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Analytic cohomology of families of L-analytic Lubin-Tate (φ_L, Γ_L)-modules

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Abstract

In this thesis we prove finiteness and base change properties for analytic cohomology of families of L-analytic (φ_L, Γ_L)-modules parametrised by affinoid algebras. To this end, we study an analogue of the Herr complex, which can be defined using p-adic Fourier theory. For technical reasons we work over a field containing the finite extension L of $\mathbb{Q}_p$ and a certain transcendental period.
In case the affinoid algebra is the base field, we prove that coadmissibility of the Iwasawa cohomology groups is sufficient for the existence of a comparison isomorphism between the Iwasawa cohomology of a (φ_L, Γ_L)-module over the Robba ring and the analytic cohomology of its Lubin-Tate deformation, which, roughly speaking, is obtained by base change to the algebra of L-analytic distributions on an open subgroup of Γ_L.
In the trianguline case we show that the complex computing Iwasawa cohomology is perfect and in particular satisfies the above condition.
Finally we describe how general perfectness results for Iwasawa cohomology can be achieved assuming conjecturally that the statement can be proved in the étale case.

Zusammenfassung

In dieser Arbeit zeigen wir Endlichkeits- und Basiswechseleigenschaften für analytische Kohomologie von Familien von L-analytischen (φ_L, Γ_L)-Moduln parametrisiert durch affinoide Algebren. Dazu untersuchen wir ein Analogon des Herr-Komplexes, welches mittels p-adischer Fouriertheorie definiert werden kann. Aus technischen Gründen arbeiten wir über einem Körper, welcher die endliche Erweiterung L von $\mathbb{Q}_p$ und eine gewisse transzendente Periode enthält.
Im Falle, dass die affinoide Algebra der Grundkörper ist, zeigen wir, dass eine hinreichende Bedingung für die Existenz eines Vergleichsisomorphismus zwischen der Iwasawakohomologie eines (φ_L, Γ_L)-Moduls über dem Robba-Ring und der analytischen Kohomologie seiner Lubin-Tate-Deformation, welche heuristisch durch Basiswechsel zur L-analytischen Distributionsalgebra einer offenen Untergruppe von Γ_L entsteht, die Kozulässigkeit der Iwasawa-Kohomologiegruppen ist.
Im triangulinen Fall zeigen wir, dass der Komplex, welcher die Iwasawa-Kohomologie berechnet, perfekt ist und insbesondere obige Bedingung erfüllt.
Zuletzt beschreiben wir, wie allgemeinere Perfektheitsergebnisse für die Iwasawa-Kohomologie erzielt werden können, unter Annahme der Vermutung, dass die Aussage im étalen Fall bewiesen werden kann.

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Contents

Introduction

Let p be a prime number. Among objects of fundamental importance in number theory are p-adic Galois representations, i.e., continuous representations of the absolute Galois group $G_{\mathbb{Q}_p}$ on finite dimensional $\mathbb{Q}_p$-vector spaces. They appear naturally in the context of p-adic étale cohomology and Fontaine's theory of (φ, Γ)-modules provides us with an equivalence of categories between p-adic Galois representations and so-called étale (φ, Γ)-modules. Let $L/\mathbb{Q}_p$ be a finite extension with absolute Galois group G_L. Fix an embedding $L \to \mathbb{C}_p$ and a uniformiser π_L of L. We denote by o_L the ring of integers of L and by q the cardinality of $o_L/\pi_L o_L$. Lubin-Tate (φ_L, Γ_L)-modules play an important role for understanding L-linear continuous representations of G_L. In the Lubin-Tate case $\varphi_L(T) \in o_L[\![T]\!]$ is a Frobenius power series for π_L, i.e., a series of the form $\varphi(T) = \pi_L T + T^q + \pi_L T^2 f$ with some $f \in o_L[\![T]\!]$. One can show that there exists a unique commutative one dimensional formal group law LT over o_L admitting φ_L as an endomorphism. The action of G_L on the π_L^n-torsion points $LT[\pi_L^n]$ defines, by passing to the limit with respect to n, a character $\chi_{LT} \colon G_L \to \mathrm{Aut}_{o_L}(\varprojlim LT[\pi_L^n]) \cong o_L^\times$. This character induces an isomorphism $\Gamma_L := \mathrm{Gal}(L_\infty/L) \cong o_L^\times$, where L_∞ is the extension obtained by adjoining to L the π_L-power torsion points. In the classical case one takes $\varphi_{\mathbb{Q}_p} = (1+T)^p - 1$. This leads to the multiplicative formal group $\mathbb{G}_m$ and the p-cyclotomic character $\chi_{cyc} \colon \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \overset{\cong}{\to} \mathbb{Z}_p^\times$. In analogy to the classical theory of cyclotomic (φ, Γ)-modules of Fontaine the category of all continuous L-linear representations is equivalent to the category of étale Lubin-Tate (φ_L, Γ_L)-modules (cf. [KR09]).

We denote by $\mathcal{R}_L^{[r,1)}$ the ring of formal Laurent series with coefficients in L converging on the half-open annulus $r \leq |T| < 1$. Their union $\mathcal{R}_L = \bigcup_{r\in[0,1)} \mathcal{R}_L^{[r,1)}$ is called the Robba ring. A crucial technique for analysing Galois representations is p-adic Hodge Theory, which uses various period rings to describe certain subcategories of representations. If one wishes to give a description purely in terms of (φ_L, Γ_L)-modules a passage to the Robba ring is required since the series in $\mathcal{R}_L$ admit a radius of convergence and can in a reasonable way be embedded into the classical period rings of p-adic Hodge Theory. One can show that the category of étale (φ_L, Γ_L)-modules over the Robba ring is equivalent to the category of so-called overconvergent étale (φ_L, Γ_L)-modules. A definition of overconvergence will be given in 1.82. For the purpose of this introduction we only need the following facts: In the cyclotomic case every Galois representation is overconvergent (cf. [Col98]) but in the case $L \neq \mathbb{Q}_p$ this is no longer true and it is not obvious how to describe Galois representations whose (φ_L, Γ_L)-modules are overconvergent in a non-tautological way. However, analyticity is a sufficient condition for overconvergence and we have in fact, by a theorem of Berger, an equivalence of categories between L-analytic representations and L-analytic (φ_L, Γ_L)-modules. A representation V is called L-analytic if for every embedding $\sigma\colon L \to \mathbb{C}_p$ that is different from the identity the semi-linear representation $\mathbb{C}_p \otimes_{\sigma,L} V$ is isomorphic to $\mathbb{C}_p^{\dim_L V}$. A (φ_L, Γ_L)-module M over $\mathcal{R}_L$ is called L-analytic if the action of Γ_L on the LF-space underlying M is differentiable and the action of $\mathrm{Lie}(\Gamma_L)$ is L-linear. This endows L-analytic (φ_L, Γ_L)-modules with additional structure. One can show that the Γ_L-action extends to an action of the algebra of L-analytic distributions $D(\Gamma_L, L)$ on M, which contains $L[\Gamma_L]$ as a dense subring and is the natural analogue of the Iwasawa algebra in this analytic situation. The classical equivalence of categories allows for a nice description of the Galois cohomology of a representation V in terms of its (φ, Γ)-module $\mathbb{D}(V)$. Assume for simplicity $p \neq 2$ such that Γ is pro-cyclic and admits a topological generator γ. In this case the Galois cohomology can be computed by the Herr complex, which is given by the total complex

$$C_{\varphi,\gamma}(\mathbb{D}(V)) := Tot \begin{pmatrix} \mathbb{D}(V) & \xrightarrow{\varphi-1} & \mathbb{D}(V) \\ \big\downarrow{\scriptstyle \gamma-1} & & \big\downarrow{\scriptstyle -(\gamma-1)} \\ \mathbb{D}(V) & \xrightarrow{\varphi-1} & \mathbb{D}(V) \end{pmatrix}.$$

In the case $L \neq \mathbb{Q}_p$ the group Γ_L is no longer pro-cyclic. Instead it contains an open subgroup $U \subset \Gamma_L$, which via the logarithm is isomorphic to the additive group $o_L \cong \mathbb{Z}_p^{[L:\mathbb{Q}_p]}$. However, thanks to the L-analytic structure on the modules, we can still describe the action of Γ_L using a single operator. A reasonable candidate is the action of $1 \in L \cong \mathrm{Lie}(\Gamma_L)$, which is commonly denoted by ∇. For an L-analytic (φ_L, Γ_L)-module this leads to the complex

$$C_{\varphi_L,\nabla}(M) := Tot \begin{pmatrix} M & \xrightarrow{\varphi_L-1} & M \\ \big\downarrow{\scriptstyle \nabla} & & \big\downarrow{\scriptstyle -\nabla} \\ M & \xrightarrow{\varphi_L-1} & M \end{pmatrix}.$$

This cohomology has been studied by Fourquaux and Xie in [FX12] and by Colmez in [Col16]. In some sense this leads to a coarser invariant than desired as can be seen in degree zero, where the cohomology is the set of elements fixed by φ_L and some open subgroup of Γ_L. A finer invariant is the cohomology theory studied by Berger and Fourquaux using L-analytic cochains of the semigroup $\varphi_L^{\mathbb{N}} \times \Gamma_L$ in [BF17]. They show that this cohomology, which they only define in degree zero and one, agrees with $H^0(\Gamma_L, H^i_{\varphi_L,\nabla}(M))$ for $i = 0, 1$. Using p-adic Fourier theory developed by Schneider and Teitelbaum we study a different candidate that allows us to describe for an open subgroup $U \subset \Gamma_L$ isomorphic to o_L the $\varphi_L^{\mathbb{N}} \times U$-cohomology using a single operator Z

that plays the role of $\gamma - 1$ in the classical case. Let $K \subset \mathbb{C}_p$ be a complete subfield containing a period Ω_L of the Lubin-Tate group in the sense of [ST01]. We denote by $D(U, L)$ (resp. $D(U, K)$) the algebra of L-analytic L-valued (resp. K-valued) distributions on U. By the work of Schneider and Teitelbaum there exists a rigid L-analytic variety $\mathfrak{X}_U$, whose points parametrise the locally L-analytic characters on U, such that the global sections of its structure sheaf are given by $D(U, L)$ via the Fourier isomorphism. They further show that over K as above $\mathfrak{X}_U$ is isomorphic to the open unit disc. This provides us with an isomorphism $D(U, K) \cong \mathcal{R}_K^{[0,1)}$ and we denote by Z the preimage of a coordinate T under this isomorphism. This allows us to define the complex

$$C_{\varphi_L,Z}(M) := Tot \begin{pmatrix} M & \xrightarrow{\varphi_L - 1} & M \\ \downarrow Z & & \downarrow -Z \\ M & \xrightarrow{\varphi_L - 1} & M \end{pmatrix}.$$

To see that this is conceptionally an analogue of the classical Herr complex recall that in the classical case (assume $p \neq 2$) the cyclotomic character induces $\Gamma_{\mathbb{Q}_p} \cong \mathbb{Z}_p^\times$ and the group $U^{(1)}$ of 1-units is isomorphic to $\mathbb{Z}_p$. For simplicity assume that the torsion subgroup of $\Gamma_{\mathbb{Q}_p}$ acts trivially on $\mathbb{D}(V)$. The Iwasawa algebra $\Lambda = \mathbb{Z}_p[\![U^{(1)}]\!]$ is isomorphic to $\mathbb{Z}_p[\![T]\!]$ via $\gamma - 1 \mapsto T$ and therefore the vertical column in the Herr complex represents $\mathbf{RHom}_\Lambda(\mathbb{Z}_p, \mathbb{D}(V))$. Similarly in our case $D(U, K)$ plays the role of the Iwasawa algebra and since

$$0 \to D(U, K) \xrightarrow{Z} D(U, K) \to K \to 0$$

is a projective resolution of K we conclude that the vertical column represents $\mathbf{RHom}_{D(U,K)}(K, M)$. This a priori algebraic invariant turns out to be of analytic nature. For the purpose of motivation we mention Kohlhaase's article [Koh11] but warn the reader that his theory is not directly applicable in our case because K is not assumed to be spherically complete. Kohlhaase defines a version of analytic cohomology and shows that these groups agree with the groups $\mathrm{Ext}^i_{D(U,K)}(K, M)$. With this in mind, up to signs, $C_{\varphi_L,Z}(M)$ can be written as the cone of $\varphi_L - 1$ acting on a complex that computes the analytic cohomology of U with coefficients in M, reinforcing that it is indeed an analytic analogue of the Herr complex. In 3.35 we show that $H^0(\Gamma_L, H^1_{\varphi_L,Z}(M))$ is isomorphic to the group of analytic extensions $\mathrm{Ext}^1_{an}(\mathcal{R}_K, M)$ and hence also isomorphic to the version considered by Berger and Fourquaux in degree one (and obviously in degree zero).
It is technically helpful to allow for more flexibility in terms of coefficients and to not restrict oneself to working only over fields. In this thesis we consider what one could call rigid-analytic families of L-analytic (φ_L, Γ_L)-modules. For an affinoid A in the sense of Tate we define these ad-hoc as (φ_L, Γ_L)-modules over the relative Robba ring $\mathcal{R}_A := \mathcal{R}_L \hat{\otimes}_L A$ whose underlying topological vector space has an L-analytic action of Γ_L. One can think of them as families of L-analytic (φ_L, Γ_L)-modules parametrised by the rigid analytic space $\mathrm{Sp}(A)$. In the classical case Berger and

Colmez construct a functor that assigns to an A-valued representation such a (φ,Γ)-module (cf. [BC08, Théorème A]). Their formalism relies on the Colmez-Sen-Tate method, which implies the overconvergence of all p-adic Galois representations as a corollary. For that abstract reason it cannot be generalised without difficulties to the Lubin-Tate case since there exist representations that are not overconvergent. We do not provide an analogous functor and instead work with the above ad-hoc description of families of (φ_L,Γ_L)-modules. Even when considering the case $A=K$ the study of families plays a crucial role for understanding another important complex namely

$$C_\Psi(M)\colon M \xrightarrow{\Psi-1} M,$$

whose cohomology we denote by $H^i_{Iw}(M)$. Here Ψ denotes the left-inverse operator of φ_L (see 1.64 for a precise definition). Informally speaking this complex (concentrated in degrees $[1,2]$) computes the Iwasawa cohomology in the étale case in both the cyclotomic situation (cf. [CC99, Proposition II.3.1, Remarque II.3.2]) and the Lubin-Tate case by [SV15, Theorem 5.13] [1].

Summary of the main results

The starting point is the study of the Γ_L-action on $M^{\Psi=0}$. By transport of structure via the isomorphism $\mathcal{R}_K^+ \cong D(U,K)$ we can define the group Robba ring $\mathcal{R}_K(U)$ (and similarly $\mathcal{R}_K(\Gamma_L)$). We denote by $\eta(1,T)$ the power series corresponding to $1 \in o_L \cong U$ under this isomorphism. Comparing the action of Z and the action of T allows us to show the following theorem:

Theorem 1 (Theorem 2.19)**.** *Let M be an L-analytic (φ_L,Γ_L)-module over $\mathcal{R}_A$ admitting a model*[2] *over $[r_0,1)$, then there exists $r_1 \geq r_0$ such that for any $r \geq r_1$ the Γ_L-action on $(M^{[r,1)})^{\psi=0}$ extends to an action of $\mathcal{R}_A^{[r,1)}(\Gamma_L)$ with respect to which $(M^{[r,1)})^{\psi=0}$ is finite projective of rank $\operatorname{rank}_{\mathcal{R}_A}(M)$. If $m_1,\dots,m_d$ generate $M^{[r,1)}$ then the elements $\eta(1,T)\varphi_L(m_1),\dots,\eta(1,T)\varphi_L(m_d)$ generate $(M^{[r,1)})^{\psi=0}$ as a $\mathcal{R}_A^{[r,1)}(\Gamma_L)$-module.*

In the cyclotomic case this is [KPX14, Theorem 3.1.1] and this theorem was proven by Schneider and Venjakob in [SV20] for free modules over $\mathcal{R}_K$. This result shows that the variable Z has properties analogous to the operator $\gamma-1$ studied in the classical case. As an immediate corollary we obtain a comparison isomorphism between the (Ψ,Z) and (φ_L,Z)-cohomology.

[1] Note that the operator used by Schneider and Venjakob differs from our Ψ by a constant. See 4.10 for a technical solution.

[2] A model is a (φ_L,Γ_L)-module $M^{[r_0,1)}$ over $\mathcal{R}_A^{[r_0,1)}$ such that $M=\mathcal{R}_A \otimes_{\mathcal{R}_A^{[r_0,1)}} M^{[r_0,1)}$.

Corollary. *Let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$. The morphism of complexes*

$$\begin{array}{ccccccc} C_{\varphi_L,Z}(M): & & M & \longrightarrow & M \oplus M & \longrightarrow & M \\ \downarrow & & \downarrow \scriptstyle{\mathrm{id}} & & \downarrow \scriptstyle{-\Psi \oplus \mathrm{id}} & & \downarrow \scriptstyle{-\Psi} \\ C_{\Psi,Z}(M): & & M & \longrightarrow & M \oplus M & \longrightarrow & M \end{array}$$

is a quasi-isomorphism.

Next we study finiteness and base change properties of $C_{\varphi_L,Z}(M)$. Here we deviate from the approach of [KPX14], who deduce finiteness as a consequence of finiteness of Iwasawa cohomology and instead opt for an approach using methods of [KL16]. The finiteness of (φ_L, Z)-cohomology can be deduced systematically from the general finiteness statements in [KL16]. We denote by $\mathbf{D}^{[0,2]}_{perf}(A)$ the full subcategory of the derived category consisting of complexes, which are quasi-isomorphic to a complex of finitely generated projective modules concentrated in degrees $[0, 2]$.

Theorem 2 (Theorem 3.22)**.** *Let A, B be K-affinoid and let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$. Let $f\colon A \to B$ be a morphism of K-affinoid algebras. Then:*

(1) $C_{\varphi_L,Z}(M) \in \mathbf{D}^{[0,2]}_{perf}(A)$.

(2) The natural morphism $C_{\varphi_L,Z}(M) \otimes^{\mathbb{L}}_A B \to C_{\varphi_L,Z}(M \hat{\otimes}_A B)$ is a quasi-isomorphism.

In particular the cohomology groups $H^i_{\varphi_L,Z}(M)$ are finite A-modules for every i.

While the preceding results are proven over the relative Robba ring $\mathcal{R}_A$ we restrict our considerations to the case $A = K$ when studying Iwasawa cohomology since already in this case we run into a series of subtleties.
However even when restricting A to a point the study of Iwasawa cohomology leads us to studying a non-trivial family of (φ_L, Γ_L)-modules, which we call the Lubin-Tate deformation of M. Roughly speaking one would like to take the completed tensor product $D(\Gamma_L, K)\hat{\otimes}_K M$ and as such define a family of (φ_L, Γ_L)-modules parametrised by the rigid analytic space $\mathfrak{X}_{\Gamma_L}$. For K'/K finite, the K' points of $\mathfrak{X}_{\Gamma_L}$ are in bijection with L-analytic K' valued characters on Γ_L and hence the Lubin-Tate deformation of M parametrises all twists of M by L-analytic characters. Some care is required because $D(\Gamma_L, K)$ is no longer affinoid. In order to work within the framework of families as above we have to work over an affinoid cover leading us to a derived limit of the corresponding Herr complexes. In more precise terms we can write $D(U, K)$ as a projective limit of affinoid algebras D_n and the deformation (on the level of U) is defined as a "sheaf" $(\mathbf{Dfm}_n(M))_{n\in\mathbb{N}}$, where each term is given as $\mathbf{Dfm}_n(M) = D_n\hat{\otimes}_K M$ and $\gamma \in U$ acts as $\delta_{\gamma^{-1}} \otimes \gamma$. The analytic cohomology of this deformation can be related to the Iwasawa cohomology of M via the following theorem.

Theorem 3 (cf. Theorem 4.24)**.** *Suppose $C_\Psi(M)$ has coadmissible cohomology groups. Then there is a canonical*[3] *isomorphism in the derived category $\mathbf{D}(D(U,K))$*

$$\mathbf{Rlim} C_{\Psi,Z}(\mathbf{Dfm}_n(M)) \simeq C_\Psi(M)$$

and, in particular, we have isomorphisms

$$\varprojlim_n H^i_{\Psi,Z}(\mathbf{Dfm}_n(M)) \cong H^i_{Iw}(M).$$

This makes it clear that the coadmissibility of the Iwasawa cohomology of M is a desirable property and we show that a sufficient condition is that $M^{\Psi=1}$ is finitely generated as a $D(U,K)$-module. We proceed to prove that this condition is satisfied by modules of rank one, which are of the form $\mathcal{R}_K(\delta)$ for a locally L-analytic character $\delta\colon L^\times \to K^\times$. By a dévissage argument this carries over to trianguline modules, i.e., successive extensions of such $\mathcal{R}_K(\delta)$. This leads us to the following theorem.

Theorem 4. *(Theorem 5.8) Let M be a trianguline L-analytic (φ_L,Γ_L)-module over $\mathcal{R}_K$. Then $C_{c\Psi}(M)$ is a perfect complex of $D(\Gamma_L,K)$-modules for any constant $c \in K^\times$.*

From the work of Schneider and Venjakob one obtains that the Iwasawa cohomology $H^i_{Iw}(\Gamma_L,V)$ of an o_L-linear representation V of G_L is computed by the complex

$$\mathbb{D}(V(\tau)) \xrightarrow{\psi_{LT}-1} \mathbb{D}(V(\tau)),$$

with some character τ. Applying this to $V(\tau^{-1})$ instead of V we obtain, in particular, that $\mathbb{D}(V)^{\psi_{LT}=1}$ is a finite $\Lambda = o_L[\![\Gamma_L]\!]$-module. If one assumes further that V is L-analytic, one can show that $\mathbb{D}(V)^{\psi_{LT}=1}$ is contained in the overconvergent submodule $\mathbb{D}^\dagger(V)$ providing us with a natural map

$$D(\Gamma_L,K)\otimes_\Lambda \mathbb{D}^\dagger(V)^{\psi_{LT}=1} \to M^{\psi_{LT}=1},$$

where M denotes the completed base change to K of the (φ_L,Γ_L)-module over $\mathcal{R}_L$ attached to V. We conjecture that this map is an isomorphism. For our purpose the following weak form is sufficient.

Conjecture. *(Conjecture 6.3) The natural map*

$$D(\Gamma_L,K)\otimes_\Lambda \mathbb{D}^\dagger(V)^{\psi_{LT}=1} \to M^{\psi_{LT}=1}$$

is surjective.

In the final chapter we explain how the étale results of Schneider and Venjakob can be implemented to show general perfectness statements for those modules that arise as a base change of some M_0 over $\mathcal{R}_L$ under the assumption of this conjecture.

[3] See Theorem 4.24 for a precise description of the maps.

The restriction to modules coming from $\mathcal{R}_L$ is rooted in the methodology of the proof. The idea is that the statement is true for a certain class of (φ_L, Γ_L)-modules, namely the étale modules studied by Schneider and Venjakob in [SV15]. An induction over Harder-Narasimhan slopes of (φ_L, Γ_L)-modules "propagates" the theorem to all L-analytic (φ_L, Γ_L)-modules over $\mathcal{R}_L$. By introducing the large extension K/L we leave the setting of Kedlayas slope filtrations due to no longer working with Bézout rings and our proofs make heavy use of the structure of $D(U, K)$ as a power series ring. For that reason we are uncertain whether the results can be descended to M_0 (as a $D(\Gamma_L, L)$-module) and whether they hold for any M (not necessarily arising as a base change from $\mathcal{R}_L$ or say $\mathcal{R}_F$ for some finite extension F/L).

Theorem 5 (Theorem 6.11). *Assume Conjecture 6.3. Let M_0 be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_L$ and let $M := K\hat{\otimes}_L M_0$. Then the complex $C_\Psi(M)$ of $D(\Gamma_L, K)$-modules is perfect.*

We go on to discuss the Euler-Poincaré characteristic formula for $C_{\Psi,Z}(M)$. We prove that the expected Euler-Poincaré characteristic formula

$$\sum_{i\in\mathbb{N}_0}(-1)^i \dim_K H^i_{\Psi,Z}(M) = -[\Gamma_L : U]\,\mathrm{rank}_{\mathcal{R}_L}(M)$$

holds in the trianguline case (see Remark 6.16). We also show that the formula in the general case would follow from $\mathcal{C}(M) = (1-\varphi_L)M^{\Psi=1}$ being projective over $D(U, K)$ of rank $[\Gamma_L : U]\,\mathrm{rank}_{\mathcal{R}_K}(M)$.
The main inspiration for our work is [KPX14]. In fact the complex considered by us specialises to theirs in the cyclotomic case since one can take $Z = \gamma - 1$. However [KPX14] work in a situation where Tate-Duality, the Euler-Poincaré formula and slope theory are known for the base case $A = K = \mathbb{Q}_p$ and can be applied point-wise. The main difference is the fact that we need to adjoin Ω_L leading to a transcendental extension K/L (which introduces functional analytic subtleties in its own right) in order to truly compare $\mathcal{R}_K^{[r,1)}$ and $D(U, K)$. While our theorems 2.19 and 3.22 are generalisations of their analogues in [KPX14], we make more restrictive assumptions in 6.11 and 4.24. Of course, one can extend coefficients even further by base changing to a spherical completion. We opt to work within $\mathbb{C}_p$ due to the advantage that arithmetic information can be recovered if we start with a (φ_L, Γ_L)-module coming from $\mathcal{R}_L$ by taking G_L-invariants with respect to the coefficient-wise action.

Applications

An application of the theory developed in [KPX14] is the study of global triangulation (cf. Section 6 in loc.cit.). Our theorem 3.22 shows that analytic cohomology is well-behaved with respect to affinoid covers, which can be used to study families parametrised by more general rigid analytic spaces. It is likely that one can explore similar avenues but an obstruction are the Euler-Poincaré characteristic formula and Tate-Duality, whose analogues are yet to be established in the L-analytic case.

A topic of great importance in global Iwasawa theory is the study of p-adic L-functions. Locally this leads to a study of local ε-constants or in more general terms local ε-isomorphism conjectures (cf. [LVZ15]). The most refined application of our results so far consists of an ongoing project (joint with Malčić, Venjakob and Witzelsperger) on ε-isomorphisms for L-analytic (φ_L, Γ_L)-modules of rank one. In order to explain, how our results fit into the picture, we give a rough outline of the key ideas. In [Nak17] Nakamura formulates a generalised local ε-conjecture for (φ, Γ)-modules and proves it in the rank one case. To each (φ, Γ)-module M he attaches a trivialisation of a certain (graded) line bundle, its fundamental line, which is essentially given as the determinant of the Herr complex. The ε-conjecture states that there is a unique such assignment satisfying certain compatibility properties (cf. [Nak17, Conjecture 3.8]).

The results of [KPX14] are a crucial technical input to define the fundamental line since the determinant of a complex only makes sense if the complex is perfect. Furthermore the perfectness of Iwasawa cohomology and its comparison to the cyclotomic deformation are crucial for the definition of ε-isomorphisms used by Nakamura in his proof in the rank one case. These results allow him to reinterpret the twist by a character, which factors over Γ as a base change to a quotient of the deformation, thereby reducing the problem to (φ, Γ)-modules $\mathcal{R}(\delta)$ such that the character δ is trivial on $\mathbb{Z}_p^\times$. Our results lay the foundations in the L-analytic case and allow us to define a fundamental line using analytic cohomology hence forming the cornerstone for a local ε-conjecture in this case.

Chapter 1

Preliminaries

1.1 Some functional analysis.

Functional analysis over non-archimedean fields is for the most part very similar to the classical theory over $\mathbb{K} = \mathbb{C}, \mathbb{R}$ but differs in some key points when the base field is not spherically complete. A simple way to enforce spherical completeness is to assume that the value group is discrete. We will be working with LF and Fréchet spaces over a complete subfield $K \subseteq \mathbb{C}_p$ with dense value group, which is why we cannot assume K to be spherically complete. For the moment let K be a field of characteristic zero that is complete with respect to a non-trivial ultra-metric absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$. Recall that a topological K-vector space is called locally convex if its topology can be defined by a family of semi-norms. For two locally convex spaces E_1, E_2 we denote by $\mathcal{L}_b(E_1, E_2)$ the space of continuous linear maps with the strong topology.

Definition 1.1. *A locally convex space E is called* ***of countable type*** *if for every continuous semi-norm on E the assosicated normed space $E/\ker(p)$ contains a countable set whose K-span is dense.*

We warn the reader that this does not imply that E itself contains a countably-dimensional dense subspace (such spaces are called strictly of countable type) unless E is metrizable. The following lemma is useful for checking whether a given space is of countable type.

Lemma 1.2. *Subspaces, continuous linear images, products, projective limits, countable locally convex direct sums and inductive limits of spaces of countable type are of countable type.*

Proof. See [PGS10, Theorem 4.2.13]. □

We denote by E' the space of continuous linear forms on E. We have the following version of the Hahn-Banach theorem.

Theorem 1.3 (Hahn-Banach)**.** *Let E be a locally convex space of countable type and $D \subset E$ a subspace.*

1. *Then any $f \in D'$ can be extended to $\overline{f} \in E'$.*

2. *If E is normed then for every $\varepsilon > 0$ and every $f \in D'$ there exists an extension of f to an element $\overline{f} \in E'$ such that*

$$||\overline{f}|| \leq (1+\varepsilon)||f||.$$

Proof. See [PGS10, Theorem 4.2.4] and its corollaries. □

Example 1.4. *Let $n \in \mathbb{N}$. The n-dimensional Tate algebra $K\langle X_1, \ldots X_n\rangle$ is of countable type.*

Proof. The polynomial algebra $K[X_1, \ldots, X_n]$ is a countably-dimensional dense subspace. □

Definition 1.5. *A K-**Banach algebra** R is a unital K-algebra endowed with a norm making it a K-Banach space space such that $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in R$. We always assume $\|1_R\| = 1$ (whenever $R \neq 0$).*

Proposition 1.6. *Let R be a Noetherian K-Banach algebra.*

1. *Every finitely generated A-module M has a unique K-Banach space structure making it an A-Banach module. The norm on M is equivalent to the one obtained by taking the quotient norm of any surjection $R^n \to M$.*

2. *Any morphism between finitely generated A-modules is continuous and strict with respect to this topology.*

Proof. See [BGR84, Section 3/3.7.3]. □

When M is a module over a Noetherian Banach algebra we implicitly endow it with its canonical topology from Proposition 1.6.

Definition 1.7. *A homomorphism of topological groups $f : G \to H$ is called **strict**, if the induced map $G/\ker f \to \operatorname{Im} f$ is a topological isomorphism with respect to the quotient topology on the left and the subspace topology on the right.*

Lemma 1.8. *Let G, H be metrizable topological groups with completions $\hat{G}$ (resp. $\hat{H}$) and let $f : G \to H$ be strict. Then*

$$\hat{f} : \hat{G} \to \hat{H}$$

is strict with kernel $\widehat{\ker(f)} = \overline{\ker(f)}$ and image $\widehat{\operatorname{Im} f} = \overline{\operatorname{Im} f}$.

Proof. See [Bou66, Chapter IX §3.1 Corollary 2 p.164]. □

Lemma 1.9. *Let G, H be Hausdorff abelian topological groups and $f : G \to H$ continuous such that f is strict on a dense subgroup $D \subset G$. Then f is strict.*

Proof. Since H is assumed to be Hausdorff $V/\ker f$ and $\operatorname{Im} f$ are Hausdorff and we can without loss of generality assume that f is a continuous bijection that is strict when restricted to a dense subgroup and are left to show that its inverse is continuous. The abelian assumption asserts that G, H admit (Hausdorff) completions $\widehat{G}$, $\widehat{H}$ (cf. [Bou66, III §3.5 Theorem 2]) and since $f : D \to f(D)$ is a topological isomorphism, the induced map $\widehat{f_{|D}} \colon \widehat{D} \to \widehat{f(D)}$ is a topological isomorphism. Since D (resp. $f(D)$) is dense in G (resp. H) we conclude that the corresponding completions agree with $\widehat{G}$ (resp. $\widehat{H}$). The restriction of the inverse $\widehat{f_{|D}}^{-1}$ to H agrees with f^{-1} by construction and is continuous. □

The following criterion will play a crucial role for checking strictness and the Hausdorff property of certain cohomology groups of (φ_L, Γ_L)-modules. The second part of 1.10 is essentially an adaptation of [Sch02, Lemma 22.2] (which only treats Fredholm operators on Fréchet spaces) (see also [Tho19, Proposition 4.1.39]).

Lemma 1.10.

1. *Any continuous linear surjection $V \to W$ between LF-spaces over K is open and in particular strict.*
2. *Any continuous linear map $f \colon V \to W$ between Hausdorff LF-spaces with finite dimensional cokernel is strict. In addition the cokernel is Hausdorff.*

Proof. 1.) See [Sch02, Proposition 8.8]. 2.) Let $X \subset W$ be a finite dimensional subspace such that (algebraically) $W = \operatorname{im}(f) \oplus X$. Since W is Hausdorff, the subspace X carries its natural norm-topology. By assumption $V/\ker f$ is Hausdorff and thus by A.1 $V/\ker f$ is itself an LF-space and we have a continuous bijection

$$h : V/\ker f \oplus X \to W$$

of LF-spaces, which by the open mapping theorem is a homeomorphism. By construction $f \colon V/\ker f \to W$ factors via $V/\ker f \to V/\ker f \oplus X \to W$ and is thus a homeomorphism onto its image. Furthermore $\operatorname{im}(f) = \ker(p_2 \circ h^{-1} : W \to X)$ is the kernel of a continuous map into a Hausdorff space and thus closed, which implies that the cokernel is Hausdorff.

□

Lemma 1.11. *Let V', V, V'' be K-Fréchet spaces that fit into a strict exact sequence*[1]

$$0 \to V' \to V \to V'' \to 0.$$

Let F be a K-Banach space of countable type over K, then the induced sequence

$$0 \to F \hat{\otimes}_K V' \to F \hat{\otimes}_K V \to F \hat{\otimes}_K V'' \to 0$$

is strict exact.

[1] Equivalently it suffices to assume that V is Fréchet and V'' is Hausdorff.

Proof. By [PGS10, Corollary 2.3.9] any infinite dimensional Banach space of countable type is isomorphic to the space of zero sequences $c_0(K)$ and observe that $c_0(K)$ can be identified with the completion of $\oplus_{n\in\mathbb{N}}K$ inside $\prod_{n\in\mathbb{N}} K$. Without loss of generality assume F is infinite dimensional and take the isomorphism $F \cong c_o(K)$ as an identification. Similarily $c_0(K)\hat{\otimes}V$ is functorially isomorphic to the completion of $\oplus_{n\in\mathbb{N}}V$ inside of $\prod_{n\in\mathbb{N}} V$. Here a map $f : V \to W$ induces the map $(v_n)_n \mapsto (f(v_n))_n$ and it is clear that strictness is preserved. Furthermore $\prod_{n\in\mathbb{N}} V$ and hence also the subspace $\oplus_{n\in\mathbb{N}}V$ is metrizable. Passing to completions and applying 1.8 gives the desired result. □

Lemma 1.11 applies in particular when F/K is a complete normed field (or more generally a K-Banach algebra) of countable type over K. Since $\mathbb{Q}_p$ has only finitely many extensions of a given degree we can conclude that $\overline{\mathbb{Q}_p} \subset \mathbb{C}_p$ is a countably-dimensional dense subspace and in particular $\mathbb{C}_p$ is of countable type over $\mathbb{Q}_p$. Slightly more subtle is the fact that any complete subfield $K \subset \mathbb{C}_p$ is of countable type over $\mathbb{Q}_p$. This follows from [IZ95, Theorem 1], which asserts that $\overline{\mathbb{Q}_p} \cap K$ is dense in K (or by applying [PGS10, Corollary 2.3.14]).

1.2 Robba rings and their (φ_L, Γ_L)-actions.

Let $\varphi_L(X) \in o_L[\![X]\!]$ be a Frobenius power series for the uniformiser $\pi_L \in o_L$, i.e., a series satisfying

$$\varphi_L(X) = \pi_L X + \text{terms of higher order}$$

and

$$\varphi_L(X) \equiv X^q \mod \pi_L.$$

There is a unique Lubin-Tate group law $F_{LT}(X, Y)$ and a unique injective homomorphism of rings

$$o_L \to \mathrm{End}(LT),$$

mapping $a \in o_L$ to a power series $[a](X)$ such that $[\pi_L](X) = \varphi_L(X)$. Denote by L_n the extension of L that arises by adjoining all π_L^n-torsion points of the LT-group. The set $LT[\pi_L^n]$ of π_L^n-torsion points carries a natural o_L-module structure which respect to which it is a free $o_L/\pi_L^n o_L$-module of rank 1. One can show that L_n is a finite Galois extension of L with Galois group isomorphic to $\mathrm{End}(LT[\pi_L^n]) \cong (o_L/\pi_L^n o_L)^\times$ and by passing to the limit one obtains a continuous character

$$\chi_{LT} : G_L \to o_L^\times$$

inducing for $L_\infty = \bigcup_{n\geq 1} L_n$ an isomorphism

$$\Gamma_L := \mathrm{Gal}(L_\infty/L) \to o_L^\times.$$

We endow $o_L[\![X]\!]$ with an action of Γ_L via $\gamma(f(X)) = f([\chi_{LT}(\gamma)](X))$ and similarly for φ_L. These actions can be extended to the p-adic completion $\mathbf{A}_L$ of the ring of formal Laurent series $o_L((X))$. Note that these actions are continuous for the weak topology, for which a fundamental system of open neighbourhoods of zero is given by

$$U_{m,n} := \pi_L^m \mathbf{A}_L + X^n o_L[\![X]\!]$$

with $m, n \in \mathbb{N}$. We frequently deal with power series or Laurent series converging on some annulus $[r, s]$ with respect to the absolute value $|\cdot|$. To avoid confusion we usually express the radii in terms of $|\pi_L|$.

Definition 1.12. *Let $K \subset \mathbb{C}_p$ be a complete field. We denote by $\mathcal{R}_K^{[r,s]}$ the ring of Laurent series (resp. power series if $r = 0$) with coefficients in K that converge on the annulus $r \leq |x| \leq s$ for $r, s \in p^{\mathbb{Q}}$ and $x \in \mathbb{C}_p$. It is a Banach algebra with respect to the norm $|\cdot|_{[r,s]} := \max(|\cdot|_r, |\cdot|_s)$. We further define the Fréchet algebra $\mathcal{R}_K^r := \mathcal{R}_K^{[r,1)} := \varprojlim_{r<s<1} \mathcal{R}_K^{[r,s]}$ and finally the **Robba ring** $\mathcal{R}_K := \varinjlim_{0 \leq r<1} \mathcal{R}_K^{[r,1)}$ endowed with the LF topology.*

We obtain similar continuous actions of Γ_L on $\mathcal{R}_L^I$ for any interval $I = [r, s] \subset [0, 1]$. For details concerning the φ_L-action we refer to [BSX20, Section 2.2]. To ensure that the action of φ_L on $\mathcal{R}_L^I$ is well-defined one has to assume that the lower boundary r of I is either 0 or $r > |u|^q =: r_L$ for a non-trivial π_L-torsion point u of the Lubin-Tate group [2]. When φ_L acts on $\mathcal{R}_L^I$ it changes the radius of convergence and we obtain a morphism

$$\varphi_L : \mathcal{R}_L^I \to \mathcal{R}_L^{I^{1/q}}.$$

We implicitly assume that r, s lie in $|\overline{\mathbb{Q}_p}|$, because in this case the algebra $\mathcal{R}_L^I$ is affinoid (cf. [Lüt16, Example 1.3.2]) and this assumption is no restriction when considering the Robba ring due to cofinality considerations. We henceforth endow the rings $\mathcal{R}_L^{[r,1)}$ (for $r = 0$ or $r > r_L$) and $\mathcal{R}_L$ with the (φ_L, Γ_L)-actions induced by the actions on $\mathcal{R}_L^I$. We also work with relative versions of these rings either defined over some affinoid A over L or more generally some complete field extension K/L contained in $\mathbb{C}_p$. Before we describe these relative Robba rings we shall discuss some generalities concerning completed tensor products of Fréchet- and LF-spaces.

Definition 1.13. *Let X, Y be (semi-)normed modules over a normed ring S. On $X \otimes_S Y$ we define the tensor product (semi-)norm*

$$|v| := \inf_r \max |x_i||y_i|,$$

where r ranges over all representations of v as a sum of elementary tensors $v = \sum_i x_i \otimes y_i$.

Definition 1.14. *The **projective completed tensor product** of normed S-modules is defined as the completion of the usual tensor product with respect to the tensor product norm. If the topologies on X and Y are defined by a family of semi-norms, we can extend this notion in the obvious way. We write*

$$X \hat{\otimes}_S Y := X \hat{\otimes}_{S,\pi} Y$$

for the projective completed tensor product.

[2] In the second case the assumptions guarantee $\varphi(T) \in (\mathcal{R}_L^{I^{1/q}})^\times$.

Remark 1.15. *Let $X = \varprojlim_{n\in\mathbb{N}} X_n$ be a Fréchet space over K with Banach spaces X_n and let W be a normed K-vector space. Assume that the transition maps $X_{n+1} \to X_n$ have topologically dense image. Then the canonical map*

$$X\hat{\otimes}_K W \to \varprojlim_n X_n \hat{\otimes}_K W$$

is a topological isomorphism.

Proof. This is a special case of Lemma 2.1.4 in [BSX20]. Note that all involved spaces are Hausdorff because they are metrizable. □

Definition 1.16. *Let V, W be locally convex K-vector spaces. The* ***inductive tensor product topology*** *is defined as the finest topology such that the bilinear map*

$$V \times W \to V \otimes_K W$$

is separately continuous. We denote the completion of the usual tensor product with respect to that topology by $V\hat{\otimes}_{K,i}W$.

Remark 1.17. *The inductive and projective tensor products agree for Fréchet spaces. The inductive tensor product and its completed version commute with countable strict locally convex inductive limits of Fréchet spaces.*

Proof. For the first statement see [Sch02, Proposition 17.6]. By [Eme17, 1.1.30] the inductive tensor product commutes with locally convex inductive limits. Let $V = \varinjlim_n V_n$ and $W = \varinjlim_n W_n$ be strict LF-spaces with Fréchet spaces V_n (resp. W_n). We already know that $V\hat{\otimes}_{K,i}W$ is the completion of $\varinjlim_n (V_n \otimes_{K,i} W_n)$. In the proof of [BSX20, 2.1.7] it is shown that for an inductive system $(E_n)_n$ of locally convex vector spaces such that $\varinjlim_{n\in\mathbb{N}} \widehat{E_n}$ is Hausdorff and complete the natural map

$$\varinjlim_n \widehat{E_n} \to \widehat{\varinjlim_n E_n}$$

is an isomorphism. By [PGS10, Theorem 11.2.4 and Theorem 11.2.5] we may apply this result to $E_n = V_n \otimes_K W_n$, which yields the desired result. □

Remark 1.18. *Let $D \subset V$ (resp. $E \subset W$) be dense subsets of locally convex spaces V, W. Then $D \otimes_K V$ is dense in $E \otimes_K W$ and $E\hat{\otimes}_K W$.*

Proof. [PGS10, Corollary 10.2.10] shows that the natural map $D \otimes_K E \to V \otimes_K W$ is a topological embedding. Applying Corollary 10.2.10(v) in loc.cit. to each seminorm defining the topology on $V \otimes_K W$ shows that $D \otimes_K E$ is dense in the $V \otimes_K W$. Because $V \otimes_K W \to V\widehat{\otimes}_K W$ is a topological embedding with dense image the statement follows. □

Definition 1.19. *An **affinoid algebra** A over a non-archimedean complete field F is an algebra that is isomorphic to*

$$T^n/I,$$

where T^n denotes the n-dimensional Tate-Algebra $F\langle X_1, \ldots, X_n\rangle$ and $I \subset T^n$ is an ideal. We always endow A with the residue norm obtained from the Gauß-norm on T_n. By [Bos14, 3.1 Proposition 20] any two residue norms are equivalent and any ideal in T^n is closed (cf. [Bos14, Section 2.3]).

Definition 1.20. *Let $F \subset K$ be a complete subfield and A be an affinoid algebra over F. We define the **relative Robba rings** $\mathcal{R}_A^{[r,s]} := \mathcal{R}_K^{[r,s]}\hat{\otimes}_F A$ and similarly $\mathcal{R}_A^{[r,1)}$ and $\mathcal{R}_A := \varinjlim_{0\leq r<1} \mathcal{R}_A^{[r,1)}$. These rings are naturally equipped with topolgies induced by the tensor product norm on $\mathcal{R}_A^{[r,s]}$.*

Definition 1.21. *A linear map $T : E \to F$ between locally convex K-vector spaces is called **compactoid** if there exists a zero neighbourhood $U \subset E$ such that $T(U)$ is **compactoid** in W meaning that for every zero neighbourhood $V \subset F$ there exists a finite set $e_1, \ldots, e_n$ such that*

$$T(U) \subset V + \sum_{i=1}^{n} o_K e_i.$$

The following is a technical subtlety and does not follow from 1.17 because (non-strict) LF-spaces are not automatically complete.

Lemma 1.22. *Let $E = \varinjlim_n E_n$ be an LB-space with L-Banach spaces E_n and compactoid steps. Let W be an L-Banach space then the natural map*

$$\varinjlim_n (E_n\hat{\otimes}_{L,\pi}W) \to E\hat{\otimes}_{L,\pi}W$$

is an isomorphism. In particular

$$E\hat{\otimes}_{L,\pi}W = E\hat{\otimes}_{L,i}W.$$

Proof. By [PGS10, 11.3.5] E is complete reflexive and its strong dual $E' := E'_b$ is Fréchet. Furthermore as an inductive limit of bornological spaces E is bornological by [Sch02, Example 2) after 6.13]. By [Sch02, 18.8] together with reflexivity we have

$$E\hat{\otimes}_{K,\pi}W = E''\hat{\otimes}_{K,\pi}W = \mathcal{L}_b(E', W).$$

Furthermore by [ST02, Proposition 1.5]

$$\varinjlim_n E_n\hat{\otimes}_{K,\pi}W = \mathcal{L}_b(E', W).$$

Combining the above and unwinding the definitions of the involved maps yields the desired claim. □

Remark 1.23. *The relative Robba ring $\mathcal{R}_A$ is complete. In particular $\mathcal{R}_A = A\hat{\otimes}_{L,i}\mathcal{R}_L$. Furthermore $\mathcal{R}_A = A\hat{\otimes}_{L,\pi}\mathcal{R}_L$.*

Proof. Recall from the proof of [BSX20, 2.1.6] that $\mathcal{R}_L$ admits a decomposition of the form $\mathcal{R}_L = \mathcal{R}_L^+ \oplus E$ with an LB-space $E = \varinjlim_n E_n$ with compactoid steps. For such spaces it is known that their inductive limit is complete by [PGS10, 11.3.5]. We obtain a corresponding decomposition $\mathcal{R}_A = \mathcal{R}_A^+ \oplus \varinjlim_n A\hat{\otimes}_{L,i}E_n$ with $\mathcal{R}_A^+ = A\hat{\otimes}_{L,i}\mathcal{R}_L^+$ and hence $\mathcal{R}_A^+ = A\hat{\otimes}_{L,\pi}\mathcal{R}_L^+$ by 1.17. The other summand is treated by the preceding Lemma 1.22.

□

Lemma 1.24. *Let $F \subset K$ be a complete subfield and A be an affinoid Algebra over F. Then the natural map induces isomorphisms*

$$\mathcal{R}_K^I\hat{\otimes}_F A \cong \mathcal{R}_K^I\hat{\otimes}_K(K\hat{\otimes}_F A)$$

and

$$\mathcal{R}_K\hat{\otimes}_{F,i} A \cong \mathcal{R}_K\hat{\otimes}_{K,i}(K\hat{\otimes}_F A).$$

Proof. The embedding $F \subset K$ is by construction isometric and thus contracting. Applying [BGR84, 2.1.7 Proposition 7] we obtain

$$\mathcal{R}_K^I\hat{\otimes}_F A = (\mathcal{R}_K^I\hat{\otimes}_K K)\hat{\otimes}_F A \cong \mathcal{R}_K^I\hat{\otimes}_K(K\hat{\otimes}_F A).$$

The second part follows by taking limits. □

Lemma 1.24 allows us to restrict ourselves to the case $F = K$ since the base change $K\hat{\otimes}_F A$ of an affinoid algebra over F is an affinoid algebra over K (cf. [BGR84, 6.1.1. Corollary 9]).

