alpha \in \mathfrak{a}^* \setminus \{0\}$ is called a *root*, if $\mathfrak{g}_\alpha \neq 0$. Then $\mathfrak{g}_\alpha$ is the *root space* of α. By Σ we denote the set of roots:

$$\Sigma := \{\alpha \in \mathfrak{a}^* \mid \alpha \text{ is a root }\}.$$ ∘

Σ is non-empty and furthermore we get the *root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \alpha} \mathfrak{g}_\alpha.$$

By the definition of the adjoint representation we have $\mathrm{ad}(H)(X) = [H, X]$ and therefore, since $\mathfrak{a}$ is abelian, it holds $\mathfrak{a} \subseteq \mathfrak{g}_0$. If α is a root, then so is $-\alpha$. Furthermore, for two roots $\alpha, \beta \in \Sigma$ it holds $[\mathfrak{g}_\alpha,\ \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta \in \Sigma$ is again a root and equals 0 otherwise ([Ebe96, 2.7.3.]).

For a root $\alpha \in \Sigma \subseteq \mathfrak{a}^*$ its kernel $\ker(\alpha) = \{H \in \mathfrak{a} \mid \alpha(H) = 0\}$ is a hyperplane which divides the vector space $\mathfrak{a}$ into several connected components. We will often identify $\mathfrak{a}$ with $\mathfrak{a}^*$ using the Killing form κ: for an element $\alpha \in \mathfrak{a}^*$ we set $H_\alpha \in \mathfrak{a}$ to be the unique element such that

$$\kappa(H_\alpha, H) = \alpha(H) \quad \forall H \in \mathfrak{a}.$$

Given two linear maps $\alpha, \beta \in \mathfrak{a}^*$ we also use the notation

$$\kappa(\alpha,\beta) := \kappa(H_\alpha, H_\beta).$$

Definition 4.1.31 The connected components of $\mathfrak{a} \setminus \bigcup_{\alpha\in\Sigma} \ker(\alpha)$ are called *Weyl chambers*. ∘

An element $H \in \mathfrak{a}$ that lies in the interior of a Weyl chamber is called *regular*, that is, $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$. Otherwise H it is called *singular*.

Now we fix one Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$. Then a root α is called *positive* (denoted by $\alpha > 0$) if it is positive on $\mathfrak{a}^+$. The Weyl chamber $\mathfrak{a}^+$ is called the *positive Weyl chamber* and the set of positive roots is denoted by Σ^+:

$$\Sigma^+ := \{\alpha \in \Sigma \mid \alpha(H) > 0 \;\; \forall H \in \mathfrak{a}^+\}.$$

Definition 4.1.32 A positive root $\alpha \in \Sigma^+$ is called *simple*, if α is not the sum of two positive roots. The set of simple roots is denoted by Δ:

$$\Delta := \{\alpha \in \Sigma^+ \mid \alpha(H) \text{ is not the sum of two positive roots}\}.$$ ∘

The simple roots form a basis of Σ in the sense that we can express every root as a linear combination of elements in Δ with integer coefficients which are either all ≥ 0 or all ≤ 0.

Example 4.1.33 As an example, let us look at the roots in $\mathfrak{sl}(n,\mathbb{C})$. An abelian subalgebra is given by the diagonal traceless matrices:

$$\mathfrak{a} := \left\{ H = \operatorname{diag}(h_1,\dots,h_n) \,\middle|\, \sum_i h_i = 0 \right\}.$$

Let $\alpha \in \mathfrak{a}^*$ be a linear map. For α to be a root, the equation

$$[H,\ X] = \alpha(H)X \quad \text{ for all } H \in \mathfrak{a}$$

has to have a non-trivial solution. As $[H,\ X]_{ij} = (h_i - h_j)X_{ij}$ this is satisfied for $\alpha = \alpha_{ij}$ with $i \neq j \in \{1,\dots,n\}$, where $\alpha_{ij} \in \mathfrak{a}^*$ is given by

$$\alpha_{ij}(H) = h_i - h_j,$$

for every $H = \operatorname{diag}(h_1,\dots,h_n) \in \mathfrak{a}$. The root space then is given as the span of the matrix E^{ij} with entry 1 exactly in the (i, j)-component:

$$\mathfrak{g}_{\alpha_{ij}} = \mathbb{C}E^{ij},$$

and $\mathfrak{g}_0 = \mathfrak{a}$. So the roots are

$$\Sigma = \{\alpha_{ij} \in \mathfrak{a}^* \mid i \neq j\}.$$

Since an element $H \in \mathfrak{a}$ lies in the kernel of a root α_{ij} if its i-th and j-th diagonal entry coincides, the Weyl chamber decomposition yields:

$$\mathfrak{a} \setminus \bigcup_{\alpha_{ij}\in\Sigma} \ker(\alpha_{ij}) = \left\{ \operatorname{diag}(h_1,\dots,h_n) \,\middle|\, \sum_i h_i = 0,\ h_i \neq h_j\ \forall i \neq j \right\}.$$

We fix a positive Weyl chamber

$$\mathfrak{a}^+ := \left\{ \mathrm{diag}(h_1, \ldots, h_n) \,\middle|\, \sum_i h_i = 0,\ h_1 > \ldots > h_n \right\},$$

then the positive roots are

$$\Sigma^+ = \{\alpha_{ij} \in \Sigma \mid i < j\}.$$

As all elements in $\mathfrak{a} \subseteq \mathfrak{p}$ have to have trace zero, the simple roots are given by

$$\Delta = \{\alpha_{i\, i+1} \mid 1 \leq i \leq n-1\}.$$

Figure 4.1 shows the Weyl chamber system in the case of $n = 4$ with the Weyl chamber walls given by the kernels of the roots. Each Weyl chamber contains diagonal matrices with pairwise

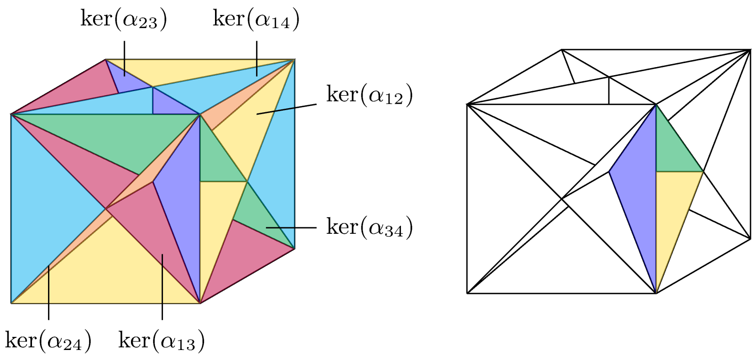


Figure 4.1: The Weyl chamber system of SL(4, ℂ) with all Weyl chamber walls (LEFT) and the positive Weyl chamber we chose (RIGHT).

distinct entries and two elements of different Weyl chambers differ form each other by the order of their entries: two matrices $H = \mathrm{diag}(h_1, \ldots, h_n)$ and $H' = \mathrm{diag}(h'_1, \ldots, h'_n) \in \mathfrak{a}$ with entries $h_1 > h_2 > h_3 > \ldots > h_n$ and $h'_2 > h'_1 > h'_3 > \ldots > h'_n$ belong to different Weyl chambers. As there are $n!$ possibilities to arrange n elements, we have $n!$ Weyl chambers in $\mathfrak{sl}(n, \mathbb{C})$. ○

Associated to the Weyl chamber division is the Weyl group, which permutes the Weyl chambers. Let $\mathcal{N}_K(\mathfrak{a}) := \{k \in K \mid Ad(k)\mathfrak{a} \subseteq \mathfrak{a}\}$ be the *normalizer* of $\mathfrak{a}$ in K and denote its *centralizer* by $C_K(\mathfrak{a}) := \{k \in K \mid Ad(k)H = H\ \forall H \in \mathfrak{a}\}$. Then $C_K(\mathfrak{a}) \trianglelefteq \mathcal{N}_K(\mathfrak{a})$ is a normal subgroup.

Definition 4.1.34 The quotient

$$\mathcal{W} := {}^{\mathcal{N}_K(\mathfrak{a})}\!\big/_{C_K(\mathfrak{a})}$$

is called the *Weyl group*. ○

The Weyl group can also be expressed as $\mathcal{N}_K(A)/C_K(A)$. Sometimes (see for example [FH91, 14.8]) the Weyl group is defined more geometrically as the reflection group generated by the reflections at the Weyl chamber walls $\Omega_\alpha := \{\beta \in \mathfrak{a}^* \mid \beta(H_\alpha) = 0\}$ with $\alpha \in \Sigma$. Then the reflection of $\beta \in \mathfrak{a}^*$ along Ω_α is given by

$$w_\alpha(\beta) = \beta - 2\frac{\kappa(\beta, \alpha)}{\kappa(\alpha, \alpha)}\alpha.$$

These different characterizations are based on the following lemma:

Lemma 4.1.35 ([Ebe96, section 2.9]) *The Weyl group $\mathcal{W}$ is discrete and finite and acts simply transitively on the set of Weyl chambers. It is generated by the reflections in the hyperplanes* $\ker(\alpha)$ *for $\alpha \in \Delta$.* ○

Example 4.1.36 We want to continue the example of $\mathrm{SL}(n, \mathbb{C})$ and compute its Weyl group. We start with the normalizer

$$\mathcal{N}_K(\mathfrak{a}) = \{k \in K \mid kHk^{-1} \in \mathfrak{a} \quad \forall H \in \mathfrak{a}\}.$$

Let $H = \mathrm{diag}(h_1, \ldots, h_n) \in \mathfrak{a}$ be a diagonal matrix with $\sum_i h_i = 0$. Then for a component of the conjugate of H by some $k \in K$ we get with $k^{-1} = k^T$:

$$(kHk^{-1})_{ij} = \sum_a h_a k_{ia} k_{ja}.$$

For the product to be an element of $\mathfrak{a}$ again, we need all non-diagonal elements to be zero. Using $h_n = -\sum_i^{n-1} h_i$ we get the condition

$$\sum_{a=1}^{n-1} h_a (k_{ia} k_{ja} - k_{in} k_{jn}) = 0 \quad \forall h_a \in \mathbb{C}.$$

Therefore

$$k_{ia} k_{ja} = k_{in} k_{jn} \quad \forall a = 1, \ldots, n-1.$$

Together with the orthonormality of k we get $0 = \sum_t^n k_{it} k_{jt} = n k_{in} k_{jn}$ which then yields

$$k_{ia} k_{ja} = 0 \quad \forall a = 1, \ldots, n; \ i \neq j.$$

So in each column of k there can only be one non-zero element. On the other hand, we know that $(kk^T)_{ii} = \sum_t k_{it}^2 = 1$, therefore not all elements of a row can be zero. As k is a square matrix, it has to be a permutation matrix with entries 1 and -1, that is, one non-zero element in each row and column. These describes the elements of the normalizer $\mathcal{N}_K(\mathfrak{a})$.

For the centralizer $C_K(\mathfrak{a})$, the defining condition restricts to $kHk^T = H$ for all $H \in \mathfrak{a}$. We get the additional condition

$$(kHk^T)_{ii} = \sum_t h_t k_{it}^2 \stackrel{!}{=} h_i \quad \forall i = 1, \ldots, n.$$

Since $C_K(\mathfrak{a}) \subseteq \mathcal{N}_K(\mathfrak{a})$, we get

$$C_K(\mathfrak{a}) = \{\mathrm{diag}(c_1, \ldots, c_n) \mid c_i \in \{\pm 1\}\}.$$

So in the quotient we can neglect the signs of the elements of the permutation matrix and get

$$\mathcal{W} = \mathcal{N}_K(\mathfrak{a}) / C_K(\mathfrak{a}) \cong S_n,$$

that is, the Weyl group is isomorphic to the permutation group on n elements. Recall that we saw in Example 4.1.33 that the Weyl chambers distinguish from each other by the order of their elements, so permuting the n diagonal elements gives the action of the Weyl group on the Weyl chambers. ∘

There is also a Cartan decomposition on the level of groups using the positive Weyl chamber $\mathfrak{a}^+$:

Lemma 4.1.37 (Cartan decomposition; [Hel78], Thm.V.6.7 and Thm.IX.1.1) *Let $\mathfrak{a}^+$ be a positive Weyl chamber. Set $A^+ := \exp(\mathfrak{a}^+) \subseteq G$ and denote by $\overline{A^+}$ its closure. Note that $\overline{A^+} = \exp(\overline{\mathfrak{a}^+})$. For every element $g \in G$ there exist some $k_1, k_2 \in K$ and a unique $a \in \overline{A^+}$ such that $g = k_1 a k_2$. We shortly write*

$$G = K\overline{A^+}K,$$

and call this a Cartan decomposition *of G.* ∘

Roots are special cases of weights, which are eigenvalues of representations. Let $\tau : \mathfrak{g} \to \mathfrak{gl}(V)$ be an irreducible finite-dimensional representation. Then the vector space V decomposes in a direct sum

$$V = \bigoplus_{\mu} V_\mu,$$

where $\mu \in \mathfrak{a}^*$ are finitely many linear maps such that $\tau(H)$ acts on each V_μ by scalar multiplication:

$$\tau(H).v = \mu(H) \cdot v \qquad \forall H \in \mathfrak{a},\ v \in V_\mu.$$

The eigenspaces V_μ are called the *weight spaces* and the eigenvalues $\mu \in \mathfrak{a}^*$ are called *weights*. So the roots are exactly the weights of the adjoint representation.

For each representation there is a distinguished weight μ_τ, called the *highest weight* of the representation τ. It has the property, that we can express all the other weights μ_i as

$$\mu_i = \mu_\tau - \sum_{\alpha \in \Delta} c_{i,\alpha} \alpha, \tag{4.4}$$

where the $c_{i,\alpha}$ are all non-negative integers. As our representation is assumed to be faithful, $\mu_\tau \not\equiv 0$ (see [GJT98, Lem. 4.16]).

Definition 4.1.38 Let μ_i be a weight of the representation τ and μ_τ the highest weight. The *support* of μ_i is the set

$$\mathrm{Supp}(\mu_i) = \{\alpha \in \Delta \mid c_{i,\alpha} > 0\} = \{\alpha \in \Delta \mid c_{i,\alpha} \neq 0\},$$

where the $c_{i,\alpha}$ are the coefficients as in Expression (4.4). ∘

Example 4.1.39 Let us continue our example of $\mathrm{SL}(4, \mathbb{C})$ with positive Weyl chamber $\mathfrak{a}^+ = \left\{\mathrm{diag}(h_1, \ldots, h_n) \,\middle|\, \sum_i h_i = 0,\ h_1 > \ldots > h_n\right\}$.

1) We consider the adjoint representation whose weights are the roots. With respect to the positive Weyl chamber $\mathfrak{a}^+$ the highest weight it α_{14}.

   Next we want to determine the support of the root α_{12}. Note that for all roots of $\mathfrak{sl}(n, \mathbb{C})$ it holds $\alpha_{ij}+\alpha_{jk} = \alpha_{ik}$ and $\alpha_{ij} = -\alpha_{ji}$ for all $i \neq j \neq k \in \{1, \ldots n\}$. Thereby $\alpha_{12} = \alpha_{14}-\alpha_{23}-\alpha_{34}$ and we see that

   $$\mathrm{Supp}(\alpha_{12}) = \{\alpha_{23}, \alpha_{34}\}.$$

   Similarly we get the supports of the other positive roots:

   $$\begin{aligned} &\mathrm{Supp}(\alpha_{14}) = \emptyset, && \mathrm{Supp}(\alpha_{24}) = \{\alpha_{12}\}, \\ &\mathrm{Supp}(\alpha_{13}) = \{\alpha_{34}\}, && \mathrm{Supp}(\alpha_{34}) = \{\alpha_{12}, \alpha_{23}\}, \\ &\mathrm{Supp}(\alpha_{23}) = \{\alpha_{12}, \alpha_{34}\}, && \mathrm{Supp}(\alpha_{12}) = \{\alpha_{23}, \alpha_{34}\}. \end{aligned}$$

   For the non-positive roots we have

   $$\alpha_{41} = \alpha_{14} - 2\alpha_{12} - 2\alpha_{23} - 2\alpha_{34}$$

   and using this we compute for example

   $$\alpha_{42} = -\alpha_{24} = -(\alpha_{14} - \alpha_{12}) = \alpha_{14} - \alpha_{12} - 2\alpha_{23} - 2\alpha_{34}.$$

   As all positive roots have $c_{i,\alpha} \in \{0, 1\}$ we see that all non-positive roots have support equal to Δ.

2) Now we consider the standard representation, which is obtained by the standard inclusion $\mathrm{SL}(n,\mathbb{C}) \hookrightarrow \mathrm{GL}(n,\mathbb{C})$ with the standard action on $\mathbb{C}^n$. From the condition $H.v \overset{!}{=} \mu(H)\cdot v$ for all $H \in \mathfrak{a}$, we obtain the weights $\beta_i \in \mathfrak{a}^*$ for $i \in \{1,\dots,n\}$ defined by

$$\mu_i(H) = h_i$$

for all $H = \mathrm{diag}(h_1,\dots,h_n) \in \mathfrak{a}$. The highest weight with respect to $\mathfrak{a}^+$ then is

$$\mu_\tau = \beta_1 \in \mathfrak{a}^*.$$

Now we go on for $n = 4$. By the identification $\mathfrak{a} \cong \mathfrak{a}^*$ we get

$$\beta_1 = \mu_\tau = \frac{1}{8}\,\mathrm{diag}\left(\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right).$$

For the support of the other weights we compute for $H = \mathrm{diag}(h_1,\dots,h_4) \in \mathfrak{a}$

$$(\beta_1 - \alpha_{12})(H) = h_1 - h_1 + h_2 = \beta_2(H),$$

which means that the support of β_2 is α_{12}. Similarly we get:

$$\begin{array}{ll} \mathrm{Supp}(\beta_1) = \emptyset, & \mathrm{Supp}(\beta_2) = \{\alpha_{12}\}, \\ \mathrm{Supp}(\beta_3) = \{\alpha_{12}, \alpha_{23}\}, & \mathrm{Supp}(\beta_4) = \{\alpha_{12}, \alpha_{23}.\alpha_{34}\} = \Delta. \end{array}$$

For the other weights we again have

$$-\beta_1 = \beta_1 - 2\alpha_{12} - 2\alpha_{23} - 2\alpha_{34}$$

and we conclude as before

$$\mathrm{Supp}(-\beta_1) = \mathrm{Supp}(-\beta_2) = \mathrm{Supp}(-\beta_3) = \mathrm{Supp}(-\beta_4) = \Delta. \qquad \circ$$

4.1.4 Groups Associated with Subsets of Simple Roots

Let Δ be the set of positive roots and $I \subseteq \Delta$ a subset. Associated to this set I we will now define analogs of the Lie algebras and Lie groups introduced before. The notations are compatible with those in [GJT98]. Let

$$a_I := \bigcap_{\alpha \in I} \ker \alpha$$

be the intersection of the hyperplanes where the roots $\alpha \in I$ vanish. $\mathfrak{a}_I$ itself is a vector space with a Weyl chamber system, which is exactly the restriction of the Weyl chamber system of $\mathfrak{a}$ to $\mathfrak{a}_I$. Let $\mathfrak{a}^I$ be the orthogonal complement of $\mathfrak{a}_I$ in $\mathfrak{a}$ with respect to the Killing form : $\mathfrak{a} = \mathfrak{a}_I \oplus \mathfrak{a}^I$. Denote by

$$A_I := \exp(\mathfrak{a}_I) \qquad \text{and} \qquad A^I := \exp(\mathfrak{a}^I)$$

the connected subgroups of A with Lie algebras $\mathfrak{a}_I$ and $\mathfrak{a}^I$, respectively. Then we similarly have $A = A_I \times A^I$.

When we denote by

$$\mathfrak{z} := C_{\mathfrak{g}}(\mathfrak{a}_I)$$

the centralizer of $\mathfrak{a}_I$ in $\mathfrak{g}$, we get a new semisimple Lie algebra $\mathfrak{g}^I$ as the derived algebra of $\mathfrak{z}$:

$$\mathfrak{g}^I := [\mathfrak{z}, \mathfrak{z}].$$

Here again we get a Cartan decomposition $\mathfrak{g}^I = \mathfrak{k}^I \oplus \mathfrak{p}^I$, where

$$\mathfrak{k}^I := \mathfrak{k} \cap \mathfrak{g}^I \qquad \text{and} \qquad \mathfrak{p}^I := \mathfrak{p} \cap \mathfrak{g}^I.$$

The set $\mathfrak{a}^I \subseteq \mathfrak{p}^I$ is a maximal abelian subalgebra. The Weyl chambers of $\mathfrak{a}^I$ are the orthogonal projections of the Weyl chambers in $\mathfrak{a}$ onto $\mathfrak{a}^I$ and we get a positive Weyl chamber $\mathfrak{a}^{I,+}$ as the projection of $\mathfrak{a}^+$. The roots then can be split up as $\Sigma = \Sigma_I \cup \Sigma^I$, where

$$\alpha \in \Sigma^I \iff \alpha(H) = 0 \quad \forall H \in \mathfrak{a}_I.$$

Corresponding to the positive Weyl chamber we get the sets of positive roots Σ_I^+ and $\Sigma^{I,+}$.

We now have the analogous structure theory as before associated to a subset $I \subseteq \Delta$. Therefore it is not surprising, that we can also construct an I-associated symmetric space X^I that is a subspace of X. This can be done the following way:

Let G^I be the Lie group associated to $\mathfrak{g}^I$, that is, G^I is the derived subgroup of the centralizer of A_I in G. The Lie group K^I of $\mathfrak{k}^I$ is a maximal compact subgroup of G^I and $K^I M$ is is the centralizer of A_I in K, where $M = C_K(A)$. Then

$$X^I := G^I / K^I$$

is a symmetric space of non-compact type that can be identified with the orbit $X^I = G^I.p_0$.

Similarly as before, $\mathcal{W}^I = N_{K^I}(A^I)/C_{K^I}(A^I)$ is the Weyl group of G^I while $\mathcal{W}_I < \mathcal{W}$ is the subgroup generated by the reflections in the hyperplanes $\ker(\alpha)$ for $\alpha \in I$.

Let $\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ be the Lie algebra obtained as the direct sum of the positive root spaces and N the associated nilpotent Lie group. It has a connected subgroup $N_I < N$ associated to I with Lie algebra

$$\mathfrak{n}_I := \bigoplus_{\alpha\in\Sigma_I^+} \mathfrak{g}_\alpha.$$

Example 4.1.40 To get an idea what these definitions mean we look at $\mathrm{SL}(4,\mathbb{C})$ and some subsets of simple roots. We know by Example 4.1.29 that the Killing form on $\mathfrak{sl}(4,\mathbb{C})$ is given by

$$\kappa(A, B) = 8\,\mathrm{tr}(AB).$$

Recall from Example 4.1.33 that the simple roots are given by

$$\Delta = \{\alpha_{12}, \alpha_{23}, \alpha_{34}\}$$

where $\alpha_{ij}(H) = h_i - h_j$ for all $H = \mathrm{diag}(h_1, \dots, h_4) \in \mathfrak{a}$.

We start with $I = \{\alpha_{12}\}$. Then

$$a_I = \ker(\alpha_{12}) = \{\mathrm{diag}(h, h, t, s) \in \mathfrak{a} \mid s = -2h - t;\ h, t \in \mathbb{C}\}$$

is a Weyl chamber wall and its orthogonal complement is the one-dimensional subspace

$$a^I = \{\mathrm{diag}(x, -x, 0, 0) \in \mathfrak{a} \mid x \in \mathbb{C}\}.$$

The set of roots then can be split up as (note that $\alpha_{ij} = -\alpha_{ji}$)

$$\Sigma^I = \{\alpha_{12}, \alpha_{21}\} \qquad \text{and} \qquad \Sigma_I = \{\pm\alpha_{13}, \pm\alpha_{14}, \pm\alpha_{23}, \pm\alpha_{24}, \pm\alpha_{34}\}.$$

Next we shortly look at the case $J = \{\alpha_{12}, \alpha_{23}\}$. We get a one-dimensional subspace

$$\mathfrak{a}_J = \ker(\alpha_{12}) \cap \ker(\alpha_{34}) = \{\operatorname{diag}(h, h, h, -3h) \mid h \in \mathbb{C}\}$$

with its orthogonal complement

$$\mathfrak{a}^J = \operatorname{span}\{\operatorname{diag}(1, -1, 00), \operatorname{diag}(0, 1, -1, 0)\}.$$

For the roots we obtain

$$\Sigma^J = \{\pm\alpha_{12}, \pm\alpha_{13}, \pm\alpha_{23}\}$$
$$\Sigma_J = \{\pm\alpha_{14}, \pm\alpha_{24}, \pm\alpha_{34}\}.$$

For $L = \{\alpha_{12}, \alpha_{34}\}$ we get

$$\mathfrak{a}_L = \ker(\alpha_{12}) \cap \ker(\alpha_{34}) = \{\operatorname{diag}(h, h, -h, -h) \mid h \in \mathbb{C}\}$$
$$\mathfrak{a}^L = \operatorname{span}\{\operatorname{diag}(1, -1, 00), \operatorname{diag}(0, 0, 1, -1)\}.$$

Although the dimensions of $\mathfrak{a}_L$ and $\mathfrak{a}^L$ are the same as before for J, the structure is quite different. The set of roots decomposes as

$$\Sigma^L = \{\pm\alpha_{12}, \pm\alpha_{34}\}$$
$$\Sigma_L = \{\pm\alpha_{13}, \pm\alpha_{14}, \pm\alpha_{23}, \pm\alpha_{24}\}. \qquad \circ$$

Generalized horocyclic decompositions

We will later make use of the generalized Iwasawa decompositions of G, respectively the generalized horocyclic decompositions of X.

Lemma 4.1.41 *For every $I \subseteq \Delta$ and $a^I \in A^I$, we have the following decomposition:*

$$X = a^I K^I a^{I^{-1}} N_I A \cdot p_0, \tag{4.5}$$

where the A component is unique up to the following condition: for every $a, a' \in A$, we have $a^I K^I a^{I^{-1}} N_I a \cdot p_0 = a^I K^I a^{I^{-1}} N_I a' \cdot p_0$ if and only if $(a^I)^{-1}a$ and $(a^I)^{-1}a'$ are conjugated by some element in $\mathcal{W}^I$.
The classical Iwasawa and horocyclic decompositions $G = NAK$ resp. $X = NA \cdot p_0$ correspond to $I = \emptyset$. $\circ$

Proof. Up to translating by $a^{I^{-1}}$, we can assume for simplicity that $a^I = e$. Recall that X^I was the relative symmetric space $X^I = G^I/K^I$ identified as the orbit $X^I = G^I \cdot p_0$ of p_0 in X. According to [GJT98, Corollary 2.16], we have the following generalized horocyclic decomposition (also called Langlands decomposition):

$$X = A_I N_I X^I \;=\; A_I N_I G^I \cdot p_0.$$

Furthermore, in this decomposition, the components in A_I, N_I and $X^I \simeq G^I \cdot p_0$ are unique.

The group K^I is a maximal compact subgroup of the semisimple group G^I, and A^I is a Cartan subgroup of G^I, so we can consider the Cartan decomposition of G^I as

$$G^I = K^I A^I K^I,$$

where the component in A^I is unique up to conjugation by some element in $\mathcal{W}^I$.

Fix some point $p \in X$. According to the two previous decompositions, we can find $a_I \in A_I$, $u_I \in N_I, k^I \in K^I$ and $a^I \in A^I$ such that

$$p = a_I u_I k^I a^I.p_0$$

where a_I and u_I are unique and a^I is unique up to conjugation by some element in $\mathcal{W}^I$. As A_I commutes with K^I, we also have

$$p = (a_I u_I a_I^{-1}) k^I a_I a^I.p_0.$$

Since A_I and $K^I M$ normalize N_I, we have $(a_I u_I a_I^{-1}) k^I \in K^I N_I$.

As a consequence, $p \in K^I N_I a_I a^I.p_0$, where $a_I a^I \in A$ is unique up to conjugation by some element in $\mathcal{W}^I$ (notice that $\mathcal{W}^I$ commutes with $a_I \in A_I$). □

4.1.5 Finsler Geometry

A Finsler metric on a smooth manifold M generalizes the concept of a Riemannian metric. It is a continuous family of (possibly asymmetric) norms on the tangent spaces, which are not necessarily induced by an inner product. See [BCS00], [Pla95] and [Run59] for a reference on Finsler geometry.

Definition 4.1.42 Let M be a smooth manifold. A *Finsler metric* on M is a continuous function

$$F : TM \longrightarrow [0, \infty)$$

such that, for each $p \in M$, the restriction $F|_{T_pM} : T_pM \longrightarrow [0, \infty)$ is a (possibly asymmetric) norm. ∘

The length and (forward) distance on a Finsler manifold can be defined in the same way as on a Riemannian manifold:

Definition 4.1.43 The *length* of a curve $\gamma : [0, 1] \subseteq \mathbb{R} \longrightarrow M$ is defined as

$$L(\gamma) := \int_I F(\gamma(t), \dot{\gamma}(t)) dt.$$

The *forward distance* between two points $p, q \in M$ is given by

$$d_F(p, q) := \inf_{\gamma} L(\gamma),$$

where the infimum is taken over all piecewise continuously differentiable curves $\gamma : [0, 1] \longrightarrow M$ with $\gamma(0) = p$ and $\gamma(1) = q$. ∘

Remark 4.1.44 As the norms on the tangent spaces do not have to be symmetric, we have in general $d_F(p, q) \neq d_F(q, p)$. ∘

The symmetric space X carries a G-invariant Riemannian metric, which is essentially unique (up to scaling on the irreducible factors). However, X also carries many G-invariant Finsler metrics.

Recall that a norm on a vector space was uniquely determined by its unit ball. We have a similar result for Finsler metrics on homogeneous spaces.

Lemma 4.1.45 ([Pla95, Ex. 6.1.2]) *Let M be homogeneous, that is, there is some topological group G which acts transitively on M by diffeomorphisms. Let $p_0 \in M$ be a point and $C \subseteq T_{p_0}M$ a convex G_{p_0}-invariant ball. Then there is exactly one G-invariant Finsler metric on M with C as unit ball of this norm $\|\cdot\|$.* ∘

The closed *unit ball* on $T_{p_0}M$ is given by

$$B^F_{p_0} := \{Y \in T_{p_0}M \mid F(p_0, Y) \leq 1\}, \tag{4.6}$$

which is a convex body. It is defined separately in each tangent space. Based on the previous lemma, Planche gives the following identification:

Proposition 4.1.46 ([Pla95, Thm. 6.2.1]) *There is a bijection between*

i) the $\mathcal{W}$-invariant convex closed balls B of $\mathfrak{a}$,

ii) the $\mathrm{Ad}(K)$-invariant convex closed balls C of $\mathfrak{p}$,

iii) the G-invariant Finsler metrics on X. ◦

In particular, any G-invariant Finsler metric on X gives rise to a (not necessarily symmetric) norm on the vector space $\mathfrak{a}$, whose unit ball is the $\mathcal{W}$-invariant convex ball B, and it is in turn completely determined by this norm. Using this equivalence, we can define a polyhedral Finsler:

Definition 4.1.47 A G-invariant Finsler metric on X is said to be *polyhedral* if its $\mathcal{W}$-invariant convex ball B in $\mathfrak{a}$ is a finite sided polytope. ◦

Polyhedral norms give Finsler metrics that are not Riemannian. To get the Riemannian metric we choose the Euclidean norm as shown in the following example:

Example 4.1.48 If we choose the Euclidean unit sphere with respect to the norm induced by the Killing form κ as the $\mathcal{W}$-invariant convex ball in $\mathfrak{a}$, then the corresponding Finsler structure on the symmetric space $X = G/K$ induces a Riemannian metric on $\mathfrak{g}$ for all $V, W \in T_{p_0}X$ by

$$g_{p_0}(V, W) := \frac{1}{2}\left[F(p_0, V + W)^2 - F(p_0, V)^2 - F(p_0, W)^2\right].$$

The other way round we have

$$F(V) := \sqrt{g_{p_0}(V, V)}.$$ ◦

4.2 The Intrinsic Compactification and the Compactification of a Flat in X

Let $X = G/K$ be a symmetric space with $G = \mathrm{Isom}(X)$ and $K = G_{p_0}$ for some base point $p_0 \in X$. Throughout this section, we will assume that the associated $\mathcal{W}$-invariant vector norm on the flat $\mathfrak{a}$ is such that every horofunction is a Busemann point. According to [Wal07, Thm. 1.2], this is equivalent to asking the set of extreme sets of the dual unit ball to be closed. This is a very mild condition, satisfied for example by every polyhedral norm.

