Inaugural-Dissertation

zur Erlangung der Doktorwürde der
Gesamtfakultät für Mathematik, Ingenieur- und Naturwissenschaften der

Ruprecht-Karls-Universität Heidelberg

vorgelegt von
Milan Malčić, M.Sc. Mathematik
geboren in
Banja Luka, Bosnien-Herzegowina

Tag der mündlichen Prüfung: ______________

A relative trace map and its compatibility with Serre duality in rigid analytic geometry

Betreuer: Prof. Dr. Otmar Venjakob

Abstract
Given a complete non-archimedean valued field K, we discuss a relative trace map attached to any finite étale morphism of smooth rigid-analytic Stein spaces over K and prove that it is compatible with the trace maps that arise in the Serre duality theory on the respective Stein spaces. Our proof builds on the technique of investigating the trace map of a rigid Stein space via a relation between algebraic local cohomology and compactly supported rigid cohomology established in the work of Beyer [Bey97a]. For this purpose we also prove a generalization of a theorem of Bosch [Bos77] which concerns the connectedness of formal fibers of a distinguished affinoid space. This closes an argumentative gap in [Bey97a].
Furthermore, we consider the behaviour of any rigid-analytic Stein space and its trace map under (completed) base change to any complete extension field K'/K and prove that there are natural base change comparison maps that yield a commutative diagram relating Serre duality over K with Serre duality over K'.
Finally we discuss the recent work of Abe and Lazda [AL20], which constructs a trace map on proper pushforwards of analytic adic spaces, and we explain how some of their results can be related to ours (via Huber's functor from the category of rigid analytic varieties to the category of adic spaces).

Zusammenfassung
Gegeben einen vollständig nicht-archimedisch bewerteten Körper K betrachten wir eine relative Spurabbildung, die sich jedem endlich étalen Morphismus glatter rigid-analytischer Steinscher Räume über K zuordnen lässt, und beweisen ihre Kompatibilität mit den in der Serre-Dualitätstheorie auf den jeweiligen Steinschen Räumen auftretenden Spurabbildungen. Unser Beweis baut auf der in Beyers Arbeit [Bey97a] verwendeten Methode der Untersuchung der Spurabbildung eines rigiden Steinschen Raumes mittels eines Zusammenhangs zwischen algebraischer lokaler Kohomologie und rigider Kohomologie mit kompakten Trägern auf. Zu diesem Zweck beweisen wir auch eine Verallgemeinerung eines Theorems von Bosch [Bos77] über Zusammenhangskomponenten formeller Fasern ausgezeichneter affinoider Räume. Damit lässt sich eine argumentative Lücke in [Bey97a] schließen.
Ferner untersuchen wir das Verhalten rigid-analytischer Steinscher Räume und ihrer Spurabbildungen unter (vervollständigtem) Basiswechsel zu jedem vollständigen Erweiterungskörper K'/K und beweisen die Existenz natürlicher Basiswechsel-Vergleichsmorphismen, welche die Serre-Dualitätspaarung über K mit jener über K' in einem kommutativen Diagramm vereinen.
Schließlich gehen wir auf eine neuere Arbeit von Abe und Lazda [AL20] ein und erklären, wie einige ihrer Ergebnisse (zur Konstruktion einer Spurabbildung auf direkten Bildern mit kompakten Trägern auf analytischen adischen Räumen) mit unseren Ergebnissen in Beziehung gesetzt werden können (über den Huber'schen Funktor von der Kategorie der rigid-analytischen Räume in die Kategorie der adischen Räume).

Acknowledgements
First and foremost, I want to express gratitude to my advisor Prof. Dr. Otmar Venjakob, who inspired me to work on these questions and provided me with the opportunity to do so. This thesis has benefited immensely from his advice and support.
Chapter 2 has resulted from an email correspondence with Prof. Dr. Werner Lütkebohmert. I wish to thank him for responding to my pointing out gaps in the literature, for suggesting the strategy of base change and descent, and for helping me clarify several technical details. I am also grateful to Prof. Dr. Peter Schneider, who, together with Otmar, showed interest in the genesis of Chapter 2, was helpful with several questions and facilitated communication with Prof. Lütkebohmert.
I thank Prof. Dr. Katharina Hübner for always taking the time to discuss questions about my research.
Moreover, I wish to acknowledge and express gratitude for the funding of my undergraduate education and my PhD research by, respectively, the Deutscher Akademischer Austauschdienst (DAAD, the German Academic Exchange Service) and the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) TRR 326 *Geometry and Arithmetic of Uniformized Structures*.
It was a pleasure to work at the Institute of Mathematics at Heidelberg University, where I had the company of many remarkable colleagues, staff members, students and friends.
Finally, I thank my parents Gordana and Predrag for their love and support throughout my life.

Contents

Introduction

Background

Cohomology theory of coherent sheaves on complex analytic spaces has been an active and fruitful area of research since the middle of the past century, one of its finest results being the *Serre duality theorem*. On the other hand, in modern number theory one often works over a *non-archimedean field*: a field that is complete with respect to a specified nontrivial absolute value satisfying the *non-archimedean triangle inequality* $|x+y| \leq \max(|x|,|y|)$. The field of p-adic numbers $\mathbb{Q}_p$ is a key example of a non-archimedean field. When working over a non-archimedean field K instead of $\mathbb{C}$, the analogue of a complex analytic space is a *rigid analytic space*. Indeed, one could argue that there is another analogue, namely the classical notion of a K-analytic manifold defined using charts and atlases, as in [Ser92]. However, these are only of limited use in non-archimedean geometry since they are totally disconnected. Therefore, Tate defined new K-analytic spaces provided with an extra "topological" structure (a certain Grothendieck topology) which allows one to say that the closed unit ball is "connected" and which yields a satisfactory theory of coherent sheaves (and hence a theory of "analytic continuation"). To distinguish them from classical K-analytic spaces, Tate called his new spaces *rigid analytic spaces*. The advancement of the theory of coherent sheaves on rigid analytic spaces in particular led to the question whether one can establish a Serre-type duality in this setting as well. This question was first answered affirmatively for rigid Stein spaces by Bruno Chiarellotto in [Chi90], who introduced a notion of cohomology with compact support for rigid spaces, a canonical topology on cohomology groups and proved:

Theorem 1 (Serre duality for smooth rigid Stein spaces). *Let K be a complete non-archimedean valued field. Let X be a smooth rigid K-space of dimension d and let $\omega_X = \wedge^d \Omega^1_{X/K}$ be the sheaf of holomorphic d-forms on X. Let $H^*_c(X,-)$ denote the cohomology with compact support. If X is Stein, then there is a canonical trace morphism*

$$t\colon H^d_c(X,\omega_X) \longrightarrow K$$

which has the following property: If $\mathcal{F}$ is a coherent sheaf on X, then the composite

$$H^{d-i}_c(X,\mathcal{F}) \times \operatorname{Ext}^i_X(\mathcal{F},\omega_X) \longrightarrow H^d_c(X,\omega_X) \xrightarrow{t} K$$

of t with the Yoneda-Cartier pairing induces an isomorphism of topological K-vector spaces

$$H_c^{d-i}(X,\mathcal{F})^\vee \xrightarrow{\sim} \operatorname{Ext}^i_X(\mathcal{F},\omega_X)$$

for all $i \geq 0$.

Here $H_c^{d-i}(X,\mathcal{F})^\vee$ denotes the space of continuous linear forms on $H_c^{d-i}(X,\mathcal{F})$, equipped with the strong dual topology. Moreover, $\operatorname{Ext}^i_X(\mathcal{F},\omega_X)$ is equipped with the canonical topology for global sections of a coherent sheaf (see Definition 1.41 below), as the discussion preceding Remark 5.4 below shows that indeed $\operatorname{Ext}^i_X(\mathcal{F},\omega_X) = H^0(X,\mathcal{E}xt^i_X(\mathcal{F},\omega_X))$.

Chiarellotto's approach was to prove the theorem in the special case of the affine rigid space and then show that the general case can be reduced to this special case by using the fact that a Stein space always admits a closed immersion into an affine rigid space. Peter Beyer subsequently provided another proof of Serre duality for rigid Stein spaces, appearing in his dissertation [Bey97b, Satz 7.1] and outlined in the abridged version of his dissertation [Bey97a, Remark 5.1.5] published as an article. Instead of relying on the embedding theorem for Stein spaces, his proof establishes and exploits a relationship between the compactly supported rigid cohomology and algebraic local cohomology. This allows him to obtain a rather explicit (albeit complicated) description of the trace map t, which is a major advantage of his approach.[1] He first constructs the trace map in the special case of so-called *special affinoid wide-open spaces* (which we review in Chapter 3 below), then he uses the fact that a Stein space can be exhausted by subspaces of this type. To prove that the trace maps in the covering are compatible and thus glue to a global trace map, he uses the aforementioned relation to local cohomology.

Structure of this thesis

The main objective of this thesis (achieved in Chapters 5 and 6) is to build upon Beyer's methods to prove companion theorems to Theorem 1. For instance, if $\alpha\colon X \longrightarrow Y$ is a finite étale morphism of smooth connected d-dimensional Stein spaces over K, there is a natural "relative trace map"

$$t_\alpha\colon \alpha_*\omega_X \longrightarrow \omega_Y$$

(see Definition 5.2) which, for an affinoid $\operatorname{Sp}(A) \subseteq Y$ and its preimage $\operatorname{Sp}(B) \subseteq X$, is given by

$$\begin{aligned} \Omega^d_{B/K} = \Omega^d_{A/K} \otimes_A B &\longrightarrow \Omega^d_{A/K} \\ \omega \otimes b &\longmapsto \operatorname{Tr}_{B/A}(b) \cdot \omega \end{aligned}$$

[1] We also mention that Beyer's technique moreover allows him to recover the results of [van92], i.e. Serre duality for proper rigid spaces. Whereas Van der Put builds upon Chiarellotto's methods by using the fact that a proper rigid space X can be covered by Stein spaces and showing that the local dualising data provided by Chiarellotto's result glues to a global dualising pair (ω_X, t), Beyer's method is again more explicit.

(see Section 5.3). Here and elsewhere, $\Omega^d_{A/K} := \wedge^d \Omega^1_{A/K}$ and $\Omega^1_{A/K}$ denotes the "universally finite" differential module, characterised by being finitely generated over A and universal for derivations into finitely generated A-modules. Moreover, $\mathrm{Tr}_{B/A}$ is the trace of the finitely generated projective A-module B. (Recall that for such a module, the canonical map $B^* \otimes_A B \xrightarrow{\sim} \mathrm{End}_A(B)$ is an isomorphism and $\mathrm{Tr}_{B/A}$ is defined as the composite $B \longrightarrow \mathrm{End}_A(B) \cong B^* \otimes_A B \longrightarrow A$ where $B^* := \mathrm{Hom}_A(B, A)$, the first map sends $b \in B$ to the endomorphism given by multiplication by b, and the last map is given by "evaluation".) If B is free of finite rank over A, then $\mathrm{Tr}_{B/A}$ coincides with the usual trace map from linear algebra. One of our major results is the following compatibility of this relative trace map with Serre duality:

Theorem A (Theorem 5.5, Proposition 5.17)**.** *Let $\alpha\colon X \longrightarrow Y$ be a finite étale morphism of smooth connected d-dimensional Stein spaces over K. Then the diagram*

$$\begin{tikzcd}
 & H^d_c(Y,\omega_Y) \arrow[dr, "t_Y"] & \\
H^d_c(Y,\alpha_*\omega_X) \arrow[ur, "H^d_c(Y{,}t_\alpha)"] \arrow[dr, "\cong"'] & & K. \\
 & H^d_c(X,\omega_X) \arrow[ur, "t_X"'] &
\end{tikzcd}$$

commutes, where $t_\alpha\colon \alpha_*\omega_X \longrightarrow \omega_Y$ is the relative trace map. In particular, letting $\mathrm{Ext}^i_X(\alpha^*\mathcal{G}, \omega_X) \longrightarrow \mathrm{Ext}^i_Y(\mathcal{G}, \omega_Y)$ denote the morphism constructed from the relative trace map and the adjunction morphism $\mathcal{G} \longrightarrow \alpha_*\alpha^*\mathcal{G}$ as in Proposition 5.17, one obtains a commutative diagram of Serre duality pairings

$$\begin{tikzcd}
H^{d-i}_c(X,\alpha^*\mathcal{G})^\vee \arrow[r, "\sim"] \arrow[d] & \mathrm{Ext}^i_X(\alpha^*\mathcal{G},\omega_X) \arrow[d] \\
H^{d-i}_c(Y,\mathcal{G})^\vee \arrow[r, "\sim"] & \mathrm{Ext}^i_Y(\mathcal{G},\omega_Y)
\end{tikzcd}$$

for every coherent sheaf $\mathcal{G}$ on Y and all $i \geq 0$.

One incentive to prove such a compatibility result was its relevance in the study of reciprocity laws for (φ_L, Γ_L)-modules over Lubin-Tate extensions done by Schneider and Venjakob, which by now has been published as a preprint [SV23]. In [SV23, §4.2], our compatibility result Theorem A is used as a conceptual way to obtain functorial properties of pairings arising from Serre duality on various rigid Stein spaces.

Chapter 5 is devoted to proving Theorem A. In a first step, carried out in Section 5.2, we prove that X and Y can be covered by special affinoid wide-open spaces in such a way that the coverings are compatible in our relative situation – in the sense that they behave well with respect to the finite étale morphism $\alpha\colon X \longrightarrow Y$. This allows us to reduce the proof of Theorem A to the case of special affinoid wide-open spaces. The second step (carried out in Section 5.4) then proves this case using the relation to local cohomology.

Concerning the first step of the proof, the relevant background (on formal fibers of rigid spaces, Stein spaces and special affinoid wide-open spaces) is given, on the one hand, in Sections 1.1 and 1.2, but also in Chapters 2 and 3. Sections 1.1 and 1.2 mostly consist of facts gathered from the literature, but we provide the occasional proof for non-obvious claims in [Bey97a] made implicitly and without proof or reference (e.g. Lemma 1.25).

Chapter 2, on the other hand, contains original material. Namely, there is a crucial technical lemma [Bey97a, Lemma 4.2.2] underlying Beyer's arguments, which asserts that special affinoid wide open spaces can be characterised in two (equivalent) ways. The proof of this lemma in turn relies on Bosch's Theorem on the connectedness of formal fibers (Theorem 2.3 below). However, this theorem contains the assumption that the affinoid algebra under consideration is *distinguished* (see Definition 1.16) and this assumption is not satisfied in the general setting of [Bey97a], resulting in an argumentative gap. Driven by the endeavour to bridge this gap, the second chapter of this thesis is devoted to proving a generalised version of Bosch's Theorem on the connectedness of formal fibers (Theorem 2.4), now valid for affinoid algebras that become distinguished after a base change to a finite Galois extension. This indeed bridges the gap in the setting of smooth special affinoid wide-open spaces, since Subsection 1.1.3 proves that the relevant affinoid algebras can be made distinguished after a base change to a finite Galois extension, allowing us to deduce Corollary 2.5.

Chapter 3 then uses the results of Chapter 2 to remedy the aforementioned gap and to lay out the necessary background on special affinoid wide-open spaces that is used in subsequent chapters. We moreover fill in some proofs that are omitted in the literature, in particular the argument in Proposition 3.10.

Returning to our outline of the proof of Theorem A in Chapter 5, we have noted that the second step of the proof involves the relation to local cohomology. In particular, this requires obtaining an explicit description of the relative trace map at the level of local cohomology, which we achieve in Lemma 5.14. The way for this is paved by Sections 1.3 and 1.4 and Chapter 4, whose purpose is to gather the relevant background facts from the literature. In Section 1.3 we recall the basics of local cohomology, whereas in Chapter 4 we review Beyer's construction of the trace map and the relation between local cohomology and compactly supported rigid cohomology.

Chapter 6 turns to investigating the behaviour of the Serre duality pairing from Theorem 1 under (completed) base change. Let K' be a complete field extension of K and, for any (separated) rigid space X over K, let

$$X' := X \widehat{\otimes}_K K'$$

denote the base change of X to K'. In Remark 6.1 and Section 6.1, we recall some background on completed base change, in particular that there is an exact "pullback" functor

$$\mathcal{F} \rightsquigarrow \mathcal{F}'$$

from coherent sheaves on X to coherent sheaves on X'. We moreover recall that the properties of being smooth, having dimension d and being Stein are all stable under base change. For a special affinoid wide-open space (resp. a Stein space) X over K, we construct comparison maps

$$H_c^j(X,\mathcal{F}) \otimes_K K' \longrightarrow H_c^j(X',\mathcal{F}')$$

in Section 6.2 (resp. Section 6.4). Our main result in Chapter 6 then reads as follows:

Theorem B (Corollary 6.18)**.** *Let X be a smooth rigid Stein K-space of dimension d. Then, for every coherent sheaf $\mathcal{F}$ on X, the diagram*

$$\begin{array}{ccccccc} H_c^{d-i}(X',\mathcal{F}') & \times & \mathrm{Ext}^i_{X'}(\mathcal{F}',\omega_{X'}) & \longrightarrow & H_c^d(X',\omega_{X'}) & \xrightarrow{t_{X'}} & K' \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ H_c^{d-i}(X,\mathcal{F}) & \times & \mathrm{Ext}^i_{X}(\mathcal{F},\omega_{X}) & \longrightarrow & H_c^d(X,\omega_{X}) & \xrightarrow{t_{X}} & K \end{array} \qquad (*)$$

commutes for all $i \geq 0$.

This result is likewise relevant in the context of [SV23, §4.2], where it is used to argue that the constructions carried out there are compatible with base change. The structure of our proof of Theorem B is as follows: we first investigate how base change affects separable Noether normalisation maps (Lemma 6.9) and how the trace of the associated extension of total rings of fractions behaves on differentials (Lemma 6.14). This allows us to prove (in Proposition 6.15) that the square on the right-hand side of the diagram $(*)$ commutes when X is a special affinoid wide-open space. Then we extend this to the case of a general Stein space by a limit argument in Theorem 6.15. Finally, in Proposition 6.17 we prove that the rest of the diagram $(*)$ commutes, i.e. that the Yoneda-Cartier pairing is compatible with base change.

In the final Chapter 7 we discuss the recent work of Abe and Lazda [AL20], and explain how some of their results can be related to our Theorem A. In [AL20], Abe and Lazda consider partially proper morphisms $f\colon X \longrightarrow Y$ of analytic adic spaces, the proper pushforward $f_!$ and its derived functor $\boldsymbol{R}f_!$, which in particular yields a notion of (relative) compactly supported cohomology. For a special class of smooth partially proper morphisms, they construct an $\mathcal{O}_Y$-linear "trace map" $\mathrm{Tr}_{X/Y}$ in the derived category of $\mathcal{O}_Y$-modules that is local on the base Y and compatible with composition (see, for instance, Proposition 7.10 below).
At the beginning of Chapter 7, we recall Huber's functor which induces an equivalence from the category of quasi-separated rigid analytic varieties over $\mathrm{Sp}(K)$ to the category of quasi-separated locally of finite type adic spaces over $\mathrm{Spa}(K,\mathfrak{o}_K)$.

Huber's functor allows us to embed our setting into the setting of Abe and Lazda. In Lemma 7.8 we compare our notion of compactly supported cohomology of a rigid Stein space to Abe and Lazda's compactly supported cohomology of the corresponding adic space. Buildung upon this, we prove in Subsection 7.3.1 that their trace map $\mathrm{Tr}_{X/Y}$ can then recover Beyer's trace map when $X \longrightarrow Y = \mathrm{Spa}(K, \mathfrak{o}_K)$ is the structure morphism of a Stein space. Finally, in Subsection 7.3.3, we prove that *assuming that Abe and Lazda's trace map* $\mathrm{Tr}_{X/Y}$ *coincides with the classical trace map* $\alpha_* \mathcal{O}_X \longrightarrow \mathcal{O}_Y$ *whenever* $\alpha\colon X \longrightarrow Y$ *is finite étale* (see Question 7.12 which is discussed thoroughly in Section 7.3.2 below), one can relate Abe and Lazda's trace map $\mathrm{Tr}_{X/Y}$ to our relative trace map t_α and recover our compatibility result (Theorem A) from their result on the compatibility of $\mathrm{Tr}_{X/Y}$ with respect to composition.

Notations and conventions

- K is a complete non-archimedean nontrivially valued field.
- $\mathfrak{o}_K$ is the ring of integers in K.
- k is the residue field of K.
- $K\langle\xi_1,\ldots,\xi_n\rangle$ is the free Tate algebra in n variables, for which we also write T_n, or $T_n(K)$ when we wish to emphasise the base field.
- We set

$$\begin{aligned}\mathbb{D}^n &:= \mathbb{D}^n_K := \operatorname{Sp} K\langle\xi_1,\ldots,\xi_n\rangle, \text{ and}\\ \mathring{\mathbb{D}}^n &:= \mathring{\mathbb{D}}^n_K := \{x \in \mathbb{D}^n \colon |\xi_i(x)| < 1 \text{ for all } i = 1,\ldots,n\}.\end{aligned}$$

  For $x \in \mathbb{D}^n$, we often write $|x| < 1$ to mean $|\xi_i(x)| < 1$ for all $i = 1,\ldots,n$.
- For an affinoid algebra R, points in the associated affinoid space $Z = \operatorname{Sp}(R)$ will usually be denoted by z and the corresponding maximal ideal in R by $\mathfrak{m}_z$.
- For a morphism $R \longrightarrow S$ of affinoid algebras, $\Omega^1_{S/R}$ denotes the universal *finite* differential module (cf. [BLR95, §1]).
- All rigid analytic spaces are assumed to be separated.[2]
- All rings are commutative and unital.
- The total ring of fractions of a ring A (i.e. the localization of A with respect to the set of all non-zero divisors in A) is denoted by $Q(A)$.
- The radical of an ideal I is denoted by $\sqrt{I}$.
- For an extension K'/K of complete fields and a (separated) rigid space X over K, the base change of X to K' is the rigid space $X \mathbin{\widehat{\otimes}}_K K'$ defined as in [BGR84, §9.3.6]. See the beginning of Chapter 6 for some basic facts on base change.

[2] Note that this assumption is superflous for Stein spaces, since they are automatically separated by [Lüt73, Satz 3.4].