Remark 1.25. *Fix a Banach norm on A. Let $f \in \mathcal{R}_A$. Then f can be expressed uniquely as a Laurent series $f = \sum_{i\in\mathbb{Z}} a_i T^i$, with $a_i \in A$ converging on some half-open disc $|T| \in [r, 1)$.*

Proof. Since f belongs to some $\mathcal{R}_A^r$ and $\mathcal{R}_A^r = \varprojlim_{r\leq s<1} \mathcal{R}_A^{[r,s]}$, it suffices to treat $\mathcal{R}_A^{[r,s]}$. In that case we can by cofinality arguments always assume $r, s \in |K^\times|$. Then the K-algebra $\mathcal{R}_K^{[r,s]}$ is K-affinoid and if the absolute values r, s are achieved by $|\rho| = r$ and $|\sigma| = s$ then the set $\{(T/\rho)^n, (\sigma/T)^m \mid n \in \mathbb{N}_0, m \in \mathbb{N}\}$ form a orthonormal basis of $\mathcal{R}_K^{[r,s]}$. Completed base change to A shows that any element in $\mathcal{R}_A^{[r,s]}$ can be uniquely written as a convergent series $\sum_{n\geq 0} a_n(T/\rho)^n + \sum_{m<0} a_m(T/\sigma)^m$. Since $\rho, \sigma \in K^\times \subset A^\times$ we get the uniqueness of the Laurent expansion. □

We have two reasonable choices for the (φ_L, Γ_L)-actions on $\mathcal{R}_K$ for a complete field $K \subset \mathbb{C}_p$. One possibility is the linear extension of the (φ_L, Γ_L) action from $\mathcal{R}_L$ using $\mathcal{R}_K = K\hat{\otimes}_L\mathcal{R}_L$. If on the other hand K is invariant under the G_L action on $\mathbb{C}_p$, we can take the semi-linear G_L-action, which factors over Γ_L if $K \subset \widehat{L_\infty}$. Unless stated otherwise we consider only the former action. This action also makes sense for more general coefficients. Another reason for studying the linear action rather than the semi-linear actions is that we would like to work with L-analytic actions. The semi-linear action on say $\widehat{L_\infty}$ will never be L-analytic by [BC16, Corollaire 4.3].

1.3 p-adic Fourier theory and $D(G, L)$ actions.

We give a survey of p-adic distributions, that play a crucial role in the study of L-analytic (φ, Γ_L)-modules. In the case $G = \mathbb{Z}_p$ and $L = \mathbb{Q}_p$ a theorem of Amice asserts that $D(G, \mathbb{Q}_p)$ is isomorphic to the holomorphic functions on the open unit disc in the variable $\mathfrak{Z} = \delta_1 - 1$. This facilitates the study of the $\Gamma_{\mathbb{Q}_p}$-action in the classical theory. If $L \neq \mathbb{Q}_p$ a similar result can only be achieved after passing to a large extension K of L that contains a certain (transcendental) period $\Omega_L \in \mathbb{C}_p$. In the notation of [ST01] the period can be taken to be $\Omega_L := \Omega_t$ for some basis t of the Tate module of the dual of the Lubin-Tate group.

Definition 1.26. *Let G be a compact L-analytic group. We denote by $D_{\mathbb{Q}_p}(G, K)$ the algebra of $\mathbb{Q}_p$-analytic distributions with values in K, which is the strong dual of $C^{\mathbb{Q}_p\text{-}an}(G, K)$ the space of locally $\mathbb{Q}_p$-analytic functions on G with values in K with multiplication given by convolution. We denote by δ_g the Dirac distribution associated to g, by which we mean the evaluation map $\delta_g \colon f \mapsto f(g)$. We denote by $D(G, K)$ the quotient of $D_{\mathbb{Q}_p}(G, K)$ corresponding to the dual of the subspace of the space $C^{an}(G, K)$ of locally L-analytic functions.*

For a detailed description of the topology on $C^{an}(G, K)$ we refer the reader to [Sch17, Chapters 10 and 12].

Theorem 1.27. *Let $G = o_L$ viewed as an L-analytic group in the natural sense and let $L \subset K \subset \mathbb{C}_p$ be a complete intermediate field. Denote by $\hat{G}$ the character variety constructed in [ST01, Section 2]. Then the Fourier transform (defined on p. 452 in loc. cit.) induces an isomorphism of K-Fréchet algebras*

$$D(G, K) \to \mathcal{O}(\widehat{G}/K).$$

If K contains a period Ω_L of the Lubin-Tate group, then $\widehat{G}$ and the open unit disc $\mathbb{B}$ are isomorphic over K and by combining the above with the Fourier isomorphism we obtain an isomorphism

$$D(G, K) \xrightarrow{\cong} \mathcal{O}(\mathbb{B}/K).$$

By choosing a coordinate T on $\mathbb{B}$ it can be described explicitly by mapping a dirac distribution δ_a to the power series

$$\eta(a, T) = \exp(a\Omega_L \log_{LT}(T)) \in o_K[[T]].$$

Proof. This follows by combining Corollary 3.7 and Theorem 2.3. in [ST01]. □

Remark 1.28. *If $H \subset G$ is an open normal subgroup then the decomposition $G = \bigcup_{g \in G/H} gH$ induces $D(G, K) \cong \bigoplus_{g \in G/H} \delta_g D(H, K) \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} D(H, K)$ algebraically and topologically.*

We are mostly interested in the case where G is abelian and contains an open subgroup isomorphic to o_L (like Γ_L). For technical purposes it is important that $C^{an}(G, K)$ can be written as a compactoid inductive limit.

Definition 1.29. *An **index** of an L-analytic group G is a family* $\mathcal{I} = (U_i, c_i, \varepsilon_i; i \in I)$ *of charts* $c_i : U_i \xrightarrow{\cong} B_{\varepsilon_i}(0)$ *mapping an open subset* $U_i \subset G$ *homeomorphically to the "closed" disc of radius* $\varepsilon_i \in \mathbb{R}_{>0}$*, such that* $G = \dot{\bigcup}_{i \in I} U_i$*. Given a collection of* ε_i*-convergent K-valued power series we can pull them back to G and glue them to a locally L-analytic K-valued function on G. We denote by* $\mathcal{F}_{\mathcal{I}}$ *the subspace of* $C^{an}(G, K)$ *of functions obtained in this way.*

Note that $C^{an}(G, K) = \varinjlim_{\mathcal{I}} \mathcal{F}_{\mathcal{I}}$, where $\mathcal{F}_{\mathcal{I}}$ is endowed with the topology induced by the ε_i-Gauss norms and the limits is taken over the set of all indices with respect to the partial order described in [Sch17, p. 77f].

Remark 1.30. *Suppose G contains an open subgroup isomorphic to* o_L*. There exists a cofinal family of indices such that the transition maps are compactoid.*

Proof. We first consider the special case $G = o_L$ and indices $\mathcal{I} = (o_L, \mathrm{id}, 1)$, $\mathcal{J} = (a + \pi_L o_L, \mathrm{id}, |\pi_L|; \overline{a} \in o_L/\pi_L o_L)$. By [PGS10, 8.1.3(i)] sums of compactoid maps are compactoid, thus it suffices to show that, denoting by

$$\mathcal{F}_{c,|\pi_L|} = \left\{ F : c + \pi_L o_L \to K \mid F(x) = \sum_{i=0}^{\infty} a_i (x - c)^i; \lim_{i \to \infty} |a_i||\pi_L^i| \to 0 \right\},$$

for each $c \in o_L$ the natural map

$$\mathcal{F}_{0,1} \to \mathcal{F}_{c,|\pi_L|}$$

is compactoid. Using [Sch17, Corollary 5.5] we may without loss of generality assume $c = 0$. In this case identify $\mathcal{F}_{0,1} \cong K\langle X \rangle \cong c_0(K)$ and $\mathcal{F}_{0,|\pi|} \cong K\langle X/\pi_L \rangle \cong c_0(K)$ in the obvious ways. Using this identification the natural map

$$K\langle X \rangle \to K\langle X/\pi_L \rangle$$

corresponds to

$$\mu \colon c_0(K) \to c_0(K) \tag{1.1}$$

$$(a_0, a_1, a_2, \dots) \mapsto (a_0, \pi_L a_1, \pi_L^2 a_2, \dots). \tag{1.2}$$

We check that this map is compactoid. Let U be the unit ball in $c_0(K)$ and let V be any neighbourhood of zero. Then V contains $\pi_L^n U$ for some $n \in \mathbb{N}$ and

$$\mu(U) \subset \pi_L^n U + \sum_{i=0}^{n-1} o_L e_i \subset V + \sum_{i=0}^{n-1} o_L e_i,$$

where e_i denotes the i-th unit vector. Analogously this result is true for any pair of radii $\varepsilon_1 > \varepsilon_2$ that lie in $|L|$. The general case is obtained from the special case above by first choosing a cover $G = \bigcup_{i \in I_0} U_i$ by disjoint open sets homeomorphic to o_L (via charts c_i), setting $\mathcal{I}_0 = (U_i, c_i, 1)$ and then defining $\mathcal{I}_n$ inductively by splitting each U_i appearing in $\mathcal{I}_{n-1}$ into q smaller circles and adjusting the radius to be $|\pi_L|$ times the previous one. This produces a cofinal set of indices with compactoid transition maps. □

Remark 1.31. *Mapping g to δ_g induces an injective group homomorphism $G \to D(G,K)^\times$ and an injection $K[G] \hookrightarrow D(G,K)$. This injection has dense image.*

Proof. For the injectivity it suffices to show that $K[G] \hookrightarrow D(G,K)$ is injective. For that purpose we show that any finite set of distinct Dirac distributions $\delta_{g_1}, \dots, \delta_{g_n}$ is K-linearly independent. Because G is Hausdorff and compact we may find a compact open subgroup $H \subset G$ such that $g_iH \cap g_jH = \emptyset$ whenever $i \neq j$. The decomposition from 1.28 show that δ_{g_i} are even $D(H,K)$-linearly independent. The inclusion $K[G] \to D(G,K)$ has dense image for K spherically complete by [ST02, Lemma 3.1]. The same proof applies to the general case since $D(G,K)$ is the strong dual of $C^{an}(G,K)$, which can be written as a compactoid inductive limit of Banach spaces by 1.30 and thus by [PGS10, 11.3.5] both $C^{an}(G,K)$ and $D(G,K)$ are strictly of countable type, reflexive and satisfy Hahn-Banach (cf. [PGS10, 4.2.6]). □

Definition 1.32. *Let $G = o_L$. After fixing a coordinate T on $\mathbb{B}$ we denote by $Z \in D(o_L,K)$ the preimage of $T \in \mathcal{O}_K(\mathbb{B})$ with respect to the isomorphism from Theorem 1.27.*

In the classical theory (assuming $p \neq 2$) we can choose explicitly $Z = \gamma - 1$ with a topological generator of $\Gamma_{\mathbb{Q}_p}$. In our situation this variable Z serves a similar purpose but is more elusive in its description. The main difficulty is reversing η since there is no (evident) connection between the exponential and the Lubin-Tate logarithm unless $L = \mathbb{Q}_p$ and $LT = \mathbb{G}_m$.

Remark 1.33. *Let $\mathrm{Aug}\colon D(o_L,K) \to K$ be the augmentation map induced by mapping each Dirac distribution to 1 and denote by ev_0 the map that evaluates a power series at $T = 0$. Then the following diagram is commutative*

$$\begin{array}{ccc} D(o_L,K) & \xrightarrow{\mathrm{Aug}} & K \\ \downarrow \cong & & \| \\ \mathcal{O}_K(\mathbb{B}) & \xrightarrow{\mathrm{ev}_0} & K \end{array}.$$

In particular

$$\ker(\mathrm{Aug}) = \overline{\mathrm{span}(\delta_a - 1, a \in G)} = ZD(o_L,K).$$

Proof. The vertical arrows are topological isomorphisms for the Fréchet-topology on the left side (resp. the valuation topology on the right side) and the Dirac distributions span a dense subspace of $D(o_L,K)$. We may therefore check the commutativity of said diagram on the Dirac distributions, where we have

$$\mathrm{Aug}(\delta_a) = 1 = \exp(a\Omega_L \log_{LT}(T))_{|T=0} = \mathrm{ev}_0(\eta(a,T)).$$

For the second statement we first remark that both maps are surjective and their kernels are mapped isomorphically to one another. Evidently $\ker(\mathrm{ev}_0) = T\mathcal{O}_K(\mathbb{B}) \cong ZD(o_L,K)$. Due to continuity the inclusion $\ker(\mathrm{Aug}) \supseteq \overline{\mathrm{span}(\delta_a - 1)}$ is clear. For the other inclusion consider the decomposition $K[o_L] \cong K\delta_0 \oplus \mathrm{span}(\delta_a - 1, a \in o_L)$ with the augmentation map restricted to the first factor mapping $\lambda\delta_0$ to λ. The left factor is clearly complete and passing to completions shows the desired result. □

Remark 1.34. *Let $a \in o_L$ and denote by a^* the map induced from the multiplication-by-a-map $a : o_L \to o_L$. Then the following diagram is commutative*

$$\begin{array}{ccc} D(o_L, K) & \xrightarrow{a^*} & D(o_L, K) \\ \Vert & & \Vert \\ \mathcal{O}_K(\mathbb{B}) & \xrightarrow{[a]} & \mathcal{O}_K(\mathbb{B}) \end{array},$$

where the vertical arrows arise from the Fourier-isomorphism.

Proof. Using $\log_{LT}([a](T)) = a \log_{LT}(T) = [a] \log_{LT}(T)$ we obtain

$$a^*(\delta_b) = \exp(\Omega_L ab \log_{LT}(T)) = [a](\exp(\Omega_L b \log_{LT}(T))) = [a](\delta_b),$$

proving the result for Dirac distributions. The general statement follows from continuity considerations. □

Lemma 1.35. *The kernel of the natural map* proj: $D(o_L, K) \to K[o_L/\pi_L^n o_L]$ *is generated by* $Z_n := [\pi_L]^n(Z) \in D(o_L, K)$.

Proof. Since Z lies in the closure of the augmentation ideal generated by the $\delta_g - 1$ with $g \in o_L$ using 1.34 we see that Z_n lies in the closure of the augmentation ideal of $D(\pi_L^n o_L, K) \subset D(o_L, K)$. We conclude $Z_n \subset \ker(\text{proj})$. By transporting the structure to $\mathcal{O}_K(\mathbb{B})$ we see that $\mathcal{O}_K(\mathbb{B})/\varphi_L^n(T)$ is free of rank q^n over K by counting the number of zeros of φ_L^n, i.e., the number of π_L^n-torsion points of the LT group. We conclude that the surjective map proj has to be injective modulo $[\pi_L]^n(Z)$. □

So far we only needed to choose a variable for $D(o_L, K)$. Analogously we can choose variables for any subgroup $\pi_L^n o_L$ since they are isomorphic to o_L. The following corollary shows that the ideal generated by such a variable is independent of any such choices. We state the result on the level of Γ_L since we use it in this particular context later on.

Corollary 1.36. *Let n_0 be minimal with the property that $\chi_{LT}(\Gamma_{n_0}) = 1 + \pi_L^{n_0} o_L$ is isomorphic to $\pi_L^{n_0} o_L$ via $\log_p$. Denote by Z_n the preimage of T under the sequence of isomorphisms*

$$D(\Gamma_n, K) \cong D(o_L, K) \cong \mathcal{O}_K(\mathbb{B})$$

viewed as an element of $D(\Gamma_{n_0}, K)$. Then there is a canonical Γ_{n_0}/Γ_n equivariant isomorphism $D(\Gamma_{n_0}, K)/Z_n \cong K[\Gamma_{n_0}/\Gamma_n]$ induced by mapping δ_g to g.

Proof. This follows from transport of structure along the isomorphisms $\chi_{LT} : \Gamma_L \to o_L^\times$ and $\log_p : 1 + \pi_L^n o_L \to \pi_L^n o_L \cong o_L$. The isomorphism is canonical in the sense that neither its definition nor its kernel depend on the particular choice of Z_n. □

For the remainder of the thesis we fix the following compatible choice of variables.

Definition 1.37. *Consider for $n \geq n_0$ the system of commutative diagrams*

$$\begin{array}{ccccc} \Gamma_n & \xrightarrow{\chi_{LT}} & 1+\pi_L^n & \xrightarrow{\log_p} & \pi_L^n o_L \\ \uparrow & & \uparrow & & \uparrow \\ \Gamma_m & \xrightarrow{\chi_{LT}} & 1+\pi_L^m & \xrightarrow{\log_p} & \pi_L^m o_L \end{array}$$

We fix as before the variable $Z_m \in D(\Gamma_m, K)$ for every $m \geq n_0$. Since $\pi_L^n o_L = \pi_L^{m-n}(\pi_L^n o_L)$ we obtain the relationship

$$Z_m = \varphi_L^{m-n}(Z_n) \tag{1.3}$$

for every $m \geq n$.

1.3.1 L-Analyticity

In this section we discuss the question of L-analyticity in families. Abstractly L-analyticity can be defined for any K-Banachspace with a continuous K-linear Γ_L-action where $L \subset K$ is a complete field extension. In our application families of (φ_L, Γ_L)-modules will be LF-spaces over K with a $\mathcal{R}_A$-semi-linear (hence a fortiori K-linear) Γ_L-action.

Definition 1.38. *Let X be a d-dimensional L-analytic manifold and let K be a complete subfield of $\mathbb{C}_p$.*

- *Let V be a K-Banach space. A map $f : X \to V$ is called **locally L-analytic** if for every $x \in X$ there exists an open neighbourhood U homeomorphic to $B_\varepsilon(0)^d$ and $(v_\mathbf{n}) \in V^{\mathbb{N}_0^d}$ such that*
$$\lim_{|\mathbf{n}| \to \infty} \varepsilon^{|\mathbf{n}|} ||v_\mathbf{n}|| = 0$$
and
$$f(x) = \sum_{\mathbf{n} \in \mathbb{N}_0^d} v_\mathbf{n} (x_1, \ldots, x_d)^\mathbf{n}$$
for every $x \in U$. Where x_i are local coordinates of x and $(x_1, \ldots, x_d)^\mathbf{n} = \prod_i x_i^{n_i}$.
- *For a K-Fréchet space $\varprojlim_j V_j$ with Banachspaces V_j a map $f : X \to \varprojlim_j V_j$ is called **pro-L-analytic** if each induced map $X \to V_j$ is locally L-analytic.*
- *For an LF-space $\varinjlim_i \varprojlim_j V_{i,j}$ with Banachspaces $V_{i,j}$ a map $f : X \to \varinjlim_i \varprojlim_j V_{i,j}$ is called **pro-L-analytic** if it factors over some $\varprojlim_j V_{i,j}$ and the induced map is pro-L-analytic in the Fréchet-sense.*

Definition 1.39. *Let $V = \varinjlim_i \varprojlim_j V_{i,j}$ be a K-LF-space and let G be a p-adic Lie group over L acting on V. The action is called **L-analytic** if for each $v \in V$ the orbit map $g \mapsto gv$ is pro-L-analytic.*

In order to treat tensor products of (φ_L, Γ_L)-modules we introduce the following auxiliary definition. This property will be satisfied in our applications.

Definition 1.40. *Let $V = \varinjlim_i \varprojlim_j V_{i,j}$ be a K-LF-space with an L-analytic action of an L-analytic group G. We say that V **admits a model** if there exists a m_0 such that the orbit map of each $v \in \varprojlim_j V_{m,j}$ already factors over $\varprojlim V_{m,j}$ whenever $m \geq m_0$.*

Lemma 1.41. *Let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$. The Γ_L action on M extends uniquely to a seperately continuous action of $D(\Gamma_L, K)$ satisfying $\delta_g m = gm$ and each morphism of L-analytic (φ_L, Γ_L)-modules is $D(\Gamma_L, K)$-equivariant.*

Proof. After reducing to the corresponding statement for the Banach space M^I for a closed interval this follows from the proof of [SV20, Proposition 2.26], which was proved for general K-Banach spaces. □

Lemma 1.42. *Let K/L be a complete field extension and V be a K-Banach space with a continuous K-linear Γ_L-action. The Γ_L-action on V is locally L-analytic if and only if the following conditions are satisfied:*

- *There exists $m \geq 2$ such that $||\gamma - 1||_V < p^{-1/(p-1)}$ for any $\gamma \in \Gamma_m$.*
- *The derived action $\mathrm{Lie}(\Gamma_L) \to \mathrm{End}_K(V)$ is L-linear.*

Proof. The first condition in 1.42 asserts that the Γ_L-action is locally $\mathbb{Q}_p$-analytic by [BSX20, Lemma 2.3.1]. The second condition then implies L-analyticity (cf. [Wit19, Folgerung 2.2.10]). □

Lemma 1.43. *Let V, W be two LF-spaces with pro L-analytic actions, that both admit models. Then the tensor product action on $E := \varinjlim_i V_i \hat{\otimes}_{K,i} W_i$ is pro L-analytic.*

Proof. Write $V = \varinjlim_i \varprojlim_j V_{ij}$ (resp. $W = \varinjlim_m \varprojlim_n W_{mn}$) with Banach spaces V_{ij} (resp. W_{mn}). For E we have $E = \varinjlim_{m,i} \varprojlim_{j,n} V_{ij} \hat{\otimes} W_{mn}$ thus using that V and W admit models we see that E also admits a model. We may therefore without loss of generality assume that V, W are Banach spaces. In this case the $\mathbb{Q}_p$-analyticity of the action follows from [FdL99, 3.3.11.]. By [FdL99, 3.3.12] the derived action is given by $D(\rho_V) \otimes \mathrm{id}_W + V \otimes D(\rho_W)$, where $D(\rho_X)$ denotes the derived action on $X \in \{V, W\}$. This action is L-linear as a sum of L-linear maps. □

Our objects of interest are projective $\mathcal{R}_A$-modules with an L-analytic Γ_L-action. The following example is a sanity check.

Example 1.44. *The action of Γ_L induced by $\gamma(T) = [\chi_{LT}(\gamma)](T)$ on the relative Robba ring $\mathcal{R}_A = \varinjlim_{0 \leq r < 1} \varprojlim_{r < s < 1} \mathcal{R}_A^{[r,s]}$ is L-analytic.*

Proof. Since $T \mapsto [\chi_{LT}(\gamma)](T)$ is an automorphism of each annulus $[r,s]$, this reduces to studying the action on $\mathcal{R}_A^{[r,s]} = A\widehat{\otimes}\mathcal{R}_K^{[r,s]}$. Here the left tensor factor carries the trivial Γ_L-action and $\mathcal{R}_K$ carries the usual action. By [BSX20, Proposition 2.3.4] the latter is L-analytic . Since the trivial action is L-analytic the statement follows from 1.43. $\square$

For certain technical arguments we require an analogue (on the distribution side) of r-convergent power series.

Definition 1.45. *Let $G \cong o_L$, let $g_1, \dots, g_d$ be a $\mathbb{Z}_p$-basis of G and let $r \in [|p|, 1)$. We define $D_{\mathbb{Q}_p,r}(G,K)$ to be the completion of $D_{\mathbb{Q}_p}(G,K)$ with respect to the norm*

$$|\sum_{\mathbf{k}\in\mathbb{N}_0^d} a_{\mathbf{k}}(\delta_{g_i}-1)_i^{\mathbf{k}}|_{D_{\mathbb{Q}_p,r}(G,K)} = \sup_{\mathbf{k}} |a_{\mathbf{k}}| r^{|\mathbf{k}|}$$

with the usual conventions for multi-indices i.e. $((\delta_{g_i}-1)_i)^{(k_1,\dots,k_d)} = \prod_{i=1}^d (\delta_{g_i}-1)^{k_i}$ and $|\mathbf{k}| = \sum_{i=1}^d k_i$. We further define $D_r(G,K)$ as the completion of $D(G,K)$ with respect to the quotient norm with respect to the natural projection $D_{\mathbb{Q}_p}(G,K) \to D(G,K)$. One can show that the algebras above are independent of the choice of basis.

Remark 1.46. *Choose an ordered $\mathbb{Z}_p$-basis $h_1, \dots, h_d$ of o_L and let $b_i := \delta_{h_i} - 1 \in D_{\mathbb{Q}_p}(o_L, K)$ and $\mathbf{b} = (b_1, \dots, b_d)$. Any element $\lambda \in D_{\mathbb{Q}_p}(o_L, K)$ admits a unique convergent expansion*

$$\lambda = \sum_{\mathbf{k}\in\mathbb{N}_0^d} a_{\mathbf{k}}\mathbf{b}^{\mathbf{k}}$$

such that $|\lambda|_{D_{\mathbb{Q}_p,r}(G,K)} = \sup |a_k| r^{|\mathbf{k}|}$ is bounded for any $0 < r < 1$. The norms $|-|_{D_{\mathbb{Q}_p,r}(G,K)}$ for $r \in p^{\mathbb{Q}}, |p| < r < 1$ are sub-multiplicative and independent of the choice of ordered basis.

Proof. See [ST03, 4.2 and discussion after 4.10]. $\square$

Lemma 1.47. *Let $I = [r,s] \subset [0,1)$ and let $\mathfrak{r} > s \geq |p|$ then the homomorphism induced by the composition of the natural projection and the LT-isomorphism*

$$D_{\mathbb{Q}_p}(o_L, K) \to D(o_L, K) \to \mathcal{O}_K(\mathbb{B})$$

extends to a continuous homomorphism

$$D_{\mathbb{Q}_p,\mathfrak{r}}(o_L, K) \to D_{\mathfrak{r}}(o_L, K) \to \mathcal{O}_K(\mathbb{B}^{[0,s]}) \subset \mathcal{O}_K(\mathbb{B}^I)$$

of operator norm 1.

Proof. Consider the composite map

$$D_{\mathbb{Q}_p}(o_L, K) \to D(o_L, K) \to \mathcal{O}_K(\mathbb{B}) \to \mathcal{O}_K(\mathbb{B}^{[0,s]}),$$

where the last arrow is the canonical inclusion. Since the target is complete (with respect to the s-Gauß norm) it suffices to show that the map is continuous with respect to the $\mathfrak{r}$-norm on the left-hand side. The series $\eta(a,T) - 1 = a\Omega T + \dots$ has no constant term and $|a\Omega| < 1$ for every $a \in o_L$. In particular $|\eta(a,T) - 1|_I < s$ by assumption. Choose as before a $\mathbb{Z}_p$-basis $h_1, \dots, h_d$ of o_L and let $\mathbf{e} = (\eta(h_1,T) - 1, \dots, \eta(h_d,T) - 1)$. Then $\lambda = \sum_{\mathbf{k}\in\mathbb{N}_0^d} a_{\mathbf{k}}\mathbf{b}^{\mathbf{k}}$ is mapped to $\sum_{\mathbf{k}\in\mathbb{N}_0^d} a_{\mathbf{k}}\mathbf{e}^{\mathbf{k}}$ and

$$\left|\sum_{\mathbf{k}\in\mathbb{N}_0^d} a_{\mathbf{k}}\mathbf{e}^{\mathbf{k}}\right|_I \leq \sup|a_{\mathbf{k}}|s^{|\mathbf{k}|} \leq |\lambda|_{D_{\mathbb{Q}_p,\mathfrak{r}}(o_L,K)}.$$

This shows that the operator norm is bounded by 1. It has to be equal to 1 because the scalars are mapped to themselves. □

Proposition 1.48. *Let $G = o_L$, denote by $Z \in D(G,K)$ the element corresponding to the variable $T \in \mathcal{O}_K(\mathbb{B})$ via the Fourier isomorphism and let $0 \neq a \in G$.*

1. *We have in $D(G,K)$*

$$(\delta_a - 1)/a\Omega Z = 1 + c_1(a)Z + c_2(a)Z^2 + \dots.$$

2. *The coefficients can be estimated with $|c_n| \leq |p|^{\frac{-n}{e(L/\mathbb{Q}_p)(q-1)}}$.*

Proof. The first part is obtained by evaluating the Fourier isomorphism at $\delta_a - 1$. One computes

$$\delta_a - 1 \mapsto \eta(a,T) - 1 = a\Omega T + \dots$$

The estimate of the coefficients can be found in the proof of Lemma 3.4 in [ST01]. Note that the authors use the normalisation $|p| = 1/p$. □

1.3.2 Lie-Elements in $D(\Gamma_L, L)$.

Proposition 1.49. *Consider Γ_L as an L-analytic Lie group via $\chi_{LT} : \Gamma_L \to o_L^\times$. Let $\Gamma \subset \Gamma_L$ be an open subgroup isomorphic to o_L.*

1. *Taking the derivative of χ_{LT} induces an isomorphism*

$$\mathfrak{g} = \mathrm{Lie}(\Gamma_L) \cong \mathrm{Lie}(\Gamma) \cong L.$$

2. *Let $\exp_\Gamma$ be the exponential for Γ, then*

$$\begin{aligned}\mathrm{Lie}(\Gamma) &\to D(\Gamma, L)\\ \mathfrak{x} &\mapsto [f \mapsto \frac{d}{dt} f(\exp_\Gamma(t\mathfrak{x})_{|t=0})]\end{aligned}$$

   is an L-linear embedding.

Proof. Since Γ is an open subgroup of the same dimension as Γ_L the first isomorphism is induced by the inclusion. The second isomorphism follows from the fact that the clopen subgroup $1 + \pi_L^n o_L \subset o_L^\times$ is isomorphic to o_L for n large enough whose Lie-algebra is L. For the second statement see [ST02, Discussion after Proposition 2.3]. □

Definition 1.50. *We denote by* $\nabla \in \mathrm{Lie}(\Gamma_L) = \mathrm{Lie}(\Gamma)$ *the preimage of* 1 *under the isomorphism* $d_1\chi_{LT} : \mathrm{Lie}(\Gamma) \xrightarrow{\cong} L$. *We denote by the same symbol the image of* ∇ *under the embedding* $\mathrm{Lie}(\Gamma) \to D(\Gamma, L)$. *We use the same symbol for the element* $1 \otimes \nabla \in K \otimes_L D(\Gamma, L) \subset D(\Gamma, K)$.

Lemma 1.51. *Let* $1 \neq \gamma \in \Gamma$ *then* $\delta_\gamma - 1$ *divides* ∇ *in* $D(\Gamma, L)$ *and* Z *divides* ∇ *in* $D(\Gamma, K)$.

Proof. The first statement is Lemma 2.10 in [SV20]. The second statement follows from the first since Z divides $\delta_g - 1$. □

This result can be surprising at first since in particular for $\Gamma = \Gamma_n$ with $n \geq n_0$ we obtain that ∇ is divisible by every Z_n or using (1.3) by every $\varphi_L^k(Z_{n_0})$. This relationship becomes more transparent by an explicit calculation.

Lemma 1.52. *Let* $n \geq n_0$ *then the image of* ∇ *under the sequence of isomorphisms*

$$D(\Gamma_n, K) \cong D(o_L, K) \cong \mathcal{O}_K(\mathbb{B})$$

induced by the isomorphism

$$\Gamma_n \cong 1 + \pi_L^n \xrightarrow{\log/\pi_L^n} o_L$$

and the Fourier isomorphism is

$$\frac{\Omega}{\pi_L^n} \log_{LT}(T).$$

In particular

$$\frac{\Omega}{\pi_L^n} \log_{LT}(Z_n) = \frac{\Omega}{\pi_L^m} \log_{LT}(Z_m)$$

for any $n, m \geq n_0$.

Proof. The computation is contained in the proof of [SV20, Remark 2.36] for a fixed index n. We already computed $Z_m = \varphi_L^{m-n}(Z_n)$ for $m \geq n$. Hence the second statement follows from the first using $\log_{LT}([a](T)) = a \log_{LT}(T)$. □

Corollary 1.53. *Let* $n \geq n_0$, *then we have a product expansion*

$$\begin{aligned}
\nabla &= \frac{\Omega}{\pi_L^n} \log_{LT}(Z_n) \\
&= \frac{\Omega}{\pi_L^n} Z_n \prod_{k \geq 1} \frac{[\pi_L^k](Z_n)}{\pi_L [\pi_L^{k-1}](Z_n)} \\
&= \frac{\Omega}{\pi_L^n} Z_n \prod_{k \geq 1} \frac{Z_{n+k}}{\pi_L Z_{n+k-1}}.
\end{aligned}$$

Proof. This follows by transport of structure from the usual expansion of $\log_{LT}(T)$. □

1.4 (φ_L, Γ_L)-modules over the Robba ring.

When studying (φ_L, Γ_L)-modules over the Robba ring $\mathcal{R}_K$ it turns out that they admit a so-called model over some half-open interval $[r, 1)$, meaning that it arises as a base extension from a module over $\mathcal{R}_K^{[r,1)}$. When working with families of such modules, we enforce the existence of such a model, which in turn allows us to view the modules (more precisely their models) as vector bundles on $\mathrm{Sp}(A) \times_K \mathbb{B}^{[r,1)}$. A suitable frame work to do so is the theory of coadmissible modules in the sense of Schneider and Teitelbaum.

Definition 1.54. *A commutative K-Fréchet algebra $\mathcal{A}$ is called* ***Fréchet-Stein algebra*** *if there is a sequence of continuous algebra seminorms $q_1 \leq \cdots \leq q_n \leq \ldots$ that define the Fréchet-topology such that*

(i) The completion $\mathcal{A}_n$ of $\mathcal{A}/\{a \mid q_n(a) = 0\}$ with respect to q_n is Noetherian and

(ii) $\mathcal{A}_n$ is flat as an $\mathcal{A}_{n+1}$-module for any $n \in \mathbb{N}$.

A ***coherent sheaf*** *is a family $(M_n)_{n\in\mathbb{N}}$ of finitely generated $\mathcal{A}_n$-modules endowed with their respective canonical topologies such that*

$$\mathcal{A}_{n+1} \otimes_{\mathcal{A}_n} M_n \cong M_{n+1}.$$

The ***global sections*** *of a coherent sheaf are defined to be the Fréchet-module*

$$\Gamma(M_n) := \varprojlim_{n\in\mathbb{N}} M_n.$$

An $\mathcal{A}$-module is called ***coadmissible*** *if it arises as the global sections of a coherent sheaf.*

Lemma 1.55. *Let $\mathcal{A}$ be a Fréchet-Stein algebra and let M be coadmissible. Then*

(i) $\mathcal{A}_n$ is flat as an $\mathcal{A}$-module.

(ii) The canonical map $M \to M_n$ has dense image and $\mathcal{A}_n \otimes_A M \cong M_n$.

(iii) $\varprojlim^i_{n\geq 1} M_n = 0$ for any $i \geq 1$.

(iv) Kernels, cokernels, images and coimages of $\mathcal{A}$-linear maps between coadmissible modules are coadmissible.

(v) Finitely generated submodules of coadmissible modules are coadmissible.

(vi) Finitely presented modules are coadmissible.

(vii) Let A be K-affinoid, then $\mathcal{R}_A^{[r,1)}$ is a Fréchet-Stein algebra for any $r \in [0,1)$.

Proof. For *(i)-(vi)* see section 3 in [ST03]. We postpone the proof of *(vii)* to section A.1 in the appendix. □

Using the theory of coadmissible modules we can deduce the following useful result.

Lemma 1.56. *Let $\mathfrak{m} \in \operatorname{Max}(A)$ be a maximal ideal, then the natural map*

$$\mathcal{R}_A \otimes_A A/\mathfrak{m} \to \mathcal{R}_{A/\mathfrak{m}}$$

is an isomorphism.

Proof. By a limit argument it suffices to show that $\mathcal{R}_A^{[r,1)} \otimes_A A/\mathfrak{m} \to \mathcal{R}_{A/\mathfrak{m}}^{[r,1)}$ is an isomorphism for every $0 < r < 1$. The right-hand side is by definition the Hausdorff completion of the left-hand side. Hence we are done if we can show that $\mathcal{R}_A^{[r,1)} \otimes_A A/\mathfrak{m} = \mathcal{R}_A^{[r,1)}/\mathfrak{m}\mathcal{R}_A^{[r,1)}$ is complete. Since A is Noetherian the ideal $\mathfrak{m}\mathcal{R}_A^{[r,1)}$ is a finitely generated submodule of a coadmissible module hence itself coadmissible. By [ST03, Lemma 3.6] the quotient in question is complete. □

We also obtain an algebraic description of $\mathcal{R}_A^{[r,1)}/(t_{LT})$.

Lemma 1.57. *The natural map $\mathcal{R}_A^{[r,1)}/(t_{LT}) \to \varprojlim_{n\geq 0} \mathcal{R}_A^{[r,1)}/(\varphi_L^n(T))$ is an isomorphism.*

Proof. Recall the product decomposition $t_{LT} = \log_{LT}(T) = T\prod_{n\geq 1} \frac{\varphi_L^n(T)}{\pi_L^{-1}\varphi_L^{n-1}(T)}$. The partial products are equal to $P_n = \pi_L^{-(n-1)}\varphi^n(T)$. In particular t_{LT} is divisible by every $\varphi^n(T)$. We can describe $\mathcal{R}_A^{[r,1)}/(t_{LT})$ as

$$\mathcal{R}_A^{[r,1)}/(t_{LT}) = \operatorname{cok}(\mathcal{R}_A^{[r,1)} \xrightarrow{t_{LT}} \mathcal{R}_A^{[r,1)})$$

and by coadmissibility we have

$$\operatorname{cok}(\mathcal{R}_A^{[r,1)} \xrightarrow{t_{LT}} \mathcal{R}_A^{[r,1)}) = \varprojlim_s \operatorname{cok}(\mathcal{R}_A^{[r,s]} \xrightarrow{t_{LT}} \mathcal{R}_A^{[r,s]})$$

Now take a sequence of radii s_n such that $[0, s_n]$ contains the π_L^n-torsion points of the LT-group (i.e. the zeroes of $\varphi^n(T)$) but no π_L^{n+1}-torsion points, that are not already π_L^n-torsion, i.e., none of the zeroes of $\varphi^{n+1}(T)/\varphi^n(T)$. In $\mathcal{R}_K^{[0,s_n]}$ and hence in $\mathcal{R}_K^{[r,s_n]}$ $\varphi^n(T)$ and t_{LT} differ by $t_{LT}/\varphi^n(T)$, which has no zeroes in the annulus $[r, s_n]$ and is therefore a unit (since it is not contained in any maximal ideal of $\mathcal{R}_K^{[r,s_n]}$ by [Bos14, 3.3 Lemma 10] and the Weierstraß Preparation Theorem), hence they differ by a unit in $\mathcal{R}_A^{[r,s_n]}$, in particular, $\mathcal{R}_A^{[r,s_n]}/\varphi^n(T) = \mathcal{R}_A^{[r,s_n]}/t_{LT}$. The statement now follows from $\mathcal{R}_K^{[r,1)}/\varphi^n(T) = \mathcal{R}_K^{[r,s_n]}/\varphi^n(T)$. □

The definition of coherent sheaves so far is too restrictive because we cannot change the lower boundary of a given half-open interval $[r, 1)$. Recall that a collection of subsets of a topological space is called locally finite if every point admits an open neighbourhood intersecting only finitely many members of the collection.

Remark 1.58. *Let $\{[r_i, s_i], i \in \mathbb{N}_0\}$ be an admissible cover of $[r, 1)$ i.e. a cover by closed intervals that admits a locally finite refinement with $r_i, s_i \in \sqrt{|K|^\times}$. For each i let $M^{[r_i,s_i]}$ be a finitely generated $\mathcal{R}_A^{[r_i,s_i]}$-module together with isomorphisms*

$$\mathcal{R}_A^{I\cap J} \otimes_{\mathcal{R}_A^I} M^I \cong \mathcal{R}_A^{I\cap J} \otimes_{\mathcal{R}_A^J} M^J$$

for any pair of intervals with non-empty intersection, satisfying the obvious compatibility conditions. Then for each $s \in [r, 1)$ there exists a unique coadmissible $\mathcal{R}_A^{[s,1)}$-module together with morphisms $M^{[s,1)} \to \mathcal{R}_A^{I\cap[s,1)} \otimes_{\mathcal{R}_A^I} M^I$ inducing

$$\mathcal{R}_A^{I\cap[s,1)} \otimes_{\mathcal{R}_A^{[s,1)}} M^{[s,1)} \cong \mathcal{R}_A^{I\cap[s,1)} \otimes_{\mathcal{R}_A^I} M^I$$

for any interval appearing in the cover above. In particular a coadmissible $\mathcal{R}_A^{[r,1)}$-module is uniquely determined by its sections along an admissible cover.

Proof. We only give a sketch of the proof. Reordering the intervals and refining the cover allows us to assume without loss of generality $r = s$ and assume $r_0 = r \le r_1 \le s_0 \le r_2 \le s_1 < \dots$. In order to construct a coadmissible $\mathcal{R}_A^{[r,1)}$-module we need to construct a compatible chain of $\mathcal{R}_A^{[r,t_i]}$-modules, with t_i converging to 1. We shall explain how to extend the module $M^{[r_0,s_0]}$ to a module $M^{[r_0,s_1]}$ satisfying $M^I \cong \mathcal{R}_A^I \otimes_{\mathcal{R}_A^{I\cap[r_0,s_1]}} M^{[r_0,s_1]}$ for $I = [r_0, s_0]$ or $I = [r_1, s_1]$. By assumption $M^{[r_0,s_0]}$ and $M^{[r_1,s_1]}$ can be glued along the isomorphism

$$\mathcal{R}_A^{[s_1,r_1]} \otimes_{\mathcal{R}_A^{[r_0,s_0]}} M^{[r_0,s_0]} \cong \mathcal{R}_A^{[s_1,r_1]} \otimes_{\mathcal{R}_A^{[r_1,s_1]}} M^{[r_1,s_1]}$$

to a coherent $\mathcal{R}_A^{[r_0,s_1]}$-module, which by Kiehl's theorem (cf. [Bos14, 6.1 Theorem 4]) is (associated to) a finitely generated $\mathcal{R}_A^{[r_0,s_1]}$-module, that we denote $M^{[r_0,s_1]}$. By construction the sections along $I \in \{[r_0, s_0], [r_1, s_1]\}$ are $\mathcal{R}_A^I \otimes_{\mathcal{R}_A^I} M^{[r_0,s_1]}$. Iterating this construction produces a sequence $M^{[r_0,s_n]}$ of $\mathcal{R}_A^{[r_0,s_n]}$-modules and a compatible sequence of isomorphisms $\mathcal{R}_A^{[r_0,s_n]} \otimes_{\mathcal{R}_A^{[r_0,s_{n+1}]}} M^{[r_0,s_{n+1}]} \cong M^{[r_0,s_n]}$ passing to global sections gives the desired result. □

Lemma 1.59. *Let $M^{[r,1)}$ be a coadmissible $\mathcal{R}_A^{[r,1)}$-module and let $\mathfrak{U} = \{[r_i, s_i], i \in \mathbb{N}\}$ be an admissible cover of $[r, 1)$.*

(i) $M^{[r,1)}$ is finitely generated if and only if there exists $n \in \mathbb{N}$ independent of $I \in \mathfrak{U}$ such that each M^I is generated by at most n elements.

(ii) $M^{[r,1)}$ is finitely presented if and only if there exist $(m, n) \in \mathbb{N}^2$ independent of $I \in \mathfrak{U}$ such that each M^I admits a presentation

$$(\mathcal{R}_A^I)^m \to (\mathcal{R}_A^I)^n \to M^I \to 0.$$

(iii) $M^{[r,1)}$ *is finitely generated projective if there exists $n \in \mathbb{N}$ independent of $I \in \mathfrak{U}$ such that each M^I is generated by at most n elements and each M^I is flat over $\mathcal{R}_A^I$.*

Proof. See Proposition 2.1.13 in [KPX14], whose proof works for any base field and the subsequent Remark 2.1.14. For the convenience of the reader we give a detailed proof of the first part in Lemma A.7 in the appendix. The second and third part are consequences of the first. □

We are now able to define L-analytic (φ_L, Γ_L)-modules. In the context of φ_L-modules over Robba rings we always assume $r_L < r_0$. Recall $r_L^{1/q} = |u|$ for $0 \neq u \in LT[\pi_L]$, which ensures that φ_L is well-defined on $\mathcal{R}_L^I$ for any interval $I \subset [r_0, 1)$.

Definition 1.60. *Let $r_L < r_0$. A (projective) φ_L-**module** over $\mathcal{R}_A^{r_0} = \mathcal{R}_A^{[r_0,1)}$ is a finite (projective) $\mathcal{R}_A^{r_0}$-module M^{r_0} equipped with an isomorphism $\varphi_L^* M^{r_0} \cong M^{r_0^{1/q}} := M^{r_0} \otimes_{\mathcal{R}_A^{r_0}} \mathcal{R}_A^{r_0^{1/q}}$. A φ_L-module M over $\mathcal{R}_A$ is defined to be the base change of a φ_L-module M^{r_0} over some $\mathcal{R}_A^{r_0}$. We call M^{r_0} a **model** of M over $[r_0, 1)$. A (φ_L, Γ_L)-**module** over the above rings is a φ_L-module whose model is projective and endowed with a semilinear continuous action of Γ_L that commutes with φ_L. Here continuous means that for every $m \in M^{[r,s]} := M^{r_0} \otimes_{\mathcal{R}_A^{r_0}} \mathcal{R}_A^{[r,s]}$ the orbit map $\Gamma_L \to M^{[r,s]}$ is continuous for the profinite topology on the left side and the Banach topology on the right-hand side.*

We will sometimes use the notation $M^{[r_0,1)}$ instead of M^{r_0} for added clarity.

Remark 1.61. *A projective φ_L-module over $\mathcal{R}_A^{r_0}$ is coadmissible. Furthermore given a coadmissible $\mathcal{R}^{r_0}$-module M^{r_0} together with an isomorphism $\varphi_L^* M^{r_0} \cong M^{r_0^{1/q}}$ it is projective if and only if it is flat.*

Proof. The first statement follows from the fact that every finitely generated projective module is finitely presented and any finitely presented module is coadmissible. For every closed interval $[r, s]$ the sections $M^{[r,s]}$ are finitely generated over the Noetherian ring $\mathcal{R}_A^{[r,s]}$ and hence finitely presented. The isomorphism $\varphi_L^* M^{r_0} \cong M^{r_0^{1/q}}$ restricts to an isomorphism $\varphi_L^*(M^I) \cong M^{I^{1/q}}$, which allows us to shift a given interval $I = [r_0, s_0]$ with $s_0 > r_0^{1/q}$ and conclude that M^{r_0} is uniformly finitely presented. Hence by 1.59(iii) M^{r_0} is projective if and only if it is flat. □

So far we worked with φ_L-modules over $\mathcal{R}_A^{[r,1)}$. In order to reduce the computation of cohomology to the level $[r, s]$ we introduce the notion of a φ_L-module over $[r, s]$. Contrary to the case $[r, 1)$, where $\varphi(\mathcal{R}_A^{[r,1)})$ can naturally be viewed as a subring of $\mathcal{R}_A^{[r^{1/q},1)}$, the interval $[r, s]$ gets shifted to $[r^{1/q}, s^{1/q}]$, which means that we can only compare a φ_L-module and its pullback after restricting to the overlap $[r^{1/q}, s]$ assuming $s \geq r^{1/q}$. This makes the following definition rather artificial.

Definition 1.62. *Let $0 < r < s < 1$ with $s \geq r^{1/q}$. A φ_L-module over $\mathcal{R}_A^{[r,s]}$ is a finitely generated $\mathcal{R}_A^{[r,s]}$ module $M^{[r,s]}$ together with an isomorphism*

$$\varphi_M^{lin} \colon \mathcal{R}_A^{[r^{1/q},s]} \otimes_{\mathcal{R}_A^{[r^{1/q},s^{1/q}]}} \varphi^*(M^{[r,s]}) \xrightarrow{\cong} M^{[r^{1/q},s]} := \mathcal{R}_A^{[r^{1/q},s]} \otimes_{\mathcal{R}_A^{[r,s]}} M^{[r,s]}.$$

A morphism $f : M^{[r,s]} \to M'^{[r,s]}$ of φ_L-modules is a $\mathcal{R}_A^{[r,s]}$-linear morphism of the underlying modules such that the diagram

$$\begin{array}{ccc} \mathcal{R}_A^{[r^{1/q},s]} \otimes_{\mathcal{R}_A^{[r,s]}} M^{[r,s]} & \xrightarrow{\mathrm{id} \otimes f} & \mathcal{R}_A^{[r^{1/q},s]} \otimes_{\mathcal{R}_A^{[r,s]}} M'^{[r,s]} \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{R}_A^{[r^{1/q},s]} \otimes_{\mathcal{R}_A^{[r^{1/q},s^{1/q}]}} \varphi^*(M^{[r,s]}) & \xrightarrow{\mathrm{id} \otimes (\mathrm{id} \otimes f)} & \mathcal{R}_A^{[r^{1/q},s]} \otimes_{\mathcal{R}_A^{[r^{1/q},s^{1/q}]}} \varphi^*(M'^{[r,s]}) \end{array}$$

commutes. We denote by $\varphi_M : M^{[r,s]} \to M^{[r^{1/q},s]}$ the semi-linear map induced by the isomorphism φ_M^{lin}. When there is no possibility of confusion we simply write φ_L instead of φ_M.