Let d be the distance function associated to a G-invariant Finsler metric on $X = G/K$, and $\psi : X \to \widetilde{C}(X)$, $z \mapsto \psi_z$ with $\psi_z(x) = d(x, z) - d(p_0, z)$ the embedding defined in Subsection 3.1.1 on page 39. Let us state some basic observations.

Lemma 4.2.1 *The function $\psi_{p_0} : X \to \mathbb{R}$ is K-invariant. Moreover, for every $g \in G$, the function $\psi_{g.p_0}$ is gKg^{-1}-invariant.* ◦

Proof. Fix $g \in G$ and $k \in K$. Then, for any $x \in X$, we have

$$\begin{aligned}\psi_{g.p_0}((gkg^{-1}).x) &= d((gkg^{-1}).x,\ g.p_0) - d(p_0,\ g.p_0)\\ &= d(x,\ gk^{-1}g^{-1}g.p_0) - d(p_0,\ g.p_0)\\ &= d(x,\ g.p_0) - d(p_0,\ g.p_0) = \psi_{g.p_0}(x).\end{aligned}$$

So $\psi_{g.p_0}$ is gKg^{-1}-invariant. □

Lemma 4.2.2 *The map $\psi : X \to \widetilde{C}(X)$ is K-equivariant, that is, $\psi_{k.z}(x) = k \cdot \psi_z(x)$ for all $x, z \in X$ and $k \in K$. Hereby the action of K on $\widetilde{C}(X)$ is given by $k \cdot f(x) := f(k^{-1}x)$.* ◦

Proof. Fix $x, z \in X$ and $k \in K$. Then

$$\begin{aligned}\psi_{k.z}(x) &= d(x,\ k.z) - d(p_0,\ k.z)\\ &= d(k^{-1}.x,\ z) - d(p_0,\ z)\\ &= \psi_z(k^{-1}.x) = k \cdot \psi_z(x).\end{aligned}$$ □

The previous two lemmas will now help us to determine the horofunction compactification of X.

Lemma 4.2.3 *Let $G = K\overline{A^+}K$ be a Cartan decomposition and $X = K\overline{A^+}.p_0$. Then*

$$\overline{X}^{hor} = \overline{\psi(X)}^{\widetilde{C}(X)} = \overline{\psi(K\overline{A^+}.p_0)}^{\widetilde{C}(X)} = K\ \overline{\psi(\overline{A^+}.p_0)}^{\widetilde{C}(X)}.$$

In particular, the horofunction compactification $\overline{\psi(X)}^{\widetilde{C}(X)}$ is determined by the horofunction compactification of the flat $F = A.p_0$, or more precisely of a closed Weyl chamber $F^+ = \overline{A^+}.p_0$. ◦

Proof. Since $\overline{\psi(\overline{A^+}.p_0)}^{\widetilde{C}(X)}$ is a compact subspace of $\widetilde{C}(X)$ and K is a compact subgroup of G which acts continuously on $\widetilde{C}(X)$, we deduce that the space $K\ \overline{\psi(\overline{A^+}.p_0)}^{\widetilde{C}(X)}$ is a compact subspace of $\widetilde{C}(X)$. Since it contains $\psi(K\overline{A^+}.p_0)$, we conclude that $\overline{\psi(K\overline{A^+}.p_0)}^{\widetilde{C}(X)} \subseteq K\ \overline{\psi(\overline{A^+}.p_0)}^{\widetilde{C}(X)}$. As the converse inclusion is clear, we conclude that

$$\overline{\psi(K\overline{A^+}.p_0)}^{\widetilde{C}(X)} = K\ \overline{\psi(\overline{A^+}.p_0)}^{\widetilde{C}(X)}.$$ □

Note that the closure of $\psi(\overline{A^+}.p_0)$ is taken in $\widetilde{C}(X)$ but not in $\widetilde{C}(\overline{A^+}.p_0)$.

In order to understand the horofunction compactification $\overline{\psi(\overline{A^+}.p_0)}^{\widetilde{C}(X)}$ of a closed Weyl chamber $\mathfrak{a}^+$ in $\widetilde{C}(X)$, we will first compare it to the closure in the so called *intrinsic horofunction compactification* in $\widetilde{C}(A.p_0)$, which we define in the next section.

4.2.1 The Closure of a Flat

The *intrinsic compactification* of the flat $F = A.p_0$ is the horofunction compactification of F within the space of continuous functions on $F = A.p_0$, i.e. $\overline{\psi(F)}^{\widetilde{C}(F)}$. That is, we see $F = A.p_0$ as a space of its own. Since the exponential map $\exp : \mathfrak{a} \to A.p_0$ is a diffeomorphism, the intrinsic compactification is homeomorphic to the horofunction compactification of the normed vector space $\mathfrak{a}$ with respect to the norm defined by the $\mathcal{W}$-invariant convex ball B. The aim of this section is to compare the intrinsic compactification of F with the closure of the flat F in the horofunction compactification of X.

In Theorem 4.2.17 we will give for any G-invariant Finsler metric on the symmetric space an explicit homeomorphism between the intrinsic compactification of a flat and the closure of a flat in the horofunction compactification of the symmetric space X. To minimize confusion, we introduce the following notation: Let

$$\begin{aligned} \psi^X : X &\longrightarrow \widetilde{C}(X) \\ z &\longmapsto \psi^X_z := d(\cdot, z) - d(p_0, z) \end{aligned} \tag{4.7}$$

be the embedding of X into the space of continuous functions on X vanishing at p_0. The closure of ψ^X in $\widetilde{C}(X)$ gives the horofunction compactification of X with respect to the G-invariant Finsler norm defining d.

We denote by d also the restriction of the distance function to the flat $F = A.p_0 \subseteq X$ and let

$$\begin{aligned} \psi^F : F &\longrightarrow \widetilde{C}(F) \\ z &\longmapsto \psi^F_z := d(\cdot, z) - d(p_0, z) \end{aligned} \tag{4.8}$$

denote the embedding of F into the space of continuous functions on F vanishing at p_0. The closure of $\psi^F(F) \subseteq \widetilde{C}(F)$ is the intrinsic compactification of F. We set $F^+ := \overline{A^+}.p_0$.

Types of Sequences and Horofunctions

We have seen in Lemma 4.2.1 that each function $\psi_{g.p_0}$ is invariant under the conjugate gKg^{-1} of the maximal compact subgroup K. In order to study the invariance properties of horofunctions, we will use the study of limits of conjugates of K (see [GJT98, Chapter IX]). In order to describe such limits, we need to introduce the notion of type of a diverging sequence of elements in A. Roughly speaking, the type of a sequence encodes the roots "along which" the sequence goes to infinity.

Definition 4.2.4 A sequence $(a_n)_{n\in\mathbb{N}}$ in $\overline{A^+}$ is said to be of *type* $(I, \hat{a}^I)$, where I is a proper subset of Δ and $\hat{a}^I \in A^I$, if

i) for $\alpha \in I$, the limit $\lim_{n\to\infty} \alpha(\log a_n)$ exists and is equal to $\alpha(\log \hat{a}^I)$,

ii) for $\alpha \in \Delta\backslash I$, there holds $\alpha(\log a_n) \to +\infty$. ◦

To minimize notational confusion, we denote elements in A^I that define the type of a sequence by a hat: $\hat{a}^I \in A^I$.

Example 4.2.5 As as example let us look at some sequences in $\mathrm{SL}(4, \mathbb{C})$ and determine their types. We start with

$$\log(a_n) = \mathrm{diag}(n + 4, n, 6, -2n - 10) \in \mathfrak{a}^+$$

with $\mathfrak{a}^+ = \{\mathrm{diag}(h_1, \dots, h_4) \mid \sum_i h_i = 0,\ h_1 > h_2 > h_3 > h_4\}$ as determined in Example 4.1.33. Then we compute

$$\begin{aligned} \alpha_{12}(\log(a_n)) &= 4; \\ \alpha_{23}(\log(a_n)) &= n - 6; \\ \alpha_{34}(\log(a_n)) &= 2n + 16. \end{aligned}$$

Therefore $(a_n)_n$ has type $(I, \hat{a}^I)$ with $I = \{\alpha_{12}\}$ and limit $\hat{a}^I = \mathrm{diag}(e^2, e^{-2}, 1, 1) \in \mathfrak{a}^I$.

Next we consider the sequence $(b_n)_{n\in\mathbb{N}}$ given by

$$\log(b_n) = \mathrm{diag}(n+1, n, n-10, -3n+9).$$

We again compute

$$\begin{aligned}\alpha_{12}(\log(b_n)) &= 1;\\ \alpha_{23}(\log(b_n)) &= 10;\\ \alpha_{34}(\log(b_n)) &= 4n-19.\end{aligned}$$

Now the limit of α_{23} is also finite and we get $J = \{\alpha_{12}, \alpha_{23}\}$ and $\hat{b}^J = \mathrm{diag}(e^4, e^3, e^{-7}, 1)$ such that $(b_n)_n$ has type $(J, \hat{b}^J)$.

The last sequence we want to look at is

$$\log(c_n) = \mathrm{diag}(n+8, n, -n+2, -n-10).$$

Here we get

$$\begin{aligned}\alpha_{12}(\log(c_n)) &= 8;\\ \alpha_{23}(\log(c_n)) &= 2n-2;\\ \alpha_{34}(\log(c_n)) &= 12.\end{aligned}$$

Therefore we know that $(c_n)_{n\in\mathbb{N}}$ has type $(K, \hat{c}^K)$ with $K = \{\alpha_{12}, \alpha_{34}\}$ and $\hat{c}^K = \mathrm{diag}(e^4, e^{-4}, e^6, e^{-6})$. ◦

The main result on limits of conjugates of K is the following.

Proposition 4.2.6 ([GJT98, Proposition 9.14]) *Let $(a_n)_{n\in\mathbb{N}}$ be an unbounded sequence in $\overline{A^+}$ of type $(I, \hat{a}^I)$. In the space of closed subgroups of G, endowed with the Chabauty topology, the sequence $(a_n K a_n{}^{-1})_{n\in\mathbb{N}}$ converges to $\hat{a}^I K^I M (\hat{a}^I)^{-1} N_I$.* ◦

Recall that we gave a short definition of the Chabauty topology after Proposition 3.1.6.

Remark 4.2.7 Since the groups $\hat{a}^I K^I M (\hat{a}^I)^{-1} N_I$ arise as limits of the maximal compact subgroups under conjugations by sequences of type I in A, the (generalized) Iwasawa decompositions can thus be seen as limits of the Cartan decomposition. ◦

We will now use this result to deduce some invariance for horofunctions.

Lemma 4.2.8 *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\overline{A^+}$ of type $(I, \hat{a}^I)$ such that $(\psi^X_{a_n.p_0})_{n\in\mathbb{N}}$ converges to ξ. Then ξ is $\hat{a}^I K^I M (\hat{a}^I)^{-1} N_I$-invariant.* ◦

Proof. For each $n \in \mathbb{N}$, the function $\psi^X_{a_n.p_0}$ is invariant under $a_n K a_n^{-1}$, because $K = G_{p_0}$ is the stabilizer of the base point p_0. Since the sequence $(a_n K a_n^{-1})_{n\in\mathbb{N}}$ converges to $\hat{a}^I K^I M (\hat{a}^I)^{-1} N_I$ in the Chabauty topology (see Proposition 4.2.6 above), for every $g \in \hat{a}^I K^I M (\hat{a}^I)^{-1} N_I$ there exists a sequence $(k_n)_n$ in K such that the sequence $(a_n k_n a_n^{-1})_n$ converges to g. Therefore, for every $p \in X$ we have

$$\begin{aligned}\xi(g.p) - \xi(p) &= \lim_{n\to+\infty} \psi^X_{a_n.p_0}(g.p) - \psi^X_{a_n.p_0}(p)\\ &= \lim_{n\to+\infty} d(g.p,\ a_n.p_0) - d(p,\ a_n.p_0)\\ &= \lim_{n\to+\infty} d(a_n k_n a_n^{-1}.p,\ a_n.p_0) - d(p,\ a_n.p_0) = 0.\end{aligned}$$

As a consequence, ξ is invariant under $\hat{a}^I K^I M (\hat{a}^I)^{-1} N_I$. □

Definition 4.2.9 A horofunction $\eta \in \partial\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$ is said to be of *type* $(I, \hat{a}^I)$, where I is a proper subset of Δ and $\hat{a}^I \in A^I$, if there exists an almost geodesic sequence $(a_n)_{n\in\mathbb{N}}$ in A of type $(I, \hat{a}^I)$ such that the sequence $(\psi^F_{a_n.p_0})_{n\in\mathbb{N}}$ converges to η in $\widetilde{C}(F)$. Note that, since we assumed that every horofunction is a Busemann point, a horofunction may have several types, but has at least one type. ∘

Lemma 4.2.10 *Let $\eta \in \partial\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$ be a horofunction which has two types $(I, \hat{a}^I)$ and $(J, \hat{b}^J)$, where $I, J \subseteq \Delta$ with $\hat{a}^I \in A^I$ and $\hat{b}^J \in A^J$. Then the horofunction η also has type $(I \cap J, \hat{c}^{I\cap J})$ for some $\hat{c}^{I\cap J} \in A^{I\cap J}$.* ∘

Proof. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two almost geodesic sequences in $\overline{A^+}$ of different types $(I, \hat{a}^I)$ and $(J, \hat{b}^J)$ respectively, such that the sequences $(\psi^F_{a_n.p_0})_{n\in\mathbb{N}}$ and $\psi^F_{b_n.p_0})_{n\in\mathbb{N}}$ both converge to η. For every $n \in \mathbb{N}$, we define

$$c_n := \exp\left(\frac{1}{2}\log(a_n) + \frac{1}{2}\log(b_n)\right).$$

The sequence $(c_n)_{n\in\mathbb{N}}$ has type $(I\cap J, \hat{c}^{I\cap J})$, where $\hat{c}^{I\cap J} \in \left(\exp\left(\frac{1}{2}\log(\hat{a}^I) + \frac{1}{2}\log(\hat{b}^J)\right)A_{I\cup J}\right) \cap A^{I\cap J}$. According to the Convexity Lemma (Lemma 3.1.16), the sequence $(\psi^F_{c_n.p_0})_n$ also converges to η. As a consequence, η has type $(I \cap J, \hat{c}^{I\cap J})$. □

Lemma 4.2.11 *Let $\eta \in \partial\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$ be a horofunction of type $(I, \hat{a}^I)$, where $I \subsetneq \Delta$ and $\hat{a}^I \in A^I$. If η is invariant under A^L for some subset $L \subseteq I$, then η also has type $(I\backslash L, \hat{c}^{I\backslash L})$ for some $\hat{c}^{I\backslash L} \in A^{I\backslash L}$.* ∘

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be an almost geodesic sequences in $\overline{A^+}$ of type $(I, \hat{a}^I)$ such that the sequence $(\psi^F_{a_n.p_0})_{n\in\mathbb{N}}$ converges to η. Fix $c \in A^{L,+}$. For each $k \in \mathbb{N}$, the sequence $(\psi^F_{c^k a_n.p_0})_{n\in\mathbb{N}}$ converges to $c^k \cdot \eta = \eta$, since η is invariant under A^L. As a consequence, there exists $n_k \in \mathbb{N}$ such that, for every $n \geq n_k$, and for every $a \in A$ such that $d(p_0, a.p_0) \leq k$, we have

$$\left|d\left(a.p_0, c^k a_n.p_0\right) - d(a.p_0, a_n.p_0)\right| \leq \frac{1}{k+1}.$$

We can also assume that the sequence $(n_k)_k$ is increasing. Fix $a \in A$. For every $k \geq d(p_0, a.p_0)$ we have $|d(a.p_0, c^k a_{n_k}.p_0) - d(a.p_0, a_{n_k}.p_0)| \leq \frac{1}{k+1}$ and $|d(p_0, c^k a_{n_k}.p_0) - d(p_0, a_{n_k}.p_0)| \leq \frac{1}{k+1}$. Therefore, we have

$$\begin{aligned}
\lim_{k\to\infty} \psi_{c^k a_{n_k}.p_0}(a.p_0) &= \lim_{k\to+\infty} d(a.p_0, c^k a_{n_k}.p_0) - d(p_0, c^k a_{n_k}.p_0) \\
&= \lim_{k\to+\infty} d(a.p_0, a_{n_k}.p_0) - d(p_0, a_{n_k}.p_0) \\
&= \eta(a.p_0) - \eta(p_0).
\end{aligned}$$

As a consequence, the sequence $\left(\psi^F_{c^k a_{n_k}.p_0}\right)_{k\in\mathbb{N}}$ converges to η.

To conclude, observe that the sequence $(c^k a_{n_k})_{k\in\mathbb{N}}$ has type $(I\backslash L, \hat{c}^{I\backslash L})$, for some $\hat{c}^{I\backslash L} \in A^{I\backslash L}$. □

4.2.2 Some Technical Lemmas

Before we come to the comparison of the compactifications of a flat in the next section, we state some technical results that will be used in the proof of Theorem 4.2.17. They are all about finding subsets of the simple roots that satisfy some orthogonality and invariance conditions.

Definition 4.2.12 Two subsets I, J of Δ are said to be *orthogonal* if the roots α and β are orthogonal for every $\alpha \in I$ and $\beta \in J$. A subset $I \subseteq \Delta$ is called *irreducible* if it is not a disjoint union of two proper orthogonal subsets. ◦

Lemma 4.2.13 *Fix a subset I of Δ and consider a linear subspace V of $\mathfrak{a}^I$ which is invariant under the action of $\mathcal{W}^I$. Then there exists a subset $J \subseteq I$ such that $V = \mathfrak{a}^J$, and J and $I \backslash J$ are orthogonal.* ◦

Proof. Let $I = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_r$ be the decomposition of I into irreducible subsets. The linear representation of $\mathcal{W}^I$ on $\mathfrak{a}^I$ decomposes as the direct sum of the irreducible representations $\mathfrak{a}^I = \bigoplus_{j=1}^{r} \mathfrak{a}^{J_j}$. Since V is a $\mathcal{W}^I$-invariant subspace, there exists $R \subseteq \{1, 2, \ldots, r\}$ such that $V = \bigoplus_{j \in R} \mathfrak{a}^{J_j}$. As a consequence, we have $V = \mathfrak{a}^J$, where $J = \bigsqcup_{j \in R} J_j$. □

Lemma 4.2.14 *Let C be a non-discrete subset of A^I. Let $L \subseteq I$ denote the smallest subset such that the following conditions are satisfied:*

i) $C \subseteq cA^L$ for all $c \in C$,

ii) L and $I \backslash L$ are orthogonal.

Then the smallest closed subgroup of $\mathcal{W}^I A$ containing all conjugates $\{c\mathcal{W}^I c^{-1} \mid c \in C\}$ is equal to $\mathcal{W}^I A^L$. ◦

Proof. In this proof, we will identify A with its Lie algebra and thus consider A as a vector space. Up to conjugating, we can assume that the affine subspace of A spanned by C contains 0. Let $\Gamma \subseteq \mathcal{W}^I A$ denote the smallest closed subgroup containing all conjugates $\{c\mathcal{W}^I c^{-1} \mid c \in C\}$. Since C is non-discrete, Γ is not discrete and the linear part of Γ is equal to $\mathcal{W}^I$. So the identity component Γ_0 of Γ is a vector subspace of A^I containing C. Since Γ_0 is invariant under $\mathcal{W}^I$, we deduce according to Lemma 4.2.13 that $\Gamma_0 = A^L$, for some $L \subseteq I$ such that L and $I \backslash L$ are orthogonal. □

Lemma 4.2.15 *Let $\eta \in \overline{\psi^F(F^+)}^{\widetilde{C}(F)}$ be a horofunction that has two types $(I, \hat{a}^I)$ and $(I, \hat{b}^I)$ with $\hat{a}^I, \hat{b}^I \in A^I$. Then there exists a subset $L \subseteq I$ such that :*

i) $\hat{a}^I \in \hat{b}^I A^L$,

ii) the roots in L and $I \backslash L$ are orthogonal, and

iii) η is $\mathcal{W}^I A^L$-invariant. ◦

Proof. For simplicity, up to translating by $(\hat{a}^I)^{-1}$, we may assume that $\hat{a}^I = e$.

Fix $\lambda \in [0, 1]$. For each $n \in \mathbb{N}$, let $c_n = \exp\left((1-\lambda)\log a_n + \lambda \log b_n\right) \in A$. According to the Convexity Lemma 3.1.16, the sequence $(\psi^F_{c_n \cdot p_0})_{n\in\mathbb{N}}$ converges to η. The sequence $(c_n)_{n\in\mathbb{N}}$ is of type $\left(I, (\hat{b}^I)^\lambda\right)$, where $(\hat{b}^I)^\lambda$ denotes $\exp(\lambda \log \hat{b}^I)$. Since the sequence $(\psi^F_{c_n \cdot p_0})_{n\in\mathbb{N}}$ converges to η, we know by Lemma 4.2.8 that η is $(\hat{b}^I)^\lambda \mathcal{W}^I \left((\hat{b}^I)^\lambda\right)^{-1}$-invariant for every $\lambda \in [0, 1]$. Let $L \subseteq I$ be the smallest subset such that $\hat{b}^I \in A^L$ and such that the roots in L and in $I \backslash L$ are orthogonal.Then by Lemma 4.2.14, η is invariant under $\mathcal{W}^I A^L$. □

Lemma 4.2.16 *Let $\eta \in \overline{\psi^F(F^+)}^{\widetilde{C}(F)}$ be a horofunction that has two types $(I, \hat{a}^I)$ and $(J, \hat{b}^J)$ where $J \subsetneq I \subseteq \Delta$. There exists a subset $L \subseteq I$ such that :*

i) $J \cup L = I$,

ii) *the roots in* L *and* $I\backslash L$ *are orthogonal, and*

iii) η *is* $\mathcal{W}^I A^L$*-invariant.* ◦

Proof. Let $(a_n)_n$ and b_n be the sequences of type I and J converging to η. For simplicity, up to translating by $(\hat{a}^I)^{-1}$, we may assume that $\hat{a}^I = e$. Up to passing to a subsequence, let us partition $I\backslash J$ into $I\backslash J = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_p$ such that:

- $\forall 1 \le i \le p, \forall \alpha, \beta \in I_i : \quad \lim_{n\to+\infty} \frac{\alpha(\log b_n)}{\beta(\log b_n)} \in (0, +\infty)$,
- $\forall 1 \le i < j \le p, \forall \alpha \in I_i, \forall \beta \in I_j : \quad \lim_{n\to+\infty} \frac{\alpha(\log b_n)}{\beta(\log b_n)} = 0$.

Fix $1 \le i \le p$ and for some $\alpha \in I_i$ define

$$t_n := \frac{1}{\alpha(\log b_n)}$$

such that $t_n \longrightarrow 0$ as $n \to +\infty$. Fix $\lambda > 0$. For each $n \in \mathbb{N}$, let

$$c_n = \exp\left((1 - \lambda t_n) \log a_n + \lambda t_n \log b_n\right) \in A.$$

According to Lemma 3.1.16, the sequence $(\psi^F_{c_n \cdot p_0})_{n\in\mathbb{N}}$ converges to η. Let us define

$$\hat{c}^{I_i} := \lim_{n\to+\infty} \left(\pi^{I_i}(b_n)\right)^{t_n} \in A^{I_i},$$

where $\pi^{I_i}(b_n)$ denotes the orthogonal projection of b_n onto A^{I_i}. This definition makes sense because the sequence converges: for any $\beta \in I_i$, we have

$$\beta\left(\log\left(\pi^{I_i}(b_n)\right)^{t_n}\right) = t_n \beta(\log b_n) = \frac{\beta(\log b_n)}{\alpha(\log b_n)},$$

so $\lim_{n\to+\infty} \beta\left(\log\left(\pi^{I_i}(b_n)\right)^{t_n}\right) \in (0, +\infty)$.

On the other hand, for any $\beta \in \Delta\backslash I_i$, we have

$$\beta\left(\log\left(\pi^{I_i}(b_n)\right)^{t_n}\right) = 0, \tag{4.9}$$

so the limit $\hat{c}^{I_i} \in A^{I_i}$ exists. Furthermore, we have $\hat{c}^{I_i} \in (A^{I_i})^+$. Let

$$J_i := J \sqcup I_1 \sqcup \cdots \sqcup I_i.$$

For every $\gamma \in \Delta\backslash J_i$ we have

$$\gamma(\log c_n) = (1 - \lambda t_n)\gamma(\log a_n) + \lambda t_n \alpha(\log b_n) \longrightarrow +\infty.$$

For every $\gamma \in J \cup I_1 \cup \cdots \cup I_{i-1}$ we have

$$\gamma(\log c_n) = (1 - \lambda t_n)\gamma(\log a_n) + \lambda t_n \gamma(\log b_n) \longrightarrow \gamma(\log \hat{a}^I) = 0.$$

For every $\gamma \in I_i$ we have

$$\begin{aligned}\gamma(\log c_n) &= (1 - \lambda t_n)\gamma(\log a_n) + \lambda t_n \gamma(\log b_n) \\ &\longrightarrow \gamma(\log \hat{a}^I) + \lambda\gamma(\log \hat{c}^{I_i}) = \lambda\gamma(\log \hat{c}^{I_i}).\end{aligned}$$

As a consequence (using Equation (4.9)) the sequence $(c_n)_{n\in\mathbb{N}}$ is of type $\left(J_i, (\hat{c}^{I_i})^\lambda\right)$, where $(\hat{c}^{I_i})^\lambda$ denotes $\exp(\lambda \log \hat{c}^{I_i})$. Since the sequence $(\psi^F_{c_n.p_0})_{n\in\mathbb{N}}$ converges to η, we deduce by Lemma 4.2.8 that η is $(\hat{c}^{I_i})^\lambda \mathcal{W}^{I_i}\left((\hat{c}^{I_i})^\lambda\right)^{-1}$-invariant.

As $c^{I_i} \in (A^{I_i})^+$, we know by Lemma 4.2.14 that η is also invariant under A^{I_i}. Because this is true for every $1 \leq i \leq p$, we conclude that η is invariant under $A^{I\backslash J}$.

Since the sequence $(a_n)_{n\in\mathbb{N}}$ is of type (I, e), and the sequence $(\psi^F_{a_n.p_0})_{n\in\mathbb{N}}$ converges to η, we know by Lemma 4.2.8 that η is $\mathcal{W}^I$-invariant. Therefore η is invariant under $\mathcal{W}^I$ and $A^{I\backslash J}$. The smallest closed subgroup of $\mathcal{W}^I A^I$ containing both $\mathcal{W}^I$ and $A^{I\backslash J}$ is $\mathcal{W}^I A^L$, where $L \subseteq I$ is the smallest subset containing $I\backslash J$ such that the roots in L and in $I\backslash L$ are orthogonal. We conclude that η is invariant under $\mathcal{W}^I A^L$. □

4.2.3 The Intrinsic Compactification versus the Closure of a Flat

Recall that on a flat F we have two compactification to consider: the intrinsic compactification of F, namely $\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$, and the closure of F in $\overline{X}^{hor}$, namely $\overline{\psi^X(F^+)}^{\widetilde{C}(X)}$. In this section we define an explicit map from the intrinsic compactification of the flat F into the horofunction compactification of X. For this we use the invariance shown in Lemma 4.2.8 and the generalized horocyclic decomposition $X = a^I K^I a^{I^{-1}} N_I A.p_0$ from Lemma 4.1.41.

For a horofunction η of type $(I, \hat{a}^I)$ we define the following map:

$$\begin{aligned} \psi^X_\eta : X &\longrightarrow \mathbb{R}, \\ \hat{a}^I k^I (\hat{a}^I)^{-1} u_I a.p_0 \in X &\longmapsto \eta(a.p_0). \end{aligned}$$

Theorem 4.2.17 *The following map*

$$\phi : \overline{\psi^F(F^+)}^{\widetilde{C}(F)} \longrightarrow \overline{\psi^X(X)}^{\widetilde{C}(X)}, \qquad \begin{cases} \psi^F_z & \longmapsto \psi^X_z \quad \text{for } z \in F^+, \\ \eta & \longmapsto \psi^X_\eta \quad \text{for } \eta \text{ of type } (I, \hat{a}^I). \end{cases}$$

is a well-defined, continuous embedding. ∘

Proof. We will first show that ϕ is well-defined and then continuity. As the restriction to F^+ is a left-inverse to ϕ, we deduce that ϕ is injective. Since $\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$ is compact, ϕ is then an embedding.

Well-definedness

We want to prove that the map ϕ is well-defined. To do so, we first show that the formula defining ϕ is independent of the choice of the component in A. Then we will show that if η has two types, then ϕ still defines the same horofunction independent of the types.

Consider first a horofunction $\eta \in \partial\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$ which has some type $(I, \hat{a}^I)$ and consider two decompositions $\hat{a}^I k^I (\hat{a}^I)^{-1} u_I a.p_0 = \hat{a}^I k'^I (\hat{a}^I)^{-1} u'_I a'.p_0$ of the same point in X as given in Equation (4.5). According to Lemma 4.1.41, there exists $w \in \mathcal{W}^I$ such that $(\hat{a}^I)^{-1} a' = w(\hat{a}^I)^{-1} a w^{-1}$. By Lemma 4.2.8 we know that η is invariant under $\hat{a}^I \mathcal{W}^I (\hat{a}^I)^{-1}$, so

$$\eta(a'.p_0) = \eta\left(\hat{a}^I w (\hat{a}^I)^{-1} a w^{-1}.p_0\right) = \eta\left((\hat{a}^I w (\hat{a}^I)^{-1}) a.p_0\right) = \eta(a.p_0).$$

This means that the formula defining ϕ does not depend on the choice of the A component in the decomposition $X = \hat{a}^I K^I (\hat{a}^I)^{-1} N_I A.p_0$.

Consider now a horofunction $\eta \in \partial\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$ which has two types $(I, \hat{a}^I)$ and $(J, \hat{b}^J)$. We will prove that the two formulas defining $\phi(\eta)$, for each type, agree. Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subseteq \overline{A^+}$ be two sequences of type $(I, \hat{a}^I)$ and $(J, \hat{b}^J)$ respectively, such that $(\psi^F_{a_n.p_0})_{n\in\mathbb{N}}$ and $(\psi^F_{b_n.p_0})_{n\in\mathbb{N}}$ both converge to η. Up to passing to a subsequence, we may assume that the sequences $(\psi^X_{a_n.p_0})_{n\in\mathbb{N}}$ and $(\psi^X_{b_n.p_0})_{n\in\mathbb{N}}$ converge to ξ and ξ' respectively.

We need to prove that $\xi = \xi'$, which will be done by induction on $|I|+|J|$. As we know (see Lemma 4.2.10) that η also has type $(I \cap J, \hat{c}^{I\cap J})$ for some $\hat{c}^{I\cap J} \in A^{I\cap J}$, we assume from now on that $J \subseteq I$.

Assume first that $|I| + |J| = 0$, so $I = J = \emptyset$. According to Lemma 4.2.8, ξ and ξ' are both N-invariant, so for every $p = ua.p_0 \in X = NA.p_0$, we have $\xi(p) = \eta(a.p_0) = \xi'(p)$. Therefore $\xi = \xi'$.

By induction, fix $m \in \mathbb{N}$ and assume that if $|I| + |J| \leq m$, then $\xi = \xi'$. Consider now I, J such that $|I| + |J| = m + 1$. We will distinguish the two cases $J = I$ and $J \subsetneq I$.

The case $J = I$ Assume that $J = I$. By Lemma 4.2.15 we know that there is a subset $L \subseteq I$ such that L and $I \backslash L$ are orthogonal and η is $\mathcal{W}^I A^L$-invariant. Therefore by Lemma 4.2.11, we know that η also has type $(I\backslash L, \hat{c}^{I\backslash L})$, for some $\hat{c}^{I\backslash L} \in A^{I\backslash L}$. Let $(c_n)_{n\in\mathbb{N}}$ denote a sequence of type $(I\backslash L, \hat{c}^{I\backslash L})$ such that the sequence $(\psi^F_{c_n.p_0})_{n\in\mathbb{N}}$ converges to η. Up to passing to a subsequence, assume that the sequence $(\psi^X_{c_n.p_0})_{n\in\mathbb{N}}$ converges to some ξ''.