Chapter 1

Preliminaries

1.1 Several special topics from the theory of rigid analytic spaces

1.1.1 The reduction of an affinoid space and formal fibers

Given a K-affinoid algebra R, an $f \in R$ and a point z in the affinoid space $Z := \mathrm{Sp}(R)$, we recall that $|f(z)|$ is defined as follows. The residue field $R/\mathfrak{m}_z$ is finite over K, hence one can choose an embedding $R/\mathfrak{m}_z \hookrightarrow \overline{K}$ into a fixed algebraic closure $\overline{K}$ of K. Then $|f(z)|$ is computed by choosing any embedding $R/\mathfrak{m}_z \hookrightarrow \overline{K}$ and taking the absolute value of the image of the residue class $f \bmod \mathfrak{m}_z$. This is well-defined since any two embeddings of $R/\mathfrak{m}_z$ into $\overline{K}$ are Galois conjugate over K and $\mathrm{Gal}(\overline{K}/K)$ acts isometrically on $\overline{K}$.

Remark 1.1. *Let $\varphi\colon R \longrightarrow S$ be a morphism of affinoid algebras and let*

$$Z_1 := \mathrm{Sp}(S) \xrightarrow{\alpha} \mathrm{Sp}(R) =: Z_2$$

be the morphism of affinoid spaces associated to φ. Then, for every point $z_1 \in Z_1$ and every element $r \in R$, we have

$$|r(\alpha(z_1))| = |\varphi(r)(z_1)|.$$

Proof. This is a well-known and straightforward fact, due to the following simple argument. Fixing an embedding $\iota\colon S/\mathfrak{m}_{z_1} \hookrightarrow \overline{K}$ and setting ι' to be the composite

$$\iota'\colon R/\varphi^{-1}(\mathfrak{m}_{z_1}) \xhookrightarrow{\varphi} S/\mathfrak{m}_{z_1} \xhookrightarrow{\iota} \overline{K}$$

we have

$$\iota'(r \bmod \varphi^{-1}(\mathfrak{m}_{z_1})) = \iota(\varphi(r) \bmod \mathfrak{m}_{z_1})$$

by design, from which the desired equality follows by taking the absolute value on both sides, because $\mathfrak{m}_{\alpha(z_1)} = \varphi^{-1}(\mathfrak{m}_{z_1})$ by definition of α. $\square$

Recall that the *supremum semi-norm* of an $f \in R$ is given by $|f|_{\sup} = \sup_{z\in Z}|f(z)|$. We have the $\mathfrak{o}_K$-algebra of all power-bounded elements in R

$$\mathring{R} = \{f \in R\colon |f|_{\sup} \leq 1\}$$

and the $\mathring{R}$-ideal of all topologically nilpotent elements in R

$$\check{R} = \{f \in R\colon |f|_{\sup} < 1\}.$$

Then

$$\widetilde{R} = \mathring{R}/\check{R}$$

is called the *reduction* of R and $\mathring{R} \longrightarrow \widetilde{R}$, $f \longmapsto \widetilde{f}$ denotes the canonical projection. In particular, $\mathring{K}$ is the ring of integers of K and $\widetilde{K}$ is the residue field of K.

Remark 1.2. (i) *In general, $\widetilde{R}$ is always reduced, since $|\cdot|_{\sup}$ is power-multiplicative (i.e. $|f^n|_{\sup} = |f|_{\sup}^n$ holds for all $f \in R$ and $n \in \mathbb{N}$).*

(ii) *$\widetilde{R}$ is a finitely generated $\widetilde{K}$-algebra by [BGR84, 6.3.4/Corollary 3] and hence in particular Noetherian.*

Definition 1.3. *The* reduction *$\widetilde{Z}$ of the affinoid space $Z := \mathrm{Sp}(R)$ is the affine algebraic variety given by the maximal spectrum of $\widetilde{R}$.*

Recall that there is a functorial reduction map

$$p\colon Z \longrightarrow \widetilde{Z},$$

defined as follows (cf. [BGR84, §7.1.5]): Given a $z \in Z$, the canonical projection map $\sigma\colon R \longrightarrow R/\mathfrak{m}_z$ induces a map $\widetilde{\sigma}\colon \widetilde{R} \longrightarrow (R/\mathfrak{m}_z)^{\sim}$ and we set $\mathfrak{m}_{p(z)} := \ker(\widetilde{\sigma})$, which indeed is a maximal ideal in $\widetilde{R}$. Letting $\widetilde{\mathfrak{m}}_z$ denote the image of $\mathfrak{m}_z \cap \mathring{R}$ under the projection $\mathring{R} \twoheadrightarrow \widetilde{R}$, we have $\mathfrak{m}_{p(z)} = \sqrt{\widetilde{\mathfrak{m}}_z}$ by [BGR84, 7.1.5/Proposition 1]. The reduction map

$$p\colon Z \twoheadrightarrow \widetilde{Z}$$

is surjective by [BGR84, 7.1.5/Theorem 4].
We often use the notation

$$p(z) \quad \text{and} \quad \widetilde{z}$$

interchangeably.

Remark 1.4. *Let $Z_1 \longrightarrow Z_2$ be a surjective map of affinoid rigid spaces. Then its reduction $\widetilde{Z_1} \longrightarrow \widetilde{Z_2}$ is surjective as well.*

Proof. We have a commutative diagram

$$\begin{tikzcd} Z_1 \arrow[r, two heads] \arrow[d, two heads] & Z_2 \arrow[d, two heads] \\ \widetilde{Z_1} \arrow[r] & \widetilde{Z_2} \end{tikzcd}$$

where all arrows with two heads are surjective. Hence the remaining arrow must also be surjective. □

Proposition 1.5. *Let $f \in \mathring{R}$ and $z \in \mathrm{Sp}(R)$. Then $|f(z)| < 1$ if and only if $\widetilde{f}(p(z)) = 0$.*

Proof. This fact, which is also recorded in [BGR84, 7.1.5/Proposition 2], follows immediately from the definition of p. □

Definition 1.6. *For $z \in Z$, the fibers*

$$Z_+(z) := p^{-1}(p(z))$$

of the reduction map $p\colon Z \longrightarrow \widetilde{Z}$ are called the formal fibers of Z, *the terminology and notation being as in [Bos77].*

For any $f_1, \dots, f_r \in \mathring{R}$ with $\sqrt{(\widetilde{f_1}, \dots, \widetilde{f_r})} = \mathfrak{m}_{p(z)}$, we have

$$Z_+(z) = \{y \in Z\colon |f_i(y)| < 1 \text{ for all } i = 1, \dots, r\}$$

as a consequence of Proposition 1.5. Thus $Z_+(z)$ is an admissible open analytic subspace of Z, an admissible cover by open affinoids being given by

$$Z_+(z) = \bigcup_{\substack{\varepsilon \in \sqrt{|K^\times|} \\ \varepsilon < 1}} \{y \in Z\colon |f_i(y)| \leq \varepsilon \text{ for all } i = 1, \dots, r\}$$

In particular, $Z_+(z)$ is quasi-Stein (see Definition 1.32). Note that $\mathbb{D}^n_+(0) = \mathring{\mathbb{D}}^n$.

Remark 1.7. *Let $Z_1 = \mathrm{Sp}(S)$ be an affinoid space. For a morphism $\pi\colon Z_1 \longrightarrow Z_2$ to an affinoid space $Z_2 = \mathrm{Sp}(R)$ and the associated ring morphism $\varphi\colon R \longrightarrow S$, the following assertions are equivalent*

(i) *π is finite and surjective.*

(ii) *φ is finite and $\ker(\varphi) \subseteq \mathrm{Nil}(R)$.*

Proof. Note that the finiteness of φ is equivalent to the finiteness of π by definition of finite morphisms between rigid spaces. To prove (ii) $\Longrightarrow$ (i), we note that the underlying sets of $\mathrm{Sp}(R)$ and $\mathrm{Sp}(R/\ker(\varphi))$ coincide because $\ker(\varphi) \subseteq \mathrm{Nil}(R)$, so we may replace φ by the induced morphism $R/\ker(\varphi) \hookrightarrow S$, i.e. assume that φ

is injective. Then we have an integral ring extension, so for every prime ideal in R there exists a prime ideal in S lying over it. Given a maximal ideal in R, any prime ideal lying over it is necessarily maximal due to the finiteness of φ, which allows us to deduce the surjectivity of π. Conversely, if we assume (i), then

$$\operatorname{Ker}(\varphi) \subseteq \bigcap_{\substack{\mathfrak{m}' \subseteq S \\ \text{max. ideal}}} \varphi^{-1}(\mathfrak{m}') = \bigcap_{\substack{\mathfrak{m} \subseteq R \\ \text{max. ideal}}} \mathfrak{m} = \operatorname{Nil}(R)$$

where the inclusion holds because $\varphi(\operatorname{Ker}(\varphi)) = \{0\} \subseteq \mathfrak{m}'$ for all $\mathfrak{m}'$, the first equality holds because π is surjective, and the second equality holds since R is a Jacobson ring (as all affinoid algebras are). $\square$

Theorem 1.8 resp. Theorem 1.10 below relates the finiteness resp. the injectivity of a morphism $\varphi\colon R \longrightarrow S$ of affinoid algebras to the finiteness resp. injectivity of its reduction $\widetilde{\varphi}\colon \widetilde{R} \longrightarrow \widetilde{S}$.

Theorem 1.8 ([BGR84, 6.3.1/Theorem 6]). *Assume that R is reduced. Then the following conditions are equivalent for any morphism $\varphi\colon R \longrightarrow S$ of affinoid algebras:*

(i) *$\varphi\colon R \longrightarrow S$ is injective and strict.*[1]

(ii) *$\varphi\colon R \longrightarrow S$ is an isometry with respect to $|\cdot|_{\sup}$.*

(iii) *$\widetilde{\varphi}\colon \widetilde{R} \longrightarrow \widetilde{S}$ is injective.*

The following remark tells us that condition (ii) in Theorem 1.8 is satisfied by any Noether normalisation morphism (or, more generally, any finite injective morphism of affinoid algebras):

Remark 1.9. *Any finite injective morphism $\varphi\colon R \hookrightarrow S$ between affinoid algebras is an isometry with respect to $|\cdot|_{\sup}$.*

Proof. The equality $|\varphi(r)|_{\sup} = |r|_{\sup}$ for all $r \in R$ is a well-known fact, where the inequality $|\varphi(r)|_{\sup} \le |r|_{\sup}$ is due to the fact that any morphism between affinoid algebras is contractive with respect to the supremum semi-norm [Bos14, 3.1/Proposition 7], and the reverse inequality is due to the following simple argument. Note that the morphism $\pi\colon \operatorname{Sp}(S) \longrightarrow \operatorname{Sp}(R)$ associated to φ is surjective since φ is finite injective. Therefore, given any $y \in \operatorname{Sp}(R)$, we can choose a preimage $x \in \operatorname{Sp}(S)$ under π and then we have $|r(y)| = |\varphi(r)(x)|$ by Remark 1.1 above. Taking the supremum over all y we obtain $|r|_{\sup} \le |\varphi(r)|_{\sup}$, which completes the proof of the assertion. $\square$

Theorem 1.10 ([BGR84, 6.3.5/Theorem 1]). *The following statements are equivalent:*

[1] A map $f\colon X \longrightarrow Y$ between topological spaces is called strict if the quotient topology on $f(X)$ coincides with the subspace topology inherited from Y.

(i) $\varphi\colon R \longrightarrow S$ *is finite.*

(ii) $\varphi\colon R \longrightarrow S$ *is integral.*

(iii) $\varphi\colon \mathring{R} \longrightarrow \mathring{S}$ *is integral.*

(iv) $\widetilde{\varphi}\colon \widetilde{R} \longrightarrow \widetilde{S}$ *is integral.*

(v) $\widetilde{\varphi}\colon \widetilde{R} \longrightarrow \widetilde{S}$ *is finite.*

Remark 1.7, Theorem 1.8, Remark 1.9 and Theorem 1.10 together imply:

Corollary 1.11. *Let $Z = \mathrm{Sp}(R)$ be an affinoid space and $\pi\colon Z \longrightarrow \mathbb{D}^n$ a morphism. Then the following are equivalent:*

(i) $\pi\colon Z \longrightarrow \mathbb{D}^n$ *is finite surjective.*

(ii) $\widetilde{\pi}\colon \widetilde{Z} \longrightarrow \mathbb{A}_k^m$ *is finite surjective.*

We end this subsection by noting that it is hard to relate the surjectivity of φ to the surjectivity of $\widetilde{\varphi}$. In fact, [BGR84, 6.3.1/Example 1] one shows that it can happen that the reduction $\widetilde{\varphi}$ is surjective (even bijective) but φ is not surjective. On the other hand, [BGR84, 6.3.1/Example 2] exhibits a surjective φ whose reduction $\widetilde{\varphi}$ isn't surjective.

1.1.2 Connected components

Definition 1.12. *Let X be a rigid space.*

(i) *X is called* connected *if every admissible open cover $X = U \cup V$ such that $U \cap V \neq \varnothing$ must have $U = \varnothing$ or $V = \varnothing$.*

(ii) *A* connected component *of X is an equivalence class of a point $x \in X$ with respect to the following equivalence relation: $x, y \in X$ are equivalent if there exist finitely many connected admissible opens $U_1, \ldots, U_n$ such that $x \in U_1, y \in U_n$ and $U_i \cap U_{i+1} \neq \varnothing$ for all $1 \leq i < n$.*

Remark 1.13. *Let X be a rigid space. Then:*

(i) *Two connected components of X are either disjoint or equal.*

(ii) *If X is connected and $X \longrightarrow\!\!\!\!\rightarrow Y$ a surjective morphism of rigid spaces, then Y is connected as well.*

Proof. Assertion (i) is immediate. To prove (ii), consider an admissible open cover $Y = U \cup V$ such that $U \cap V \neq \varnothing$. Then $X = f^{-1}(U) \cup f^{-1}(V)$ is an admissible open cover such that $f^{-1}(U) \cap f^{-1}(V) \neq \varnothing$. Since X is connected, it follows that, say, $f^{-1}(U) = \varnothing$. Due to the surjectivity of f, we then have $f(f^{-1}(U)) = U$, i.e. $U = \varnothing$ as well. $\square$

We list additional facts regarding connected components of rigid spaces, all of which can be found in [Con99, §2.1], [BGR84, §9.1.4] or [Bos14, §5.3]:

Remark 1.14. *Let X be a rigid space. Then:*

(i) *If $\{X_i\}_{i\in\Sigma}$ is a subset of the set of connected components of X, then $\cup_{i\in\Sigma}X_i$ is admissible open and $\{X_i\}_{i\in\Sigma}$ is an admissible covering thereof. Moreover, $\{X_i\}_{i\in\Sigma}$ are precisely the connected components of $\cup_{i\in\Sigma}X_i$.*

(ii) *X is connected if and only if it has exactly one connected component.*

(iii) *The admissible opens of X obtained as in (i) are precisely the admissible opens of X that are also analytic sets, i.e. "open and closed".*[2] *We refer to such subsets as* clopen.

(iv) *Any connected admissible open in X is contained in a connected component of X. In particular, the connected components of X are precisely the maximal connected admissible opens in X.*

(v) *X is connected if and only if $\Gamma(X, \mathcal{O}_X)$ has no non-trivial idempotents. In particular, an affinoid* $\mathrm{Sp}(A)$ *is connected if and only if the affine scheme* $\mathrm{Spec}(A)$ *is Zariski-connected.*

Remark 1.14 (v) will allow us to deduce the following:

Remark 1.15. *Let $Z = \mathrm{Sp}(R)$ be a connected affinoid space whose local rings $\mathcal{O}_{Z,z}$ are integral domains for all $z \in Z$. Then R is an integral domain. In particular, if $Z = \mathrm{Sp}(R)$ is a connected smooth affinoid space, then R is an integral domain.*

Proof. We will show that the affine scheme $\mathrm{Spec}(R)$ is integral (i.e. irreducible and reduced), whence R is an integral domain by algebraic geometry. The local rings of Z are integral domains and therefore reduced, so Z is a reduced rigid space (by definition) which by [BGR84, 7.3.2/Corollary 9] is equivalent to $\mathrm{Nil}(R) = 0$. But $\mathrm{Nil}(R) = 0$ implies that $\mathrm{Spec}(R)$ is a reduced affine scheme. It remains to show that $\mathrm{Spec}(R)$ is irreducible for the Zariski topology. Since R is Noetherian, $\mathrm{Spec}(R)$ has finitely many irreducible components, say $Z_1, \dots, Z_r$ where the Z_i correspond to minimal prime ideals $\mathfrak{p}_i$ in R. By assumption and Remark 1.14 (v), we know that $\mathrm{Spec}(R)$ is connected for the Zariski topology, so it suffices to prove that $Z_i \cap Z_j = \varnothing$ for $i \neq j$, because then $\mathrm{Spec}(R) = \coprod_{i=1}^r Z_i$ is a covering by pairwise disjoint closed subsets, whence connectedness forces $r = 1$. Suppose there exists an $x \in Z_i \cap Z_j$ with $i \neq j$. By [Stacks, Tag 02IL], we may assume that x is a closed point, i.e. corresponds to a maximal ideal $\mathfrak{m}_x$ in R. Then the fact that $x \in Z_i \cap Z_j$ implies that $R_{\mathfrak{m}_x}$ contains two different minimal prime ideals ($\mathfrak{p}_i$ and $\mathfrak{p}_j$). But this contradicts the fact that $R_{\mathfrak{m}_x}$ is an integral domain (being a subring $R_{\mathfrak{m}_x} \longrightarrow \mathcal{O}_{Z,x}$ of the integral domain $\mathcal{O}_{Z,x}$). $\square$

[2] Recall that any analytic subset of X can canonically be given the structure of a reduced closed analytic subvariety of X by [BGR84, 9.5.3/Proposition 4].

If X is a rigid space and $C \subseteq X$ a connected component, then C is admissible open in X and hence an open rigid subspace. On the other hand, C is an analytic subset of X, so it can also be endowed with the structure of a reduced closed rigid subspace. If X is reduced, then these two canonical structures on C coincide[3], in which case we can (and will) often consider the inclusion $C \hookrightarrow X$ simultaneously as an open immersion and a closed immersion.

1.1.3 Distinguished affinoid algebras and absolutely reduced algebras

The following definition is as in [BGR84, 6.4.3/Definition 2] and in [Bos69, §6, after Satz 1]:

Definition 1.16. *Let R be a K-affinoid algebra. A surjective morphism $\alpha\colon K\langle\xi_1,\dots\xi_m\rangle \twoheadrightarrow R$ is called* distinguished *if the residue norm $|\cdot|_\alpha$ coincides with the supremum semi-norm $|\cdot|_{\sup}$ on R. A K-affinoid algebra R is called* distinguished *if for some $m \geq 0$ it admits a distinguished $\alpha\colon K\langle\xi_1,\dots\xi_m\rangle \twoheadrightarrow R$.*

In particular, if R is distinguished, then $|R|_{\sup} = |R|_\alpha = |K\langle\xi_1,\dots\xi_m\rangle| = |K|$, where the middle equality holds by [BGR84, 5.2.7/Corollary 8]. Moreover, since $|\cdot|_{\sup} = |\cdot|_\alpha$ is a norm (and not only a semi-norm), R is reduced.
Conversely, these two conditions are sufficient for R to be distinguished if we assume that K is a stable field (in the sense of [BGR84, 3.6.1/Definition 1]):

Theorem 1.17 ([BGR84, 6.4.3/Theorem 6])**.** *Assume that K is stable, and let R be an K-affinoid algebra. Then the following two statements are equivalent:*

a) R is distinguished.

b) R is reduced and $|R|_{\sup} = |K|$.

We mention that K is stable if it is spherically complete [BGR84, 3.6.2/Proposition 15] or algebraically closed [BGR84, 3.6.2/Proposition 12]. In the latter case, the second condition in b) of Theorem 1.17 is automatic, i.e. we have $|R|_{\sup} = |K|$ for all affinoid K-algebras R. (In general we have $|R|_{\sup} \subseteq |\overline{K}|$ because of the Maximum Principle.)
For example, a finite field extension L of $K := \mathbb{Q}_p$ with $e(L/\mathbb{Q}_p) > 1$ is a reduced $\mathbb{Q}_p$-affinoid algebra where $|L|_{\sup} = |L|$ is strictly larger than $|\mathbb{Q}_p|$, hence L is not distinguished, according to Theorem 1.17.

[3] In general, for any analytic subset $Y \subseteq X$, the structure sheaf $\mathfrak{O}_Y$ of a reduced closed subvariety on Y is obtained from the following local building blocks: Given an affinoid open $U \subseteq X$, one has $(\mathfrak{O}_Y)_{|U\cap Y} := \mathcal{O}_U/\mathfrak{I}_{U\cap Y}$ where $\mathfrak{I}_{U\cap Y} \subseteq \mathcal{O}_U$ is the radical ideal associated to the analytic subset $U \cap Y$ of U. In our case, $Y := C$ is also admissible open in X, meaning that it can be covered by open affinoids $U \subseteq X$ that are contained in C (i.e. $U \cap C = U$). Then $(\mathfrak{O}_C)_{|U} = (\mathfrak{O}_C)_{|U\cap C} = \mathcal{O}_U/\mathfrak{I}_{U\cap C} = \mathcal{O}_U/\mathfrak{I}_U = \mathcal{O}_U/\operatorname{Nil}(\mathcal{O}_U) = \mathcal{O}_U$ since $\operatorname{Nil}(\mathcal{O}_U) = 0$, proving our claim.