Lemma 1.63. *Let $r \in [0,1)$ and let $s \in (r^{1/q},1)$. The functor that assigns to a projective φ_L-module over $\mathcal{R}_A^{[r,1)}$ its section $M^{[r,s]} = \mathcal{R}_A^{[r,s]} \otimes_{\mathcal{R}_A^{[r,1)}} M^{[r,1)}$ is an exact equivalence of categories between projective φ_L-modules over $\mathcal{R}_A^{[r,1)}$ and projective φ_L-modules over $[r,s]$.*

Proof. We first show essential surjectivity. Let $M^{[r,s]}$ be a φ_L-module over $\mathcal{R}_A^{[r,s]}$. By assumption $\varphi^*(M^{[r,s]}) = \mathcal{R}_A^{[r^{1/q},s^{1/q}]} \otimes_{\mathcal{R}_A^{[r,s]},\varphi_L} M^{[r,s]}$ is a finitely generated $\mathcal{R}_A^{[r^{1/q},s^{1/q}]}$-module and we have an isomorphism

$$\mathcal{R}_A^{[r^{1/q},s]} \otimes_{\mathcal{R}_A^{[r^{1/q},s^{1/q}]}} \varphi^*(M^{[r,s]}) \cong \mathcal{R}_A^{[r^{1/q},s]} \otimes_{\mathcal{R}_A^{[r,s]}} M^{[r,s]} = M^{[r^{1/q},s]}.$$

The right-hand side being the restriction of $M^{[r,s]}$ to $\mathrm{Sp}(A) \times_K \mathbb{B}^{[r^{1/q},s]}$, which is precisely the overlap $\mathrm{Sp}(A) \times_K \mathbb{B}^{[r,s]} \cap \mathrm{Sp}(A) \times_K \mathbb{B}^{[r^{1/q},s^{1/q}]}$ by our assumption on s, and thus allows us to glue $M^{[r,s]}$ and $\varphi^*(M^{[r,s]})$ to a coherent sheaf on $\mathrm{Sp}(A) \times_K \mathbb{B}^{[r,s^{1/q}]}$, which by Kiehl's theorem is associated to a finitely generated $\mathcal{R}_A^{[r,s^{1/q}]}$-module that we denote by $M^{[r,s^{1/q}]}$. It remains to construct an isomorphism

$$\mathcal{R}_A^{[r^{1/q},s^{1/q}]} \otimes_{\mathcal{R}_A^{[r^{1/q},s^{1/q^2}]}} \varphi^*(M^{[r,s^{1/q}]}) \cong \mathcal{R}_A^{[r^{1/q},s^{1/q}]} \otimes_{\mathcal{R}_A^{[r,s^{1/q}]}} M^{[r,s^{1/q}]}.$$

Restricting $M^{[r,s^{1/q}]}$ to $[r^{1/q}, s^{1/q}]$ gives us

$$\varphi^*(M^{[r,s]}) \cong \mathcal{R}_A^{[r^{1/q},s^{1/q}]} \otimes_{\mathcal{R}_A^{[r,s^{1/q}]}} M^{[r,s^{1/q}]} \tag{1.4}$$

by construction. To simplify notation let $I := [r, s]$ and $J := [r, s^{1/q}]$. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{R}_A^{J^{1/q}} & \longrightarrow & \mathcal{R}_A^{I^{1/q}} \\ {\scriptstyle\varphi_L}\uparrow & & {\scriptstyle\varphi_L}\uparrow \\ \mathcal{R}_A^{J} & \longrightarrow & \mathcal{R}_A^{I} \end{array}$$

with the horizontal arrows being the natural maps. We can interpret restriction as a pullback along the canonical inclusion. The diagram tells us that the restriction to $I^{1/q}$ of the φ_L-pullback of M^J is the φ_L-pullback of the restriction of M^J to I. In formulae

$$\mathcal{R}_A^{I^{1/q}} \otimes_{\mathcal{R}_A^{J^{1/q}}} \varphi^*(M^J) = \varphi^*(M^I)$$

and plugging in (1.4) gives the desired

$$\mathcal{R}_A^{I^{1/q}} \otimes_{\mathcal{R}_A^{J^{1/q}}} \varphi^*(M^J) \cong \mathcal{R}^{I^{1/q}} \otimes_{\mathcal{R}_A^J} M^J.$$

Iterating this construction we obtain a φ_L-module over $\mathcal{R}_A^{[r,1)}$. Note that this module is finitely generated by 1.59 since each $M^{[r^{1/q^n}, s^{1/q^n}]}$ is generated by the same number of elements as $M^{[r,s]}$ by construction. To conclude the projectivity of $M^{[r,1)}$ one uses 1.59(iii). We next show that the functor is fully faithful. Given a morphism $M^{[r,s]} \to N^{[r,s]}$ we can apply the previous construction to both modules at the same time and take as a morphism between their φ-pullbacks the φ-pullback of f. These morphisms glue together due to the φ-compatibility of f. If $f = f^{[r,s]}$ is the restricition of a morphism $f^{[r,1)} : M^{[r,1)} \to N^{[r,1)}$, then this construction yields a morphism $g^{[r,1)}$ such that $g - f = 0$ on every $M^{[r^{q^{-k}}, s^{q^{-k}}]}$ but then $g = f$ by the coadmissiblity of $M^{[r,1)}$ and the condition on r, s that ensures that these intervals cover $[r, 1)$. Because $\mathcal{R}_A^{[r,1)} \to \mathcal{R}_A^{[r,s]}$ is flat, the functor $M^{[r,1)} \mapsto M^{[r,s]}$ is exact. Consider an exact sequence of φ_L-modules over $\mathcal{R}_A^{[r,s]}$

$$0 \to M_1^{[r,s]} \to M_2^{[r,s]} \to M_3^{[r,s]} \to 0.$$

Taking the pullback along $\varphi : \mathcal{R}_A^{[r,s]} \to \mathcal{R}_A^{[r^{1/q}, s^{1/q}]}$ remains exact because the $M_i^{[r,s]}$ are flat. Hence the induced sequence

$$0 \to M_1^{[r,1)} \to M_2^{[r,1)} \to M_3^{[r,1)} \to 0$$

is exact when restricted to each $[r^{q^{-k}}, s^{q^{-k}}]$. Finally the global section sequence remains exact by Lemma 1.55*(iii)*. □

Definition 1.64. *Following [SV15, Section 2] we define $\psi_{col} \colon o_L[\![T]\!] \to o_L[\![T]\!]$ to be the unique o_L-linear endomorphism satisfying*

$$\varphi_L \circ \psi_{col}(f)(T) = \sum_{a \in LT[\pi_L]} f(a +_{LT} T)$$

for all $f \in o_L[\![T]\!]$, where $+_{LT}$ denotes the addition via the Lubin-Tate group law. This endomorphism can be extended to a continuous endomorphism of the Robba ring $\mathcal{R}_L$ (cf. [FX12, Section 2.1]), which we denote by the same symbol. We define $\psi_{LT} := \pi_L^{-1}\psi_{col}$. We use the same symbol for the endomorphism $1 \otimes \psi_{LT}$ of $\mathcal{R}_A = A\hat{\otimes}_L\mathcal{R}_L$.

Note that we have $\psi_{LT} \circ \varphi_L = \frac{q}{\pi}$. In particular $\Psi = \frac{\pi}{q}\psi_{LT}$ is a continuous left-inverse of φ_L.

Definition 1.65. *Using the isomorphism $\varphi_M^* M^{r_0} \cong M^{r_0^{1/q}}$ we define*

$$\psi_M : M^{r_0^{1/q}} \cong \mathcal{R}_A^{r_0^{1/q}} \otimes_{\mathcal{R}_A^{r_0},\varphi_L} M^{r_0} \to M^{r_0}$$

by mapping $f \otimes m$ to $\psi_{LT}(f)m$.

Definition 1.66. *A (φ_L, Γ_L)-module over $\mathcal{R}_A$ is called L-**analytic** if its Γ_L-action is L-analytic.*

Remark 1.67. *Let M be a (φ_L, Γ_L)-module over $\mathcal{R}_A$. The dual*

$$M^* := \mathrm{Hom}_{\mathcal{R}_A}(M, \mathcal{R}_A)$$

endowed with the contragradient semi-linear (φ_L, Γ_L)-action is a (φ_L, Γ_L)-module.

Proof. For any model M^r we have canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{R}_A}(M, \mathcal{R}_A) = \mathrm{Hom}_{\mathcal{R}_A^{[r.1)}}(M^{[r,1)}, \mathcal{R}_A) = \mathcal{R}_A \otimes_{\mathcal{R}_A^{[r,1)}} \mathrm{Hom}_{\mathcal{R}_A^{[r.1)}}(M^{[r,1)}, \mathcal{R}_A^{[r,1)}).$$

The first one is given by the adjunction of restriction of scalars and base change while the second one follows from the projectivity of M^r. One can then check that $\mathrm{Hom}_{\mathcal{R}_A^{[r.1)}}(M^{[r,1)}, \mathcal{R}_A^{[r,1)})$ is a model for M^*. □

The problem of analyticity is more subtle. Hence we only treat it in the free case.

Remark 1.68. *Let M be a free (φ_L, Γ_L)-module of rank d over $\mathcal{R}_A$. Then the map assigning to $\gamma \in \Gamma_L$ its representation matrix $\mathrm{Mat}(\gamma) \in \mathrm{GL}_d(\mathcal{R}_A)$ with respect to any basis of M is L-analytic (with respect to the subspace topology of the product topology on the space $M_n(\mathcal{R}_A) \cong \mathcal{R}_A^{n^2}$) if and only if M is L-analytic. In this case the dual*

$$M^* := \mathrm{Hom}_{\mathcal{R}_A}(M, \mathcal{R}_A)$$

endowed with the contragradient semi-linear (φ_L, Γ_L)-action is also L-analytic.

Proof. We deduce from the definitions that given a locally L-analytic map $\Gamma_L \to W$ into a K-Banach space W and a continuous K-linear map $W \to E$ into another Banach space E, the composed map $\Gamma_L \to E$ is again L-analytic. Fix a model $M^{[r,1)}$ of M and a basis $e_1, \dots e_d$. These e_i are then also a basis of each $M^{[r,s]}$ for any $s \in [r, 1)$. In the remainder fix some s and a $\mathcal{R}_A^{[r,s]}$-Banach norm on M. If M is assumed to be L-analytic the orbit map of (the image of) each e_i in $M^{[r,s]}$ is locally L-analytic. I.e.

we have a locally L-analytic map $\Gamma_L \to M \cong \mathcal{R}_A^d$, mapping γ to $\gamma e_i = \sum_j a_{ij} e_j(\gamma)$. Projecting down to the component of e_j we see that each of these projections is locally L-analytic. In particular the map $\Gamma_L \to \mathrm{GL}_d(\mathcal{R}_A^{[r,s]})$ is L-analytic with respect to the subspace topology of the product topology on $\mathrm{GL}_d(\mathcal{R}_A^{[r,s]})) \subset M_d(\mathcal{R}_A^{[r,s]})$. Now suppose that the matrix describing the Γ_L action with respect to some basis $e_1, \dots, e_d$ defines an L-analytic map $\gamma \mapsto \mathrm{Mat}(\gamma)$. Then in particular the orbit map of each basis vector e_i is L-analytic and we are reduced to showing that the orbit map of $\mathcal{R}_A$-linear combinations of the e_i is L-analytic. Let $m = \sum_i f_i e_i$. Since we can instead consider the map

$$\Gamma_L \to M^d \to M$$

given by the composite of $\gamma \mapsto (\gamma(f_i e_i)_i)$ and $(m_i)_i \mapsto \sum_i m_i$ it suffices to show that $\gamma \mapsto \gamma(f_i e_i)$ is L-analytic for each i which can be seen by expanding the orbit maps of f_i and e_i using that the orbit map of f_i is locally L-analytic by example 1.44 and restricting to a joint radius of convergence using that the norm on $M^{[r,s]}$ is $\mathcal{R}_A$-submultiplicative. For the final statement recall that the inversion map $\gamma \mapsto \gamma^{-1}$ is locally L-analytic and the map sending a matrix to its transpose is continuous and K-linear, which implies that the contragradient action on the dual can be represented with respect to the dual basis of a given basis of M by a locally L-analytic matrix-valued function (namely $\gamma \mapsto \gamma(\mathrm{Mat}(\gamma^{-1}))^t$) and thus is L-analytic by the first part. □

For an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$ the fibre at $z \in \mathrm{Sp}(A)$, i.e., the reduction $M_z = M/\mathfrak{m}_z M$ is an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_{A/\mathfrak{m}_z}$ and it is natural to ask in what sense analyticity can be checked at the fibres. In Proposition A.27 in the appendix we show that this is the case if A is reduced.

1.4.1 A standard estimate for (φ_L, Γ_L)-modules

A recurring theme in [KPX14] is the fact than upon restricting M to a closed interval $I = [r, s]$ we have for the operator norm

$$||\gamma - 1||_{M^I} \xrightarrow{\gamma \to 1} 0.$$

This remains true for our case but in order to study the action of the distribution algebra via the operators $Z_n = \varphi^{n-n_0}(Z_{n_0}) \in D(\Gamma_n, K)$ we need to estimate the operator norm of these variables. Note that since $D(\Gamma_n, K)$ is a Fréchet space and M^I is a Banach space the a priori separately continuous action $D(\Gamma_n, K) \times M^I$ is in fact jointly continuous.

Remark 1.69. *The induced map $\rho : D(\Gamma_n, K) \to \mathcal{E}nd_K(M^I)$, that maps λ to the map mapping $x \to \lambda x$, is continuous with respect to the operator norm on $\mathcal{E}nd_K(M^I)$.*

Proof. Let $\varepsilon > 0$. Since the multiplication map $D(\Gamma_n, K)\hat{\otimes}_K M^I \to M^I$ is continuous with respect to the projective tensor product, there is a continuous semi-norm p and a constant c such that the ball $\{v \in D(\Gamma_n, K)\hat{\otimes}_K M^I, p \otimes |-|_I(v) \leq c\}$ is mapped into $\{m \in M^I \mid |m| \leq \varepsilon.\}$ If $\lambda \in D(\Gamma_n, K)$ satisfies $p(\lambda) \leq c$, then $p \otimes |-|_I(\lambda \otimes n) \leq c$ for any $n \in M^I$ with $|n|_{M^I} \leq 1$. In conclusion the ball $\{\lambda \in D(\Gamma_n, K) \mid p(\lambda) \leq c\}$ is mapped into $\{F \in \mathcal{E}nd_K(M^I) \mid ||F||_{M^I} \leq \varepsilon\}$, because for any $x \in M^I$ with $|x| \leq 1$ we have $|\rho(\lambda)(x)| \leq \varepsilon$. □

Remark 1.70. *The sequence $T, \varphi(T), \varphi^2(T), \dots$ converges to 0 with respect to the Fréchet-topology on $\mathcal{R}_K^{[0,1)}$.*

Proof. By [Sch17, Lemma 1.7.7] we have $\varphi^{2k}(T) \in T\pi_L^k o_L[\![T]\!] + T^k o_L[\![T]\!]$. A small calculation further shows $\varphi^{2k+1}(T) \in T\pi_L^k o_L[\![T]\!] + T^k o_L[\![T]\!]$. Observe that the r-Gauß norm of any element in $T\pi_L^k o_L[\![T]\!] + T^k o_L[\![T]\!]$ is at most $\max(|\pi_L|^k r, r^k)$ and hence tends to 0 for every $r \in (0, 1)$. □

Lemma 1.71. *Let M be a finitely generated module over $\mathcal{R}_A^r$ with an L-analytic semi-linear Γ_L-action. Fix any closed interval $I = [r, s] \subset [r, 1)$ and any Banach norm on M^I.*

(i) We have $||\gamma - 1||_{M^I} \xrightarrow{\gamma \to 1} 0$.

(ii) Furthermore $||Z_n||_{M^I} \xrightarrow{n \to \infty} 0$.

Proof. For the first statement let $\varepsilon > 0$ and let $m_1, \dots, m_d$ be a system of generators of M^I. We will show that there exists an open subgroup U such that $||\gamma - 1|| < \varepsilon$ for every $\gamma \in U$. By the continuity of the action we have $\lim_{\gamma \to 1}(\gamma - 1)m_i = 0$ for every i. Furthermore given $m = \sum_i f_i m_i \in M^I$ we can treat each factor $f_i m_i$ seperately and get

$$(\gamma - 1)f_i m_i = (\gamma - 1)(f_i)m_i + \gamma(f_i)(\gamma - 1)(m_i).$$

We know that $(\gamma - 1)(m_i)$ can be made arbitrarily small. It remains to show that $(\gamma - 1)(f_i)$ converges to 0 uniformly as $\gamma \to 1$ and that $\gamma(f_i)$ is bounded. We have $|\gamma(f_i)| = |\gamma(f_i) - f_i + f_i| \leq \max(|\gamma(f_i) - f_i|, |f_i|)$. It thus suffices to show the statement for $M = \mathcal{R}_A$. Since $\mathcal{R}_K \otimes_K A$ is dense in $\mathcal{R}_A$ it suffices to check the corresponding statement there. The case $\mathcal{R}_K$ is in [BSX20, Lemma 2.3.5] hence we may find an open subgroup U such that the result holds for $A = K$ and every $\gamma \in U$. Let $v = \sum_i f_i \otimes a_i$ be some representation of $v \in \mathcal{R}_K^I \otimes_K A$. We get $(\gamma - 1)v = \sum_i (\gamma - 1)f_i \otimes a_i$ and thus $|(\gamma - 1)v| \leq \max \varepsilon |f_i||a_i|$. Since this holds for any representation of v we get $|(\gamma - 1)v| \leq \varepsilon |v|$, which proves the statement. For the second statement we combine Remark 1.70 and Remark 1.69 to conclude that given $\varepsilon > 0$ there exists k_0 such that $||\varphi^k(Z_{n_0})||_{M^I} < \varepsilon$ for any $k \geq k_0$. Since $Z_n = \varphi^{n - n_0}(Z_{n_0})$ the conclusion follows. □

Remark 1.72. *In the classical case one works with the variable $\gamma - 1$ and 1.71(ii) is an immediate consequence of the continuity of the Γ_L action since $\varphi_{cyc}(\gamma - 1) = (1 + (\gamma - 1))^p - 1 = \gamma^p - 1$.*

1.4.2 Duality

Definition 1.73. *Let* $f = \sum_i a_i T^i \in \mathcal{R}_A$ *we define* $\operatorname{res}(f) := a_{-1} \in A$*. We obtain an* A*-linear map* $\mathcal{R}_A \xrightarrow{\operatorname{res}} A$*.*

This result is well-covered in the literature when A is a (discretely valued) field (see for instance [Cre98, Chapter 5]). The case of general A is similar due to 1.25.

Proposition 1.74. *Consider the bilinear map* $\mathcal{R}_A \times \mathcal{R}_A \to A$*, mapping* (f,g) *to* $\operatorname{res}(fg)$*. This induces topological isomorphisms (for the strong topologies*[3]*)*

$$\operatorname{Hom}_{A,cts}(\mathcal{R}_A, A) \cong \mathcal{R}_A,$$

$$\operatorname{Hom}_{A,cts}(\mathcal{R}_A^+, A) \cong \mathcal{R}_A/\mathcal{R}_A^+$$

and

$$\operatorname{Hom}_{A,cts}(\mathcal{R}_A/\mathcal{R}_A^+, A) \cong \mathcal{R}_A^+.$$

Proof. Let $\mu \in \operatorname{Hom}_{A,cts}(\mathcal{R}_A, A)$. We define a Laurent series $f_\mu := \sum_{n\in\mathbb{Z}} \mu(T^{-1-n})T^n$. On the other hand to $h \in \mathcal{R}_A$ we assign the functional $\operatorname{can}(h) : g \mapsto \operatorname{res}(hg)$. We first show that f_μ is well-defined. Because μ is continuous it remains continuous when we restrict to any $\mathcal{R}^{[r,1)}$ with respect to the Fréchet topology. In particular we may find some $r < s < 1$ and $\varepsilon > 0$ such that $|\mu(U_{\varepsilon,[r,s]})| \leq 1$ where $U_{\varepsilon,[r,s]}$ denotes the "closed" ball with radius ε with respect to the Banach norm $|\cdot|_{[r,s]}$. Assume without loss of generality that $\varepsilon, r, s \in |K^\times|$ and let $\eta, \rho, \sigma \in K$ be elements satisfying $|\eta| = \varepsilon$ (resp. for r, s). Then $T^n\eta\sigma^{-n}$ and $T^{-n}\eta\rho^n$ belong to $U_{\varepsilon,[r,s]}$ for every $n \geq 0$. We deduce

$$|\mu(T^n)| \leq \max\{\varepsilon^{-1}r^n, \varepsilon^{-1}s^n\} \tag{1.5}$$

for every $n \in \mathbb{Z}$. Next we need to refine the estimate for negative n. Note that for any ρ' with $|\rho'| > |\rho| = r$. We also get $T^n\eta(\rho')^{-n} \in U_{\varepsilon,[r,s]}$ for negative n. If we fix n and let $(\rho'_m)_m$ converge to 1 we obtain $|\mu(T^n)| \leq \limsup_m \varepsilon^{-1}|(\rho'_m)^n| = \varepsilon^{-1}$. It suffices to prove that the series $\sum_i \mu(T^{-i})T^i$ belongs to $\mathcal{R}_A$ since it differs from f_μ by the unit $T \in \mathcal{R}_K^\times$. For $n \geq 0$ we have $|\mu(T^{-n})| \leq \varepsilon^{-1}$ by the improved estimate. Therefore the non-principal part of the Laurent series converges for $|T| < 1$. For the principal part we have $|\mu(T^n)| \leq \varepsilon^{-1}|\sigma^n| = \varepsilon^{-1}s^n$. This shows $f_\mu \in \mathcal{R}_A^{[t,1)}$ for any $t > s$. Next we show that $\mu \mapsto f_\mu$ is inverse to the map $\operatorname{can}(-)$. On the one hand given $g = \sum_n a_nT^n$ we have $f_{\operatorname{can}} = \sum_n \operatorname{res}(gT^{-1-n}T^n) = \sum a_nT^n = g$. On the other hand given $\operatorname{can}(f_\mu)(g) = \sum_n \operatorname{can}(f_\mu)(a_nT^n) = \sum_n a_n\mu(T^n) = \mu(g)$. Futhermore f_μ belongs to $\mathcal{R}_A^+$ if and only if the coefficients of the principal part (i.e. $\mu(T^n)$ for $n \geq 0$) vanish. This is the case if and only if $\mu(\mathcal{R}_A^+) = 0$. Regarding the topologies we sketch how to deduce the general case from the case $A = L$ (treated by [Cre98]) by showing that for $E \in \{\mathcal{R}_L, \mathcal{R}_L/\mathcal{R}_L^+, \mathcal{R}_L^+\}$ we have a canonical isomorphism $\operatorname{Hom}_{A,cts}(A\hat{\otimes}_{L,\pi}E, A) \cong \operatorname{Hom}_{L,cts}(E, A) = \mathcal{L}_b(E, A)$, which allows us to deduce the

[3]By the strong topology we mean the subspace topology of the space of continuous K-linear operators $\mathcal{L}_{K,b}(\mathcal{R}_A, A)$ with the strong topology.

general statement from the known case using $E\hat{\otimes}_{L,\pi}A \cong \mathcal{L}_b(E', A)$ (as was seen in 1.23).[4] We define the maps as follows: Let $f : A\hat{\otimes}_{L,\pi}E \to A$ be a continuous A-linear homomorphism. Set $\tilde{f} : E \to A \otimes_{L,\pi} E \to A\hat{\otimes}_{L,\pi}E \to A$. Given by mapping e to $1 \otimes e$ and post-composing with the natural map. By [PGS10, Theorem 10.3.9] since $\|1\|_A = 1$ the first map is a homeomorphic embedding (in particular continuous). On the other hand let $h : E \to A$ be continuous L-linear then h extends uniquely to an A-linear map $A \otimes_L E \to A$ and it remains to check that it is continuous. As we saw in 1.23 it suffices to check separate continuity, which is clear. Due to A being complete h extends uniquely to a continuous A-linear map $h_A : A\hat{\otimes}_{L,\pi}E \to A$. Clearly $f \to \tilde{f}$ and $h \to h_A$ are inverse to one another. It remains to check continuity with respect to the corresponding strong topologies. For that purpose we denote by A_0 the unit ball inside A. Let $B \subset A\hat{\otimes}E$ be a bounded set and suppose $f(B) \subset A_0$, then (again using [PGS10, Theorem 10.3.9]) the preimage $\tilde{B}$ of $B \cap 1 \otimes E$ in E is bounded and by construction $\tilde{f}(\tilde{B}) \subset A_0$. On the other hand let $B' \subset E$ be bounded and suppose $h(B') \subset A_0$. Then the closure B_A of $\mathrm{span}_{o_L}\{x \otimes y \mid x \in A_0, y \in B\}$ is bounded in $A\hat{\otimes}_{L,\pi}E$ and by construction $h_A(B_A) \subset A_0$. □

Proposition 1.75. *Let M be a (φ_L, Γ_L)-module over $\mathcal{R}_A$ and consider the dual $\check{M} := \mathrm{Hom}_{\mathcal{R}_L}(M, \mathcal{R}_A(\chi_{LT}))$. The canonical pairing*

$$\check{M} \times M \to \mathcal{R}_A(\chi_{LT})$$

is perfect and by post-composing with the residue map gives a bilinear pairing

$$\check{M} \times M \xrightarrow{\langle \cdot,\cdot \rangle} A$$

identifying $\check{M}$ (resp. M) with $\mathrm{Hom}_{A,cts}(\check{M}, A)$ ($\mathrm{Hom}_{A,cts}(M, A)$) with respect to the strong topology, satisfying

1. $\langle \varphi_L(\check{m}), \varphi_L(m) \rangle = \frac{q}{\pi_L} \langle \check{m}, m \rangle$
2. $\langle \sigma\check{m}, \sigma m \rangle = \langle \check{m}, m \rangle$
3. $\langle \psi_{LT}(\check{m}), m \rangle = \langle \check{m}, \varphi_L(m) \rangle$

For all $\sigma \in \Gamma_L$, $\check{m} \in \check{M}$ and $m \in M$.

Proof. The perfectness of the first pairing is well-known since M is finitely generated and projective. The properties of the pairing are proved in [SV15, Section 3] for the ring $\widehat{o_L((T))}^{p-\mathrm{adic}}$ and their proofs carry over to this case. By writing M as a direct summand of a finitely generated free module the identification of duals follows from the free case by induction over the rank from 1.74. □

[4]Alternatively one can redo the proof with A replacing L.

1.4.3 Sen-Theory

We do not give a conceptual treatment of Sen-Theory and instead present some ad-hoc results that allow us to view a (φ_L, Γ_L)-module M^r over $\mathcal{R}_K^r$ as a Γ_L-submodule of a finite projective $(L_n \otimes K)[\![t_{LT}]\!]$-module $D^+_{\mathrm{dif},n}(M)$ for a suitable $n \in \mathbb{N}$, which is technically useful because the latter module is a projective limit of finite-K-dimensional Γ_L-representations namely $D^+_{\mathrm{dif},n}(M)/(t^k_{LT})$. The name "Sen-Theory" stems from the fact that these modules are the analogues of their counterparts in classical Sen-Theory (more precisely its extension to B_{dR}-representations) for (φ_L, Γ_L)-modules which arise from Fontaine's equivalence of categories. For $n \in \mathbb{N}$ let $K_n := K \otimes_L L_n$ and fix a non-trivial compatible system u_n of π_L^n-torsion points of the Lubin-Tate group. We endow K_n with its canonical K-Banach space topology and endow $K_n[\![t_{LT}]\!] = \varprojlim_k K_n[\![t_{LT}]\!]/(t^k_{LT})$ with the projective limit topology of the canonical topologies on each term. Since $t_{LT} = \log_{LT}(T)$ has no constant term and non-vanishing derivative in 0 the induced maps

$$K_n[\![t_{LT}]\!]/(t^k_{LT}) \to K_n[\![T]\!]/(T^k)$$

are isomorphisms of finite dimensional K-vector spaces and hence topological for the respective canonical topologies. We obtain that the natural map

$$K_n[\![t_{LT}]\!] \to K_n[\![T]\!]$$

is an isomorphism if we endow the right-hand side with its weak topology. The definition for ι_n below is taken from [Col16, Section 1.4.2].

Lemma 1.76. *Let $n \in \mathbb{N}$ and let $r^{(n)} = |u_n|$ then*

$$\begin{aligned}\iota_n : \mathcal{R}_K^{[r^{(n)},1)} &\to K_n[\![t_{LT}]\!]\\ T &\mapsto \iota_n(T),\end{aligned}$$

given by $\iota_n(T) := u_n +_{LT} \exp_{LT}(\pi_L^{-n} \log_{LT}(T))$, is well-defined, injective and Γ_L-equivariant, where Γ_L acts on the right-hand side via the trivial action on K, the Galois action on L_n and the usual action (via χ_{LT}) on t_{LT}.

Proof. The convergence, injectivity and Γ_L-equivariance of ι_n in the cyclotomic case with $K = L = \mathbb{Q}_p$ is known (see [Ber02, Proposition 2.25 and the remark before 2.35]) and can be analogously proved for the Robba ring $\mathcal{R}_L$ over L. The map over $\mathcal{R}_K^{[r^{(n)},1)}$ arises by applying $K\hat{\otimes}_{L,\pi}$ to the version over the spherically complete L. We apply [Eme17, 1.1.26] to conclude that the induced map remains injective using that K as an L-Banach space is automatically bornological. The compatibility with the actions is preserved, since we let Γ_L act trivially on K. □

Definition 1.77. *Let M be a (φ_L, Γ_L)-module over $\mathcal{R}_K$ with model $M^{r^{(n)}}$ over $[r^{(n)}, 1)$ with $r^{(n)}$ as in 1.76. We define $\mathbb{D}^+_{dif,n}(M) := K_n[\![t_{LT}]\!] \otimes_{\iota_n, \mathcal{R}_K^{[r(n),1)}} M^{r^{(n)}}$.*

Remark 1.78. *Let M be a (φ_L, Γ_L)-module over $\mathcal{R}_K$ with model $M^{r^{(n)}}$ over $[r^{(n)}, 1)$ with $r^{(n)}$ as in 1.76. Then the natural map*

$$M^{r^{(n)}} \to \mathbb{D}^+_{dif,n}(M)$$

is injective and Γ_L-equivariant.

Proof. Using that $M^{r^{(n)}}$ is projective the statement follows by tensoring the injective and Γ_L-equivariant map ι_n with $M^{r^{(n)}}$. □

1.5 Étale (φ_L, Γ_L)-modules

In this section we give a brief overview on étale modules over the p-adic completion $\mathbf{A}_L$ of $o_L((T))$. We denote by $\mathrm{Rep}_{o_L}(G_L)$ the category of finitely generated o_L-modules with o_L-linear continuous (with respect to the p-adic topology) G_L-action. Similary let $\mathrm{Rep}_L(G_L)$ be the category of finite dimensional L-vector space with continuous L-linear G_L-action. We denote by $\mathbb{C}_p^\flat$ the tilt of $\mathbb{C}_p$ and for an o_L-algebra R we write $W(R)_L$ for the ring of ramified Witt vectors.

Definition 1.79. *A (φ_L, Γ_L)-module D over $\mathbf{A}_L$ is a finitely generated $\mathbf{A}_L$-module with a φ_L-semi-linear map φ_D and a semi-linear action of Γ_L commuting with φ_D, such that Γ_L acts continuously with respect to the weak topology. D is called **étale** if the linearised map $\mathbf{A}_L \otimes_{\mathbf{A}_L, \varphi_L} D \to D$, mapping $a \otimes d$ to $a\varphi_D(d)$ is an isomorphism. Analogously we define (φ_L, Γ_L)-modules over $\mathbf{B}_L := \mathbf{A}_L[1/p]$. Such a module is called **étale** if it arises as a base change of an étale (φ_L, Γ_L)-module over $\mathbf{A}_L$.*

Remark 1.80. *Fixing a choice of generator of $\varprojlim_n LT[\pi_L^n]$ leads to a canonical continuous Γ_L-equivariant embedding*

$$\iota\colon \mathbf{A}_L \to (W(\mathbb{C}_p^\flat)_L)^{H_L}$$

such that $\iota\varphi_L(f) = \mathrm{Frob}_q(\iota(f))$, whose image is independent of the choice. This embedding extends to an embedding of the p-adic completion of the maximal unramified extension $\mathbf{A}$ of $\mathbf{A}_L$ into $W(\mathbb{C}_p^\flat)_L$.

Proof. See [Sch17, Proposition 2.1.16, Remark 2.1.17 and Remark 3.1.4]. □

By abuse of notation we write φ_L for the Frobenius operator on $W(\mathbb{C}_p^\flat)_L$, which is justified due to 1.80.

Theorem 1.81. *The functors*

$$V \mapsto \mathbb{D}(V) := (\mathbf{A} \otimes_{o_L} V)^{H_L}$$

and

$$D \mapsto \mathbb{V}(D) := (\mathbf{A} \otimes_{\mathbf{A}_L} D)^{\varphi_L = 1}$$

are quasi-inverse and give an equivalence of categories between $\mathrm{Rep}_{o_L}(G_L)$ and the category of étale (φ_L, Γ_L)-modules over $\mathbf{A}_L$. This equivalence is exact and respects duals, torsion-sub-objects and tensor products and inverting p induces an equivalence of categories between $\mathrm{Rep}_L(G_L)$ and the category of étale (φ_L, Γ_L)-modules over $\mathbf{B}_L$.

Proof. See [Sch17, Theorem 3.3.10]. □

In more generality one has for F/L finite a similar equivalence for $\mathrm{Rep}_{o_L}(G_F)$ but throughout this thesis only concern ourselves with the case $F = L$. The rings $\mathbf{B}_L$ and $\mathcal{R}_L$ are not directly comparable because the series in $\mathbf{A}_L$ do not necessarily converge on some annulus and on the other hand the coefficients of a series in $\mathcal{R}_L$ are not necessarily bounded. However both rings contain the rings

$$\mathbf{B}_L^{\dagger,r} := \left\{ f = \sum_{k\in\mathbb{Z}} a_k T^k \mid \lim_{k\to-\infty} |a_k| r^k = 0 \text{ and } \sup_k |a_k| < \infty \right\}$$

and their union $\mathbf{B}_L^\dagger$.

Definition 1.82. *A (φ_L, Γ_L)-module D over $\mathbf{B}_L$ is called **overconvergent** if it admits a basis such that the matrices of φ_L and all $\gamma \in \Gamma_L$ have entries in $\mathbf{B}_L^\dagger$. The $\mathbf{B}_L^\dagger$-span of this basis is denoted by $D^\dagger$. A representation $V \in \mathrm{Rep}_L(G_L)$ is called **overconvergent** if $\mathbb{D}(V)$ is overconvergent. V is called L-**analytic** if $\mathbb{C}_p \otimes_{L,\sigma} V$ is isomorphic to the trivial $\mathbb{C}_p$-semilinear representation $\mathbb{C}_p^{\dim_L(V)}$ for every embedding $\sigma : L \to \mathbb{C}_p$ with $\sigma \neq \mathrm{id}$. If V is overconvergent, we set $\mathbb{D}_{rig}^\dagger(V) := \mathcal{R}_L \otimes_{\mathbf{B}_L^\dagger} \mathbb{D}^\dagger(V)$.*

By [FX12, Proposition 1.6] étale (φ_L, Γ_L)-modules over $\mathcal{R}_L$ always descend to $\mathbf{B}_L^\dagger$. But contrary to the classical cyclotomic situation there exist representations that are not overconvergent (cf. [FX12, Theorem 0.6]) and hence the category of (φ_L, Γ_L)-modules over $\mathcal{R}_L$ is in some sense insufficient to study representations. Remarkably restricting to L-analytic objects on both sides alleviates this problem.

Definition 1.83. *Let $\rho : L^\times \to L^\times$ be a continuous character. We define $\mathcal{R}_L(\rho) = \mathcal{R}_L e_\rho$ as the free rank 1 (φ_L, Γ_L)-module with basis e_ρ and action given as $\varphi_L(e_\rho) = \rho(\pi_L) e_\rho$ and $\gamma(e_\rho) = \rho(\chi_{LT}(\gamma)) e_\rho$. For $i \in \mathbb{Z}$ we write $\mathcal{R}_L(x^i)$ as shorthand for the (φ_L, Γ_L)-module associated to the character $x \mapsto x^i$.*

Remark 1.84. *Let M be a (φ_L, Γ_L)-module over $\mathcal{R}_L$ of rank 1. Then there exists a character $\rho : L^\times \to L^\times$ such that M is isomorphic to $\mathcal{R}_L(\rho)$. The module is L-analytic if and only if $\rho_{|o_L^\times}$ is locally L-analytic.*

Proof. See [FX12, Proposition 1.9]. □

Definition 1.85. *Let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_L$. If $M \cong \mathcal{R}_L(\rho)$ has rank 1 we define*

$$\deg(M) := \mathrm{val}_{\pi_L}(\rho(\pi_L)).$$

*In general we define $\deg(M) := \deg(\Lambda^{\mathrm{rank}(M)} M)$ and finally the slope $\mu(M) := \deg(M)/\mathrm{rank}(M)$[5] M is called **isoclinic** if $\mu(N) \geq \mu(M)$ for every subobject N of M. M is called **étale** if it is isomorphic to the (φ_L, Γ_L)-module $\mathbb{D}_{rig}^\dagger(V)$ attached to some L-analytic G_L-representation V.*

[5]With our conventions $\mathcal{R}_L(x^i)$ has slope i.

Theorem 1.86 (Kedlaya/ Berger)**.** *Every non-zero (L-analytic)* (φ_L, Γ_L)*-module* M *possesses a unique functorial filtration*

$$0 = M_0 \subsetneq M_1 \cdots \subsetneq M_d = M$$

such that the succesive quotients M_i/M_{i-1} *are isoclinic (L-analytic)* (φ_L, Γ_L)*-modules and* $\mu(M_1/M_0) < \cdots < \mu(M_d/M_{d-1})$*. Furthermore the functor* $V \mapsto D^{\dagger}_{rig}(V)$ *defines an equivalence of categories between the category of L-analytic L-linear* G_L*-representations and the full subcategory of L-analytic* (φ_L, Γ_L)*-modules that are isoclinic of slope* 0*.*

Proof. From [Ked05, Theorem 6.4.1] we obtain a unique filtration by (so-called) saturated φ_L-modules M_i. By the uniqueness of the filtration the additional Γ_L-structure is inherited by the modules in the filtration. Note that if M is assumed to be L-analytic, then so are the M_i in its Harder-Narasimhan filtration. The requirement in loc. cit. for the sub-modules $M_i \subseteq M$ to be saturated is equivalent to the requirement that $M_i \subset M_{i+1}$ is a $\mathcal{R}_L$-direct summand (c.f. [Pot20, §8.1]), which is equivalent to requiring that M_i/M_{i+1} is again projective[6] i.e. a (φ_L, Γ_L)-module. The statement regarding étale modules is [Ber16, Theorem 10.4]. Berger defines the notion of étale differently. The fact that being étale is equivalent to being isoclinic of slope 0 is implicit in the proof of 10.1. ibidem. □

[6]Note that $\mathcal{R}_L$ is a Bézout domain hence being finitely generated projective is equivalent to being finite free.

Chapter 2

The kernel of ψ

In this chapter we study the Γ_L-action on the kernel of the operator $\psi = \psi_M$ (defined in 1.64) for an L-analytic (φ_L, Γ_L)-module M. For the whole chapter fix $r_0 \in (0,1)$ such that M comes from a (φ_L, Γ_L)-module over $\mathcal{R}_A^{[r_0,1)}$. We wish to show that $M^{\psi=0}$ is a finite projective module over the relative group Robba ring $\mathcal{R}_A(\Gamma_L)$. For a closed interval I we always view the continuous K-linear endomorphisms $\mathcal{E}nd_K(M^I)$ as a Banach space endowed with the operator norm. We denote by $\mathcal{E}nd_A(M^I)$ the A-subalgebra of continuous A-linear endomorphisms.

2.1 Some technical preparation.

Lemma 2.1. *Let V be a K-Banach space and let $F, G \in \mathcal{E}nd_K(V)$ such that G is invertible and*

$$||F - G|| < ||G^{-1}||^{-1}.$$

Then F is invertible.

Proof. By assumption the operator

$$(1 - FG^{-1}) = -(F - G)G^{-1}$$

has operator norm < 1 hence the series

$$\sum_{k \geq 0} (1 - FG^{-1})^k$$

converges to an inverse of $F \circ G^{-1}$ with respect to the operator norm. Using that G is invertible we conclude that F has to be invertible as well. □

Lemma 2.2. *Let R be an A-Banach algebra, i.e. a complete normed A-Algebra. Let B be a K-Banach algebra and let $H : B \to R$ be a continuous K-algebra homomorphism. Then it extends to a continuous A-linear homomorphism*

$$A \hat{\otimes}_K B \to R.$$

Proof. Let $a \in A$ and define a K-bilinear map $A \times B \to R$ by mapping (a, b) to $aH(b)$. Using that R is a topological A-module and the map H is continuous one verifies that this map is separately continuous. Since A and B are Banach-spaces the inductive and projective tensor product topology agree and due to the completeness of R we obtain an extension

$$A\hat{\otimes}_K B \to R.$$

This extension is a K-algebra homomorphism because $\lambda H(b) = H(\lambda b)$ for any $b \in K$ and furthermore it is A-linear by construction. □

Remark 2.3. *Let I be closed then $\mathcal{E}nd_A(M^I)$ is an A-Banach algebra*

Proof. It suffices to prove that $\mathcal{E}nd_A(M^I)$ is a closed subspace of the K-Banach algebra $\mathcal{E}nd_K(M^I)$. For any $a \in A$ denote by the same symbol the multiplication-by-a-map. Then $\theta_a : \mathcal{E}nd_K(M^I) \to \mathcal{E}nd_K(M^I)$ mapping f to $af - fa$ is continuous with respect to the operator norm and an endomorphism is A-linear if and only if it lies in the closed subspace $\bigcap_{a\in A} \ker(\theta_a)$. □

2.2 The group Robba ring and the structure of $M^{\psi=0}$.

A key observation is the following decomposition.

Lemma 2.4. *We have*

$$\mathcal{R}_K^{[r,s]} \cong \bigoplus_{a\in o_L/\pi_L^n} \varphi_L^n\left(\mathcal{R}_K^{[r^{q^n},s^{q^n}]}\right)\eta(a,T).$$

Proof. See [Col16, Proposition 1.4]. □

We now define the relative Robba group ring $\mathcal{R}_A(\Gamma_L)$.

Definition 2.5. *From the isomorphism $\chi_{LT} : \Gamma_L \to o_L^\times$ we get a canonical filtration $\Gamma_L = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \dots$ by defining $\Gamma_n := \chi_{LT}^{-1}(1 + \pi_L^n o_L)$ for $n \geq 1$. For n large enough we have an isomorphism $\Gamma_n \cong \pi_L^n o_L$ by mapping γ to $\log(\chi_{LT}(\gamma))$. Let n_0 be minimal with this property. Define charts $l_n : \Gamma_n \xrightarrow{\log(\chi_{LT}(\cdot))} \pi_L^n o_L \xrightarrow{\cong} o_L$, where the second arrow is given by dividing by π_L^n. This induces an isomorphim of Fréchet algebras*

$$D(o_L, K) \cong D(\Gamma_n, K).$$

Using the isomorphism from 1.27 we can view the right-hand side as a ring of convergent power series in the variable Z_n from 1.37. By transport of structure we define the ring extensions $\mathcal{R}_K(\Gamma_n) \cong \mathcal{R}_K$ and $\mathcal{R}_K^I(\Gamma_n) \cong \mathcal{R}_K^I$. Denoting by $i_{n+m} : \Gamma_{n+m} \to \Gamma_n$ the natural inclusion for $m \geq 0$ we get a commutative diagram

$$\begin{array}{ccc} o_L & \xrightarrow{l_{n+m}^{-1}} & \Gamma_{n+m} \\ \big\downarrow{\scriptstyle \pi_L^m} & & \big\downarrow{\scriptstyle \iota_{n+m}} \\ o_L & \xrightarrow{l_n^{-1}} & \Gamma_n, \end{array}$$

which induces a commutative diagram

$$\begin{array}{ccc} D(o_L,K) & \xrightarrow{(l_{n+m}^{-1})_*} & D(\Gamma_{n+m},K) \\ \downarrow{\scriptstyle (\pi_L^m)_*} & & \downarrow{\scriptstyle (\iota_{l+m})_*} \\ D(o_L,K) & \xrightarrow{(l_n^{-1})_*} & D(\Gamma_n,K). \end{array}$$

Using the isomorphism $\mathcal{R}^+ \cong D(o_L,K)$ *and the fact that* $(\pi_L)_*$ *corresponds to the map* φ_L *we get a commutative diagram*

$$\begin{array}{ccc} \mathcal{R}_K^{I^{q^m}} & \xrightarrow{(l_{n+m}^{-1})_*} & \mathcal{R}_K^{I^{q^m}}(\Gamma_{n+m}) \\ \downarrow{\scriptstyle \varphi_L^m} & & \downarrow{\scriptstyle (\iota_{l+m})_*} \\ \mathcal{R}_K^I & \xrightarrow{(l_n^{-1})_*} & \mathcal{R}_K^I(\Gamma_n) \end{array}$$

and by taking limits a corresponding diagram

$$\begin{array}{ccc} \mathcal{R}_K & \xrightarrow{(l_{n+m}^{-1})_*} & \mathcal{R}_K(\Gamma_{n+m}) \\ \downarrow{\scriptstyle \varphi_L^m} & & \downarrow{\scriptstyle (\iota_{l+m})_*} \\ \mathcal{R}_K & \xrightarrow{(l_n^{-1})_*} & \mathcal{R}_K(\Gamma_n). \end{array}$$

By mapping $\gamma \in \Gamma_n$ *to its Dirac distribution we obtain a canonical map* $\Gamma_n \to D(\Gamma_n,K)^\times$. *By transport of structure from 2.4 we see that the natural maps induce topological*[1] *isomorphisms*

$$\mathcal{R}_K^{I^{q^m}}(\Gamma_{n+m}) \otimes_{\mathbb{Z}[\Gamma_{n+m}]} \mathbb{Z}[\Gamma_n] \to \mathcal{R}_K^I(\Gamma_n)$$

and

$$\mathcal{R}_K(\Gamma_{n+m}) \otimes_{\mathbb{Z}[\Gamma_{n+m}]} \mathbb{Z}[\Gamma_n] \to \mathcal{R}_K(\Gamma_n).$$

This allows us to extend our definitions to $0 \leq n \leq n_0$ *by setting*

$$\mathcal{R}_K^I(\Gamma_n) := \mathcal{R}_K^{I^{q^{n_0-n}}}(\Gamma_{n_0}) \otimes_{\mathbb{Z}[\Gamma_{n_0}]} \mathbb{Z}[\Gamma_n]$$

and

$$\mathcal{R}_K(\Gamma_n) := \mathcal{R}_K(\Gamma_{n_0}) \otimes_{\mathbb{Z}[\Gamma_{n_0}]} \mathbb{Z}[\Gamma_n],$$

where the topology is given by the product topology with respect to the decomposition of $\mathbb{Z}[\Gamma_n]$. *Finally in the relative case we define* $\mathcal{R}_A^I(\Gamma_n)$ *as the completed tensor product* $\mathcal{R}_K^I(\Gamma_n)\hat{\otimes}_K A$ *endowed with the tensor product norm and* $\mathcal{R}_A(\Gamma_n)$ *via*

$$\mathcal{R}_A(\Gamma_n) := \varinjlim_{0\leq r<1} \varprojlim_{r<s<1} \mathcal{R}_A^{[r,s]}(\Gamma_n).$$

[1] We endow the left-hand side with the maximum norm with respect to the decomposition $\mathbb{Z}[\Gamma_n] = \bigoplus_{\gamma\in\Gamma_n/\Gamma_{n+m}} \gamma\mathbb{Z}[\Gamma_{n+m}]$.

Remark 2.6. *$\mathcal{R}_A^{[r,1)}(\Gamma_n)$ is a Fréchet-Stein algebra.*

Proof. If $n \geq n_0$ the statement is clear by transport of structure. If $n < n_0$ we recall the decomposition

$$\mathcal{R}_A^I(\Gamma_n) = \mathcal{R}_A^{I^{q^{n_0-n}}}(\Gamma_{n_0}) \otimes_{\mathbb{Z}[\Gamma_{n_0}]} \mathbb{Z}[\Gamma_n].$$

Take a sequence $r = r_0, r_1, \dots$ converging to 1, let $I_k := [r_0, r_k]$ and let $\tilde{I}_k := I_k^{q^{n_0-n}}$. We know that $\mathcal{R}_A^{[r,1)^{q^{n_0-n}}}$ is Fréchet-Stein and hence the maps $\mathcal{R}_A^{\tilde{I}_{k+1}} \to \mathcal{R}_A^{\tilde{I}_k}$ are flat with dense image. Since $\mathbb{Z}[\Gamma_n]$ is free over $\mathbb{Z}[\Gamma_{n_0}]$ and hence flat these properties remain for the induced maps

$$\mathcal{R}_A^{\tilde{I}_{k+1}}(\Gamma_{n_0}) \otimes_{\mathbb{Z}[\Gamma_{n_0}]} \mathbb{Z}[\Gamma_n] \to \mathcal{R}_A^{\tilde{I}_k}(\Gamma_{n_0}) \otimes_{\mathbb{Z}[\Gamma_{n_0}]} \mathbb{Z}[\Gamma_n].$$

The resulting rings are finite modules over the Noetherian rings $\mathcal{R}_A^{\tilde{I}_{k+1}}(\Gamma_{n_0})$ hence themselves Noetherian. □

Note that we change the radius of convergence while also changing the group. This stems from the fact that, using suitable charts for $o_L \cong \Gamma_n \subset \Gamma_L$, the subgroup Γ_{n+1} corresponds to the index q subgroup $\pi_L o_L$ and multiplication by π_L corresponds to φ_L via the Fourier isomorphism. This rule of thumb does not apply at the level $\Gamma_1 \subset \Gamma_L = \Gamma_0$, where $[\Gamma_L : \Gamma_1] = q - 1$. In Proposition 2.8 it will become apparent, why this convention on radii makes sense for studying the action on $\ker(\psi)$. Another caveat, we would like to point out, is the fact that contrary to the cyclotomic case the notions of r-convergent distributions (introduced in 1.45) and "I-convergent" distributions $\mathcal{R}_A^I(\Gamma_L)$ are not related in an obvious way (outside of certain special cases). See [BSX20, Section 1.3] for a precise description of the relationship.

