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Martin Rabel

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Advisor:

Prof. Dr. Markus Banagl

Abstract

This thesis studies the geometric properties related to certain transversality statements on singular spaces, in a purely topological setting. These enter in the main part – the construction of a generalized homology theory realized via bordism of such singular spaces – through the inverse of the excision-isomorphism, the most difficult aspect of that problem. The relevancy of this homology theory is due to the unification of both, possessing a geometric description, establishing geometric fundamental-classes, and at the same time being well-suited to study inherently topological phenomena, like homeomorphism-invariance of said fundamental-classes, even in the absence of pl-structures. As an application, the invariance of Goresky–MacPherson L-classes under certain homeomorphisms is demonstrated.

Zusammenfassung

Technische Grundlage dieser Arbeit bildet die Untersuchung von, gewissen Transversalitätsaussagen zugrundeliegenden, geometrischen Eigenschaften singulärer Räume in einem rein topologischen Kontext. Für die hier vorgestellte Konstruktion einer verallgemeinerten Homologietheorie, realisiert als Bordismustheorie solcher singulären Räume, besteht die Hauptschwierigkeit in der Invertierbarkeit des Ausschneideisomorphismus. Ein Problem, dessen Lösung sich in eben solchen Transversalitätsaussagen findet. Die Relevanz einer solchen Homologietheorie liegt im Vorhandensein einer geometrischen Beschreibung und damit von Fundamentalklassen, bei gleichzeitiger Kompatibilität mit topologischen Phänomenen, wie etwa der Invarianz jener Fundamentalklassen unter Homöomorphismen, ohne die Notwendigkeit einer PL-Struktur. Als Anwendungsbeispiel wird die Invarianz von Goresky–MacPherson L-Klassen unter bestimmten Homöomorphismen gezeigt.

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0 Introduction

Transversality and general-position techniques – i. e. the ability to avoid geometrically degenerate special cases "as good as possible" while maintaining full generality, for example by moving things around a little bit without changing the problem[1] – play a central role in the answer to many geometric questions.

These questions include the correspondence of geometry and algebra in high-dimensional h-cobordisms [Sma62], the s-cobordism-theorem (by Barden, Mazur, Stallings, see [Ker65]) or end-theory [Fre31; Sie65], the realization of excision (or suspension) isomorphisms in (reduced) generalized homology theories given as bordism-theories [Tho54], (closely related) the Thom–Pontryagin-construction, or characteristic classes, for example L-classes, see e. g. [Hir71] (which is most apparent from a viewpoint introduced by Thom [Tho58], see [Ran95, Prop. 2.6 (p. 7)] or [Ban07, §5.7 (p. 120–122)]).

Vice versa, geometric structures allow for certain transversality constructions.[2] The interplay of transversality, geometry and algebraic topology is combined in this thesis to construct an oriented bordism-theory, in a singular (non-manifold, but manifold-stratified), *inherently topological* context (to be explained below).

By (manifold-) stratified spaces, we mean singular (non-manifold) spaces, that can be divided into "reasonable" (e. g. manifold-) parts, called the "strata", that fit together in some controlled way. One of the motivating examples for the study of such objects are algebraic varieties, that is, zero-sets of polynomials [Whi65; Tho69; Mat]. Other examples are orbit-spaces of group-actions [Qui88a], or stratifications of mapping-cylinders [CS95].

Besides these being of independent interest by themselves and through their natural occurrence in such examples, stratified spaces additionally proved useful in the study of manifolds and simplicial sets. As an example, the so-called "pinch-bordism" [Sie83] allows for a very elegant proof of Novikov-additivity (of intersection-forms) on manifolds, by using a singular bordism-theory. On the other hand, [Sie72] studies self-homeomorphisms of finite simplicial sets via stratified spaces.

As always, provided such a new setting, the question of classification, or at least useful invariants, was (and certainly is) an important aspect of development in this area. A rather natural (and quite successful) approach to this problem is, by generalizing manifold-invariants to manifold-stratified spaces, see for example [GM80; Ban07] and references therein.

[1] This viewpoint, generalizing simplicial approximation to the smooth category, without explicit use of triangulations [Whi40], seems to originate from [Tho54, chapitre I].

[2] Note, that also the treatment of topological transversality in high dimensions given by Kirby and Siebenmann [KS77], uses a (smooth or pl) product-structure-theorem, a geometric result based in turn on the (smooth or pl) h-cobordism-theorem.

Classically, manifold classification relies heavily on surgery-theory [Mil61a; Wal70], with (simply-connected) obstructions found in intersection-form-signatures (or more generally in properties of intersection-pairings and L-theory [Ran92]). These invariants rely heavily on Poincaré-duality, which is *not* readily available for singular spaces when using "standard" (e. g. singular) homology. A suitable replacement was found in intersection-homology [GM80], by allowing for certain controlled deviations from transversality of cycles to strata. Other approaches to restore Poincaré-duality are L^2-cohomology [Che80] or intersection-spaces [Ban10]. Having restored a (suitable form of) Poincaré-duality, one can also obtain, for example for Siegel's [Sie83] Witt-spaces, non-degenerate intersection forms. These allow in turn the definition of a (bordism-invariant) signature and L-classes [GM80].

While in differential topology (cohomological) L-classes, and many other characteristic classes of smooth manifolds, are defined via tangent-bundles, for topological manifolds, and especially for singular spaces, another approach is required. As it turns out, one can sometimes rely on transversality instead, for example for homological L-classes, the Poincaré-duals of the standard ones, such a construction was already used by Thom [Tho58], and then transferred to the singular case by Goresky and MacPherson [GM80].

One should note however, that already the transfer of transversality-ideas to *topological manifolds* [KS77; Qui88b] was quite hard to achieve and involves many of the techniques developed for example in [Nov65; Kir69]. Hence it is not actually too surprising, that finding a singular topological setting with suitable (and accessible) transversality properties will turn out to be the central problem of this thesis.

In a singular context, there is the additional issue of "inhomogeneities" where strata meet. The usual approach on stratified spaces, via induction over skeleta, requires some way of extending geometric structures from one stratum (or skeleton) into the next, adjacent one. This suggests the use of some sort of normal-structure hypothesis – as is usually built into the definitions of stratified spaces for such inductions. However geometric normal-structure (e. g. bundles or compatible triangulations) is not readily available in the topological category, not even for locally flat submanifolds (see e. g. [FQ90, §9.5 (p. 150f)]). So artificially "enforcing" it (as a hypothesis), may render the result incompatible with topological constructions (see below).

The existence of some – suitable for our purposes by a transversality-theorem of [CV99] – geometric structure, namely mapping-cylinder neighborhoods of approximate fibrations, can be guaranteed (in high dimensions) by topological hypotheses through "controlled topology" constructions [Qui79; Qui02; Cha83]. This allows for fixing a *topological* hypothesis, to then obtain a geometric structure "on-the-fly" whenever needed for a transversality-construction. So in this sense, it constitutes the core element for realizing a geometric theory, within the topological setting used. Much more details and discussion on suitable (topological) normal structure is provided in Chapter 1 ("Background").

The treatment presented here "encapsulates" the transversality properties in a generalized homology theory realized by bordism of stratified spaces, more precisely in (the inverse of) the excision-isomorphism. Similar theories using strong geometric hypotheses [Min04] or piecewise linear (pl) structures [Sie83; Par90; BLM19] are quite well-understood. The interesting new aspect of the approach presented here is, as outlined above, its compatibility with "inherently topological" problems.

We illustrate the "inherently topological setting" via an example: To study the "transport-behavior" of geometric structures (e. g. of certain, L-theory-type, fundamental classes) under *topological* homeomorphisms, the natural geometric object to start from, is the mapping-cylinder of that homeomorphism (which could then serve e. g. as a bordism). Now, restricting to the case of pl-pseudomanifolds for the moment, such a homeomorphism, even if stratified, need not be pl[3], so generally we do not even know if its mapping-cylinder can be compatibly (with the stratification) triangulated while keeping its "ends" fixed. Similarly, strong geometric structures, like certain bundles, do not "usually" arise on that mapping-cylinder.

We will be working with a type of stratified spaces known as "Quinn"- or "homotopy"-stratified [Qui88a]. Cappell and Shaneson [CS95] found this type of stratified spaces to "arise naturally" (see p. 59 of the reference) in the context of mapping-cylinder stratifications, even though they studied smooth maps on smooth stratifications. At the same time, these spaces are well-suited (essentially by construction) for the use of controlled topology techniques [Qui88a, e. g. Thm. 1.7 (p. 446)], [CV99].

We construct a bordism-theory of such spaces, which has both nice geometric properties – for example fundamental-classes and a signature-invariant (see below) – while also being able to describe their transport under certain homeomorphisms. So, there are geometric fundamental-classes (close to) being homeomorphism / topological invariants.

What, so far, we loosely referred to as "transport-behavior" under homeomorphisms can be understood as the combination of two questions: The behavior under "isomorphisms", i. e. stratified homeomorphisms as we are working in a stratified category, and, on the other hand, dependence on the choice of stratification, i. e. behavior under *un*stratified homeomorphisms. As we work in the realm of Quinn's [Qui88a] manifold homotopy-stratified spaces, where certain intrinsic stratifications are available [Qui87], even *unstratified* homeomorphisms seem tractable. We demonstrate this for spaces with at most two strata. *Stratified* homeomorphisms can be treated more generally.

However, the transversality requirements come at the cost of additional hypotheses on the (fundamental-groups of) links, and a "dimensional gap" between strata, that is related to certain low-dimensional problems. This will become more apparent later. It is not entirely clear as of now, to what degree the hypotheses used are strictly necessary – this is discussed further later on – but it seems quite possible, that "suitable" (for constructing a generalize-homology-bordism-theory) transversality results are indeed obstructed in the setting observed. Similar "K-theoretic" problems [Qui04] were found by [RY06], see below, which suggests, that (some of) the obstructions might be "genuine".

[3] Even for pl-manifolds (i. e. spaces with only one stratum, where any map is stratified) topological homeomorphisms are not necessarily pl [Mil61b].

On the other hand, this setup can be combined with a Witt-hypothesis [Sie83; Fri09], to obtain a well-defined signature-invariant. Then the theory encodes (bordism-invariant) intersection-form- and thus signature-information. This is the typical "L-theory"-type information used for example in surgery theory of manifolds very successfully. Also note, that introducing some "rigidity" on bordisms is particularly important for singular theories, as, otherwise, it might happen that the cone on any allowable space is an allowable null-bordism, which would render the theory trivial. A signature invariant certainly prevents that.[4]

As an application of our construction, we show, that for suitable stratified spaces, the Goresky–MacPherson L-classes [GM80] are invariant under certain (stratified, or, of spaces with at most two strata) homeomorphisms. There are many previous results concerning this problem: The manifold-case (of the invariance of rational Pontryagin-classes, thus of L-classes) has been shown for the pl-case by Thom [Tho58], and for the topological case, famously, by Novikov [Nov65]. A treatment of this problem for pl-homeomorphisms, using ad-theories – which are a little stronger than, but closely related to, bordism-theories – can be found in [BLM19]. Another treatment, also answering some questions on generalizations of rational Pontryagin-classes (i. e. including a treatment via characteristic classes of certain generalized bundle-theories) can be found in [RW10]. As other "algebra-side" treatments (see next paragraph), the geometric control provided by this reference's treatment seems insufficient for fundamental-classes beyond the manifold-case. The invariance of stratified L-classes has also been treated by [CSW91] and [Wei94, p. 209f].

It should be mentioned, that there is a "complementary" / "algebraic" view-point, know as "controlled L-theory" [RY06]. This approach also seems to "see" K-theoretic obstructions (to generalized homology-theories), but seems well-suited for still incorporating the obstructed cases into a useful treatment. However in exchange, it provides less geometric control, and "fundamental-classes" only in the manifold-case.

The contents of this thesis can be outlined as follows: Initially, Chapter 1 ("Background") introduces and organizes known results from literature, as relevant to the later development. Much of the introduction, and discussion motivating the choices of which techniques to use, is also included there. Next, Chapter 2 ("Bordism Constructions") analyzes the requirements for building a bordism-theory (as a generalized homology-theory; similar to for example [Aki75; Fri15]) and connects these to the material of the background-chapter, in order to establish a first "preliminary" setup, demonstrating the relevant ideas in the simplest non-trivial case. Subsequently, Chapter 3 ("Multiple Strata") gives some generalizations, for broader applicability of the theory, but also to set the grounds for Chapter 4 ("Homeomorphisms") which uses those results to study mapping-cylinders of certain homeomorphisms. Further extensions are discussed in Section 2.6 ("Improvements") and Chapter 6 ("Conclusion"). In Chapter 5 ("The Main Theorem and its Applications"), implications, applicability,

[4] The theory presented here allows arbitrary topological manifolds, and on these, the signature-invariant calculates the standard-signature, so for example the complex projective spaces $\mathbb{CP}^{2n}$ cannot be trivial.

and a formulation mostly independent of the details behind the implementation of the bordism-theory, are given. Finally Chapter 6 ("Conclusion") discusses difficulties and further aspects of the problems studied.

Contents

1 Background

This chapter summarizes known results from the literature and organizes them in a way, applicable to the subject of constructing a bordism theory. It provides the basic definitions and results used throughout this thesis.

The general progression of this chapter develops around finding the “correct” notion of transversality in a stratified *topological* setting. The first clue is, that for spaces with manifold-strata, transversality *within* individual strata is well-understood. So the root of the problem really is in compatibility-conditions where the strata meet. This requires some notion of “normal-structure”. However, in a topological setting, strong geometric structures, like normal-vector-bundles, typically do not exist. Rather, a natural notion was found in “approximate fibrations” [Edw, Def. 2 (p. 8), Thm. 2.1 (p. 11), Prop. 3.1 (p. 13)]. For cylinders of these, Connolly and Vajiac [CV99] give indeed a transversality-theorem.[1] This answers only half the question however: As will be seen, most difficult issues have merely been moved – not resolved – and are now hidden in the existence of (suitable) cylinder neighborhoods. Luckily, this type of problem, known as controlled “end-theory” or “ends of maps” [Qui79; Qui82a; Qui82b; Qui86; HR96] has received a lot of attention in the past, and is well-understood (at least in high dimensions). An adequate framework of stratified spaces to work with controlled ends, is found in Quinn’s [Qui88a] “(Manifold) Homotopy Stratified Spaces” or (M)HSS for short.

The remaining sections of the chapter are concerned with properties of these spaces as required for the present task. Namely, there is a well-defined intersection-homology theory [Qui87], and a treatment of Poincaré-duality [Fri09], which also includes a Witt-condition. This in turn allows for the definition of a signature-invariant of such Witt-spaces. Finally there are intrinsic stratifications [Qui87] that are very close to being topologically-intrinsic – i. e. in most cases only depend on the topology of the underlying (unstratified) space – stressing the topological nature of the setting.

Throughout this chapter, we are working with spaces with at most two strata (or, for gluing results, with spaces of at most two meeting strata), as this considerably simplifies notation. Where relevant, the respective versions for spaces with more than two strata are included in Chapter 3 (“Multiple Strata”).

Even if results are associated directly to a given reference, these are not to be considered to be “at verbatim”. The statement given may be reformulated to integrate with the notation and to incorporate further discussion, that is given by the source, but not included in the statement itself. Any alterations should be clear from the

[1] They are working on Quinn-Spaces already, see Section 1.4 (“Quinn Spaces / (M)HSS”) for details, but their construction mostly is “separate” from controlled topology and mapping-cylinder existence, see Section 1.9 (“Mapping-Cylinder Neighborhoods”).

context or additionally explained if not so.

1.1 Topological Normal Structure

This section summarizes some of the problems and requirements, that lead to the choice of structures and techniques being used in the remainder of this thesis.

For realizing a transversality statement, we want to use manifold-transversality in the lower stratum, extend the resulting geometric structure into a neighborhood by using some sort of normal structure, and finally use relative manifold-transversality in the upper stratum to finish the construction.

This seems to – and in fact does[2] – require a *geometric* notion of normal-structure. The usual candidates, (vector) bundles, can be used for this task [Min04], but typically will *not exist* in a topological setting. Even in the "simple" case, where the total space X is itself a manifold, and we are investigating the normal-structure of a (locally flat) submanifold B, normal-*bundles* may not exist. A noteworthy special case, where they *do* exist[3] are manifold-boundaries:

Definition 1.1.1: *A* collar *of $Y \subset X$ is homeomorphism $c : Y \times [0, \infty) \to N$, where N is an open neighborhood of $Y \subset X$ and $c(y, 0) = y$ for all $y \in Y$. If such a collar c exists, $Y \subset X$ is* collared (by c).

Theorem 1.1.2: *[Bro62]: Let $(M, \partial M)$ be a manifold with boundary.[4] Then $\partial M \subset M$ is collared.*

The arguments given in this thesis require conditions on local fundamental-groups – basically fundamental-groups of "links" from the stratified-spaces point of view taken here – and one might suspect, that *in these cases* bundle-structures could exist. However, again even for submanifolds, correct fundamental-group structure, does *not* even imply local flatness (for an example, see [FQ90, p. 151]).

It would be possible, of course, to state a result with a geometric structure as part of the hypothesis. This does however not work well with inherently topological problems like constructing a bordism from (the mapping cylinder of) a homeomorphism. This is exactly the type of question we want to answer, therefore we need to use an other notion of normal-structure.

[2] See the "backward"-direction of [CV99, p. 536], and Section 1.6 ("Stratified Transversality") below.

[3] Other examples where they *do* exist are for low-dimensional submanifolds (for $\dim(B) \leq 3$ and $\dim(X) \geq 5$) as well as small codimensions, see [FQ90, p. 150]

[4] I. e. points have neighborhoods homeomorphic to either a euclidean space $\mathbb{R}^n$ or a "half-plane" $\mathbb{R}^{n-1} \times [0, \infty)$, meaning the boundary is *locally* collared, and M is second-countable and Hausdorff.

Returning to the manifold-case, it is in fact known, that there are normal-structures which *always* exist. On the one hand, there are Milnor's [Mil64] (stable) microbundles – there does not seem to be an obvious way of generalizing these from submanifolds to more general stratifications, however, and "stable" existence seems not to be enough either. On the other hand, in high dimensions, (certain) submanifolds do have neighborhoods of the form of the mapping-cylinder of an "approximate fibration" [Edw] (Section 1.2 ("Approximate Fibrations")).

This notion of normal-structure has been extended to stratified spaces by [Hug+00], in the case of two strata, and, based on that result, in a series of papers by Hughes [Hug99c; Hug99a; Hug99b; Hug02; Hug04] (see [Hug96] for an overview), in the general case (for compactly dominated local homotopy links, see below). There is a known connection to transversality problems [CV99] (Section 1.6 ("Stratified Transversality")), via controlled end-theory [Qui82a] (Section 1.7 ("Excursion: Controlled Topology"), Section 1.9 ("Mapping-Cylinder Neighborhoods")). These take place in the setting of Quinn's [Qui88a] "homotopy-stratified" spaces (Section 1.4 ("Quinn Spaces / (M)HSS")).

As they come up repeatedly, we also fix notation for mapping-cylinders (with the convention, that the cylinder-coordinate measures the distance to the "base"):

Definition 1.1.3: *Given a (continuous) map $f : X \to Y$, the* mapping-cylinder $\mathrm{cyl}(f)$ of f *is the topological space given by the equivalence classes in $X \times I \sqcup Y$, where $I = [0, 1]$, of the equivalence-relation generated by $(x, 0) \sim f(x)$ for $x \in X$ with the quotient topology.*

If f is surjective, points in the cylinder can always be denoted as $[x, t]$, with $x \in X$ and $t \in I$. We will sometimes refer to t as the cylinder-coordinate *and to $Y \subset \mathrm{cyl}(f)$ as the* base.

There is a canonical embedding of X as the cylinder-cap *at $X \times \{1\} \subset \mathrm{cyl}(f)$, where no identifications via $\sim$ take place nearby.*

The open cylinder $\mathring{\mathrm{cyl}}(f)$ *is the cylinder without cap, i. e.* $\mathring{\mathrm{cyl}}(f) := \mathrm{cyl}(f) - (X \times \{1\})$.

The cone $c(X)$ on X *is the cylinder of the (unique) map to the point* $c(X) := \mathrm{cyl}(X \to \{\mathrm{pt}\})$*, similarly* $c^\circ(X) := \mathring{\mathrm{cyl}}(X \to \{\mathrm{pt}\})$. *The "base" is in this case referred to as* vertex v.

1.2 Approximate Fibrations

Approximate fibrations are continuous maps, that approximate the lifting properties of fibrations. There are different ways of making this statement precise: Most notably, via "smallness" with respect to open covers, smallness in a metric sense (if the base-space is metric) or via continuity of "open" lifts when composed into the base. See for example [HTW90, §12 (p. 43–52)] for a detailed analysis. We are particularly interested in the case where total- and base-space are metric manifolds (thus metric-ANR). In this case, the commonly used definitions agree (for the direction needed here, see [HTW90, Thm. 12.13 (p. 50)]), and we will adopt the following intuitive definition:

Definition 1.2.1: *Let E, B be metric (metric-)*[5] *ANR, $p : E \to B$ continuous.*

(i) *p has the ϵ*-lifting property *with respect to A for $\epsilon : B \to (0, \infty)$, if given a commutative square (where arrows are continuous) of the form*

$$\begin{tikzcd} A \times \{0\} \arrow[r, "F_0"] \arrow[d, "\mathrm{incl}"'] & E \arrow[d, "p"] \\ A \times I \arrow[r, "f"] \arrow[ur, dashed, "F"] \arrow[ur, phantom, shift right=3ex, "\approx \epsilon"] & B \end{tikzcd}$$

the continuous "lift" F (dashed arrow) exists, such that the upper-left triangle commutes, and the bottom-right triangle "commutes up to ϵ", i. e. for all $(a, t) \in A \times I$: $\mathrm{dist}_B(p \circ F(a, t), f(a, t)) < \epsilon(f(a, t))$. p is called an ϵ-fibration *if it has the ϵ-lifting property with respect to all A.*

(ii) *p is an* approximate fibration (AF) *if it is an ϵ-fibration for all $\epsilon > 0$.*

(iii) *p is a* manifold approximate fibration (MAF) *if it is a proper map, an AF and E, B are manifolds.*

The difference to an "exact" fibration is probably best illustrated by examples:

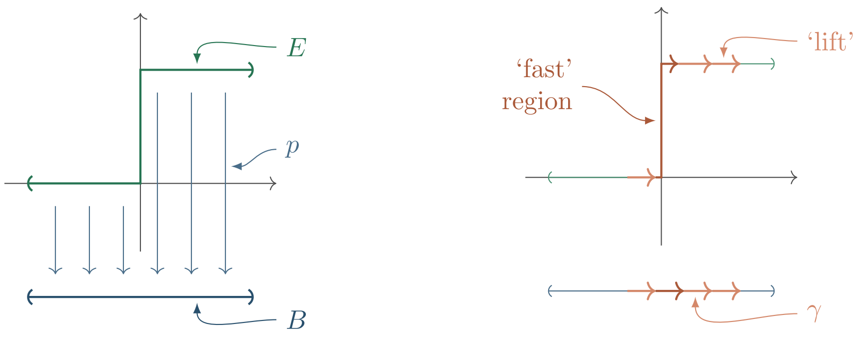


Figure 1.1: The MAF of Example 1.2.2.

[5] Metric-ANR here is supposed to mean ANR w. r. t. metric spaces. We will only work with manifolds, which are metric-ANR, so for our purposes, we may as well replace "ANR" by "manifold".

Example 1.2.2: *Let* $E := (-1,0] \times \{0\} \cup \{0\} \times [0,1] \cup [0,1) \times \{1\} \subset \mathbb{R} \times \mathbb{R}$ *and* $B := (-1,1)$*, define* $p : E \to B$ *as the projection to the first coordinate. See Figure 1.1.*

Then p *is* not *a fibration, because given* $f : \{\mathrm{pt}\} \times I \to B, t \mapsto t - 1/2$ *and* $F_0 : \mathrm{pt} \mapsto (-1/2, 0)$ *no exact lift can be continuous at* $t = 1/2$*, i. e. over* $0 \in B$*.*

But p is *a MAF, because given any* $\epsilon(0) > 0$*, a lift may run "*$\frac{1}{\epsilon}$*-fast" through the vertical segment.*

This behavior is "representative" for AFs $\mathbb{R} \to \mathbb{R}$ in the following sense.

Example 1.2.3: *Given a map* $p : E \to B$*:*

(a) Let $E = B = [0,1]$*. Then*

(i) p *is an AF* $\Leftrightarrow$ p *is monotonic and surjective*

(ii) p *is a fibration* $\Leftrightarrow$ p *is strictly monotonic and surjective*

(b) If $E = B = \mathbb{R}$ *and* $p : \mathbb{R} \to \mathbb{R}$ *is a proper map, then*

(i) p *is a MAF* $\Leftrightarrow$ p *is monotonic and surjective*

(ii) p *is a fibration* $\Leftrightarrow$ p *is strictly monotonic and surjective*

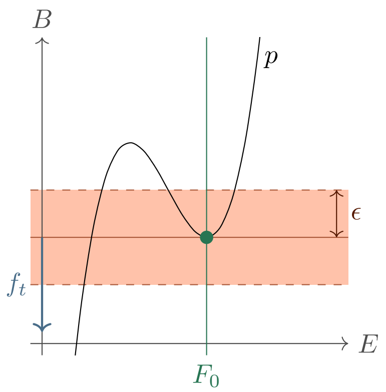


Figure 1.2: An MAF from $\mathbb{R}$ to $\mathbb{R}$ must be monotonic, see Example 1.2.3.

Proof: "(a) $\Rightarrow$": (i) See also Figure 1.2 for a picture of the contradiction found in the following proof. Assume p where not monotonic, i. e. there are $t_0 < t_1' < t_2 \in [0,1]$ s. t. $p(t_0), p(t_2) > p(t_1')$. Pick $t_1 \in p^{-1}\big(\min\{p(t) | t \in [t_0, t_2]\}\big) \neq \emptyset$ (the minimum on the compact $[t_0, t_2]$ is attained, thus this preimage is non-empty), then $t_0 < t_1 < t_2$ and $p(t_0), p(t_2) > p(t_1') \geq p(t_1)$ still hold. Let $\epsilon := 1/2 \min(p(t_2) - p(t_1), p(t_0) - p(t_1))$ and

$$f : \{\mathrm{pt}\} \times I \to B = [0,1], \qquad t \mapsto p(t_1) - 2\epsilon t$$
$$F_0 : \{\mathrm{pt}\} \to E = [0,1], \qquad \mathrm{pt} \mapsto t_1$$

Then, there must be an approximate lift $F : \{\mathrm{pt}\} \times I \to E = [0,1]$, with $p \circ F$ ϵ-close to f. By ϵ-closeness, $p(F(t')) < p(t_1)$ for $t' := 3/4$, and by $p(t_1)$ being the minimum of p on $[t_0, t_2]$ by definition, $F(t') \notin [t_0, t_2]$. W. l. o. g. (the case $F(t') < t_0$ can be treated analogously) $F(t') > t_2$, thus $t_2 \in \mathrm{im}(F)$, because the continuous image of the connected $\{\mathrm{pt}\} \times I$ is connected (and $t_1 = F(0), F(t') \in \mathrm{im}(F)$). But this is a contradiction to $p \circ F$ being ϵ-close to f, since $p(t_2) > p(t_1) + 2\epsilon$ by the choice of ϵ, while all points in the image of f, are at $\leq p(t_1)$.

Surjectivity can be seen as follows: The continuous image $\mathrm{im}(p)$ of the compact $[0, 1]$ is compact, thus closed. It is also open, because given $t_0 = p(\tau) \in \mathrm{im}(p)$, let $\epsilon := \min(1 - t_0, t_0)$ (or $\epsilon = 1/2$ in the case $t_0 = 0$ or 1, where we only have to check one direction), $\epsilon/2$-lift $f_\pm(t) := t_0 \pm \epsilon t$ with starting-point τ. Then the connected, continuous images $\mathrm{im}(p \circ F_\pm)$ must contain $(t_0 - \epsilon/2, t_0 + \epsilon/2)$.

(ii) If p is an exact fibration, it is a AF, thus monotonic and surjective. It can not "linger" over a single point, for the reasons given in the previous example, thus is strictly monotonic.

"(a) $\Leftarrow$": We start by showing (ii): If p is strictly monotonic, it is injective, thus, being surjective by hypothesis, bijective. It is a well-known fact, that a continuous bijection from a compact space to a Hausdorff-space is a homeomorphism (thus a fibration).

(i) Given $\epsilon > 0$ (the minimum of $\epsilon : [0, 1] \to (0, \infty)$), subdivide $[0, 1]$ into $N > \frac{1}{2\epsilon}$ pieces (of equal length). By surjectivity, each $\frac{i}{N}$ has a preimage t_i. Define p' by linearly interpolating $t_i \mapsto \frac{i}{N}$. By monotonicity, $t \in [t_i, t_{i+1}] \Rightarrow p(t) \in [\frac{i}{N}, \frac{i+1}{N}]$, which is of course also true for p'. Thus p and p' are ϵ-close. p' is strictly monotonic and surjective by construction, thus a fibration by (ii). Produce an exact lift w. r. t. p'. This is automatically ϵ-close to p, because p' is.

"(b)": The only modifications for the statement about proper maps are: For $\Rightarrow$, part (i), the closedness of the image, and thus surjectivity, follows from proper maps of locally compact Hausdorff spaces being closed. The $\Leftarrow$-direction of (ii) does not require compactness for p to be a homeomorphism because p is proper (thus closed, see above, thus p^{-1} is continuous, see for example the proof of [Bre97, Thm. 7.8 (p.19)]). Finally, the mesh of the subdivision for (i) can be made "adaptive", as ϵ is not just a number anymore, but this does not considerably change the argument. For example, for $n \in \mathbb{N}$ use the previous argument on $[n, n + 1]$ inductively. □

Such "monotonic directions" exist more generally in the sense that, given an AF $p : E \to B = B_0 \times \mathbb{R}$ the translation-isotopy $(x, t) \mapsto (x, t+1)$ can be ϵ-lifted (starting at the identity). This differs substantially from smooth (gradient) flows, not only by introduction of ϵ, but also by their homotopic (as opposed to geometric) nature: One obtains (thin) h-cobordisms rather than products in the total-space E. See also the discussion on [Hug85, "Approximation Theorem" (p. 168)] below.

The following observation and example are from [HTW90, (p. 45)]: For path-connected[6] B, any fibration must be surjective (if a point were not in the image, try lifting a path from one that is, to obtain a contradiction). This must not necessarily

[6] If B has multiple components, each one is either contained in the image or disjoint from the image.

be true for (non-proper) approximate fibrations: They only have *dense* image in general.

Example 1.2.4: *Let $E := (0,1)$ and $B := [0,1]$, define $p : E \to B$ as the inclusion.*

Then p is not *a fibration, as indicated above.*

But p is *an AF, because any lift can be "clipped" to $[\epsilon, 1-\epsilon]$: Just compose with the projection to $[\epsilon, 1-\epsilon]$ (interpreted as a quotient of the interval), and "lift" by the* id. *The result is ϵ-close. (We may assume $\epsilon > 0$ is a number, as opposed to a map, because B is compact.)*

Evidently this p is not a proper map, so not a MAF.

On locally compact Hausdorff spaces (thus on manifolds) proper maps are closed, so density of the image – which can be characterized as closure(im(p)) $= B$ – implies surjectivity for MAFs to connected spaces.

We continue by examining some of the properties AFs *do* have. Notably the definition is local (as stated above, that is, on ANRs) as is shown for example in [HTW90, Cor. 12.14 (p. 51)].

Lemma 1.2.5: *If $U \subset B$ is open, then*

(i) if p is an AF, then so is the restriction $p|_U : p^{-1}(U) \to U$.

(ii) if p is a MAF, then so is the restriction $p|_U : p^{-1}(U) \to U$.

(iii) if $\mathcal{U}$ is an open cover of B, and $\forall U \in \mathcal{U} : p|_U$ is a MAF, then f is a MAF.

Proof: Part (i) follows immediately from [HTW90, Cor. 12.14 (p. 51)], (ii) is a consequence of (i): The continuous preimage $p^{-1}(U)$ is also open, and open subsets of manifolds are manifolds, further properness is local (compactness is an absolute property: A compact $K \subset U$ is compact in B). (iii) is shown later as Cor. 1.4.20–1 as a consequence of Hughes' cylinder-theorem and locality of MHSS. □

Apparently AFs are neither locally products, in the sense of local trivialization (cf. Example 1.2.2), nor do they have well-defined (up to homotopy-equivalence) fibers as fibrations do (cf. Example 1.2.4). However, in high dimensions, they are close to both: By Hughes [Hug85, "Approximation Theorem" (p. 168)] there are, over simplices, ϵ-close approximations by products[7], which can, for example, be used to obtain certain lifting results – or in the context of stratified spaces (as defined later) extension results – e. g. for isotopies [Hug+00, Cor. 2.4 (p. 6)] (see also [HR96, Thm. 17.4 (p. 201)]) or h-cobordisms [Hug+00, Cor. 9.7 (p. 57)].

More important for the following: The mapping-cylinder of an AF has a "homotopy-link" that is a fibration – so in turn has a well-defined fiber-homotopy-type

[7]See the reference for more details, we do not need this result in the following.

[Qui04, Lemma 6.5.2 (p. 54)]. According to Quinn [Qui88a, footnote on p. 452], the holink-construction was introduced by Fadell [Fad65] to "homotopically locate the normal bundle of a submanifold", which is a first indication of the relation between AF-structures and normal structure as advertised above.

Definition 1.2.6: *Given $Y \subset X$, the* homotopy link $\operatorname{holink}(X, Y) \to Y$ *is the path-space (with compact-open topology)*

$$\operatorname{holink}(X, Y) := \left\{ \gamma : I \to X \middle| \gamma(0) \in Y \text{ and } \gamma((0,1]) \subset X - Y \right\} \xrightarrow{ev_0} Y$$

where $ev_0(\gamma) := \gamma(0)$ is evaluation at zero.

There are a number of modifications of this construction, giving smallness conditions on the length of paths [Qui88a, "holink$^\delta$" (p. 453)], induced stratifications [Hug99c, "holink$_S$" (p. 5)], or local versions ("$\operatorname{holink}(X, x)$"; see [Hug99b, §5 (p. 317ff)] or [Fri09, (p. 2172)]). Precise definitions will be given where they are used. The connection to AFs through holinks of mapping-cylinders indicated above, is (reformulated from the original source for the case of only two strata; the basic argument is – to my knowledge – due to Hughes [Hug99a]):

Lemma 1.2.7: *See e. g. [Qui04, Lemma 6.5.2 (p. 54)]: Given a continuous map $p : E \to B$, where E, B are metric ANR,*

$$p \text{ is AF} \quad \Leftrightarrow \quad \operatorname{holink}(\operatorname{cyl}(p), B) \to B \text{ is a fibration}$$

Homotopy-links turn out to be useful, even if no mapping-cylinder neighborhoods exist. In fact they can be used used in Quinn's [Qui82a] "controlled end-theory", to study the question of existence of such neighborhoods. Before returning to this point of view in Section 1.9 ("Mapping-Cylinder Neighborhoods"), we will first introduce "homotopy stratified spaces" (in Section 1.4 ("Quinn Spaces / (M)HSS")) – also known as Quinn-spaces – because these will also fix a notion of "tameness", that is easily seen to be a necessary requirement for the existence of cylinder neighborhoods.

The implications of the existence of those cylinder-neighborhoods on the problem of extending transversality-constructions "upwards" from the lower stratum will then be studied in Section 1.6 ("Stratified Transversality").

Finally, we note some basic properties of (M)AFs:

Lemma 1.2.8: *Properties of (M)AFs*

(i) Given two AF $p : E \to B$ and $p' : E' \to B'$, with B and B' compact, the product $p \times p' : E \times E' \to B \times B'$ is a AF.

(ii) Given two MAF $p : E \to B$ and $p' : E' \to B'$, the product $p \times p' : E \times E' \to B \times B'$ is a MAF.

(iii) Given two (M)AF $p : E \to B$ and $q : B \to X$, the composition $q \circ p : E \to X$ is a (M)AF.

(iv) Product-projections $\pi_X : X \times Y \to X$ are AF, if X and Y are ANR and MAF if X and Y are manifolds and Y is compact.

Proof: "(i)": Given $\epsilon : B \times B' \to (0, \infty)$ and a lifting problem of the form

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{F_0} & E \times E' \\ {\scriptstyle \mathrm{incl}}\downarrow & & \downarrow{\scriptstyle p \times p'} \\ A \times I & \xrightarrow{f} & B \times B' \end{array}$$

There are lifting problems given by $\pi_E F_0$ and $\pi_B f$ for p (and similarly for p'). Using $\epsilon_B(b) := \min(\epsilon(\{b\} \times B'))$, there are thus ϵ_B-lifts g and (an $\epsilon_{B'}$-lift) g' with $g|_{A \times \{0\}} = \pi_E F_0$ and $\mathrm{dist}_B(pg, \pi_B f) < \epsilon_B$ (and similarly for g'). Then $(g, g') : A \times I \to E \times E'$ is an ϵ-solution to the lifting problem, if we are for example using the maximum-metric to measure distances on the product.

"(ii)": MAFs are proper, so we do not need the compactness-hypothesis: For every point p in the manifold $B \times B'$ there is chart φ_p centered at p. The open images of open unit-discs $\varphi_p(\mathring{D}_1)$ thus cover $B \times B'$, and by Lemma 1.2.5 (iii), it is enough to show, that the restrictions to $\varphi_p(\mathring{D}_1)$ are MAFs. But the *closed* unit-disks $\varphi_p(\bar{D}_1)$ are compact, so we can choose ϵ_B and $\epsilon_{B'}$ as before (on these compact sets, thus on their subsets $\varphi_p(\mathring{D}_1)$. The proof of part (i) shows, that these restrictions are indeed MAFs.

"(iii)": Given $\epsilon : X \to (0, \infty)$ and a lifting problem

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{F_0} & E \\ & & \downarrow{\scriptstyle p} \\ {\scriptstyle \mathrm{incl}}\Big\downarrow & & B \\ & & \downarrow{\scriptstyle q} \\ A \times I & \xrightarrow{f} & X \end{array}$$

By continuity of q, there is $\delta : B \to (0, \infty)$, such that whenever $\mathrm{dist}_B(b, b') < \delta$ holds, then $\mathrm{dist}_X(q(b), q(b')) < {}^{\epsilon}\!/_2$. Because q is an AF, there is a ${}^{\epsilon}\!/_2$-lift $\tilde{F}$ of f starting at pF_0. Because p is an AF, there is a δ-lift F of $\tilde{F}$ starting at F_0. By choice of δ, qpF is ${}^{\epsilon}\!/_2$-close to $q\tilde{F}$, which is also ${}^{\epsilon}\!/_2$-close to f.

"(iv)": Product-projections are fibrations, thus (M)AF, if the spaces are suitable and if the map is proper (in the case of a MAF). □

This concludes, for the moment, the discussion of (M)AFs.

1.3 Excursion: Stratified Spaces

In this section, basic concepts of “stratified spaces” are briefly reviewed. Further, some commonly used classes of stratified spaces are recalled. The reader familiar with the world of stratified spaces may safely skip this excursion, potentially referring back if necessary. More detailed treatments can be found for example in [Whi65; Tho69; Mat; GM80; Kin85; Qui88a; Hug99c; Ban07; Kre10]. Often there is no “fully canonical” choice for how to define certain details, if substantial to the argument, such differences are pointed out in the place where they occur throughout this thesis.

We restrict ourselves to spaces with a finite number of strata. This removes most differences between the “filtered” and the “stratified” point of view (see below). Also, boundaries often require some special treatment, see for example Def. 1.4.13.

Definition 1.3.1: *“Filtered point of view”:*

A filtered space *is a topological space X together with a series of subspaces*

$$X = X^n \supset X^{n-1} \supset \ldots \supset X^0 \supset X^{-1} = \emptyset$$

If the filtration is by closed subspaces*, i. e. if the X^i are closed (in X), we call X^i the* i-skeleton *and $X_i := X^i - X^{i-1}$ the* i-stratum*.*

X is manifold-stratified *if it is filtered by closed subspaces, and the strata X_i are manifolds. In this case, it is filtered* by dimension *if for all non-empty strata* $\dim(X_i) = i$.

A manifold-stratified X is orientable (oriented)*, if its top-stratum is orientable (oriented) in the manifold-sense.*

There is also the alternate point of view, starting from a partition into “strata” indexed by a (finite) set $\mathcal{I}$, that are locally closed, and satisfy a frontier condition:

Definition 1.3.2: *“Stratified point of view”:*

A partition of a topological space X into locally closed, disjoint strata $(X_{(i)})_{i\in\mathcal{I}}$ *satisfies the* frontier condition *if*

$$X_{(i)} \cap \mathrm{closure}(X_{(j)}) \neq \emptyset \quad \Rightarrow \quad X_{(i)} \subset \mathrm{closure}(X_{(j)})$$

In this case, there is a partial order on $\mathcal{I}$ (thus on skeleta and strata) given by

$$i \leq j \quad :\Leftrightarrow \quad X_{(i)} \subset \mathrm{closure}(X_{(j)})$$

Extend this partial order to a total order $\leq^*$ (on a finite set this does not require the axiom of choice), then skeleta *can be defined as*

$$X^{(i)} := \mathrm{closure}(X_{(i)}) \cup_{j \leq^* i} X^{(j)}$$

Remark 1.3.3: We will usually only require the frontier condition component-wise, i. e. *components* of $X_{(i)}$ that intersect closure$(X_{(j)})$ must be contained entirely in closure$(X_{(j)})$. This provides a partial order on components of strata rather than the index-set $\mathcal{I}$, but does not otherwise differ much from the “global” version above. See also Rmk. 1.4.20.

The “extension of the order” part simply means: If there are i, j such that $X_{(i)} \cap \operatorname{closure}(X_{(j)}) = \emptyset$ and $\operatorname{closure}(X_{(i)}) \cap X_{(j)} = \emptyset$, i. e. if the *partial* order “$\leq$” is not total – because things get added in disjoint places – define $X^{(i)} := \operatorname{closure} X_{(i)}$ but $X^{(j)} := \operatorname{closure} X_{(j)} \cup X^{(i)}$ and so on.

For our requirements these viewpoints coincide (see e. g. [Hug99c, §2 (p. 3f)]):

Lemma 1.3.4: *A finite filtration by closed subspaces X^i defines a partition into locally closed, disjoint strata $X_{(i)} := X_i$, that satisfy the frontier condition (component-wise).*

A partition into disjoint strata $X_{(i)}$ that satisfies the frontier condition, defines a filtration by closed subspaces $X^i := X^{(i)}$.

In these cases the notions of “strata” and “skeleta” coincide.

Given a filtered / stratified space, there are some evident constructions

Definition 1.3.5: *Given X filtered by closed subspaces (thus stratified), V a manifold, define:*

(i) The induced filtration / stratification *on a subspace $A \subset X$ is the filtration by closed subspaces given by $A^i := X^i \cap A$.*

(ii) The product filtration / stratification *on a $X \times V$ is the filtration by closed subspaces given by $(X \times V)^i := X^i \times V$. A shift in the indexing may be used to possibly maintain the property of being filtered by dimension.*

(iii) The stratified cone *$c(X)$ is filtered by $c(X)^{i+1} = \pi(X^i \times I)$ (where $\pi : X \times I \to c(X)$ is the quotient-projection) and $c(X)^0 = \{v\}$. Here v is the cone-point / vertex.*

(iv) The mapping cylinder $\operatorname{cyl}(f)$ *of a map $f : M \to B$ between manifolds M and B, is naturally filtered as a space with two strata by* $\operatorname{cyl}(f) \supset B$. *For similar constructions where domain / target are themselves stratified spaces, see for example [CS95].*

A more meaningful distinction of filtered / stratified can be made for mappings:

Definition 1.3.6: *Stratified and filtered maps*

(i) *A continuous map $f : X \to Y$ of stratified spaces X, Y (both indexed by $\mathcal{I}$) is* stratum-preserving *if $f(X_i) \subset Y_i$.*

(ii) *A continuous map $f : X \to Y$ of filtered spaces X, Y (both indexed by $\mathcal{I}$) is a* filtered map *if $f(X^i) \subset Y^i$.*

(iii) *A continuous map $f : X \to Y$ of stratified spaces X, Y is a* coarsening *if for all components of strata Y_j the preimage $f^{-1}(Y_j)$ is a union of components of strata of X.*

(iv) *A continuous map $f : X \to Y$ of filtered spaces X, Y (both indexed by $\mathcal{I}$) is a* isomorphism / stratified homeomorphism *if it is filtered and a homeomorphism.*

While clearly a stratum-preserving map is filtered, a filtered map may send larger strata to smaller ones in general. It is of course true however, that stratified homeomorphisms are indeed stratified maps, and their inverses are again stratified homeomorphisms. Coarsenings appear for example in [Qui87].

Further we use the following (strong choice) for homotopy-equivalences:

Definition 1.3.7: *Given stratified spaces X, Y (both indexed by $\mathcal{I}$), define:*

(i) *A map $H : X \times I \to Y$ is a* stratified homotopy *if it is "stratum preserving along I" i. e. if $\forall x \in X : \gamma_x : I \to Y, t \mapsto H(x,t)$ maps to a single stratum: $\exists i : \mathrm{im}(\gamma_x) \subset Y_i$.*

(ii) *Two maps $f, g : X \to Y$ are* stratified homotopic $f \simeq_{\text{strat}} g$ *if there is a stratified homotopy from f to g (in the usual sense $H(-,0) = f$ and $H(-,1) = g$).*

(iii) *A continuous map $f : X \to Y$ of is a* stratum-preserving homotopy equivalence *if f is stratum-preserving, and there is a stratum-preserving map $f^{-1} : Y \to X$ such that $f \circ f^{-1} \simeq_{\text{strat}} \mathrm{id}_Y$ and $f^{-1} \circ f \simeq_{\text{strat}} \mathrm{id}_X$.*

(iv) *A subspace $A \subset X$ is a* stratum-preserving neighborhood deformation retract *if, giving A the induced stratification, there is a neighborhood $A \subset U \subset X$ where the inclusion $A \to U$ is a stratum-preserving homotopy-equivalence rel A (i. e. the "retraction" / inverse of i is rel A).*

Finally, returning to spaces, we give some examples for classes of stratified spaces. A basic geometric property that is often part of the definition, and in a certain sense replaces the notion of "charts" in a manifold setting, is being "locally conelike", as illustrated by the following class of manifold-stratified spaces:

Definition 1.3.8: *A manifold-stratified metric space X (filtered by dimension) is a* conelike-stratified (CS) set *in the sense of Siebenmann [Sie72], if every point $x \in X$, say in the i-stratum $x \in X_i$, has a neighborhood of the form (i. e. stratified[8]-homeomorphic to) $\mathbb{R}^i \times c^\circ(L_x)$, for some compact space L_x (of finite formal dimension) filtered by closed subsets, which intersects X_i in $\mathbb{R}^i \times \{v\}$. The spaces L_x are called the "links".*

The dimension of X is the largest n such that $X^n \neq X^{n-1}$. This agrees with the manifold-dimension of the top-stratum if X is filtered by dimension.

The links are in general *neither* manifold-stratified *nor* unique (up to stratified-homeomorphism). Nevertheless, these are often well-behaved. For example CS sets have an underlying "intrinsic" stratification as CS set, that only depends on topology (*not* on the stratification) [Han78, Thm. 2.4 (p. 169)].

Example 1.3.9: *[Sie72, Examples 1.3 No. 4 (p. 128)]: Let (M, N) be a manifold-pair, where $N \neq \emptyset$ is locally flat in M. Then the stratification with strata $M - N$ and N is CS.*

More consistency of links is provided for example by pseudomanifolds (see e. g. [GM80] or [Ban07, Def. 4.1.1 (p. 72)][9])

Definition 1.3.10: *A* topological stratified pseudomanifold *X of dimension n is a CS set of dimension n, such that either*

(i) X is a countable discrete point-set or

(ii) there exist choices of links L_x such that L_x is a topological pseudomanifold of dimension $n - i - 1$, where $x \in X_i$.

Further, require there is no codimension 1 stratum $X^{n-1} = X^{n-2}$, and the top-stratum X_n is dense in X.

A piecewise linear (pl)-stratified pseudomanifold *is additionally a pl-space, such that the filtration is compatible with the pl-structure and the restriction of the pl-structure turns strata into pl-manifolds (see [Ban07, Def. 4.1.2 (p. 73)]).*

This definition makes sense (inductively), because the dimension of L_x (if manifold-stratified ones exist, as required for being themselves CS) is automatically lower than the dimension of X and the "start of induction" is provided by the discrete point-sets.

Finally we give some examples:

[8] $\mathbb{R}^i \times c^\circ(L_x) \subset \mathbb{R}^i \times c(L_x)$ is given the topology induced from the product-topology with the stratified cone, cf. Def. 1.3.5.

[9] Replacing "metric" by "paracompact Hausdorff" seems to be sufficient as a point-set-topological requirement on the underlying space.

Example 1.3.11: *See [Ban07, Examples 4.1.3 (p. 82)]: Many algebraic varieties (for example pure-dimensional complex ones) are topological pseudomanifolds. These algebraic varieties are even Whitney-stratified [Whi65], which is a smooth notion of stratification.*

Example 1.3.12: *Orbit filtrations of topological group-actions provide an important class of examples. Depending on properties of the groups and actions, the technical details tend to get quite involved rather quickly however, see for example [Qui88a, p. 442]: Smooth actions yield smooth orbit-filtrations. Already the pl-case gets much harder (in general) however, as here the orbit-filtrations need not be compatibly pl themselves.*

The next section will introduce a weaker notion of "homotopy stratified spaces" due to Quinn [Qui88a], where neighborhoods are conelike only up to stratified homotopy-equivalence.

1.4 Quinn Spaces / (M)HSS

Here, we introduce the "homotopy stratified sets" [Qui88a], which are used throughout the remainder of this thesis. They are actually doing double-duty: We use them in the "expected" way – as a quite general class of stratified spaces – but also as a setup to study the end of the top-stratum controlled over the bottom-stratum (see Section 1.9 ("Mapping-Cylinder Neighborhoods") below). For controlled ends of this form there is – in high dimensions – a correspondence of algebra and geometry [Qui82a] very similar to what is found for "classical" ends and h-cobordisms. (see Section 1.7 ("Excursion: Controlled Topology") and Section 1.9 ("Mapping-Cylinder Neighborhoods"))

We start by the definition of "tameness".

Definition 1.4.1: *[Qui88a, p. 452]: A subset $B \subset X$ is* (forward) tame *in X iff it is a nearly stratum-preserving neighborhood deformation retract (NDR).*

This means, if there is a neighborhood N of B in X and a deformation-retraction $R : N \times I \to B$, that is, it is "$\operatorname{rel} B$": $\forall b \in B : R(b, t) = b$, it starts at the identity: $\forall x \in N : R(x, 0) = x$ and it ends at a retraction R_1 to B: $\forall x \in N : R(x, 1) \in B$, while being stratum preserving "until the last moment", that is, $\forall x \in X - B, t < 1 : R(x, t) \in X - B$.

Tameness is a local property (see e. g. the proof below for an indication why) so for example:

Example 1.4.2: *Let $B \subset M$ be a locally flat closed submanifold. Then $B \subset M$ is tame.*

Proof: Local flatness (for $\partial B = \emptyset$) means, that each $b \in B$ has a neighborhood U in M, where $(U \cap M, U \cap B) \cong (\mathbb{R}^{\dim(M)}, \mathbb{R}^{\dim(B)})$ (as a pair). By compactness of B, we can cover B by the unit-balls (under this identification) of a finite number of such neighborhoods $U_1, \ldots, U_n$, and corresponding homeomorphisms h_i.

Clearly, there is a deformation R of $(\mathbb{R}^{\dim(M)}, \mathbb{R}^{\dim(B)})$ that nearly strictly retracts the unit-ball to the plane $\mathbb{R}^{\dim(B)}$ (rel this plane), and is the identity outside the two-ball. Define $R^i : M \times I \to M$ as $h_i^{-1} \circ R \circ h_i$ on U_i and as the identity on $M - U_i$ (these are continuous, because R is already the identity outside the two-ball).

Define $R^M : M \times I \to M$ as $R^M_t(x) = R^0_t \circ R^1_t \circ \ldots \circ R^n_t$. Then R^M retracts a neighborhood of B nearly strictly to B: Set $U :=$ interior $\big((R^M_1)^{-1}(B)\big)$, where clearly R^M retracts U to B, and $(R^n_1)^{-1}(B) \supset h_n^{-1}(\text{unit-ball}) \cup B$, and $(R^{n-1}_1)^{-1}\big((R^n_1)^{-1}(B)\big) \supset h_{n-1}^{-1}(\text{unit-ball}) \cup V_n^{n-1} \cup B$, with V_n^{n-1} an open neighborhood of $h_n^{-1}(\text{unit-ball}) \cap B$ (by R^{n-1}_1 being continuous and rel B). Finally we thus end up with $(R^M_1)^{-1}(B)$ containing an open neighborhood of $(h_1^{-1}(\text{unit-ball}) \cap B) \cup \ldots \cup (h_n^{-1}(\text{unit-ball}) \cap B) = B$, where the last equality holds by the choice of the cover $U_1, \ldots, U_n$. So the interior U is open, and still contains a neighborhood of B. Further, R^M is nearly-strict and rel B, because the R^i are. □

Note however, that there are a number of other commonly used variations of "tameness" (see also Appendix A ("Ends in MHSS")). For this reason, the requirement given above is sometimes referred to as "forward tameness" – as opposed to "reverse tameness", where things can instead be pulled *away* from B [Qui79]. The term "tame" is sometimes used in the sense of "forward *and* reverse tame". This is not quite as bad as it may sound, because for MHSS (see below) by [Qui88a, Prop. 2.14 (p. 466)] these notions coincide (by a homological characterization of tameness as a consequence of Poincaré-duality on *manifold*-strata).

Tameness is obviously necessary for the existence of cylinder-neighborhoods:

Example 1.4.3: *Let $f : E \to B$ be a continuous map. Then $B \subset \mathrm{cyl}(f)$, embedded as the "base" (see Def. 1.1.3), is tame.*

A nearly stratum-preserving (neighborhood) deformation retraction is obtained by pushing along the cylinder-coordinate:

$$R : \mathrm{cyl}(f) \times I \to \mathrm{cyl}(f), \quad \big((x, s), t\big) \mapsto \big(x, s(1-t)\big)$$

This is combined with the normal-structure provided by certain homotopy-links (holinks were defined in Def. 1.2.6, where they were understood to characterize AFs as those maps having holink-*fibrations*):

Definition 1.4.4: *[Qui88a, Def. 3.1 (p. 30)]: A* manifold homotopy stratified set (MHSS)*, with at most two strata, is a separated, metric space $X \supset B$ filtered by closed subsets, s. t. (using $M := X - B$ the "upper stratum"):*

(i) M and B are topological manifolds

(ii) $B \subset X$ is tame

(iii) the homotopy-link $\operatorname{holink}(X, B) \to B$ *is a fibration*

(iv) local homotopy-links are compactly dominated (see below)

The dimension *of X is the manifold-dimension of the* top-stratum *M, i. e.* $\dim(X) := \dim(M)$.

If only (ii) and (iii) hold, we call X an HSS.

The last condition (to be explained in a moment) is *not part of Quinn's original work* [Qui88a], it was only introduced later by [Hug99b] to obtain a better geometric description of neighborhoods (see Example 1.4.5 below). Spaces with more than two strata are defined in Chapter 3 ("Multiple Strata").

The "local homotopy-link" $\operatorname{holink}(X, b)$ (see e. g. [Hug99b, §5 (p. 317ff)] or [Fri09, (p. 2172)]) at a point $b \in B$ is – in the case of at most two strata – just the fiber of the holink(-fibration). Compact domination, means, there is a homotopy from the identity on the local homotopy-link into a compact subspace (i. e. a continuous $R : \operatorname{holink}(X, b) \times I \to \operatorname{holink}(X, b)$, with $R_0 = \mathrm{id}$ and $\operatorname{im}(R_1) \subset K$ where $K \subset \operatorname{holink}(X, b)$ is compact).

Example 1.4.5: *"Teardrops" (see e. g. [Hug+00]): Let $p : E \to B \times \mathbb{R}$ be a MAF. As a set, define $E \cup_p B := E \sqcup B$ (the disjoint union), and give $E \cup_p B$ the minimal topology, such that $E \to E \cup_p B$ (the set-inclusion) is an open (topological) inclusion, and such that the "collapse-map"*

$$c : E \cup_p B \to B \times (-\infty, \infty], \quad x \mapsto \begin{cases} p(x) & \text{if } x \in E \\ (x, \infty) & \text{if } x \in B \end{cases}$$

is continuous. Then $E \cup_p B$ is a MHSS by [Hug99b, 'Main Theorem' (p. 306)].

These "teardrops" (or very closely related: "local approximate tubular neighborhoods", cf. [Fri09]) seem to be the natural geometric neighborhood-structure for high-dimensional MHSS (with compact singular set), in the sense, that they always exist, *if X also satisfies condition (iv) of Def. 1.4.4* [Hug02, Thm. 7.1 (p. 887)] as long as the dimensions of non-minimal[10] strata are ≥ 5. The case of two strata was

[10]whenever two strata "meet" in the sense, that their closures intersect, then at least one of them must be of dimension ≥ 5, see Def. 1.4.23.

already contained in [Hug+00]. This type of neighborhoods does *not*[11] suffice to give geometric transversality constructions however.

A teardrop[12] is a mapping-cylinder if and only if it is possible to write p as a product $p = p' \times \mathrm{id}_{\mathbb{R}}$. Why this may fail to be the case is explained in Section 1.9 ("Mapping-Cylinder Neighborhoods").

We will *not* be much concerned with this weaker notion of neighborhood, for our purposes require the stronger notion of cylinder neighborhoods (which imply (iv), the high-dimensionality in Hughes' Existence Theorem [Hug02, Thm. 7.1 (p. 887)] only affects the other direction). So we may think of these as forming a hierarchy

"spaces we are interested in / mapping-cylinders"

$\cap$ (obstructed by q_0 in high dimensions)

"spaces with teardrop-neighborhoods"

$\cap$ ("=" in high dimensions)

MHSS with (iv)

$\cap$

Quinn's original definition

See Section 1.9 ("Mapping-Cylinder Neighborhoods") for details on the obstruction q_0. The only place where we are (implicitly) using the weaker neighborhood structures is in Section 1.12 ("Witt-Condition and Signature of MHSS") through Friedman's [Fri09] treatment of Poincaré-duality in intersection-homology.

Another potentially interesting question is the following:

Example 1.4.6: *Given a "tame" (in the sense of knot-theory, i. e. framed by a 2-disc) knot $S^1 \times D^2 \subset S^3$, then $X = S^3 \supset S^1 \times \{0\}$ is stratified as MHSS.*

This raises the interesting question, do "wild" (non-tame) knots $S^1 \subset S^3$, that stratify S^3 as a MHSS exist?

Interestingly, there is some indication, that some geometric structure could exist near any knot that is stratified as a MHSS, but the "full" backwards-direction of the initial statement seems unlikely:

At least the uncontrolled end-theorem (see Section 1.8 ("Excursion: End-Theory") and Section 1.9 ("Mapping-Cylinder Neighborhoods")) works (again) in 3 *dimensions [FQ90, "So if the Poincaré conjecture is true then the strong form of the end theorem is true for 3-manifolds." (p. 216)], since problems arising from low dimensionality often are similar in the controlled case (see*

[11] There is a "backwards" direction in the argument of [CV99], see Section 1.6 ("Stratified Transversality").

[12] Generally, the teardrop and quotient topology may *not* agree on the cylinder, but Example 1.4.5 defined teardrops explicitly for MAFs which are proper and between manifolds (thus locally compact Hausdorff spaces), where they *do* agree [Hug99a, p. 130].

e. g. [Qui82b] and Rmk. 1.9.4), at least a cylinder-neighborhood of a MAF might exist (assuming obstructions vanish, in the uncontrolled case, by the restrictions of possible fundamental groups of oriented surfaces, the obstruction-groups are automatically trivial over the point, so for the controlled case, the arguments of Lemma 3.1.11 should apply).

But what is known about classification of such MAF (e. g. [HTW90, Example 1.5 (p. 6)]), indicates, that the resulting mapping-cylinder might well be different from $S^1 \times D^2$.

We continue by giving some properties of MHSS: Typically local cone-like structure plays an important role for the study of stratified spaces (see Section 1.3 ("Excursion: Stratified Spaces")). MHSS may *not* have locally conelike structure up to homeomorphism. But they *do* have locally conelike structure *up to stratified homotopy-equivalence.*

Theorem 1.4.7: *[Qui87, Thm. 2 (p. 239) and "Converse" (p. 240)]*[13]

Suppose X is an HSS, then for $U \subset B$ open and contractible, there is a stratified space L and a map $U \times c^\circ(L) \to X$, which is a stratum-preserving (the left-hand-side is stratified in the obvious way: The bottom stratum is $U \times \{\text{cone-point}\}$, the higher strata are $U \times (0,1) \times L_i$ for the strata L_i of L) homotopy-equivalence near U.

If X has locally contractible skeleta and given $b \in B$ there is a neighborhood $b \in U \subset B$, a stratified space L and a map $U \times c^\circ(L) \to (X - B) \cup U$, which is a stratum-preserving homotopy-equivalence near U, then X is an HSS.

Corollary 1.4.7–1: *Given a (sep. metric) manifold-stratified space $X \supset B$ with two strata and B connected, then:*

X is an MHSS, if and only if for any manifold-chart $\mathbb{R}^j \cong U \subset B$, there is a space L such that there is a map $\mathbb{R}^j \times c^\circ(L) \to X$, which is a stratum-preserving homotopy-equivalence near U.

Proof: Manifold-charts are of course open and contractible, so the "⇒"-direction is clear. To see "⇐", note that by the cone-like property, local contractibility of skeleta follows from local contractibility of strata, which are manifolds by hypothesis. □

This is particularly useful for Friedman's [Fri09] treatment of Poincaré-duality.[14] Also, this implies that MHSS are a quite large class of stratified spaces:

[13] Note, that while the statement given in the reference about *existence* of L is correct, the concrete construction of L is not (in general), as remarked (also by Quinn) in [Qui88a, footnote 2 (p. 445)]. The (weakened) phrasing given here is correct. The wording was also changed to fit the nomenclature of [Qui88a] used here, and the statement was simplified for the two-stratum case.

[14] Up to complications due to intersection-homology with *closed* support not being invariant under stratified homotopy-equivalence in general. See Section 1.12 ("Witt-Condition and Signature of MHSS").

Example 1.4.8: *A (separable) CS set in the sense of Siebenmann [Sie72] (see Def. 1.3.8 in the preceding excursion) is an MHSS. The local homotopy-links are homotopy-equivalent to the links of the CS set.*

Many algebraic varieties (the main concern are non-"pure-dimensional" parts) are topological pseudomanifolds (see e. g. [Ban07, Examples 4.1.3 (p. 82)]; this was mentioned before in Example 1.3.11) and topological pseudomanifolds are CS, thus are MHSS.

These claims remain true in the case of more than two strata, treated in Chapter 3 ("Multiple Strata"), However, while here local homotopy-links are the fibers of homotopy-links, some care has to be taken, if more than two strata are involved, as the two concepts do not *coincide anymore.*

Proof: CS sets have manifold strata, and manifolds are locally contractible. Further, CS sets are locally cone-like up to *homeomorphism*, thus clearly also up to homotopy-equivalence. So by Thm. 1.4.7 they are MHSS.

Homotopy-links, and thus their fibers, are determined locally (via ϵ-holinks), so we may restrict our attention to local trivializations, and thus to $\operatorname{holink}(\mathbb{R}^i \times c^\circ(L), \mathbb{R}^i \times \{v\})$. The fiber of the holink-evaluation (evaluation at zero), over $0 \in \mathbb{R}^i$, is

$$\left\{ \quad \gamma : I \to \mathbb{R}^i \times c^\circ(L) \quad \middle| \quad \gamma(0) = (0, v) \text{ and } \gamma((0,1]) \subset \mathbb{R}^i \times (L \times (0,1)) \quad \right\}$$

which can be deformed into

$$F_0 := \left\{ \quad \gamma : I \to \mathbb{R}^i \times c^\circ(L) \quad \middle| \quad \gamma(t) = (0, [x,t]) \quad \right\}$$

by a deformation via paths mapping zero to $(0, v)$ and $t \neq 0$ to outside the vertex, given as

$$R_s(\gamma)(t) := \begin{cases} \Big((1-s)\gamma_{\mathbb{R}^i}(t), [\gamma_L(1), (1-s)\gamma_{\mathrm{cyl}}(t) + st]\Big) & \text{if } t \geq 1-s \\ \Big((1-s)\gamma_{\mathbb{R}^i}(t), [\gamma_L(\tfrac{t}{1-s}), (1-s)\gamma_{\mathrm{cyl}}(t) + st]\Big) & \text{if } t < 1-s \end{cases}$$

where $\gamma(t) =: \Big(\gamma_{\mathbb{R}^i}(t), [\gamma_L(t), \gamma_{\mathrm{cyl}}(t)]\Big) \in \mathbb{R}^i \times c(L)$. Now $L \to F_0, x, \mapsto \gamma(t) = (0, [x,t])$ is a homeomorphism (with inverse $\pi_L \mathrm{ev}_1(\gamma) = \pi_L \gamma(1)$), thus a homotopy-equivalence. So local homotopy-links are homotopy-equivalent to the links L of the CS set.

The definition of pseudomanifolds Def. 1.3.10 was actually formulated as pseudomanifolds being special CS sets (with "consistent" links). □

By Example 1.3.9, this includes flat submanifolds, and, more precisely it holds:

Example 1.4.9: *[Hug99a, Thm 6.1 (p. 140) and Cor. 6.2 (p. 141)]: Let (M^m, N^n) be a manifold-pair, with $m := \dim(M) \geq 6$, $n := \dim(N) \geq 5$. Then*

$$N \text{ is locally flat in } M \quad \Leftrightarrow \quad \begin{cases} M \supset N \text{ is a MHSS, and} \\ \text{the holink-fiber is homotopy-equivalent to } S^{m-n-1} \end{cases}$$

In this (either) case N has a cylinder-neighborhood (of an MAF) in M.

Proof: There are two minor modifications compared to [Hug99a, Cor. 6.2 (p. 141)]: For once, we additionally claim, that if $M \subset N$ is a MHSS with holink-fiber homotopy-equivalent to S^{m-n-1}, then a cylinder-neighborhood exists. This is a consequence of Quinn's end-theorem (Thm. 1.9.3) and $\pi_1(S^{m-n-1}) = 0$ (or $\mathbb{Z}$ if $m = n + 2$) and $\mathrm{Wh}(\mathbb{Z}^k) = 0$ by the Bass–Heller–Swan theorem (see Example 2.2.4). $\dim(M) \geq 6$ is sufficiently large (by hypothesis), so the end-theorem applies. The cylinder is the cylinder of an MAF by Hughes' cylinder theorem (Thm. 1.4.19).

The other difference is, that the reference, given a cylinder-neighborhood of N homeomorphic to $\mathrm{cyl}(p)$ for some p, requires, that the homotopy-fiber of p be $\simeq S^{m-n-1}$, while we required this for the holink-fiber instead. These agree however: For example use [Hug99a, Thm. 3.1 (p. 131)], see also the proof of [Hug99a, Thm 6.1 (p. 140)]. □

Further, being MHSS is a local property in the sense that

Lemma 1.4.10: *If X is an MHSS and $U \subset X$ is open, then U, with the induced filtration (i. e. $U^i := X^i \cap U$), is an MHSS.*

Given a filtered space X and an open cover $\mathcal{U}$, s. t. $\forall U \in \mathcal{U}$, the induced filtration makes U into an MHSS, then X is an MHSS.

Proof: This follows from Thm. 1.4.7, but can actually also be seen "directly" from the definition: Forward tameness is a local property on metric spaces by [Qui88a, Lemma 2.5 (p. 455)]. The homotopy link is determined locally by [Qui88a, Lemma 2.4 (p. 454)] via its "δ-holink". This also implies that local homotopy-links are determined locally. Clearly, the strata being manifolds is also a local condition. □

For closed subspaces, the situation is slightly more complicated. To be able to discuss this case in some detail, we first introduce "p-NDRs":

Definition 1.4.11: *See e. g. [Qui88a, p. 469]: Given a continuous map $p : E \to B$, a subset $A \subset B$ is a p-NDR if there is a deformation retraction r of a neighborhood of A in B, that is covered by a retraction R of a neighborhood of $p^{-1}(A) \subset E$, i. e. $p \circ R(x, t) = r(p(x), t)$.*

This relates, for HSS, stratum-wise NDRs to stratified NDRs:

Lemma 1.4.12: *[Qui88a, Prop. 3.5 (p. 472)]: Suppose X is a HSS, and $Y \subset X$ is closed. Give Y the induced stratification. Then*

$$Y \text{ is a stratified NDR} \quad \Leftrightarrow \quad \begin{cases} Y \text{ is a HSS and} \\ Y \cap M \subset M \text{ a NDR and} \\ Y \cap B \subset B \text{ is a } p\text{-NDR} \end{cases}$$

Proof: [Qui88a, Prop. 3.5 (p. 472)] applies, because the definition of "homotopy-transverse" [Qui88a, Def. 3.4 (p. 469)] is satisfied by Y: It can be seen by the discussion after that definition, that "homotopy-transverse" in this case means $\mathrm{incl} : \mathrm{holink}(Y, Y \cap B) \to \mathrm{holink}(M \cup (Y \cap B), Y \cap B)$ needs to be a fibered ho-equivalence over $Y \cap B$. By locality of holinks (ϵ-holinks) this clearly is the case if $Y \cap B \subset B$ is a p-NDR.

Further, the p-NDR requirement also implies that $Y \cap B$ is a NDR in B. □

Another important property is an observation of Quinn, that (in high dimensions) homotopy-collared boundaries are already collared. To discuss this, we want to first define, what we mean by a space with boundary:

Definition 1.4.13: *A* MHSS with boundary (∂-MHSS) of dimension n *is a pair* $(X, \partial X)$ *where* $X - \partial X$ *and* ∂X *are* MHSS *of dimension* n *and* $n-1$ *respectively*[15]*, with at most two strata such that* $(M, \partial M)$ *and* $(B, \partial B)$ *are manifolds-with-boundary, where* ∂M *is the top stratum of* ∂X *and* M *is the union of the top-strata of* $X - \partial X$ *and* ∂X*, and similarly for* B*.*

Further, we require that ∂X *has a stratified collar-neighborhood in* X*. This means, there is a map* $\varphi : \partial X \times [0, \infty) \to X$*, which is a stratified homeomorphism to its image, and is the identity on* $\partial X \times \{0\}$*.*

Remark 1.4.14: Thus $\partial X = \partial M \cup \partial B$, and further, collaring implies $\partial X \subset X$ is automatically closed, because $\partial X \times \{0\} \subset \partial X \times [0, \infty)$ is.

Quinn [Qui88a] uses a "homotopy-transverse" condition [Qui88a, Def. 3.4 (p. 472)] – see also proof of Lemma 1.4.12 – on the inclusion of the boundary, instead of collaring [Qui88a, Def. 5.1 (p. 491)].

But as indicated above, there is a strong connection of these "homotopy-collars" and actual collars: A "homotopy-transverse" boundary is collared, up to issues in dimension 4 and below.

[15] The dimension of any non-empty boundary is automatically $n-1$, if the top-stratum is dense (by boundary-collaring), so this requirement is typically fulfilled automatically.

Theorem 1.4.15: *[Qui88a, Thm. 1.2 (p. 444)]: Given a "MHSS with weak boundary" (X, B) in the following sense: Let X be a compact space filtered by closed subsets, such that its strata are manifolds with boundary. Let B be the subspace $B := \cup_i \partial(X_i)$ given by the union of boundaries of strata with the induced stratification, i. e. with strata $B_i = \partial(X_{i+1})$. Further, assume, that the top-stratum of B is $(n-1)$-dimensional, where $n := \dim(X)$, and its closure in X is B, i. e. $\mathrm{closure}_X(B_{n-1}) = B$. Furthermore $X - B$ and B are MHSS.*

Then homotopy-collaring of this "weak boundary" implies geometric (homeomorphism) collaring away from dimension 4: Suppose the 4-skeleton $B^4 \subset X^4$ has a stratum preserving collar (this 4-skeleton may be empty). Then, if $B \subset X$ is a stratum-preserving NDR, $B \subset X$ has a stratum preserving collar.

Relative version [Qui88a, 'Lemma' (p. 492)]: Suppose the k-skeleton $B^k \subset X^k$ has a stratum preserving collar b_k and $k \geq 4$. Then, if $B^{k+1} \subset X^{k+1}$ is a stratum-preserving NDR, B^{k+1} has a stratum preserving collar b_{k+1} in X^{k+1}, extending b_k, i. e. $b_{k+1}|_{X^k} = b_k$.