A significant result for us will be Proposition 1.20 below, which tells us that, when regarding an affinoid algebra that is not necessarily distinguished, we can achieve that the algebra becomes distinguished after a base change to a finite Galois extension. First we recall the following definition as in [Bos70]:

Definition 1.18. *A K-affinoid algebra R is called* absolutely reduced *(or* analytically separable *in [Ber+67, Definition 4.2.1]) over K, if for every complete field extension K' of K, the algebra $R \widehat{\otimes}_K K'$ is reduced.*

The condition in Definition 1.18 is an "analytic version" of MacLane's separability criterion [Eis13, Theorem A1.3], which explains why the algebras that satisfy it are sometimes also called analytically separable.
Next, we recall from [Kie67b, (1.11.1) in Definition 1.11] that, if $\mathrm{Sp}(R)$ is smooth over K, then $(R \widehat{\otimes}_K K')_{\mathfrak{m}}$ is a regular ring for every complete field extension K' of K and every maximal ideal $\mathfrak{m}$ of $R \widehat{\otimes}_K K'$. In particular, this implies the following remark:

Remark 1.19. *The affinoid algebra of a smooth affinoid space is absolutely reduced.*

We are now ready to prove the following:

Proposition 1.20. *Let R be an absolutely reduced K-affinoid algebra. Then there exists a finite* Galois *field extension K'/K such that the algebra $R \otimes_K K'$ is distinguished.*

Proof. In the proof of [Bos71, Lemma 2.7], Bosch constructs a finite field extension K'/K such that the algebra $R \otimes_K K'$ is distinguished. (Note that we don't have to complete the tensor product since K' is finite over K.) We go through Bosch's proof and argue why K' can be taken to be Galois. Let $C = \widehat{\overline{K}}$ be the completed algebraic closure of K. Since R is absolutely reduced, $R_C = R \widehat{\otimes}_K C$ is reduced and hence a distinguished affinoid algebra over the algebraically closed field C. Let $x = (x_1, \ldots, x_m)$ be an affinoid generating system of R and $y' = (y'_1, \ldots, y'_n)$ a distinguished affinoid generating system of R_C, i.e.

$$\Phi' \colon C\langle \xi_1 \ldots, \xi_n \rangle \longrightarrow R_C, \qquad \xi_i \longmapsto y'_i$$

is a distinguished epimorphism. Then the morphism $\mathring{\Phi}' \colon \mathfrak{o}_C\langle \xi_1 \ldots, \xi_n \rangle \longrightarrow \mathring{R}_C$ obtained by restriction is surjective by [BGR84, 6.4.3/Proposition 3]. The property of being an affinoid generating system is preserved under completed base change to C [BGR84, 2.1.8/Proposition 6], so x is an affinoid generating system of R_C. Hence there exist elements $c'_{i\nu} \in C$ (with $i = 1, \ldots, n$ and $\nu \in \mathbb{N}_0^m$) such that

$$y'_i = \sum_{\nu \in \mathbb{N}_0^n} c'_{i\nu} x^\nu .$$

The separable algebraic closure K^{sep} is dense in $\overline{K}$ by [BGR84, 3.4.1/Proposition 6]. Since $\overline{K}$ is dense in C, it follows that K^{sep} is also dense in C, meaning that we can find $c_{i\nu} \in K^{\mathrm{sep}}$ which are zero for all save finitely many indices i and ν and such that the

$$y_i := \sum_{\nu \in \mathbb{N}_0^n} c_{i\nu} x^\nu$$

satisfy

$$|y_i - y_i'|_{\alpha'} < 1.$$

The argument in [Bos14, 4.2/Lemma 8] can then be applied both to

$$\Phi \colon C\langle \xi_1 \ldots, \xi_n \rangle \longrightarrow R_C, \qquad \xi_i \longmapsto y_i$$

and $\mathring{\Phi}$ to deduce the surjectivity of Φ resp. $\mathring{\Phi}$ from the surjectivity of Φ' resp. $\mathring{\Phi}'$. Thus Φ is a distinguished epimorphism by [BGR84, 6.4.3/Corollary 6]. We set K' to be the Galois closure of the finite separable extension $K(c_{i\nu} \mid i = 1, \ldots, n, \nu \in \mathbb{N}_0^m)$ of K. Then $y_i \in R_{K'} = R \otimes_K K'$ and the rest of the argument in the proof of [Bos71, Lemma 2.7] can be applied verbatim to show that

$$\phi \colon K'\langle \xi_1 \ldots, \xi_n \rangle \longrightarrow R_{K'}, \qquad \xi_i \longmapsto y_i$$

is a distinguished epimorphism (since its base change $\phi_C = \Phi$ is a distinguished epimorphism). □

1.1.4 Noether normalisation with specific properties

We recall two refined versions of the Noether normalisation theorem for affinoid algebras, namely Lemma 1.21 and Lemma 1.24 below:

Lemma 1.21 (Noether normalisation map with prescribed zeros)**.** *Let $Z = \mathrm{Sp}(R)$ be an affinoid space. Given finitely many points $z_1, \ldots, z_n \in Z$, there exists a finite surjective morphism $\pi \colon Z \longrightarrow \mathbb{D}^d$ such that $\pi(z_i) = 0$ for all $i = 1, \ldots, n$. Equivalently, there exists a Noether normalisation $T_d = K\langle \xi_1, \ldots, \xi_d \rangle \hookrightarrow R$ such that all the $\mathfrak{m}_{z_i}$ lie over the maximal ideal $(\xi_1, \ldots, \xi_d)$.*

Proof. This is [Ber+67, Lemma 1.1.4]. □

Recall that, given any field F and any ring A, a ring map $F \longrightarrow A$ is étale if and only if A is isomorphic as an F-algebra to a finite product of finite separable field extensions of F, see [Stacks, Tag 00U3].

Definition 1.22 (Separable ring map)**.** *Let A be an integral domain. A torsion-free morphism $A \longrightarrow B$ is called* separable *if the induced map $Q(A) \hookrightarrow Q(B)$ of the fraction field $Q(A)$ of A into the total ring of fractions $Q(B)$ of B is étale.*

Accordingly, a morphism $\mathrm{Sp}(S) \longrightarrow \mathrm{Sp}(R)$ of affinoid spaces is called separable if the ring morphism $R \longrightarrow S$ is separable.

Definition 1.23. (i) *A morphism $A \longrightarrow B$ of local rings is said to be* unramified *if the maximal ideal $\mathfrak{m}_B$ of B is generated by the maximal ideal $\mathfrak{m}_A$ of A and $B/\mathfrak{m}_B$ is a separable algebraic extension of $A/\mathfrak{m}_A$.*

(ii) *A morphism $A \longrightarrow B$ of rings is said to be* unramified at a prime ideal $\mathfrak{p} \subseteq B$, *if the induced morphism $A_{\mathfrak{p}\cap A} \longrightarrow B_{\mathfrak{p}}$ of local rings is unramified.*

In particular, an injective morphism $A \longrightarrow B$ of integral domains is separable if it is finite and unramified at $(0) \subseteq B$. Being unramified at (0) is also called being "generically unramified".

Lemma 1.24 (Separable Noether normalisation)**.** *If $Z = \mathrm{Sp}(R)$ is a smooth connected affinoid space, then there exists a finite surjective morphism $\pi\colon Z \longrightarrow \mathbb{D}^d$ that is separable (meaning that the associated morphism of affinoid algebras is separable).*
More generally, for any affinoid algebra A that is an integral domain, the following statements are equivalent:

(i) *A is absolutely reduced.*

(ii) *There exists a Noether normalisation $T_d \hookrightarrow A$, such that $Q(A)$ is separable over $Q(T_d)$.*

Proof. See [Ber+67, Satz 4.2.4]. □

We need something akin to a combined version of Lemma 1.21 and Lemma 1.24:

Lemma 1.25 (Separable Noether normalisation with controlled behaviour on points)**.** *Given a smooth connected affinoid space $Z = \mathrm{Sp}(R)$ and points $z_1, \dots, z_n \in Z$, there exists a finite surjective separable morphism $\pi\colon Z \longrightarrow \mathbb{D}^d$, such that $|\pi(z_i)| < 1$ for all $i = 1, \dots, n$.*

Proof. We start by choosing a Noether normalisation morphism $\pi\colon Z \longrightarrow \mathbb{D}^d$ such that $\pi(z_i) = 0$ for all $i = 1, \dots, n$, which is possible due to Lemma 1.21. Let

$$\varphi\colon T_d = K\langle \xi_1, \dots, \xi_d\rangle \hookrightarrow R$$

be the corresponding finite injective morphism of affinoid algebras. Setting $x_i := \varphi(\xi_i)$, one can then modify the x_i by finding certain $y_i \in R, q \in \mathbb{N}$ and defining

$$x_i' := x_i^q + y_i$$

as in the proof of [Ber+67, Satz 4.1.9], to achieve that the morphism

$$\varphi'\colon T_d \hookrightarrow R, \qquad \xi_i \longmapsto x_i'$$

is finite, injective and separable, i.e. the associated morphism of affinoid spaces $\pi'\colon Z \longrightarrow \mathbb{D}^d$ is finite, surjective and separable. At the same time, we claim that one can achieve that

$$|x_i'(z_j)| < 1 \tag{1.1}$$

for all $i = 1, \ldots, d$ and $j = 1, \ldots, n$, which means that $|\pi'(z_j)| < 1$ for all $j = 1, \ldots, n$ by Remark 1.1. To see that this can be achieved, we outline the gist of the argument in the proof of [Ber+67, Satz 4.1.9] and do the necessary modification at a certain step. In the first step there, it is argued that it suffices to prove that, given any maximal ideal $\mathfrak{m} \subseteq R$, there exists a Noether normalisation $\varphi' \colon T_d \hookrightarrow R$ that satisfies (1.1) and is unramified at $\mathfrak{m}$, since then φ' is necessarily generically unramified (=separable). The minimal number of generators of $M := \Omega^1_{R_\mathfrak{m}/K}$ (i.e. $\dim_k(M/\mathfrak{m}M)$) is d due to the smoothness assumption on Z and the morphism $(T_d)_{\mathfrak{m}\cap T_d} \longrightarrow R_\mathfrak{m}$ will be unramified if we can choose the x_i' in such a way that

$$dx_1', \ldots, dx_d'$$

is a minimal system of generators of M. On a side note, we observe that due to Lemma 1.21 we can assume that all the x_i lie in $\mathfrak{m}$, which will allow us to apply Nakayama's lemma later. Now we choose $y_1, \ldots, y_d \in R$ such that $dy_1, \ldots, dy_d$ is a minimal system of generators of M. Since we have $R = K \cdot \mathring{R}$, we can assume that

$$y_i \in \mathring{R} \text{ and } |y_i(z_j)| < 1 \tag{1.2}$$

for all $i = 1, \ldots, d, j = 1, \ldots, n$. If we choose a non-unit $t \in \mathfrak{o}_K$ and set $\overline{\mathring{R}} = \mathring{R}/t\mathring{R}$ and $B = \operatorname{im}\varphi$, then $\overline{\mathring{R}}$ is finite over $\overline{\mathring{B}} = k[\overline{x_1}, \ldots, \overline{x_d}]$. Taking an integral equation

$$\overline{y_1}^r + p_1(\overline{x})\overline{y_1}^{r-1} + \ldots + p_r(\overline{x}) = 0 \qquad (p_i(\overline{x}) \in k[\overline{x_1}, \ldots, \overline{x_d}]) \tag{1.3}$$

for $\overline{y_1}$ over $\overline{\mathring{B}}$, we choose $q \in \mathbb{N}$ large enough so that defining $x_1' := y_1 + x_1^q$ and plugging in $\overline{y_1} = \overline{x_1'} - \overline{x_1^q}$ into (1.3) and regrouping terms yields an integral equation for $\overline{x_1}$ over $k[\overline{x_1'}, \overline{x_2}, \ldots, \overline{x_d}]$. Therefore, $\overline{\mathring{B}} = k[\overline{x_1}, \ldots, \overline{x_d}]$ is finite over $k[\overline{x_1'}, \overline{x_2}, \ldots, \overline{x_d}]$ and by transitivity $\overline{\mathring{R}}$ is then finite over $k[\overline{x_1'}, \overline{x_2}, \ldots, \overline{x_d}]$. By [Tat71, Proposition 4.2], this implies that R is finite over $K\langle x_1', x_2, \ldots x_d\rangle$. Moreover, the latter is in fact a free affinoid algebra since the substitution homomorphism $T_d \longrightarrow K\langle x_1', x_2, \ldots x_d\rangle$ is a surjective morphism from a domain to a ring of the same finite Krull dimension and hence necessarily an isomorphism. Finally, since $dx_1' = dy_1 + q \cdot x_1^{q-1}dx_1$ and $x_1 \in \mathfrak{m}$, we see that the system $dx_1', dy_2, \ldots, dy_d$ reduces modulo $\mathfrak{m}$ to a basis of $M/\mathfrak{m}M$, so by Nakayama's lemma it is a minimal generating system of M. Proceeding in the same fashion for x_2' and x_3' and so forth, we eventually obtain the Noether normalisation morphism

$$\varphi' \colon T_d \hookrightarrow R, \qquad \xi_i \longmapsto x_i'$$

that satisfies all the desired properties, including

$$|x_i'(z_j)| = |y_i(z_j) + x_i^q(z_j)| \leq \max\{|y_i(z_j)|, |x_i(z_j)|^q\} < 1$$

due to (1.2) and since $x_i(z_j) = 0$ (because $\pi(z_j) = 0$). □

Remark 1.26. *We stress that the separability of a Noether normalisation morphism is not automatic even if K is perfect. Indeed, if K is a perfect non-archimedean field of characteristic $p > 0$ (e.g. if K is the t-adic completion of the separable algebraic closure $\mathbb{F}_p((t))^{\mathrm{sep}}$ of the formal Laurent series $\mathbb{F}_p((t))$), then $K\langle \xi^p \rangle \longrightarrow K\langle \xi \rangle$ is a Noether normalisation morphism that isn't separable.*

1.1.5 A note on pure dimension

If R is a K-affinoid algebra, then R is Noetherian and hence contains only finitely many minimal prime ideals, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. As in [Bos70], we say that R *has pure dimension* (or *is equidimensional*) if

$$\dim R/\mathfrak{p}_1 = \dim R/\mathfrak{p}_2 = \ldots = \dim R/\mathfrak{p}_s.$$

Note that there exists a chain of prime ideals of length $\dim R$ in R and it necessarily starts with a minimal prime ideal, say $\mathfrak{p}_j$, so it projects to a chain of the same length in $R/\mathfrak{p}_j$, hence $\dim R = \dim R/\mathfrak{p}_j$ for some j. Thus the above condition is equivalent to the condition

$$\dim R = \dim R/\mathfrak{p}_1 = \dim R/\mathfrak{p}_2 = \ldots = \dim R/\mathfrak{p}_s.$$

Lemma 1.27. *Let R be a reduced K-affinoid algebra and $d = \dim R$. The following are equalent:*

(i) *R has pure dimension d.*

(ii) *Every finite injective morphism $T_d \longhookrightarrow R$ is torsion-free.*

(iii) *There exists a finite injective morphism $T_d \longhookrightarrow R$ that is torsion-free.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal prime ideals of R.
(i) $\Longrightarrow$ (ii). Let $T_d \longhookrightarrow R$ be a finite injective morphism. We have to show that no element of T_d, different from zero, is a zero divisor in R. Since R is reduced, [Stacks, Tag 00EW] tells us that the set of zero divisors in R is $\bigcup_{i=1}^s \mathfrak{p}_i$, so we have to show that $\mathfrak{p}_i \cap T_d = 0$ for each i. The finite injective morphism $T_d \longhookrightarrow R$ induces a finite injective morphism $T_d/(\mathfrak{p}_i \cap T_d) \longhookrightarrow A/\mathfrak{p}_i$, so in particular $\dim T_d/(\mathfrak{p}_i \cap T_d) = \dim A/\mathfrak{p}_i$. By assumption (i), the latter is equal to d, so there exists a chain of prime ideals $0 \subsetneq \overline{\mathfrak{q}}_1 \subsetneq \ldots \subsetneq \overline{\mathfrak{q}}_d$ of length d in $T_d/(\mathfrak{p}_i \cap T_d)$ and this lifts to a chain of prime ideals $\mathfrak{p}_i \cap T_d \subsetneq \mathfrak{q}_1 \subsetneq \ldots \subsetneq \mathfrak{q}_d$ of length d in T_d. Thus $\mathfrak{p}_i \cap T_d = 0$, as otherwise we would have the chain $0 \subsetneq \mathfrak{p}_i \cap T_d \subsetneq \mathfrak{q}_1 \subsetneq \ldots \subsetneq \mathfrak{q}_d$ of length $d+1$ in T_d.
(ii) $\Longrightarrow$ (iii). By Noether normalisation, there exists a finite injective morphism $T_d \longhookrightarrow A$. This morphism is torsion-free by assumption (ii).
(iii) $\Longrightarrow$ (i). If we have a finite injective morphism $T_d \longhookrightarrow A$ which is torsion-free, then the set $\bigcup_{i=1}^s \mathfrak{p}_i$ of zero divisors in R pulls back to 0 in T_d, so in particular $\mathfrak{p}_i \cap T_d = 0$ for each i. Hence the morphism $T_d \longhookrightarrow A$ induces a finite injective morphism $T_d \longhookrightarrow A/\mathfrak{p}_i$, so we have $\dim A/\mathfrak{p}_i = \dim T_d = d$ for all $i = 1, \ldots, s$. □

If R is a reduced K-affinoid algebra which has pure dimension, then Lemma 1.27 shows that every finite injective morphism $T_d \hookrightarrow R$ induces a morphism $Q(T_d) \hookrightarrow Q(R)$ of the fraction field $Q(T_d)$ of T_d into the total ring of fractions $Q(R)$ of R. Bosch makes extensive use of this fact in [Bos77, §§4-6], where he works exclusively with affinoid algebras that are distinguished (and hence reduced) and have pure dimension.
In our applications, $Z = \mathrm{Sp}(R)$ is a connected smooth affinoid space, so R is an integral domain by Remark 1.15 and hence (0) is its only minimal prime ideal, so in particular R has pure dimension.

1.2 Compactly supported cohomology of rigid spaces

1.2.1 Cohomology and Stein spaces: definitions and basic facts

Lemma 1.28 ([Chi90, Proposition 1.2]). *Let X be a rigid analytic variety and $Z \subseteq X$ a finite union of admissible affinoids. Then $X \setminus Z$ is admissible open in X.*

Definition 1.29 ([Bey97a, §1.1], [Chi90, §1]). *Let X be a rigid analytic variety and $Z \subseteq X$ a finite union of admissible affinoids. For a sheaf $\mathcal{F}$ of abelian groups, set*

$$\Gamma_Z(X, \mathcal{F}) := \ker(\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X \setminus Z, \mathcal{F})). \tag{1.4}$$

Then

$$\Gamma_c(X, \mathcal{F}) := \varinjlim_Z \Gamma_Z(X, \mathcal{F})$$

where the limit is taken over all subspaces Z of the above form. We use the following notation for its right derived functors:

$$H^j_c(X, -) := R^j\Gamma_c(X, -).$$

If we denote the j-th right derived functor of the left exact functor $\Gamma_Z(X, -)$ by $H^j_Z(X, -)$, then

$$H^j_c(X, \mathcal{F}) = \varinjlim_Z H^j_Z(X, \mathcal{F})$$

by [Chi90, the discussion preceding Remark 1.4]. We note that there is the standard long exact sequence

$$\ldots \longrightarrow H^j_Z(X, \mathcal{F}) \longrightarrow H^j(X, \mathcal{F}) \longrightarrow H^j(X \setminus Z, \mathcal{F}) \longrightarrow H^{j+1}_Z(X, \mathcal{F}) \longrightarrow \ldots \tag{1.5}$$

obtained in the same way as in the topological situation (cf. [Stacks, Tag 0A39]).

Definition 1.30. *A rigid analytic space X is said to be* quasi-compact *if it satisfies one of the following equivalent conditions:*

(i) *Every admissible cover of X admits a finite subcover.*

(ii) *X admits an admissible covering consisting of finitely many affinoid opens.*

Remark 1.31. *If X is quasi-compact, then we can take $Z = X$ in Definition 1.29 which is a cofinal object, hence $H^j_c(X, \mathcal{F}) = H^j_X(X, \mathcal{F}) = H^j(X, \mathcal{F})$, the last equality being true since $\Gamma_X(-, \mathcal{F}) = \Gamma(-, \mathcal{F})$.*
If X is proper or affinoid, then in particular X is quasi-compact by Definition 1.30 (ii), hence $H^j_c(X, \mathcal{F}) = H^j(X, \mathcal{F})$.

In complex analysis, the complex spaces X that satisfy *Cartan's Theorem B* (i.e. for every coherent analytic sheaf $\mathcal{F}$ on X, $H^q(X, \mathcal{F})$ vanishes for all $q \geq 1$) are called *Stein* [GR04, IV.§1.1 Definition 1]. By [GR04, V.§4.2 Theorem 3], these are precisely the complex spaces that are "holomorphically complete" [GR04, IV.§3.3 Definition 8], i.e. the spaces that can be exhausted by "analytic blocks", where an analytic block is defined as a compact set D in X that can be mapped by a finite proper holomorphic map (which is defined on a neighbourhood of D) into an m-fold product of rectangles in some $\mathbb{C}^m$. By [GR04, IV.§4.5 Theorem 5], such an exhaustion $\{D_i\}_{i \in \mathbb{N}}$ is necessarily a "Stein exhaustion" [GR04, IV.§1.5 Definition 6], in particular it satisfies the property that restriction map $\mathcal{O}_X(D_{i+1}) \longrightarrow \mathcal{O}_X(D_i)$ has dense image for all i. This characterisation is taken as the inspiration for the definition of (quasi-)Stein spaces in the non-archimedean world:

Definition 1.32 ([Kie66, Definition 2.3])**.** *Let X be a rigid space.*

(i) *X is said to be* quasi-Stein *if it admits an an admissible affinoid cover $\{D_i\}_{i \in \mathbb{N}}$ such that, for $i > 0$, $D_i \subseteq D_{i+1}$ and the restriction map $\mathcal{O}_X(D_{i+1}) \longrightarrow \mathcal{O}_X(D_i)$ has dense image.*

(ii) *A* Stein space *is a quasi-Stein space which admits an admissible affinoid cover as in (i) that additionally satisfies $D_i \Subset D_{i+1}$ for all i (see Definition 1.33 below).*

Definition 1.33 (Relative compactness)**.** *Let $X \longrightarrow Y$ be a morphism of rigid spaces with an affinoid base space Y and let $Z_1 \subseteq Z_2$ be open affinoid subspaces of X. We say that Z_1* is relatively compact in Z_2 *over Y and write*

$$Z_1 \Subset_Y Z_2$$

if there is a closed immersion $i \colon Z_2 \hookrightarrow \mathbb{D}^r_Y$ over Y such that Z_1 lands in $\mathring{\mathbb{D}}^r_Y$. (∗)
Equivalently, if there exists an affinoid generating system $f_1, \ldots, f_r$ of $\mathcal{O}(Z_2)$ over $\mathcal{O}(Y)$ such that

$$Z_1 \subseteq \{z \in Z_2 \colon |f_i(z)| < 1 \text{ for all } i = 1, \ldots, r\}$$

or, equivalently (by the Maximum Modulus Principle for affinoids) such that there is an $\varepsilon \in \sqrt{|K^\times|}$, $0 < \varepsilon < 1$, satisfying

$$Z_1 \subseteq \{z \in Z_2 \colon |f_i(z)| \leq \varepsilon \text{ for all } i = 1, \ldots, r\}.$$

If $Y = \mathrm{Sp}(K)$*, then we just write*

$$Z_1 \Subset Z_2.$$

It is obvious from the definition that any affinoid space is quasi-Stein (see Remark 1.35 below for a sharper statement). Therefore, Kiehl's Theorem B (Theorem 1.37 below) and Remark 1.31 together imply that Serre Duality doesn't hold for affinoid spaces (except for zero-dimensional ones). Since, on the other hand, Serre Duality does hold for Stein spaces, we see that no affinoid space of positive dimension is Stein.