Lemma 2.7. *Let M be a φ_L-Module over $\mathcal{R}_A$ together with a model M_0 over $\mathcal{R}_A^{r_0}$. Then for $r, s \geq r_0^{1/q^n}$ we have*

$$M^{[r,s]} \cong \bigoplus_{a \in o_L/\pi_L^n} \eta(a,T)\varphi_M^n M^{[r^{q^n}, s^{q^n}]}.$$

Proof. Because the linearised map is an isomorphism we get

$$\begin{aligned} M^{[r,s]} &\cong \mathcal{R}_A^{[r,s]} \otimes_{\mathcal{R}_A^{[r,s]},\varphi_L^n} M^{[r^{q^n},s^{q^n}]} && (2.1)\\ &\cong \Big(\bigoplus_{a \in o_L/\pi_L^n} \varphi_L^n(\mathcal{R}^{[r^q,s^q]})\eta(a,T)\Big) \otimes_{\mathcal{R}_A^{[r^{q^n},s^{q^n}]},\varphi_L^n} M^{[r^{q^n},s^{q^n}]} && (2.2)\\ &\cong \bigoplus_{a \in o_L/\pi_L^n} \eta(a,T)\varphi_M^n M^{[r^{q^n},s^{q^n}]}. && (2.3) \end{aligned}$$

□

In order to prove Theorem 2.19 we will need a several base change formulae. These allow us, roughly speaking, to change the interval $[r,s]$ to an interval $[r,s]^{q^n}$, by replacing $M^{[r,s]}$ with $\varphi^n M^{[r^{q^n},s^{q^n}]}$.

Proposition 2.8. *Let $I = [r, s]$ be an interval such that ψ_M is defined on $M^{I^{1/q^n}}$. We have*

$$(M^{I^{1/q^n}})^{\psi=0} \cong \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_n]} \eta(1,T)\varphi_M^n M^I$$

and

$$(M^{[r^{1/q^n},1)})^{\psi=0} \cong \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_n]} \eta(1,T)\varphi_M^n M^{[r,1)}.$$

Proof. From 2.7 we get a decomposition

$$M^{I^{1/q^n}} \cong \bigoplus_{a \in o_L/\pi_L^n} \eta(a,T)\varphi_M^n M^I.$$

Recall that $\frac{\pi_L}{q}\psi_{LT}(\eta(i,T)) = \eta(\frac{i}{\pi_L})$ if $i \in \pi_L o_L$ and 0 otherwise. Furthermore for $a \in o_L^\times \cong \Gamma_L$ we have $\eta(a,T) = \eta(\chi_{LT}(\chi_{LT}^{-1}(a)),T) = \chi_{LT}^{-1}(a)\eta(1,T)$. Because $\chi_{LT}^{-1}(a) \in \Gamma_L$ induces an automorphism of M^I and commutes with φ_M we get $\chi_{LT}^{-1}(a)(\eta(1,T)\varphi_M^n M^I) = \eta(a,T)\varphi^n M^I$. Combining everything we get

$$\begin{aligned}(M^{I^{1/q^n}})^{\psi=0} &\cong \bigoplus_{a \in (o_L/\pi_L^n o_L)^\times} \eta(a,T)\varphi^n M^I \\ &= \bigoplus_{a \in (o_L/\pi_L^n o_L)^\times} \chi_{LT}^{-1}(a)(\eta(1,T)\varphi^n M^I) \\ &= \bigoplus_{\gamma \in \Gamma_L/\Gamma_n} \gamma(\eta(1,T)\varphi^n M^I) \\ &\cong \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_n]} \eta(1,T)\varphi_M^n M^I.\end{aligned}$$

This proves the first formula. The second formula follows by passing to the limit $s \to 1$. ☐

These results show that we have to understand the Γ_L-action on $\eta(1,T)\varphi^n(M^I)$. Let $\gamma \in \Gamma_n$ such that $(\chi_{LT}(\gamma) - 1)$ is divisible by π_L^n. We compute

$$\gamma(\eta(1,T)\varphi^n(m)) = \eta(\chi_{LT}(\gamma),T)\varphi^n(\gamma m)) = \eta(1,T)\varphi^n\left(\eta\left(\frac{\chi_{LT}(\gamma)-1}{\pi_L^n},T\right)\gamma m\right).$$

We may thus equivalently study the action of Γ_n on M^I given by

$$H_n : \Gamma_n \to \mathcal{E}nd_A(M^I) \tag{2.4}$$

$$\gamma \mapsto [m \mapsto \left(\eta\left(\frac{\chi_{LT}(\gamma)-1}{\pi_L^n},T\right)\gamma m\right] \tag{2.5}$$

Leaning on the results of [SV20] we shall extend this action to an action of $A\hat{\otimes}_K D(\Gamma_n, K)$. Note that we deviate from their notation. We always use T for the variable of $\mathcal{R}_A$ acting via multiplication on M and use Z_n for the variable of $D(\Gamma_n, K)$ whenever $n \geq n_0$ acting on M via continuous extension of the $K[\Gamma_L]$-action.

2.3 The action via H_n.

We explain how to extend the H_n-action to $A\hat{\otimes}_K D(\Gamma_n, K)$. Since the Γ_L-action is A linear it is natural to expect that this extension arises as a base change of an extension to $D(\Gamma_n, K)$. Such an extension is constructed in [SV20, p. 50, Discussion after 2.29] for the case $A = K$. If we forget the A-action and think of modules over $\mathcal{R}_A$ by considering their underlying topological vector space we arrive at a similar situation. Strictly speaking the underlying $\mathcal{R}_K$-module of a (φ_L, Γ_L)-module over $\mathcal{R}_A$ is not a (φ_L, Γ_L)-module over $\mathcal{R}_K$ since it is not finitely generated. For this reason the results of [SV20] are not directly applicable and we sketch the construction for the convenience of the reader.

In the following I denotes one of the intervals $[r_0, r_0], [r_0, r_0^{1/q}]$. For $m \geq 0$ set $\mathfrak{r}_\mathfrak{m} := p^{\frac{-1}{p^m}}$. We fix an integer m_0 such that for any $m \geq m_0$ we have $r_0^{1/q} < \mathfrak{r}_m$ and

$$|\eta(x, T) - 1|_I < \mathfrak{r}_m. \tag{2.6}$$

This is possible due to Proposition 1.48. Furthermore Lemma 1.71 (i) allows us to choose $n_1 \geq n_0$ such that

$$||\gamma - 1||_{M^I} < \mathfrak{r}_{m_0}. \tag{2.7}$$

for any $\gamma \in \Gamma_{n_1}$ with respect to a fixed Banach-module norm on M^I.

Proposition 2.9. *The Γ_n-action on M^I via H_n extends to a continuous ring homomorphism*

$$H_n : D_{\mathfrak{r}_m}(\Gamma_n, K) \to \mathcal{E}nd_A(M^I)$$

for any $m \geq m_0$ and $n \geq n_1$.

Proof. We first construct an extension that we denote by the same symbol

$$H_n : D_{\mathfrak{r}_\mathfrak{m}}(\Gamma_n, K) \to \mathcal{E}nd_K(M^I),$$

which is induced by mapping a $\gamma - 1$ to $H_n(\gamma) - 1$. Since M is assumed to be L-analytic and the map $\gamma \mapsto \pi_L^{-n}(\eta(\chi_{LT}(\gamma) - 1), T)$ (as a function $\Gamma_n \to \mathcal{R}_K^+$) is locally L-analytic one checks that the action via H_n is L-analytic. It thus suffices to extend the action to an action of $D_{\mathbb{Q}_\mathfrak{p}, \mathfrak{r}_\mathfrak{m}}(\Gamma_n, K)$ as the latter will factor as desired

$$H_n : D_{\mathbb{Q}_\mathfrak{p}, \mathfrak{r}_\mathfrak{m}}(\Gamma_n, K) \xrightarrow{\text{can}} D_{\mathfrak{r}_\mathfrak{m}}(\Gamma_n, K) \to \mathcal{E}nd_K(M^I).$$

Let $\lambda \in D_{\mathbb{Q}_\mathfrak{p}, \mathfrak{r}_\mathfrak{m}}(\Gamma_n, K)$ and let $\mathbf{b} = (\gamma_1 - 1, \ldots, \gamma_d - 1)$, where $\gamma_1, \ldots, \gamma_d$ is a $\mathbb{Z}_p$-basis of Γ_n. Recall that λ admits a convergent expansion

$$\lambda = \sum_{\mathbf{k} \in \mathbb{N}_0^d} a_\mathbf{k} \mathbf{b}^\mathbf{k}.$$

We are reduced to showing that the operator

$$H_n(\lambda) := \sum_{\mathbf{k} \in \mathbb{N}_0^d} a_\mathbf{k} H_n(\mathbf{b}^\mathbf{k})$$

converges with respect to the operator norm on M^I. Knowing that $|a_{\mathbf{k}}|\mathfrak{r}_m^{|\mathbf{k}|}$ tends to zero it suffices due to the sub-multiplicativity of the operator norm to show $\|H_n(\mathbf{b}^{\mathbf{k}})\|_{M^I} \leq \mathfrak{r}_m$ for any $\mathbf{k}$ with $|\mathbf{k}| = 1$, in particular, it suffices to show

$$\|H_n(\gamma - 1)\|_{M^I} \leq \mathfrak{r}_m$$

for any $\gamma \in \Gamma_m$. We write out

$$H_n(\gamma - 1)m = \eta\left(\frac{\chi_{LT}(\gamma) - 1}{\pi_L^n}, T\right)(\gamma m - m) + (\eta\left(\frac{\chi_{LT}(\gamma) - 1}{\pi_L^n}, T\right) - 1)m.$$

The assumptions (2.6) and (2.7) assert that both summands are bounded above by $\mathfrak{r}_m\|m\|_{M^I}$. We conclude that the series defining $H_n(\lambda)$ converges with respect to the operator norm. Our proof also shows that $\lambda \mapsto H_n(\lambda)$ is bounded with operator norm bounded by 1, which shows that the extension we constructed is continuous. The assumption that γ acts $\mathcal{R}_A$-semi-linearly guarantees in particular that $\gamma - 1$ acts A-linearly for any $\gamma \in \Gamma_L$ using 2.3 we conclude that the image of this extension is contained in $\mathcal{E}nd_A(M^I)$. □

Corollary 2.10. *The Γ_n-action on M^I via H_n extends to a continuous ring homomorphism*

$$H_n : A\hat{\otimes}_K D_{\mathfrak{r}_m}(\Gamma_n, K) \to \mathcal{E}nd_A(M^I)$$

for any $m \geq m_0$ and $n \geq n_1$. Passing to the limit with respect to m we obtain the desired extension

$$H_n : A\hat{\otimes}_K D(\Gamma_n, K) \to \mathcal{E}nd_A(M^I).$$

Proof. For the first part apply 2.2 in conjunction with 2.9. We have $D(\Gamma_n, K) = \varprojlim_m D_{\mathfrak{r}_m}(\Gamma_n, K)$ and using 1.15 we conclude that

$$A\hat{\otimes}_K D(\Gamma_n, K) \cong \varprojlim_m A\hat{\otimes} D_{\mathfrak{r}_m}(\Gamma_n, K).$$

□

We may increase m such that $\mathfrak{r}_m > r_0^{1/q}$. In this case using 1.47 we can extend the scalar action of $\mathcal{O}_K(\mathbb{B})$ to an action of $D_{\mathbb{Q}_p,\mathfrak{r}_m}(\Gamma_n, K)$ that we call scalar action via

$$\mathfrak{S}_n : D_{\mathbb{Q}_p,\mathfrak{r}_m}(\Gamma_n, K) \xrightarrow{l_n*} D_{\mathbb{Q}_p,\mathfrak{r}_m}(o_L, K) \xrightarrow{\text{proj}} D_{\mathfrak{r}_m}(o_L, K) \xrightarrow{LT} \mathcal{O}_K(\mathbb{B}^I).$$

If we denote by Z_n a preimage of T in $D(\Gamma_n, K)$ and by X_n a lift to $D_{\mathbb{Q}_p,\mathfrak{r}_m}(\Gamma_n, K)$ then by construction $\mathfrak{S}_n(X_n)$ acts as multiplication by T on M^I. Since $0 \notin I$ we know that the action of T on M^I is invertible and the goal is to compare the action of $H_n(X_n) = H_n(Z_n)$ with $\mathfrak{S}_n(X_n) = T$. The following lemma allows us to choose a sequence of lifts whose $\mathfrak{r}_m$-norms do not depend on n.

Lemma 2.11. *Fix a lift X_{n_1} of Z_{n_1} to $D_{\mathbb{Q}_p,\mathfrak{r}_m}(\Gamma_{n_1},K)$. Then there exists a sequence $X_{n_1+l}\in D_{\mathbb{Q}_p,\mathfrak{r}_m}(\Gamma_{n_1+l},K)$ such that X_{n_1+l} is a lift of $Z_{n_1+l}\in D(\Gamma_{n_1+l},K)$ and*

$$|X_{n_1+l}|_{D_{\mathbb{Q}_p,\mathfrak{r}_m}(\Gamma_{n_1+l},K)}=|X_{n_1}|_{D_{\mathbb{Q}_p,\mathfrak{r}_m}(\Gamma_{n_1},K)}$$

for every $l\geq 0$.

Proof. The charts satisfy $l_n=\pi_L l_{n+1}$ for every $n\geq n_0$. Transporting the problem to o_L and arguing inductively it suffices to show that given $\lambda\in D_{\mathbb{Q}_p,\mathfrak{r}_m}(o_L,K)$ there exists an element $\tilde{\lambda}\in D_{\mathbb{Q}_p,\mathfrak{r}_m}(\pi_L o_L,K)$, whose projection to $D(\pi_L o_L,K)$ is equal to $\pi_{L*}(\overline{\lambda})$, such that $\tilde{\lambda}$ satisfies

$$|\tilde{\lambda}|_{D_{\mathbb{Q}_p,\mathfrak{r}_m}(\pi_L o_L,K)}=|\lambda|_{D_{\mathbb{Q}_p,\mathfrak{r}_m}(o_L,K)}.$$

We claim that $\pi_{L*}(\lambda)$ has the desired properties. Given a $\mathbb{Z}_p$-basis $b_1,\dots,b_d$ of o_L the elements $\pi_L b_1,\dots,\pi_L b_d$ form a basis $\pi_L o_L$. Since the $\mathfrak{r}_m$-norm is independent of the choice of basis we see that the isomorphism

$$D_{\mathbb{Q}_p}(o_L,K)\to D_{\mathbb{Q}_p}(\pi_L o_L,K)$$

induced by the isomorphism $o_L\cong\pi_L o_L$ given by multiplication-by-π_L is isometric with respect to the respective $\mathfrak{r}_m$-norms hence extends to an isometric isomorphism of the respective completions. □

Lemma 2.12. *There exists $n_2\geq n_1$ such that the map H_n constructed above extends to continuous ring homomorphism*

$$\mathcal{R}_A^I(\Gamma_n)\to\mathcal{E}nd_A(M^I)$$

for any $n\geq n_2$.

Proof. Lemma 2.11 allows us to fix a sequence of elements X_n lifting Z_n such that $C=|X_n|_{D_{\mathbb{Q}_p,\mathfrak{r}_m(\Gamma_n,K)}}$ is independent of $n\geq n_1$. Let $0<\varepsilon<\min(r_0/C,1)$. Having fixed such a sequence we assume that n_2 is large enough such that the following assumptions are satisfied:

A.1 $\|\gamma-1\|_{M^I}<\varepsilon r_0^{1/q}$ for every $\gamma\in\Gamma_{n_2}$.

A.2 Choose $l=l(\varepsilon)$ such that $\|[a]-\mathrm{id}\|_{\mathcal{R}_A^I}<\varepsilon$ for every $a\in 1+\pi_L^l o_L$.

A.3 $\frac{1+\pi_L^{n_2}x}{x\pi_L^{n_2}}=1-\frac{\pi_L^{n_2}x}{2}+\dots$ belongs to $1+\pi_L^l o_L$.

The first two conditions can be achieved by using 1.71 applied to M^I and $\mathcal{R}_A^I$ respectively. The third one can be achieved by making n_2 large enough after having chosen l. Let $n\geq n_2$ and fix a $\mathbb{Z}_p$-basis $\gamma_1,\dots,\gamma_d$ of Γ_n. Write

$$X_n=\sum_{\mathbf{k}\in\mathbb{N}_0^d}a_{\mathbf{k}}(\delta_{\gamma_i}-1)^{\mathbf{k}}$$

by construction we have $C = \sup_{\mathbf{k}} |a_{\mathbf{k}}| \mathfrak{r}_m^{|\mathbf{k}|}$. We claim

$$\|H_n(X_n) - \mathfrak{S}_n(X_n)\|_{M^I} < r_0 = |T^{-1}|_I^{-1} \leq \|T^{-1}\|_{M^I}^{-1}.$$

We abbreviate $\alpha(\gamma) = l_n(\gamma) = \log(\chi_{LT}(\gamma))/\pi_L^n$ and denote by β the chart $\beta(\gamma) = \frac{\chi_{LT}(\gamma)-1}{\pi_L^n}$. By construction

$$H_n(\gamma - 1) = \eta(\beta(\gamma), T)(\gamma - 1) + (\eta(\beta(\gamma), T) - 1)$$

and

$$\mathfrak{S}_n(\gamma - 1) = \eta(\alpha(\gamma), T) - 1.$$

We first show

$$\|H_n(\gamma - 1)^k - \mathfrak{S}_n(\gamma - 1)^k\|_{M^I} < \varepsilon(r_0^{1/q})^k.$$

We have

$$\|H_n(\gamma - 1)\|_{M^I} \leq \sup(\|\eta(\beta(\gamma), T)(\gamma - 1)\|_{M^I}, \|\eta(\beta(\gamma), T) - 1\|_{M^I}) < r_0^{1/q}$$

and

$$\|\eta(\alpha(\gamma), T) - 1\|_{M^I} \leq |\eta(\alpha(\gamma), T) - 1|_I < r_0^{1/q}$$

using that $|\eta(a, T) - 1|_I < r_0^{1/q}$ and $|\eta(a, T)|_I = 1$ for any $a \in o_L$, the assumption **A.1** together with $\varepsilon < 1$. This allows us to reduce the claim to the case $k = 1$ by writing for $x = \gamma - 1$

$$\begin{aligned} &H_n(x)^k - \mathfrak{S}_n(x)^k \\ &= H_n(x)(H_n(x)^{k-1} - \mathfrak{S}_n(x)^{k-1}) + \mathfrak{S}_n(x)^{k-1}(H_n(x) - \mathfrak{S}_n(x)). \end{aligned} \tag{2.8}$$

If $k = 1$ we have

$$H_n(\gamma - 1) - \mathfrak{S}_n(\gamma - 1) = \eta(\beta(\gamma), T)(\gamma - 1) + (\eta(\alpha(\gamma), T) - 1) - (\eta(\beta(\gamma), T) - 1) \tag{2.9}$$

$$= \eta(\beta(\gamma), T)(\gamma - 1) + ([u(\gamma)] - 1)(\eta(\alpha(\gamma), T - 1). \tag{2.10}$$

Where $u(\gamma) = \beta(\gamma)/\alpha(\gamma)$, which due **A.3** belongs to $1 + \pi_L^l o_L$ using that $\eta(\beta(\gamma), T) = \eta(\alpha(\gamma)u(\gamma), T) = [u(\gamma)](\eta(\alpha(\gamma), T))$ and the fact that 1 is fixed by the Γ_L-action. The assumptions **A.1** and **A.2** ensure that both terms can be estimated by $\varepsilon r_0^{1/q}$. Let $\mathbf{b} = (\gamma_1 - 1, \ldots, \gamma_d - 1)$. We next prove

$$\|H_n(\mathbf{b})^{\mathbf{k}} - \mathfrak{S}_n(\mathbf{b})^{\mathbf{k}}\| < \varepsilon(r_0^{1/q})^{|\mathbf{k}|}$$

for any multi-index $\mathbf{k} \in \mathbb{N}_0^d$ by induction on the number h of non-zero components of $\mathbf{k}$. We already treated the case $h = 1$ and may therefore split $\mathbf{k} = (k_1, 0, \ldots, 0) + (0, k_2, k_3, \ldots, k_d)$ and assume that the corresponding estimate holds for $\mathbf{i} = (k_1, 0, \ldots, 0)$ and $\mathbf{j} = (0, k_2, k_3, \ldots, k_d)$. Using the same trick as in (2.8) we rewrite

$$\begin{aligned} &H_n(\mathbf{b})^{\mathbf{k}} - \mathfrak{S}_n(\mathbf{b})^{\mathbf{k}} \\ &= H_n(\mathbf{b})^{\mathbf{j}}(H_n(\mathbf{b})^{\mathbf{i}} - \mathfrak{S}_n(\mathbf{b})^{\mathbf{i}}) + \mathfrak{S}_n(\mathbf{b})^{\mathbf{i}}(H_n(\mathbf{b})^{\mathbf{j}} - \mathfrak{S}_n(\mathbf{b})^{\mathbf{j}}) \end{aligned}$$

and use the estimates $\|H_n(\mathbf{b})^{\mathbf{l}}\| < (r_0^{1/q})^{|\mathbf{l}|}$ (resp.$\|\mathfrak{S}_n(\mathbf{b})^{\mathbf{l}}\| < (r_0^{1/q})^{|\mathbf{l}|}$) that can be obtained in the same way as in the case $h = 1$. Putting everything together we obtain the final estimate

$$\|H_n(X_n) - \mathfrak{S}_n(X_n)\| \leq \sup_{\mathbf{k}}|a_{\mathbf{k}}|\|H_n(\mathbf{b})^{\mathbf{k}} - \mathfrak{S}_n(\mathbf{b})^{\mathbf{k}}\| \tag{2.11}$$

$$< \varepsilon \sup_{\mathbf{k}}|a_{\mathbf{k}}|(r_0^{1/q})^{|\mathbf{k}|} \tag{2.12}$$

$$< \varepsilon \sup_{\mathbf{k}}|a_{\mathbf{k}}|\mathfrak{r}_m^{|\mathbf{k}|} = \varepsilon C < r_0. \tag{2.13}$$

Using 2.1 we conclude that $H_n(Z_n)$ is invertible and its inverse given by

$$H_n(Z_n)^{-1} = T^{-1}((T^{-1}H_n(Z_n) - 1) + 1)^{-1} = T^{-1}\sum_{k\geq 0}(1 - T^{-1}H_n(Z_n))^k$$

has operator norm

$$\|H(Z_n)^{-1}\|_{M^I} \leq \|T^{-1}\|_{M^I}$$

and satisfies

$$\|H_n(Z_n)^{-1} - T^{-1}\|_{M^I} < \|T^{-1}\|_{M^I},$$

which follows from the estimate (2.13), which asserts that the expression in the geometric series has operator norm less than 1. From (2.13) and the strict triangle inequality we further conclude

$$\|H_n(Z_n)\|_{M^I} = \|H_n(Z_n) - T + T\|_{M^I} \leq \|T\|_{M^I},$$

which means that given $f(T) \in \mathcal{R}_A^I$ the operator $f(H_n(Z_n))$ converges to an operator on M^I of operator norm bounded by $|f|_I$. In particular we obtained the desired continuous homomorphism

$$\mathcal{R}_A^I(\Gamma_n) \to \mathcal{E}nd_A(M^I)$$

given by mapping Z_n to $H_n(Z_n)$. □

Remark 2.13. *Let $n \geq n_2$ and let $f \in \mathcal{R}_A^I$. We have*

$$\|f(H_n(Z_n)) - f(T)\|_{M^I} < \|f(T)\|_I.$$

Proof. We show that

$$\|H_n(Z_n)^{\pm k} - T^{\pm k}\|_{M^I} < \|T^{\pm k}\|_I$$

holds for every $k \in \mathbb{N}$. The case $k = 0$ is trivial and the case $k = 1$ has been treated in the proof of 2.12. We proceed inductively by expressing

$$H_n(Z)^{\pm 1^k} - T^{\pm 1^k} = H_n(Z)^{\pm 1(k-1)}(H_n(Z)^{\pm 1} - T^{\pm 1}) + T^{\pm 1}(H_n(Z)^{\pm 1(k-1)} - T^{\pm 1(k-1)})$$

and using the estimates $\|H_n(Z_n)^{\pm j}\|_{M^I} \leq \|T^{\pm}\|_{M^I}^j \leq |T^{\pm}|_I^j = |T^{\pm j}|_I$ for $j \in \{1, k-1\}$ that were obtained implicitly in the proof of 2.12. Note that $|T^{\pm j}|_I = |T^{\pm}|_I^j$ by definition of the I-norm. □

Lemma 2.14. *M^I is finite projective with respect to the $\mathcal{R}_A^I(\Gamma_n)$-module structure induced by H_n of the same rank as M^I over $\mathcal{R}_A^I$. Any system of generators of M^I as a $\mathcal{R}_A^I$-module also generates M^I as a $\mathcal{R}_A^I(\Gamma_n)$-module (via H_n).*

Proof. Choose N^I (and a Banach norm on N^I) such that $M^I \oplus N^I = (\mathcal{R}_A^I)^d = \bigoplus_{i=1}^d \mathcal{R}_A^I e_i$ endowed with the sup-norm of the norms on M^I and N^I. We endow N^I with a tautological $\mathcal{R}_A^I(\Gamma_n)$-module structure by letting the variable $Z = Z_n \in D(\Gamma_n, K)$ act as multiplication by $T \in \mathcal{R}_A^I$. Then the estimate from Remark 2.13 remains valid for N^I, since $H_n(Z) - T$ acts as zero on N^I by construction. Fix a basis of $M^I \oplus N^I$ and define

$$\Phi : M^I \oplus N^I \to (\mathcal{R}_A^I)^d \tag{2.14}$$

$$\sum f_i(T)v_i \mapsto f_i(T)e_i \tag{2.15}$$

and

$$\Psi : (\mathcal{R}_A^I)^d \to (\mathcal{R}_A^I(\Gamma_n))^d \to M^I \oplus N^I$$
$$\sum f_i(T)e_i \mapsto \sum f_i(Z_n)e_i \mapsto \sum f_i(H_n(Z))(v_i).$$

By construction Φ is a topological isomorphism and $\Psi \circ \Phi$ is an endomorphism of $M^I \oplus N^I$ leaving both M^I and N^I invariant. We claim

$$||\Psi \circ \Phi - 1||_{M^I \oplus N^I} < 1.$$

This implies that $\Psi \circ \Phi$ is an automorphism, but then Ψ has to be an isomorphism. In particular the map $\mathcal{R}_A^I(\Gamma_n)^d \to M^I \oplus N^I$ has to be an isomorphism, which shows the projectivity and the second part of the statement. For the estimation observe that $\Psi \circ \Phi - 1$ is 0 on N^I, hence we only need to concern ourselves with M^I, where the estimate follows from 2.13. Regarding the rank we compute

$$\begin{aligned}
\operatorname{rank}_{\mathcal{R}_A^I(\Gamma_n)}(M^I) &= \operatorname{rank}_{\mathcal{R}_A^I(\Gamma_n)}(M^I \oplus N^I) - \operatorname{rank}_{\mathcal{R}_A^I(\Gamma_n)}(N^I) \\
&= \operatorname{rank}_{\mathcal{R}_A^I}(M^I \oplus N^I) - \operatorname{rank}_{\mathcal{R}_A^I}(N^I) \\
&= \operatorname{rank}_{\mathcal{R}_A^I}(M^I),
\end{aligned}$$

using additivity of ranks in the first and third equation. The second equality follows by construction since on the one hand $M^I \oplus N^I$ is isomorphic to $\mathcal{R}_A^I(\Gamma_n)^d$ and on the other hand N^I is viewed as a $\mathcal{R}_A^I(\Gamma_n)$-module via transport of structure along the isomorphism $\mathcal{R}_A^I(\Gamma_n) \cong \mathcal{R}_A^I$. The statement about the generators follows from the fact that $\Psi \circ \Phi$ respects the decomposition $M^I \oplus N^I$ in the sense that M^I (resp. N^I) is mapped into itself. Hence if M^I admits a system of generators that lifts to a basis v_i of $M^I \oplus N^I$ the statement becomes clear since we have shown that these form a basis of the $\mathcal{R}_A^I(\Gamma_n)$-module $M^I \oplus N^I$ (with action via H_n). Having chosen some system of generators $(m_1, \ldots, m_d)$ of M^I we can always find a suitable N^I such that the m_i are projections of a basis of $(\mathcal{R}_A^I)^d$ by taking the surjection $(\mathcal{R}_A^I)^d \to M^I$ given by mapping $e_i \to m_i$ and splitting it using the projectivity of M^I. □

2.4 Interpreting the results

We now explain how these results translate to the original module. We found it convenient to introduce the following abstract notation.

Definition 2.15. *Let $I \in \{[r,s],[r,1)\}$ with $r \geq r_0$ and let $n \in \mathbb{N}$. We say that M satisfies* ***property*** $\mathcal{P}(n,I)$ ***with respect to a system of generators*** $m_1,\ldots,m_d \in M^I$ *if the $A[\Gamma_n]$-action on $\eta(1,T)\varphi^n(M^I)$ extends to an action of $\mathcal{R}_A^I(\Gamma_n)$ with respect to which M^I is projective and finitely generated by the elements $\eta(1,T)\varphi^n(m_i), i = 1,\ldots,d$.*

This notion depends on the choice of system of generators. Since we assumed that M admits a model over $[r_0,1)$ we may fix a system of generators $m_1,\ldots,m_d$ of $M^{[r_0,1)}$. For any $I \subset [r_0,1)$ we take the images of m_i as a choice of system of generators for M^I. To keep notation simple in the following we refer to $\mathcal{P}(n,I)$ with respect to this choice of generating system.

Proposition 2.16. *Let $I \subset [r_0,1)$ be an interval, $l,n \in \mathbb{N}$ and let $(I_k)_k$ be an admissible covering of $[r_0,1)$. Then*

1.) $\mathcal{P}(n+l,I)$ implies $\mathcal{P}(n,I^{1/q^l})$.

2.) If M satisfies $\mathcal{P}(n,I_k)$ is for every k then M satisfies $\mathcal{P}(n,[r_0,1))$.

Proof. The first statement follows from the decomposition

$$\eta(1,T)\varphi^n(M^{I^{1/q^l}}) = \eta(1,T)\varphi^{n+l}(M^I) \otimes_{\mathbb{Z}[\Gamma_{n+l}]} \mathbb{Z}[\Gamma_n].$$

For the second statement our assumptions guarantee that each $\eta(1,T)\varphi^n M^{I_k}$ is flat and finitely generated by at most d elements. If $n \geq n_0$ such that $\mathcal{R}_A^{[r_0,1)}(\Gamma_n) \cong \mathcal{R}_A^{[r_0,1)}$ the statement follows from Lemma 1.59. In the case $n < n_0$ one can adapt Lemma 1.59 since any covering of $[r_0,1)$ provides a system of algebras defining the Fréchet-Stein structure on $\mathcal{R}_A^{[r_0,1)}(\Gamma_n)$ as explained in Remark 2.6. □

Our results so far translate as follows.

Lemma 2.17. *Let $I = [r_0,r_0]$ or $I = [r_0,r_0^{1/q}]$. Then there exists $n_1 \in \mathbb{N}$ such that for any $n \geq n_1$ the property $\mathcal{P}(n,I)$ is satisfied.*

Proof. This translates to the assertion of 2.14. □

Lemma 2.18. *There exists r_2 such that M satisfies $\mathcal{P}(1,[r,1))$ for any $r \geq r_2$.*

Proof. By Lemma 2.17 we have $\mathcal{P}(n,I)$ for any $n \geq n_1$. Applying Proposition 2.16 1.) we obtain $\mathcal{P}(n_1,I^{1/q^l})$ for any $l \geq 0$. Notice that the intervals I^{1/q^l} with I as in Lemma 2.17 cover $[r_0,1)$. Using Proposition 2.16 2.) we conclude that $\mathcal{P}(n_1,[r_0,1))$ is satisfied and applying Proposition 2.16 1.) yet again we conclude that $\mathcal{P}(1,[r_2,1))$ holds with $r_2 = r_0^{1/q^{n_1-1}}$. □

Theorem 2.19. *Let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$ admitting a model over $[r_0, 1)$, then there exists $r_1 \geq r_0$ such that for any $r \geq r_1$ the Γ_L-action on $(M^{[r,1)})^{\psi=0}$ extends to an action of $\mathcal{R}_A^{[r,1)}(\Gamma_L)$ with respect to which $(M^{[r,1)})^{\psi=0}$ is finite projective of rank $\operatorname{rank}_{\mathcal{R}_A}(M)$. If $m_1, \dots, m_d$ generate $M^{[r,1)}$ then the elements $\eta(1,T)\varphi(m_1), \dots, \eta(1,T)\varphi(m_d)$ generate $(M^{[r,1)})^{\psi=0}$ as a $\mathcal{R}_A^{[r,1)}(\Gamma_L)$-module.*

Proof. Using the decomposition $(M^{[r,1)})^{\psi=0} = \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_1]} \eta(1,T)\varphi(M^{[r^q,1)})$ this follows from Lemma 2.18 by taking $r_1 = r_2^{1/q}$. □

We have implicitly proved the following result.

Theorem 2.20. *Let M be a an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$ admitting a model over $[r_0, 1)$. Let $n \geq n_0$ then there exists r_1 such that the action of $Z_n \in D(\Gamma_n, K)$ is invertible on $(M^{[r,1)})^{\psi=0}$ for any $r \in [r_1, 1)$.*

Proof. We have seen that the action of $D(\Gamma_n, K)$ extends to an action of $\mathcal{R}_A^{[r,1)}(\Gamma_n)$ $r \geq r_1$ with a suitable r_1. Note that the variable Z_n is a unit in every $\mathcal{R}_A^J(\Gamma_n)$ for any interval $J \subset (0,1)$. □

If M is free we can further sharpen the results.

Corollary 2.21. *Let M be a free L-analytic (φ_L, Γ_L)-module with a model over $[r_0, 1)$ such that $m_1, \dots, m_d$ are a basis of $M^{[r_0,1)}$. Then there exists $r_1 \geq r_0$ such that the action of Γ_L on $(M^{[r,1)})^{\psi=0}$ extends to a $\mathcal{R}_A^{[r_1,1)}(\Gamma_L)$-action with respect to which $\eta(1,T)\varphi(m_1), \dots, \eta(1,T)\varphi(m_d)$ form a basis.*

Proof. We use the notation of the proof of Lemma 2.14. Using that M is free one can choose $N = 0$. This shows that M^I is free over $\mathcal{R}_A(\Gamma_n)$ with basis $\eta(1,T)\varphi^n(m_i)$. Tracing through the definitions we conclude that $\eta(1,T)\varphi(m_1), \dots, \eta(1,T)\varphi(m_d)$ are global sections of the projective and hence coadmissible $\mathcal{R}_A^{[r,1)}(\Gamma_L)$-module $(M^{[r,1)})^{\psi=0}$ that form a basis of $(M^J)^{\psi=0}$ for J in a suitable cover of $[r,1)$. Then the map $\mathcal{R}_A^{[r,1)}(\Gamma_L)^d \to (M^{[r,1)})^{\psi=0}$ mapping e_i to $\eta(1,T)\varphi(m_i)$ is an isomorphism of coadmissible modules, hence the claim. □

Chapter 3

Analytic cohomology via Herr complexes

In this chapter we introduce the analytic Herr complex, which serves as an analogue of the classical Herr complex with the operator $\gamma^{p^n} - 1$ replaced by Z_n and prove finiteness and base change properties similar to [KPX14]. In order to obtain the operators Z_n on M we need to choose an open subgroup of Γ_L that is isomorphic to o_L, which in general is not a direct summand in Γ_L. Before circumventing this difficulty we discuss the split case.

3.1 Analytic Herr complex for $e < p - 1$

Assume for the moment $e(L/\mathbb{Q}_p) < p - 1$[1]. Therefore we have

$$\Gamma_L \cong o_L^\times \cong \kappa_L^\times \times U_1 \cong \text{torsion} \times o_L,$$

where the isomorphism $o_L \cong U_1$ is induced by $\exp(\pi_L \cdot)$. We denote by $\Delta \subset \Gamma_L$ the torsion subgroup.

Definition 3.1. *Let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$ and let f be an A-linear continuous operator that commutes with the action of Γ_L. We define*

$$C_{f,D(\Gamma_L,A)} := [0 \to M^\Delta \xrightarrow{(f-1,Z)} M^\Delta \oplus M^\Delta \xrightarrow{Z \oplus 1-f} M^\Delta \to 0]$$

concentrated in $[0,2]$. We denote by $H^*_{f,D(\Gamma_L,A)}(M)$ the cohomology of this complex.

Remark 3.2. *The morphism*

$$\begin{array}{ccccc} M^\Delta & \longrightarrow & M^\Delta \oplus M^\Delta & \longrightarrow & M^\Delta \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle -\frac{\pi_L}{q}\psi_{LT} \oplus \mathrm{id}} & & \downarrow{\scriptstyle -\frac{\pi_L}{q}\psi_{LT}} \\ M^\Delta & \longrightarrow & M^\Delta \oplus M^\Delta & \longrightarrow & M^\Delta \end{array}$$

is a quasi-isomorphism between $C_{\varphi_L,D(\Gamma_L,A)}$ and $C_{\frac{\pi_L}{q}\psi_{LT},D(\Gamma_L,A)}$.

[1] This also forces $p \neq 2$.

Proof. The cokernel complex is 0 because $\frac{\pi_L}{q}\psi_{LT}$ is surjective since it is the left inverse of φ_L. The kernel complex is given by

$$M^{\Delta^{\psi_{LT}=0}} \xrightarrow{Z} M^{\Delta^{\psi_{LT}=0}},$$

which is quasi-isomorphic to 0 since by 2.20 the action of Z is bijective on the kernel of ψ_{LT}. □

3.2 The case of general e.

Let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$.

Definition 3.3. *Let $n \geq n_0$ such that $\chi_{LT} \circ \log$ induces an isomorphism $\Gamma_{n_0} \cong \pi_L^{n_0} o_L \cong o_L$. We define*

$$C_{f,Z_n} := [M \xrightarrow{(f-1,Z_n)} M \oplus M \xrightarrow{Z_n \oplus 1-f} M].$$

We denote the cohomology of this complex (concentrated in degrees $[0,2]$) by H^i_{f,Z_n}.

Remark 3.4. *The complexes C_{φ_L,Z_n} and $C_{\frac{\pi_L}{q}\psi_{LT},Z_n}$ are quasi-isomorphic.*

Proof. We may define the quasi-isomorphism analogously to 3.2 and invoke Theorem 2.20 to deduce that the action of Z_n is invertible on the kernel of ψ_{LT}. □

Definition 3.5. *Let $m \geq n \geq n_0$. We define the restriction $\mathrm{res}_{n,m} : C_{f,Z_n} \to C_{f,Z_m}$ as*

$$\begin{array}{ccccc} M & \longrightarrow & M \oplus M & \longrightarrow & M \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id} \oplus \mathfrak{Q}_{m-n}} & & \downarrow{\scriptstyle \mathfrak{Q}_{m-n}} \\ M & \longrightarrow & M \oplus M & \longrightarrow & M, \end{array}$$

where $\mathfrak{Q}_{m-n}$ is defined using $Z_m = \varphi_L^{m-n}(Z_n) = \mathfrak{Q}_{m-n} Z_n$.

Lemma 3.6. *Let $m \geq m' \geq n \geq n_0$. Then*

1. *We have $\mathrm{res}_{n,m} = \mathrm{res}_{m',m} \circ \mathrm{res}_{n,m'}$.*
2. *For each i we have $\mathrm{im}(\mathrm{res}_{n,m})(H^i_{f,Z_n}(M)) \subset (H^i_{f,Z_m}(M))^{Z_n=0}$.*

Proof. The first statement follows by transport of structure from the corresponding computation in $\mathcal{R}_K^+ \cong D(\Gamma_n, K)$. There we have for any pair $a, b \in \mathbb{N}$

$$\varphi_L^{a+b}(T) = \varphi_L^a(\varphi_L^b(T)) = Q_a(\varphi_L^b(T))\varphi_L^b(T).$$

For the second statement we consider each degree individually. Z_m is divisible by Z_n in the distribution algebra, which implies the statement in degree 0. In degree 1 consider a class $\overline{(a,b)} \in H^1_{f,Z_n}$ with $Z_n a = (f-1)b$. We compute

$$Z_n \,\mathrm{res}_{n,m}(a,b) = Z_n(a, \mathfrak{Q}_{m-n} b) = (Z_n a, Z_m b) = (f - 1b, Z_m b),$$

which lies in the image of the first differential of C_{f,Z_m}. In degree 2 let $b \in M$. Then $Z_n \mathfrak{Q}_{m-n} b = Z_m b = \partial(b,0) \equiv 0$. □

Corollary 3.7. *Applying the above for $m = n$ shows that H^i_{f,Z_n} has trivial Γ_n-action.*

Contrary to the classical theory we cannot define the analytic Herr complex for Γ_L directly unless $e < p-1$. Our observations so far show that the cohomology groups of the restricted Herr complex for any subgroup Γ_n isomorphic to o_L carry a residual Γ_L/Γ_n action. This allows us to define the Herr cohomology for Γ_L after choosing such a subgroup.

Definition 3.8. *Choose $n \geq n_0$. We define $H^i_{f,D(\Gamma_L,K)}(M) := (H^i_{f,Z_n}(M))^{\Gamma_L}$.*

Lemma 3.9. *$H^i_{f,D(\Gamma_L,K)}$ is independent of the choice of $n \geq n_0$.*

Proof. We are reduced to showing that $\mathrm{res}_{n,m} : (H^i_{f,Z_n}(M)) \to (H^i_{f,Z_m}(M))^{Z_n=0}$ is an isomorphism.
The case $i = 0$ follows from $H^0_{f,Z_n}(M) = M^{f=1,\Gamma_n} = (M^{f=1,\Gamma_m})^{\Gamma_n}$.
For $i = 1$ consider a pair (a, b) with $Z_n a = (f-1)b$ that is mapped to 0 in H^1_{f,Z_m}. Meaning that there exists a $v \in M$ satisfying $((f-1)v, Z_m v) = (a, \mathfrak{Q}_{m-n} b)$. We compute the image of v in H^1_{f,Z_n} and obtain $((f-1)v, Z_n v) - (a, b) = (0, Z_n v - b)$, which is an element in the kernel of the multiplication-by-$\mathfrak{Q}_{m-n}$-map. But on the cohomology Z_n acts as zero and hence $\mathfrak{Q}_{m-n}$ is invertible as it has a non-zero constant term, which proves injectivity. For surjectivity consider $(a, b) \in M^2$ satisfying $Z_m a = (f-1)b$. By the same argument as above the operator $\mathfrak{Q}_{m-n}$ is invertible on $H^1_{f,Z_m}(M)^{Z_n=0}$ and we may find a cocycle (c, d) such that $\mathfrak{Q}_{m-n}(c, d) \equiv (a, b)$ in $H^1_{f,Z_m}(M)$. Then $(\mathfrak{Q}_{m-n} c, d)$ satisfies $Z_n(\mathfrak{Q}_{\mathfrak{m}-\mathfrak{n}} c) = Z_m c = (f-1)d$ and is mapped to the class of (a, b) by the restriction map.
It remains to treat the case $i = 2$. We first prove injectivity. Let $c \in M$ such that $\mathfrak{Q}_{m-n} c$ vanishes in $H^2_{f,Z_m}(M)$ i.e. $\mathfrak{Q}_{m-n} c \in Z_m M + (f-1)(M)$. This means $\mathfrak{Q}_{m-n} m \in Z_n M + (f-1)M$ and hence vanishes in $H^2_{f,Z_n}(M)$, but then c already has to vanish in $H^2_{f,Z_n}(M)$. For surjectivity let $\overline{d} \in H^2_{f,Z_m}(M)^{Z_n=0}$. By the preceeding arguments we may find an element $\overline{c} \in H^2_{f,Z_m}(M)$ satisfying $\mathfrak{Q}_{m-n}(\overline{c}) = \overline{d}$. We can lift $\overline{c}$ to an element of M and take its projection to $H^2_{f,Z_n}(M)$ in order to find a preimage of d in $H^2_{f,Z_n}(M)$. □

Remark 3.10. *Let $n \geq n_0$. The action of Γ_L on M induces a natural action on $C_{f,Z_n}(M)$, given by letting $\gamma \in \Gamma_L$ act in the usual way on each component in each degree. If $\gamma \in \Gamma_n$ then this action is homotopic to the identity. In other words, the image of $C_{f,Z_n}(M)$ in the derived category $\mathbf{D}(A)$ carries an action of Γ_L/Γ_n.*

Proof. The action is well-defined because f commutes with $\gamma \in \Gamma_L$. If $\gamma \in \Gamma_n$ then the action of $\gamma - \mathrm{id}$ is given by $\eta(a, Z_n) - 1 = Z_n H(Z_n)$ with some $a \in o_L$ and $H(Z_n) \in o_L[\![Z_n]\!]$. A small calculation shows that the maps

$$\begin{aligned} M^2 &\to M \\ (m, m') &\mapsto H(Z_n)m' \end{aligned}$$

and

$$\begin{aligned} M &\to M^2 \\ m &\mapsto (0, H(Z_n)m) \end{aligned}$$

define a homotopy between γ and id. □

3.3 Finiteness of analytic Herr cohomology

The results of the previous section allow us to flexibly change between the open subgroups Γ_n used to define the analytic Herr complex and to simplify the notation we fix some $n \geq n_0$ and write $Z := Z_n$. The goal of this section is to prove that for any analytic (φ_L, Γ_L)-module M over $\mathcal{R}_A$ the cohomology groups $H^i_{\varphi_L, Z}(M)$ are finitely generated over A. We follow the strategy of [Bel21] using the result from the previous chapter regarding the Z action on the kernel of ψ to arrive at a situation that allows us to apply results from [KL16]. This approach differs from [KPX14] who first prove the finiteness of the Iwasawa cohomology of M and compare it to the (φ, Γ)-cohomology of the cyclotomic deformation of M. The finiteness of the cohomology of M is obtained in [KPX14] as a corollary by writing M as a base change of its deformation. Similar arguments to the ones of Bellovin already appear in [KP18]. We denote by $\mathbf{D}^b_{perf}(A), \mathbf{D}^-_{perf}(A), \mathbf{D}^{[a,b]}_{perf}(A)$ the full subcategory of the derived category consisting of objects which are quasi-isomorphic to bounded (resp. bounded above, resp. concentrated in degree $[a, b]$) complexes of finite projective A-modules.

Definition 3.11. *Let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$ with model over $[r_0, 1)$. For any $1 > r \geq r_0$ we define*

$$C_{\varphi_L, Z, [r,1)}(M) : M^{[r,1)} \xrightarrow{(\varphi_L - 1, Z)} M^{[r^{1/q},1)} \oplus M^{[r,1)} \xrightarrow{Z \oplus (1-\varphi_L)} M^{[r^{1/q},1)}.$$

For $r < s < 1$ we define the complex

$$C_{\varphi_L, Z, [r,s]}(M) : M^{[r,s]} \xrightarrow{(\varphi_L - 1, Z)} M^{[r^{1/q},s]} \oplus M^{[r,s]} \xrightarrow{Z \oplus (1-\varphi_L)} M^{[r^{1/q},s]}$$

for any $s \in [r^{1/q}, 1)$. We obtain canonical morphisms of complexes (of A-modules)

$$C_{\varphi_L, Z, [r,1)}(M) \to C_{\varphi_L, Z}(M)$$

and

$$C_{\varphi_L, Z, [r,1)}(M) \to C_{\varphi_L, Z, [r,s]}(M).$$

We say that the cohomology of M is computed on the level of $C_{\varphi_L, Z, [r,1)}(M)$ (resp. $C_{\varphi_L, Z, [r,s]}(M)$) if the first (resp. both) maps are quasi-isomorphisms.

When working with φ_L-modules over $\mathcal{R}_A$ we can identify the cohomology of the complex $[M \xrightarrow{\varphi_L - 1} M]$ with the Yoneda extension groups in the category of modules over the twisted polynomial ring $\mathcal{R}_A[X;\varphi_L]$ with X acting as φ_L on $\mathcal{R}_A$ (cf. [KL13, Definition 1.5.4]). The analogous result holds for φ_L-modules over $[r,1)$ (resp. $[r,s]$) but requires some care, since $\varphi_L : M^{[r,1)} \to M^{[r^{1/q},1)}$ changes the ring over which the module is defined.

Definition 3.12. *Let M be a φ_L-module over $\mathcal{R}_A^I$, where $I=[r,1)$ or $I=[r,s]$ with $s\in[r^{1/q},1)$. We denote by $\operatorname{Ext}^1_\varphi(\mathcal{R}_A^I, M)$ the group of extensions (as φ_L-modules) of $\mathcal{R}_A^I$ by M.*

Remark 3.13. *Let M be a φ_L-module over $\mathcal{R}_A^I$ with I as in 3.12. The natural map*

$$\begin{aligned}\ker(\varphi_M - 1) &\to \operatorname{Hom}_\varphi(\mathcal{R}_A^I, M^I)\\ x &\mapsto (f\mapsto fx)\end{aligned}$$

is an isomorphism.