The result will now follow by two inductions. Since $\hat{a}^I \in \hat{b}^I A^L$ (see Lemma 4.2.15) and $\hat{a}^I \neq \hat{b}^I$, we know that $L \neq \emptyset$. Therefore (recall that $I = J$) we have $|I| + |I\backslash L| < |I| + |I| = m + 1$, so $|I| + |I\backslash L| \leq m$. By induction applied to the sequences $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$, we deduce that $\xi = \xi''$. By induction applied to the sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$, we deduce that $\xi' = \xi''$. In conclusion, we have $\xi = \xi'$. This concludes the induction, and finishes the proof that $\xi = \xi'$ in the case where $J = I$.

The case $J \subsetneq I$ Assume that $J \subsetneq I$. Similarly to the case before, we first observe an extra invariance of η: By Lemma 4.2.16 we know that there is a subset $L \subseteq I$ with $I = J \cup L$ such that L and $I\backslash L$ are orthogonal and η is $\mathcal{W}^I A^L$-invariant. To conclude the result by induction, we have to distinguish again two cases depending on whether $I \setminus L = J$ or not.

Since η is invariant under A^L, we deduce again by Lemma 4.2.11 that η has type $(I\backslash L, \hat{c}^{I\backslash L})$, for some $\hat{c}^{I\backslash L} \in A^{I\backslash L}$.

If $I\backslash L \subsetneq J$, then $|I| + |I\backslash L| < |I| + |J|$ and $|J| + |I\backslash L| < |I| + |J|$, so by applying the induction twice, we know that $\xi = \xi'$.

We are left with the case $I\backslash L = J$. In this case J and $L = I\backslash J$ are orthogonal and η is $A^{I\backslash J}$-invariant. We show that $\xi = \xi'$.

By the orthogonality of J and $L = I\backslash J$ we have the orthogonal decomposition $A^I = A^J A^L$. Let us decompose $\hat{a}^I = \hat{a}^J \hat{a}^L \in A^J A^L$. Up to translating by $(\hat{a}^J \hat{a}^L)^{-1}$, we can assume that $\hat{a}^J = e$ and $\hat{a}^I = \hat{a}^L \in A^L$.

As Σ^J and Σ^L are orthogonal, we have the decomposition $K^I = K^J K^L$, with K^J and K^L commuting. Furthermore K^J and A_J are commuting. Since $A^L \subseteq A_J$, we deduce that $\hat{a}^L$ commutes with K^J. In particular,

$$\hat{a}^L K^I (\hat{a}^L)^{-1} = K^J \hat{a}^L K^L (\hat{a}^L)^{-1}. \tag{4.10}$$

Let $p \in X$ be any fixed point. We will show that $\xi'(p) = \xi(p)$. This will be done by showing that $\xi'(p) = \eta(c_L.p_0) = \xi(p)$ for some $c_L \in A_L$ that will be defined on the way. We start with $\xi'(p)$.

Using Equation (4.10) in the generalized horocyclic decomposition given in Lemma 4.1.41, we get $X = \hat{a}^L K^I (\hat{a}^L)^{-1} N_I A.p_0 = K^J N_I \hat{a}^L K^L (\hat{a}^L)^{-1} A.p_0$. Write

$$p = k^J u_I \hat{a}^L k^L (\hat{a}^L)^{-1} c.p_0 \in X, \tag{4.11}$$

where $k^J \in K^J$, $u_I \in N_I$, $k^L \in K^L$ and $c \in A$. According to Lemma 4.2.8 and because we assumed $\hat{a}^J = e$, we know that ξ' is invariant under $K^J M N_J$. Since $N_I \subseteq N_J$, we conclude that

$$\xi'(p) = \xi'\left(\hat{a}^L k^L (\hat{a}^L)^{-1} c.p_0\right). \tag{4.12}$$

In the decomposition $A = A_L A^L$, let us write $c = c_L c^L$. Then

$$\hat{a}^L k^L (\hat{a}^L)^{-1} c = c_L \hat{a}^L k^L (\hat{a}^L)^{-1} c^L \in c_L G^L. \tag{4.13}$$

By the Iwasawa decomposition it is $G^L = N^L A^L K^L$, and therefore we can find $u^L \in N^L$ and $d^L \in A^L$ such that $u^L \hat{a}^L k^L (\hat{a}^L)^{-1} c^L \in d^L K^L$. As $K = G.p_0$ we get from Equation (4.13):

$$u^L \cdot \left(\hat{a}^L k^L (\hat{a}^L)^{-1} c.p_0\right) = c_L d^L.p_0. \tag{4.14}$$

We claim that

$$\xi'(p) = \xi'\left(c_L d^L.p_0\right),$$

or equivalently by Equation (4.12) and (4.14), that

$$\xi'\left(\hat{a}^L k^L (\hat{a}^L)^{-1} c.p_0\right) = \xi'\left(u^L \hat{a}^L k^L (\hat{a}^L)^{-1} c.p_0\right).$$

Since the sequence $(b_n K b_n^{-1})_{n\in\mathbb{N}}$ converges to $K^J M N_J$ in the Chabauty topology (see Proposition 4.2.6), and as $u^L \in N^L \subseteq N_J$, there exists a sequence $(k_n)_{n\in\mathbb{N}}$ such that the sequence $(b_n k_n b_n{}^{-1})_{n\in\mathbb{N}}$ converge to u^L. Therefore:

$$\begin{aligned}
&\xi'\left(u^L \hat{a}^L k^L (\hat{a}^L)^{-1} c.p_0\right) - \xi'\left(\hat{a}^L k^L (\hat{a}^L)^{-1} c.p_0\right)\\
&\quad = \lim_{n\to+\infty} d\left(u^L \hat{a}^L k^L (\hat{a}^L)^{-1} c.p_0,\ b_n.p_0\right) - d\left(\hat{a}^L k^L (\hat{a}^L)^{-1} c.p_0,\ b_n.p_0\right)\\
&\quad = \lim_{n\to+\infty} d\left(b_n k_n b_n^{-1} \hat{a}^L k^L (\hat{a}^L)^{-1} c.p_0,\ b_n.p_0\right) - d\left(\hat{a}^L k^L (\hat{a}^L)^{-1} c.p_0,\ b_n.p_0\right)\\
&\quad = 0.
\end{aligned}$$

Hence $\xi'\left(u^L \hat{a}^L k^L (\hat{a}^L)^{-1} c.p_0\right) = \xi'\left(\hat{a}^L k^L (\hat{a}^L)^{-1} c.p_0\right)$, so

$$\xi'(p) = \xi'\left(c_L d^L.p_0\right).$$

By assumption, η is invariant under $A^{I\setminus J} = A^L$. Since we have $c_L \in A_L$ as well as $d^L \in A^L$ and since ξ' and η coincide on A, we have

$$\xi'(p) = \xi'\left(c_L d^L.p_0\right) = \eta\left(c_L d^L.p_0\right) = \eta(c_L.p_0).$$

Next we want to show that also $\xi(p) = \eta(c_L.p_0)$. According to Equation (4.11) and Lemma 4.2.8, we have

$$\xi(p) = \xi\left(k^J u_I \hat{a}^L k^L (\hat{a}^L)^{-1} c.p_0\right) = \xi(c.p_0) = \eta(c.p_0),$$

because $c \in A$. Since $c = c_L c^L$ and η is invariant under A^L, we conclude that $\xi(p) = \eta(c_L.p_0)$. Therefore $\xi'(p) = \xi(p)$, and as p was some arbitrary point, we get $\xi = \xi'$.

We have shown that in any case, if η has two types $(I, \hat{a}^I)$ and $(J, \hat{b}^J)$ with associated sequences $(a_n)_n$ and $(b_n)_n$ respectively, then the limits $\xi = \lim_n \psi^X_{a_n.p_0}$ and $\xi' = \lim_n \psi^X_{b_n.p_0}$ coincide. This shows that the map ϕ is well-defined.

Continuity

We want to prove that the map ϕ is continuous. It is clear that ϕ is continuous on the interior $\psi^F(F^+)$. So we need to show that ϕ is continuous at η for some fixed $\eta \in \partial\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$.

Claim 1 Let $(a_n)_{n\in\mathbb{N}}$ be an almost geodesic sequence in $\overline{A^+}$ such that the sequence $(\psi^F_{a_n.p_0})_{n\in\mathbb{N}}$ converges to η. Then $(\psi^X_{a_n.p_0})_{n\in\mathbb{N}}$ converges to $\phi(\eta)$.

Proof of Claim 1. Up to passing to a subsequence, we may assume that the sequence $(a_n)_{n\in\mathbb{N}}$ has some type $(I, \hat{a}^I)$ and that the sequence $(\psi^X_{a_n.p_0})_{n\in\mathbb{N}}$ converges to some ξ. By Lemma 4.2.8, ξ is invariant under $\hat{a}^I K^I M (\hat{a}^I)^{-1} N_I$, so for every $p = \hat{a}^I k^I (\hat{a}^I)^{-1} u_I a.p_0 \in X = \hat{a}^I K^I (\hat{a}^I)^{-1} N_I A.p_0$, we have $\xi(p) = \xi(a.p_0) = \eta(a.p_0)$.

Furthermore, since ϕ is well-defined and η has type $(I, \hat{a}^I)$, we can use this type in the definition of $\phi(\eta)$, and thus $\phi(\eta)(p) = \eta(a.p_0) = \xi(p)$. In conclusion, $\xi = \phi(\eta)$, so $(\psi^X_{a_n.p_0})_{n\in\mathbb{N}}$ converges to $\phi(\eta)$ in $C(X)$. □

Claim 2 Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in $\partial\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$ converging to η in $\widetilde{C}(F)$. Then $(\phi(\eta_n))_{n\in\mathbb{N}}$ converges to $\phi(\eta)$ in $\widetilde{C}(X)$.

Proof of Claim 2. Up to passing to a subsequence, we may assume that the sequence $(\phi(\eta_n))_{n\in\mathbb{N}}$ converges to some horofunction ξ in $\widetilde{C}(X)$. Up to passing again to a subsequence, we may assume that there exists $I \subsetneq \Delta$ such that for each $n \in \mathbb{N}$, η_n is of type $(I, \hat{a}^I_n)$ for some $\hat{a}^I_n \in A^I$. For each $n \in \mathbb{N}$, consider a sequence $(a_{n,m})_{m\in\mathbb{N}}$ of type $(I, \hat{a}^I_n)$ converging to η_n. Up to passing to a subsequence, we may assume that the sequence $(\hat{a}^I_n)_{n\in\mathbb{N}}$ is of type $(J, \hat{a}^J)$ for some $J \subseteq I$ and some $\hat{a}^J \in A^J$. For each $n \in \mathbb{N}$, one can find some $m_n \in \mathbb{N}$ such that the sequence $(a_{n,m_n})_{n\in\mathbb{N}}$ is of type $(J, \hat{a}^J)$ and converges to η.

Fix

$$p = \hat{a}^J k^J (\hat{a}^J)^{-1} u_J c.p_0 \in X = \hat{a}^J K^J (\hat{a}^J)^{-1} N_J A.p_0.$$

Since the sequence $(\hat{a}^I_n K^I M (\hat{a}^I_n)^{-1} N_I)_{n\in\mathbb{N}}$ converges to $\hat{a}^J K^J M (\hat{a}^J)^{-1} N_J$ in the Chabauty topology (see Proposition 4.2.6), there exist sequences $(k^I_n)_{n\in\mathbb{N}}$ in $K^I M$ and $(u_{n,I})_{n\in\mathbb{N}}$ in N_I such that the sequence $(\hat{a}^I_n k^I_n (\hat{a}^I_n)^{-1} u_{n,I})_{n\in\mathbb{N}}$ converges to $\hat{a}^J k^J (\hat{a}^J)^{-1} u_J$. Hence

$$\begin{aligned}
\xi(p) &= \lim_{n\to+\infty} \phi(\eta_n)(\hat{a}^I_n k^I_n (\hat{a}^I_n)^{-1} u_{n,I} c.p_0) \\
&= \lim_{n\to+\infty} \eta_n(c.p_0) \\
&= \eta(c.p_0)
\end{aligned}$$

$$
\begin{aligned}
&= \phi(\eta)(\hat{a}^J k^J (\hat{a}^J)^{-1} u_J c.p_0)\\
&= \phi(\eta)(p).
\end{aligned}
$$

As a consequence, we have $\xi = \phi(\eta)$, so the sequence $(\phi(\eta_n))_{n\in\mathbb{N}}$ converges to $\phi(\eta)$. □

So we have proven that the map ϕ in Theorem 4.2.17 is continuous. This concludes the proof of the theorem. □

The just proven embedding shows that the horofunction compactification of the flat $F = A.p_0$ in $\overline{X}^{hor}$ is the same as the intrinsic compactification of the flat:

Theorem 4.2.18 *Let $X = G/K$ be a symmetric space of non-compact type. Consider a G-invariant Finsler metric on X such that the dual unit ball belongs to one of the cases I) - IV) and such that its set of extreme sets is closed. Let $\overline{X}^{hor}$ be the horofunction compactification of X with respect to this Finsler metric. Then the closure of a maximal flat F in $\overline{X}^{hor}$ is isomorphic to the horofunction compactification of F with respect to the induced metric.* ◦

4.3 The Satake Compactification of Symmetric Spaces

In this section we want to introduce the generalized Satake compactification of a symmetric space $X = G/K$ of non-compact type. There are several ways to define this compactification, we will follow the one Satake went in his paper [Sat60] and like it is done in [BJ06, I.4].

4.3.1 Satake Compactifications

The Satake compactification is defined in two steps. In the first one, we construct the Standard-Satake compactification $\overline{\mathcal{P}_n}^S$ of a space $\mathcal{P}_n$, which is independent of X. The second step is based on an embedding of X into $\mathcal{P}_n$ as totally geodesic submanifold and depends on a representation τ of G.

The first step Define the space

$$\mathcal{P}_n := {}^{\textstyle \mathrm{PSL}(n,\mathbb{C})}\big/_{\textstyle \mathrm{PSU}(n)}$$

and identify it via the map $m\,\mathrm{PSU}(n) \longmapsto m\, m^*$ (for $m \in \mathrm{PSL}(n,\mathbb{C})$) with the space of positive definite Hermitian matrices, where $m^* := \overline{m}^T$ denotes the conjugate transpose of $m \in \mathrm{PSL}(n,\mathbb{C})$. Let $\mathcal{H}_n$ be the real vector space of Hermitian matrices and $\mathrm{P}(\mathcal{H}_n)$ the corresponding compact projective space. For $A \in \mathcal{H}_n$ we denote the corresponding equivalence class in $\mathrm{P}(\mathcal{H}_n)$ by $[A]$. As $\mathcal{P}_n \subseteq \mathcal{H}_n$ is a subset, the map

$$
\begin{aligned}
i : \mathcal{P}_n &\longrightarrow \mathrm{P}(\mathcal{H}_n)\\
A &\longmapsto [A], \qquad (4.15)
\end{aligned}
$$

is a $\mathrm{PSL}(n,\mathbb{C})$-equivariant embedding. Therefore we define

$$\overline{\mathcal{P}_n}^S := \overline{i(\mathcal{P}_n)} \subseteq \mathrm{P}(\mathcal{H}_n)$$

to be the *Standard-Satake compactification*. Note that this is a general construction independent of X.

The second step Let $\tau : G \longrightarrow \mathrm{PSL}(n, \mathbb{C})$ be a faithful irreducible projective representation of G. With the map

$$\begin{aligned} i_\tau : X = G/K &\longrightarrow \mathcal{P}_n \\ gK &\longmapsto \tau(g)\tau(g)^* \end{aligned} \tag{4.16}$$

we can embed X into $\mathcal{P}_n$ as totally geodesic submanifold. There is a 1-to-1-correspondence between such embeddings and faithful projective representations of G into $\mathrm{PSL}(n, \mathbb{C})$ with the additional condition $\tau(\theta(g)) = (\tau(g)^*)^{-1}$ for all $g \in G$, where θ denotes the Cartan involution on G. Indeed, this additional conditions assures that $\tau(K) \subseteq \mathrm{PSU}(n)$ and thereby $\tau(k)\tau(k)^* = id$ for all $k \in K$. With this we define

$$\overline{X}_\tau^S := \overline{i_\tau(X)} \subseteq \overline{\mathcal{P}_n}^S$$

as the *Satake compactification* of X with respect to the representation τ.

The action of G on $\mathcal{P}_n$ is given by

$$g \cdot A = \tau(g)\, A\, \tau(g)^*, \qquad \text{for } g \in G,\ A \in \mathcal{P}_n.$$

Therefore the first embedding i_τ is G-equivariant and $\overline{X}_\tau^S$ is a G-compactification, that is, the G-action on X extends to a continuous action on $\overline{X}_\tau^S$.

Note that there are finitely many isomorphism classes of Satake compactifications, one associated to any proper subset $I \subseteq \Delta$, see [BJ06, Prop. I.4.35] for details. Equivalently, the isomorphism class only depends on the Weyl chamber face of $\overline{\mathfrak{a}^+}$ containing the highest weight μ_τ. If μ_τ is generic, that is, it is contained in the interior of $\mathfrak{a}^+$ (that corresponds to $I = \emptyset$), then the resulting compactification dominates all other Satake compactifications. Therefore, it is called the *maximal Satake compactification* of X and denoted by $\overline{X}_\tau^S$. A Satake compactification that is dominated by all the others is called *minimal*. In this case $|I| = |\Delta - 1|$. We will see more about this later when talking about isomorphic Satake compactifications in Proposition 4.3.20.

Remark 4.3.1 The same construction also works when τ is not irreducible. Then we obtain the generalized Satake compactification $\overline{X}_\tau^S$ as introduced and described in [GKW15]. For generalized Satake compactifications there are infinitely many isomorphism classes. ○

Other ways to construct the Satake compactification of X are to use parabolic subgroups and boundary components that are glued together appropriately. Descriptions for this can be found in [BJ06].

4.3.2 The Compactification of a Flat in a Satake Compactification

We now compare the Satake compactification with the horofunction compactification of X with respect to an appropriate polyhedral G-invariant Finsler metric.

With the Cartan decomposition (see Lemma 4.1.37 on page 119) we can write $X = K\overline{A^+}.p_0$, where K is compact. By the previous section and Theorem 4.2.17, it is sufficient to show that we have an $\mathcal{W}$-equivariant homeomorphism between the closures of $A.p_0$ in the horofunction compactification and the Satake compactification respectively.

For the closure of the flat $F = A.p_0$ in the Satake compactification we use the following result:

Proposition 4.3.2 ([Ji97, Prop.4.1]) *Let $\tau : G \to \mathrm{PSL}(n, \mathbb{C})$ be a faithful irreducible projective representation. Let $\mu_1, \ldots, \mu_k$ be the weights of τ. Then the closure of the flat $A.p_0$ in the Satake compactification $\overline{X}_\tau^S$ is $\mathcal{W}$-equivariantly isomorphic to* $\mathrm{conv}(2\mu_1, \ldots, 2\mu_k) \subseteq \mathfrak{a}^*$. ○

The factor 2 in the convex hull comes from the construction of the isomorphism using the moment map in [Ji97]. As we are interested in horofunction compactifications which are invariant under scaling of the (dual) unit ball, we will from now on omit this extra factor.

Remark 4.3.3 Because of the symmetry of the weights with respect to the Weyl chambers, the convex hull of all weights is the same as the convex hull of the Weyl-group orbit of $\chi_1, \cdots, \chi_l$, where χ_i are the highest weights of the irreducible components τ_i of τ:

$$\operatorname{conv}(\mu_1, \ldots, \mu_k) = \operatorname{conv}\left(\mathcal{W}(\chi_1), \cdots, \mathcal{W}(\chi_l)\right). \qquad \circ$$

Example 4.3.4 Let us look at an example. Take $X = \mathrm{SL}(3, \mathbb{C}) / \mathrm{SU}(3)$ and recall (see Example 4.1.33) that the roots $\alpha_{ij} \in \mathfrak{a}^*$ with $1 \leq i \neq j \leq 3$ are given by

$$\alpha_{ij}(H) = h_i - h_j$$

for any diagonal matrix $H = \operatorname{diag}(h_1, h_2, h_3) \in \mathfrak{a}$. The positive Weyl chamber we chose is

$$\mathfrak{a}^+ := \left\{ \operatorname{diag}(h_1, h_2, h_3) \in \mathfrak{sl}(3, \mathbb{C}) \,\middle|\, h_1 > h_2 > h_3, \ \sum_i h_i = 0 \right\}$$

and the simple roots are $\Delta = \{\alpha_{12}, \alpha_{23}\}$. Now let us look at different representations of $\mathrm{SL}(3, \mathbb{C})$ and the corresponding convex hulls of the highest weights.

We start with the adjoint representation ad of $\mathfrak{g}$, which induces a representation on G. The weights of ad are exactly the roots and the highest weight with respect to the positive Weyl chamber $\mathfrak{a}^+$ is α_{13}. Here the highest weight is regular, that is, it lies in the interior of $\mathfrak{a}^+$, and we get a hexagon as the convex hull of its Weyl group orbit, see Figure 4.2 on the left.

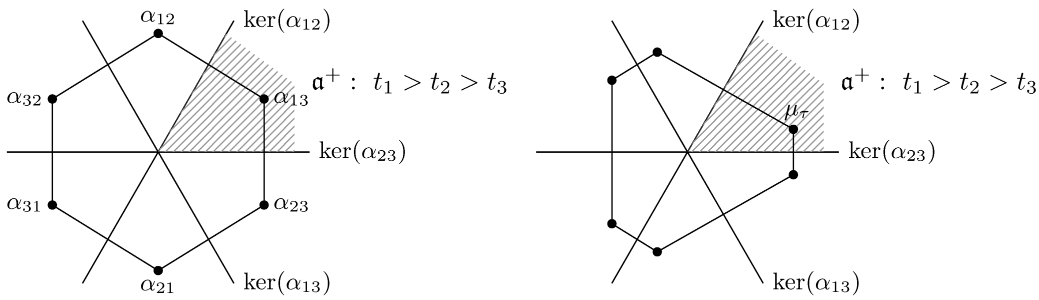


Figure 4.2: LEFT: $\operatorname{conv}(\mu_1, \ldots, \mu_6)$ for the representation $\tau = \mathrm{ad}$ in $\mathfrak{a}^*$. RIGHT: The convex hull of a representation with regular highest weight μ_τ.

If we had taken another representation where the highest weight is regular, we would also have obtained a hexagon, for example as in Figure 4.2 on the right. All compactifications with respect to a regular highest weight give the maximal Satake compactification.

In the case of $\mathrm{SL}(3, \mathbb{C})$ there are two minimal Satake compactifications. To get them, the highest weight has to lie on a singular direction, see Figure 4.3 for a picture. The representations here are the standard and the dual standard representation (call them τ and τ^*) obtained by the inclusion $\mathrm{SL}(3, \mathbb{C}) \hookrightarrow \mathrm{GL}(3, \mathbb{C})$. The highest weight with respect to $\mathfrak{a}^+$ is

$$\mu_\tau := \beta_1 \in \mathfrak{a}^*.$$

By the identification $\mathfrak{a} \cong \mathfrak{a}^*$ via the Killing form we also have

$$H_{\beta_1} = \frac{1}{6} \operatorname{diag}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right).$$

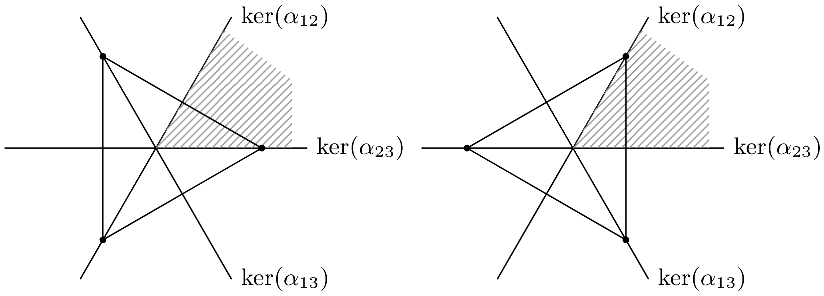


Figure 4.3: These two convex hulls correspond to the standard (LEFT) and the dual standard (RIGHT) representations.

This shows, that $H_{\beta_1} \in \ker(\alpha_{23})$ as given in the left picture of Figure 4.3.

The second minimal Satake compactification τ^* has highest weight $\mu_{\tau^*} = -\beta_3$ with $H_{-\beta_3} \in \ker(\alpha_{12})$. ○

Example 4.3.5 All representations we considered so far were irreducible and the corresponding compactifications therefore classical Satake compactifications. If we now take the convex hull of the two triangles from the previous example, we again obtain a hexagon but now with its vertices on the singular directions, see Figure 4.4.

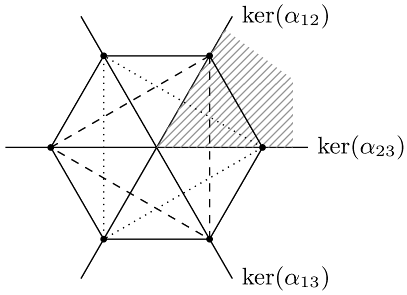


Figure 4.4: The convex hull of the two balls above give a hexagon with vertices on the singular directions.

This compactification of the flat corresponds to a generalized Satake compactification associated to the direct sum of the standard and the dual standard representation. ○

Our goal of this section is to compare the Satake compactification $\overline{X}^S_\tau$ with the horofunction compactification $\overline{X}^{hor}$. By Proposition 4.3.2 and Theorem 3.2.6 we already know that

$$\begin{aligned}\overline{X}^S_\tau &\simeq D := \operatorname{conv}(\mathcal{W}(\mu_\tau)) \simeq -D\\ \overline{X}^{hor}_B &\simeq B^\circ.\end{aligned}$$

Additionally we know (see Proposition 4.1.46) that any G-invariant Finsler norm on X is defined by the choice of a $\mathcal{W}$-invariant unit ball B in $\mathfrak{a}$. The only requirement for B is $\mathcal{W}$-invariance, so we have a lot of freedom here. Given a representation τ of G and from this a polytope $D = \operatorname{conv}(\mathcal{W}(\mu_\tau)) \subseteq \mathfrak{a}^*$, we can choose $B := -D^\circ$. Then B is $\mathcal{W}$-invariant and we get a homeomorphism between $\overline{X}^S_\tau$ and $\overline{X}^{hor}$ that realizes $\overline{X}^S_\tau$ as a horofunction compactification. Note that we choose $B = -D^\circ$ and not $B = D^\circ$ because of the topology defined by the convergence behavior, which we consider in the next section.

4.3.3 The Topology

In Theorem 3.2.6 and the remark thereafter we explained explicitly the convergence behavior of sequences in the horofunction compactification. We now want to state a similar result for the Satake compactification to finally see in Theorem 4.3.18 that with the appropriate choice of B, convergence in $\overline{X}^S_\tau$ is equivalent to convergence in $\overline{X}^{hor}$. We follow [BJ06, I.4.15.ff].

As in the decomposition $X = Ke^{\overline{\mathfrak{a}^+}} p_0$ the group K is compact, we will consider limits of the form $i_\tau(e^{H_m} p_0)$ with i_τ as in (4.16) and $H_m \in \overline{\mathfrak{a}^+}$ unbounded. We choose a suitable basis of $\mathbb{C}^n$ such that

$$\tau(e^H) = \operatorname{diag}\left(e^{\mu_1(H)}, \ldots, e^{\mu_n(H)}\right)$$

is a diagonal matrix for all $H \in \mathfrak{a}$. Then

$$i_\tau(e^{H_m} p_0) = \left[\operatorname{diag}\left(e^{2\mu_1(H_m)}, \ldots, e^{2\mu_n(H_m)}\right)\right].$$

It will turn out later, that convergent sequences can be characterized by special subsets of the simple roots, so-called μ_τ-connected subsets:

Definition 4.3.6 Let μ_τ be the highest weight of the representation τ. Then a subset $I \subseteq \Delta$ is called *μ_τ-connected*, if the set $I \cup \{\mu_\tau\}$ is connected, that is, it is not the union of two subsets orthogonal to each other with respect to the Killing form κ. ∘

Remark 4.3.7 An easy way to decide graphically whether a subset $I \subseteq \Delta$ is μ_τ-connected or not is to consider the Dynkin diagram[3] of the roots. Add $\{\mu_\tau\}$ as a vertex to the diagram and join it with an edge to all those vertices of simple roots that are not perpendicular to μ_τ with respect to κ. Then I is μ_τ-connected if and only if $I \cup \{\mu_\tau\}$ is a connected subset in the diagram. ∘

There is a close connection between μ_τ-connected subsets of Δ and the support of the weights as it was defined in Definition 4.1.38 on page 120:

Lemma 4.3.8 ([BJ06, Prop. I.4.18])

(1) Let μ_i be a weight of the representation τ. Then its support $\operatorname{Supp}(\mu_i)$ *is a μ_τ-connected subset of Δ.*

(2) Let on the other hand $I \subseteq \Delta$ be μ_τ-connected. Then there is a weight μ_j of τ such that $I = \operatorname{Supp}(\mu_j)$. ∘

Example 4.3.9 We consider again $SL(4, \mathbb{C})$ with the adjoint and the standard representation. The Dynkin diagrams of both representations are shown in Figure 4.5. For notations see the Examples 4.1.33, 4.1.40 and 4.2.5 before.

Figure 4.5: The Dynkin diagram of $\mathfrak{sl}(4, \mathbb{C})$ for the adjoint representation (LEFT) and the standard representation (RIGHT).

[3]The Dynkin diagram is a graph whose vertices are given by the set of simple roots Δ. Two vertices are connected with (up to three directed) edges, if the corresponding roots are not orthogonal. The number of edges depends on the angle between the roots.

1) Let us start with the adjoint representation. With the highest weight $\mu_\tau = \alpha_{14}$ the μ_τ-connected subsets are:

$$\emptyset,\ \{\alpha_{12}\},\ \{\alpha_{34}\},\ \{\alpha_{12},\alpha_{23}\},\ \{\alpha_{12},\alpha_{34}\},\ \{\alpha_{23},\alpha_{34}\},\ \Delta.$$

This can either be calculated or seen by the Dynkin-diagram of $\mathfrak{sl}(4,\mathbb{C})$. By Example 4.1.39 we know that

$$\begin{aligned}
&\operatorname{Supp}(\alpha_{14}) = \emptyset, && \operatorname{Supp}(\alpha_{24}) = \{\alpha_{12}\},\\
&\operatorname{Supp}(\alpha_{13}) = \{\alpha_{34}\}, && \operatorname{Supp}(\alpha_{34}) = \{\alpha_{12},\alpha_{23}\},\\
&\operatorname{Supp}(\alpha_{23}) = \{\alpha_{12},\alpha_{34}\}, && \operatorname{Supp}(\alpha_{12}) = \{\alpha_{23},\alpha_{34}\}.
\end{aligned}$$

So the proper μ_τ-connected subsets of Δ are in 1-to-1 correspondence with the positive roots. When looking for a root with support $I = \Delta$ we get all the negative roots.

2) For the standard representation, the μ_τ-connected subsets are (see Figure 4.5 on the right):

$$\emptyset,\ \{\alpha_{12}\},\ \{\alpha_{12},\alpha_{23}\},\ \Delta.$$

The support of the weights was computed in Example 4.1.39 as

$$\begin{aligned}
&\operatorname{Supp}(\beta_1) = \emptyset, && \operatorname{Supp}(\beta_2) = \{\alpha_{12}\},\\
&\operatorname{Supp}(\beta_3) = \{\alpha_{12},\alpha_{23}\}, && \operatorname{Supp}(\beta_4) = \{\alpha_{12},\alpha_{23},\alpha_{34}\} = \Delta.
\end{aligned}$$

and all non-positive roots have support Δ. Here again all weights have a μ_τ-connected subset as support and we find a weight with support I for all μ_τ-connected subsets I of Δ. $\circ$

Recall that a positive chamber in $\mathfrak{a}^I$ was given by $\mathfrak{a}^{I,+} := \left\{H \in \mathfrak{a}^I \,\middle|\, \alpha(H) > 0\ \forall \alpha \in I\right\}$. Its closure will be denoted by $\overline{\mathfrak{a}^{I,+}}$.

Let $(H_m)_{j\in\mathbb{N}} \subseteq \overline{\mathfrak{a}^+}$ be an unbounded sequence of type $(J, \hat{a}^J)$, that is, it satisfies the following conditions:

(1) for $\alpha \in J$ the limit $\lim_{j\to\infty} \alpha(H_m) = \alpha(\log \hat{a}^J)$ exists and is finite,

(2) for $\alpha \in \Delta \setminus J$ there holds $\alpha(H_m) \longrightarrow +\infty$.