Remark 1.34. (i) *Both in complex analysis and in rigid geometry, Stein spaces satisfy embedding theorems into the affine space. In particular, a complex manifold is Stein if and only if it embeds into some* $\mathbb{C}^N$ *as a closed complex submanifold [GR04, V §1.1] and a smooth rigid analytic* K*-space is Stein if and only if admits a closed immersion into some* $\mathbb{A}_K^{N,\mathrm{an}}$ *[Lüt73, Theorem 4.25].*

(ii) *Rigid analytifications of affine* K*-schemes of finite type are rigid Stein spaces. This is due to (i), since the analytification functor respects closed immersions (which is clear from the explicit constructions in [Bos14, §5.4]).*

From the definitions we immediately deduce the following:

Remark 1.35. *A rigid analytic space is affinoid if and only if it is quasi-Stein and quasi-compact.*

Lemma 1.36. *Let* X *be a rigid space that admits a closed immersion* $\iota\colon X \hookrightarrow S$ *into a quasi-Stein space* S*. Then* X *is also a quasi-Stein space.*

Proof. Let $\{D_i\}_{i\in\mathbb{N}}$ be a cover of S as in Definition 1.32. By [BGR84, 9.5.3/Proposition 2], each $\iota^{-1}(D_i)$ is again affinoid and each morphism $\mathcal{O}_S(D_i) \longrightarrow \mathcal{O}_X(\iota^{-1}(D_i))$ is surjective. Thus the commutative diagrams

$$\begin{tikzcd}
\mathcal{O}_X(\iota^{-1}(D_{i+1})) \arrow[d] & \mathcal{O}_S(D_{i+1}) \arrow[l, two heads] \arrow[d, "\text{dense}"] \\
\mathcal{O}_X(\iota^{-1}(D_i)) & \mathcal{O}_S(D_i) \arrow[l, two heads]
\end{tikzcd}$$

imply that the vertical maps on the left have dense image. Indeed, the morphism $\mathcal{O}_S(D_i) \longrightarrow \mathcal{O}_X(\iota^{-1}(D_i))$ being surjective, it pulls back non-empty open sets to non-empty open sets, all of which have non-trivial intersection with the image of $\mathcal{O}_S(D_{i+1}) \longrightarrow \mathcal{O}_S(D_i)$. Due to the commutativity of the diagram, this implies that all non-empty open sets in $\mathcal{O}_X(\iota^{-1}(D_i))$ have non-trivial intersection with the image of $\mathcal{O}_X(\iota^{-1}(D_{i+1})) \longrightarrow \mathcal{O}_X(\iota^{-1}(D_i))$. □

The following analogue of Cartan's Theorem B is due to Kiehl and is often called *Kiehl's Theorem B*:

Theorem 1.37 ([Kie66, Satz 2.4]). *Let S be a quasi-Stein Space and $\mathcal{F}$ a coherent sheaf on S. Then*

$$H^n(S, \mathcal{F}) = 0 \qquad \text{for all } n \geq 1.$$

Contrary to Stein spaces in complex analysis, it turns out that rigid quasi-Stein spaces are not characterised by the vanishing of higher coherent cohomology. This was proved in [Liu90], where a rigid space is called "Stein" if it satisfies Kiehl's Theorem B, and according to [Liu90, Théorème 4] there exists such a space that is moreover quasi-compact yet isn't affinoid (and therefore isn't quasi-Stein, by Remark 1.35). Hence we rather follow [Sig17] in calling such spaces *cohomologically Stein*, to avoid confusion:

Definition 1.38. *A rigid space X is called* cohomologically Stein *if, for every coherent analytic sheaf $\mathcal{F}$ on X, $H^q(X, \mathcal{F})$ vanishes for all $q \geq 1$.*

Recall the following *Theorem of Leray*, which tells us that the Čech cohomology on an acyclic admissible covering computes derived functor cohomology:

Theorem 1.39 ([Bos14, 6.2/Theorem 5]). *Let $\mathfrak{U}$ be an admissible covering of a rigid space X and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Suppose that $\mathfrak{U}$ is a* Leray covering, *i.e. that every finite intersection of sets in $\mathfrak{U}$ is cohomologically Stein. Then the canonical map from Čech- to derived cohomology*

$$H^j(\mathfrak{U}, \mathcal{F}) \longrightarrow H^j(X, \mathcal{F})$$

is an isomorphism for all $j \geq 0$.

Since the intersections of two affinoid opens in a separated rigid space is again affinoid and affinoids are quasi-Stein, we obtain:

Corollary 1.40. *Let X be a rigid space and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathfrak{U}$ be an admissible covering of X by affinoids. Then the canonical map from Čech- to derived functor cohomology*

$$H^j(\mathfrak{U}, \mathcal{F}) \longrightarrow H^j(X, \mathcal{F})$$

is an isomorphism for all $j \geq 0$.

1.2.2 The topology on $H^j(X, \mathcal{F})$

In this and the following subsection, we introduce canonical topologies on the cohomology groups, as in [Bey97a, §1.3] and [van92, 1.6]. We assume that X is a (separated) rigid analytic space of countable type, i.e. it has a countable (or finite) admissible covering by affinoids.

Definition 1.41 (Topology on $H^j(X, \mathcal{F})$). *Let $\mathcal{F}$ be a coherent sheaf on X. Then:*

(i) *$\mathcal{F}(X)$ receives the following canonical structure of a Fréchet space.*
Choose a countable admissible covering $\{X_i\}_i$ of X by affinoids. For each finite intersection V of the X_i, $\mathcal{F}(V)$ is a finitely generated module over the affinoid algebra $\mathcal{O}(V)$ and thus canonically a Banach space. Hence $\prod_i \mathcal{F}(X_i)$ and $\prod_{i,j} F(X_i \cap X_j)$ are countable products of Banach spaces and in particular Fréchet spaces. Then

$$\mathcal{F}(X) = \ker\left(\prod_i \mathcal{F}(X_i) \longrightarrow \prod_{i,j} F(X_i \cap X_j)\right)$$

is a closed subspace of a Fréchet space and hence a Fréchet space. This topology is independent of the choice of the covering $\{X_i\}_i$.

(ii) *Given a countable admissible covering $\mathfrak{U} = \{U_i\}_i$ of X, the Čech cohomology $H^j(\mathfrak{U}, \mathcal{F})$ is topologised as follows.*
For each finite intersection U of the U_i, $\mathcal{F}(U)$ is a Fréchet space by (i). Then the space of cochains

$$\check{C}^j(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < \ldots < i_j} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_j})$$

is a countable product of Fréchet spaces and hence a Fréchet space. The subspaces of cocycles $\ker \partial^p$ and of coboundaries $\operatorname{im} \partial^{j-1}$ are endowed with the subspace topology and

$$H^j(\mathfrak{U}, \mathcal{F}) = \ker \partial^j / \operatorname{im} \partial^{j-1}$$

with the quotient topology.

(iii) *To topologise $H^j(X, \mathcal{F})$, we choose a countable Leray covering $\mathfrak{U}$ of X, so that*

$$H^j(\mathfrak{U}, \mathcal{F}) = H^j(X, \mathcal{F})$$

and we let it carry the topology from (ii). To see that this is independent of the choice of $\mathfrak{U}$, we note that, given a countable Leray covering $\mathfrak{V}$ of X which refines $\mathfrak{U}$, the continuous maps between Fréchet spaces

$$\check{C}^j(\mathfrak{U}, \mathcal{F}) \longrightarrow \check{C}^j(\mathfrak{V}, \mathcal{F})$$

induce continuous bijections in cohomology due to Corollary 1.40, so Lemma 1.43 below then asserts that these bijections are in fact homeomorphisms

$$H^j(\mathfrak{U}, \mathcal{F}) \xrightarrow{\sim} H^j(\mathfrak{V}, \mathcal{F}).$$

Remark 1.42. *The differentials $\partial^j : \check{C}^j(\mathfrak{U}, \mathcal{F}) \longrightarrow \check{C}^{j+1}(\mathfrak{U}, \mathcal{F})$ are continous, hence $\ker \partial^j$ is closed in $\check{C}^j(\mathfrak{U}, \mathcal{F})$ and thus a Fréchet space. On the other hand, $\operatorname{im} \partial^{j-1}$ is in general not closed in $\ker \partial^j$, which means that $H^j(\mathfrak{U}, \mathcal{F})$ and $H^j(X, \mathcal{F})$ are seldom Hausdorff, let alone Fréchet spaces.*

Lemma 1.43. *Let $C^\bullet$ and $C'^\bullet$ be complexes of Fréchet spaces (with continuous linear differentials) and $f\colon C^\bullet \longrightarrow C'^\bullet$ a continuous linear map. If f induces bijections on cohomology, then these bijections are necessarily homeomorphisms for the natural (quotient) topologies on the cohomology groups (even though these are not necessarily Hausdorff).*

Proof. This is [RR70, Lemme 1]. We give the full proof, since we wish to fill in the details in the proof outline in loc. cit. Let Z^j, B^j and $H^j = Z^j/B^j$ denote the cocycles resp. coboundaries resp. cohomology of $C^\bullet$ and Z'^j, B'^j, H'^j the corresponding objects for $C'^\bullet$. Suppose for a moment that we have shown that

$$f(\overline{B^j}) = \overline{B'^j}, \tag{1.6}$$

where the overline denotes topological closure. This would mean that the induced map $H^j \longrightarrow H'^j$ (also denoted by f) maps the closure of $\{0\}$ onto the closure of $\{0\}$. Thus f induces a continuous linear bijection

$$H^j/\overline{\{0\}} \longrightarrow H'^j/\overline{\{0\}} \tag{1.7}$$

between Fréchet spaces, which is necessarily a homeomorphism due to the Open Mapping Theorem. To show that the continuous bijection $f\colon H^j \longrightarrow H'^j$ is also a homeomorphism, we show that it maps closed subsets to closed subsets. Let $W \subseteq H^j$ be a non-empty closed subset. We may assume that W contains 0 (by choosing a $w \in W$ and replacing W by $W - w$), hence also $\overline{\{0\}} \subseteq W$. We claim that then necessarily $W + \overline{\{0\}} = W$. Indeed, the image of $W \times \{0\}$ under the continuous map $+\colon H^j \times H^j \longrightarrow H^j$ is W. As with any continous map, the closure $\overline{W \times \{0\}} = \overline{W} \times \overline{\{0\}} = W \times \overline{\{0\}}$ is then mapped into the closure $\overline{W} = W$ of the image, i.e. we have $W + \overline{\{0\}} \subseteq W$. Thus $W + \overline{\{0\}} = W$, which in particular implies that the image of W in $H^j/\overline{\{0\}}$ is closed in $H^j/\overline{\{0\}}$. The map (1.7) being a homeomorphism, it follows that the image of $f(W)$ in $H'^j/\overline{\{0\}}$ is closed in $H'^j/\overline{\{0\}}$, i.e. $f(W) + \overline{\{0\}}$ is closed in H'^j. But then

$$f(W) = f(W + \overline{\{0\}}) = f(W) + f(\overline{\{0\}}) = f(W) + \overline{\{0\}}$$

is closed in H'^j, where the first equality is due to $W + \overline{\{0\}} = W$, the second equality is clear and the third equality is a consequence of (1.6). Thus $f\colon H^j \longrightarrow H'^j$ is a homeomorphism, as desired.

It remains to show (1.6). The map $f\colon H^j \longrightarrow H'^j$ being surjective implies that the continuous linear map between Fréchet spaces

$$\alpha\colon C'^{i-1} \oplus Z^i \longrightarrow Z'^i, \qquad (x, y) \longmapsto \partial'(x) + f(y)$$

is surjective, hence a quotient map due to the Open Mapping Theorem. We claim that the closed subset $A := C'^{i-1} \oplus \overline{B^i}$ of $C'^{i-1} \oplus Z^i$ is saturated with respect to α, i.e. we have $\alpha^{-1}(\alpha(A)) = A$. Indeed, let $(a, b) \in \alpha^{-1}(\alpha(A))$, so $\partial'(a) + f(b) = \partial'(x) + f(y)$ for some $x \in C'^{i-1}, y \in \overline{B^i}$; we need to show that $b \in \overline{B^i}$. Note that

$f(b-y) = \partial'(x-a) \in \partial'(C'^{i-1}) = B'^i$, so $b - y \in f^{-1}(B'^i) = B^i$ (this last equality being guaranteed by the assumption that f induces a bijection $H^j \longrightarrow H'^j$) and thus $b \in y + B^i \subseteq \overline{B^i} + B^i = \overline{B^i}$, as desired. Here $\overline{B^i} + B^i = \overline{B^i}$ holds since the closure of the vector space B^i is again a vector space, and contains B^i. Quotient maps send saturated closed sets to closed sets, so $\alpha(A) = B'^i + f(\overline{B^i})$ is closed in Z'^i. Since it contains B'^i, this implies $\overline{B'^i} \subseteq B'^i + f(\overline{B^i})$. The equality $B'^i = f(B^i)$, which is due to the assumption that f induces a bijection $H^j \longrightarrow H'^j$, then allows us to conclude

$$\overline{B'^i} \subseteq B'^i + f(\overline{B^i}) = f(B^i) + f(\overline{B^i}) = f(B^i + \overline{B^i}) = f(\overline{B^i}).$$

The reverse inclusion $\overline{B'^i} = \overline{f(B^i)} \supseteq f(\overline{B^i})$ is clear due to the continuity of f, so (1.6) is shown and the proof of the lemma is complete. □

Proposition 1.44. *Let $f\colon X \longrightarrow Y$ be a finite morphism of rigid spaces and $\mathcal{F}$ a coherent sheaf on X. Then $f_*\mathcal{F}$ is coherent and the induced morphism on cohomology*

$$H^j(Y, f_*\mathcal{F}) \xrightarrow{\sim} H^j(X, \mathcal{F})$$

is a topological isomorphism for all $j \geq 0$.

Proof. The sheaf $f_*\mathcal{F}$ is coherent by [BGR84, 9.4.4/Proposition 3]. For any admissible covering $\mathfrak{U}$ of Y and any morphism $f\colon X \longrightarrow Y$ (not necessarily finite), we have the obvious equality of Čech-complexes $\check{C}^j(\mathfrak{U}, f_*\mathcal{F}) = \check{C}^j(f^{-1}\mathfrak{U}, \mathcal{F})$ which induces an equality of cohomologies

$$H^j(\mathfrak{U}, f_*\mathcal{F}) = H^j(f^{-1}\mathfrak{U}, \mathcal{F}). \tag{1.8}$$

Now if $\mathfrak{U}$ consists of affinoids (so $H^j(\mathfrak{U}, f_*\mathcal{F}) = H^j(Y, f_*\mathcal{F})$) and f is finite, then $f^{-1}(\mathfrak{U})$ also consists of affinoids by [BGR84, 9.4.4/Corollary 2], so $H^j(f^{-1}\mathfrak{U}, \mathcal{F}) = H^j(X, \mathcal{F})$. Hence the assertion follows from the established identity (1.8). □

Remark 1.45. *In algebraic geometry, a morphism of schemes $f\colon X \longrightarrow Y$ is called* affine *if for every affine open subset $U \subset X$, $f^{-1}(U)$ is affine. This is equivalent to the condition that there exists an open affine cover $\mathfrak{U}$ of Y such that $f^{-1}(\mathfrak{U})$ is an open affine cover. The same argument as in our proof of Proposition 1.44 then shows that an affine morphism $f\colon X \longrightarrow Y$ of separated schemes induces isomorphisms $H^j(Y, f_*\mathcal{F}) \xrightarrow{\sim} H^j(X, \mathcal{F})$ on cohomology, for any quasi-coherent sheaf $\mathcal{F}$ on X. The reason we need f to be a finite morphism in Proposition 1.44 rather than merely an "affinoid" morphism, is, on the one hand, not to leave the class of coherent sheaves by applying f_* and, on the other hand, that there is no good notion of an affinoid morphism in rigid geometry. Indeed, [SW20, Example 9.1.2] constructs a morphism $f\colon V \longrightarrow X$ of rigid spaces where X is affinoid and $V = f^{-1}(X)$ is not affinoid, even though there exists a certain affinoid open cover $\mathfrak{U}$ of X such that $f^{-1}(\mathfrak{U})$ is an affinoid open cover of V.*

An immediate consequence of Proposition 1.44 is the following:

Remark 1.46. *The preimage of a cohomologically Stein rigid space under a finite morphism is itself cohomologically Stein.*

1.2.3 The topology on $H^j_c(X, \mathcal{F})$

Definition 1.47 (Topology on $H^j_c(X, \mathcal{F})$)**.** *Let $Z \subseteq X$ be a finite union of admissible affinoids. We endow $\Gamma_Z(X, \mathcal{F}) \subseteq \Gamma(X, \mathcal{F})$ with the subspace topology and $H^j_Z(X, \mathcal{F})$, $j \geq 1$, with the finest topology making the connecting homomorphism*

$$H^{j-1}(X \setminus Z, \mathcal{F}) \longrightarrow H^j_Z(X, \mathcal{F})$$

from the long exact sequence (1.5) continuous, where $H^{j-1}(X \setminus Z, \mathcal{F})$ is topologised as in Definition 1.41. Note that each map $H^j_Z(X, \mathcal{F}) \longrightarrow H^j(X, \mathcal{F})$ in (1.5) is then also continuous, since its continuity is equivalent to the continuity of the composite map $H^{j-1}(X \setminus Z, \mathcal{F}) \longrightarrow H^j(X, \mathcal{F})$ which indeed is continuous (being the zero map). Finally,

$$H^j_c(X, \mathcal{F}) = \varinjlim_{Z} H^j_Z(X, \mathcal{F})$$

receives the direct limit topology.

Remark 1.48. *$H^j_c(X, \mathcal{F})$ is in general not Hausdorff, let alone a Fréchet space.*

Using the long exact sequence (1.5) and Definition 1.47, one obtains:

Remark 1.49. *Let S be a rigid space that is cohomologically Stein, $Z \subseteq S$ a finite union of admissible affinoids and $\mathcal{F}$ a coherent sheaf on S. Then (1.5) yields topological isomorphisms*

$$H^n_Z(S, \mathcal{F}) \cong H^{n-1}(S \setminus Z, \mathcal{F}) \qquad \text{for } n \geq 2$$

and

$$H^1_Z(S, \mathcal{F}) \cong H^0(S \setminus Z, \mathcal{F})/H^0(S, \mathcal{F}).$$

Lemma 1.50. *Let $f \colon X \longrightarrow Y$ be a finite morphism of rigid spaces. Let $\mathcal{Z}$ be a cofinal subfamily of the family of all finite unions of admissible affinoids in Y. Then $f^{-1}(\mathcal{Z})$ is a cofinal subfamily of the family of all finite unions of admissible affinoids in X.*

Proof. The preimage of an affinoid subset under f is again affinoid because f is finite, so $f^{-1}\mathcal{Z}$ is indeed a subfamily of the family of all finite unions of admissible affinoids in X. To show cofinality, suppose that W is an affinoid subset of X. Take an admissible affinoid cover $\mathfrak{U}$ of Y. Then $f^{-1}\mathfrak{U}$ is an admissible affinoid cover of X. Since W is affinoid and in particular quasi-compact, it is covered by finitely many members of $f^{-1}\mathfrak{U}$, the union of which is obviously a member of $f^{-1}\mathcal{Z}$. Hence every affinoid subset of X is contained in a member of $f^{-1}\mathcal{Z}$. But then every finite union of affinoids in X is contained in a finite union of members of $f^{-1}\mathcal{Z}$, and the latter finite union is itself a member of $f^{-1}\mathcal{Z}$. $\square$

Proposition 1.51. *Let $f\colon X \longrightarrow Y$ be a finite morphism of cohomologically Stein rigid spaces and $\mathcal{F}$ a coherent sheaf on X. Then $f_*\mathcal{F}$ is coherent and we have an induced morphism on compactly supported cohomology*

$$H^j_c(Y, f_*\mathcal{F}) \xrightarrow{\sim} H^j_c(X, \mathcal{F})$$

which is in fact a topological isomorphism for all $j \geq 0$.

Proof. If Z is a finite union of affinoids in Y, then f restricts to a finite morphism

$$X \setminus f^{-1}(Z) = f^{-1}(Y \setminus Z) \longrightarrow Y \setminus Z$$

which induces isomorphisms on cohomology

$$H^j(Y \setminus Z, f_*\mathcal{F}) \xrightarrow{\sim} H^j(X \setminus f^{-1}(Z), \mathcal{F}) \qquad \text{for } j \geq 0 \tag{1.9}$$

due to Proposition 1.44. By Remark 1.49, these yield isomorphisms

$$H^j_Z(Y, f_*\mathcal{F}) \xrightarrow{\sim} H^j_{f^{-1}(Z)}(X, \mathcal{F}) \qquad \text{for } j \geq 1. \tag{1.10}$$

Moreover, the isomorphisms (1.9) allow us to apply the *5-Lemma* to the commutative diagram obtained from the long exact sequences (1.5)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\cong"] & 0 \arrow[r] \arrow[d, "\cong"] & H^0_Z(Y, f_*\mathcal{F}) \arrow[r] \arrow[d] & H^0(Y, f_*\mathcal{F}) \arrow[r] \arrow[d, "\cong"] & H^0(Y \setminus Z, f_*\mathcal{F}) \arrow[d, "\cong"] \\
0 \arrow[r] & 0 \arrow[r] & H^0_{f^{-1}(Z)}(X, \mathcal{F}) \arrow[r] & H^0(X, \mathcal{F}) \arrow[r] & H^0(X \setminus f^{-1}(Z), \mathcal{F})
\end{tikzcd}$$

and deduce the hitherto missing case

$$H^0_Z(Y, f_*\mathcal{F}) \xrightarrow{\sim} H^0_{f^{-1}(Z)}(X, \mathcal{F}). \tag{1.11}$$

By passing to the limit in (1.10) and (1.11) we obtain

$$H^j_c(Y, f_*\mathcal{F}) = \varinjlim_Z H^j_Z(Y, f_*\mathcal{F}) \xrightarrow{\sim} \varinjlim_Z H^j_{f^{-1}(Z)}(X, \mathcal{F}) = H^j_c(X, \mathcal{F}) \qquad \text{for } j \geq 0$$

where the limit on the right-hand side indeed produces $H^j_c(X, \mathcal{F})$ due to Lemma 1.50. ☐

1.2.4 Compactly supported cohomology of the open unit ball

We set

$$\mathbb{D}^n_K(\varepsilon) := \mathbb{D}^n(\varepsilon) := \{x \in \mathbb{D}^n : |x_i| \leq \varepsilon \text{ for all } i = 1, \dots, n\}.$$

Any admissible open affinoid $W \subseteq \mathring{\mathbb{D}}^n$ is contained in a $\mathbb{D}^n(\varepsilon)$ for some $\varepsilon \in (0, 1)$. To see this, note that, setting $\varepsilon_m := 1 - 1/m$, we have that $\mathring{\mathbb{D}}^n = \bigcup_m \mathbb{D}(\varepsilon_m)$ is an admissible affinoid cover of $\mathring{\mathbb{D}}^n$. Since W is affinoid and in particular quasi-compact, it is covered by finitely many $\mathbb{D}(\varepsilon_m)$. Choosing ε to be the largest among these finitely many m's, we obtain our claim. This proves the following remark:

Remark 1.52. *The $\mathbb{D}^n(\varepsilon)$, for varying $0 < \varepsilon < 1$, form a cofinal subfamily of the family of all finite unions of admissible affinoids in $\mathring{\mathbb{D}}^n$. In particular, we have*

$$H^n_c(\mathring{\mathbb{D}}^n, \mathcal{O}_{\mathbb{D}^n}) = \varinjlim_{0<\varepsilon<1} H^n_{\mathbb{D}^n(\varepsilon)}(\mathring{\mathbb{D}}^n, \mathcal{O}_{\mathbb{D}^n})$$

as is recorded in [Bey97a, Lemma 1.2.2].