Proof: By [Qui88a, Prop. 3.5 (p. 472)] the formulation of [Qui88a, Thm. 1.2 (p. 444)] applies. Compare this also to the proof of Lemma 1.4.12.

The relative version [Qui88a, 'Lemma' (p. 492)] is used in the proof of the absolute version [Qui88a, as 'Corollary' (p. 494) in §5.3 (p. 492–495)] via an induction over skeleta. This 'Lemma' uses closed subsets $Y \times [0, \infty), \partial X \subset X$, with $\partial X \cap Y \times [0, \infty) = Y \times \{0\}$ and $X - (Y \times [0, \infty))$ a manifold which intersects ∂X in its boundary.

We identify these as follows: The collar $U^k := \mathrm{im}(b_k)$ is an open neighborhood of $B^k \subset X^k$, pick an open neighborhood U^{k+1} of B^{k+1} in X^{k+1} which has a stratified deformation retraction to B^{k+1} (which exists by B^{k+1} being a stratified NDR) such that $U \cap X^k = \mathrm{im}(b_k)$. As an open subset of the MHSS X^{k+1}, this U^{k+1} is again a MHSS. Define $X = U^{k+1}$, $\partial X = B^{k+1}$ and $Y = B^k$. Note, that indeed $B^k \times [0, \infty) \cong \mathrm{im}(b_k) \subset U^{k+1}$ is closed in U^{k+1}. Further $B^{k+1} \cap \mathrm{im}(b_k) = B^k \times \{0\}$ and $U^{k+1} - (B^k \times [0, \infty))$ is a manifold (indeed, this is the $(k+1)$-stratum of the MHSS U^{k+1}), which intersects B^{k+1} in its boundary.

Next we check the hypotheses of the 'Lemma' [Qui88a, p. 492]: $X = U^{k+1}$ is locally compact (it is an open subset of X^{k+1}), $Y \times [0, \infty) = \mathrm{im}(b_k)$ is tame (because it is the k-skeleton of the MHSS U^{k+1}, thus tame see Lemma 3.1.8 part (ii) for the multi-stratum case, in the case of at most two meeting strata (see Def. 1.4.23), this is a minimal stratum, thus tame by definition of MHSS), $Y = B^k$ is a finite dimensional ANR (because it is a MHSS with a finite number of strata which are finite dimensional manifolds), $\mathrm{holink}(X, Y \times [0, \infty)) = \mathrm{holink}(U^{k+1}, U^k)$ is fiber homotopy equivalent to $\mathrm{holink}(\partial X, Y \times \{0\}) = \mathrm{holink}(B^{k+1}, B^k)$ (because B^{k+1} is a stratified NDR, see construction of U^{k+1}), and $\mathrm{holink}(\partial X, Y \times \{0\}) = \mathrm{holink}(B^{k+1}, B^k)$ is a fibration (a stratified system of fibrations in the multi-stratum case, see Chapter 3 ("Multiple Strata"); because B^{k+1} is a MHSS). Further $\dim(X - Y \times [0, \infty)) = \dim(U^{k+1} - U^k) = k + 1 \geq 4 + 1 = 5$ (by hypothesis).

In this case, the 'Lemma' [Qui88a, p. 492] states: Then the open collar $Y \times [0, \infty) = \mathrm{im}(b_k)$ extends to an open collar of $\partial X = B^{k+1}$. Here, it "extends to" means, there is a homeomorphism of a neighborhood of $(B^{k+1} - B^k) \subset (U^{k+1} - U^k)$ to $(B^{k+1} - B^k) \times [0, \infty)$ (a choice of manifold boundary-collar), which extends by the identity on $Y \times [0, \infty) \cong \mathrm{im}(b_k)$ to a stratified one of $B^{k+1} \subset U^{k+1}$ [Qui88a, see p. 494 in the proof of 'Lemma' (p. 493f)], which is an extension in the claimed sense. $\square$

Note, that this is the same as answering the question, if the mapping-cylinder $\mathring{\mathrm{cyl}}(\mathrm{incl} : \partial X \to X)$ is an MHSS (hence the label as "transverse", see [Qui88a, discussion on the bottom of p. 472]. It is thus not surprising that Quinn's proof uses controlled topology techniques (a controlled h-cobordism-theorem; see Section 1.7 ("Excursion: Controlled Topology"), specifically the discussion after Thm. 1.7.7) similar to what will be discussed in Section 1.9 ("Mapping-Cylinder Neighborhoods") to study the existence of mapping-cylinder neighborhoods of the bottom stratum.

This may be formulated in simpler terms, by "absorbing" the boundary into the stratification (it could be recovered [Qui88a, p. 491]). However, that would break the "at most two strata" condition we are using for simplicity. Details on this are provided in Section 3.2 ("Absorbing Boundary"). We note, that for the cases we are most interested in ("pure-dimensional" without codimension 1 stratum), there is no ambiguity in the choice of ∂X:

Definition 1.4.16: *We say a MHSS X of dimension n is* proper*, if its top-stratum X_n is dense in X and X does not have a codimension 1 stratum.*

A MHSS with boundary $(X, \partial X)$ is proper *of dimension n, if its interior $X - \partial X$ and boundary ∂X are proper.*

Example 1.4.17: *Topological stratified pseudomanifolds are proper MHSS in this sense.*[16]

Witt-spaces (see Section 1.12 ("Witt-Condition and Signature of MHSS")) do not have a codimension 1 stratum (Example 1.12.8), so that properness becomes a question about density of the top-stratum only.

Disallowing codimension 1 strata ensures there is a well-defined boundary of the top-stratum and combined with the density of the top-stratum fixes a unique (canonical) choice for the boundary of the entire space.

Lemma 1.4.18: *Let $(X, \partial X)$ be a proper MHSS with boundary. Then $\partial X = \mathrm{closure}(\partial X_n)$.*

[16] Originally formulated on pl-spaces (which can always be stratified as CS sets [Sie72, Examples 1.3 (p. 128)]), this was considered the defining property of a "pseudomanifold".

Proof: The boundary ∂X is closed (see Rmk. 1.4.14), and $\partial X_n \subset \partial X$ (because, by definition, $\partial X = \cup_i \partial X_i$, see again Rmk. 1.4.14), so the closure of ∂X_n in X is just the closure in ∂X. Since ∂X is proper, with dense top-stratum ∂X_n, we find $\text{closure}(\partial X_n) = \text{closure}_{\partial X}(\partial X_n) = \partial X$. □

Now, we return to the connection to AFs. The following was already (partially) indicated in Lemma 1.2.7.

Theorem 1.4.19: *[Hug99a, Thm. 5.11 (p. 140)] (simplified for two strata): Let $p : V \to B$ be a proper map between manifolds.*

Then p is MAF if and only if *the mapping-cylinder* $\mathring{\text{cyl}}(p)$ *with the natural partition (Def. 1.3.5) is a MHSS.*

Remark 1.4.20: We replaced AF and HSS by MAF and MHSS, which is possible, because p is assumed to be proper and V, B are manifolds. Further, Hughes assumes strata of B (i. e. B itself in the case of two strata) are path connected. This is required to satisfy the frontier-condition (see [Hug99a, proof of Cor. 5.8 (p. 137)]): Obviously, if a stratum is not path connected, if one of its components intersects the closure of another stratum, there is no reason for its other components to be contained in the same closure, e. g. think of $B = \{\text{pt}, \text{pt}'\}$, $p : V \to B, x \mapsto \text{pt}$, where certainly $\emptyset \neq B \cap \text{closure}(V \times (0, 1]) \subset \text{cyl}(p)$, but $\text{pt}' \notin \text{closure}(V \times (0, 1])$ is a disjoint point. See also Rmk. 1.3.3.

For "proper" (Def. 1.4.16) MHSS with only two strata, this cannot happen: By density of the top-stratum, its closure already contains the entire (only) other stratum. More generally for spaces with more than two strata, we may content ourselves with a component-wise version of the frontier-condition (or a filtration by closed subsets) as we will not need a "global" version.

Corollary 1.4.20–1: *Part (iii) of Lemma 1.2.5: If $f : M \to B$ is a map, and $\mathcal{U}$ an open cover of B, such that $f|_U : f^{-1}(U) \to U$ is a MAF $\forall U \in \mathcal{U}$, then f is a MAF.*

Proof: By Thm. 1.4.19, $\text{cyl}(f|_U)$ are MHSS, thus, by locality of MHSS Lemma 1.4.10, $\text{cyl}(f)$ is a MHSS. Again by Thm. 1.4.19, f is a MAF. □

The "natural partition" in this case (of two strata) is the filtration $\text{cyl}(p) \supset B$. It still remains to answer the question of when cylinder-neighborhoods *do* exist. Section 1.9 ("Mapping-Cylinder Neighborhoods") will complement the present picture with regard to this problem.

But first, we briefly summarize some constructions using MHSS:

Lemma 1.4.21: *Let X be a MHSS (with at most two strata) without boundary, $A \subset X$ a closed stratified NDR. Further let $(Y, \partial Y)$ be a MHSS with boundary and at most two strata. Then:*

(i) $(X \times Y, X \times \partial Y)$ with the product-stratification is a MHSS with boundary (possibly with more than two strata).

(ii) Given a manifold with boundary $(V, \partial V)$, the product $(X \times V, X \times \partial V)$ is a MHSS with boundary and at most two strata.

(iii) The subspace A with the induced stratification is a HSS.

(iv) The fibers of the homotopy-links of A are homotopy-equivalent to those of X.

(v) The fibers of the homotopy-links of ∂Y are homotopy-equivalent to those of Y

(vi) An orientation of (the top-stratum of) Y, canonically induces an orientation of (the top-stratum of) ∂Y.

Proof: "(i)": For a more detailed treatment of MHSS with more than two strata, see Chapter 3 ("Multiple Strata"). The product-statement for MAFs (given in Lemma 1.2.8), together with Thm. 1.4.19, shows (i). (There are also more direct ways to prove this.)

"(ii)": By (i) we only need to count strata of the product. These are generally up to *number of strata of* $X \times$ *number of strata of* $Y \leq 2 \times 1$.

"(iii)": This is part of Lemma 1.4.12.

"(iv)": By Lemma 1.4.12, $A \cap B$ is a p-NDR, thus fibers are homotopy-equivalent.

"(v)": The boundary $\partial Y \subset Y$ is collared, thus a stratified NDR, so (iv) applies.

"(vi)": This is clear from the definition of boundaries as stratum-wise boundaries Def. 1.4.13 and the manifold-case. □

The relevancy of the "stability" of holink-fibers (that they are the same for ∂X and X etc.) will become more apparent in Chapter 2 ("Bordism Constructions"), where this property will "automatically" ensure that the excision-isomorphism does respect the class of "allowable" spaces, if being allowable depends on holink-fibers (and codimension) only.

In order to obtain a gluing result, some care should be taken with regard to the restriction to spaces with at most two strata. Until we introduce spaces with an arbitrary (finite) number of strata in Chapter 3 ("Multiple Strata"), we use the following technical modification (see Def. 2.1.4):

Definition 1.4.22: *A* MHSS with at most two meeting strata *is a separated, metric space X filtered by closed subsets, such that the non-top (i. e.* $\dim(B_i) < \dim(X)$*) strata B_i are closed and disjoint ($i \neq j \Rightarrow B_i \cap B_j = \emptyset$). Further, for all of the B_i, and the top-stratum M, we require the same properties as in Def. 1.4.4:*

(i) M and all B_i are topological manifolds
(ii) $B_i \subset X$ is tame for all i
(iii) the homotopy-links $\operatorname{holink}(X, B_i) \to B_i$ *are fibrations*
(iv) local homotopy-links are compactly dominated

In this case, we call the B_i minimal strata*.*

Definition 1.4.23: *More generally (also for spaces with more than two meeting strata, see Chapter 3 ("Multiple Strata")), we call a stratum B_i* minimal*, if all other strata S, whose closure intersects the closure of B_i, are of higher dimension, i. e.* $\operatorname{closure}(S) \cap \operatorname{closure}(B_i) \neq \emptyset \Rightarrow \dim(S) > \dim(B_i)$.

Remark 1.4.24: Both the previous Def. 1.4.4 and the present Def. 1.4.22 are local. Locality, combined with normality (of the metric X) and disjointness of the B_i, means that spaces satisfying Def. 1.4.22 locally look like spaces with exactly two strata (or manifolds), and the above results apply automatically if they are local. Only boundary-collaring (Thm. 1.4.15) may not be obvious, however, the statement of this result given above, did not actually put a limit on the number of strata, so still applies here.

Of course, MHSS with at most two meeting strata as defined here, are MHSS in the sense of [Qui88a] and Chapter 3 ("Multiple Strata"). In this sense, spaces with at most two meeting strata, are spaces with one top-stratum and otherwise only minimal strata.

The reason for the appearance of this definition, is, that gluing spaces with two strata, of the same (top-stratum) dimension, but with minimal strata of different dimensions far away from the "gluing interface", would not technically yield a two-stratum space. One might either both drop the requirement that the space be stratified by dimension and weaken strata being manifolds to components of strata being manifolds, or use a definition allowing for disjoint additional strata as in Def. 1.4.22.

Combining the definition of boundaries (Def. 1.4.13) with locality of being MHSS (Lemma 1.4.10) and the constructions above (Lemma 1.4.21), one easily obtains the following gluing-result:

Lemma 1.4.25: *Given $(X, \partial X)$ and $(Y, \partial Y)$ both MHSS, of (the same) dimension n, with boundary, at most two meeting strata and with subspaces $X_0, X_1 \subset \partial X$ such that*

(i) X_0, X_1 with the induced stratification are MHSS of dimension $n-1$ with boundary.
(ii) $X_0 \cup X_1 = \partial X$.
(iii) $X_0 \cap X_1 = \partial X_0 = \partial X_1$.

and $Y_0, Y_1 \subset \partial Y$ *satisfying the same conditions. Further suppose there is a stratified homeomorphism* $h : X_0 \to Y_0$*.*

Then the adjoint pair $(X \cup_h Y, X_1 \cup_{h|_{\partial X_1}} Y_1)$*, with the stratification given by adjoining stratum-wise, is an MHSS, of dimension* n*, with boundary and at most two meeting strata.*

Proof: We show the lemma for the case $X_1, Y_1 = \emptyset$ first. Being a MHSS is a local property (Lemma 1.4.10), thus it suffices to show the statement on the elements of an open cover. Let c_X and c_Y be stratified boundary-collars, and define the open cover

$$\mathcal{U} = \Big\{ \quad X - \partial X, \quad Y - \partial Y, \quad \operatorname{im}(c_X) \cup_{h|_{\partial X_1}} \operatorname{im}(c_Y) \quad \Big\}$$

where $X - \partial X$ and $Y - \partial Y$ are MHSS by hypothesis. All identifications by $h : \partial X \to \partial Y$ are in the collars at $c_X(\partial X \times \{0\})$ thus, using $c_Y^{-1} \circ h \circ c_X|_{\partial X \times \{0\}} = h \times 0$:

$$\begin{aligned} \operatorname{im}(c_X) \cup_h \operatorname{im}(c_Y) &\cong_{\text{strat}} \Big(\partial X \times [0, \infty)\Big) \cup_{h \times 0} \Big(\partial Y \times [0, \infty)\Big) \\ &\cong_{\text{strat}} \Big(\partial X \times [0, \infty)\Big) \cup_{\operatorname{id} \times 0} \Big(h^{-1}(\partial Y) \times [0, \infty)\Big) \\ &\cong_{\text{strat}} \partial X \times \mathbb{R} \end{aligned}$$

This is stratified by the product-stratification, because the collars were stratified. Since ∂X is a MHSS by hypothesis, this is a MHSS with at most two (meeting) strata by Lemma 1.4.21 (ii) (disjointness of different minimal strata B_i is obviously preserved). Minimal strata B_i of far away from (outside the boundary-collar of) X_0 also remain disjoint from of those of Y far away from Y_0.

In the case that X_1 or $Y_1 \neq \emptyset$, the previous case applies to the boundary part: $X_1 \cup_{h|_{\partial X_1}} Y_1$ is a MHSS. Note, that near the "corners" $T := \partial X_1 = \partial X_0$, the collaring in X is three-fold: (X_0, T) is a MHSS with boundary by hypothesis, thus there are collars $c_X^0 : T \times [0, \infty) \to X_0$, and $c_X^1 : T \times [0, \infty) \to X_1$, and additionally $(X, X_0 \cup X_1)$ is a MHSS with boundary by hypothesis, thus there is a collar $c_X : (X_0 \cup X_1) \times [0, \infty) \to X$.

Putting the first two "back to back" we obtain a bi-collar $c_X^0 \cup c_X^1 : T \times \mathbb{R} \to X_0 \cup X_1$, which is the identity at $T \times \{0\}$. Restricting the third one, c_X, to the image of this bi-collar we get

$$c_X| \circ c_X^0 \cup c_X^1 : T \times \mathbb{R} \times [0, \infty) \to X$$

a stratified homeomorphism to its image and the identity at $T \times \{0\} \times \{0\}$. So we can refine the gluing-procedure in the boundary, to produce a "double-product" stratification in T, and glue away from T (thus away from X_1, Y_1) as before. $\square$

The ability to glue spaces is of course quite important for bordism-theories – for example it will ensure transitivity of the bordism-equivalence-relation. One should note, that collaredness of the boundary is a very important ingredient in establishing this property. This is why rather strong transversality / cutting results (ensuring bi-collaredness) must be available for later constructions.

1.5 Excursion: Manifold Transversality

Here, we will first discuss "transversality" in a manifold-setting, to then, in Section 1.6 ("Stratified Transversality"), discuss the an approach based on a theorem by Connolly and Vajiac [CV99], where mapping-cylinder neighborhoods of MAFs are used to give transversality constructions on MHSS.

Typically "transversality" (or "general position") refer to generalizations of the observation, that, in the euclidean n-space $\mathbb{R}^n$, given two planes $\mathbb{R}^m \cong A \subset \mathbb{R}^n$ and $\mathbb{R}^k \cong B \subset \mathbb{R}^n$ of dimensions m and k, after an arbitrarily small move of either one, one may assume that they intersect in an (at most) $((m+k)-n)$-dimensional plane $A \cap B$. (With the convention that $\dim(\emptyset) = -\infty$.) There are (at least) two paths along which such statements can be formalized:

On the one hand, there are statements about "isotoping subspaces into general position" – referred to as embedded transversality below, see Thm. 1.5.11 – which is basically a direct formalization of the above procedure. E. g. for the manifold-case one would obtain something like "Given a manifold M^n and two submanifolds U^m, V^k, there is a small isotopy of either one, moving it such that $\dim(U \cap V) \leq (m+k)-n$ in a way compatible with (fixed in advance) normal-bundles" [KS77; Qui88b]. Some technical details for dealing with normal-bundles are given below, but only a much easier special case is needed in the remainder of the thesis.

On the other hand, there is the following well-known result from differential topology, sometimes referred to as "map transversality":

Definition 1.5.1: *"Smooth map-transversality": Given smooth manifolds M^m, N^n and a smooth map $f : M \to N$, and further a smooth submanifold $V^k \subset N$.*

If, for any $m \in M$ with $f(m) \in V$ the differential $df_m : T_mM \to T_{f(m)}N$ is such that its image $\mathrm{im}(df_m)$*, together with $T_{f(m)}V \subset T_{f(m)}N|_V$, spans $T_{f(m)}N = \mathrm{im}(df_m) + T_{f(m)}V$, then f is said to be* transverse *to V.*

In this case, the preimage $f^{-1}(V)$ is a smooth submanifold of dimension $(m+k)-n$, with normal-bundle the pullback of the normal bundle ${}^{TN|_V}/_{TV}$ of V in N.

Theorem 1.5.2: *"Smooth Transversality" (see e. g. [Bre97, Ch. II, Thm. 15.2 (p. 114)], the original idea of such triangulation-independent replacements of simplicial approximation[17] seems to come from [Tho54]):*

Given smooth manifolds M^m, N^n and a smooth map $f : M \to N$, and further a smooth submanifold $V^k \subset N$.

Then f can be homotoped (by a small homotopy) to a smooth map transverse to V.

[17] A simplicial map automatically has the transversality-property indicated above on interior points of top-dimensional simplices of the target. Since smooth manifolds can be triangulated [Whi40], there is an indirect approach using simplicial approximation instead. See discussion in the introduction of [Tho54].

So the theorem essentially says, this transversality-property can always be ensured by small "distortion" of f. This can be understood as a special case of moving subspaces into general position: The graph of f in $M \times N$ being transverse to $M \times V$ implies the result for the preimage (see Cor. 1.5.11–1). Outside of the smooth category, this also requires fixed normal-bundles. For example Milnor's [Mil64] micro-bundles may be used [KS77].

One should note however, that the difficulty in giving a topological transversality result goes well beyond normal-bundle problems: While locally (on charts or handles) there is an obvious smooth structure (the standard one on the $\mathbb{R}^n$), thus transversality can be realized locally (via smooth transversality), this has to be done "relatively", so that the obtained structure glues (in an inductive proof) to the previously obtained one on other (overlapping) charts. This requires, for example, some sort of "product-structure-theorem", based on the h-cobordism-theorem [Sma62], but also techniques from [Nov65] and the torus trick of [Kir69] and a topological handle-body theory [KS77].

It is the second case, "map transversality", that we will be mostly interested in. More specifically we study, how maps into $\mathbb{R}$ can be made transverse to 0. To some degree more general cases can be reconstructed by repeated application of such a result, however, the normal-bundles (pulled back from a point) will end up being trivial (see also Section 5.4 ("Singular Transversality")). The main advantage of this specific case is, that oriented (two-sided) normal-(micro-)bundles of co-dimension 1 manifolds are just "bi-collars" (in a sense to be explained below), i. e. they always have strong geometric structure, even in the topological category (cf. boundary collaring [Bro62] / Thm. 1.1.2, see also [FQ90, p. 150] and MHSS boundary collaring [Qui88a, Thm. 1.2 (p. 444)] / Thm. 1.4.15). Hence the definition of transversality given at the beginning of the next section is essentially formulated as a statement about the existence of bi-collars.

A formal statement of a (general) transversality-theorem for topological manifolds requires some microbundle-theory [Mil64], but the results that we will need later, can be reformulated in terms of bi-collars. So the reader not (currently) interested in technical details or a general theorem may safely restrict their reading to Def. 1.5.5, the last paragraph of Def. 1.5.7, Rmk. 1.5.8 and the corollaries Cor. 1.5.9–1 and Cor. 1.5.9–2.

We briefly define microbundles and normal microbundles as well as their morphisms:

Definition 1.5.3: *[Mil64, 'Definition' (p. 54)] Microbundles:*

A microbundle ξ *of* fiber-dimension n *consists of a* base space B, *a* total space E *a* zero-section $i : B \to E$ *and a* projection $p : E \to B$ *(all topological spaces and continuous maps), satisfying* $p \circ i = \mathrm{id}_B$ *and which is locally-trivial:*

"Local triviality": For any $b \in B$*, there are open neighborhoods* $U \subset B$ *of* b *and* $V \subset E$ *of* $i(b)$*, with* $i(U) \subset V$ *and* $p(V) \subset U$*, such that there is a homeomorphism* $h : V \xrightarrow{\sim} U \times \mathbb{R}^n$*, compatible with the obvious zero-section and projection on* $U \times \mathbb{R}^n$*:* $h \circ i|_U(x) = (x, 0)$ *and* $\pi_U \circ h = p|_V$*.*

[KS77, p. 84] Normal Microbundles:

A normal microbundle *of a closed $Y \subset X$ is microbundle ν, with base-space Y, total-space an open neighborhood of Y in X and zero-section the identity on Y.*

A microbundle map *$f : E(\xi) \to E(\xi')$ is a continuous map, such that the restriction to a fiber, maps to a single fiber in the target, and is an open topological embedding there, formally: Given $b \in B(\xi)$, the restriction $f| : p(\xi)^{-1}(b) \to p(\xi')^{-1}(f(b))$ is well-defined and an open embedding.*

Example 1.5.4: *See [Mil64, Examples 1 to 3 (p. 55)]:*

(1) There are trivial microbundles $B \xrightarrow{\times 0} B \times \mathbb{R}^n \xrightarrow{\pi_B} B$.

(2) Vector-bundles have an underlying microbundle.

(3) Topological manifolds have a tangent microbundle $M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$. For (paracompact) smooth manifolds, this agrees (up to isomorphism) with the underlying microbundle of the smooth tangent-bundle [Mil64, Thm. 2.2 (p. 56)].

We will typically restrict our attention to the codimension 1 case, where bi-collars may be used instead.

Definition 1.5.5: *Given a codimension 1 submanifold $N \subset M - \partial M$, then a* bi-collar *of N in M is a map $c : N \times (-\epsilon, \epsilon) \to M$, which is a homeomorphism to its image and restricts to the identity on $N \times \{0\}$, i. e. it maps $(n, 0) \mapsto n$. Here, for non-compact N, ϵ may be chosen as a function $\epsilon : N \to (0, \infty)$, in which case $N \times (-\epsilon, \epsilon) := \{(n, t) \in N \times \mathbb{R} \,|\, |t| < \epsilon(n)\}$.*

We call N bi-collared in *M, if a bi-collar exists.*

Given a continuous map $f : M \to M'$ and $N \subset M$, $N' \subset M'$ bi-collared by c, c', we call f compatible with c and c', *if $\mathrm{im}(fc) \subset \mathrm{im}(c')$ and $(c')^{-1} fc = f' \times \mathrm{id}_{(-\epsilon,\epsilon)}$ is a product.*

For $U \subset N$ or $V \subset M$, the restrictions will be abbreviated as $c|_U := c|_{U \times (-\epsilon,\epsilon)}$ and $c|_V := c|_{(V \cap N) \times (-\epsilon,\epsilon)}$.

We will not explicitly distinguish the notion of a germ of bi-collars from plain bi-collars, so for a bi-collar c' restricting to c (or extending c) we will typically allow c' to be "thinner" (i. e. defined for ϵ' potentially smaller than ϵ where c was defined).

Example 1.5.6: *A locally flat codimension 1 submanifold $N \subset M$ with M closed, is bi-collared if it is two-sided, i. e. if there are subsets M_+ and M_-, such that $M = M_+ \cup M_-$ and $N = M_+ \cap M_-$.*

Proof: Both $(M_\pm, N)$ are manifolds with boundary, because N is locally flat. By [Bro62] (see Thm. 1.1.2), the boundary N of $M_\pm$ is collared by $c_\pm$, glue these back-to-back to obtain

$$c : N \times \mathbb{R} \to M, \quad (x,t) \mapsto \begin{cases} c_+(x,t) & \text{if } t \geq 0 \\ c_-(x,t) & \text{if } t \leq 0 \end{cases}$$

which is a bi-collar of N in M. □

Using either microbundles or bi-collars, (map-)transversality can be defined as:

Definition 1.5.7: *[KS77, p. 84]: Let M^n be a (topological) n-manifold, (X, Y) a topological pair, $Y \subset X$ closed and with a normal microbundle ξ of fiber-dimension k.*

A continuous map $f : M \to X$ is transverse to ξ at ν, *if $L := f^{-1}(Y) \subset M$ is a (topological) submanifold of* $\dim(L) = n - k$ *and ν is a normal microbundle of fiber-dimension k of L in M such that $f| : E(\nu) \to E(\xi)$ is a microbundle map.*

Given $U \subset M$ open, f is transverse to ξ on U at ν, *if $f|_U$ is transverse to ξ at ν.*

We call f transverse to ξ, *if a microbundle ν exists, such that f is transverse to ξ at ν.*

Further, given $U \subset M$ open, we will call a map $f : M \to \mathbb{R}$ (bi-collar) transverse to 0 on U at c, *if $f^{-1}(\{0\}) \cap U$ is a submanifold of U bi-collared by c, such that f is compatible with c and the trivial bi-collar of $\{0\} \subset \mathbb{R}$, i. e. $fc(x,t) = t$.*

Similarly, we will call a map $f : M \to J$ to a real interval J (bi-collar) transverse to $t_0 \in \mathrm{interior}(J)$ on U at c, if $f^{-1}(\{t_0\}) \cap U$ is a submanifold and bi-collared by c, such that $fc(x,t) = t_0 + t$. Typically $J = [-1, 1]$ and $t_0 = 0$ in our applications.

Remark 1.5.8: The subsequent statements are typically given for a map $f : M \to X$, which will be "made transverse on D rel C" for closed subsets $C, D \subset M$. The formal statement may be easier to read with some explanation on what the intent of the inputs C and D is. The result is usually a transverse map $f_\perp \simeq f$, with the homotopy respecting C, D in this (explained below) sense.

The "relative part" C: It is assumed, that f already *is* transverse near (on a neighborhood of) C. Typically, applying a result (theorem, corollary, ...) to obtain a transverse $f_\perp \simeq f$, will then leave f unchanged ($f = f_\perp$, with trivial homotopy) near (on a potentially smaller neighborhood of) C. So the existing transverse preimage $f^{-1}(Y)$ and its normal-bundle ν_0 near C are really *extended* into the remainder of M.

The "active part" D: The map f will be changed to a transverse map $f_\perp$ near D. A very useful, although somewhat trivial, choice is $D = M$, which will produce a map $f_\perp$ transverse everywhere on M. A potential reason for choosing $D \neq M$ is, that f will be *changed* to a transverse map near D, but it will, indeed, *only* be changed near D. So whatever properties f might have away from D, they are maintained by $f_\perp = f$ away from D.

One may assume $C \subset D$: If $C \not\subset D$, then the result is transverse near $C \cup D$ (because it was transverse near $C - D$ and remains unchanged there). However, some caution should be taken, when working rel boundary (see Cor. 1.5.9–2).

Clearly, the case of mapping to a real interval J is essentially equivalent to mapping to $\mathbb{R}$, by application of transversality-theorems to suitable subsets.

The following is the famous transversality theorem proofed (in high[18] dimensions) by Kirby and Siebenmann:

Theorem 1.5.9: *[KS77, Thm. 1.1 (p. 85)] Map Transversality Theorem (on D rel C):*

Let M^n be a (topological) n-manifold, (X, Y) a topological pair, $Y \subset X$ closed and with a normal microbundle ξ of fiber-dimension k.

Further suppose the following dimensional restrictions (these can be removed completely [Qui88b], see Cor. 1.5.11–1): $n \neq 4$ and $n-k \neq 4$ and either $\partial M \subset C$ or both $(n-1) \neq 4$ and $(n-1) - k \neq 4$.

Let $C, D \subset M$ be closed subsets with open neighborhoods $U, V \subset M$.

Given a continuous map $f : M \to X$, transverse to ξ on U at a (given) microbundle ν_0, then there are open neighborhoods[19] U', V', W' of $C, D, M - V$, a microbundle ν and a homotopy $f \simeq f_\perp \operatorname{rel} U' \cup W'$, such that $f_\perp$ is transverse to ξ on $U' \cup V'$ at ν where $\nu|_U = \nu_0$.

Furthermore, this can be done by an arbitrarily small homotopy: If X is a metric space (with metric d), and $\epsilon : M \to (0, \infty)$ continuous, then U', V', W', ν, $f_\perp$ and the homotopy $H : M \times I \to X$ of $H_0 = f$ to $H_1 = f_\perp$ can be chosen such that H is ϵ-small in the sense of $d(H(x, 0), H(x, t)) < \epsilon(x)$ for all $x \in M$ and all $t \in I$.

This can be reformulated in terms of bi-collar transversality to 0:

Corollary 1.5.9–1: *Let M^n be a (topological) n-manifold, and $C, D \subset M$ closed subsets with open neighborhoods $U, V \subset M$.*

Given a continuous map $f : M \to \mathbb{R}$, transverse to 0 on U at c, then there are open neighborhoods U', V', W' of $C, D, M - V$ and $f \simeq f_\perp \operatorname{rel} U' \cup W'$ such that $f_\perp$ is transverse to 0 on $U' \cup V'$ at c' with $c'|_{U'} = c|_{U'}$.

[18] In the dimensions below ≤ 3, smooth and topological category agree, so the result was known to hold there. (Co-)Dimension 4 was supplemented later, see [Qui88b].

[19] Obtaining these, as stated here, i. e. with the same ones for the different parts of the result, requires intersecting the (unnamed) ones of the reference.

Again, the entire construction can be made small to obtain an ϵ-homotopy. One may replace $\mathbb{R}$ by a real interval J and 0 by $t_0 \in \operatorname{interior}(J)$.

Proof: By definition (Def. 1.5.7), f being transverse to 0 on U at c, means, $L_0 := f^{-1}(\{0\}) \cap U$ is a manifold, compatibly bi-collared by $c : L_0 \times (-\epsilon, \epsilon) \to U$. This c defines a normal microbundle ν_0 of $L_0 \subset U$ by $L_0 \xrightarrow{\text{incl}} \operatorname{im}(c) \xrightarrow{\text{incl}\, \pi_{L_0} c^{-1}} L_0$.

The bi-collar being compatible, means $fc(x,t) = t$, so f is a microbundle map from ν to the trivial normal microbundle ξ of $\{0\} \subset \mathbb{R}$: This requires that for $y \in L_0$, the fiber $F_y := (\text{incl}\, \pi_{L_0} c^{-1})^{-1}(\{y\})$ is embedded into the fiber $\mathbb{R}$ over $f(y) = 0$. The fiber is $F_y = c(\pi_{L_0}^{-1}(\text{incl}^{-1}(\{y\}))) = c(\{y\} \times (-\epsilon, \epsilon))$ so by compatibility of the bi-collar, $f| : F_y = c(\{y\} \times (-\epsilon, \epsilon)) \to \mathbb{R}, c(y,t) \mapsto fc(x,t) = t$ is the standard-embedding $(-\epsilon, \epsilon) \subset \mathbb{R}$.

So Thm. 1.5.9 applies to f, with C, D and U, V and ν_0 to obtain $f_\perp \simeq f \operatorname{rel} U' \cup W'$ transverse to ξ at ν on $U' \cup V'$, where $\nu|_{U'} = \nu_0|_{U'}$. We construct, what will become the inverse of the bi-collar c' using $L := f_\perp^{-1}(\{0\})$ as

$$\alpha : E_\nu \to L \times \mathbb{R}, \quad x \mapsto (p_\nu(x), f_\perp(x)).$$

Note, that $c^{-1}(x) = (p_{\nu_0}(x), f(x))$, by compatibility with c (i. e. by $fc(x,t) = t$) and by the definition of ν_0. So by $\nu|_{U'} = \nu_0|_{U'}$ and $f|_{U'} = f_\perp|_{U'}$, on U', α agrees with c^{-1} on U'.

Transversality of $f_\perp$ to ξ at ν, means by definition, that $f_\perp$ embeds fibers of p_ν into $\mathbb{R}$. This makes α injective: If $\alpha(x) = \alpha(x')$, then $p_\nu(x) = p_\nu(x')$, so x and x' are in the same fiber of p_ν, which gets embedded into $\mathbb{R}$ by $f_\perp$. Since also $f_\perp(x) = f_\perp(x')$, it thus follows $x = x'$.

Clearly α is continuous, and one-to-one onto its image. It is also proper: Given a compact $K \times [-N, N] \subset L \times \mathbb{R}$ (any compact subset of $L \times \mathbb{R}$ is a closed subset of one of this form, so its preimage is a closed subset of the preimage of the form $\alpha^{-1}(K \times [-N, N])$ and it suffices to show, that these are compact) cover K by finitely many trivialization-charts h_i of the microbundle ν, where we may assume, these trivializations are additionally over manifolds-charts $h_i : E_\nu \to \mathbb{R}^{\dim(L)} \times \mathbb{R}^k$ (where k is the fiber-dimension of ν), s. t. the images $D_i := h_i^{-1}(\bar{D}_1) \times \{0\} \subset L$ of closed unit-disks cover K. The pre-image $\alpha^{-1}(D_i \times [-N, N]) \cong \bar{D}_1 \times [-N', N']$ by $f_\perp$ being a microbundle-map and the local trivialization preserving fibers (of p_ν). Here $N' : \bar{D}_1 \to \mathbb{R}$ is a map, but we may replace it by its maximum N_i (on the compact one-disk) for the remaining proof. So $\alpha^{-1}(K \times [-N, N])$ is a closed subset of the union of these finitely many compact $\alpha^{-1}(D_i \times [-N_i, N_i])$, thus compact. Hence α is a proper map, one-to-one onto its image, between locally compact Hausdorff spaces, so it is a homeomorphism to its image (the inverse being continuous is the same as α being open; as a proper map of locally compact Hausdorff spaces α is closed, it is also one-to-one, when restricted to its image, thus open). Since $L \subset E_\nu$, further $L \times \{0\} \subset \operatorname{im}(\alpha)$. Hence there is a suitable ϵ' with $c' := \alpha^{-1}| : L \times (-\epsilon', \epsilon') \to M$ a homeomorphism to its image.

Remark: It may seem quite surprising, that a (oriented) microbundle of fiber-dimension 1 "contains" a bi-collar. Note however, that by a theorem of Kister and Mazur, every microbundle (of fiber-dimension k) contains (in a similar sense) a $(\mathbb{R}^k, 0)$-fiber-bundle! So the trivial $\nu = f^*\xi$ (a pullback of the bundle over a point) "should" actually look like a product.

Here (in the case $k = 1$), one could also use the techniques of [Bro62] (see Thm. 1.1.2) to see this: Both $L_{\geq} := f_{\perp}^{-1}([0,\infty)) \cap (U' \cup V')$ and $L_{\leq} := f_{\perp}^{-1}((-\infty, 0]) \cap (U' \cup V')$ are manifolds with boundary L, which is collared. Gluing the collars "back-to-back" yields a bi-collar of L (see Example 1.5.6).

As we have seen above, $c|_{U'} = c'|_{U'}$. The (small if necessary) homotopy already has the correct properties (it is rel $U' \cup W'$). What remains to be checked, is, that $f_{\perp}$ is indeed transverse to 0 on $U' \cup V'$ at c'. This consists of $L = f_{\perp}^{-1}(\{0\}) \cap (U' \cup V')$ being a manifold (which we already established above), and of the compatibility with c', which follows immediately from the definition of α (and c').

For the replacement of $\mathbb{R}$ by a real interval J and 0 by $t_0 \in \text{interior}(J)$ (see Rmk. 1.5.8), apply the statement for $\mathbb{R}$ to a suitable neighborhood of t_0 and rescale the bi-collar. □

For this case, transversality to 0, linear interpolation / a straight-line-homotopy in $\mathbb{R}$ are available, and using that manifold-boundaries are collared, a result rel boundary can be understood rather easily (technically, Thm. 1.5.9 includes the case of non-trivial boundary, but we want to be a little more precise on the relative hypotheses):

Corollary 1.5.9–2: *"Bi-collar-transversality rel boundary" (rel $C_\partial \cup C$ on D):*

Let M^n be a (topological) n-manifold, and $C_\partial \subset \partial M$ closed, with an open neighborhood $U_\partial \subset \partial M$ in the boundary and fix a boundary collar $b : \partial M \times [0,\infty) \to M$. Let $C, D \subset M$ be closed with open neighborhoods U, V in M. Assume $C_\partial, C \subset D$ and $C \cap \partial M = \emptyset$.

Given a continuous map $f : M \to \mathbb{R}$, with $f|_{\partial M}$ transverse to 0 on U_∂ at c_∂, and f transverse to 0 on U at c, then there are open neighborhoods $U'_\partial, U^M_\partial$ of C_∂ in $\partial M, M$ and U', V', W' of $C, D, M - V$ in M, and a homotopy $f \simeq f_{\perp} \operatorname{rel} U'_\partial \cup U' \cup W'$, such that $f_{\perp}$ is transverse to 0 on V' at a bi-collar c'.

The bi-collar c' in the result restricts to c on U' and extends c_∂ in the following sense: b restricts to b_0 on the transverse preimage $L := f_{\perp}^{-1}(\{0\})$ and c' restricts to c'_∂ on the boundary, such that $c'_\partial = c_\partial$ on U'_∂. Further, there is a $\delta > 0$ (or a map in the non-compact case), such that the following diagram commutes (all arrows except $f_{\perp}$ are homeomorphisms to their image):

$$\begin{tikzcd}
\partial L \times [0,\delta) \times (-\epsilon,\epsilon) \arrow[r, "b_0 \times \mathrm{id}_{(-\epsilon,\epsilon)}"] \arrow[d, "c_\partial \times \mathrm{id}_{[0,\delta)}"'] & L \times (-\epsilon,\epsilon) \arrow[d, "c"] \arrow[dr, "\pi_{(-\epsilon,\epsilon)}"] & \\
\partial M \times [0,\delta) \arrow[r, "b"] & M \arrow[r, "f_\perp"] & \mathbb{R}
\end{tikzcd}$$

Again, the entire construction can be made small to obtain a ϵ-homotopy. Further, one may replace $\mathbb{R}$ by a real interval J and 0 by $t_0 \in \operatorname{interior}(J)$.

Proof: Apply Cor. 1.5.9–1 to $f|_{\partial M} : \partial M \to \mathbb{R}$ on $D_\partial := D \cap \partial M$ rel C_∂ at c_∂ (with neighborhoods U_∂ and $V_\partial := V \cap \partial M$) to obtain $U'_\partial, V'_\partial, W'_\partial$ open neighborhoods of $C_\partial, D_\partial, \partial M - V_\partial$, and $f|_{\partial M} \simeq f_\perp^\partial \operatorname{rel} V'_\partial \cup W'_\partial$ transverse to 0 on $U'_\partial \cup V'_\partial$ at c'_∂, where $c'_\partial = c_\partial$ on U'_∂.

Next we use the fixed boundary collar $b : \partial M \times [0, \infty) \to M$ to extend this structure to $\operatorname{im}(b)$. First, we ensure, that our construction takes place near D and away from C: Because $C_\partial \subset D$, there is δ' and an open neighborhood U''_∂ of C_∂ in ∂M with $b(U''_\partial \times [0, \delta')) \subset V'$ for an open neighborhood $V' \subset V$ of D in M. Further, $C \cap \partial M = \emptyset$, so there are smaller U'''_∂ and δ'' with $b(U'''_\partial \times [0, \delta'')) \cap U' = \emptyset$ for an open neighborhood U' of C in M.

By regularity of the metrizable M, there is $\eta : M \to [0,1]$ with $\eta = 1$ near (on an open neighborhood in M of) C_∂ and $\eta = 0$ near the complement of $b(U'''_\partial \times [0, \delta''))$. Extend $f_\perp^\partial$ to a map on all of M by interpolation in $\mathbb{R}$:

$$f'_\perp : M \to \mathbb{R}, \quad x \mapsto \begin{cases} f(x) & \text{if } x \notin b(U'''_\partial \times [0, \delta'')) \\ \eta(x) f_\perp^\partial(\pi_{\partial M} b^{-1}(x)) + (1 - \eta(x)) f(x) & \text{otherwise} \end{cases}$$

This extends $f_\perp^\partial$ in the sense that it agrees with it on a neighborhood (in ∂M) of C_∂, because $\eta = 1$ there, it is continuous, because $\eta = 0$ (near) where they meet.

A homotopy can be obtained via linear interpolation / the straight-line-homotopy in $\mathbb{R}$: Define $H : M \times I \to \mathbb{R}, (x,t) \mapsto (1-t)f(x) + t f'_\perp$. By the choices of U'''_∂ and δ'', and by application of the transversality-theorem on the boundary rel C_∂, this homotopy is $\operatorname{rel} W' \cup U' \cup U'_\partial$ with W' an open neighborhood of $M - V'$.

Next, we show, that the result $f'_\perp$ is transverse to 0 at c' on an open neighborhood U_δ of C_∂ in M. First, $\eta^{-1}(\{1\})$ contains an open neighborhood of C_∂, this open neighborhood contains one of the form $U_\delta := b(U'''_\partial \times [0, \delta))$.

By definition of $f'_\perp$, on $\eta^{-1}(\{1\})$ we find $f'_\perp|^{-1}(\{0\}) = b(\pi_{\partial M}^{-1}((f_\perp^\partial)^{-1}(\{0\})))$ where $L_\partial := (f_\perp^\partial)^{-1}(\{0\})$ is a submanifold of ∂M by construction of the transverse $f_\perp^\partial$. So $(f'_\perp)^{-1}(\{0\}) = b(L_\partial \times [0, \infty))$ is a submanifold of $\operatorname{im}(b)$, and so is the open restriction to U_δ, i. e. the pre-image of the restriction $L' := f'_\perp|_{U_\delta}^{-1}(\{0\})$.

We also need the bi-collar $c' : L' \times (-\epsilon, \epsilon) \to U_\delta$, which is supposed to be compatible with $f'_\perp|_{U_\delta}$ and b. First, note, that by $L' \subset (f'_\perp)^{-1}(\{0\}) = b(L_\partial \times [0, \infty))$, b can be restricted to $b_0 = b|_{L_\partial}$ to get a (compatible) collar of the boundary L_∂ of L'. Use this, to write $x \in L'$ as $x = b_0(y, s)$, with $y \in L_\partial$, $s \in [0, \infty)$ and define $c'(b_0(y,s), t) := b(c_\partial(y,t), s)$, which automatically makes the rectangle-part of the compatibility-diagram in the statement of the corollary commute. It is also compatible with $f'_\perp$ by definition of $f'_\perp$ (the right triangle of the diagram also commutes).

Finally, apply Cor. 1.5.9–1 again, this time to $f'_\perp$ on D rel $C \cup C_\partial$ to obtain $f_\perp \simeq f'_\perp \operatorname{rel} U'' \cup W''$ (where we may assume $U'' \subset U'$ and $W'' \subset W'$, e. g. by intersecting them), thus $f_\perp \simeq f'_\perp \simeq f \operatorname{rel} U'' \cup W'' \cup U'_\partial$ at c''. Since c'' agrees with c' only on U'', which is potentially smaller than $b(\partial M \times [0, \delta)$, we may need to choose a smaller δ for the diagram to commute (which exists, however, by U'' containing a neighborhood of C_∂ in M).

After renaming U'', W'' etc. this finishes the proof of the corollary.

Smallness of the homotopy, can be achieved by the applications of Cor. 1.5.9–1 both producing a $\epsilon/3$-small homotopy, and making the straight-line-homotopy $f \simeq f'_\perp$ also $\epsilon/3$-small in $\mathbb{R}$, by choosing the collar $\operatorname{im}(b)$ thin enough.

Again, for the replacement of $\mathbb{R}$ by a real interval J and 0 by $t_0 \in \operatorname{interior}(J)$ (see Rmk. 1.5.8), apply the statement for $\mathbb{R}$ to a suitable neighborhood of t_0 and rescale the bi-collar. □

Finally, for the sake of completeness, we give an "embedded" (in the original reference "imbedded") statement, and show, how it implies map-transversality.

Definition 1.5.10: *[KS77, Def. 1.4 (p. 91)]: Let M be a (topological) manifold, (W, N) a manifold pair, $N \subset W$ closed and with a normal microbundle ξ.*

Given a proper inclusion onto a proper[20] submanifold $f : M \to W$, then M is embedded-transverse to ξ in W, *if f is transverse to ξ at some ν, and $f(M) \cap N$ is a submanifold of W.*

Given $C \subset W$, then M is embedded-transverse to ξ in W near C, *if there is a neighborhood U of C in W and $U \cap M$ is embedded-transverse to $\xi \cap U$ in U.*

Theorem 1.5.11: *[KS77, Thm. 1.5 (p. 91)] (with dimensional restrictions), see also [Qui88b, 'Theorem' (p. 145)] (all dimensions); Embedded Transversality Theorem (on D rel C):*

Suppose M and N are proper submanifolds of W, N has a normal microbundle ξ in W, and closed subsets $C \subset D \subset W$ and M is transverse to ξ near C.

Then, there is an isotopy of M, supported in any (given, but arbitrary small) neighborhood of $(D - C) \cap M \cap N$, to a submanifold transverse to ξ near D.

Corollary 1.5.11–1: *The embedded transversality theorem implies map-transversality for manifold-targets:*

For Thm. 1.5.11, in the reference [Qui88b, 'Theorem' (p. 145)], a version without dimensional restrictions is explicitly given, so this formally verifies the claim (at least for a manifold target-space), that Thm. 1.5.9 also applies without dimensional restrictions (which is of course also implicit in [Qui88b]).

[20] Here, $M \subset W$ proper means closed (as a subspace, not as a manifold) and such that the boundary is intersected precisely in the boundary: $\partial W \cap M = \partial M$.

Proof: Suppose, we are given a continuous map $f : M \to W$ and $N \subset W$, all (sub)manifolds, and a normal microbundle ξ with $N \xrightarrow{i_\xi} E_\xi \xrightarrow{p_\xi} N$ of N in W, i. e. $E_\xi \subset W$ and $i_\xi = i_N$ is the inclusion of N. The proof is essentially by making the graph of f embedded-transverse to $N \times M$.

"Case $C = \emptyset$, $D = M$": First, note, that $\xi \times M$ given by $N \times M \xrightarrow{i_\xi \times \mathrm{id}_M} E_\xi \times M \xrightarrow{p_\xi \times \mathrm{id}_M} N \times M$, is a normal microbundle of $N \times M \subset W \times M$.

Define (the proper embedding) $g : M \to W \times M, x \mapsto (f(x), x)$, then use the embedded transversality theorem (Thm. 1.5.11) to obtain $g_\perp \simeq g$, embedded-transverse to $\xi \times M$ at some ν. This means by definition, that $\mathrm{im}(g_\perp) \cap (N \times M)$ is a submanifold of W. As $g_\perp$ is a homeomorphism to its image, this is homeomorphic to $L := g_\perp^{-1}(N \times M)$, which is a submanifold of M.

Define $f_\perp := \pi_W g_\perp : M \to W$. We claim, that $f_\perp$ is already transverse to ξ at ν. First, we note, that indeed $f_\perp^{-1}(N) = g_\perp^{-1}(\pi_W^{-1}(N)) = g_\perp^{-1}(N \times M) = L$ is a submanifold of M (see above). Next, we have to verify, that f restricts to a microbundle map on E_ν.

This means, we have to show, that f embeds, for any $y \in L$, the fiber $F_y^\nu := p_\nu^{-1}(\{y\})$ of ν into the fiber $F_{f_\perp(y)}^\xi := p_\xi^{-1}(f_\perp(y))$ of ξ. But $g_\perp$ is transverse to ξ at ν by construction, so $g_\perp| : F_y^\nu \to (p_\xi \times \mathrm{id}_M)^{-1}(g_\perp(y)) = F_{f_\perp(y)}^\xi \times \{\pi_M g_\perp(y)\}$ is an embedding. Since also $\pi_W : W \times M \to W$ restricts to an embedding on $F_{f_\perp(y)}^\xi \times \{\pi_M g_\perp(y)\} \to F_{f_\perp(y)}^\xi$, and $f_\perp = \pi_W g_\perp$ by definition, $f_\perp$ restricts to a microbundle map on E_ν.

Finally, we use the (isotopy, thus) homotopy H^g of $g \simeq g_\perp$ to construct one $f \simeq f_\perp$ from f to $f_\perp$. Since $\pi_W g = f$, and $\pi_W g_\perp = f_\perp$, we may do so, simply by composition with π_W as $H^f := \pi_W H^g$.

"Case of general C and D": If near C, f is already transverse to ξ at a ν_0, then g is embedded transverse to $\xi \times M$ at ν_0 near C. The only thing that needs checking, is that $\mathrm{im}(g) \cap (N \times U)$ is indeed a submanifold of $W \times U$ for some neighborhood $U \subset M$ of C. We know, that $L_U := f^{-1}(N) \cap U \subset U$ is a submanifold (or rather, we know, that a U exists such that this is true, by f being transverse to ξ near C),

$$\mathrm{im}(g) \cap (N \times U) \quad = \quad \left\{ \quad (f(x), x) \in N \times U \quad \middle| \quad x \in U \cap f^{-1}(N) = L_U \quad \right\}$$

is the graph of the restriction $f|_{L_U}$ to the submanifold L_U, thus a submanifold of $W \times U$.

We may assume $C \subset D$ (Rmk. 1.5.8), so we can apply the embedded transversality theorem (Thm. 1.5.11) rel $W \times C$ on $W \times D$. The resulting $f_\perp$ of the construction above is then transverse to ξ at ν near D with $\nu = \nu_0$ near C. □

Remark 1.5.12: If the target is *not* a manifold pair, but some compact metric $Y \subset X$ with normal microbundle ξ, cover Y by (finitely many) trivializations h_i of this normal microbundle ξ, that is $h_i : V_i \xrightarrow{\sim} U_i \times \mathbb{R}^k$, inductively make $\pi_{\mathbb{R}^k} h_i f$ transverse to the trivial normal microbundle of $\{0\} \subset \mathbb{R}^k$ rel where it is already transverse from previous induction-steps as in Cor. 1.5.11–1. Since the h_i preserve fibers, the result is transverse to ξ.

The "other direction" has no such elementary argument, as map-transversality gives no hint at how to one could obtain an *isotopy* for the embedded theorem.

In the next section, the bi-collar transversality-statements Cor. 1.5.9–1 (without boundary) and Cor. 1.5.9–2 (including boundary) are generalized to those MHSS which posses mapping-cylinder neighborhoods of skeleta.

1.6 Stratified Transversality

This section summarizes transversality results on MHSS admitting cylinder-neighborhoods of the lower stratum. It does *not* treat the question of existence of such neighborhoods, this is deferred to the subsequent sections. The results presented here rely heavily on Connolly and Vajiac's [CV99] transversality theorem (see Thm. 1.6.4).

The setting used in [CV99] is in the context of Quinn's [Qui88a] "homotopy stratified spaces" (see Section 1.4 ("Quinn Spaces / (M)HSS")), combined with controlled end-theory. The aspect of how to obtain suitable transversality results, or at least certain precursors[21], is independent of these notions in principle. For the sake of presentation, controlled end-theory is discussed later. To this end, some statements are formulated slightly different than in the original source. From the following sections Section 1.7 ("Excursion: Controlled Topology") and Section 1.9 ("Mapping-Cylinder Neighborhoods") the connections will become apparent. The relevant hypothesis is the existence of a cylinder neighborhood in the following sense:

Definition 1.6.1: *Given a filtered space $X^n \supset \ldots \supset X^0$, a cylinder-neighborhood of X^k in X^{k+1} is an open neighborhood N of $X^k \subset X^{k+1}$, which is (given the induced stratification) stratified homeomorphic to a mapping-cylinder* $\mathring{\text{cyl}}(f)$ *(Def. 1.1.3) of a map $f : V \to X^k$, with the stratification (cf. Def. 1.3.5) defined by the filtration* $\mathring{\text{cyl}}(f) \supset X^k \supset \ldots \supset X^0$*, where X^k is identified with the cylinder-base (at 0 in the convention used here).*

Remark 1.6.2: We are working with MHSS, thus by Hughes' cylinder-theorem (Thm. 1.4.19), we may assume, that f is a MAF (or a suitable generalization thereof in the case of more than two strata (cf. Chapter 3 ("Multiple Strata")), also implying that V is a manifold.[22]

[21] By Quinn's collaring theorem Thm. 1.4.15, constructing homotopy-collars is already enough.

[22] The top-stratum $V \times (0, 1)$, as an open subset of X_{k+1}, is automatically a manifold. In the topological category, this does not imply that V is one however [Bin58].

We begin the treatment of transversality by giving a formal definition of transversality in the stratified setting (for spaces with boundary, see Def. 1.6.5 below).

Definition 1.6.3: *Let X be a MHSS, and $g : X \to \mathbb{R}$ a continuous map. Then g is* transverse to 0 *at c, if $X_0 := g^{-1}(\{0\})$ is stratified bi-collared by $c : X_0 \times (-\epsilon, \epsilon) \to X$ compatible with g, that is*

(i) c is a stratified homeomorphism to its image

(ii) $g \circ c(x,t) = t \quad \forall (x,t) \in X_0 \times (-\epsilon, \epsilon)$

(iii) X_0 is a MHSS

If X_0 is non-compact, $\epsilon : X_0 \to (0, \infty)$ may be considered a map.

If the bottom stratum has a cylinder neighborhood, then any continuous $g : X \to \mathbb{R}$ will turn out to be close to a transverse map. The most difficult part of proving this is provided by the following result of Connolly and Vajiac [CV99]:

Theorem 1.6.4: *[CV99, Thm. 2.2 (p. 529)]: Let X be a MHSS of dimension n, with two strata B and $M := X - B$, without boundary, such that*

(i) $B = B_0 \times (-1, 1)$ for some closed manifold B_0 and

(ii) there is $p = (n, s) : \partial M' \to B = B_0 \times (-1, 1)$ surjective, continuous, such that $X = \mathrm{cyl}(p)$.

Here, M' is a manifold with boundary $\partial M'$ such that $M \subset M'$ and $M' - M = \partial M'$.[23] *Set (see Figure 1.3)*

$$\begin{array}{rlcll} V := \{ & [x,t] \in \mathrm{cyl}(p) & \mid & s(x) \le t < 1 & \} \\ W := \{ & [x,t] \in \mathrm{cyl}(p) & \mid & 0 \le t \le s(x) & \} \\ U := \{ & [x,t] \in \mathrm{cyl}(p) & \mid & 0 \le s(x) = t < 1 & \} \end{array}$$

Then:

(a) U is a stratified (strong) deformation retract of both V and W

(b) $V \cup W = X$; $V \cap W = U$;
V and W are closed subsets of X;
The lower strata are $V_B = B_0 \times (-1, 0]$, $W_B = B_0 \times [0, 1)$ and $U_B = B_0 \times \{0\}$

(c) U, V and W are stratified subspaces of X (i. e. ∂-MHSS when given the induced stratification), with boundaries $\partial V = \partial W = U$ and $\partial U = \emptyset$.

[23] The last equality – instead of the usual inclusion, see Section 1.8 (“Excursion: End-Theory”) – holds, because $\partial M = \emptyset$ by hypothesis.

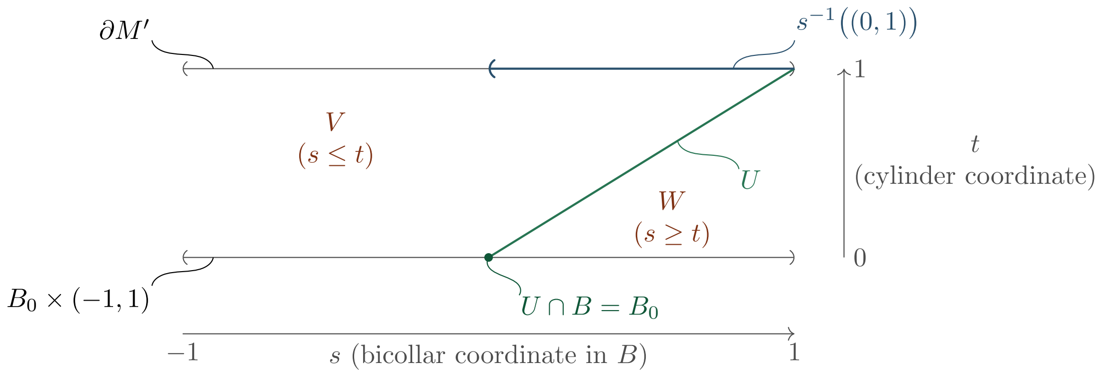


Figure 1.3: Pictorial representation of the construction by Connolly and Vajiac [CV99], similar to their Fig. 4 (p. 530).

"Reverse direction" (see Connolly and Vajiac [CV99, p. 536]): Given a codimension 1, stratified bi-collared MHSS $X_0 \subset X$, then the end-obstruction $q_0\big(B_0 \times (-\tilde{\epsilon}, \tilde{\epsilon})\big)$ over a (potentially "thinner") bi-collar of the lower stratum B_0 of X_0 vanishes (see also Section 1.9 ("Mapping-Cylinder Neighborhoods")). If $\dim(X) \geq 6$*, this implies existence of a cylinder-neighborhood over $B_0 \times (-\tilde{\epsilon}, \tilde{\epsilon})$ by Quinn's end-theorem (see Thm. 1.9.3).*

Proof: We briefly outline the proof as given in [CV99]: There are two particularly interesting facts to check.

First, the upper stratum of U is homeomorphic to $s^{-1}\big((0,1)\big)$, an open subset of the manifold $\partial M'$, hence a manifold. Compare this to Section 1.7 ("Excursion: Controlled Topology") and Rmk. 1.7.5 in particular, where it is seen, that obtaining the "correct" $\partial M'$ is actually one of the most difficult parts – and in fact the obstructed step – in ensuring the existence of a suitable mapping-cylinder to apply the theorem above. Much of the difficulty of the transversality-problem is hidden in the existence of $\partial M'$.

Second, the "stratified (strong) deformation retract" part ensures (in high dimensions) the existence of a bi-collar, by Quinn's collaring theorem Thm. 1.4.15. The reference [CV99] uses the following elegant trick to construct these deformation retractions: By Hughes' cylinder theorem Thm. 1.4.19, p is a MAF. This implies by Lemma 1.2.8 that also

$$p \times \mathrm{id} : \partial M' \times (0,1) \to B \times (0,1) = B_0 \times (-1,1) \times (0,1) \quad \text{is a MAF.}$$

Retracting the images (see Figure 1.4)

$$\begin{aligned} V_{\mathrm{img}} &:= \{(b,s,t) \in B_0 \times (-1,1) \times (0,1) | s \leq t\} \quad \text{and} \\ W_{\mathrm{img}} &:= \{(b,s,t) \in B_0 \times (-1,1) \times (0,1) | s \geq t\} \quad \text{to} \\ U^+_{\mathrm{img}} &:= \{(b,s,t) \in B_0 \times (-1,1) \times (0,1) | 2s = t\} \quad \text{or} \\ U^-_{\mathrm{img}} &:= \{(b,s,t) \in B_0 \times (-1,1) \times (0,1) | {}^{3s}\!/\!_4 = t\} \end{aligned}$$

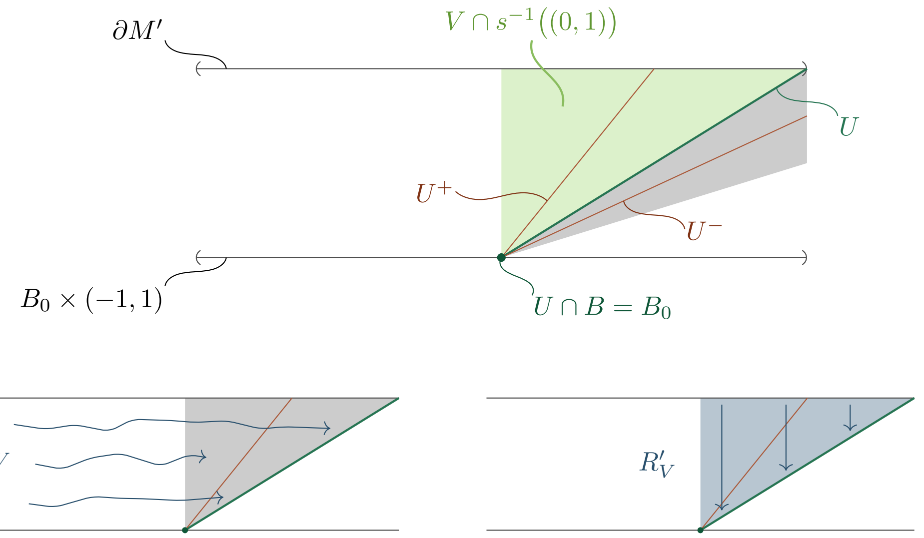


Figure 1.4: Pictorial representation of the retractions, similar to [CV99, Fig. 5 (p. 532)].

is easily done (in $(-1,1) \times (0,1)$, and then $\times \mathrm{id}_{B_0}$) by, say, R^V_{img} (for V, all of the following can be done similarly for W). Now, one defines

$$\epsilon : B \times (0,1) \to (0,\infty), \quad (b,t) \mapsto t\epsilon_0$$

for some fixed $\epsilon_0 > 0$ (chosen suitably small, see below) and produces an ϵ-lift R^V of R^V_{img} w. r. t. the MAF $p \times \mathrm{id}$ starting at the identity (see Figure 1.4). By choice of ϵ, specifically by $\epsilon \searrow 0$ as $t \searrow 0$ thus near the lower stratum, this can be extended by the obvious retraction of $B_0 \times (-1,0]$ to $B_0 \times \{0\}$ to a stratified one.

Further, if ϵ_0 is chosen small enough, this ends up somewhere in

$$V' := \{[x,t] \in \mathrm{cyl}(p) | 0 < s(x) \leq t < 1\} = V \cap s^{-1}\big((0,1)\big)$$

which can in turn be retracted to U using (see Figure 1.4)

$$R'_V : V' \times I \to V', \quad ([x,t],\tau) \mapsto [x,(1-\tau)t + \tau s(x)]$$

Hence the composition of first R^V then R'_V yields the required stratified deformation retraction of V to U. With some slight modifications (see picture) the same can be done on W.

"Reverse direction" (this part of the proof can be found near the bottom of p. 536 in [CV99]): X_0 is a MHSS, and therefore (by Lemma 1.4.21 (ii)) $X_0 \times S^1$ is a MHSS. The control-map (holink-evaluation) of the end of the upper stratum is a product $p_{X_0} \times \mathrm{id}_{S^1}$ (where p_{X_0} is the holink-evaluation of X_0). There are a number of ways, to see, that the obstruction q_0 of such a product with S^1 vanishes. The (probably) most elementary and geometric argument is found along the lines of [Fer81], but this is actually a manifestation of a more general phenomenon: A control-map, that can be written as a product by a finite complex $p' = p \times \mathrm{id}_K$ has end-obstruction $q_0(p') = \chi(K) q_0(p)$ [Qui82a, Prop. 1.8c (p. 359)] (if $q_0(p)$ is defined, which it is here, because X_0 is a MHSS). Here, χ is the Euler-characteristic of K (thus 0 in the case of $K = S^1$). This vanishing-result is also stated similarly in [CV99, Cor. 1.15 (p. 525)].

In any case, we next embed the bi-collar $X_0 \times (-\epsilon, \epsilon)$ into $X_0 \times S^1$ by $j := \mathrm{id}_{X_0} \times g$ where g is a universal cover of S^1 mapping $(-\epsilon, \epsilon)$ to $S^1 - \{\mathrm{pt}\}$. By continuity of the map f creating the cylinder(-neighborhood) $X_0 \times S^1 \cong \mathring{\mathrm{cyl}}(f)$, we find $0 < \tilde{\epsilon} < \epsilon$, such that $f^{-1}\Big(B_0 \times (-\tilde{\epsilon}, \tilde{\epsilon})\Big) \subset \mathrm{im}(j)$. Since j is a homeomorphism to its image, j^{-1} restricts to a homeomorphism $\mathring{\mathrm{cyl}}\Big(f| : f^{-1}\Big(B_0 \times (-\tilde{\epsilon}, \tilde{\epsilon})\Big) \to B_0 \times (-\tilde{\epsilon}, \tilde{\epsilon})\Big) \cong N$, where $N \subset X_0 \times (-\epsilon, \epsilon)$ is a neighborhood of $B_0 \times (-\tilde{\epsilon}, \tilde{\epsilon})$ as required. □

A suitable transversality theorem for high dimensions, can be obtained from this, combined with manifold transversality.

Corollary 1.6.4–1: *See also [CV99, Cor. 1.17 (p. 527)]: Let X be a MHSS of dimension $n \geq 6$ with at most two meeting strata (without boundary), with disjoint cylinder-neighborhoods of the minimal strata B_i (see Def. 1.4.23), further $g : X \to \mathbb{R}$ continuous and $\epsilon > 0$.*

Then there is a transverse to 0 map $g_\perp \simeq_\epsilon g$ rel $g^{-1}(\mathbb{R} - (-\epsilon, \epsilon))$. I. e. there is a homotopy H with $H_0 = g$, $H_1 = g_\perp$, $\mathrm{diam}(H(\{x\} \times I)) < \epsilon$ for all $x \in X$ and $g(x) \notin (-\epsilon, \epsilon) \Rightarrow H(x, t) = x$.

Relative Version: If g is already transverse to 0 on (some of the) B_i (let B be the union of these) at a bi-collar c_B, then the construction can be done $\mathrm{rel}\, B$*, i. e. the homotopy can be chosen to additionally be* $\mathrm{rel}\, B$*, and $c|_B = c_B$.*

Proof: The homotopy $g \simeq g_\perp$ will be added in the end (we can simply use the straight-line homotopy in $\mathbb{R}$), also size-estimates (making the homotopy ϵ-small) are provided, after the construction was outlined.

In the absolute case (or if g is not yet transverse on *all* the B_i), use manifold-transversality Cor. 1.5.9–1 individually on those B_i to obtain $g_\perp^B \simeq g|_{B_i}$ on B_i, transverse to 0 at a bi-collar c_{B_i}. On those B_i, where $g|_B$ is already transverse to 0 at c_B (in the relative case), define $g_\perp^B := g|_B$. So $g_\perp^B$ is actually defined on the union of all minimal strata B_i.

Next we want to apply the transversality-theorem by Connolly and Vajiac (see Thm. 1.6.4). By hypothesis each $B_i \subset X$ has a mapping-cylinder neighborhood (disjoint from those of other B_i), i. e. there is $\mathring{\mathrm{cyl}}(f_i : \partial M_i' \to B_i) \cong N_i \,\mathrm{rel}\, B_i$ for an open neighborhood N_i of $B_i \subset X$ with $i \neq j \Rightarrow N_i \cap N_j = \emptyset$. Thus we can (for now) work on the B_i independently of each other.

Let $p_i := \Big(f_i| : f_i^{-1}(\mathrm{im}(c_B) \cap B_i) \to \mathrm{im}(c_B) \cap B_i\Big)$, the restriction to the image of the bi-collar in B_i. Now, $\mathring{\mathrm{cyl}}(p_i) \subset X$ is a MHSS (as an open subset of the MHSS X), its lower stratum is a product of the form $\mathrm{im}(c_B) \cap B_i \cong B_0 \times (-1, 1)$ using $B_0 := g_\perp^B(\{0\}) \cap B_i$ with a cylinder neighborhood in X given by p_i, so Thm. 1.6.4 applies to yield stratified, as MHSS with "weak boundary" (i. e. boundaries are uncollared as of yet, see Thm. 1.4.15), subspaces U, V, W, where $U = V \cap W$ is a stratified NDR of both V and W, together $V \cup W = \mathring{\text{cyl}}(p_i)$ and with lower strata $U_B = B_0 \times \{0\}$ and $V_B = B_0 \times (-1, 0]$, $W_B = B_0 \times [0, 1)$. By Quinn's boundary-collaring theorem Thm. 1.4.15 (its relative version [Qui88a, 'Lemma' (p. 492)]), U is stratified bi-collared by $c_U : U \times (\epsilon', \epsilon') \to N$ extending c_B near B_i. This is, because the two "sides" V and W are both MHSS, collared by the individual sides of c_B in the $\dim(X) - 1 \geq 5$-skeleton, with homotopy-collared (stratified NDR) boundary U, which is therefore collared in both individually, extending the sides of c_B. The bi-collar c_U is then obtained by gluing both these collars "back-to-back".

By regularity of the (metric) X, there is $\eta : X \to [0, 1]$ such that $\eta = 1$ near (on an open neighborhood of) B_i, and $\eta = 0$ near the complement of N_i (the complement of the cylinder-neighborhood). Define:

$$g'_\perp : X \to [-1, 1], \quad x \mapsto \begin{cases} g(x) & \text{if } x \notin N \\ \eta(x)\pi_{\mathbb{R}} c_U^{-1}(x) + (1 - \eta(x))g(x) & \text{if } x \in N \end{cases}$$

This is continuous, by the choice of η and $g'_\perp|_B = g_\perp^B$ agree on B_i by compatibility of $g_\perp^B$ with c_B, meaning $g_\perp^B c_B(b, t) = t$, and c_U extending c_B near B_i.

This $g'_\perp$ is transverse near B_i at c_U: Since $\eta = 1$ near B_i, there is an open neighborhood U'_i of B_i in X, with $\eta|_{U'_i} = 1$. The preimage $(g'_\perp|_{U'_i})^{-1}(\{0\}) = U \cap U'_i \subset U$ is an open subset of the MHSS U (as obtained above by application of Thm. 1.6.4), thus an MHSS, and stratified bi-collared in U'_i by the restriction of c_U.

By definition of η, $g'_\perp = g$ outside N_i and, given another minimal stratum B_j, then $N_i \cap N_j = \emptyset$, so this can be done on all the minimal strata to obtain a single $g'_\perp$, transverse to 0 near all the B_i.

The closed complement $X - \cup U'_i$ of the U'_i is disjoint from all the closed B_i (by the choice of η). By normality of the metric X, it can thus be separated by open neighborhoods W of $X - \cup U'_i$ and W_i of B_i with $W \cap W_i = \emptyset$. We set $C_X := X - W$, which is closed in X, such that its interior contains $\cup B_i$ and with $C_X \subset \cup U'_i$. In the subspace-topology on $M := X - \cup B_i$, the intersection $C := C_X - \cup B_i = C_X \cap M$ is thus closed. Next apply, on the top-stratum M, manifold-transversality Cor. 1.5.9–1 again, to $g'_\perp|_M$ on (all of) M rel C. This can be done, because $g'_\perp$ is already transverse on $(\cup U'_i) \cap M$ (which is an open neighborhood of C in M by construction of C) at c_U. This yields $g_\perp^M \simeq g'_\perp \operatorname{rel} U''$, where U'' is a (smaller) neighborhood of C, with $g_\perp^M$ transverse to 0 (everywhere on M) at c_M, and $c_M|_{U'' \cap M} = c_U|_{U'' \cap M}$.