Proof. The map is well-defined because $\varphi_M(fx)=\varphi_L(f)\varphi_M(x)=\varphi_L(f)x$ and an inverse is given by mapping $\alpha\in\operatorname{Hom}_\varphi(\mathcal{R}_A^I, M^I)$ to $\alpha(1)$. □

Lemma 3.14. *Let M be a φ_L-module over $\mathcal{R}_A^I$, with I as in 3.12 and let $J := I^{1/q}\cap I$. The map that assigns to $E\in\operatorname{Ext}^1_\varphi(\mathcal{R}_A^I, M)$ the element $\varphi_E(e)-e\in M^J$, where e is any preimage of $1\in\mathcal{R}_A^I$, induces an isomorphism*

$$\operatorname{Ext}^1_\varphi(\mathcal{R}_A^I, M)\cong M^J/(\varphi_M-1)(M^I).$$

Proof. One checks that $\varphi_E(e)-e$ is mapped to zero in $\mathcal{R}_A^J$ and hence $\varphi_E(e)-e$ indeed belongs to M^J. Let $\tilde{e}$ be another preimage of $1\in\mathcal{R}_A^I$, then $e-\tilde{e}$ lies in M^I and therefore $\varphi_E(e)-e=\varphi_E(\tilde{e})-\tilde{e}+\varphi_M(e-\tilde{e})-(e-\tilde{e})$. This proves that the map in question is well-defined. Let $v\in M^J/(\varphi_M-1)(M^I)$ and define E_v to have $M^I\times\mathcal{R}_A^I$ as its underlying $\mathcal{R}_A^I$-module with $\varphi_{E_v}(m,r):=(\varphi_M(m)+\varphi_L(r)v,\varphi_L(r))$. The module E_v is finitely generated over $\mathcal{R}_A^I$ and φ_{E_v} is φ_L-semi-linear. The linearised map is an isomorphism by the five lemma and hence $E_v\in\operatorname{Ext}^1_\varphi(\mathcal{R}_A^I, M)$. The element $e_1:=(0,1)$ is a preimage of 1 and satisfies $\varphi(e_1)-e_1=(v,0)$, which proves the surjectivity. It remains to show injectivity. A computation shows that E_v is the trivial extension if $v\equiv 0$ and hence it suffices to show that any extension E is isomorphic to E_v with $v=\varphi_E(e)-e$. Since $\mathcal{R}_A^I$ is free and hence projective any extension of $\mathcal{R}_A^I$ by M^I is split as a $\mathcal{R}_A^I$-module. Let $s:\mathcal{R}_A^I\to E$ be any $\mathcal{R}_A^I$-linear section. And write $x\in E$ as $x=m+s(f)=m+fs(1)$. Then $\varphi_E(x)=\varphi_M(m)+\varphi_L(f)\varphi_E(s(1))=\varphi_M(m)+\varphi_L(f)(\varphi_E(s(1))-s(1))+\varphi_L(f)s(1)$. In particular, E is isomorphic to E_v with $v=\varphi_E(s(1))-s(1)$. □

Lemma 3.15. *Let M be a projective φ_L-module over $\mathcal{R}_A$ with model over $[r,1)$ and let $s\in[r^{1/q},1)$. Then the canonical morphism*

$$[M^{[r,1)}\xrightarrow{\varphi_L-1}M^{[r^{1/q},1)}]\to[M^{[r,s]}\xrightarrow{\varphi_L-1}M^{[r^{1/q},s]}]$$

induced by the restrictions $M^{[r,1)}\to M^{[r,s]}$ (resp. $M^{[r^{1/q},1)}\to M^{[r^{1/q},s]}$) is a quasi-isomorphism.

Proof. The statement follows from 1.63 together with 3.13 and 3.14. □

Lemma 3.16. *Let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$ with model over $[r_0, 1)$. Then there exists $1 > r_1 \geq r_0$ such that for any $1 > r \geq r_1$ the cohomology $H^i_{\varphi_L,Z}(M)$ is computed by the complex*

$$C_{\varphi_L,Z,[r,s]} : M^{[r,s]} \xrightarrow{(\varphi_L-1,Z)} M^{[r^{1/q},s]} \oplus M^{[r,s]} \xrightarrow{Z\oplus(1-\varphi_L)} M^{[r^{1/q},s]}$$

for any $s \in [r^{1/q}, 1)$.

Proof. The complex computing the cohomology of M is the direct limit of the corresponding complexes for $M^{[r,1)}$ as $r \to 1$. By a cofinality argument we may take the colimit over ${r_0}^{1/q^n}$ with $n \to \infty$. We hence need to show that the restriction maps (labeled as id below)

$$\begin{array}{cccccc} C_{\varphi,Z,[r,1)}: & M^{[r,1)} & \longrightarrow & M^{[r^{1/q},1)} \oplus M^{[r,1)} & \longrightarrow & M^{[r^{1/q},1)} \\ \downarrow \mathrm{id} & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\ C_{\varphi,Z,[r^{1/q},1)}: & M^{[r^{1/q},1)} & \longrightarrow & M^{[r^{1/q^2},1)} \oplus M^{[r^{1/q},1)} & \longrightarrow & M^{[r^{1/q^2},1)} \end{array}$$

induce quasi-isomorphisms. Following [Bel21] we do so in two steps. We show that the restriction above is homotopic to the map

$$\begin{array}{cccccc} C_{\varphi,Z,[r,1)}: & M^{[r,1)} & \longrightarrow & M^{[r^{1/q},1)} \oplus M^{[r,1)} & \longrightarrow & M^{[r^{1/q},1)} \\ \downarrow \varphi_L & \downarrow \varphi_L & & \downarrow \varphi_L & & \downarrow \varphi_L \\ C_{\varphi,Z,[r^{1/q},1)}: & M^{[r^{1/q},1)} & \longrightarrow & M^{[r^{1/q^2},1)} \oplus M^{[r^{1/q},1)} & \longrightarrow & M^{[r^{1/q^2},1)} \end{array}$$

and the latter induces the desired quasi-isomorphism. One can check that the maps

$$\mathrm{pr}_1 \colon M^{[r^{1/q},1)} \oplus M^{[r,1)} \to M^{[r^{1/q}]}$$

and

$$(0, -\mathrm{id}) \colon M^{[r^{1/q},1)} \to M^{[r^{1/q},1)} \oplus M^{[r,1)}$$

induce a homotopy between φ_L and id. In order to see that φ_L induces a quasi-isomorphism consider the left-inverse $\Psi : C_{\varphi_L,Z,[r^{1/q},1))} \to C_{\varphi_L,Z,[r,1)}$ obtained by applying $\frac{\pi_L}{q}\psi_{LT}$ in each degree of the complex. By 2.20 there exists r_1 such that for any $r \geq r_1$ the action of Z on the kernel of Ψ is invertible with continuous inverse (note that the constant $\frac{\pi_L}{q}$ does not change the kernel). In particular we obtain a decomposition $C_{\varphi_L,Z,[r^{1/q},1)} = \varphi_L(C_{\varphi_L,Z,[r,1)}) \oplus \ker(\Psi)$ as complexes of A-modules. We claim that the (Z, φ_L)-cohomology of the second summand vanishes, which implies the desired result. The vanishing of H^0 and H^2 is an immediate consequence of Z being invertible on the kernel of ψ_{LT}. We prove that $H^1_{\varphi_L,Z}((M^{[r,1)})^{\psi_{LT}=0})$ vanishes.

Consider (a,b) that satisfy $Za+(1-\varphi_L)b=0$ then $a=Z^{-1}(\varphi_L-1)b=(\varphi_L-1)Z^{-1}b$ and obviously $b=ZZ^{-1}b$. Setting $x:=Z^{-1}b$ we see that $(a,b)=((\varphi_L-1)x,Zx)$ vanishes in H^1. So far we proved that the cohomology is computed by the complex $C_{\varphi_L,Z,[r,1)}$ $r\geq r_1$. Consider the canonical morphism $C_{\varphi_L,Z,[r,1)}\to C_{\varphi_L,Z,[r,s]}$. Up to signs these complexes are the total complexes of the double complexes

$$\begin{array}{ccc} M^{[r,1)} & \xrightarrow{\varphi_L-1} & M^{[r^{1/q},1)} \\ \downarrow{\scriptstyle Z} & & \downarrow{\scriptstyle -Z} \\ M^{[r,1)} & \xrightarrow{\varphi_L-1} & M^{[r^{1/q},1)} \end{array}$$

(resp. for $M^{[r,s]}$)). By the Acyclic Assembly Lemma A.15 together with 3.15 we conclude that these total complexes are quasi-isomorphic. □

Definition 3.17. *Let R be a topological ring complete with respect to a submultiplicative semi-norm containing a topologically nilpotent unit. A map between two R-Banach modules $f\colon M\to N$ is called **completely continuous** if there exists a sequence of finitely generated R-submodules $N_i\subset N$ such that operator norms of $f_i\colon M\xrightarrow{f} N\to N/N_i$ converge to zero with respect to fixed Banach norms on M,N and the quotient seminorm on N/N_i.*

Lemma 3.18. *Let $R\to S$ be a bounded morphism of Banach-algebras over A, that is completely continuous when viewing R,S as Banach modules over A. Let M be a finitely generated R-Banach module such that $S\otimes_R M$ is an S-Banach module, then*

$$M\to M\otimes_R S$$

is completely continuous as a morphism of A-Banach modules.

Proof. See [KL16, Remark 1.7]. □

Lemma 3.19. *Let $[r',s']\subset[r,s]\subset(0,1)$ with $r'>r$, $s'<s$ and $r,r',s,s'\in|K^\times|$. The natural inclusions*

$$\mathcal{R}_A^{[0,s]}\to\mathcal{R}_A^{[0,s']}$$

and

$$\mathcal{R}_A^{[r,s]}\to\mathcal{R}_A^{[r',s']}$$

are completely continuous. [2]

Proof. The first case is proved analogously to the second case with a slightly simpler proof. By assumption there exist elements $\rho,\sigma,\rho',\sigma'\in K$ attaining the absolute values r,s,r',s' respectively. Write $\mathcal{R}_A^{[r,s]}=A\langle T/\sigma,\rho/T\rangle$ and take as $N_i\subset\mathcal{R}_A^{[r',s']}$ the subspace generated by the monomials $T^{-i},\dots,1,T,\dots T^i$. By expressing a series

[2] The assumption $r,s,r',s'\in|K^\times|$ can be weakened to $r,s,r',s'\in\sqrt{|K^\times|}$ without any difficulties. In our application K carries a non-discrete valuation and cofinality arguments allow us to choose suitable intervals.

$f \in \mathcal{R}_A^{[r,s]}$ as an element of $A\langle T/\sigma', \rho'/T\rangle$ and projecting modulo N_i we obtain the estimate

$$|f|_{A\langle T/\sigma',\rho'/T\rangle/N_i} \leq \max\left(\frac{r}{r'}, \frac{s'}{s}\right)^{i+1} |f|_{A\langle T/\sigma,\rho/T\rangle}.$$

By our assumption on the intervals $\max\left(\frac{r}{r'}, \frac{s'}{s}\right) < 1$ and therefore the operator norms of the composed maps $\mathcal{R}_A^{[r,s]} \to \mathcal{R}_A^{[r',s']}/N_i$ tend to zero. □

Lemma 3.20. *Let $M^{[r,1)}$ be a (φ_L, Γ_L)-module over $\mathcal{R}_A^{[r,1)}$ and let $0 < r < r' \leq s' < s < 1$. Then the restriction $M^{[r,s]} \to M^{[r',s']}$ is completely continuous.*

Proof. By definition $M^{[r,s]}$ is a finitely generated module over the Banach algebra $\mathcal{R}_A^{[r,s]}$ and $M^{[r',s']} = \mathcal{R}_A^{[r',s']} \otimes_{\mathcal{R}_A^{[r,s]}} M^{[r,s]}$. The result follows from 3.18 because the natural inclusion $\mathcal{R}_A^{[r,s]} \to \mathcal{R}_A^{[r',s']}$ is completely continuous. □

Theorem 3.21. *Let A be K-affinoid and let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$. Then the cohomology groups*

$$H^i_{\varphi_L,Z}(M)$$

are finitely generated over A.

Proof. By Lemma 3.16 the cohomology can be computed on the level of $M^{[r,s]}$ for $r \geq r_1$ and any $s \in [r^{1/q}, 1)$. Choose any subinterval $[r', s'] \subset [r, s]$ like in Lemma 3.20 satisfying in addition $s \in [r'^{1/q}, 1)$. The restriction $M^{[r,s]} \to M^{[r',s']}$ induces completely continuous maps in each degree of $C_{\varphi_L,Z,[r,s]} \to C_{\varphi_L,Z,[r',s']}$ that are quasi-isomorphisms by Lemma 3.16. By Lemma 1.10 in [KL16] the cohomology groups are contained in a finitely generated A-module and hence themselves finitely generated because A is Noetherian.

□

Theorem 3.22. *Let A, B be K-affinoid and let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$. Let $f\colon A \to B$ be a morphism of K-affinoid algebras. Then:*

(1) $C_{\varphi_L,Z}(M) \in \mathbf{D}^{[0,2]}_{perf}(A)$.

(2) The natural morphism $C_{\varphi_L,Z}(M) \otimes^{\mathbb{L}}_A B \to C_{\varphi_L,Z}(M \hat{\otimes}_A B)$ is a quasi-isomorphism.

Proof. By 3.21 the cohomology groups of $C_{\varphi_L,Z}(M)$ are finitely generated and because A is Noetherian quasi-isomorphic to a bounded above complex of finitely generated projective A-modules by [Sta21, Tag 05T7] using that the category of finitely generated modules over a Noetherian ring is abelian. By A.5 $\mathcal{R}_A$ is flat over A and hence $C_{\varphi_L,Z}(M)$ consists of flat A-modules. Combining both points we see that $C_{\varphi_L,Z}(M)$ is quasi-isomorphic to a complex

$$X \to P_1 \to P_2,$$

where P_1, P_2 are projective and $X = \ker(P_1 \to P_2)$ is flat using A.16. Then X is a finitely generated submodule of a finitely generated module over a Noetherian ring and hence even finitely presented. We conclude that X is finitely generated and projective by [Sta21, Tag 00NX] and therefore $C_{\varphi_L,Z}(M) \in \mathbf{D}^{[0,2]}_{\mathrm{perf}}(A)$. For the second statement the proof of [KPX14] carries over verbatim using 1.56. □

Remark 3.23. *More precisely 3.22, 3.21 hold for $C_{\varphi_L,Z,I}(M)$ for any $I = [r,s]$ or $I = [r,1)$ with $r \geq r_1$ and $s \in [r^{1/q}, 1)$.*

Proof. The analogues of 3.21 and 3.22 (1) were proved implicitly. For 3.22 (2) the same proof works when we replace M by M^I the only subtlety being that in order to apply [KPX14, 4.1.5] we need $C_{\varphi_L,Z,I}(M\hat{\otimes}_A B) \in \mathbf{D}^-_{perf}(B)$, which we deduce by using 3.21 requiring $r \geq r_1$, with r_1 depending on M! One can check that the same r_1 works for $M\hat{\otimes}_A B$ using that the (φ_L, Γ_L)-action on B is trivial. □

For the full analytic Herr cohomology we obtain a variant of 3.22. Because the cohomology is defined by taking the invariants of the (φ_L, Z)-cohomology we cannot formulate similar perfectness results (outside of the case $e < p-1$) and we only obtain a base change result in the flat case.

Remark 3.24. *Let $R \to S$ be a flat morphism of commutative rings, G be a finite group and W an $R[G]$-module. Then $(S \otimes_R W)^G = S \otimes_R W^G$*

Proof. We can rewrite W^G as $W^G = \ker(W \xrightarrow{\bigoplus(g-1)} \bigoplus_{g\in G} W)$ and apply the exact functor $S \otimes_R -$. □

Corollary 3.25. *Let A, B be K-affinoid and let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$. Let $f\colon A \to B$ be a flat morphism of K-affinoid algebras. Then:*

(1) The groups $H^i_{\varphi_L,D(\Gamma_L,A)}(M) = H^i_{\varphi_L,Z}(M)^{\Gamma_L}$ are finitely generated and vanish for $i \neq 0,1,2$.

(2) The natural morphism $H^i_{\varphi_L,D(\Gamma_L,A)}(M) \otimes_A B \to H^i_{\varphi_L,D(\Gamma_L,A)}(M\hat{\otimes}_A B)$ is an isomorphism.

Proof. The first statement follows from 3.21 because A is Noetherian. For the second statement we use 3.22 to conclude $H^i_{\varphi_L,Z}(M) \otimes_A B = H^i_{\varphi_L,Z}(M\hat{\otimes} B)$ and taking Γ_L-invariants we obtain

$$H^i_{\varphi_L,D(\Gamma_L,A)}(M\hat{\otimes}_A B) = (B \otimes_A H^i_{\varphi_L,Z}(M))^{\Gamma_L},$$

using 3.24 and the fact that the Γ_L action factors over Γ_n we have

$$(B \otimes_A H^i_{\varphi_L,Z}(M))^{\Gamma_L} = B \otimes_A H^i_{\varphi_L,Z}(M)^{\Gamma_L} = B \otimes_A H^i_{\varphi_L,D(\Gamma_L,A)}(M).$$

□

Corollary 3.26. *Let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$. Assigning to an affinoid subdomain $Sp(A') \subset Sp(A)$ the cohomology groups $H^i_{\varphi_L,Z}(M\hat{\otimes}_A A')$ (resp. $H^i_{\varphi_L,D(\Gamma_L,A)}(M\hat{\otimes}_A A')$) defines a coherent sheaf on $Sp(A)$.*

Proof. By [Bos14, 4/Corollary 5 p. 68] the map $A' \to A$ is flat. Theorem 3.21 and the base change property 3.22(2) assert that the sheaf is associated to the finitely generated module $H^i_{\varphi_L,Z}(M)$ and hence coherent. The second case is treated in the same way with the base change formula from Corollary 3.25. □

3.4 ψ-cohomology of $M/t_{LT}M$.

The following section is based on Sections 3.2 and 3.3 in [KPX14]. We adapt their methods to our situation. The main results are Lemmas 3.29 and 3.30. The proofs are adaptations of their counterparts Propositions 3.2.4 and 3.2.5 in [KPX14]. They turn out to be more complicated due to the implicit nature of "the" variable Z and the fact that by extending scalars to K some care is required when studying the quotients $\mathcal{R}_K^+/\varphi^n(T)$ since K could contain non-trivial π_L^n-torsion points while having the trivial Γ_L-action. In the appendix we elaborate on crucial parts of the argument whose details are left out in loc. cit.. Let $Q_1(T) = \varphi_L(T)/T$ and $Q_n(T) := \varphi^{n-1}(Q_{n-1}(T))$ for $n \geq 1$. We will show in $A.11$ that $\mathcal{R}_L^{[r,s]}/Q_n$ can be identified with the field L_n whenever the zeroes of Q_n (i.e. the π_L^n-torsion points of the LT group that are not already π_L^{n-1}-torsion) lie in the annulus $[r,s]$. This is Γ_L-equivariant for the Galois action on L_n and $\varphi : \mathcal{R}_L^{[r,s]}/Q_n \to \mathcal{R}_L^{[r^q,s^q]}/Q_{n+1}$ corresponds to the inclusion $L_n \hookrightarrow L_{n+1}$. When we extend coefficients to K we let Γ_L act trivially on the coefficients on $\mathcal{R}_K$. One has to be careful since $\mathcal{R}_K^{[r,s]}/Q_n$ is not necessarily a field extension of K. It is in general only some finite étale K-algebra, which we denote by E_n. It carries a Γ_L-action induced from the action on $\mathcal{R}_K$. We can write $Q_n = G_nU_n$ with some polynomial G_n, that is necessarily irreducible[3] over L with splitting field L_n and a unit $U_n \in (\mathcal{R}_K^{[r,s]})^\times$. E_n can now be explicitly described as $E_n = \mathcal{R}_K^{[r,s]}/Q_n \cong K[T]/G_n \cong K \otimes_L L[T]/G_n \cong K \otimes_L L_n$. Here $\mathcal{R}_K^{[r,s]}$ carries its usual Γ_L-action while Γ_L acts on the right-hand side via the right factor. We also need similar elements on the level of $D(\Gamma, K)$. For $n \geq n_0$ we define

$$\mathfrak{L}_n := Z_n \prod_m \pi_L^{-1} Q_m(Z_n) = \log_{LT}(Z_n).$$

Corollary 1.36 asserts that the ideal generated by $\varphi_L(Z_n)$ in $D(\Gamma_n, K)$ does not depend on the choice of variable and hence neither does the ideal generated by $\varphi_L^k(Z_n) = Z_n \prod_{m=1}^k Q_m(Z_n)$ nor the ideal generated by $\mathfrak{L}_n$.

Lemma 3.27. *Let M^{r_0} be a (φ, Γ)-module over $\mathcal{R}_A^{r_0}$. Let $n_1 = n_1(r_0) \in \mathbb{N}_0$ be minimal among $n \in \mathbb{N}$ such that $|\pi_L|^{\frac{1}{q^{n-1}(q-1)}} \geq r_0$. then:*

- *$M^{r_0}/t_{LT}M^{r_0} \cong \prod_{n \geq n_1} M^{r_0}/Q_nM^{r_0}$.*
- *φ^m induces an $A[\Gamma]$-linear isomoprhism $\varphi^m \otimes 1 : M^{r_0}/Q_n \otimes_{E_n} E_{n+m} \to M^{r_0}/Q_{n+m}$*

Proof. The zeros of Q_n are precisely the π_L^n-torsion points of the LT-group, which are not already π_L^{n-1}-torsion. Hence Q_n is a unit in $\mathcal{R}_K^{[r,s]}$ if and only if

$$v(n) := |\pi_L|^{\frac{1}{q^{n-1}(q-1)}} \notin [r, s].$$

[3] Since it is the minimal polynomial of the π_L^n-torsion points of the Lubin-Tate group.

Let $w(s)$ be the largest integer n satisfying $v(n) \leq s$. For a closed interval we obtain from the Chinese remainder theorem

$$\mathcal{R}_K^{[r_0,s]}/t_{LT}\mathcal{R}_K^{[r_0,s]} = \bigoplus_{n_1 \leq n \leq w(s)} \mathcal{R}_K^{[r_0,s]}/Q_n\mathcal{R}_K^{[r_0,s]}$$

and

$$\mathcal{R}_A^{[r_0,s]}/t_{LT}\mathcal{R}_A^{[r_0,s]} = \bigoplus_{n_1 \leq n \leq w(s)} \mathcal{R}_A^{[r_0,s]}/Q_n\mathcal{R}_A^{[r_0,s]}.$$

The first statement follows by passing to the limit $s \to 1$. The second statement follows inductively from the case $m = 1$. Because the linearised map is an isomorphism we have

$$\begin{aligned} M^{r_0}/Q_{n+1} &= M^{r_0^{1/q}}/Q_{n+1} && (3.1)\\ &\cong \varphi_L^*(M^{r_0})/\varphi_L(Q_n) && (3.2)\\ &\cong M^{r_0}/Q_n \otimes_{E_n} E_{n+1}, && (3.3) \end{aligned}$$

where for the last isomorphism we use $\mathcal{R}_K^{r_0^{1/q}}/Q_{n+1} = E_{n+1}$ since the zeros of Q_{n+1} are the preimages of the zeros of Q_n under φ and we assumed that the latter are contained in $[r_0, 1)$. We further used that φ induces the canonical inclusion $E_n \to E_{n+1}$ and that the identification $\mathcal{R}_K^{[r,s]}/Q_n \cong E_n$ is Γ-equivariant as described in the beginning of the chapter. □

To keep notation light we define $M_n := M_n^r := M^r/Q_nM^r$ and suppress the dependence on r. This poses no problem as long as M admits a model over $[r, 1)$ and $n \geq n_0(r)$ satisfying the conditions of 3.27.

Corollary 3.28. *With respect to the decomposition*

$$M^{r_0}/t_{LT}M^{r_0} \cong \prod_{n \geq n_1} M^{r_0}/Q_nM^{r_0} = \prod_{n \geq n_1} M_n$$

the map $\varphi_L : M^{r_0}/t_{LT}M^{r_0} \to M^{r_0^{1/q}}/t_{LT}M^{r_0^{1/q}}$ *takes* $(x_n)_n$ *to* $(x_{n-1})_n$. *The map* $\psi_{LT} : M^{r_0^{1/q}}/t_{LT}M^{r_0^{1/q}} \to M^{r_0}/t_{LT}M^{r_0}$ *takes* $(x_n)_n$ *to* $(\pi^{-1}\operatorname{tr}_{E_n/E_{n-1}}(x_n))_n$. *Where*

$$\operatorname{tr}_{E_n/E_{n-1}} : M_n^{r_0} = M_{n-1}^{r_0} \otimes_{E_{n-1}} E_n \to M_{n-1}^{r_0}$$

is given by the trace induced from the trace $L_n \to L_{n-1}$ *on the second factor of* $E_n \cong K \otimes_L L_n$.

Proof. The statement for φ_L follows by combining both points in 3.27. The statement for ψ_{LT} follows from A.11. □

Lemma 3.29. *Let* M *be an* L*-analytic* (φ_L, Γ_L)*-module over* $\mathcal{R}_A$. *Then the cohomology groups* $H^i_{\varphi,Z}(M/tM)$ *vanish outisde of degrees* 0 *and* 1 *and are finitely generated* A*-modules.*

Proof. Assume M is defined over $[r_0, 1)$ and let $n_1 \in \mathbb{N}$ such that $M^{r_0}/tM^{r_0} \cong \prod_{n \geq n_1} M_n$. We claim that $\varphi - 1$ is surjective which implies the vanishing of H^2. Let $(x_n) \in \prod_{n \geq n_1+1} M_n$. Set $y_n := -x_n + x_{n-1} - x_{n-2} + \cdots + (-1)^{n-n_1} x_{n_1+1}$. Then $(\varphi_L - 1)(y_{n-1}) = x_n$. On the other hand $\ker((\varphi - 1)_{[r,1)}) = (M^r/tM^r)^{\varphi=1} \cong M_{n_1(r)}$ and thus

$$\ker(\varphi - 1) = \varinjlim_{r>0} (M^r/tM^r)^{\varphi=1} \cong \varinjlim_n M_n.$$

Next consider the complex

$$(\varinjlim_n M_n) \xrightarrow{Z} (\varinjlim_n M_n).$$

Using 3.27 we can explicitly describe each module appearing in the direct limit as $M_{n_1} \otimes_{L_{n_1}} L_m$, where $W := M_{n_1}$ is an A-module of finite type with a continuous L-analytic A-linear Γ_L-action and Γ_L acts on the right factor via its natural action. We claim that for $m \gg 0$ the natural map

$$[M_m \xrightarrow{Z} M_m] \to [(\varinjlim_n M_n) \xrightarrow{Z} (\varinjlim_n M_n)]$$

is a quasi-isomorphism. By the normal basis theorem and Maschke's theorem there is for any pair $m \geq m'$ an isomorphism of representations $L_m \cong L_{m'}[\Gamma_{m'}/\Gamma_m] \cong \prod_\eta L_{m'}(\eta)$, where the product runs over all characters of $\mathrm{Gal}(L_m/L_{m'})$. Hence for $n \geq m$ as representations

$$M_{n_1} \otimes_{L_{n_1}} L_n \cong \bigoplus_\rho M_{n_1}(\rho) = \bigoplus_{\rho(\Gamma_m)=1} M_{n_1}(\rho) \oplus \bigoplus_{\rho(\Gamma_m)\neq 1} M_{n_1}(\rho),$$

where ρ runs through the characters of Γ_{n_1}/Γ_n.
It suffices to show that there exists a m such that for any ρ with $\rho(\Gamma_m) \neq 1$ the action of Z is invertible meaning that the only contribution to the cohomology comes from the components corresponding to characters vanishing on Γ_m hence the claim. It suffices to show that the action of Z_m is invertible for some $m \gg 0$ hence we may assume that M_{n_1} satisfies the estimates of A.14 with respect to the action of Γ_m. But then $A.14$ asserts that the action of Z_m on $M_{n_1}(\rho)$ is invertible for any ρ that is not trivial on Γ_m. Since M_m is finitely generated over A we conclude that the complex computing $H^i_{\varphi,Z}(M/t_{LT}M)$ is quasi-isomorphic to a complex of finitely generated A-modules and because A is Noetherian we conclude that the cohomology groups are finitely generated.

□

Lemma 3.30. *Let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$. Then*

$$\frac{\pi_L}{q}\psi_{LT} - 1 : M/t_{LT}M \to M/t_{LT}M$$

is surjective and its kernel viewed as a $D(\Gamma_n, A)$-module admits a 2-term finite projective resolution for any $n \geq n_0$.

Proof. Let M^{r_0} be a model of M with $r_0 > \left|\pi_L^{\frac{1}{q^{n_0-1}(q-1)}}\right|$ such that $n_1(r_0) \geq n_0$. Consider the decomposition $M^{r_0}/t_{LT}M^{r_0} \cong \prod_{n\geq n_1} M_n$ from 3.27 and let $x = (x_n)_{n\geq n_1} \in \prod_{n\geq n_1} M_n$. If $z \in M_{n+1}$ belongs to the image of M_n then its trace is qz since due to our assumptions on r_0 we have $[L_{n+1} : L_n] = q$ for every $n \geq n_1$. In particular $\frac{\pi_L}{q}\pi_L^{-1}\operatorname{Tr}(z) = z$ which via the explicit description of the ψ_{LT}-action in Corollary 3.28 should be read as "$\frac{\pi_L}{q}\psi_{LT}(z) = z$". Using the explicit description of $M^{r_0}/t_{LT}M^{r_0}$ from 3.28 we shall in the following define a tuple $y = (y_n)_{n\geq n_0(r_0^{1/q})} \in M^{r_0^{1/q}}/t_{LT}M^{r_0^{1/q}}$ with $y_{n_0(r^{1/q})} = 0$ such that $(\frac{\pi_L}{q}\psi_{LT} - 1)y = x$. This notation is abusive since in order to make sense of $\frac{\pi_L}{q}\psi_{LT} - 1$ we need to view ψ_{LT} as a map

$$M^{r_0^{1/q}}/t_{LT}M^{r_0^{1/q}} \xrightarrow{\frac{\pi_L}{q}\psi_{LT}} M^{r_0}/t_{LT}M^{r_0} \xrightarrow{\text{res}} M^{r_0^{1/q}}/t_{LT}M^{r_0^{1/q}}$$

i.e. via the description from 3.28 a map

$$\prod_{n\geq n_0(r_0^{1/q})} M_n \xrightarrow{(\frac{\pi_L}{q}\pi_L^{-1}\operatorname{Tr})_n} \prod_{n\geq n_1(r_0)=n_0(r_0^{1/q})-1} M_n \xrightarrow{\text{res}} 0 \times \prod_{n\geq n_1(r_0^{1/q})} M_n \to \prod_{n\geq n_1(r_0^{1/q})} M_n.$$

The restriction map is given by mapping $(x_n \mod Q_n)_n$ to $(\operatorname{res}(x_n) \mod Q_n)_n$ and one can see that by the choice of $n_1(r_0)$ the element Q_{n_1} becomes invertible when restricted to $[r_0^{1/q}, 1)$. In particular the $n_1(r_0)$-th component is mapped to zero and the last map is given by ommiting this component. We define $y_{n_1(r_0)} = 0$ and $y_n = \sum_{j=n_1}^{n-1} x_j$. One can see inductively that $(\frac{\pi_L}{q}\psi_{LT} - 1)(y) = x$. Indeed the map $\frac{\pi}{q}\psi_{LT}$ corresponds to shifting indices and applying $\frac{1}{q}$-times the trace map by 3.28. Hence we obtain $(\frac{\pi_L}{q}\psi_{LT} - 1)(y) = (\frac{1}{q}\operatorname{Tr}(y_{n+1}) - y_n)_n$, which turns out to be $(x_n)_n$ since in every component M_{n+1} we apply $1/q\operatorname{Tr}$ to elements in the image of M_n such that all terms but x_{n+1} cancel out. Using the description of $M^{r_0}/t_{LT}M^{r_0}$ and the ψ_{LT}-action we obtain

$$\ker\left(\frac{\pi_L}{q}\psi_{LT} - 1\right) = \left\{(m_n)_n \mid \frac{1}{q}\operatorname{Tr}(x_{n+1}) = x_n\right\} = \varprojlim_{\frac{1}{q}\operatorname{Tr}(L_{n+1}/L_n)} M_n \otimes_{L_n} L_{n+1}.$$

By 1.36 (after base change from K to A) we have $D(\Gamma_{n_1}, A)/Z_n \cong A[\Gamma_{n_1}/\Gamma_n]$ and we obtain

$$\varprojlim_{\frac{1}{q}\operatorname{Tr}(L_{n+1}/L_n)} M_n \otimes_{L_n} L_{n+1} \cong \varprojlim_n M_{n_1} \otimes_A D(\Gamma_{n_1}, A)/Z_n \tag{3.4}$$

$$= M_{n_1} \otimes_A \varprojlim_n D(\Gamma_{n_1}, A)/Z_n \tag{3.5}$$

$$= M_{n_1} \otimes_A D(\Gamma_{n_1}, A)/\mathfrak{L}_{n_1}. \tag{3.6}$$

using that M_{n_1} is finitely presented over A due to being finitely generated over a Noetherian ring and that $(A[\Gamma_{n_1}/\Gamma_n])_{n\geq n_1}$ is Mittag-Leffler due to having surjective transition maps consisting of free and hence flat A-modules to apply A.12. In the last equality we use the relationship (1.3) and use 1.57 via transport of structure along $\mathcal{R}_A^+ \cong D(\Gamma_{n_1}, A)$ since under the map $T \mapsto Z_{n_1}$ the element t_{LT} is mapped precisely to $\mathfrak{L}_{n_1}$. This isomorphism is Γ_{n_1}-equivariant with respect to the diagonal action. We have a naive resolution

$$M_{n_1} \otimes_A D(\Gamma_{n_1}, A)/\mathfrak{L}_{n_1} = \mathrm{cok}(M_{n_1} \otimes_A D(\Gamma_{n_1}, A) \xrightarrow{\mathrm{id}\otimes\mathfrak{L}_{n_1}} M_{n_1} \otimes_A D(\Gamma_{n_1}, A)).$$

We shall prove that each factor of this resolution is finite projective with respect to the diagonal action, which will complete the proof. Using A.21 applied to the algebras $D(\Gamma_m, A) \subset D(\Gamma_{n_1}, A) \subset D(\Gamma_{n_0}, A)$ we are reduced to proving that each factor is finite projective over $D(\Gamma_m, A)$ for some $m \geq n_0$. We remark at this point that M_{n_1} is projective over $\mathcal{R}_A^{r_0}/Q_n$ due to the projectivity of M^{r_0} and hence projective over A since $\mathcal{R}_A^{r_0}/Q_n$ is free over A. Because M_{n_1} is finitely generated projective over A, we can choose some finitely generated projective complement N_{n_1} which we view with the trivial Γ_L-action, such that $M_{n_1} \oplus N_{n_1} \cong A^d$ and we endow the left-hand side with the norm corresponding to the sup-norm of the Banach norm on the right side with respect to some basis $e_1, \ldots, e_d$. By an analogue of Lemma 1.71 for finitely generated A-modules with A-linear L-analytic Γ_L-action we may assume that $||Z_m||_{M_{n_1}\oplus N_{n_1}} < \varepsilon < 1$ after eventually enlarging m and the $D(\Gamma_m, K)$-action extends to an action of $D_{\mathfrak{r}_l}(\Gamma_m, K)$ for any $l \geq l_0$ with a suitable $l_0 \in \mathbb{N}$. By the same reasoning as in the proof of 2.10 the action extends to a continuous action of $D_{\mathfrak{r}_l}(\Gamma_m, A) := A\hat{\otimes}_K D_{\mathfrak{r}_l}(\Gamma_m, K)$.[4] Consider the maps

$$\bigoplus_{i=1}^{d} D(\Gamma_m, A)e_i \to (M_{n_1} \oplus N_{n_1}) \otimes_A D(\Gamma_m, A) \tag{3.7}$$

$$\Phi : f(Z_m)e_i \to f(Z_m) \cdot (e_i \otimes 1) \tag{3.8}$$

$$\Psi : f(Z_m)e_i \to e_i \otimes f(Z_m) \tag{3.9}$$

By construction Φ is equivariant for the diagonal action while Ψ is a topological isomorphism. It remains to conclude that Φ is an isomorphism and for that purpose it suffices to show that

$$\Phi \circ \Psi^{-1} : D(\Gamma_m, A)^d \to D(\Gamma_m, A)^d$$

is an isomorphism. By passing to the limit it suffices to show that for any $\mathfrak{r}_l > \varepsilon$ the induced map

$$\Phi \circ \Psi^{-1} : D_{\mathfrak{r}_l}(\Gamma_m, A)^d \to D_{\mathfrak{r}_l}(\Gamma_m, A)^d$$

is an isomorphism. We henceforth assume $\mathfrak{r}_l > \varepsilon$ in particular we may find $\delta < 1$ such that $\varepsilon = \mathfrak{r}_l\delta$.[5] Let $\overline{\lambda} = (\overline{\lambda_1}, \ldots, \overline{\lambda_d}) \in D_{\mathfrak{r}_l}(\Gamma_m, A)^d$ and let $\lambda_i \in D_{\mathbb{Q}_p,\mathfrak{r}_l}(\Gamma_m, A)$

[4] The precise value $\mathfrak{r}_l = p^{-\frac{1}{p^l}}$ is not relevant in the following. One could replace $\mathfrak{r}_l$ by any sequence converging to 1, that is bounded below by $\mathfrak{r}_{l_0}$.

[5] This choice of δ is a technicality in order to obtain a strict bound with respect to the quotient topology in (3.15).

be lifts. We wish to show $\|\Phi \circ \Psi^{-1} - \mathrm{id}\|_{D_{\mathfrak{r}_l}(\Gamma_m, A)^d} < 1$. For that purpose we view the $D(\Gamma_m, A)$-action as a $D_{\mathbb{Q}_p}(\Gamma_m, A)$ action that factors over the natural projection $D_{\mathbb{Q}_p}(\Gamma_m, A) \to D(\Gamma_m, A)$. This allows us to define analogously

$$\Phi_{\mathbb{Q}_p}, \Psi_{\mathbb{Q}_p} : \bigoplus_{i=1}^{d} D_{\mathbb{Q}_p}(\Gamma_m, A)e_i \to A^d \otimes_A D_{\mathbb{Q}_p}(\Gamma_m, A),$$

given on Dirac distributions by $\Phi_{\mathbb{Q}_p}(\gamma e_i) = \gamma(e_i \otimes 1)$ and $\Psi_{\mathbb{Q}_p}(e_i \otimes \gamma) = \gamma e_i$. Evidently $\Psi_{\mathbb{Q}_p}$ is an isomorphism. We denote by $\|\cdot\|_{\mathfrak{r}_l}$ the norm introduced in 1.45. Let $\gamma_1, \ldots, \gamma_h$ be a $\mathbb{Z}_p$-Basis of o_L and let $\mathbf{b} = (\gamma_j - 1)_j$. Recall that we have $\|\gamma_j - 1\|_{\mathfrak{r}_l} = \mathfrak{r}_l < 1$ by definition and hence also $\|\gamma_j\|_{\mathfrak{r}_l} = 1$. We first show

$$\|\Phi_{\mathbb{Q}_p} \circ \Psi_{\mathbb{Q}_p}^{-1}(e_i \otimes \mathbf{b}^{\mathbf{k}}) - e_i \otimes \mathbf{b}^{\mathbf{k}}\| \leq \varepsilon \mathfrak{r}_l^{|\mathbf{k}|-1} \leq \delta \mathfrak{r}_l^{|\mathbf{k}|} < \mathfrak{r}_l^{|\mathbf{k}|}$$

for any $\mathbf{k} \in \mathbb{N}_0^h$. The assumption on ε guarantees

$$(\gamma_i - 1)(x \otimes y) = (\gamma_i - 1)x \otimes \gamma_i y + x \otimes (\gamma_i - 1)y$$

has operator norm $\leq \mathfrak{r}_l$, which allows us to reduce the computation by induction on $|\mathbf{k}|$ (the case $k = 0$ being trivial) . Assume $\mathbf{b}^{\mathbf{k}} = (\gamma_j - 1)\mathbf{b}^{\mathbf{k}'}$ with $|\mathbf{k}'| + 1 = |\mathbf{k}|$. We compute

$$\Phi_{\mathbb{Q}_p} \circ \Psi_{\mathbb{Q}_p}^{-1}(e_i \otimes \mathbf{b}^{\mathbf{k}}) - e_i \otimes \mathbf{b}^{\mathbf{k}} \tag{3.10}$$
$$= \mathbf{b}^{\mathbf{k}}(e_i \otimes 1) - e_i \otimes \mathbf{b}^{\mathbf{k}} \tag{3.11}$$
$$= (\gamma_j - 1)\mathbf{b}^{\mathbf{k}'}(e_i \otimes 1) - e_i \otimes (\gamma_j - 1)\mathbf{b}^{\mathbf{k}'} \tag{3.12}$$
$$= (\gamma_j - 1)(\Phi_{\mathbb{Q}_p} \circ \Psi_{\mathbb{Q}_p})(e_i \otimes \mathbf{b}^{\mathbf{k}'}) - (\gamma_j - 1)(e_i \otimes (\mathbf{b}^{\mathbf{k}'})) - (\gamma_j - 1)e_i \otimes \gamma_j \mathbf{b}^{\mathbf{k}'} \tag{3.13}$$
$$= (\gamma_j - 1)([\Phi_{\mathbb{Q}_p} \circ \Psi_{\mathbb{Q}_p}^{-1} - \mathrm{id}](e_i \otimes \mathbf{b}^{\mathbf{k}'})) - ((\gamma_j - 1)e_i) \otimes \gamma_j \mathbf{b}^{\mathbf{k}'}. \tag{3.14}$$

Assuming that the corresponding estimate holds for $\mathbf{b}^{\mathbf{k}'}$ we obtain

$$\|(\Phi_{\mathbb{Q}_p} \circ \Psi_{\mathbb{Q}_p}^{-1} - \mathrm{id})(e_i \otimes \mathbf{b}^{\mathbf{k}})\| \leq \sup(\mathfrak{r}_l \|(\Phi_{\mathbb{Q}_p} \circ \Psi_{\mathbb{Q}_p}^{-1} - \mathrm{id})(e_i \otimes \mathbf{b}^{\mathbf{k}'})\|, \varepsilon \mathfrak{r}_l^{|\mathbf{k}'|}) \leq \delta \mathfrak{r}_l^{|\mathbf{k}|}.$$

In conclusion for each $\overline{\lambda_i}$ and any lift λ_i thereof we have

$$\|(\Phi_{\mathbb{Q}_p} \circ {\Psi_{\mathbb{Q}_p}}^{-1} - \mathrm{id})(e_i \otimes \lambda_i)\| < \|\lambda_i\|_{\mathfrak{r}_l}.$$

More precisely our proof shows

$$\|(\Phi_{\mathbb{Q}_p} \circ {\Psi_{\mathbb{Q}_p}}^{-1} - \mathrm{id})(e_i \otimes \lambda_i)\| \leq \delta \|\lambda_i\|_{\mathfrak{r}_l}.$$

Hence the corresponding estimate with respect to the quotient norm $\|\cdot\|_{\overline{\mathfrak{r}_l}}$ of $\|\cdot\|_{\mathfrak{r}_l}$ on $D(\Gamma_m, A)$ namely

$$\|(\Phi \circ \Psi^{-1} - \mathrm{id})(e_i \otimes \lambda_i)\| \leq \delta \|\lambda_i\|_{\overline{\mathfrak{r}_l}} < \|\lambda_i\|_{\overline{\mathfrak{r}_l}} \tag{3.15}$$

remains valid. By a geometric series argument $\Phi \circ \Psi^{-1}$ is an isomorphism forcing Ψ to also be an isomorphism.

□

Lemma 3.31. *Let $m \in \mathbb{N}$, $C_0 := |p|^{q/(q-1)e}$ and $C_0 \leq r \leq s < 1$. We denote by $|\cdot|_s$ the s-Gauss norm on $\mathcal{R}_K^{[r,1]}$. Then there exists a constant such that*

$$|\psi(T^{-m})|_s \leq C|T^{-m/q}|_s.$$

Proof. By [FX12, Lemma 2.7] there exists a constant C such that $|\psi(T^{-m})|_s \leq C|T^{-m}|_{s^{1/q}} \leq C|T|_s^{-m/q}$. □

Theorem 3.32. *Let M be a (φ, Γ)-module over $\mathcal{R}_A$. Then $M/(\psi - 1)$ is a finitely generated A-module.*

Proof. Let M^{r_0} be a model of M over $[r_0, 1)$ with $r_0 \geq C_0$. Define $r := r_0^{1/q^2}$. Let $e_1', \ldots e_n'$ be generators of M^r. By assumption M^{r^q} is a projective module generated by $e_i := \varphi(e_i')$. We can hence find N^{r^q} satisfying $N^{r^q} \oplus M^{r^q} \cong \bigoplus_i \mathcal{R}_A^{r^q} e_i =: E$. We denote by $|\cdot|_s$ the s-Gauss norm on the free module E and denote by the same symbol the restriction of said norm to M^{r^q} viewed as a submodule of E. Let C_s be the operator norm of the canonical projection $E \to M^{r^q}$ with respect to the s-Gauss norm. Because the e_i and e_i' generate M^{r^q} we may find a $\mathcal{R}_A^{r^q}$ valued matrix F' satisfying $e_j = \sum_i F_{ij}' e_i'$. Similarily we can chose a matrix G with values in $\mathcal{R}_A^{r^q}$ satisfying $e_j' = \sum_i G_{ij} e_i$. Setting $F := \varphi(F')$ we compute

$$\varphi\left(\sum_i c_i e_i\right) = \sum_i \varphi(c_j)\varphi(e_i) = \sum_i \left(\varphi(c_i)\varphi\left(\sum_j F_{ij}' e_i'\right)\right) = \sum_i \left(\sum_j F_{ij}\varphi(c_j)\right) e_i \tag{3.16}$$

$$\psi\left(\sum_i c_i e_i\right) = \sum_i \psi\left(c_i\left(\varphi\left(\sum_j G_{ij} e_i\right)\right)\right) = \sum_i \left(\sum_j \psi(c_i) G_{ij} e_i\right). \tag{3.17}$$

Let $0 < \varepsilon$, let $a, b \in \mathbb{R}$ and let $I \subset \mathbb{R} \cup \{\pm\infty\}$ be an interval. Any element of the free module E can be written as a convergent series $v = \sum_{j=1}^n \sum_{i \in \mathbb{Z}} a_{ij} T^i e_j$. This allows us to define the A-linear map

$$P_I(v) := \sum_j \sum_{i \in I \cap \mathbb{Z}} a_{ij} T^i e_j.$$

By abuse of notation we denote the induced map

$$M^{r^q} \overset{\iota}{\hookrightarrow} E \xrightarrow{P_{[a,b]}} E \xrightarrow{\text{proj}} M^{r^q}$$

by the same symbol. The image of $P_{[a,b]}$ is clearly a finitely generated A-submodule of M^{r^q} generated by the elements $T^i e_j$ with $j = 1, \ldots, n$ and $i \in [a, b] \cap \mathbb{Z}$. Given $v \in F$ we can write

$$\begin{aligned} v &= \sum_j \sum_{i<a} a_{ij} T^i e_j + \sum_j \sum_{i \in [a,b] \cap \mathbb{Z}} a_{ij} T^i e_j + \sum_j \sum_{i>b} a_{ij} T^i e_j \\ &= P_{[-\infty,a)}(v) + P_{[a,b]}(v) + P_{(b,\infty]}(v). \end{aligned}$$

Define $v^- := P_{[-\infty,a)}(v)$ and $v^+ := P_{(b,\infty]}(v)$. The decomposition above becomes

$$v = v^- + P_{[a,b]}(v) + v^+.$$

We now define $\lambda : M^{r^q} \to M^{r^q}$ via

$$\lambda(v) := \operatorname{proj} v^- - \frac{\pi}{q} \operatorname{proj}(\varphi(v^+)).$$

We compute

$$v - P_{[a,b]}(v) + (\psi - 1)(\lambda(v)) \tag{3.18}$$
$$= \operatorname{proj}(v^- + v^+) + (\psi - 1)(\operatorname{proj} v^- - \frac{\pi}{q} \operatorname{proj}(\varphi(v^+))) \tag{3.19}$$
$$= \operatorname{proj} \psi((v^-)) + \operatorname{proj}(\frac{\pi}{q}\varphi(v^+)) \tag{3.20}$$
$$= \operatorname{proj}\left(\sum_{lj}\sum_{i<a} G_{lj}\psi(a_{ij}T^i)e_j\right) + \operatorname{proj}\left(\sum_{lj}\sum_{i>b} \frac{\pi}{q} F_{lj}\varphi(a_{ij}T^i)e_j\right). \tag{3.21}$$

Next we shall estimate both summands of (3.21). For the left-hand side

$$C_r \max_{lj} \sup_{i<a} |G_{lj}\psi(a_{ij}T^i)|_r \leq \max_{j} \sup_{i<a} C||G||_r |a_{ij}\psi(T^i)| \tag{3.22}$$
$$\leq C \max_{j} \sup_{i<a} |a_{ij}||q^2 T^{i/q}|_r \tag{3.23}$$
$$\tag{3.24}$$
$$\leq CC_1 \max_{j} \sup_{i<a} |a_{ij}T^i T^{i(q^{-1}-1)}| \tag{3.25}$$
$$\leq CC_1 |v|_r |T^{a(q^{-1}-1)}|_r \tag{3.26}$$
$$\leq \tilde{C}|v|_r r^{a(q^{-1}-1)} \tag{3.27}$$

where C is a suitable constant, C_1 is the constant from 3.31 and $\tilde{C} = CC_1$. For a small enough we have

$$\tilde{C}|v|_r r^{a(q^{-1}-1)} \leq |v|_r \epsilon. \tag{3.28}$$

For the right-hand side of (3.21) we have

$$\begin{aligned} C_r \max_{lj} \sup_{i>b} \left|\frac{\pi}{q} F_{lj}\varphi(a_{ij}T^i)\right|_r &\leq \max_{j} \sup_{i>b} C||F||_r |a_{ij}\varphi(T^i)|_r \\ &\leq C \max_{j} \sup_{i>b} ||F||_r |a_{ij}T^i|_r \\ &\leq C||F||_r |v|_r |T^b|_r \end{aligned}$$

where again C is some suitable constant (not necessary equal to the previous constant) and we use the estimate $|\varphi(T^i)|_r \leq |T^i|_{r^q} \leq |T^{iq}|_r$ since $\varphi(T) \in T + o_L[[T]]$. For b large enough we can have

$$C||F||_r |v|_r |T^b|_r \leq |v|_r \epsilon. \tag{3.29}$$

We now show that the series $w := \sum_k v_k$ with $v_0 = v$ and $v_{n+1} = v_n - P_{[a,b]}(v_n) + (\psi - 1)\lambda(v_n)$ converges and satisfies

$$v - P_{[a,b]}(w) + (\psi - 1)(\lambda(w)) = 0.$$

By choosing a small enough and b large enough we can ensure that $|v_{n+1}|_r \leq |v_n|_r \varepsilon$. Because we chose $\varepsilon < 1$ this means that v_k is a zero-sequence with respect to the r-Gauss norm. We next show that this series converges with respect to any s-Gauß norm for $s \in (r^{1/q}, 1)$. Arguing analogously for the constants C_s and the operator norms $\|G\|_s$ (resp. $\|F\|_s$) one concludes that we may find $a' < a$ and $b' > b$ (depending on s but independent of v) such that the analogues of (3.28) and (3.29) hold. By splitting the summands in (3.21) into four parts namely the partial sums $i < a', a' \leq i < a$ and respectively $i > b'$ and $b' \geq i > b$, it remains to estimate the summands corresponding to the intervals $[a', a)$ and $[b, b')$. One checks that the s-Gauß norm of these summands can be bounded by $C^{(s)}|v|_r$ with a suitable constant $C^{(s)}$ independent of v. Hence we obtain $|v_{n+1}|_s \leq \max\{\varepsilon|v_n|_s, C^{(s)}|v_n|_r\}$, which means that v_n is also a zero-sequence with respect to the s-Gauß norm for $s \in (r^{1/q}, 1)$. This means that v_n tends to zero with respect to the $[r, s]$-norms for any $s \in (r^{1/q}, 1)$. Therefore w converges with respect to the Fréchet topology on M^{r^q}. We compute

$$P_{[a,b]}(w) - (\psi - 1)\lambda(w) = \sum_{n \geq 0} P_{[a,b]}(v_n) - (\psi - 1)\lambda(\sum_{n \geq 0} v_n) \tag{3.30}$$

$$= \sum_{n \geq 0} v_n - v_{n+1} \tag{3.31}$$

$$= v_0 = v. \tag{3.32}$$

This in turn implies that any $v \in M^{r^q}$ can be represented modulo $\psi - 1$ by an element in the image of $P_{[a,b]}$. This implies that $M^{r^q}/(\psi - 1)$ is finite over A. Finally let $v \in M$. Then v belongs to some M^s with $1 > s \geq r_0$. Then take m large enough such that $\psi^m v$ belongs to M^{r^q}. We deduce that $\psi^m v$ and v are congruent modulo $\psi - 1$ and $\psi^m v$ is represented by an element in the image of $P_{[a,b]}$. Therefore $M/(\psi - 1)$ is finite over A. □

Remark 3.33. *The same result holds for $c\psi$ for any constant c.*

Proof. Apply 3.32 to a module with φ_L-action twisted by c^{-1}. □

Lemma 3.34. *There exists $n \gg 0$ such that $\psi - 1 : t_{LT}^{-n} M \to t_{LT}^{-n} M$ is surjective.*

Proof. Because M is projective and hence torsion-free, $t^{-n}M$ can be viewed as a $\mathcal{R}_A$ submodule (isomorphic to M) of $M[1/t]$. Furthermore the formula $\varphi(t_{LT}) = \pi_L t_{LT}$ allows us to extend φ and ψ to $t^{-n}M$. We use the notation from the proof of theorem 3.32. Note that replacing M with $t^{-n}M$ replaces the matrices F and G with $\pi_L^{-n}F$ and $\pi_L^n G$. In the proof of 3.32 we implicitly showed that M is generated by $\#\mathbb{Z} \cap [a, b]$ elements. This number is evidently zero, if $a = b \notin \mathbb{Z}$. Hence it suffices to show that there exists a choice of a such that

$$C_1 |\pi_L^n| ||G|| r^{a(q^{-1}-1)} \leq \epsilon \tag{3.33}$$

and

$$C_2|\pi_L^{-n}|||F||_r r^a \leq \epsilon \tag{3.34}$$

with some suitable constants C_i independent of n and a. We can replace the first inequality by

$$C_1^q|\pi_L^{qn}|||G||^q r^{a(1-q)} \leq \epsilon^q \tag{3.35}$$

and multiply it with the second inequality to obtain

$$C_1^q C_2|\pi_L|^{(q-1)n}||G||^q||F||r^{a(2-q)} \leq \varepsilon^{q+1}. \tag{3.36}$$

Chosing any $a \notin \mathbb{Z}$ such that (3.33) is satisfied for any $n \in \mathbb{N}$, we can see that (3.36) is satisfied for $n \gg 0$, because $|\pi_L|^{(q-1)n}$ converges to 0 for $n \to \infty$ and all other terms are independent of n. Obviously (3.33) implies (3.35) while (3.36) and (3.35) imply (3.34). □

3.5 Comparison between $H^1_{\varphi_L,\Gamma_n}$ and $\mathrm{Ext}^1_{\mathrm{an}}$

In this section we simplify the notation and consider $U = \Gamma_n$ for an $n \geq n_0$ and write Z for the variable $Z_n \in D(\Gamma_L, K)$. We denote by $\alpha(-) := \log(\chi_{LT}(-))/\pi_L^n$ the chart used to indentify U with o_L.