Hence we wish to understand $H^n_{\mathbb{D}^n(\varepsilon)}(\mathring{\mathbb{D}}^n, \mathcal{O}_{\mathbb{D}^n})$. Remark 1.49 thus leads us to study $\mathring{\mathbb{D}}^n \setminus \mathbb{D}^n(\varepsilon)$. Consider the admissible open cover

$$\mathring{\mathbb{D}}^n \setminus \mathbb{D}^n(\varepsilon) = \bigcup_{i=1}^n U_{i,\varepsilon} \quad \text{where} \quad U_{i,\varepsilon} := \{x \in \mathring{\mathbb{D}}^n \colon \varepsilon < |x_i|\}. \tag{1.12}$$

The intersection $\bigcap_{i=1}^n U_{i,\varepsilon}$ is precisely the annulus

$$\bigcap_{i=1}^n U_{i,\varepsilon} = \{x \in \mathbb{D}^n \colon \varepsilon < |x_i| < 1 \text{ for all } i = 1, \ldots, n\},$$

whose ring of holomorphic functions is given by

$$\mathcal{R}_{K,n,\varepsilon} := H^0(\bigcap_{i=1}^n U_{i,\varepsilon}, \mathcal{O}_{\mathbb{D}^n}) = \{\sum_{\alpha\in\mathbb{Z}^n} a_\alpha X^\alpha \colon \lim_{|\alpha|\to\infty} |a_\alpha| \cdot \varepsilon^\alpha = 0\}.$$

Note that (1.12) is a Leray cover (i.e. it satisfies the conditions of Theorem 1.39) since the $U_{i,\varepsilon}$ and their finite intersections are quasi-Stein, so Čech cohomology computes

$$H^{n-1}(\mathring{\mathbb{D}}^n \setminus \mathbb{D}^n(\varepsilon), \mathcal{O}_{\mathbb{D}^n}) = \frac{\mathcal{O}_{\mathbb{D}^n}(\bigcap_{i=1}^n U_{i,\varepsilon})}{\operatorname{im} \partial_\varepsilon^{n-1}} = \mathcal{R}_{n,\varepsilon}/\operatorname{im} \partial_\varepsilon^{n-1} \qquad \text{for } n \geq 1$$

where $\partial_\varepsilon^{n-1}$ is the differential in degree $n-1$ of the Čech complex under consideration. For $n \geq 2$, every element of $\mathcal{R}_{n,\varepsilon}/\operatorname{im} \partial_\varepsilon^{n-1}$ is represented by a Laurent series $\sum_{\alpha<0} a_\alpha X^\alpha$ consisting only of the principal part, because the regular part (that is holomorphic on the whole disk) lives in $H^{n-1}(\mathring{\mathbb{D}}^n, \mathcal{O}_{\mathbb{D}^n})$ which vanishes for $n \geq 2$ due to Kiehl's Theorem B.

Remark 1.49 then yields

$$H^n_{\mathbb{D}^n(\varepsilon)}(\mathring{\mathbb{D}}^n, \mathcal{O}_{\mathbb{D}^n}) = H^{n-1}(\mathring{\mathbb{D}}^n \setminus \mathbb{D}^n(\varepsilon), \mathcal{O}_{\mathbb{D}^n}) = \mathcal{R}_{n,\varepsilon}/\operatorname{im} \partial_\varepsilon^{n-1} \qquad \text{for } n \geq 2$$

and

$$H^1_{\mathbb{D}^1(\varepsilon)}(\mathring{\mathbb{D}}^1, \mathcal{O}_{\mathbb{D}^1}) = \frac{H^0(\mathring{\mathbb{D}}^1 \setminus \mathbb{D}^1(\varepsilon), \mathcal{O}_{\mathbb{D}^1})}{H^0(\mathring{\mathbb{D}}^1, \mathcal{O}_{\mathbb{D}^1})} = \frac{\mathcal{R}_{1,\varepsilon}}{\operatorname{im} \partial_\varepsilon^0 + H^0(\mathring{\mathbb{D}}^1, \mathcal{O}_{\mathbb{D}^1})}. \tag{1.13}$$

Thus, for all $n \geq 1$, every element in $H^n_{\mathbb{D}^n(\varepsilon)}(\mathring{\mathbb{D}}^n, \mathcal{O}_{\mathbb{D}^n})$ is represented by a Laurent Series $\sum_{\alpha<0} a_\alpha X^\alpha$ consisting only of the principal part. Indeed, for $n \geq 2$ this follows from our discussion above and for $n = 1$ it is true because $H^0(\mathring{\mathbb{D}}^1, \mathcal{O}_{\mathbb{D}^1}) = \mathcal{O}(\mathring{\mathbb{D}}^1)$ is divided out on the right-hand side of (1.13). In fact:

Lemma 1.53 ([Bey97a, Corollary 1.2.5]). *Letting* $K\langle X_1^{-1}, \ldots, X_n^{-1}\rangle^\dagger := \varinjlim_{0<\varepsilon<1} \mathcal{R}_{n,\varepsilon}^-$ *where* $\mathcal{R}_{n,\varepsilon}^- \subseteq \mathcal{R}_{n,\varepsilon}$ *denotes the subring of Laurent series of the form* $\sum_{\alpha \leq 0} a_\alpha X^\alpha$, *we have an isomorphism of topological* K*-vector spaces:*

$$H_c^n(\mathring{\mathbb{D}}^n, \mathcal{O}_{\mathbb{D}^n}) \cong K\langle X_1^{-1}, \ldots, X_n^{-1}\rangle^\dagger \cdot \frac{1}{X_1 \cdots X_n}.$$

1.3 Local cohomology

1.3.1 Definitions and basic properties

Let R be a commutative ring and $\mathfrak{a}$ an ideal of R.

Definition 1.54. *For a* R*-module* M*, set*

$$\Gamma_{\mathfrak{a}}(M) := \{m \in M \colon \mathfrak{a}^t M = 0 \text{ for some } t \in \mathbb{N}\}.$$

We denote the j*-th right derived functor of the left exact functor* $\Gamma_{\mathfrak{a}}(-)$ *by* $H_{\mathfrak{a}}^j(-)$, *i.e.*

$$H_{\mathfrak{a}}^j(M) = H^j(\Gamma_{\mathfrak{a}}(I^\bullet))$$

where $I^\bullet$ *is an injective resolution of* M*. The* R*-module* $H_{\mathfrak{a}}^j(M)$ *is called* the j-th local cohomology of M with support in $\mathfrak{a}$*.*

Local cohomology commutes with finite direct sums: If $\{M_\lambda\}$ is a finite family of R-modules, then

$$H_{\mathfrak{a}}^j(\bigoplus_\lambda M_\lambda) = \bigoplus_\lambda H_{\mathfrak{a}}^j(M_\lambda). \tag{1.14}$$

Indeed, if $I_\lambda^\bullet$ is an injective resolution of M_λ, then $\bigoplus_\lambda I_\lambda^\bullet$ is an injective resolution of $\bigoplus_\lambda M_\lambda$ since every direct sum of finitely many injective modules is injective. Moreover, $\Gamma_{\mathfrak{a}}(-)$ obviously commutes with arbitrary direct sums so we have the equality $\Gamma_{\mathfrak{a}}(\bigoplus_\lambda I_\lambda^\bullet) = \bigoplus_\lambda \Gamma_{\mathfrak{a}}(I_\lambda^\bullet)$ which then yields (1.14) after we take cohomologies. We add that a ring R is Noetherian if and only if arbitrary direct sums of injective R-modules are injective [Lam99, Theorem 3.46] and therefore, if the base ring R is Noetherian, the above argument shows that local cohomology commutes with arbitrary direct sums.

Remark 1.55. *Suppose* R *is Noetherian. Then*

$$H_{\mathfrak{a}}^j(M) = H_{\sqrt{\mathfrak{a}}}^j(M).$$

for each $j \geq 0$

Proof. Since R is Noetherian, $\mathfrak{a}$ contains a power of its radical $\sqrt{\mathfrak{a}}$, hence we even have $\Gamma_{\mathfrak{a}}(-) = \Gamma_{\sqrt{\mathfrak{a}}}(-)$. □

The following description of local cohomology as a limit of Ext-modules will be useful in the proof of Proposition 1.59 below: If $\{\mathfrak{a}_n\}$ is any nested system of ideals which are cofinal with the powers of $\mathfrak{a}$, then

$$H^j_{\mathfrak{a}}(M) = \varinjlim_n \operatorname{Ext}^j_R(R/\mathfrak{a}_n, M) \tag{1.15}$$

for each $j \geq 0$. Indeed, this identification comes from the fact that we have a canonical identification

$$\{m \in M \colon \mathfrak{a}_n M = 0\} = \operatorname{Hom}_R(R/\mathfrak{a}_n, M)$$

and hence $\Gamma_{\mathfrak{a}}(M) = \bigcup_n \{m \in M \colon \mathfrak{a}_n M = 0\} = \varinjlim_n \operatorname{Hom}_R(R/\mathfrak{a}_n, M)$, which extends to higher cohomology groups as well, by a standard argument (cf. [Hun07, §2.1]).
Next we recall the notion of a system of parameters for a local ring:

Definition 1.56 ([Stacks, Tag 07DU, Tag 00KU]). *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. An ideal $J \subseteq R$ whose radical $\sqrt{J}$ satisfies $\sqrt{J} = \mathfrak{m}$ is called an* ideal of definition *of R. A* system of parameters *for R is a sequence of d elements $x_1, \dots, x_d \in \mathfrak{m}$ that generates an ideal of definition in R.*
If $\mathfrak{m}$ can be generated by d elements, R is said to be a regular local ring *and a system of parameters generating $\mathfrak{m}$ is called a* regular system of parameters.

We note that every Noetherian local ring has a system of parameters by [Stacks, Tag 00KQ].

1.3.2 A description obtained via Koszul- and Čech-complexes

Lemma 1.57. *Let R be Noetherian, $\mathfrak{a} \subseteq R$ an ideal and M a R-module. Let $t_1, \dots, t_d \in R$ be such that $\sqrt{\mathfrak{a}} = \sqrt{(t_1, \dots, t_d)}$. Write $t = (t_1, \dots, t_d)$. For each $\rho \in \mathbb{N}$, set $t^\rho := (t_1^\rho, \dots, t_d^\rho)$. Then there are canonical isomorphisms*

$$\varinjlim_\rho M/t^\rho M \xrightarrow{\sim} H^d_{\mathfrak{a}}(M) \tag{1.16}$$

and

$$M_{t_1 \cdots t_d} / \sum_{i=1}^d \operatorname{image}(M_{t_1 \cdots t_{i-1} t_{i+1} \cdots t_d}) \xrightarrow{\sim} H^d_{\mathfrak{a}}(M), \tag{1.17}$$

where M_x denotes the localisation of M at $x \in R$. Furthermore, if we let $\left[\frac{m}{t^\rho}\right]$ denote the image of $m + t^\rho M$ under $M/t^\rho M \longrightarrow \varinjlim_j M/t^j M \xrightarrow{\sim} H^d_{\mathfrak{a}}(M)$, then $\left[\frac{m}{t^\rho}\right]$ coincides with the image of the residue class of $\frac{m}{(t_1 \cdots t_d)^\rho}$ under then map (1.17).

Proof. The assertion of the lemma is standard material that can be found in the literature ([Iye+07, Theorem 7.11 and Theorem 7.13]), but we spell out some details for the convenience of the reader. Note that we can assume $\mathfrak{a} = (t_1, \dots, t_d)$ by Remark 1.55. Then we claim that we obtain (1.16) by specializing to degree $j = d$ in the isomorphism

$$\varinjlim_{\rho} H^j(\mathrm{Hom}_R(K^\bullet(t^\rho; R), M)) \xrightarrow{\sim} H^j_{\mathfrak{a}}(M) \tag{1.18}$$

from [Iye+07, Theorem 7.11], where $K^\bullet(t^\rho; R)$ is the *Koszul complex* on t^ρ as in [Iye+07, Definition 6.1]. Indeed, to see that

$$H^d(\mathrm{Hom}_R(K^\bullet(t^\rho; R), M)) = M/t^\rho M, \tag{1.19}$$

we first note that we have a commutative diagram of complexes of R-modules

$$\begin{tikzcd}
K^{-(d+1)}(t^\rho; R) \arrow[r] & K^{-d}(t^\rho; R) \arrow[r, "\partial"] & K^{-(d-1)}(t^\rho; R) \\
0 \arrow[u, equal] \arrow[r] & \bigwedge^d R = R \arrow[u, "\cong"] \arrow[r, "1 \mapsto \sum_{j=1}^d t_j^\rho E_j"] & \bigwedge^{d-1} R \arrow[u, "\cong"']
\end{tikzcd}$$

where the formal symbols

$$E_j := e_1 \wedge \ldots \wedge \widehat{e_j} \wedge \ldots \wedge e_d.$$

are an R-basis of $\bigwedge^{d-1} R$. By the definition of the complex $\mathrm{Hom}_R(K^\bullet(t^\rho; R), M)$, we have for the object in degree j that $(\mathrm{Hom}_R(K^\bullet(t^\rho; R), M))^j = \mathrm{Hom}_R(K^{-j}(t^\rho; R), M)$. Therefore, applying $\mathrm{Hom}_R(-, M)$ to the right-hand square in the above diagram yields

$$\begin{tikzcd}
(\mathrm{Hom}_R(K^\bullet(t^\rho; R), M))^d & (\mathrm{Hom}_R(K^\bullet(t^\rho; R), M))^{d-1} \arrow[l] \\
M \arrow[u, "\cong"] & M^d \arrow[l, "\sum_{j=1}^d t_j^\rho m_j \mapsfrom (m_1, \dots, m_d)"] \arrow[u, "\cong"']
\end{tikzcd}$$

where we have used the standard identification $\mathrm{Hom}_R(R^j, M) \cong M^j$ in the bottom row. Now (1.19) is obtained by taking cokernels of the horizontal maps.
In [Iye+07, Construction 7.12], an isomorphism of complexes

$$\check{C}^\bullet(t; M) \cong \varinjlim_{\rho} \mathrm{Hom}_R(K^\bullet(t^\rho; R), M) \tag{1.20}$$

is constructed, where $\check{C}^\bullet(t; M)$ is the *Čech complex*[4] on t as in [Iye+07, Definition 6.25]:

$$M \longrightarrow \bigoplus_{1 \le i \le d} M_{t_i} \longrightarrow \bigoplus_{1 \le i < j \le d} M_{t_i t_j} \longrightarrow \ldots \longrightarrow M_{t_1 \cdots t_d}$$

[4]This Čech complex is sometimes also called the "(stable) Koszul complex" in the literature, a collision of nomenclature which is justified by the isomorphism (1.20). Indeed, in the discussion preceding Proposition 2.13 in [Hun07], the complex $\check{C}^\bullet(t; M)$ is denoted by $K^\bullet(t_1, t_2, \dots, t_d; M)$ and is called the "Koszul cohomology complex".

with $\bigoplus_{1\le i_1<\ldots<i_k\le d} M_{t_{i_1}\cdots t_{i_k}}$ sitting in degree k and the differentials being alternating sums of localization maps. We mention that the construction of the isomorphism (1.20) is reduced to the case $d = 1$ and $M = R$. Then, given a single $x \in R$, $\varinjlim_\rho \operatorname{Hom}_R(K^\bullet(x^\rho; R), M)$ is isomorphic to the direct limit of the following direct system of complexes that are concentrated in degree $[0, 1]$

$$\begin{tikzcd}
 & & \cdots \arrow[d] \\
{[R} \arrow[r, "\cdot x^\rho"] \arrow[d, "\mathrm{id}"] & {R]} \arrow[r, equal, "\sim"] \arrow[d, "\cdot x"] & \operatorname{Hom}_R(K^\bullet(x^\rho; R), R) \arrow[d] \\
{[R} \arrow[r, "\cdot x^{\rho+1}"] & {R]} \arrow[r, equal, "\sim"] & \operatorname{Hom}_R(K^\bullet(x^{\rho+1}; R), R) \arrow[d] \\
 & & \cdots
\end{tikzcd} \tag{1.21}$$

The fact that the maps $\varphi_\rho \colon R \longrightarrow R_x, a \longmapsto \frac{a}{x^\rho}, \rho \in \mathbb{N}$, are compatible with the transition maps in the direct system

$$R \xrightarrow{\cdot x} R \xrightarrow{\cdot x} R \xrightarrow{\cdot x} \ldots \tag{1.22}$$

and realise R_x as the direct limit of (1.22), implies that the direct limit of the left-hand side of (1.21) is the complex $[R \longrightarrow R_x] = \check{C}^\bullet(x; R)$, proving our claim.
Specializing the isomorphism (1.20) to degree j in cohomology and then composing with (1.18) yields the isomorphism

$$H^j(\check{C}^\bullet(t; M)) \xrightarrow{\sim} H^j_{\mathfrak{a}}(M) \tag{1.23}$$

-- which for $j = d$ is the isomorphism (1.17) -- and proves the rest of Lemma 1.57. □

1.3.3 Permanence properties under completion and base change

With a straightforward application of the isomorphism (1.23), one can prove the following:

Proposition 1.58. *Let $\varphi \colon R \longrightarrow S$ be a morphism between Noetherian rings and $\mathfrak{a} \in R$ an ideal. Let M be an R-module and N an S-module.*

(i) *(Flat base change) If φ is flat, then $H^j_{\mathfrak{a}}(M) \otimes_R S \cong H^j_{\mathfrak{a}S}(M \otimes_R S)$.*

(ii) *(Independence of base) $H^j_{\mathfrak{a}}(N) \cong H^j_{\mathfrak{a}S}(N)$, where $H^j_{\mathfrak{a}}(N)$ is computed over the base ring R.*

Proof. For the convenience of the reader, we reproduce the proof from [Hun07, Proposition 2.14] here, since the argument will be needed in Lemma 5.14 later. We have $\check{C}^\bullet(t;M) \otimes_R S = \check{C}^\bullet(\varphi(t); M \otimes_R S)$ and hence

$$\begin{aligned} H^j_{\mathfrak{a}S}(M \otimes_R S) &\cong H^j(\check{C}^\bullet(\varphi(t); M \otimes_R S)) \cong H^j(\check{C}^\bullet(t;M) \otimes_R S) \\ &\cong H^j(\check{C}^\bullet(t;M)) \otimes_R S \cong H^j_{\mathfrak{a}}(M) \otimes_R S \end{aligned}$$

where the third isomorphism holds because φ is flat, proving Assertion (i). Assertion (ii) follows from the fact that

$$\begin{aligned} \check{C}^\bullet(t;N) &= \check{C}^\bullet(t;R) \otimes_R N = (\check{C}^\bullet(t;R) \otimes_R S) \otimes_S N \\ &= \check{C}^\bullet(\varphi(t); S) \otimes_S N = \check{C}^\bullet(\varphi(t); N) \end{aligned}$$

and (1.23). ☐

Lemma 1.57 is often used in the case where R is a local ring and $t_1, \ldots, t_d$ is a system of parameters for R. We may often assume without loss of generality that we are in this case, by passing over to the completion with respect to a maximal ideal $\mathfrak{m} = (t_1, \ldots, t_d)$, due to the insensitivity of local cohomology to completion:

Proposition 1.59. *Let R be a Noetherian ring and let $\mathfrak{a} \subseteq R$ be an ideal. Let M be a finitely generated R-module. Let $\widehat{R}$ be the $\mathfrak{a}$-adic completion of R; it is a Noetherian ring that is complete with respect to the ideal $\widehat{\mathfrak{a}} = \mathfrak{a}\widehat{R}$ and the ring map $R \longrightarrow \widehat{R}$ is flat. The $\widehat{R}$-module $\widehat{M} = \widehat{R} \otimes_R M$ is finite and we have*

$$H^j_{\mathfrak{a}}(M) \cong H^j_{\widehat{\mathfrak{a}}}(\widehat{M}).$$

for all $j \geq 0$.