Then, define $g_\perp$ on all X by

$$g_\perp : X \to [-1, 1], \quad x \mapsto \begin{cases} g_\perp^B(x) & \text{if } x \in B \\ g_\perp^M(x) & \text{if } x \in M \end{cases}$$

which is continuous, because on the open neighborhood U'' of C (which is also an open neighborhood of $B \subset C$), it is $g_\perp = g'_\perp$ and $g'_\perp$ is continuous and well-defined on X (it agrees with $g_\perp^B$ on B). Similarly

$$c : g_\perp^{-1}(\{0\}) \times (-\epsilon, \epsilon) \to X, \quad (x_0, t) \mapsto \begin{cases} c_U(x_0, t) & \text{if } x_0 \in U'' \\ c_M(x_0, t) & \text{otherwise} \end{cases}$$

is continuous, because $c_M|_{U''\cap M} = c_U|_{U''\cap M}$ (which also implies that $c|_B = c_U|_B = c_B$ as claimed). Further $g_\perp$ is transverse to 0 at c, because $(g'_\perp|_{U''})^{-1}(\{0\})$ is an MHSS (see above), and $(g_\perp^M)^{-1}(\{0\})$ is a manifold (thus an MHSS) and both together form an open cover of $g_\perp^{-1}(\{0\})$, which is thus an MHSS (which is a local property). Also, c is a stratified bi-collar, because $c_U|_{U''}$ and c_M are (and they overlap on the open $U'' - \cup B_i$).

For the homotopy H from g to $g_\perp$, we use the straight-line homotopy in $\mathbb{R}$, i. e. $H_t(x) = (1-t)g(x) + tg_\perp(x)$, which is indeed $\operatorname{rel} B$ for the relative case, because then g and $g_\perp$ agree there.

This construction can be modified to make the homotopy ϵ-small and additionally $\operatorname{rel}(X - g^{-1}((-\epsilon,\epsilon)))$ as follows (for $\gamma = \epsilon/4$): For the two applications of manifold-transversality use Cor. 1.5.9–1 on $D = g^{-1}((-\epsilon,\epsilon))$, more precisely, first on B, use $D_B = D \cap B$ and make $g|_B$ transverse to 0 on D_B (on those B_i where it is not already transverse) such that the resulting homotopy is γ-small (see Cor. 1.5.9–1). Then also $g|_B$ and $g_\perp^B$, the ends of the homotopy, differ by at most γ. This will later suffice to ensure, that the straight-line homotopy is also γ-small on B. Similarly we will later, in the third step, on M use $D_M = D \cap M$ to obtain $g_\perp \simeq_\gamma g'_\perp$ rel the complement of a (arbitrarily small) neighborhood of D_M.

It remains to see, that the step from g to $g'_\perp$ (extending along the construction of Connolly and Vajiac), can be made 3γ-small (it is γ-small on B, as $g'_\perp = g_\perp^B$ there). To ensure this, we use the following observations: On $g^{-1}((-2\gamma, 2\gamma))$, obviously $|g| < 2\gamma$, and since $g_\perp^B$ is γ-close to $g|_B$, $g^{-1}((-2\gamma,2\gamma))$ contains an open neighborhood W (in X) of $(g_\perp^B)^{-1}(\{0\}) \subset B$. We restrict the extension-step to this W. Note, that $D_M = g^{-1}((-\epsilon,\epsilon)) \cap M$ will be a useful choice for the application of manifold-transversality on M in the third step, as $(g'_\perp)^{-1}(\{0\}) \subset \operatorname{interior}(D_M)$. This is "useful", in the sense, that the manifold-transversality result Cor. 1.5.9–1 will yield a $g_\perp$ transverse near D, $\operatorname{rel} W'$ for an open neighborhood W' of the complement of D, which means all zeros of $g_\perp$ are inside of D, thus $g_\perp$ is transverse to 0 *everywhere* on X (trivially so away from D).

Further, we can replace the bi-collar $c_U : U \times (-\epsilon', \epsilon') \to N$, by a (potentially thinner, $\gamma' = \min(\epsilon', \gamma)$-small) one $c_U : U \times (-\gamma', \gamma') \to N$. Then $g'_\perp$, where it differs from g (on a subset where $\eta \neq 0$, see above), is defined as a linear interpolation (by η) of $\pi_{(-\gamma',\gamma')} c_U^{-1}$ and g, thus $|g'_\perp - g| \le |\pi_{(-\gamma',\gamma')} c_U^{-1} - g|$ here, where $|\pi_{(-\gamma',\gamma')} c_U^{-1}| < \gamma' \le \gamma$ and $|g| < 2\gamma$ (see above), so $|g - g'_\perp| < 3\gamma$ where they differ (and, of course, also where they agree).

As indicated above, in the third step, $g'_\perp$ and $g_\perp$ can be obtained γ-close (from the γ-homotopy obtained by manifold-transversality on M), so g and $g_\perp$ are $4\gamma = \epsilon$-close. The choice of $D = g^{-1}((-\epsilon,\epsilon))$ implies (together with the choice of W ensuring D contains all zeros), that $g = g_\perp$ outside $g^{-1}((-\epsilon,\epsilon))$. These together imply, that the straight-line homotopy $H_t(x) = (1-t)g(x) + tg_\perp(x)$ is both ϵ-small and $\operatorname{rel}(X - g^{-1}((-\epsilon,\epsilon)))$. It is also $\operatorname{rel} B$ in the relative case (as before). $\square$

We will next investigate two different modifications of the proof to spaces with boundary. First, we say what transversality on a space with boundary means precisely and then give further properties, which are additionally ensured by our results.

Definition 1.6.5: *Let $(X, \partial X)$ be a MHSS with boundary, and $g : X \to \mathbb{R}$ a continuous map.*

Then g is transverse to 0 at c, *if $X_0 := g^{-1}(\{0\})$ is stratified bi-collared by $c : X_0 \times (-\epsilon, \epsilon) \to X$ compatible with g, i. e. $gc(x,t) = t$ as before.*

Further, $(X_0, X_0 \cap \partial X)$ is required to be a MHSS with boundary and that c is strict with respect to the boundary, in the sense, that $c(\partial X \times (-\epsilon, \epsilon)) \subset \partial X$ and $c((X - \partial X) \times (-\epsilon, \epsilon)) \subset (X - \partial X)$, thus the restrictions $g|_{X-\partial X}$ and $g|_{\partial X}$ are transverse to 0 at the corresponding restrictions of c.

Definition 1.6.6: *Given a MHSS X of the form $\mathring{\mathrm{cyl}}(p : \partial M' \to B_0 \times (-1, 1))$, and a transverse to 0 map g on X, we say g is* standard with respect to p, *if $g^{-1}(\{0\}) \cap N = U \cap N$, for an open neighborhood N of B in X and the stratified NDR $U := \{[x,t] \in \mathrm{cyl}(p) | 0 \le \pi_{(-1,1)} p(x) = t < 1\}$ of Thm. 1.6.4.*

Given a MHSS with boundary $(X, \partial X)$, and a fixed boundary collar $b : \partial X \times [0, \gamma) \to X$, we call $g : X \to \mathbb{R}$ transverse to 0 at c compatible with b, *if b restricts to a collar b_0 of $\partial L := \partial X \cap g^{-1}(\{0\}) \subset g^{-1}(\{0\}) =: L$ and c restricts to a bi-collar c_∂ of $\partial L \subset \partial X$ such that (cf. Cor. 1.5.9–2)*

$$\begin{tikzcd}
\partial L \times [0,\gamma) \times (-\epsilon,\epsilon) \arrow[r, "b_0 \times \mathrm{id}_{(-\epsilon,\epsilon)}"] \arrow[d, "c_\partial \times \mathrm{id}_{[0,\gamma)}"'] & L \times (-\epsilon,\epsilon) \arrow[d, "c"] \arrow[dr, "\pi_{(-\epsilon,\epsilon)}"] & \\
\partial M \times [0,\gamma) \arrow[r, "b"] & M \arrow[r, "f_\perp"] & \mathbb{R}
\end{tikzcd}$$

Remark 1.6.7: The compatibility with the boundary condition may seem quite strong, but actually, since our boundary is collared anyway, we can just glue a "trivial" collar to the "outside" to obtain a stratified homeomorphic space $X' \cong_h X$, and define $g : \partial X \times [-1, 0] \to \mathbb{R}, (y,t) \mapsto g(c_\partial(y, 0))$, which certainly fits together with g on $X \subset X'$ (and by strictness also c extends).

Choosing the outside collar "thin enough" one can make h arbitrarily small, so any transversality-theorem that yields an arbitrarily close, transverse map $g_\perp$ without the last condition also provides arbitrarily close transverse maps $g'_\perp = h^{-1} \circ g_{\text{extended}}$ that satisfies the last condition. Vice versa, compatibility with the boundary-collar implies of course strictness in the sense of Def. 1.6.5.

The "standard" condition will usually be satisfied automatically, e. g. if g was made transverse using Cor. 1.6.4–1 (which is how we will typically obtain transverse maps in the first place): Inspecting the proof of Cor. 1.6.4–1, we note, that $g_\perp$ is constructed to agree with $g'_\perp$ near B, which in turn is defined as $\pi_{\mathbb{R}} c_U^{-1}$ near B, where c_U is a bi-collar of U and clearly $(\pi_{\mathbb{R}} c_U^{-1})^{-1}(\{0\}) = U$.

The first idea to prove a transversality-theorem including boundary is, to use spaces with more than two strata (see Chapter 3 ("Multiple Strata")), "absorb the boundary" into the stratification (see Section 3.2 ("Absorbing Boundary")), and then treat the space as if it were a space without boundary. For example if $(X, \partial X)$ has two strata $(M, \partial M)$ and $(B, \partial B)$, then there is also a stratification (as MHSS as it will turn out) with four strata $M, \partial M, B, \partial B$ without boundary, in which case a multi-stratum analogue of Cor. 1.6.4–1 suffices for an inductive argument over skeleta. Since, for now, we specialize to spaces of at most two strata, we defer this argument to Cor. 3.2.3–1.

For now, we use a different approach, that does not use spaces with more than two strata, but requires a bit more effort to get the bi-collars right, because they need to be constructed in the interior and the boundary at the same time. To this end, we strengthen the requirement on the mapping-cylinder-neighborhood, so that the one of the boundary actually "fits together" with the one on the interior. See Section 1.7 ("Excursion: Controlled Topology") and Section 1.9 ("Mapping-Cylinder Neighborhoods") for a treatment of cylinder-neighborhood-existence. It is possible to construct such compatible cylinder-neighborhoods using the relative end-theorem of [Qui79, Thm. 2.1 (p. 282)], where the same obstruction-groups, as otherwise used in this thesis, are encountered.[24]

Thus we can consistently maintain the requirement of at most two-strata, if we want to. The argument used is similar to the one in the gluing lemma (Lemma 1.4.25), and indeed, we do not need such a strong result here (it would suffice to show, that corners are "homotopy-corners", i. e. that $X_{\geq}$ etc. are MHSS with boundary, because the gluing result – the only argument that requires geometric corners – already deals with this). However, we can (even) show the following (geometric corner) result:

Corollary 1.6.4–2: *Let $(X, \partial X)$ be a MHSS with a boundary of $\dim(\partial X) \geq 6$ with at most two meeting strata, with disjoint cylinder-neighborhoods of the minimal strata B_i (see Def. 1.4.23) in the interior $\mathring{\text{cyl}}(F_i : \partial M_i' \to (B_i - \partial B_i))$ and in the boundary $\mathring{\text{cyl}}(f_i : \partial N_i' \to \partial B_i)$, where, given a boundary collar $b : \partial X \times [0, \gamma) \to X$, there are identifications $\partial N_i' \times (0, \gamma) \subset \partial M_i'$ such that*

$$F_i|_{\partial N' \times (0,\gamma)} = (f_i \times \mathrm{id}_{(0,\gamma)})$$

and $b([x, t], s) = [(x, t), s] \in \mathring{\text{cyl}}(f_i \times \mathrm{id})$. Then $\mathring{\text{cyl}}(\bar{F}_i = F_i \cup (f_i \times 0))$ is an open neighborhood of B_i in X. Further, let $g : X \to \mathbb{R}$ be continuous and $\epsilon > 0$.

Then there is a transverse to 0 map $g_\perp \simeq_\epsilon g$ rel $g^{-1}(\mathbb{R} - (-\epsilon, \epsilon))$, which is standard with respect to f_i on ∂X and with respect to F_i on $X - \partial X$ and compatible with b (both in the sense of Def. 1.6.6).

Relative Version: If g is already transverse to 0 on B (a union of B_i) at c_B, compatible with $b|_B$, then the construction can additionally be achieved rel B. *If*

[24] Implicitly, this makes use of the vanishing of the controlled h-cobordism obstruction-groups (which is also guaranteed by our hypotheses) and the ensuing uniqueness of completions (cylinder-neighborhoods), see Rmk. 4.1.2.

g is already transverse to 0 on ∂X at c_∂, and is standard (Def. 1.6.6) on ∂X with respect to $f_i|_{\mathrm{im}(c_B)\cap\partial B_i}$ for all i, then the construction can additionally be made $\mathrm{rel}\,\partial X$*. Both can be achieved at the same time, i. e. if g is already transverse to 0 on B and ∂X, then the construction can be done* $\mathrm{rel}\,B\cup\partial X$*.*

Proof: As in the proof of the previous Cor. 1.6.4–1, the closed disjoint B_i can be treated individually. For ease of notation and readability, we assume there is only one $B_i =: B$.

If X has only one stratum, the result is given by manifold-transversality with boundary Cor. 1.5.9–2, so we may assume, that X has two non-trivial strata.

If $\partial X = \emptyset$, this is Cor. 1.6.4–1, with the addition of the last claim, that the result be standard with respect to F. This can be seen from the proof of Cor. 1.6.4–1, as described in Rmk. 1.6.7.

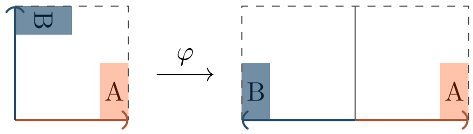
Otherwise, and if g is not yet transverse to 0 on B, obtain $g^B_\perp \simeq g|_B$ transverse to 0 in B at c_B, compatible with the boundary-collar $b|_B$, using manifold-transversality with boundary Cor. 1.5.9–2 (on $D^B = B$ and $C^B = \emptyset$ and $\mathrm{rel}\,C^B_\partial = \emptyset$ if g is also not given transverse on ∂X, or $\mathrm{rel}\,C^B_\partial = \partial B$ if g is already transverse on ∂X).

If g is not yet transverse on the boundary ∂X, apply the version for empty boundary to ∂X, to obtain $g^\partial_\perp \simeq g|_\partial$, transverse to 0 at c_∂ and standard with respect to $p_\partial = f|_{\mathrm{im}(c_B)\cap\partial B}$.

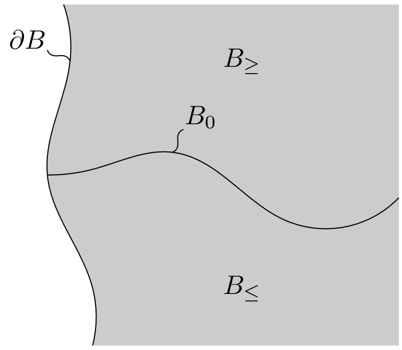
The crucial new observation is, that, the results of Thm. 1.6.4 applied to interior and boundary fit together by compatibility of f and F: On the one hand, using $p = F_{\mathrm{im}(c_B)\cap(B-\partial B)}$, Thm. 1.6.4 yields $U, V, W \subset X - \partial X$, which depend only on the form of F. On the other hand, applying Thm. 1.6.4 to p_∂ yields corresponding $U^\partial, V^\partial, W^\partial$ in ∂X, which depend only on the form of f. The compatibility condition on F and f means, that near the boundary, these overlap as $p_\partial \times \mathrm{id}\,(0,\gamma) = F|$ and hence $U = U^\partial \times [0,\gamma)$ (where we suppressed writing the identifying b in the notation) and similar for V, W (as is immediate from the explicit form of these subset as given in the statement of Thm. 1.6.4).

Further, $g^\partial_\perp$ being standard w. r. t. p_∂, additionally $U_\partial \times [0,\gamma) = (g^\partial_\perp)^{-1}(\{0\}) \times [0,\gamma)$. Note, that these form an open cover by MHSS (with boundary) of (U, U_∂), which is thus also a MHSS with boundary. The boundary-collar can be chosen as a restriction of b, since in the definition, b was implicitly used in defining $U_\partial \times [0,\gamma)$. We call this subspace of the cylinder-neighborhood $(X_0, \partial X_0) := (U, \partial U)$.

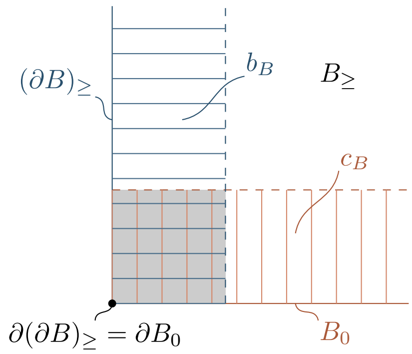
Extending the bi-collars from the lower stratum and boundary requires a little more effort, because without absorbing the boundary, see Section 3.2 ("Absorbing Boundary"), the boundary requires some special treatment as compared to the previous Cor. 1.6.4–1. We will work with the following setup (see also Figure 1.5 (b)): The transverse on B map $g^B_\perp$ divides B in two parts $B_\geq := (g^B_\perp)^{-1}([0,1])$ and $B_\leq := (g^B_\perp)^{-1}([-1,0])$. These are separated by $B_0 := (g^B_\perp)^{-1}(\{0\})$. all of these are manifolds with boundary. ∂B_0 are the corners $B_0 \cap \partial B$, while the boundary $\partial B_\geq$ (and correspondingly of $B_\leq$) consists of two parts: B_0 and $B_\geq \cap \partial B =: (\partial B)_\geq$. Indeed, also $(\partial B)_\geq$ is a manifold with boundary also given by the corners ∂B_0. Now, B_0 is collared in $B_\geq$ by one half of the bi-collar $c^B = c|_B$ of $B_0 \subset B$, because $g^B_\perp c^B(x,t) = t$.



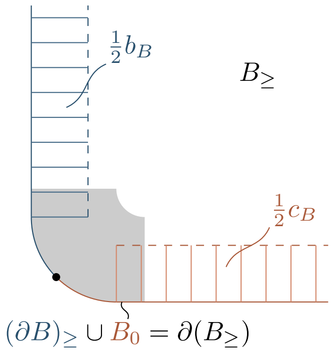
(a) The homeomorphism φ fixes the colored parts (see main text).



(b) The decomposition of B used, the corner of the upper "half" $B_{\geq}$ at its bottom left is shown in (c) and (d).



(c) The corner, where the one side of the bi-collar c_B of B_0 and the boundary-collar b_B (of ∂B restricted to $(\partial B)_{\geq}$) meet.



(d) The straightened corner, with a boundary-collar of $\partial(B_{\geq})$ determined by b_B, c_B and the choice of φ.

Figure 1.5: Corner-Straightening as used in the proof of Cor. 1.6.4–2.

Further, also the other part of the boundary of $B_{\geq}$, given by $(\partial B)_{\geq}$ is collared in $B_{\geq}$, this time by restriction of the collar $b^B = b|_B$ of $\partial B \subset B$, because c^B and b^B are also compatible.

Briefly, the construction can be outlined as follows (see also Figure 1.5 (c), (d)): These two collars of the individual parts of $\partial B_{\geq} = B_0 \cup (\partial B)_{\geq}$ will be combined (by straightening the corners) to a collar $c^B_{\geq}$ of $\partial B_{\geq} \subset B_{\geq}$. As this is a manifold with boundary, *some* collar exists anyway, but the point is, that we can fix one, relative to the given parts. Next, we will use Quinn's boundary-collaring theorem (Thm. 1.4.15), to extend $c^B_{\geq}$ to $c_{\geq}$, a collar of $\partial X_{\geq} \subset X_{\geq}$ (see below for the precise definition of the partitioning of X). From there we will unstraighten the corners, such that the result fits together with b extending (its restriction) $b^B = b|_B$ near the corners, and with c^B on the lower stratum B. Then, the construction can be continued as in Cor. 1.6.4–1.

We start by partitioning X. The pairs $(B_{\geq}, \partial B_{\geq})$ etc. above do not have immediate analogues in X, but they do in the cylinder-neighborhood $N := \mathring{\text{cyl}}(\bar{F})$ of B in X, by restricting the cylinder(-map) to the corresponding parts of B. We call these spaces $X_{\geq}, \ldots$, and note, that b extends (its restriction) b^B. Also b restricts (see above) to b_0 collaring $\partial X_0 \subset X_0$. The corners ∂X_0 are also stratified NDR (thus collared) in $(\partial X)_{\geq}$, because they have a neighborhood (W^∂, U^∂) from Thm. 1.6.4 (see definition of X_0 above). The boundary-collar b is strict (keeping $W - U$ in $W - U$ and U in U) with respect to the stratified NDR pair (W, U), which was obtained by Thm. 1.6.4. So the (stratified) deformation of W to U can be combined with $b|_{X_{\geq}}$ to see, that $\partial X_{\geq} \subset X_{\geq}$ is also a stratified NDR. Thus, once we have constructed a collar $c^B_{\geq}$ in B, it can be extended using Quinn's boundary-collaring theorem (Thm. 1.4.15).

Next, we construct this collar $c^B_{\geq}$. First we fix a homeomorphism straightening the corner (see Figure 1.5 (a)):

$$\varphi : [0, \epsilon) \times [0, \epsilon) \xrightarrow{\sim} (-\epsilon, \epsilon) \times [0, \epsilon)$$

such a φ certainly exists such that $\varphi(s, 0) = (s, 0)$ and $\varphi(0, t) = (-t, 0)$ and further, the lower half close to the "right" side is fixed: $\varphi(s, t) = (s, t)$ for $s > {}^{3\epsilon}/4$ and $t < {}^{\epsilon}/2$, and finally, close to the other side of the corner $t > {}^{3\epsilon}/4$, the "left" half $s < {}^{1}/2$ is simply rotated down to become the lower half near the left of the straitened corner as $\varphi(s, t) = (-t, s)$.

Then define near the "corners" ∂B_0, on $\operatorname{im}(c^B_\partial) \cup \operatorname{im}(b^B_0) \subset \partial X_{\geq}$, the collar $c^B_{\geq}$ as the composition (see Figure 1.5 (c), (d); with $c^B_\partial \cup b^B_0$ the collars of ∂B_0 in $(\partial B)_{\geq}$ and B_0 glued back-to-back along $\partial B_0 \times \{0\}$):

$$\begin{aligned} c^B_{\geq}| : (\operatorname{im}(c^B_\partial) \cup \operatorname{im}(b^B_0)) \times [0, \epsilon) &\xrightarrow{(c^B_\partial \cup b^B_0)^{-1} \times \mathrm{id}} \partial B_0 \times (-\epsilon, \epsilon) \times [0, \epsilon) \\ &\xrightarrow{\mathrm{id} \times \varphi^{-1}} \partial B_0 \times [0, \epsilon) \times [0, \epsilon) \\ &\xrightarrow{b^B_0 \times \mathrm{id}} \operatorname{im}(b^B_0) \times [0, \epsilon) \\ &\xrightarrow{c^B|} B_{\geq} \end{aligned}$$

By the choice of φ, after restriction to $[0, {}^{\epsilon}/2)$, this fits together with c^B on the rest of $\operatorname{im}(c^B)$: On $b^B_0(B_0 \times ({}^{3\epsilon}/4, \epsilon)) \times [0, {}^{\epsilon}/2)$ the composition is $c^B \circ (b^B_0 \times \mathrm{id}) \circ (\mathrm{id} \times \mathrm{id}) \circ ((b^B_0)^{-1} \times \mathrm{id}) = c_B$. By compatibility of c with the boundary, $c(b_0 \times \mathrm{id}) = b(\mathrm{id} \times c_\partial)$ (flipping the sides of the corner), so similarly it fits together with b^B on the rest of $\operatorname{im}(b^B)$: On $c^B_\partial(B_0 \times (-\epsilon, -{}^{3\epsilon}/4)) \times [0, {}^{\epsilon}/2)$ the composition is $b^B \circ (\mathrm{id} \times c^B_\partial) \circ$ (rotate around $(0, 0)$ by 90 degrees) $\circ\, ((c^B_\partial)^{-1} \times \mathrm{id}) = b^B$ where rotation and flipping sides cancel each other.

The resulting collar of $\partial B_{\geq} \subset B_{\geq}$ can be extended to one of $\partial X_{\geq} \subset X_{\geq}$ using Quinn's boundary-collaring theorem (Thm. 1.4.15), see above. We call the result $c_{\geq}$. Finally, we construct c^X as a collar of X_0 on the corners $\operatorname{im}(b_0) \subset X_0$ (see again Figure 1.5 (c), (d), this time we go in the other direction – constructing a half of the bi-collar of X_0 as fixed by the collar of $\partial(X_{\geq})$ and the choice of φ – and work in X instead of B, but the picture remains the same; also as before, at the corner we can glue $b_0 \cup c_\partial$ back-to-back, i. e. $b_0 \cup c_\partial(x,t)$ is $b_0(x,t)$ if $t \geq 0$ and $c_\partial(x,-t)$ if $t \leq 0$):

$$\begin{aligned} c^X| : \mathrm{im}(b_0) \times [0,\epsilon) &\xrightarrow{b_0^{-1}\times \mathrm{id}} \partial X_0 \times [0,\epsilon) \times [0,\epsilon) \\ &\xrightarrow{\mathrm{id}\times\varphi} \partial X_0 \times (-\epsilon,\epsilon) \times [0,\epsilon) \\ &\xrightarrow{b_0\cup c_\partial \times \mathrm{id}} \mathrm{im}(b_0 \cup c_\partial) \times [0,\epsilon) \\ &\xrightarrow{c_\geq|} X_\geq \end{aligned}$$

By the choice of φ this can be extended into $X_0 - \mathrm{im}(B_0) \times [0, \epsilon/2)$ by $c_\geq$, because close to the inside-end $b^0(\partial X_0 \times (3\epsilon/4, \epsilon))$, φ is given by the identity (on the lower half $[0, \epsilon/2)$), so the above composition reads $c_\geq \circ b_0 \circ (\mathrm{id} \times \mathrm{id}) \circ b_0^{-1}$. On the other hand, for $x_0 \in \partial X_0$ (thus $b_0^{-1}(x_0) = (x_0, 0)$), the composition is given as (using $\varphi(0,t) = (-t, 0)$, see above) $c^X(x_0,t) = c_\geq(c_\partial \times \mathrm{id}(x_0, \varphi(0,t))) = c_\geq(c_\partial(x_0,t), 0) = c_\partial(x_0,t)$. Thus $c^X = c_\partial$ on ∂X_0.

Furthermore, this also fits together with c_B on B: Restricted to B, the above composition reads (near corners) $c^X|_B| = c^B_\geq| \circ ((b^B_0 \cup c^B_\partial) \times \mathrm{id}) \circ (\mathrm{id} \times \varphi) \circ ((b^B_0)^{-1} \times \mathrm{id})$, plugging in $c^B_\geq| = c^B| \circ (b^B_0 \times \mathrm{id}) \circ (\mathrm{id} \times \varphi^{-1}) \circ ((c^B_\partial \cup b^B_0)^{-1} \times \mathrm{id})$ as defined above, everything except $c^B|$ cancels (near corners). Away from corners, $c^X = c^X_\geq$ by definition, extending $c^B_\geq$, which (away form corners) is given by $c^B_\geq = c^B$.

The same construction applied to the other side $X_\leq$ yields the other side of a bi-collar, which we will also call c^X, of $X_0 \subset X$. As of yet, this c^X agrees with c_∂ on ∂X, but may not be compatible with b. So, to make c^X compatible with the collar b of the boundary $\partial X \subset X$, glue an outside-collar (Rmk. 1.6.7), where $c^X = c_\partial \times \mathrm{id}$. (This does not change $g|_B$ or $c|_B$, because these are compatible with b already, and thus we are only gluing two products together, which is homeomorphic to the original product.) Outside of the cylinder-neighborhood N, this can then be extended by $c_\partial \times \mathrm{id}$ near ∂X (in the open neighborhood $\mathrm{im}(b)$).

As in the proof of Cor. 1.6.4–1, use a continuous $\eta : X \to [0,1]$ with $\eta = 1$ near $B \cup \partial X$ and $\eta = 0$ near the complement $X - (N \cup \mathrm{im}(b))$ of cylinder-neighborhood and boundary, to define

$$g'_\perp : X \to [-1,1], \quad x \mapsto \begin{cases} g(x) & \text{if } x \notin N \cup \mathrm{im}(b) \\ \eta(x)\pi_\mathbb{R} c^{-1}(x) + (1-\eta(x))g(x) & \text{if } x \in N \cup \mathrm{im}(b) \end{cases}$$

The restriction $g'_\perp|M$ to the upper stratum is thus transverse to 0 at c near $M \cap (B \cup \partial X)$. So we can apply manifold transversality (without boundary, as $C \supset \partial M$) Cor. 1.5.9–1, on M and $\mathrm{rel}\, C = \partial X \cup N'$, where N' is a neighborhood of $B \subset X$ intersected with M. This yields $g^M_\perp$ transverse to 0 on M at c', with $c' = c$ near $M \cap (\partial X \cup B)$. Thus c' can be extended to B by c and $g^M_\perp$ can be extended by $g'_\perp$ to $g_\perp$, also as in the proof of Cor. 1.6.4–1.

Near B this agrees with c_B and $g^B_\perp$, by construction of c, near ∂X it is a product (thus compatible with b) of $c_\partial \times \mathrm{id}$, thus extends c_∂.

Smallness of the construction (i. e. that H be an ϵ-homotopy and additionally $\mathrm{rel}\, g^{-1}(\{0\})$), can be achieved by the same arguments as in the proof of Cor. 1.6.4–1. □

From the construction of Connolly and Vajiac [CV99], it is evident how transversality is very closely related to end-problems ("end-problems" here means "cylinder-neighborhood-existence", see Section 1.8 ("Excursion: End-Theory")), an observation that is also discussed for example in [Qui82a, §2.2 (p. 365ff), "transversality is an end problem" (p. 367)].

We finish the present section, by illustrating, how also the opposite direction, transversality solving end-problems, seems to be true, at least for the problems studied here (excision in a bordism theory must be applicable repeatedly, so the "cut" X_0 must somehow be ensured to posses a cylinder neighborhood again).

Lemma 1.6.8: *Let X be a MHSS of dimension n (with two strata B and $M := X - B$ and without boundary), s. t.*

(i) $B = B_0 \times (-1, 1)$ (for some closed manifold B_0) and

(ii) there is $p = (n, s) : \partial M' \to B = B_0 \times (-1, 1)$, s. t. $X = \mathrm{cyl}(p)$.

Further assume, that $s : \partial M' \to (-1, 1)$ is transverse to zero. Then $X_0 := \mathrm{cyl}(p|_{s^{-1}(\{0\})})$ is a MHSS with cylinder-neighborhood of the lower stratum, and is stratified bi-collared in X.

Proof: See the more general Lemma 1.6.9 below.

Note, that this does not have any dimensional restrictions. One may also combine this with the construction of Connolly and Vajiac [CV99], if s is not transverse "everywhere". Then boundary-collaring (from stratified NDRs, Thm. 1.4.15) will reintroduce some dimensional-requirements into the construction of a "$g_\perp$", however.

Lemma 1.6.9: *"Modification of [CV99, Thm. 2.2 (p. 529)]": Let X be a MHSS of dimension n (with two strata B and $M := X - B$ and without boundary), s. t.*

(i) $B = B_0 \times (-1, 1)$ (for some closed manifold B_0) and

(ii) there is $p = (n, s) : \partial M' \to B = B_0 \times (-1, 1)$, s. t. $X = \mathrm{cyl}(p)$.

(M' is a manifold with boundary $\partial M'$ such that $M \subset M'$ and $M' - M \subset \partial M'$.) Further assume, that there is $\Sigma \subset \partial M'$, such that the restriction $s| : (\partial M' - \Sigma) \to (-1, 1)$ is transverse to 0 (in the manifold sense).

Set (as before; see Figure 1.6)

$$
\begin{aligned}
V &:= \{ \quad [x,t] \in \mathrm{cyl}(p) \quad | \quad s(x) \le t < 1 \quad \} \\
W &:= \{ \quad [x,t] \in \mathrm{cyl}(p) \quad | \quad 0 \le t \le s(x) \quad \} \\
U &:= \{ \quad [x,t] \in \mathrm{cyl}(p) \quad | \quad 0 \le s(x) = t < 1 \quad \}
\end{aligned}
$$

Then (as before)

(a) U is a stratified (strong) deformation retract of both V and W

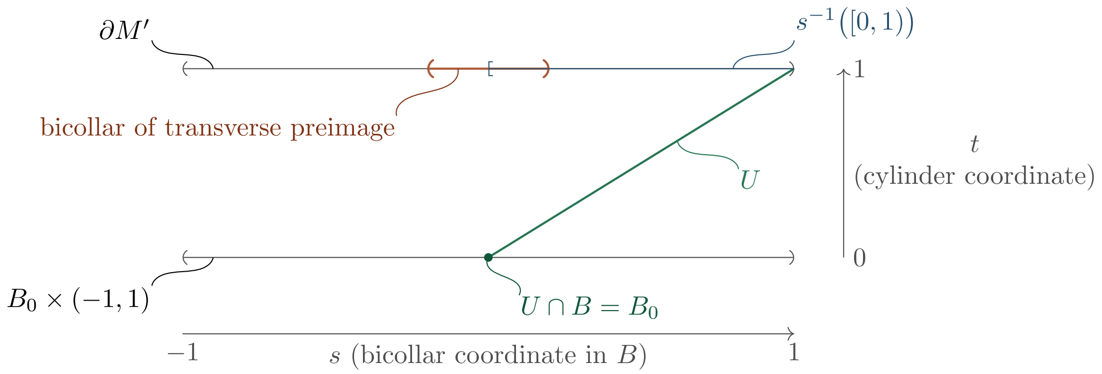


Figure 1.6: Pictorial representation of the construction by Connolly and Vajiac [CV99], similar to their Fig. 4 (p. 530), but now (away from Σ) the end of $s^{-1}\big((0,1)\big)$, and thus the upper stratum of U, has an "obvious" completion by $s^{-1}\big([0,1)\big)$, where before, it had to be chosen open to ensure that it is a manifold.

(b) $V \cup W = X$*;* $V \cap W = U$*;*
V *and* W *are closed subsets of* X*;*
The lower strata are $V_B = N \times (-1, 0]$*,* $W_B = N \times [0, 1)$ *and* $U_B = N \times \{0\}$

(c) U*,* V *and* W *are stratified subspaces of* X *(i. e.* ∂-MHSS *when given the induced stratification), with boundaries* $\partial V = \partial W = U$ *and* $\partial U = \emptyset$*.*

and further:

(d) The lower stratum $B \cap U$ *of* U *has a cylinder-neighborhood in* U *away from* $p(\Sigma)$ *stratified homeomorphic to* $\mathring{\text{cyl}}(p|_{s^{-1}(\{0\})-\Sigma})$ *rel* $B - \Sigma$*.*

Proof: We only need to verify the additional part (d), i. e. we need to show, that the stratified homeomorphism exists. To this end, let $c : (s^{-1}(\{0\}) - \Sigma) \times (-\epsilon, \epsilon) \to \mathring{\text{cyl}}(p)$ a stratified bi-collar with $s \circ c(x, t) = t$, which exists by the transversality hypothesis on s. Recall that there is a homeomorphism to its image of the upper-stratum

$$h : U \cap M \to \partial M' \cap s^{-1}\big((0, 1)\big), \quad (x, t) \mapsto x$$

which was used before to see that this is a manifold. Note that by definition of U implying $s(h(x, t)) = t$ the inverse is given by $h^{-1}(x) = (x, s(x))$. Define

$$\varphi : \mathring{\text{cyl}}(p|_{s^{-1}(\{0\})-\Sigma}) \to U \quad [x, t] \mapsto \begin{cases} h^{-1} \circ c(x, t) & \text{if } t > 0 \\ p(x) & \text{if } t = 0 \end{cases}$$

This is well-defined, because if $t > 0$, then by $s \circ c(x, t) = t$ automatically $c(x, t) \in \text{im}(h)$. If $t = 0$, then by $x \in s^{-1}(\{0\}) - \Sigma$, also $p(x) \in B_0 \times \{0\} \subset U$. Note that this also shows, that φ is stratum-preserving.

Further φ is continuous, which needs to be checked for $t \to 0$. By $s \circ c(x, t) = t$ an the form of $h^{-1}(x) = (x, s(x))$, it holds that

$$h^{-1} \circ c(x, t) = (c(x, t), t)$$

By continuity of p, the cylinder-identification in U given by $p(x) \sim \varphi(x,0) = (c(x,0),0)$ and in $\mathring{\text{cyl}}(p|_{s^{-1}(\{0\})-\Sigma})$ given by $p|(x) \sim (x,0)$ agree.

φ is a homeomorphism to its image, because, on the upper-stratum it is the composition of two homeomorphisms (to their images), and on the lower-stratum it is the identity of $B_0 \times \{0\}$. $\square$

These results might be useful for future work, to weaken the "dimensional-gap-hypothesis" that later will be encountered in the construction of bordism theories (see for example the hypothesis of the main theorem Thm. 5.1.2).

It remains to give a satisfactory answer to the question of existence of mapping-cylinder neighborhoods. Even more so, as Connolly and Vajiac [CV99, p. 536] in fact give a kind of "backwards"-direction (see Thm. 1.6.4), so these are (at least locally) *necessary.*

"Controlled Topology", as briefly explained in the in the following sections, does provide reasonable answers, at least in high dimensions, to these questions.

1.7 Excursion: Controlled Topology

This section briefly summarizes some ideas and results from controlled topology (see e. g. [Qui79; Qui02; Cha83]), with a focus on controlled end-theory (existence of mapping-cylinder neighborhoods), also known as "ends of maps" [Qui79; Qui82a; Qui82b; Qui86].

We start by recalling some classical ("uncontrolled") results, namely the h-cobordism-theorem (i. e. $\pi_1 = 0$) [Sma62] and the s-cobordism-theorem proved by Barden, Mazur and Stallings (see for example [Ker65; Coh73]) and – very closely related – (uncontrolled) end-theory, initiated[25] by Freudenthal [Fre31] and later developed further by Siebenmann [Sie65] (see also [HR96] and Section 1.8 ("Excursion: End-Theory")). We will only discuss the high-dimensional cases.

We start by recalling what an h-cobordism is:

Definition 1.7.1: *A* (manifold-)cobordism *is a triple* $(W^{n+1}; M_0^n, M_1^n)$ *where* W *is a compact oriented* $(n+1)$*-manifold with boundary* $\partial W = M_0 \sqcup -M_1$*.*

An (uncontrolled) $(n+1)$*-dimensional* h-cobordism $(W^{n+1}; M_0^n, M_1^n)$ *is a cobordism which deformation-retracts to both* M_0 *and* M_1*.*[26]

A h-cobordism is trivial *if it is a product* $W \cong M_0 \times I$*.*

The question, which high-dimensional, meaning $\dim(W) \geq 5+1$, h-cobordisms are trivial is answered by the "s-cobordism-theorem". The formal starting point is

[25] according to [HR96, p. 'x']

[26] So the inclusions of M_0 and M_1 are homotopy-equivalences, and W has the homotopy type of a cylinder on M_0, hence the "h-".

usually the observation, that this is the case iff the retractions are *simple* (in the sense of simple homotopy-theory), hence the “s-” in the name. The problem can be approached using handle-body decompositions of topological manifolds – which exist *except* for non-smoothable 4-manifolds (see e. g. [FQ90, §9.2 (p. 136)]), hence always exist in “high dimensions”. By a handle-body we mean (technical details will not be discussed here, but the following definition should provide sufficient understanding of the concept for the ideas sketched below)

Definition 1.7.2: *See e. g. [KS77], [FQ90, §9.1 (p. 134–136)] or [Wal71]: Given an n-manifold W and a codimension 0 submanifold $N \subset \partial W$ of its boundary, then:*

A k-handle *is an embedding i of $D^k \times D^{n-k}$ into W. It is* attached to $i((\partial D^k) \times D^{n-k})$. *Further $i((\partial D^k) \times 0)$ is called the* attaching-sphere (a-sphere), $i(0 \times (\partial D^{n-k}))$ *is called the* belt-sphere (b-sphere).

A (relative) handlebody-structure *on (W, N) is a filtration $W_0 \subset W_1 \subset \ldots \subset W$, where:*

(1) W_0 is a collar $N \times I$ on N.
(2) W_{i+1} is obtained from W_i by attaching a handle to $\partial W_i - N \times \{0\}$.
(3) Handles are locally finite.

The details of the following discussion are beyond the scope of this overview, see for example [Coh73; Hud69] for a more complete treatment of the matter. We want to focus our discussion on two important questions encountered in the proof of the s-cobordism-theorem.

(1) “Geometric Connectivity” ([Wal71]): A pair (W, M_0) is (relatively) “homotopically i-connected” if $\forall j \leq i : \pi_j(W, M_0) = 0$. It is “geometrically i-connected” if W can be obtained from (a collar on) M_0 by adding only j-handles for $j > i$. Clearly geometric connectivity implies homotopic connectivity, but when does the converse hold?
(2) “Algebraic Classification”: How can the geometric data, of the (simplified by (2)) handle-body structure of an h-cobordism, be “classified” algebraically?

The answer to the first question is (roughly): One can eliminate handles of dimension $\leq i$ by potentially adding i and $i+1$ dimensional ones if i is not “too close to $\dim(W)$”. Now, h-cobordisms are not only relatively homotopically i-connected for all i, but they can also be “turned around”: An i-handle $D^i \times D^{n-i}$ can be regarded as a $(n-i)$-handle, the handle-decomposition of (W, M_0) then becomes one of (W, M_1) with handles in complementary dimensions. But the problem is symmetric in M_0 and M_1, so one can start eliminating handles from one direction, then turn around and eliminate those previously “too close to $\dim(W)$” which are now in low dimensions. This allows elimination of all handles except for those in two adjacent dimensions k and $k+1$ on a collar of M_0. For connectivity reasons, these must appear in pairs:

$$W = M_0 \times I \quad \cup \quad \text{pairs of the form (} k\text{-handle, } (k+1)\text{-handle)}$$

Clearly the h-cobordism is trivial iff all the remaining handles can be "absorbed" into the collar on M_0 (if the collar with the pair attached is again homeomorphic to a collar).

More generally, answering the second question, one observes: A pair of a k-handle and a $(k+1)$-handle is homeomorphic to a n-disc attached along a $(n-1)$-disc in its boundary (an "elementary expansion" in the sense of simple homotopy theory) and can thus be absorbed into the collar *if and only if* the "attaching-sphere" (a-sphere) of the $k+1$ handle intersects the "belt-sphere" (b-sphere) of the k handle transversality in exactly one point. In high dimensions, this can in turn be arranged by general position arguments *if and only if* the *algebraic* intersection "number" is $1 \in \mathbb{Z}\pi_1(W)$.

Algebraic intersection-data is in the group-ring of the fundamental groups, because given an intersection-point, since handles are simply-connected, after fixing paths (from a basepoint of W) to the basepoints of handles, there is a well-defined element in $\pi_1(W)$ obtained by moving to the base-point of the first handle, moving to the intersection-point (within that handle), switching into the other (intersecting) handle, moving in that handle to its base-point, and finally back to the basepoint of W. Then formally sum up all intersections with sign depending on orientation. See for example [FQ90, §1.7 (p. 21–23)].

The intersection data of all the k and $(k+1)$-handles can be arranged into a square matrix (of all possible pairs), with entries in $\mathbb{Z}\pi_1(W)$. Note that adding or removing trivial pairs does not alter the geometry (the homeomorphism-type). So any classifying algebraic invariant must allow stabilization and destabilization (of these matrices). Further, one may rearrange the order of handles without changing the geometric problem, corresponding to row or column swaps. Slightly more general, certain "sliding" operations will not change the geometry, algebraically, these manifest themselves as "elementary matrices" (beyond row or column swaps). Moving along loops, one can further "multiply by π_1-elements on the diagonal" (also, the "transverse intersection in one point" condition above is clearly insensitive to the π_1-element associated to that intersection). Accounting for all of these identifications on the algebraic side, one obtains an element in the "Whitehead-group" $\mathrm{Wh}\big(\pi_1(M_0)\big)$. Noteworthy, this depends *only* on the fundamental-group $\pi_1(M_0) = \pi_1(M_1) = \pi_1(W)$.

The choice of base-point fixing π_1 should be irrelevant from the geometric perspective and indeed can be eliminated by the algebraic identifications indicated above. Hence we will simply write $\pi_1(W)$, without reference to a base-point. For non-path-connected spaces this should be read "component-wise".

This is summarized by the well-known s-cobordism-theorem:

Theorem 1.7.3: *"s-Cobordism Theorem (Barden, Mazur, Stallings)": A (path-connected) h-cobordism $(W; M_0, M_1)$ of dimension at least $5+1$ is classified by an invariant – the "Whitehead-torsion" – in the abelian group $\tau \in \mathrm{Wh}(\pi)$, where $\pi = \pi_1(M_0) = \pi_1(M_1) = \pi_1(W)$, in the sense that:*

(i) Two h-cobordisms $(W; M_0, M_1)$ and $(W'; M_0, M_1')$ on M_0 are homeomorphic rel M_0 if and only if *they have the same Whitehead-torsion $\tau = \tau'$.*

(ii) Given M_0 and $\tau \in \mathrm{Wh}(\pi_1(M_0))$, there is an h-cobordism $(W; M_0, M_1)$ with Whitehead-torsion τ.

(iii) Two h-cobordisms $(W; M_0, M_1)$ and $(W'; M_1, M_2)$ with Whitehead-torsion τ and τ' can be glued to an h-cobordism $(W \cup W'; M_0, M_2)$ with Whitehead-torsion $\tau + \tau'$

Further, the trivial h-cobordism $(M_0 \times I; M_0, M_0)$ has Whitehead-torsion $\tau = 0$.

Hence, by (i), a (high-dimensional) h-cobordism is trivial *if and only if* it has vanishing Whitehead-torsion. To see how and why this is relevant to the treatment of cylinder-neighborhoods of strata in MHSS, we first have a look at an example. The subsequent proof also illustrates how mapping-cylinders can be constructed given certain "nice" neighborhoods (and neighborhood-boundaries specifically).

Example 1.7.4: *Let $(W^{n+1}; M_0, M_1)$ be a cobordism of dimension $n+1 \geq 5+1$. Define $N := {}^{W}\!/_{M_0} \supset {}^{M_0}\!/_{M_0} = \{\mathrm{pt}\}$. Then:*

(i) N is a stratified cone $c(M_1)$ if and only if *$(W^{n+1}; M_0, M_1)$ is a h-cobordism.*

(ii) Given a manifold-stratified space $X \supset \{\mathrm{pt}\}$, and two cone-neighborhoods N, N' of $\{\mathrm{pt}\}$, then:
If *the h-cobordisms have the same Whitehead-torsion $\tau = \tau'$, there are $t, t' \in (0, 1]$ and an isotopy of id_X to $h : X \to X$ such that $h(N_t) = N'_{t'}$*

Here N_t is the image of ${}^{M_1 \times [0,t]}\!/_{\text{identifications at } 0} \subset c(M_1) = N$ in X under the embedding of N (as a neighborhood of $\{\mathrm{pt}\}$).

Remark: Part (ii) could be formulated as an if and only if statement, if the isotopy is required to be level [Qui86, Prop. 2.4 (p. 1150)] – in the sense of not only matching (all $x \in M_0$ at) at one specific t to t' but to match all t to $t'(t)$ – this is much harder however (see proof for further information).

Proof: Part (i): "$\Rightarrow$": Manifold boundaries are collared [Bro62] (see Thm. 1.1.2) so things can be "pulled away" and "pushed into" both M_0 and M_1 for ϵ-far along the collar coordinate. But the interior of the cone is a product, so it is clear how to build a retraction there. Compose pushing, interior and pulling into a retraction.

"$\Leftarrow$" (sketch; a complete proof is a easy consequence of the end-theorem quoted later): This is based on the argument Quinn [Qui02, §3.3 (p. 479)] calls a "swindle"[27] (see also [Qui79]): If W is a h-cobordism, then $W - M_0$ is a manifold with boundary with the "correct homotopy-type" of an open collar: $(W - M_0, M_1) \simeq (M_1 \times (0, 1], M_1 \times \{1\})$.

So we start with a "neighborhood of the end", that is "nice" in the sense, that on the side away from the end (the "end" here is the open direction of $W - M_0$, where "M_0 is missing") it is admits a *manifold boundary* $M_1 \times \{1\}$, and has the

[27]Probably in reference to what is known as the "Mazur-swindle" (or on the algebraic side as "Eilenberg-swindle"), of organizing terms in infinite sums in a way to make them trivially cancel each other (locally).

"homotopy-type" of an open collar. For now, by a "neighborhood of the end", we mean: An open neighborhood U of $M_0 \subset W$ intersected with $W - M_0$ (for the general definition see below or Appendix A ("Ends in MHSS")).

Assume for a moment, that knowing about this (homotopical) "niceness" of our initial "neighborhood of the end", such "nice" neighborhoods could be constructed inside any open "neighborhood of the end". The explanation of when or why this would be possible is deferred to later in this section, see question (b) below.

If the above assertion holds, we may pick a "neighborhood of the end" $U_0 \subset W - (M_0 \cup M_1)$ and find a "nice" neighborhood $(N_0, \partial N_0)$ within U_0. Then we can continue by picking $U_1 \subset N_0 - \partial N_0$ and so on. Clearly, we can choose the U_j to be closer than $\frac{1}{j}$ to M_0, e.g. as $U_j := (N_{j-1} - \partial N_{j-1}) \cap \{x \in W - M_0 | \operatorname{dist}_W(x, M_0) < \frac{1}{j}\}$.

Since the $(N_j, \partial N_j)$ are manifolds with boundary, there are collars $c_j : \partial N_j \times [0, \infty) \to N_j$ of ∂N_j. By possibly making the collars and U_{j+1} smaller, we may assume $\operatorname{im}(c_j) \cap N_{j+1} = \emptyset$. The central idea is, that the regions "between neighborhood-boundaries" (up to "gaps" at $[0, 1]$), see Figure 1.7

$$N'_j := N_j - c_j(\partial N_j \times [0, 1)) - (N_{j+1} - \partial N_{j+1})$$

are h-cobordisms. This can be seen from the homotopical part of the "niceness" assumption, but will be skipped in this proof-sketch, as it is rather technical, see for example [Qui79].

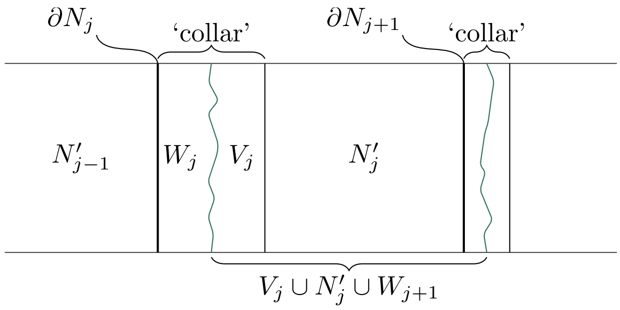


Figure 1.7: Illustration of the "swindle" argument, cf. [Qui82a, p. 410]. The parts labeled by "collar", are $c_j(\partial N_j \times [0, 1])$.

These h-cobordisms have Whitehead-torsions, say τ_j. By the realization-part of the s-cobordism-theorem, there is a h-cobordism W_j with torsion $\tau(W_j) = -\sum_{i \le j} \tau_i$, and V_j with torsion $\tau(V_j) = \sum_{i \le j} \tau_i$. Since $\tau(W_j \cup V_j) = 0$ we can embed this *product* in the "boundary-collar" at $c_j(\partial N_j \times [0, 1])$. Now, $\tau(V_j \cup N'_j \cup W_{j+1}) = 0$ so this is a product. Then repeat the argument inductively. See again Figure 1.7.

By the "$1/j$-smallness" of U_j, we will approach M_0. Gluing all these products gives a global product $W - M_0 \cong M_1 \times (0, 1]$. Thus $N \cong_{\text{strat}} c(M_1)$.

Part (ii): We assume $N' \subset N$ (a reference for the general case is given at the end of the proof). Let $N'' := N - (N' - \partial N')$ be the "region between" $\partial N'$ and ∂N. This is a h-cobordism (see above) with invariant τ''. But $N = N' \cup N''$, hence $\tau = \tau' + \tau''$.

So if $\tau = \tau'$, then $\tau'' = 0$ and N'' is a product. h maps a collar of ∂N to N'', and the isotopy is constructed in the evident way.

Note that this only aligns the cones at one "slice". There is a more general statement (giving a level-wise homeomorphism) [Qui86, Prop. 2.4 (p. 1150)]. However this requires a "swindle" argument very similar to the one above, but "one tier[28] higher" using "pseudo-isotopies" in the place of h-cobordisms. □

So (at least over points) h-cobordisms do *not* answer the above question about *existence* of mapping-cylinder neighborhoods, but they do explain the matter of *uniqueness* (see Thm. 1.8.5 (iii) and Thm. 1.9.3 (iii)).

This example immediately raises two further questions however:

(a) What happens for bottom stratum $\neq \{\mathrm{pt}\}$, i. e. for "actual" mapping-cylinders, as opposed to cones?

(b) The manifold-neighborhoods N of the example already have the "homotopically correct" boundary $\partial N = M_1$, in the sense that $(N - \{\mathrm{pt}\}, \partial N)$ has the homotopy-type of an open collar $(M_1 \times (0,1], M_1 \times \{1\})$. How are such neighborhoods constructed?

The first one (a) will be answered by controlled topology. The second one (b) is answered by supplementing the concept of an h-cobordism with the – very closely related – concept of an "end".

Remark 1.7.5: We have seen when discussing the proof of Connolly and Vajiac's [CV99] transversality theorem Thm. 1.6.4, that finding the "correct boundary" incorporates much of the difficulty inherent to the construction. This only further emphasizes the relevance of (b). Indeed the end-*obstruction* can be understood as the obstruction to the existence of one (and as it turns out, thus any) such (small-enough) "nice" neighborhood.

We start by treating the first question (a). To this end, some of the basic ideas of "controlled topology" are reviewed next. One such idea is to add "size-control" to the constructions related to (e. g.) h-cobordisms. For example, one may ask: Given a trivial h-cobordism $M_0 \times I$, the "rays" $t \mapsto (x,t)$ are constant when projected to M_0. If we were given some map $f : W \to M_0$, with $f|_{M_0} = \mathrm{id}$, taking the place of the product-projection, but for a general h-cobordism $(W; M_0, M_1)$, under which hypotheses is W an ϵ-product? I. e. when is there a homeomorphism $h : M_0 \times I \to W$ rel M_0, such that additionally the "rays" $t \mapsto h(x,t)$ are ϵ-small when "projected" by f, that is $\mathrm{diam}(t \mapsto f \circ h(x,t)) < \epsilon \ \forall x \in M_0$. This example would add "control over M_0" in the sense of "controlled topology".

[28] There is a very similar theory for these in [Qui86]: Basically Quinn gives obstructions q_0, q_1, q_2 in degrees 0, 1 and 2 of a "homology-theory" for completions of ends, h-cobordisms, and pseudo-isotopies respectively (see also next section).

Clearly, this idea is not limited to using M_0 for measuring sizes. All that is really needed is a metric space X and a proper[29] map $f : W \to X$. To get a better grasp of what kind of hypotheses will be required, we briefly look at how the arguments for the situation over the point – the "classical / uncontrolled case" is, from this viewpoint, just the special case of $X = \{\mathrm{pt}\}$ – need to be modified to accommodate control.

The key idea is, to make the constructions outlined above "small" on a low level.[30] Of the two steps (in the proof of the "classical" s-cobordism theorem) outlined initially, the "geometric connectivity" part does not reveal anything fundamentally new, so we will focus on the translation to algebra instead. The reader interested in geometric connectivity may consult [Qui79, §6 (p. 308–314)].

Firstly, however, note that if we were to successfully obtain a small product-structure, this can in turn be used to construct small retractions (by pushing along the $\times I$-coordinate). Hence, necessarily, the retractions of the h-cobordism can be made small. The typical result will be phrased something like "$\forall \epsilon > 0 \exists \delta > 0$ such that a (δ,h)-cobordism with … is an ϵ-product if and only if … ". Where (δ,h)-cobordism is taken to mean

Definition 1.7.6: *[Qui79, Def. 2.6 (p. 284)]: Suppose $(W, \partial W)$ is a manifold with boundary, X a (compact)[31] metric space, $f : W \to X$ proper and $\delta > 0$. Let $M_0 \subset \partial W$ be a codimension 0 submanifold, and $M_1 := \mathrm{closure}(\partial W) - M_0$.*

- *(i)* (W, M_0) *is a* (δ, h)-cobordism over X, *if, for $i = 0, 1$, there are homotopies R_i of W rel M_i into M_i of diameter $< \delta$ when measured in X, i. e.* $\mathrm{diam}(fR_i(\{w\} \times I)) < \delta$ *for all $w \in W$.*
- *(ii)* (W, M_0) *is a* δ-product over X, *if there is a homeomorphism $h : M_0 \times I \to W$, which is the identity on $M_0 \times \{0\}$ and has diameter $< \delta$ over X, i. e.* $\mathrm{diam}(fh(\{m_0\} \times I)) < \delta$ *for all $m_0 \in M_0$.*
- *(iii)* (W, M_0) *is a (controlled) h-cobordism over X, if it is a (δ, h)-cobordism over X for any $\delta > 0$ (see [Qui82a, p. 357]).*

Having fixed a notion of controlled h-cobordisms, we return to the handlebody arguments, that gave a correspondence of geometry and algebra in the uncontrolled case. Assuming all handles are subdivided "fine enough" that they themselves are small, the algebraic description needs to capture, that any "rearrangements" of handles must be achieved by small moves. Some of these problems are relatively straight-forward, for example stabilization via a pair of large handles can be replaced by a "chain" of many pairs of small handles. But some of the constructions need to

[29] Ordinary / uncontrolled h-cobordisms are assumed compact, here this is naturally weakened to properness of f.

[30] A "low level" in the sense of already very elementary operations (on handles) are controlled, then assembled into controlled higher-level results.

[31] See the end of this section for a more general notion.

be shrunk by a torus-trick (similar to Kirby's [Kir69], see also [Qui10]). This is done inductively over handles[32] of the control-space X. Roughly speaking, for each handle, things over the "center" part are pulled back to an immersed punctured i-torus, the puncture is "filled in", next everything is lifted to a universal covering $\mathbb{R}^i$ (for an i-handle), then some "large" disk is "compressed" into the unit-disk, and finally all is pushed back down (and is now smaller than before). The trace this procedure leaves is most apparent in the formulation of [Qui79], where theorems are stated using a $\mathrm{Wh}(\pi \times \mathbb{Z}^i) = 0\ \forall i \geq 0$ condition (the group π will be identified in the next paragraph). Note the fundamental group $\pi_1(i\text{-torus}) = \mathbb{Z}^i$ of the i-torus. Giving actual obstructions and an "if and only if"-type statement as in [Qui82a] requires the use of either hard to calculate pseudo-isotopy-spectra or rather sophisticated homology-like theories ("homology with spectral-cosheaf coefficients", see for example [Qui82a, §8 (p. 419ff)] or slightly more detailed [Qui04, §6 (p. 50ff)]; some statements about this theory are also given in the background-section of Chapter 3 ("Multiple Strata")).

While it makes sense, that some group "π" related to W or f should appear, it is not quite so evident, what the correct choice for π is supposed to be. Clearly if there is a "small" solution to problems, this should be something local over X – often called a "local fundamental group". The basic idea is that it consists of small loops up to small homotopies (a detailed treatment is given in Appendix A ("Ends in MHSS")).

There is an other point of view: The "fiber" of the map f should also describe such local π_1-properties. Indeed, over the point $X = \{\mathrm{pt}\}$, we know, this is related to the "fiber" of the map $f : W \to \{\mathrm{pt}\}$, as $\pi = \pi_1(W)$. In general, f may, of course, not be a fibration and there may not be a well-defined homotopy-type of the fiber. *But*, ultimately, we are interested in MHSS, which have holink-fibrations by definition. It turns out, that these can be used to describe the obstruction-groups for the completion of controlled ends, see Section 1.9 ("Mapping-Cylinder Neighborhoods"), details are given in Appendix A ("Ends in MHSS"). We have seen a similar effect before: By Hughes' cylinder-theorem (Thm. 1.4.19), mapping-cylinders of MAF are MHSS, thus have homotopy-link-fibrations with well-defined fiber. So, also here, approximate behavior and holink-fibrations are connected.

A precise statement can be given, for π the local fundamental group (calculated for example as fundamental-group of the fiber of the homotopy-link), as

Theorem 1.7.7: *"Thin h-Cobordism Theorem" [Qui79, Thm. 2.7 (p. 284)]: Given a locally 1-connected[33] compact metric space X and $\epsilon > 0$, then there is $\delta > 0$,*

[32]The problem can be reduced via Hilbert-cube-arguments from general metric X to manifold control spaces. We are interested in MHSS, where control is provided over the lower stratum, thus a manifold, so the Hilbert-cube-arguments are only needed, if B is a non-smoothable 4-manifold.

[33]The "locally 1-connected" hypothesis is not necessary when this is formulated using $(\delta, 1)$-connected control maps (see the end of this section). However, we are interested in manifold base spaces anyway, which are locally 1-connected.

such that any (δ, h)-cobordism of dimension $\geq 5+1$ *with* $\mathrm{Wh}(\pi \times \mathbb{Z}^i) = 0\ \forall i \geq 0$ *(for* π *the local fundamental-group) is an* ϵ*-product.*

Relative version: If additionally given a $D \subset X$*, then there is* δ*, such that given a* δ*-product-structure on* $D^\epsilon = \{x \in X \mid \mathrm{dist}(D, x) < \epsilon\}$*, the resulting product-structure can be chosen to agree with the given one on* D*.*

If X *is not compact, but* F *a controlled h-cobordism and proper, the absolute result is still true after replacing* ϵ *and* δ *by maps* $X \to (0, \infty)$*. [Qui82a, Thm. 1.2 (p. 357)]*

Corollary 1.7.7–1: *(Cf. [Qui79, Thm. 1.5 (p. 280) or Thm. 3.1.1 (p. 286)]) Given two MAF* $f : M \to B$ *and* $g : M' \to B$ *with* $\mathrm{Wh}(\pi \times \mathbb{Z}^i) = 0$ *for* $\pi = \pi_1(L_f)$ *the fundamental group of the fiber of the homotopy-link of* $B \subset \mathring{\mathrm{cyl}}(f)$*, further a proper controlled h-cobordism* $F : W \to B$ *from* f *to* g*, for dimension* $\geq 5+1$*.*

Then $\mathring{\mathrm{cyl}}(f) \cong_{strat} \mathring{\mathrm{cyl}}(g)\,\mathrm{rel}\,B$*.*

Proof of the corollary: The question of why this is the correct "π" for applying the theorem is discussed in Appendix A ("Ends in MHSS") and [Qui82a] (see also paragraph below).

Note that also $F \times \mathrm{id} : W \times (0,1] \to X \times (0,1]$ is a proper controlled h-cobordism. Let $\epsilon : X \times (0,1] \to (0,\infty), (x,t) \mapsto {}^1\!/t$. The ϵ-product structure on $W \times (0,1]$ induces an ϵ-homeomorphism $\varphi : M \times (0,1] \to M' \times (0,1]$. Note that by the choice of ϵ and the control being just id in the second coordinate, this extends by id_B to a stratified homeomorphism $\mathrm{cyl}(f) \cong_{\mathrm{strat}} \mathrm{cyl}(g)$. □

There are other formulations, see e. g. [Qui82a, Thm. 1.6 (p. 359)]. Much of the power of the statement comes from the early choice of δ – *before* any control-map or even a h-cobordism is fixed. For an interesting example of how to use this type of theorem, and the "late choice" of control-maps in particular, see Quinn's proof [Qui88a, §5.3 (p. 492–495)] of his boundary-collaring theorem (Thm. 1.4.15 presented earlier when discussing MHSS).

This notion of "approximate" seems to describe for AF, what transition-functions are for bundles, in the following sense:

Example 1.7.8: *Let* $(W; M_0, M_1)$ *be a cobordism (neither "h-" nor "controlled"). Further, let* $f : W \to B$ *be a MAF and* $\epsilon > 0$*. There are (automatically) collars* c_0 *and* c_1 *of the manifold-boundary* M_0 *and* M_1*, we may assume they are "*${}^\epsilon\!/3$*-thin" (in the sense of* $\mathrm{diam}(c_i(\{x\} \times I)) < {}^\epsilon\!/3$*).*

Suppose there is an embedded product-structure in $B_0 \times \mathbb{R} \subset B$*, such that* $f(\partial W) \cap B_0 \times \mathbb{R} = \emptyset$ *and for some* $N \in \mathbb{N}$*,* $f^{-1}(B_0 \times (-\infty, -N)) \subset \mathrm{im}(c_0)$ *and*

$f^{-1}(B_0 \times (N, \infty)) \subset \mathrm{im}(c_1)$. Pick

$$\gamma := \min\Big(\epsilon/3, \quad \mathrm{dist}\big(f^{-1}([-6N, 6N]), \quad W - f^{-1}(B_0 \times \mathbb{R}) \big), \quad \mathrm{dist}\big(f^{-1}([-N, N]), \quad W - f^{-1}(B_0 \times (-2N, 2N)) \big) \Big)$$

and use the approximate lifting property to obtain a γ-lift R_0^W starting at the identity of the "isotopic pushes" $R_0^B : B \times I \to B$ given by id *outside $B_0 \times \mathbb{R}$ and $(b_0, t) \mapsto (b_0, t - 4N)$ on $B_0 \times \mathbb{R}$. (Similarly for R_1^W.) Then, define $R_0 : W \times I \to W$ by composing, "pushing away" from M_1 along c_1 then R_0^W and finally "pushing into" M_0 along c_0.*

The resulting R_0, R_1 give $f' : (W; M_0, M_1) \to B_0$ the structure of an (ϵ,h)-cobordism where $f' = \pi_{B_0} \circ f$ outside $\mathrm{im}(c_0) \cup \mathrm{im}(c_1)$ *and $f'(c_i(x,t)) :=$* $\lim_{s\to\infty} \pi_{B_0} \circ f(c_i(x,s))$ *on* $\mathrm{im}(c_i)$ *respectively.*

Some care needs to be taken to understand the relation of the local fundamental groups of f' and those of f. Since f is an AF, the holink-fiber (of its mapping-cylinder) provides a natural starting-point. Using a formulation via (δ,h)-connectivity of the homotopy-link (see below and Appendix A ("Ends in MHSS")), it is relatively straight-forward to see, that this is a "thin" h-cobordism.

This example also illustrates why "strong bundle-structure" (vector-bundles and the like) are not expected in a topological context: The natural "transition-functions / structures" that occur between "local trivializations" are not products but rather h-cobordisms. For vanishing $\mathrm{Wh}(\pi \times \mathbb{Z}^k)$ this may seem fine, but *in low dimensions* there is not much hope for finding geometric structures. To get "nice transitions", one needs some sort of inductive argument (e. g. over handle-dimensions in the base) making more than just one direction into a $\times\mathbb{R}$- coordinate. It seems like any such procedure must handle the transition to low dimensions – including dimension 4, which is notoriously "ill-suited" for these constructions.

We can now extend the initial Example 1.7.4 to more general base $\neq \{\mathrm{pt}\}$. Note that a subtle, but non-trivial, new problem appears in the "swindle"-argument:

Example 1.7.9: *Let $(W^{n+1}; M_0, M_1)$ with $f : W \to B$ proper be a controlled cobordism of dimension $n + 1 \geq 5 + 1$. Define $N := W \cup_{f|_{M_0}} B \supset B$. Then:*

(i) N is a stratified cylinder cyl$(f|_{M_1})$ if and only if *$(W^{n+1}; M_0, M_1)$ is a controlled h-cobordism.*

(ii) Given a manifold-stratified space $X \supset B$ and two such cylinder-neighborhoods N, N' of B, there are $t, t' \in (0, 1]$ and an isotopy of id_X *to $h : X \to X$ such that $h(N_t) = N'_{t'}$* if *the h-cobordisms have the same controlled obstruction $q_1 = q_1'$.*

Proof: The proof is mostly the one sketched in Example 1.7.4, with control added. See also [Qui82a, §6.5 (p. 410f)] for a similar argument.

We only indicate required modifications. These occur in the "$\Leftarrow$"-direction of part (i): Before, the successive N_i were only moved closer to M_0, now we also require, that the successive *products* become smaller and smaller (over B). Fix $\epsilon_j \searrow 0$ to ensure this (the h-cobordisms N'_j need be δ_j-small). This makes "cylinder-rays" $f(\{x\} \times (0,1])$ "Cauchy" over B. Then continuity of f near M_0 is enough to guarantee "convergence", in the sense that X is actually the cylinder as claimed.

There is, however, one major caveat in this argument: Before, it was enough to find some h-cobordisms W_j and V_j with invariants $-q_1^{j-1}$ and q_1^{j-1} to glue in a product part of a collar of ∂N_j. The existence of V_j – which was trivial before (one could just have used the "original" N'_j) – suddenly faces a problem: It needs to be δ_{j+1} small, while the original h-cobordism N'_j was just δ_j-small, because we will replace $N'_{j+1} \mapsto V_j \cup N'_{j+1}$ for the next step.

This problem is solved by a "stability theorem" [Qui82a, §4 (p. 381–388)], which roughly states, that once the geometric objects become small enough, the algebraic obstructions become "stable" in the sense, that they do not change when the geometric sizes are further reduced. So there is a (fixed initially) $\delta_0 > 0$ such that, if we additionally require all the h-cobordisms to be $< \delta_0$ – this is really only a change for the first "few" (finitely many) in the sequence – then, the V_j *can* in fact be realized δ_{j+1}-small.

In other words: It is still true, that the construction works as before, but this now requires the highly non-trivial stability-theorem mentioned above. $\square$

So, for the most part, controlled topology seems to work just fine for dealing with mapping-cylinder neighborhoods and it remains to understand question (b) from above: How to find the correct neighborhood(-boundary)?

Before discussing this question in detail in the next two sections, finally, we remark, that it is possible to formulate the hypothesis used for the thin h-cobordism theorem using $(\delta, 1)$-connected maps to fibrations (similar to how we will control ends in MHSS via holink-fibrations). Here $(\delta, 1)$-connected means

Definition 1.7.10: *[Qui79, Def. 2.3 (p. 282f)] Given $f : M \to X$ and $p : E \to X$, a control-map $F : M \to E$ with $pF = f$ is $(\delta, 1)$-connected, if given a relative 2-complex (R, S) and a commutative diagram*

$$\begin{tikzcd}
S \arrow[r, "s"] \arrow[d, "\mathrm{incl}"'] & M \arrow[r, "F"] \arrow[d, "f"'] & E \arrow[dl, "p"] \\
R \arrow[r, "r"] & X &
\end{tikzcd}$$

there is a δ-lift $g : R \to E$, i. e. $g \circ \mathrm{incl} = F \circ s$ and $p \circ g$ is δ-close to r.

Example 1.7.11: *Given $X = \mathrm{cyl}(f)$, where f is an MAF, then X is a MHSS (by Thm. 1.4.19), and there is a natural $(\delta, 1)$-connected map from f to the holink-fibration (Lemma A.2.4).*

This is particularly useful when dealing with spaces with more than two strata and will be explained as needed in Chapter 3 ("Multiple Strata"). It is also briefly illustrated in Appendix A ("Ends in MHSS").

1.8 Excursion: End-Theory

We will return to controlled topology results in the next Section 1.9 ("Mapping-Cylinder Neighborhoods"), but first, we want to approach the other question (b) posed before in the last section: How do we obtain information about the existence of neighborhoods, rather than uniqueness? This in turn requires some background on "end-theory".

There are a number of different definitions as to what exactly the "end" of a space is. Here we *illustrate* the problem[34] using a definition given by Hughes and Ranicki [HR96]. We will later work with controlled ends, where we will formulate the precise hypothesis on MHSS (see Section 1.9 ("Mapping-Cylinder Neighborhoods")).