Theorem 3.35. *Let M, E be L-analytic (φ_L, Γ_L)-modules over $\mathcal{R}_A$ that fit in an exact sequence*

$$0 \to M \to E \to \mathcal{R}_A \to 0.$$

Let e be a preimage of 1 in E then $e \mapsto ((\varphi_L - 1)e, Ze)$ gives a well-defined injection

$$\Theta : \mathrm{Ext}^1_{an}(\mathcal{R}_A, M) \to H^1_{\varphi_L,Z}(M),$$

whose image is contained in $H^1_{\varphi_L,D(\Gamma_L,K)}(M)$ inducing a bijection

$$\mathrm{Ext}^1_{an}(\mathcal{R}_A, M) \to H^1_{\varphi_L,D(\Gamma_L,K)}(M).$$

Proof. Since all modules involved are L-analytic, the morphisms between them are even $D(\Gamma_L, K)$-linear. Because 1 is invariant under the action of φ_L and U the element $((\varphi_L - 1)e, Ze)$ lies in $M \times M$ and it is clear that $\partial_2((\varphi_L - 1)e, Ze) = 0$. If $\tilde{e}$ is another preimage of 1 then $e - \tilde{e}$ lies in M and therefore $((\varphi_L - 1)e, Ze) - ((\varphi_L - 1)\tilde{e}, Z\tilde{e}) = ((\varphi_L - 1)(e - \tilde{e}), Z(e - \tilde{e})) = \partial_1(e - \tilde{e})$. Note that for any $\gamma \in \Gamma_L$ the element γe is yet another preimage of 1 and the above computation shows that the cocycles $(\varphi_L - 1)e, Ze)$ and $\gamma(\varphi_L - 1)e, Ze)$ differ by a coboundary, which proves the Γ_L-invariance of $\overline{((\varphi_L - 1)e, Ze)}$. For the injectivity let E be an extension such that $((\varphi_L - 1)e, Ze)$ vanishes in H^1. Then there exists $d \in M$ such that $((\varphi_L - 1)(e - d), Z(e - d)) = 0$. That means $e - d \in E$ is a preimage of $1 \in \mathcal{R}_A$ fixed by φ_L and U, which can be modified to be (φ_L, Γ_L)-invariant by replacing it with $\frac{1}{[\Gamma_L:U]}\sum_{\gamma \in \Gamma_L/U} \gamma(e - d)$ and therefore induces a section of $E \to \mathcal{R}_A$, which shows injectivity.

For the surjectivity let $\overline{(a,b)} \in H^1_{\varphi_L, D(\Gamma_L,K)}(M)$. We first explain how to construct an L-analytic coycle of the semigroup $\varphi_L^{\mathbb{N}_0} \times \Gamma_L$ with values in M. In the following we identify σ with its corresponding Dirac distribution. We obtain $\sigma - 1 = ZG_\sigma(Z)$ with a suitable power series $G_\sigma \in o_K[\![Z]\!]$. Note that the map $\sigma \mapsto G_\sigma(Z)b$ defines a 1-cocycle $c : U \to M$ as for $\tau \in U$ we have

$$\sigma\tau - 1 = \sigma(\tau - 1) + (\sigma - 1) = Z\delta_\sigma G_\tau(Z) + ZG_\sigma(Z).$$

We extend it to the whole group Γ_L using $\frac{1}{[\Gamma_L:U]}$-times the corestriction (defined in [BF17, Definition 2.1.2]) such that the restriction of c to U is the cocycle we started with. Finally we define an extension of M by $\mathcal{R}_A$ by setting $E = M \times \mathcal{R}_A$ as $\mathcal{R}_A$-modules with actions $\sigma((m,r)) = (\sigma m + (\sigma r)c(\sigma), \sigma r)$ and $(\varphi_E(m,r) = (\varphi_M(m) + \varphi_L(r)a, \varphi_L(r)))$. In order to show that this extension is L-analytic we need to show that the function $\sigma \mapsto c(\sigma)$ is L-analytic. It suffices to show that for $m \in M^{[r,s]}$ for any interval $[r,s]$ and a sufficiently small open subgroup $U' \subset U$ the orbit map $\sigma \mapsto \sigma m$ restricted to U' is L-analytic. Recall that σ acts on m via the operator

$$\eta(\alpha(\sigma), Z)$$

for some fixed $n \in \mathbb{N}$ depending on U and we abbreviate $x := \alpha(\sigma)$. We wish to show that the series

$$G_\sigma(Z)m = (\eta(x,Z) - 1)/Zm = \sum_{k=1}^{\infty} \frac{(x\Omega \log_{LT}(Z))^k}{k!Z} m$$

converges on some ball $|x| \leq \pi_L^j$, which via the chart χ_{LT} corresponds to the desired subgroup U'. For that purpose choose j such that $||\frac{(\Omega \log_{LT}(Z)\pi_L^j/Z)^k}{k!}||_{M^{[r,s]}}$ converges to zero for $k \to \infty$. The choice of j is possible because the series $\exp(T)$ has non-trivial radius of convergence and the operator norm of $\log_{LT}(Z)/Z \in D(U,K)$ acting on $M^{[r,s]}$ is bounded. It remains to show that the image of E is the original element $\overline{(a,b)}$. We may choose $e = (0,1)$ as an explicit preimage of 1. By construction $(\varphi - 1)e = (a,0)$. It remains to show $Ze = (b,0)$. For $\sigma \in U$ we compute $(\sigma - 1)(e) = (G_\sigma(Z)b, 0)$. Recall that the action of $D(\Gamma_L, K)$ is obtained by continuous extension of the $K[\Gamma_L]$ action and that the action of $Z \in D(U,K)$ agrees with the action of any $X \in D_{\mathbb{Q}_p}(U,K)$ that projects to Z. Choose such an element and express it as a convergent series

$$X = \sum_{\mathbf{k} \in \mathbb{N}_0^d} a_{\mathbf{k}} \mathbf{b}^{\mathbf{k}},$$

where $\mathbf{b} = (\gamma_i - 1, \ldots, \gamma_d - 1)$ is a $\mathbb{Z}_p$-basis of U. The series defining X converges a fortiori in $D(U,K)$ by definition of the quotient topology and since $\gamma_i - 1$ is given by the power series $\eta(\alpha(\gamma_i), Z) - 1$ evaluating at $Z = 0$ shows that necessarily $a_0 = 0$. We set $G_i(Z) := G_{\gamma_i}(Z)$. Let $\mathbf{G} := (G_1, \ldots, G_d)$ such that the image of $\mathbf{b}^{\mathbf{k}}$ under the projection proj : $D_{\mathbb{Q}_p}(U,K) \to D(U,K)$ is $(ZG_1(Z),\ldots,ZG_d(Z))^{\mathbf{k}}$. We compute

$$\begin{aligned} Z &= \mathrm{proj}(X) && (3.37)\\ &= \sum_{0\neq\mathbf{k}\in\mathbb{N}_0^d} a_{\mathbf{k}} Z^{|\mathbf{k}|}\mathbf{G}^{\mathbf{k}} && (3.38)\\ &= Z(\sum_{0\neq\mathbf{k}\in\mathbb{N}_0^d} a_{\mathbf{k}} Z^{|\mathbf{k}|-1}\mathbf{G}^{\mathbf{k}}) && (3.39)\end{aligned}$$

We claim that the inner sum $\mu := \sum_{0\neq\mathbf{k}\in\mathbb{N}_0^d} a_{\mathbf{k}} Z^{|\mathbf{k}|-1}\mathbf{G}^{\mathbf{k}}$ converges with respect to the Fréchet topology to an element of $D(U,K)$ which satisfies $\mu Z = Z$ by construction and hence has to be 1 since $D(U,K)$ is a domain. For the convergence we remark that the map $\lambda \mapsto Z\lambda$ is an injective continuous operator with finite-dimensional (hence Hausdorff by 1.10) cokernel.

In particular $ZD(U,K)$ is itself a Fréchet space and we conclude that $\lambda \mapsto \lambda Z$ is a continuous surjection $D(U,K) \to ZD(U,K)$ between Fréchet spaces and thus a homeomorphism by the open mapping theorem. We further compute for $\mathbf{k} \neq 0$

$$\mathbf{b}^{\mathbf{k}}(e) = (Z^{|\mathbf{k}|-1}\mathbf{G}^{\mathbf{k}}b, 0),$$

which can be seen as follows. Without loss of generality assume $k_1 \neq 0$. Since U is commutative we may first apply $\gamma_1 - 1$ and obtain $(\gamma_1 - 1)(e) = (G_1(Z)b, 0)$ by construction. The resulting element belongs to the image of M under the natural inclusion $M \to E$ and hence $\gamma_i - 1$ acts via multiplication by $ZG_i(Z)$. Putting everything together we conclude

$$Ze = \left(\sum_{0\neq\mathbf{k}\in\mathbb{N}_0^d} a_{\mathbf{k}} Z^{|\mathbf{k}|-1}\mathbf{G}^{\mathbf{k}}b, 0\right) = (\mu b, 0) = (b, 0).$$

□

Chapter 4

Lubin-Tate deformations and Iwasawa cohomology

In this chapter we study deformations of (φ_L, Γ_L)-modules with the distribution algebra and their relationship to Iwasawa cohomology. We shall establish a comparison between the (Ψ, Z)-cohomology of $\mathbf{Dfm}(M)$ and the Iwasawa cohomology of M which is defined as the cohomology of the complex

$$C_\Psi(M) : M \xrightarrow{\Psi - 1} M$$

concentrated in degrees $[1, 2]$. This is motivated by the fact that for an étale module (coming from a representation V) this cohomology is closely related to the Iwasawa cohomology of V. Roughly speaking the deformation is a family of (φ_L, Γ_L)-modules parametrised by the Fréchet-Stein algebra $D(\Gamma_L, K)$ and specialising to a point $\mathfrak{m}_x \in \mathrm{Sp}(D(\Gamma_L, K))$ corresponds to twisting the module at $x = 0$ by an analytic character. As before we run into the problem that, contrary to the cyclotomic case, the inclusion $\Gamma_{n_0} \subset \Gamma_L$ does not split and hence we restrict a priori to this subgroup.

4.1 Basic definitions

For now fix $U := \Gamma_m$ for some $m \geq n_0$. We shorten our notation and write $D := D(U, K)$ and pick an affinoid cover $D = \varprojlim_n D_n$ that arises as a base change of an affinoid cover of $D(U, L)$ with U-stable terms (e.g. $D_n := D_{r_n}(U, K)$ for a non trivial sequence r_n converging to 1 from below). We further abbreviate $D(\Gamma_L) := D(\Gamma_L, K) = \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[U]} D(U, K)$ and $D_n(\Gamma_L) := \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[U]} D_n$. In this chapter let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$ with a model over $[r_0, 1)$. As before we denote by Z the preimage of a coordinate T under the Fourier isomorphism. We denote by Ψ the left-inverse operator to φ_L i.e. $\Psi = \frac{\pi_L}{q} \psi_{LT}$. We henceforth omit the subscript ψ_{LT} and write $\psi := \psi_{LT}$.

Definition 4.1. *We define for* $r \geq r_0$

$$\mathbf{Dfm}_n(M^r) := D_n \hat{\otimes}_{K,\pi} M^r$$

and

$$\mathbf{Dfm}(M^r) := \varprojlim_n \mathbf{Dfm}_n = D\hat{\otimes}_{K,\pi} M^r.$$

We endow $\mathbf{Dfm}_n(M^r)$ *(resp.* $\mathbf{Dfm}(M^r)$*) with an action of* φ_L *and* U *via* $\gamma(a\otimes m) = \delta_{\gamma^{-1}} a \otimes \gamma m$ *and* $\varphi_L(a\otimes m) := a\otimes \varphi_L(m)$*. We further define* $\mathbf{Dfm}_n^{\Gamma_L}(M^r)$ *(resp.* $\mathbf{Dfm}_n^{\Gamma_L}(M^r)$*) as* $D_n(\Gamma_L)\hat{\otimes} M^r$ *(resp.* $D(\Gamma_L)\hat{\otimes} M^r$*) with analogously defined actions.*

In particular $\mathbf{Dfm}(M^r)$ can be viewed as a sheaf on (the rigid analytic space associated to) D. We can also define

$$\mathbf{Dfm}_n(M) := D_n\hat{\otimes}_{K,i} M$$

but passing to **Dfm** poses a problem, since inductive tensor product topologies do not necessarily commute with projective limits. However the inductive tensor product topology is the most reasonable choice for a tensor product of an LF-space and a Fréchet space. We shall avoid this problem by working on the level of models.

Proposition 4.2. $\mathbf{Dfm}_n^{\Gamma_L}(M^r)$ *is an* L*-analytic family of* (φ_L,Γ_L)*-modules over* $\mathrm{Sp}(D_n(\Gamma_L)) \times_K \mathrm{Sp}(A)$ *i.e. an* L*-analytic* (φ_L,Γ_L)*-module over* $\mathcal{R}^{[r,1)}_{D_n(\Gamma_L)\hat{\otimes} A}$.

Proof. By construction $\mathbf{Dfm}_n^{\Gamma_L}(M^r)$ is finite projective of the same rank as M. The actions are semi-linear (in particular $D_n\hat{\otimes}A$-linear) because

$$\gamma(\lambda\mu\otimes am) = (\lambda\otimes a)(\delta_\gamma^{-1}\mu\otimes m)$$

for any $\lambda\in D, \mu\in D_n, a\in A$ and $m\in M^r$. Note that semi-linearity refers to the action of $\lambda\otimes a\otimes f\in\mathcal{R}_{D_n\hat{\otimes}A}$ via multiplication $(\lambda\otimes a\otimes f)(\mu\otimes m) = \lambda\mu\otimes afm$, where $\lambda\in D, \mu\in D_n, a\in A, f\in\mathcal{R}_K, m\in M^r$. L-analyticity (and hence continuity) follows from 1.43 once we establish that the U-action on D_n (via inverted multiplication) is locally L-analytic. By our assumptions the Dirac distribution corresponding to $u\in U$ admits an expansion $\delta_u = \eta(a(u), Z)$ with a suitable $a(u)\in o_L$. Fixing $\lambda\in D_n$ its orbit map is given by $u\mapsto \eta(-a(u), Z)\lambda$. Expanding out the terms shows that the orbit map is locally L-analytic. The φ_L and Γ_L-actions clearly commute and the linearised φ_L-map is invertible, since it is the linear extension of φ_{M^r}. □

Remark 4.3. $\mathbf{Dfm}(M^r), \mathbf{Dfm}_n(M^r)$ *carry two different* D*-actions. One induced by the scalar action in the left tensor component, that we shall call scalar action. The other one induced by the* L*-analytic action defined in 4.1, which we shall call diagonal action. Note that* $\mathbf{Dfm}(M^r)$ *is only a* (φ_L, U)*-module but* ***not*** *a* (φ_L,Γ_L)*-module. (Because there is no action of the full group* Γ_L*.) Instead we can restrict the action to the subgroup* $U\subset\Gamma_L$ *which is still enough to make sense of the operator* $Z\in D(U,K)$ *and the complex* $C_{\varphi,Z}(\mathbf{Dfm}(M^r))$ *(resp.* $C_{\varphi,Z}(\mathbf{Dfm}_n(M^r))$*). We use a subscript* $(-)_{diag}$ *to emphasize that the action is given diagonally when ambiguity can arise.*

Another subtlety is the fact that the induced diagonal action of D on $\mathbf{Dfm}(M^r)$, in particular, the diagonal action of Z is harder to understand. It is neither given by $Z \otimes Z$ nor by $Z^i \otimes Z$, where Z^i denotes the action induced by the inversion on U. This problem already occurs in the cyclotomic case where $Z = \gamma - 1$ and

$$(\gamma - 1)_{\mathrm{diag}}(a \otimes m) = \delta_{\gamma^{-1}} a \otimes \gamma m - a \otimes m$$

while

$$(\gamma - 1)a \otimes (\gamma - 1)m = \delta_{\gamma^{-1}} a \otimes \gamma m - a \otimes \gamma m - \delta_{\gamma^{-1}} a \otimes m + a \otimes m.$$

4.2 Coadmissibility of Iwasawa cohomology $C_\Psi(M)$ over a field.

For technical reasons that will become clear in the proof of 4.24 we require that the complex of D-modules $C_\Psi(M)$ has coadmissible cohomology groups to obtain a comparison between the Iwasawa cohomology and the cohomology of $\mathbf{Dfm}(M)$. Our Theorem 5.8 asserts that this perfectness holds for so-called trianguline modules. In chapter 6 we shall explore conjecturally how the étale case can be incorporated into the picture. More precisely we show that it suffices to proof the statement in the étale case to conclude that it holds for every (φ_L, Γ_L)-module coming from $\mathcal{R}_L$.

Remark 4.4. *The rings $\mathcal{R}_K^+, \mathcal{R}_K^{[r,1)}, \mathcal{R}_K, D(o_L, K), D(o_L, L)$ are Prüfer-domains (i.e. every finitely generated ideal is invertible). In particular a module over the above rings is flat if and only if it is torsion-free and any finitely generated torsion-free module is projective.*

Proof. Using the Fourier isomorphism [ST01, Theorem 2.3] this is [BSX20, Corollary 1.1.8]. □

Definition 4.5. *We define the **heart** of M as*

$$\mathcal{C}(M) := (\varphi_M - 1)M^{\Psi=1}.$$

If there is no possibility of confusion we omit M and simply write $\mathcal{C} := \mathcal{C}(M)$. For each $c \in K^\times$ we define a variant of the heart as

$$C_c(M) := (\varphi_M - c)M^{c\Psi=1}.$$

Remark 4.6. *$\mathcal{C}$ is a $D(\Gamma_L, K)$-submodule of $M^{\psi_{LT}=0}$, in particular $\mathcal{C}$ is $D(\Gamma_L, K)$-torsion-free. Furthermore we have for every $c \in K^\times$ an exact sequence*

$$0 \to M^{\varphi=c} \xrightarrow{\iota} M^{c\Psi=1} \xrightarrow{\varphi-c} \mathcal{C}_c(M) \to 0,$$

where ι is the inclusion.

Proof. $\mathcal{C}_c(M)$ is a $K[\Gamma_L]$-submodule since the actions of Γ_L and φ (resp. Ψ) commute and by continuity considerations it is also a $D(\Gamma_L, K)$-submodule of M. Using $\Psi \circ \varphi = \mathrm{id}$ one concludes that $\mathcal{C}_c(M)$ is contained in $M^{\Psi=0}$ which is projective over $\mathcal{R}_K(\Gamma_L)$ by 2.19 and, in particular, $D(\Gamma_L, K)$-torsion-free. The exactness of the sequence on the right is given by definition. It remains to see $M^{\varphi=c} \cap M^{c\Psi=1} = M^{\varphi=c}$ i.e. $M^{\varphi=c} \subset M^{c\Psi=1}$ for this purpose let $m \in M^{\varphi=c}$ then $cm = \varphi(m)$ implies $\Psi(\varphi(m)) = c\Psi(m)$ but then $c\Psi(m) = m$. □

The following lemma is a strengthening of [KPX14, Lemma 4.1.6]. We elaborate on the proof for the convenience of the reader.

Lemma 4.7. *Let $r \in [0,1)$. Any $\mathcal{R}_K^{[r,1)}$-module V of finite K-dimension admits a resolution of the form*

$$0 \to F_1 \to F_2 \to V \to 0,$$

where F_i are finite free $\mathcal{R}_K^{[r,1)}$-modules.

Proof. Let $v_1, \ldots, v_d$ be a K-basis of V. Consider the surjection $(\mathcal{R}_K^{[r,1)})^d \to V$ mapping e_i to v_i. Since for each i the elements $v_i, Tv_i, T^2v_i, \ldots$ have to be linearly dependent we observe that there exists a polynomial $f \in K[T]$ such that the structure map $\mathcal{R}_K^{[r,1)} \to \mathrm{End}(V)$ factors over $\mathcal{R}_K^{[r,1)}/(f)$. Now take a factorisation of f viewed as an element in $K\langle T\rangle$ of the form ug, where u is a unit and g is a Weierstraß polynomial and rewrite this decomposition as ug_1g_2, such that the zeroes of g_1 lie outside the annulus $[r,1)$ (i.e. g_1 becomes a unit in $\mathcal{R}_K^{[r,1)}$) and the zeroes of g_2 are contained inside the annulus. Without loss of generality we assume $g_2 = g$. By the coadmissibility of $\mathcal{R}_K^{[r,1)}$ and $g\mathcal{R}_K^{[r,1)} \cong \mathcal{R}_K^{[r,1)}$ we have $\mathcal{R}_K^{[r,1)}/(g) = \mathrm{cok}(\mathcal{R}_K^{[r,1)} \xrightarrow{g} \mathcal{R}_K^{[r,1)}) = \varprojlim_s \mathrm{cok}(\mathcal{R}_K^{[r,s]} \xrightarrow{g} \mathcal{R}_K^{[r,s]})$. For s large enough (such that the zeroes of g are contained in the annulus $[r,s]$) we have by [Bos14, 3.3 Lemma 10] and the chinese remainder theorem $\mathcal{R}_K^{[r,s]}/(g) \cong K\langle T\rangle/(g)$. In particular the limit stabilises for s large enough and we obtain $\mathcal{R}_K^{[r,1)}/(g) \cong K\langle T\rangle/(g)$. Recall that $K\langle T\rangle$ is a principal ideal domain by [Bos14, 2.2 Corollary 10]. By the elementary divisor theorem we may find a free resolution of V as a $K\langle T\rangle$-module of the form

$$0 \to K\langle T\rangle^{d_1} \to K\langle T\rangle^{d_2} \to V \to 0$$

with some $d_i \in \mathbb{N}$. Since $K\langle T\rangle$ is a principal ideal domain and $\mathcal{R}_K^{[r,1)}$ is torsion free we get via base change along the flat map $K\langle T\rangle \to \mathcal{R}_K^{[r,1)}$ a resolution

$$0 \to (\mathcal{R}_K^{[r,1)})^{d_1} \to (\mathcal{R}_K^{[r,1)})^{d_2} \to \mathcal{R}_K^{[r,1)} \otimes_{K\langle T\rangle} V \to 0.$$

Because V is a $\mathcal{R}_K^{[r,1)}/(g)$-module we obtain $\mathcal{R}_K^{[r,1)} \otimes_{K\langle T\rangle} V = \mathcal{R}_K^{[r,1)}/(g) \otimes_{K\langle T\rangle} V = K\langle T\rangle/(g) \otimes_{K\langle T\rangle} V = V$. We have therefore constructed the desired resolution of V. □

Remark 4.8. *Let N be a not necessarily L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_K$ and let $n \geq n_0$ such that $\Gamma_n \cong o_L$. Then*

1. $N^{c\varphi=1}$ *has finite* K*-dimension for any* $c \in K^\times$.

2. *If* N *is* L*-analytic and* $c \in K^\times$ *then* $N^{c\varphi=1}[0]$ *belongs to* $\mathbf{D}^b_{perf}(D(\Gamma_n, K))$.

3. *If* N *is* L*-analytic and* $c \in K^\times$ *then* $N/(c\psi_{LT}-1)[0]$ *belongs to* $\mathbf{D}^b_{perf}(D(\Gamma_n, K))$.

Proof. Restricting the residue pairing from 1.75 to $N^{c\varphi=1}$ we see that the pairing factors over $\check{N}/(c\psi_{LT}-1)$ which is finite-dimensional by 3.32. If N is L-analytic, then $N^{\varphi=1}$ and $N/(c\psi_{LT}-1)$ carry natural $D(\Gamma_n, K)$-module structures and are finite dimensional over K by the above (resp. 3.32). Hence 2. and 3. follow from 4.7 by transport of structure along $\mathcal{R}^+_K \cong D(\Gamma_n, K)$. □

Proposition 4.9. *Let* $c \in K^\times$. *The following are equivalent.*

(i.) $C_{c\Psi}(M) \in \mathbf{D}^b_{perf}(D(U, K))$.

(ii.) $M^{c\Psi=1}$ *is coadmissible and finitely generated as a* $D(U, K)$*-module.*

(iii.) $M^{c\Psi=1}$ *is finitely generated as a* $D(U, K)$*-module.*

(iv.) $\mathcal{C}_c(M)$ *is finitely generated as a* $D(U, K)$*-module.*

Proof. (i.) $\implies$ (ii.) follows from the fact that finite projective modules are automatically coadmissible and the latter form an abelian category. Hence $M^{\Psi=1}$ is coadmissible as a cohomology group of a complex of coadmissible modules. Finite generation follows from [BSX20, Lemma 1.1.9]. The implication (ii.) $\implies$ (iii.) is trivial. (iv.) follows immediately from (iii.) via the exact sequence from 4.6. Lastly assume (iv.) then $\mathcal{C}_c(M)$ is a torsion-free module which is finitely generated. Since $D(U, K)$ is a Prüfer-domain we conclude that $\mathcal{C}_c(M)$ has to be finitely generated projective by Remark 4.4. From the exact sequence in 4.6 and by 4.8 2.) we conclude that the bounded complex $C_{c\Psi}(M)$ has cohomology groups belonging to $\mathbf{D}^b_{\mathrm{perf}}(D)$. Then [Sta21, Tag 066U] implies that $C_{c\Psi}(M)$ itself belongs to $\mathbf{D}^b_{\mathrm{perf}}(D)$.

□

In chapter 6 we will require some flexibility concerning the constant c.

Lemma 4.10. *Let* $c \in K^\times$ *and define a character* $\rho : L^\times \to K^\times$ *by* $\rho(\pi_L) = c$ *and* $\rho_{|o_L^\times} = 1$. *Then the identity induces a* Γ_L*-equivariant isomorphism.*

$$C_\Psi(M) \cong C_{c\Psi}(M(\rho)).$$

Furthermore if M *is* L*-analytic then so is* $M(\rho)$ *and the isomorphism above is* $D(\Gamma_L, K)$*-equivariant.*

Proof. Since the character ρ is trivial on $o_L^\times$ the identity is Γ_L-equivariant. Furthermore we have $\varphi_{M(\rho)}(m) = c\varphi_M(m)$ and hence $\Psi_{M(\rho)} = c^{-1}\Psi_M$ which shows that the identity induces a morphism of complexes. The second part of the statement follows from the fact that ρ does not change the Γ_L-action and hence it remains L-analytic on $M(\rho)$. The $D(\Gamma_L, K)$-equivariance follows from continuity. □

Corollary 4.11. *Let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_K$. Then the complex $C_{c\Psi}(M/t_{LT}) = [M/t_{LT} \xrightarrow{c\Psi-1} M/t_{LT}]$ is perfect.*

Proof. Since twisting by a character that is trivial on $o_L^\times$ preserves the property that M is L-analytic by 4.10 we may without loss of generality assume $c = 1$. Because the complex is bounded it suffices to show that the cohomology groups are perfect. The cohomology groups are precisely kernel and cokernel of $\Psi - 1$ for which the statement has been shown in 3.30. □

4.3 Consistent complexes

When studying the Iwasawa cohomology $C_\Psi(M)$ the conceptual approach is to view the cohomology groups $H^i_{Iw}(M)$ as coherent sheaves on $D(U, K)$. We do however not know if M itself can be viewed as a sheaf on $D(U, K)$ in a suitable sense and we instead study the "sheaf" $D_n \mapsto C_\Psi(D_n \otimes_D M)$. We describe a framework for studying complexes of D modules whose cohomology groups are coadmissible D modules. A similar situation was studied by Berthelot and Ogus (cf. Appendix B in [BO78]) in the case where A is a noetherian ring that is I-adically complete and $A_n := A/I^n$. We adapt their setup for our purpose following [Pot13]. One can view a projective system of D_n modules as a sheaf on the ringed site $\mathbb{N}$ (where $\mathcal{O}(n) = D_n$) with the indiscrete topology (such that only isomorphism are coverings and thus every presheaf is a sheaf). In overblown terms given a projective system (A_n) of rings with $A := \varprojlim_n A_n$ we have a canonical morphism of topoi $f : \mathrm{Sh}(\mathbb{N}, A_\bullet) \to \mathrm{Sh}(\mathrm{pt}, A)$. Where f_* is just $\varprojlim_n$ and f^* will be described below. In order to describe the cohomology groups of complexes of sheaves on both sides we need to understand the respective derived functors. Fix a countable projective system $(A_n)_{n\in\mathbb{N}}$ of rings and denote by A their limit. We denote by $\mathrm{mod}(\mathbb{N}, A)$ the abelian category of inverse systems $(M_n)_n$ of abelian groups indexed by $\mathbb{N}$ such that each M_n is a A_n-module and the transition maps $M_{n+1} \to M_n$ are A_{n+1}-linear. Denote by $\mathbf{Rlim}$ the right-derived functor of the functor $(M_n) \mapsto \varprojlim_n M_n$ taking values in $\mathbf{D}(A)$. Observe that a morphism of complexes in $\mathrm{mod}(\mathbb{N}, A)$ is a quasi-isomorphism if and only if this is the case on every level and hence the projection to the n-th degree of the projective system induces a functor $\mathbf{D}(\mathrm{mod}(\mathbb{N}, A)) \to \mathbf{D}(A_n)$. We denote the image of $C \in \mathbf{D}(\mathrm{mod}(\mathbb{N}, A))$ by C_n.

Lemma 4.12. *Given $C = (C_n^\bullet)_n \in \mathbf{D}(\mathrm{mod}(\mathbb{N}, A_n))$ we have a canonical distinguished triangle*

$$\mathbf{Rlim} C \to \prod_n C_n^\bullet \to \prod_n C_n^\bullet \to \mathbf{Rlim} C[1]$$

in $\mathbf{D}(A)$. Its long exact sequence splits into short exact sequences

$$0 \to \mathbf{R}^1 \varprojlim_n H^{i-1}(C_n^\bullet) \to H^i(\mathbf{Rlim} C) \to \varprojlim_n H^i(C_n^\bullet) \to 0$$

Proof. See [Sta21, Tag 0CQD] together with [Sta21, Tag 0CQE]. □

Now let M be an A module and consider $S(M) := (A_n \otimes_A M)_n$. By taking the component wise base change of a morphism along $A \to A_n$ this defines a right-exact functor to $\mathrm{mod}(\mathbb{N}, A)$ and we denote by $\mathbf{L}S : \mathbf{D}^-(A) \to \mathbf{D}^-(\mathrm{mod}(\mathbb{N}, A))$ its left-derived functor. Here we remark that a complex in $\mathrm{mod}(\mathbb{N}, A)$ is bounded above if for some i_0 the cohomology groups vanish in degree $i \leq i_0$ in every step of the projective system. By construction $(\mathbf{L}SC)_n \simeq A_n \otimes^{\mathbb{L}}_A C_n$. We have constructed functors

$$\mathbf{D}^-(A) \underset{\overleftarrow{\mathbf{Rlim}}}{\overset{\mathbf{L}S}{\longrightarrow}} \mathbf{D}^-(\mathrm{mod}(\mathbb{N}, A)).$$

One can check that $\varprojlim$ and $S(-)$ are adjoint and hence by [Sta21, Tag 0DVC] so are their derived functors (restricted to the respective $\mathbf{D}^-$). Assume henceforth that each A_n (but not necessarily A) is Noetherian. In this case it makes sense to speak of the full triangulated subcategory $\mathbf{D}_{ft}(A_n)$ of objects in $\mathbf{D}(A_n)$ whose cohomology groups are A_n-finitely generated (cf. [Sta21, Tag 06UQ]).

Definition 4.13. *Let $C \in \mathbf{D}^-(\mathrm{mod}(\mathbb{N}, A))$.*

1. *We call C **quasi-consistent** if $A_n \otimes^{\mathbb{L}}_{A_{n+1}} C_{n+1} \to C_n$ is an isomorphism (in $\mathbf{D}(A_n)$) for every $n \in \mathbb{N}$.*
2. *C is called **consistent** if it is quasi-consistent and $C_n \in \mathbf{D}_{ft}(A_n)$.*

We denote by $\mathbf{D}^-_{con}(\mathrm{mod}(\mathbb{N}, A))$ the full subcategory of $\mathbf{D}^-(\mathrm{mod}(\mathbb{N}, A))$ of consistent objects.

The following result is [BO78, Corollary B.9].

Remark 4.14. *If A is Noetherian and I-adically complete for some ideal I, then $\mathbf{L}S$ and $\mathbf{Rlim}$ induce an equivalence of categories*

$$\mathbf{D}^-_{ft}(A) \cong \mathbf{D}^-_{con}(\mathrm{mod}(\mathbb{N}, A)).$$

We now specialise to the situation where A is a Fréchet-Stein algebra. We denote by $\mathbf{D}^-_{\mathcal{C}}(A)$ the full triangulated subcategory of objects in the bounded above derived category whose cohomology groups are coadmissible A-modules. This makes sense by [Sta21, Tag 06UQ] because an extension of two coadmissible modules is again coadmissible and hence the category of coadmissible modules is a weak Serre-subcategory of the category of all A-modules.

Proposition 4.15. *Let $A = \varprojlim A_n$ be a Fréchet-Stein algebra then the functor S is exact and the adjoint pair $S \dashv \mathbf{Rlim}$ restricts to an equivalence of categories*

$$\mathbf{D}^-_{\mathcal{C}}(A) \cong \mathbf{D}^-_{con}(\mathrm{mod}(\mathbb{N}, A))$$

Proof. By Lemma 1.55 $A \to A_n$ is flat and hence S is exact. Flatness also implies that the functors are well-defined since for a complex C of modules with coadmissible cohomology we have $A_n \otimes_A H^i(C) \cong H^i(A_n \otimes_A C)$ and the left-hand side is finitely generated by assumption. On the other hand if we have $(C_n)_n$ representing an object in $\mathbf{D}^-_{con}(\mathrm{mod}(\mathbb{N}, A))$ then due to flatness of A_n over A_{n+1} and quasi-consistency the natural morphism $A_n \otimes_{A_{n+1}} H^i(C_n) \to H^i(C_{n+1})$ is an isomorphism for every n. The assumption that $(C_n)_n$ is consistent asserts that $(H^i(C_n))_n$ is a coherent sheaf in the sense of Schneider and Teitelbaum and thus $\varprojlim_n (H^i(C_n))$ is a coadmissible A-module. The key observation of the proof is that $\lim^1(H^i(C_n)) = 0$ for any quasi-consistent C and any i by 1.55. This applied to the exact sequence in 4.12 shows that $H^i\mathbf{Rlim}(C_n)$ is coadmissible and hence the functor is well-defined. The same observation allows us to conclude that the natural maps (obtained from the adjunction) $S(\mathbf{Rlim}C_n) \to (C_n)_n$ and $M \to \mathbf{Rlim}(S(M))$ are quasi-isomorphism. We have

$$H^i(S(\mathbf{Rlim}C_m)) = (A_n \otimes_A H^i\mathbf{Rlim}C_m)_n = (A_n \otimes_A \varprojlim_m H^i(C_m))_n = (H^i(C_n))_n$$

using flatness in the first, 4.12 in the second and 1.55 in the last equation. For the second quasi-isomorphism we have

$$H^i(\mathbf{Rlim}(S(C))) = \varprojlim_n H^i(A_n \otimes_A C) = \varprojlim_n A_n \otimes_A H^i(C) = H^i(C)$$

using similar arguments and coadmissibility in the last equation.

□

4.4 Comparison to Iwasawa cohomology.

Proposition 4.15 gives us the correct framework to describe a comparison between Iwasawa cohomology and analytic cohomology of the Lubin-Tate deformation.

Remark 4.16. *The projective system $(C_{\Psi,Z}(\mathbf{Dfm}_n(M)))_n$ defines a consistent object in $\mathbf{D}(\mathrm{mod}(\mathbb{N}, D_n))$. In particular the cohomology groups $H^i(\mathbf{Rlim}(C_{\Psi,Z}(\mathbf{Dfm}_n(M))))$ are coadmissible D-modules for every i and*

$$H^i(\mathbf{Rlim}(C_{\Psi,Z}(\mathbf{Dfm}_n(M)))) \cong \varprojlim_n H^i_{\Psi,Z}(\mathbf{Dfm}_n(M)).$$

Proof. Consistency follows from 3.22 together with the fact that D is a Fréchet-Stein algebra. The latter cohomology groups are coadmissible by 4.15. The isomorphism follows from 4.12 using again 3.22. □

Lemma 4.17. *Let V be a finite dimensional K-linear U-representation. Then for $W = D_n \otimes_K V$ we have $H^0(U, W) = 0$ with respect to the U-action via $\gamma(1 \otimes m) = \delta_{\gamma^{-1}} \otimes \gamma m$.*

Proof. Fix a basis $w_1, \ldots w_d$ of W let $w \in W^U$ and write $w = \sum_{i=1}^d \lambda_i \otimes w_i$. Let $g \in U$ and define $G \in M_{\dim_K V}(K)$ via $gw_j = \sum_i G_{ij} w_i$. We compute

$$gw = \sum_{j=1}^{d} (g^{-1}\lambda_j \otimes gw_j) = \sum_{j=1}^{d} \sum_{i=1}^{d} G_{ij}(g^{-1}\lambda_j \otimes w_i).$$

Since we assumed that w is fixed by g and the decomposition with respect to the basis w_i is unique we conclude

$$\lambda_j = \sum_{i=1}^{d} G_{ji} g^{-1} \lambda_i.$$

Multiplying both sides by g we see $g\lambda_j \in \operatorname{span}_K(\lambda_i)$. Since this works for any choice of g we conclude that the λ_i span a finite-dimensional U-stable subspace of D_n. This is only possible if $\lambda_1 = \cdots = \lambda_d = 0$. □

Lemma 4.18. *Let $m \in \mathbb{N}$ then for the diagonal action of U we have $H^0(U, D_m \hat{\otimes}_K M^r) = 0$. In particular the kernel of Z acting diagonally is trivial.*

Proof. It suffices to show that there are no non-trivial U-invariant elements. By 1.78 there exists $n \gg 0$ such that M^r can be embedded into $D^+_{\mathrm{dif},n}(M^r)$ which is a projective finitely generated $(K \otimes_L L_n)[\![t_{LT}]\!]$-module. We claim that we have an injection

$$D_m \hat{\otimes}_K M^r \to \mathbb{D} := D_m \hat{\otimes}_K D^+_{\mathrm{dif},n}(M^r).$$

Since we do not know whether ι_n is strict we instead make use of [Eme17, 1.1.26] by rewriting $D_m \hat{\otimes}_K - = D_m^{(L)} \hat{\otimes}_L K \hat{\otimes}_K -$ with a suitable Banach algebra $D_m^{(L)}$ over L and using the associativity of projective tensor products from [BGR84, 2.1.7 Proposition 7] after reducing to the Banach case via 1.15. By applying again 1.15 it suffices to show that $\mathbb{D}/t_{LT}^k \mathbb{D}$ has no non-trivial U-invariants for each $k \geq 0$. Dévissage using the exact sequence

$$0 \to \mathbb{D}/t_{LT}\mathbb{D} \to \mathbb{D}/t_{LT}^k \mathbb{D} \to \mathbb{D}/t_{LT}^{k-1}\mathbb{D} \to 0,$$

induction on k and passing to the limit $\mathbb{D} = \varprojlim_k \mathbb{D}/t_{LT}^k \mathbb{D}$ shows that it suffices to prove the statement for $\mathbb{D}/t_{LT}\mathbb{D} = D_m \otimes_K (D^+_{\mathrm{dif},n}(M^r)/t_{LT} D^+_{\mathrm{dif},n}(M^r))$, where we use 1.11 and can omit the completion since the right-hand side is finite-dimensional Hausdorff. The statement now follows from 4.17.

□

Lemma 4.19. *The natural map*

$$\mathfrak{I} : \mathbf{Dfm}(M^r) \to M^r$$
$$\lambda \otimes m \mapsto \lambda m$$

is surjective and its kernel is the image of $Z \in D_n$ (acting diagonally).