Proof. The assertions regarding $\widehat{R}$ and $\widehat{M}$ are standard facts from commutative algebra [Stacks, Tag 05GH, Tag 00MB, Tag 00MA]. The isomorphism between the local cohomology groups is shown in [Hun07, Proposition 2.15] for the case that R is local and $\mathfrak{a} = \mathfrak{m}$ is its unique maximal ideal. We show that the proof of [Hun07, Proposition 2.15] carries over verbatim to our (more general) setting: We have

$$\begin{aligned} H^j_{\widehat{\mathfrak{a}}}(\widehat{M}) &\cong H^j_{\mathfrak{a}\widehat{R}}(M \otimes_R \widehat{R}) \cong H^j_{\mathfrak{a}}(M) \otimes_R \widehat{R} \cong \varinjlim_n \operatorname{Ext}^j_R(R/\mathfrak{a}^n, M) \otimes_R \widehat{R} \\ &\cong \varinjlim_n (\operatorname{Ext}^j_R(R/\mathfrak{a}^n, M) \otimes_R \widehat{R}) \cong \varinjlim_n \operatorname{Ext}^j_R(R/\mathfrak{a}^n, M) = H^j_{\mathfrak{a}}(M), \end{aligned}$$

the second isomorphism being due to Proposition 1.58 (i) since $R \longrightarrow \widehat{R}$ is flat, the third due to (1.15), and the fifth due to the fact that each $\operatorname{Ext}^j_R(R/\mathfrak{a}^n, M)$ is $\mathfrak{a}$-adically complete since it is annihilated by a power of $\mathfrak{a}$. Indeed, to see that $\operatorname{Ext}^j_R(R/\mathfrak{a}^n, M)$ is annihilated by $\mathfrak{a}^n$, take an injective resolution $M \longrightarrow I^\bullet$ and note that every module in the complex $\operatorname{Hom}_R(R/\mathfrak{a}^n, I^\bullet)$ is already annihilated by $\mathfrak{a}^n$. ☐

1.3.4 The Mayer-Vietoris sequence

Another basic result that we recall is the Mayer-Vietoris sequence in local cohomology: Let $\mathfrak{a}' \supseteq \mathfrak{a}$ be ideals in a Noetherian ring R and M an R-module. Then the inclusion $\Gamma_{\mathfrak{a}'}(M) \subseteq \Gamma_{\mathfrak{a}}(M)$ induces a homomorphism

$$\theta^j_{\mathfrak{a}',\mathfrak{a}} \colon H^j_{\mathfrak{a}'}(M) \longrightarrow H^j_{\mathfrak{a}}(M).$$

for each j. We abuse notation and write just θ for $\theta^j_{\mathfrak{a}',\mathfrak{a}}$ since the sub- and superscripts of θ will always be implicitly clear from the context. For ideals $\mathfrak{a}, \mathfrak{b} \subseteq R$, we define

$$\begin{aligned} \iota^j_{\mathfrak{a},\mathfrak{b}} \colon H^j_{\mathfrak{a}+\mathfrak{b}}(M) &\longrightarrow H^j_{\mathfrak{a}}(M) \oplus H^j_{\mathfrak{b}}(M) \\ z &\longmapsto (\theta(z), \theta(z)), \\ \text{and } \pi^j_{\mathfrak{a},\mathfrak{b}} \colon H^j_{\mathfrak{a}}(M) \oplus H^j_{\mathfrak{b}}(M) &\longrightarrow H^j_{\mathfrak{a}\cap\mathfrak{b}}(M) \\ (x,y) &\longmapsto \theta(x) - \theta(y) \end{aligned}$$

where we will again abuse notation and simply write ι and π.

Proposition 1.60 (Mayer-Vietoris sequence)**.** *Let $\mathfrak{a}, \mathfrak{b}$ be ideals in a Noetherian ring R and M an R-module. Then there exists a long exact sequence of the form*

$$\begin{aligned} 0 \longrightarrow H^0_{\mathfrak{a}+\mathfrak{b}}(M) &\xrightarrow{\iota} H^0_{\mathfrak{a}}(M) \oplus H^0_{\mathfrak{b}}(M) \xrightarrow{\pi} H^0_{\mathfrak{a}\cap\mathfrak{b}}(M) \\ \xrightarrow{\partial} H^1_{\mathfrak{a}+\mathfrak{b}}(M) &\xrightarrow{\iota} H^1_{\mathfrak{a}}(M) \oplus H^1_{\mathfrak{b}}(M) \xrightarrow{\pi} H^1_{\mathfrak{a}\cap\mathfrak{b}}(M) \xrightarrow{\partial} \dots \end{aligned}$$

that is functorial in M.

Proof. [Iye+07, Theorem 15.1]. ☐

Corollary 1.61. *Let $\mathfrak{m}_1, \dots, \mathfrak{m}_s$ be different maximal ideals in a Noetherian ring R and M an R-module. Then the map*

$$\begin{aligned} \bigoplus_{i=1}^s H^j_{\mathfrak{m}_i}(M) &\xrightarrow{\sim} H^j_{\bigcap_{i=1}^s \mathfrak{m}_i}(M) \\ (z_1, \dots, z_s) &\longmapsto \sum_{i=1}^s \theta(z_i) \end{aligned}$$

is an isomorphism for all $j \geq 0$.

Proof. Assume for the moment that $s = 2$. Since $\mathfrak{m}_1 + \mathfrak{m}_2 = R$ and $\Gamma_R(-)$ is the zero functor (so $H^j_R(M)$ is always zero), the Mayer-Vietoris sequence tells us that

$$\begin{aligned} \pi \colon H^j_{\mathfrak{m}_1}(M) \oplus H^j_{\mathfrak{m}_2}(M) &\xrightarrow{\sim} H^j_{\mathfrak{m}_1\cap\mathfrak{m}_2}(M) \\ (z_1, z_2) &\longmapsto \theta(z_1) - \theta(z_2) \end{aligned}$$

is an isomorphism. Precomposing with the isomorphism $(z_1, z_2) \longmapsto (z_1, -z_2)$ proves the assertion for $s = 2$, more generally for any two (not necessarily maximal) ideals $\mathfrak{m}_1, \mathfrak{m}_2$ that are coprime in the sense that they satisfy $\mathfrak{m}_1 + \mathfrak{m}_2 = R$. Now we proceed by induction on s: The map

$$\begin{aligned} \bigoplus_{i=1}^{s} H^j_{\mathfrak{m}_i}(M) &\xrightarrow{\sim} H^j_{\bigcap_{i=1}^{s-1}\mathfrak{m}_i}(M) \oplus H^j_{\mathfrak{m}_s}(M) \\ (z_1, \dots, z_s) &\longmapsto (\sum_{i=1}^{s-1} \theta(z_i), z_s) \end{aligned} \tag{1.24}$$

is then an isomorphism by the induction hypothesis. Moreover, the map

$$\begin{aligned} H^j_{\bigcap_{i=1}^{s-1}\mathfrak{m}_i}(M) \oplus H^j_{\mathfrak{m}_s}(M) &\xrightarrow{\sim} H^j_{\bigcap_{i=1}^{s}\mathfrak{m}_i}(M) \\ (x, y) &\longmapsto (\theta(x) + \theta(y)) \end{aligned} \tag{1.25}$$

is an isomorphism by the case $s = 2$ for two coprime ideals, where we note that $\mathfrak{m}_s$ and $\bigcap_{i=1}^{s-1}\mathfrak{m}_i$ are indeed coprime, since otherwise $\mathfrak{m}_s + \bigcap_{i=1}^{s-1}\mathfrak{m}_i$ would be a proper ideal containing $\mathfrak{m}_s$ i.e. we would have $\mathfrak{m}_s + \bigcap_{i=1}^{s-1}\mathfrak{m}_i = \mathfrak{m}_s$ and hence $\bigcap_{i=1}^{s-1}\mathfrak{m}_i \subseteq \mathfrak{m}_s$, which would, by [AM69, Proposition 1.11. ii)], necessarily imply $\mathfrak{m}_i \subseteq \mathfrak{m}_s$ for some $i < s$, a contradiction. Composing the two isomophisms (1.24) and (1.25) yields the desired isomorphism, proving the assertion. □

1.3.5 Relation to sheaf cohomology

Finally, we review the geometric interpretation of the local cohomology groups $H^\bullet_{\mathfrak{a}}(M)$, namely that they are isomorphic to the sheaf cohomology groups $H^\bullet_Z(\operatorname{Spec} R, \widetilde{M})$ of the quasi-coherent sheaf $\widetilde{M}$ with support in $Z := V(\mathfrak{a}) \subseteq \operatorname{Spec} R$. This is used in the construction of the map from local cohomology to compactly supported rigid cohomology (Lemma 4.6).

Definition 1.62. *Let X be a topological space and $Z \subseteq X$ a closed subspace. For a sheaf $\mathcal{F}$ of abelian groups, the* sections with support in Z are

$$\Gamma_Z(X, \mathcal{F}) := \ker(\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X \setminus Z, \mathcal{F})).$$

We denote the j-th right derived functor of the left exact functor $\Gamma_Z(X, -)$ by $H^j_Z(X, -)$ and call $H^j_Z(X, \mathcal{F})$ the j-th local cohomology of $\mathcal{F}$ with support in Z.

Remark 1.63. *For a sheaf $\mathcal{F}$ of abelian groups on X and a section $s \in \mathcal{F}(U)$ over an open set U, the support of s is defined to be* $\operatorname{Supp}(s) := \{p \in U : s_p \neq 0\}$*, where s_p is the germ of s in the stalk $\mathcal{F}_p$. We have*

$$\Gamma_Z(X, \mathcal{F}) = \{s \in \Gamma(X, \mathcal{F}) : \operatorname{Supp}(s) \subseteq Z\},$$

since $s_{|(X \setminus Z)} = 0$ if and only if $s_p = 0$ for all $p \in X \setminus Z$, i.e. if and only if $\operatorname{Supp}(s) \subseteq Z$.

There is a long exact sequence

$$\ldots \longrightarrow H^j_Z(X,\mathcal{F}) \longrightarrow H^j(X,\mathcal{F}) \longrightarrow H^j(X\setminus Z,\mathcal{F}) \longrightarrow H^{j+1}_Z(X,\mathcal{F}) \longrightarrow \ldots \tag{1.26}$$

Remark 1.64. *Let R be a Noetherian ring and $\mathfrak{a} \subseteq R$ an ideal. For $Z = V(\mathfrak{a})$ in $X = \operatorname{Spec} R$, one has*

$$H^j_{\mathfrak{a}}(M) \cong H^j_Z(X, \widetilde{M})$$

for any R-module M and any $j \in \mathbb{N}_0$.

Proof. This is well-known, but we summarise the simple argument for the convenience of the reader. First of all, one has $\Gamma_Z(\widetilde{M}) = \Gamma_{\mathfrak{a}}(M)$, for the following reason: Choosing generators $f_1, \ldots, f_k$ of $\mathfrak{a}$, the subsets $U_i := \operatorname{Spec} A_{f_i}$ yield a finite open covering of $X \setminus Z$. Hence, an element of $\Gamma(X, \widetilde{M}) = M$ becomes zero in $\Gamma(X\setminus Z, \widetilde{M})$ if and only if it becomes zero in $\Gamma(U_i, \widetilde{M}) = M_{f_i}$ for all i, i.e. if and only if it is annihilated by a power $f_i^{n_i}$ of f_i for all i. This last condition is obviously equivalent to the element being annihilated by a suitable power of $\mathfrak{a}$, which proves that $\Gamma_Z(\widetilde{M}) = \Gamma_{\mathfrak{a}}(M)$.
Now choose an injective resolution $M \longrightarrow I^\bullet$ of M and consider $\widetilde{M} \longrightarrow \widetilde{I}^\bullet$. Then we have $\Gamma_{\mathfrak{a}}(I^\bullet) = \Gamma_Z(\widetilde{I}^\bullet)$ by the above, so it remains to show that $\Gamma_Z(\widetilde{I}^\bullet)$ computes $H^\bullet_Z(X, \widetilde{M})$. Note that $\widetilde{M} \longrightarrow \widetilde{I}^\bullet$ is an injective resolution of $\widetilde{M}$ in the category of $\mathcal{O}_X$-modules, by [Har66, Corollary II.7.14]. Moreover, any injective $\mathcal{O}_X$-module is flasque by [Har77, Lemma III.2.4] and any flasque sheaf is acyclic for Γ_Z by [Har66, IV.1 Motif C]. Hence we are done by [Har77, Proposition III.1.2A]. □

From Remark 1.64, we deduce using the long exact sequence (1.26) and Serre's cohomological characterization of affineness:

Proposition 1.65 (Local cohomology and sheaf cohomology)**.** *Let R be a Noetherian ring and $\mathfrak{a} = (x_1, \ldots, x_d)$ an ideal. Set $X = \operatorname{Spec}(R)$ and $U = X \setminus V(\mathfrak{a})$. Let M be an R-module and $\widetilde{M}$ the corresponding quasi-coherent sheaf on X. Then we have an isomorphism*

$$H^j(U, \widetilde{M}) \xrightarrow{\sim} H^{j+1}_{\mathfrak{a}}(M) \qquad \text{for each } j \geq 1$$

and a surjection

$$H^0(U, \widetilde{M}) \twoheadrightarrow H^1_{\mathfrak{a}}(M)$$

which sits in an exact sequence

$$0 \longrightarrow H^0_{\mathfrak{a}}(M) \longrightarrow H^0(X, \widetilde{M}) = M \longrightarrow H^0(U, \widetilde{M}) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0.$$

It is worth noting that an alternative proof of Proposition 1.65 is given by the fact that, by (1.23), $H^\bullet_{\mathfrak{a}}(M)$ can be computed using the Čech complex $\check{C}^\bullet(t; M)$ on $x = (x_1, \ldots, x_d)$ which in degree k is the R-module

$$\bigoplus_{1 \leq i_1 < \ldots < i_k \leq d} M_{x_{i_1} \cdots x_{i_k}},$$

and the fact that the sheaf cohomology $H^\bullet(U, \widetilde{M})$ can be computed using the topological Čech complex with respect to the open cover $\mathfrak{U} = \{U_i\}$ of U consisting of the open sets $U_i := X \setminus V(x_i)$, which in degree k is

$$\prod_{1 \leq i_1 < \ldots < i_{k+1} \leq d} \Gamma(U_{i_1} \cap \ldots \cap U_{i_{k+1}}, \widetilde{M})$$

while observing that $U_{i_1} \cap \ldots \cap U_{i_{k+1}} = X \setminus V(x_{i_1} \cdots x_{i_{k+1}}) = \operatorname{Spec}(R_{x_{i_1} \cdots x_{i_{k+1}}})$ and thus $\Gamma(U_{i_1} \cap \ldots \cap U_{i_{k+1}}, \widetilde{M}) = M_{x_{i_1} \cdots x_{i_{k+1}}}$.

1.4 Commutative algebra

1.4.1 A note on completions and systems of parameters

For a ring A and an ideal $\mathfrak{a} \subseteq A$, we write $A^{\wedge \mathfrak{a}}$ for the $\mathfrak{a}$-adic completion of A. If A is local, we write $\widehat{A}$ for the completion with respect to the maximal ideal. The purpose of this section is to justify the claim in Lemma 4.5 (ii) below that the maps $T_n{}^{\wedge \mathfrak{m}} \longrightarrow R^{\wedge \mathfrak{m}_i}$ send any system of parameters to a system of parameters (see Definition 1.56). This is the content of Corollary 1.68 below.

Lemma 1.66. *Let $\varphi\colon (A, \mathfrak{m}_A) \longrightarrow (B, \mathfrak{m}_B)$ be a local ring homomorphism between Noetherian local rings of dimension d. Suppose that φ is integral. Then any system of parameters is mapped to a system of parameters under φ.*

Proof. Let $a_1 \ldots, a_d \in A$ be a system of parameters in A, meaning that the ideal $I := (a_1, \ldots, a_d)$ satisfies $\sqrt{I} = \mathfrak{m}_A$. Since the ideal $\varphi(I)B \subseteq B$ is generated by d elements $\varphi(a_1), \ldots \varphi(a_d)$ and B has dimension d, it suffices to show that $\sqrt{\varphi(I)B} = \mathfrak{m}_B$. Let $\mathfrak{q}$ be a prime ideal containing $\varphi(I)B$. Then $\mathfrak{p} := \varphi^{-1}(\mathfrak{q})$ is a prime ideal in A containing I and hence also containing $\sqrt{I} = \mathfrak{m}_A$. Thus $\mathfrak{p} = \mathfrak{m}_A$. Now $B/\mathfrak{q}$ is an integral domain and $A/\mathfrak{p} \hookrightarrow B/\mathfrak{q}$ a subring such that $B/\mathfrak{q}$ is integral over $A/\mathfrak{p}$, so we have

$$A/\mathfrak{p} \text{ is a field} \iff B/\mathfrak{q} \text{ is a field.}$$

But $A/\mathfrak{p}$ is indeed a field since $\mathfrak{p} = \mathfrak{m}_A$, so we conclude that $\mathfrak{q}$ is a maximal ideal and hence necessarily $\mathfrak{q} = \mathfrak{m}_B$. We have shown that $\mathfrak{m}_B$ is the only prime ideal containing $\varphi(I)B$, whence $\sqrt{\varphi(I)B} = \mathfrak{m}_B$. □

Proposition 1.67. *Let A and B be Noetherian rings, $A \longrightarrow B$ a ring morphism. Let $\mathfrak{m}$ be a maximal ideal in A and $\mathfrak{n}$ a maximal ideal in B that pulls back to $\mathfrak{m}$. If $A \longrightarrow B$ is finite, then the induced morphism on the completions*

$$A^{\wedge \mathfrak{m}} \longrightarrow B^{\wedge \mathfrak{n}}$$

is finite as well. If $A \longrightarrow B$ is moreover flat, then $A^{\wedge \mathfrak{m}} \longrightarrow B^{\wedge \mathfrak{n}}$ is flat as well.

Proof. This is surely a well-known fact but we give a proof due to lack of an adequate reference. Since A is Noetherian and B is finite over A, we have $A^{\wedge\mathfrak{m}} \otimes_A B = B^{\wedge\mathfrak{m}B}$. Hence the induced morphism $A^{\wedge\mathfrak{m}} \longrightarrow B^{\wedge\mathfrak{m}B}$ arises by tensoring the given finite (resp. finite flat) morphism $A \longrightarrow B$ with $A^{\wedge\mathfrak{m}}$ over A and is therefore also finite (resp. finite flat). Thus we are reduced to proving that $B^{\wedge\mathfrak{m}B} \longrightarrow B^{\wedge\mathfrak{n}}$ is finite (resp. finite flat). Let $\mathfrak{n}_1, \dots, \mathfrak{n}_r$ be all the prime ideals of B that pull back to $\mathfrak{m}$, so in particular $\mathfrak{r} := \sqrt{\mathfrak{m}B} = \mathfrak{n}_1 \cap \dots \cap \mathfrak{n}_r$. Since $A \longrightarrow B$ is finite, the $\mathfrak{n}_i$ are maximal ideals. We claim that the natural maps $B^{\wedge\mathfrak{m}B} \longrightarrow B^{\wedge\mathfrak{n}_i}$ together induce an isomorphism

$$B^{\wedge\mathfrak{m}B} = B^{\wedge\mathfrak{n}_1} \times \dots \times B^{\wedge\mathfrak{n}_r}. \tag{1.27}$$

Indeed, as a special case of [Bou72, III §2.13 Proposition 1.7], we see that the natural maps $B^{\wedge\mathfrak{r}} \longrightarrow B^{\wedge\mathfrak{n}_i}$ together yield an isomorphism $B^{\wedge\mathfrak{r}} = B^{\wedge\mathfrak{n}_1} \times \dots \times B^{\wedge\mathfrak{n}_r}$ Since moreover $\mathfrak{m}B$ and $\mathfrak{r} = \sqrt{\mathfrak{m}B}$ define the same topology on B, our claim follows. Now, because $\mathfrak{n}$ appears among the $\mathfrak{n}_i$, we see from (1.27) that the map $B^{\wedge\mathfrak{m}B} \longrightarrow B^{\wedge\mathfrak{n}}$ can be realised as a projection onto a factor in a product, hence is finite flat. (Indeed, for any ring product $R = R_1 \times R_2$, the projection map $R \longrightarrow R_1$ is finite because it is surjective and it is flat because it can be identified with the localisation map $R \longrightarrow R[(1,0)^{-1}]$.) ☐

Corollary 1.68. *Let R be an affinoid algebra and $T_n \longrightarrow R$ a finite injective morphism (thus necessarily $\dim T_n = \dim R$), let $\mathfrak{m}$ be a maximal ideal in T_n and $\mathfrak{n}$ a maximal ideal in R that pulls back to $\mathfrak{m}$. Then*

$${T_n}^{\wedge\mathfrak{m}} \longrightarrow R^{\wedge\mathfrak{n}}$$

is a finite morphism between Noetherian local rings of the same dimension (so by Lemma 1.66 it sends any system of parameters to a system of parameters).

Proof. Since $T_n \longrightarrow R$ is finite, ${T_n}^{\wedge\mathfrak{m}} \longrightarrow R^{\wedge\mathfrak{n}}$ is finite by Proposition 1.67. Moreover, ${T_n}^{\wedge\mathfrak{m}}$ and $R^{\wedge\mathfrak{n}}$ indeed have the same Krull dimension, since

$$\dim {T_n}^{\wedge\mathfrak{m}} = \dim \widehat{(T_n)_\mathfrak{m}} = \dim (T_n)_\mathfrak{m} = \dim T_n = \dim R = \dim R_\mathfrak{n} = \dim \widehat{R_\mathfrak{n}} = \dim R^{\wedge\mathfrak{n}}$$

where the first and seventh equalities hold since "completion with respect to a maximal ideal commutes with localisation"[5], the second and sixth equalities hold since the dimension of a Noetherian local ring is invariant under completion [Stacks, Tag 07NV], the third resp. fifth equality holds since all maximal ideals in T_n resp. in R have the same height by [Con99, Lemma 2.5], and the fourth equality holds since there exists a finite injective ring morphism $T_n \longrightarrow R$. ☐

[5]Meaning that, if A is a ring and $\mathfrak{m}$ a maximal ideal in A, then the natural map $A^{\wedge\mathfrak{m}} \xrightarrow{\sim} \widehat{A_\mathfrak{m}}$ is an isomorphism. Indeed, since $\mathfrak{m}$ is maximal, $A/\mathfrak{m}^i$ is equal to its localisation $(A/\mathfrak{m}^i)_\mathfrak{m}$, which in turn is equal to $A_\mathfrak{m}/\mathfrak{m}^i_\mathfrak{m}$ since localisation is exact. The claim now follows from the fact that $A^{\wedge\mathfrak{m}}$ resp. $\widehat{A_\mathfrak{m}}$ is the projective limit of the $A/\mathfrak{m}^i$ resp. $A_\mathfrak{m}/\mathfrak{m}^i_\mathfrak{m}$.

Chapter 2

A generalisation of Bosch's theorem on the connectedness of formal fibers

The following result will play an important role in the proof of Theorem 2.4 below.

Proposition 2.1. *Let R be an K-affinoid algebra and K'/K a finite Galois extension. Consider*

$$S := R \otimes_K K'.$$

Let $\iota\colon R \longrightarrow S$ be the canonical inclusion and

$$\phi\colon Z' = \mathrm{Sp}(S) \longrightarrow Z = \mathrm{Sp}(R)$$

the associated morphism of rigid spaces over K, which fits into the commutative diagram

$$\begin{tikzcd} Z' \arrow[r, two heads] \arrow[d, "\phi"'] & \widetilde{Z}' \arrow[d, "\widetilde{\phi}"] \\ Z \arrow[r, two heads] & \widetilde{Z}. \end{tikzcd}$$

Then the reduction map $Z' \longrightarrow \widetilde{Z}'$ is fiberwise surjective in the sense that for every $z \in Z$, the induced map

$$\phi^{-1}(z) \longrightarrow \widetilde{\phi}^{-1}(\widetilde{z})$$

is surjective.