Definition 1.8.1: *[HR96, Def. 1 (p. 'x')] Let W be a non-compact space.*

(i) A neighborhood U of an end of W *is a $U \subset W$ which contains a component of $W - K$ for a non-empty compact $K \subset W$.*

(ii) An end ϵ of W *is an equivalence class of sequences of connected open neighborhoods (of an end of W): $W \supset U_1 \supset U_2 \supset \ldots$ such that*

$$\bigcap_{i=1}^{\infty} \mathrm{closure}(U_i) = \emptyset$$

w. r. t. the equivalence relation

$$\begin{aligned} & (W \supset U_1 \supset U_2 \supset \ldots) \sim (W \supset V_1 \supset V_2 \supset \ldots) \\ :\Leftrightarrow \quad & \forall i \exists j \text{ with } U_i \subset V_j \text{ and } \forall j' \exists i' \text{ with } V_{j'} \subset U_{i'} \end{aligned}$$

(iii) The fundamental group $\pi_1(\epsilon)$ of an end ϵ *is the inverse limit*

$$\pi_1(\epsilon) := \overleftarrow{\lim_i}\, \pi_1(U_i) \quad \textit{(when well-defined).}$$

(iv) The fundamental group of ϵ is stable *[Sie65, p. 1] if the inverse limit over $\tilde{K}_0(\mathbb{Z}\pi_1(U_i))$ is well-behaved. In this case define $\tilde{K}_0(\mathbb{Z}\pi_1(\epsilon))$ as this inverse limit.*

[34] this "definition" is not intended for actual use, but really just to illustrate the problem. It is a bit fuzzy on some aspects for the sake of brevity.

Most of the time we will be concerned with the following example:

Example 1.8.2: *Let $(M, \partial M)$ be a compact manifold with boundary. Given a codimension 0 submanifold $A \subset \partial M$, the space $M - A$ has one end for every connected component of A.*

In fact, the central question of "end-theory" (for manifolds, as treated here; see [CV99] for a discussion of ends of MHSS) is to determine, given a non-compact manifold with an end, if it is of this form. The same question for multiple ends can typically be decided on individual ends independently. Formally (compare to e. g. [Qui79]):

Definition 1.8.3: *Let W be a non-compact manifold (possibly with boundary) with one end ϵ. Then W' is a* completion of ϵ (and of W) *if W' is a compact manifold with boundary $\partial W'$ containing $W \subset W'$, such that $W' - W \subset \partial W'$.*

If a completion exists, boundary-collaring [Bro62] implies the existence of a product-structure – an open collar – at the end. Hence this does, indeed, (mostly) answer our question:

Example 1.8.4: *Let $X \supset \{\mathrm{pt}\}$ be a manifold-stratified space, i. e. $M := X - \{\mathrm{pt}\}$ is a manifold. Then $\{\mathrm{pt}\}$ has a stratified cone-neighborhood – which is the same as a cylinder-neighborhood over the point –* if *the end of M near $\{\mathrm{pt}\}$ has a completion.*

The other direction is true in the piecewise-linear (PL) or smooth (DIFF) category but may fail topologically.

Proof: "$\Leftarrow$": If there is a completion $(M', \partial M')$, then, because the boundary is collared [Bro62], collapsing the "new" component $M' - M$ to a point is a cone.

"$\Rightarrow$": There is $c(Y) \cong_{\mathrm{strat}} N \subset X$. Thus $N \cap M \cong Y \times (0, 1]$. If Y is a manifold, then "glue" $Y \times [0, 1)$ to M along $Y \times (0, 1)$. The result is then a manifold-with-boundary (M', Y).

We *do* know, that $Y \times (0, 1) \cong Y \times \mathbb{R}$ is an open subset of M, thus a manifold. However, in the topological category, this does *not* imply that Y itself is a manifold. A "classical" counter-example is Bing's [Bin58] "dogbone"-space. For PL or DIFF, Y must be a manifold. $\square$

The somewhat unexpected twist in the "only if" direction does not need to concern us: The transversality-theorem of Connolly and Vajiac [CV99] (see Section 1.6 ("Stratified Transversality")) uses the manifold boundary (or rather part of it) as the "new" upper stratum of the "cut", so we are truly working in the world of completions, and are *not* introducing too strong of a hypothesis when demanding that Y be a manifold[35] in the above example.

[35]This is implicit in the definition of MAF requiring the cylinder-cap to be a manifold.

The boundary-collaring shows also, however, that similar to the case of h-cobordisms, also ends must necessarily satisfy some homotopy-condition if a completion exists: Clearly one can provide homotopies “pushing towards” the end or “pulling away” from the end along the boundary collar. This leads to a homotopical “tameness” condition. A precise definition will be given for controlled ends.

The fact, that end-problems are closely related to h-cobordisms, is also reflected in a similar relation to algebra (via handle-bodies). The main difference is, that a “swindle”-argument as in the proof-sketch of Example 1.7.4 is used, effectively pushing obstructions away to infinity. The difficulty is in constructing the “correct” (so that $(N_i, \partial N_i)$ is homotopically highly connected) ∂N_i to “close the construction” on one side (see Rmk. 1.7.5). Algebraically this leads to the following modification: A h-cobordism corresponds to a square matrix (up to certain identifications) representing an isomorphism, and it is trivial (in high dimensions) if this matrix is equivalent to the identity-matrix. A manifold end corresponds to a (generally non-square) matrix (up to certain identifications) representing a projection, and it admits a completion (in high dimensions) if this matrix is equivalent to a “standard”-projection (i. e. a matrix with only ones on a “diagonal” and zeros otherwise).

Further, note that there is a connection between projections and projective modules. In fact it turns out, the obstruction group is indeed $\tilde{K}_0(\mathbb{Z}[\pi])$, the group of projective modules over the group-ring $\mathbb{Z}[\pi]$ with certain identifications, replacing $\mathrm{Wh}(\pi)$ when compared to the h-cobordism case. Siebenmann [Sie65] finds the following “End-Theorem”.

Theorem 1.8.5: *Let M^n be a (smooth[36]) open manifold of dimension $n \geq 6$.*

(i) [Sie65, (p. 1)]: If an end ϵ of M is tame with stable fundamental group, then there is a well-defined invariant $\sigma(\epsilon) \in \tilde{K}_0(\mathbb{Z}[\pi_1(\epsilon)])$.

(ii) [Sie65, “Main Theorem” (p. 2)]: M admits a (smooth) completion if and only if *M has finitely many connected components, and each end ϵ of M is tame with stable fundamental group and has invariant $\sigma(\epsilon) = 0$.*

(iii) Cf. [Sie65, Thm. 9.1 (p. 70)], [HR96, Thm. 10.2 (p. 110f)] and [Qui82a, Thm. 1.1 b (p. 357)]: Completions are unique up to h-cobordism, in the sense, that given two completions with boundary-collars, one can be embedded in the other, with the “difference” being a h-cobordism.

The combination of both ideas discussed in this section and the previous one – namely adding control and using ends to supplement h-cobordisms – is where we leave this “excursion” and return to the main line of argumentation.

[36] Siebenmann works with smooth manifolds, but the results apply also topologically – important developments in topological transversality (and handle-body) theory occurred only later, see for example [KS77]. The topological case also is implicit in the results of [Qui82a], by taking the control-space to be a point.

1.9 Mapping-Cylinder Neighborhoods

Here, results concerning the existence of mapping-cylinder neighborhoods in MHSS are discussed. The underlying theory is about "controlled end-completions" – controlled in the sense of "controlled topology" – also known as "ends of maps". The results do not require prior knowledge of the subject, but a basic understanding may help, so reading Excursion 1.7 ("Excursion: Controlled Topology") and Excursion 1.8 ("Excursion: End-Theory") above could clarify (and "demystify") some results.

While there are more general ways to define controlled ends and their properties [Qui82a], it suffices for our purposes to promote the following example to a definition:

Example 1.9.1: *Let X be a MHSS (with two strata B and $M := X - B$), with tameness retraction $R : N \times I \to N$, which, by definition, (nearly-strictly) deforms a neighborhood N of B into B. It thus defines a map $r = R_1 : N \to B$.*

M has an end (where B was taken away from X). This end is "tame", controlled over B by r, with "local fundamental group" $\pi = \pi_1(L)$ where L is the fiber of the homotopy-link.[37]

See also Appendix A ("Ends in MHSS").

Definition 1.9.2: *A non-compact manifold M has a "tame" end, controlled over B by $r : N \to B$ – where N is a neighborhood of the end (i. e. an open subset of M such that $M - N$ is compact) – with "local fundamental group" π if there is a MHSS X with strata B and M, and a tameness-retraction R with $R_1 = r$, such that the holink has fiber L with $\pi_1(L) \cong \pi$ (where the isomorphism is by conjugation).*

A "completion" of this end is a manifold-with-boundary $(M', \partial M')$ – with $M \subset M'$ and $M' - M \subset \partial M'$ – together with a map $r' : \partial M' \to B$, such that there is a neighborhood $N' \subset X$ of B, with $N' \cong_{strat} \mathring{\text{cyl}}(r')$. Here $\mathring{\text{cyl}}(r')$ is stratified with two strata B (the target / cylinder-base) and $\mathring{\text{cyl}}(r') - B$, see also Def. 1.3.5 (iv).

The existence of a completion / mapping-cylinder-neighborhood is obstructed in general, but in high dimensions, an obstruction theory was developed by Quinn [Qui82a]. We use the following formulation[38] for MHSS[39]:

[37] Technically this depends on a choice of basepoint for L (so is only well-defined up to isomorphism by conjugation), but similarly to how the Whitehead-group does not depend on this choice for h-cobordisms, the actual obstruction groups here will be well-defined as well.

[38] This is not the way end-theorems are usually formulated, but lends itself well to the present context. A more "standard" formulation can be found in [Qui79; Qui82a], the relation of the two versions is explained in more detail in Appendix A ("Ends in MHSS").

[39] In the sense of Quinn, without the additional compact-domination property for local holinks. But the definition used here otherwise is strictly stronger, so the result clearly still applies.

Theorem 1.9.3: *[Qui88a, Thm. 1.7 (p. 446)]: Suppose $X \supset B$ is a MHSS of dimension ≥ 6 (with two strata) and holink-fibration $p := ev_0 : \mathrm{holink}(X, B) \to B$, then:*

(i) There is an invariant $q_0(X, B) \in H_0^{\mathrm{lf}}(B; \mathcal{S}(p))$.

(ii) There is a mapping cylinder neighborhood of B in X (i. e. a completion of the controlled end in the sense defined above), if and only if $q_0(X, B) = 0$.

The reader of Excursion 1.7 ("Excursion: Controlled Topology") may want to know:

(iii) [Qui82a, Thm. 1.1 b (p.357)]: Completions are unique up to controlled h-cobordism.

We also note the following interesting refinement:

(iv) [Qui79, Thm. 1.4 (p. 280)]: If the upper stratum $M := X - B$ is a smooth (or pl) manifold, then also M' can be chosen smooth (or pl).

Remark 1.9.4: For "good" (see below) fundamental groups, the dimensional requirement can probably be weakened to $\dim(X) \geq 5$, a detailed treatment is hard to find in literature however, see [Qui79, Thm. 2.1.2 (p. 505)] (for "good= trivial", but without explicit mention of more general "goodness" being sufficient; the proof seems to nevertheless apply using more general good groups) [FQ90, 'End theorem' (p. 214)] (more general "good", but $B = \{\mathrm{pt}\}$; see also Example 1.9.6), [Qui88a, §1.13 (p. 451)] (concerning cylinder-neighborhoods, as formulated here, but without proof or formal statement):

If $\dim(X) = 5$ and the local fundamental group π (the fundamental group of the fiber of the holink, see Appendix A ("Ends in MHSS")) is "good" (see below), then there is a mapping cylinder neighborhood of B in X (i. e. a completion of the controlled end in the sense defined above), *if and only if* $q_0(X, B) = 0$.

Definition 1.9.5: *A group π is* good *if given a 4-manifold M with $\pi_1(M) = \pi$, the 4-dimensional disk-embedding theorem holds for M, i. e. (roughly): An immersion of a disjoint union of 2-disks in M with vanishing algebraic (self-) intersections, can be deformed to an embedding [FQ90, p. 99].*

A group is poly-(finite / cyclic)*, if there is a normal-series (a filtration by subgroups, normal in the respective next one) with quotients, which are either finite groups or infinite-cyclic (see e. g. [FH81, p. 308]).*

Example 1.9.6: *Poly-(finite or cyclic) groups are good [FQ90, §5 (p. 86ff)].*

Remark 1.9.7: Technically, the end-theorem seems to hold again in low dimensions ≤ 3 (unconditionally, which is consistent with the possible fundamental groups of surfaces etc.), see ["So if the Poincaré conjecture is true then the strong form of the end theorem is true for 3-manifolds." FQ90, (p. 216)] for the uncontrolled case.

Some explanations on the obstruction-groups seem in place: The H_*^{lf} refers to locally finite homology with spectral cosheaf coefficients in the sense of [Qui82a, §8 (p. 419ff)] and [Qui04, §6 (p. 50ff)]. The coefficient "spectral-cosheaf" $\mathcal{S}(p)$ in this case, is the one associated (by Def. 3.1.20) to $\mathcal{S}(X;p)$ (the finite-structure-spectrum of [Qui82a]), which is itself a functor from fibrations over ANR (in the case of two strata) to spectra (both with suitable morphisms), with the (fixed) control map p given by the holink-fibration.

Instead of going into details of its definition, we will content ourselves with giving enough information about the theory, to be able to apply the result above. Some more information is summarized in Chapter 3 ("Multiple Strata"). Most importantly, we will use the special case treated in [Qui79]:

Lemma 1.9.8: *If a fibration $p : E \to B$, with B a path connected manifold, is such that its fiber F satisfies $\forall k \geq 0$ $\mathrm{Wh}(\pi_1(F) \times \mathbb{Z}^k) = 0$, then $H_i^{\mathrm{lf}}(B; \mathcal{S}(p)) = 0$ for $i \leq 1$.*

For non-path connected B, the conclusion holds, if the hypothesis holds component-wise.

Remark 1.9.9: Given a path connected topological space F, the abelian group $\mathrm{Wh}\left(\pi_1(F, \{*_F\}) \times \mathbb{Z}^k\right)$ does not depend on the choice of base-point $*_F$, which we can therefore safely omit in the notation. Further, over a connected component of B, the homotopy-type (and thus the fundamental group) of the fiber of the *fibration* p is well-defined.

Proof of the lemma: A direct proof of this statement (for ends, i. e. $i = 0$ as required later on) is given in Appendix A ("Ends in MHSS") (see Lemma A.2.4), by identifying the local fundamental groups of [Qui79] as the fundamental-groups of the fiber of the homotopy-link and showing that the homotopy-link-evaluation is $(\delta, 1)$-connected to the end. This then implies existence of a completion by the Whitehead-group hypothesis of [Qui79], which means the obstruction of [Qui82a] always vanishes for these ends, so by the realization results of [Qui82a], the obstruction-*groups* of [Qui82a] must vanish as claimed.

Alternatively, the lower K-groups that occur as coefficients in Lemma 1.9.10 below, are constructed [Bas68, p. 664], such that $\mathrm{Wh}(\pi \times \mathbb{Z}^k)$ can be decomposed into a direct sum of lower Wh- (thus K-) groups (in particular, it occur exactly those Wh_j for $j = -k + 1, \ldots, 1$) and certain nil-groups. Hence if $\mathrm{Wh}(\pi \times \mathbb{Z}^k)$ vanishes for all $k \geq 0$, then the coefficients in Lemma 1.9.10 vanish for the degrees $i \leq 1$, and thus by the Atiyah–Hirzebruch-type spectral-sequence of Quinn [Qui82a, Thm. 8.7 (p. 423)] the obstruction-groups $H_i^{\mathrm{lf}} = 0$ vanish for $i \leq 1$.

For non-path connected B, if the hypothesis holds component-wise, then the conclusion hold by component-wise application of the result for path connected B. □

The only ones of these obstruction-groups we need are $H_0^{\mathrm{lf}}(B;\mathcal{S}(p))$ (for end-obstructions) and potentially / implicitly (Rmk. 4.1.2) $H_1^{\mathrm{lf}}(B;\mathcal{S}(p))$ (for h-cobordism obstructions), both of which vanish under the hypothesis of the lemma.

It is also useful to know, that H_*^{lf} behaves much like a homology-theory – for example there is a "characterization theorem" [Qui82a, Thm. 8.5 (p. 422)] in the spirit that Eilenberg–Steenrod-axioms characterize "conventional" homology-theories, so coefficients, i. e. groups of the point (in low degrees) already contain (most of) the information about the theory (there is also an Atiyah–Hirzebruch-type spectral-sequence [Qui82a, Thm. 8.7 (p. 423)] making this statement precise). The coefficients (in low degrees) are known:

Lemma 1.9.10: *The coefficients are* $H_j^{\mathrm{lf}}(\{\mathrm{pt}\};\mathcal{S}(F\to\{\mathrm{pt}\}))=\mathrm{Wh}_j(\pi_1(F))$ *for* $j\le 1$, *where*

$$\mathrm{Wh}_1(\pi_1(F))=\mathrm{Wh}(\pi_1(F))$$
$$\mathrm{Wh}_j(\pi_1(F))=\tilde{K}_j(\mathbb{Z}[\pi_1(F)])\text{ for } j\le 0$$

where the $\tilde{K}_j$ *are the "lower K-groups" of Bass [Bas68].*

Proof: See e. g. [Qui82a, p. 356] combined with the assembly $A:\mathbb{H}^{\mathrm{lf}}(X;\mathcal{S}(p))\to\mathcal{S}(X;p)$ being a homotopy-equivalence of spectra [Qui82a, Thm. 8.5 (p. 422)]. Further H_*^{lf} are defined as the homotopy-groups of the spectrum $\mathbb{H}^{\mathrm{lf}}$. □

This fits together nicely with the results seen in the last two sections for the obstruction-groups over the point – $\tilde{K}_0(\mathbb{Z}[\pi_1(F)])$ for ends and $\mathrm{Wh}(\pi_1(F))$ for h-cobordisms.

We conclude this section with an example where the $\times\mathbb{Z}^k$ term in the obstruction-groups actually matters, for examples of "strange ends" (ones that cannot be completed, although being tame etc., because they have non-vanishing obstruction) see for example [Sie65, §8 (p. 56–70)] or the realization part for controlled structures [Qui82a, p. 411].

Example 1.9.11: *Farrell and Hsiang [FH67] use a product of a lens-space with fundamental-group* $\mathbb{Z}/p^2\mathbb{Z}$ *and a torus to construct certain non-trivial h-cobordisms.*

Notice, that this also illustrates the obstruction-groups encountered here (e. g. fix $p=2$*): While* $\mathrm{Wh}(\mathbb{Z}/4\mathbb{Z})=0$, *for any* $k\ge 1$, *one finds* $\mathrm{Wh}(\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}^k)\ne 0$ *is* non-*trivial.*

Proof: "$\mathrm{Wh}(\mathbb{Z}/4\mathbb{Z})=0$": For the cyclic group $\mathbb{Z}/q\mathbb{Z}$ of finite order q, the Whitehead-group $\mathrm{Wh}(\mathbb{Z}/q\mathbb{Z})$ is free of rank $\lfloor q/2\rfloor+1-\delta(q)$ (see e. g. [Coh73, 11.5 (p. 45)]), where $\delta(q)$ is the number of divisors of q. Since 4 has divisors $\{1,2,4\}$, $\lfloor 4/2\rfloor+1-\delta(4)=2+1-3=0$.

"$\mathrm{Wh}(\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}^k) \neq 0$": This follows for example from [BM67, Thm. 10.8 (p. 68)], as the direct summand, called "X" in the their theorem, of $\mathrm{Wh}(\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}^k)$ is not even finitely generated (by part (d) of the reference) for non square-free order p^2. See also [FH67]. □

This finishes the bulk of the background-material. The remaining sections introduce mostly independent concepts extending the strongly interconnected material presented up to this point.

1.10 Intrinsic Stratifications

A reason that makes MHSS particularly useful for the study of inherently topological phenomena – e. g. transport of stratified invariants along *un*stratified homeomorphisms – is their "closeness" to the underlying (unstratified) topological space. This "closeness" manifests itself, for example, in Quinn's [Qui87] "intrinsic skeleta".

First, we fix a notion of what "topologically intrinsic" is supposed to mean:

Definition 1.10.1: *[Qui87, p. 234]: Given a topological space X, define an equivalence relation $\sim$ on points as $x \sim x'$ if and only if $\exists$ neighborhoods U, U' of x, x' and a homeomorphism $h : U \to U'$ with $h(x) = x'$.*

Let $\mathcal{I}$ be the set of equivalence classes. The topologically intrinsic stratification *of X has strata*

$$\left(|X|_i := \left\{x \in X \middle| x \text{ is in the equivalence class } i\right\}\right)_{i \in \mathcal{I}}$$

If the stratification is equivalent to a filtration (for example if it satisfies the frontier-condition, and the number of strata is finite), we denote the topologically intrinsic skeleta *by $|X|^i$.*

Given a stratified space X, we denote the underlying topological space *by $|X|$, i. e. the image under the forgetful functor from stratified spaces to topological spaces.*

In general, this stratification may of course not have any reasonable properties. If X can be stratified as a MHSS (in the sense of Quinn) however, it holds that:

Theorem 1.10.2: *[Qui87, Thm. 1.1 (p. 235)]: Suppose X has a filtration (by dimension) as MHSS. Then there is a filtration $X_{0,0}$ of the underlying topological space $|X|$ as HSS, such that:*

(1) For any filtration Y of $|X|$ as MHSS, the identity $\mathrm{id} : Y \to X_{0,0}$ *is a coarsening (preimages of components of strata are unions of components of strata, see Def. 1.3.6 (iii)).*

(2) *If* $k \geq 5$ *or* X^4 *is locally conelike, then:*
   (i) *The* k*-skeleton is the topologically intrinsic one* $(X_{0,0})^k = |X|^k$.
   (ii) *The* k*-stratum* $(X_{0,0})_k$ *is a manifold.*

(3) *If* $(X_{0,0})^0 = \emptyset$ *then* $(X \times \mathbb{R})^{k+1}_{0,0} = (X_{0,0})^k \times \mathbb{R}$.

Proof: We briefly sketch the proof given in [Qui87]: Define $X_{n,n} := X$ (where $n = \dim(X)$), then, for each component of X_{n-1}, check, if "promoting" this component into X_n (i. e. changing the stratification such that this component is now part of the n-stratum) maintains the property, that X_n is a manifold (This can be decided from homotopy-link-fiber being a homology-sphere![40]) and if the result would still be a HSS. Define $X_{n,n-1}$ as $X_{n,n}$ with all such "promotable" components of X_{n-1} actually promoted into X_n.

Next, repeat the same by promoting components of X_{n-2} into X_n to obtain $X_{n,n-2}$ and so on. Finally we obtain $X_{n,0}$.

Then, define $X_{n-1,n-1} := X_{n,0}$, and promote components from X_{n-2} into X_{n-1} to obtain $X_{n-1,n-2}$, then from X_{n-3} into X_{n-1} to get $X_{n-1,n-3}$ and so on. Once we get to $X_{n-1,0}$, we define again $X_{n-2,n-2} := X_{n-1,0}$ and in the end this yields the $X_{0,0}$ of the theorem.

The properties stated in the theorem can then (mostly) be proven by contradiction (if they did not hold, the component, where they do not hold, would have been promoted). □

Note, that part (1) implies both $\mathrm{id} : Y_{0,0} \to X_{0,0}$ and $\mathrm{id} : X_{0,0} \to Y_{0,0}$ are coarsenings, thus $X_{0,0} = Y_{0,0}$. So even in the (rare) case that $X_{0,0}$ is *not* the topologically intrinsic stratification, it still holds that $X_{0,0}$ is "intrinsic" in the sense, that it does *not* actually matter, which stratification of $|X|$ we started with.

If X has only strata of high dimension (≥ 5), then (2) implies $X_{0,0}$ is the topologically intrinsic stratification and a MHSS. The last statement also holds for our definition (including compact domination of local holinks), because with $X_{0,0}$ being a coarsening of X, its local (stratified in the case of more than two strata) holinks are coarsenings of local holinks of X, which are compactly dominated. Compact domination requires a stratified deformation, but being stratified with respect to a *finer* stratification implies it is also stratified with respect to the *coarser* stratification.

1.11 Excursion: Intersection Homology and Poincaré-Duality

This section gives a brief overview of the concepts behind intersection homology. For a more in-depth treatment, see for example [GM80; GM83; Kin85; KW06; Ban07; Fri09].

[40]Except for zero-strata, where a homotopy-sphere is needed, hence the requirement $(X_{0,0})^0 = \emptyset$ in part (3).

Intersection forms, as originally introduced and studied by Poincaré [Poi95] and Lefschetz [Lef26] play an important role, for example via the signature-invariant, in the theory of manifolds, e. g. as a surgery-obstruction. The question of how to generalize a bordism-invariant signature to singular spaces was (according to [GM80]) originally posed by D. Sullivan.

An answer was discovered independently by Goresky and MacPherson [GM80] – by introducing "perversities" to control deviation from general-position of cycles and strata – and by Cheeger [Che80] – "L^2-cohomology", using a suitable integrability-condition on the de-Rham-complex. A third such generalization is through "intersection-spaces" [Ban10].

We will employ the first one, "intersection homology", which has been developed further, for example in [GM83] (extending from pl to topological pseudomanifolds, using sheaves), [Kin85] (showing topological invariance on pseudomanifolds, using singular chains), [Qui87] (showing topological invariance on MHSS). Such invariants can in fact even be established on structures generalizing intersection-homology sheaves [Ban02] (using Lagrangian structures and self-dual sheaves "between" the lower- and upper-middle-perversity), see also [Ban07, §9]. Particularly relevant to the present task is also [Fri09] (giving a Witt-condition, Poincaré-duality, and signature on MHSS, using sheaves).

The goal of this section is to merely outline the important ideas, so, to keep things simple, we will avoid sheaves and use a chain based approach as in [GM80; Kin85].

First, we define "perversities", which will measure the deviation from normality / transversality.

Definition 1.11.1: *A* perversity $\bar{p}$ *is a mapping* $\mathbb{N} \to \mathbb{N}$*, which is:*

(i) Monotonically increasing, in steps of at most one:
$\bar{p}(k+1) \in \{ \quad \bar{p}(k), \quad \bar{p}(k)+1 \quad \}$.

(ii) Satisfies $\bar{p}(2) = 0$.

There are some particularly relevant special cases:

- *The "top-perversity"* $\bar{t}(k) = k-2$.
- *Two perversities* $\bar{p}$ *and* $\bar{q}$ *are "complementary" if* $\bar{p}(k) + \bar{q}(k) = \bar{t}(k)$.
- *The "lower-middle-perversity"* $\bar{m}(k) = \lfloor \frac{k-2}{2} \rfloor$ *and the "upper-middle-perversity"* $\bar{n}(k) = \lceil \frac{k-2}{2} \rceil$*, which are complementary.*

The introduction of perversities to allow chains non-transverse to strata in a controlled codimension-dependent way was the main breakthrough in [GM80].

Definition 1.11.2: *Using singular chains, [Kin85, p. 151]: Let X be a filtered (by closed subsets) space, $\bar{p}$ a perversity.*

A singular i-simplex $\sigma_i : \Delta_i \to X$ is $\bar{p}$-allowable*, if* $\forall k$

$$\sigma_i^{-1}(X_{n-k}) \subset \Delta_i^{(i-k+\bar{p}(k))}$$

where $\Delta_i^{(i-k+\bar{p}(k))}$ *is the* $(i-k+\bar{p}(k))$*-skeleton of* Δ_i*, i. e. the union of all faces of dimension* $\leq i-k+\bar{p}(k)$*.*

The singular $\bar{p}$-intersection-chain complex $\mathrm{IC}_*^{\bar{p}}(X)$ of X *in degree* i *is the free abelian group on generators the* $\bar{p}$*-allowable chains with* $\bar{p}$*-allowable boundary:*

$$\mathrm{IC}_i^{\bar{p}}(X) := \langle \quad \sigma_i : \Delta_i \to X \quad | \quad \begin{array}{l} \sigma_i \text{ is a } \bar{p}\text{-allowable } i\text{-simplex and} \\ \partial\sigma_i \text{ is a } \bar{p}\text{-allowable } (i-1)\text{-simplex} \end{array} \quad \rangle$$

the boundaries are given by the restrictions of the standard ∂_i*.*

The singular $\bar{p}$-intersection-homology groups $\mathrm{IH}_*^{\bar{p}}(X;G)$ of X *(with compact support) are the homology-groups of* $\mathrm{IC}_*^{\bar{p}}(X) \otimes G$*.*

Remark 1.11.3: Using simplicial chains instead of singular ones, for a triangulation compatible with the stratification (i. e. such that skeleta are subcomplexes), a simplex *transverse* to strata would intersect them in dimension $i + \dim(\text{stratum}) - n = i - k$, so in this context, it is more obvious in what sense $\bar{p}$ describes a deviation from transversality through a correction-term in this formula.

Note, that this is *not* a homology-theory in the sense of Eilenberg–Steenrod, for example it is not generally homotopy-invariant (even though it is, for example, invariant under stratified homotopy-equivalences [Fri03], see below). Also concerning functoriality, some care has to be taken, as (non-stratified) maps do not generally induce homomorphisms on IH. While this definition seems to depend on a choice of stratification, these intersection-homology groups are (on reasonable spaces) independent of this choice.

Theorem 1.11.4: *[GM83] (pseudomanifold case), [Kin85, Thm. 9 (p. 157)] (CS case), [Qui87, Cor. (p. 243) of Thm. 3 (p. 242)] (MHSS): Given an MHSS* X *and a perversity* $\bar{p}$*, the intersection-homology groups* $\mathrm{IH}_*^{\bar{p}}(X)$ *are topological invariants, i. e. they do not depend on the choice of stratification of the underlying topological space* $|X|$*.*

[Fri03, Prop. 2.1 (p. 73) and §2 (p. 71–78)]: Given a filtered (by closed subsets) space X*, then intersection homology (with compact support) is invariant under stratum-preserving homotopy-equivalences (with stratum-preserving homotopy-inverses; which is implicit in our definition Def. 1.3.7).*

As indicated above, intersection-homology was constructed to have the following property:

Theorem 1.11.5: *[GM80] (pseudomanifold case), [Fri09] (MHSS): Given a closed, orientable MHSS X of dimension n and complementary perversities $\bar{p} + \bar{q} = \bar{t}$, there is a "Poincaré Duality-Isomorphism"*

$$\mathrm{Hom}(\mathrm{IH}_i^{\bar{p}}(X;\mathbb{Q}),\mathbb{Q}) \xrightarrow{\sim} \mathrm{IH}_{n-i}^{\bar{q}}(X;\mathbb{Q})$$

If we know that, for some reason, $\mathrm{IH}_*^{\bar{p}}(X;\mathbb{Q}) \cong \mathrm{IH}_*^{\bar{q}}(X;\mathbb{Q})$, typically for the middle-perversities $\bar{p} = \bar{m}$ and $\bar{q} = \bar{n}$, then this yields a non-singular pairing and thus a signature invariant. This happens for example for X with only even-codimensional strata (e. g. a complex algebraic variety), where the condition on $\sigma_i^{-1}(X_{n-k})$ is void for k odd, while $\bar{m}(k) = \bar{n}(k)$ for k even, thus already $\mathrm{IC}_*^{\bar{m}}(X) = \mathrm{IC}_*^{\bar{n}}(X)$ on the chain-level in this case. An other example of such spaces are those satisfying a "Witt-condition" [Sie83], as described in the next section. For now we fix the preliminary result (to be revisited in Thm. 1.12.10 and Prop. 1.12.11):

Theorem 1.11.6: *[GM80]: Given a closed, orientable pl-pseudomanifold X of dimension n with only even-codimensional strata, then there is a well-defined signature invariant $\sigma(X) \in \mathbb{Z}$, given by the signature of the middle-dimensional, middle-perversity intersection-form.*

This $\sigma(X)$ is invariant under orientable pl-pseudomanifold bordisms with only even-codimensional strata.

The relevant modifications to extend the signature-invariant to a suitable bordism-theory of MHSS are reviewed in the next section.

1.12 Witt-Condition and Signature of MHSS

While it is relatively straight-forward to see, how a non-singular intersection form can be restored in intersection homology for (suitable) stratified spaces with *only even-codimensional* strata, that seems like a very "brute force" way to attain this goal. Clearly it should be possible to find more subtle ways of ensuring their existence.

The first such generalization are Siegel's [Sie83] "Witt-spaces" – he studies their bordism-groups, identifying them with "Witt-groups", hence the name. These are orientable pl-pseudomanifolds with a condition imposed on the middle-dimensional intersection-homology of links of odd-codimensional strata (for details see below), to ensure, that the middle-dimensional middle-perversity intersection-form of the total space is non-singular.[41]

The key ingredient is, as for spaces with only even-codimensional strata, that the duality of $\mathrm{IH}^{\bar{m}}$ and $\mathrm{IH}^{\bar{n}}$ implies self-duality, in the case where the inclusion $\mathrm{IC}^{\bar{m}} \subset \mathrm{IC}^{\bar{n}}$

[41] Non-singular for rational coefficients, hence these are also often referred to $\mathbb{Q}$-Witt-spaces. Extensions to more general fields an rings can be found for example in [Fri09].

induces an isomorphism in homology. So the contents of this section mostly fall in two categories:

The (quite general; not specific to MHSS) study of what “Witt-condition” makes $\mathrm{IC}^{\bar{m}} \subset \mathrm{IC}^{\bar{n}}$ a quasi-isomorphism on closed spaces, and how the “difference” on compact spaces with boundary relates to the boundary, ultimately implying cobordism-invariance. This part is mostly based on arguments as found in [Ban07] and [GM80].

And on the other hand, there is the problem of understanding such a “Witt-condition” in terms of properties of links on MHSS, and finding a generalization of duality to MHSS. This also requires some non-trivial arguments, as strong geometric normal structure may *not* exists in general. This part is based on [Fri09].

Both lines of argument can be “easily” connected, as Friedman [Fri09, Thm. 5.1 (p. 2177)] shows that his “intersection-chain sheaf” $\mathcal{I}^{\bar{p}}\mathcal{S}$ is the Deligne-sheaf [GM83], while the arguments used by Banagl [Ban07, §6.1 (p. 123–127) and §6.4 (p. 133–135)] use an axiomatic description of the Deligne-sheaf (actually the intersection-chain sheaves on topological pseudomanifolds are defined via these axioms here, see [Ban07, p. 94]).

We have to formally work in the derived category (see e. g. [Ban07]), but may think of the resulting distinguished triangle as the long exact sequence in the proof of [GM80, ‘Theorem’ (p. 155)] (which uses a pl-chain-description). Some care has to be taken with the different indexing-conventions (basically $\mathcal{I}^{\bar{p}}\mathcal{S}^k \cong \mathrm{IC}_{\bar{p}}^{-k}$ for the cohomological viewpoint)

Lemma 1.12.1: *See [Ban07, Lemma 6.4.1 (p. 94)] and discussion thereafter: Let* $\mathrm{IC}^*_{\bar{m}}$, $\mathrm{IC}^*_{\bar{n}}$ *satisfy the Deligne-sheaf axioms for the same orientation (local system) on the top-stratum (for the middle-perversities* $\bar{m}$ *and* $\bar{n}$*). Given a distinguished triangle* $\mathrm{IC}^*_{\bar{m}} \to \mathrm{IC}^*_{\bar{n}} \to S^* \xrightarrow{[1]}$ *on the canonical morphism (corresponding to the inclusion* $\mathrm{IC}_{\bar{m}} \subset \mathrm{IC}_{\bar{n}}$ *and its long exact sequence in [GM80]), then*

$$\mathbb{H}^i(S^*)_x = \begin{cases} \mathbb{H}^i(\mathrm{IC}^*_{\bar{n}})_x & \text{if } x \in X_{n-k} \text{ where } i = \bar{n}(k) - n \\ 0 & \text{otherwise} \end{cases}$$

This rather technical result has multiple important consequences: First, vanishing of $\mathbb{H}^i(\mathrm{IC}^*_{\bar{n}})_x$ clearly is equivalent to $\mathrm{IC}^{\bar{m}} \subset \mathrm{IC}^{\bar{n}}$ being a quasi-isomorphism. This is the Witt-condition we are looking for. The condition can be brought into a more appealing form:

Lemma 1.12.2: *See comment below [Fri09, Def. 8.1 (p. 2197)]: If* X *is a MHSS with sufficiently many local approximate tubular neighborhoods, then*

$$\mathbb{H}^k\Big(\mathcal{I}^{\bar{n}}\mathcal{S}^*(\mathbb{Q})_x\Big) \cong \mathrm{IH}^{\bar{n}}_k(\mathcal{L}_x;\mathbb{Q})$$

where $\mathrm{IH}_*^{\bar{n}}$ *is (compactly supported; as defined in Def. 1.11.2 for filtered spaces in general) intersection homology, and where* k *is fixed by the stratum* $x \in X_{n-(2k+1)}$ *containing* x *with local homotopy-link*[42]

$$\mathcal{L}_x := \operatorname{holink}(X, x) = \{\gamma \in \operatorname{holink}_S(X, X_{n-(2k+1)}) | \gamma(0) = x\}.$$

Remark 1.12.3: Having "sufficiently many local approximate tubular neighborhoods" (see [Fri09, §4 (p. 2174–2176)]) means (for spaces with at most two meeting strata, see Def. 1.4.23), points in the minimal strata have neighborhoods that are teardrops (Example 1.4.5). Mapping-cylinder-neighborhoods are a special case of such teardrop-neighborhoods, hence this is not a restriction for our purposes, as we will study spaces with mapping-cylinder neighborhoods. Also, in high dimensions, teardrop-neighborhoods (of pure subsets, thus of skeleta, cf. Chapter 3 ("Multiple Strata")) exist by a theorem of Hughes [Hug02, Thm. 7.1 (p. 887)] under mild hypothesis (for example if the singular set is compact, see also [Fri09, Thm. 3.2 (p. 2174)]).

Proof of the lemma: The proof of [Fri09, Prop. 5.2 (p. 2178)] needs only be modified slightly to show this:

Let $x \in X_{n-l}$. The stalks $H^i\Big(\mathcal{I}^{\bar{n}}\mathcal{S}^*(\mathbb{Q})_x\Big) = \varinjlim_{x \in U} \mathbb{H}^i\Big(U; \mathcal{I}^{\bar{n}}\mathcal{S}^*(\mathbb{Q})\Big)$ are direct limits over open neighborhoods U of x, where the sheaf $\mathcal{I}^{\bar{n}}\mathcal{S}$ is constructed such that, on open sets, it is quasi-isomorphic to the (closed support IH_*^{∞}) intersection-chain sheaf, i. e. $\mathbb{H}^i\Big(U; \mathcal{I}^{\bar{n}}\mathcal{S}^*(\mathbb{Q})\Big) \cong I^{\bar{n}}H_{n-i}^{\infty}(U)$. By [Fri09, Lemma 4.3 (p. 2176)], there is a cofinal sequence of local approximate tubular neighborhoods, hence the limit can be calculated assuming U is a local approximate tubular neighborhood. This means $U \cap X_{n-l} \cong \mathbb{R}^{n-l}$ and [Fri07, Thm. 6.15 (p. 55)] applies to yield a spectral-sequence calculating the closed-support intersection-homology $I^{\bar{n}}H_{n-(p+q)}^{\infty}(U)$ from the compactly-supported one in $E_2^{p,q} = H^p\Big(\mathbb{R}^{n-l}, \mathrm{IH}_{l-q}^{\bar{n}}(c\mathcal{L}_U, \mathcal{L}_U \times \mathbb{R})\Big)$ where $\mathcal{L}_U = \operatorname{holink}_S(U, x)$.

The intersection-homology of the cone appearing in the E^2-terms can be calculated from the one of the "link" $\mathcal{L}_U$ as in the pseudomanifold case (see for example [Ban07, Example 4.1.15 (p. 80)]; the potential "wildness" of $\mathcal{L}_U$ formally complicates things; for details see the proof of [Fri09, Prop. 5.2 (p. 2178)])

$$\mathrm{IH}_j^{\bar{n}}\left(c\mathcal{L}_U, \mathcal{L}_U \times \mathbb{R}\right) \cong \begin{cases} 0 & : j < l - \bar{n}(l) \\ \mathrm{IH}_{j-1}^{\bar{n}}(\mathcal{L}_U) & : j \geq l - \bar{n}(l) \end{cases}$$

The reference [Fri09, Prop. 5.2 (p. 2178)] uses this result, to establish, that $H^i(\mathcal{I}^{\bar{n}}\mathcal{S}_x^*)$ vanishes for high enough i, so that the sheaf $\mathcal{I}^{\bar{n}}\mathcal{S}$ satisfies axiom (2) (in the numbering used in the reference) of the Deligne-Sheaf.

[42] For spaces with only two strata, this is the fiber of the homotopy-link, for the general case, see Chapter 3 ("Multiple Strata")

What we want to understand is, why $H^k\Big(\mathcal{I}^{\bar{n}}\mathcal{S}^*(\mathbb{Q})_x\Big) \cong \mathrm{IH}^{\bar{n}}_k(\mathcal{L};\mathbb{Q})$. But this follows immediately: In our case, the E^2-terms are non-zero only for $p=0$, so the spectral sequence collapses immediately at $E^\infty = E^2$.

$$\begin{aligned}\mathbb{H}^q\Big(U;\mathcal{I}^{\bar{n}}\mathcal{S}^*(\mathbb{Q})\Big) &\cong I^{\bar{n}}H^\infty_{n-q}(U) \cong \mathrm{IH}^{\bar{n}}_{l-q}(c\mathcal{L}_U, \mathcal{L}_U\times\mathbb{R})\Big)\\ &\cong \begin{cases}0 & : q > \bar{n}(l)\\ \mathrm{IH}^{\bar{n}}_{l-q-1}(\mathcal{L}_U) & : q \le \bar{n}(l)\end{cases}\end{aligned}$$

We are interested in $l = 2k+1$ and $q = k$, with $\bar{n}(l) = \bar{n}(2k+1) = \lceil\frac{2k+1-2}{2}\rceil = k \ge q$ resulting in

$$\mathbb{H}^k\Big(U;\mathcal{I}^{\bar{n}}\mathcal{S}^*(\mathbb{Q})\Big) \cong \mathrm{IH}^{\bar{n}}_k(\mathcal{L}_U)$$

There is one last thing to check, namely $\mathcal{L}_x \simeq_{\text{strat}} \mathcal{L}_U$, then, by stratified-homotopy-invariance of *compactly supported* intersection homology on filtered spaces [Fri03] (see Thm. 1.11.4), the claim follows. This equivalence to $\mathcal{L}_x$ is a consequence of locality of holinks: U is an open neighborhood of x, so for example using Quinn's ϵ-holinks ([Qui88a, p. 453]) with ϵ small enough that all $\gamma \in \mathrm{holink}^\epsilon$ have image in U can be used to obtain this result. □

Remark 1.12.4: See the discussion in [Fri09, p. 2197]: Other than in the pseudo-manifold case, where the links are again pseudo-manifolds, the spaces $\mathcal{L}_x$ may not be such that the duality-isomorphism exists. As a result, $\mathrm{IH}^{\bar{n}}_k(\mathcal{L};\mathbb{Q})$ *cannot* generally be replaced by $\mathrm{IH}^{\bar{m}}_k(\mathcal{L};\mathbb{Q})$, which seems to be the more common choice for formulating the Witt-condition in literature.

Now, the Witt-condition can be formulated as:

Definition 1.12.5: *[Fri09, Def. 8.1 (p. 2197)][43]: A MHSS X with sufficiently many local approximate tubular neighborhoods (see Rmk. 1.12.3) is a* ($\mathbb{Q}$-)Witt space, *if for each point of an odd-codimensional stratum $x \in X_{n-(2k+1)}$ we have:*

$$\mathrm{IH}^{\bar{n}}_k(\mathcal{L}_x;\mathbb{Q}) = 0$$

where $\mathcal{L}_x = \mathrm{holink}(X,x) = \{\gamma \in \mathrm{holink}_S(X, X_{n-(2k+1)}) | \gamma(0) = x\}$ is the local holink at x and $\mathrm{IH}^{\bar{n}}_*$ is intersection-homology (with compact support) utilizing the upper-middle perversity $\bar{n}$.

A large and important class of examples are:

[43] This is a simplified version of what this reference defines: We do not want to account for more general base-rings than $\mathbb{Q}$ here. See also the next lemma.

Example 1.12.6: *Pseudomanifolds with only even-codimensional strata are Witt. This includes pure-dimensional complex algebraic varieties.*

By construction, the so-defined Witt-spaces have the following property:

Lemma 1.12.7: *See discussion around [Fri09, Thm. 8.2 (p. 2198)] and the Lemma 1.12.2 above: For a closed oriented MHSS X with sufficiently many local approximate tubular neighborhoods (see Rmk. 1.12.3), without codimension 1 stratum*[44]*, it holds that:*

$$\begin{aligned} &X \text{ is } (\mathbb{Q}\text{-})\text{Witt} \\ &\quad\Leftrightarrow \\ &\mathrm{IC}^{\bar{m}}_*(X) \subset \mathrm{IC}^{\bar{n}}_*(X) \text{ induces an isomorphism } \mathrm{IH}^{\bar{m}}_*(X;\mathbb{Q}) \cong \mathrm{IH}^{\bar{n}}_*(X;\mathbb{Q}) \end{aligned}$$

The following example demonstrates, how this definition is (mostly) compatible with "properness" (the absence of a codimension one stratum) and the manifold-case.

Example 1.12.8: *Let X be a compact MHSS, then*

$$X \text{ is Witt} \qquad \Rightarrow \qquad X \text{ does } \textit{not} \text{ have a codimension 1 stratum}$$

further, if the underlying topological space $|X|$ is a manifold, then

$$X \text{ is Witt} \qquad \Leftrightarrow \qquad X \text{ does } \textit{not} \text{ have a codimension 1 stratum}$$

Proof: "$\Rightarrow$": Witt-spaces never have codimension 1 strata ([Fri09, (p. 2197)]), because the condition on the link (for $2k+1=1$, i. e. $k=0$) reads $\mathrm{IH}^{\bar{n}}_0(\mathcal{L};\mathbb{Q})=0$, which would require a ordinary homology group in degree 0 to vanish as well (by arguments similar to the proof of [Fri09, Prop. 5.2 (p. 2178)] / Lemma 1.12.2).

"$\Leftarrow$": We give two alternative proofs for this direction. A brief one explaining the statement itself, and a sketch of a second one to shed a little more light on the relation to links and giving an interesting connection to intrinsic stratifications (see Section 1.10 ("Intrinsic Stratifications")).

(1) Proof of "$\Leftarrow$": Intersection-homology on MHSS is topologically invariant [Qui87], and on the manifold $|X|$ clearly $\mathrm{IC}^{\bar{m}}_*(|X|) = \mathrm{IC}^{\bar{n}}_*(|X|)$. The Witt-condition was chosen to be equivalent to these being quasi-isomorphic (Lemma 1.12.7). So the only thing that could go wrong would be, if the orientation of the top-stratum of X would *not* induce a (compatible) orientation of $|X|$ (so the Deligne-sheaf

[44] What we really need is: There is an orientation of the top-stratum $M_{0,0} \supset M$ of the intrinsic skeleton $X_{0,0}$ (see Section 1.10 ("Intrinsic Stratifications")), that restricts to the one chosen for M. See the proof of Example 1.12.8 below.

argument in Lemma 1.12.2 may fail, because the orientation for X and $|X|$ may not agree). Obviously if X has a codimension 1 stratum, the top-stratum may have more connected components than $|X|$ has, thus more choices of orientations (some of which must be incompatible). If it does not, the orientation of $|X|^*$ induces one on X. See for example the discussion around [Fri15, Lemma 2.9 (p. 10)], which treats the pl-case, but the argument is by extension (the intrinsic stratification is a coarsening by Thm. 1.10.2) of sheaf-isomorphisms – defining an orientation as an isomorphism of the orientation-sheaf to the constant sheaf in the "usual" way – which does not rely on the space being pl.

(2) Sketch of alternative proof: Investigating the construction of the intrinsic skeleta of Quinn [Qui87, Thm. 1.2 (p. 235)] (see Thm. 1.10.2), we find that for the component B to be "promotable" (i. e. B is part of a larger stratum in $X_{0,0}$), its links (homotopy-link fibers) in X must necessarily be homology-spheres. Since $|X|$ is a manifold by hypothesis, writing $Y = |X| \supset \emptyset$, this is a stratification as MHSS. So by Thm. 1.10.2 $X_{0,0} = Y_{0,0} = Y$. Thus holinks must be homology-spheres (possibly stratified in a non-trivial way when working with spaces with more than two strata, if certain larger strata were not promoted either; see also [Hug99a, Cor. 6.2 (p. 141)], where a similar result for submanifolds being *locally flat* iff they have fibers that are *homotopy*-spheres is given). By invariance under stratified homotopy-equivalence, the (mapping cylinder of) the holink being homotopy equivalent to a neighborhood (thus a MHSS), and topological invariance of IH on MHSS [Qui88a], $\mathrm{IH}_k^{\bar{n}}(\mathcal{L};\mathbb{Q}) = H_k(S^{2k};\mathbb{Q}) = 0$ for all $k > 0$. □

Remark 1.12.9: Generally, even for a manifold with boundary $(M, \partial M)$, we may think of M as a MHSS with one stratum and $\partial X = \partial M$, or as a MHSS with two strata, filtered as $M \supset \partial M$, without boundary. This seems to be bad news for "Thom's theorem" / bordism invariance of the signature (Prop. 1.12.11) to make any sense.

It is the fact, that Witt-spaces cannot have codimension 1 strata, together with boundary-collaring, that saves the day and ensures there is no ambiguity in identifying what is a $\partial X \subset X$.

The typical use-case of the Witt-condition is, of course, the definition of a signature-invariant:

Theorem 1.12.10: *[Fri09, Thm. 8.2 (p. 2198)]: Let X be a closed*[45] *oriented MHSS Witt-space of dimension $4k$, then X has a well-defined signature $\sigma(X) \in \mathbb{Z}$, given by the signature of the middle-dimensional, middle-perversity intersection-homology (with rational coefficients) non-degenerate intersection-form.*

If the dimension is not divisible by 4, define $\sigma(X) = 0$.

[45] While the reference states results for "compact" spaces, it defines MHSS explicitly as MHSS without boundary (in our sense) (see [Fri09, p. 2172]), so "compact" in the notation of the reference, actually means closed in the notation used here.

See also [Fri09, Cor. 7.3 (p. 2193)] for the relevant (middle-dimension, middle perversity) pairing.

Next we want to understand bordism-invariance, known as "Thom's theorem" in the manifold case. This is expected to work "as usual" because the relevant argument is on the algebraic side, thus unaffected by the change of underlying geometry. It requires to account for boundaries however (for obvious reasons):

Proposition 1.12.11: *"Thom's Theorem for MHSS": Let X be a compact oriented MHSS Witt-space with boundary ∂X closed. Then $\sigma(\partial X) = 0$.*

Proof: The orientation of X uniquely induces an orientation of ∂X, because it is defined as an orientation of the top-stratum, and for the manifold-case this is a well-known fact. We may assume $\dim(X) = 4k+1$ with $k > 0$, since the other cases are trivial.

The proof (for $k > 0$) proceeds basically as for [GM80, 'Theorem' (p. 155)] or [Ban07, Thm. 6.1.4 (p. 125)]:

Define $Y := X \cup_{\partial X} c(\partial X)$, which is a compact MHSS with sufficiently many local approximate tubular neighborhoods and without boundary. This is, because this is a local property, and true for X and therefore (since the boundary ∂X is collared in X by definition) this only needs to be checked near the cone-point. But the cone-point has a cone-like neighborhood, so is a MHSS with sufficiently many local approximate tubular neighborhoods. Hence [Fri09, Cor. 7.3 (p. 2193)] applies and there are duality-isomorphisms $\mathrm{PD} : \mathrm{IH}^{\bar{m}}_l(Y) \xrightarrow{\sim} (\mathrm{IH}^{\bar{n}}_{n-l}(Y))^* := \mathrm{Hom}(\mathrm{IH}^{\bar{n}}_{n-l}(Y), \mathbb{Q})$. If coefficients for IH are not explicitly stated, we assume them to be the rationals $\mathbb{Q}$.

By Lemma 1.12.2 $\mathrm{IC}^{\bar{m}} \subset \mathrm{IC}^{\bar{n}}$ differs near x from a quasi-isomorphism by stalks $\cong \mathrm{IH}^{\bar{n}}_k(\mathcal{L}_x; \mathbb{Q})$. Away from the cone-point, these vanish, since X is Witt. The cone-point v has $\mathcal{L}_x \simeq_{\mathrm{strat}} \partial X$ (by pushing along the cone coordinate, see Example 1.4.8), and co-dimension n. Thus the only difference is in degree $i = \bar{n}(n) = \bar{n}(4k+1) = 2k$.

Combining the long-exact-sequences / distinguished-triangles of Lemma 1.12.2 with the duality-isomorphisms on Y one obtains the following commutative ladder-diagram with exact rows (using $n - 2k = 2k+1$):

$$\begin{tikzcd}
\mathrm{IH}^{\bar{m}}_{2k+1}(Y) \arrow[r] \arrow[d, "\mathrm{PD}"', "\wr"] & \mathrm{IH}^{\bar{n}}_{2k+1}(Y) \arrow[r] \arrow[d, "\mathrm{PD}"', "\wr"] & \mathrm{IH}^{\bar{n}}_{2k}(\partial X) \arrow[r] & \mathrm{IH}^{\bar{m}}_{2k}(Y) \arrow[r] \arrow[d, "\wr"', "\mathrm{PD}"] & \mathrm{IH}^{\bar{n}}_{2k}(Y) \arrow[d, "\wr"', "\mathrm{PD}"] \\
(\mathrm{IH}^{\bar{n}}_{2k}(Y))^* \arrow[r] & (\mathrm{IH}^{\bar{m}}_{2k}(Y))^* \arrow[r] & (\mathrm{IH}^{\bar{n}}_{2k}(\partial X))^* \arrow[r] & (\mathrm{IH}^{\bar{n}}_{2k+1}(Y))^* \arrow[r] & (\mathrm{IH}^{\bar{m}}_{2k+1}(Y))^*
\end{tikzcd}$$

where $(\ldots)^* := \mathrm{Hom}(\ldots, \mathbb{Q})$. Since ∂X is Witt (its strata have the same holinks and codimensions as those in its collar in X, and X is Witt), $\mathrm{IH}^{\bar{n}}_{2k}(\partial X) \cong \mathrm{IH}^{\bar{m}}_{2k}(\partial X)$. Further, using the canonical isomorphism (Lemma 1.12.7) induced by the inclusion $\mathrm{IC}^{\bar{m}} \subset \mathrm{IC}^{\bar{n}}$ we define the composition (of isomorphisms)

$$\varphi : \mathrm{IH}^{\bar{m}}_{2k}(\partial X) \to \mathrm{IH}^{\bar{n}}_{2k}(\partial X) \xrightarrow{\mathrm{PD}} \left(\mathrm{IH}^{\bar{m}}_{2k}(\partial X)\right)^*$$

Using naturality of the constructions, we can replace the middle-column of the above diagram by this to get (working in the derived category, the existence of φ follows from the axioms of triangulated categories):

$$\begin{tikzcd}
\mathrm{IH}^{\bar m}_{2k+1}(Y) \arrow[r] \arrow[d, "\mathrm{PD}"', "\wr"] & \mathrm{IH}^{\bar n}_{2k+1}(Y) \arrow[r, "j"] \arrow[d, "\mathrm{PD}"', "\wr"] & \mathrm{IH}^{\bar m}_{2k}(\partial X) \arrow[r] \arrow[d, "\wr"', "\varphi"] & \mathrm{IH}^{\bar m}_{2k}(Y) \arrow[r] \arrow[d, "\wr"', "\mathrm{PD}"] & \mathrm{IH}^{\bar n}_{2k}(Y) \arrow[d, "\wr"', "\mathrm{PD}"] \\
(\mathrm{IH}^{\bar n}_{2k}(Y))^* \arrow[r] & (\mathrm{IH}^{\bar m}_{2k}(Y))^* \arrow[r, "j^*"] & (\mathrm{IH}^{\bar m}_{2k}(\partial X))^* \arrow[r] & (\mathrm{IH}^{\bar n}_{2k+1}(Y))^* \arrow[r] & (\mathrm{IH}^{\bar m}_{2k+1}(Y))^*
\end{tikzcd}$$

The usual arguments – see for example [Ban07, '1.' (p. 124)] and the proof of [Ban07, Thm. 6.1.2 (p. 124)] – apply, to show, that $\mathrm{im}(j)$ is a Lagrangian (i. e. an, under this pairing φ, maximally self-annihilating) subspace in $\mathrm{IH}^{\bar m}_{2k}(\partial X)$. Thus the signature of φ, which is the middle-dimensional ($\dim \partial X = \dim X - 1 = 4k$), middle-perversity intersection-form, is zero as claimed. ☐

To summarize: Using Friedman's [Fri09] formulation, the usual sheaf-theoretic treatment [GM83] (see also [Ban07]) applies. In consequence, for MHSS with sufficiently many local approximate tubular neighborhoods, there is a "Witt-Condition", that can be formulated on the intersection-homology of local-homotopy-links – in essentially the same way as on links for pseudomanifolds (only the choice of $\bar n$ cannot be replaced by $\bar m$ anymore) – to obtain a class of spaces with well-defined, Witt-bordism invariant signature.

1.13 Summary

The central "line of argument", of this background-chapter can be summarized as follows (for spaces with at most two strata):

Stratified transversality requires some normal-structure, to extend geometric constructions into the upper stratum. Such normal structure in a topological setting seems to be best described by mapping-cylinders of MAFs, where the transversality-construction of Connolly and Vajiac [CV99] applies.

A suitable setup for the study of the existence and uniqueness of such neighborhoods is identified in Quinn's [Qui88a] (M)HSS, where classical h-cobordism and end-problems can be treated in a controlled-topology (generalizing from lower stratum $B = \{\mathrm{pt}\}$) setting. Here, the existence / uniqueness question can be treated by methods of [Qui79].

Finally, this class of spaces that rather "naturally" arose in the treatment of transversality-questions in an inherently topological setting, does indeed have (topologically) "intrinsic" stratifications. Further it is suitable for the definition of intersection-homology [GM80; Qui87] satisfying Poincaré duality and, in the Witt-case [Sie83], having non-singular intersection-forms, thus signature-invariants (see [Fri09]).

This "two-stratum" picture is supplemented in Chapter 3 ("Multiple Strata") by a treatment of cases with more than two-strata.

2 Bordism Constructions

Given suitable transversality properties, bordism theories often provide generalized homology-theories. At the same time, for "allowable" spaces X^n – i. e. those constituting the "probing" objects of the theory – there is natural notion of a "fundamental-class" $[X] := [X \xrightarrow{\mathrm{id}} X] \in \Omega_n(X)$. This makes them interesting structures for the study of characteristic classes.

We want to construct a bordism-theory, whose "allowable" spaces are certain MHSS, with the motivation, that MHSS-stratifications are characterized topologically, thus will often be available on purely topological constructions, for example on mapping-cylinders of stratified homeomorphisms. This in turn allows, for example, the study of the "transport-behavior" of said fundamental classes $[X]$ under such stratified homeomorphisms.

Additionally, being close to (topologically) intrinsic ones (see Section 1.10 ("Intrinsic Stratifications")), these stratifications even allow for the study of the transport-behavior under unstratified homeomorphisms. This case – *unstratified* homeomorphisms – is very closely related to the question, whether $[X]$ depends on the stratification of X or just on its topology.

We start by the construction of bordism theories, not necessarily generalized homology theories yet, in Section 2.1 ("Bordism Theories"). Then it is formalized what "suitable transversality properties" is supposed to mean in Section 2.2 ("Transversality Properties"). The actual verification of homology-axioms (of an unreduced theory) in the sense of Eilenberg–Steenrod is carried out in Section 2.3 ("Generalized Homology Theory"). For certain MHSS with at most two strata, the transversality properties can be seen to hold relatively directly. This is shown in Section 2.4 ("An Example-Theory") and introduces an important condition on Whitehead-groups related to the fundamental groups of "links". Some stratified homeomorphisms can be studied with this basic theory already. The reduced homology theory, associated with the homology theories produced before, has certain nice geometric properties, which can be formalized through the form / realization of the suspension-isomorphism. This is done in Section 2.5 ("Reduced Theories") and used in Section 5.3 ("L-Classes"). Finally, it is outlined in Section 2.6 ("Improvements"), how the initial "basic theory" can be extended to remove some unpleasant restrictions. Such "extensions" are carried out in the next chapters (Chapter 3 ("Multiple Strata") and Chapter 4 ("Homeomorphisms")) and constitute the bulk of technical work presented in this thesis. For this reason, their relevancy and limitations are discussed in this dedicated section. Further discussion can be found in Section 6.3 ("Outlook and Further Ideas").

In this thesis, the terms "bordism" and "cobordism" are used synonymous. We do *not* treat “cohomological” theories, e. g. in the sense of Buonchristiano, Rourke, and Sanderson’s [BRS76] “mock-bundles”, so no confusion should arise from this convention.

2.1 Bordism Theories

Bordism theories have been studied intensely in the past, originally on (smooth) manifolds [Tho54; Wal60], with additional structure via bundles [Ati61], for stratified spaces [Sie83; Par90], for stratified spaces and additional structure [Min06; Ban06] or for abstract classes of spaces satisfying certain requirements [Aki75; Fri15] (similar to the approach here). It turns out, that quite generally, many (co)homology theories can be realized geometrically via suitable bordism-constructions (and mock-bundles) [BRS76].

Bordisms play an important role e. g. in surgery (as the “trace”) to establish (surgery-)invariants, or in (tangent-bundle-independent) constructions of characteristic classes, such as L-classes [Tho58]. Also the Pontryagin–Thom-construction (the construction of the bordism-spectrum MSO) was quite influential.

As of the contents of this section, first, the specifics of “bordism-theories” are defined, relations to generalized homology theories will then be discussed in Section 2.3 (“Generalized Homology Theory”).

In a stratified setting, bordism-theories typically need some “rigidity” beyond the compactness hypothesis used, for example, for manifolds. This is, because otherwise, for a space X the cone cX is typically a “null-bordism” (see below), rendering the theory trivial. An important example of such “rigidity” is a Witt-condition, because a bordism-invariant signature invariant prevents a “collapse” to the trivial theory. Further, to get a generalized homology theory later, we will need some hypothesis to ensure sufficiently nice transversality properties.

For these two reasons, the theories as formulated here use certain subclasses $\mathcal{C} \subset \partial$-MHSS, rather than *all* MHSS. Formally, one may use as $\mathcal{C}$ a category with forgetful functor to ∂-MHSS, meaning probing objects could contain certain additional information (if that information glues etc.), but we do not use this freedom (see however Rmk. 4.1.2).

Clearly such subclasses need to be chosen “consistently” – in the sense, that bordism is indeed an equivalence-relation – which will be ensured by the following property (similar approaches can be found for example in [Aki75; Fri15])

Definition 2.1.1: *We call a class $\mathcal{C}$ of compact proper (the top-stratum is dense, and there is no codimension 1 stratum) orientable MHSS with boundary* stable*, iff*

(i) For $(X, \partial X) \in \mathcal{C}$ also $\partial X \in \mathcal{C}$.

(ii) For $X \in \mathcal{C}$, given a decomposition of X into $X_\geq$, $X_\leq$ such that $X = X_\geq \cup X_\leq$ and such that $X_0 := X_\geq \cap X_\leq$ is codimension 1 *and stratified*

bi-collared in X, with the restriction of the bi-collar to both $X_{\geq}$ and $X_{\leq}$ a collar, then $(X_{\geq}, X_0) \in \mathcal{C}$ and $(X_{\leq}, X_0) \in \mathcal{C}$.

(iii) For $X, Y \in \mathcal{C} \Rightarrow X \cup Y \in \mathcal{C}$, whenever gluing along boundary-components in the sense of Lemma 1.4.25 applies.

(iv) Given $X \in \mathcal{C}$ and $(M, \partial M)$ a compact manifold $\Rightarrow \Big(X \times M, (X \times \partial M) \cup (\partial X \times M)\Big) \in \mathcal{C}$

(v) The $\emptyset_n \in \mathcal{C}$ is an element[1] in any dimension $n \geq 0$.

Part (iv) can be weakened to the case $M = I$, which would only remove the statement about being a module over the manifold-cobordism-ring from the final result.

The "properness" assumption ensures that dimension is "well-defined" (independent of the connected component), because MHSS are assumed to be stratified by dimension. Note, that clearly:

Lemma 2.1.2: *If $\mathcal{C}$ and $\mathcal{C}'$ are two stable classes, then $\mathcal{C} \cap \mathcal{C}'$ is also stable.*

The very closely related concept of fixing a bordism-theory through allowable links has proven successful in the past (see e. g. [Sie83; Ban11; Fri15]), we have already seen an example of such a class

Example 2.1.3: *A condition that depends only on holink-fibers and codimension is stable. This also holds for local (stratified) holinks, see Chapter 3 ("Multiple Strata").*

Thus, the class $\mathcal{C} = \mathrm{Witt}$ of MHSS satisfying the Witt-condition of Section 1.12 ("Witt-Condition and Signature of MHSS") is stable.

The class of proper MHSS is stable.

Proof: This follows from Lemma 1.4.21, because stratified (bi-)collared subsets are stratified NDRs. Also $(X_{\geq}, X_0)$ and $(X_{\leq}, X_0)$ are automatically MHSS with boundary, by locality of being MHSS, the collar of X_0 given by hypothesis, and X_0 being a stratified NDR.

Codimensions are preserved (see above) so for the stability of being proper only density of the top-stratum remains to be checked. But again, a stratified (bi-)collar shows, that a boundary (or cut X_0) also has dense top-stratum. □

We had also seen, that gluing works for spaces with at most two meeting strata (Def. 1.4.22), which – in anticipation of the multi-stratum theory in Chapter 3 ("Multiple Strata") – motivates the following definition and subsequent observation:

[1] So we formally consider $\emptyset_n$ as a n-dimensional MHSS.

Definition 2.1.4: *A MHSS X has* at most k meeting strata*, if for all $x \in X$, there are at most k different strata X_i with $x \in \operatorname{closure}(X_i)$.*

Example 2.1.5: *For fixed $k \geq 1$, the class of spaces with at most k meeting strata is stable.*

One may also notice the following famous non-example:

Non-Example 2.1.6: *The class consisting of the n-disk and $\emptyset_n$ in dimension n is* not *stable: Cutting a disk, in the sense of condition (ii), such that the result is* not *a disk is easy, e. g. let X_0 be a $(n-1)$-sphere, $X_\leq$ its "inside" and $X_\geq$ its "outside".*

This is (closely related to) why (unstable) homotopy-groups do not *form a homology-theory. (They do not satisfy excision.)*

So "stability" as defined here is not needed for well-defined bordism-groups (clearly homotopy-groups are well-defined and functorial), but already anticipates the incorporation of excision. By Lemma 2.2.2 it will become clear, why we call such conditions "stable", and do not content ourselves with a weaker requirement here.

Next, we define an equivalence relation over topological pairs

Definition 2.1.7: *Fix a pair (A, B) of topological spaces. Given a stable class $\mathcal{C}$, two oriented spaces $(X, \partial X)$ and $(Y, \partial Y)$ in $\mathcal{C}$, with orientations[2] O_X, O_Y, and continuous maps of pairs $f : (X, \partial X) \to (A, B)$ and $g : (Y, \partial Y) \to (A, B)$ define:*

f and g are bordant over (A, B)*, $f \sim_{\mathcal{C}} g$, iff there is an oriented $W \in \mathcal{C}$ with orientation O_W and $\partial W = W_0 \cup W_1 \cup W_*$ together with $F : W \to A$ such that*

(i) $F(W_*) \subset B$

(ii) $\partial W_0 = W_0 \cap W_*$ *and* $\partial W_1 = W_1 \cap W_*$ *and* $\partial W_* = \partial W_0 \cup \partial W_1$ *while* $W_0 \cap W_1 = \emptyset$.

(iii) $W_0, W_1, W_* \in \mathcal{C}$

(iv) $W_0 =^{3} X$ *and* $W_1 = Y$

(v) $F|_{W_0} = f$ *and* $F|_{W_1} = g$

[2] We may define an orientation as a choice of isomorphism from the orientation-sheaf to the constant sheaf, a viewpoint which lends itself well to sheaf-theoretic treatments of intersection-homology (see Section 1.12 ("Witt-Condition and Signature of MHSS")). But really, we are free to use any (meaningful) formal definition of orientation.

[3] One may require orientation-preserving, stratified homeomorphic instead, although this is not entirely trivial here, and is discussed more in-depth in Rmk. 4.1.2, after discussing stratified homeomorphisms and their mapping cylinders.

(vi) $O_X = O_{\partial W}|_{W_0}$ *and* $O_Y = -O_{\partial W}|_{W_1}$

We will usually drop the explicit reference to $\mathcal{C}$*, where no confusion arises. Instead we will often write* $\sim_W$ *or* $\sim_F$ *in explicit reference to the* bordism W*.*

Remark 2.1.8: As is most evident from (ii), the restriction $F|_{W_*}$ to W_* is a bordism from $f|_{\partial W_0}$ to $g|_{\partial W_1}$ over B. This observation will be useful to construct well-defined boundary-maps.

It is, of course, also possible to define an "unoriented" theory by simply dropping the orientation-requirements.

Before we continue to study further structure on equivalence-classes, we want to check, that this is indeed an equivalence relation.

Lemma 2.1.9: *Bordism, as defined above (Def. 2.1.7), is an equivalence relation.*

Proof: "Reflexivity": Given $X \in \mathcal{C}$ and $f : (X, \partial X) \to (A, B)$, since $\mathcal{C}$ is stable by hypothesis, $W := \big(X \times I, (X \times \partial I) \cup (\partial X \times I)\big) \in \mathcal{C}$. Let $W_0 := X \times \{0\}$, $W_1 := X \times \{1\}$ and $W_* := \partial X \times I$, further, $F : W \to A, (x,t) \mapsto f(x)$. Then F is a bordism $f \sim f$, because $F(W_*) = f(\partial X) \subset B$.

"Symmetry": Given $f \sim_F g$, by $F : W \to A$, using the inverse orientation $-O_W$ and exchanging W_0 with W_1, we obtain a bordism "$-F$" that realizes $g \sim f$.

"Transitivity": Given $f \sim_F g \sim_{F'} h$, using that $\mathcal{C}$ is stable, thus compatible with the gluing construction of Lemma 1.4.25, we obtain $F \cup F' : W \cup_{W_1 = W_0'} W' \to A$, a bordism $f \sim_{F \cup F'} h$. The orientations of W and W' fit together, because of the sign $O_{W_0'} = O_Y = -O_{W_1}$ in the definition. □

Further the equivalence-classes define an abelian group for each dimension:

Definition 2.1.10: *Fix a pair* (A, B) *of topological spaces. Given a stable class* $\mathcal{C}$*, the* bordism groups $\Omega_n^{\mathcal{C}}(A, B)$ *in degree* $n \geq 0$ *are the equivalence classes* $[f : (X, \partial X) \to (A, B)]$ *with* $\dim(X) = n$ *of the equivalence-relation defined by bordism. For* $n < 0$*, set* $\Omega_n^{\mathcal{C}}(A, B) = 0$*.*

The group-operation is given by disjoint union:

$$\begin{aligned} &[f : (X, \partial X) \to (A, B)] + [g : (Y, \partial Y) \to (A, B)] \\ &\quad := \quad [f \sqcup g : (X \sqcup Y, \partial X \sqcup \partial Y) \to (A, B)] \end{aligned}$$

If $B = \emptyset$*, we write* $\Omega_*^{\mathcal{C}}(A)$ *for* $\Omega_*^{\mathcal{C}}(A, \emptyset)$*.*

We required "stability" not only for $X \times I$ but for any compact manifold M, so this is actually a module over $\Omega_*^{\mathrm{STOP}} = \Omega_*^{\mathrm{STOP}}(\{\mathrm{pt}\})$, the oriented manifold-bordism-groups (see below, Cor. 2.1.13–1).

Next, we check that this defines, in fact, abelian groups:

Lemma 2.1.11: *The so defined bordism "groups" are abelian groups.*

Proof: The operation is clearly well-defined (the bordism-relation $\sim$ can be treated component-wise), and associative (because disjoint union is).

The neutral element in degree n is $\emptyset_n \in \mathcal{C}$. We formally allow continuous maps *from* the empty set.

The inverse of an element $[f : (X, \partial X) \to (A, B)]$ is "$[-f]$", obtained by inverting the orientation-class of X. A null-bordism $[f] + [-f] \sim [\emptyset] = 0$ is the product with the closed interval $f \times \mathrm{id}_I$ as described in the "reflexivity"-part of the proof of Lemma 2.1.9 ($\sim$ is an equivalence relation), but with $W_0 := X \times \partial I$ and $W_1 = \emptyset$.