Proof. Surjectivity is clear by definition. Observe that for any $y \in \mathbf{Dfm}(M^r)$ and any $\gamma \in U$ the element $(\gamma - 1)y$ lies in the kernel of $\mathfrak{I}$. Since Z lies in the closure of the augmentation ideal in D we conclude that $\operatorname{Im}(Z) \subseteq \ker(\mathfrak{I})$. In order to show $\ker(\mathfrak{I}) \subseteq \operatorname{Im}(Z)$ we will reduce to the case of elementary tensors via a series of technical arguments. We will show that an element of the form $\lambda \otimes m - 1 \otimes \lambda m$

belongs to the image of Z and its preimage can be bounded for each norm defining the Fréchet topology of $\mathbf{Dfm}(M^r)$ which, in particular, implies strictness with respect to the Fréchet topology. Note that the map $\mathfrak{I}$ admits a section $\mathfrak{S} : m \mapsto 1 \otimes m$ and any element y of the kernel can be written as $z - \mathfrak{S}\mathfrak{I}(z)$ for some z. As an intermediate step we consider an element of the form $\lambda \otimes m \in \ker(\mathfrak{I})$ with $\lambda \in D$. Fix a $\mathcal{R}^{[r,s]}$-module norm on $M^{[r,s]}$ and consider the tensor product norm induced by the norm on D_m and said norm on $D_m \hat{\otimes}_K M^{[r,s]}$ for $m \in \mathbb{N}_0$ We will show that there exists a constant C depending only on m and $\|-\|_{M^{[r,s]}}$ such that $\lambda \otimes m - 1 \otimes \lambda m = Z_{diag} x$ and $\|x\| \leq C \|\tilde{\lambda} \otimes m\|_{D_{m,\mathbb{Q}_p} \hat{\otimes}_K M^{[r,s]}}$, where $\tilde{\lambda}$ is any lift of λ in $D_{\mathbb{Q}_p,m}(U,K)$. Since Z is injective by 4.18 we obtain that x is uniquely determined and hence satisfies this bound with respect to the quotient norm on D_m. Let $\varepsilon = \sup_{\gamma \in U} \|\gamma - 1\|_{D_m}$. Choose $n \gg 0$ such that $\|\gamma - 1\|_{M^{[r,s]}} < \varepsilon$ for $\gamma \in \Gamma_n$. Before treating the general case assume that λ belongs to $D(\Gamma_n, K)$. Fix a $\mathbb{Z}_p$-Basis γ_i of Γ_n and set $\mathbf{b} := (\delta_{\gamma_i} - 1)_i$. By taking a preimage in $D_{\mathbb{Q}_p}(\Gamma_n, K)$ we can express λ as a convergent series

$$\lambda = \sum_{\mathbf{k} \in \mathbb{N}_0^d} a_k \mathbf{b}^{\mathbf{k}}$$

We compute

$$\lambda \otimes m - 1 \otimes \lambda m = \sum_{\mathbf{k} \in \mathbb{N}_0^d} a_k (\mathbf{b}^{\mathbf{k}} \otimes m - 1 \otimes \mathbf{b}^{\mathbf{k}} m)$$

The terms in degree $\mathbf{k} = 0$ cancel out and we shall estimate $\mathbf{b}^{\mathbf{k}} \otimes m - 1 \otimes \mathbf{b}^{\mathbf{k}} m$. Without loss of generality we can assume $\mathbf{k} \neq 0$ i.e. at least one term $\delta_{\gamma_i} - 1$ appears in $\mathbf{b}^{\mathbf{k}}$. We will show that each summand is in the image of the diagonal Z-map and estimate the norm of its Z-preimage. We first explain a dévissage procedure to arrive at a situation where we estimate terms of the form

$$\begin{aligned} (\gamma - 1)a \otimes b - a \otimes (\gamma - 1)b &= \gamma a \otimes b - a \otimes \gamma b && (4.1) \\ &= (\gamma^{-1} - 1)(a \otimes \gamma b) && (4.2) \\ &= ZG(Z)(a \otimes \gamma b) && (4.3) \end{aligned}$$

Where $\gamma \in \{\gamma_1, \ldots \gamma_d\}$ and the assumption on the operator norm of $\gamma - 1$ acting on $M^{[r,s]}$ asserts that $\|\gamma b\|_{[r,s]} = \|b\|_{[r,s]}$ and hence

$$\|G(Z)(a \otimes \gamma b)\| \leq C(\gamma) \|(\gamma - 1)a \otimes b\|$$

with $C(\gamma)$ depending on γ and $[r, s]$. Without loss of generality assume $\mathbf{k} = (k_1, \ldots, k_d)$ with $k_1 \neq 0$ and let $\mathbf{k}' = (k_1 - 1, \ldots, k_d)$. We rewrite

$$\begin{aligned} \mathbf{b}^{\mathbf{k}} \otimes m - 1 \otimes \mathbf{b}^{\mathbf{k}} m &= (\gamma_1 - 1)\mathbf{b}^{\mathbf{k}'} \otimes m - 1 \otimes (\gamma_1 - 1)\mathbf{b}^{\mathbf{k}'} m \\ &= (\gamma_1 - 1)\mathbf{b}^{\mathbf{k}'} \otimes m - \mathbf{b}^{\mathbf{k}'} \otimes (\gamma_1 - 1)m \\ &+ \mathbf{b}^{\mathbf{k}'} \otimes (\gamma_1 - 1)m - 1 \otimes (\gamma_1 - 1)\mathbf{b}^{\mathbf{k}'} m && (4.4) \end{aligned}$$

We see that $(\gamma_1 - 1)\mathbf{b}^{\mathbf{k}'} \otimes m - \mathbf{b}^{\mathbf{k}'} \otimes (\gamma_1 - 1)m$ is an expression of the form (4.1). While the remainder i.e. $\mathbf{b}^{\mathbf{k}'} \otimes (\gamma_1 - 1)m - 1 \otimes (\gamma_1 - 1)\mathbf{b}^{\mathbf{k}'}m$ can be rewritten as

$$\mathbf{b}^{\mathbf{k}'} \otimes m' - 1 \otimes \mathbf{b}^{\mathbf{k}'}m',$$

where $m' = (\gamma_i - 1)m$ and by the assumption on the operator norm we have

$$\|\mathbf{b}^{\mathbf{k}'} \otimes m'\| \leq \|\mathbf{b}^{\mathbf{k}} \otimes m\|.$$

The remainder vanishes as soon as $\mathbf{k}' = 0$ and if $\mathbf{k}' \neq 0$ we may again find an index, that is not zero and apply the same procedure to, in the end, express $\mathbf{b}^{\mathbf{k}} \otimes m - 1 \otimes \mathbf{b}^{\mathbf{k}}m$ as a finite sum of elements of the form from (4.1) more explicitly we can group them as

$$\mathbf{b}^{\mathbf{k}} \otimes m - 1 \otimes \mathbf{b}^{\mathbf{k}}m = \sum_{i=1}^{d} \sum_{j=1}^{e_i} (\gamma_i - 1)a_{ij} \otimes b_{ij} - a_{ij} \otimes (\gamma_i - 1)b_{ij},$$

Where the elements a_{ij}, b_{ij} are not canonical and depend on the order in which we reduce the components of $\mathbf{k}$ in the inductive procedure. Nonetheless our construction asserts that each $(\gamma_i - 1)a_{ij} \otimes b_{ij}$ is bounded above by $\mathbf{b}^k \otimes m$. Using (4.1) we can write

$$\mathbf{b}^{\mathbf{k}} \otimes m - 1 \otimes \mathbf{b}^{\mathbf{k}}m = Zx_{\mathbf{k}},$$

where

$$\|x_{\mathbf{k}}\|_{D_n \hat{\otimes} M^{[r,s]}} \leq C \|\mathbf{b}^{\mathbf{k}} \otimes m\|_{D_n(\mathbb{Q}_p) \hat{\otimes} M^{[r,s]}}$$

with a suitable constant C depending only on $[r, s]$. By Lemma 4.18 Z is injective and hence the element x_k is uniquely determined. Note that a priori (4.1) produces constants for each γ_i but we can take the supremum over all those constants. In particular

$$x := \sum_{\mathbf{k} \in \mathbb{N}_0^d} a_k x_{\mathbf{k}}$$

converges to an element in $D_n \hat{\otimes}_K M^r$ satisfying $Zx = \lambda \otimes m - 1 \otimes \lambda m$ and $\|x\|_{D_n \hat{\otimes} M^{[r,s]}} \leq C \|\lambda \otimes m\|_{D_n(\mathbb{Q}_p) \hat{\otimes} M^{[r,s]}}$ and because this estimate holds for any preimage of λ and x is uniquely determined by the injectivity of Z we also obtain $\|x\|_{D_n \hat{\otimes} M^{[r,s]}} \leq C \|\lambda \otimes m\|_{D_n \hat{\otimes} M^{[r,s]}}$. Now assume $\lambda \in D(U, K)$ and decompose $\lambda = \sum_{g \in U/\Gamma_n} g\lambda_g$ with $\lambda_g \in D(\Gamma_n, K)$ we obtain

$$\begin{aligned} g\lambda_g \otimes m - 1 \otimes g\lambda_g m &= g\lambda_g \otimes m - \lambda_g \otimes gm + \lambda_g \otimes gm - 1 \otimes \lambda_g gm \\ &= (g^{-1} - 1)_{diag}(\lambda_g \otimes gm) + \lambda_g \otimes gm - 1 \otimes \lambda_g gm \qquad (4.5) \end{aligned}$$

The case $\lambda_g \otimes gm - 1 \otimes \lambda_g gm$ has been treated above and recall that $g^{-1} - 1$ is divisible by Z in D (and hence also in every D_m). Combining the case treated above with an argument similar to the one after (4.1) we conclude that there exists a unique element x such that $\lambda \otimes m - 1 \otimes \lambda m = Zx$ and a constant C depending on $[r, s]$ such that $\|x\|_{D_n \hat{\otimes} M^{[r,s]}} \leq C \|\lambda \otimes m\|_{D_n \hat{\otimes} M^{[r,s]}}$. If we pass from $M^{[r,s]}$ to $M^{[r,s']}$ the n that we chose before might no longer satisfy the desired bound on the operator norm and we might be required to pass to a subgroup satisfying the corresponding bound. Similarly if we pass from D_m to D_{m+1} we have $\|\gamma - 1\|_{D_{m+1}} \le \|\gamma - 1\|_{D_m}$ and given $[r, s]$ we might need to pass to a smaller subgroup $\Gamma_{\tilde{n}} \subset U$ in order to achieve the estimate $\|\gamma - 1\|_{M^{[r,s]}} < \|\gamma - 1\|_{D_{m+1}}$ that we used in the preceding computations. In both those cases a consideration analogous to (4.5) leads to the existence of a constant $C(m, [r, s'])$ such that

$$\|x_{\mathbf{k}}\|_{D_m \hat{\otimes} M^{[r,s']}} \le C(m, [r, s'])\|\mathbf{b}^{\mathbf{k}} \otimes m\|_{D_m(\mathbb{Q}_p)\hat{\otimes} M^{[r,s']}}$$

implying the convergence of x with respect to the Fréchet topology and the estimate

$$\|x\|_{D_m \hat{\otimes} M^{[r,s]}} \le C(m, [r, s'])\|\lambda \otimes m\|_{D_m \hat{\otimes} M^{[r,s]}}. \tag{4.6}$$

Now consider a general element y of the kernel of $\mathfrak{I}$ and write it as a convergent series

$$y = \sum_{i=0}^{\infty} \lambda_i \otimes m_i.$$

Since it belongs to the kernel we have

$$y = y - \mathfrak{S}(\mathfrak{I}(y)) = \sum_{i=0}^{\infty} (\lambda_i \otimes m_i - 1 \otimes \lambda_i m_i)$$

and the preceding discussion shows that each $\lambda_i \otimes m_i - 1 \otimes \lambda_i m_i$ belongs to the image of Z_{diag} and can be written as

$$\lambda_i \otimes m_i - 1 \otimes \lambda_i m_i = Z x_i.$$

Clearly if $x = \sum_{i=0}^{\infty} x_i$ converges then it satisfies $Zx = y$. The convergece of x with respect to the Fréchet topology defined by the tensor product norms on $D_m \hat{\otimes}_K M^{[r,s]}$ follows from the convergence of the series defining y and the estimates (4.6). □

Definition 4.20. *Let F be a topological D-module whose underlying K-vector space is Fréchet. We define*

$$D_n \hat{\otimes}_D F$$

as the completion of $D_n \otimes_D F$ with respect to the quotient topology of $D_n \otimes_{K,\pi} F$. For a D-module whose underlying K-vector space is an LF-space $E = \varinjlim E_n$ we define

$$D_n \tilde{\otimes}_D E := \varinjlim_r D_n \hat{\otimes}_D E^r.$$

We do not know whether $D_n \tilde{\otimes}_D M$ is complete for a (φ_L, Γ_L)-module M. Even if we knew that it was complete, we would run into subtleties concerning commutation of completion and cohomology since these spaces are in general not metrizable.

Lemma 4.21. *We have for each $r \in [r_0, 1)$ a strict exact sequence of D-modules*

$$0 \to D\hat{\otimes}_K M^r \xrightarrow{Z} D\hat{\otimes}_K M^r \xrightarrow{\mu} M^r \to 0, \tag{4.7}$$

where the $D\hat{\otimes}_K M^r$ is viewed as D-module via the left tensor component and μ is given by $\mu(\lambda \otimes m) = \lambda m$. Which induces for every m compatible exact sequences

$$0 \to D_m \otimes_D (D\hat{\otimes}_K M^r) \xrightarrow{\mathrm{id}\otimes Z} D_m \otimes_D (D\hat{\otimes}_K M^r) \xrightarrow{\mathrm{id}\otimes\mu} D_m \otimes_D M^r \to 0, \tag{4.8}$$

$$0 \to D_m\hat{\otimes}_K M^r \xrightarrow{Z} D_m\hat{\otimes}_K M^r \xrightarrow{\mu} D_m\hat{\otimes}_D M^r \to 0 \tag{4.9}$$

and

$$0 \to D_m\hat{\otimes}_K M \xrightarrow{Z} D_m\hat{\otimes}_K M \xrightarrow{\mu} D_m\tilde{\otimes}_D M \to 0. \tag{4.10}$$

The sequence 4.9 is strict for every m.

Proof. One first checks that all maps are D-linear. The exactness of the first sequence was proved in 4.19. On the one hand the operator μ is a continuous surjection of Fréchet spaces and hence strict. On the other hand, since the image of Z is the kernel of a continuous map of Fréchet spaces, it is closed and hence itself a Fréchet space. By the same argument Z is a continuous surjection onto $\mathrm{Im}(Z)$ and hence a homeomorphism on its image. The exactness of the second sequence is clear because D_n is flat over D. The modules appearing in (4.8) endowed with the quotient topology from a surjection from $D_m \otimes_{K,\pi} (D\hat{\otimes}_K M^r)$ are not necessarily hausdorff (hence in particular not necessarily metrizable). Nonetheless the Hausdorff completion of $D_m \otimes_D (D\hat{\otimes}_K M^r)$ can naturally be identified with $D_m\hat{\otimes}_K M^r$ and we can argue using the maximal Hausdorff quotients [1] as follows. The Hausdorff quotient $\mathcal{X}$ of $D_m \otimes_D (D \otimes M^r)$ can be embedded into $D_m\hat{\otimes}_K M^r$ and the map induced by $\mathrm{id}\otimes Z$ is continuous and strict on the dense subset $D\hat{\otimes}_K M^r$ by the preceding (4.7) and hence strict by 1.9. As a quotient of a metrizable space by a closed space $\mathcal{X}$ is again metrizable and we conclude using 1.8 that

$$D_m\hat{\otimes}_K M^r \xrightarrow{Z} D_m\hat{\otimes}_K M^r$$

is strict, injective and its cokernel is the Hausdorff completion of $D_m \otimes_D M^r$ since a strict map of Fréchet spaces has closed image. This gives the desired (4.9). Finally passing to direct limits produces the sequence 4.10 using 1.23 to see that $\varinjlim_r D_m\hat{\otimes}_K M^r$ is complete.

Definition 4.22. *We define the complex*

$$C_\Psi(M) : M \xrightarrow{\Psi-1} M$$

concentrated in degrees $1, 2$ and we call its cohomology groups the ***Iwasawa cohomology of*** *M. Analogously we define $C_{c\Psi}(M)$ and $C_{c\Psi}(M^r)$ for $r \in [0, 1)$, $c \in K^\times$.*

[1] For a topological group G the maximal Hausdorff quotient is defined as $G/\overline{\{1_G\}}$.

□

Lemma 4.23. *Assume $C_\Psi(M)$ has coadmissible cohomology groups. Then the natural map*

$$D_n \otimes_D M \to D_n \tilde{\otimes}_D M$$

induces a quasi-isomorphism

$$D_n \otimes_D C_\Psi(M) = C_\Psi(D_n \otimes_D M) \to C_\Psi(D_n \tilde{\otimes}_D M)$$

Proof. Taking cohomology commutes with colimits and thus it suffices to show that

$$D_n \otimes_D M^r \to D_n \hat{\otimes}_D M^r$$

is a quasi-isomorphism for sufficiently large r. The groups $D_n \hat{\otimes}_D M^r$ are metrizable and by 1.8 and strictness of $\Psi - 1$ taking kernels and cokernels commutes with completion i.e. $H^i_\Psi(D_n \otimes_D M^r) \cong H^i_\Psi(D_n \hat{\otimes}_D M^r)$. By assumption the cohomology groups are coadmissible and thus $D_n \otimes_D H^i_\Psi(M)$ is D_n-finite, complete and any submodule is itself D_n finite and complete by 1.6 because D_n is Noetherian. This implies $D_n \otimes_D H^0_\Psi(M^r) \cong D_n \hat{\otimes}_D H^0_\Psi(M^r)$. Regarding H^1 the proof of 3.32 shows that for all sufficiently large r we have that $M^r/(\Psi - 1)$ is K-finite therefore D-finitely generated and is in addition Hausdorff by the strictness of $\Psi - 1$. We conclude $D_n \otimes_D H^1_\Psi(M^r) \cong D_n \hat{\otimes}_D H^1_\Psi(M^r)$ (for $r \gg 0$ as in the proof of 3.32). □

Theorem 4.24 (Comparison between Herr- and Iwasawa-cohomology)**.** *Consider the complexes*

$$(C_{\Psi,Z}(\mathbf{Dfm}_n(M)))_{n\in\mathbb{N}}$$

and

$$(C_\Psi(D_n \tilde{\otimes}_D M))_{n\in\mathbb{N}}$$

in $\mathrm{mod}(\mathbb{N}, D)$. *There is a canonical compatible family of morphisms*

$$\mathrm{Comp}_{IW}(C_{\Psi,Z}(\mathbf{Dfm}_n(M)))_{n\in\mathbb{N}} \to (C_\Psi(D_n \tilde{\otimes}_D M))_{n\in\mathbb{N}}$$

induced by the exact sequences (4.8). If the cohomology groups of $C_\Psi(M)$ are coadmissible as D-modules we further obtain canonical compatible quasi-isomorphisms

$$C_{\Psi,Z}(\mathbf{Dfm}_n(M)) \simeq C_\Psi(D_n \tilde{\otimes}_D M),$$

which together with the maps induced by the natural maps $C_\Psi(M) \to \mathbf{Rlim}(C_\Psi(D_n \otimes_D M))$ and $(C_\Psi(D_n \otimes_D M))_n \to (C_\Psi(D_n \tilde{\otimes}_D M))_n$ induce an isomorphism in $\mathbf{D}(D)$

$$\mathbf{Rlim} C_{\Psi,Z}(\mathbf{Dfm}_n(M)) \simeq C_\Psi(M)$$

and, in particular, isomorphisms

$$\varprojlim_n H^i_{\Psi,Z}(\mathbf{Dfm}_n(M)) \cong H^i_{Iw}(M).$$

Proof. We have seen in 4.9 that there exists a compatible family of surjections $\mu_n : \mathbf{Dfm}_n(M) \to D_n\tilde{\otimes}_D M$ whose kernel is the image of the diagonal Z-map. Rewrite $C_{\Psi,Z}(\mathbf{Dfm}_n(M))$ as a total complex of the double complex

$$\begin{array}{ccc} \mathbf{Dfm}_n(M) & \xrightarrow{\Psi-1} & \mathbf{Dfm}_n(M) \\ \downarrow{\scriptstyle Z} & & \downarrow{\scriptstyle -Z} \\ \mathbf{Dfm}_n(M) & \xrightarrow{\Psi-1} & \mathbf{Dfm}_n(M) \end{array} \tag{4.11}$$

and consider $C_\Psi(D_n\tilde{\otimes}_D M)$ as the total complex of the "double complex"

$$\begin{array}{ccc} 0 & \xrightarrow{\Psi-1} & 0 \\ \downarrow{\scriptstyle Z} & & \downarrow{\scriptstyle -Z} \\ D_n\tilde{\otimes}_D M & \xrightarrow{\Psi-1} & D_n\tilde{\otimes}_D M \end{array}$$

Applying μ_n in the lower row and zero in the upper row of (4.11) induces a surjective morphism of double complexes with kernel

$$\begin{array}{ccc} \mathbf{Dfm}_n(M) & \xrightarrow{\Psi-1} & \mathbf{Dfm}_n(M) \\ \downarrow{\scriptstyle Z} & & \downarrow{\scriptstyle -Z} \\ \mathrm{Im}(Z) & \xrightarrow{\Psi-1} & \mathrm{Im}(Z) \end{array}.$$

The kernel double-complex has exact columns and hence acyclic total complex by A.15. Since passing to total complexes is exact we obtain the desired compatible family of quasi-isomorphisms. By 4.23 the natural map

$$D_n \otimes_D C_\Psi(M) \to C_\Psi(D_n\tilde{\otimes}_D M)$$

is a quasi-isomorphism.
Composing the first quasi-isomorphism with the inverse of the latter gives isomorphism in $\mathbf{D}(D)$

$$\mathbf{Rlim} C_{\Psi,Z}(\mathbf{Dfm}_n(M)) \simeq \mathbf{Rlim} C_\Psi(D_n \otimes_D M).$$

Using the coadmissibility assumption on $C_\Psi(M)$ by 4.15 the natural map

$$C_\Psi(M) \to \mathbf{Rlim}((C_\Psi(D_n \otimes_D M))_n)$$

is an isomorphism in $\mathbf{D}(D)$.
By Remark 4.16 we have $H^i(\mathbf{Rlim} C_{\Psi,Z}(\mathbf{Dfm}_n(M))) \cong \varprojlim_n H^i_{\Psi,Z}(\mathbf{Dfm}_n(M))$ and putting everything together we obtain

$$H^i(\mathbf{Rlim} C_{\Psi,Z}(\mathbf{Dfm}_n(M))) \cong \varprojlim_n H^i_{\Psi,Z}(\mathbf{Dfm}_n(M)) \cong \varprojlim_n D_n \otimes_D H^i_\Psi(M) \cong H^i_\Psi(M).$$

□

Chapter 5

Explicit results in the rank one and trianguline case

In this chapter we study $M^{\Psi=1}$ for rank one modules of the form $\mathcal{R}_K(\delta)$. The ideas are based on [Col08], [Col16], [FX12] and [Che13]. Some small adjustments are required in order to incorporate 2.19. We are mostly interested in showing that $M^{\Psi=1}$ is finitely generated and coadmissible.

Definition 5.1. *An L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_K$ is called trianguline if it is a successive extension of modules of the form $\mathcal{R}_K(\delta)$ with locally L-analytic characters $\delta\colon L^\times \to K^\times$.*

Fix as usual a subgroup $U \subset \Gamma_L$ isomorphic to o_L. Recall that the space $C^{an}(U, K)$ of locally L-analytic functions $U \to K$ is reflexive (see proof of 1.31) and its strong dual is by definition $D(U, K)$. On the other hand $\mathcal{R}_K^+$ is reflexive and its strong dual is $\mathcal{R}_K/\mathcal{R}_K^+$ via the residue pairing from Proposition 1.75.

Lemma 5.2. *By transport of structure along $\mathcal{R}_K^+ \cong D(U, K)$ we obtain a (φ_L, U)-semi-linear strict exact sequence*

$$0 \to D(U, K) \to \mathcal{R}_K \to C^{an}(U, K) \otimes \chi^{-1} \to 0,$$

where $\chi(\pi) = \frac{\pi}{q}$ and $\chi(a) = a$ for $a \in o_L^\times$.

Proof. See [Col16, Théorème 2.3]. □

Remark 5.3. *Let $\delta : L^\times \to K^\times$ be a locally L-analytic character and $\mathcal{R}_K(\delta) = \mathcal{R}_K e_\delta$ the corresponding free rank 1 module. Then $\mathcal{R}_K^+(\delta) := \mathcal{R}_K^+ e_\delta$ is a (φ_L, Γ_L)-stable submodule and fits into a short exact sequence*

$$0 \to \mathcal{R}_K^+(\delta) \to \mathcal{R}_K(\delta) \to C^{an}(U, K) \otimes \chi^{-1}\delta \to 0,$$

where $\chi(\pi) = \frac{\pi}{q}$ and $\chi(a) = a$ for $a \in o_L^\times$.

Proof. The stability follows because the image of δ is contained in $(\mathcal{R}_K^+)^\times$. The sequence is obtained by twisting the sequence from 5.2. □

Lemma 5.4. *Any element $\zeta \in C^{an}(o_L, K)$ admits a unique expansion of the form*

$$\zeta = \sum_{k\geq 0} a_k \begin{bmatrix} x \\ k \end{bmatrix},$$

where $\begin{bmatrix} x \\ k \end{bmatrix} : o_L \to K$ is the polynomial function in x defined via

$$\eta(x, T) = \sum_{k\geq 0} \begin{bmatrix} x \\ k \end{bmatrix} T^k$$

and the coefficients satisfy $\lim_{k\to\infty}|a_k|r^k$ for some $r > 1$.

Proof. See [ST01, Theorem 4.7]. □

The following lemmas are essentially an L-analytic version of [Che13, Lemme 2.9]. We let Ψ act on $C^{an}(o_L, K)$ as $\Psi(f)(x) = f(\pi_L x)$.

Lemma 5.5. *Let $\alpha \in K^\times$. Denote by $x \in C^{an}(o_L, K)$ the function $x \mapsto x$.*

1. *If N is such that $|\alpha\pi_L^N| < 1$ then $1 - \alpha\Psi$ is bijective on $x^N C^{an}(o_L, K)$.*
2. *The cokernel of $1 - \alpha\Psi$ acting on $\mathcal{R}_K^+$ is at most one-dimensional over K.*
3. *The cokernel of the inclusion $(\mathcal{R}_K^+)^{\alpha\Psi=1} \to \mathcal{R}_K^{\alpha\Psi=1}$ is finite dimensional over K.*

Proof. For $h \in \mathbb{N}_0$ denote by C_h^{an} the subspace of functions that are globally analytic on $a + \pi_L^h o_L$ for each $a \in o_L/\pi_L^h o_L$. Every element $f \in C^{an}(o_L, K)$ belongs to some C_h^{an} and from the definitions one has $\Psi(x^N C_h^{an}) \subset x^N C_{h-1}^{an}$ for $h \geq 1$. If $f \in x^N C_0^{an}$ then expanding f as a power series yields

$$|\Psi(f)|_{o_L} \leq |\pi_L^N||f|_{o_L} \tag{5.1}$$

with respect to the sup-norm on o_L. Let $f \in x^N C_h^{an}$ then $\Psi^h(f) \in C_0^{an}$ and the estimate (5.1) together with the assumption on N shows that the series

$$\sum_{k=0}^{\infty} \alpha^m \Psi^m$$

converges to an inverse of $1-\alpha\Psi$ on $x^N C_h^{an}$. The claim follows because $C^{an}(o_L, K) = \varinjlim_h C_h^{an}$.

For the second statement choose $N \gg 0$ such that

$$F := \sum_{i=0}^{\infty} \alpha^{-i}\varphi^i$$

converges to a continuous operator on $T^N\mathcal{R}_K^+$. The existence of such N can be seen using that T^N tends to zero as $N \to \infty$ and that φ_L is contractive with respect to the norms $|-|_{[0,r]}$ for any $0 < r < 1$ as a consequence of Remark 1.70. Given $h \in T^n\mathcal{R}_K^+$ we observe that $(1-\alpha\Psi)(F(h)) = \alpha\Psi(h)$. Using that Ψ is surjective on $\mathcal{R}_K^+$, we can deduce that h belongs to the image of $1-\alpha\Psi$. By writing $\mathcal{R}_K^+ = \bigoplus_{k=0}^{N-1} Kt_{LT}^k \oplus T^K\mathcal{R}_K^+$ we conclude that it remains to show that at most one t_{LT}^k is not contained in the image of $1-\alpha\Psi$. Because $\Psi(t_{LT}^i) = \pi_L^{-i}t_{LT}^i$ there can be at most one i_0 such that $(1-\alpha\Psi)(t_{LT}^{i_0}) = 0$. For every $i \neq i_0$ we have $(1-\alpha\Psi)(t_{LT}^i) = (1-\alpha\pi_L^{-i})t_{LT}^i$ with $(1-\alpha\pi_L^{-i}) \in K^\times$.
For the third statement observe that Lemma 5.2 and the Snake Lemma gives a short exact sequence

$$0 \to (\mathcal{R}_K^+)^{\alpha\Psi=1} \to \mathcal{R}_K^{\alpha\Psi=1} \to C^{an}(U,K)^{\alpha\frac{q}{\pi}\Psi=1}.$$

The image of the last map is finite dimensional by 1.) because any element in C^{an} can be written as a sum of a polynomial of degree $\leq N-1$ and an element of $x^NC^{an}(o_L,K)$ by Lemma 5.4. □

Lemma 5.6. *Let $\delta : L^\times \to K^\times$ be a locally L-analytic character and $M := \mathcal{R}_K(\delta)$ the corresponding free rank 1 module. Let $M^+ := \mathcal{R}_K^+(\delta)$.*

1. *We have $(M^+)^{\psi=0} = D(\Gamma_L,K)\eta(1,T)\varphi(e_\delta)$. In particular $(M^+)^{\Psi=0}$ is free of rank 1 over $D(\Gamma_L,K)$.*
2. *The map $1-\varphi\colon (M^+)^{\Psi=1} \to (M^+)^{\Psi=0}$ has finite dimensional kernel and cokernel.*

Proof. 1. follows from the explicit description in Corollary 2.21.
For 2. let $\eta(1,T)\varphi(m) \in (M^+)^{\Psi=0}$ with some $m = fe_\delta \in M^+$. Let $N \in \mathbb{N}$ and write $f = f_0 + T^Ng$ according to the decomposition $\mathcal{R}_K^+ = \bigoplus_{k=0}^{N-1} t_{LT}^k + T^N\mathcal{R}_K^+$. Choosing N large enough (like in the proof of Lemma 5.5) we can ensure that

$$h := \sum_{k=0}^{\infty} \delta(\pi_L)^k\varphi^k(\eta(1,T)\varphi((T^Ng))$$

converges independently of $g \in \mathcal{R}_K^+$. The element $m' := he_\delta \in M^+$ satisfies $(\varphi - 1)(m') = \eta(1,T)(\varphi(T^Ng)e_\delta)$. In particular $(\varphi_L-1)(m') \in M^{\Psi=0}$ and hence necessarily $\Psi(m') = m'$. Finite dimensionality of the kernel is immediate from Remark 4.8. For the codimension of the image $(M^+)^{\Psi=1}$ in $(M^+)^{\Psi=0}$ our proof thus far shows that any element in $\eta(1,T)(\varphi(T^NM^+))$ lies in the image of $\varphi_L - 1$. We use the analogue of the decomposition from Proposition 2.8 for M^+ to conclude that the codimension of the image is at most $N[\Gamma_L : \Gamma_1] = N(q-1)$. □

Proposition 5.7. *Let $\delta\colon L^\times \to K^\times$ be a locally L-analytic character and let $M = \mathcal{R}_K(\delta)$ then $M^{\Psi=1}$ is a finitely generated coadmissible $D(U,K)$-module of rank $[\Gamma_L : U]$. In particular, $C_\Psi(M)$ is perfect.*

Proof. Applying Lemma 5.5 and Lemma 5.6 we conclude that $M^{\Psi=1}$ fits into an exact sequence

$$0 \to (M^+)^{\Psi=1} \to M^{\Psi=1} \to V \to 0,$$

with a $D(U,K)$-module V whose underlying K-vector space is finite-dimensional. By Lemma 4.7 V is coadmissible and evidently torsion as a $D(U,K)$-module. Because the category of coadmissible module is abelian we conclude that it suffices to show that $(M^+)^{\Psi=1}$ is coadmissible of the desired rank. For M^+ we have by Lemma 5.6 an exact sequence of the form

$$0 \to V_1 \to (M^+)^{\Psi=1} \xrightarrow{\varphi-1} (M^+)^{\Psi=0} \to V_2 \to 0$$

with two $D(U,K)$-modules whose underlying K-vector-spaces are finite-dimensional. From this exact sequence and Lemma 5.6 it is clear that the rank is precisely $[\Gamma_L : U]$. Again V_2 is coadmissible by Lemma 4.7 and the image of $(M^+)^{\Psi=1}$ is the kernel of a map between coadmissible modules and hence coadmissible. By [ST03, Lemma 3.6] the image is closed in the canonical topology. Because $(M^+)^{\Psi=0}$ is finitely generated projective we obtain that $(1-\varphi)((M^+)^{\Psi=1})$ is finitely generated by [BSX20, Lemma 1.1.9]. Now the short exact sequence

$$0 \to V_1 \to (M^+)^{\Psi=1} \to (1-\varphi)((M^+)^{\Psi=1}) \to 0$$

proves that $(M^+)^{\Psi=1}$ is finitely generated and coadmissible. Perfectness follows from Proposition 4.9. □

Theorem 5.8. *Let M be a trianguline L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_K$. Then $C_{c\Psi}(M)$ is a perfect complex of $D(\Gamma_L, K)$-modules for any constant $c \in K^\times$.*

Proof. By applying Lemma A.21 to the cohomology groups of $C_\Psi(M)$ we conclude that perfectness as a complex of $D(\Gamma_L, K)$-modules follows from perfectness as a complex of $D(U,K)$-modules. For $c = 1$ this is a corollary of Proposition 5.7. Because twisting by a character preserves the property of M being trianguline the general statement follows from Lemma 4.10. □

Chapter 6

Towards perfectness of $C_\Psi(M)$ and a remark on the Euler-Poincaré formula

6.1 Perfectness of $C_\Psi(M)$.

In the cyclotomic case the perfectness of $C_\Psi(M)$ as a complex of $D(U,K)$-modules is obtained by an inductive procedure from the étale case. In the cyclotomic étale case the heart $C = (1-\varphi)(M)$ is free over $D_{\mathbb{Q}_p}(\Gamma_{\mathbb{Q}_p}, \mathbb{Q}_p)$ of the same rank as M (cf. [Col10, Proposition V.1.18]) and an induction over the Harder-Narasimhan slopes implies the general result. In our situation we run into several problems. Most notably the corresponding result concerning the heart is not known. Furthermore the theory of slope filtrations makes heavy use of the Bézout property of the coefficient rings. By passing to the large extension K/L we leave the situation of Kedlaya's slope theory and also do not have the equivalence of categories at our disposal for modules of slope zero. We interpret the passage to K as a technical tool to understand (φ_L, Γ_L)-modules coming from $\mathcal{R}_L$ employed for example in [BSX20] and [FX12]. Hence we put a special emphasis on those coming from $\mathcal{R}_L$. We show that the induction over Harder-Narasimhan slopes as in [KPX14] works in essentially the same way. To do so we require some technical lemmas concerning the (Ψ, ∇)-cohomology of M that might be interesting in their own right.

Conjecture 6.1. *Let M_0 be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_L$. Then the complex $C_\Psi(K\hat{\otimes}_L M_0)$ is a perfect complex of $D(\Gamma_L, K)$-modules.*

So far we have not concerned ourselves with the étale picture. Schneider and Venjakob showed the following

Theorem 6.2. *Let $V \in \operatorname{Rep}_{o_L}(G_L)$, let $H^i_{Iw}(L_\infty/L, V) := \varprojlim_{L\subset F\subset L_\infty} H^i(F,V)$ and let M_{LT} be the étale Lubin-Tate-(φ_L, Γ_L)-module over $\mathbf{A}_L$ attached to $V(\chi_{LT}\chi_{cyc}^{-1})$ endowed with its integral ψ-operator (i.e. $\psi \circ \varphi = \frac{q}{\pi}$).*

1. $H^i(L_\infty/L, V)$ *vanishes for* $i \neq 1, 2$.

2. $H^2(L_\infty/L, V)$ *is* o_L*-finite.*

3. $H^1(L_\infty/L, V)$ *is* $\Lambda := o_L[\![\Gamma_L]\!]$*-finite.*

4. $H^i_{Iw}(L_\infty/L, V)$ *is computed by the complex*

$$M_{LT} \xrightarrow{\psi-1} M_{LT}$$

concentrated in degrees 1 *and* 2.

Proof. See [SV15, Lemma 5.12 and Theorem 5.13]. □

In order to connect this result to the present situation assume first that M_{LT} is L-analytic and thus a fortiori overconvergent. An analogue of [CC99, Proposition III.3.2] (cf. [SV20, A.53]) allows us to view $M_{LT}^{\psi=1}[1/p]$ as a Λ-submodule of $\mathbb{D}^\dagger(V)$[1] and thus as a Λ-submodule of $\mathbb{D}^\dagger_{rig}(V)^{\psi=1}$. Let $M_{rig} := \mathbb{D}^\dagger_{rig}(V)$. Since M_{rig} is L-analytic we obtain that $M_{rig}^{\psi=1}$ is even a $D(\Gamma_L, L)$-module and thus a natural map

$$D(\Gamma_L, L) \otimes_\Lambda M_{LT}^{\psi=1} \xrightarrow{\text{comp}} {M_{rig}}^{\psi=1}.$$

This leads us to the following natural conjecture.

Conjecture 6.3. *Let* $M_{LT} = \mathbb{D}(V)$ *be an étale* L*-analytic* (φ_L, Γ_L)*-module over* $\mathbf{A}_L$ *and let* $M_{rig} := \mathbb{D}^\dagger_{rig}(V)$ *then the natural map*

$$D(\Gamma_L, L) \otimes_\Lambda M_{LT}^{\psi=1} \xrightarrow{comp} {M_{rig}}^{\psi=1}$$

is surjective.

A stronger form of this conjecture would be to require bijectivity. In the classical case the map above is bijective by [Col10, Proposition V.1.18]. We will show that this weaker version is sufficient for conjecture 6.1. Observe that Theorem 3.32 works over any base field and hence $M_{rig}/(\psi - 1)$ is finite L-dimensional. This implies that $\psi - 1$ is strict by Lemma 1.10 and because K is an L-Banach space of countable type over L we conclude using Lemma 1.11

$$(M_{rig} \hat{\otimes}_L K)^{\psi=1} = (M_{rig}^{\psi=1}) \hat{\otimes}_L K.$$

The results of Schneider and Venjakob in the étale case suggest that $\psi - 1$ is the "correct" operator to study Iwasawa cohomology. This leads us to believe that the complex defined using ψ is well-behaved while in order to obtain a quasi-isomorphism to the (φ_L, Z)-complex we need to work with the Ψ-complex with the left-inverse operator. Our philosophy is that the $c\Psi$-complex for some constant c can be reinterpreted as the Ψ-complex of a module twisted by a (non-étale) character. In particular, if

[1] $\mathbb{D}^\dagger(V)$ was defined in 1.82.

the analogue of Conjecture 6.1 concerning the ψ-complex is true for all analytic (not necessarily étale) (φ_L, Γ_L)-modules coming from $\mathcal{R}_L$, then Lemma 4.10 asserts that Conjecture 6.1 itself is true.
We require an analogue of the fact that there exists an isoclinic module E of slope $1/d$ that is a successive extension of $\mathcal{R}_L(x) \cong \mathcal{R}_L t_{LT}$ by $d-1$ copies of $\mathcal{R}_L$ requiring in addition that the latter module is L-analytic. This allows us to argue inductively over the slopes of a module. While the (φ_L, Z)-cohomology enjoys a number of nice properties, its biggest downfall is the fact that it can only be defined over the large field K that is not discretely valued and hence $\mathcal{R}_K$ does not fit into the framework of Kedlayas slope theory. In order to translate known results from the étale case to more general modules we use the Lie-algebra cohomology of M that can be defined over L using either the operators $(\nabla, \varphi - 1)$ or $(\nabla, \Psi - 1)$. We do not know whether they give the same cohomology groups in degree 2 and since the cokernel of $\Psi - 1$ is better behaved than that of $\varphi_L - 1$, we shall work with the Ψ-version.

Definition 6.4. *Let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_L$. Let ∇ be the operator defined in 1.50. We define the complex*

$$C_{Lie}(M) := [M \xrightarrow{(\nabla, \Psi-1)} M \oplus M \xrightarrow{(\Psi-1)(\mathrm{pr}_1)-\nabla(\mathrm{pr}_2)} M]$$

concentrated in degrees $[0,2]$. Denote by $H^i_{Lie}(-)$ the cohomology groups of the complex $C_{Lie}(-)$ and define $H^i_{an}(M) := H^i_{Lie}(M)^{\Gamma_L}$. Analogously we define $H^i_{an}(M/t_{LT}M)$.

Remark 6.5. *One can show that the residual Γ_L-action on $H^i_{Lie}(M)$ is discrete (i.e. every element has open stabiliser). Since the cohomology groups are L-vector spaces we can deduce that $H^i_{an}(M)$ takes short exact sequences to long exact sequences in cohomology.*

Remark 6.6. *Analogously one can define a version with $\varphi - 1$ instead of $\Psi - 1$. Copying the proof of [FX12, Proposition 4.1] together with the comparison isomorphism in [BF17, Theorem 2.2.2] shows that the H^0 and H^1 agree for the φ_L and Ψ-versions and agree with the corresponding cohomology groups for L-analytic cocycles studied by [BF17].*

The main reason to use the Ψ-version is the following Lemma.

Lemma 6.7. *Let M be an L-analytic (φ_L, Γ_L)-module then $H^2_{Lie}(M)$ is finite dimensional and Hausdorff.*

Proof. The continuous map $\Psi - 1 \colon M \to M$ has finite dimensional cokernel by Theorem 3.32. The statement follows from Lemma 1.10. □

Lemma 6.8. *For the L-analytic rank one (φ_L, Γ_L)-modules $\mathcal{R}_L(x^i)$ we have*

$$H^2_{an}(\mathcal{R}_L(x^i)) = 0$$

for every $i \in \mathbb{Z}$.

Proof. Since $H^2_{\mathrm{Lie}}(\mathcal{R}_L(x^i))$ is finite-dimensional Hausdorff, every linear form is continuous. Suppose

$$H^2_{\mathrm{Lie}}(\mathcal{R}_L(x^i)) = (\mathcal{R}_L(x^i)/(\nabla, (\Psi - 1)))$$

admits a (continuous) functional $H^2_{\mathrm{Lie}}(\mathcal{R}_L(x^i)) \to L$, then pre-composing it with the canonical map $\mathcal{R}_L(x^i) \to H^2_{\mathrm{Lie}}(\mathcal{R}_L(x^i))$ gives a continuous functional $\mathcal{R}(x^i) \to L$ which by the duality described in Proposition 1.75 corresponds to a unique element in $m \in \mathcal{R}_L(x^{-i})(\chi_{LT})$ killed by $\varphi_L - 1$ and ∇_ι, where ∇_ι denotes the adjoint of ∇. A computation using that the adjoint of $\gamma \in \Gamma_L$ is γ^{-1} shows $\nabla_\iota = -\nabla$ and that m is also killed by ∇. Hence m is an element in $\mathcal{R}_L(x^{-i})(\chi_{LT})^{\nabla=0,\varphi=1}$, but $\mathcal{R}_L(x^{-i})(\chi_{LT})^{\nabla=0,\varphi=1} = 0$ by [FX12, Proposition 5.6] and thus $m = 0$. We conclude that $H^2_{\mathrm{Lie}}(\mathcal{R}_L(x^i))$ admits no non-zero functionals and therefore has to be zero. □

Lemma 6.9. *Let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_L$. Then:*

1. *There is a canonical bijection between $H^1_{an}(M)$ and isomorphism classes of extensions*

$$0 \to M \to E \to \mathcal{R}_L \to 0$$

   of L-analytic (φ_L, Γ_L)-modules.

2. *For $i \in \mathbb{N}_0$ the L-dimension of $H^j_{an}(\mathcal{R}_L(x^{-i}))$ in degrees $j = 0, 1, 2$ is $1, 2, 0$ respectively.*

3. *For $i \in \mathbb{N}$ the L-dimension of $H^j_{an}(\mathcal{R}_L(x^i))$ in degrees $j = 0, 1, 2$ is $0, 1, 0$ respectively.*

Proof. For the corresponding results using the φ_L-version see [FX12, Theorem 0.1 (resp. Theorem 4.2)] for the first statement and [Col16, Théorème 5.6] for 2 and 3 under the assumption that the field contains Ω_L. The dimensions were computed without this assumption by Fourquaux and Xie in degrees $0, 1$ and agree with the results of Colmez. By [BF17, Theorem 2.2.2] they can be translated to the Ψ-version. The computation of H^2 over L was done in Lemma 6.8. □

The proof of the following Lemma is based on Liu's proof in [Liu07, Lemma 4.2], but we need to make some adjustments since we do not know in general whether the Euler-Poincaré-Characteristic formula holds for analytic cohomology.

Lemma 6.10. *There exist an L-analytic (φ_L, Γ_L)-module E of rank d that is isoclinic of slope $1/d$ and a successive extension of $\mathcal{R}(x^i)$, where $i = 0, 1$. It can be chosen such that $H^1_{an}(E) \neq 0$ and $H^2_{an}(E) = 0$.*

Proof. We shall construct a sequence $(E_d)_d$ of L-analytic modules of rank $d \in \mathbb{N}$ isoclinic of slope $1/d$ such that $H^1_{\mathrm{an}}(E_d) \neq 0$ and $H^2_{\mathrm{an}}(E_d) = 0$. Clearly $E_1 := \mathcal{R}(x)$ is L-analytic of rank 1 and isoclinic of slope 1. By Lemma 6.9 E_1 satisfies $H^1_{\mathrm{an}}(E_1) \neq 0$ and $H^2_{\mathrm{an}}(E_1) = 0$. Suppose E_d has been constructed. Take a non-trivial extension E_{d+1} corresponding to a non-zero element in $H^1_{\mathrm{an}}(E_d)$ and consider the exact sequence

$$0 \to E_d \to E_{d+1} \to \mathcal{R}_L \to 0.$$

Passing to the long exact cohomology sequence we obtain by the second point of Lemma 6.9 an exact sequence

$$\cdots \to H^2_{\mathrm{an}}(E_d) \to H^2_{\mathrm{an}}(E_{d+1}) \to 0$$

and the vanishing of $H^2_{\mathrm{an}}(E_d)$ implies the vanishing of $H^2_{\mathrm{an}}(E_{d+1})$. Due to the exactness of

$$\cdots \to H^1_{\mathrm{an}}(E_{d+1}) \to H^1_{\mathrm{an}}(\mathcal{R}_L) \to H^2_{\mathrm{an}}(E_d) = 0$$

we conclude that $H^1_{\mathrm{an}}(E_{d+1})$ surjects onto a non-zero space and hence has to be non-zero. The slope of E_{d+1} is $1/(d+1)$ by construction. It remains to see that E_{d+1} is isoclinic. For the convenience of the reader we reproduce Liu's argument. Suppose $P \subset E_{d+1}$ is a non-zero proper subobject of slope $\mu(P) = \frac{\deg(P)}{\mathrm{rank}(P)} < 1/(d+1)$. Then its rank is bounded above by $d+1$ and hence $\deg(P) \leq 0$ is necessary which implies $\mu(P) \leq 0$. Denote by X the image of P in $\mathcal{R}_L$. The exact sequence

$$0 \to P \cap E_d \to P \to X \to 0$$

implies by general Harder-Narasimhan-Theory (cf. [Pot20, 4.4]) that the corresponding slopes are given either in ascending or descending order. Since $\mu(P) \leq 0$ and $\mu(X) \geq 0$ due to $\mathcal{R}_L$ being isoclinic of slope 0 we conclude that we have $\mu(P \cap E_d) \leq 0$ which together with the fact that E_d is isoclinic of slope $1/d$ implies that $P \cap E_d = 0$ holds. This in turn means P is a subobject of $\mathcal{R}_L$ with slope ≤ 0 and hence isomorphic to $\mathcal{R}_L$. This contradicts the assumption that the extension E_{d+1} is not split. □

Theorem 6.11. *Assume Conjecture 6.3 is true. Let M_0 be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_L$ and let $M := K\hat{\otimes}_{L,i}M_0$. Then the complex $C_{\psi_{LT}}(M)$ of $D(U,K)$-modules is perfect.*

Proof. We abbreviate $\psi = \psi_{LT}$. By abuse of language we say M is étale if M_0 is étale. Similarily we mean the slopes of M_0 when refering to the slopes of M. Since the complex is bounded it suffices to prove the statement for the cohomology groups by [Sta21, Tag 066U]. More precisely Proposition 4.9 shows that finite generation of $M^{\psi=1}$ is sufficient. Hence if M is étale the statement follows by combining Conjecture 6.3 with Proposition 4.9. Suppose M is isoclinic and has integral slopes. Then Mt_{LT}^n is étale for some power $n \in \mathbb{Z}$ and we argue inductively via the exact sequence of complexes induced by the sequence

$$0 \to Mt_{LT} \to M \to M/t_{LT} \to 0.$$

For negative integers we set $N = t_{LT}^{-1}M$ and use the corresponding sequence for N. We obtain a short exact sequence of complexes

$$0 \to C_\psi(Mt_{LT}) \to C_\psi(M) \to C_\psi(M/t_{LT}) \to 0.$$

Here we make implicit use of the exactness of $K\hat{\otimes}_L-$ for strict sequences of Fréchet spaces given by Lemma 1.11 and the exactness of filtered colimits to reduce to the

Fréchet case. The perfectness of the rightmost term holds unconditionally by Corollary 4.11. By induction hypothesis either the middle or the leftmost term are perfect. In both cases we conclude from [Sta21, Tag 066R] that the third term is also perfect. This concludes the case of integral slopes. We first show that the theorem holds for any isoclinic M such that $M/(\psi-1)$ and $M(x)/(\psi-1)$ vanishes. Recall that in the proof of Lemma 6.10 we produced a family of exact sequences

$$0 \to E_i \to E_{i+1} \to \mathcal{R}_L \to 0$$

such that E_i is isoclinic of slope $1/i$ starting with $E_1 = \mathcal{R}(x)$. Tensoring this sequence with M_0 and applying $\psi - 1$ we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_i \otimes_{\mathcal{R}_L} M_0 & \longrightarrow & E_{i+1} \otimes_{\mathcal{R}_L} M_0 & \longrightarrow & M_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi-1} & & \downarrow{\scriptstyle \psi-1} & & \downarrow{\scriptstyle \psi-1} & & \\ 0 & \longrightarrow & E_i \otimes_{\mathcal{R}_L} M_0 & \longrightarrow & E_{i+1} \otimes_{\mathcal{R}_L} M_0 & \longrightarrow & M_0 & \longrightarrow & 0 \end{array}$$

and induction on i together with the Snake Lemma shows that $(E_i \otimes_{\mathcal{R}_L} M_0)/(\psi-1)$ vanishes for every i. For $i = d-1$ we obtain a surjection $(E_d \otimes_{\mathcal{R}_L} M_0)^{\psi=1} \to (M_0)^{\psi=1}$ and $E_d \otimes_{\mathcal{R}_L} M_0$ is pure of slope $c/d + 1/d = (c+1)/d$. By Proposition 4.9 we are done if we can show that $M^{\psi=1}$ is finitely generated. Due to strictness of $\psi - 1$ we have $M^{\psi=1} = K\hat{\otimes}_L M_0^{\psi=1}$ and it suffices to show that $M_0^{\psi=1}$ is finitely generated. Our argument thus far shows that given an isoclinic module M_0 of some slope $\frac{c}{d}$ with vanishing $(\psi-1)$-cokernel there exists an isoclinic module of slope $\frac{c+i}{d}$ say X_i with vanishing $(\psi-1)$-cokernel such that $X_i^{\psi=1}$ surjects onto $M_0^{\psi=1}$. We fix d and argue inductively "in reverse", i.e. start with the base case $\frac{d}{d}$ and from the above deduce the statement for $\frac{(d-i)}{d}$ for $i \in \{1, \dots, d-1\}$.
Now let M_0 be arbitrary isoclinic then $t_{LT}^{-n} M_0$ satisfies $t_{LT}^{-n} M_0/(\psi-1) = 0$ by Lemma 3.34 and we can thus apply the preceding result and Corollary 4.11. Finally assume M_0 is arbitrary then either M_0 is isoclinic or it fits into an exact sequence $0 \to N \to M_0 \to M_0/N \to 0$ with M_0/N isoclinic and $\operatorname{rank}(N) < \operatorname{rank}(M)$. Since every rank 1 module is automatically isoclinic we deduce the general statement by induction over the rank of M. □

Corollary 6.12. *In the situation of 6.11 the complex $C_{c\Psi}(M)$ is perfect as a complex of $D(\Gamma_L, K)$-modules for any $c \in K^\times$.*

Proof. Apply Lemma 4.10 and Theorem 6.11 to conclude perfectness as a complex of $D(U, K)$-modules. An application of Lemma A.21 to the cohomology groups implies perfectness as a complex of $D(\Gamma_L, K)$-modules. The cohomology groups are coadmissible because finite projective modules are coadmissible and coadmissible modules form an abelian category. □

Remark 6.13. *We do not know whether the corresponding statement is true for the complex of $D(\Gamma_L, L)$-modules $C_\psi(M_0)$ since the results 3.30 and 4.7 made use of the explicit description of $D(U, K)$ as a power-series ring.*

6.2 Towards the Euler-Poincaré formula

In this section we discuss the Euler-Poincaré characteristic formula for the analytic Herr complex $C_{\Psi,Z}(M)$ for L-analytic (φ_L, Γ_L)-modules. From formal arguments one can deduce a variant of the formula involving the heart $\mathcal{C}(M)$ for all L-analytic (φ_L, Γ_L)-modules over $\mathcal{R}_K$. Our computations in the trianguline case are sufficient to prove the expected formula in this case.