Let $I \subseteq J$ be the largest μ_τ-connected subset contained in J. It exists uniquely because the property of being μ_τ-connected is closed under unions and the empty set is also μ_τ-connected.

Let $H_\infty = \log(\hat{a}^I) \in \overline{\mathfrak{a}^{I,+}}$ be the unique vector in $\overline{\mathfrak{a}^{I,+}}$ such that $\alpha(H_\infty) = \lim_{j\to\infty} \alpha(H_m)$ for $\alpha \in I$. If $I = J$ then $H_\infty = \log(\hat{a}^J)$. Otherwise, since $\mathfrak{a}^I \subseteq \mathfrak{a}^J$, note that H_∞ does not depend on the limits of $\alpha(H_m)$ for $\alpha \in J\backslash I$. For simplicity of notation let the weights of τ be ordered as follows: $\mu_1 = \mu_\tau$ and $\mu_1, \ldots, \mu_k$ are all the weights with $\operatorname{Supp}(\mu_i) \subseteq I$. Let $\mu_{k+1}, \ldots, \mu_r$ be the other weights. As H_m and therefore also $i_\tau(e^{H_m} p_0)$ is not bounded, we have $k \leq r-1$ because $I = \Delta$ is not possible by the unboundedness of $(H_m)_m$. With this ordering we obtain as limit

$$\begin{aligned}
i_\tau(e^{H_m} p_0) &= \left[\operatorname{diag}\left(e^{2\mu_1(H_m)}, \ldots, e^{2\mu_r(H_m)}\right)\right]\\
&= \left[e^{2\mu_\tau(H_m)} \operatorname{diag}\left(1, e^{-2\sum_{\alpha\in\Delta} c_{2,\alpha}\alpha(H_m)}, \ldots, e^{-2\sum_{\alpha\in\Delta} c_{r,\alpha}\alpha(H_m)}\right)\right]\\
&= \left[\operatorname{diag}\left(1, e^{-2\sum_{\alpha\in\Delta} c_{2,\alpha}\alpha(H_m)}, \ldots, e^{-2\sum_{\alpha\in\Delta} c_{r,\alpha}\alpha(H_m)}\right)\right]\\
&\longrightarrow \left[\operatorname{diag}\left(1, e^{-2\sum_{\alpha\in I} c_{2,\alpha}\alpha(H_\infty)}, \ldots, e^{-2\sum_{\alpha\in I} c_{k,\alpha}\alpha(H_\infty)}, 0, \ldots, 0\right)\right],
\end{aligned}$$

where we used the notation of Equation (4.4) in the first step and in the last step we used the fact that for each $i \geq k+1$ there is an $\alpha \notin I$ such that $c_{i,\alpha} > 0$ and that therefore the whole expression goes to zero.

Example 4.3.10 We look at some of the sequences in $\mathrm{SL}(4,\mathbb{C})$ already considered in Example 4.2.5 on page 127.

Let

$$H_m = \mathrm{diag}(m+4,\ m,\ 6,\ -2m-10) = \log(a_m).$$

Then (see Example 4.2.5) H_m has type $(I = \{\alpha_{12}\}, \hat{a}^I)$ with $H_\infty = \log(\hat{a}^I) = \mathrm{diag}(2,-2,0,0)$. The set I is μ_τ-connected with respect to the adjoint representation, where $\mu_\tau = \alpha_{14}$. By Example 4.3.9 we know that apart from α_{14} also the root α_{24} has support contained in I. We enumerate the weights (i.e. the positive roots) in the following order:

$$\begin{array}{lllll} \mu_1 = \alpha_{14} & \mu_2 = \alpha_{24} & & & \\ \mu_3 = \alpha_{12} & \mu_4 = \alpha_{13} & \mu_5 = \alpha_{23} & \mu_6 = \alpha_{34} & \mu_7 = \alpha_{41} \\ \mu_8 = \alpha_{42} & \mu_9 = \alpha_{21} & \mu_{10} = \alpha_{31} & \mu_{11} = \alpha_{32} & \mu_{12} = \alpha_{43}. \end{array}$$

For the non-positive roots $\mu_7, \ldots, \mu_{12}$ we have $\mu_j(H_m) < 0$ for m big enough and

$$e^{2(\mu_j(H_m)-\mu_\tau(H_m))} \longrightarrow 0 \qquad \forall j = 7, \ldots, 12.$$

Therefore we will omit their explicit expression in the following calculation and just indicate their presence by some dots. Then we get

$$\begin{aligned} i_\tau(e^{H_m}.p_0) &= \left[\mathrm{diag}\left(e^{2\alpha_{14}(H_m)}, e^{2\alpha_{24}(H_m)}, e^{2\alpha_{12}(H_m)}, e^{2\alpha_{13}(H_m)}, e^{2\alpha_{23}(H_m)}, e^{2\alpha_{34}(H_m)}, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot\right)\right] \\ &= \left[\mathrm{diag}\left(e^{2(3m+14)}, e^{2(3m+10)}, e^{2\cdot 4}, e^{2(m-2)}, e^{2(m-6)}, e^{2(2m+16)}, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot\right)\right] \\ &= \left[e^{6m+28}\,\mathrm{diag}\left(1, e^{-8}, e^{-6m-20}, e^{-4m-32}, e^{-4m-40}, e^{-2m+4}, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot\right)\right] \\ &\longrightarrow \left[\mathrm{diag}\left(1, e^{-8}, 0,0,0,0,0,0,0,0,0,0\right)\right] \\ &= \left[\mathrm{diag}\left(1, e^{-2\alpha_{12}(H_\infty)}, 0, \ldots, 0\right)\right]. \end{aligned}$$

Next we look at the sequence

$$H'_m = \mathrm{diag}\,(m+8,\ m,\ -m+2,\ -m-10)$$

with type associated to $K = \{\alpha_{12}, \alpha_{34}\}$ and $H'_\infty = \mathrm{diag}(4,-4,6,-6)$. The set K is μ_τ-connected with respect to the adjoint representation and the roots with support contained in K are

$$\mu_1 = \alpha_{14}, \quad \mu_2 = \alpha_{24}, \quad \mu_3 = \alpha_{13}, \quad \mu_4 = \alpha_{23}.$$

The remaining two positive roots are $\mu_5 = \alpha_{12}$ and $\mu_6 = \alpha_{34}$. As above (omitting non-positive roots) we calculate:

$$\begin{aligned} i_\tau(e^{H'_m}.p_0) &= \left[\mathrm{diag}\left(e^{2\alpha_{14}(H'_m)}, e^{2\alpha_{24}(H'_m)}, e^{2\alpha_{13}(H'_m)}, e^{2\alpha_{23}(H'_m)}, e^{2\alpha_{12}(H'_m)}, e^{2\alpha_{34}(H'_m)}, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot\right)\right] \\ &= \left[\mathrm{diag}\left(e^{2(2m+18)}, e^{2(2m+10)}, e^{2(2m+6)}, e^{2(2m-2)}, e^{2\cdot 8}, e^{2\cdot 12}, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot\right)\right] \\ &= \left[e^{4m+36}\,\mathrm{diag}\left(1, e^{-16}, e^{-24}, e^{-40}, e^{-4m-20}, e^{-4m-12}, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot\right)\right] \\ &\longrightarrow \left[\mathrm{diag}\left(1, e^{-16}, e^{-24}, e^{-40}, 0,0,0,0,0,0,0,0\right)\right] \\ &= \left[\mathrm{diag}\left(1, e^{-2\alpha_{12}(H_\infty)}, e^{-2\alpha_{34}(H_\infty)}, e^{-2(\alpha_{12}+\alpha_{34})(H_\infty)}, 0, \ldots, 0\right)\right]. \end{aligned}$$

Next we take now the same sequence but consider the standard representation. It has highest weight $\mu_\tau = \beta_1$ as given in Example 4.2.5 and the set K is not μ_τ-connected. The largest μ_τ-connected subset contained in K is $I = \{\alpha_{12}\}$. Then the weights $\mu_1 = \beta_1$ and $\mu_2 = \beta_2$ have support contained in I whereas $\mu_3 = \beta_3$ and $\mu_4 = \beta_4$ do not. Here now we get (with dots instead of the terms with non-positive weights):

$$\begin{aligned} i_\tau(e^{H'_m}.p_0) &= \left[\operatorname{diag}\left(e^{2\beta_1(H'_m)}, e^{2\beta_2(H'_m)}, e^{2\beta_3(H'_m)}, e^{2\beta_4(H'_m)}, \cdot, \cdot, \cdot, \cdot\right)\right] \\ &= \left[\operatorname{diag}\left(e^{2(m+8)}, e^{2\cdot m}, e^{2(-m+2)}, e^{2(-m-10)}, \cdot, \cdot, \cdot, \cdot\right)\right] \\ &= \left[e^{2m+16}\operatorname{diag}\left(1, e^{-16}, e^{-4m-12}, e^{-4m-36}, \cdot, \cdot, \cdot, \cdot\right)\right] \\ &\longrightarrow \left[\operatorname{diag}\left(1, e^{-16}, 0, 0, 0, 0, 0, 0\right)\right] \\ &= \left[\operatorname{diag}\left(1, e^{-2\alpha_{12}(H_\infty)}, 0, \dots, 0\right)\right]. \end{aligned}$$ ◦

In their book [BJ06, §I.4.20], Borel and Ji show that the map

$$\begin{aligned} i_I : e^{\overline{\mathfrak{a}^{I,+}}} &\longrightarrow \overline{X}^S_\tau, \\ H_\infty &\longmapsto \left[\operatorname{diag}\left(1, e^{-2\sum_{\alpha\in I} c_{2,\alpha}\alpha(H_\infty)}, \dots, e^{-2\sum_{\alpha\in I} c_{k,\alpha}\alpha(H_\infty)}, 0, \dots, 0\right)\right]. \end{aligned} \tag{4.17}$$

is well defined and an embedding. With the conditions for $(H_m)_m$ as stated above we have already the important aspects of the proposition about converging sequences:

Proposition 4.3.11 ([BJ06, Prop. I.4.23]) *Let $(H_m)_m \in \overline{\mathfrak{a}^+}$ be an unbounded sequence. Then $\left(e^{H_m}p_0\right)_m$ converges in $\overline{X}^S_\tau$ if and only if there is a μ_τ-connected subset $I \subseteq \Delta$ satisfying*

(S1) for all $\alpha \in I$ the limit $\lim_{m\to\infty}\alpha(H_m)$ exists and is finite,

(S2) for all μ_τ-connected subsets $I' \subseteq \Delta$ properly containing I, there is an $\alpha \in I' \setminus I$ with $\alpha(H_m) \longrightarrow \infty$.

Let H_∞ be the unique vector in $\overline{\mathfrak{a}^{I,+}}$ such that $\alpha(H_\infty) = \lim_{m\to\infty}\alpha(H_m)$ for all $\alpha \in I$. Then, with the above ordering of the weights,

$$i_\tau(e^{H_m}p_0) \longrightarrow i_I(e^{H_\infty}) \;=\; \left[\operatorname{diag}\left(1, e^{-2\sum_{\alpha\in I} c_{2,\alpha}\alpha(H_\infty)}, \dots, e^{-2\sum_{\alpha\in I} c_{k,\alpha}\alpha(H_\infty)}, 0, \dots, 0\right)\right].$$

The closure of the positive chamber $e^{\overline{\mathfrak{a}^+}}p_0$ in $\overline{X}^S_\tau$ is given by

$$\overline{i_\tau(e^{\overline{\mathfrak{a}^+}}p_0)} \;=\; i_\tau\left(e^{\overline{\mathfrak{a}^+}}p_0\right) \cup \coprod_{\substack{I\subseteq\Delta \\ \mu_\tau-\text{conn.}}} i_I\left(e^{\overline{\mathfrak{a}^{I,+}}}\right) \;\cong\; \overline{\mathfrak{a}^+} \cup \coprod_{\substack{I\subseteq\Delta \\ \mu_\tau-\text{conn.}}} \overline{\mathfrak{a}^{I,+}}.$$ ◦

Before we come to the identification of convergent sequences in $\overline{X}^{hor}$ and $\overline{X}^S_\tau$, we show some lemmas about the connection of μ_τ-connected subsets of I to faces of B the boundary of $\overline{\mathfrak{a}^{I,+}}$. Both results will be needed in the proof of Theorem 4.3.18. We start with some notations: Let $\mathfrak{a}_I^+ = \mathfrak{a}_I \cap \mathfrak{a}^+$ be the restricted positive Weyl chamber and $\overline{\mathfrak{a}_I^+} = \mathfrak{a}_I \cap \overline{\mathfrak{a}^+}$ its closure $\mathfrak{a}_I$. By $B_{\mathfrak{a}_I} = B \cap \mathfrak{a}_I$ we denote the restriction of B to $\mathfrak{a}_I$.

Lemma 4.3.12 *Let $I \subseteq \Delta$ be a μ_τ-connected subset. Then*

$$\dim(\mathfrak{a}_I) = \dim\mathfrak{a} - \#I.$$ ◦

Proof. Assume the statement would not be true for some $I \subseteq \Delta$ which we choose minimal. As we are taking intersections of hyperplanes for constructing $\mathfrak{a}_I$, we know that $\dim\mathfrak{a}_I > \dim\mathfrak{a} - \#I$, as we loose maximal one dimension with each element of I. In other words

$$\#I > \dim\mathfrak{a} - \dim\mathfrak{a}_I.$$

Take $\mathfrak{a}^I$, the orthogonal complement of $\mathfrak{a}_I$ in $\mathfrak{a}$. Then $\dim \mathfrak{a}^I = \dim \mathfrak{a} - \dim \mathfrak{a}_I$. With the identification $\mathfrak{a} \simeq \mathfrak{a}^*$ via the Killing form κ, a root $\alpha \in \mathfrak{a}^*$ is orthogonal to its own kernel $\ker(\alpha)$ and therefore $I \subseteq \mathfrak{a}^I$. But that means, that we have a $(\dim \mathfrak{a} - \dim \mathfrak{a}_I)$-dimensional subspace in which lie $\#I$ simple roots. As $\#I > \dim \mathfrak{a} - \dim \mathfrak{a}_I$, this is a contradiction to the fact that simple roots are linearly independent. □

Lemma 4.3.13 *Let the notations be as before. Then the relative boundary of $\overline{\mathfrak{a}^+}$ is given as*

$$\partial_{rel}\, \overline{\mathfrak{a}^+} = \left(\bigcup_{\emptyset \neq J \subseteq \Delta} \mathfrak{a}_J \right) \cap \overline{\mathfrak{a}^+}. \qquad \circ$$

Proof. Recall that for a subset $I \subseteq \Delta$, we defined $\mathfrak{a}_I = \bigcap_{\alpha \in I} \ker(\alpha)$. As every positive root in Σ^+ can be written as a linear combination of simple roots with positive integer coefficients, we have

$$\mathfrak{a}^+ = \{H \in \mathfrak{a} \mid \alpha(H) > 0\ \forall \alpha \in \Sigma^+\} = \{H \in \mathfrak{a} \mid \alpha(H) > 0\ \forall \alpha \in \Delta\}.$$

The closure of the positive Weyl chamber is given by $\overline{\mathfrak{a}^+} = \{H \in \mathfrak{a} \mid \alpha(H) \geq 0\ \forall \alpha \in \Delta\}$. Therefore

$$\begin{aligned} \partial_{\mathrm{rel}} \overline{\mathfrak{a}^+} &= \{H \in \overline{\mathfrak{a}^+} \mid \exists \alpha \in \Delta : \alpha(H) = 0\} \\ &= \{H \in \mathfrak{a} \mid \exists \alpha \in \Delta : \alpha(H) = 0\} \cap \overline{\mathfrak{a}^+} \\ &= \left(\bigcup_{\emptyset \neq J \subseteq \Delta} \mathfrak{a}_J \right) \cap \overline{\mathfrak{a}^+}. \end{aligned} \qquad □$$

Now we want to give the correspondence between proper faces of B and μ_τ-connected proper subsets $I \subsetneq \Delta$. Let us first look at the face structure of $D = \mathcal{W}(\mu_\tau)$ and its negative dual $B = -D^\circ$. Note that B is polyhedral. We again identify $\mathfrak{a} \cong \mathfrak{a}^*$ via the Killing form and denote the unique element of $\mathfrak{a}$ associated to an element $\alpha \in \mathfrak{a}^*$ again by H_α. As D is the orbit of the highest weight μ_τ under the Weyl group $\mathcal{W}$, which acts by reflection at the Weyl chamber walls, the facets of D are orthogonal to one-dimensional intersections of Weyl chamber walls. Therefore all vertices of $B \subseteq \mathfrak{a}$ lie on such one-dimensional Weyl chamber faces.

Let $F = \operatorname{conv}\{b_1, \ldots, b_k\} \subseteq \partial B$ be a convex polyhedral subset with vertices $b_j \in \partial B$. Then for each b_j there is a subset $I_j \subseteq \Delta$ such that $b_j \in \mathfrak{a}_{I_j}$ and $\dim(\mathfrak{a}_{I_j}) = 1$. Now we set

$$I := I_1 \cap \ldots \cap I_k \subseteq \Delta.$$

Then I contains exactly all those roots, such that F is contained in $\mathfrak{a}_I$. Since $\mathfrak{a}_J \cap \mathfrak{a}_K = \mathfrak{a}_{J \cup K}$ for all $J, K \subseteq \Delta$, the set I is the maximal one satisfying

$$F \subseteq \mathfrak{a}_I$$

and it holds

$$\dim(F) = \dim(\mathfrak{a}_I) - 1.$$

Given a subset $I \subseteq \Delta$, let $F_I \subseteq \overline{\mathfrak{a}^+}$ be the unique face of B determined by

$$B_{\mathfrak{a}_I} \cap \overline{\mathfrak{a}^+} = F_I \cap \overline{\mathfrak{a}^+}, \tag{4.18}$$

where $B_{\mathfrak{a}_I} = B \cap \mathfrak{a}_I$ is the restricted unit ball.

When restricting $F = \operatorname{conv}\{b_1, \ldots, b_k\}$ to intersect $\overline{\mathfrak{a}^+}$, then the above two definitions are inverse to each other.

The face $F_\emptyset$ associated to $\emptyset \subseteq \Delta$ is the facet of B that covers $\mathfrak{a}^+$ (and probably more). By Lemma 4.3.13, all cones in the boundary of $\overline{\mathfrak{a}^+}$ correspond to non-empty subsets $I \subseteq \Delta$. Now we have to characterize those subsets $I \subseteq \Delta$ that correspond to faces of $B \cap \overline{\mathfrak{a}^+}$ and not only convex polyhedral sets.

Lemma 4.3.14 *F is a face of B if and only if I is μ_τ-connected.* ◦

Proof. Let $I \subseteq \Delta$ be not μ_τ-connected. Then there are $J, K \subseteq \Delta$ orthogonal to each other with $I \cup \{\mu_\tau\} = J \cup K$. Without loss of generality assume $\mu_\tau \in J$. Then $K \subseteq I$ and for all roots $\alpha \in K$ it holds

$$\kappa(\alpha, \mu_\tau) = 0.$$

Therefore H_{μ_τ} is contained in the subspace $\mathfrak{a}_K$. This means that the polyhedral set $D \subseteq \mathfrak{a}$ has a vertex (namely H_{μ_τ}) in the subspace $\mathfrak{a}_K$ and $-H_{\mu_\tau} \in \mathfrak{a}_K$ is a vertex of $-D$. By duality, $B = -D^\circ$ then has a facet orthogonal to $\mathfrak{a}_K$. As $H_{\mu_\tau} \in \overline{\mathfrak{a}^+}$, and thereby also $-H_{\mu_\tau} \in \overline{\mathfrak{a}^+}$, this facet is exactly $F_\emptyset$. Let $F_I \subseteq B$ be the subset associated to I (see Equation (4.18)) with $F_I \cap \overline{\mathfrak{a}^+} \neq \emptyset$. The Weyl group $\mathcal{W}$ acts by the reflections at the hyperplanes and especially also at those containing $\mathfrak{a}_I$ and $\mathfrak{a}_K$, where $\mathfrak{a}_I \subseteq \mathfrak{a}_K$. As $F_I \subseteq F_\emptyset \cap \mathfrak{a}_I$ and $F_\emptyset$ is orthogonal to $\mathfrak{a}_K$, the set F is not a face but lies in the interior of $F_\emptyset$.

Let now on the other hand $I \subseteq \Delta$ be a μ_τ-connected subset. Then $\mu_\tau \notin \mathfrak{a}_I$ as I is not orthogonal to μ_τ. Since $\mathfrak{a}_I^+ = \mathfrak{a}_I \cap \overline{\mathfrak{a}^+} \neq \emptyset$ and $\mu_\tau \in \overline{\mathfrak{a}^+}$, no element in the $\mathcal{W}$-orbit of μ_τ lies in $\mathfrak{a}_I^+$, which means that D does not have a vertex in $\mathfrak{a}_I^+$. Consequently also $-D$ has no vertex lying in $-\overline{\mathfrak{a}^+}_I$ and $F_\emptyset$ is not orthogonal to $\mathfrak{a}_I^+$. This is equivalent to $F_I \subseteq \mathfrak{a}_I$ (given by $B_{\mathfrak{a}_I} \cap \overline{\mathfrak{a}^+} = F_I \cap \overline{\mathfrak{a}^+}$) being a face of B. □

The correspondence now follows directly:

Corollary 4.3.15 Let $\mathcal{F}$ be the set of proper faces of B and Δ the set of simple roots. Let $\overline{\mathfrak{a}^+}$ denote the closure of the positive Weyl chamber in $\mathfrak{a}$. Then there is a 1-to-1 correspondence

$$\{F \in \mathcal{F} \mid F \cap \overline{\mathfrak{a}^+} \neq \emptyset\} \xleftrightarrow{1-1} \{I \subsetneq \Delta \mid I \text{ is } \mu_\tau - \text{connected}\}. \qquad \circ$$

Using the above result, it is now easy to describe the relative boundary of the cone over a face of B by intersection of Weyl chamber walls associated to μ_τ-connected subsets:

Lemma 4.3.16 *Let $F \subseteq B$ be a face and $I \subseteq \Delta$ the associated μ_τ-connected subset. Then*

$$\partial_{rel} K_F \cap \overline{\mathfrak{a}^+} = \Bigg(\bigcup_{\substack{I \subsetneq I' \subseteq \Delta \\ \mu_\tau - con.}} \mathfrak{a}_{I'} \Bigg) \cap \overline{\mathfrak{a}^+}. \qquad \circ$$

Proof. In the subspace $\mathfrak{a}_I$ the face F is a facet of the restricted unit ball $B_{\mathfrak{a}_I}$. So it plays the role $F_\emptyset$ did before with respect to $B \subseteq \mathfrak{a}$. Because of the same orthogonality and reflecting argument for not-μ_τ-connected subsets as in the proof of Lemma 4.3.14, we only have to consider μ_τ-connected subset I' of Δ to be faces in the relative boundary of F. As everything is happening in $\mathfrak{a}_I$ now, the respective subsets I' have to contain I as a proper subset. As F might reach out over the positive Weyl chamber, intersecting with $\overline{\mathfrak{a}^+}$ gives the result. □

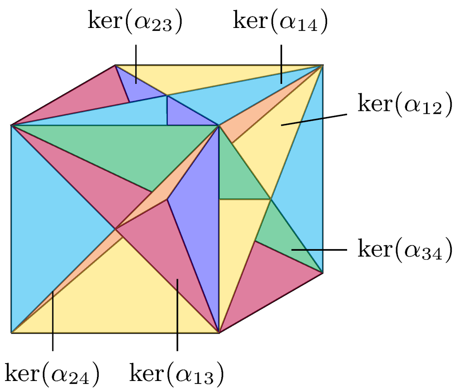


Figure 4.6: LEFT: The Weyl chamber system of $\mathrm{SL}(4,\mathbb{R})$ with all Weyl chamber walls. RIGHT: the positive Weyl chamber we chose.

Example 4.3.17 For the sake of clarity we have a look at an example. We again take the space $X = \mathrm{SL}(4,\mathbb{C})/\,\mathrm{SU}(4)$ with the same notations as in Example 4.1.33 on page 117. We choose the usual positive Weyl chamber $\mathfrak{a}^+ = \{\mathrm{diag}(t_1,\ldots,t_4) \in \mathfrak{sl}(4,\mathbb{C}) \mid t_1 > \cdots > t_4;\ \sum_{i=1}^4 t_i = 0\}$. The Weyl chamber system is again shown in Figure 4.6: the picture on the left illustrates the structure of the Weyl chamber walls while the one on the right shows the positive Weyl chamber we chose.

Let $\tau = \mathrm{ad}$ be the representation considered first. Then the highest weight with respect to $\mathfrak{a}^+$ is $\mu_\tau = \alpha_{14}$. Let H_τ be the element of $\mathfrak{a}$ corresponding to μ_τ by identifying $\mathfrak{a}$ and $\mathfrak{a}^*$ with the Killing form. Note that as $\alpha_{23}(H_\tau) = 0$, H_τ lies on the Weyl chamber wall $\mathfrak{a}_{\alpha_{13}}$.

The μ_τ-connected subsets are

$$\emptyset,\ \{\alpha_{12}\},\ \{\alpha_{34}\},\ \{\alpha_{12},\alpha_{34}\},\ \{\alpha_{12},\alpha_{23}\},\ \{\alpha_{23},\alpha_{34}\},\ \Delta.$$

The convex hull D of the weights is a regular polyhedral ball with 12 vertices and 14 maximal dimensional faces. Accordingly, the unit ball $B = -D^\circ$ has 14 vertices and 12 faces, a picture of both is given in Figure 4.7. The dashed lines in the right picture are to indicate, that some pairs of triangles form together a rhombus.

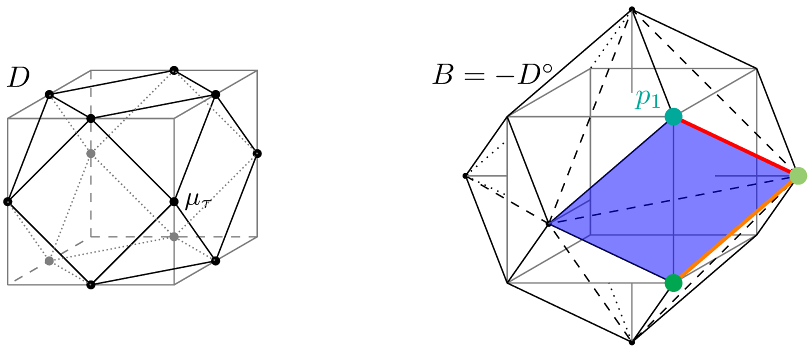


Figure 4.7: The convex hull $D = \mathrm{conv}(\mathcal{W}(\mu_\tau))$ (LEFT) and the dual unit ball $B = -D^\circ$ (RIGHT) for the adjoint representation.

The colored parts are the extreme sets of B that have non-empty intersection with $\overline{\mathfrak{a}^+}$. We know by Lemma 4.3.15, that to each of them corresponds a μ_τ-connected subset of Δ. By dimensional reasons the blue extreme set has to correspond to $I = \emptyset$. For the others we have to consider in which Weyl chamber walls they lie in. The red line for example lies in $\ker(\alpha_{34})$ and as it holds $\dim(F) = \dim(\mathfrak{a}_I) - 1$, we know that in this case $I = \{\alpha_{34}\}$, which is μ_τ-connected as required. For

the orange extreme set we obtain $I = \{\alpha_{12}\}$. For the extreme points, let us consider the point p_1, the one on top. It lies in the intersection of $\ker(\alpha_{23})$ and $\ker(\alpha_{34})$ and therefore $I = \{\alpha_{23}, \alpha_{34}\}$. It lies obviously also in the intersection with $\ker(\alpha_{24})$, but as α_{24} is not a simple root, we do not have to consider it here.
It strikes that $\{\alpha_{23}\}$ is not a μ_τ-connected subset. This corresponds to the picture, because the extreme set that would belong to $I = \{\alpha_{23}\}$ is not an extreme set of B, as it is entirely contained in the blue extreme set. This means that the blue extreme set is orthogonal to $\ker(\alpha_{23})$.
Let us have a look at the relative boundary of the cones K_F. For the red extreme set, the relative boundary is $(\mathfrak{a}_I \cup \mathfrak{a}_J \cup \mathfrak{a}_\Delta) \cap \overline{\mathfrak{a}^+}$ with $I = \{\alpha_{12}, \alpha_{34}\}$ and $J = \{\alpha_{23}, \alpha_{34}\}$. Just as expected, these are the only μ_τ-connected subsets containing α_{34}. The relative boundary of the cone over an extreme point is $\mathfrak{a}_\Delta = \{0\}$.

Let us now consider other representations. If the representative H_τ of the highest weight lies completely inside of $\mathfrak{a}^+$ we get D and B as shown in Figure 4.8. All subsets of Δ are μ_τ-connected. The polyhedron $D = \mathrm{conv}(W(\mu_\tau))$ is then called the permutohedron of dimension 3. More generally, if τ is an irreducible faithful representation of $\mathrm{SL}(n, \mathbb{C})$ with regular H_τ, then the polyhedron $D = \mathrm{conv}(W(\mu_\tau))$ is the $(n-1)$-dimensional permutohedron.

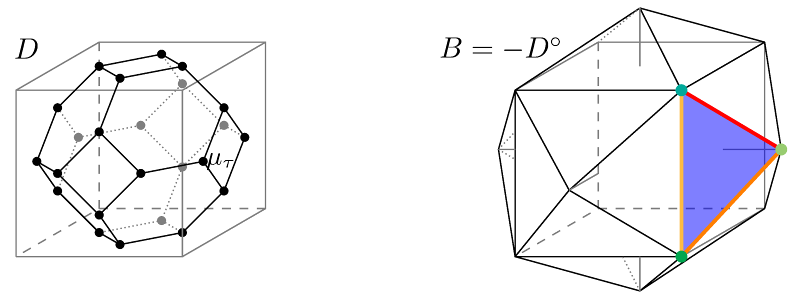


Figure 4.8: D (LEFT) and $B = -D^\circ$ (RIGHT) for a representation with highest weight in a regular direction.

If H_τ lies in more than one Weyl chamber wall, the convex hull D of the weights and the unit ball B of the Finsler norm are like shown in Figure 4.9 or a rotated version of it, depending on the Weyl chamber face H_τ lies in.

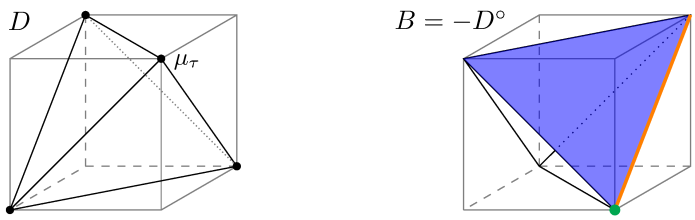


Figure 4.9: D (LEFT) and $B = -D^\circ$ (RIGHT) for a representation with highest weight in a singular direction.

In this example, D and $-D$ do not coincide and it makes even as sets a difference whether we take $B = D^\circ$ or $B = -D^\circ$. ◦

Let us finally show the equivalence of convergence of sequences in $\overline{X}^{hor}$ and $\overline{X}^S_\tau$.

Theorem 4.3.18 *Let $X = G/K$ be a symmetric space of non-compact type. Let τ be a faithful irreducible projective representation of G, $\mu_1, \dots, \mu_n$ the weights and μ_τ the highest weight of τ. With the Weyl group $\mathcal{W}$ let $D := \mathrm{conv}(\mu_1, \dots, \mu_n) = \mathrm{conv}(\mathcal{W}(\mu_\tau))$ be the $\mathcal{W}$-orbit of the highest weight. Let $B = -D^\circ$ define a unit ball in the maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{g}$.*

Then a sequence converges in the Satake compactification $\overline{X}^S_\tau$ if and only if it converges in the horofunction compactification $\overline{\mathfrak{a}}^{hor}$. ◦

Proof. We will the theorem by comparing the conditions on converging sequences given in Remark 3.2.7 and in Proposition 4.3.11 on page 59 and 143 respectively.

⇒ Let $(H_m)_m \in \overline{\mathfrak{a}^+}$ be an unbounded sequence such that $(e^{H_m} p_0)_m$ converges in $\overline{X}^S_\tau$. Then we know by Proposition 4.3.11 on page 143, that there is a μ_τ-connected subset $I \subsetneq \Delta$ satisfying the following two conditions:

(S1) For all $\alpha \in I$ the limit $\lim_{m\to\infty} \alpha(H_m)$ exists and is finite.