There is no inherent ordering of components in the theory, so disjoint union, and thus "+", is commutative. □

This construction, with induced group-homomorphisms defined by composition, is functorial

Lemma 2.1.12: *The assignment* $(A, B) \mapsto \Omega_*(A, B)$ *and*

$$\Big(\alpha : (A, B) \to (A', B')\Big) \quad \mapsto \quad \Big(\alpha_* : \Omega_*(A, B) \to \Omega_*(A', B'), [f] \mapsto [\alpha \circ f]\Big)$$

is a covariant functor from pairs of topological spaces to abelian groups.

Proof: It has been shown above that $\Omega_*(A, B)$ are abelian groups, the composition of a disjoint union (of maps) is the disjoint union of compositions (so induced mappings are group-homomorphisms) and composition of morphisms is clear from the definition (composition of maps is associative). □

While the definition of "stable classes" (Def. 2.1.1) required that products with manifolds still lie in the class $\mathcal{C}$, we only used this for the interval $(I, \partial I)$. However, more generally, this allows for the following product-constructions:

Lemma 2.1.13: *Let* Ω_*^{STOP} *denote the oriented topological manifold bordism-groups (i. e. in the notation above:* STOP *is the class containing all compact, orientable manifolds) and* $\Omega_*^{\mathcal{C}}$ *those of another stable class* $\mathcal{C}$*. Then there is a "product":*

$$\begin{aligned} &\Omega_i^{\mathrm{STOP}}(A', B') \otimes \Omega_j^{\mathcal{C}}(A, B) \to \Omega_{i+j}^{\mathcal{C}}((A', B') \times (A, B)), \quad \text{induced by} \\ &\quad (\quad [f' : (M, \partial M) \to (A', B')], \quad [f : (X, \partial X) \to (A, B)] \quad) \\ &\quad \mapsto [f' \times f : \quad (M, \partial M) \times (X, \partial X) \to (A', B') \times (A, B)] \end{aligned}$$

where $(A', B') \times (A, B) := (A' \times A, \quad A' \times B \cup B' \times A)$*.*

Corollary 2.1.13–1: *$\Omega_*^{\mathcal{C}}(A, B)$ is a module over $\Omega_*^{\mathrm{STOP}} := \Omega_*^{\mathrm{STOP}}(\{\mathrm{pt}\})$.*

Proof: By Def. 2.1.1 $f' \times f$ is allowed as a representative and bilinearity are clear. It remains to show that the equivalence-class $[f' \times f]$ does not depend on the choices of representatives of $[f]$ and $[f']$. Let $g \sim_F f$ and $g' \sim_{F'} f'$. Then $F' \times F$ is an allowable bordism in $\Omega_*^{\mathcal{C}}$ (again by Def. 2.1.1), because $F' : W' \to A'$ is defined on a manifold W'. So $g' \times g \sim_{F' \times F} f' \times f$. □

Note, that a more general ring-structure on all of $\Omega_*^{\mathcal{C}}$ (instead of Ω_*^{STOP}) does not work well with restrictions on the number of strata. The example given in the next sections below, is for spaces with at most two strata, where products may have up to four strata.

So far, this is not fundamentally different from the well-known theory on manifolds. The main difficulty is in constructing an excision-isomorphism to obtain a homology theory, for a class $\mathcal{C}$, that is still "interesting" to study.

2.2 Transversality Properties

The properties studied here identify classes of spaces with "good enough" transversality-properties to construct an excision-isomorphism (in the sense of Eilenberg–Steenrod axioms).

We will mostly define purely abstract properties. It will not immediately be evident that stable classes with these properties actually exist, but an example will be provided later in this section. The next chapter, Chapter 3 ("Multiple Strata")), is primarily concerned with providing more involved examples.

Finally we show, that the bordism-groups defined for classes of spaces with such transversality-properties satisfy the excision-axiom of a generalized homology-theory.

The main problem of constructing an excision-isomorphism is to "cut" spaces in the following sense:

Definition 2.2.1: *Given a MHSS X and closed disjoint subspaces $B_+, B_- \subset X$, we say, that X* can be cut between B_+ and B_-, *iff there is a map $g : X \to [-1, 1]$ transverse to 0 (in the sense of Def. 1.6.3), mapping $g(B_\pm) \subset \{\pm 1\}$. We will, in this case, refer to the bi-collared MHSS $X_0 := g^{-1}(\{0\})$ as the* cut. *The spaces*

$$(X_\leq, X_0 \cup \partial_- X) := \Big(g^{-1}([-1, 0]), \quad g^{-1}(\{0\}) \cup \Big(g^{-1}([-1, 0]) \cap \partial X\Big)\Big)$$
$$(X_\geq, X_0 \cup \partial_+ X) := \Big(g^{-1}([0, +1]), \quad g^{-1}(\{0\}) \cup \Big(g^{-1}([0, +1]) \cap \partial X\Big)\Big)$$

are automatically MHSS with boundary (by bi-collaredness of X_0 and locality of being MHSS).

We call X a weak transversality-space (weak t-space)*, iff given any closed disjoint $B_\pm \subset X$, it can be cut between $B_\pm$.*

*We call a subclass $\mathcal{C} \subset \partial$-*MHSS *a* weak transversality-class (weak t-class)*, iff any $X \in \mathcal{C}$ is a weak t-space.*

*We call a subclass $\mathcal{C} \subset \partial$-*MHSS *a* strong transversality-class (strong t-class)*, iff it is a weak t-class and stable (Def. 2.1.1), thus for any such "cut", the spaces $X_\leq$ and $X_\geq$ are again in $\mathcal{C}$.*

These are compatible with further restriction to stable subclasses in the sense of

Lemma 2.2.2: *Given a weak / strong t-class $\mathcal{T}$ and a stable class $\mathcal{S}$, the combination $\mathcal{T} \cap \mathcal{S}$ is a weak / strong t-class.*

Proof: $\mathcal{T} \cap \mathcal{S}$ is a weak t-class, because its elements are in $\mathcal{T}$, so can be cut.

If $\mathcal{T}$ is a strong t-class, then it is stable, so by Lemma 2.1.2, $\mathcal{T} \cap \mathcal{S}$ is stable. Any cutting of a space $X \in \mathcal{T} \cap \mathcal{S}$ yields spaces $X_\geq$, $X_\leq$ in $\mathcal{T}$. As a cut, by definition, they satisfy the hypothesis of part (ii) in the definition of stable classes (Def. 2.1.1; essentially this means, they intersect in a bi-collared X_0), so $X_\geq$ and $X_\leq$ are also in $\mathcal{S}$ (which is stable by hypothesis). □

In view of Quinn's controlled-end theorem as stated in Section 1.9 ("Mapping-Cylinder Neighborhoods") and the transversality- theorem of Connolly and Vajiac as stated in Section 1.6 ("Stratified Transversality"), we define the following property (and thus class) of MHSS:

Definition 2.2.3: *Given a MHSS X (with at most two meeting strata), we say X has* simple links *if: For all components B_i of the (disjoint) minimal strata and all components L_{ij} of the fiber of the respective homotopy-link, it holds that $\forall k \geq 0$ $\mathrm{Wh}(\pi_1(L_{ij}) \times \mathbb{Z}^k) = 0$. Since the Whitehead-groups do not (within connected components) depend on a choice of base-point, we suppress base-points in the notation of π_1 (see Rmk. 1.9.9).*

See Def. 3.1.10 for a more general definition, including spaces with more than two strata.

This condition is for example satisfied, if links are simply connected:

Example 2.2.4: *"Supernormal" spaces*[4] *in the sense of [Wei94, §12.1 (p. 202f)], with at most two meeting strata, i. e. MHSS with simply connected local holinks, have simple links.*

In fact, any MHSS X with holink-fibers, whose (component-wise) fundamental-groups are torsion-free and poly-(finite / cyclic), has simple links. Poly-(finite / cyclic) means, there is a normal-series (a filtration by subgroups, normal in the respective next one) with quotients, which are either finite groups or infinite-cyclic (see Def. 1.9.5).

Proof: It is non-trivial, but true, that $\mathrm{Wh}(\mathbb{Z}^k) = 0$, this is famously known as the Bass–Heller–Swan theorem [BHS64].

Farrell and Hsiang [FH81, Thm. 3.2 (p. 308)] show, that for any torsion-free and poly-(finite / cyclic) group Γ it holds that $\mathrm{Wh}(\Gamma) = 0$. If G is torsion-free and poly-(finite / cyclic), then so is $G \times \mathbb{Z}^k$. Note, that also $\mathbb{Z}^k$ is torsion-free and poly-(finite / cyclic). □

And using this condition, we can give the first examples of t-classes:

Example 2.2.5: *For spaces with at most two meeting strata, it holds that*

(i) The class Manifolds $\subset$ MHSS *of compact, orientable manifolds is a strong t-class.*

(ii) The class $\mathrm{Wh}_2^{\geq 6} \subset$ MHSS *given by all compact, orientable and proper (without codimension 1 stratum and with dense top-stratum) MHSS X with at most two meeting strata, where* $\dim(\partial X) \geq 6$ *or* $\partial X = \emptyset$ *and* $\dim(X) \geq 6$*, with simple links is a weak t-class.*

(iii) The class $\mathrm{Wh}_2^{\mathrm{gap}\,5} \subset$ MHSS *given by all compact, orientable and proper MHSS X with at most two meeting strata where the codimensions of minimal strata B_i in X are ≥ 5 with simple links is a strong t-class.*

Proof: Part (i) follows from manifold-transversality: By regularity of the (metric) X, there is a continuous $g : X \to [-1, 1]$ such that $g(B_\pm) \subset \{\pm 1\}$, apply Cor. 1.5.9–2 with $\epsilon = 1/2$ (any $\epsilon < 1$ will do), on $D = X$ and $\mathrm{rel}\, C \cup C_\partial = \emptyset$ to obtain $g_\perp$ transverse to 0 at c and $1/2$-close to g. Define $X_0 := g_\perp^{-1}(\{0\})$ and $X_\geq := g_\perp^{-1}([0, 1])$ and $X_\leq := g_\perp^{-1}([-1, 0])$. By closeness to g, this decomposition satisfies $B_+ \subset X_\geq$ and $B_- \subset X_\leq$, while X_0 is bi-collared by c. It is immediately clear from the definition, that $X_\geq \cup X_\leq = X$ and $X_\geq \cap X_\leq = X_0$, and that these are manifolds with boundary (compact, with induced orientation).

Part (ii): As before, there is a continuous $g : X \to [-1, 1]$ such that $g(B_\pm) \subset \{\pm 1\}$. We can (by normality of the metric X), pick disjoint open neighborhoods N_i of the (disjoint) minimal strata B_i, which are MHSS (as open subsets of the MHSS X) with dim ≥ 6 (by density of the top-stratum of X), so the controlled end-theorem

[4] A space is called normal, if its links are path connected, that is, if $\pi_0(L)$ is trivial. This led to the term "super-normal", referring to spaces where even $\pi_1(L) = 0$ is trivial.

Thm. 1.9.3 applies (on each N_i) and because X has simple links, by Lemma 1.9.8, the obstruction-group, and thus the obstruction q_0 vanish.

If X has no boundary, we thus obtain mapping-cylinder neighborhoods of the B_i (individually and component-wise), within these N_i, thus disjoint in X, so Cor. 1.6.4–1 applies to make g transverse to 0.

If X has non-empty boundary, we may either apply the end-theorem to the boundaries first (these have simple links as stratified NDR, and $\dim(\partial X) \geq 6$ by hypothesis), then extend (the cylinder-neighborhood) into X using a boundary-collar to extend them as a product to an open neighborhood of the boundary, and the relative end-theorem [Qui79, Thm. 2.1 (p. 282)] to then extend them further to cylinder-neighborhoods of parts of the B_i outside of the boundary-collar. Next, apply Cor. 1.6.4–2 to make g transverse to 0.

Alternatively, we may use a formulation using spaces with more than two strata to "absorb the boundary" as will be detailed in Section 3.2 ("Absorbing Boundary") to then apply Cor. 3.2.3–1 to make g transverse to 0. These two possibilities were also discussed in Section 1.6 ("Stratified Transversality").

In any case we obtain a close to g, transverse to 0 map $g_\perp$. This means precisely, defining $X_0, X_\geq, X_\leq$ as before, that X can be "cut" according to Def. 2.2.1 so is a "weak t-space". As X was arbitrary in $\mathrm{Wh}_2^{\geq 6}$, this is a weak t-class.

Part (iii): The "stability" part follows, because the operations required to be "stable" do not change the homotopy-type of holink-fibers (see Lemma 1.4.21) or the codimension, see also Example 2.1.3 and by Example 2.1.5 restricting to spaces with at most two meeting strata is also stable.

The difficulty in choosing the class of spaces "correctly", so that it is a *strong* t-class, is repeated application of the cutting property, eventually leading to low (absolute) dimensions, where mapping-cylinder-existence (Thm. 1.9.3) and boundary-collaring (Thm. 1.4.15) may not apply anymore. We treat four cases separately, and we assume there is only one minimal stratum $B_i =: B$, as different B_i (being closed, disjoint) can be treated individually (in exactly the same way as in the proof of Cor. 1.6.4–1 and in part (ii) above).

"Case $\dim(\partial X) \geq 6$ or $\partial X = \emptyset$ and $\dim(X) \geq 6$": X satisfies the hypothesis of part (ii), i. e. is in $\mathrm{Wh}_2^{\geq 6}$ and can thus be cut. Both $X_\geq$ and $X_\leq$ are still in $\mathrm{Wh}_2^{\mathrm{gap}\,5}$, by stability of the holink-fiber and codimension conditions: The hypotheses of (iii) are local, so we only need to check them near the "new" boundary-segment X_0, but X_0 is bi-collared in X (thus stratified NDR, thus p-NDR), so has the same homotopy-link fibers (up to homotopy) and either only one stratum or a stratum of the same codimension as $B \subset X$, which is ≥ 6 (see again Example 2.1.3).

"Case $\dim(\partial X) = 5$": The following is a slight modification of the proof of Cor. 1.6.4–1, which showed transversality on spaces with at most two meeting strata without boundary.

By the codimension-requirement, the boundary of the lower stratum ∂B is of dimension ≤ 0 (a finite point-set), so we can absorb ∂B into the rel parts B_+ and B_-, e. g. by $B_- \mapsto B_- \cup (\partial B - B_+)$, with $B_\pm$ still being closed and disjoint. Since $\partial X \cap B = \partial B$ and $\partial B \subset B_+ \cup B_-$, there is $\epsilon > 0$ such that for $x \in \partial X \cup B$ with

$\mathrm{dist}(x, \partial B) < \epsilon$: $g(x) \neq 0$. We apply manifold-transversality Cor. 1.5.9–2 for (the manifolds) $\partial X - \partial B$ and B on $D_\partial := \partial X - (\partial B)^\epsilon$ and on $D_B := B - (\partial B)^\epsilon$, using $(\partial B)^\epsilon := \{x \in X | \mathrm{dist}(X, \partial B) < \epsilon\}$, to g (for the "new" $B_\pm$) with $\epsilon' := \min(\epsilon/2, 1/2)$. The resulting $g_\perp^B$ and $g_\perp^\partial$ are (by choice of ϵ') such that the transverse preimages are disjoint $(g_\perp^B)^{-1}(\{0\}) \cap (g_\perp^\partial)^{-1}(\{0\}) = \emptyset$ and $B \cap B_+ \subset (g_\perp^B)^{-1}([0,1])$, similar for B_- and for ∂X. On the open neighborhood $N := B^\delta$ of B, with $\delta > 0$ such that $N \cap (g_\perp^\partial)^{-1}(\{0\}) = \emptyset$, there is a cylinder-neighborhood of $(B - \partial B) \cap N \subset (X - \partial X) \cap N$ by Thm. 1.9.3 (because $\dim(X) = \dim(\partial X) + 1 \geq 6$ and X having simple links, thus via Lemma 1.9.8 vanishing obstruction-group, thus $q_0 = 0$), so by Cor. 1.6.4–1, there is $g'_\perp$ extending $g_\perp^B$ to a transverse to 0 map on $N - \partial X$. By making this smaller than $\epsilon'/2$, and potentially replacing N by a smaller open neighborhood of B, we may assume $\mathrm{dist}\left((g_\perp^\partial)^{-1}(\{0\}), \partial X\right) > \epsilon'/4$.

Next, construct $g''_\perp : X \to [-1, 1]$ such that $g''_\perp = g'_\perp$ near $(g_\perp^B)^{-1}(\{0\})$ and $g''_\perp = g_\perp^\partial$ near $(g_\perp^\partial)^{-1}(\{0\})$, by picking a $\epsilon'/8$-thick (i. e. $\mathrm{dist}(X - \mathrm{im}(b), \partial X) < \epsilon'/8$) boundary-collar $b : \partial X \times [0, 2\gamma) \to X$ and η_B, η_∂ with $\eta_B = 1$ near $(g'_\perp)^{-1}(\{0\})$ and $\eta_B = 0$ on a neighborhood of the complement of N, and $\eta_\partial = 1$ near $b((g_\perp^\partial)^{-1}(\{0\}) \times [0, \gamma))$ and $\eta_\partial = 0$ near $X - b((g_\perp^\partial)^{-1}(\{0\})^{\epsilon'/4} \times [0, 2\gamma))$. Then define, using the subset $L_b := b((g_\perp^\partial)^{-1}(\{0\})^{\epsilon'/4} \times [0, 2\gamma))$ of the boundary-collar (near the zeros of $g_\perp^\partial$):

$$g''_\perp(x) := \begin{cases} g_\perp^\partial(\pi_{\partial X} b^{-1}(x))\eta_\partial(x) + g(x)(1 - \eta_\partial(x)) & \text{if } x \in L_b \\ g'_\perp(x)\eta_B(x) + g(x)(1 - \eta_B(x)) & \text{if } x \in N \\ g(x) & \text{otherwise} \end{cases}$$

which is transverse to 0 near $B \cup \partial X$: Away from $(g_\perp^\partial)^{-1}(\{0\}) \cup (g_\perp^B)^{-1}(\{0\})$, obviously $g \neq 0$, making it trivially transverse to 0 (if $g''_\perp|^{-1}(\{0\}) = \emptyset$, then $g''_\perp|$ is transverse to 0), and near $(g_\perp^\partial)^{-1}(\{0\}) \cup (g_\perp^B)^{-1}(\{0\})$ by construction of $g''_\perp$.

Finally, use manifold-transversality Cor. 1.5.9–1, to make this transverse on the top-stratum, rel a neighborhood of $B \cup \partial X$ intersected with the top-stratum. The construction of $X_0, X_\geq$ and $X_\leq$ remains the same as before.

"Case $\dim(X) = 5$": Now $\dim(\partial B) < 0$ and thus $\partial B = \emptyset$, while $\dim(B) \leq 0$, so B consists of (finitely many) points and can again be "absorbed" into B_-. Hence the cut must be produced in the manifold-with-boundary $(X - B, \partial X)$, which can be done using manifold transversality as in part (i). The resulting space X_0 is a manifold with boundary, hence again in $\mathrm{Wh}_2^{\mathrm{gap}\,5}$, so are $X_\geq, X_\leq$, because near X_0 (inside the respective half of the bi-collar), they are manifolds with boundary, and away from X_0 they did not change, so they possess an open cover by spaces in $\mathrm{Wh}_2^{\mathrm{gap}\,5}$, with all the required properties being local.

"Case $\dim(X) \leq 4$": By the codimension-requirement, X is a manifold-with-boundary. So by part (i) there is nothing to show. $\square$

This example (iii) will be employed to construct a bordism-theory in Section 2.4 ("An Example-Theory"). It is slightly unsatisfactory with regard to multiple problems, which will be elaborated on in Section 2.6 ("Improvements") and the next chapters.

Finally, we use these transversality-properties, to construct (the inverse of) the excision-isomorphism in the sense of the Eilenberg–Steenrod axioms.

Lemma 2.2.6 ("Excision"): *Let $\mathcal{C}$ be a strong t-class. Given a pair (A, B) and $D \subset B$ with* $\operatorname{closure}(D) \subset \operatorname{interior}(B)$*, the map induced (in the sense of Lemma 2.1.12) by the inclusion* $\operatorname{incl} : (A-D, B-D) \to (A, B)$ *is an isomorphism in all degrees j:*

$$\operatorname{incl}_* : \Omega_j^{\mathcal{C}}(A - D, B - D) \xrightarrow{\sim} \Omega_j^{\mathcal{C}}(A, B)$$

Proof: We explicitly construct an inverse ψ. Let $[f : (X, \partial X) \to (A, B)] \in \Omega_j^{\mathcal{C}}(A, B)$. Define $B_+ := X - \operatorname{interior}(f^{-1}(B))$ and $B_- := f^{-1}(\operatorname{closure}(D))$. Then B_+ and B_- are closed (in the case of B_- by continuity of f, in the case of B_+ by definition) and disjoint: By hypothesis, $\operatorname{closure}(D) \subset \operatorname{interior}(B)$, so that $B_- \subset f^{-1}(\operatorname{interior}(B)) \subset \operatorname{interior}(f^{-1}(B)) = X - B_+$, where the second inclusion is due to continuity of f. Further, any $y \in X - (B_+ \cup B_-)$ is mapped to $f(y) \in B - D$: Firstly, $y \notin B_-$ implies $f(y) \notin \operatorname{closure}(D)$, so $f(y) \in A - \operatorname{closure}(D) \subset A - D$. Secondly, $y \notin B_+$ implies $y \in \operatorname{interior}(f^{-1}(B)) \subset f^{-1}(B)$, so $f(y) \in B$.

Next, we check, that the boundary (or the corners, see below) does not "get in the way":

Claim: There is a representative f' of the equivalence class $[f]$ of f and, defining $B'_\pm$ as before but now with f' in the place of f, a $\tilde{B}'_- \supset B'_-$ closed and disjoint from B'_+ s. t. $\partial X \subset \tilde{B}'_-$.

Proof of claim: We would want to set $\tilde{B}_- := B_- \cup \partial X$, which, on first sight, seems to work, because $f(\partial X) \subset B$. However, B need not be open, thus $f(\partial X) \not\subset \operatorname{interior}(B)$ in general. Thus, such a $\tilde{B}_-$ may not be disjoint from B_+.

This problem can be avoided, by using collaredness of ∂X to "push" a neighborhood of ∂X into B: Let $c : \partial X \times [0, \infty) \to X$ be a collar of ∂X. Define

$$T : \Big(\partial X \times [0, \infty)\Big) \times I \to \partial X \times [0, \infty), \quad ((x, s), t) \mapsto \begin{cases} (x, s - t) & : \text{if } s \geq t \\ (x, 0) & : \text{if } s \leq t \end{cases}$$

$$H : X \times I \to A, \quad (x, t) \mapsto \begin{cases} f(x) & : \text{if } x \notin \operatorname{im}(c) \\ fc\Big(T(c^{-1}(x), t)\Big) & : \text{if } x \in \operatorname{im}(c) \end{cases}$$

This is continuous, because by the choice of the collar-coordinate as $[0, \infty)$, the translation T approaches the identity near $X - \operatorname{im}(c)$. Then $W := X \times I \xrightarrow{H} A$ satisfies $W \in \mathcal{C}$ by stability, and setting $W_0 := X \times \{0\}$, $W_1 := X \times \{1\}$ and $W_* := \partial X \times I$, it holds that $H(W_*) = f(\partial X) \subset B$ and $H|_{W_0} = f$. So this is a bordism $f \sim f'$ to $f' := H_1$.

For this f' the choice $\tilde{B}'_- := B'_- \cup \partial X$ is valid (disjoint from B'_+), because f' maps an open neighborhood $c(\partial X \times [0, 1))$ of ∂X into B. This finishes the proof of the claim.

Next, we use that $\mathcal{C}$ is a weak t-class (because strong implies weak), to obtain $X_\geq \supset B'_+$ and $X_\leq \supset \tilde{B}'_-$, where

(1) $X_\leq \subset X - B'_+$ thus $f'(X_\leq) \subset \text{interior}(B)$

(2) $X_\geq \subset X - \tilde{B}'_- \subset X - B'_-$ thus $f'(X_\geq) \subset A - D$

(3) $\partial X_\geq = X_0 = X_\geq \cap X_\leq \subset X - (B'_+ \cup \tilde{B}'_-) \subset X - (B'_+ \cup B'_-)$ thus, combining this with (2), $f'(\partial X_\geq) \subset B - D$

We define the "inverse" ψ of incl_* on $[f]$ as:

$$\psi\big([f]\big) := \Big[(X_\geq, X_0) \xrightarrow{f'|_{X_\geq}} (A - D, B - D)\Big]$$

There is a lot to check:

(a) $[f'|]$ is allowed as an element of $\Omega_j^{\mathcal{C}}(A - D, B - D)$

(b) The bordism class of $\psi([f])$ does not depend on the choice of $X_\geq, X_\leq$ or X_0 for a given representative f (thus a fixed X) and fixed f'.

(c) The bordism class of $\psi([f])$ does not depend on the choice of the representative f (or X) or the collar c used to construct f'.

(d) ψ is a group-homomorphism

(e) $\psi \circ \text{incl}_* = \text{id}$

(f) $\text{incl}_* \circ \psi = \text{id}$

The individual statements together imply the statement claimed by the lemma.

(a) $f'|$ is a map of pairs, by (2) and (3). $(X_\geq, X_0) \in \mathcal{C}$, because $\mathcal{C}$ is a strong t-class, thus stable.

(b) We first reduce this to the case "$X_\geq \supset X'_\geq$": For fixed f', given two choices $X_\geq, X_\leq, X_0$ and $X'_\geq, X'_\leq, X'_0$, define $B''_+ := B'_+ \cup X_\geq \cup X'_\geq$. This is closed (as finite union of closed subsets), and it is disjoint from $\tilde{B}'_-$, because all parts in the union individually are. As before, use that $\mathcal{C}$ is a weak t-class to obtain $X''_\geq, X''_\leq, X''_0$, a cut between B''_+ and $\tilde{B}'_-$. By choice of B''_+, this satisfies $X''_\geq \supset X_\geq$ and $X''_\geq \supset X'_\geq$.

We show that $f|_{X_\geq} \sim f|_{X''_\geq} \sim f|_{X'_\geq}$. Part (a) applies to $X''_\geq$, so this makes sense. The claim then follows by transitivity of $\sim$. The argument for both bordisms is the same, so we only explicitly give the first one. Define $W := X''_\geq \times I$ and $F : W \to A - D, (x, t) \mapsto f(x)$. With $W_0 := X''_\geq \times \{0\}$, $W_1 := X_\geq \times \{1\}$ (which works by $X''_\geq \supset X_\geq$), and $W_* := X''_0 \cup \Big(X''_\geq - (X_\geq - X_0)\Big) \times \{0\}$ this is a allowable bordism $f|_{X_\geq} \sim f|_{X''_\geq}$, because W is allowable ($\mathcal{C}$ is a strong t-class), $F(W_*) \subset f(X''_\geq) \subset A - D$, by $(2'')$, hence $\text{im}(F) \subset A - D$ and $F(W_*) \subset f(X_\leq) \subset B$ by (1).

(c) Let f, g be two representatives of $[f]$. I. e. there is a bordism $F : W \to A$. By stability of $\mathcal{C}$, W is a weak t-space. A neighborhood of $W_* \subset \partial W$ can be "pushed into B" to yield F', so "corners" are in B_- in the same way as in the claim at the beginning of this proof. This can be done along a collar extending a given collar c in X and c' in Y, the domains of f', g'.

Remark: On elements, by the initial claim of the proof, we never need to cut spaces with boundary (or at least the cut can always be constructed away from the boundary). This is *not* true for the well-definedness part anymore! This is why the definition of "t-classes" requires the existence of cuts even for spaces with boundary. But at least corners can be "pushed away" into B_-, so we do not need to worry about places where W_0, W_1 or W_* meet each other.

Cutting W produces $W_\geq$ (again allowable by $\mathcal{C}$ being a strong t-class), with boundary allowed as bordism (because corners are in B_-) of $f' \sim g'$.

Choosing $f = g$ shows, that the choice of collar to construct f' does not matter. This then also reduces (c) to (b) shown above.

(d) Cutting a disjoint union will yield a disjoint union of cuts. Ordering does not matter, so this is a matter of well-definedness of individual cuts.

(e) Given $[f] \in \Omega^{\mathcal{C}}(A - D, B - D)$, the image $\mathrm{incl}_*([f])$ maps to $\mathrm{im}(f) \subset A - D$, so $X_\geq := X$ is a valid choice of "cut", thus by well-definedness of ψ, (c), there is nothing to check, $\psi(\mathrm{incl}_*([f])) = [f'] = [f]$.

(f) Given $[f] \in \Omega^{\mathcal{C}}_j(A, B)$, we have the representative given by $\psi([f]) = [f'|_{X_\geq}]$ and by definition $\mathrm{incl}_*(\psi([f])) = [\mathrm{incl} \circ f'|_{X_\geq}]$. We use again that $W = X \times I$ is allowable, this time with boundary-subdivision $W_0 := X_\geq \times \{0\}$, $W_1 := X \times \{1\}$ and $W_* := X_\leq \times \{0\} \cup \partial X \times I$. This is a valid bordism, because $f'(X_\leq) \subset B$ and $f'(\partial X) \subset B$. It shows $\mathrm{incl} \circ f'|_{X_\geq} \sim f'$, but by construction $f' \sim f$. Therefore $\mathrm{incl}_*(\psi([f])) = [f]$.

This finishes the proof of the lemma. □

This concludes the treatment of aspects reliant on transversality. The other axioms of a generalized homology-theory are checked in the next section.

2.3 Generalized Homology Theory

Here, it is shown, that the bordism-groups as defined in Section 2.1 ("Bordism Theories") satisfy the axioms of a generalized homology theory, except for excision, which was seen to hold for certain "strong transversality-classes" $\mathcal{C}$ in the previous section, see Lemma 2.2.6.

By a generalized bordism-theory we mean:

Definition 2.3.1: *A generalized*[5] *homology theory H_* is a collection of functors H_k for $k \in \mathbb{Z}$, from the category of pairs of topological spaces (and continuous maps of pairs) to the category of abelian groups, together with natural transformations $\partial_k : H_{k+1}(X, A) \to H_k(A, \emptyset)$ called the* boundary homomorphisms *such that the Eilenberg–Steenrod-Axioms hold:*

[5] The original treatment by Eilenberg and Steenrod normalized the homology theories by requiring $H_0(\{\mathrm{pt}\}) = G$, $H_{k\neq 0}(\{\mathrm{pt}\}) = 0$, in which case, they showed, the axioms "characterize" a theory (uniquely on reasonable spaces), which was their main point of interest.

(1) Homotopy*: Homotopic maps (of pairs*[6]*) $f \simeq g$ induce the same homomorphisms $H_k(f) = H_k(g)$ on homology $\forall k$.*

(2) Exactness*: Given a pair of spaces (X, A), using the inclusions $i : (A, \emptyset) \to (X, \emptyset)$ and $j : (X, \emptyset) \to (X, A)$, with induced maps $i_* := H_*(i)$ and $j_* := H_*(j)$, there is a long exact sequence in homology:*

$$\dots \xrightarrow{\partial_k} H_k(A, \emptyset) \xrightarrow{i_k} H_k(X, \emptyset) \xrightarrow{j_k} H_k(X, A) \xrightarrow{\partial_{k-1}} H_{k-1}(A, \emptyset) \xrightarrow{i_{k-1}} \dots$$

(3) Excision*: Given a pair (X, A) and $B \subset A$ with* closure$(B) \subset$ interior(A)*, then the inclusion $i : (X - B, A - B) \to (X, A)$ induces an isomorphism in homology (i. e. it induces an isomorphism in each degree).*

(4) Additivity*: Co-products are mapped to co-products, i. e. given an (arbitrary) disjoint union $X = \sqcup_\alpha X_\alpha$, then the inclusions $i_\alpha : X_\alpha \to X$ induce an isomorphism (in each degree k):*

$$\oplus_\alpha H_k(X_\alpha) \xrightarrow{+_\alpha H_k(i_\alpha)} H_k(\sqcup_\alpha X_\alpha)$$

First we define the boundary-homomorphisms for the case of bordism-theories:

Definition 2.3.2: *The* boundary homomorphisms *in degree k of $\Omega_*(A, B)$ are*

$$\begin{aligned} \partial_k : \quad & \Omega_{k+1}(A, B) \to \Omega_k(B), \\ & [(X, \partial X) \xrightarrow{f} (A, B)] \mapsto [\partial X \xrightarrow{f|_{\partial X}} B] \end{aligned}$$

Lemma 2.3.3: *The so-defined boundary-homomorphisms are well-defined and natural.*

Proof: "Well-definedness:" By stability, ∂X is allowed as a "probing-space" (even if it is empty, as we explicitly allowed the empty set as a space in any dimension). Given two representatives $f : (X, \partial X) \to (A, B)$ and $g : (Y, \partial Y) \to (A, B)$ of the same bordism-class $[f] = [g]$, there is a bordism $F : W \to A$ with $\partial W = X \cup Y \cup W_*$, which restricts to f and g respectively and where $F(W_*) \subset B$. Note (cf. Rmk. 2.1.8), that $F|_{W_*}$ is a bordism from $f|_{\partial X}$ to $g|_{\partial Y}$ in B. But these are just $\partial_*(f)$ and $\partial_*(g)$, so these are bordant in $\Omega_*(B)$.

This is a group-homomorphism, because it clearly maps disjoint unions to disjoint unions.

[6] Requiring this only for "absolute" groups (groups of pairs of the form $(X, \emptyset)$) is equivalent (when combined with the other axioms) to requiring it for all pairs (X, A), because the long-exact sequence plus the "5-lemma" imply the second from the first (the other direction is trivial).

"Naturality:" We have to show: $\partial_k : \Omega_{k+1}(A, B) \to \Omega_k(B)$ is a natural transformation of the indicated functors (mapping a pair (A, B) to either $\Omega_{k+1}(A, B)$ or $\Omega_k(B, \emptyset)$ respectively).

Given $\alpha : (A, B) \to (A', B')$, we thus have to show, that the diagram

$$\begin{tikzcd}
\Omega_{k+1}(A, B) \arrow[r, "\alpha_{k+1}"] \arrow[d, "\partial_k"'] & \Omega_{k+1}(A', B') \arrow[d, "\partial'_k"] \\
\Omega_k(B) \arrow[r, "\alpha_k"] & \Omega_k(B')
\end{tikzcd}$$

commutes. Given $[f : (X, \partial X) \to (A, B)] \in \Omega_{k+1}(A, B)$, by definition

$$\begin{aligned}
\partial'_k(\alpha_{k+1}([f])) &= \partial'_k([\alpha \circ f]) &&= [(\alpha \circ f)|_{\partial X}] \\
\alpha_k(\partial_k([f])) &= \alpha_k([f|_{\partial X}]) &&= [\alpha \circ (f|_{\partial X})]
\end{aligned}$$

and, of course, $(\alpha \circ f)|_{\partial X} = \alpha \circ (f|_{\partial X})$. $\square$

We continue by the homotopy axiom (a corresponding statement for pairs follows from the long exact sequence of pairs and the five-lemma)

Lemma 2.3.4 ("Homotopy"): *Given two homotopic maps $\alpha \simeq \beta : A \to A'$, the induced maps $\alpha_*, \beta_* : \Omega_*(A) \to \Omega_*(A')$ agree $\alpha_* = \beta_*$.*

Proof: Let $H : A \times I \to A'$ be a homotopy from α to β. Let $[f : X \to A] \in \Omega_*(A)$, thus $\partial X = \emptyset$. By stability of the underlying class $\mathcal{C}$, $W := X \times I$ is an allowable space. Subdividing the boundary as $W_0 := A \times \{0\}$, $W_1 := A \times \{1\}$ and $W_* = \emptyset$, this defines a bordism $H \circ (f \times \mathrm{id}_I) : W \to A'$ of $\alpha \circ f \sim \beta \circ f$. By definition, this means $\alpha_*([f]) = \beta_*([f])$, and the lemma follows because $[f]$ was arbitrary. $\square$

Next we show, there is a long exact sequence of a pair:

Lemma 2.3.5 ("Exactness"): *Given a pair of spaces (A, B), there is a long exact sequence*

$$\ldots \to \Omega_{k+1}(A, B) \xrightarrow{\partial_k} \Omega_k(B) \xrightarrow{i_*} \Omega_k(A) \xrightarrow{j_*} \Omega_k(A, B) \to \ldots$$

where $i : (B, \emptyset) \to (A, \emptyset)$ and $j : (A, \emptyset) \to (A, B)$ are the inclusions (of pairs).

Proof: We need to check exactness at each individual term:

Step 1: The sequence is exact at $\Omega_k(B)$.

"$\ker(i_*) \supset \operatorname{im}(\partial_*)$": Let $[f : (X, \partial X) \to (A, B)] \in \Omega_{k+1}(A, B)$. Then $i_*\partial_k([f]) = i_*([f|_{\partial X}]) = [i \circ f|_\partial] = 0$ via the null-bordism $F = f : W = X \to A$.

"$\ker(i_*) \subset \operatorname{im}(\partial_*)$": Let $[f : X \to B] \in \ker(i_*) \subset \Omega_k(B)$. Thus there is $F : W \to A$, with $\partial W = X$ (as a valid bordism in $\Omega_*(B) = \Omega_*(B, \emptyset)$, necessarily $F(W_*) \subset \emptyset$ so $W_* = \emptyset$) and $F|_{\partial W} = i \circ f$. Because $F(\partial W) = i \circ f(X) \subset B$, we may define $[F : (W, \partial W) \to (A, B)] \in \Omega_{k+1}(A, B)$, a valid element, with $\partial_k([F]) = [F|_{\partial W}] = [f]$.

Step 2: The sequence is exact at $\Omega_k(A)$.

"$\ker(j_*) \supset \operatorname{im}(i_*)$": Let $[f : X \to B] \in \Omega_k(B)$. Then $j_*(i_*([f])) = [j \circ i \circ f] = 0 \in \Omega_k(A, B)$, because $F := (f \circ \pi_X) : W := X \times I \to A$ with $W_0 = X \times \{0\}$, $W_1 = \emptyset$ and $W_* = X \times \{1\}$ is a null-bordism, as $F(W_*) = f(X) \subset B$.

"$\ker(j_*) \subset \operatorname{im}(i_*)$": Let $[f : X \to A] \in \ker(j_*) \subset \Omega_k(A)$. Thus there is a null-bordism $F : W \to A$ of $[j \circ f]$ in $\Omega_k(A, B)$, i. e. $W_1 = \emptyset$ and $F(W_*) \subset B$. This null-bordism in $\Omega_k(A, B)$ can be used as a bordism $f \sim F|_{W_*}$ in $\Omega_k(A)$, by swapping $W_1 \leftrightarrow W_*$. This is allowed in $\Omega_*(A) = \Omega_*(A, \emptyset)$, because $W_1 = \emptyset$ is mapped to $\emptyset$ by F and $[f] \in \Omega_k(A) \Rightarrow \partial X = \emptyset \Rightarrow W_0 \cap W_* = \partial W_0 = \partial X = \emptyset$ (so after swapping $W_1 \leftrightarrow W_*$, still $W_0 \cap W_1 = \emptyset$ holds, as required for part (ii) of Def. 2.1.7). But $\operatorname{im}(F|_{W_*}) = F(W_*) \subset B$, so $[F|_{W_*}]$ is in the image of i_*.

Step 3: The sequence is exact at $\Omega_k(A, B)$.

"$\ker(\partial_*) \supset \operatorname{im}(j_*)$": $\partial_{k-1} j_* = 0$: Let $[f : X \to A] \in \Omega_k(A)$. Then $\partial_{k-1}([j \circ f]) = [(j \circ f)|_{\partial X}] = [(j \circ f)|_\emptyset] = 0$ and there is nothing to show.

"$\ker(\partial_*) \subset \operatorname{im}(j_*)$": Let $[f : (X, \partial X) \to (A, B)] \in \ker(\partial_{k-1}) \subset \Omega_k(A, B)$. Thus there is a null-bordism $F : W \to B$ of $\partial_{k-1}([f]) = [f|_{\partial X}]$ in $\Omega_{k-1}(B)$, i. e $W_1 = \emptyset$ and $W_* = \emptyset$ (because $F(W_*) \subset \emptyset$), hence $\partial W = W_0 = \partial X$ and $F|_{\partial W} = f|_{\partial X}$. Glue $f \cup F : X \cup_{W_0} W \to A$ by Lemma 1.4.25. Since $\partial(X \cup_{W_0} W) = (\partial X - W_0) \cup (\partial W - W_0) = \emptyset \cup \emptyset$, we have an element $[f \cup F] \in \Omega_k(A)$. This gets mapped by j_* to $j \circ (f \cup F)$, which is bordant to f in $\Omega_k(A, B)$ by the bordism

$$F' := (f \cup F) \circ \pi_{X \cup_{W_0} W} : W' := (X \cup_{W_0} W) \times I \to A$$

with the boundary-subdivision given by $W_0' = (X \cup_{W_0} W) \times \{0\}$, $W_1' = X \times \{1\}$ and $W_*' = W \times \{1\}$. This is allowed, because $F'(W_*') = F(W) \subset B$, as F is a bordism in $\Omega_{k-1}(B)$. The result is hence, that $j_*([f \cup F]) = [f]$, and we have found a preimage.

This finishes the proof of "exactness". □

Also additivity holds:

Lemma 2.3.6 ("Additivity"): *Coproducts are preserved, i. e. given a disjoint union over a family of topological spaces $\{A_i\}_{i \in I}$, applying Ω_k yields a direct sum of abelian groups:*

$$\Omega_k(\coprod_{i \in I} A_i) \cong \bigoplus_{i \in I} \Omega_k(A_i)$$

Proof: Any $[f : X \to A]$, by f being continuous, thus mapping connected components to connected images, can be described by its components, so the left hand side is a subgroup of the product. Because X is compact, there is only a finite number of components, and the image in the product is contained in the direct sum, so the homomorphism (addition / disjoint union can be done component-wise) indicated above is well-defined. It is injective, because given null-bordisms on all components, the disjoint union of these null-bordisms is a null-bordism in $\Omega_k(\coprod_{i\in I} A_i)$. It is surjective, because any finite collection of $([f_i : X_i \mapsto A_i])_{i\in I}$ is the image of $[\coprod_{i\in I} f_i]$ (by finiteness, $\coprod_{i\in I} X_i$ is compact if the X_i are). □

Finally, all put together, we have proven the main result of this chapter:

Proposition 2.3.7: *Given a strong t-class $\mathcal{C} \subset \partial$-MHSS, the bordism-theory $\Omega_*^{\mathcal{C}}(-,-)$ is a generalized homology theory.*

Next, we will have a brief look at the bordism-theory defined by the strong t-class of Example 2.2.5, i. e. the class $\mathcal{C} = \mathrm{Wh}_2^{\mathrm{gap}\,5}$.

2.4 An Example-Theory

Combining the Example 2.2.5 and the strong t-class $\mathcal{C} = \mathrm{Wh}_2^{\mathrm{gap}\,5}$ it defines with Prop. 2.3.7, one gets a first example for a topologically stratified bordism-theory, with only algebraic-topological and homotopy-theoretical requirements to transitions between strata. I. e. no bundle- or pl-hypothesis is needed.

Proposition 2.4.1: *The bordism-theory $\Omega_*^{\mathrm{Wh}_2^{\mathrm{gap}\,5}}(-,-)$, which allows as "probing"- and "bordism"-spaces all compact, orientable, proper MHSS with boundary X, with at most two meeting strata – the minimal ones having codimension ≥ 5 in X – with simple links, i. e. holink-fibers L with $\mathrm{Wh}(\pi_1(L) \times \mathbb{Z}^k) = 0\ \forall k \geq 0$, is a generalized homology theory.*

An "allowable" space $X \in \mathrm{Wh}_2^{\mathrm{gap}\,5}$ of dimension n has a canonical "fundamental class" $[X] \in \Omega_n^{\mathrm{Wh}_2^{\mathrm{gap}\,5}}(X)$ given by

$$[X] := [\mathrm{id} : X \to X] \in \Omega_n^{\mathrm{Wh}_2^{\mathrm{gap}\,5}}(X).$$

Using Lemma 2.2.2 with Example 2.1.3, the same results hold after replacing $\mathrm{Wh}_2^{\mathrm{gap}\,5}$ by $\mathrm{Wh}_2^{\mathrm{gap}\,5} \cap \mathrm{Witt}$, i. e. we may additionally require the allowable spaces to be Witt.

A discussion of implications and applications is deferred until after the generalizations provided by Chapter 3 (“Multiple Strata”) and Chapter 4 (“Homeomorphisms”). See Chapter 5 (“The Main Theorem and its Applications”).

Remark 2.4.2: Note, that by the gap / codimension ≥ 5 requirement,

$$\Omega_i^{\mathrm{Wh}_2^{\mathrm{gap}\,5}}(-,-) = \Omega_i^{\mathrm{STOP}}(-,-) \quad \text{for all } i < 4$$

(in degree 4, the equivalence relation of bordism differs, as singular bordisms of manifolds are allowed in $\Omega_i^{\mathrm{Wh}_2^{\mathrm{gap}\,5}}(-,-)$), this theory agrees with manifold-bordism in low degrees.

One might also fix a codimension $k \geq 5$ and allow only one minimal stratum of codimension exactly k (this is consistent with gluing). This produces bordism groups $\Omega_*^{\mathrm{Wh}_{2,k}}$, that agree with manifold-bordism in degrees $i < k-1$, and with natural transformations $\Omega_i^{\mathrm{Wh}_{2,k}}(-,-) \xrightarrow{\phi_k} \Omega_i^{\mathrm{Wh}_2^{\mathrm{gap}\,5}}(-,-)$.

These arguments work of course also, if we restrict to Witt-spaces, as manifolds are trivially Witt.

While the bordism theory constructed so far is already quite interesting, there are some evident improvements to be made. An overview of these will be discussed in Section 2.6 (“Improvements”) and Section 6.3 (“Outlook and Further Ideas”).

2.5 Reduced Theories

This section describes, how to construct a reduced homology theory in a standard way (see for example [Ati67; Hat02; Bre97]), but also gives a certain “geometric description” of the suspension isomorphism.

The material of this section is relevant to the application to L-classes in Section 5.3 (“L-Classes”). The “geometric description” will give a connection to the transversality-based construction of Goresky–MacPherson L-classes [GM80].

There are different (equivalent) ways to axiomatize reduced (generalized) homology theories, the main “choice” to be made is, whether or not one combines long exact sequence and excision into a “Mayer–Vietoris” sequence (or a MV-like long exact sequence, see e. g. [Hat02, §2.3 (p. 160ff)] for a CW treatment), or treats excision separately via a suspension-isomorphism. Since transversality / excision will usually be the center of our attention, we use the latter one. This additionally allows for a formulation analogously to the one given for (unreduced) theories in Def. 2.3.1. We briefly show near the end of this section, how the axioms given here induce a “Mayer–Vietoris-like” long exact sequence (and an actual Mayer–Vietoris-sequence). The opposite-direction, deducing the suspension-isomorphism from the Mayer–Vietoris-sequence, is also easily obtained.

Definition 2.5.1: *A* reduced (generalized) homology theory $\tilde{h}_*$ *is a sequence of covariant functors from the category of well-pointed*[7] *topological spaces (with pointed maps) to the category of abelian groups with a sequence of natural transformations* $\mathrm{susp} : \tilde{h}_k(-) \to \tilde{h}_{k+1}(\Sigma -)$*, called the* suspension-homomorphism*, such that*

(1) Homotopy*: Homotopic maps (of pointed spaces) $f \simeq g$ induce the same homomorphisms $\tilde{h}_k(f) = \tilde{h}_k(g)$ on reduced homology $\forall k$.*

(2) Exactness*: Given a pair of pointed spaces (X, A), using the inclusions $i : A \to X$ and $j : X \to c(i)$, where $c(i) = {}^{\mathrm{cyl}(i)}/_{A}$ is the mapping-cone, with induced maps $i_* := \tilde{h}_*(i)$ and $j_* := \tilde{h}_*(j)$, there is an exact sequence in reduced homology (for all k):*

$$\tilde{h}_k(A) \xrightarrow{i_k} \tilde{h}_k(X) \xrightarrow{j_k} \tilde{h}_k(c(i)).$$

(3) Excision / Suspension*: The suspension homomorphisms* $\mathrm{susp}_k(X)$ *are isomorphisms for all k and X.*

(4) Wedge / Milnor-Additivity*: Co-products are mapped to co-products, i. e. given an (arbitrary) one-point-union $X = \vee_\alpha X_\alpha$, then the inclusions $i_\alpha : X_\alpha \to X$ induce an isomorphism (in each degree k):*

$$\oplus_\alpha \tilde{h}_k(X_\alpha) \xrightarrow{+_\alpha \tilde{h}_k(i_\alpha)} \tilde{h}_k(\vee_\alpha X_\alpha)$$

When talking about pointed spaces, these are supposed to be well-pointed. One can translate between reduced and unreduced homology-theories (see also Def. 2.5.15 for the "other direction")

Definition 2.5.2: *Given a generalized homology theory $H_*(-,-)$, the* associated reduced theory $\tilde{H}_*(-)$ *is the covariant functor from pointed spaces to abelian groups, given degree-wise on $(X, *)$ as the cokernel (in the category of abelian groups) of the homomorphism* $\mathrm{incl}_* = H_*(\mathrm{incl} : \{*\} \to X)$ *induced by the base-point inclusion. Thus (degree-wise)*

$$0 \to H_i(\{*\}) \xrightarrow{\mathrm{incl}_*} H_i(X) \xrightarrow{q} \tilde{H}_i(X) \to 0$$

is a short exact sequence. It splits (on the left side) via the homomorphism induced by the unique map $X \to \{\mathrm{pt}\}$, so

$$H_i(X) \cong \tilde{H}_i(X) \oplus H_i(\{\mathrm{pt}\})$$

with $\mathrm{incl} : \{*\} \to X$ *inducing the summand-inclusion.*

We write $[\![f]\!]$ for the equivalence class of $[f]$, that is $[\![f]\!] := q([f])$.

Induced maps are given as $\tilde{H}_*(f)([\![x]\!]) = [\![H_*(f)([x])]\!]$, which is well-defined, because the pointed map f factors the inclusion of the base-point into its target.

[7] Well-pointed means pointed, with the base-point-inclusion a co-fibration, see proof of Lemma 2.5.7.

Evidently this "kills" the homology of contractible spaces.

Example 2.5.3: *Given a contractible space $X \simeq \{\mathrm{pt}\}$, we have $\tilde{H}_*(X) = 0$.*

Proof: The inclusion-induced homomorphism incl_* is an isomorphism by homotopy-invariance and long exact sequence (of unreduced homology). □

This $\tilde{H}$ automatically has the properties of a reduced homology-theory. For homotopy-invariance, and additivity implying the wedge-axiom, this is clear. The exact sequence of a pair can be deduced from the (unreduced) long exact sequence as follows (for a "long" version, see the Mayer–Vietoris-sequence Lemma 2.5.13 and Cor. 2.5.13–1)

Lemma 2.5.4: *Given a pointed pair (X, A) (i. e. $A \subset X$ are pointed spaces, with the same base-point $*_A = *_X$), there is a long exact sequence*

$$\ldots \to \tilde{H}_i(A) \to \tilde{H}_i(X) \xrightarrow{\tilde{j}_*} H_i(X, A) \xrightarrow{\tilde{\partial}_*} \tilde{H}_{i-1}(A) \to \ldots$$

where $\tilde{\partial}_*$ is the composition $\tilde{\partial}_* : H_i(X, A) \xrightarrow{\partial_*} H_{i-1}(A) \xrightarrow{q} \tilde{H}_{i-1}(A)$ and (see proof) $\tilde{j}_*(q(x)) := j_*(x)$.

Proof: Any $y \in \tilde{H}_i(X)$ can be written as $q(x) + \eta$ for $x \in H_i(X)$ and $\eta \in H_i(\{*\})$, using $j : (X, \emptyset) \to (X, A)$, define $\tilde{j}_*(q(x) + \eta) := j_*(x) := H_*(j)(x)$. This is well-defined, because $\mathrm{incl}_* : H_i(\{*\}) \to H_i(X)$ factors through $H_i(A)$, as the topological inclusion does, and by the long exact sequence (for the unreduced theory) of the pair (X, A), thus $j_*(\mathrm{incl}_*(\eta)) = 0$.

Note that in the diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
\cdots \arrow[r] & \tilde{H}_i(A) \arrow[r] \arrow[u] & \tilde{H}_i(X) \arrow[r] \arrow[u] & H_i(X, A) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & H_i(A) \arrow[r] \arrow[u] & H_i(X) \arrow[r] \arrow[u] & H_i(X, A) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & H_i(\{*\}) \arrow[r] \arrow[u] & H_i(\{*\}) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

the middle row is the long exact sequence (for the unreduced theory) of the pair (X, A) and bottom row is trivially exact. The first two columns are short exact sequences by the definition of the reduced theory, the last column is trivially short exact. By construction of $\tilde{j}_*$ and $\tilde{\partial}_*$ the diagram commutes. By a "standard" diagram-chase argument (see Lemma 2.5.5 below), the first row is a long exact sequence. This proofs the lemma. □

Corollary 2.5.4–1: *Given a pointed pair (X, A), using the inclusions $i : A \to X$ and $j : X \to c(i)$, there is a exact sequence*

$$\tilde{H}_i(A) \xrightarrow{i_*} \tilde{H}_i(X) \xrightarrow{j_*} \tilde{H}_i(c(i))$$

Proof: The inclusion of the base-point of A into the contractible $c(A)$ induces an isomorphism while factoring (with i) the inclusion of the base-point into $c(i)$. So in the (unreduced) long exact sequence of the pair $(c(i) = X \cup_A c(A), c(A))$, we have $\partial_* = 0$, thus by uniqueness of the cokernel, $H_i(X, A) \cong \tilde{H}_i(c(i))$. Composing this with $\tilde{j}_*$ is just j_*, because the $\eta \in H_i(\{*\})$ (see proof above) gets mapped to the other summand (of $H_i(c(i)) \cong \tilde{H}_i(c(i)) \oplus H_i(\{*\})$), since, as was remarked in Def. 2.5.2, this splitting is induced by the base-point-inclusion. □

This uses the following technical result (this is certainly well-know; see for example "Nine-Lemma")

Lemma 2.5.5: *Given a commutative diagram*

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
\cdots \arrow[r, "c_0"] & C_1 \arrow[u] \arrow[r, "c_1"] & C_2 \arrow[u] \arrow[r, "c_2"] & C_3 \arrow[u] \arrow[r, "c_3"] & \cdots \\
\cdots \arrow[r, "b_0"] & B_1 \arrow[u, "\beta_1"'] \arrow[r, "b_1"] & B_2 \arrow[u, "\beta_2"'] \arrow[r, "b_2"] & B_3 \arrow[u, "\beta_3"'] \arrow[r, "b_3"] & \cdots \\
\cdots \arrow[r, "a_0"] & A_1 \arrow[u, "\alpha_1"'] \arrow[r, "a_1"] & A_2 \arrow[u, "\alpha_2"'] \arrow[r, "a_2"] & A_3 \arrow[u, "\alpha_3"'] \arrow[r, "a_3"] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

with long exact second "B_*" and third "A_*" row and short exact columns, then the first row "C_*" is long exact.

Proof: By translational symmetry in the horizontal direction, we need only to check this at C_2. Note that by exactness of the columns, the β_* are surjective, and the α_* are injective. To maintain some amount of readability, variables are named with a small letter corresponding to their row, and one, zero or two primes respectively for the different columns (e. g. $c \in C_2$, $c' \in C_3$ or $a'' \in A_1$ etc.)

"im $\subset$ ker": Let $c = c_1(c') \in \mathrm{im}(c_1)$. Because β_1 is surjective, there is $b' \in B_1$ with $\beta_1(b') = c'$. So $c_2(c) = c_2c_1(c') = c_2c_1\beta_1(b')$, and by commutativity, $c_2c_1\beta_1(b') = \beta_3b_2b_1(b')$. But the second row being exact implies $b_2b_1(b') = 0$, thus $c_2(c) = 0$.

"ker $\subset$ im": Let $c \in \ker(c_2)$, i. e. $c_2(c) = 0$. By surjectivity of β_2, there is $b \in B_2$ with $\beta_2(b) = c$. Set $b'' := b_2(b)$, then by commutativity $\beta_3(b'') = c_2(c) = 0$. By exactness of the third column, there is hence $a'' \in A_3$ with $\alpha_3(a'') = b''$. Note that $b_3(b'') = b_3b_2(b) = 0$ by exactness of the middle row, so that by injectivity of α_4 and commutativity $a_3(a'') = 0$. Hence by exactness of the bottom row, there is $a \in A_2$ with $a_2(a) = a''$. Set $\tilde{b} := \alpha_2(a)$. By commutativity, $b_2(\tilde{b}) = b'$ and therefore $b_2(b - \tilde{b}) = b' - b' = 0$, so that by exactness of the middle row, there is $b'' \in B_1$ with $b_1(b'') = b - \tilde{b}$. Define $c'' := \beta_1(b'')$. Then by commutativity, $c_1(c'') = \beta_2(b - \tilde{b}) = \beta_2(b) - \beta_2\alpha_2(a)$ (by definition of $\tilde{b}$) $= \beta_2(b)$ (by exactness of the middle column) $= c$ (by definition of b). So c'' is the required preimage. $\square$

Remark 2.5.6: From an abstract point of view, the axiomatization of a reduced theory should not feature the unreduced groups $H(-,-)$. While "$\tilde{H}_*(X, A) := H_*(X, A)$ for $A \neq \emptyset$" would solve this, one may want to keep the description of $\tilde{H}_*(-)$ as a functor with one argument. This can be achieved rather directly, by restricting the category of spaces on which $\tilde{H}$ is defined, to (e. g.) CW-complexes, where, for a (closed) subcomplex $A < X$ (whose inclusion is automatically a cofibration), it holds that $H_*(X, A) \cong \tilde{H}({}^{X}\!/\!{}_{A})$. Hence in this case, the replacement "$H_*(X, A) \mapsto \tilde{H}({}^{X}\!/\!{}_{A})$" in the formulation of the long exact sequence works. More generally, we have to work with something like $\tilde{H}_*(X \cup c(A))$. Such constructions make sense, as long as spaces are at least well-pointed (see proof of Lemma 2.5.7 below).

There is a natural suspension, which is an isomorphism by excision

Lemma 2.5.7: *There are natural suspension-isomorphisms*

$$\mathrm{susp} : \tilde{H}_i(X) \to \tilde{H}_{i+1}(\Sigma X)$$

Proof: First note, that X is supposed to be well-pointed, thus so are ΣX and cX and further reduced and unreduced suspensions are homotopy equivalent $\Sigma X \simeq SX$ (see e. g. [Bre97, Thm. VII.1.9 (p. 436)]. Write $SX = c_+X \cup c_-X$. Then by the (reduced) "long" exact sequence (Lemma 2.5.4) of the pair $(\Sigma X, c_-X)$ and $c_-X \simeq \{\mathrm{pt}\}$ contractible, there is an isomorphism

$$\tilde{j}_* : \tilde{H}_{i+1}(\Sigma X) \to H_{i+1}(\Sigma X, c_-X).$$

By the (reduced) "long" exact sequence (Lemma 2.5.4) of the pair (c_+X, X) and $c_+X \simeq \{\mathrm{pt}\}$ contractible, there is an isomorphism

$$\tilde{\partial} : H_{i+1}(c_+X, X) \to \tilde{H}_i(X).$$

Further by excision (of a small neighborhood N of the cone-point of c_+X) and homotopy-invariance, there is an isomorphism

$$\mathrm{exc} : H_{i+1}(c_+X, X) \to H_{i+1}(\Sigma X, c_-X),$$

so that, all combined, we can define susp as a composition of these isomorphisms

$$\begin{aligned}\mathrm{susp} : \tilde{H}_i(X) &\xrightarrow{\tilde{\partial}_*^{-1}} H_{i+1}(c_+X, X) \\ &\xrightarrow{\mathrm{exc}} H_{i+1}(\Sigma X, c_-X) \xrightarrow{\tilde{j}_*^{-1}} \tilde{H}_{i+1}(\Sigma X)\end{aligned}$$

□

In fact, for the case of bordism-theories, the inverse is given by the following geometric construction (very similar to the inverse homomorphism of excision):

Example 2.5.8: *If $\tilde{H}$ is the reduced homology theory associated to a bordism theory in the sense of Section 2.1 ("Bordism Theories"), then, for $X = S^n$ a sphere, the inverse of the suspension*

$$\mathrm{susp}^{-1} : \tilde{H}_{i+1}(S^{n+1}) \to \tilde{H}_i(S^n)$$

applied to $[\![f : Y \to S^{n+1}]\!] \in \tilde{H}_{i+1}(S^{n+1})$ is represented by the cut of Y over a small deformation-retract-neighborhood U of the "equator" $S^n \subset S^{n+1}$, pushed homotopically over the equator along the deformation-retraction.

Proof: By the proof of the lemma, $\mathrm{susp}^{-1} = \tilde{\partial}_* \circ \mathrm{exc}^{-1} \circ \tilde{j}_*$. From the definition of $\tilde{j}_*$ (and the well-definedness arguments), $\tilde{j}_*([\![f]\!])$ merely "interprets" the representative f as representative in $H_{i+1}(\Sigma X, c_-X)$, so does not change f or Y.

The inverse of exc was constructed in Chapter 2 ("Bordism Constructions") by cutting (using $\Sigma X \simeq SX$, see proof of Lemma 2.5.7) between closure$(SX - c_-X)$ and closure(N), where N is the small neighborhood of the cone-point of c_+X from the proof of the lemma. Clearly N can be chosen, such that $c_+X - N$ deformation retracts to X, for example by restricting the cone-coordinate $N := c_\epsilon X$. This produces $\mathrm{exc}^{-1}([f]) = [f|_{Y_\geq}]$, where the boundary Y_0 of $Y_\geq$ is the "cut", because Y was closed initially.

Finally $\tilde{\partial}_*^{-1}([f|_{Y_\geq}]) = [f|_{Y_0}]$, interpreted as the equivalence class $[\![f|_{Y_0}]\!] = q([f|_{Y_0}])$ in the reduced theory $\tilde{H}_i(S^n)$. Choose U to contain $c_+X - N$, e. g. by removing a small cone $(c_\epsilon X)$ on both sides, then the cut is over U. The deformation-retraction of U to the equator pushes $f|_{Y_0}$ over the equator.

Hence $\mathrm{susp}^{-1}([\![f]\!])$ is indeed represented by the cut $f|_{Y_0}$ pushed over the equator.
□

For later reference, we want to formalize this connection of the geometric construction of susp^{-j} to transversality. The example given afterwards should clarify the intent / use-case of this definition.

Definition 2.5.9: *A reduced homology-theory, induced by a bordism-theory, has a* geometric description of suspension *if, given $f : X \to \Sigma^j S^k$ and $f_\perp \simeq f$ transverse to $S^k \subset \Sigma^j S^k = S^{j+k}$, in the sense, that $f_\perp^{-1}(S^k)$ is a MHSS and there is a map (generalizing a stratified bi-collar) $h : f_\perp^{-1}(S^k) \times \mathbb{R}^j \to X$, which is a stratified homeomorphism to its image, such that $f_\perp \circ h$ is compatible with the (trivial) normal-bundle of $S^k \subset S^{j+k}$ (i. e. it is given by $f_\perp \circ h|_{S^k} \times \mathrm{id}_{\mathbb{R}^j}$ on a trivialization). Then*

$$\mathrm{susp}^{-j} : \tilde{H}_{i+j}(S^{k+j}) \to \tilde{H}_i(S^k), [\![f]\!] \mapsto [\![f_\perp]\!]$$

i. e. the image of susp^{-j} *can be represented by* $[\![f_\perp]\!]$.

Note that such a $f_\perp$ always exists by repeated cutting, but the point is that *any* choice is valid, so for example:

Example 2.5.10: *Let X be a pl-pseudomanifold with $X \in \mathrm{Wh}_2^{\mathrm{gap}\,5} \cap \mathrm{Witt}$. Then a map $\varphi : X \to S^j = \Sigma^j S^0$ can be made transverse, in the sense of the definition above, but additionally, the transverse preimage can be chosen to be a pl-pseudomanifold.*

So, given a "geometric description of suspension" in the sense of Def. 2.5.9, $\mathrm{susp}^{-j}([\![\varphi]\!])$ *can again be represented by a pl-pseudomanifold, if $[\varphi]$ can, i. e. desuspension properly restricts to geometric sub-classes where transversality is available (potentially by other means).*

Proof: For pl-compatibly stratified spaces, for example Akin's [Aki75] arguments apply to construct such cuts (see e. g. [Sie83]).

The reason this works in PL is due to the following observation: pl-objects always (locally) have a "smallest scale", and, by constructing structures "half that size" one can always guarantee transversality. For example, given $A_\pm < X$ closed disjoint subcomplexes, to cut between them, let N_+ be a regular neighborhood of A_+ in the barycentric subdivision. Then ∂N_+ is "between" $A_\pm$ as required, but it is also transverse to the compatibly stratified strata (because it is on a simplex level). Further, strata of the cut being manifolds, and the cut being a pseudomanifold, can be decided by the links (which do not "considerably" change in the bi-collared cut). For example in PL, knowing that the bi-collar $M_0 \times \mathbb{R}$ in a stratum is a manifold, implies that M_0, the stratum of the cut, is a manifold. In the topological category this is not true (cf. [Bin58]).

By the "geometric description of suspension", such a transverse (in the pl-sense) $\varphi_\perp|$ represents $\mathrm{susp}^{-j}([\![\varphi]\!])$. □

The motivating Example 2.5.8 given above for our bordism-theories (induced by strong t-classes), shows, that the associated reduced theories have this "geometric" property:

Lemma 2.5.11: *Given a strong t-class $\mathcal{C}$, then $\tilde{\Omega}^{\mathcal{C}}_*(-)$ has a geometric description of suspension.*

Proof: From the geometric construction of desuspension Example 2.5.8, we find that $\mathrm{susp}^{-j}(\llbracket\varphi\rrbracket)$ is obtained by cutting near the equator, then near the equator of the equator and so on j times. Finally we obtain a codimension j subspace of X, say $X_{[j]}$ of X and $\psi(\llbracket f : X \to S^{k+j}\rrbracket) = \llbracket f| : X_{[j]} \to S^k\rrbracket$. The repeated bi-collars of the cuts give a local structure of the form $(\dots((X_{[j]} \times \mathbb{R}) \times \mathbb{R}) \dots \times \mathbb{R})$, i. e. a trivial normal-bundle (of rank j).

By well-definedness of the desuspension homomorphism, if we are given a transverse $f_\perp : X \to \Sigma^j S^k \simeq f$, we may successively pick these cuts (near M) as hyperplanes in the normal-bundle. Then the image $\mathrm{susp}^{-j}(\llbracket f_\perp\rrbracket)$ is represented by $f_\perp|_{X_{[j]}}$, where $X_{[j]} = f_\perp^{-1}(S^k)$ and $f \simeq f_\perp$ implies $\llbracket f\rrbracket = \llbracket f_\perp\rrbracket$.

These cuts are all "eligible" for use in the bordism-theory, because strong t-classes were required to be stable classes, thus persist on a bi-collared cut automatically. □

In such cases, the suspension-isomorphisms and "suspension of maps" are are compatible, at least for sufficiently "nice" target-space:

Lemma 2.5.12: *Let $\tilde{H}$ be a reduced homology theory, given as bordism of a strong t-class (so that there is a geometric description of the suspension-isomorphism and the "multiplication" below is well-defined). Given a map $\varphi : X \to S^k$, define (not quite in the standard way, as we multiply the sphere on the left-hand-side, rather than smashing it, hence the notation with a lower cross)*

$$\Sigma^j_\times \varphi : X \times S^j \xrightarrow{\varphi\times\mathrm{id}} S^k \times S^j \to {}^{S^k \times S^j}\!/\!{}_{S^k \vee S^j} = S^k \wedge S^j = \Sigma^j S^k = S^{k+j}$$

and the "multiplication" (Lemma 2.1.13) with the (manifold) fundamental-class $[S^j \xrightarrow{\mathrm{id}} S^j] \in \Omega^{\mathrm{STOP}}_j(S^j)$

$$-\times[S^j] : \tilde{H}_i(X) \to \tilde{H}_{i+j}(X \times S^j), \quad [Y \xrightarrow{f} X] \mapsto [Y \times S^j \xrightarrow{f\times\mathrm{id}} X \times S^j]$$

for the induced homomorphisms on reduced homology:

$$(\Sigma^j_\times\varphi)_* \circ (-\times[S^j]) = \mathrm{susp}^j \circ \varphi_* \quad : \tilde{H}_i(X) \to \tilde{H}_{i+j}(S^{k+j})$$

Proof: For well-definedness of the multiplication, see Lemma 2.1.13.

"Case $j = 1$": Given $\llbracket f : Y \to X\rrbracket \in \tilde{H}_i(X)$, by definition $(\Sigma^1_\times\varphi)_*(\llbracket f\rrbracket \times [S^1]) = \llbracket\Sigma^1_\times\varphi \circ (f \times \mathrm{id}_{S^1})\rrbracket$. The geometric description of desuspension Lemma 2.5.11 shows, that $\mathrm{susp}^{-1}(\llbracket\Sigma^1_\times\varphi\circ(f\times\mathrm{id}_{S^1})\rrbracket)$ is represented by restriction to a cut near the equator. By well-definedness, we may use any such cut, we pick $\llbracket\Sigma^1_\times\varphi \circ (f \times \mathrm{id}_{S^1})|_{Y\times S^0}\rrbracket$, which is clearly stratified bi-collared. But $\Sigma^1_\times\varphi \circ (f \times \mathrm{id}_{S^1})|_{Y\times S^0} = \varphi \circ f$, so $\mathrm{susp}^{-1} \circ(\Sigma^1_\times\varphi)_*(\llbracket f\rrbracket \times [S^1]) = \varphi_*(\llbracket f\rrbracket)$.

"General j" (see also Section 5.4 ("Singular Transversality")): Desuspending repeatedly is the same as cutting repeatedly (Lemma 2.5.11; because $\mathrm{susp}^{-j} = \mathrm{susp}^{-1} \circ \ldots \circ \mathrm{susp}^{-1}$, j-times, by definition), and we can do so over the respective equators ($S^{j-1} \subset S^j$, then $S^{j-2} \subset S^{j-1}$ etc.). So we find $\mathrm{susp}^{-j} \circ (\Sigma^j_\times \varphi)_*([\![f]\!] \times [S^j]) = [\![\Sigma^j_\times \varphi \circ (f \times \mathrm{id}_{S^j})|_{Y \times S^0}]\!]$, and as before $\Sigma^j_\times \varphi \circ (f \times \mathrm{id}_{S^j})|_{Y \times S^0} = \varphi \circ f$. □

Finally we return to the axiomatizations to indicate, how the transversality can alternatively be encapsulated in suitable "boundaries", by combining exact sequence and suspension / excision into a long exact and a Mayer–Vietoris-sequence:

Lemma 2.5.13: *Let $\tilde{h}_*$ be a reduced homology theory. Given a pair of pointed spaces (X, A), there is a long exact sequence, using the inclusions $i : A \to X$ and $j : X \to c(i)$*

$$\ldots \to \tilde{h}_i(A) \xrightarrow{i_*} \tilde{h}_i(X) \xrightarrow{j_*} \tilde{h}_i(c(i)) \xrightarrow{\partial_*} \tilde{h}_{i-1}(A) \to \ldots$$

Proof: The argument is the same one as commonly used for the constructing homotopy exact sequences / Puppe-sequences. By repeated use of the exact sequence for the pair (X, A) then for $(c(i), X)$ one gets an exact sequence

$$\tilde{h}_i(A) \xrightarrow{i_*} \tilde{h}_i(X) \xrightarrow{j_*} \tilde{h}_i(c(i)) \xrightarrow{k_*} \tilde{h}_i(c(j)) \xrightarrow{l_*} \tilde{h}_i(c(k))$$

Note that $c(j) \simeq \Sigma A$ (see for example [Bre97, Proof of Cor. 5.4 (p. 446)]). By this homotopy-equivalence indicated above and the (inverse) suspension-isomorphism

$$\tilde{h}_i(c(j)) \cong \tilde{h}_i(\Sigma X) \cong \tilde{h}_{i-1}(X)$$

thus defining $\partial_* := \mathrm{susp}^{-1} \circ k_*$, we are only left with checking exactness at A. This follows by $l \simeq \Sigma i$ (after identifying domains and targets up to the homotopy-equivalences above; see again [Bre97, Proof of Cor. 5.4 (p. 446)]). □

Corollary 2.5.13–1: *Let $\tilde{h}_*$ be a reduced homology theory. Given X and subsets $A, B \subset X$ with $X \subset \mathrm{interior}(A) \cup \mathrm{interior}(B)$ and $A \cap B \neq \emptyset$ (with a common base-point in $A \cap B$), there is a Mayer–Vietoris-sequence, i. e. a long exact sequence*

$$\ldots \to \tilde{h}_i(A \cap B) \xrightarrow{(i^A_*, i^B_*)} \tilde{h}_i(A) \oplus \tilde{h}_i(B) \xrightarrow{j^A_* - j^B_*} \tilde{h}_i(X) \xrightarrow{\partial^{\mathrm{MV}}_*} \tilde{h}_{i-1}(A \cap B) \to \ldots$$

where $i^A : A \cap B \to A$ and $j^A : A \to X$, similarly for B, are the inclusions.