Definition 6.14. *Let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_A$. We define the **Euler-Poincaré characterstic** of M as*

$$\chi(M) := \sum_{i \in \mathbb{N}_0} (-1)^i \operatorname{rank}_A(H^i_{\psi,Z}(M)).$$

Theorem 3.22 asserts that $\chi(M)$ is well-defined. Note that this formula depends on Z or more precisely on the index of the group $U \subset \Gamma_L$ used to define Z. We expect

$$\chi(M) = -[\Gamma_L : U] \operatorname{rank}_{\mathcal{R}_A}(M).$$

When considering modules over relative Robba rings $\mathcal{R}_A$, the validity of such a formula can be checked on each fibre $z \in \operatorname{Sp}(A)$ and thus there is no harm in assuming $A = K$. The classical methods of Herr (cf. [Her98, Section 4.2]) show that the heart $\mathcal{C}(M)$ of M plays an integral role. The following proposition is from [MSVW].

Proposition 6.15. *Let M be an L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_K$. Then $\mathcal{C}(M)/Z$ is finite K-dimensional and $\chi(M) = -\dim_K(\mathcal{C}(M)/Z)$.*

Proof. By the exact sequence from 4.6 it suffices to show that $M^{\Psi=1}/Z$ is finite K-dimensional. Let $m \in M^{\Psi=1}$. One checks that $(0, m)$ is a 1-cocycle for $C_{\Psi,Z}(M)$ and the coboundary condition $(0, m) = ((\Psi - 1)(n), Zn)$ for some $n \in M$ implies $n \in M^{\Psi=1}$ and therefore $m \in ZM^{\Psi=1}$. We thus have an injection $M^{\Psi=1}/Z \hookrightarrow H^1_{\Psi,Z}(M)$ forcing the left-hand side to be finite K-dimensional by Theorem 3.22 as a subspace of a finite-dimensional space. Consider the finite filtration of $\mathcal{F}^0 C_{\Psi,Z}(M) := C_{\Psi,Z}(M)$ by the complexes

$$\mathcal{F}^1 C_{\Psi,Z}(M) = [M^{\Psi=1} \xrightarrow{Z} M^{\Psi=1}],$$

$$\mathcal{F}^2 C_{\Psi,Z}(M) = [M^{\varphi_L=1} \xrightarrow{Z} M^{\varphi_L=1}]$$

concentrated in degrees $[0, 1]$ and $\mathcal{F}^3 C_{\Psi,Z}(M) = 0$. Clearly

$$\operatorname{gr}^2 C_{\Psi,Z}(M) = [M^{\varphi_L=1} \xrightarrow{Z} M^{\varphi_L=1}]$$

and the exact sequence from Remark 4.6 shows

$$\operatorname{gr}^1 C_{\Psi,Z}(M) = [\mathcal{C}(M) \xrightarrow{Z} \mathcal{C}(M)].$$

An argument analogous to [Her98, Lemme 4.2] shows that $\mathrm{gr}^0 C_{\Psi,Z}(M)$ is quasi-isomorphic to the complex

$$M/(\Psi-1) \xrightarrow{Z} M/(\Psi-1)$$

concentrated in degrees $[1,2]$. From the associated convergent spectral sequence (cf. [Sta21, Tag 012W])

$$E_1^{p,q} = H^{p+q}(gr^p(C_{\Psi,Z}(M))) \implies H^{p+q}(C_{\Psi,Z}(M))$$

we conclude, using that all terms on the first page are finite-dimensional,

$$\chi(M) = \sum_{p,q}(-1)^{p+q}\dim_K H^{p+q}(gr^p(C_{\Psi,Z}(M))).$$

By remark 4.8 the terms for $\mathrm{gr}^2\, C_{\Psi,Z}(M)$ and $\mathrm{gr}^0\, C_{\Psi,Z}(M)$ cancel out. Because $\mathcal{C}(M)$ is a subspace of $\ker(\Psi)$, the bijectivity of Z on the kernel of Ψ implies that the only remaining term is $-\dim_K(\mathcal{C}(M)/Z)$. □

The expected Euler-Poincaré characteristic formula holds in the trianguline case.

Remark 6.16. *Let M be a trianguline L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_K$, then*

$$\chi(M) = -[\Gamma_L : U]\,\mathrm{rank}_{\mathcal{R}_K}(M).$$

Proof. By induction it suffices to treat the case where $M = \mathcal{R}_K(\delta)$ for some locally L-analytic character $\delta\colon L^\times \to K^\times$. In this case by Proposition 5.7 $M^{\Psi=1}$ is finitely generated and hence $\mathcal{C}(M)$ is projective by Remark 4.4. The proof of 5.7 shows further that $\mathrm{rank}_{D(U,K)}(\mathcal{C}(M))$ is precisely $[\Gamma_L : U]$. Since $D(U,K)$ is a domain with maximal ideal (Z), the rank of $\mathcal{C}(M)$ is equal to $\dim_K(\mathcal{C}(M)/Z)$. The result now follows from Proposition 6.15. □

Proposition 6.15 shows that the Euler-Poincaré formula would follow from $\mathcal{C}(M)$ being projective as a $D(U,K)$-module of rank $[\Gamma_L : U]\,\mathrm{rank}_{\mathcal{R}_L} M$. In the cyclotomic case one uses slope theory to reduce to the étale case where it follows from corresponding results for (φ,Γ)-modules over $\mathbf{A}_{\mathbb{Q}_p}$ (cf. [Col10, V.1.13, V.1.18]). We run into various problems because our theory requires the passage to K in order to be able to define the cohomology groups in the first place and some key structural results like Theorem 2.19 do not have an analogue over the base field L. Furthermore the dimension of $o_L[\![U]\!]$ poses a problem when working with an étale module M_{LT} over $\mathbf{B}_L$. The projectivity of $(1-\frac{\pi_L}{q}\varphi_L)M_{LT}^{\psi_{LT}=1}$ is unknown to us and does not follow from a reflexivity argument like in [Col10, I.5.2], because $o_L[\![U]\!]$ can be of dimension greater than two.

Appendix

Lemma A.1. *([Eme17]) Let $V \to W$ be a strict surjection with W Hausdorff and V an LF-space (resp. LB-space) then W is an LF-space (resp. LB-space). In particular any quotient of an LF-space (resp. LB-space) by a closed subspace is an LF- (resp. LB-space).*

Proof. Let $V = \varinjlim_n V_n$ with Fréchet spaces V_n (resp. Banach spaces) and denote by X_n the kernel of $V_n \to V \to W$. Since W is assumed to be Hausdorff X_n is closed in V_n and therefore $V_n/X_n =: W_n$ is a Fréchet space (resp. Banach space). As a result $V \to W$ factors as $V \xrightarrow{\alpha} \varinjlim_n W_n \xrightarrow{\beta} W$ and with a continuous surjection of LF-spaces α which has to be open by the open mapping theorem and a continuous bijection β such that $\beta \circ \alpha$ is strict forcing β to be strict as well hence the claim. □

A.1 Coherent Sheaves

We would like to prove that $\mathcal{R}_A^{[r_0,1)}$ is a Fréchet-Stein algebra in the sense of Schneider-Teitelbaum. For that purpose we fix a sequence $0 \leq r_0 < r_1 < \dots$ in $[0,1) \cap |\overline{\mathbb{Q}_p}|$ converging to 1. We have to show that $\mathcal{R}_A^{[r_0,r_n]}$ is Noetherian und $\mathcal{R}_A^{[r_0,r_{n+1}]} \to \mathcal{R}_A^{[r_0,r_n]}$ is flat and has topologically dense image. Recall that a coadmissible module $M^{[r_0,1)}$ over $\mathcal{R}_A^{[r_0,1)}$ is uniquely determined by its sections along an admissible cover by Remark 1.58.

Lemma A.2. $\mathcal{R}_A^{[r_0,r_{n+1}]} \to \mathcal{R}_A^{[r_0,r_n]}$ *is flat and has topologically dense image.*

Proof. Assume $r_0 \neq 0$ since the case $r_0 = 0$ can be treated analogously. Let $I = [r_0, r_n]$. The Laurent polynomials $K[T, T^{-1}]$ are a dense subset of $\mathcal{R}_K^I$ and thus the image of $K[T, T^{-1}] \otimes_F A$ is dense in both rings. It remains to show flatness. Let $I \subset \mathcal{R}_A^{[r_0,r_n]}$ be a finitely generated ideal and consider the map $I \otimes_{\mathcal{R}_A^{[r_0,r_n]}} \mathcal{R}_A^{[r_0,r_{n+1}]} \to \mathcal{R}_A^{[r_0,r_n]}$. It remains to show flatness. We can view $U := Sp(\mathcal{R}_K^{[r_0,r_n]})$ as an affinoid subdomain of $X := Sp(\mathcal{R}_K^{[r_0,r_{n+1}]})$. Consider the canonical map $p_1 : Y = X \times_K Sp(A) \to X$ then by [BGR 7.2.2. prop 4]. $U \times_X Y$ is an affinoid subdomain of Y. By Definition

$$U \times_X Y = U \times_X X \times_K Sp(A) = U \times_K Sp(A).$$

We conclude that $\mathcal{R}_A^{[r_0,r_{n+1}]} \to \mathcal{R}_A^{[r_0,r_n]}$ is the canonical map induced from the inclusion of the affinoid subdomain $U \times_K Sp(A) = Sp(\mathcal{R}_A^{[r_0,r_{n+1}]})$ in $X \times_K Sp(A)$. Then by [BGR Corollary 6 Section 7.3.2.] the coressponding map on the level of rings is flat. □

Lemma A.3. $\mathcal{R}_A^{[r_0,r_n]}$ *is Noetherian.*

Proof. This follows from the fact that the completed tensor product of two affinoid algebras is again affinoid by [BGR84, 7.1.3. Proposition 4] and hence Noetherian by [BGR84, 5.2.6 Theorem 1]. □

Lemma A.4. *(cf. [KPX14, 2.1.12]) Let M be a coadmissible module over $\mathcal{R}_A^r$. Suppose M^I is generated by $f_1, \dots, f_n$. Then there exists an $\varepsilon > 0$ such that any $g_1, \dots, g_n$ satisfiying $|f_i - g_i| < \varepsilon$ also generates M^I.*

Proof. We have a continuous surjection of Banach modules $(\mathcal{R}_A^I)^n \to M^I$. Where e_i is mapped to f_i. By the open mapping theorem the image of the ball $\{v \in (\mathcal{R}_A^I)^n \mid |v| < 1/2\}$ contains some ball $\{m \in M^I \mid |m| < \varepsilon\}$. Given g_i we can write $g_i - f_i = \sum_j a_{ij} f_j$ with $|a_{ij}| < 1/2$. In other words $g_i = f_i + \sum_j a_{ij} f_j$. By a geometric-series-argument the matrix $E_n + (a_{ij})_{ij}$ is invertible. We conclude that the g_i generate M^I. □

Lemma A.5. *Let $r_0 \in (0,1)$ and let $f \in \mathcal{R}_K^{r_0}$. Then $\mathcal{R}_A/f$, and $\mathcal{R}_A^{[r_0,1)}/f$ are flat as A-modules.*

Proof. This is analogous to [KPX14, 2.1.5]. The proof carries over to our situation using that $\mathcal{R}_K^{[r,s]}$ is of countable type over K as a quotient of a Tate-Algebra over K. □

The following Lemma is [KPX14, Proposition 2.1.13(i)]. We elaborate on the proof.

Lemma A.6. *Let M be a coadmissible module over $\mathcal{R}_A^r$. Suppose there exist global sections $f_1, \dots, f_n \in M$ that generate each M^I for any closed interval I. Then the f_i generate M.*

Proof. Consider the morphism

$$\theta : (\mathcal{R}_A^r)^n \to M$$

mapping e_i to f_i. Then $\ker(\theta)$ is coadmissible by 1.55(iv). By assumption we have an exact sequence

$$0 \to \ker(\theta\mid_{[r,s]}) \to (\mathcal{R}_A^{[r,s]})^d \to M^{[r,s]} \to 0.$$

Passing to the limit $s \to 1$ and using the fact that $\varprojlim^1$ vanishes for coadmissible modules by 1.55(iii) we obtain that θ is surjective. □

Lemma A.7. *Let M be coadmissible module over $\mathcal{R}_A^{[r_0,1)}$. Then M is finitely generated if and only if there exists a natural number n and an admissible cover $[r_i, s_i], i \in \mathbb{N}_0$ of $[r_0, 1)$ such that each $M^{[r_i,s_i]}$ is generated by at most n elements.*

Proof. Define $I_i := [r_i, s_i]$. We treat the non-trivial implication. After eventually refining a given admissible cover of $[r_0, 1)$ we can assume that $[r_i, s_i]$ and $[r_{i+2}, s_{i+2}]$ do not intersect for any $i \in \mathbb{N}_0$ and that the intervals are ordered in ascending order i.e. $r_i < r_{i+1}$ and $s_i < s_{i+1}$. By construction the intervals with even indices $[r_{2i}, s_{2i}]$ are pairwise disjoint. The same applies for the intervals $[r_{2i+1}, s_{2i+1}]$ with odd indices. We first show that there exist global sections $f_1, \dots, f_n$ and $g_1, \dots, g_n$ such that the f_j generate each $M^{[r_{2i}, s_{2i}]}$ and the g_j generate each $M^{[r_{2i+1}, s_{2i+1}]}$. Since the argument is symmetric we restrict ourselves to the intervals with even indices. By assumption for each i we can find sections $f_{2i,1}, \dots, f_{2i,n}$ that generate $M^{[r_{2i}, s_{2i}]}$. When $i = 0$ we can assume that the sections are global sections because M is dense in M^I by using Lemma 1.55(i). When $i \neq 0$, we can a priori assume that the sections lie in $M[1/T]$. But in this case T becomes a unit in M^I hence even in this case we can assume that the sections are global sections. For every interval I_{2i} we can find $\varepsilon(2i)$ such that any $f'_{2i,j}$ satisfying $|f_{2i,j} - f'_{2i,j}| \leq \varepsilon(2i)$ generates the same module by A.4. The idea is to define a convergent series

$$f_j := \sum_k p^{a(k)} T^{b(k)} f_{2k,j}.$$

Satisfying $a(0) = 0, b(0) = 0$ and the following three conditions

1. for any $k < k_0$ we have $|p^{a(k_0)-a(k)} T^{b(k_0)-b(k)} f_{2k_0,j}|_{[r_{2k}, s_{2k}]} \leq \varepsilon(2k)$
2. for any $k < k_0$ we have $|p^{a(k)-a(k_0)} T^{b(k)-b(k_0)} f_{2k,j}|_{[r_{2k_0}, s_{2k_0}]} \leq \varepsilon(2k_0)$
3. for any $k < 2k_0 - 1$ we have $|p^{a(k_0)} T^{b(k_0)} f_{2k_0,j}|_{[r_k, s_k]} \leq p^{-k_0}$

The third condition guarantees that f_j converges with respect to the Fréchet topology. We now explain how to chose $a(k), b(k)$ inductively. By our conventions we have for any interval I : $|T|_I = \sup I, |T^{-1}|_I = \inf I^{-1}$ and $|p^n|_I = p^{-n}$. As noted before we define $a(0) = b(0) = 0$. Next assume $a(2k), b(2k)$ have been defined up to some $k_0 \in \mathbb{N}$. We rewrite [2] the conditions as follows

1. for any $k < k_0$ we have $p^{-a(k_0)} s_{2k}^{b(k_0)} \leq s_{2k}^{b(k)} |f_{2k_0,j} p^{-a(k)}|^{-1}_{[r_{2k}, s_{2k}]}$
2. for any $k < k_0$ we have $p^{-a(k_0)} r_{2k_0}^{b(k_0)} \geq r_{2k_0}^{b(k)} |p^{a(k)} f_{2k,j}|_{[r_{2k_0}, s_{2k_0}]}$
3. for any $k < 2k_0 - 1$ we have $p^{-a(k_0)} s_k^{b(k_0)} \leq |f_{2k,j}|^{-1}_{[r_k, s_k]} p^{-k_0}$.

Because the intervals are disjoint, we have $s_{2k} < r_{2_{k_0}}$ and $s_k < r_{2_{k_0}}$. After replacing the right-hand side in each inequality with a suitable constant by taking the infimum (in the first and third inequality) or supremum (in the second inequality) over all k we can reduce to the subsequent Lemma A.8. It remains to show, that the $f_j's$ generate M^I for every interval $I = [r_{2k_0}, s_{2k_0}]$. By A.4 it suffices to show

$$\left| f_j \frac{1}{p^{a(k_0)} T^{b(k_0)}} - f_{2i,j} \right|_I \leq \varepsilon(2k)$$

[2] With the convention $0^{-1} = +\infty$.

We compute

$$\left|f_j \frac{1}{p^{a(k_0)}T^{b(k_0)}} - f_{2i,j}\right|_{[r_{2k_0}, s_{2k_0}]} \tag{A.1}$$

$$= \left|\sum_{k<k_0} p^{a(k)-a(k_0)}T^{b(k)-b(k_0)} f_{2k,j} + \sum_{k>k_0} p^{a(k)-a(k_0)}T^{b(k)-b(k_0)} f_{2k,j}\right|_{[r_{2k_0}, s_{2k_0}]} \tag{A.2}$$

$$\leq \max\left(\max_{k<k_0} \left|p^{a(k)-a(k_0)}T^{b(k)-b(k_0)} f_{2k,j}\right|_{[r_{2k_0}, s_{2k_0}]}, \max_{k>k_0} \left|p^{a(k)-a(k_0)}T^{b(k)-b(k_0)} f_{2k,j}\right|_{[r_{2k_0}, s_{2k_0}]}\right) \tag{A.3}$$

$$\leq \varepsilon(2k_0). \tag{A.4}$$

Here we use the second inequality to estimate the left-hand side of (A.3) and apply the first inequality with the roles of k and k_0 interchanged to estimate the right summand of (A.3). Finally we use lemma A.6 to conclude that M is finitely generated. □

Lemma A.8. *Let $x_1, x_2, x_3 \in [0, 1)$ with $x_2 > x_1, x_3$ and let $0 < C_1, C_2, C_3$ be real constants. Then there exists a natural number $b \gg 0$ such that there exists a natural number a satisfying the following inequalities*

1. $p^a x_1^b \leq C_1$
2. $p^a x_2^b \geq C_2$
3. $p^a x_3^b \leq C_3$.

Proof. By assumption $x_i/x_2 < 1$ for $i = 1, 3$ and thus $Y_1 := x_1^b x_2^{-b}$ converges to 0 for large b. The same is true for $Y_2 := x_3^b x_2^{-b}$. Choosing b large enough we can ensure that $Y_1 \leq p^{-1}C_1/C_2$ and $Y_2 \leq p^{-1}C_3/C_2$ next choose a to be minimal such that $p^a \geq C_2 x_2^{-b}$. By construction $p^a \leq pC_2 x_2^{-b}$. The second inequality is satisfied by construction. For the other inequalities we compute

$$p^a x_1^b \leq pC_2 x_1^b x_2^{-b} = pC_2 Y_1 \leq C_2(C_1/C_2) = C_1.$$

The computation for the third inequality is analogous. □

A.2 The action on $\mathcal{R}_L^r/Q_n$.

Recall the product formula

$$t_{LT} = \log_{LT}(X) = X \prod_{n\geq 1} \frac{Q_n(X)}{\pi_L},$$

where $Q_1(X) = \varphi(X)/X$ and $Q_n := \varphi(Q_{n-1})$. The zeroes of Q_n are precisely the π_L^n-torsion points of the LT-group that are not already π_L^{n-1} torsion points. Contrary to the cyclotomic situation these Q_n are not necessarily polynomials. We denote by G_n the polynomial

$$G_n := \prod_{a \in LT[\pi_L^n] \setminus LT[\pi_L^{n-1}]} (X - a).$$

By construction $G_n \in o_L[[X]]$ and $G_n \mid Q_n$ in $o_L[[X]]$ and therefore in every $\mathcal{R}_L^{[r,s]}$. We also observe that Q_n/G_n is a unit in $\mathcal{R}_L^{[r,s]}$ since it does not have any zeroes in $\mathbb{C}_p$ and therefore is not contained in any maximal ideal of $\mathcal{R}_L^{[r,s]}$ by an argument like in the proof of 1.57.

Lemma A.9. *Let $z \in LT[\pi_L^n] \subset L_n$ and let $\sigma \in \Gamma_L$ then*

$$\sigma(z) = [\chi_L(\sigma)]_\phi(z).$$

Proof. See [Sch17, 1.3.12]. □

Lemma A.10. *Fix a Basis $z = (z_n)$ of $\varprojlim_n LT[\pi_L^n]$. By mapping X to $z_n \in LT[\pi_L^n]$ we obtain a Γ_L-equivariant isomorphism*

$$\mathbf{B}_L^+/Q_n \cong L_n$$

furthermore the following diagrams commute

$$\begin{array}{ccc} \mathbf{B}_L^+/Q_n & \longrightarrow & L_n \\ \downarrow{\scriptstyle \varphi_L} & & \downarrow \\ \mathbf{B}_L^+/Q_{n+1} & \longrightarrow & L_{n+1} \end{array} \qquad \begin{array}{ccc} \mathbf{B}_L^+/Q_{n+1} & \longrightarrow & L_{n+1} \\ \downarrow{\scriptstyle \psi_{LT}} & & \downarrow{\scriptstyle \pi_L^{-1}\operatorname{tr}_{L_{n+1}/L_n}} \\ \mathbf{B}_L^+/Q_n & \longrightarrow & L_n \end{array}$$

Proof. Note that L is flat over o_L and because L_n over L is totally ramified with uniformiser z_n we have $o_{L_n} = o_L[z_n]$. Hence it suffices to show the corresponding statement for $\mathbf{A}_L^+ = o_L[[X]]$. On the one hand the constant term of Q_n is π_L, on the other hand the constant term of G_n has valuation $\operatorname{val}(z)[L_n : L] = \operatorname{val}(\pi_L) = 1$ Therefore the quotient Q_n/G_n is already a unit in $o_L[[X]]$. Evaluation at z_n induces $o_L[[X]]/Q_n \cong o_L[[X]]/G_n \cong o_L[X]/G_n \cong o_L[z]$ using that G_n is a distinguished polynomial because it reduces to the monic polynomial $X^{\deg(G_n)}$ modulo π_L. This isomorphism is Γ_L-equivariant by $A.9$. The desired isomorphism is obtained by tensoring both sides with L. For the commutativity of the first diagram we observe that $\varphi_L(X)$ is mapped to $[\pi_L](z_n) = z_{n-1}$ by assumption. For the second diagram we first remark that ψ_{LT} is well-defined. Let $f = g + hQ_{n+1}$ then $\psi_{LT}(f) = \psi_{LT}(g) + \psi_{LT}(\varphi_L(Q_n)h) = \psi_{LT}(g) + Q_n\psi_{LT}(h)$ by the projection formula. The commutativity of the diagram follows from $\varphi_L\psi_{LT}(x) = \frac{1}{\pi_L}\operatorname{tr}_{\mathbf{B}_L/\varphi(\mathbf{B}_L)}(x)$ using that the latter trace can be computed as the trace of the multiplication-by-x-matrix M_x with respect to the basis $1, X, \ldots, X^{q-1}$ and said matrix has coefficients in $\mathbf{B}_L^+$ if $x \in \mathbf{B}_L^+$. Because of the additivity of matrix traces we have $\operatorname{trace}(M_x) \mod Q_n = \operatorname{trace}(M_x \mod Q_n)$. Finally we use that φ_L induces the inclusion $L_n \hookrightarrow L_{n+1}$ modulo Q_n. □

Corollary A.11. *Let $[r,s]$ be an interval with $r_L < r < s < 1$ such that the corresponding annulus contains $LT[\pi_L^n]$. Fix a Basis $z = (z_n)_n$ of $\varprojlim LT[\pi_L^n]$. By mapping X to $z_n \in LT[\pi_L^n]$ we obtain a Γ_L-equivariant isomorphism*

$$\mathcal{R}_L^{[r,s]}/Q_n \cong L_n$$

and the following diagrams commute:

$$\begin{array}{ccc} \mathcal{R}_L^{[r,s]}/Q_n & \longrightarrow & L_n \\ \downarrow{\scriptstyle \varphi_L} & & \downarrow \\ \mathcal{R}_L^{[r^q,s^q]}/Q_{n+1} & \longrightarrow & L_{n+1} \end{array} \qquad \begin{array}{ccc} \mathcal{R}_L^{[r^q,s^q]}/Q_{n+1} & \longrightarrow & L_{n+1} \\ \downarrow{\scriptstyle \psi_{LT}} & & \downarrow{\scriptstyle \pi_L^{-1}\operatorname{tr}_{L_{n+1}/L_n}} \\ \mathcal{R}_L^{[r,s]}/Q_n & \longrightarrow & L_n \end{array}$$

Proof. By assumption the map in question is well-defined. The commutativity follows from A.10 by first considering the map $\mathbf{B}_L^+/Q_n \to L_n$, which factors over $\mathcal{R}_L^{[r,s]}$ since L_n is a field and z_n lies in the annulus $\mathbb{B}_{[r,s]}(\mathbb{C}_p)$. Note that a priori $\mathcal{R}_L^{[r,s]}/Q_n \to L_n$ is merely surjective. By writing $Q_n = U_n G_n$ with a unit U_n and G_n like in the proof of A.10 one can see, that the left hand-side is at most $[L_n : L]$-dimensional over L, which implies the injectivity. □

Lemma A.12. [3] *Let M be a finitely presented R-module and $(N_n)_{n\in\mathbb{N}}$ either a countable projective Mittag-Leffler system of flat R-modules or a countable projective system of Artinian modules. Then the natural map*

$$M \otimes_R \varprojlim_n N_n \to \varprojlim_n M \otimes_R N_n$$

is an isomorphism.

Proof. We first treat the flat case. Take a finite presentation $R^s \to R^r \to M \to 0$ of M. Tensoring with $N := \varprojlim_n N_n$ (resp. N_n)) allows us to express $M \otimes_R N$ (resp. $M \otimes_R N_n$) as the cokernel of the induced map $N^s \to N^r$ (resp. $N_n^s \to N_n^r$). But then the statement follows if we can show $\operatorname{cok}(N^s \to N^r) = \varprojlim_n \operatorname{cok}(N_n^s \to N_n^r)$. Consider the extended exact sequence

$$0 \to C \to R^s \to R^r \to M \to 0,$$

where $C := \ker(R^s \to R^r)$. Since N_n is assumed to be flat we have an exact sequence

$$0 \to C \otimes_R N_n \to N_n^s \to N_n^r \to M \otimes N_n \to 0$$

and one checks that $C \otimes N_n$ is again Mittag-Leffler. Splitting the above sequence into two short sequences shows the vanishing of $\lim^1 N_n^s/(C \otimes N_n)$ since $\lim^1 N_n^s = 0$

[3] This is an adaption of an answer to [Cyr]. Commonly this result is stated requiring N_n to be of finite length, which does not suffice to treat our desired application to $R = A$ affinoid over K unless A is Artinian.

surjects onto it, which via the isomorphism between coimage and image implies the vanishing of $\lim^1(\mathrm{im}(N_n^s \to N_n^r))$ hence the desired $\mathrm{cok}(N^s \to N^r) = \varprojlim_n \mathrm{cok}(N_n^s \to N_n^r)$. In the Artinian case it is well-known that $(N_n)_n$ is Mittag-Leffler and since submodules of Artinian modules are Artinian, we see that $(\ker(N_n^s \to N_n^r))_n$ is also Mittag-Leffler. Hence one may proceed with the same arguments as in the first case. $\square$

Lemma A.13. *Let $n \geq n_0$ and let $\rho : \Gamma_n \to K^\times$ be a non-trivial character of finite order. We denote by $K(\rho)$ the corresponding one-dimensional K-linear representation. Then*

$$\|Z\|_{K(\rho)} \geq |p|^{\frac{1}{p-1}}.$$

Proof. First of all we remark that any such character is automatically continuous by [NS03, Théorème 0.1] because Γ_n is topologically of finite type. $K(\rho)$ is a locally L-analytic representation since the orbit maps are even locally constant. Since $\Gamma_n \cong o_L$ by assumption, we have $\rho(\Gamma_n^{p^m}) = 1$ for some $m \gg 0$. By assumption there exists some $\gamma \in \Gamma_n$ such that $\rho(\gamma) \neq 1$. In this case $\rho(\gamma)$ is some non-trivial p-power root of unity. In particular $|\rho(\gamma) - 1| \geq |p|^{\frac{1}{p-1}}$. Since $K(\rho)$ is one-dimensional the operator norm is multiplicative and because $\sum_{k\geq 0} Z^k \in D(\Gamma_n, K)$ converges to a well-defined operator on $K(\rho)$ we necessarily have $\|Z\|_{K(\rho)} < 1$. By expressing $\delta_\gamma - 1$ as a power series $\delta_\gamma - 1 = ZF(Z)$, we conclude $\|Z\|_{K(\rho)} \geq \|ZF(Z)\|_{K(\rho)} = \|\gamma - 1\|_{K(\rho)} \geq |p|^{1/(p-1)}$, where for the first estimate we use $ZF(Z) \in Zo_K[\![Z]\!]$. $\square$

Lemma A.14. *Let $r \in (|p|^{\frac{1}{p-1}}, 1)$ and fix a lift X_n of Z_n to $D_{\mathbb{Q}_p}(\Gamma_n, K)$ of norm $|X|_r = C(r)$. Let W be a finite A-module with an L-analytic Γ_n-action such that*

$$\|\gamma - 1\|_W < \varepsilon := C(r)^{-1}|p|^{\frac{1}{p-1}}$$

for any $\gamma \in \Gamma_n$. Then the action of $Z_n \in D(\Gamma_n, K)$ is invertible on $W(\rho) = W \otimes_K K(\rho)$ for any non-trivial character $\rho : \Gamma_n \to K^\times$ of finite order.

Proof. It suffices to show that the action of Z_m is invertible for some $m \geq n$ since $Z_{n+1} = \varphi(Z_n) = Q(Z_n)Z_n$. If we fix a $\mathbb{Z}_p$-basis $\gamma_1, \ldots, \gamma_d$ of Γ_n the images $\rho(\gamma_i)$ have to be p-power roots of unity and by replacing Γ_n with a small enough subgroup $\Gamma_m \subset \Gamma_n$ we may and do assume that $\rho(\gamma_i) \neq 1$ for at least one i and $\rho(\gamma_i)$ is a p-th root of unity for every i, i.e. ρ is a non-trivial finite-order character whose values lie in the goup of p-th roots of unity. By Lemma 2.11 we can replace Z_n by Z_m and replace the lift X_n by a lift X_m of Z_m whose r-norm is the same. Since K is one-dimensional the K-linear action of Z on $K(\rho)$ is either zero or invertible. Since ρ is assumed to be non-trivial it has to be invertible. Let $X = \sum_{\mathbf{k}\in\mathbb{N}_0^d} a_{\mathbf{k}}\mathbf{b}^{\mathbf{k}}$ be a preimage of Z in $D_{\mathbb{Q}_p}(\Gamma_m, K)$ of norm $\|X\|_r \leq C(r)$. Our assumptions guarantee that $\gamma_i - 1$ acts on $W \otimes K(\rho)$ with operator norm bounded above by $|p|^{\frac{1}{p-1}} = |\zeta_p - 1|$ and Lemma A.13 asserts that the invertible operator $\mathrm{id} \otimes Z$ has operator norm bounded below by

$|p|^{\frac{1}{p-1}}$. We use the notation Z_{diag} to emphasize that Z acts diagonally and compute

$$\begin{aligned} &\|Z_{\mathrm{diag}}(a \otimes b) - (\mathrm{id} \otimes Z)(a \otimes b)\|_{W(\rho)} && \text{(A.5)}\\ &\leq \sup_{\mathbf{k}} |a_{\mathbf{k}}| \|\mathbf{b}^{\mathbf{k}}(a \otimes b) - a \otimes \mathbf{b}^{\mathbf{k}}(b)\|_{W(\rho)} && \text{(A.6)}\\ &< \sup_{\mathbf{k}} |a_{\mathbf{k}}| \varepsilon |p|^{\frac{1}{p-1}^{|\mathbf{k}|}} \|a \otimes b\|_{W(\rho)} && \text{(A.7)}\\ &< \sup_{\mathbf{k}} |a_{\mathbf{k}}| \varepsilon r^{|\mathbf{k}|} \|a \otimes b\|_{W(\rho)} && \text{(A.8)}\\ &= C(r)^{-1} C(r) |p|^{\frac{1}{p-1}} \|a \otimes b\|_{W(\rho)}, && \text{(A.9)} \end{aligned}$$

□

where we use in (A.7) the estimate

$$\|(\gamma - 1)(a \otimes b) - a \otimes (\gamma - 1)b\| = \|(\gamma - 1)a \otimes \gamma b\| < \varepsilon \|a \otimes b\|$$

and the same inductive argument that we used in the proof of Lemma 2.12 to treat general multi-indices. We conclude $\|Z_{\mathrm{diag}} - (id \otimes Z)\| < \|id \otimes Z\|_{W(\rho)} = (\|\mathrm{id} \otimes Z\|^{-1})^{-1}_{W(\rho)}$ and using 2.1 that the diagonal action of Z is invertible on $W(\rho)$. Note that $K(\rho)$ is one-dimensional hence the action of $1 \otimes Z$ is given by multiplication by a constant in $K^{\times}$ and hence satisfies the last equality $\|id \otimes Z\|_{W(\rho)} = (\|\mathrm{id} \otimes Z\|^{-1})^{-1}_{W(\rho)}$.

A.3 Homological Algebra

In the following R always denotes a commutative unital ring. We recall a useful result on double complexes which we intend to apply to the Herr complex by viewing it as a total complex of a double complex. By a double complex of R-modules we mean a system of R-modules $C^{\bullet,\bullet}$ with horizontal differentials $d^h_{p,q} : C^{p,q} \to C^{p,q+1}$ and vertical differentials $d^v_{p,q} : C^{p,q} \to C^{p+1,q}$ satisfying $0 = (d^h)^2 = (d^v)^2$ and $d^v d^h + d^h d^v = 0$ (i.e. the squares are anti-commutative). The (sum)-total complex of C is the complex $\mathrm{Tot}(C)^n := \bigoplus_{p+q=n} C^{p,q}$ with differentials $d = d^h + d^v$.

Lemma A.15 (Acyclic Assembly Lemma)**.** *Let C be a bounded double complex with exact rows or exact columns. Then* $\mathrm{Tot}(C)$ *is exact.*

Proof. Cf. [Wei95, Lemma 2.7.3 p. 59f.] □

The following Lemma is [KPX14, 4.1.3]. For the convenience of the reader we elaborate on the argument.

Lemma A.16. *Let $C^\bullet, D^\bullet$ be complexes of projective (resp. flat) R-modules with $C^\bullet$ concentrated in degrees $[0, d]$ and $D^\bullet$ bounded above. Suppose we have a quasi-isomorphism $D \to C$ or $C \to D$. Then D is quasi-isomorphic to the complex*

$$\mathrm{coker}(d_{-1}) \to D_1 \to \dots$$

and $\mathrm{coker}(d_{-1})$ *is projective (resp. flat).*

Proof. Since C and D are quasi-isomorphic the cohomology of D has to vanish in degrees < 0 which proves first claim. Without loss of generality we replace D by the complex $\mathrm{coker}(d_{-1}) \to D_1 \to \dots$. Since the complexes C and D are quasi-isomorphic the mapping fibre $\mathrm{cone}(D^\bullet \to C^\bullet)[-1]$ or $\mathrm{cone}(C^\bullet \to D^\bullet)[-1]$ (depending on the direction of the quasi-isomorphism) is acyclic. In the first case the fibre is a complex of the form

$$0 \to \mathrm{coker}(d_{-1}) \to X^1 \to \cdots \to X^n \to 0,$$

where all $X^i = C^i \oplus D^{i-1}$. are projective (resp. flat) by assumption. Because the complex is acyclic we can split it into short exact sequences in particular we have a short exact sequence

$$0 \to \ker(X^{n-1} \to X^n) \to X^{n-1} \to X^n \to 0.$$

Since X^{n-1} and X^n are projective (resp. flat) the module $\ker(X^{n-1} \to X^n) = \mathrm{im}(X^{n-2} \to X^{n-1})$ has to be projective (resp. flat). Inductively we conclude that $\mathrm{coker}(d_{-1})$ has to be projective (resp. flat).
In the second case the fibre is a complex of the form

$$0 \to Y^0 \to Y^1 \oplus \mathrm{coker}(d_{-1}) \to Y^2 \to \cdots \to Y^n \to 0,$$

where $Y^0 = C^0, Y^1 = C^1, Y^i = C^i \oplus D^{i-1}$ are projective (resp. flat) by assumption. By the same argument as before we arrive at a situation where we have a short exact sequence $0 \to Y^0 \to Y^1 \oplus \mathrm{coker}(d_{-1}) \to M \to 0$ with a projective (resp. flat) module M. Then $Y^1 \oplus \mathrm{coker}(d_{-1})$ has to be projective (resp. flat) which implies that $\mathrm{coker}(d_{-1})$ has the same property as a direct summand.

□

Lemma A.17. *Let $\Psi : C^\bullet \to D^\bullet$ be a morphism of complexes in $D^-_{perf}(R)$. Then Ψ is a quasi-isomorphism if and only if the induced morphism*

$$C^\bullet \otimes^{\mathbb{L}}_R R/\mathfrak{m} \to D^\bullet \otimes^{\mathbb{L}}_R R/\mathfrak{m}$$

is a quasi-isomorphism for every maximal ideal $\mathfrak{m} \in R$.

Proof. This is [KPX14, 4.1.5]. □

Lemma A.18. *Let S be a ring and N an S-module. Then $pd_S(M) \leq pd_S(N)$ for any direct summand $M \subset N$.*

Proof. By [Sta21, Tag 065R] we have

$$\mathrm{Ext}^i_S(N, T) = 0$$

for any S-module T and any $i > n$ if and only if $pd_S(M) \leq n$. Writing $N = M \oplus M'$ we conclude that $\mathrm{Ext}^i_S(M, T) = 0$ for $i > pd_S(N)$, proving that $pd_S(M) \leq pd_S(N)$ using again the characterisation above. □

Definition A.19. *A complex of R-module is called* ***pseudo-coherent*** *if it is quasi-isomorphic to a bounded above complex of finite free modules. An R-module is called* ***pseudo-coherent*** *if $M[0]$ is pseudo-coherent.*

Lemma A.20. *Let M be a module over a ring R.*

1. *M is pseudo-coherent if and only if M admits a resolution*

$$\cdots \to F_1 \to F_0 \to M \to 0$$

of finite free modules.

2. *Direct summands of pseudo-coherent complexes are pseudo-coherent.*

Proof. See [Sta21, Tag 064T an Tag 064X]. □

Lemma A.21. *Let S be a finite R algebra, that is free as an R-module such that the inclusion $R \to S$ admits an R-linear section and let M be an S-module that admits (as an R-module) a d-term projective resolution by finitely generated projective R-modules. Then M admits (as an S-module) a d-term projective resolution by finitely generated S modules.*

Proof. By assumption we have a resolution of M with finitely generated projective R-modules

$$0 \to X_n \to \cdots \to X_0 \to M \to 0.$$

Since S is assumed flat over R we obtain an exact sequence of finitely generated projective S-modules

$$0 \to X_n \otimes_R S \to \cdots \to X_0 \otimes_R S \to M \otimes_R S \to 0.$$

In particular $M \otimes_R S$ is a pseudo-coherent S-module. The R-linear split $S = R \oplus \tilde{S}$ gives a $R \otimes_R S = S$ linear split $M \otimes_R S = M \oplus \tilde{M}$ hence M is a direct summand in $M \otimes_R S$. By A.18 the projective dimension of M is bounded by that of $M \otimes_R S$ and by A.20 M is pseudo-coherent and hence admits a (potentially infinte) resolution

$$\ldots F_1 \to F_0 \to M \to 0$$

by finite free S-modules. By A.18 the kernel of $F_d \to F_{d-1}$ is projective and we may truncate the sequence by replacing F_d with $\ker(F_d \to F_{d-1})$. The latter remains finitely generated because it is equal to the image of $F_{d+1} \to F_d$ by construction. □

A.4 Analyticity in fibres

In this section we discuss the property of being analytic in Families. As before let $L \subset K \subset \mathbb{C}_p$ be a complete field extension and let A be K-affinoid. For each maximal ideal $\mathfrak{m}_z \subset A$ corresponding to $z \in \mathrm{Sp}(A)$ we obtain a (φ_L, Γ_L)-module M_z over $\mathcal{R}_{A/\mathfrak{m}_z}$ (cf. Lemma 1.56) and one checks that the latter is L-analytic if the original module is L-analytic by projecting the orbit maps down to M_z.

Definition A.22. *Let M be a (φ_L, Γ_L)-module over $\mathcal{R}_A$. We call M **fibre-wise L-analytic** if M_z is L-analytic for each $z \in \mathrm{Sp}(A)$.*

We will show that for reduced A fibre-wise L-analyticity is equivalent to L-analyticity. We collect some preliminaries from commutative algebra.

Definition A.23. *Let R be a commutative ring. Let M be an R-module. We define $\mathrm{Rad}(M)$ as the intersection of all maximal submodules of M. As usual we denote by $J(R)$ the (Jacobson) radical of R.*

Lemma A.24. *Let R be a commutative ring. Let $M, (M_i)_{i\in I}$ be R-modules. We have*

1. $\mathrm{Rad}(M) = \bigcap_{\mathfrak{m}\in \mathrm{Max}(R)} \mathfrak{m}M$
2. $\mathrm{Rad}(\bigoplus_{i\in I} M_i) = \bigoplus_{i\in I} \mathrm{Rad}(M_i)$.
3. $\mathrm{Rad}(M) = J(R)M$ *for every free R-module.*
4. $\mathrm{Rad}(M) = J(R)M$ *for every projective R-module.*

Proof. 1.) and 2.) are well-known, 3.) is clear from 2.), 4.) follows by writing a projective module as a direct summand of a free module and applying 3.). □

Lemma A.25. *Let A be K-affinoid then A is Jacobson. In particular if A is reduced we have $J(A) = 0$.*

Proof. See [BGR84, 3.8 Lemma 9 and 6.1 Proposition 3]. □

Lemma A.26. *Let A, B be K-affinoid and reduced. Then*

$$\mathrm{Sp}(A) \times_K \mathrm{Sp}(B) = \mathrm{Sp}(A \hat{\otimes}_K B)$$

is reduced.

Proof. See [Duc09, Théorème 8.1] for the statement concerning the corresponding Berkovich spaces using that over a perfect field geometrically reduced and reduced are equivalent. By [BGR84, 7.3.2. Corollary 9] an affinoid is reduced if and only if the ring of global sections is reduced. On the other hand for a K-affinoid $\mathcal{A}$ (in the sense of Berkovich) the corresponding Berkovich space $\mathcal{M}(\mathcal{A})$ is reduced if and only if $\mathcal{A}$ is reduced as a ring by combining [Ber93, Theorem 2.2.1] and [Ber93, Theorem 2.1.1]. □

Proposition A.27. *Let A be a reduced K-affinoid and M be a fibre-wise L-analytic (φ_L, Γ_L)-module M over $\mathcal{R}_A$. Then M is L-analytic.*

Proof. After choosing a model M^r we can view M^r as a vector bundle on $\operatorname{Sp} A \times_K \operatorname{Sp}(\mathcal{R}_K^{[r,1)})$. By 1.42 it suffices to show that the derived action Lie $\to \operatorname{End}(M^r)$ is L-linear. Equivalently it suffices to show that for every $m \in M^r$ and $\lambda \in L$ we have $\lambda\nabla(m) - \nabla(\lambda m) = 0$. This vanishing can be checked on each $M^{[r,s]}$. In this case $M^{[r,s]}$ is a projective module over the reduced (by A.26) affinoid $X := \operatorname{Sp} A \times_K \operatorname{Sp}(\mathcal{R}_K^{[r,s]})$. Let $m \in M^{[r,s]}$ and $\lambda \in L$. Then $v := \lambda\nabla(m) - \nabla(\lambda m)$ belongs by assumption to $\mathfrak{m}_x M^{[r,s]}$ for every $x \in X$. More precisely v belongs a priori to $\bigcap_{z \in \operatorname{Sp}(A)} \mathfrak{m}_z M^{[r,s]}$. By 1.56 $\mathfrak{m}_z \mathcal{R}_A^{[r,s]}$ is prime because $\mathcal{R}_{A/\mathfrak{m}_z}^{[r,s]}$ is a domain as a subring of $\mathcal{R}_{\mathbb{C}_p}^{[r,s]}$. Thus the ideal generated by $\mathfrak{m}_z$ is a radical ideal and because $\mathcal{O}_X(X)$ is Jacobson it is the intersection of all maximal ideals containing it. However every $x \in \operatorname{Sp}(X)$ lies above some $z \in \operatorname{Sp}(A)$ which implies that the intersection is taken over all of X. From A.24 and A.25 we conclude $v = 0$. $\square$

Errata

Lemma 1.11: The proof of the lemma contains a false claim. Evidently $c_o(K)$ is not the completion of $\bigoplus_{n\in\mathbb{N}} K$ with respect to the product topology, but rather the completion with respect to the sup-norm topology (which can be viewed as a subset of the product). Nevertheless $c_o(K)\hat{\otimes}_K V$ can be functorially identified with the space of zero sequences in V and strict maps are preserved (cf. proof of [Eme17, Proposition 2.1.23]).

Lemma 1.43: In the proof there is an inaccuracy. In order to commute the projective limit and the completed projective tensor product, one has to represent the Fréchet spaces V_i, W_i as projective limits of Banach spaces *with dense transition maps*. This can be done without loss of generality by replacing a countable system of semi-norms inducing the topology on a given Fréchet space by an ordered system and taking the projective limit over the corresponding Hausdorff completions.

Proposition 1.74. The line on the top of page 36: *As was seen in 1.23* should be replaced by *as was seen in the proof of 1.22*. More precisely, the claim follows from [Sch02, Corollary 18.8] using the reflexivity of E in the known case.

Definition 2.5. The definition in question contains some ambiguities which we would like to clarify. For $n \geq n_0$ we define $\mathcal{R}_K^I(\Gamma_n)$ as the ring of I-convergent Laurent series in the variable Z_n. The map $(l_n^{-1})_*$ is thus given by mapping $T \in \mathcal{R}_K^+ \cong D(o_L, K)$ to Z_n. We opted for the complicated notation to keep track of the charts as the commutativity of the diagrams involved is a consequence of our specific choices of charts. The commutative diagrams on page 43 contain a typo: ι_{l+m} should be called ι_{n+m}.

Lemma A.21 is incorrect in the generality stated by us. For our intended application the following form is sufficient.

Lemma E.1. *Let G be a compact L-analytic group, let $H \subset G$ be a (finite index) normal open subgroup, let A be K-affinoid, let $S := D(G, A)$*[4]*, $R := D(H, A)$ and let M be an S-module that admits (as an R-module) a $d+1$-term ($d \geq 0$) projective resolution by finitely generated projective R-modules. Then M admits (as an S-module) a $d+1$-term projective resolution by finitely generated S-modules.*

Proof. Let T be an S-module. Using that the Dirac distributions δ_g, where g runs through a system of representatives of G/H, form a basis of S as an R-module,

[4] We take $D(G, A) := A\hat{\otimes}_K D(G, K)$ as a convention.

we have $\mathrm{Hom}_S(M,T) = \mathrm{Hom}_R(M,T)^{G/H}$, where G/H acts on a homomorphism f via $(gf)(-) = gf(g^{-1}-)$. Because A contains a field of characteristic 0 the functor $(-)^{G/H}$ is exact on $A[G/H]$-modules and we obtain corresponding isomorphisms

$$\mathrm{Ext}^i_S(M,T) \cong \mathrm{Ext}^i_R(M,T)^{G/H}.$$

In particular the assumption on the length of the resolution asserts

$$\mathrm{Ext}^i_S(M,T) = 0 \tag{$*$}$$

for every $i > d$. This implies that the projective dimension of M is bounded by d. By assumption we have a resolution of M with finitely generated projective R-modules. From [Sta21, Tag 064U] we obtain that M is in particular pseudo-coherent as an R-module. The ring S is finite free as an R module and hence pseudo-coherent. Applying [Sta21, Tag 064Z] we can conclude that M is pseudo-coherent as an S-module. By Lemma [Sta21, Tag 064T] admits a (potentially infinte) resolution

$$\dots F_1 \to F_0 \to M \to 0$$

by finite free S-modules.
Using $(*)$ the kernel of $F_d \to F_{d-1}$ is projective and we may truncate the sequence by replacing F_d with $\ker(F_d \to F_{d-1})$. The latter remains finitely generated because it is equal to the image of $F_{d+1} \to F_d$ by construction. □

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