(S2) For all μ_τ-connected subsets $I' \subsetneq \Delta$ properly containing I, there is an $\alpha' \in I' \setminus I$ with $\alpha'(H_m) \longrightarrow \infty$.

Recall that with $D = \mathrm{conv}(\mu_1, \ldots, \mu_k)$, the set $B := -D^\circ$ defines a $\mathcal{W}$-invariant unit ball in $\mathfrak{a}$. This gives us a G-invariant Finsler structure on X. Note that as the representation is finite, B is a convex polyhedral unit ball. Denote by B° its dual in $\mathfrak{a}^*$, that is, $B^\circ = -D$. Let $\mathcal{F}$ be the set of proper extreme sets of B. For notational reasons define

$$z_m := H_m,$$

and consider $(z_m)_{m\in\mathbb{N}}$ as a sequence in $\overline{\mathfrak{a}^+}$. To show that $(\psi_{z_m})_m$ converges to some horofunction $h_{E,p}$ we have to show that $(z_m)_{m\in\mathbb{N}}$ fulfills all four conditions of Remark 3.2.7 on page 3.2.7.

As H_m was assumed to be an unbounded converging sequence, $\|z_m\|_B \longrightarrow \infty$. Therefore the first condition is fulfilled.

For the next three conditions we have to choose an extreme set $F \in \mathcal{F}$ and a point $p \in V(F)^\perp$. Choose $F \in \mathcal{F}$ corresponding to I from $(S1)$ as defined by the correspondence in Lemma 4.3.15. As F is a proper extreme set of a unit polyhedral ball, it does not contain the origin. Hence $\dim V(F) = \dim \mathfrak{a} - \#I$, where $V(F)$ is the subspace generated by F. With $\dim \mathfrak{a}_I = \dim \mathfrak{a} - \#I$ and and since $F \subseteq \mathfrak{a}_I$ we conclude that

$$V(F) = \mathfrak{a}_I.$$

Now we split our sequence $(z_m)_m$ into two parts depending on I, according to the splitting of $\mathfrak{a}$ into $\mathfrak{a}_I$ and $\mathfrak{a}^I$. For every $m \in \mathbb{N}$ let $z_m = z_{m,I} + z_m^I$ with $z_{m,I} = z_{m,V(F)} \in \mathfrak{a}_I$ and $z_m^I = z_m^{V(F)} \in \mathfrak{a}^I$ be the projections of z_m to $\mathfrak{a}_I$ and $\mathfrak{a}^I$ respectively.

For the second conditions of Remark 3.2.7 we have to show that the projected sequence $(z_{m,I})_m$ lies in K_F for n big enough. K_F is a $(\dim \mathfrak{a}_I)$-dimensional subset of $\mathfrak{a}_I$ and entirely contains F. As $F \cap \overline{\mathfrak{a}^+} \neq \emptyset$, we also have $K_F \cap \overline{\mathfrak{a}^+} \neq \emptyset$. By the construction of F in Equation (4.18) we know that F is the maximal dimensional extreme set of $B_{\mathfrak{a}_I}$ in $\mathfrak{a}_I$ and therefore covers at least $\overline{\mathfrak{a}_I^+}$. Hence

$$\mathfrak{a}_I \cap \overline{\mathfrak{a}^+} = K_F \cap \overline{\mathfrak{a}^+}.$$

Since $z_m \in \overline{\mathfrak{a}^+}$ for all $m \in \mathbb{N}$ and Weyl chamber walls of a common Weyl chamber have angle $\leq \frac{\pi}{2}$ between them because of the $\mathcal{W}$-action, we have

$$z_{m,I} \in \overline{\mathfrak{a}^+}.$$

Therefore

$$z_{m,I} \in \mathfrak{a}_I \cap \overline{\mathfrak{a}^+} = K_F \cap \overline{\mathfrak{a}^+} \subseteq K_F.$$

We know by $(S\,1)$ that the limit $\lim_{m\longrightarrow\infty}\alpha(H_m)$ exists and is finite for all $\alpha\in I$. It is $\alpha(H_m)=\alpha(z_m^I)$ for $\alpha\in I$. As $\alpha(H)=\kappa(H_\alpha,H)$ for all $H\in\mathfrak{a}$, also the limit $\lim_m z_m^I$ exists and is finite. Set

$$p:=\lim_{m\longrightarrow\infty} z_m^I\in\mathfrak{a}^I=V(F)^\perp.$$

Then

$$\|z_m-z_{m,I}-p\|_B\longrightarrow 0$$

as $m\longrightarrow\infty$ and the last condition of the criterion in Remark 3.2.7 is shown.

Now we want to show that the distance between $(z_{m,I})_m$ and $\partial_{\mathrm{rel}}K_F$ goes to infinity as $m\to\infty$. By $(S\,2)$, in every μ_τ-connected subset $I'\subseteq\Delta$ properly containing I, there is an $\alpha'\in I'\setminus I$ such that $\alpha'(H_m)\longrightarrow\infty$. Assume $d(\partial_{\mathrm{rel}}K_F,z_{m,I})<\infty$ and let $M:=\sup_{m\in\mathbb{N}}d(\partial_{\mathrm{rel}}K_F,z_{m,I})$. As the relative boundary $\partial_{\mathrm{rel}}K_F\cap\overline{\mathfrak{a}^+}$ is the union of cones of the form $\mathfrak{a}_{I'}\cap\overline{\mathfrak{a}^+}$ with $I\subsetneq I'\subseteq\Delta$ is μ_τ-connected, there has to be a μ_τ-connected subset $J\subseteq\Delta$ with $I\subseteq J$ such that $d(\mathfrak{a}_J,z_{m,I})\le M$ for all $m\in\mathbb{N}$. Set $h_m:=z_{m,I}$. We can split h_m up again to $h_m=h_{m,J}+h_m^J$ with $h_{m,J}\in\mathfrak{a}_J\cap\mathfrak{a}_I=\mathfrak{a}_J$ and h_m^J in the orthogonal complement of $\mathfrak{a}_J$ in $\mathfrak{a}_I$. Then

$$d(\mathfrak{a}_J,z_{m,I})=d(\mathfrak{a}_J,h_{m,J}+h_m^J)=d(\mathfrak{a}_J,h_m^J)<M \tag{4.19}$$

because $h_{m,J}\in\mathfrak{a}_J$. As $\alpha^J(z_m)=\alpha^J(h_m^I)+\alpha^J(z_m^I)$ for all $\alpha^J\in J\backslash I$ and by the boundedness of $(z_m^I)_m$, Equation (4.19) is a contradiction to the requirement of (S2), namely that there is an $\alpha^J\in J$ with $\alpha^J(H_m)\longrightarrow\infty$. Therefore the third condition of the remark is shown.

$\Leftarrow$ Now let $(z_m)_{m\in\mathbb{N}}$ be a sequence converging in the horofunction compactification with unit ball $B=-D^\circ$. Let $F\in\mathcal{F}$ be the extreme set of B that corresponds to this sequence. As by Remark 3.2.7 $\|z_m\|_B\longrightarrow\infty$, we have an unbounded sequence. The convergence in the Satake compactification requires $(H_m)_m:=(z_m)_m\subseteq\overline{\mathfrak{a}^+}$, but we only know that $(z_{m,V(F)})_m\subseteq K_F$ for m large enough. As described in the proof of Lemma 4.3.15, K_F might be bigger than $\overline{\mathfrak{a}^+}$. That is, F belongs not only to $\overline{\mathfrak{a}^+}$ but also to other Weyl chambers. Remember that we chose the positive Weyl chamber arbitrarily. So we could choose another of those chambers F belongs to as positive Weyl chamber. Then the μ_τ-connected subset $I\subseteq\Delta$ corresponding to F would still be μ_τ-connected with respect to this new $\overline{\mathfrak{a}^+}$. This is because F is orthogonal to the Weyl chamber wall between these Weyl chambers and therefore I remains μ_τ-connected. So by passing to a subsequence we can assume $(H_m)_m=(z_m)_m\in\overline{\mathfrak{a}^+}$ for a suitable choice of a positive Weyl chamber. For the first of the two conditions for a sequence to converge in the Satake compactification we have to show, that for all $\alpha\in I$ the limit $\lim_{m\to\infty}\alpha(H_m)$ exists and is finite. As $(z_{m,I})_m\in\mathfrak{a}_I$ it is

$$\alpha(H_m)=\alpha(z_{m,I}+z_m^I)=\alpha(z_m^I)$$

with $z_{m,I}\in\mathfrak{a}_I$ and $z_m^I\in\mathfrak{a}^I$ for all $m\in\mathbb{N}$. Then as $\|z_m^I-p\|_B=\|z_m-z_{m,I}-p\|_B\longrightarrow 0$, the limit of $(z_m^I)_m$ exists and is finite. As $\alpha\in\mathfrak{a}^*$ is a linear form, also $\alpha(z_m^I)$ exists and is finite.

The second condition to show is that in every μ_τ-connected subset $I'\subsetneq\Delta$ properly containing I such that for all $\alpha'\in I'\setminus I$ there holds $\alpha'(H_m)\longrightarrow\infty$. The $\overline{\mathfrak{a}^+}$-part of the relative boundary of K_F contains only those Weyl chamber walls $\mathfrak{a}_{I'}$ where I' is μ_τ-connected and properly contains I, see also Lemma 4.3.16. By Remark 3.2.7, $d(z_{m,I},\partial_{\mathrm{rel}}K_F)\longrightarrow\infty$. Assume there is a μ_τ-connected subset $I'\subseteq\Delta$ with $I\subsetneq I'$ such that there is an $\alpha'\in I'\setminus I$ with $\alpha'(H_m)\nrightarrow\infty$. Take a maximal of those subsets. This means in the relative boundary of K_F there is an $\mathfrak{a}_{I'}\cap\overline{\mathfrak{a}^+}=\bigcap_{\beta'\in I'}\ker(\beta')\cap\overline{\mathfrak{a}^+}$ with $\alpha'(H_m)$ is bounded for all $\alpha'\in I'$. But this is a contradiction to the requirement that $d(z_{m,I},\partial_{\mathrm{rel}}K_F)\longrightarrow\infty$. $\square$

Definition 4.3.19 Two compactifications $\overline{X}^1, \overline{X}^2$ are called *isomorphic*, if the identity map on X extends to a homeomorphism between them. ∘

As stated above, there are only finitely many isomorphism classes of Satake compactifications, depending on the Weyl chamber face the highest weight belongs to. The explicit statement goes as follows:

Proposition 4.3.20 ([BJ06, Prop. I.11.15]) *Let $\tau_1, \tau_2 : G \longrightarrow \mathrm{PSL}(n, \mathbb{C})$ be two irreducible faithful projective representations of G whose highest weights $\mu_{\tau_1}, \mu_{\tau_2}$ belong to the same Weyl chamber face. Then the corresponding Satake compactifications $\overline{X}^S_{\tau_1}, \overline{X}^S_{\tau_2}$ are isomorphic.* ∘

Let us explain next how this isomorphism also works for the corresponding horofunction compactifications. Two horofunction compactifications of X with two different polyhedral unit balls B_1 and B_2 are isomorphic, if B_1 and B_2 are combinatorially the same and if additionally their extreme sets lie in the same directions. Otherwise we could find a sequence going through an extreme set of B_1 but not of B_2 and which would therefore converge to different boundary points, and the identity could not extend to an homeomorphism on the boundaries.

Let $D = \mathrm{conv}(\mathcal{W}(\mu_\tau))$ for some representation τ and let $B = -D^\circ$ be the negative polar of D. Each extreme point of B corresponds to a facet of D and B is the convex hull of all its vertices. So let E be a face of D. By the construction of D via reflections with respect to the Weyl chamber walls, there is a (probably empty) subset $J \subseteq \Delta$ such that E is invariant under reflections at $\mathfrak{a}_J$. This means that E is orthogonal to each Weyl chamber wall $\mathfrak{a}_\alpha$ with $\alpha \in J$. The corresponding extreme point p_E will lie on $\mathfrak{a}_J$ when $\mathfrak{a}$ is identified with $\mathfrak{a}^*$ via κ.

Let now τ_1, τ_2 be two different representations of G with highest weights $\mu_{\tau_1}, \mu_{\tau_2}$ in the same Weyl chamber face. Then the convex hulls D_1 and D_2 are combinatorially the same. Let $E_1 \subseteq D_1, E_2 \subseteq D_2$ be two corresponding extreme faces. Then they are invariant with respect to reflections at the same Weyl chamber walls. This means that they are parallel to each other. Because of this parallelism and the discussion before, the corresponding extreme points p_{E_1} and p_{E_2} of B_1 and B_2, respectively, lie in the same Weyl chamber face and therefore in the same direction. With this, all extreme sets of B_1 and B_2 are pairwise in the same direction and so the corresponding horofunction compactifications are isomorphic.

4.3.4 Realization of a Satake Compactification as a Horofunction Compactification

Proposition 4.3.21 *Let $X = G/K$ be a symmetric space of non-compact type. Let τ be a faithful irreducible projective representation of G, and $\mu_1, \dots, \mu_n$ its weights. Let $D := \mathrm{conv}(\mu_1, \dots, \mu_n) \subseteq \mathfrak{a}^*$. Let $B = -D^\circ$ the dual closed convex set in the maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{g}$.*
Then the closure of the flat $A.p_0$ in the Satake compactification is $\mathcal{W}$-equivariantly isomorphic to the closure of the flat $A.p_0$ in the horofunction compactification of X with respect to the Finsler metric defined by B. ∘

Proof. By Theorem 4.2.17, it suffices to compare the closure of $A^+.p_0$ in the Satake compactifications with the closure of $A^+.p_0$ in the flat compactification of $A.p_0$ with respect to the norm defined by B. By Proposition 4.3.2 and Theorem 3.2.6, both are $\mathcal{W}$-equivariantly homeomorphic to the closed convex $\mathrm{conv}(\mu_1, \dots, \mu_n) = D = -B^\circ$.

As by Theorem 4.3.18 a sequence $(H_m)_m \in \mathfrak{a}$ converges in the Satake compactification $\overline{X}^S_\tau$ if and only if it converges in the horofunction compactification $\overline{X}^{hor}$ with respect to the G-invariant Finsler metric defined by B, the statement follows. □

Theorem 4.3.22 *Let $X = G/K$ be a symmetric space of non-compact type. Let τ be a faithful irreducible projective representation of G and $\mu_1, \ldots, \mu_n$ its weights. Let $D := \mathrm{conv}(\mu_1, \ldots, \mu_n)$. Let $B = -D^\circ$ define a unit ball in the maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{g}$. Then the Satake compactification $\overline{X}^S_\tau$ is G-equivariantly isomorphic to the horofunction compactification of X with respect to the Finsler metric defined by B.* ○

Proof. We show that a sequence converges in the Satake compactification $\overline{X}^S_\tau$ if and only if it converges in the horofunction compactification $\overline{X}^{hor}$ with respect to the G-invariant Finsler metric defined by B. Let $x_n \in X$ be a sequence. Then we can write $x_n = k_n \cdot a_n p_0$, where $k_n \in K$ and $a_n \in \overline{A^+}$ is uniquely determined. Up to passing to a subsequence we can assume that x_n converges in $\overline{X}^S_\tau$ and that k_n converges to an element $k \in K$. Therefore Theorem 4.3.22 is a consequence of Proposition 4.3.21. □

Remark 4.3.23 Note that Theorem 4.3.22 describes explicitly the convex unit ball of the Finsler metric which induces the horofunction compactification realizing the Satake compactifications. For classical Satake compactifications the convex D (and hence also the unit ball B) has a particularly simple description as it is just the convex hull of the Weyl group orbit of the highest weight vector of τ. In order to obtain the Satake compactification determined by a subset $I \subseteq \Delta$ one has to choose a representation τ, whose highest weight vector has support equal to I. ○

Remark 4.3.24 For generalized Satake compactifications, the same result as Proposition 4.3.2 holds: $\overline{X}^S_\tau$ is $\mathcal{W}$-equivariantly homeomorphic to the convex hull D of the Weyl group orbit of the highest weights $\mu_{\tau_1}, \ldots, \mu_{\tau_k}$ of the irreducible components in $\mathfrak{a}^*$. Therefore the convex hull D can have more than one vertex in a Weyl chamber and its negative dual $B := -D^\circ$ has some of its vertices on Weyl chamber walls but not all of them. Therefore a different criterion for the convergence of sequences is needed, one that does not depend on μ_τ-connected subset of Δ but chooses subsets of roots in a different way. Such a criterion will be given in a new version of [GKW15]. ○

4.4 The Martin Compactification of Symmetric Spaces

The Martin compactification can be defined for any complete Riemannian manifold X using the spectrum of the Laplace-Beltrami operator and has no direct geometric interpretation. We will give a short introduction on the basic idea first and then give a geometric characterization of the Martin compactification of a symmetric space X of non-compact type in terms of the maximal Satake and the geodesic compactification. Finally we show how we can realize the Martin compactification as a horofunction compactification with respect to an appropriate norm.

We start with a short description of the basic construction of the Martin compactification. More details can be found in [BJ06, §I.7], [Ji97] or [GJT98, §VI]. For us let X be a symmetric space of non-compact type. Consider the Laplace-Beltrami operator Δ which is a generalization of the Laplacian on $\mathbb{R}^n$ and is also defined as the divergence of the gradient of a function: $\Delta f = \nabla \cdot \nabla f$. In local coordinates, Δ is given by

$$\Delta f = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j f \right),$$

where $\partial_i := \frac{\partial}{\partial x_i}$ is the frame of the tangent bundle TM and g is the metric tensor with $|g| := |\det(g_{ij})|$.

Let the operator be normalized such that $\Delta \geq 0$. We want to look at the spectrum of Δ, that is, we consider the eigenvalue equation

$$-\Delta u = \lambda u.$$

Let $\lambda \in \mathbb{R}$ be an eigenvalue and

$$C_\lambda(X) := \{u \in C^\infty(X) \mid \Delta u = \lambda u,\ u > 0\}$$

the cone of associated eigenfunctions. If $\lambda = 0$, then $C_0(X)$ is the set of harmonic functions. Let $\lambda_0 = \lambda_0(X)$ denote the bottom of the spectrum of $-\Delta$. By a result of Cheng and Yau [CY75], the space of eigenfunctions $C_\lambda(X)$ is non-empty if and only if $\lambda \leq \lambda_0(X)$. To each such $\lambda \leq \lambda_0$ one can associate a Martin compactification $X \cup \partial_\lambda(X)$, defined by the asymptotic behavior of the Green's function[4] $G_\lambda(x, y)$ of $\Delta - \lambda$. Each boundary point $\xi \in \partial_\lambda(X)$ then corresponds to a positive eigenvalue function $K_\lambda(x, \xi) \in C_\lambda(X)$, called the *Martin kernel function*. It is given as the limit of the normalized Green functions $K_\lambda(x, y_m) := G_\lambda(x, y_m)/G_\lambda(x_0, y_)$. Note that $K_\lambda(x, y)$ is smooth on $X\backslash\{y\}$, satisfies the eigenvalue equation $\Delta K_\lambda(x, y) = \lambda K_\lambda(x, y)$ and for the basepoint x_0 it holds $K_\lambda(x_0, y) = 1$.

Although the Martin compactification is not particular geometric, it can be described in terms of the maximal Satake and the geodesic compactification. Before we give the identification explicitly, let us shortly recall the construction of the geodesic compactification (see [BJ06, §I.2] for details).

Geodesic compactification Let γ_1, γ_2 be two geodesic rays in X. Then they are called *equivalent* and denoted by $\gamma_1 \sim \gamma_2$, if

$$\limsup_{t\to\infty} d(\gamma_1(t), \gamma_2(t)) < \infty.$$

This gives an equivalence relation with equivalence classes denoted by $[\gamma]$. The *sphere at infinity* $X(\infty)$ is the set of all equivalence classes:

$$X(\infty) := \{[\gamma] \mid \gamma \text{ is a geodesic in } X\}.$$

It can be canonically identified with the unit sphere (with respect to the Riemannian metric on X) in $\mathfrak{p}$. Attaching the sphere at infinity to X gives us the *geodesic compactification* $X \cup X(\infty)$.

Let us look at the geodesic compactification in $\mathfrak{a}$. The unit sphere of the Riemannian metric is a Euclidean ball. By our results in the previous section the geodesic compactification is homeomorphic to the horofunction compactification of X with respect to the Riemannian metric, that is, the Euclidean norm on $\mathfrak{a}$. This can also be seen by a direct comparison with the convergence behavior of sequences we discussed in Example 3.2.9 in Chapter 3: parallel sequences have bounded distance and thereby converge to the same limit function. But sequences following straight lines through the origin in different directions converge to different horofunctions.

The Martin compactification then can be characterized in the following way:

Proposition 4.4.1 ([BJ06, Prop. I.7.15]) *The Martin compactification $X \cup \partial_{\lambda_0}(X)$ is isomorphic to the maximal Satake compactification $\overline{X}^S_{\max}$.*

For $\lambda \leq \lambda_0(X)$, the Martin compactification $X \cup \partial_\lambda(X)$ is the least common refinement of the maximal Satake compactification $\overline{X}^S_{\max}$ and the geodesic compactification $X \cup X(\infty)$:

$$X \cup \partial_\lambda(X) = \overline{X}^S_{\max} \vee X \cup X(\infty). \qquad \circ$$

[4] $G_\lambda(x, y)$ is a positive symmetric function such that $\Delta G_\lambda(x, y) - \lambda G_\lambda(x, y) = \delta(x - y)$ and $G_\lambda(x, y_m) \to 0$ for $y_m \to \infty$.

In Section 3.4 we showed that we get the least common refinement of two horofunction compactifications $\overline{X}_1, \overline{X}_2$ by adding their norms $\|\cdot\|_1, \|\cdot\|_2$, respectively. In terms of unit balls, the refinement is the horofunction compactification with respect to the unit ball whose dual is the Minkowski sum of the associated dual unit balls B_1° and B_2°. Therefore the dual unit ball of $X \cup \partial_\lambda(X)$ is the Minkowski sum of the Euclidean unit sphere and the dual unit ball of the maximal Satake compactification $\overline{X}^S_{\max}$. As the unit ball of the Euclidean norm is smooth and the one of the maximal Satake compactification is polyhedral, the Martin compactification is homeomorphic to the dual unit ball. Therefore we have the following Theorem.

Theorem 4.4.2 *Let $X = G/K$ be a symmetric space of non-compact type. Let τ be a faithful irreducible projective representation of G with generic highest weight $\mu_\tau \in \mathfrak{a}^+$. With the Weyl group $\mathcal{W}$ let $D := \mathrm{conv}(\mathcal{W}(\mu_\tau))$ be the $\mathcal{W}$-orbit of the highest weight. Denote the norm on $\mathfrak{a}$ defined by the unit ball $B_S := -D^\circ$ by $\|\cdot\|_S$. Let $\|\cdot\|_E$ be the Euclidean norm on $\mathfrak{a}$.*

Then for $\lambda = \lambda_0$, the Martin compactification $X \cup \partial_{\lambda_0}(X)$ is homeomorphic to the horofunction compactification of X with respect the Finsler norm given by $\|\cdot\|_S$ on $\mathfrak{a}$.

For $\lambda < \lambda_0(X)$, the Martin compactification $X \cup \partial_\lambda(X)$ is homeomorphic to the horofunction compactification of X with respect to the Finsler norm given by the sum $\|\cdot\| = \|\cdot\|_S + \|\cdot\|_E$ on $\mathfrak{a}$. ◦

For the convergence behavior of sequences in the Martin compactification we have the following result:

Lemma 4.4.3 ([Ji05, S. 96]) *An unbounded sequence $(e^{H_m}.p_0)_{m\in\mathbb{N}} \in e^{\overline{\mathfrak{a}^+}}.p_0 \subseteq X$ converges in the Martin compactification $X \cup \partial_\lambda(X)$ if and only if there is a proper subset $I \subsetneq \Delta$ such that*

- *$(H_m)_m$ is of type $(I, \hat{a}^I)$ for some $\hat{a}^I \in A^I$*
- *the direction $\frac{H_m}{\|H_m\|}$ converges to a limit L.* ◦

In [GJT98, Def. 8.6] such a sequence is called *I-directional*. The convergence condition is a combination of the one for the maximal Satake compactification (the type) and the geodesic compactification (the direction) which is exactly the result of Lemma 3.4.6. The limit depends on both the type and the limiting direction.

Example 4.4.4 Let us consider a Martin compactification of $X = \mathrm{SL}(3, \mathbb{C})/SU(3)$. The maximal Satake compactification can be obtained for example for the adjoint representation $\tau = \mathrm{ad}$.

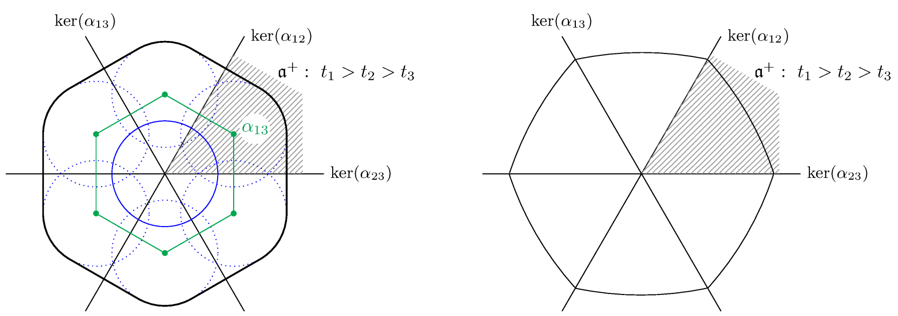


Figure 4.10: LEFT: The sum of the $\mathcal{W}$-orbit of the highest weight $\mu_\tau = \alpha_{13}$ for the maximal Satake compactification (green hexagon) and the Euclidean unit ball for the geodesic compactification (blue circle) is a blown-up hexagon with rounded corners. RIGHT: The associated unit ball is a blow-up hexagon with corner points on the Weyl chamber walls.

Then the convex hull of the highest weight $\mu_\tau = \alpha_{13}$ is a hexagon with regular vertices. For the geodesic compactification we get a unit Euclidean circle as dual to the norm. The Minkowski sum M of these two convex sets is a hexagon with rounded corners and facets orthogonal to the

Weyl chamber walls as pictured in Figure 4.10 on the left. In the end we want the horofunction compactification of $\mathfrak{a}$ to be homeomorphic to this rounded hexagon, so we choose $B := -M^\circ \subseteq \mathfrak{a}$ as unit ball. B is a blown-up hexagon with corner points on the Weyl chamber wall, see the right picture in Figure 4.10. When we take the horofunction compactification of X with respect to the Finsler norm defined by B, the compactification is homeomorphic to the Martin compactification $X \cup \partial_\lambda(X)$.

Next we want to look at the convergence behavior of sequences. Let us first consider the sequence

$$H_m = \operatorname{diag}(2m+k, -m, -m-k) = \operatorname{diag}\left(0, \frac{k}{2}, -\frac{k}{2}\right) + \operatorname{diag}\left(2m+k, -\frac{k}{2}, -\frac{k}{2}\right)$$

for some $k > 0$. It is of type $I = (\{\alpha_{23}\}, H^I)$ with $H^I = \exp\left(\operatorname{diag}\left(0, -\frac{k}{2}, -\frac{k}{2}\right)\right) \in A^I$. The limiting direction is $L = \operatorname{diag}(2, -1, -1) \in \mathfrak{a}_I$. So we have the same limiting direction for all $k \in \mathbb{R}$ but their limits in the Martin compactification differ because each k defines a different type. This behavior coincides well with the convergence behavior in the horofunction compactification with respect to B. The limiting direction lies in $\mathfrak{a}_I$ where B has a corner point F and it determines the extreme set F° to which the horofunction is associated. The extreme sets are shown in Figure 4.11. The type of $(H_m)_m$ corresponds to the parameter $p \in \mathfrak{a}^I$ which is bounded.

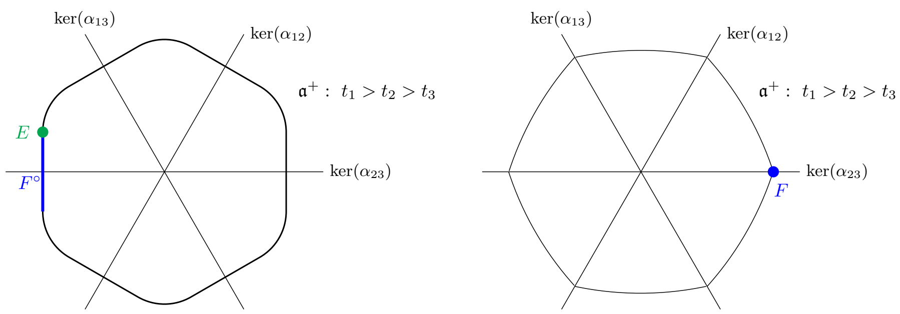


Figure 4.11: LEFT: The dual unit ball with the extreme sets F° and E. RIGHT: The unit ball with the extreme point F in the limiting direction.

Let us now change the sequence by replacing k with $\log(m)$ which now is unbounded:

$$H'_m = \operatorname{diag}\left(2m + log(m), -m, -m - \log(m)\right).$$

Then the new sequence $(H'_m)_m$ has type $I = \emptyset$ which is exactly what we want because that corresponds to a smooth extreme point of B. To know which one, we look at the limit of directions and get $L' = \operatorname{diag}(2, -1, -1)$ as before. This limit in the Martin compactification corresponds to the horofunction associated to the non-exposed extreme point E in the relative boundary of F°. $\circ$

5 | Toric Varieties

Toric varieties provide a basic class of algebraic varieties which are relatively simple. Many algebro-geometric properties of projective toric varieties X_{Σ_P} can be described in terms of a defining polytope P in $\mathbb{R}^n$. Important for us is a homeomorphism between the non-negative part $X_{\Sigma,\geq 0}$ of a projective toric variety and the dual polytope P°. We will use this correspondence to explain how the non-negative part $X_{\Sigma,\geq 0}$ is homeomorphic to the horofunction compactification of $\mathbb{R}^n$ with respect to a suitable norm. To do so, we introduce a topological model $\overline{T}_\Sigma$ of the toric variety with a characterization of convergent sequences. A key point then will be an identification between the topological model $\overline{T}_\Sigma$ and the toric variety X_{Σ_P} constructed in the usual way.

5.1 Background Knowledge about Toric Varieties and Fans

We now give a summary of several results on toric varieties which are needed to understand and prove Theorem 5.3.8. The basic references for this section are [Ful93], [CLS11], [Oda88], [Oda78], [AMRT10], [Cox03], and [Sot03]. To minimize notational confusion, we will in this Chapter denote more-dimensional elements in $\mathbb{R}^n$ or $\mathbb{Z}^n$ (or in their dual spaces) by bold letters.

5.1.1 Affine, Projective and Toric Varieties

Originally, varieties were introduced as the solution set of a set of equations. The easiest examples are affine and projective varieties:

An *affine variety* is the zero-locus of a set S of polynomials,

$$V = \mathbf{V}(S) = \{x \in \mathbb{C}^n \mid f(x) = 0 \ \forall f \in S\} \subseteq \mathbb{C}^n.$$

A *projective variety* is the zero-locus of a set S of homogeneous polynomials:

$$V = \mathbf{V}(S) = \{x \in \mathbb{P}^n \mid F(x) = 0 \ \forall F \in S\} \subseteq \mathbb{P}^n.$$

By the Hilbert Basis Theorem, the set S can be assumed to be finite. So affine (or projective) varieties can be embedded into $\mathbb{C}^n$ (or $\mathbb{P}^n$) for some n. Morphisms between varieties are regular maps, that is, they are maps which are locally given by polynomials. There are also *abstract varieties* that can not be embedded. For us an abstract variety X is obtained by gluing finitely many affine varieties V_α along common Zariski open (definition see below) subsets: Let V_α be finitely many affine varieties such that for all α, β there are Zariski open subsets $V_{\beta\alpha} \subseteq V_\alpha$ and isomorphisms $g_{\beta\alpha} : V_{\beta\alpha} \cong V_{\alpha\beta}$ such that the following three conditions are satisfied for all α, β, γ:

1) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$,
2) $g_{\beta\alpha}(V_{\beta\alpha} \cap V_{\gamma\alpha}) = V_{\alpha\beta} \cap V_{\gamma\beta}$,
3) $g_{\gamma\alpha} = g_{\gamma\beta} \circ g_{\beta\alpha}$ on $V_{\beta\alpha} \cap V_{\gamma\alpha}$.