Proof: We combine the sequences obtained via the lemma for the pairs $(B, A \cap B)$ and (X, A) into a commutative ladder (with vertical arrows inclusion-induced homomorphisms):

$$\begin{tikzcd}
\ldots \arrow[r] & \tilde{h}_{i+1}(B \cup c(A \cap B)) \arrow[r] \arrow[d, "\cong"] & \tilde{h}_i(A \cap B) \arrow[r] \arrow[d] & \tilde{h}_i(B) \arrow[r] \arrow[d] & \tilde{h}_i(B \cup c(A \cap B)) \arrow[r] \arrow[d, "\cong"] & \ldots \\
\ldots \arrow[r] & \tilde{h}_{i+1}(X \cup c(A)) \arrow[r] & \tilde{h}_i(A) \arrow[r] & \tilde{h}_i(X) \arrow[r] & \tilde{h}_i(X \cup c(A)) \arrow[r] & \ldots
\end{tikzcd}$$

The arrows marked by "$\cong$" are indeed isomorphisms, because by $A \cap B \neq \emptyset$ and $X \subset \text{interior}(A) \cup \text{interior}(B)$ we find, that (X, B) is excisive with respect to $X - A$ in the sense, that $\text{closure}(X - A) \subset \text{interior}(B)$, thus $B \cup c(A \cap B) \simeq B \cup c(A)$, where the right-hand-side is simply $X \cup c(A)$ (by $X = A \cup B$).

Now, define ∂_*^{MV} as the following composition: Starting at $\tilde{h}_i(X)$ in the bottom row, go one step to the right, then up the inverse-isomorphism to the top-row, then one step further to the right to $\tilde{h}_{i-1}(A \cap B)$. Then fitting this in a long sequence with the "direct sum" of the two arrows leaving $\tilde{h}_{i-1}(A \cap B)$ and then subtracting arrows from there in $\tilde{h}_{i-1}(X)$ yields indeed a long exact sequence, as can be checked by a diagram-chase. □

Conversely, the Mayer–Vietoris-sequence can be used to obtain suspension-isomorphisms

Lemma 2.5.14: *Let $\tilde{h}_*$ be a functor satisfying the reduced homology-theory axioms as stated in Def. 2.5.1, except for the suspension-isomorphisms, but such that (natural) Mayer–Vietoris-sequences exist. Then there are (natural) suspension-isomorphisms.*

Proof: Given X, write $\Sigma X = c_+(X) \cup c_-(X)$, but choose the cones such that they overlap as $c_+(X) \cap c_-(X) = X \times (-\epsilon, \epsilon)$ in the middle. Apply the Mayer–Vietoris-sequence to $\Sigma X = c_+(X) \cup c_-(X)$. Since the cones are contractible $\tilde{h}_i(c_\pm(X)) = 0$ by Example 2.5.3, so $\partial_*^{\mathrm{MV}} : \tilde{h}_i(\Sigma X) \to \tilde{h}_{i-1}(c_+(X) \cap c_-(X))$ is an isomorphism. Further $c_+(X) \cap c_-(X) \simeq X$, so this is the (inverse of the) suspension-isomorphism already. □

For completeness, we note, that of course the other direction, reduced theories defining unreduced ones also works (and the constructions are easily seen to be "inverse" to each other).

Definition 2.5.15: *Given a reduced homology-theory $\tilde{h}_*$, there is an associated unreduced theory given by $h(X, A) := \tilde{h}_*(X_+ \cup c(A_+))$, where X_+ and A_+ are the union with a disjoint base-point. Boundary-homomorphisms can be constructed from Lemma 2.5.13.*

To summarize, we have seen, that excision, suspension-isomorphisms and the boundary-map of a Mayer-Vietoris-sequence are very similar concepts. In the case of bordism-theories, they are further very closely related to transversality, as was formalized for the suspension-isomorphisms in Def. 2.5.9, to allow for certain "restrictions to geometric subclasses", cf. Example 2.5.10.

2.6 Improvements

Here, possible and desirable improvements of the example given in Section 2.4 ("An Example-Theory") are discussed. These concern

(a) The large-codimension requirement.
(b) The discussion of homeomorphisms.
(c) The restriction to spaces with at most two meeting strata.
(d) The obstructions involved.

(a) The codimension requirement: The issue becomes clear from the proof of Example 2.2.5, where repeated "cutting" of spaces eventually renders controlled end-theory inapplicable through violation of the high-dimensionality requirement.

Still, this seems "accidental" / unnecessary from the point of view that we are interested mostly in high-dimensional spaces. However, we are ultimately interested in a strong geometric description (e. g. we want a signature-invariant), so working "up to suspension" (in a "theory-of-spectra" sense) seems not to be the solution we want.

It might be possible to transition into a smooth or pl theory on low skeleta, as illustrated for the manifold case in the following example.

Example 2.6.1: *Let M be a closed 4-manifold, $g : M \to [-1,1]$ a continuous map. The* high-dimensional *arguments of [KS77] do not apply to make g transverse to* 0.

However 4-manifolds are close to being smooth, in the following sense [FQ90, §8.8 (p. 131–133)]: There is a discrete point-set $\Sigma \subset M$ such that $M - \Sigma$ has a smooth structure. By compactness Σ is finite, and $g(\Sigma)$ can easily be "moved away from" 0*, for example by adding bump-functions supported only close to Σ. Further, there is a close, smooth approximation of $g|_{M-\Sigma}$, which in turn is close to a (smoothly) transverse to* 0 *map (near the complement of a small neighborhood of Σ). Here, "close" can be taken to mean half the minimum* $1/2 \min(g(\Sigma) \cup \{\epsilon\})$*, for some fixed (target-size) ϵ. Thus g* can *be made transverse to* 0 *by small modification – near Σ, it is trivially transverse to* 0*, because $g|^{-1}(\{0\}) = \emptyset$. A small homotopy is given by the straight-line homotopy in* $[-1,1]$.

Of course, the problem for manifolds has long been well-understood and there are much stronger transversality-theorems known [Qui88b], but this example still *serves well to see how "closeness" to the smooth category can help to understand low-dimensional transversality problems in the topological category.*

As it turns out, some aspects of this approach seem to apply to the MHSS case. For example, mapping-cylinder neighborhoods away from a set of points would clearly suffice for the cutting arguments, and Sard's theorem allows to carry over *smooth* cylinders to the cut (because the map into $\mathbb{R}$ is "usually" transverse on the *boundary / completion* of the upper stratum, see Lemma 1.6.9). "Smooth enough" (manifold) structures can be obtained through "almost smoothing" arguments [FQ90] (for 5-manifolds[8]), so it would remain to smooth certain MAFs (with, for example, a Whitehead-condition) without changing the stratified homeomorphism-type of the mapping-cylinders. One may want to construct a (5+1)-dimensional h-cobordism controlled by a smooth submersion (Ehresmann's lemma provides most of the "small-retractions") to achieve this, but while smooth refinements of end- and h-cobordism theorems (in dimension 5+1) seem to allow for a suitable smoothing of individual "monotonic directions" (see Example 1.2.3 and discussion thereafter; use end-completions to "divide" the space in h-cobordism-slices getting thinner towards one side of the "large-scale" h-cobordism being constructed, see above, then use the thin-h-cobordism-theorem on these slices), however making them independent[9] seems to be hard. This may be due to closeness to 4- or 5-dimensional h-cobordisms (even though certain singularities could be allowed, see footnote above).

It is unclear as of now, whether this issue can be resolved (e. g. for low dimension of the lower stratum / "few directions" to be made independent) or if a connection to 4- or 5-dimensional h-cobordisms can be made precise (in which case the difficulty of the problem could at least be understood to be rooted in the extraordinarily hard problem of low dimensional h-cobordisms). Note, that nevertheless, this problem is considerably "easier" than extending bundle structures (for example from the "sides" of a mapping-cylinder of a homeomorphism), in that, for the "difficult cuts", in dimensions 4 and 5, one can allow for 1-dimensional, and 0-dimensional respectively, singularities, where the "nice" structure can be interrupted (cf. Example 2.6.1). Additionally, working with bundles, there is no easy way to preempt the issue by a codimension-condition.

We briefly return to the dimensional-gap problem in Section 3.3 ("Special Gaps"), where a very simple, but still useful special case (adding a minimal stratum, such that the union with the next stratum one dimension higher, is a manifold with boundary) is treated.

(b) (Un)stratified Homeomorphisms: One would hope, that intrinsic stratifications (cf. Section 1.10 ("Intrinsic Stratifications")) can be used to deal even with

[8] The non-smooth set is 1-dimensional for 5-manifolds, so intersects the codimension 1 cut "generally" in a point-set, which could be ignored for the subsequent cuts.

[9] Even up to, say, 1-dimensional a subspace, which would be tolerable, as it would ("generally") intersect the cut in a 0-dimensional, thus for subsequent cuts ignorable, region.

unstratified homeomorphisms. There is a difficulty however: Stratifications of the mapping-cylinder constructed using intrinsic stratifications (see [Ban07, (p. 136f)] and [Fri15, (p. 14f)]) typically introduce "additional" strata, so even given a homeomorphism $h : X \xrightarrow{\sim} Y$, with both X and Y having only two strata, a stratification of $\mathrm{cyl}(h) \cong X \times I \cong Y \times I$ will typically have more than two strata (see Section 4.2 ("Spaces With at Most Two Strata")). So a treatment of unstratified homeomorphisms requires the material in Chapter 3 ("Multiple Strata"). A discussion is hence given afterwards in Chapter 4 ("Homeomorphisms").

(c) Multiple Strata: While statements and results become much harder to read, the "underlying technology" used up until now – especially Quinn's controlled end-theory and the transversality-theorem by Connolly and Vajiac – is available, with almost no further restrictions, on MHSS with more than two strata. This will be used to extend our results in Chapter 3 ("Multiple Strata").

Multiple strata – besides being desirable by their own right – are also important to understand certain two-stratum properties, that are naturally connected to multiple-stratum questions. These are, for example, unstratified homeomorphisms (see (b) above), transversality on spaces with boundary (see Cor. 3.2.3–1) or products (see Lemma 2.1.13).

(d) Obstructions: While extremely interesting, the questions discussed under this point are far beyond the scope of this thesis.

The connection of transversality and end-theory goes in both directions, for example Connolly and Vajiac [CV99] give a "backward direction" showing, that near a bi-collared cut, there is also a mapping-cylinder (in high dimensions). This allows them to state a one-to-one relation of the existence of extensions of bi-collared cuts into higher strata and certain local end-obstructions near the cut in the lower skeleton.

This is not quite what we want however. The transversality-question relevant to bordism-theories as described here, fundamentally differs from this notion in at least two ways. First, we need not only be able to extend one *fixed in advance* cut, rather, it must be possible to extend *any* cut – up to what is the second difference: Extending a fixed *choice* of cut in the lower stratum is not necessary for the transversality properties we want. Really, what is required is the existence of *some* cut between closed disjoint subsets $D_\pm$ that can be extended. This seems to be a "cohomological" question, because we are asking for an obstruction defined on something closer to an element of cohomotopy of the bottom-stratum $\pi^1(B)$ or framed manifold-bordism $\Omega^{\mathrm{SO,\,framed}}_{n-1}(B)$ than to a "rigid choice" of cut.

Quinn's [Qui82a] obstruction theory, while it extends well beyond the "point-wise" / fiber-wise vanishing of obstruction-groups – with regard to both providing groups depending on B and obstruction-*elements* within those groups – does not seem to answer this type of question. It is not event quite evident to what degree this actually changes the problem, as illustrated by this example:

Example 2.6.2: *Given a MHSS $X \supset B$ with $\dim(B) = 1$ and either a cylinder-neighborhood of B in X, or, both vanishing end-obstruction [Qui82a] $q_0(X, B) = 0$ and $\dim(X) \geq 6$. For example, if the holink-evaluation is of the form $p = p' \times \mathrm{id}_K$ with K a finite complex, then q_0 is multiplicative with the euler-characteristic $\chi(K)$ of K by $q_0(X, B) = \chi(K) q_0(X, B')$ [Qui82a, Prop. 1.8c (p. 359)], so if $X = Y \times S^1$ (with $\chi(S^1) = 0$), then these are satisfied.*

In this case, transversality (in the sense of Section 1.6 ("Stratified Transversality")) by itself is not an issue, because a cylinder-neighborhood exists. Further any resulting cut X_0 has as lower stratum a finite point-set, which can be avoided for further cuts.

Stability would, however, require that $X_0 \times I$ be allowable, which is not evident from this argument. In fact, the (end-)obstruction here is given by an element over the point, thus again "fiber-wise".

The "regression" of the last paragraph may indeed always happen (by "cutting down" to B only points – automatically maintaining holink-fiber ho-types – and then crossing with I). Note however, that even then, we would have an *element* q_0 over the point, that we need to vanish – for *any* cut again however, so this advantage may not persist either.

Since (compatibly) pl-spaces are "easy" to cut (see example below), but there is no obvious reason why they would generally satisfy the Whitehead-condition as stated here, one would expect some generalization to be possible.

Example 2.6.3: *Cf. [Aki75]: Given a compatibly (skeleta are subcomplexes) triangulated compact stratified space X, and two disjoint subcomplexes $B_+, B_- < X$, one can cut between B_+ and B_- (near B_-) by defining X_0 as the boundary of a regular neighborhood of B_- in the first barycentric subdivision of (the triangulation of) X.*

This is (essentially) the same as picking a map $g : X \to [-1, 1]$, mapping $B_\pm$ to ± 1, triangulate $[-1, 1]$ such that 0 is in the interior of some 1 simplex, and use simplicial approximation on g to obtain $g_\perp$. Since $g_\perp$ is linear on simplices, the preimage of 0 is bi-collared in X (and thus automatically inherits some properties – for example concerning links – from the ambient X). One may want to compare this to the usual proof of the excision-axiom for singular homology, which works similarly.

While one could, of course just "brute force" this into the theory, by additionally allowing skeleta if they have a compatibly pl neighborhood. However, this would not help the treatment of *non-pl* homeomorphisms of such spaces.

3 Multiple Strata

This chapter generalizes the treatment, so far given for spaces with at most two strata, to spaces with arbitrarily (finitely) many strata.

First, in Section 3.1 (“Background”), generalized statements of the techniques and definitions involved are given. This section is mostly technical. Afterwards, in Section 3.2 (“Absorbing Boundary”), corresponding results for spaces with boundaries are given, by (temporarily) “absorbing” the boundary as a codimension 1 stratum. Then, Section 3.3 (“Special Gaps”) discusses a simple, but useful for Chapter 4 (“Homeomorphisms”), special case where the codimension-requirement can be weakened. Finally, the results are combined to construct a more general transversality-class in Section 3.4 (“Transversality-Class”), which will be used in the main theorem (Thm. 5.1.2) to obtain a corresponding generalized homology theory realized as bordism-theory.

3.1 Background

The introductory Chapter 1 (“Background”) only introduced MHSS with two strata to simplify notation. The relevant results from controlled topology and on transversality remain valid in this case. Here, the formal statements and references are given, preparing for the further treatment of the “multi-stratum” case.

We start by giving a definition of general MHSS

Definition 3.1.1: *[Qui88a, Def. 3.1 (p. 30)]: A* manifold homotopy stratified space (MHSS) X *is a separated, metric space, filtered by closed subsets* $X^n \supset \cdots \supset X^0$*, such that:*

(i) Strata $X_i := X^i - X^{i-1}$ *are topological manifolds.*

(ii) Strata meet such that $\forall i < k : X_i \subset X_k \cup X_i$ *is tame and*

(iii) the homotopy-links $ev_0 : \mathrm{holink}(X_k \cup X_i, X_i) \to X_i$ *are fibrations.*

(iv) Local homotopy-links (see below) are compactly dominated. [Hug99b]

The dimension *of* X *is the manifold-dimension of the* top-stratum $X_n := X^n - X^{n-1}$, *i. e.* $\dim(X) := \dim(X_n)$. *We assume* X *is filtered by dimension, i. e.* $\dim(X_i) = i$ *for non-empty* X_i.

If only (ii) and (iii) hold, we call X *an HSS.*

The “local homotopy-links” in this case are themselves stratified spaces:

Definition 3.1.2: *Let X be a topological space, $Y \subset X$, and X filtered by closed subsets $X^n \supset \cdots \supset X^0$.*

[Hug99b, §4 (p. 311f)]: The stratified homotopy-link $\mathrm{holink}_S(X,Y)$ *is the filtered space given by the filtration (for $i = 0, \ldots, n$)*

$$\mathrm{holink}^i_S(X,Y) := \Big\{\gamma \in \mathrm{holink}(X,Y) \Big| \gamma\Big((0,1]\Big) \subset X_i\Big\}.$$

[Hug99b, §5 (p. 317ff)]: Given $x \in X$, there is a unique k with $x \in X_k$, and the local homotopy-link at x *is the filtered space given by the filtration (for $i = 0, \ldots, n$)*

$$\mathrm{holink}^i(X,x) := \Big\{\gamma \in \mathrm{holink}^i_S(X,X_k) \Big| \gamma(0) = x\Big\}.$$

[Hug99b, Def. 5.1 (p. 318)]: X has compactly dominated local holink at $x \in X$, *if there exist a compact $K \subset \mathrm{holink}(X,x)$ and a stratum-preserving deformation of* $\mathrm{holink}(X,x)$ *into K. I. e. there is a stratum-preserving homotopy (Def. 1.3.7) $d : \mathrm{holink}(X,x) \times I \to \mathrm{holink}(X,x)$ with $d_0 = \mathrm{id}$ and $\mathrm{im}(d_1) \subset K$.*

Note, that this means we have two different notions of "links" / fibers: The fibers of the homotopy-links of meeting strata $X_i \subset X_k \cup X_i$, and the "local homotopy-links", which are the fibers of the *stratified* homotopy-links of strata $X_i \subset X$.

While this definition is straight-forward from the two-stratum case, it raises a couple of questions on the normal structure of *skeleta* – which one would expect to be the relevant objects for inductive arguments. Before, the normal structure of the lower stratum was controlled in the higher stratum by the holink-evaluation being a fibration – which was seen to agree with cylinder-neighborhoods, if they exist, being cylinders of MAFs.

To describe the normal structure of a skeleton – consisting itself of multiple strata – in the next higher stratum, we need suitable notions replacing fibrations and MAFs when mapping to stratified spaces rather than manifolds. There are different modifications for generalizing to maps . . .

(a) . . . with stratified domain: "Stratified fibrations" [Hug99c, §5 (p. 6)] are defined using lifting properties for stratified homotopies.

(b) . . . with stratified range: "Stratified systems of fibrations" [Qui88a, p. 469], which is what we need for inductive arguments, and is thus defined formally below.

(c) . . . with stratified domain and range: "stratified systems of stratified fibrations" [Hug99c, §7 (p. 12)], are defined like stratified systems of fibrations (b), but being stratified fibrations (a) over strata.

Corresponding "approximate" versions can be defined by replacing lifting-properties by approximate lifting-properties, or – equivalently, but easier to formalize, for the more complicated versions – by enforcing Hughes' cylinder-theorem, that is, by defining $p : E \to X$ is an "approximate stratified systems of approximate stratified fibrations" if and only if $\mathrm{ev}_0 : \mathrm{holink}(\mathrm{cyl}(p), X) \to X$ is a stratified systems of stratified fibrations (this stance is taken for example in [Qui04]).

We formalize the case of "stratified systems of fibrations", where the range is stratified, which we need for an inductive step X^i to X^{i+1} (the domain in this case is $X_{i+1} = X^{i+1} - X^i$, hence a ("unstratified") manifold if X is a MHSS).

Definition 3.1.3: *See e. g. [Qui88a, p. 469] Stratified Systems of Fibrations:*

*(i) Recall Def. 1.4.11: Given a continuous map $p : E \to X$, a subset $Y \subset X$ is a p-*NDR *if there is a deformation retraction r of a neighborhood of Y in X, that is covered by a retraction R of a neighborhood of $p^{-1}(Y) \subset E$, i. e. $p \circ R(x, t) = r(p(x), t)$.*

(ii) p is a a stratified system of fibrations *if, for all i, its restrictions to strata $p| : p^{-1}(X_i) \to X_i$ are fibrations and skeleta $X^i \subset X$ are p-NDR.*

Remark 3.1.4: These "stratified system of fibrations" do have an "approximate lifting property" for *stratified* homotopies. [Qui88a, Lemma 3.3 (p. 469)]

Example 3.1.5: *A stratified NDR $Y \subset X$ is a p-NDR for $p : \mathrm{holink}(X, Y) \to Y$. See Lemma 1.4.12 for a more precise relation of stratified NDRs and p-NDRs.*

A property of skeleta relevant to the question about normal-structure is the following:

Definition 3.1.6: *[Qui88a, p. 469]: Let X be a filtered space. A* pure *subset $Y \subset X$ is a closed subset, which is a union of components of strata.*

Example 3.1.7: *Suppose X is stratified by closed subsets, then skeleta $X^i \subset X$ are pure.*

This property allows to formalize the notion of HSS defined with normal structure on a stratum-to-stratum level (Def. 3.1.1), having appropriate normal structure on a skeleton-to-skeleton level:

Lemma 3.1.8: *[Qui88a, Prop. 3.2 (p. 469)]: Suppose X is a HSS and $Y \subset X$ a pure subset (given the induced filtration). Then:*

(i) If $X^i \subset Y$, then X^i is a p-NDR, where $p = ev_0 : \mathrm{holink}(Y, X^i) \to X^i$.

(ii) There is a nearly strict and nearly stratum-preserving deformation retraction of a neighborhood of Y in X to Y.

(iii) $ev_0 : \mathrm{holink}(X, Y) \to Y$ is a stratified system of fibrations.

(iv) Y is a HSS. If X is a MHSS, then Y is a MHSS.

Proof: Parts (ii) and (iii) are immediate consequences of [Qui88a, Prop. 3.2 (p. 469)]. Part (i) follows from the inductive "conclusion (1)" in Quinn's proof [Qui88a, Proof of 3.2 (p. 471)] of that result.

It remains to validate part (iv). Clearly Y is a HSS, as their definition only depends on conditions of pairs of (components of) strata, and the "pairs of (components of) strata" of Y are a subset of the "pairs of (components of) strata" of X. If X is an MHSS, Y is also separable, metric, because X is. Its strata consist of components of manifolds (the strata of X), thus are manifolds. Local holinks are compactly dominated, because, given $y \in Y$, $\mathrm{holink}(Y, y) \subset \mathrm{holink}(X, x)$ is pure, so a stratified deformation d of $\mathrm{holink}(X, x)$ into a compact $K \subset \mathrm{holink}(X, x)$ restricts to a deformation of $\mathrm{holink}(Y, y)$ into the compact $K \cap \mathrm{holink}(Y, y)$. The well-definedness of the restriction follows from the initial d being stratified. □

Part (iii) uses an *un*stratified holink, hence the domain is indeed unstratified (but may be complicated). However, for an inductive step X^i to X^{i+1}, this is fine, as X^{i+1} is a MHSS, so applying the lemma to $X^i \subset X^{i+1}$ provides a "more reasonable" difference $X_{i+1} = X^{i+1} - X^i$.

This is actually enough to provide the "full" end-theorem (the obstruction-groups are explained below):

Theorem 3.1.9: *[Qui88a, Thm. 1.7 (p. 446)]: Suppose X is a MHSS (stratified by dimension) and $i \geq 6$, then:*

(i) There is an invariant $q_0(X^i, X^{i-1}) \in H_0^{\mathrm{lf}}(X^{i-1}; \mathcal{S}(p_i))$.

(ii) There is a mapping cylinder neighborhood of X^{i-1} in X^i (i. e. a completion of the controlled end of X_i), if and only if $q_0(X^i, X^{i-1}) = 0$.

(iii) [Qui82a, Thm. 1.1 b (p.357)]: Completions are unique up to controlled h-cobordism.

Here the stratified system of fibrations $p_i = ev_0 : \mathrm{holink}(X^i, X^{i-1}) \to X^{i-1}$ is the holink-evaluation.

We will extend the discussion of the obstruction-groups $H_0^{\mathrm{lf}}(X^{i-1}; \mathcal{S}(p_i))$ beyond what has been said in Section 1.9 ("Mapping-Cylinder Neighborhoods") below, but first we briefly recall the previous results and give a suitable (as it will turn out) generalization of the "simple links"-condition.

First note, that $\mathcal{S}$ as defined in [Qui82a] is indeed a functor from *stratified systems of* fibrations to spectra – which includes the special case of fibrations, that appeared earlier when discussing spaces with at most two strata. Before, the spectral-"cosheaves" where trivial, in the sense, that given a fibration, at least on individual connected components of the lower stratum, the holink-fibers did not change. Thus in the two-stratum case, a condition on links (holink-fibers) L of the form $\forall k \geq 0 :$

$\mathrm{Wh}(\pi_1(L) \times \mathbb{Z}^k) = 0$, did suffice to show vanishing of the obstruction groups (see Lemma 1.9.8).

For all (more than two) strata to fit together, it seems reasonable, that at least *all pairs* of two strata should fit together in this sense. This motivates to the following definition:

Definition 3.1.10: *Let X be a MHSS. X has* simple links *if and only if for all $j > i$, and for all connected components L of the fiber of the fibration* $\mathrm{holink}(X_j \cup X_i, X_i) \to X_i$ *over all connected components of X_i it holds that* $\forall k \geq 0 : \mathrm{Wh}(\pi_1(L) \times \mathbb{Z}^k) = 0$.

A suitable generalization of Lemma 1.9.8 is then:

Lemma 3.1.11: *Let X be a MHSS (stratified by dimension) with simple links. Then $\forall i : H_j^{\mathrm{lf}}(X^{i-1}; \mathcal{S}(p_i)) = 0$ for $j \leq 1$, where $p_i : \mathrm{holink}(X^i, X^{i-1}) \xrightarrow{ev_0} X^{i-1}$ is the holink evaluation map.*

Proof: The proof of this lemma is given below on page 126.

To prove this and to provide some understanding of the obstruction groups, we very briefly outline some of the constructions involved to obtain $H_*^{\mathrm{lf}}(X; \mathcal{S}(p))$.

First, we briefly outline the construction of the “finite structure spectrum” $\mathcal{S}$ of [Qui82a, §5 (p. 388–402)] as a pseudo-isotopy spectrum.

Definition 3.1.12: *Given a compact manifold M, a* pseudo-isotopy θ of M *is a homeomorphism $\theta : M \times I \xrightarrow{\sim} M \times I$ with $\theta|_{M\times\{0\}\cup\partial M\times I} = \mathrm{id}$ and $\theta(M \times \{1\}) \subset M \times \{1\}$.*

Given a polyhedron K, pseudo-isotopy θ of M parametrized by K *is a homeomorphism $\theta : M \times I \times K \xrightarrow{\sim} M \times I \times K$, that commutes with projection to K and such that for fixed $T \in K$ each $\theta(-,-,T)$ is a pseudo-isotopy.*

[Qui82a, Def. 5.1 (p. 388)]: Given maps $E \xrightarrow{p} X$ and $U \xrightarrow{r} E$ on a codimension 0 submanifold $U \subset \mathbb{R}^n$ and X metric, with proper composition pr, one can add size-control and limit the support $C \subset X$ (where C being compact suffices to ensure U being compact by properness) as follows: Require θ to be defined on $(pr)^{-1}(C) \times I \times K$, and weaken $\theta|_{M\times\{0\}\cup\partial M\times I} = \mathrm{id}$ to $\theta|_{M\times\{0\}\cup\partial M\times I} = \mathrm{incl}$, require that the image of θ contains $(pr)^{-1}(C^{-\epsilon}) \times I \times K \subset \mathrm{im}(\theta)$ (here: $C^{-\epsilon}$ is $X - (X - C)^\epsilon$ and $D^\epsilon = \{x \in X | \operatorname{dist}(x, D) < \epsilon\}$), and further, that both θ and θ^{-1} (where defined) are ϵ-small (as a homotopy, i. e. rays $pr\theta(x,-,T)$ are ϵ-small), when measured in X, i. e. after composition with pr.

In the controlled case, there are certain operations, under which one wishes to identify pseudo-isotopies (see [Qui82a, 5.2 (p. 389)]). These are "reduction" (to smaller $D \subset C$ and greater $\delta \geq \epsilon$), "deletion" (of relative parts where θ is trivial) and suspension (crossing U with I^k and θ with id; this is not quite canonical[1]). Quinn [Qui82a] does so, by constructing a Δ-set $\mathcal{P}$ of pseudo-isotopies with these identifications built into boundaries:

Definition 3.1.13: *[Qui82a, Def. 5.3 (p. 389)]: Given $E \xrightarrow{p} X$, and $C \subset X$ compact, $\epsilon > 0$, there is a Δ-set $\mathcal{P}(X, p; C, \epsilon)$ with k simplices the pseudo-isotopies controlled by pr, with support C and size $< \epsilon$, parametrized by the standard-simplex Δ^k. Boundary maps are defined up to suspension, reductions and deletions. We denote by $\mathcal{P}(X, p)$, the Δ-set constructed in this way, allowing any support C and size ϵ.*

At this point, one can define a homotopy-limit which will become the space at index -2 of the finite structure spectrum $\mathcal{S}$:

Definition 3.1.14: *[Qui82a, Def. 5.4 (p. 390)]: Given $E \xrightarrow{p} X$, define $\mathcal{S}_{-2}$ as the homotopy-limit $\mathcal{S}_{-2}(X, Y; p) := \operatorname{holim}_{C \supset X-Y, \epsilon \to 0} \mathcal{P}(p; C, \epsilon)$. This can be done explicitly via paths to infinity: Give $[0, \infty)$ the simplicial (or Δ-set) structure with vertices at integers and 1-simplices between them, then $\mathcal{S}_{-2}(X, Y; p)$ is given by simplicial maps (really maps of Δ-sets) $[0, \infty) \to \mathcal{P}(X, p)$, such that simplices in the image of $\Delta^k \times [N, \infty)$ are in in $\mathcal{P}(X, p; C_N, \epsilon_N)$, where $\epsilon_i \searrow 0$ is a sequence monotonically falling to zero, and C_i are compact subsets with $C_i \subset \operatorname{interior}(C_{i+1})$ and $\cup_i C_i \supset X - Y$.*

Using suitable stability / shrinking results [Qui82a, Thm. 5.6 (p. 391) and 'shrinking lemma' 5.8 (p. 395)] (cf. discussion in the proof of Example 1.7.9) to compose ϵ-small objects into ϵ-small (rather than 2ϵ-small) objects (see e. g. [Qui82a, 'homotopy lemma' 5.7 (p. 394)]), one obtains certain homotopy-equivalences:

Lemma 3.1.15: *[Qui82a, Thm. 5.9 (p. 397)]: For X a locally compact metric ANR, p a stratified system of fibrations, there are homotopy-equivalences*

$$T : \mathcal{S}_{-2}(X; p) \to \Omega\mathcal{S}_{-2}(X \times \mathbb{R}; p \times \mathrm{id}).$$

These T can be used to define an Ω-spectrum in the obvious way:

[1]One needs to fix the "rel boundary" requirement, by a suitable homeomorphism h_k of I^k depending on the integer k where one cannot, in general, ensure, that $h_k \times h_l = h_{k+l}$.

Definition 3.1.16: *[Qui82a, Def. 5.10 (p. 400)]: Given $E \xrightarrow{p} X$, a stratified system of fibrations over a locally compact metric ANR X, there is an Ω-spectrum $\mathcal{S}(X;p)$ with spaces*

$$\mathcal{S}_j(X;p) := \begin{cases} \Omega^{2-j}\mathcal{S}_{-2}(X;p) & \text{if } j < -2 \\ \mathcal{S}_{-2}(X \times \mathbb{R}^{j+2}, p \times \mathrm{id}) & \text{if } j \geq -2 \end{cases}$$

and structure-maps (adjoints of) T (of the previous lemma).

This construction turns out have reasonable properties, as long p is a stratified system of fibrations (Def. 3.1.3). This can be formalized as follows (the overview given here is mostly based on [Qui82a, §8 Appendix (p. 419–423)], a more detailed treatment can for example be found in [Qui04, §6 (p. 50–64)]):

Definition 3.1.17: *[Qui82a, p. 421]: A functor $\mathcal{S}(X;p)$ from stratified systems of fibrations p over locally compact σ-compact ANR X to spectra is a* locally finite homology-theory with spectral cosheaf coefficients *if it satisfies the following axioms (we will only need the first one, and just indicate the other two):*

(1) "Restriction": Given an open $W \subset X$, there is a natural restriction-map $\mathcal{S}(X;p) \to \mathcal{S}(W;p|_W)$, such that if $Y \subset X$ is a closed p-NDR, then $\mathcal{S}(Y;p|_Y) \xrightarrow{\text{incl}_*} \mathcal{S}(X;p) \xrightarrow{\text{restr}} \mathcal{S}(X-Y;p|_{X-Y})$ has composition the point-map and is a homotopy-fibration.

(2) "Continuity": The assignment from morphisms of stratified systems of fibrations (as diagrams with compact-open-topology) to morphism of spectra (as degree-wise maps of spaces with compact-open-topology) is continuous.

(3) "Inverse limit": Restrictions define a homotopy-equivalence of $\mathcal{S}(X;p)$ to $\operatorname{holim}_Y \mathcal{S}(X-Y;p_{X-Y})$.

Example 3.1.18: *[Qui82a, Thm. 5.11 (p. 400) and its proof]: The finite structure spectrum $\mathcal{S}(X;p)$ is a locally finite homology-theory with spectral cosheaf coefficients.*

Remark 3.1.19: A slightly different formulation of these axioms can be found for example in the appendix of [Qui04].

Such theories are characterized by these axioms in the sense of Eilenberg–Steenrod, i. e. they are determined block- / fiber-wise in a sense outlined below.

To make this precise (see Def. 3.1.20), we need to first say, what "determined block- / fiber-wise" is supposed to mean. To this end, given $\mathcal{S} = \mathcal{S}(X;p)$, one defines a functor of spaces $\mathcal{S}(F)$ (also denoted by $\mathcal{S}$), that only encodes the information contained "fiber-wise" in $\mathcal{S}(X;p)$ (i. e. forgets the remainder of the structure and data). Then, one further defines, for fixed p, a "spectral cosheaf" (see below) $\mathcal{S}(p)$, by block-wise application. Such spectral cosheaves also determine a locally finite homology-theory with spectral cosheaf coefficients, see Example 3.1.21 (similar to the choice of coefficient groups in ordinary homology).

Quinn's characterization-theorem (Thm. 3.1.22) then essentially says, that this homology theory determined by coefficients (that contain only the block- / fiber-wise information) is in fact the same as the original theory $\mathcal{S}(X;p)$. This means, no information is *actually* lost, in forgetting the remainder of the structure, all of it can be recovered.

Definition 3.1.20: *See [Qui82a, p. 419f]:*

There is a homotopy-invariant (homotopy-equivalences induce homotopy-equivalences of spectra) functor $\mathcal{S}(F)$ from spaces F to spectra induced by $\mathcal{S}(X;p)$ as $\mathcal{S}(F) := \mathcal{S}(\{\mathrm{pt}\}; F \to \{\mathrm{pt}\})$.

Given a map p, a functor from spaces to spectra $\mathcal{S}(F)$ induces $\mathcal{S}(p)$ by "block-wise" application: If $X = |K|$ is a polyhedron, define $\mathcal{S}(p) := \left(\sqcup_{\sigma\in K}\mathcal{S}(\sigma)\times\sigma\right)/\sim \xrightarrow{p_*} |K|$, where $\sim$ is the evident equivalence-relation on faces, and p_* maps $(y,x) \in \mathcal{S}(\sigma)\times\sigma$ to $x \in |K|$. For general X, one can later (see below) use a suitable holimit over neighborhoods U of the diagonal in $X \times X$ and $K = S_U(X) := \{\sigma|\text{singlular simplices with } \operatorname{supp}(\sigma\times\sigma) \subset U\}$.

Further, if p is a stratified system of fibrations, then $\mathcal{S}(p) \xrightarrow{p_*} X$ is a "spectral cosheaf" (in the sense of e. g. [Qui04, Def. 6.7.3 (p. 58)]; in particular, the structure maps glue in a suitable way to turn this into a spectrum again), and there is a Ω-spectrum, called the homology-spectrum*, defined, for $X = |K|$ a polyhedron as $\mathbb{H}(K,L;\mathcal{S}(p)) := \varinjlim \Omega^j\left(\mathcal{S}_j(p)/i(K) \cup p_*^{-1}(L)\right)$, where $i: K \to \mathcal{S}(p)$ is the inclusion $\sigma \mapsto \text{basepoint}\times\sigma$. For general X, define $\mathbb{H}(X;\mathcal{S}(p)) := \operatorname{holim}\mathbb{H}(S_U(X);\mathcal{S}(p))$ as the homotopy inverse limit over neighborhoods U of the diagonal in $X\times X$ (see above).*

The locally finite homology-spectrum *is $\mathbb{H}^{\mathrm{lf}}(X;\mathcal{S}(p)) := \operatorname{holim}\mathbb{H}(X,Y;\mathcal{S}(p))$ the homotopy inverse limit over $Y \subset X$ with $\operatorname{closure}(X-Y)$ compact. An explicit form can be given for example if X is σ-compact, by taking the limit over the complements of a compact exhaustion in a sense similar to the definition of $\mathcal{S}_{-2}$ (Def. 3.1.14).*

The locally finite homology groups *$H_k^{\mathrm{lf}}(X;\mathcal{S}(p)) := \pi_k(\mathbb{H}^{\mathrm{lf}}(X;\mathcal{S}(p)))$ are the (stable) homotopy-groups of this spectrum in the usual sense.*

Example 3.1.21: *[Qui82a, Prop. 8.4 (p. 421)]: If p is a stratified system of fibrations, and $\mathcal{S}(F)$ a homotopy-invariant functor from spaces to spectra, then $\mathbb{H}^{\mathrm{lf}}(X;\mathcal{S}(p))$ is a locally finite homology-theory with spectral cosheaf coefficients in the axiomatic sense Def. 3.1.17.*

The characterization / uniqueness can then be formulated as follows:

Theorem 3.1.22: *[Qui82a, Thm. 8.5 (p. 421)] "Characterization Theorem":*
Given a locally finite homology-theory with spectral cosheaf coefficients $\mathcal{S}(X;p)$, then there is a homotopy-equivalence of spectra $A : \mathbb{H}^{\mathrm{lf}}(X;\mathcal{S}(p)) \to \mathcal{S}(X;p)$.

This essentially means, that any such $\mathcal{S}(X;p)$ is determined by its block-wise / fiber-wise properties encoded in $\mathcal{S}(p)$.

To continue with the proof of simple links implying vanishing of end-obstruction groups (Lemma 3.1.11), we quickly observe:

Lemma 3.1.23: *Let X be a MHSS (stratified by dimension) and for $i < j$*

$$p_i^j := ev_0 : \mathrm{holink}(X_j \cup X^i, X^i) \to X^i$$

Then (restrictions are to be read as restrictions to the respective preimages)

(i) $(p_i^j)|_{X^{i-1}} = p_{i-1}^j$ *and*

(ii) $(p_i^j)|_{X_i} = ev_0 : \mathrm{holink}(X_j \cup X_i, X_i) \to X_i$.

Proof: The maps p of all holinks are evaluation at zero, so it suffices to check, that the total-spaces restrict (map zero) correctly.

"Part (i)":

$$\begin{aligned}
&\mathrm{holink}\left(X_j \cup X^i, X^i\right)|_{X^{i-1}} \\
&= \left\{ \gamma : I \to X_j \cup X^i \;\middle|\; \gamma(0) \in X^{i-1} \text{ and } \gamma((0,1]) \subset X_j \cup X^i - X^i \right\} \\
&= \left\{ \gamma : I \to X_j \cup X^{i-1} \;\middle|\; \gamma(0) \in X^{i-1} \text{ and } \gamma((0,1]) \subset X_j \cup X^{i-1} - X^{i-1} \right\} \\
&= \mathrm{holink}\left(X_j \cup X^{i-1}, X^{i-1}\right)
\end{aligned}$$

"Part (ii)":

$$\begin{aligned}
&\mathrm{holink}\left(X_j \cup X^i, X^i\right)|_{X_i} \\
&= \left\{ \gamma : I \to X_j \cup X^i \;\middle|\; \gamma(0) \in X_i \text{ and } \gamma((0,1]) \subset X_j \cup X^i - X^i \right\} \\
&= \left\{ \gamma : I \to X_j \cup X_i \;\middle|\; \gamma(0) \in X_i \text{ and } \gamma((0,1]) \subset X_j \cup X_i - X_i \right\} \\
&= \mathrm{holink}\left(X_j \cup X_i, X_i\right)
\end{aligned}$$

□

Note, that the definition of "simple links" is modeled on holinks of the form (ii). If $X^i = B$ is the lowest non-trivial skeleton (thus a stratum) then also (i) is of this form.

Now, we can finally proof the vanishing result for spaces with simple links:

Proof of Lemma 3.1.11: We will use the notation of the previous Lemma 3.1.23. In this notation $p_i = p^i_{i-1}$, so we need to proof, that for any $i > 0$, $H^{\mathrm{lf}}_*(X^i; \mathcal{S}(p^{i+1}_i)) = 0$ in degrees $* \leq 1$. Actually, we show the slightly stronger statement

$$H^{\mathrm{lf}}_*(X^i; \mathcal{S}(p^j_i)) = 0 \quad \text{in degrees } * \leq 1 \text{ for all } j > i,$$

inductively over i, thus over skeleta.

First note, that we have already seen the two-stratum case, that

$$H^{\mathrm{lf}}_*(X_i; \mathcal{S}(p^j_i|_{X_i})) = 0 \quad \text{in degrees } * \leq 1 \text{ for all } j > i,$$

i. e. vanishing of restrictions to strata in Lemma 1.9.8. This was essentially a consequence of the coefficients (Lemma 1.9.10) vanishing under the simple links hypothesis.

For once, this gives the start of induction over the skeleta, because the lowest non-trivial skeleton is the lowest non-trivial stratum (say X_0), so $H^{\mathrm{lf}}_*(X_0; \mathcal{S}(p^i_0|_{X_0})) = 0$ in degrees $* \leq 1$, for all $i > 0$.

On the other hand we will use it in the inductive step: Assume the claim is true on the k-skeleton X^k, that is $H^{\mathrm{lf}}_*(X^k; \mathcal{S}(p^j_k)) = 0$ in degrees $* \leq 1$ for all $j > k$. Then, use that $\mathbb{H}^{\mathrm{lf}}(X^{k+1}; \mathcal{S}(p^l_{k+1}))$ is, for all $l > k+1$, a homology-theory in the sense of Def. 3.1.17 by Example 3.1.21, because of holinks of skeleta being stratified systems of fibrations (by Lemma 3.1.8, which applies by Example 3.1.7).

Hence by the restriction axiom, and the skeleton X^k being a p^l_{k+1}-NDR in the (pure in $X^{k+1} \cup X_l$) skeleton X^{k+1} (again by Lemma 3.1.8), there is a homotopy-fibration

$$\mathbb{H}^{\mathrm{lf}}_*(X^k; \mathcal{S}(p^l_{k+1}|_{X^k})) \xrightarrow{\mathrm{incl}_*} \mathbb{H}^{\mathrm{lf}}_*(X^{k+1}; \mathcal{S}(p^l_{k+1})) \xrightarrow{\mathrm{restr}} \mathbb{H}^{\mathrm{lf}}_*(X^{k+1} - X^k; \mathcal{S}(p^l_{k+1}|_{X^{k+1}-X^k}))$$

where $X^{k+1} - X^k = X_{k+1}$ is a stratum and (by Lemma 3.1.23 part (ii)) the two-stratum result (see above) applies to the last term, i. e. for $* \leq 1$:

$$\pi_*\Big(\mathbb{H}^{\mathrm{lf}}(X_{k+1}, \mathcal{S}(p^l_{k+1}|_{X_{k+1}}))\Big) = H^{\mathrm{lf}}_*(X_{k+1}, \mathcal{S}(p^l_{k+1}|_{X_{k+1}})) = 0$$

and in the first term, using $p^l_{k+1}|_{X^k} = p^l_k$ (Lemma 3.1.23, part (i)), the inductive hypothesis appears (for $* \leq 1$):

$$\pi_*\Big(\mathbb{H}^{\mathrm{lf}}(X^k, \mathcal{S}(p^l_k))\Big) = H^{\mathrm{lf}}_*(X^k, \mathcal{S}(p^l_k)) = 0$$

so that, by the diagram being a homotopy-fibration, we find that also (the homotopy-groups of) the middle-term vanish (for $* \leq 1$):

$$H^{\mathrm{lf}}_*(X^{k+1}; \mathcal{S}(p^l_{k+1})) = \pi_*\Big(\mathbb{H}^{\mathrm{lf}}(X^{k+1}; \mathcal{S}(p^l_{k+1}))\Big) = 0$$

This is exactly what we wanted to show. $\square$

Remark 3.1.24: The reader may have noticed the following: By the characterization-theorem, knowledge of coefficients in low degrees (Lemma 1.9.10) is already enough to fully determine the relevant (degree ≤ 1) obstruction-groups. The role that the finite structure spectrum plays is essentially only that of showing the existence of such a theory (extension to higher degrees is required to obtain a well-defined theory in this sense).

This immediately raises the question about uniqueness (in higher degrees), and indeed it turns out, that other suitable theories, for example controlled K-theory in the sense of [Qui04] exist, that are not equivalent to the finite structure spectrum in higher degrees.

One may use any such (suitable) theory, as it automatically yields the correct relevant (degree ≤ 1) obstruction-groups by the characterization theorem, see also [Qui02].

This result provides enough understanding of the obstruction-groups for our purposes. So we return to the discussion of further (multi-stratum) results.

The transversality theorem remains valid without (beyond formal) changes, leading to the evident inductive argument of the corollary below (for a sketch of the proof of the theorem given by [CV99, Thm. 2.2 (p. 529)], see also Thm. 1.6.4):

Theorem 3.1.25: *[CV99, Thm. 2.2 (p. 529)]: Let X be a MHSS of dimension n (without boundary), s. t.*

(i) $X^{n-1} = X_0^{n-2} \times (-1,1)$ *(for some closed MHSS X_0^{n-2}) and*

(ii) there is $p = (n,s) : \partial M' \to X^{n-1} = X_0^{n-2} \times (-1,1)$ *surjective, continuous, s. t.* $X = \mathrm{cyl}(p)$.

(M' is a manifold with boundary $\partial M'$ such that $X_n \subset M'$ and $M' - X_n \subset \partial M'$.) Set

$$\begin{aligned} V &:= \{ \quad [x,t] \in \mathrm{cyl}(p) \quad | \quad s(x) \leq t < 1 \quad \} \\ W &:= \{ \quad [x,t] \in \mathrm{cyl}(p) \quad | \quad 0 \leq t \leq s(x) \quad \} \\ U &:= \{ \quad [x,t] \in \mathrm{cyl}(p) \quad | \quad 0 \leq s(x) = t < 1 \quad \} \end{aligned}$$

with the induced stratifications. Then:

(a) U is a stratified (strong) deformation retract of both V and W

(b) $V \cup W = X$; $V \cap W = U$;
V and W are closed subsets of X;
The next-to-top skeleta are $V^{n-1} = X_0^{n-2} \times (-1,0]$, $W^{n-1} = X_0^{n-2} \times [0,1)$ *and* $U_B^{n-2} = X_0^{n-2} \times \{0\}$

(c) U, V and W are stratified subspaces of X (i. e. ∂-MHSS with the induced stratification), with boundaries $\partial V = \partial W = U$ and $\partial U = \emptyset$.

Corollary 3.1.25–1: *See also [CV99, Cor. 1.17 (p. 527)]:*[2] *Let X be a MHSS of dimension $n \geq 6$ (without boundary) with simple links and $g : X \to [-1, 1]$ a map, s. t. on the $(n-1)$-skeleton $g|_{X^{n-1}}$ is transverse to 0 at c_{n-1}.*

Then, given $\epsilon > 0$, there is $g_\perp \simeq_\epsilon g \operatorname{rel} X^{n-1} \cup g^{-1}\Big(\mathbb{R} - (-\epsilon, \epsilon)\Big)$ transverse to 0 at c_n with $c_n|_{X^{n-1}} = c_{n-1}$.

Proof of the corollary: X has simple links and thus vanishing end-obstruction groups by Lemma 3.1.11, so that $q_0(X^n, X^{n-1}) = 0$ in Thm. 3.1.9 which in turn (by $n \geq 6$), implies that there is a cylinder-neighborhood of $X^{n-1} \subset X^n$.

The remainder of the proof, is essentially the same as the one of Cor. 1.6.4–1: Apply Thm. 3.1.25 together with (the relative version of) Thm. 1.4.15, which is again applicable by $n \geq 6$, to extend c_{n-1} into the cylinder-neighborhood as c'_n. Define $g'_\perp$ on X as the projection to the cylinder-coordinate $\pi_\mathbb{R}(c'_n)^{-1}$ near X^{n-1} and interpolate to g near the complement of the cylinder-neighborhood of X^{n-1} (e. g. by using a helper-function $\eta : X \to [0, 1]$, mapping a neighborhood of X^{n-1} to 1 and a neighborhood of the complement of the cylinder to 0, to define $g'_\perp(x) := \eta(x)\pi_\mathbb{R}(c'_n)^{-1}(x) + (1 - \eta(x))g(x)$ within the cylinder neighborhood, and $g'_\perp = g$ outside).

Then, on the top-stratum $X_n = X - X^{n-1}$ (since $\dim(X) = n$, $X = X^n$ this is a single stratum) obtain $g^M_\perp \simeq g'_\perp|_{X-X^{n-1}}$ transverse to 0 rel a neighborhood of X^{n-1} in X intersected with this top-stratum at c_M, using manifold-transversality Cor. 1.5.9–1.

Finally, by $g^M_\perp = g'_\perp|_{X-X^{n-1}}$ on a neighborhood of X^{n-1}, this fits together with $g'_\perp|_{X^{n-1}} = g|_{X^{n-1}}$ and so does the bi-collar. Call this glued map $g_\perp := g^M_\perp \cup g|_{X^{n-1}}$, which is transverse to 0 at the glued bi-collar $c_n = c'_n \cup c_M$.

Smallness can also be achieved in the same way as in Cor. 1.6.4–1: When constructing $g'_\perp$, choose the complement of the cylinder-neighborhood (where $g'_\perp = g$) large enough, that $g'_\perp$ is close to g (or equivalently: choose η small enough; this works, because $g'_\perp = \pi_\mathbb{R}(c'_n)^{-1} = g$ on X^{n-1}). Further, when applying manifold-transversality, apply the size-controlling version thereof.

The homotopy $g \simeq g_\perp$ can be chosen as the straight-line homotopy in $\mathbb{R}$, which is small if g and $g_\perp$ are close, and $\operatorname{rel} X^{n-1}$ (where they agree). □

The corollary can serve as an inductive step, because for any MHSS X with simple links the n-skeleton X^n is again a MHSS with simple links. Since $X^{n-1} = \emptyset$ is allowed (this is really the manifold case), this result may also be used as the start of induction. However, manifold transversality does not require a dimensional hypothesis, so as long as the 5-skeleton is a disjoint union of manifolds (of potentially different dimensions), using manifold transversality directly as the start of induction gives slightly stronger results.

[2] The obstruction-theory used by the reference is also based on Quinn's [Qui82a], but is refined to get an "if and only if"-statement, rather than a "stable" (in our sense) condition.

With this formalization of the situation for more than two strata, the previous arguments mostly carry over. First, the situation of obstructions for the case of "cutting" MHSS with boundary is reviewed from the vantage point of spaces with more than two strata by absorbing the boundary into the stratification (Section 3.2 ("Absorbing Boundary")). The next section (Section 3.3 ("Special Gaps")) proceeds to give a slight modification to allow for a construction needed in Chapter 4 ("Homeomorphisms"). Then, the generalized homology theory defined for two strata in Section 2.4 ("An Example-Theory"), or rather, the underlying "strong t-class", is extended to the present context.

3.2 Absorbing Boundary

This sections analyzes, how a MHSS with boundary can be turned into a homeomorphic MHSS without boundary, and how a boundary can be "detected". This idea appears already in Quinn's original paper [Qui88a] on HSS. We focus on high (absolute) dimension, as this is the case we will need, as low (absolute) dimensions can / must be treated by other means: For the most part, the "gap" / codimension hypothesis on transversality classes makes sure, that low (absolute) dimension skeleta are manifolds (see Section 3.3 ("Special Gaps") for more details).

We start by defining what "absorbing the boundary" is supposed to mean precisely.

Definition 3.2.1: *See [Qui88a, p. 491]: Let X be a MHSS of dimension n with boundary (filtered by dimension). The* stratification with absorbed boundary $X_\#$ of X *is the stratification of the underlying space $|X|$ given by the filtration (by closed subsets)*

$$\begin{aligned} X^n_\# &= X^n && \text{(the total space)} \\ X^k_\# &= X^k \cup \partial(X_{k+1}) && \text{for } k \neq n \end{aligned}$$

thus with strata

$$\begin{aligned} (X_\#)_n &= X_n - \partial(X_n) && \text{(the top-stratum)} \\ (X_\#)_k &= X_k \cup \partial(X_{k+1}) - \partial(X_k) && \text{for } k \neq n \end{aligned}$$

Remark 3.2.2: By definition of MHSS with boundary, and stratifications being by dimension, $\partial(X_{k+1}) = (\partial X)_k$, i. e. the (manifold-)boundary of the $(k+1)$-stratum of X is the k stratum of the (MHSS-)boundary ∂X. Similarly for skeleta $\partial(X^{k+1}) = (\partial X)^k$ (as a MHSS this time).

This stratification does have reasonable properties:

Lemma 3.2.3: *Let X be a MHSS of dimension n with boundary.*

(i) The "stratification with absorbed boundary" $X_\#$ is an MHSS without boundary.

(ii) If X has simple links then $X_\#$ has simple links.

(iii) If X is proper (in the sense of Def. 1.4.16, i. e. the top-stratum is dense, and there is no codim 1 stratum), then X can be recovered from $X_\#$ in a natural way.

Proof: "(i)": To see, that $X_\#$ is an HSS, we need to check the defining conditions on *pairs* of components of strata. For pairs, where both meeting strata are in $X - \partial X$ or both in ∂X this is clear: $X - \partial X$ and ∂X is an MHSS by definition of MHSS with boundary.

For pairs $A_i \subset (X - \partial X)_i$ and $B_j \subset (\partial X)_j$, if $i \leq j$, then closure$(A_i) \subset X^i$ so that closure$(A_i) \cap B_j = \emptyset$ and closure$(B_j) \subset \partial X$ so that closure$(B_j) \cap A_i = \emptyset$ so all conditions are fulfilled trivially.

So it remain pairs $A_i \subset (X - \partial X)_i$ and $B_j \subset (\partial X)_j$, with $i > j$. Still closure$(B_j) \cap A_i \subset \partial X \cap A_i = \emptyset$, but $B_j \cap X^i$ may not be empty. Being an HSS is a local property, so we may restrict to a collar of the boundary, thus $A_i \subset (\partial X)_{i-1} \times [0, \infty)$ and $B_j \subset (\partial X)_j \times \{0\}$. The only case where $A_i, B_j \subset A_i \cup B_j$ are not disjoint connected components occurs if $j = i - 1$ and $(A_i \cup B_{i-1}, B_{i-1}) \subset (X_i, (\partial X)_{i-1})$ is a manifold with boundary.

In this case boundary-collaring for manifolds implies tameness, and the holink is holink$(B_j \times [0, \infty), B_j \times \{0\}) \to B_j \times \{0\}$ which is a fibration with holink-fiber $\simeq \{\text{pt}\}$. So this also proves part (ii), via Example 2.2.4 (simply connected links), because all other holink-fibers are those of pairs of strata in $X - \partial X$ or ∂X, so have vanishing $\mathrm{Wh}(\pi_1(L) \times \mathbb{Z}^k)$ by the simple links hypothesis on X.

Strata are manifolds (by construction) and compact domination of local homotopy-links is a consequence of boundary-collaring (of ∂X in X): For a point $x \in \partial X$, the local homotopy-link can be "pushed into" the corresponding local homotopy-link in ∂X by using this stratified boundary collar. Then by ∂X being a MHSS, it can be further deformed into a compact subset. For a point $x \in (X - \partial X)$ this (local property) is immediate from $X - \partial X$ being a MHSS.

"(iii)": Set $(X_n, \partial(X_n)) := ((X_\#)_n, (X_\#)_{n-1})$. Because, X did not have a codimension 1 stratum, this is the only possible choice. By Lemma 1.4.18, $\partial X =$ closure$(\partial(X_n))$ is uniquely determined by $\partial(X_n)$. Hence we can simply promote components of strata $(X_\#)_i \subset$ closure$(\partial(X_n))$ in this boundary back into X_{i+1} as manifold-boundaries. □

Corollary 3.2.3–1: *Extending*[3] *[CV99, Cor. 1.17 (p. 527)]: Let $(X, \partial X)$ be a MHSS of dimension $n \geq 6$ with boundary, with simple links and $g : X \to [-1, 1]$*

[3] Connolly and Vajiac where evidently aware of the possibility of absorbing boundary (as indicated above, this idea appears already in [Qui88a, p. 491]) to obtain a theorem for spaces with boundary [CV99, "if one wishes to apply this theorem to a manifold-stratified space with boundary, one should first absorb this boundary into the stratification", p. 523]. For their purposes – *finding*

a map, s. t. on the $(n-1)$-skeleton X^{n-1} and the boundary ∂X the restriction $g|_{X^{n-1}\cup\partial X}$ is transverse to 0 at c compatible with a restriction of a boundary collar b of $\partial X \subset X$ to one of $\partial(X^{n-1}) \subset X^{n-1}$.

Then, given $\epsilon > 0$, there is $g_\perp \simeq_\epsilon g \operatorname{rel} X^{n-1} \cup \partial X \cup g^{-1}\Big(\mathbb{R} - (-\epsilon, \epsilon)\Big)$ transverse to 0 at c', with $c'|_{X^{n-1}\cup\partial X} = c$ and compatible with the boundary collar b.

Proof: First, we show, that $g|_{X_\#^{n-1}}$ is transverse to 0 as a map on the stratification with absorbed boundary (see above). We start by the observation, that $X_\#^{n-1} = X^{n-1} \cup \partial X$. Further, we know, that $g_{X^{n-1}\cup\partial X}$ is transverse to 0 at c, which means, writing $L := g^{-1}(\{0\}) \cap (X^{n-1} \cup \partial X)$, there is a stratified bi-collar $c : L \times (-\epsilon', \epsilon') \to X^{n-1} \cup \partial X$ compatible with g, i. e. $gc(x,t) = t$. Now, the induced (from X) stratification of the underlying space $|X^{n-1} \cup \partial X|$ is coarser than (has strictly larger skeleta than) $X_\#^{n-1}$ (the identity $\mathrm{id} : X_\#^{n-1} \to X^{n-1} \cup \partial X$ is a coarsening Def. 1.3.6 (iii)). But the only difference is that boundaries of strata are now part of the next smaller stratum. Since c was compatible with the boundary collar $b|$ (actually the strictness requirement built into our definition of stratified transversality Def. 1.6.5 is already enough), we find $c((L \cap \partial X) \times (-\epsilon', \epsilon')) \subset \partial X$ and $c((L - \partial X) \times (-\epsilon', \epsilon')) \subset X^{n-1} - \partial X$, so this bi-collar c is already stratified with respect to $X_\#^{n-1}$.

Thus, g, as a map on $X_\#$ is transverse to 0 on $X_\#^{n-1}$ at c. At this point, we can apply the result without boundary Cor. 3.1.25–1, to obtain $g'_\perp \simeq g \operatorname{rel} X_\#^{n-1}$ transverse to 0 on $X_\#$ at c''.

Next, we glue an outside collar to X in the sense of Rmk. 1.6.7: Fix a (stratified w. r. t. X) homeomorphism $\nu : \partial X \times [0, \gamma) \to \partial X \times [-\gamma, \gamma), (y,t) \mapsto (y, 2t - \gamma)$, then define

$$g_\perp : X \to [-1,1], x \mapsto \begin{cases} g'_\perp(y) & \text{if } x \in \operatorname{im}(b) \text{ and } \nu b^{-1}(x) = (y,t) \in \partial X \times [-\gamma, 0] \\ g'_\perp(x) & \text{if } x \in \operatorname{im}(b) \text{ and } \nu b^{-1}(x) \in \partial X \times [0, \gamma) \\ g'_\perp(x) & \text{otherwise} \end{cases}$$

Then $g_\perp = g'_\perp$ on X^{n-1} (because g is compatible with b) and on ∂X (trivially, by $b(y,0) = y$). The same construction can be applied to c'' producing c'. The result $g_\perp$ is transverse to 0 at c' with respect to the stratification X, because it is on ∂X (where the induced stratification from $X_\#$ is the original one), and on the interior (where, again, both stratifications agree), and it is compatible with the boundary by the construction using an outside-collar.

Smallness can be achieved by producing a small $g'_\perp$ by the small version of Cor. 3.1.25–1, and by using a thin enough boundary-collar (small enough γ). As a (small) homotopy, the straight-line homotopy can be used (as always). □

Again, this can be used inductively: Start with manifold transversality (with boundary) on minimal strata, make the map transverse on the next higher skeleton

a boundary of an MHSS in the sense of end-theory – this aspect did not require a detailed treatment.

of the boundary by Cor. 3.1.25–1 (some care should be taken, as this may require $\dim(\partial X) \geq 6$, as opposed to $\dim(X) \geq 6$), then on the entire next skeleton by Cor. 3.2.3–1. As we are dealing with spaces with finitely many strata only, these contribute a finite number of small steps, so the construction can be made small (by making steps ϵ/N-small).

Before we continue by defining a transversality class, we first give a slight generalization of the gap condition, that will turn out to be useful in Chapter 4 ("Homeomorphisms").

3.3 Special Gaps

A general treatment of the "gap-hypothesis" (strata differing in dimension by at least 5), for example by transitioning to a pl or smooth theory on low (absolute dimension) skeleta might be possible, but seem rather hard to achieve technically (see Section 2.6 ("Improvements")).

Nevertheless, there are certain special cases, that can be included "trivially": For example, if "close" (in the sense of having a small dimension-gap) strata are (compatibly) pl, one could certainly still cut those. This is, however, *not* really helpful for treating intrinsically topological problems like (topological) homeomorphism-invariance (Chapter 4 ("Homeomorphisms")).

There is another, even topologically simple, special case: Manifold-boundaries are collared ([Bro62], cf. Thm. 1.1.2), so transversality on manifolds with boundary is not an issue (beyond "standard" manifold-transversality). Further manifolds with boundary are stable under products and cutting.

This can be used directly to incorporate "special gaps" into the theory: We first demonstrate this for spaces with three strata $X \supset B \supset B'$, where (B, B') is a manifold-with-boundary and $\dim(X) - \dim(B) \geq 5$.

Example 3.3.1: *Given a MHSS with boundary $(X, \partial X)$, with at most three strata $X \supset B \supset B'$, such that $(B, \partial B := (B \cap \partial X) \cup B')$ is a manifold-with-boundary, $B' \subset \partial B$ is a codimension 0 submanifold, $\dim(X) - \dim(B) \geq 5$ and X has simple links.*[4] *Automatically, by X being manifold-stratified, $(B - B', B \cap \partial X)$ is a manifold-with boundary, so this really means, that B' can be added to $B - B'$ as a disjoint (from $B \cap \partial X$) part of the boundary of a completion B of the end of $B - B'$.*

Then X can be cut, and further the resulting cut X_0 and the parts $X_{\leq}$ and $X_{\geq}$ of Def. 2.2.1 again satisfy the hypothesis of this example.

Proof: The corners $(B \cap \partial X) \cap B'$ can be straightened, as we are working in the topological category.

Cut the manifold-with-boundary $(B, \partial B)$, using manifold-transversality Cor. 1.5.9–2. The resulting cut $(B_0, \partial B_0)$ is again a manifold with boundary, and so are $B_{\geq}$ and

[4] The link of B' in B is automatically "simple", as its fiber is homotopy-equivalent to a point.

$B_{\leq}$. Next, extend the cut to ∂X using Cor. 3.1.25–1, then to X using Cor. 3.2.3–1. The result is again of the form $X_0 \supset B_0 \supset B_0'$, with $B_0' := B_0 \cap B'$ (and for $X_{\leq}$, $X_{\geq}$ correspondingly) satisfying the hypothesis of the example. □

Remark 3.3.2: Such spaces do have sufficiently many local approximate tubular neighborhoods (hence the Poincaré-duality and Witt-space treatment of [Fri09] applies, see Section 1.12 (“Witt-Condition and Signature of MHSS”)): The skeleton $B \subset X$ has a cylinder-neighborhood, by the vanishing of the end-obstruction-groups (as before), while $B' \subset B$, as a manifold-boundary, is collared, thus has a cylinder-neighborhood, too. Since a cylinder-neighborhood restricts to a local approximate tubular neighborhood, this establishes the claim.

This may not seem like a particularly useful result, but it has two important applications:

(a) Absorbing boundaries, for a treatment of spaces with boundary without relying on relative[5] end-theorems, see previous section. However, absorption of boundary is only used in high *absolute* dimensions, and “undone” before any potential further cuts, so the arguments in Section 3.4 (“Transversality-Class”) do *not* actually rely on this argument. This is why we will restrict our special treatment to only the bottom stratum.

(b) “Terminating” certain strata in stratifications of mapping-cylinders of homeomorphisms, see Chapter 4 (“Homeomorphisms”).

There is nothing special about having exactly *three* strata (see Section 3.4 (“Transversality-Class”)), it is really the relation between B and B' that is special.

3.4 Transversality-Class

We extend the Example 2.2.5, which gave transversality-conditions based on a “simple links” condition for spaces with at most two strata. For the multi-stratum case, Section 3.1 (“Background”) identified the “correct” – in the sense of giving suitable transversality results Cor. 3.1.25–1 and Cor. 3.2.3–1 – extension of what simple links should mean (Def. 3.1.10).

To ensure a well-defined notion of boundaries, we assume spaces are “proper” (Def. 1.4.16), i. e. have dense top-stratum and no codim 1 stratum.

Putting these together yields the following result:

Example 3.4.1: *With the methods introduced in this chapter we obtain the following transversality-classes:*

[5] These are available, see e. g. [Qui79, Thm. 2.1 (p. 282)], but are not particularly well-documented in the literature for cases of more than two strata.

(i) The class $\mathrm{Wh}_*^{\geq 6} \subset \partial$-MHSS *given by all compact, orientable, proper MHSS* X*, with boundary, with simple links and where at most two strata* $B \supset B'$ *are of dimension* < 6*, and (in that case are) such that* $B' = \emptyset$ *or* $(B, (B \cap \partial X) \cup B')$ *is a manifold with boundary,* $B' \subset (B \cap \partial X) \cup B'$ *is a codimension* 0 *submanifold, is a weak t-class.*

(ii) The class $\mathrm{Wh}_*^{\mathrm{gap}\,5} \subset \partial$-MHSS *given by all compact, orientable, proper MHSS* X *with boundary, with simple links and where for strata* M *and* B *that meet, i. e. those with* $B \cap \mathrm{closure}(M) \neq \emptyset$*, pairwise either*

   (a) $\dim(M) - \dim(B) \geq 5$ *or*

   (b) B *is a minimal stratum (it meets only higher-dimensional strata, cf. Def. 1.4.23) and* $(M \cup B, \partial M \cup B)$ *is a manifold with boundary,* $B \subset \partial M \cup B$ *is a codimension* 0 *submanifold.*

is a strong t-class.

Proof: Part (i): See also Example 3.3.1. Cut $(B, (B \cap \partial X) \sqcup B')$ using manifold-transversality Cor. 1.5.9–2. Next, inductively extend the cut over skeleta of ∂X using Cor. 3.1.25–1, and over skeleta of X using Cor. 3.2.3–1.

Part (ii): We start by analyzing the low-dimensional $X^5 \cup \partial(X_6)$ part.

First, we do so, for X^5. There can be at most two (meeting) strata in X^5: Either, there is a pair of bottom-strata of the form (b), in which case one of these two is at least of dimension 1, so the next (meeting) stratum must be of the form (a), thus have dimension $\geq 1 + 5 = 6$, hence is not in the 5-skeleton. Cut this pair using manifold-transversality Cor. 1.5.9–2. Or, if two meeting strata in X^5 are no such pair, in which case they must be of the form (a), thus differ in dimension by 5, which is possible only if one is of dimension 0 (and the other one is of dimension 5). In this case the arguments of Example 2.2.5 (the two-stratum case) apply, i. e. the lower stratum (which is a finite point-set) can essentially be ignored and manifold-transversality Cor. 1.5.9–2 solves the problem. In either case, the 5-skeleton can be cut.

If $\partial(X^6) \neq \emptyset$, then on these components of X^6, either X_6 and X_5 meet in the form (b), so X^6 can be cut as a manifold with boundary Cor. 1.5.9–2 (after straightening corners). Or, on these components $X^5 = \ldots = X^2 = \emptyset$. Further $\dim(\partial(X^1)) = 0$, so this is a finite point-set, thus can be essentially ignored (as in Example 2.2.5) for cutting the boundary $\partial(X^6)$ (which is a manifold away from these finitely many points). Similarly, cut X^1 away from the finite point-set $X^0 \cup \partial(X^1)$. This yields $g'_\perp$ transverse to 0 on $\partial(X^6) \cup X^1$. Apply Cor. 3.2.3–1 to extend this to the next skeleton X^6. The same argument is used (and additional details are provided) in the proof of Example 2.2.5 part (iii) in the case "$\dim(\partial X) = 5$".

Then continue, by inductively extending the cut over skeleta of ∂X (a space without boundary) using Cor. 3.1.25–1 and then over X using Cor. 3.2.3–1.

Finally, strata in $X_\geq, X_\leq$ and X_0 "meet" only if they come from strata in X, that meet in X and by construction they meet in the respective cases (a) or (b) again. Hence the cut is again in $\mathrm{Wh}_*^{\mathrm{gap}\,5}$. □

The important part of the statement is of course that $\mathrm{Wh}_*^{\mathrm{gap}\,5}$ is a *strong transversality-class*, thus the constructions of Chapter 2 ("Bordism Constructions") apply to yield a bordism-theory. This is further discussed in Chapter 5 ("The Main Theorem and its Applications").

4 Homeomorphisms

First, we briefly investigate stratified homeomorphisms, and their connection to gluing properties and definitions of bordism theories, in Section 4.1 (“Stratified Homeomorphisms”).

With a theory of spaces with multiple strata at hand, we can also study certain *unstratified* homeomorphisms, which we do for spaces with at most two strata (the bordism will need a third stratum). These are considerably simpler, not only formally, but also in the form of intrinsic stratifications, than the general case, which will become apparent in Section 4.2 (“Spaces With at Most Two Strata”).

The arguments both use stratifications of mapping-cylinders as bordisms. The construction is similar to the “half-intrinsic suspensions” of [Fri15, §4 (p. 14–19)] (in the pl-pseudomanifold-case). Also stratifications as MHSS of (more general) mapping-cylinders – although not as bordisms / spaces with (collared) boundary – have also been studied by e. g. [CS95; Hug99a].

4.1 Stratified Homeomorphisms

Many authors define a bordism $a \sim b$ by requiring an isomorphism (e. g. an orientation-preserving stratified homeomorphism in our setup) from the boundary pieces W_0, W_1 to a and b instead of equality (and compositions for F instead of restrictions).

For our setup, the choice of isomorphisms is “weak”, in that it conserves stratifications, but *no* (immediate) “compatibility data” (like cylinder-neighborhoods) between strata beyond homeomorphism-type. So it is not actually obvious that a formulation via isomorphisms on boundary pieces (see above) will work. In fact, for the classes of spaces considered here, it *does* work as a consequence of Prop. 4.1.1 below. This is discussed further in Rmk. 4.1.2.

All the defining properties of being in $\mathrm{Wh}_*^{\mathrm{gap}\,5}$ are stratified-homeomorphism-invariant, so one easily obtains:

Proposition 4.1.1: *Given $X, Y \in \mathrm{Wh}_*^{\mathrm{gap}\,5}$ of dimension n and an orientation-preserving stratified homeomorphism $h : X \to Y$, then the “fundamental-classes” $[X] := [X \xrightarrow{\mathrm{id}_X} X]$ are invariant in the sense of*

$$h_*([X]) = [Y] \in \Omega_n^{\mathrm{Wh}_*^{\mathrm{gap}\,5}}(Y)$$

If X, Y are additionally Witt, then equality holds in $\Omega_n^{\mathrm{Wh}\cap\mathrm{Witt}}(Y)$. Here $\mathrm{Wh}\cap\mathrm{Witt} := \mathrm{Wh}_*^{\mathrm{gap}\,5} \cap \mathrm{Witt}$ are the spaces in $\mathrm{Wh}_*^{\mathrm{gap}\,5}$, that are also Witt (Def. 1.12.5).