Proof. The morphism ϕ is finite, so $\phi^{-1}(z)$ is a finite set, say

$$\phi^{-1}(z) = \{z'_1, \dots, z'_n\}.$$

Then $\widetilde{\phi}$ is also finite by Theorem 1.10, so the complement $\widetilde{\phi}^{-1}(\widetilde{z}) \setminus \{\widetilde{z}', \dots, \widetilde{z}'_n\}$ is a finite set and we have to show that it is in fact empty. Suppose that it is non-empty, say

$$\widetilde{\phi}^{-1}(\widetilde{z}) \setminus \{\widetilde{z}', \dots, \widetilde{z}'_n\} = \{\widetilde{z}'_{n+1}, \dots, \widetilde{z}'_s\}$$

for some elements $\widetilde{z}'_{n+1}, \ldots, \widetilde{z}'_s \in \widetilde{Z}'$ with lifts $z'_{n+1}, \ldots, z'_s \in Z'$. Based upon this assumption, we will construct an $f \in \mathring{R}$ such that

$$|f(z)| = 1 \quad \text{and} \quad |f(\phi(z'_{n+1}))| < 1.$$

But then, by Proposition 1.5, $|f(z)| = 1$ means that $\widetilde{f}(\widetilde{z}) \neq 0$ and $|f(\phi(z'_{n+1}))| < 1$ means that $\widetilde{f}(\widetilde{z}) = 0$ since $\phi(z'_{n+1})$ also reduces to $\widetilde{z}$. Thus we arrive at a contradiction and the assertion is proved. It remains to construct such an $f \in \mathring{R}$. For this, we first choose a $\widetilde{g} \in \widetilde{S}$ such that

$$\widetilde{g}(\widetilde{z}'_j) = 1 \text{ for } j = 1, \ldots n \quad \text{and} \quad \widetilde{g}(\widetilde{z}'_{n+1}) = 0. \tag{2.1}$$

This is possible since

$$\widetilde{S} \longrightarrow \prod_{j=1}^{n+1} \widetilde{S}/\mathfrak{m}_{\widetilde{z}'_j}$$

is surjective by the Chinese Remainder Theorem. Then we choose a lift $g \in \mathring{S}$ of $\widetilde{g}$. Note that (2.1) then yields

$$|g(z'_j)| = 1 \text{ for } j = 1, \ldots n \quad \text{and} \quad |g(z'_{n+1})| < 1 \tag{2.2}$$

by Proposition 1.5. Next, the Galois group $G = \mathrm{Gal}(K'/K)$ acts in an obvious way on $R \otimes_K K' = S$ by R-algebra homomorphisms, and it is easy to prove that for the fixed elements we have

$$S^G = R.$$

Therefore,

$$f := \prod_{\sigma \in G} \sigma(g) \in R.$$

Moreover, any morphism $S \longrightarrow S$ is contractive with respect to $|\cdot|_{\sup}$ by [Bos14, 3.1/Proposition 7], so in particular we have $|\sigma(g)|_{\sup} \leq |g|_{\sup} \leq 1$ for all $\sigma \in G$. Hence

$$|f|_{\sup} \leq \prod_{\sigma \in G} |\sigma(g)|_{\sup} \leq 1. \tag{2.3}$$

Due to Remark 1.9, the $|\cdot|_{\sup}$ of S restricts to the $|\cdot|_{\sup}$ of R. Thus the inequality (2.3) shows that in fact

$$f \in \mathring{R}.$$

Next we claim that

$$|\sigma(g)(z'_1)| = 1 \quad \text{for all } \sigma \in G. \tag{2.4}$$

To see this, note that each $\sigma \in G$ permutes $\{\mathfrak{m}_{z'_1}, \ldots, \mathfrak{m}_{z'_n}\}$ so we have $\sigma^{-1}(\mathfrak{m}_{z'_1}) = \mathfrak{m}_{z'_j}$ for some $j \in \{1, \ldots n\}$ and thus σ induces an isomorphism

$$\sigma \colon S/\mathfrak{m}_{z'_j} \xrightarrow{\sim} S/\mathfrak{m}_{z'_1}$$

mapping $g \bmod \mathfrak{m}_{z'_j}$ to $\sigma(g) \bmod \mathfrak{m}_{z'_1}$, which means that $|g(z'_j)| = |\sigma(g)(z'_1)|$. Since $|g(z'_j)| = 1$ by (2.2), this proves (2.4). Finally, we compute

$$\begin{aligned}
|f(z)| &= |f(\phi(z'_1))| \\
&= |\iota(f)(z'_1)| \\
&= |\prod_\sigma \sigma(g)(z'_1)| \\
&= \prod_\sigma |\sigma(g)(z'_1)| \\
&= \prod_\sigma 1 \\
&= 1
\end{aligned}$$

and similarly

$$\begin{aligned}
|f(\phi(z'_{n+1}))| &= \prod_\sigma |\sigma(g)(z'_{n+1})| \\
&= |g(z'_{n+1})| \cdot \prod_{\sigma \neq \mathrm{id}} |\sigma(g)(z'_{n+1})| \\
&< 1
\end{aligned}$$

since $|g(z'_{n+1})| < 1$ and moreover $|\sigma(g)(z'_{n+1})| \leq 1$ because $|\sigma(g)|_{\sup} \leq 1$. ☐

Corollary 2.2. *Let Z be an K-affinoid space and K'/K a finite Galois extension. Consider the base change*

$$Z' := Z \otimes_K K'$$

of Z to K' and the associated morphism $\phi \colon Z' \longrightarrow Z$ of rigid spaces. Then we have the following relation between formal fibers

$$Z_+(z) = \bigcup_{i=1}^{n} \phi(Z'_+(z'_i)), \tag{2.5}$$

where $\{z'_1, \ldots, z'_n\} = \phi^{-1}(z)$.

Proof. The inclusion "$\supseteq$" in (2.5) is clear due to the commutativity of the diagram

$$\begin{tikzcd}
Z' \arrow[r, two heads] \arrow[d, "\phi"'] & \widetilde{Z}' \arrow[d, "\widetilde{\phi}"] \\
Z \arrow[r, two heads] & \widetilde{Z}.
\end{tikzcd} \tag{2.6}$$

To show the reverse inclusion, let $y \in Z_+(z)$ and take a preimage $y' \in Z'$ under ϕ. We will show that $y' \in Z'_+(z'_i)$ for some i. By the commutativity of (2.6) and since $y \in Z_+(z)$, we have $\widetilde{\phi}(\widetilde{y}') = \widetilde{y} = \widetilde{z}$, so $\widetilde{y}' \in \widetilde{\phi}^{-1}(\widetilde{z})$. Now $\widetilde{\phi}^{-1}(\widetilde{z}) = \{\widetilde{z}'_1, \ldots, \widetilde{z}'_n\}$ by Proposition 2.1, whence $\widetilde{y}' = \widetilde{z}'_i$ for some i, i.e. $y' \in Z'_+(z'_i)$. ☐

The following Theorem 2.3, which is due to Bosch, is an important technical result concerning the connectedness of formal fibers:

Theorem 2.3 ([Bos77, Satz 6.1])**.** *Let R be a distinguished K-affinoid algebra which has pure dimension and let $Z = \mathrm{Sp}(R)$. Then, for every $z \in Z$, the formal fiber $Z_+(z) = p^{-1}(p(z))$ of the reduction map $p\colon Z \longrightarrow \widetilde{Z}$ is connected.*

We generalise Bosch's theorem to the case of a not necessarily distinguished affinoid algebra:

Theorem 2.4. *Let R be an K-affinoid algebra which has pure dimension and let $Z = \mathrm{Sp}(R)$. Suppose that there exists a finite Galois extension K'/K such that $S := R \otimes_K K'$ is distinguished. Then, for every $z \in Z$, the formal fiber $Z_+(z) = p^{-1}(p(z))$ of the reduction map $p\colon Z \longrightarrow \widetilde{Z}$ is connected.*

We postpone the proof for a moment to record the following corollary:

Corollary 2.5. *Let R be an K-affinoid algebra such that $Z = \mathrm{Sp}(R)$ is smooth and connected. Then, for every $z \in Z$, the formal fiber $Z_+(z) = p^{-1}(p(z))$ of the reduction map $p\colon Z \longrightarrow \widetilde{Z}$ is connected.*

Proof of Corollary 2.5. Since $Z = \mathrm{Sp}(R)$ is smooth and connected, R is an integral domain and hence has pure dimension. Remark 1.19 and Proposition 1.20 show that the other condition of Theorem 2.4 is satisfied as well. ☐

Proof of Theorem 2.4. Let $\iota\colon R \longrightarrow S$ be the canonical inclusion and $\phi\colon Z' = \mathrm{Sp}(S) \longrightarrow Z = \mathrm{Sp}(R)$ the associated morphism of rigid spaces, which fits into the commutative diagram

$$\begin{tikzcd} Z' \arrow[r, two heads] \arrow[d, "\phi"'] & \widetilde{Z}' \arrow[d, "\widetilde{\phi}"] \\ Z \arrow[r, two heads] & \widetilde{Z}. \end{tikzcd} \tag{2.7}$$

The morphism $\iota\colon R \longrightarrow S$ is finite flat and injective, so ϕ is finite flat and surjective. Since the morphism ϕ is finite, $\phi^{-1}(z)$ is a finite set, say

$$\phi^{-1}(z) = \{z'_1, \ldots, z'_n\}.$$

Since R has pure dimension, the base change $S = R \otimes_K K'$ also has pure dimension by [Bos70, Lemma 2.5]. Because S is moreover distinguished, the formal fibers $Z'_+(z'_i), i = 1, \ldots, n$ are connected by Theorem 2.3. Being a flat map between quasi-compact rigid F-spaces, ϕ is open by [BL93, Corollary 5.11], so $\phi(Z'_+(z'_i))$ is open in Z and hence a rigid (sub)space. In particular, the restriction $Z'_+(z'_i) \twoheadrightarrow \phi(Z'_+(z'_i))$ is a surjective map of rigid spaces whose domain is connected, whence the codomain $\phi(Z'_+(z'_i)$ is also connected. On the other hand, ϕ is finite and hence proper, so ϕ maps closed analytic subsets to closed analytic subsets by [Kie67a, Satz 4.1 and its proof], which is why $\phi(Z'_+(z'_i))$ is clopen in Z. Due to the commutativity of the diagram (2.7) above, $\phi(Z'_+(z'_i))$ is contained in $Z_+(z)$, i.e. we can regard it as a (clopen) subset of $Z_+(z)$. Therefore, being clopen and connected, each $\phi(Z'_+(z'_i))$ is a connected component of $Z_+(z)$ by Remark 1.14. On the other hand, we have

$$Z_+(z) = \bigcup_{i=1}^{n} \phi(Z'_+(z'_i)) \tag{2.8}$$

by Corollary 2.2. Next we will show that $\phi(Z'_+(z'_i)) = \phi(Z'_+(z'_1))$ for all $i = 1, \ldots, n$, which then implies that $Z_+(z) = \phi(Z'_+(z'_1))$ due to (2.8) and thus completes the proof of the theorem. Since $\phi(z'_i) = z = \phi(z'_1)$, we have $z \in \phi(Z'_+(z'_1)) \cap \phi(Z'_+(z'_i))$, so $\phi(Z'_+(z'_1)) \cap \phi(Z'_+(z'_i)) \neq \varnothing$ for all i which means that the connected components $\phi(Z'_+(z'_1))$ and $\phi(Z'_+(z'_i))$ must coincide. $\square$

Chapter 3

Special affinoid wide-open spaces and a gap in Beyer's article

3.1 Remedying a gap regarding "Special affinoid wide-open spaces"

Definition 3.1 ([Bey97a, Definition 4.2.1]). *Let Z be an affinoid space. A subset $\mathring{W} \subseteq Z$ is called a* special affinoid wide-open space *if $\mathring{W}$ is the preimage of a finite set of points of the reduction $\widetilde{Z}$ under the reduction map $p\colon Z \longrightarrow \widetilde{Z}$*

There is a gap in the proof of the crucial result [Bey97a, Lemma 4.2.2] on equivalent characterisations of special affinoid wide-opens: Bosch's Theorem on the connectedness of formal fibers of distinguished affinoid algebras (Theorem 2.3 above) is applied to a not necessarily distinguished affinoid algebra. Our generalisation of Bosch's Theorem (Corollary 2.5) can be used to remedy this gap.

Lemma 3.2. *Let $Z = \mathrm{Sp}(R)$ be a smooth and connected affinoid space, $p\colon Z \longrightarrow \widetilde{Z}$ the reduction map and let $\mathring{W} \subseteq Z$ be a special affinoid wide-open space. Then there exists a finite surjective morphism $\pi\colon Z \longrightarrow \mathbb{D}^m$ such that $\mathring{W}$ is a union of connected components of $\pi^{-1}(\mathring{\mathbb{D}}^m)$. For any such morphism π, $\pi^{-1}(\mathring{\mathbb{D}}^m)$ consisty of only finitely many connected components.*
More precisely: For $\mathring{W} = p^{-1}(\{\widetilde{z}_1, \dots, \widetilde{z}_r\})$, there exists a finite surjective morphism $\widetilde{\pi}\colon \widetilde{Z} \longrightarrow \mathbb{A}^m_k$ that maps all the $\widetilde{z}_i$ to 0, and any lift π of any such $\widetilde{\pi}$ satisfies the desired properties above. In this setting, if $\widetilde{\pi}^{-1}(0) = \{\widetilde{z}_1, \dots, \widetilde{z}_r, \widetilde{z}_{r+1}, \dots, \widetilde{z}_s\}$ is the full fiber over zero, then the connected components of $\pi^{-1}(\mathring{\mathbb{D}}^m)$ are precisely the $p^{-1}(\widetilde{z}_1), \dots, p^{-1}(\widetilde{z}_s)$ and $\mathring{W}$ is the union of the components $p^{-1}(\widetilde{z}_1), \dots, p^{-1}(\widetilde{z}_r)$.

Proof. Choosing a π as in Lemma 1.21 and taking its reduction, we obtain a $\widetilde{\pi}$ as in the assertion. On the other hand, given any $\widetilde{\pi}$ as in the assertion, then any lift π of $\widetilde{\pi}$ is finite surjective by Corollary 1.11 and, as we will now prove, satisfies the fact that $\mathring{W}$ is a finite union of connected components of $\pi^{-1}(\mathring{\mathbb{D}}^m)$.

Indeed, we have

$$\mathring{W} \subseteq p^{-1}(\widetilde{\pi}^{-1}(0)) = \pi^{-1}(\mathbb{D}^m_+(0)) = \pi^{-1}(\mathring{\mathbb{D}}^m)$$

where the first equality holds by the commutativity of

$$\begin{tikzcd} Z \arrow[r, "\pi"] \arrow[d, "p"] & \mathbb{D}^m \arrow[d] \\ \widetilde{Z} \arrow[r, "\widetilde{\pi}"] & \mathbb{A}^m_k \end{tikzcd} \tag{3.1}$$

and the second equality holds because $\mathbb{D}^m_+(0) = \mathring{\mathbb{D}}^m$. Let $\widetilde{z}_{r+1}, \ldots, \widetilde{z}_s \in \widetilde{Z}$ be such that $\widetilde{\pi}^{-1}(0) = \{\widetilde{z}_1, \ldots, \widetilde{z}_r, \widetilde{z}_{r+1}, \ldots, \widetilde{z}_s\}$. Then $Z_+ := \pi^{-1}(\mathring{\mathbb{D}}^m)$ as an analytic space is the disjoint union of the $p^{-1}(\widetilde{z}_i)$. Finally, each $p^{-1}(\widetilde{z}_i)$ is connected by Corollary 2.5. Thus the connected components of $\pi^{-1}(\mathring{\mathbb{D}}^m)$ are precisely the $p^{-1}(\widetilde{z}_1), \ldots, p^{-1}(\widetilde{z}_s)$. Since moreover $\mathring{W} = p^{-1}(\{\widetilde{z}_1, \ldots, \widetilde{z}_r\})$ by assumption, it follows that $\mathring{W}$ is the union of the components $p^{-1}(\widetilde{z}_1), \ldots, p^{-1}(\widetilde{z}_r)$. □

A very important technical result is the fact that the morphism π in Lemma 3.2 can be chosen to be separable (see Definition 1.22):

Remark 3.3. *Let $Z = \mathrm{Sp}(R)$ be a smooth and connected affinoid space, $p\colon Z \longrightarrow \widetilde{Z}$ the reduction map and let $\mathring{W} \subseteq Z$ be a special affinoid wide-open space. Then there exists a finite surjective* separable *morphism $\pi\colon Z \longrightarrow \mathbb{D}^m$ such that $\mathring{W}$ is a union of connected components of $\pi^{-1}(\mathring{\mathbb{D}}^m)$.*
More precisely: For $\mathring{W} = p^{-1}(\{\widetilde{z}_1, \ldots, \widetilde{z}_r\})$, there exists a finite surjective morphism $\widetilde{\pi}\colon \widetilde{Z} \longrightarrow \mathbb{A}^m_k$ that maps all the $\widetilde{z}_i$ to 0, and that admits a separable lift π.

Proof. Choose a finite surjective separable π as in Lemma 1.25 and note that the property $|\pi(z_i)| < 1$ means that $\widetilde{\pi}(\widetilde{z}_i) = 0$ (due to Proposition 1.5). □

Now we show that the converse to Lemma 3.2 holds as well:

Lemma 3.4. *Let Z be a smooth connected affinoid space and let $\mathring{W} \subseteq Z$ be a subset. Suppose that there exists a finite surjective morphism $\pi\colon Z \longrightarrow \mathbb{D}^m$ such that $\mathring{W}$ is a union of connected components*[1] *of $\pi^{-1}(\mathring{\mathbb{D}}^m)$. Then $\mathring{W} \subseteq Z$ is special affinoid wide-open.*

Proof. Applying the reduction functor, we obtain a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\pi"] \arrow[d, "p"] & \mathbb{D}^m \arrow[d] \\ \widetilde{Z} \arrow[r, "\widetilde{\pi}"] & \mathbb{A}^m_k. \end{tikzcd}$$

[1] The rigid space $\pi^{-1}(\mathring{\mathbb{D}}^m)$ has only finitely many connected components, as we will see in the proof.

Knowing that π and hence $\widetilde{\pi}$ is finite and surjective by Corollary 1.11, we have that $\widetilde{\pi}^{-1}(0)$ is finite and non-empty, say $\widetilde{\pi}^{-1}(0) = \{\widetilde{z}_1, \dots, \widetilde{z}_s\}$ for suitable points $\widetilde{z}_i \in \widetilde{Z}$. Since $\pi(\mathring{W}) \subseteq \mathring{\mathbb{D}}^m$ and $\mathring{\mathbb{D}}^m$ reduces to $\{0\} \subseteq \mathbb{A}_k^m$, we have $p(\mathring{W}) \subseteq \widetilde{\pi}^{-1}(0)$, say $p(\mathring{W}) = \{\widetilde{z}_1, \dots, \widetilde{z}_r\}$. Then $Z_+ := \pi^{-1}(\mathring{\mathbb{D}}^m)$ as an analytic space is the disjoint union of the $p^{-1}(\widetilde{z}_i)$, $i = 1, \dots, s$, and each $p^{-1}(\widetilde{z}_i)$ is connected by Corollary 2.5, so the connected components of $\pi^{-1}(\mathring{\mathbb{D}}^m)$ are precisely the $p^{-1}(\widetilde{z}_1), \dots, p^{-1}(\widetilde{z}_s)$. Thus $\mathring{W}$, being the union of some connected components, must satisfy $\mathring{W} = p^{-1}(\{\widetilde{z}_1, \dots, \widetilde{z}_r\})$. □

Lemma 3.2, Remark 3.3 and Lemma 3.4 together yield:

Proposition 3.5 (Equivalent characterisation of special affinoid wide-opens)**.** *Let Z be a smooth connected affinoid space. For a subset $\mathring{W} \subseteq Z$, the following conditions are equivalent:*

(i) *$\mathring{W} \subseteq Z$ is special affinoid wide-open.*

(ii) *There exists a finite surjective morphism $\pi \colon Z \longrightarrow \mathbb{D}^n$ such that $\mathring{W}$ is a union of connected components of $\pi^{-1}(\mathring{\mathbb{D}}^n)$.*

(iii) *There exists a finite surjective separable morphism $\pi \colon Z \longrightarrow \mathbb{D}^n$ such that $\mathring{W}$ is a union of connected components of $\pi^{-1}(\mathring{\mathbb{D}}^n)$.*

Recall that in the setting of Proposition 3.5 (ii) and (iii) above, $\pi^{-1}(\mathring{\mathbb{D}}^n)$ consists of finitely many connected components.
We note that a special affinoid wide-open space need not be affinoid. For example, $\mathring{\mathbb{D}}^n$ with $n > 0$ is a special affinoid wide-open space that isn't affinoid. In fact, one can show that an affinoid space which is special affinoid wide-open is necessarily zero-dimensional.

3.2 "Special affinoid wide-open spaces" are "affinoid wide-open"

We digress to provide a proof for an important claim made implicitly and without proof in [Bey97a], namely that special affinoid wide-open spaces are "affinoid wide-open spaces" in the following sense ([Bey97a, Definition 4.1.1 and Remark 4.1.2]):

Definition 3.6. *Let $Z = \mathrm{Sp}(R)$ be an affinoid space. A subset $\mathring{W} \subseteq Z$ is called an* affinoid wide-open space *if there exists a closed immersion $i \colon Z \hookrightarrow \mathbb{D}^m$ such that $\mathring{W}$ lands in $\mathring{\mathbb{D}}^m$ via i and the diagram*

$$\begin{tikzcd} \mathring{W} \arrow[r, hook, "i"] \arrow[d, "\subseteq"] & \mathring{\mathbb{D}}^m \arrow[d, "\subseteq"] \\ Z \arrow[r, hook, "i"] & \mathbb{D}^m \end{tikzcd} \tag{3.2}$$

is cartesian.