Then we define the abstract variety X by

$$X := \coprod_{\alpha} V_\alpha / \sim,$$

where the equivalence relation is given by $x \sim y$ if and only if $x \in V_{\beta\alpha}$, $y \in V_{\alpha\beta}$ such that $y = g_{\beta\alpha}(x)$. With $W_\alpha := \{[x] \in X \mid x \in V_\alpha\}$ we get an open cover of X. As $W_\alpha \cong V_\alpha$ by the projection on the equivalence class, X locally looks like an affine variety. This definition is due to [CLS11, Def. 3.0.5] and corresponds to the gluing of schemes. We will use this construction later in Subsection 5.1.3 to construct the toric variety X_Σ associated to a fan Σ out of affine varieties U_σ.

Sometimes the affine/projective varieties as we described them above are named *affine/projective algebraic sets* and only called varieties, if they are irreducible, that is, if they can not be written as the union $V_1 \cup V_2$ of two (smaller) algebraic sets $V_1 \neq V, V_2 \neq V$. We will call both kinds varieties and distinguish between general and irreducible ones.

The topology on algebraic varieties is the Zariski topology. Closed sets in this topology are the general algebraic varieties as defined above. So for two varieties $V_2 \subseteq V_1$ the complement $W := V_1 \setminus V_2$ is called a *Zariski open subset of* V_1. Considering the set S_2 of (respectively homogeneous) polynomials defining V_2, then W are exactly those points of V_1 where not all equations of S_2 vanish.

Definition 5.1.1 A toric variety over $\mathbb{C}$ is an irreducible variety V over $\mathbb{C}$ such that

1) the complex torus $T := (\mathbb{C}^\times)^n$ can be embedded as a Zariski open subset of V and
2) the action of T on itself by multiplication extends to an algebraic action of T on V. ○

Example 5.1.2 We collect some examples of toric varieties:

1) $(\mathbb{C}^\times)^n$ itself is a toric variety.
2) Consider
$$W := \mathbf{V}(x_1 \cdot \ldots \cdot x_n) = \{z \in \mathbb{C}^n \mid z_i = 0 \text{ for some } i \in \{1, \ldots, n\}\},$$
where z_i denotes the i-component of z. The identity $(\mathbb{C}^\times)^n = \mathbb{C}^n \setminus W$ tells us that $(\mathbb{C}^\times)^n \subseteq \mathbb{C}^n$ is a Zariski open subset. Therefore $\mathbb{C}^n$ is an affine toric variety.
3) $\mathbb{P}^n$ is also a toric variety. To see this, consider the algebraic set
$$U := \mathbb{P}^n \setminus W,$$
where $W = \mathbf{V}(x_0 x_1 \cdots x_n)$. Then under the embedding
$$\begin{aligned} (\mathbb{C}^\times)^n &\longrightarrow \mathbb{P}^n \\ (t_1, \ldots, t_n) &\longmapsto (1 : t_1 : \ldots : t_n) \end{aligned}$$
$(\mathbb{C}^\times)^n$ can be identified with U and is therefore a Zariski open subset of $\mathbb{P}^n$. The torus action is given by
$$(t_1, \ldots, t_n).(x_0 : x_1 : \ldots : x_n) = (x_0 : t_1 x_1 : \ldots : t_n x_n)$$
and shows that $\mathbb{P}^n$ is a projective toric variety.

4) If V, W are toric varieties, so is $V \times W$. Therefore $\mathbb{P}^1 \times \mathbb{P}^1$ is a toric variety to which we will come back again later in Example 5.1.15. ∘

Associated to the torus $(\mathbb{C}^\times)^n$, there are two important classes of maps: the characters, which are regular group homomorphisms $\chi : (\mathbb{C}^\times)^n \longrightarrow \mathbb{C}^\times$, and the one-parameter subgroups (or co-characters), which are regular group homomorphisms $\lambda : \mathbb{C}^\times \longrightarrow (\mathbb{C}^\times)^n$.

A point $\mathbf{m} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ defines a *character* $\chi^{\mathbf{m}}$ by

$$\begin{aligned} \chi^{\mathbf{m}} : (\mathbb{C}^\times)^n &\longrightarrow \mathbb{C}^\times \\ \mathbf{t} &\longmapsto \chi^{\mathbf{m}}(\mathbf{t}) = \mathbf{t}^{\mathbf{m}} = \prod_{j=1}^{n} t_j^{a_j}. \end{aligned} \tag{5.1}$$

All characters of $(\mathbb{C}^\times)^n$ arise this way, therefore the group $M := \mathrm{Hom}_{\mathrm{reg}}((\mathbb{C}^\times)^n, \mathbb{C}^\times)$ of characters is isomorphic to $\mathbb{Z}^n$.

The *one-parameter-subgroup* associated to $\mathbf{u} = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ is

$$\begin{aligned} \lambda^{\mathbf{u}} : \mathbb{C}^\times &\longrightarrow (\mathbb{C}^\times)^n \\ r &\longmapsto \lambda^{\mathbf{u}}(r) = (r^{b_1}, \ldots, r^{b_n}). \end{aligned}$$

All one-parameter-subgroups of $(\mathbb{C}^\times)^n$ are defined this way, so we again have an isomorphism of the group of one-parameter-subgroups to $\mathbb{Z}^n$, $N := \mathrm{Hom}_{reg}(\mathbb{C}^\times, (\mathbb{C}^\times)^n) \cong \mathbb{Z}^n$.

When we combine these two maps, we get a map $\chi^{\mathbf{m}} \circ \lambda^{\mathbf{u}} : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ given by $r \mapsto r^{\langle \mathbf{m}|\mathbf{u}\rangle}$, where $\langle \mathbf{m}|\mathbf{u}\rangle = \sum_j a_i b_i$ is the normal dot product on $\mathbb{Z}^n$.

Remark 5.1.3 Most literature does not work directly with $\mathbb{Z}^n$ and $\mathbb{R}^n$ but in a more general setting where the algebraic torus $T = \mathbb{G}_m^n$ is not equal but isomorphic to $(\mathbb{C}^\times)^n$. Then characters are regular group homomorphisms $\chi : T \longrightarrow \mathbb{C}^\times$ with $M = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{C}^\times) \cong \mathbb{Z}^n$ and one-parameter-subgroups are elements of $N = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{C}^\times, T) \cong \mathbb{Z}^n$. The isomorphism $T \cong (\mathbb{C}^\times)^n$ induces dual bases of M and N such that even in the general setting, characters are Laurent monomials and one-parameter-subgroups are monomial curves. ∘

5.1.2 Rational Polyhedral Cones, Fans and Polytopes

We fix the *standard lattice* $\mathbb{Z}^n$ in $\mathbb{R}^n$, which gives $\mathbb{R}^n$ an integral structure and also a $\mathbb{Q}$-structure, $\mathbb{Q}^n \subseteq \mathbb{R}^n$. As cones and their faces play an important role when building toric varieties, we recall the definition:

Definition 5.1.4 A *rational polyhedral cone* $\sigma \subseteq \mathbb{R}^n$ is a cone generated by finitely many elements $u_1, \cdots, u_m$ of $\mathbb{Z}^n$, or equivalently of $\mathbb{Q}^n$:

$$\sigma = \{\lambda_1 u_1 + \ldots + \lambda_m u_m \in \mathbb{R}^n \mid \lambda_1, \cdots, \lambda_m \geq 0\}.$$ ∘

Usually, σ is assumed to be *strongly convex*: $\sigma \cap -\sigma = \{0\}$, that is, σ does not contain any line through the origin. As we require a cone σ to be generated by only finitely many elements, every extreme face of σ is exposed and will be just called a *face*.

For each strongly convex rational polyhedral cone σ, we can define its dual cone:

Definition 5.1.5 Let $\sigma \subseteq \mathbb{R}^n$ be a strongly convex rational polyhedral cone. Its *dual cone* $\sigma^\vee$ is given by

$$\sigma^\vee := \{v \in \mathbb{R}^n \mid \langle v|u\rangle \geq 0 \text{ for all } u \in \sigma\}. \tag{5.2}$$

Then $\sigma^\vee$ is also a convex rational polyhedral cone, though it is not strongly convex anymore unless $\dim \sigma = n$ (see [Oda88, Prop. 1.3] for a proof). ◦

Remark 5.1.6 Recall that in Definition 2.4.1 on page 21 we defined the dual of a non-empty compact convex set $B \subseteq \mathbb{R}^n$ as $B^\circ = \{y \in (\mathbb{R}^n)^* \mid \langle y|x\rangle \geq -1\ \forall x \in B\}$. Our cone here now is closed but not bounded. Applying the same defining equation $\langle y|u\rangle \geq -1$ for all $u \in \sigma$ to the cone σ leads to the above definition with ≥ 0 since σ is unbounded: let $y \in (\mathbb{R}^n)^*$ and $u \in \sigma$ be given such that $0 > \langle y|u\rangle \geq -1$. Then with $\alpha > 0$ we can find some $w = \alpha u \in \sigma$ such that $\langle y|w\rangle = \alpha\langle y|u\rangle < -1$ if we only choose $\alpha \in \mathbb{R}_{\geq 0}$ big enough. Therefore we use the name *dual* in both cases but take different notations: P° for dual unit balls of polytopes and $\sigma^\vee$ for dual cones. ◦

Similarly to the theory of polytopes, we can describe the faces of a cone $\sigma \subseteq \mathbb{R}^n$ as those subsets on which the pairing with an element of the dual cone vanishes:

Remark 5.1.7 ([Oda88, Prop 1.3.]) A subset $\tau \subseteq \sigma$ is a face (often denoted by $\tau < \sigma$), if there is an $m_0 \in \sigma^\vee \cap \mathbb{Z}^n$ such that

$$\tau = \sigma \cap \{m_0\}^\perp = \{y \in \sigma \mid \langle m_0|y\rangle = 0\},$$

where as usual $\{m_0\}^\perp$ denotes the orthogonal subspace. ◦

Remark 5.1.8 One face of the dual cone $\sigma^\vee$ is for example given by the *orthogonal cone*

$$\sigma^\perp = \{v \in \mathbb{R}^n \mid \langle v|u\rangle = 0 \text{ for all } u \in \sigma\}. \qquad ◦$$

Example 5.1.9 Let us look at some examples of cones and their duals

1) We consider a two-dimensional cone in $\mathbb{R}^2$ as given in Figure 5.1 on the left.

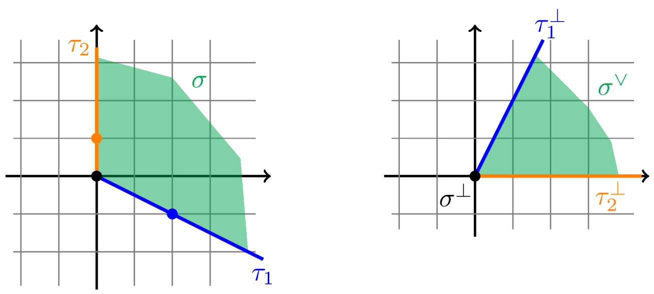


Figure 5.1: A two-dimensional cone σ in $\mathbb{R}^2$ (LEFT) and its dual (RIGHT).

Its two one-dimensional faces τ_1, τ_2 are determined by the points $(2, -1)$ and $(0, 1)$ and each of them defines an orthogonal hyperplane. The dual cone $\sigma^\vee$ then is the intersection of the two positive half-spaces determined by these hyperplanes and given as in Figure 5.1 on the right. The orthogonal cone $\sigma^\perp$ is the intersection of the two hyperplanes and therefore only the origin.

2) We look again at $\mathbb{R}^2$ but now our cone σ is only one-dimensional, see Figure 5.2 on the left.

Then every point $v = (v_1, v_2) \in \mathbb{R}^2$ with $v_1 = 0$ has pairing equal to 0 with σ. If $v_1 \geq 0$ then the pairing is also nonnegative, so the orthogonal cone $\sigma^\perp$ is the y-axis and $\sigma^\vee$ are those points with $v_1 \geq 0$. Note that now $\sigma^\vee$ is not strongly convex and more. ◦

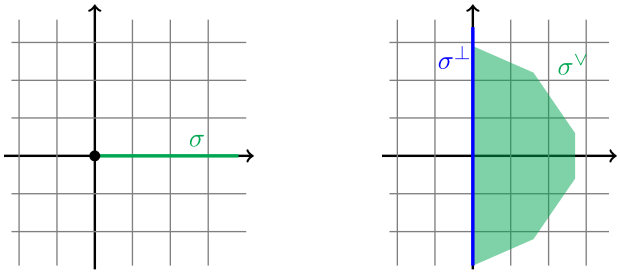


Figure 5.2: A one-dimensional cone σ in $\mathbb{R}^2$ (LEFT) and its dual (RIGHT). As the cone σ is not two-dimensional, its dual cone is not strongly convex.

Putting several cones together in the same way as a simplicial complex is build out of simplices, we get a fan, which will play an important role for constructing toric varieties.

Definition 5.1.10 A *fan* Σ in $\mathbb{R}^n$ is a collection of strongly convex rational polyhedral cones such that

1) if $\sigma \in \Sigma$, then every face of σ also belongs to Σ;
2) if $\sigma, \tau \in \Sigma$, then their intersection $\sigma \cap \tau$ is a common face of both of them, and hence belongs to Σ. ∘

In this thesis, we only deal with fans which consist of finitely many polyhedral cones. Note that relative interiors of the cones of Σ do not intersect each other.

An important class of fans for us are those that can be constructed from a rational convex polytope. Hereby a convex polytope $P \subseteq \mathbb{R}^n$ is called a *rational convex polytope*[1], if all vertices of P are contained in $\mathbb{Z}^n$. As a more detailed reference for this construction see [Ful93, Section 1.5]. Now assume that P is such a rational convex polytope in $\mathbb{R}^n$ and contains the origin as an interior point. Then each face F of P spans a rational polyhedral cone

$$\sigma_F = \mathbb{R}_{\geq 0} \cdot F,$$

that is, the face F is a section of the cone σ_F, and these cones σ_F form a fan in $\mathbb{R}^n$, denoted by Σ_P. See Figure 5.3 below for an example. For any $k \in \mathbb{Z}$, the scaled polytope kP is also a rational polytope and gives the same fan:

$$\Sigma_{kP} = \Sigma_P.$$

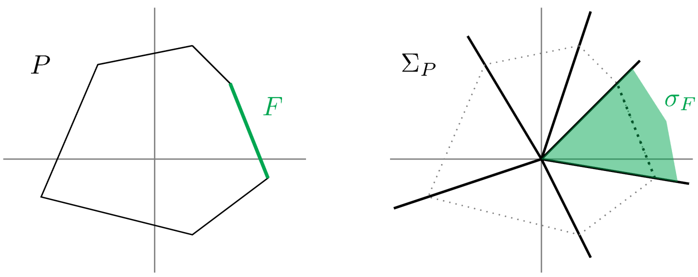


Figure 5.3: A rational convex polytope P and its corresponding fan Σ_P in $\mathbb{R}^2$

Remark 5.1.11 The fan Σ_P as we defined it here must not be confused with the *normal fan* of a polytope: Given a polytope Δ (this is the notation used in [Cox03]), the fan Σ_Δ associated to the

[1] This definition is due to [Ful93, p. 24]. At some other places, P is called a rational convex polytope if the vertices of P are contained in $\mathbb{Q}^n$, and P is called an integral convex polytope if the vertices of P are contained in $\mathbb{Z}^n$.

faces of the dual polytope Δ° is called the normal fan of the polytope Δ (it is defined for example in [Cox03, pp. 217-218] or [Ful93, Proposition, p. 26]). Is is usually constructed by normal cones of the Δ, see also [Wei07] for more details. $\circ$

It is known that not every fan Σ in $\mathbb{R}^n$ comes from such a rational convex polytope P, as the following example shows.

Example 5.1.12 ([Ful93, p.25]) Take the fan generated as follows: at first consider the eight half-lines through the origin and one of the vertices of a standard cube. Now replace the vertex $(1, 1, 1)$ by $(1, 2, 3)$. Then it is not possible to find eight points, one on each of the half-lines, such that for each of the six cones, the four corresponding generating points lie on one affine hyperplane. This can be seen by trying to solve the corresponding system of linear equations. $\circ$

In [Ful93, p. 26] and [Cox03, p. 219], the polar P° of a rational polytope P is used to construct a toric variety. Recall the definition of the polar of a convex polytope:

$$P^\circ = \{v \in \mathbb{R}^n \mid \langle v, u\rangle \geq -1, \text{ for all } u \in P\}. \tag{5.3}$$

When P is a rational convex polytope containing the origin as an interior point, then so is P°.

By Remark 2.4.14 on page 27 (see also [Ful93, p. 24] or [HSWW18, Lemma 3.7]), we know that there is a duality between P and P° which is given by an one-to-one correspondence between the set of faces of P and the set of faces of P° which reverses the inclusion relation. Explicitly, to each face F of P there is a unique dual face F° of P° satisfying:

1) For any $x \in F$ and $y \in F^\circ$ it holds: $\langle x, y\rangle = -1$,
2) $\dim(F) + \dim(F^\circ) = n - 1$.

5.1.3 Constructing a Toric Variety from a Fan

The way how a toric variety X_Σ is constructed from a fan Σ and a description of its topology in terms of Σ is crucial to the proof of Theorem 5.3.8. Given a fan Σ in $\mathbb{R}^n$, its associated abstract *toric variety* X_Σ is constructed as follows in two steps:

Step 1: Build affine toric varieties U_σ

We first construct an affine toric variety U_σ for each cone $\sigma \in \Sigma$. By Gordan's Lemma, the affine semigroup,

$$S_\sigma := \sigma^\vee \cap \mathbb{Z}^n = \{\mathbf{m} \in \mathbb{Z}^n \mid \langle \mathbf{m}|\mathbf{u}\rangle \geq 0 \ \forall \mathbf{u} \in \sigma\}$$

is finitely generated and it contains the origin, see also [CLS11, p. 30]. Let $\mathbf{m}_1, \ldots, \mathbf{m}_k \in \sigma^\vee \cap \mathbb{Z}^n$ be a set of generators of this semigroup, that is,

$$S_\sigma = \left\{ \sum_{i=1}^{k} a_i \mathbf{m}_i | a_i \in \mathbb{Z}_{\geq 0} \right\}.$$

Consider the map

$$\begin{aligned} \varphi : (\mathbb{C}^\times)^n &\longrightarrow \mathbb{C}^k, \\ \mathbf{t} &\longmapsto (\chi^{\mathbf{m}_1}(\mathbf{t}), \cdots, \chi^{\mathbf{m}_k}(\mathbf{t})) \end{aligned} \tag{5.4}$$

based on the characters as defined in Equation (5.1). Taking the Zariski closure of the image we get the affine toric variety U_σ:

$$U_\sigma := \overline{\varphi((\mathbb{C}^\times)^n)} \subseteq \mathbb{C}^k. \tag{5.5}$$

This means that U_σ is the smallest variety containing the image $\varphi((\mathbb{C}^\times)^n)$. See [CLS11, Thm. 1.1.8] for a proof that is actually is an affine toric variety.

Remark 5.1.13 From the algebraic geometry side of the story, U_σ is often defined as the spectrum of the coordinate ring $\mathbb{C}[S_\sigma]$, that is, the set of its prime ideals:

$$U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma]). \tag{5.6}$$

A subset of the set of prime ideals are the maximal ideals, $\operatorname{Specm}(\mathbb{C}[S_\sigma])$ which correspond to closed points of U_σ. Since S_σ is a semigroup, closed elements in (i.e.elements of $\operatorname{Specm}(\mathbb{C}[S_\sigma])$) correspond to semigroup homomorphisms from S_σ to $\mathbb{C}$, where $\mathbb{C}$ is regarded a group under multiplication. See [Ful93, Sec. 1.3] for details on this construction.

From these two ways to describe the affine variety U_σ there are also two ways to describe closed points of U_σ: as points in $\mathbb{C}^k$ or as semigroup homomorphisms $\gamma : S_\sigma \to \mathbb{C}$. We will give this bijective correspondence explicitly in Section 5.1.4 on page 164. For more details see the remark after Proposition 1.2 in [Oda88] or Theorem 1.2.18 and Proposition 1.1.17 in [CLS11], where $Y_{\mathcal{A}}$ denotes the Zariski closure of the image of the map (5.4) in a slightly more general setting (see Definition 1.1.7 in [CLS11]).

The coordinate ring of U_σ is

$$\mathbb{C}[S_\sigma] = \operatorname{span}\{\mathbf{t}^{\mathbf{m}} \mid \mathbf{m} \in S_\sigma\} \subseteq \mathbb{C}[t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots, t_n^{-1}]. \qquad \circ$$

Step 2: Glue the U_σ together to X_Σ

For any two cones σ_1, σ_2 in Σ, if σ_1 is a face of σ_2, then U_{σ_1} is a Zariski open subvariety of U_{σ_2}. Since for any two cones $\tau, \sigma \in \Sigma$ the intersection $\tau \cap \sigma$ is a common face of both τ and σ, $U_{\tau\cap\sigma}$ can be identified with a subvariety of both U_τ and U_σ. The isomorphism $g_{\sigma\tau} : U_\tau \supset U_{\sigma\cap\tau} \cong U_{\tau\cap\sigma} \subseteq U_\sigma$ satisfies the compatibility conditions given on page 156.

So we glue the affine toric varieties U_σ together along these common subvarieties (i.e. with respect to the relation $\sim$ coming from the inclusion relation of faces) to obtain the abstract variety X_Σ:

$$X_\Sigma := \bigcup_{\sigma\in\Sigma} U_\sigma / \sim .$$

This abstract variety actually is toric. To see this, we need an embedded torus and a continuous extension of the torus action on itself to X_Σ. For the embedded torus, note that every fan contains the origin $\{0\}$ as a face and (see Example 5.1.15(1) below) it is $U_{\{0\}} = T$. Therefore X_Σ canonically contains T as a Zariski open subset. To get the torus action, we see elements of U_σ as maps $\gamma : S_\sigma \to \mathbb{C}$. For some $t \in T$ we then get $t.\gamma \in U_\sigma$ given by:

$$\begin{aligned} t.\gamma : S_\sigma &\longrightarrow \mathbb{C} \\ \mathbf{m} &\longmapsto \chi^{\mathbf{m}}(t)\gamma(\mathbf{m}). \end{aligned}$$

This gives us an action on the affine varieties U_σ and by gluing also on X_Σ. For the trivial cone $\{0\}$, the action coincides with the usual group multiplication on T

The toric variety X_Σ is compact if and only if the support of Σ is equal to $\mathbb{R}^n$, that is, if Σ gives a rational polyhedral decomposition of $\mathbb{R}^n$ ([Cox03, Thm. 9.1]).

Definition 5.1.14 We denote the toric variety defined by the fan Σ_P by X_{Σ_P}. ◦

Since the support of Σ_P is equal to $\mathbb{R}^n$, X_{Σ_P} is compact.

Example 5.1.15

1) The easiest example to look at is the trivial cone $\{0\} \subseteq \mathbb{R}^n$. Its dual cone is $\mathbb{R}^n$ itself, so $S_{\{0\}} = \mathbb{Z}^n$, generated by the standard basis $e_1^*, -e_1^*, e_2^*, \ldots, -e_n^*$. As $\mathbf{t}^{e_j} = t_j$ and $\mathbf{t}^{-e_j} = t_j^{-1}$ the coordinate ring of $U_{\{0\}}$ is given by $\mathbb{C}[t_1, t_1^{-1}, t_2, \ldots, t_n^{-1}]$, which is the coordinate ring of $(\mathbb{C}^\times)^n$. Therefore
$$U_{\{0\}} = T = (\mathbb{C}^\times)^n.$$

2) In $\mathbb{R}$ we consider the fan Σ given in Figure 5.4, which consists of the two cones σ_1, σ_2 and the trivial cone τ at the origin, which is a face of both former ones. Then $\sigma_1^\vee = \sigma_1$ and S_{σ_1}

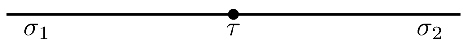


Figure 5.4: A simple fan which corresponds to the toric variety $\mathbb{P}^1$.

is generated by e_1^*1. So the map $\varphi_1 : \mathbb{C}^\times \longrightarrow \mathbb{C}$ is the identity with coordinate ring $\mathbb{C}[t]$, which gives us $U_{\sigma_1} = \mathbb{C}$. Similarly we conclude $U_{\sigma_2} = \mathbb{C}$ with coordinate ring $\mathbb{C}[t^{-1}]$. The gluing then identifies t with t^{-1} and we get
$$X_\Sigma = \mathbb{P}^1.$$

3) Now we go on to $\mathbb{R}^2$ with the fan shown in Figure 5.5. Here we have nine cones: four

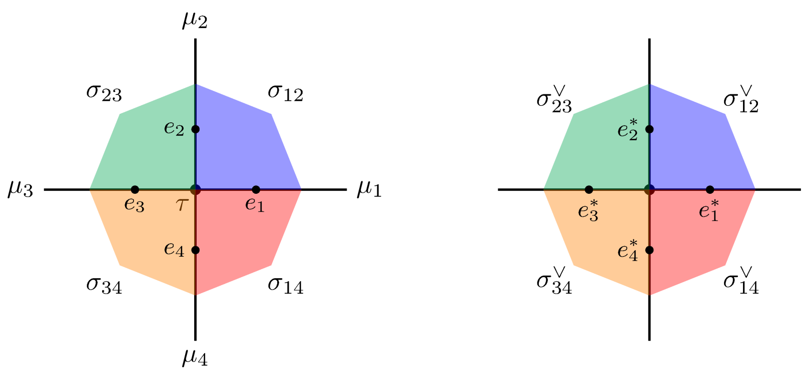


Figure 5.5: LEFT: This fan Σ consists of 9 cones and gives us the toric variety $\mathbb{P}^1 \times \mathbb{P}^1$; RIGHT: The dual cones of the two-dimensional cones σ_{ij}.

two-dimensional ones (σ_{ij}), four one-dimensional ones (μ_j) and the origin (τ). The four two-dimensional cones σ_{ij} satisfy $\sigma_{ij}^\vee = \sigma_{ij}$, and are generated by e_i^* and e_j^*. Because $e_3^* = -e_1^*$ and $e_4^* = -e_2^*$, we get for the affine varieties associated to the two-dimensional cones:

$$\begin{aligned}
U_{\sigma_{12}} &= \mathrm{Spec}(\mathbb{C}[S_{\sigma_{12}}]) \simeq \mathbb{C}[x, y] \\
U_{\sigma_{23}} &= \mathrm{Spec}(\mathbb{C}[S_{\sigma_{23}}]) \simeq \mathbb{C}[x^{-1}, y] \\
U_{\sigma_{34}} &= \mathrm{Spec}(\mathbb{C}[S_{\sigma_{34}}]) \simeq \mathbb{C}[x^{-1}, y^{-1}] \\
U_{\sigma_{14}} &= \mathrm{Spec}(\mathbb{C}[S_{\sigma_{14}}]) \simeq \mathbb{C}[x, y^{-1}].
\end{aligned}$$

So we have $U_{\sigma_{ij}} \simeq \mathbb{C}^2$ for all four of them but with different coordinate rings. Gluing $U_{\sigma_{12}}$ and $U_{\sigma_{23}}$ yields $\mathbb{P}^1 \times \mathbb{C}$ with coordinates $([t_1 : t_2], y)$. Similarly we have $U_{\sigma_{34}}$ and $U_{\sigma_{14}}$ glued together to $\mathbb{P}^1 \times \mathbb{C}$ with coordinates $([t_1 : t_2], y^{-1})$. For X_Σ we have to glue these two pieces together identifying the second components. Therefore we obtain

$$X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$$

with coordinates $([t_1 : t_2], [s_1 : s_2])$. Another way to describe this toric variety is as a quotient by a group action, see [CLS11, §5] for the general theory and the Examples 5.1.3, 5.1.8 and 12.2.2 in [CLS11] for this specific example. It is not hard to see that

$$\mathbb{P}^1 \times \mathbb{P}^1 = {\mathbb{C}^4 \setminus \left((\mathbb{C} \times \{0\}) \cup (\{0\} \times \mathbb{C}^2)\right)}\Big/{(\mathbb{C}^\times)^2},$$

with the torus action on $\mathbb{P}^1 \times \mathbb{P}^1$ given by $(t_1, t_2).(a, b, c, d) = (t_1 a, t_1 b, t_2 c, t_2 d)$ for all $(t_1, t_2) \in T = (\mathbb{C}^\times)^2$ and $(a, b, c, d) \in \mathbb{C}^4$.

4) We look again at the cone given in the first example of Example 5.1.9. Its dual cone is generated over $\mathbb{Z}_{\geq 0}$ by the three elements

$$\mathbf{m}_1 = (1, 0), \quad \mathbf{m}_2 = (1, 1), \quad \mathbf{m}_3 = (1, 2).$$

The map φ then maps some $\mathbf{t} = (t_1, t_2) \in (\mathbb{C}^\times)^2$ to the point $(t_1, t_1 t_2, t_1 t_2^2) \in \mathbb{C}^3$. For the coordinate ring we obtain

$$\mathbb{C}[S_\sigma] = \mathbb{C}[t_1, t_1 t_2, t_1 t_2^2] = {\mathbb{C}[X, Y, Z]}\Big/{\langle Y^2 - XZ \rangle}$$

and thereby the affine variety is

$$U_\sigma = \{(x, y, z) \in \mathbb{C}^3 \mid y^2 - xz = 0\} \subseteq \mathbb{C}^3,$$

which is a cone over a conic. ◦

Coming from the above construction of X_Σ, there is a strong correspondence between fans $\Sigma \subseteq \mathbb{R}^n$ and toric varieties which are normal[2], namely:

1) For every fan Σ of $\mathbb{R}^n$, there is an associated toric variety X_Σ, which is a normal algebraic variety.

2) If a toric variety X is a normal variety, then X is of the form X_Σ for some fan Σ in $\mathbb{R}^n$.

Because of this correspondence, toric varieties are often required to be normal, for example in [Ful93]. In this thesis, we follow this convention and require all toric varieties to be normal. The following lemma shows that this is not a strong restriction:

Lemma 5.1.16 ([CLS11, Thm. 1.3.5]) *$U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$ is normal if and only if $\sigma \subseteq \mathbb{R}^n$ is a strongly convex rational polyhedral cone.* ◦

Remark 5.1.17 In [Cox03, Ex. 7.1 and Thm. 7.2] Cox gives the following characterization of normality for affine toric varieties: only if we take all generators $\mathbf{m}_i$ such that S_σ is generated by them over the nonnegative integers $\mathbb{Z}_{\geq 0}$, then the affine variety U_σ defined as the Zariski closure of the map φ (Equation (5.5)) is the normal affine toric variety defined by σ and the lattice $\mathbb{Z}^n$ as given in Equation (5.6). If we use less generators $\mathbf{m}_i$ then we change the lattice (i.e. get another toric variety) or loose normality. ◦

[2] Normality is a quite algebraic condition for general varieties, Remark 5.1.17 shows what it means in our context of toric varieties.

In Example 5.1.12 we saw that not every fan comes from a rational convex polytope. But those toric varieties X_Σ defined by fans $\Sigma = \Sigma_P$ which do come from rational convex polytopes P as above have a simple characterization:

Proposition 5.1.18 *A toric variety X_Σ is a projective variety if and only if the fan Σ is equal to the fan Σ_P induced from a rational convex polytope P containing the origin as an interior point as above.* ◦

Proof. This statement and its proof are given as Theorem 12.3 in [Cox03]. There the result is given it terms of normal fans of rational polytopes. As we constructed the toric variety Σ_P using the fan over the faces of the rational polytope P (and not over the faces of the dual polytope P° as for the normal fan), we can adapt the proof given in [Cox03] by setting $P = \Delta^\circ$, using that $(P^\circ)^\circ = P$. □

Example 5.1.19 Take a square in $\mathbb{R}^2$ with integral vertices as the polytope P. Then its dual polytope P° is a square with vertices on the axes of the coordinate system. The fan Σ_{P° over the dual polytope then is exactly the fan of $\mathbb{P}^1 \times \mathbb{P}^1$ as given in Example 5.1.15. In our notation and

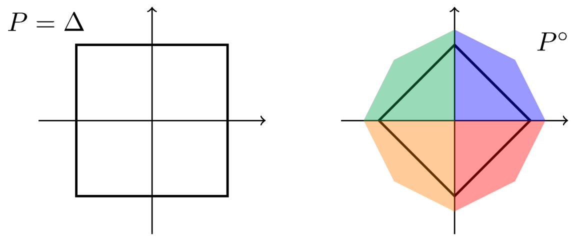


Figure 5.6: The square P has a diamond as dual polytope P°. The fan over P° is the normal fan of Δ.

the notation used in [Cox03, CLS11, Ful93] we have:

$$X_{\Sigma_{P^\circ}} = X^{Cox}_{\Sigma_\Delta} = \mathbb{P}^1 \times \mathbb{P}^1. \qquad \circ$$

5.1.4 The Orbit-Cone-Correspondence

Many properties of X_Σ can be expressed in terms of the combinatorial properties of the fan Σ. The Orbit-Cone-Correspondence (see Proposition 5.1.23 below) gives a bijective correspondence between orbits under the action of the torus $(\mathbb{C}^\times)^n$ in X_Σ and cones σ in the fan Σ. Before we state the statement explicitly, we have to explain in detail the two ways how elements of U_σ can be see and then introduce the distinguished point x_σ, which exists for any cone $\sigma \in \Sigma$ and lives in U_σ.