Proof: The mapping-cylinder $\operatorname{cyl}(h) \cong X \times I \cong Y \times I$ is homeomorphic to the actual cylinders, e. g. by $\operatorname{cyl}(h) \to X \times I, [x,t] \mapsto (x,t)$. As h is surjective, every element of $\operatorname{cyl}(h)$ can be written as $[x,t]$, since h is injective, this map is well-defined. Its inverse is simply given by $(x,t) \mapsto [x,t]$, and both are continuous by definition of the quotient topology. Similarly, $\operatorname{cyl}(h) \to Y \times I, [x,t] \mapsto (h(x),t)$ (the collapse-map plus the identity on I) is a homeomorphism (obviously compatible with the one to $X \times I$ fixed above via $X \times I \to Y \times I, (x,t) \mapsto (h(x),t)$). Define a stratification of $\operatorname{cyl}(h)$ by pulling back the product one from $X \times I$ (or, equivalently by h being stratified, from $Y \times I$) and call this the stratified mapping-cylinder.

This stratified mapping-cylinder $\operatorname{cyl}(h)$ is by construction orientation-preserving stratified homeomorphic to $X \times I$ which is an allowable bordism (X is allowable, and $\mathrm{Wh}_*^{\mathrm{gap}\,5}, \mathrm{Wh} \cap \mathrm{Witt}$ are stable). Since the property of being allowable depends only on topology and stratification, $\operatorname{cyl}(h)$ is allowable as a bordism.

Thus the collapse-map $p_h : \operatorname{cyl}(h) \to Y$, mapping $[x,t] \mapsto h(x)$ (again, since h is surjective, every element of $\operatorname{cyl}(h)$ can be written as $[x,t]$) provides a bordism $[\mathrm{id} : Y \to Y] \sim [h : X \to Y] = h_*([\mathrm{id} : X \to X])$. $\square$

This may seem a little tautological, the underlying problem is actually a bit more involved:

Remark 4.1.2: Instead of using a "subclass" $\mathcal{C}$ we may use a category with forgetful functor, as long as gluing etc. can be done *within that category*. For example, we could investigate spaces with a fixed cylinder-neighborhood of the lower stratum as part of the data. This complicates cutting and gluing (but under certain circumstances is feasible, because our Whitehead condition also makes h-cobordisms trivial; this is closely related to the treatment of stratified homeomorphisms above) but gives a "finer" theory: Given a stratified homeomorphism, the question of stratifiability of the cylinder, now becomes a question about uniqueness of cylinder-neighborhoods!

We know these are unique up to h-cobordism (Thm. 1.9.3), but for, say $X \supset B$, this only yields a MHSS $X \times I \supset B$. Or in the notion of MAFs (using Hughes' cylinder-theorem Thm. 1.4.19): For the initial (fixed) neighborhoods $\mathring{\operatorname{cyl}}(f : \partial M_f \to B)$ and $\mathring{\operatorname{cyl}}(g : \partial M_g \to B)$ there is a MAF $F : \partial M_F \to B$ which is a controlled h-cobordism from f to g. What we need is however a space with lower stratum $B \times I$ (to get boundaries of the correct dimension in the lower stratum to be a bordism). *If* the h-cobordism is trivial, i. e. $\partial M_F \cong \partial M_f \times I$ (controlled over B), then $F \times \mathrm{id}_I$ is a MAF, and defines (again by Thm. 1.4.19) a bordism with the "correct" lower stratum.

In high dimensions, our Whitehead-group requirements also make such h-cobordisms trivial. This may be an "accident", by having chosen too strong of a hypothesis. However, gluing also relies (implicitly) on this effect: The "strong" Whitehead-group hypothesis allows the condition we use to be formulated "pointwise" (on fibers), so gluing need only respect this local condition, which it does along a collared boundary.

A "weaker" hypothesis would probably have to rely on certain global properties of the lower stratum B. For example the existence of a cylinder (note that this alone leads at most to a weak t-class, as the cut may not have a cylinder-neighborhood) has an obstruction $q_0 \in H_0^{\mathrm{lf}}(B;\mathcal{S}(p))$ (see Chapter 3 ("Multiple Strata")), where $H_*^{\mathrm{lf}}(B;\mathcal{S}(p))$ behaves much like a homology-theory. However, for a condition that requires global properties of B, gluing becomes much more challenging (see also Example 2.6.2).

We also want to point out a particular problem, which arises when one tries to treat topological (stratified) homeomorphisms in a pl-space context, that we can avoid by working in a inherently topological setting. (For difficulties that arise when using geometric bundle hypotheses, see paragraph (a) of Section 2.6 ("Improvements").)

Example 4.1.3: *Given two stratified homeomorphic MHSS with two (high-dimensional) strata and simple links $h : X \xrightarrow{\simeq} Y$, such that X and Y are compatibly pl (the lower stratum is a subcomplex). This h need not be a pl-homeomorphism, actually h can already be non-pl when restricted to strata [Mil61b]. However, on manifolds this problem is well-understood, and this is not the issue we are currently interested in. Rather, we want to see, what kind of new issues may arise through the compatibility of strata.*

Suppose a compatibly pl stratification of $\mathrm{cyl}(h)$ *does exist. Then by the pl-refinement of the (relative) end-theorem (Thm. 1.9.3, part (iv)), we can extend pl-cylinder-neighborhoods of the lower strata in X and Y to the cylinder (by $X \times I \cong \mathrm{cyl}(h)$ having simple links). We further specialize to isolated singularities (lower strata being points), in which case the normal structure in X and Y near the lower stratum is given by the choice of manifold-boundary $\partial M'_X$ and $\partial M'_Y$ of the upper stratum (the cylinder-neighborhood is the cone / the cylinder of the unique map to the point). The pl-cylinder-neighborhood in* $\mathrm{cyl}(h)$ *consists of a manifold-boundary $\partial M'_{\mathrm{cyl}}$ and a map to $\{\mathrm{pt}\} \times I$, which restricts to $\partial M'_X$ and $\partial M'_Y$ at 0 and 1. This $\partial M'_{\mathrm{cyl}}$ is a pl-s-cobordism, because it is a topological product, its (topologically invariant) Whitehead-torsion vanishes (cf. e. g. [KS77, Thm. 4.1 (p. 25)]), so it is a pl-product, and $\partial M'_X \cong_{\mathrm{PL}} \partial M'_Y$, i. e. X and Y have the same (pl) normal structure, suggesting, that h was already pl (if it was pl on strata).*

Of course, the mapping-cylinder of h is only one possible choice for a bordism, but it is not clear, how a non-pl-homeomorphism would otherwise induce a bordism, probably other ideas would be required.

So the pointwise / stratified-homeomorphism-invariant formulation of our Whitehead condition is actually pivotal to the realization of the theory, as becomes particularly apparent from the "gluing issues" discussed in Rmk. 4.1.2.

4.2 Spaces With at Most Two Strata

The construction of Quinn's [Qui87] "intrinsic skeleta" $X_{0,0}$ (see Section 1.10 ("Intrinsic Stratifications")) leads to an interesting observation: The space $X_{0,0}$ is obtained from X by "promoting" connected components into higher strata. If $X \supset B$ has only two strata, we can treat components of B separately, hence may assume B connected. Then there are only two possibilities: $X_{0,0}$ is a manifold – the (only) one component of the (only) lower stratum B was "promoted" into the top-stratum – or $X_{0,0} = X$ remains unchanged – if the (only) one component of the (only) lower stratum B was *not* "promoted". More formally:

Lemma 4.2.1: *Let $X \supset B$ be a MHSS with a most two strata. Then the intrinsic stratification $X_{0,0}$ is given as $X \supset B'$, where $B' \subset B$ is open and closed (i. e. consists of connected components of B).*

Let $X \supset B$ be a MHSS with a most two meeting strata (Def. 1.4.22). Then such a B'_j exists for each of the disjoint minimal strata B_j.

Proof: By Thm. 1.10.2, the identity $\mathrm{id} : X \to X_{0,0}$ is a coarsening, meaning, the preimage of the component of a stratum of $X_{0,0}$ is a union of components of strata of X. Applied to the top-stratum, this implies $B' \subset B$. Applied to B', it implies, that components of B' are unions of components of strata of X, which by the above must lie in B, so are components of B.

For a MHSS with at most two meeting strata, the same argument applies to disjoint open neighborhoods of the (disjoint closed) minimal strata B_j (in the metric, thus normal X). □

Corollary 4.2.1–1: *Let X be a MHSS with a most two meeting strata. Homotopy-links (also stratified and local ones) that appear in $X_{0,0}$ also appear as ones in X.*

If X is Witt, then $X_{0,0}$ is Witt.

Proof: There are fewer (meeting) strata in $X_{0,0}$, so there are fewer (local / stratified) holinks to check on, those, that remain, are already (local / stratified) holinks in X. □

Since homeomorphic MHSS $h : |X| \to |Y|$ have the same (stratified homeomorphic via h) "intrinsic skeleton" $X_{0,0} \cong_{\text{strat}} Y_{0,0}$, and stratified homeomorphisms have been described in Prop. 4.1.1, to show $h_*([X]) = [Y] \in \Omega_*(Y)$ – for some suitable bordism-theory Ω_* and if "fundamental classes" $[X], [Y]$ exist – it remains only to understand the relation between X and $X_{0,0}$. Further, the case $X_{0,0} = X$ is clear, so that the problem can be reduced to understanding how, for $X \supset B$ with B connected and $|X|$ a manifold, X can be seen to be bordant to the filtration $X_{0,0} = (|X| \supset \emptyset)$.

Similar intrinsic stratifications are available for pl-pseudomanifolds, and have been used for similar constructions by [Fri15] and [BLM19].

Before going into the details of this case in Lemma 4.2.3, we briefly note the following technical result:

Lemma 4.2.2: *If X is a metric space, $B \subset X$, then* $\operatorname{holink}(X,B)$ *is metrizable.*
If X is further separable, then $\operatorname{holink}(X,B)$ *is paracompact.*
If $X \in$ MHSS, $B \subset X$, then $\operatorname{holink}(X,B)$ *is metrizable and paracompact. In particular, this implies, that partitions of unity exist.*

Proof: Because I is compact and Hausdorff, and X is metric, the compact-open-topology on X^I is induced by the uniform metric (by the supremum of pointwise distances), see e. g. [Bre97, Thm. VII.2.12 (p. 440)]. Hence the subspace $\operatorname{holink}(X,B) \subset X^I$ is also metrizable.

Further if X is separable, i. e. there is a countable dense subset $\tilde{X} \subset X$, then $\cup_{n\in\mathbb{N}} \tilde{X}^{\{0,\frac{1}{n},\ldots,\frac{n-1}{n},1\}}$ is countable, and when included into X^I (by linear interpolation), is dense. Hence X^I is separable. Subspaces of separable metric spaces are separable (and of course metric), so $\operatorname{holink}(X,B) \subset X^I$ is also separable. By Stone's Theorem $\operatorname{holink}(X,B)$ is thus paracompact.

Finally, $X \in$ MHSS implies, by definition, that X is both metric and separable. Metric spaces are Hausdorff, and paracompact Hausdorff spaces admit partitions of unity subordinate to arbitrary open covers. □

The main line of argument starts with the important special case outlined above. The "bordism" constructed has strata, whose dimensions differ only by one, however they have the form of a manifold with boundary.

Lemma 4.2.3: *Let X be a closed MHSS with at most two meeting strata and* $\dim(X) \geq 5$*. Then there is a MHSS bordism W from X to $X_{0,0}$. This bordism W has the underlying topological space $|W| = |X \times I|$.*
The homotopy-links of W have fibers that are either homotopy-equivalent to the fiber of a homotopy-link of X or to a point.
Further W has strata in dimensions $\dim(X)+1$, $\dim(B_j)+1$ *and* $\dim(B_j)$*, with the latter ones disjoint for different minimal strata B_j, and for the same minimal stratum B_j such that their union is a manifold with boundary (cf. Section 3.3 ("Special Gaps")).*
If X is Witt, then W is Witt.

Proof: By Thm. 1.10.2 and $\dim(X) \geq 5$, the intrinsic stratification $X_{0,0}$ has manifold strata, thus is a MHSS. By Lemma 4.2.1, $X_{0,0}$ there are B'_j with $B'_j \subset B_j$ a union of components of B_j. Define, using $B := \cup_j B_j$ and $B' := \cup_j B'_j$,

$$W \quad = \quad X\times[0,1] \quad \supset \quad \Big((B-B')\times[0,1/2] \quad \cup \quad B'\times[0,1]\Big) \quad \supset \quad (B-B')\times\{1/2\}$$

with boundary $\partial W = X \times \{0\} \sqcup (-X_{0,0}) \times \{1\}$, where the orientation-part can be seen as follows: The top-stratum (other than B) is not separated by a codimension

1 stratum, so has the expected (uniquely fixed by X) orientation. Further, this is evidently a filtration by closed subsets, and the (open) left and right halves $X \times [0, {}^1\!/\!_2)$ and $X_{0,0} \times ({}^1\!/\!_2, 1]$ are open subsets of the MHSS $X \times I$ and $X_{0,0} \times I$, thus MHSS, with holinks those of X (cf. Cor. 4.2.1–1 in the case of $X_{0,0}$). Additionally, given an open neighborhood U' of B' with $U' \cap B \subset B'$, then $U' \times I$ is a MHSS (because $U' \subset X$ is open in the MHSS X) and a neighborhood of $B' \times I$ in W (also with "correct" holinks).

The conditions that have to be checked for the lemma (MHSS and form of holinks) are local, so it is only left to check everything near $(B - B') \times \{{}^1\!/\!_2\}$. Again by locality, this can be done independently near each component of $(B - B')$.

For the remainder of the proof, we will thus assume, that $W = X \times [0, 1] \supset B \times [0, {}^1\!/\!_2] \supset B \times \{{}^1\!/\!_2\}$ with B connected, and $X_{0,0} =: N$ a manifold, since the remaining part of the proof can always be reduced to this form (see above).

Here, it remains to check – for all (combinations of) strata – that: Strata are manifolds, tame and homotopy-links are fibrations with fibers L corresponding either to those in X or trivial.

Step 1: The strata of W are manifolds.

The top-stratum is an open subset of the manifold $W \cong N \times I$, hence a manifold. The middle-stratum $B \times [0, {}^1\!/\!_2)$ is a manifold because B is. So is the bottom-stratum $B \times \{{}^1\!/\!_2\}$.

Step 2: The following subsets are tame:

(i) $B \times [0, {}^1\!/\!_2) \subset X \times I - B \times \{{}^1\!/\!_2\}$

(ii) $B \times \{{}^1\!/\!_2\} \subset B \times [0, {}^1\!/\!_2]$

(iii) $B \times \{{}^1\!/\!_2\} \subset X \times I - B \times [0, {}^1\!/\!_2)$

(i) By hypothesis X is a MHSS, thus there is an open neighborhood $U \subset X$ of $B \subset X$, which deformation-retracts nearly-strictly to B by $R : U \times I \to U$. Define $U' := U \times [0, {}^1\!/\!_2) \subset X \times I - B \times \{{}^1\!/\!_2\}$, an open neighborhood of $B \times [0, {}^1\!/\!_2) \subset X \times I - B \times \{{}^1\!/\!_2\}$. Then $R' : U' \times I \to U'$ given by $R \times \mathrm{id}_{[0, {}^1\!/\!_2)}$ is a nearly-strict deformation retraction to $B \times [0, {}^1\!/\!_2)$ as required.

(ii) Let $U' := B \times ({}^1\!/\!_4, {}^1\!/\!_2]$ and $R' : U' \times I \to U'$, $\big((x, s), t\big) \mapsto \big(x, (1 - t)s + {}^t\!/\!_2\big)$.

(iii) Let U and R as in (i). Let

$$
\begin{aligned}
&U' := U \times ({}^1\!/\!_4, {}^3\!/\!_4) - B \times ({}^1\!/\!_4, {}^1\!/\!_2) \quad \text{and} \\
&R' : U' \times I \to U', \big((x, s), t\big) \mapsto \big(R_t(x), (1 - t)s + {}^t\!/\!_2\big).
\end{aligned}
$$

Then R' is well-defined, because R is nearly-strict, and it deforms U' into $B \times \{{}^1\!/\!_2\}$. It is nearly-strict, because for any $t < 1$, in the case $x \notin B$ nearly-strictness of R implies $R'_t(x, s) \notin B \times ({}^1\!/\!_4, {}^3\!/\!_4)$ while in the case of $x \in B$ (and $(x, s) \notin B \times \{{}^1\!/\!_2\}$), by the form of U', $s > {}^1\!/\!_2$, hence $(1 - t)s + {}^t\!/\!_2 > {}^1\!/\!_2$.

For the question of holinks being fibrations, Suppose we are given a lifting-problem of the form

$$\begin{tikzcd}
A \times \{0\} \arrow[r, "F_0"] \arrow[d, "\mathrm{incl}"'] & \mathrm{holink}(\dots, \mathrm{stratum}) \arrow[d, "\mathrm{ev}_0"] \\
A \times I \arrow[r, "f"] \arrow[ur, dotted, "F"] & \mathrm{stratum}
\end{tikzcd}$$

for the individual choices of subsets / strata indicated below.

Step 3: $\mathrm{holink}\Big(X \times I - B \times \{1/2\}, B \times [0, 1/2)\Big) \xrightarrow{\mathrm{ev}_0} B \times [0, 1/2)$ is a fibration with fiber $F \simeq F_X$ where F_X is the fiber of $\mathrm{holink}(X, B)$.

We want to use the hypothesis that $X \in \mathrm{MHSS}$ to lift the X-coordinate, to then "trivially" construct lifts in the I-coordinate. However, generally some of the $F_0(a)(t)$ (since $F_0(a) \in$ holink, it is a path, t denotes the path-coordinate) may reach into the "right-hand-side", where their first coordinate may take values in B for some $t \neq 0$, so do not "project" to well-defined elements of $\mathrm{holink}(X, B)$ as is. We us an approach similar to Quinn's ϵ-holinks – or rather his proof of their equivalence to the ordinary ones [Qui88a, Proof of Lemma 4.2 (p. 454)] – to alleviate this complication: Start by defining, for $\gamma \in$ holink a set

$$I_\gamma := \quad \{ \quad t \in I \quad | \quad \gamma(t) \in X \times [1/2, 1] \quad \}$$

and a number $\epsilon_\gamma := 1/2 \min(I_\gamma)$ if $I_\gamma \neq \emptyset$, $\epsilon_\gamma := 1$ otherwise. The minimum is defined, because $\gamma^{-1}(X \times [1/2, 1])$ is closed in the compact I. Further, $\epsilon_\gamma > 0$, because $\gamma \in$ holink implies $\gamma(0) \in B \times [0, 1/2)$ and $\gamma^{-1}(X \times [0, 1/2))$ is open. Note also, that by construction

$$\gamma \in M_{\epsilon_\gamma} := \quad \Big\{ \quad \eta \in \mathrm{holink} \quad \Big| \quad \eta([0, \epsilon_\gamma]) \subset X \times [0, 1/2) \quad \Big\},$$

which is an open subset of holink by definition of the compact-open-topology. So $\{M_{\epsilon_\gamma}\}_{\gamma \in \mathrm{holink}}$ is an open cover of holink. By paracompactness of the holink (Lemma 4.2.2), there is a locally finite sub-cover, say indexed by α, and a partition of unity θ_α (subordinate to that cover). Hence we may define (pointwise, the sum has only a finite number of non-zero summands):

$$\epsilon : \mathrm{holink} \to (0, \infty), \gamma \mapsto \sum_\alpha \theta_\alpha(\gamma)\epsilon_\alpha$$

By the choice of the M_γ in the cover, $\gamma([0, \epsilon(\gamma)]) \in X \times [0, 1/2)$, because ϵ is smaller than the maximum of the finitely many ϵ_α with $\gamma \in M_{\epsilon_\alpha}$, and we only need to find one, because all will satisfy the bounds. So there is a well-defined map

$$\pi : \mathrm{holink} \to \mathrm{holink}(X, B), \gamma \mapsto \Big(t \mapsto \pi_X \circ \gamma(t\epsilon(\gamma))\Big).$$

Because $\mathrm{holink}(X, B)$ is a fibration by hypothesis, and $\pi(F_0(a))(0) = \pi_B \circ f(a, 0)$ (as π does not change the B-coordinate at 0), there is a lift $G : A \times I \to \mathrm{holink}(X, B)$

of $\pi_B \circ f$ starting at $\pi \circ F_0$, i. e.

$$G(a,s)(0) = \pi_B \circ f(a,s)$$
$$G(a,0)(t) = \pi\big(F_0(a)\big)(t) = \pi_X \circ F_0(a)\big(t\epsilon(F_0(a))\big)$$

Use this to define $F : A \times I \to \text{holink}$ (see Figure 4.1) with X-coordinate

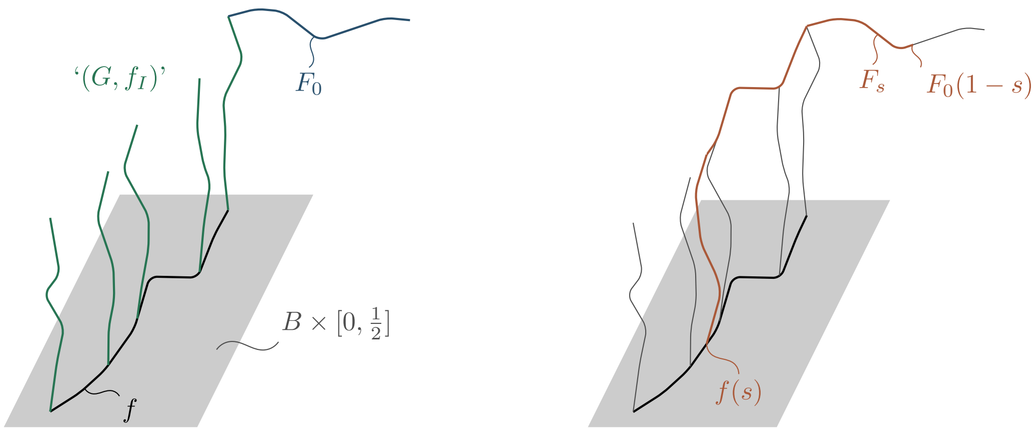


Figure 4.1: Sketch of the construction of the lift F in case (i). Here, A is drawn as a point. F_0 may "puncture" the B-plane to the right of $1/2$.

$$F_X(a,s) = t \mapsto \begin{cases} G\big(a,s\big)\big(t/\epsilon(a)\big) & \text{if} \quad t \in [0, \epsilon(a)] \\ G\big(a, s-t+\epsilon(a)\big)\big(1\big) & \text{if} \quad t \in [\epsilon(a), \epsilon(a)+s] \\ \pi_X \circ F_0(a)(t-s) & \text{if} \quad t \in [\epsilon(a)+s, 1] \end{cases}$$

where $\epsilon(a) := \epsilon(F_0(a))$, and I-coordinate

$$F_I(a,s)(t) = \begin{cases} \psi\Big(\quad \pi_I \circ f(a,s), \quad \pi_I \circ F_0(a)(t-st') \quad \Big) & \text{if} \quad t \in [0, \epsilon(a)+s] \\ \pi_I \circ F_0(a)(t-s) & \text{if} \quad t \in [\epsilon(a)+s, 1] \end{cases}$$

where, the terms t', τ and ψ are given by

$$t' := \frac{t}{\epsilon(a)+s} \qquad \tau := \frac{t'-s}{t'+s} \qquad \psi(x,y) = \Big(\frac{1-\tau}{2}x + \frac{1+\tau}{2}y\Big)\,(1-t') + yt'$$

which is chosen, s. t. for $t = 0$ it is $\tau = -1$ and $t' = 0$, thus $\psi(x,y) = x$, and for $s = 0$ it is $\tau = +1$, thus $\psi(x,y) = y$. Finally, for $t' = 1$, clearly $\psi(x,y) = y$. With the choices of x, y as above, they fit together at $s = 0 = t$ (because $f(a,0) = F_0(a)$), so that the singularity of τ is not an issue. We may assume $\psi(x,t) \in [0,1]$, otherwise replace ψ by $\max(\min(\psi,1),0)$.

We will first validate, that F is continuous, then, that F is well-defined as a mapping into the holink, and finally, that it is indeed a solution of the initial lifting-problem.

"Continuity of the X-coordinate": At $t \to \epsilon$:

$$\begin{aligned}&\text{"Case 1":} \quad G(a,s)(\epsilon/\epsilon) = G(a,s)(1)\\ &\text{"Case 2":} \quad G(a,s)(a, s-\epsilon+\epsilon)(1) = G(a,s)(1)\end{aligned}$$

while at $t \to \epsilon + s$:

$$\begin{aligned}&\text{"Case 2":} \quad G(a,s)(a, s-(\epsilon+s)+\epsilon)(1) = G(a,0)(1) = \pi_X \circ F_0(a)(\epsilon)\\ &\text{"Case 3":} \quad \pi_X \circ F_0(a)((\epsilon+s)-s) = \pi_X \circ F_0(a)(\epsilon)\end{aligned}$$

"Continuity of the I-coordinate": At $t \to \epsilon + s$, i. e. $t' \to 1$ thus $\psi = y$:

$$\begin{aligned}&\text{"Case 1":} \quad \pi_I \circ F_0(a)((\epsilon+s)-s1) = \pi_I \circ F_0(a)(\epsilon)\\ &\text{"Case 2":} \quad \pi_I \circ F_0(a)((\epsilon+s)-s) = \pi_I \circ F_0(a)(\epsilon)\end{aligned}$$

"F maps into holink": For $t \in (0, \epsilon+s]$, the X coordinate $F_X(a,s)(t)$ is given by $G(\ldots)(t>0) \in X - B$, because $\mathrm{im}(G) \subset \mathrm{holink}(X,B)$. Thus F is not in $B \times [0, 1/2]$ for these t. For $t \in [\epsilon+s, 1]$, $F(a,s)(t) = F_0(a)(t-s)$, which, by $F_0(a) \in$ holink and $t - s \geq \epsilon + s - s > 0$ is not in $B \times [0, 1/2]$. Finally, the case $t = 0$, will follow from $F(a,s)(0) = f(a,s) \in B \times [0, 1/2)$ being a lift as shown below.

"F is a solution of the lifting-problem (X-coordinate)": For $t \to 0$:

$$F_X(a,s)(0) = G(a,s)(0/\epsilon) = G(a,s)(0) = \pi_B \circ f(a,s)$$

For $s \to 0$:

$$\begin{aligned}F_X(a,0)(t) &= \begin{cases} G(a,0)(t/\epsilon) = \pi_X(F_0(a)(t/\epsilon\epsilon)) & \text{if } t \leq \epsilon \\ \pi_X(F_0(a)(t)) & \text{if } t \geq \epsilon \end{cases}\\ &= \pi_X \circ F_0(a)(t)\end{aligned}$$

"F is a solution of the lifting-problem (I-coordinate)": For $t \to 0$ (by choice of ψ):

$$F_I(a,s)(0) = \pi_I \circ f(a,s)$$

For $s \to 0$:

$$\begin{aligned}F_I(a,0)(t) &= \begin{cases} \pi_I \circ F_0(a)(t - 0t') & \text{if } t \leq \epsilon + s \\ \pi_I \circ F_0(a)(t-0) & \text{if } t \geq \epsilon + s \end{cases}\\ &= \pi_I \circ F_0(a)(t)\end{aligned}$$

So F solves the original lifting problem.

By Quinn's ϵ-holink construction, holink $\simeq_{\text{fibered}}$ $\text{holink}^\delta = \text{holink}(X \times I, B \times I)^\delta \simeq_{\text{fibered}} \text{holink}(X \times I, B \times I)$ for a suitable δ, e. g. $\delta(b,s) = 1/2 - s$, the statement about the fiber follows from $\text{holink}(X \times I, B \times I)$ having the same (homotopy-type of) fiber as $\text{holink}(X,B)$. This finishes Step 3 of the proof.

Step 4: holink $\big(B \times [0, 1/2], B \times \{1/2\}\big) \xrightarrow{\mathrm{ev}_0} B \times \{1/2\}$ is a fibration with fiber $F \simeq \{\mathrm{pt}\}$.

This is just a (much) simpler version of (i): Define the path $F(a,s)$ as starting at $f(a,s)$, moving away from B in the I-coordinate for a "time" proportional to s, then move back along f (in a plane $B \times \{\text{something} \propto s\}$ parallel to $B \times \{1/2\}$) and then continue "parallel" to the original F_0, possibly rescaling the value in the I-coordinate (proportionally to s) to avoid values < 0 (cf. (iii) below). The equivalence $F \simeq \{\mathrm{pt}\}$ follows by the same argument as the final step of (iii) below, in fact we can use the same deformation, with the I-coordinate mirrored at $1/2$.

Step 5: holink $\big(X \times I - B \times [0, 1/2), B \times \{1/2\}\big) \xrightarrow{\mathrm{ev}_0} B \times \{1/2\}$ is a fibration with fiber $F \simeq \{\mathrm{pt}\}$.

We employ the intuition, that F_0 may be "folded around" $B \times [0, 1/2)$, but cannot actually intersect it. Hence we can "untangle" it from the left side, simply by pushing along the I coordinate to the right. This leads to the following definition (whose properties are checked below):

$$F(a,s)(t) := \begin{cases} \Big(\pi_B f(a,s), \quad \frac{1}{2} + \frac{t}{1+4s} \Big) & \text{if} \quad t \in [0,s] \\ \Big(\pi_B f(a, 2s-t), \quad \frac{1}{2} + \frac{s}{1+4s} \Big) & \text{if} \quad t \in [s, 2s] \\ \Big(\pi_X \circ F_0(a)(t-2s), \quad \frac{1}{2} + \frac{\pi_I \circ F_0(a)(t-2s) + s - 1/2}{1+4s} \Big) & \text{if} \quad t \in [2s, 1] \end{cases}$$

where, to ensure that the second coordinate stays within the interval I, we formally replace it by $\min(1, \pi_I F)$. This does not change continuity, well-definedness or the property of being a lift.

Then F is a well-defined lift (see below) and continuous: Continuity of the individual pieces, and at $t = s$ follows from continuity of f (and $t = s$ being interchangeable at that point). At $t = 2s$ pieces fit together by $f(a,0) = \mathrm{ev}_0 \circ F_0(a) = F_0(a)(0)$ for the X-coordinate and by $\pi_I F_0(a)(0) = 1/2$ for the I-coordinate. Clearly $F(a,s)(0) = f(a,s)$, because $\pi_I f \in \{1/2\}$, and $F(a,0) = F_0(a)$, so this is a lift.

For well-definedness, it is to check that F indeed maps into the holink. First, $F(a,s)(0) \in B \times \{1/2\}$ is clear (this is just $f(a,s)$, see above), so we need to show $t > 0 \Rightarrow F(a,s)(t) \notin B \times [0, 1/2]$. For the first two cases, the second coordinate is always $> 1/2$ and this is evidently true, except, possibly, if $s = 0$. But for $s = 0$, automatically $t \in [2s, 1]$, so it is enough to check the third case.

In the third case, we know $F_0(a) \in$ holink thus $F_0(a)(t > 0) \notin B \times [0, 1/2]$. This implies $\big(\pi_I F_0(a)(t) \leq 1/2 \Rightarrow \pi_X F_0(a)(t) \notin B\big)$. Since, the I-coordinate of F is always larger than the one of some F_0, in the sense of $\pi_I F(a,s)(t) \geq \pi_I F_0(a)(t-2s)$ it follows $1/2 \geq \pi_I F(a,s)(t) \geq \pi_I F_0(a)(t-2s) \Rightarrow \pi_X F(a,s)(t) = \pi_X F_0(a)(t-2s) \notin B$ and hence $F(a,s)(t > 0) \notin B \times [0, 1/2]$.

Finally, we have the check, that F is indeed a solution to the lifting problem: This requires, that $\mathrm{ev}_0 \circ F = f$, which is clearly the case, as setting $t = 0$ in the definition of F implies, we are always in the first case, and $f : A \times \{0\} \to B \times \{1/2\}$ certainly maps the second coordinate to $1/2$. On the other hand, we need to check, that the lift starts at F_0, i. e. for $s = 0$ we should find $F(a, 0) = F_0(a)$. Here, we are always in the last case, and the X-coordinate evidently is correct. Further $s = 0$ implies that the denominator $1 + 4s = 1$, thus the second coordinate is $\pi_I \circ F_0$ as required.

Next, we analyze the fiber of this fibration. Let $b_0 \in B$ and $F_{b_0} := \mathrm{ev}_0^{-1}(\{(b_0, {}^1\!/\!_2)\})$. First, deform this into

$$D := \left\{ \quad \gamma \in F_{b_0} \quad \middle| \quad (x, s) \in \mathrm{im}(\gamma) \Rightarrow \min\Big(\mathrm{d}(x, B), {}^1\!/\!_4\Big) \le s - {}^1\!/\!_2 \quad \right\}$$

by $R : F_{b_0} \times I \to F_{b_0}, (\gamma, u) \mapsto \gamma'_u$, where

$$\gamma'_u(t) := \Big(\pi_X \circ \gamma(t) \quad , \quad \pi_I \circ \gamma(t)(1-u) + u\Big(\min\Big(\mathrm{d}(\pi_X \circ \gamma(t), B), {}^1\!/\!_4\Big) + {}^1\!/\!_2\Big)\Big)$$

Again, where the second coordinate is $\le {}^1\!/\!_2$, it is larger than $\pi_I\gamma(t)$, and $\gamma \in$ holink implies $\gamma'_u \in$ holink as above. Clearly $\gamma'_0 = \gamma$, i. e. $R_0 = \mathrm{id}$ and $\gamma'_1 \in D$, i. e. $\mathrm{im}\, R_1 \subset D$.

Note, that by construction for $\gamma \in D$ it holds that $\pi_I\gamma(t) \le {}^1\!/\!_2 \Rightarrow \gamma(t) \in B \times \{{}^1\!/\!_2\} \Rightarrow t = 0$.

By hypothesis $B \subset X$ is tame, so there is $R^X : X \times I \to X$ retracting X into B rel B.[1] We use this to further deform D into

$$D' := \left\{ \quad \gamma \in F_{b_0} \quad \middle| \quad \gamma\Big((0, 1]\Big) \subset B \times \Big({}^1\!/\!_2, 1\Big] \quad \right\}$$

by composition $R' : D \times I \to D, (\gamma, u) \mapsto \Big(R^X_u \circ \pi_X \circ \gamma, \pi_I \circ \gamma\Big)$.

From here, the final step is to deform D' into a single point $\{t \mapsto (b_0, {}^1\!/\!_2 + t\,{}^1\!/\!_4)\}$. Let $R'' : D' \times I \to D', (\gamma, u) \mapsto \gamma''_u$, where

$$\gamma''_u(t) := \Big(\quad \pi_X \circ \gamma\Big(t(1-s)\Big) \quad , \quad \pi_I \circ \gamma(t)(1-s) + \Big({}^1\!/\!_2 + {}^t\!/\!_4\Big)s \quad \Big)$$

As noted above $\pi_I\gamma(t) \le {}^1\!/\!_2 \Rightarrow t = 0$, so this stays in D'.

In conclusion, we have constructed $F = F_{b_0} \simeq_R D \simeq_{R'} D' \simeq_{R''} \{\mathrm{pt}\}$ as required. Finally, we also note, that the homotopy-equivalence $F_{b_0} \simeq \{\mathrm{pt}\}$ is continuous in b_0 (the only part in the composition, that depends on b_0 is R', which clearly is continuous in b_0), so we actually have constructed a fiber-homotopy-equivalence of the homotopy-link fibration to the identity on B. This finishes Step 5 of the proof.

Finally, it only remains to check, that W is Witt if X is Witt. This can be checked locally (as it only depends on local holinks). Because $|X - B'| = |X_{0,0} - B'|$ is the top-stratum of the MHSS (see above) $X_{0,0}$, it is (homeomorphic to) a manifold, so it is Witt by Example 1.12.8, as W has no codimension 1 stratum (because the Witt space X does not have one, see also Example 1.12.8). So it remains to check, that W is Witt near $B' \times I$. But a (small enough, see e. g. U' at the beginning of

[1] Even nearly-strictly so, which would allow to integrate this step into the previous one, if we wanted to.

the proof) neighborhood of $B' \times I$ agrees with (is stratified homeomorphic to) a neighborhood of $B' \times I \subset X \times I$ (see definition of W), which is Witt, since X is and the Witt-condition is stable (Example 2.1.3). □

Corollary 4.2.3–1: *Let X be a closed proper MHSS Witt space with simple links and with at most two meeting strata, with minimal strata of* $\dim(B_j) \leq \dim(X) - 5$.

Then there is a bordism $W \in \mathrm{Wh}_*^{\mathrm{gap}\,5}$ (in the strong t-class of Example 3.4.1) from X to $X_{0,0}$. This bordism W has the underlying topological space $|W| = |X \times I|$. Further, also $X_{0,0} \in \mathrm{Wh}_*^{\mathrm{gap}\,5}$, and W and $X_{0,0}$ are Witt if X is Witt.

Proof: This X satisfies the requirements of the lemma. The W obtained by the lemma is a MHSS, and has simple links, because X has simple links, and W has only holink-fibers that appear in X or have trivial fundamental group (by the lemma). The (meeting) strata appearing in W either differ in dimension by 5, or are of the "special" form Section 3.3 ("Special Gaps"), so W is allowed in $\mathrm{Wh}_*^{\mathrm{gap}\,5}$.

Similarly the claim follows for $X_{0,0}$ from Cor. 4.2.1–1. □

Additionally including a Witt-condition as well as maps (elements of bordism groups not over the point), we obtain:

Proposition 4.2.4: *Let X and Y be closed proper MHSS Witt spaces with simple links and with at most two meeting strata, with minimal strata of* $\dim(B_j) \leq \dim(X) - 5$. *Let $h : X \xrightarrow{\sim} Y$ be a (not necessarily stratified) homeomorphism.*

Then $h_*([X \xrightarrow{\mathrm{id}} X]) = [X \xrightarrow{h} Y] = [Y \xrightarrow{\mathrm{id}} Y]$ in $\Omega_*^{\mathrm{Wh}\,\cap\,\mathrm{Witt}}(Y)$. Here $\mathrm{Wh} \cap \mathrm{Witt}$ *denotes the strong t-class of spaces that are both in the strong t-class* $\mathrm{Wh}_*^{\mathrm{gap}\,5}$ *of Example 3.4.1 and are additionally Witt.*

Proof: By Cor. 4.2.3–1 there are bordisms $W_X, W_Y \in \mathrm{Wh}_*^{\mathrm{gap}\,5}$ from X to $X_{0,0}$ and from Y to $Y_{0,0}$ respectively, which are Witt, because X and Y are. These have underlying topological spaces $|W_X| = |X \times I|$ and $|W_Y| = |Y \times I|$, so we can define maps $|W_X| = |X \times I| \xrightarrow{h\pi_X} |Y|$ and $|W_Y| = |Y \times I| \xrightarrow{\pi_Y} |Y|$, that restrict to h and id_Y on either side, making them into bordisms from $X \xrightarrow{h} |Y|$ to $X_{0,0} \xrightarrow{h} |Y|$ and from $Y \xrightarrow{\mathrm{id}} |Y|$ to $Y_{0,0} \xrightarrow{\mathrm{id}} |Y|$ in $\Omega_*^{\mathrm{Wh}\,\cap\,\mathrm{Witt}}(Y)$.

By Thm. 1.10.2, the homeomorphism h is stratified with respect to the (topologically intrinsic) $X_{0,0}, Y_{0,0}$. So by Prop. 4.1.1, the stratified mapping-cylinder $\mathrm{cyl}(h)$ is a bordism from $X_{0,0} \xrightarrow{h} |Y|$ to $Y_{0,0} \xrightarrow{\mathrm{id}} |Y|$ in $\Omega_*^{\mathrm{Wh}\,\cap\,\mathrm{Witt}}(Y)$. Note, that this stratified mapping-cylinder $\mathrm{cyl}(h) \cong_{\mathrm{strat}} X \times I$ (see Prop. 4.1.1), thus is Witt (because X is, and the Witt-condition is stable Example 2.1.3).

Composing these, we find

$$(X \xrightarrow{h} Y) \sim_{W_X} (X_{0,0} \xrightarrow{h} Y) \sim_{\mathrm{cyl}(h)} (Y_{0,0} \xrightarrow{\mathrm{id}} Y) \sim_{-W_Y} (Y \xrightarrow{\mathrm{id}} Y) \quad \text{in } \Omega_*^{\mathrm{Wh}\,\cap\,\mathrm{Witt}}(Y)$$

which finishes the proof. □

A very similar approach is to cone off the lower stratum in the middle (cf. [Ban07, p. 137]). This could avoid the “special gaps” (see Section 3.3 (“Special Gaps”)), but would turn B into a “link”, thus require B to satisfy a Whitehead condition on its fundamental group. Further, a treatment beyond the coefficient case $\Omega_*(\{\mathrm{pt}\})$ would require some contractibility hypothesis on B to be able define the map of the bordism.

This means in a bordism-theory, fundamental classes of such two-stratum spaces are invariant under *unstratified* homeomorphisms. See the Main Theorem (Section 5.1 (“The Main Theorem”)).

5 The Main Theorem and its Applications

In this chapter, the results of the thesis are collected together into a single theorem (Section 5.1 ("The Main Theorem")). Its range of applicability is briefly explored in Section 5.2 ("Satisfying the Hypotheses"). Then this is applied to study the transport-behavior of Goresky–MacPherson L-classes under homeomorphisms (Section 5.3 ("L-Classes")). Finally, we give a separate transversality-statement in Section 5.4 ("Singular Transversality"), which is technically slightly stronger than what is used in the the bordism-theory.

5.1 The Main Theorem

We start by a brief summary of hypotheses required in the theorem, and the reasons why they appear.

The probably most evident requirement, is, that spaces have "simple links". This means, homotopy-links of pairs of strata must have fibers (more precisely: Path components of fibers over path components of the lower stratum) with fundamental groups π, such that $\forall k \geq 0$: $\mathrm{Wh}(\pi \times \mathbb{Z}^k) = 0$. The reason for this condition to appear, is, that the geometric objects which need to be constructed for the (inverse of the) excision-isomorphisms, rely on the existence of certain manifold boundaries [CV99], that are in turn constructed as (controlled) end-completions, in the sense of e. g. [Qui79], of strata over the next lower skeleton. Excision may occur in different places and potentially multiple times in sequence, so vanishing of an end-obstruction(-group) or existence of a (single) cylinder-neighborhood do not suffice.

Next, there is a "dimensional gap" requirement, that any pair of (meeting) strata differ in dimension by at least 5. This is related to problems occurring at low (absolute) dimension: The end-theorem (being rather similar to the s-cobordism theorem) does not work well in intermediate dimensions, in dimension 5 fundamental groups must additionally be "good" (e. g. poly-(finite / cyclic)) [FQ90]. In dimension 4 not much is known. Low (absolute) dimensions may always be reached by repeated application of excision, the codimension requirement ensures however, that low-dimensional skeleta are manifolds (up to finitely many points of a potential zero-stratum).

Remark 5.1.1: It seems plausible, that for "good" fundamental groups (in the sense of [FQ90], see Def. 1.9.5), co-dimension 4 strata can be allowed, so that the gap hypothesis may be reduced to ≤ 4 (see Rmk. 1.9.4).

This applies for example to poly-(finite / cyclic) groups (see Example 1.9.6), thus many examples (e. g. links having free abelian Example 5.2.5, or more generally torsion-free poly-(finite / cyclic) Example 5.2.7, thus poly-(finite / cyclic) fundamental groups) still apply.

Further, there is the implicit choice of working with MHSS [Qui88a] as probing spaces. This is, because they both lend themselves well the study to end-completions (see above), while also being defined purely topologically (without the requirement of pl or geometric-bundle hypotheses) and have intrinsic stratifications, thus are well-suited for the study of topological homeomorphisms.

Additionally, spaces are required to be Witt [Sie83; Fri09], providing enough rigidity to ensure, that the theory obtained is non-trivial (for example the complex projective spaces $\mathbb{CP}^{2k}$ allowed over the point and are not null-bordant), and supplementing a signature-homomorphism, which allows, for example, for the study of L-classes (see Section 5.3 ("L-Classes")).

Finally, by having "a geometric description of suspension" (in the sense of Def. 2.5.9), representatives of a desuspension (in the reduced theory) can by chosen "universally" with certain transversality properties, which allows for example to conclude, that an element which can be represented as a pl-stratified pseudomanifold, also has a representative of its desuspension which is a pl-stratified pseudomanifold (see Example 2.5.10).

One may note, that transversality-constructions are done inductively over skeleta, and only strata that meet (i. e. have common points in their closures $\mathrm{closure}(S_1) \cap \mathrm{closure}(S_2) \neq \emptyset$) will require normal structure for the inductive step. This means, that also codimension (gap) conditions, and requirements on the number of strata concern only meeting strata. So e. g. a MHSS with top-stratum M and more than two minimal strata B_i, with all B_i closed and disjoint (from each other), is allowed as a space with at most two *meeting* strata.

Theorem 5.1.2 ("The Main Theorem"): *There is a generalized homology theory $\Omega_*^{\mathrm{Wh}\,\cap\,\mathrm{Witt}}(-,-)$ (realized as oriented bordism of certain MHSS Witt-spaces), defined on topological pairs, with the following properties:*

(1) $\Omega_*^{\mathrm{Wh}\,\cap\,\mathrm{Witt}}$ is a module over Ω_*^{STOP}, the oriented bordism-groups of topological manifolds.

(2) For any topological space X, there is a group-homomorphism

$$\sigma : \Omega_*^{\mathrm{Wh}\,\cap\,\mathrm{Witt}}(X) \to \mathbb{Z}.$$

(cf. 3. a for a normalization)

(3) An oriented closed "gapped" – in the sense, that meeting strata $\mathrm{closure}(M) \cap B \neq \emptyset$ must be "gapped" by $\dim(M) - \dim(B) \geq 5$ – MHSS Witt-space X with dense top-stratum, of dimension n, with "simple links" (that is, with homotopy-links whose fibers over all connected components have connected components with fundamental groups π, such that $\forall k \geq 0$ $\mathrm{Wh}(\pi \times \mathbb{Z}^k) = 0$), has a "fundamental class" $[X] \in \Omega_n^{\mathrm{Wh}\,\cap\,\mathrm{Witt}}(X)$, such that:

(a) *The group-homomorphism σ of part (2) is normalized by* $\sigma([X]) = \operatorname{sign}(X)$, *where* $\operatorname{sign}(X)$ *is the (middle-dimension, middle-perversity) intersection-homology pairing signature of the MHSS Witt-space* X.

(b) *Let* $h : X \xrightarrow{\sim} Y$ *be an orientation-preserving stratified homeomorphism of such (satisfying the hypothesis of (3)) spaces* X *and* Y. *Then* $h_*([X]) = [Y] \in \Omega_n^{\mathrm{Wh}\cap\mathrm{Witt}}(Y)$.

(c) *Let* $h : X \xrightarrow{\sim} Y$ *be an orientation-preserving homeomorphism (*not *necessarily stratified) of such spaces* X *and* Y, *where* X, Y *both have at most two meeting strata. Then* $h_*([X]) = [Y] \in \Omega_n^{\mathrm{Wh}\cap\mathrm{Witt}}(Y)$.

(4) *The reduced theory* $\tilde{\Omega}_*^{\mathrm{Wh}\cap\mathrm{Witt}}(-)$ *has a geometric description of suspension (in the sense of Def. 2.5.9).*

Proof: We obtain a bordism-theory by applying Prop. 2.3.7 to the strong transversality-class obtained from combining the strong t-class $\mathrm{Wh}_*^{\mathrm{gap}\,5}$ from Example 3.4.1 with the stable class Witt from Example 2.1.3 by Lemma 2.2.2 into $\mathrm{Wh}\cap\mathrm{Witt} := \mathrm{Wh}_*^{\mathrm{gap}\,5}\cap\mathrm{Witt}$. By Prop. 2.3.7, this is a generalized homology-theory and satisfies "(1)".

"(2)": For $[Y \to X] \in \Omega_*^{\mathrm{Wh}\cap\mathrm{Witt}}(X)$, since Y is a MHSS Witt-space, there is a well-defined signature $\operatorname{sign}(Y)$ by Thm. 1.12.10. By Thom's theorem (for MHSS) Prop. 1.12.11 this signature does not depend on the choice of representative of $[Y \to X]$, so $\sigma([Y \to X]) := \operatorname{sign}(Y)$ is well-defined. The group-operation on $\Omega_*^{\mathrm{Wh}\cap\mathrm{Witt}}(X)$ is defined by disjoint union (see Section 2.1 ("Bordism Theories")), and by the definition of the signature $\operatorname{sign}(Y \sqcup Y') = \operatorname{sign}(Y) + \operatorname{sign}(Y')$, therefore σ is group-homomorphism.

"(3)": The spaces in (3) are allowed as "probing spaces" in the bordism-theory, so $[X] := [X \xrightarrow{\mathrm{id}} X] \in \Omega_n^{\mathrm{Wh}\cap\mathrm{Witt}}(X)$ is a well-defined element. Part (a) follows directly from the definition of σ (see above). Parts (b) and (c) follow from Chapter 4 ("Homeomorphisms"), more precisely from Prop. 4.1.1 and Prop. 4.2.4 respectively.

"(4)": This was proved in Section 2.5 ("Reduced Theories") as Lemma 2.5.11. □

In the subsequent sections, we first give a number of examples where the theorem applies. Then the theorem is used to study the transport-behavior of singular L-classes under certain (stratified) homeomorphisms.

5.2 Examples of Classes of Spaces Satisfying the Hypotheses of the Main Theorem

This section collects a number of examples for spaces satisfying the different hypotheses of the main-theorem.

First, all conditions rely on how strata fit together, so:

Example 5.2.1: *Let M^n be a closed oriented topological manifold. Then M has a fundamental-class $[M] \in \Omega_n^{\mathrm{Wh}\cap\mathrm{Witt}}(M)$.*

Further, given an element $[Y \xrightarrow{f} X] \in \Omega_n^{\mathrm{Wh}\cap\mathrm{Witt}}(X)$ with $n := \dim(Y) \neq 5$, there is a homological orientation $[Y]_H \in H_n(Y)$, which is the manifold orientation-class if Y is a manifold, such that the evident map

$$\iota : \Omega_*^{\mathrm{STOP}}(X) \to \Omega_*^{\mathrm{Wh}\cap\mathrm{Witt}}(X), [f : M \to X] \mapsto [f : M \to X]$$

factors the Hurewicz-map from manifold-bordism

$$\mathrm{Hur}^{\mathrm{Mfld}} : \Omega_*^{\mathrm{STOP}}(X) \to H_*(X), [f : M \to X] \mapsto H_*(f)([M]_H)$$

via another Hurewicz-map in degrees $* \neq 5$

$$\mathrm{Hur}^{\mathrm{MHSS}} : \Omega_*^{\mathrm{Wh}\cap\mathrm{Witt}}(X) \to H_*(X), [f : Y \to X] \mapsto H_*(f)([Y]_H)$$

i. e. $\mathrm{Hur}^{\mathrm{Mfld}} = \mathrm{Hur}^{\mathrm{MHSS}} \circ \iota$.

Remark 5.2.2: In dimension 5, one could likely use a teardrop-neighborhood of the next-to-top skeleton (see Example 1.4.5 and discussion thereafter) instead of a cylinder-neighborhood to remove the dimensional requirements.

Proof: Given $[Y \xrightarrow{f} X] \in \Omega_n^{\mathrm{Wh}\cap\mathrm{Witt}}(X) = \Omega_n^{\mathrm{Wh}\cap\mathrm{Witt}}(X, \emptyset)$, thus $\partial Y = \emptyset$, a homological orientation $[Y]_H$ may be defined as follows: If $n < 5$, then Y is a closed manifold and has an orientation-class. Otherwise, if $\dim(Y) \geq 6$, since Y has simple links, (by Lemma 3.1.11, the end-theorem Thm. 3.1.9 and the dimension-requirement) there is cylinder-neighborhood, such that $Y = M' \cup_{\partial M'} \mathring{\mathrm{cyl}}(g : \partial M' \to Y^{n-5})$ rel Y^{n-5} for some MAF g (or a suitable replacement thereof in the case of more than two strata, see Chapter 3 ("Multiple Strata")), mapping from the boundary $\partial M'$ of a compact manifold M' to the next-to-top skeleton of Y.

Because Y is oriented, there is an orientation $[M', \partial M'] \in H_n(M', \partial M')$, identifying the cylinder-coordinate with the collar-coordinate of a boundary collar of $\partial M' \subset M'$ we obtain a map $\pi_g : (M', \partial M') \to (Y, Y^{n-5})$ inducing $H_*(\pi_g)([M', \partial M']) \in H_n(Y, Y^{n-5})$, but $H_{n-1}(Y^{n-5}) = 0$ (for dimensional reasons), so by the long exact sequence of the pair (Y, Y^{n-5}) this comes from some unique, by $H_n(Y^{n-5}) = 0$, element $[Y]_H \in H_n(Y)$. If Y is in the image of ι, it is a manifold, it must be closed, by $[f] \in \Omega_*^{\mathrm{Wh}\cap\mathrm{Witt}}(X, \emptyset)$, and oriented, so $M' = Y$ and $[Y]_H$ is the usual manifold-orientation, thus indeed $\mathrm{Hur}^{\mathrm{Mfld}} = \mathrm{Hur}^{\mathrm{MHSS}} \circ \iota$ holds.

Next we check, that the homological fundamental-class is well-defined (does not depend on the choice of cylinder-neighborhood), because by the end-theorem (Thm. 3.1.9) completions are unique up to (controlled) h-cobordism, thus by vanishing of the obstruction-groups (again by Lemma 3.1.11), are unique up to homeomorphism rel lower skeleton. The construction of the homological fundamental-class was natural, so gets mapped "correctly" by this homeomorphism.

Further, it remains to show, that these mappings are well-defined. For ι, this is the case, because a manifold-bordism is also allowed as bordism in $\Omega_*^{\mathrm{Wh}\cap\mathrm{Witt}}$.

We briefly recall the manifold-case $\mathrm{Hur}^{\mathrm{Mfld}}$: First, note, that $[f : M \to X] \sim [f' : N \to X]$ means, there is there is a manifold-with-boundary W and a continuous $F : W \to X$ that restricts to $F|_{\partial W} = f \sqcup f'$, where the orientations on the components of ∂W are $[M]_H$ and $-[N]_H$, i. e. the orientation $[W, \partial W] \in H_{n+1}(W, \partial W)$ is such that $\partial_*([W, \partial W]) = [\partial W] = i_*^M[M]_H - i_*^N[N]_H \in H_n(\partial W)$. Thus (since $f = F \circ i^M$ and $f' = F \circ i^N$), we find $(F|_{\partial W})_*(\partial_*[W, \partial W]) = f_*([M]_H) - f'_*([N]_H)$ and it suffices to show, that the left-hand-side vanishes in $H_n(X)$. But F also induces a map of pairs $(W, \partial W) \to (X, X)$ and by naturality of the boundary, $(F|_{\partial W})_*(\partial_*[W, \partial W]) = \partial_*(F_*([W, \partial W])) = 0$, because $F_*([W, \partial W]) \in H_n(X, X) = 0$.

The MHSS case $\mathrm{Hur}^{\mathrm{MHSS}}$ works exactly the same, as long as we can apply the manifold argument to the (completed) top-stratum $(W_{n+1})'$ of a bordism. Since the bordism has itself a cylinder-neighborhood (as it has simple links) of its top-stratum, and the homological orientation does not depend on the choice of cylinder-neighborhood by uniqueness of end-completions (by having simple links, making also the h-cobordism obstructions vanish, see Lemma 3.1.11; see above), such a manifold-bordism of completed top-strata exists. □

More generally, stratified pseudomanifolds[1] with suitable links can be used:

Example 5.2.3: *If X^n is a closed stratified pseudomanifold and $\mathbb{Q}$-Witt[2], with simple links and satisfying the gap-hypothesis, then X has a fundamental class $[X] \in \Omega_n^{\mathrm{Wh}\cap\mathrm{Witt}}(X)$ and part (3) of the main theorem applies.*

Proof: X is a CS set in the sense of Siebenmann [Sie72] (see Def. 1.3.10), hence a MHSS (Example 1.4.8).

The Witt-hypothesis on MHSS and stratified pseudomanifolds agrees (see Example 5.2.8 below). □

For a space with more than two strata, the simple links condition (as of now) cannot be checked on the stratified pseudomanifold-links, but rather concerns the stratum-to-stratum links individually, see part (f) of Section 6.3 (“Outlook and Further Ideas”). This corresponds to checking the condition on *strata* of links:

[1] By a stratified pseudomanifold, we mean one in the sense of Def. 1.3.10, that is, manifold-stratified, locally-conelike, with no codimension 1 stratum, dense top-stratum and with links themselves compact stratified pseudomanifolds of lower dimension.

[2] Either in the pseudomanifold or in the MHSS sense, both agree here by Example 5.2.8.

Example 5.2.4: *If X^n is a closed stratified pseudomanifold with links*[3] *L_i (in the pseudomanifold-sense) with strata $(L_i)_j$, such that $\mathrm{Wh}(\pi_1((L_i)_j) \times \mathbb{Z}^k) = 0$ for all i, j, k, then X has simple links.*

Proof: Again, X is a CS set in the sense of Siebenmann [Sie72] (see Def. 1.3.10), hence a MHSS. Note, that it has local homotopy-links (see Example 1.4.8) L_i.

For example in the two stratum case local homotopy-links are homotopy-link fibers and nothing more would be required to be checked. In this case the L only have one stratum.

In general, strata of the local homotopy-links are (by definition) those paths, mapping into a single stratum (except at 0), so are the homotopy-links of pairs of strata. The homotopy-link fibers are the fibers of these "homotopy-links of pairs", so satisfy the Whitehead-hypothesis for the main theorem, if they satisfy the hypothesis of the example.

This can also be checked directly: By locality of holinks and local-conelikeness of stratified pseudomanifolds, we may check this near a point $p \in X_i$ on $\mathbb{R}^i \times c^\circ(L_i)$ (with the stratification induced by L_i, which is itself a stratified pseudomanifold, thus stratified). The homotopy link of $X_i = \mathbb{R}^i \times \{c\}$ in $X_i \cup X_{i+j+1} = X_i \cup \mathbb{R}^i \times ((L_i)_j \times (0,1)) = \mathbb{R}^i \times c^\circ((L_i)_j)$ has fiber $F_{ij} = (L_i)_j$ by pushing along the cone-coordinate (see Example 1.4.8), so the Whitehead-hypothesis implies the "simple links"-condition. □

As already mentioned in Example 2.2.4 (the restriction to two strata is again required for local holinks and pairwise-holinks to agree):

Example 5.2.5: *If X is a "supernormal" ([Wei94, §12.1 (p. 202f)]) MHSS with at most two meeting strata, then it has simple links.*

One may, however, apply the algebraic knowledge (of $\mathrm{Wh}(\mathbb{Z}^k) = 0$, [BHS64]) stratum-wise to links, e. g. for stratified pseudomanifolds (combining Example 5.2.4 and Example 5.2.5):

Example 5.2.6: *If X is a closed stratified pseudomanifold and $\mathbb{Q}$-Witt, satisfying the gap-hypothesis, and with links L_i such that the strata $(L_i)_j$ of those links are simply-connected, then X has a fundamental class $[X] \in \Omega_n^{\mathrm{Wh} \cap \mathrm{Witt}}(X)$ and part (3) of the main theorem applies.*

Or slightly more general:

[3] In the case, where X is not pl, the links may not be known to be well-defined, but their homotopy-types are (they are homotopy-equivalent to the homotopy-well-defined holink-fibers).

Example 5.2.7: *If X is a closed stratified pseudomanifold and $\mathbb{Q}$-Witt, satisfying the gap-hypothesis, and with links L_i such that the strata $(L_i)_j$ of those links have torsion-free poly-(finite / cyclic) (e. g. finitely generated free abelian) fundamental-groups (see Example 2.2.4) then X has a fundamental class $[X] \in \Omega_n^{\mathrm{Wh} \cap \mathrm{Witt}}(X)$ and part (3) of the main theorem applies.*

Proof: This is a consequence of the algebraic structure, see [BHS64; FH81, Thm. 3.2 (p. 308)] and Example 2.2.4 □

Also, not very surprisingly, but nevertheless quite useful:

Example 5.2.8: *If X is a closed stratified pseudomanifold, then X is Witt as stratified pseudomanifold if and only if it is Witt as MHSS.*

Proof: X is a CS set in the sense of Siebenmann [Sie72] (see Def. 1.3.10), hence a MHSS (see Example 1.4.8), so this makes sense.

The condition that $\mathrm{incl} : \mathrm{IH}^{\bar{m}} \to \mathrm{IH}^{\bar{n}}$ be an isomorphism is the same, since the intersection-homology-theories agree (they are determined by the Deligne-sheaf, see [Fri09, Thm. 5.1 (p. 2177)] and [GM83]). The claim follows by Lemma 1.12.7 and its analogue in the pseudomanifold-case. □

Now, having seen some possibilities to validate the hypotheses appearing in the main theorem, we continue by an application to L-classes.

5.3 L-Classes

We start by some background on L-classes. Originally, L-classes on smooth manifolds were introduced by Hirzebruch [Hir56, §1.5 (p. 13f)] to formulate, what is today known as the "Hirzebruch Signature-Theorem" [Hir56, 'Hauptsatz' 8.2.2 (p. 85)] (see also [Hir71]): It seems, that it had been known before, that the signature of a (smooth) manifold can be written as a polynomial of its Pontryagin numbers [Tho53], but using "multiplicative sequences" to explicitly construct these "L-polynomial" (see e. g. [Hir71] or [Ban07, §5.6 (p. 117–119)]), Hirzebruch could actually prove the general case of this "index-theorem":

Theorem 5.3.1: *"Hirzebruch Signature-Theorem" [Hir56, 'Hauptsatz' 8.2.2 (p. 85)]: The L-Polynomials, given as the multiplicative sequence associated to ${}^{x}/_{\tanh(x)}$, define the L-Genus $L(M) := \langle L^n(p_1(TM), \ldots, p_n(TM)), [M] \rangle$ of a smooth manifold M^n by formal application of L^n to the Pontryagin-classes p_i of the tangent-bundle TM of M, evaluated via the Kronecker-product[4] on the orientation-class $[M]$ of M.*

[4] Since M is smooth, using de-Rham cohomology, this can of course also be expressed as an integral.

It then holds, that this is the signature of (the middle-dimension Poincaré- / intersection-pairing) of M, that is $L(M) = \operatorname{sign}(M)$.

Note, that these L-polynomials have rational coefficients, while the signature is an integer, so for *smooth manifolds* certain “divisibility”-rules must be satisfied (which can be used to construct / verify certain examples of non-smooth manifolds [MS74, p. 247f]).

It was shown by Novikov [Nov65], that (rational) Pontryagin-classes (and thus L-classes) of manifolds are in fact *topological* invariants, i. e. even though they are explicitly constructed as characteristic classes of the smooth tangent-bundle, the choice of smooth structure does *not* actually matter:

Theorem 5.3.2: *Novikov [Nov65, Thm. 1 (p. 921)]:*
Given two smooth manifolds M_1^n and M_2^n, and a homeomorphism $h : M_1 \to M_2$, the rational Pontryagin-classes are invariant: $h_*(p_i(TM_2; \mathbb{Q})) = p_i(TM_1; \mathbb{Q})$.

This result is actually far from trivial, since at that time not much was known about transversality in the topological category. In fact, the methods of the proof given by Novikov [Nov65] were an important ingredient in the development of such transversality-methods [KS77].

While homeomorphism-invariant (on manifolds) the L-classes are *not* homotopy-invariant, so it is meaningful to ask, which homology-equivalences (maps that induce isomorphisms on homology-groups) leave L-classes invariant or which part of the information contained in L-classes *is* homotopy-invariant (for example the top L-class / signature is; more generally certain higher signatures can be introduced, see e. g. [Ran95], the question about their invariance is famously known as the “Novikov Conjecture”).

While generalizations of tangent-bundles to topological manifolds exist [Mil64], and the Pontryagin-classes can be defined as characteristic-classes of topological tangent-bundles in a suitable sense [RW10, Prop. 9.4 (p. 340)], this is not the approach Novikov [Nov65] takes. This suggests a definition independent of tangent-bundles should exist. To study invariants of stratified spaces, this is an important observation, because tangent bundles are not readily available.

Such an alternative description was given by Thom [Tho58] (see also Ranicki [Ran95, Prop 2.6 (p. 7)]) to handle the pl-case[5] by general-position arguments, rather than through tangent-bundles. The idea is, roughly-speaking, to invoke Hirzebruch’s signature-theorem “backwards” to see, that the signatures of certain “special” sub-manifolds combined contain exactly the information of the L-classes. Indeed, this is also the perspective taken by [GM80] to define their “Goresky–MacPherson L-classes” on pl-stratified pseudomanifolds.

[5] Again, transversality / general-position arguments, became available for the topological category only later [KS77].

The invariance of L-classes (and Pontryagin-classes), and its proof, has been studied intensively, see for example [RW10]. An approach particularly relevant to the present discussion of singular spaces, is the construction of Banagl, Laures, and McClure [BLM19], even giving a fundamental class in $L^\bullet$-homology (see [Ran92]) for pl intersection-homology Poincaré (IP) spaces [GS83, §7 (p. 103f)] (these are similar to Witt-spaces, but constructed for *integral* non-singular forms, see also [Fri09] for a discussion of generalizations of base-rings), which, among other things, implies invariance of Goresky–MacPherson L-classes under pl-homeomorphisms. The argument is based on the use of ad-theories (these are similar to bordism, and in this case based on Pardon's [Par90] IP-bordism). Since it is known how to stratify mapping-cylinders of pl-homeomorphisms, invariance ultimately traces back to having a bordism-theory, which contains bordisms corresponding to pl-homeomorphisms. The bordism-theory introduced in this thesis is designed to be compatible with *topological* (stratified) homeomorphisms, so one may wonder, to what degree this can be fit into the logic of [BLM19]. These ideas are also related to the treatment of the manifold case given in [Ran92, Prop. 16.16 (p. 188)].

Treatments of the topological invariance of Goresky–MacPherson L-classes can also be found in [CSW91] and [Wei94, p. 209f].

Before going into details, we briefly recall the (Thom-)construction as laid out in [Ban07, §5.7 (p. 120–122)] of Goresky–MacPherson L-classes [GM80], see also [Ran95] for a similar (non-singular) treatment.

This construction will yield homological L-classes, which are the Poincaré-duals of the "usual" cohomological ones, where those are defined (on smooth manifolds). We do not review the proof of this connection, this is, for example, done in [Ban07, Prop. 5.7.2 (p. 122)].

Our transversality-statement is encapsulated in the homology-theory $\Omega_*^{\mathrm{Wh}\cap\mathrm{Witt}}$ via excision, or, in the reduced case, via the (de-)suspension. This close relationship is most apparent in the proof of excision and in the "geometric description" of the suspension. Formally we could make elements of cohomotopy-groups transverse directly (see Section 5.4 ("Singular Transversality")), but well-definedness is easier to check when using a detour through our bordism-theory (this also shows, that the "weaker" – in that it allows moving things in the lower stratum while extending – transversality-statement embedded in the bordism-theory is "good enough" for this conclusion):

Definition 5.3.3: *Let X be a closed oriented MHSS Witt-space. Let*

$$\psi : \tilde{\Omega}_i^{\mathrm{Wh}\cap\mathrm{Witt}}(S^k) \xrightarrow{\mathrm{susp}^{-k}} \tilde{\Omega}_{i-k}^{\mathrm{Wh}\cap\mathrm{Witt}}(S^0) \cong \Omega_{i-k}^{\mathrm{Wh}\cap\mathrm{Witt}}(\{\mathrm{pt}\})$$

and, using $q([X]) \in \tilde{\Omega}_*^{\mathrm{Wh}\cap\mathrm{Witt}}(X)$, the reduced equivalence-class of the "fundamental class" $[X]$, see Section 2.5 ("Reduced Theories"), we can compose this with the Hurewicz-map to obtain:

$$l_k(X) : \pi^k(X) \to \mathbb{Z}, \quad [\varphi] \mapsto \sigma\Big(\psi(\varphi_*(q([X])))\Big)$$

Note that $\pi^k(X)$ may not be a group for small k, these cases can be avoided by a stability argument however, see below.

These l_k are stable under suspension in the sense, that:

Lemma 5.3.4: *This l is stable in the sense that $\forall [\varphi] \in \pi^k(X)$*

$$l_{k+N}(X \times S^N)([\Sigma^N_\times \varphi]) = l_k(X)([\varphi])$$

where, given $\varphi : X \to S^k$, we use the "suspension" (see Lemma 2.5.12)

$$\Sigma^N_\times \varphi : X \times S^N \xrightarrow{\varphi \times \mathrm{id}_{S^N}} S^k \times S^N \xrightarrow{\pi} {}^{S^k \times S^N}/_{S^k \vee S^N} = S^k \wedge S^N \cong S^{k+N}$$

Proof: Let $n := \dim(X)$. We need to check commutativity of the following diagram

$$\begin{tikzcd}
\pi^k(X) \arrow[r, "{[\varphi] \mapsto \varphi_*([X])}"] \arrow[d, "\Sigma^N_\times(-)"'] & \tilde{\Omega}^{\mathrm{Wh} \cap \mathrm{Witt}}_n(S^k) \arrow[r, "\mathrm{susp}^{-k}"] & \tilde{\Omega}^{\mathrm{Wh} \cap \mathrm{Witt}}_{n-k}(S^0) \\
\pi^{k+N}(X \times S^N) \arrow[r, "{[\psi] \mapsto \psi_*([X \times S^N])}"] & \tilde{\Omega}^{\mathrm{Wh} \cap \mathrm{Witt}}_{n+N}(S^{k+N}) \arrow[u, "\mathrm{susp}^{-N}"'] \arrow[ur, "\mathrm{susp}^{-k-N}"'] &
\end{tikzcd}$$

The triangle to the right clearly commutes, as all maps are compositions of the (inverse) suspension maps. For the left-hand side rectangle, we need

$$\mathrm{susp}^N(\varphi_*([X])) = (\Sigma^N_\times \varphi)_*([X \times S^N])$$

i. e. both notions of suspensions must coincide. This has been shown in Lemma 2.5.12. □

By the same construction as for (pseudo-)manifolds, these "stabilized" l_k can be represented in homology (essentially through the stable Hurewicz-map being an isomorphism rationally / "Serre's theorem"):

Lemma 5.3.5: *For N large enough, rationally, there is a natural isomorphism*

$$\lambda_N : \mathrm{Hom}\left(\pi^{k+N}(X \times S^N) \otimes \mathbb{Q}, \mathbb{Q}\right) \to H_k(X; \mathbb{Q})$$

Given another such $N' > N$, then $\lambda'_N([\Sigma^{N'-N}\varphi]) = \lambda_N([\varphi])$.

Proof: Let χ denote the universal-coefficient isomorphism (the ext-groups vanish rationally)

$$\chi : \mathrm{Hom}\left(H^{k+N}(X \times S^N; \mathbb{Q}), \mathbb{Q}\right) \to H_{k+N}(X \times S^N; \mathbb{Q})$$

For N large enough the rational Hurewicz-map is an isomorphism

$$\eta : \pi^{k+N}(X \times S^N) \otimes \mathbb{Q} \to H^{k+N}(X \times S^N; \mathbb{Q}), \quad [\varphi] \otimes q \mapsto q[\varphi^*([S^N])]$$

so that its inverse induces

$$(\eta^{-1})^* : \mathrm{Hom}\left(\pi^{k+N}(X \times S^N) \otimes \mathbb{Q}, \mathbb{Q}\right) \to \mathrm{Hom}\left(H^{k+N}(X \times S^N; \mathbb{Q}), \mathbb{Q}\right)$$

Finally the Künneth-theorem provides an isomorphism (for $k < N$)

$$\kappa : H_{k+N}(X \times S^N; \mathbb{Q}) \to H_k(X; \mathbb{Q})$$

Setting $\lambda := \kappa \circ \chi \circ (\eta^{-1})^*$ has the claimed properties (by Lemma 5.3.4). $\square$

We define the "L-classes" as those homology-representatives:

Definition 5.3.6: *Let X be a closed oriented MHSS Witt-space with simple links and satisfying the gap-hypothesis. Define*

$$L_k(X) := \lambda_N(l_{k+N}(X \times S^N) \otimes \mathbb{Q}) \in H_k(X; \mathbb{Q})$$

where N is large enough that Lemma 5.3.5 applies.

This is well-defined by Lemma 5.3.4 and Lemma 5.3.5. Showing, that l_k are indeed group-homomorphisms (for large enough k, when $\pi^k(X)$ are actually groups) would require a bit more work, however, it does not differ significantly from the "usual" treatment, so we omit it here. Also, for the interesting cases (see below), we can identify things on the "l_*-level" (before applying λ) with the classes by Goresky–MacPherson, so, for the results concerning Goresky–MacPherson L-classes, we may work with l_* directly.