Coming from the construction and definition of the affine toric variety U_σ as the Zariski closure of the map φ in $\mathbb{C}^k$ or as $\mathrm{Specm}(\mathbb{C}[S_\sigma])$, there is a bijective correspondence (see [CLS11, Prop. 1.3.1]) between closed points $p \in U_\sigma$ and semigroup homomorphisms $\gamma_p : S_\sigma \longrightarrow \mathbb{C}$, where $\mathbb{C}$ is seen as a semigroup under multiplication. Explicitly, the correspondence is given as follows:

Let $p \in U_\sigma \subseteq \mathbb{C}^k$ be a closed point. Then we associate to it the map

$$\begin{aligned} \gamma_p : S_\sigma &\longrightarrow \mathbb{C} \\ \mathbf{m} &\longmapsto \chi^{\mathbf{m}}(p), \end{aligned}$$

which is reasonable, as $\mathbb{C}[S_\sigma] = \mathbb{C}[U_\sigma]$ is the algebra of regular functions on U_σ and $\chi^{\mathbf{m}} \in \mathbb{C}[S_\sigma]$. Recall that $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n \subseteq \mathbb{Z}^n$, so elements of S_σ are characters of $T = (\mathbb{C}^\times)^n$. They are exactly those characters, that can be extended regularly to the (partial) compactification of T by U_σ. More details can be found in [CLS11, Thm. 1.1.17] and its proof.

For the other way, let $\mathbf{m}_1, \dots, \mathbf{m}_k \in S_\sigma$ be a set of generators. Then the point associated to a map $\gamma : S_\sigma \longrightarrow \mathbb{C}$ is given by

$$(\gamma(\mathbf{m}_1), \dots, \gamma(\mathbf{m}_k)) \in U_\sigma \subseteq \mathbb{C}^k.$$

We now define the distinguished point x_σ, more details about it can be found in [CLS11, p.116], where the distinguished point is denoted by γ_σ:

Definition 5.1.20 For each cone σ, the *distinguished point* x_σ is defined to be the semigroup homomorphism

$$\begin{aligned} x_\sigma : S_\sigma = \sigma^\vee \cap \mathbb{Z}^n &\longrightarrow \{0, 1\} \\ \mathbf{m} &\longmapsto \begin{cases} 1 & \text{if } \mathbf{m} \in S_\sigma \cap \sigma^\perp = \sigma^\perp \cap \mathbb{Z}^n \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \qquad \circ$$

As $\sigma^\perp$ is a face of $\sigma^\vee$, for $\mathbf{m}, \mathbf{m}' \in S_\sigma$ we have $\mathbf{m} + \mathbf{m}' \in S_\sigma \cap \sigma^\perp$ if and only if both $\mathbf{m}$ and $\mathbf{m}'$ are already elements of $S_\sigma \cap \sigma^\perp$. This shows that the map x_σ as defined above is a semigroup homomorphism.

The smallest cone $\{0\}$ of the fan Σ has $\{0\}^\vee = \{0\}^\perp = \mathbb{R}^n$ as dual cone and the distinguished point is $(1, \cdots, 1)$ in this case. The T-orbit through this point gives the embedding of T into U_σ. Recall that any one-parameter subgroup $\lambda : \mathbb{C}^\times \to (\mathbb{C}^\times)^n$ was of the form

$$\lambda_{\mathbf{u}}(z) = (z^{b_1}, \cdots, z^{b_n}),$$

where $\mathbf{u} = (b_1, \cdots, b_n) \in \mathbb{Z}^n$. Let $\mathbf{u}$ be an integral vector contained in the relative interior of the cone σ. By [CLS11, Proposition 3.2.2] (see also [Cox03, p. 212] or [Ful93, p. 37]), the distinguished point is given by

$$x_\sigma = \lim_{z \to 0} \lambda_{\mathbf{u}}(z) \in U_\sigma.$$

This limit exists and the two definitions of x_σ are equivalent by the following argument: For $\mathbf{u} \in \sigma \cap \mathbb{Z}^n$, the limit point $\lim_{z\to 0} \lambda_{\mathbf{u}}(z)$ corresponds to the semigroup homomorphism given by

$$\begin{aligned} \alpha_{\mathbf{u}} : S_\sigma &\longrightarrow \mathbb{C} \\ \mathbf{m} &\longmapsto \lim_{t \to 0} t^{\langle \mathbf{m}, \mathbf{u} \rangle}. \end{aligned} \tag{5.7}$$

As $\mathbf{u}$ was chosen to be in the relative interior of σ, we have $\langle \mathbf{m}, \mathbf{u} \rangle \geq 0$ for all $\mathbf{m} \in S_\sigma$ and the limit exists. The pairing $\langle \mathbf{m}, \mathbf{u} \rangle$ vanishes for exactly those $\mathbf{m} \in S_\sigma$ that are elements of $\sigma^\perp$, otherwise the pairing is positive. As the parameter t has the pairing in the exponent, the homomorphism from Equation (5.7) corresponds to the map given in Definition 5.1.20.

Before we state the Orbit-Cone-Correspondence, we have a look at some examples about the correspondence stated above.

Example 5.1.21

1) We first consider the fan given in Example 5.1.15 (3). It has nine cones and it gives us $X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$ as toric variety. The action of $T = (\mathbb{C}^\times)^2$ is given by

$$(s, t) . ([1 : x], [1 : y]) = ([1 : sx], [1 : ty])$$

for $(s,t) \in T$ and $([1:x],[1:y]) \in \mathbb{P}^1 \times \mathbb{P}^1$. In other words, T is contained in X_Σ via $(t_1,t_1) \longmapsto ([1:t_1],[1:t_2])$. For $\mathbf{u} = (a,b) \in \operatorname{relint}(\sigma_1)$ in the first quadrant, we have $a,b > 0$ and therefore

$$\lambda_{\mathbf{u}}(t) = ([1:t^a],[1:t^b]) \xrightarrow{t\to 0} ([1:0],[1:0]).$$

For the face $\mu_1 < \sigma_1$, a is still positive while $b = 0$. Then $t^b = 1$ for any $t \in \mathbb{C}$ and we get

$$\lambda_{\mathbf{u}}(t) = ([1:t^a],[1:t^b]) \xrightarrow{t\to 0} ([1:0],[1:1]).$$

A similar calculation using homogeneous coordinates can be done for any of the nine cones. The results are listed in Table 5.7.

cone	conditions for $\mathbf{u} = (a,b)$	$\lim_{t\to 0} \lambda_{\mathbf{u}}(t)$
σ_{12}	$a,b > 0$	$([1:0],[1:0])$
σ_{23}	$a < 0, b > 0$	$([0:1],[1:0])$
σ_{34}	$a,b < 0$	$([0:1],[0:1])$
σ_{14}	$a > 0, b < 0$	$([1:0],[0:1])$
μ_1	$a > 0, b = 0$	$([1:0],[1:1])$
μ_2	$a = 0, b > 0$	$([1:1],[1:0])$
μ_3	$a < 0, b = 0$	$([0:1],[1:1])$
μ_4	$a = 0, b < 0$	$([1:1],[0:1])$
$\tau = \{0\}$	$a = b = 0$	$([1:1],[1:1])$

Figure 5.7: The distinguished points x_σ for all cones σ of the fan Σ of $\mathbb{P}^1 \times \mathbb{P}^1$ as given in Figure 5.5.

Note that there are three kinds of limits, depending on the dimension of the cone. Accordingly the dimension of a T-orbit of such a limit for a cone σ is $n - \dim(\sigma)$ over $\mathbb{C}$.

Now we want to calculate the distinguished point by using Definition 5.1.20. As the two-dimensional cones σ_{ij} are generated by e_i^*, e_j^* and are all self-dual, the distinguished point is $(0,0)$ for all four of them. Note that we get the same for all four cones because these are the coordinates in a local chart. By the action of the torus and the gluing of the $U_{\sigma_{ij}}$ to X_Σ, we get the same results as in Table 5.7 above. Now consider the one-dimensional cone μ_1 generated by e_1. It is a face of σ_{12} and σ_{14}. Its orthogonal cone is the y-axis, whose positive part is a face of $\sigma_{12}^\vee$ and its negative part is a face of $\sigma_{14}^\vee$. Therefore the second component now is mapped to 1 instead of 0 by the distinguished point x_{μ_1} which then gives us $x_{\mu_1} = (0,1)$ in local coordinates or $([1:0],[1:1])$ as above. The other one-dimensional cones go the same. For the trivial cone $\{0\}$ the distinguished point maps all generators to 1, as its dual cone is $\mathbb{R}^2$. This then gives us $([1:1],[1:1])$.

2) We come back to Example 5.1.15(4), which was the affine variety associated to the cone given in Example 5.1.9(1). In Figure 5.8 we show again the picture we had before: S_σ was generated by the three elements

$$\mathbf{m}_1 = (1,0), \quad \mathbf{m}_2 = (1,1), \quad \mathbf{m}_3 = (1,2)$$

and we already know that

$$\mathbb{C}[S_\sigma] = \mathbb{C}[t_1, t_1t_2, t_1t_2^2] = \mathbb{C}[X,Y,Z]\big/\langle Y^2 - XZ\rangle.$$

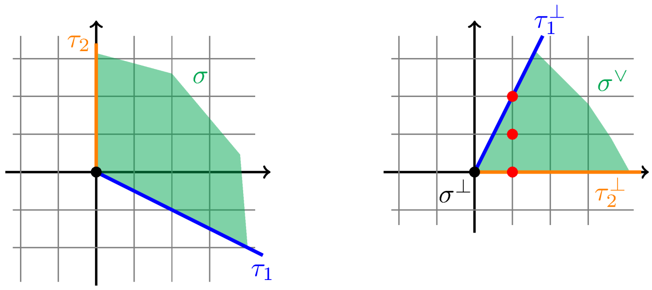


Figure 5.8: The cone σ in $\mathbb{R}^2$ (LEFT) and its dual with the points generating S_σ (RIGHT).

We first determine the distinguished point directly as a semigroup homomorphism as given in Definition 5.1.20. The cone σ is two-dimensional and $\sigma^\perp = \{0\}$, so we immediately know that x_σ sends every generator of S_σ to 0:

$$x_\sigma = (0, 0, 0).$$

For the face τ_1 of σ, the orthogonal cone is generated by $\mathbf{m}_3$, it is the blue line in the right picture above. The other face τ_2 has the positive x-axis as its orthogonal cone (orange in the picture), which is generated by $\mathbf{m}_1$. Together we get

$$\mathbf{x}_{\tau_1} = (0, 0, 1), \quad \mathbf{x}_{\tau_2} = (1, 0, 0).$$

The origin, which is also a face of σ, has the whole space $\mathbb{R}^2$ as orthogonal complement, so

$$x_{\{0\}} = (1, 1, 1).$$

The torus $T = (\mathbb{C}^\times)^2$ is contained in X_Σ via

$$(t_1, t_2) \longmapsto (t_1, t_1 t_2, t_1 t_2^2).$$

An point $\mathbf{u} = (a, b) \in \mathbb{R}^2$ is contained in the relative interior of σ, if and only if $a > 0$ and $b > -\frac{a}{2}$. Then in X_Σ we have

$$\lambda_{\mathbf{u}}(t) = (t^a, t^{a+b}, t^{a+2b}) \xrightarrow{t \to 0} (0, 0, 0),$$

as all exponents are positive. For the face τ_1 we have $\mathbf{u} = (a, b) \in \operatorname{relint}(\tau_1)$ if and only if $a > 0$ and $b = -\frac{a}{2}$. Using this we calculate

$$\lambda_{\mathbf{u}}(t) = (t^a, t^{a/2}, t^{a-a}) \xrightarrow{t \to 0} (0, 0, 1),$$

For the last cone τ_2, $a = 0, b > 0$ are the conditions for $\mathbf{u} = (a, b)$ to lie in the relative interior of τ_2. Then

$$\lambda_{\mathbf{u}}(t) = (1, t^b, t^2 b) \xrightarrow{t \to 0} (1, 0, 0).$$

When $a = b = 0$ then the limit is $(1, 1, 1)$. As expected, the limits of the one-parameter subgroups coincide with the distinguished points determined before. ◦

In both examples above, the distinguished points of the two-dimensional cones σ are fixed points of the torus action. This is indeed a general result: x_σ is a fixed point under the action of T, if and only if $\dim(\sigma) = n$. Now we also look at the T-orbits of the other distinguished points.

Definition 5.1.22 For $\sigma \in \Sigma$ we denote the orbit of the associated distinguished point x_σ under the action of the torus by $\operatorname{orb}(\sigma)$:

$$\operatorname{orb}(\sigma) := T.x_\sigma \subseteq X_\Sigma. \qquad \circ$$

The previously announced bijective correspondence between T-orbits and cones now states as follows:

Proposition 5.1.23 (Orbit-Cone-Correspondence) *For every toric variety X_Σ of the fan Σ, there is a bijective correspondence between cones $\sigma \in \Sigma$ and $T = (\mathbb{C}^\times)^n$-orbits in X_Σ given by the orbit of the distinguished point $x_\sigma \in U_\sigma$. The orbit $\operatorname{orb}(\sigma) = T.x_\sigma$ is a complex torus isomorphic to $(\mathbb{C}^\times)^{n-\dim\sigma}$. In particular, the open and dense orbit $(\mathbb{C}^\times)^n$ corresponds to the trivial cone $\{0\}$.* $\circ$

More about this correspondence can be found for example in [Ful93, §3.1], [CLS11, Thm. 3.2.6] and [Cox03, §9]

5.2 A Topological Model of Toric Varieties

In order to better understand the toric variety X_Σ as a compactification of $(\mathbb{C}^\times)^n$, we now want to give a topological description of X_Σ which clearly shows its dependence on the fan Σ and also describes explicitly sequences in $(\mathbb{C}^\times)^n$ which converge to points in the complement $X_\Sigma \setminus (\mathbb{C}^\times)^n$. The topological model $\overline{T}_\Sigma$ will be constructed as the complex torus T to which we attach some boundary components $O(\sigma)$. Later in this section we will then show that the topological model $\overline{T}_\Sigma$ and the usual construction of X_Σ as the variety obtained from a fan are homeomorphic as T-topological spaces. The key point of this homeomorphism will be an identification of the boundary components $O(\sigma)$ of $\overline{T}_\Sigma$ with the T-orbits $\operatorname{orb}(\sigma)$ in X_Σ. The identification is based on the fact that both are associated to the same cone $\sigma \in \Sigma$ in the fan and each of them contains a distinguished point $x_\sigma \in U_\sigma$ and $0_\sigma \in O(\sigma)$, respectively, that correspond to each other.

5.2.1 Definition of the Topological Model $\overline{T}_\Sigma$

Note that in terms of the standard integral structure $i\mathbb{Z} \subseteq \mathbb{C}$, we have an identification $(i\mathbb{Z})\backslash\mathbb{C} \cong \mathbb{C}^\times$ via the map $z \mapsto e^{-2\pi z}$. When $\operatorname{Re}(z) \longrightarrow +\infty$, it holds $e^{-2\pi z} \to 0$. Figure 5.9 shows a picture how the identification works.

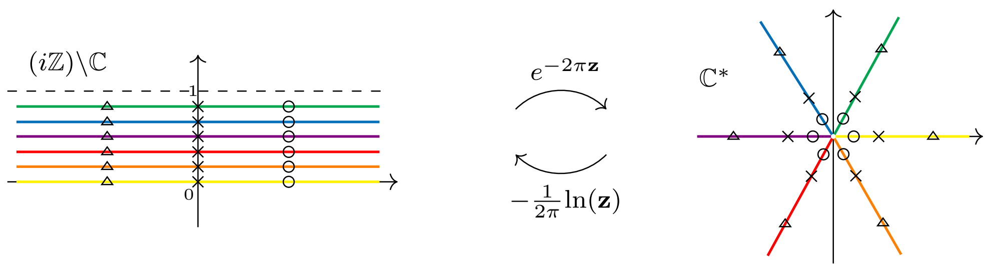


Figure 5.9: The identification $(i\mathbb{Z})\backslash\mathbb{C} \cong \mathbb{C}^\times$.

Then the n-dimensional exponential map gives an identification

$$\begin{aligned} (i\mathbb{Z}^n)\backslash\mathbb{C}^n &\cong (\mathbb{C}^\times)^n \\ \mathbf{z} &\longmapsto e^{-2\pi\mathbf{z}} = (e^{-2\pi z_1}, \cdots, e^{-2\pi z_n}). \end{aligned} \tag{5.8}$$

Conversely, using the complex logarithm $-\frac{1}{2\pi}\ln$, we get an identification

$$\begin{aligned}(\mathbb{C}^\times)^n &\cong (i\mathbb{Z}^n)\backslash\mathbb{C}^n \\ \mathbf{a} &\longmapsto -\frac{1}{2\pi}\ln(\mathbf{a}),\end{aligned} \tag{5.9}$$

where for $\mathbf{a} = (r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})$ the image is given by

$$-\frac{1}{2\pi}\ln(\mathbf{a}) = \frac{1}{2\pi}(\ln(r_1) + i\theta_1, \dots, \ln(r_n) + i\theta_n).$$

Given a fan $\Sigma \subseteq \mathbb{R}^n$, we will now define a bordification $\overline{T}_\Sigma$ of the torus $T = (\mathbb{C}^\times)^n$ and show in Proposition 5.2.11 that $\overline{T}_\Sigma$ is homeomorphic to the toric variety X_Σ as T-topological spaces.

Definition 5.2.1 For each cone $\sigma \in \Sigma$, define a *boundary component*

$$O(\sigma) := (i\mathbb{Z}^n)\backslash\mathbb{C}^n / \operatorname{span}_{\mathbb{C}}(\sigma). \qquad \circ$$

By π_σ we denote the projection map

$$\pi_\sigma : (i\mathbb{Z}^n)\backslash\mathbb{C}^n \longrightarrow (i\mathbb{Z}^n)\backslash\mathbb{C}^n / \operatorname{span}_{\mathbb{C}}(\sigma) = O(\sigma).$$

Note that $O(\sigma)$ is a complex torus $(\mathbb{C}^\times)^{n-\dim\sigma}$ of dimension $n - \dim\sigma$. When $\sigma = \{0\}$, then $O(\sigma) = T$. Later we will identify $O(\sigma)$ with the T-orbits $\operatorname{orb}(\sigma)$ as defined in Definition 5.1.22.

Definition 5.2.2 Define a *topological bordification* $\overline{T}_\Sigma$ by

$$\overline{T}_\Sigma := T \cup \coprod_{\substack{\sigma\in\Sigma,\\ \sigma\neq\{0\}}} O(\sigma) \tag{5.10}$$

with the following topology: A sequence

$$\mathbf{z}_m = \mathbf{x}_m + i\mathbf{y}_m \in T = (i\mathbb{Z}^n)\backslash\mathbb{C}^n,$$

where $\mathbf{x}_m \in \mathbb{R}^n$ and $\mathbf{y}_m \in \mathbb{Z}^n\backslash\mathbb{R}^n$, converges to a point $\mathbf{z}_\infty \in O(\sigma)$ for some $\sigma \in \Sigma$ if and only if the following conditions hold:

1) The real part $\mathbf{x}_m$ can be split up as $\mathbf{x}_m = \mathbf{x}'_m + \mathbf{x}''_m$ such that for $m \longrightarrow +\infty$ it holds:

   a) $\mathbf{x}'_m$ is contained in the relative interior of the cone σ and its distance to the relative boundary of σ goes to infinity,

   b) $\mathbf{x}''_m$ is bounded.

2) the image of $\mathbf{z}_m$ in $O(\sigma) = i\mathbb{Z}^n\backslash\mathbb{C}^n / \operatorname{span}_{\mathbb{C}}(\sigma)$ under the projection $\pi_\sigma : (i\mathbb{Z}^n)\backslash\mathbb{C}^n \to O(\sigma)$ converges to the point $\mathbf{z}_\infty$:

$$\pi_\sigma(\mathbf{z}_m) \longrightarrow \mathbf{z}_\infty. \qquad \circ$$

Note that the imaginary part $\mathbf{y}_m$ of $\mathbf{z}_m$ lies in the compact torus $\mathbb{Z}^n\backslash\mathbb{R}^n = (\mathbb{S}^1)^n$, and the second condition controls both the imaginary part $\mathbf{y}_m$ and the bounded component $\mathbf{x}''_m$ of the real part $\mathbf{x}_m$. It is clear from the definition that the bordification $\overline{T}_\Sigma$ is a compactification of T if and only if the support of Σ is equal to $\mathbb{R}^n$.

Remark 5.2.3 The above definition of $\overline{T}_\Sigma$ and the identification of X_Σ with $\overline{T}_\Sigma$ in Proposition 5.2.11 follows the construction and discussion in [AMRT10, pp. 1-6]. We note that there is one difference with the convention there: On page 2 in [AMRT10], the complex torus $(\mathbb{C}^\times)^n$ is identified with $\mathbb{Z}^n\backslash\mathbb{C}^n$, the real part is the compact torus $(\mathbb{S}^1)^n$ and the imaginary part is $i\mathbb{R}^n$, which can be identified with $\mathbb{R}^n$. $\circ$

Example 5.2.4 Once again we consider the cone σ as in Examples 5.1.9(1) and 5.1.15(4). In Figure 5.10 the cone is shown again.

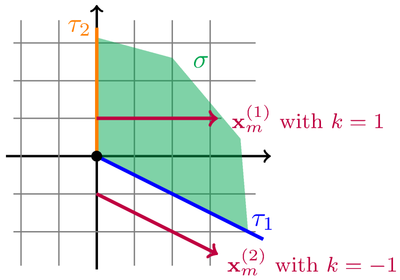


Figure 5.10: The cone σ in $\mathbb{R}^2$ and its dual. The real part $\mathbf{x}_m^{(1)}$ of the first sequence $\mathbf{z}_m = \mathbf{z}_m^{(1)}$ remains in the cone and the distance to the boundary of σ goes to infinity. The real part $\mathbf{x}_m^{(2)}$ of the second sequence $\mathbf{z}_m = \mathbf{z}_m^{(2)}$ can be split up such that one part lies in τ_1 and the other part is bounded.

The two cones τ_1 and τ_2, spanned by $(2, -1)$ and $(0, 1)$ respectively, are the one-dimensional faces of σ. Then $\operatorname{span}_{\mathbb{C}}(\sigma) = \mathbb{C}^2$ whereas $\operatorname{span}_{\mathbb{C}}(\tau_j) \cong \mathbb{C}$. We now want to examine the above given topology by looking at sequences and their converging behavior. The crucial part will be to find the appropriate face of σ such that the sequence converges to a point in the boundary component of this face.

1) For some $k \in \mathbb{R}$ consider the sequence

$$\mathbf{z}_m = \begin{pmatrix} m \\ k \end{pmatrix} + i \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \subseteq \mathbb{C}^2,$$

with $y_1, y_2 \in \mathbb{Z}\backslash\mathbb{R}$. We start with σ as a face of itself. As k is constant, we can write the real part $\mathbf{x}_m$ of $\mathbf{z}_m$ as

$$\mathbf{x}_m = \underbrace{\begin{pmatrix} m \\ 0 \end{pmatrix}}_{\mathbf{x}'_m} + \underbrace{\begin{pmatrix} 0 \\ k \end{pmatrix}}_{\mathbf{x}''_m}$$

such that the conditions are all satisfied. As $\operatorname{span}_{\mathbb{C}}(\sigma) = \mathbb{C}^2$, the projection is trivial, so independent of k and all sequences of this type converge to the same boundary point. If we had taken one of the faces τ_j instead, we could not have found a splitting of $\mathbf{x}_m$ with $\mathbf{x}'_m \in \operatorname{relint}(\tau_j)$ and $\mathbf{x}''_m$ bounded, as the distance of $\mathbf{x}_m$ to τ_j grows linearly in m. The same holds for the trivial face $\{0\}$. Therefore $\mathbf{z}_m \to 0 \in O(\sigma)$.

2) Now we look at the sequence

$$\mathbf{z}_m = \begin{pmatrix} 2m \\ -m + k \end{pmatrix} + i \begin{pmatrix} \frac{1}{m} \\ y_2 \end{pmatrix} \in \mathbb{C}^2,$$

where again $k \in \mathbb{R}$ and $y_2 \in \mathbb{Z}\backslash\mathbb{R}$. Then as

$$\mathbf{x}_m = \underbrace{\begin{pmatrix} 2m \\ -m \end{pmatrix}}_{\in \tau_1} + \begin{pmatrix} 0 \\ k \end{pmatrix}, \tag{5.11}$$

the distance of $\mathbf{x}_m$ to the relative boundary of σ is bounded. So we can not fulfill condition 1a) of Definition 5.2.2 with respect to the cone σ. Indeed, we could construct $\mathbf{x}'_m \in \operatorname{relint}(\sigma)$ with unbounded distance to $\partial_{\mathrm{rel}}(\sigma)$, but then the second part $\mathbf{x}''_m$ can not be bounded.

So we try τ_1 next. We already have a suitable splitting given in Equation (5.11). Now we have to determine the image of $\mathbf{z}_m$ under the projection π_{τ_1}, where $\operatorname{span}_{\mathbb{C}}(\tau_1) = \mathbb{C}(2, -1)$. A system of linear equations yields

$$\mathbf{z}_m = \left[m - \frac{k}{5} + i\left(-\frac{1}{5}y_2 + \frac{2}{5m}\right)\right]\begin{pmatrix}2\\-1\end{pmatrix} + \left[\frac{2k}{5} + i\left(\frac{2}{5}y_2 + \frac{1}{5m}\right)\right]\begin{pmatrix}1\\2\end{pmatrix},$$

and we conclude

$$\mathbf{z}_m \longrightarrow \mathbf{z}_\infty = \left[\frac{2k}{5} + i\frac{2}{5}y_2\right]\begin{pmatrix}1\\2\end{pmatrix} \in O(\tau_1). \qquad \circ$$

So far we have $\overline{T}_\Sigma$ constructed as T to which we attached some boundary components $O(\sigma)$ and with a topology how sequences on T converge to these boundary parts. What is still missing is a continuous action of T on $\overline{T}_\Sigma$ compatible with the topology. Note that by the identification $(i\mathbb{Z}^n)\backslash\mathbb{C}^n \cong (\mathbb{C}^\times)^n$, $\mathbb{C}^n$ and $i\mathbb{Z}^n\backslash\mathbb{C}^n$ act on T and on every boundary component $O(\sigma)$ by translation. These translations are compatible in the following sense.

Lemma 5.2.5 *Let $\mathbf{z}_m \in T = i\mathbb{Z}^n\backslash\mathbb{C}^n$ be as sequence convergent in $O(\sigma) \subseteq \overline{T}_\Sigma$. Then for any vector $\mathbf{z} \in \mathbb{C}^n$, or rather its image in $i\mathbb{Z}^n\backslash\mathbb{C}^n$, the shifted sequence $\mathbf{z}_m + \mathbf{z}$ is also convergent. Furthermore,*

$$\lim_{n\to+\infty} (\mathbf{z} + \mathbf{z}_m) = \pi_\sigma(\mathbf{z}) + \lim_{n\to+\infty} \mathbf{z}_m. \qquad \circ$$

Proof. The proof follows directly from Definition 5.2.2 using the linearity of the projection and the fact that $\mathbf{z}$ is constant. □

This implies the following result about the torus action on $\overline{T}_\Sigma$:

Proposition 5.2.6 *The action of $T = (\mathbb{C}^\times)^n$ on itself by multiplication extends to a continuous action on $\overline{T}_\Sigma$, and the decomposition of $\overline{T}_\Sigma$ into $O(\sigma)$ as given in Equation (5.10) gives the orbit decomposition of $\overline{T}_\Sigma$ with respect to the action of T.* ◦

Proof. We note that the multiplication of the torus T on itself and on the boundary components $O(\sigma)$ corresponds to translation in $\mathbb{C}^n$ and $i\mathbb{Z}^n\backslash\mathbb{C}^n$ by Equation (5.9). Then the statement follows from Lemma 5.2.5. □

5.2.2 The Identification of the Models: $X_\Sigma \cong \overline{T}_\Sigma$

We know want to identify the topological model $\overline{T}_\Sigma$ with the abstract variety X_Σ, which both contain the torus T. We will do this by identifying the boundary components $O(\sigma) \subseteq \overline{T}_\Sigma$ with the T-orbits $\operatorname{orb}(\sigma) \subseteq X_\Sigma$ and show compatibility with the convergence of sequences. This will happen in Proposition 5.2.11. Before we come to that we define a distinguished point 0_σ in the bordification $\overline{T}_\Sigma$ which will play the same role under convergence as the distinguished point $x_\sigma \in U_\sigma \subseteq X_\Sigma$ which we defined in Definition 5.1.20.

Definition 5.2.7 Consider the projection $\pi'_\sigma : \mathbb{C}^n \to O(\sigma) = (i\mathbb{Z}^n)\backslash\mathbb{C}^n/\operatorname{span}_{\mathbb{C}}(\sigma)$. Then the distinguished point 0_σ in $O(\sigma) \subseteq \overline{T}_\Sigma$ is defined to be the image of the origin of $\mathbb{C}^n$ under π'_σ:

$$0_\sigma := \pi'_\sigma(0) \in O(\sigma). \qquad \circ$$

Remark 5.2.8 Note that the projection π_σ after Definition 5.2.1 was only given for $(\mathbb{C}^\times)^n$. The projection π'_σ arises from π_σ by continuous extension. ◦

Lemma 5.2.9 *For any cone $\sigma \in \Sigma$, a sequence $\mathbf{z}_m$ in $(\mathbb{C}^\times)^n = T$ converges to the distinguished point x_σ in the toric variety X_Σ if and only if it converges to the distinguished point 0_σ in the topological model $\overline{T}_\Sigma$.* ◦

Proof. Recall that by Definition 5.1.20 under the embedding $\varphi : (\mathbb{C}^\times)^n \to \mathbb{C}^n$, the coordinates $\mathbf{t}^{\mathbf{m}_i}$ of the distinguished point x_σ are either 1 or 0, depending on whether the element $\mathbf{m}_i \in \sigma^\vee \cap \mathbb{Z}^n$ is zero or positive on σ. This implies that a sequence $\mathbf{z}_m \in (\mathbb{C}^\times)^n$ converges to the distinguished point x_σ if and only if the following conditions are satisfied:

1) $\mathbf{z}_m^{\mathbf{m}} \xrightarrow{m\to\infty} 0$ if $\mathbf{m} \in \sigma^\vee \cap \mathbb{Z}^n$ with $\mathbf{m}|_\sigma > 0$,

2) $\mathbf{z}_m^{\mathbf{m}} \xrightarrow{m\to\infty} 1$ if $\mathbf{m} \in \sigma^\vee \cap \mathbb{Z}^n$ with $\mathbf{m}|_\sigma = 0$.