Lemma 5.3.7: *Let X be a closed oriented pl-stratified pseudomanifold Witt-space with simple links and satisfying the gap-hypothesis. Then $L_k(X) = L_k^{\mathrm{GM}}(X)$ is the Goresky–MacPherson L-class ([GM80]).*

Proof: By Example 5.2.3 the main theorem (including part (3) applies).

First, the construction of L_* from l_* via λ is the same as in the pseudomanifold-case. So we only need to check, that the l_k are actually “doing the right thing” with $[\varphi] \in \pi^k(X)$.

Since X is a pl-stratified pseudomanifold, there is $\varphi_\perp \simeq \varphi$ transverse to $S^0 \subset S^k$ (see Example 2.5.10).

Because the reduced theory has a geometric description of suspension in the sense of Def. 2.5.9, by Lemma 2.5.11, and using $\varphi_*(q([X])) = [\![\varphi]\!]$, the desuspension $\operatorname{susp}^{-k}([\![\varphi]\!])$ can be represented by $[\![\varphi_\perp|_{X_{[k]}}]\!]$, where $X_{[k]}$ is the transverse preimage of $\{\mathrm{pt}\} \subset S^0$.

Further l_k is simply taking the signature of $X_{[k]}$, and Friedman's [Fri09] intersection-homology treatment on MHSS agrees with the one on stratified pseudomanifolds Goresky and MacPherson [GM80] use (it is the Deligne-sheaf). Hence, this is the signature of the transverse preimage of a transverse representative $\varphi_\perp$ of $[\varphi]$ in the sense of [GM80], so that indeed $l_k(X)(q([\varphi]))$ is the same as the integer used in the construction of Goresky–MacPherson L-classes. □

Finally, we use the Main Theorem to study transport of the L_* under homeomorphisms. Note, that the homeomorphisms in the following statement are *not* pl. The case of pl-homeomorphisms has been treated in great generality by [BLM19].

Proposition 5.3.8: *These L-classes are invariant whenever the main-theorem applies:*

Let X and Y be closed oriented MHSS Witt-spaces with simple links and satisfying the gap-hypothesis.

Let $h : X \to Y$ be a stratified homeomorphism. Then $h_*\big(L_k(X)\big) = L_k(Y)$.

*If X and Y have at most two meeting strata, and $h : X \to Y$ is a homeomorphism (*not *necessarily stratified). Then $h_*\big(L_k(X)\big) = L_k(Y)$.*

Corollary 5.3.8–1: *On suitable pl-stratified pseudomanifolds, this holds for Goresky–MacPherson L-classes:*

Let X and Y be closed oriented pl-stratified pseudomanifold Witt-spaces with simple links and satisfying the gap-hypothesis.

Let $h : X \to Y$ be a stratified homeomorphism. Then $h_*\big(L_k^{\mathrm{GM}}(X)\big) = L_k^{\mathrm{GM}}(Y)$.

*If X and Y have at most two meeting strata, and $h : X \to Y$ is a homeomorphism (*not *necessarily stratified). Then $h_*\big(L_k^{\mathrm{GM}}(X)\big) = L_k^{\mathrm{GM}}(Y)$.*

Proof of the proposition: By the Main Theorem, $h_*([X]) = [Y]$ in both cases and thus $h_*(q([X])) = q([Y])$ on the reduced theory (by naturality of q). So

$$\begin{aligned}
l_k(Y)([\varphi]) &= \sigma\Big(\psi(\varphi_*(q([Y])))\Big) && \text{(by definition)}\\
&= \sigma\Big(\psi(\varphi_*(h_*(q([X]))))\Big) && \text{(by the Main Theorem)}\\
&= \sigma\Big(\psi((\varphi \circ h)_*(q([X])))\Big) && \text{by functoriality of } \Omega\\
&= l_k(X)([\varphi \circ h]) && \text{(by definition)}\\
&= l_k(X)(h^*[\varphi]) && \text{(def. of pullback on } \pi^k)
\end{aligned}$$

and thus (using stability Lemma 5.3.4)

$$\begin{aligned}
L_k(Y) &= \lambda(l_{k+N}(Y \times S^N)) && \text{(by definition)}\\
&= \lambda(l_k(Y)) && \text{(by stability)}\\
&= \lambda\Big(l_k(X) \circ h^*\Big) && \text{(previous equation)}\\
&= \lambda\Big(h_*(l_k(X))\Big) && \text{(def. of pushforward on Hom)}\\
&= \lambda\Big(h_*(l_{k+M}(X \times S^M))\Big) && \text{(by stability)}\\
&= h_*\Big(\lambda(l_{k+M}(X \times S^M))\Big) && \text{(by naturality of } \lambda)\\
&= h_*\Big(L_k(X)\Big) && \text{(by definition)}
\end{aligned}$$

This is the result claimed by the proposition. □

Proof of the corollary: By Example 5.2.8 the proposition applies. By Lemma 5.3.7, further $L_k(X) = L_k^{\mathrm{GM}}(X)$. □

5.4 Singular Transversality

Transversality was used to construct (the inverse of) the excision-isomorphism. But conversely one can also deduce a transversality statement from the bordism-theory, as has been indicated in the previous section and Section 2.5 ("Reduced Theories"), e. g. through the "geometric description" of (de)suspension.

However, the transversality statement, relative skeleta, that we use, is stronger than what would be needed for the construction of a generalized homology-theory. And conversely, the transversality statement, that is implied by the "geometric description" of (de)suspension is weaker than the initial transversality statement.

This is, because for excision / suspension, some movement in the lower skeleton would be acceptable when extending into the next stratum (as already discussed in paragraph (d) in Section 2.6 ("Improvements")), while the transversality result can actually keep the lower skeleton (and also the boundary) fixed "exactly".

Hence, not only because it is of independent interest, but also because it is slightly stronger (and can probably be improved further to allow for more general relative statements and normal micro-bundles) than the version implicit in the main theorem, we additionally provide a separate statement about transversality for trivial normal (micro-)bundle in the target:

Definition 5.4.1: *Given a continuous $g = (g_0, g_\nu) : X \to Y_0 \times \mathbb{R}^m$, where X is a MHSS, we call g* transverse to $Y_0 \times \{0\}$, *if $X_0 := g^{-1}(Y_0 \times \{0\})$ is a MHSS and there is a map $\nu : X_0 \times \mathbb{R}^m \to X$, which is a stratified homeomorphism to its image and such that there is a neighborhood of $Y_0 \times \{0\} \subset Y_0 \times \mathbb{R}^m$, where $g \circ \nu(x_0, t) = (g_0(x_0), t)$.*

If $Y_0 = \{\mathrm{pt}\}$, we omit Y_0 from the notation and call g transverse to 0.

Remark 5.4.2: This is a stratified version (because ν is stratified) of micro-bundle map-transversality (see Section 1.5 ("Excursion: Manifold Transversality")), to the trivial normal micro-bundle of $Y \times \{0\} \subset Y_0 \times \mathbb{R}^m$.

One could formulate the next theorem on the basis of the "strong t-classes" of Section 2.2 ("Transversality Properties"), making it a little bit more general.[6] We give a more direct version instead.

Proposition 5.4.3: *Let $(X, \partial X)$ be a compact MHSS with boundary, meeting strata differing in dimension by at least 5 and with simple links.*

Further, suppose $g : X \to \mathbb{R}^m$ is continuous, $g|_{X^k \cup \partial X^{k+1}}$ transverse to 0 and $\epsilon > 0$. Then $g \simeq g_\perp \operatorname{rel} X^k \cup \partial X \cup W$, where W is the complement of an arbitrary small open neighborhood of $g^{-1}(\{0\}) \cap X^{k+1}$ in X, by an ϵ-homotopy, with $g_\perp|_{X^{k+1}}$ transverse to 0.

Proof: This follows by similar arguments as given in the proof of Example 3.4.1, by inductive application of Cor. 3.2.3–1.

We construct $g_\perp$ on X^{k+1}, and extend the result to the remainder of X at the end of the proof. We do so, by induction over m (the dimension of the target space $\mathbb{R}^m$), so the start of induction is provided by Cor. 3.2.3–1, dealing with zero-strata as in Example 3.4.1 to construct a transverse to 0 map on X^{k+1}.

For the inductive step, we are given $g : X^{k+1} \to \mathbb{R}^{m+1}$, and make the projection to the first m coordinates $g_\perp^m \simeq \pi_m g$ transverse to 0 by the inductive hypothesis. This yields a MHSS X_0, with meeting strata differing in dimension by at least 5 and with simple links, and a "normal-bundle" $\nu^m : X_0 \times (-\epsilon, \epsilon)^m \to X^{k+1}$, a stratified homeomorphism to its image, with $\pi_m g_\perp^m \nu^m(x, T) = T$ for all $x \in X_0$ and $T \in (-\epsilon, \epsilon)^m$. Let π_{m+1} be the projection to the last coordinate, and make

[6] Our strong t-classes also do not seem to be the "correct" concept for this particular case, because they require stability under products with manifolds, which is not needed here.

$\pi_{m+1}g|_{X_0} : X_0 \to \mathbb{R}$ transverse to 0 by Cor. 3.2.3–1, again dealing with zero-strata as in Example 3.4.1. This yields $g_\perp^{m+1} : X_0 \to \mathbb{R}$, a MHSS X_0', with simple links and meeting strata differing in dimension by at least 5, further a stratified bi-collar $c' : X_0' \times (-\epsilon', \epsilon') \to X_0$, such that $g_\perp^{m+1} c'(x', t) = t$, for all $x' \in X_0'$ and $t \in (-\epsilon', \epsilon')$. We define (using $\gamma := 1/2 \min(\epsilon, \epsilon')$)

$$g_\perp' : \nu^m\Big(\operatorname{im}(c') \times (-\epsilon, \epsilon)^m\Big) \to \mathbb{R}^{m+1}, \quad x \mapsto \Big(g_\perp^m(x), g_\perp^{m+1}\pi_{X_0}(\nu^m)^{-1}(x)\Big)$$
$$\nu : X_0' \times (-\gamma, \gamma)^{m+1} \to X^{k+1}, \quad (x_0, T, t) \mapsto \nu^m(c(x_0, t), T)$$

and further, using a continuous $\eta : X \to [0,1]$ such that $\eta|_{\operatorname{im}(\nu)} = 1$ and $\eta = 0$ on the complement of $\nu^m\Big(\operatorname{im}(c') \times (-\epsilon, \epsilon)^m\Big)$ (which exists by regularity of the metric X^{k+1} and the choice of γ) to define

$$g_\perp : X^{k+1} \to \mathbb{R}^{m+1}, x \mapsto \begin{cases} \eta(x)g_\perp'(x) + (1 - \eta(x))g(x) & \text{if } x \in \nu^m\big(\operatorname{im}(c') \times (-\epsilon, \epsilon)^m\big) \\ (g_\perp^m, g_{m+1})(x) & \text{otherwise} \end{cases}$$

where g_{m+1} is close to $\pi_{m+1}g$ (see below). This $g_\perp$ is transverse to 0 at ν: First $g_\perp^{-1}(\{0\}) = X_0'$ (see below), thus a MHSS, with simple links and meeting strata differing in dimension by at least 5, and secondly

$$\begin{aligned} g_\perp\nu(x_0, T, t) &= \Big(g_\perp^m(\nu^m(c(x_0, t), T)), \quad g_\perp^{m+1}\pi_{X_0}(\nu^m)^{-1}(\nu^m(c(x_0, t), T))\Big) \\ &= \Big(T, \quad g_\perp^{m+1}(c(x_0, t))\Big) \quad = (T, t) \end{aligned}$$

Previously, when using such interpolations (in the proofs of earlier transversality-results) we only needed the result to be transverse to 0, where $\eta = 1$, and $g_\perp$ agrees with $g_\perp'$. This is not entirely true here, as the claim $g_\perp^{-1}(\{0\}) = X_0'$, requires additionally, that $g_\perp$ does not have "new" zeros, in places, where $g_\perp'$ did not have zeros and which could spoil transversality to 0 away from $\nu^m(\operatorname{im}(c') \times (-\epsilon, \epsilon)^m)$. However, we can ensure, that this does never actually happen by replacing $\pi_{m+1}g$ by a δ-close g_{m+1} as follows: First, note, that by inductive hypothesis, $g_\perp^m$ is transverse to 0 (on all of X^{m+1}) at ν^m, so all the zeros of $g_\perp^m$ are in $\nu^m(X_0 \times \{0\})$. Since zeros of $(g_\perp^m, g_{m+1})$ are also zeros of $g_\perp^m$, this is also true for $(g_\perp^m, g_{m+1})$, so we can focus on $\nu^m(X_0 \times \{0\})$ for now – extending g_{m+1} to the rest of X^{k+1} is then works as in the cases treated before, because we need not worry about introducing new zeros for $g_\perp$ outside of $\nu^m(X_0 \times \{0\})$ through the choice g_{m+1}.

To this end, we separate $N := \nu^m\Big(\operatorname{im}(c') \times (-\epsilon, \epsilon)^m\Big)$ in two parts (the "sides" of the new cut) $N_+ := (\pi_{m+1}g_\perp')^{-1}([0, \infty))$, and $N_- := (\pi_{m+1}g_\perp')^{-1}((-\infty, 0])$. Then, given $\delta > 0$, define

$$g_{m+1} : N - X_0' \to \mathbb{R}, \quad x \mapsto \begin{cases} \max(\quad \delta, \quad \pi_{m+1}g \quad) & \text{on } N_+ - X_0' \\ \min(\quad -\delta, \quad \pi_{m+1}g \quad) & \text{on } N_- - X_0' \end{cases}$$

which is undefined at $N_+ \cap N_- = (\pi_{m+1}g_\perp')^{-1}(\{0\}) = X_0'$, but $\eta = 1$ near X_0', so $g_\perp = g_\perp'$ near X_0' independently of g_{m+1}, i. e. $g_\perp$ remains well-defined (and continuous).

Further, we may assume $g^{\perp}_{m+1}$ is δ-close to $\pi_{m+1}g$, because it is constructed as an ϵ-close transverse map (where ϵ may be chosen arbitrarily small, thus smaller than this δ, which can be chosen beforehand, see its use to ensure smallness below). Thus g_{m+1} and $\pi_{m+1}g$ are 2δ-close: At points where $g_{m+1} = \pi_{m+1}g$ trivially so, at points of N_+, where $g_{m+1} = \delta$, since $g^{\perp}_{m+1} > 0$ on N_+ and δ-close to $\pi_{m+1}g$, we find $\pi_{m+1}g > -\delta$, so that indeed $|g_{m+1} - \pi_{m+1}g| < |\delta + \delta| = 2\delta$. Similarly at points of N_-, where $g_{m+1} = -\delta$.

Further, the relevancy of this construction is in avoiding new zeros of $g_\perp$, which it indeed does: On $N_+ - X'_0$, both $\pi_{m+1}g > 0$ (by definition of N_+) and $g_{m+1} > 0$ (by construction), so their interpolation (by η) is positive as well. Similarly on $N_- - X'_0$ both are negative. On X'_0, $\eta = 1$, and thus $g_\perp = g'_\perp = 0$. So indeed, on N, $g_\perp^{-1}(\{0\}) = X'_0$. Outside of $\nu^m(X_0 \times \{0\})$, as explained above, we need not worry to accidentally introduce new zeros, so we may chose any suitable η' to interpolate (linearly) from g_{m+1} to $\pi_{m+1}g$ (which is between both, thus also still 2δ-close to $\pi_{m+1}g$, e. g. by triangle-inequality).

As a homotopy in $\mathbb{R}^{m+1}$, use the straight-line homotopy, which is small, if g and $g_\perp$ are close, which in turn can be guaranteed by Cor. 3.2.3–1 and suitable choice of δ in the construction of g_{m+1} (see above). In the same way, it can also be guaranteed, that g and $g_\perp$ agree outside of $g^{-1}((-\epsilon, \epsilon)^m)$.

Finally, we need to extend this construction into the remainder (i. e. the higher than $k+1$ skeleta) of X. Apply the construction above small enough, to take place in $W \cap X^{k+1}$ (by continuity of g, there is ϵ such that $W \cap X^{k+1} \subset g^{-1}((-\epsilon, \epsilon)^m)$, to obtain $g_\perp$ on X^{k+1}.

Next, pick $\eta_W : X \to [0,1]$ such that $\eta_W = 1$ on $g^{-1}(\{0\}) \cap X^{k+1}$ and $\eta_W = 0$ on W (recall, that W, is given as the complement of an open neighborhood of $g^{-1}(\{0\}) \cap X^{k+1}$, so is closed, disjoint from $g^{-1}(\{0\}) \cap X^{k+1}$, thus by regularity of X such an η_W exists). Further let $R_1 : N \to X^{k+1}$ the tameness-retraction of X to X^{k+1} (skeleta of MHSS are tame by Lemma 3.1.8 part (ii), this applies to the boundary, whose retraction can be extended as a product – thus strict with respect to the boundary – on the boundary-collar, and it applies to the interior, to produce a retraction everywhere). We may assume, that $X - W \subset N$, otherwise replace W by $W \cap N$. Further, let $d_\partial : X \to [0,1], x \mapsto \min(\mathrm{dist}_X(x, \partial X), 1)$. Then, define

$$g''_\perp : X \to \mathbb{R}^{m+1}, \quad x \mapsto \begin{cases} \eta_W(x) d_\partial(x) g_\perp(R_1(x)) + (1 - \eta_W(x) d_\partial(x)) g(x) & \text{on } N \\ g(x) & \text{on } X - N \end{cases}$$

This agrees with $g_\perp$ on $X^{k+1} - \partial X$ by R_1 being $\operatorname{rel} X^{k+1}$ and the choice of η_W, on ∂X $g_\perp = g$ was transverse by hypothesis, thus $g''_\perp$ is transverse to 0 on X^{k+1} at c. It agrees with g on ∂X, because $d_\partial(x) = 0$ on ∂X, thus the straight line-homotopy to g is $\operatorname{rel} \partial X$. It is continuous at $\partial(X^{k+1})$, because $g_\perp$ was constructed rel this boundary, i. e. it agrees with g there. By choice of η_W the straight line-homotopy to g is $\operatorname{rel} W \cup \partial X$ and by construction of $g_\perp$ (see above) it is additionally $\operatorname{rel} X^k$. $\square$

Corollary 5.4.3–1: *Let $(X, \partial X)$ be a compact MHSS with boundary, meeting strata differing in dimension by at least 5 and simple links.*

Let $Y_0 \subset Y$ be a subspace with trivial normal micro-bundle (see Def. 1.5.3), i. e. there is a neighborhood $U \cong_h Y_0 \times \mathbb{R}^m$ rel $Y_0 \times \{0\}$.

Given $g : X \to Y$ continuous, with $g|_{X^k \cup \partial X^{k+1}}$ transverse to Y_0, then there is an ϵ-homotopic rel $X^k \cup \partial \cup W$ transverse to Y_0 on X^{k+1} map $g_\perp$, where W is again an arbitrarily small open neighborhood of $g^{-1}(\{0\}) \cap X^{k+1}$.

Proof: Over U, write $hg = (g_0, g_\nu)$ with $g_0 : X \to Y_0$ and $g_\nu : X \to \mathbb{R}^m$. Make g_ν transverse to 0 on X^{k+1} by the proposition above (w. l. o. g. $\epsilon < 1/2 \operatorname{diam}(U)$), then, set

$$g_\perp : X \to Y, \quad x \mapsto$$
$$\begin{cases} h^{-1}\Big(g_0 \pi_{X_0} c^{-1}(x), (g_\nu)_\perp(x)\Big) & \text{if } x \in \operatorname{im}(c) \text{ and } c^{-1}(x) \in X_0 \times B_1(0) \\ h^{-1}\Big(g_0 T_1(x), (g_\nu)_\perp(x)(2 - R(x)) + g_\nu(x)(R(x) - 1)\Big) & \\ \qquad\qquad \text{if } x \in \operatorname{im}(c) \text{ and } c^{-1}(x) \in X_0 \times (B_2(0) - B_1(0)) & \\ h^{-1}\Big(g_0 T_1(x), g_\nu(x)\Big) & \text{if } x \in \operatorname{im}(c) \text{ and } c^{-1}(x) \notin X_0 \times B_2(0) \\ g(x) & \text{if } x \notin \operatorname{im}(c) \end{cases}$$

where $R(x) := |\pi_{\mathbb{R}^m} c^{-1}(x)|$ is the distance to the "center" X_0 of c and $T_1(x) := c\Big(\pi_{X_0} c^{-1}(x), \pi_{\mathbb{R}^m} c^{-1}(x) \frac{R(x)-1}{R(x)}\Big)$ translates x by 1 towards the "center" (h was the homeomorphism $U \to Y_0 \times \mathbb{R}^m$, see statement). Then $g_\perp$ is transverse to Y_0 (using $B_1(0) \cong \mathbb{R}^m$ and restricting c accordingly) and all constructions can easily be made small (for example by replacing c by c' with small enough image; $(g_\nu)_\perp$ is already arbitrary close to g_ν by the proposition). □

The evident inductive argument (over skeleta of the boundary, then skeleta) recovers the transversality-statement as it is implicit in the bordism-theory.

Example 5.4.4: *Let $(X, \partial X)$ be a compact MHSS with boundary, meeting strata differing in dimension by at least 5 and simple links and $g : X \to \mathbb{R}^n$ with $n = \dim(X)$ continuous.*

Then Prop. 5.4.3 (inductively) yields $g_\perp$ transverse to 0 at ν, where $g_\perp^{-1}(\{0\}) =: \Sigma$ is a finite set of points in the interior of the top-stratum $(X - \partial X)_n$ and for $p \in \Sigma$, the restriction $\nu|_{\{p\} \times \mathbb{R}^n}$ is a manifold-chart of this top-stratum near p.

Proof: Inductively apply Prop. 5.4.3 to skeleta of the boundary, than to skeleta. During each such step, say from k- to $(k+1)$-skeleton, the induction over m (the target-dimension) in the proof of Prop. 5.4.3 will successively reduce the dimension of the intersection of $(g_\perp^m)^{-1}(\{0\})$ with the next stratum (here: the $(k+1)$-stratum) by 1 for $m = 1, \ldots, n$, so if $k + 1 < n$, after $k + 2$ of these steps, this intersection is empty. This matches the intuition, that the image of a $(k+1)$-dimensional stratum can be pushed off the origin in a $(k+2)$-dimensional target-space by an arbitrarily small move. So the only surviving points in $(g_\perp^m)^{-1}(\{0\})$ are in (the interior of) the top-stratum, and the intersection must be of dimension $n - n = 0$, so Σ is a 0-submanifold of the compact top-stratum and framed (by ν), thus a finite point-set.

The restrictions $\nu|_{\{p\}\times\mathbb{R}^n}$ are stratified homeomorphisms, but the stratification of $\{p\} \times \mathbb{R}^n$ is the trivial one, so $\mathrm{im}(\nu) \subset (X - \partial X)_n$, i. e. these must have image in the top-stratum, thus are manifold-charts mapping 0 to p. $\square$

The reader may want to compare Prop. 5.4.3 to [CV99, Cor. 1.17 (p. 527)], which treats the case $m = 1$, but gives a much more precise description of obstructions for that case. Note, that for the result above, *repeated* cutting is used, so a gap hypothesis (as opposed to an absolute-dimension hypothesis, cf. Cor. 3.1.25–1) and "simple links" (as opposed to vanishing end-obstruction / a cylinder-neighborhood, cf. Thm. 3.1.25) are required.

6 Conclusion

We conclude, by a summary of the technical results, a brief discussion of difficulties encountered and their essentiality (why they are hard to circumvent), to finally point out a number of open questions.

6.1 Summary of Results

We have seen, that under suitable hypotheses, a generalized homology theory can be realized via certain stratified spaces, such that there are geometric fundamental-classes, containing signature-information, which are invariant under stratified homeomorphisms and – at least in the case of at most two meeting strata – independent of the choice of stratification.

These properties were seen to be suitable for the study of the invariance of Goresky–MacPherson L-classes on such spaces.

As detailed in Chapter 1 (“Background”), Quinn’s MHSS seem to provide a good context to study transversality in a topologically stratified setting, and thereby invariance properties of geometric structures.

6.2 Encountered Problems

Even though we were interested in the study of high-dimensional spaces, certain low-dimensional problems appeared, manifesting themselves in the gap-hypothesis of the main-theorem. This is because, by repeated application of excision, absolute dimensions can always become arbitrarily small. The geometric structures used for the transversality-results are ultimately obtained using controlled topology versions of end-theory (and h-cobordisms), which are not well-understood in low dimensions.

Note, that, while we construct a generalized homology-theory – which, from a “theory of spectra” perspective is a stable limit anyway – there does not seem to be an easy way to “stabilize” the problem while maintaining the geometric (fundamental-class) properties, including the signature. The underlying issue seems to stem from both, (fundamental groups of) links and co-dimensions (the hypotheses of the main theorem), remaining unchanged under suspension (essentially crossing with an S^1 for probing spaces). It seems difficult to improve from this, as, while end-obstructions *do* vanish when crossing with a 1-sphere (see Example 2.6.2), this *only* implies, that after suspending, one can desuspend *once*, which is however trivial and does not help the cause.

Further, even in high dimensions, given geometric structures (vanishing obstructions), these do not seem to "survive" through cutting / gluing operations, forcing us to use pointwise conditions on links (vanishing obstruction-groups) instead (see also Example 2.6.2 and the discussion there).

6.3 Outlook and Further Ideas

Some of the potential paths forward have been outlined in Section 2.6 ("Improvements") already, after giving the baseline / simplest possible useful theory, because the subsequent chapters are organized around answering some of the questions arising there. So, here we will discuss only those possible "improvements" beyond what was discussed before. Continuing the list from Section 2.6, we are left with the following directions for further extension:

(e) A generalization of products to a ring-structure.
(f) The "obstruction-groups" used in the multi-stratum-case are unwieldy for constructions like products (a ring-structure on Ω_*) and formulated for a notion of "links" different from what typically is used in the theory of stratified spaces (see Section 5.2 ("Satisfying the Hypotheses")), see e. g. the Witt-condition (Def. 1.12.5).
(g) The discussion of unstratified homeomorphisms relied heavily on two-stratum-specific properties, although intrinsic stratifications exist well beyond that case.
(h) The treatment of [BLM19] uses ad-theories to obtain (via Quinn-spectra) a more general result on $L^\bullet$-homology.
(i) Is there a dual "mock-bundle"-theory in the sense of [BRS76]?

In more detail, these comprise:

(e) Ring-Structure: For the multi-stratum case, other than for the two-stratum case, there is no fundamental reason, why products and thus a ring-structure should be impossible to define. However, the "simple link" condition requires some care. Some of the links, of say $X \times Y$, are those of a holink $\times$stratum, thus have the fibers of one of X or Y. These are the holinks of $X_i \times Y_j$ in $(X_k \cup X_i) \times Y_j$ and so on. There are also those of $X_i \times Y_j$ in $(X_k \times Y_l) \cup (X_i \times Y_j)$ with $i \neq k$ and $j \neq l$, where it is not quite obvious what happens to fibers. Also, Whitehead groups behave well only with respect to free products (i. e. coproducts in the category of groups) where $\mathrm{Wh}(G * H) = \mathrm{Wh}(G) \oplus \mathrm{Wh}(H)$ [Sta65], but the obstructions used here rely on direct sum (i. e. coproducts in the category of *abelian* groups) with (free abelian) torus-groups $\mathbb{Z}^k$ in the argument of $\mathrm{Wh}(-)$.

(f) Multi-Stratum Obstructions: Quinn [Qui04, §6.10 (p. 62–64)] discusses Leray–Serre-type spectral sequences for the obstruction-theory that also applies to controlled ends. It seems plausible, that these arguments allow for a reformulation of the link-conditions in the multi-stratum case in terms of local homotopy-links, this seems technically rather demanding, but might help the understanding of the obstructions themselves and also could potentially make problems like the ring-structure (see above) more approachable.

(g) Dependency on Stratification for more than two Strata: Note that, away from zero-strata, the intrinsic stratifications of [Qui87], are compatible with bi-collars in the sense that $(X \times \mathbb{R})_{0,0} = X_{0,0} \times \mathbb{R}$. So it seems to be the case, that if X can be cut, also $X_{0,0}$ can be cut: Simply cut X, then take the intrinsic stratification of the cut $(X_0)_{0,0}$, this is bi-collared in $X_{0,0}$. A separate treatment of zero-strata (or, easier to realize technically: of *minimal* strata) is required, but seems possible in principle, e. g. using embedded manifold-transversality Thm. 1.5.11 to isotope a (submanifold) 1-stratum of X (which goes away in $X_{0,0}$) into general position with a bi-collared cut. Thus, one could probably "complete" strong t-classes, by additionally making the intrinsic stratifications of allowable spaces themselves allowable spaces again. Similarly, one might allow for spaces that are obtained by gluing along coarsenings to intrinsic stratifications (instead of along stratified homeomorphisms) – e. g. allowing $X \times I \cup X_{0,0} \times I$, a bordism $X \sim X_{0,0}$ – which can be cut, by cutting the gluing-interface (as above, with a cut which is bi-collared both for the given stratification and for the intrinsic stratification), and then extending this cut rel boundary(-component) into the interiors of the glued parts (the cut is again of this form). However, this requires some care, concerning for example the definition of spaces with boundary: Generally, given a MHSS with (collared) boundary $(X, \partial X)$, the intrinsic skeleta $(X_{0,0}, (\partial X)_{0,0})$ may not be such that $(\partial X)_{0,0} \subset X_{0,0}$ is collared (if there are non-trivial, not 1-LC (see Appendix A ("Ends in MHSS")), points in the zero-skeleton $(\partial X)^0_{0,0}$, that are boundaries of 1-strata in X which got promoted in $(X - \partial X)_{0,0}$). However, using embedded manifold-transversality as indicated above, it seems possible to ensure, that cuts always have $(\partial X)^0_{0,0} = (\partial X)^0$, so it may be possible to introduce this as an additional (consistent) requirement.

An alternative approach would be to put additional hypotheses similar to the simple links condition on stratified homotopy-links. In the language of the approach above, one might thereby ensure, that the "completion" of such a strong t-class *is* the original (unchanged) t-class, which of course implies that the "completion" is again a strong t-class, i. e. this is a strictly stronger statement than what was discussed above. Thus it might be (considerably) harder to show, for example finding a bordism $X \sim X_{0,0}$ might be difficult.

(h) Ad-Theories: Typically, corners (as they occur in ad-theories during gluing) are less of a concern in the topological category (than for example in the smooth category). So it might be possible to generalize the theory as presented here to a treatment similar to the one given by [BLM19]. This treatment uses (pl) IP-spaces [GS83] and is closely related to their bordisms [Par90], but technically quite involved.

IP-spaces are constructed to have non-singular *integral* intersection-pairings. The argument does not rely on integral coefficients, and also coefficients other than $\mathbb{Q}$ are treated by [Fri09], so could be integrated into the treatment given here rather "easily" if need should be.

(i) Mock-Bundles: A mock-bundle theory in the sense of Buonchristiano, Rourke, and Sanderson [BRS76] could also be an interesting subject to study, especially since the classification approach of Hughes, Taylor, and Williams [HTW90; HTW91] suggests, that there might be ways to factor the duality-isomorphism [BRS76, Thm. 3.2 (p. 30)]. That duality occurs by "reinterpreting" a mock-bundle as a bordism, while for MHSS, there is a description via (stratified) MAF and one through their teardrops. For more than two strata, one could successively transition from MAFs to their teardrops, starting in the bottom skeleton, going upwards, through multiple intermediate stages.

A Ends in MHSS

The main concern of this appendix is to illustrate, how the definition of MHSS can be translated into a more intuitive description of ends, as found for example in [Qui79].

The results of this chapter are also implicit in [Qui04] and in the cylinder-existence / end-theorem [Qui88a, Thm. 1.7 (p. 446)] and are certainly known to people working in that field, however it is hard to find a detailed treatment in the literature, thus some details are given here.

A.1 Background on Controlled End-Theory

The following definitions and results (barring the examples) are from [Qui79]. Instead of via MHSS and cylinder-neighborhoods (Section 1.9 ("Mapping-Cylinder Neighborhoods")), ends and completions are described via non-compact manifolds and finding a boundary (Section 1.8 ("Excursion: End-Theory")) with control added as for h-cobordisms (Section 1.7 ("Excursion: Controlled Topology")). For a treatment of spaces with more than two strata see [Qui82a].

This first section identifies which properties of skeleta in MHSS have to be understood, to formulate cylinder-neighborhood existence as an end-problem. A treatment of said properties is given in the subsequent sections.

We start by formalizing what adding "control" to the definitions relevant to an end-problem is supposed to mean:

Definition A.1.1: *Let M be a manifold, X a locally compact space, $e : M \to X$ a map. A* completion of (the end of) *e is a manifold-with-boundary $M' \supset M$ with $M' - M \subset \partial M'$, together with an extension of e to a proper map $e' : M' \to X$.*

A neighborhood of the end *is an open subset $U \subset M$ s. t. $e|_{M-U}$ is proper.*

This is, of course, closely related to the MHSS and cylinder-neighborhood based description used in the main text:

Example A.1.2: *Let X be a compact* MHSS *with two strata M and B. There exists a neighborhood N of B in M which deformation retracts to B by R (by the forward-tameness condition). By* $\operatorname{im}(R_1) \subset B$*, we may set $r = R_1 : N \to B$ the induced retraction. Then a completion r' of $r|_{N-B}$ provides a mapping-cylinder neighborhood of B in X, namely* $\mathring{\text{cyl}}(r'|_{\partial N'})$*, where N' is the manifold with boundary associated to the completion, s t.* interior $N' = N - B$.

Further, neighborhoods of the end are (open) neighborhoods of B intersected with $N - B$.

Proof: The boundary of N' is collared in N' [Bro62], let $c_\partial : \partial_0 N' \times [0,1) \to N'$ be such a collar. Define as (the cylinder identifies to B at 0):

$$\phi : \mathrm{cyl}(r'|_{\partial N'}) \to X$$
$$[x,t] \mapsto \begin{cases} c_\partial(x,t) \in (N' - \partial_0 N') = N - B \subset X & \text{if } t > 0 \\ r'(x) \in B & \text{if } t = 0 \end{cases}$$

This ϕ is clearly continuous, one-one and onto a neighborhood of B, by compactness (and the spaces being Hausdorff) ϕ is homeomorphism to its image (which is an open neighborhood of B in X).
We may assume N is closed[1] (thus compact) in X, and $N - B$ an open manifold-with-boundary (use manifold-transversality on M to "cut away" the part of X far from B). Open neighborhoods U of B then have closed (therefore compact) complement in N, making $r|_{(N-B)-U}$ proper. On the other hand, if $r|_{(N-B)-U}$ is proper, then $(N - B) - U \subset N - B$ is compact (B is closed as set, thus compact). Thus $\tilde{U} := N - ((N - B) - U)$ is open and contains B. $\square$

As has been indicated before (in the main text), there are other reasonable and useful variations of tameness, like "reverse tameness".

Definition A.1.3: *A subset $B \subset X$ (metric) is* reverse tame *in X iff there is a retraction $r : N \to B$ of a neighborhood and $\forall \epsilon : Y \to (0, \infty)$ and neighborhood U of B there exist a neighborhood V of B and a homotopy $h : (X - B) \times I \to X - B$ with*

(i) $h = \mathrm{id}$ *on* $(X - B) \times \{0\} \cup (X - U) \times I$
(ii) $h((U - B) \times I) \subset U - B$
(iii) $h((X - B) \times \{1\}) \subset X - V$
(iv) the radius of $r \circ h$ is $< \epsilon$

This means, there is a homotopy pulling the complement of B away from B rel U (cf. [Qui88a, p. 465]).
The following properties describe "nice" ends. These are the properties we want to establish in MHSS.

Definition A.1.4: *Compare [Qui79, Def. 1.1 & Def. 1.2 (p. 279f)]:*

[1] Finding closed neighborhoods *with the correct homotopy type* is very hard in general, see Section 1.7 ("Excursion: Controlled Topology"), but this is *not* what we are doing here, the cylinder is just a subset of N. The difficulty is contained in the assumption, that the completion r' exists.

(i) *The end* $e : M \to X$ *is* tame *if for every neighborhood* U *of the end of* e*, and for every* $\epsilon : X \to (0, \infty)$*, there is a neighborhood* $V \subset U$ *of the end of* e *and a homotopy* $h : M \times I \to M$ *with* $h_0 = \mathrm{id}_M$*,* $h_1(M) \subset M - V$ *and* $h_t(M - U) \subset M - V$*,* $\mathrm{diam}(e \circ h) < \epsilon$

(ii) *The end* $e : M \to X$ *is* 0-LC *if for* $x \in X$*,* $V \subset X$ *a neighborhood of* x *and* $U \subset M$ *a neighborhood of the end, there exist smaller such neighborhoods* U' *and* V' *s. t. any* $y, y' \in U' \cap e^{-1}(V')$ *can be joined by an arc in* $U \cap e^{-1}(V)$*.*

(iii) *The end* $e : M \to X$ *is* 1-LC *if additionally loops in* $U' \cap e^{-1}(V')$ *can be contracted in* $U \cap e^{-1}(V)$*.*

(iv) *The end* $e : M \to X$ *has* locally constant fundamental group *if it is 0-LC and it exist neighborhoods* U *of the end of* e *and* V *of* $x \in X$ *s. t. there is a covering space of* $U \cap e^{-1}(V)$ *whose end is 1-LC.*
The local fundamental group $\pi_1(e)$ *is the group of covering-transformations of this covering (well-defined up to isomorphism, see [Qui79] or below).*

(v) *The end* $e : M \to X$ *is* onto *if for any neighborhood* U *of the end* $e(U) = X$*.*

Remark A.1.5: It is not quite clear from this definition, what “the end of the covering space” is supposed to mean. To fix notation, call the covering $\pi : \tilde{N} \to N := U \cap e^{-1}(V)$. Naïvely one may simply look at the end of $e \circ \pi$. This prompts two issues:
Once, for a non-finite covering, already the preimage of a (compact) point is non-compact. The definition of a neighborhood $\tilde{U}$ of the end of $e \circ \pi$ would require $e \circ \pi|_{\tilde{N} - \tilde{U}}$ to be proper. Thus $\tilde{U}$ needs to contain “most of” $\tilde{N}$, while we are actually interested in behavior “close to the end”.
Second, U is required to be open (as a neighborhood of the end of e), hence N has “two ends”, the one we are interested in, and one “on the outside”, that we don’t really care about.

We will adopt the following “interpretation”:
The end $e : M \to X$ has *locally constant fundamental group* if it is 0-LC and $\forall x \in X$ there are neighborhoods U_0 of the end of e and V_0 of $x \in X$ and there is a regular covering $\pi : \tilde{N} \to N := U_0 \cap e^{-1}(V_0)$ s. t. for $x \in V_0$, neighborhoods $V \subset V_0$ of x and $U \subset M$ of the end of e, there are smaller such neighborhoods U' and V' s. t.

(i) Given two points in $\pi^{-1}\Big(e^{-1}(V') \cap U'\Big)$, there exists an arc in $\pi^{-1}\Big(e^{-1}(V) \cap U\Big)$ joining them.

(ii) Given a loop in $\pi^{-1}\Big(e^{-1}(V') \cap U'\Big)$, it contracts in $\pi^{-1}\Big(e^{-1}(V) \cap U\Big)$.

Inspecting the proof of the End-Theorem in [Qui79, p. 284f], the local fundamental group hypothesis is used in the construction of “tameness structures” implicitly required for application of the “approximate End-Theorem” (cf. [Qui79, §7 (p. 314f)]).

These are in turn constructed in [Qui79, §5 (p. 301ff)]. But the tameness structures consist of subsets of M (the domain of e), the regular coverings are only needed to see, that certain deformations are "$(\delta, 1)$-connected", which follows from the interpretation above. Therefore *this interpretations is really sufficient for the application of the End-Theorem*, and thus for all we intend to use it for.

This definition also fits together with the "locally constant fundamental group in the complement" of [Qui88a], see the proof of Cor. A.3.1–1.

As indicated above, the examples relevant to our treatment of ends with such properties are neighborhoods of skeleta in MHSS. Much of the remainder of this appendix collects the results necessary to prove the claims made in these examples (see Section A.4 ("Previously Deferred Proofs of Examples")).

Example A.1.6: *A retraction to a subspace $B \subset X$ (restricted to $X - B$) has tame end if and only if B is forward and reverse tame in X.*

Example A.1.7: *Let X be a MHSS with two strata M, B and M dense in X. Let r as before. Then the end of r is tame, 0-LC and onto, with locally constant fundamental group $\pi_1(r) = \pi_1(L)$, where L is the fiber of* $\operatorname{holink}(X, B) \to B$.

Proof: This is shown below in Section A.4 ("Previously Deferred Proofs of Examples").

Quinn [Qui79] shows both an existence and a uniqueness theorem for completions of such "nice" ends:

Theorem A.1.8: *"Existence Theorem" [Qui79, Thm. 1.4]: Suppose X is locally compact, locally 1-connected, metric and $e : M \to X$ is proper on ∂M, the end of e is onto, 0-LC, with locally constant fundamental group s. t. $\forall k : \mathrm{Wh}(\pi_1(e) \times \mathbb{Z}^k) = 0$ and $\dim(M) \geq 6$.*

Then e has a completion $e' : M' \to X$.

Remark A.1.9: There are also a relative version [Qui79, Thm. 2.1 (p. 282)] and a version for $\dim(M) = 5$ if $\pi_1(e)$ is "good" [Qui82b, Thm. 2.1.2 (p. 505)], see Rmk. 1.9.4.

So, based on the Example A.1.7 given above (to be proven below), we can formulate a point-wise (dependent only on the fiber of the holink-fibration) version of [Qui88a, Thm. 1.7 (p. 446)], recovering the version modeled on MHSS given in the main text at Thm. 1.9.3).

Example A.1.10: *Let X be a MHSS with two strata M (dense in X) and B and homotopy-link-fiber L, with connected components L_{ij} over the components B_i of B, s. t.* $\mathrm{Wh}\left(\pi_1(L_{ij}) \times \mathbb{Z}^k\right) = 0$ *for all* $k \geq 0$ *and* $\dim(M) \geq 6$. *Then B has a mapping-cylinder neighborhood in X.*

Proof: Let r be a retraction of a neighborhood of B as before. By Example A.1.7 we can apply Quinn's End-Theorem (Thm. A.1.8) to get a completion of $r|_{N-B}$. By Example A.1.2 this yields a mapping-cylinder neighborhood of B in X. $\square$

So we have seen, how Example A.1.7 lets us recover (in Example A.1.10) the formulation of the end-theorem as provided in the main text, from the more "direct" approach of adding control to end-completions. It remains to prove the results claimed by said example.

A.2 Local Fundamental Groups

This section analyzes, how local fundamental groups, in the sense of end-theory, are connected to fundamental groups of fibers of homotopy-links.

The first, rather technical, result tells us, that given an MHSS $X \supset B$, over a neighborhood $U \subset X$ very close to B, the "rays" R_t stay close to R_1 in B. This follows directly from continuity of R through basic arguments.

Lemma A.2.1: *Let (X, d) be compact metric, a subset $B \subset X$ with a nearly-strict deformation retraction R, given $x \in B$ and open neighborhoods $U_2 \subset X$ of B, $V_1 \subset V_2 \subset B$ of x, s. t.* $\mathrm{dist}(V_1, B - V_2) > 0$, *there exists an open neighborhood $U_1 \subset U_2$ of B such that $R(N_1 \times I) \subset N_2$, where $N_i := R_1^{-1}(V_i) \cap U_i$, $i = 1, 2$.*

Proof: Let $\epsilon_0 := \mathrm{dist}(V_1, B - V_2) > 0$. By compactness of B there is an $\epsilon_1 > 0$ such that $B^{\epsilon_1} := \{x \in X | \,\mathrm{dist}(x, B) < \epsilon_1\} \subset U_2$. Put $\epsilon := \min(\epsilon_0/4, \epsilon_1/2)$.

Let $N' \subset X$ be closed such that $U_2 \subset N' \subset N$. Making U_2 smaller strengthens the statement, so we may replace it by a slightly smaller choice.) By compactness of $N' \times I$ (and the Heine-Cantor-Theorem / uniform continuity, δ should depend on (x, t), but compactness allows for the choice $\min(\delta_{x,t}) > 0$), there is a $\delta > 0$, such that $\forall (x, t), (x', t') \in N' \times I$ with $d\big((x, t), (x', t')\big) < \delta \Rightarrow \big(R(x, t), R(x', t')\big) < \epsilon$.
Let $U_1 := B^\delta$. Then, for $x \in U_1$ there is $b \in B$, with $d(x, b) < \delta$ and hence (using $R(b, t) = b$ independently of t for $b \in B$) $d(R(x, t), b) = d(R(x, t), R(b, t)) < \epsilon$, thus $d(R(x, t), R(x, t')) \leq d(R(x, t), b) + d(b, R(x, t')) < 2\epsilon$. This already shows $d(R(x, t), R(x, 1)) < 2\epsilon \leq \epsilon_1$, i. e. $x \in U_1 \Rightarrow R(x, t) \in U_2$.

Further $d\Big(R(R(x, t), 1), R(x, t)\Big) = d\Big(R(R(x, t), 1), R(R(x, t), 0)\Big) < 2\epsilon$, implying $d\Big(R_1(R(x, t)), R_1(x)\Big) \leq d\Big(R(R(x, t), 1), R(x, t)\Big) + d\Big(R(x, t), R(x, 1)\Big) < 2\epsilon + 2\epsilon \leq \epsilon_0$.
Hence $x \in R_1^{-1}(V_1) \Rightarrow R(x, t) \in R_1^{-1}(V_2)$.
Combining both results shows $R(N_1 \times I) \subset N_2$ as required. $\square$

This forms the technical basis for applicability of the next result. We will stick to the notation $N_i := R_1^{-1}(V_i) \cap U_i$ in the following. The tameness-retraction relates homotopy-links of different small neighborhoods in the following sense:

Lemma A.2.2: *Given a forward-tame pair (X, B) (with $R : N \times I \to I$ the nearly strict tameness deformation retraction), a point $x \in B$ and open neighborhoods $V_1 \subset V_2 \subset B$ of x and $U_1 \subset U_2 \subset X$ of B, s. t. $R(N_1 \times I) \subset N_2$, there is a homotopy-commutative diagram (the lower-right triangle actually commutes "exactly"):*

$$\begin{tikzcd}
\operatorname{holink}(N_1, V_1) \arrow[r, "\simeq", "\mathrm{incl}"'] \arrow[d, "ev_1"'] & \operatorname{holink}(N_2 - (V_2 - V_1), V_1) \arrow[d, "ev_1"] \\
N_1 - V_1 \arrow[ur, "\tilde{R}"] \arrow[r, "\mathrm{incl}"] & N_2 - V_2
\end{tikzcd}$$

The inclusion of homotopy-links (the top row) is a homotopy-equivalence.

Proof: First define $\tilde{R}$ by $\tilde{R}(x)(t) := R(x, 1-t)$ this maps to the homotopy-link by nearly-strictness of R and to $\operatorname{holink}(N_2 - (V_2 - V_1), V_1)$ by the choice of $N_1 \subset R_1^{-1}(V_1)$ and the hypothesis $R(N_1 \times I) \subset N_2$.
The lower-right triangle thus commutes, as $\mathrm{ev}_1(\tilde{R}(x)) = \tilde{R}(x, 1-1) = x$.
To show commutativity up to homotopy of the upper-left triangle, define
$H : \operatorname{holink}(N_2 - (V_2 - V_1), V_1) \times I \times I \to N_2$ as

$$(\gamma, t, s) \mapsto \begin{cases} \gamma(t) & \text{if} \quad t \geq s \\ R\Big(\gamma(s), \frac{s-t}{s+t}\Big) & \text{if} \quad t \leq s \quad \text{and} \quad (s,t) \neq (0,0) \end{cases}$$

which induces $\tilde{H} : \operatorname{holink}(N_2 - (V_2 - V_1), V_1) \times I \to \operatorname{holink}(N_2 - (V_2 - V_1), V_1), (\gamma, s) \mapsto \big(t \mapsto H(\gamma, t, s)\big)$. We have to check multiple things: $\tilde{H}$ must be well-defined, i. e. actually map to the homotopy-link, H must be continuous (then $\tilde{H}$ is continuous in the compact open-topology), and this must actually define a homotopy of the maps as claimed.
For $\tilde{H}$ to map into the homotopy-link, we need $\forall \gamma, s$ that $H(\gamma, 0, s) \in V_1$, while $\forall t > 0 : H(\gamma, t, s) \notin V_1$. We have $H(\gamma, 0, s \neq 0) = R(\gamma(s), 1) \in V_1$, and $H(\gamma, 0, 0) = \gamma(0) \in V_1$, while $H(\gamma, s \geq t > 0) = R(\gamma(s), \frac{s-t}{s+t}) \notin V_1$ because of $\gamma(s > 0) \notin V_1$ and $t \neq 1 \Rightarrow \frac{s-t}{s+t} \neq 1$ together with nearly-strictness of R. Further $H(\gamma, t > 0, s \leq t) = \gamma(t > 0) \notin V_1$.
Continuity must be checked where $s = t$ (the individual cases are continuous, as evaluation is continuous in compact-open topology). If $s = t \neq 0$ we have in the first case $\gamma(t) = \gamma(s)$ and in the second case $R(\gamma(s), 0) = \mathrm{id}(\gamma(s)) = \gamma(s)$. It remains to check, that case one at $(t, s) = (0, 0)$ is a continuous extension of case two. This follows from $\gamma(t) \xrightarrow{t \to 0} \gamma(0) \in B$ by continuity of γ and $R|_{V_1} = \mathrm{id}$. Formally a

uniform-continuity argument in the t variable is required, but by compactness of I this follows via the Heine–Cantor theorem.
Finally $H(\gamma, t, 0) = \gamma(t)$ and $H(\gamma, t, 1) = R(\gamma(1), \frac{1-t}{1+t})$ which is a reparametrization of $\tilde{R}(\gamma(1))(t)$. Hence composing H with a homotopy realizing this reparametrization proves the claim about the commutativity up to homotopy.

The second statement concerning the inclusion of homotopy-links in the top row being a homotopy-equivalence follows from [Qui88a, Corollary (3), p. 455]. The argument given there is based on [Qui88a, Lemma 2.4, p. 454] which asserts that the inclusion of the "δ-holink" – consisting of paths γ in the "ordinary" holink of length smaller $\delta(\gamma(0)) > 0$ – into the ordinary holink is a homotopy-equivalence (by "contracting" γ, i. e. running through the path faster, thus reaching 1 closer to the base-space, and finding a uniform "cutoff" larger zero). Putting $\delta(b) := \mathrm{dist}(b, X - N_1) > 0$ the resulting δ-holink is a subset of both $\mathrm{holink}(N_1, V_1)$ and $\mathrm{holink}(N_2 - (V_2 - V_1), V_1)$. Since the inclusions commute, the statement follows. □

Further, we will need to combine multiple such diagrams ("glue them side-to-side", see proof of Lemma A.2.4), for which we need the right-hand side to "look like" the left-hand side of the next diagram. We are interested in MHSS, where B is a manifold, and thus has nice local properties, so that, for example, the following lemma applies:

Lemma A.2.3: *Let $B' \subset B$ be a (strict) neighborhood deformation retract, (X, B) s. t. $ev_0 : \mathrm{holink}(X, B) \to B$ is a fibration, then the inclusion $\mathrm{holink}(X - (B - B'), B') \to \mathrm{holink}(X, B)$ is a homotopy-equivalence.*

Proof: Let $r : B \times I \to B$ be the deformation ($r_0 = \mathrm{id}$, $\mathrm{im}(r_1) \subset B'$, $\mathrm{rel}\, B'$). It induces $\tilde{r} : \mathrm{holink}(X, B) \times I \to B, (\gamma, t) \mapsto r(\gamma(0), t)$, with $\tilde{r}_0 = \mathrm{ev}_0 \circ \mathrm{id}$, hence it lifts to $\psi : \mathrm{holink}(X, B) \times I \to \mathrm{holink}(X, B)$ with $\psi_0 = \mathrm{id}$ and $\mathrm{ev}_0 \circ \psi = \tilde{r}$.
We note that the restriction $\psi|$ to $\mathrm{holink}(X - (B - B'), B') \times I$ maps to $\mathrm{holink}(X - (B - B'), B')$ because r is $\mathrm{rel}\, B'$ and that ψ_1 maps to $\mathrm{holink}(X - (B - B'), B')$ because of $\mathrm{im}(r_1) \subset B'$. We can therefore factor $\psi_1 = \mathrm{incl} \circ \bar{\psi}_1$.
Then, $\mathrm{incl} \circ \bar{\psi}_1 \simeq \mathrm{id}$ by ψ and $\bar{\psi}_1 \circ \mathrm{incl} \simeq \mathrm{id}$ by $\psi|$. Thus $\bar{\psi}_1$ is a homotopy inverse of the inclusion, finishing the proof. □

Now we have the tools at hand to prove the central result of this section, identifying "nice" local-fundamental-group properties in MHSS:

Lemma A.2.4: *Let (X, B) be a metric pair, with $B \subset X$ tame, B a manifold and $\mathrm{holink}(X, B) \xrightarrow{ev_0} B$ a fibration (e. g. $X \supset B$ is a manifold homotopy stratified set in the sense of Quinn). Let $R : N \times I \to N$ be a nearly-strict tameness-retraction, $L := ev_0^{-1}(\{*\})$ the fiber of the homotopy-link. We assume L to be path-connected, since its path-components can be treated independently. Then*

(i) The end of R_1 has locally constant fundamental group $\pi_1(L)$.
(ii) The end of R_1 is 0-LC.
(iii) The "control-map" $N - B \to \operatorname{holink}(X, B), x \mapsto \Big(t \mapsto R_{1-t}(x)\Big)$ is $(\delta, 1)$-connected.

Proof: Fix $x \in B$, $\phi : \mathbb{R}^n \to B$ a manifold chart with $0 \mapsto x$. Fix $V_0 := \operatorname{im}(\phi)$, $U_0 := N$ (a neighborhood of the end of R_1) and $V_1 := \phi(B_1(0))$.
Lemma A.2.1 yields $U_1 \subset U_0$ with $R(N_1 \times I) \subset N_0$. Apply Lemma A.2.2 to $U_1 \subset U_0$ and $V_1 \subset V_0$ and replace the top-right object $\operatorname{holink}(N_0 - (V_0 - V_1), V_1)$ by $\operatorname{holink}(N_0, V_0)$ using Lemma A.2.3 to get a homotopy-commutative diagram:

$$\begin{tikzcd}
\operatorname{holink}(N_1, V_1) \arrow[rr, "\simeq", "\text{incl}"'] \arrow[d, "\mathrm{ev}_1^1"'] & & \operatorname{holink}(N_0, V_0) \arrow[d, "\mathrm{ev}_1^0"] \\
N_1 - V_1 \arrow[urr, "R_{10}"] \arrow[rr, "\text{incl}"] & & N_0 - V_0
\end{tikzcd}$$

There exists (by classifications of coverings) a regular covering $\pi : \tilde{N} \to N_1 - V_1$ of $N_1 - V_1$ with $\operatorname{im}(\pi_*) = \ker((R_{10})_*)$.
Claim: The covering-transformations of π are $\operatorname{Deck}(\pi) \cong \pi_1(L)$. Further $(\mathrm{ev}_1^1)_*$ is injective and $\operatorname{im}((\mathrm{ev}_1^1)_*) \cong \pi_1(L)$.
The isomorphisms are canonical, so we can identify the deck transformations with $\operatorname{im}((\mathrm{ev}_1^1)_*)$.

Proof of claim: The diagram above induces a diagram of fundamental groups (by homotopy-commutativity), where the arrow in the top becomes an isomorphism. Thus $(R_{10})_*$ is surjective, $(\mathrm{ev}_1^1)_*$ is injective. We know (by classifications of coverings) that
$\operatorname{Deck}(\pi) \cong \pi_1(N_1 - V_1)/\ker((R_{10})_*) \cong \operatorname{im}((R_{10})_*) \cong \pi_1\Big(\operatorname{holink}(N_0 - (V_0 - V_1), V_1)\Big)$
$\cong \pi_1\Big(\operatorname{holink}(N_1, V_1)\Big)$.
The open restriction of the holink-fibration to V_1 is a fibration. Further $V_1 = \operatorname{im}(\phi)$ is contractible by construction. The long exact homotopy-sequence thus shows $\pi_1\Big(\operatorname{holink}(N_1, V_1)\Big) \cong \pi_1(L)$. Finally by injectivity of $(\mathrm{ev}_1^1)_*$, also $\operatorname{im}((\mathrm{ev}_1^1)_*) \cong \pi_1\Big(\operatorname{holink}(N_1, V_1)\Big)$, finishing the proof of the claim.

We have to show, that for given $y \in V_1$, and open neighborhoods $U \subset N_1 - V_1$ of the end of R_1 and $V \subset V_1$ of y there are open neighborhoods $U' \subset U$ and $V' \subset V$ such that:

(1) Given two points in $e^{-1}(V') \cap U'$, there exists an arc in $e^{-1}(V) \cap U$ joining them.
(2) Given a loop in $e^{-1}(V') \cap U'$, it contracts in $e^{-1}(V) \cap U$.

That is $\operatorname{im}(\pi)$ was chosen "large enough" for (1) to hold (a larger image means fewer deck-transformations, which must have small representatives), and "small enough" for (2) to hold (all the loops that come from the covering-space must be trivial in

the slightly larger neighborhood).
We start by giving $U' := U_3$ and $V' := V_3$ for x, U, V.
V_1 was defined as $\phi(B_1(0))$, and $V \subset V_1$ is open, so we can assume there are a and $r > 0$ with $V_2 = \phi(B_{2r}(a)) \subset V$ (with $B_{2r}(a)$ the open ball of radius $2r$, centered at a with $\phi(a) = y$).
By Lemma A.2.1 there is $U_2' \subset U_1$ with $R(N_2' \times I) \subset N_1$. Let $U_2 := U \cap U_2' \subset U$.
Define $V_3 := \phi(B_r(a))$, then again Lemma A.2.1 provides U_3 s. t. $R(N_3 \times I) \subset N_2$. Certainly $V_1, \dots, V_3$ satisfy the contractibility and deformation hypothesis of Lemma A.2.3.
We find two additional diagrams from Lemma A.2.2 and by Lemma A.2.3 we can "glue" them to the left of the one shown earlier in the proof (inclusions in the top row are homotopy-equivalences):

$$\begin{tikzcd}
\mathrm{holink}(N_3, V_3) \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}_1^3"'] & \mathrm{holink}(N_2, V_2) \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}_1^2"'] & \mathrm{holink}(N_1, V_1) \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}_1^1"'] & \mathrm{holink}(N_0, V_0) \arrow[d, "\mathrm{ev}_1^0"'] \\
N_3 - V_3 \arrow[r] \arrow[ur, "R_{32}"] & N_2 - V_2 \arrow[r] \arrow[ur, "R_{21}"] & N_1 - V_1 \arrow[r] \arrow[ur, "R_{10}"] & N_0 - V_0
\end{tikzcd}$$

Claim: Let $z, z' \in N_3 - V_3$, then there is a path in $N_2 - V_2$ connecting them. Thus the end of R_1 is 0-LC.

Proof of claim: By the homotopy sequence of the holink-fibration and V_2, L path-connected, the homotopy-link $\mathrm{holink}(N_2, V_2)$ is also path-connected. Thus there is a path $\tilde{p}$ connecting $R_{32}(z)$ to $R_{32}(z')$. The composition $p := \mathrm{ev}_1 \circ \tilde{p}$ connects z to z' in $N_2 - V_2$ because the lower-right triangle actually commutes (cf. Lemma A.2.2, and compositions from "gluing" / Lemma A.2.3 are only inclusions).

Claim: Let $z, z' \in \pi^{-1}(N_3 - V_3)$, then there is a path in $\pi^{-1}(N_2 - V_2)$ connecting them.

Proof of claim: The previous claim yields a path p in $N_2 - V_2$ from $\pi(z)$ to $\pi(z')$. Lift this path to $\tilde{p} : I \to \tilde{N}$ with starting point z and endpoint $z'' \in \pi^{-1}(\pi(z'))$. This path $\tilde{p}$ is in $\pi^{-1}(N_2 - V_2)$ as required.
It is left to show, that there is a path connecting z' to z''. Since the covering is regular ($\mathrm{Deck}(\pi)$ acts transitively), and z', z'' are in the same fiber, there is $[\gamma] \in \mathrm{im}((\mathrm{ev}_1^1)_*)$ lifting to $\tilde{\gamma} : I \to \tilde{N}$ with $\tilde{\gamma}(0) = z'$ and $\tilde{\gamma}(1) = z''$.
It remains to show, that $[\gamma]$ has a small representative: In the "triple-diagram" above, maps in the top row are homotopy-equivalences, and $(\mathrm{ev}_1^1)_*$ is injective (see first claim above), thus $[\gamma]$ comes from the top-left group, and by commutativity of the diagram there is $[\gamma'] \in \mathrm{im}(\mathrm{ev}_1^3)$ mapped to $[\gamma]$ by the inclusions in the bottom row. In other words, there is $\gamma' \simeq \gamma$ with $\mathrm{im}(\gamma') \subset N_3 - V_3 \subset N_2 - V_2$. This lifts to $\tilde{\gamma}' : I \to \pi^{-1}(N_2 - V_2)$ (by $\pi^{-1}(N_2) \subset U$ as before) with $\tilde{\gamma}'(0) = \tilde{\gamma}(0) = z'$ and $\tilde{\gamma}'(1) = \tilde{\gamma}(1) = z''$.

Claim: Let γ be a loop in $\in \pi^{-1}(N_3 - V_3)$, then it contracts in $\pi^{-1}(N_2 - V_2)$.

Proof of claim: Observe, that a representative γ of $[\gamma] \in \pi_1(\pi^{-1}(N_3 - V_3))$ maps under π_* to a loop in $N_3 - V_3$ and we start with $\pi \circ \gamma$ in the bottom left of the "triple-diagram". On the other hand $\mathrm{im}(\pi_*) = \ker((R_{10})_*)$ and thus maps to $0 \in \pi_1\Big(\mathrm{holink}(N_0, V_0)\Big)$ in the top-right. Since the inclusions in the top row induce isomorphisms and the diagram commutes $(R_{32})_*$ already maps $\pi_*([\gamma])$ to 0. Thus under the map induced by the inclusion $N_3 - V_3 \to N_2 - V_2$ (which factors as $(\mathrm{ev}_1^2)_*(R_{32})_*$) it maps to 0 as well, meaning $\pi \circ \gamma$ contracts in $N_2 - V_2$. Lifting the null-homotopy to $\tilde{N}$ yields one in $\pi^{-1}(N_2 - V_2)$ as required.

As indicated above, using $U' = U_3$ and $V' := V_3$, these last two claims finish the proof of the Lemma, because $U_2 \subset U$ and $V_2 \subset V$ hence points (loops) over $N_3 - V_3$ connect (contract) over $N_2 - V_2 \subset R_1^{-1}(V) \cap U$.

Finally, it remains to show (iii):

Claim: The control-map $f : N - B \to \mathrm{holink}(X, B), x \mapsto \Big(t \mapsto R_{1-t}(x)\Big)$ is $(\delta, 1)$-connected.

Proof of claim: Recall, that "$(\delta, 1)$-connected" means, that given $\delta > 0$, and a lifting problem for a relative 2-complex (R, S)

$$\begin{tikzcd}
S \arrow[r, "s"] \arrow[d, "\mathrm{incl}"'] & N - B \arrow[r, "f"] \arrow[d, "R_1"'] & \mathrm{holink}(X, B) \arrow[dl, "\mathrm{ev}_0"] \\
R \arrow[r, "r"] & B &
\end{tikzcd}$$

there is a δ-lift $g : R \to \mathrm{holink}(X, B)$, i. e. $g \circ \mathrm{incl} = f \circ s$ and $\mathrm{ev}_0 \circ g$ is δ-close to r (see Def. 1.7.10)

We construct this map g. To this end, cover B by open δ-balls, having local fundamental group $\pi_1(L)$, there are ϵ-balls, and by the end being 0-LC further γ-balls, such that the following construction works:

We start by lifting the zero-skeleton of $R - S$: By surjectivity of ev_0, there is at least one preimage for each vertex, pick an arbitrary one. This is zero-small when measured in B. Next, by the 0-LC conditions, 1-simplices of $R - S$ can be lifted, such that they remain in the ϵ-balls. The lifts start at the "correct" vertex and end at a point, that differs by a path γ from the correct one (we may assume L is path-connected, and treat components independently), where, $\mathrm{ev}_0 \circ \gamma$ is a loop in the ϵ-ball. We can pick the "balls" contractible, if we pick them for example as $\varphi(\mathring{D}_r)$, where φ is a manifold-chart and r is the maximal (euclidean) distance from 0 on the chart, such that $\mathrm{diam}\Big(\varphi(\mathring{D}_r)\Big) < \epsilon$. By the homotopy-sequence of the fibration ev_0 and the local fundamental group being $\pi_1(L)$ (see above for the actual mapping), γ can be homotoped to become small. Similarly, 2-cells are small disks, bounding (thus contractible) 1-spheres, which can again be lifted by the homotopy-sequence of the fibration ev_0 and the local fundamental group being $\pi_1(L)$.

This finishes the proof of the lemma. $\square$

Essentially, this means, that we understand local-fundamental-groups of neighborhood-retractions in MHSS good enough, to know, how to apply "standard" controlled end-theory, whenever we are able to establish suitable tameness-properties.

A.3 Tameness Properties

This section treats tameness, especially the reverse-tameness condition used in Quinn's controlled end-theorem and its relation to forward tameness as used e. g. in the definition of MHSS.

If the top-stratum $X - B$ is a manifold, Quinn shows – using a homological characterization of forward / reverse tameness and Poincaré-duality – that both notions of tameness coincide for the MHSS-related ends:

Lemma A.3.1: *[Qui88a, Prop. 2.14 (p. 466)]: Let X be locally compact, $B \subset X$ closed with locally constant fundamental group in the complement (see below), $X - B$ a manifold without boundary. Then:*
$B \subset X$ forward tame $\Leftrightarrow$ $B \subset X$ reverse tame

Corollary A.3.1–1: *Let $X \supset B$ be a MHSS with two strata. $R : N \times I \to N$ the nearly strict deformation-retraction of a neighborhood N of B provided by the forward tameness of B.*
Then the end of R_1 is tame.

Proof: The "locally constant fundamental group π in the complement" is defined at [Qui88a, p. 464] as:

For $x \in B$ there is a neighborhood $N_0 \subset X$ of x and $\alpha : \pi_1(N_0 - B) \to \pi$, and a smaller neighborhood N with the property: Given $\epsilon > 0$, there is $\delta > 0$ s. t. a loop γ in $N - B$ of diameter smaller δ contracts in $N_0 - B$ by a homotopy of radius less than ϵ if and only if $\alpha([\gamma]) = 0$.

The corollary follows by Lemma A.2.4 and Quinn's Lemma above, if we show: B has locally constant fundamental group π in the complement, if R_1 has locally constant fundamental group π.

Remark: The other direction also follows from the proof of Lemma A.2.4, up to the "0-LC" part included in the definition of locally constant fundamental groups of ends, but is not needed in the following anyway.

Assume R_1 has locally constant fundamental group π and let $x \in B$. There are neighborhoods U_0 of B, and V_0 of x and a regular covering p, with $\mathrm{Deck}(p) = \pi = \pi_1(N_0 - V_0) / \mathrm{im}(p_*)$ (since p is regular), where $N_0 := R_1^{-1}(V_0) \cap U_0$. Let $\alpha : \pi_1(N_0 - V_0) \to \pi$ be the quotient map.

Choose a slightly smaller $V \subset V_0$, Lemma A.2.1 yields a U such that $R(N \times I) \subset N_0$, where $N := R_1^{-1}(V) \cap U$.

Let $\epsilon > 0$. We may assume closure(N) is compact, as X is locally compact, and we are allowed to choose N arbitrarily small. Define for all $x \in V_0$ a ball V_x s. t. $R_1^{-1}(V_i) \cap B^{\epsilon/2}$ has diameter less than ϵ. By the definition of locally constant fundamental group of the end, for $B^{\epsilon/2}$ (as a neighborhood of the end) and V_x there are U'_x and V'_x such that loops in the covering over $R_1^{-1}(V'_x) \cap U'_x$ contract over $R_1^{-1}(V_x) \cap B^{\epsilon/2}$. Let $\delta_x > 0$ be the largest possible number, such that $B_{2\delta}(x) \subset R_1^{-1}(V'_x) \cap U'_x$. Clearly already the $B_\delta(x)$ cover the compact closure(V_0). Thus there is a finite collection $x_1, \ldots, x_n$ with $B_\delta(x_i)$ covering V_0. Let $\tilde{\delta} := \min(\delta_{x_1}, \ldots, \delta_{x_n})$.

By a uniform continuity argument (using compactness of closure(N) $\times I$, see previous section) there is a $\delta > 0$, s. t. $d(x, x') < \delta \Rightarrow \forall t \in I : d(R(x,t), R(x',t)) < \min(\tilde{\delta}, \epsilon/4)$.

Let γ be a loop in $N - B$ of diameter smaller δ. The choice of δ, together with $R(x,0) = x$ and $R(N \times I) \subset N_0$, allows us to pull γ arbitrary close to B in N_0 using R, without increasing the size beyond $\min(\tilde{\delta}, \epsilon/4)$, thus assume γ is in $B^{\tilde{\delta}}$, thus in some $B_{2\delta}(x_i)$, so a lift $\tilde{\gamma}$ contracts over $R_1^{-1}(V_i) \cap B^{\epsilon/2}$, which was chosen with diameter less than ϵ.

If $[\gamma] \in \mathrm{im}(p_*) \Leftrightarrow \alpha([\gamma]) = 0$, then γ also contracts in $R_1^{-1}(V_i) \cap B^{\epsilon/2}$, thus by a homotopy of diameter less than ϵ. Conversely, if $[\gamma] \notin \mathrm{im}(p_*) \Leftrightarrow \alpha([\gamma]) \neq 0$, assume that ϵ small enough that γ is contained in an evenly covered neighborhood. Then $[\gamma] \neq 0 \in \mathrm{Deck}(p)$ lifts to a path leaving the evenly covered neighborhood. □

This shows, how tameness of strata in MHSS implies the tameness properties required to apply "standard" controlled end-theory. Finally, it remains to assemble the proofs, on which the first section, and the "rewriting" of the end-theorem in terms of MHSS and cylinder-neighborhoods, were based.

A.4 Previously Deferred Proofs of Examples

In the first section it has been claimed in Example A.1.7, that for X a MHSS with two strata M (dense in X), B and r as before, the end of r is tame, 0-LC and onto, with locally constant fundamental group $\pi_1(r) = \pi_1(L)$, where L is the fiber of $\mathrm{holink}(X, B) \to B$.

Proof of Example A.1.7: The locally constant fundamental group and 0-LC parts have been shown in Lemma A.2.4, the tameness has been shown in Cor. A.3.1–1.

For the onto part, assume $U \subset X$ were an open neighborhood of B with $r(U - B) \neq B$. Let $b \in B - r(U - B)$. Then $i|_{B-\{b\}} \to N - \{b\}$ and $r' : N - \{b\} \to B - \{b\}$ are homotopy-inverse to each other, because $R : N \times I - \{b\} \times I \to N - \{b\}$ is well-defined by nearly-strictness and r is rel B. Thus $i : (B, B - \{b\}) \to (N, N - \{b\})$ is a strict homotopy-equivalence and therefore

$$H_*(N, N - \{b\}) \cong H_*(B, B - \{b\}) \cong \begin{cases} \mathbb{Z} & \text{if} \quad * = \dim(B) \text{ or } 0 \\ 0 & \text{otherwise} \end{cases}$$

However by [Qui88a, Lemma 2.4 (p. 454)] (X, B) is also strict homotopy-equivalent to the pushout of the homotopy-link (rel B; using evaluation at 0 and at 1 as maps). Using that ev_0 is a fibration by hypothesis (with non-empty total-space over every component of B by density of $M \subset X$), for any $\dim(B) = i$-simplex $\alpha \in H_i(B, B - \{b\})$, we can lift its cone $b * \alpha$ to β in the homotopy-link. Map the cone-parameter to the cylinder of ev_0 such that the base of the cone maps to B and the rest of the cone to inside the cylinder of ev_0, thus obtaining $\gamma : b * \Delta^i \to \mathring{\mathrm{cyl}}(\mathrm{ev}_0)$ with the boundary $\partial\gamma = \alpha \cup \{\text{something in N-B}\}$, thus $\alpha = 0$ in $H_{\dim(B)}(N, N - \{b\})$. Hence $H_{\dim(B)}(N, N - \{b\}) = 0$, in contradiction with the conclusion above, thus there is $x \in N - B$ with $R(x) = b$.

By excision, and U being an open neighborhood of B, the local-homology conclusion remains true for N replaced by U, and using δ-holinks (cf. [Qui88a, Lemma 2.4 (p. 454)]) the construction leading to the contradiction can also be "pushed into" U. $\square$

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