Note that the vectors in $\sigma^\vee \cap \mathbb{Z}^n$ with $\mathbf{m}|_\sigma \geq 0$ span the dual cone $\sigma^\vee$, i.e., linear combinations of these vectors with nonnegative coefficients give $\sigma^\vee$. In terms of the identification $(\mathbb{C}^\times)^n = i\mathbb{Z}^n\backslash\mathbb{C}^n$, write $\mathbf{z}_m = \mathbf{x}_m + i\mathbf{y}_m$ with $\mathbf{x}_m, \mathbf{y}_m \in \mathbb{R}^n$ as in the definition of the topology of $\overline{T}_\Sigma$. Then $(\mathbf{z}_m)^{\mathbf{m}} = e^{-2\pi\langle \mathbf{z}_m|\mathbf{m}\rangle}$. As the dual cone $\sigma^\vee$ is the intersection of all positive half-spaces generated by the faces $\tau < \sigma$, the above conditions for $\mathbf{z}_m$ are equivalent to the following conditions:

1) The real part $\mathbf{x}_m$ can be written as $\mathbf{x}_m = \mathbf{x}'_m + \mathbf{x}''_m$ such that when $m \to +\infty$,
   a) the first part $\mathbf{x}'_m$ is contained in the interior of the cone σ and its distance to the relative boundary of σ goes to infinity,
   b) the second part $\mathbf{x}''_m$ is bounded.
2) The image of $\mathbf{z}_m$ in $O(\sigma) = i\mathbb{Z}^n\backslash\mathbb{C}^n / \operatorname{span}_{\mathbb{C}}(\sigma)$ under the projection
$$i\mathbb{Z}^n\backslash\mathbb{C}^n \to i\mathbb{Z}^n\backslash\mathbb{C}^n / \operatorname{span}_{\mathbb{C}}(\sigma)$$
   converges to the image in $O(\sigma)$ of the zero vector in $\mathbb{C}^n$.

By the definition of $\overline{T}_\Sigma$, this is exactly the conditions for the sequence $\mathbf{z}_m$ to converge to the distinguished point 0_σ in $\overline{T}_\Sigma$. □

Example 5.2.10 In Example 5.2.4 we had two sequences converging to different cones with respect to the topology of $\overline{T}_\Sigma$. Now we look at the same sequences again to illustrate the equivalence of converging conditions given in the proof before. The generators of S_σ were

$$\mathbf{m}_1 = (1,0), \quad \mathbf{m}_2 = (1,1), \quad \mathbf{m}_3 = (1,2).$$

We first look at
$$z_m = \begin{pmatrix} m \\ k \end{pmatrix} + i\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$
with $y_1, y_2 \in \mathbb{Z}\backslash\mathbb{R}$, from which we know that it converges with respect to σ to $\mathbf{z}_\infty = 0 = 0_\sigma$. We want to verify that also $z_m \to x_\sigma$. To do so, we have to show that $\mathbf{z}_m^{\mathbf{m}_j} \to 0$ for all $j \in \{1,2,3\}$. By the identification $(i\mathbb{Z}^n)\backslash\mathbb{C}^n \cong (\mathbb{C}^\times)^n$ in Equation 5.8 we get
$$z_m = \begin{pmatrix} e^{-2\pi m} e^{-2\pi i y_1} \\ e^{-2\pi k} e^{-2\pi i y_2} \end{pmatrix},$$
and therefore for the limits when $\mathbf{m} \to \infty$:
$$(z_m)^{\mathbf{m}_1} = e^{-2\pi m} e^{-2\pi i y_1} \longrightarrow 0$$

$$\begin{aligned}(z_m)^{\mathbf{m}_2} &= e^{-2\pi m}e^{-2\pi k}e^{-2\pi i(y_1+y_2)} \longrightarrow 0\\ (z_m)^{\mathbf{m}_3} &= e^{-2\pi m}e^{-4\pi k}e^{-2\pi i(y_1+2y_2)} \longrightarrow 0.\end{aligned}$$

So for this class of sequences, the convergence behavior in $\overline{T}_\Sigma$ and X_Σ coincides. For the second sequence

$$\mathbf{z}_m = \begin{pmatrix} 2m \\ -m+k \end{pmatrix} + i \begin{pmatrix} \frac{1}{m} \\ y_2 \end{pmatrix}$$

we already know that in the topological model it converges to $\mathbf{z}_\infty = \left[\frac{2}{5}k + i\frac{2}{5}y_2\right](1,2)$. So we set $k = y_2 = 0$ such that the sequence converges to the distinguished point $0(\tau_1)$. Now we have to show that it also converges to the distinguished point x_{τ_1} in X_Σ. The identification then yields

$$z_m = \begin{pmatrix} e^{-4\pi m}e^{-2\pi i\frac{1}{m}} \\ e^{2\pi m} \end{pmatrix}$$

and so

$$\begin{aligned}(z_m)^{\mathbf{m}_1} &= e^{-4\pi m}e^{-2\pi i\frac{1}{m}} \longrightarrow 0\\ (z_m)^{\mathbf{m}_2} &= e^{-4\pi m}e^{2\pi m}e^{-2\pi i(\frac{1}{m})} \longrightarrow 0\\ (z_m)^{\mathbf{m}_3} &= e^{-4\pi m}e^{4\pi m}e^{-2\pi i(\frac{1}{m})} \longrightarrow 1.\end{aligned}$$

This was exactly the distinguished point x_{τ_1} we calculated before in Example 5.1.21. ∘

With these examples and Lemma 5.2.9 in mind, we are now ready to show the identification $X_\Sigma \simeq \overline{T}_\Sigma$. This description of the toric variety X_Σ as a *topological T-space* will be one key result in the proof of Theorem 5.3.8. Although the proposition is well known in literature (see for example [AMRT10, pp. 1-6], [Oda78, §10], [Cox03, p. 211] or [Ful93, p. 54]), it is usually not explicitly written down or proven.

Proposition 5.2.11 *The identity map on $T = (\mathbb{C}^\times)^n$ extends to a homeomorphism $X_\Sigma \to \overline{T}_\Sigma$, which is equivariant with respect to the action of $T = (\mathbb{C}^\times)^n$, and the T-orbits* $\mathrm{orb}(\sigma)$ *in the toric variety X_Σ are mapped homeomorphically to the boundary components $O(\sigma)$.* ∘

Proof. The idea of the proof is to use the continuous actions of T on X_Σ and $\overline{T}_\Sigma$ to extend the equivalence of convergence of interior sequences to the distinguished point $x_\sigma = 0_\sigma$ (Lemma 5.2.9) to other boundary points.

Under the action of T, the orbit $T \cdot 0_\sigma$ in $\overline{T}_\Sigma$ gives $O(\sigma)$. As pointed out in the Orbit-Cone-Correspondence (Prop. 5.1.23 on page 168), the orbit $\mathrm{orb}(\sigma) = T \cdot x_\sigma$ in X_Σ gives the orbit corresponding to σ. It can be seen that the stabilizer of the distinguished point $x_\sigma \in \mathrm{orb}(\sigma)$ in $T = i\mathbb{Z}^n \backslash \mathbb{C}^n$ is equal to the subgroup $i(\mathrm{span}_{\mathbb{C}}(\sigma) \cap \mathbb{Z}^n)\backslash \mathrm{span}_{\mathbb{C}}(\sigma)$ (see [CLS11, Lemma 3.2.5]). By the definition of $\overline{T}_\Sigma$, the stabilizer of the point $0_\sigma \in O(\sigma)$ is also equal to $i(\mathrm{span}_{\mathbb{C}}(\sigma) \cap \mathbb{Z}^n)\backslash \mathrm{span}_{\mathbb{C}}(\sigma)$. Therefore, there is a canonical identification between $\mathrm{orb}(\sigma)$ and $O(\sigma)$.

By Lemma 5.2.9, for any sequence $\mathbf{z}_m$ in T, $\mathbf{z}_m \to x_\sigma$ in X_Σ if and only if $\mathbf{z}_m \to 0_\sigma$ in $\overline{T}_\Sigma$. Take any such sequence $\mathbf{z}_m \in (\mathbb{C}^\times)^n$ with $\lim_{m\to+\infty} \mathbf{z}_m = x_\sigma$. and let $\mathbf{t}_m \in (\mathbb{C}^\times)^n$ be any converging sequence with $\lim_{m\to\infty} \mathbf{t}_m = \mathbf{t}_\infty$. For both the toric variety X_Σ and the bordification $\overline{T}_\Sigma$, the continuous actions of T on X_Σ and $\overline{T}_\Sigma$ in Proposition 5.2.6 imply that the sequence $\mathbf{t}_m\mathbf{z}_m$ converges to $\mathbf{t}_\infty \cdot x_\sigma$ in X_Σ, and to $\mathbf{t}_\infty \cdot 0_\sigma \in O(\sigma)$ in $\overline{T}_\Sigma$ respectively. This implies that a sequence of interior points $\mathbf{z}_m$ in T converges to a boundary point in the orbit $\mathrm{orb}(\sigma) \subseteq X_\Sigma$ if and only if it converges to a corresponding point in $O(\sigma) \subseteq \overline{T}_\Sigma$. Since σ is an arbitrary cone in Σ and $\mathbf{t}_m$ is an arbitrary convergent sequence in $(\mathbb{C}^\times)^n$, this proves the topological description of toric varieties. □

Remark 5.2.12 The identification between X_Σ and $\overline{T}_\Sigma$ allows one to see that when a sequence $(\mathbf{x}_m)_m$ of points in the real part $\mathbb{R}^n$ of the complex torus $i\mathbb{Z}^n\backslash\mathbb{C}^n$ goes to infinity along the directions contained in a cone σ of the fan Σ, the sequence $\mathbf{x}_m$ will converge to a point of a complex torus $i\mathbb{Z}^n\backslash\mathbb{C}^n/\operatorname{span}_{\mathbb{C}}(\sigma)$ of smaller dimension. Hence the compact torus $(\mathbb{S}^1)^n$, which is the fiber over x_m in the toric variety, will collapse to a torus of smaller real dimension $\dim\sigma$. The behavior of converging sequences is schematically shown in Figure 5.11 and Figure 5.12.

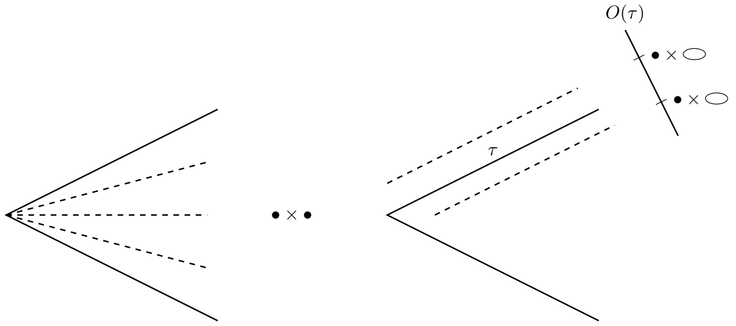


Figure 5.11: Within a chamber all fibers collapse in the same way: Both circles are collapsed to points (LEFT). Fibers parallel to a wall collapse differently, depending on the wall and the distance to it (RIGHT). Only one circle is collapsed to a point.

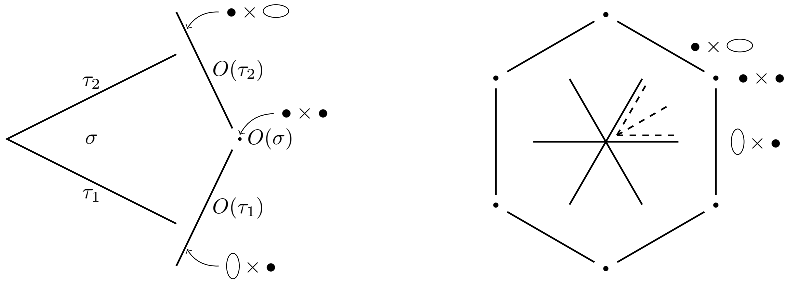


Figure 5.12: LEFT: The collapsing behavior of the fibers when the base point moves to infinity. Depending on the direction of movement either one or both circles are collapsed. RIGHT: The global picture of collapsing of a whole toric variety.

Such a picture of toric varieties including also the compact part of the torus $(\mathbb{C}^\times)^n$ is often described in connection with the moment map of toric varieties (for a reference see [Ful93, p. 79] or [Mil08]). We will come back to this map later. ◦

Lemma 5.2.13 *Under the identification of X_Σ with $\overline{T}_\Sigma$ in Proposition 5.2.11, the distinguished point x_σ in X_Σ corresponds to the image 0_σ of the origin of $\mathbb{C}^n$ in $\overline{T}_\Sigma$ under the projection π'_σ. When the orbit $O(\sigma)$ is identified with $(\mathbb{C}^\times)^r$, where $r = \operatorname{codim}(\sigma) = \dim_{\mathbb{C}}(\mathbb{C}^n/\operatorname{span}_{\mathbb{C}}(\sigma))$, then 0_σ corresponds to $(1, \cdots, 1)$.* ◦

Proof. As mentioned before on page 165, the distinguished point $x_{\{0\}}$ for the trivial cone $\sigma = \{0\}$ of the fan Σ is $(1, \cdots, 1)$. Under the identification $(\mathbb{C}^\times)^n \cong i\mathbb{Z}^n\backslash\mathbb{C}^n$ in Equation (5.9) on page 169, the distinguished point $x_{\{0\}}$ corresponds to the image of the origin of $\mathbb{C}^n$ under the projection $\mathbb{C}^n \to i\mathbb{Z}^n\backslash\mathbb{C}^n = O(\{0\})$. So for $\sigma = \{0\}$, x_σ corresponds to 0_σ in $O(\sigma) \subseteq \overline{T}_\Sigma$.

For any nontrivial cone $\sigma \subseteq \Sigma$, the distinguished point x_σ in the orbit $\operatorname{orb}(\sigma)$ is equal to the limit $\lim_{t\to 0}\lambda_{\mathbf{u}}(t)$ in X_Σ, where $\mathbf{u}$ is an integral vector contained in the relative interior of the cone σ. We need to determine the limit $\lim_{t\to 0}\lambda_{\mathbf{u}}(t)$ in the bordification $\overline{T}_\Sigma$. When we identify $(\mathbb{C}^\times)^n$ with $\mathbb{R}^n \times i\mathbb{Z}^n\backslash\mathbb{R}^n = i\mathbb{Z}^n\backslash\mathbb{C}^n$ using the logarithm as above in Equation (5.9), the complex curve $t \mapsto \lambda_{\mathbf{u}}(t) = t^{\mathbf{u}}$ for $t \in \mathbb{C}$ is mapped to a complex line in $i\mathbb{Z}^n\backslash\mathbb{C}^n$ with slope given by $\mathbf{u}$.

Hence its real part is a straight line in $\mathbb{R}^n$ through the origin with slope $\mathbf{u}$, that is, it is of the form $s \mapsto (m_1 s, \cdots, m_n s)$ for some $s \in \mathbb{R}$, and $\lambda_{\mathbf{u}}(t)$ is contained in $\mathrm{span}_{\mathbb{C}}(\sigma)$.

By the definition of the topology of $\overline{T}_\Sigma$ above, $\lim_{t\to 0} \lambda_{\mathbf{u}}(t)$ converges to the distinguished point 0_σ in $O(\sigma)$, that is, to the image of the origin of $\mathbb{C}^n$ in $O(\sigma)$. □

5.3 The Nonnegative Part of Toric Varieties and the Moment Map

Every toric variety X_Σ has a nonnegative part $X_{\Sigma,\geq 0}$. In this last section about toric geometry, we state our main result, namely that the nonnegative part $X_{\Sigma,\geq 0}$ is homeomorphic to the horofunction compactification $\overline{\mathbb{R}^n}^{hor}$ with respect to a suitable norm. We start with a description of the nonnegative part $X_{\Sigma,\geq 0}$. See also [Ful93, p. 78], [Oda88, §1.3], [CLS11, §12.2] and [Sot03, §6] for more details.

5.3.1 The Nonnegative Part $X_{\Sigma,\geq 0}$ of a Toric Variety X_Σ

In one dimension, the real part of $\mathbb{C}^\times$ is $\mathbb{R}^\times = \mathbb{R}_{>0} \cup \mathbb{R}_{<0}$ and its positive part is $\mathbb{R}_{>0}$. Similarly we have $(\mathbb{R}^\times)^n$ as the real part of $(\mathbb{C}^\times)^n$, which has 2^n-connected components. The positive part of $(\mathbb{C}^\times)^n$ is $(\mathbb{R}_{>0})^n$. Under the identification $(\mathbb{C}^\times)^n \cong i\mathbb{Z}^n\backslash\mathbb{C}^n = \mathbb{R}^n \times i\mathbb{Z}^n\backslash\mathbb{R}^n$ (Equation (5.9) on page 169) the positive part $(\mathbb{R}_{>0})^n$ corresponds to $\mathbb{R}^n \times i0 \cong \mathbb{R}^n$.

The basic idea for the construction of the real or the nonnegative part of a toric variety is to replace all complex numbers $\mathbb{C}$ by real numbers $\mathbb{R}$ or nonnegative numbers $\mathbb{R}_{\geq 0}$. So instead of $(\mathbb{C}^\times)^n$ we consider the action and embedding of the real torus $(\mathbb{R}^\times)^n$. Doing so, we denote the real part of X_Σ by $X_{\Sigma,\mathbb{R}}$ and get

Definition 5.3.1 [Sot03, Definition 6.2] For any toric variety X_Σ, the closure of the positive part $(\mathbb{R}_{>0})^n$ in the real part $X_{\Sigma,\mathbb{R}}$ is called the *nonnegative part* of X_Σ, denoted by $X_{\Sigma,\geq 0}$. ∘

Remark 5.3.2 A more constructive definition of $X_{\Sigma,\geq 0}$ is given in [CLS11, Prop. 12.2.3] and [Ful93, §4.2] and goes similarly as we did for X_Σ in Section 5.1.3: Let Σ be a fan and $\sigma \in \Sigma$. For the affine variety $U_\sigma = \mathrm{Hom}_{\mathbb{Z},\mathrm{reg}}(S_\sigma, \mathbb{C})$ we define its nonnegative part $U_{\sigma,\geq 0}$ by restricting the possible image of the semigroup homomorphism:

$$U_{\Sigma,\geq 0} := \mathrm{Hom}_{\mathbb{Z},\mathrm{reg}}(S_\sigma, \mathbb{R}_{\geq 0}).$$

The real part is given by

$$U_{\Sigma,\mathbb{R}} := \mathrm{Hom}_{\mathbb{Z},\mathrm{reg}}(S_\sigma, \mathbb{R}).$$

Gluing the $U_{\Sigma,\geq 0}$ ($U_{\Sigma,\mathbb{R}}$) together along common open subsets given by common faces of cones as we did before, we get the subset $X_{\Sigma,\geq 0} \subseteq X_\Sigma$ ($X_{\Sigma,\mathbb{R}} \subseteq X_\Sigma$) that is closed in the classical topology. When we replace the semigroup homomorphism $\gamma : S_\sigma \to \mathbb{C}$ with its absolute value $|\gamma|$, we get a retraction $U_\sigma \to U_{\sigma,\geq 0}$. Gluing these retractions together accordingly to the gluing to obtain X_Σ, we get a retraction $X_\Sigma \to X_{\Sigma,\geq 0}$.

As a subgroup of the complex torus T, the compact real torus $S_N \simeq (\mathbb{S}^1)^n$ acts on X_Σ. The retraction from X_Σ to $X_{\Sigma,\geq 0}$ then yields an homomorphism

$$X_{\Sigma,\geq 0} \simeq {}^{X_\Sigma}\big/_{S_N}. \qquad \circ$$

Example 5.3.3 We consider once again the example of $\mathbb{P}^1 \times \mathbb{P}^1$ as given in Example 5.1.15 on page 162. Replacing complex with real numbers yields

$$(\mathbb{P}^1 \times \mathbb{P}^1)_{\geq 0} = \mathbb{R}^4_{\geq 0} \setminus \left((\mathbb{R}^2 \times \{0\}) \cup (\{0\} \times \mathbb{R}^2)\right) \big/ \mathbb{R}^2_{>0}.$$

By the action $(r_1, r_2).(a, b, c, d) = (r_1 a, r_1 b, r_2 c, r_2 d)$ for $(r_1, r_2) \in \mathbb{R}^2_{>0}$ and $(a, b, c, d) \in \mathbb{R}^4$ we can scale the first two and the second two entries such that

$$(\mathbb{P}^1 \times \mathbb{P}^1)_{\geq 0} = \{(a, b, c, d) \in \mathbb{R}^4_{\geq 0} \mid a + b = 1;, c + d = 1\}.$$

In a two-dimensional affine subspace of $\mathbb{R}^4_{\geq 0}$ this is a square. So combinatorially the positive part $(\mathbb{P}^1 \times \mathbb{P}^1)_{\geq 0}$ it is the same as the unit square in $\mathbb{R}^2$ and this was exactly the dual polytope P° when constructing $\mathbb{P}^1 \times \mathbb{P}^1$ from the fan over a polytope P as in Example 5.1.19. Such a correspondence is the result of Theorem 5.3.8. ◦

If we denote $\mathbb{R}^n / \operatorname{span}_{\mathbb{R}}(\sigma)$ by $O_{\mathbb{R}}(\sigma)$, then for any two cones σ_1, σ_2, it holds that $O_{\mathbb{R}}(\sigma_1)$ is contained in the closure of $O_{\mathbb{R}}(\sigma_2)$ if and only if σ_2 is a face of σ_1. Under the identification in Proposition 5.2.11, $X_{\Sigma,\geq 0}$ can be described as follows:

Proposition 5.3.4 *For any fan Σ of $\mathbb{R}^n$, the nonnegative part $X_{\Sigma,\geq 0}$ is homeomorphic to the space*

$$\overline{\mathbb{R}^n}_\Sigma := \mathbb{R}^n \cup \coprod_{\substack{\sigma \in \Sigma, \\ \sigma \neq \{0\}}} O_{\mathbb{R}}(\sigma) \tag{5.12}$$

with the following topology: An unbounded sequence $\mathbf{x}_m \in \mathbb{R}^n$ converges to a boundary point $\mathbf{x}_\infty$ in $O_{\mathbb{R}}(\sigma) = \mathbb{R}^n / \operatorname{span}_{\mathbb{R}}(\sigma)$ for a cone σ if and only if one can write $\mathbf{x}_m = \mathbf{x}'_m + \mathbf{x}''_m$ such that the following conditions are satisfied:

1) *when $m \to +\infty$, $\mathbf{x}'_m$ is contained in the cone σ and its distance to the relative boundary of σ goes to infinity,*
2) *$\mathbf{x}''_m$ is bounded,*
3) *the image of $\mathbf{x}_m$ in $O_{\mathbb{R}}(\sigma)$ under the projection $\mathbb{R}^n \to \mathbb{R}^n / \operatorname{span}_{\mathbb{R}}(\sigma) = O_{\mathbb{R}}(\sigma)$ converges to $\mathbf{x}_\infty$.* ◦

This proposition was explained in detail and proved in [AMRT10, pp. 2-6] and motivated Proposition 5.2.11 above.

This yields the following result:

Corollary 5.3.5 The translation action of $\mathbb{R}^n$ on $\mathbb{R}^n$ extends to a continuous action on $X_{\Sigma,\geq 0}$, and the decomposition of $X_{\Sigma,\geq 0}$ in Equation (5.12) is the decomposition into $\mathbb{R}^n$-orbits. This decomposition of the nonnegative part of the toric variety $X_{\Sigma,\geq 0}$ is a cell complex dual to the fan Σ. If $\Sigma = \Sigma_P$ for a rational convex polytope P containing the origin as an interior point, then this cell complex structure is isomorphic to the cell structure of the polar set P°. ◦

5.3.2 How the Nonnegative Part $X_{\Sigma,\geq 0}$ is Homeomorphic to a Polar Polytope P°

The main result of this section in to show that the nonnegative part of a projective toric variety is homeomorphic to the horofunction compactification of $\mathbb{R}^n$ with respect to a suitable norm. The connection between these objects will be the bounded convex polytope $P \subseteq \mathbb{R}^n$, that gives on the one hand side the fan defining the toric variety and which on the other hand side is the unit ball of the norm determining the horofunction compactification. More precisely, the nonnegative part $X_{\Sigma,\geq 0}$ and the compactification $\overline{\mathbb{R}}^{hor}$ are both homeomorphic to the dual polytope P°.

Let P be a rational convex polytope containing the origin as an interior point and let X_{Σ_P} be the associated projective variety. The homeomorphism between the nonnegative part of the toric variety $X_{\Sigma_P,\geq 0}$ and the dual polytope $P^\circ = \operatorname{conv}\{\mathbf{m}_1, \dots \mathbf{m}_k\}$ is given by the algebraic moment map. Let P° be scaled such that the vertices $\mathbf{m}_j$ of P° are all integer points. Then the algebraic moment map is given by

$$\begin{aligned} \mu : X_{\Sigma_P} &\longrightarrow \mathbb{R}^s \\ x &\longmapsto \frac{1}{\sum_{j=1}^k |\chi^{\mathbf{m}_j}(x)|} \sum_{j=1}^k |\chi^{\mathbf{m}_j}(x)| \mathbf{m}_j . \end{aligned}$$

So the image of the map is a convex combination of the vertices of P° and thereby contained in P°. It is not only contained as some subset, but also the structure of $X_{\Sigma_P,\geq 0}$ with its boundary components coming from the orbits $O_{\mathbb{R}}(\sigma)$ corresponds to the face structure of the convex polytope P°. Recall that by definition, each cone σ of Σ_P corresponds via μ to a unique face F_σ of P, which in turn corresponds (see Remark 2.4.14) to a dual face F_σ° of the polar set P°. Then the result on the homeomorphism coming from the moment map can be stated as follows:

Proposition 5.3.6 *The moment map induces a homeomorphism*

$$\mu : X_{\Sigma_P,\geq 0} \to P^\circ$$

such that for every cone $\sigma \in \Sigma_P$, the positive part of the orbit $O(\sigma)$ as a complex torus, or equivalently the orbit $O_{\mathbb{R}}(\sigma)$ in Proposition 5.3.4, is mapped homeomorphically to the relative interior of the face F_σ° corresponding to the cone σ. ∘

Proof. The proof of the proposition can be found in [CLS11, Prop. 12.2.5] and [Ful93, §4.2] but we again have to be careful with the notations. As mentioned before in the proof of Proposition 5.1.18, our main references [CLS11, Cox03, Ful93] use the normal fan (i.e. the fan over the dual polytope) to construct the toric variety, whereas we use directly the fan over the polytope. But as we assume our polytopes to be maximal dimensional and containing the origin as an interior point, it holds $(P^\circ)^\circ = P$ and therefore the statement is true as stated above. □

The homeomorphism between the horofunction compactification $\overline{X}^{hor}$ of a finite-dimensional normed vector space X and the dual unit ball B° we constructed in Section 3.3 was inspired by this moment map.

For more details about the moment map and the induced homeomorphism see [Oda88, p. 94], [Ful93, §4.2], [Sot03, §8] and [JS16, Thm. 1.2].

Remark 5.3.7 A similar convexity result about the image of the (symplectic) moment map is also well know in symplectic geometry. Consider a symplectic toric manifold (M^{2n}, ω), that is, M is a compact connected symplectic manifold with a faithful action of the torus T^n of dimension n and a moment map $\mu : M \to \mathbb{R}^n$. It was shown independently by Atiyah [Ati82] and Guillemin-Sternberg [GS82] that the image of the moment map is the convex hull of the fixed points of the action of the torus and therefore a convex polytope. ∘

It remains now to put all ingredients about the homeomorphism together to the main result of this section:

Theorem 5.3.8 *Let $X = X_{\Sigma_P}$ be a projective toric variety of dimension n. Then the following are homeomorphic:*

1) *the nonnegative part $X_{\geq 0}$ of the toric variety X*

2) *the image of the moment map of the toric variety X*

3) *the horofunction compactification $\overline{\mathbb{R}^n}^{hor}$ of $\mathbb{R}^n$ with respect to the norm $||\cdot||_P$*

These homeomorphisms give a bijective correspondence between projective toric varieties X of dimension n and rational polyhedral norms $||\cdot||$ on $\mathbb{R}^n$ up to scaling in every dimension $n \geq 1$. ◦

Proof. Let P be a rational polytope containing the origin as an interior point. Let $\Sigma = \Sigma_P$ be the fan obtained by taking cones over the faces of P. By Propositions 5.3.4 and 3.2.7 on page 59, an unbounded sequence of $\mathbb{R}^n$ converges to a boundary point in the compactification $\overline{\mathbb{R}^n}_{\Sigma}$ if and only if it converges in the horofunction compactification $\overline{\mathbb{R}^n}^{hor}$ with respect to the polyhedral norm $||\cdot||_P$. Therefore, the two compactifications $\overline{\mathbb{R}^n}_{\Sigma}$ and $\overline{\mathbb{R}^n}^{hor}$ of $\mathbb{R}^n$ are homeomorphic. Proposition 5.3.4 again implies that the nonnegative part $X_{\Sigma,\geq 0}$ of the toric variety is homeomorphic to the horofunction compactification $\overline{\mathbb{R}^n}^{hor}$. □

Remark 5.3.9 Given the one-to-one correspondence between the toric varieties and rational polyhedral norms in Theorem 5.3.8 and the fact that each polytope P also determines a Hilbert metric $d_H(\cdot,\cdot)$ on the interior int(P) of P, one natural question is whether there exists a similar relation between $X_{\Sigma_P,\geq 0}$ and the horofunction compactification of $(\text{int}(P), d_H(\cdot,\cdot))$. The results in [Wal14b] and [Wal14a] show that besides the Hilbert metric, Funk metric and reverse Funk metric should also be considered, and that the horofunction compactifications of the Funk metric seems to be related to $X_{\Sigma_{P^\circ},\geq 0}$, the toric variety associated to the polar set P°, and the horofunction compactification of the Hilbert metric is more complicated. ◦

6 | Outlook

In the previous chapters we have seen how to interpret the horofunction compactification of a finite-dimensional normed space as the dual unit ball of the norm. Additionally we saw how to realize other compactifications of symmetric spaces of non-compact type as horofunction compactifications of the space with an appropriate norm. We now discuss some open problems and questions for future research work.

Generalization of Theorem 3.2.6

In Theorem 3.2.6 we determined the topology of the compactification by the convergence behavior of sequences. Thereby we had to restrict ourselves to unit balls that have a particular nice shape. It may be possible to generalize this theorem to any norm for which the set of extreme sets of the dual unit ball is closed. We gave an idea of how to deal with this problem in Section 3.2.7. Our approach is based on a conjecture about the right rate of convergence of a sequence of sets in the dual unit ball B°. Future work in this direction could either lead to a proof of Conjecture 3.2.12 or to a different approach of determining the point p.

Playing around with norms

Apart from taking the Minkowski sum of two compact convex sets in $\mathbb{R}^n$, there are many more natural operations on the set of convex sets, like taking the intersection or the convex hull of two or more compact convex sets. The only restriction for us on such an operation is that the new set is still compact convex and has the origin as an interior point. Then we can determine the horofunction compactification of the space with respect to this norm. If its set of extreme sets is still closed (which is not necessarily true) we can calculate all its horofunctions with the techniques shown before. But it remains to interpret this new compactification in terms of the previous ones.

Question 6.0.1 What operations on compact convex sets extend to operations on the corresponding horofunction compactifications? ◦

We already saw that the Minkowski sum of the dual unit balls lead to the least common refinement of the compactifications. And in Section 4.3 we already mentioned that taking the convex hull of the Weyl group orbits of each irreducible component leads to a generalized Satake compactification.

In their book [BJ06] Borel and Ji discuss many more of compactifications of symmetric and locally symmetric spaces. They also present a uniform approach to construct them by adding boundary components and show how these compactifications are related to each other in terms of refinements or quotients. This is the starting point for many interesting questions: How does the horofunction compactification fit into the picture? Are there other compactifications than the Satake and the Martin compactification that can be realized as a horofunction compactification? How is corresponding unit ball then obtained?

(Dual) generalized Satake compactifications

It seems plausible that any generalized Satake compactification as defined in [GKW15] can be realized as the horofunction compactification for a G-invariant polyhedral Finsler metric, but it remains to verify this. This correspondence would then allow us to define the dual generalized Satake compactification $\overline{X}_\tau^{S^*}$ like this:

Definition 6.0.2 Let $\tau : G \to \mathrm{PSL}(n, \mathbb{C})$ be a faithful projective representations and $\overline{X}_\tau^S$ the associated generalized Satake compactification. The dual generalized Satake compactification $\overline{X}_\tau^{S^*}$ is defined to be the horofunction compactification of X with respect to the polyhedral G-invariant Finsler metric defined by the unit ball $B = D = \mathrm{conv}(\mu_1, \ldots, \mu_k)$. ◦

Question 6.0.3 Is there a geometric way to interpret the duality between $\overline{X}_\tau^S$ and $\overline{X}_\tau^{S^*}$? ◦

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