Remark 3.7. *Let Z be an affinoid space. For a subset $\mathring{W} \subseteq Z$, the following conditions are equivalent:*

(i) $\mathring{W} \subseteq Z$ *is affinoid wide-open.*

(ii) *There exists an affinoid generating system $f_1, \dots, f_m$ of R over K with*

$$\mathring{W} = \{z \in Z \colon |f_i(z)| < 1 \text{ for all } i = 1, \dots, m\}.$$

Proof. To give a closed immersion $i \colon Z \hookrightarrow \mathbb{D}^m$ is the same as to give an epimorphism $\varphi \colon K\langle \xi_1, \dots \xi_m \rangle \twoheadrightarrow R$. Then the $f_j := \varphi(\xi_j)$ are an affinoid generating system of R over K (by definition), and we have $|\xi_j(i(z))| = |f_j(z)|$ for all $z \in Z$ and $j = 1, \dots, m$ by Remark 1.1. Thus we have

$$|i(z)| < 1 \iff |\xi_j(i(z))| < 1 \text{ for all } j \iff |f_j(z)| < 1 \text{ for all } j$$

from which it follows that $\mathring{W} := \{z \in Z \colon |f_j(z)| < 1 \text{ for all } j = 1, \dots, m\}$ is precisely the subspace of Z that makes the diagram (3.2) cartesian. □

Lemma 3.8. *Let R be an affinoid algebra. Given finitely many points $z_1, \dots, z_r$ in $\mathrm{Sp}(R)$, there exists an affinoid generating system $g_1, \dots, g_n$ of R that satisfies $|g_i(z_j)| < 1$ for all i and j.*

We postpone the proof for a moment to discuss the following:

Remark 3.9. *A naive idea for the proof of Lemma 3.8 would be to take an arbitrary affinoid generating system and rescale it. This doesn't work, because the rescaled system needn't be an affinoid generating system of R anymore. For instance, the variable ξ is an affinoid generating system of $\mathbb{Q}_p\langle\xi\rangle$ but $p\xi$ isn't, for the map $\mathbb{Q}_p\langle\xi\rangle \longrightarrow \mathbb{Q}_p\langle\xi\rangle, \xi \longmapsto p\xi$ isn't surjective (it merely has dense image). Indeed, the image is the subalgebra $\mathbb{Q}_p\langle p\xi\rangle = \{\sum_{i=0}^{\infty} b_i \xi^i \in \mathbb{Q}_p[\![\xi]\!] \colon \lim_{i\to\infty} b_i p^{-i} = 0\}$ and $\mathbb{Q}_p\langle p\xi\rangle \subsetneqq \mathbb{Q}_p\langle\xi\rangle$ since $\sum_{i=0}^{\infty} p^i \xi^i \in \mathbb{Q}_p\langle\xi\rangle \setminus \mathbb{Q}_p\langle p\xi\rangle$.*

Proof of Lemma 3.8. Lemma 1.21 guarantees the existence of a finite injective morphism $\varphi \colon K\langle \xi_1, \dots, \xi_d \rangle \hookrightarrow R$ such that the $\varphi(\xi_i) =: g_i$ satisfy $g_i(z_j) = 0$ for all i, j. Now let $f_1, \dots, f_s \in R$ generate R as a module over $\operatorname{im}\varphi$. Choose an $a \in F$ with $|a| \ll 1$ such that

$$g_{m+i} := a \cdot f_i \in \mathring{R}, \quad i = 1, \dots, s$$

and also $|g_{m+i}(z_j)| < 1$ for all $j = 1, \dots, r$. The elements $g_{m+1}, \dots, g_{m+s}$ also generate R as a module over $\operatorname{im}\varphi$. Consequently, the morphism

$$\varphi' \colon K\langle \xi_1, \dots \xi_m, \xi_{m+1}, \dots, \xi_{m+s} \rangle \longrightarrow R, \qquad \xi_i \longmapsto g_i$$

is surjective, because its image contains both $\operatorname{im}\varphi$ and $g_{m+1}, \dots, g_{m+s}$, so it contains $\sum_{i=1}^{s} \operatorname{im}\varphi \cdot g_{m+i} = R$. □

Proposition 3.10 (Special affinoid wide-open spaces are "affinoid wide-open spaces"). *Let $\mathring{W} \subseteq Z = \mathrm{Sp}(R)$ be a special affinoid wide-open space. Then $\mathring{W} \subseteq Z$ is "affinoid wide-open" in the sense of Definition 3.6.*

Proof. Let
$$\mathring{W} = \bigcup_{j=1}^{n} Z_+(z_i).$$
Choose an affinoid generating system $g_1, \ldots, g_s$ of R, so necessarily $g_i \in \mathring{R}$ for all i. By Lemma 3.8, we may assume $|g_i(z_j)| < 1$ for all $i = 1, \ldots, s$ and all $j = 1, \ldots, n$. By Proposition 1.5, this means that $\widetilde{g}_1, \ldots, \widetilde{g}_s \in \mathfrak{m}_{p(z_j)}$ for all $j = 1, \ldots, n$. Since $\widetilde{R}$ is Noetherian, we can choose $\widetilde{g}_{s+1}, \ldots, \widetilde{g}_r \in \widetilde{R}$ such that
$$(\widetilde{g}_1, \ldots, \widetilde{g}_s, \widetilde{g}_{s+1}, \ldots, \widetilde{g}_r) = \bigcap_{j=1}^{n} \mathfrak{m}_{p(z_j)}.$$
Since $\widetilde{g}_{s+1}, \ldots, \widetilde{g}_r \in \mathfrak{m}_{p(z_j)}$, any system of lifts $g_{s+1}, \ldots, g_r \in \mathring{R}$ satisfies $|g_i(z_j)| < 1$ for all $i = s+1, \ldots, r$, again by Proposition 1.5. Moreover, $g_1, \ldots, g_s, g_{s+1}, \ldots, g_r$ is an affinoid generating system of R, since this is already true for $g_1, \ldots, g_s$. We claim that
$$\mathring{W} = \{w \in Z \colon |g_i(w)| < 1 \text{ for all } i = 1, \ldots, r\}.$$
Indeed, let $z' \in Z_+(z_j)$ for some j. Then we have $p(z') = p(z_j)$ and hence $\widetilde{g}_i(p(z')) = \widetilde{g}_i(p(z_j)) = 0$, so $|g_i(z')| < 1$ for all $i = 1, \ldots, r$. Conversely, if $|g_i(w)| < 1$ for all $i = 1, \ldots, r$, then $\widetilde{g}_i(p(w)) = 0$. In particular, $\bigcap_{j=1}^{n} \mathfrak{m}_{p(z_j)} = (\widetilde{g}_1, \ldots, \widetilde{g}_r) \subseteq \mathfrak{m}_{p(w)}$. In general, if an intersection of finitely many ideals is contained in a prime ideal, then one of the ideals is already contained in that prime ideal, hence $\mathfrak{m}_{p(z_j)} \subseteq \mathfrak{m}_{p(w)}$ for some j. But then $\mathfrak{m}_{p(z_j)} = \mathfrak{m}_{p(w)}$ i.e. $p(w) = p(z_j)$. □

Remark 3.11. *Let Z be a connected smooth affinoid space. Let $\mathring{W} \subseteq Z$ be a special affinoid wide-open space with associated finite surjective morphism*
$$\pi \colon Z = \mathrm{Sp}\, R \longrightarrow \mathbb{D}^n = \mathrm{Sp}\, K\langle \xi_1, \ldots, \xi_n \rangle.$$
Let $\mathring{Z} := \pi^{-1}(\mathring{\mathbb{D}}^n)$. Then the natural map
$$H^j_c(\mathring{\mathbb{D}}^n, \pi_* \omega_Z) \xrightarrow{\sim} H^j_c(\mathring{Z}, \omega_Z)$$
is an isomorphism for all $j \geq 0$.

Proof. Since $\pi \colon \mathring{Z} = \pi^{-1}(\mathring{\mathbb{D}}^n) \longrightarrow \mathring{\mathbb{D}}^n$ is a finite morphism between quasi-Stein spaces, the assertion follows from Proposition 1.51. □

For a disjoint union $X = X_1 \amalg X_2$ of rigid spaces and any coherent sheaf $\mathcal{F}$ on X, we have a canonical isomorphism

$$\mathcal{F} \xrightarrow{\sim} j_{1*}j_1^*\mathcal{F} \oplus j_{2*}j_2^*\mathcal{F}$$

where $j_i \colon X_i \longrightarrow X$ is the inclusion. In particular, we have

$$H_c^p(X, \mathcal{F}) = H_c^p(X, j_{1*}j_1^*\mathcal{F}) \oplus H_c^p(X, j_{2*}j_2^*\mathcal{F}) \tag{3.3}$$

for all $p \geq 0$.

Remark 3.12. *In the setting of Remark 3.11, we have a natural embedding*

$$H_c^n(\mathring{W}, \omega_Z) \hookrightarrow H_c^n(\mathring{Z}, \omega_Z), \tag{3.4}$$

whose construction we wish to explain precisely. First of all, we explain the abuse of notation in (3.4): Let ι denote the clopen immersion $\mathring{W} \longrightarrow \mathring{Z}$ (see Remark 1.14 (iii)) and $\mathcal{F}$ denote the restriction of ω_Z to $\mathring{Z}$. Then the pullback $\iota^*\mathcal{F}$ is the restriction of ω_Z to $\mathring{W}$ and the left-hand side of (3.4) is just notation for $H_c^n(\mathring{W}, \iota^*\mathcal{F})$. Then (3.4) is defined as the composite

$$H_c^n(\mathring{W}, \iota^*\mathcal{F}) \xrightarrow{\sim} H_c^n(\mathring{Z}, \iota_*\iota^*\mathcal{F}) \hookrightarrow H_c^n(\mathring{Z}, \mathcal{F}),$$

where the first map is the inverse of $H_c^n(\mathring{Z}, \iota_*\iota^*\mathcal{F}) \longrightarrow H_c^n(\mathring{W}, \iota^*\mathcal{F})$ (which is an isomorphism by Proposition 1.51, since $\mathring{W} \longrightarrow \mathring{Z}$ is a closed immersion and thus a finite map) and the second map is the inclusion of a direct summand as in (3.3).

The natural map $H_c^n(\mathring{\mathbb{D}}^n, \pi_*\omega_Z) \longrightarrow H_c^n(\mathring{Z}, \omega_Z)$ being an isomorphism, we can consider its inverse and the composite.

$$H_c^n(\mathring{W}, \omega_Z) \hookrightarrow H_c^n(\mathring{Z}, \omega_Z) \xrightarrow{\sim} H_c^n(\mathring{\mathbb{D}}^n, \pi_*\omega_Z) \tag{3.5}$$

which later plays a role in the definition of the trace map for a special affinoid wide open space (cf. Definition 4.10).

We finish this section by proving that the converse to Proposition 3.10 is in fact also true, so that the notions of "special affinoid wide open spaces" and "affinoid wide open spaces" are equivalent:

Proposition 3.13 (All affinoid wide-open spaces are "special")**.** *Let $Z = \mathrm{Sp}(R)$ be an affinoid space and $\mathring{W} \subseteq Z$ be an affinoid wide-open space. Then $\mathring{W} \subseteq Z$ is "special affinoid wide-open" in the sense of Definition 3.1.*

Proof. Suppose that $\mathring{W}$ is affinoid wide open in Z, i.e. there exists an affinoid generating system $f_1, \ldots, f_m$ of R over K with

$$\mathring{W} = \{z \in Z \colon |f_i(z)| < 1 \text{ for all } i = 1, \ldots, m\}.$$

Then

$$\mathring{W} = p^{-1}(\{y \in \widetilde{Z} \colon \widetilde{f}_i(y) = 0 \text{ for all } i = 1, \ldots, m\})$$

by Proposition 1.5. Thus it remains to show that the set $\{y \in \widetilde{Z} \colon \widetilde{f}_i(y) = 0 \text{ for all } i\}$ is finite. The morphism $\varphi \colon K\langle \xi_1, \ldots \xi_m \rangle \longrightarrow R$ mapping ξ_i to f_i is surjective and hence finite, so $\widetilde{\varphi} \colon k[\xi_1, \ldots \xi_m] \longrightarrow \widetilde{R}$ is finite by Theorem 1.10. Setting $\widetilde{\pi} := \mathrm{Sp}(\widetilde{\varphi})$, we have that $\widetilde{\pi}^{-1}(0)$ is finite, so in particular there exist only finitely many maximal ideals $\mathfrak{m}$ in $\widetilde{R}$ such that $\widetilde{\varphi}^{-1}(\mathfrak{m}) = (\xi_1, \ldots, \xi_m)$. In other words, there exist only finitely many maximal ideals $\mathfrak{m}$ in $\widetilde{R}$ such that $\widetilde{f}_i \in \mathfrak{m}$, i.e. the set $\{y \in \widetilde{Z} \colon \widetilde{f}_i(y) = 0 \text{ for all } i = 1, \ldots, m\}$ is finite. □

Chapter 4

Review of residue maps and trace maps

4.1 The residue map on local cohomology

Let $Z = \mathrm{Sp}(R)$ be a connected smooth affinoid space of dimension n and $z \in Z$ a point with corresponding maximal ideal $\mathfrak{m}_z \subseteq R$. Following Beyer, we often shorten the notation as follows: Given a coherent sheaf $\mathcal{F}$ on Z, we set $M = \Gamma(Z, \mathcal{F})$ and

$$H_z^j(\mathcal{F}) := H_{\mathfrak{m}_z}^j(M) = H_{\widehat{\mathfrak{m}_z}}^j(\widehat{M}).$$

The latter identification (due to Proposition 1.59) enables use of Lemma 1.57 in a relative situation (where a system of parameters is mapped to a system of parameters) -- which yields the isomorphisms (4.4) and (4.5) in Lemma 4.5 (ii) below.
In [Bey97a, Definition 4.2.7], Beyer defines a canonical *residue map*

$$\mathrm{res}_z \colon H_z^n(\omega_Z) = H_{\mathfrak{m}_z}^n(\Omega_{R/K}^n) \longrightarrow K. \tag{4.1}$$

on the local cohomology group $H_z^n(\omega_Z)$. This is not to be confused with the residue map (4.8) on the compactly supported rigid cohomology group $H_c^n(\mathring{\mathbb{D}}^n, \omega_{\mathring{\mathbb{D}}^n})$, although there is a relationship between these two that is encoded in Proposition 4.5 below.
In Definition 4.2 below, we recall the construction of the local residue map (4.1), but first we need to make some preparations. We recall that a local ring $(A, \mathfrak{m})$ is said to be *equicharacteristic* if $\operatorname{char} A = \operatorname{char} A/\mathfrak{m}$. We mention that this is equivalent to saying that A contains a field.

Theorem 4.1 (Structure of complete local rings). *Let $(A, \mathfrak{m})$ be an equicharacteristic local ring.*

(i) *If A is complete, then A has a "coefficient field" (i.e. a subfield $K' \subseteq A$ that maps isomorphically to $A/\mathfrak{m}$ under the natural map $A \longrightarrow A/\mathfrak{m}$).*

(ii) *If A is complete and $A/\mathfrak{m}$ is separable over a subfield $k \subseteq A$, then A has a coefficient field containing k.*

(iii) *Suppose that A is Noetherian, complete and regular. Then A has a coefficient field K' and choosing any regular system of parameters $x_1, \dots, x_d$ of A yields an isomorphism*

$$K'[\![X_1, \dots, X_d]\!] \xrightarrow{\sim} A, \qquad X_i \longmapsto x_i. \tag{4.2}$$

If we moreover assume that $A/\mathfrak{m}$ is separable over a subfield $k \subseteq A$, then K' can be chosen to contain k by (ii), so in particular (4.2) is an isomorphism of k-algebras.

Proof. Assertion (i) is [Mat87, Theorem 28.3 (ii)] and Assertion (ii) follows from [Mat87, Theorem 28.3 (iii) and (iv)]. For an argument that (4.2) is an isomorphism, see [Mat87, Proof of the converse statement in Lemma 1 subsequent to Theorem 28.3]. $\square$

Definition 4.2 ([Bey97a, Definition 4.2.7])**.** *Let $Z = \mathrm{Sp}(R)$ be a connected smooth affinoid space of dimension n and $z \in Z$ a point with corresponding maximal ideal $\mathfrak{m}_z \subseteq R$. Let $K' = R/\mathfrak{m}_z$ and let $\widehat{R}$ be the $\mathfrak{m}_z$-adic completion of R. Then K' is also the residue field of $\widehat{R}$. Suppose that K' is separable over K, so in particular Theorem 4.1 (iii) yields an isomorphism $K'[\![X_1, \dots, X_d]\!] \xrightarrow{\sim} \widehat{R}$ of K-algebras (depending on a choice of a regular system of parameters $t_1, \dots, t_d$ of $\widehat{R}$). Let $\mathfrak{m}$ be the maximal ideal in $K'[\![X_1, \dots, X_d]\!]$. To shorten the notation, we write $K'[\![X]\!]$ for $K'[\![X_1, \dots, X_d]\!]$. The universal finite differential module $\Omega^1_{K'[\![X]\!]/K}$ is free over $K'[\![X]\!]$, with basis $dX_1, \dots, dX_d$, hence $\Omega^d_{K'[\![X]\!]/K} = \bigwedge^d \Omega^1_{K'[\![X]\!]/K}$ is free of rank one, with basis $dX = dX_1 \wedge \ldots \wedge dX_d$. Setting $X^\nu = X_1^{\nu_1} \cdots X_n^{\nu_n}$ for $\nu \in \mathbb{N}_0^n$, the local residue map*

$$\mathrm{res}_z \colon H^n_z(\omega_Z) = H^n_{\mathfrak{m}_z}(\Omega^n_{R/K}) \longrightarrow K$$

is then defined as the composite

$$H^n_{\mathfrak{m}_z}(\Omega^n_{R/K}) \xrightarrow{\sim} H^n_{\widehat{\mathfrak{m}_z}}(\widehat{\Omega^n_{R/K}}) \xrightarrow{\sim} H^n_{\widehat{\mathfrak{m}_z}}(\Omega^n_{\widehat{R}/K}) \xrightarrow{\sim} H^n_{\mathfrak{m}}(\Omega^n_{K'[\![X]\!]/K}) \to K \tag{4.3}$$
$$\left[\begin{matrix} \sum_\nu a_\nu X^\nu \cdot dX \\ X^\rho \end{matrix}\right] \mapsto \mathrm{Tr}_{K'/K}(a_{\rho-1,\dots,\rho-1})$$

which is in fact independent of the choice of $t_1, \dots, t_d$ by [Bey97a, Proposition 3.2.1]. We note that the second isomorphism in (4.3) is due to Remark 4.4 below. Moreover, we have used the notation of Lemma 1.57 in the last map in (4.3).

Convention 4.3. To be able to define res_z, we need the residue field of z to be separable over K, which is why we henceforth tacitly assume that K is a perfect field in every result where res_z is mentioned for arbitrary points $z \in Z$. This convention stands throughout, unless otherwise stated.

Remark 4.4. *In the setting of Definition 4.2 above, we have $\widehat{\Omega^n_{R/K}} \cong \Omega^n_{\widehat{R}/K}$.*

Proof. We have $\widehat{R} = \widehat{R_{\mathfrak{m}_z}}$ since "completion with respect to a maximal ideal commutes with localisation" (see the footnote in the proof of Corollary 1.68). Moreover, we claim that we have $R_{\mathfrak{m}_z} \otimes_R \Omega^n_{R/K} \cong \Omega^n_{R_{\mathfrak{m}_z}/K}$ and $\widehat{R_{\mathfrak{m}_z}} \otimes_{R_{\mathfrak{m}_z}} \Omega^n_{R_{\mathfrak{m}_z}/K} \cong \Omega^n_{\widehat{R_{\mathfrak{m}_z}}/K}$. Indeed, this is true for $n = 1$ by [Ber+67, page 56] resp. by [Ber+67, Satz 2.4.2] and therefore also for higher n, since the tensor product commutes with exterior powers. Thus

$$\begin{aligned}\widehat{\Omega^n_{R/K}} &\cong \widehat{R} \otimes_R \Omega^n_{R/K} \cong R_{\mathfrak{m}_z} \otimes_{R_{\mathfrak{m}_z}} \widehat{R} \otimes_R \Omega^n_{R/K} \cong \widehat{R} \otimes_{R_{\mathfrak{m}_z}} R_{\mathfrak{m}_z} \otimes_R \Omega^n_{R/K} \\ &\cong \widehat{R_{\mathfrak{m}_z}} \otimes_{R_{\mathfrak{m}_z}} \Omega^n_{R_{\mathfrak{m}_z}/K} \cong \Omega^n_{\widehat{R_{\mathfrak{m}_z}}/K} \cong \Omega^n_{\widehat{R}/K}.\end{aligned}$$

□

Lemma 4.5. *Let $Z = \mathrm{Sp}(R)$ be a connected smooth affinoid space of dimension n. Write $T_n = K\langle \xi_1, \dots, \xi_n\rangle$. Let $\mathring{W} \subseteq Z$ be a special affinoid wide-open space with associated finite surjective morphism*

$$\pi\colon Z = \mathrm{Sp}\, R \longrightarrow \mathbb{D}^n = \mathrm{Sp}\, T_n.$$

with corresponding finite injective ring morphism

$$\varphi\colon T_n \longrightarrow R.$$

Let $\{z_1, \dots, z_r\} = \pi^{-1}(0) \cap \mathring{W}$ and $\{z_1, \dots, z_r, z_{r+1}, \dots z_s\} = \pi^{-1}(0)$. Denote by $\mathfrak{m}_1, \dots \mathfrak{m}_s \subseteq R$ the corresponding maximal ideals in R. Let $M = \Gamma(Z, \omega_Z)$ and let $\mathfrak{m}$ denote the maximal ideal corresponding to $0 \in \mathbb{D}^n$. Then:

(i) *For every coherent sheaf $\mathcal{F}$ on Z one has a canonical isomorphism*

$$\gamma\colon \bigoplus_{i=1}^{s} H^n_{z_i}(\mathcal{F}) \xrightarrow{\sim} H^n_0(\pi_*\mathcal{F}).$$

We write $\gamma = \gamma_{\mathcal{F},\pi}$ when we want to stress the dependence on $\mathcal{F}$ and π.

(ii) *Let $X_1, \dots, X_n$ be a system of parameters for the $\mathfrak{m}$-adic completion ${T_n}^{\wedge\mathfrak{m}}$. One has canonical isomorphisms*

$$H^n_0(\pi_*\omega_Z) \cong \varinjlim_{\rho} \widehat{M_{\mathfrak{m}}}/(X_1^\rho, \dots, X_n^\rho) \tag{4.4}$$

and

$$\bigoplus_{i=1}^{s} H^n_{z_i}(\omega_Z) \cong \varinjlim_{\rho} \bigoplus_{i=1}^{s} \widehat{M_{\mathfrak{m}_i}}/(X_1^\rho, \dots, X_n^\rho), \tag{4.5}$$

where we have denoted the image of X_j under each map ${T_n}^{\wedge\mathfrak{m}} \longrightarrow R^{\wedge\mathfrak{m}_i}$ on completions induced by φ again by X_j; these form a system of parameters in

$R^{\wedge\mathfrak{m}_i}$ by the results of Subsection 1.4.1. Then the isomorphism γ^{-1} can be identified with the map

$$\begin{aligned} \widetilde{\gamma}^{-1}\colon \varinjlim_{\rho} \widehat{M_{\mathfrak{m}}}/(X_1^\rho,\dots,X_n^\rho) &\xrightarrow{\sim} \varinjlim_{\rho} \bigoplus_{i=1}^{s} \widehat{M_{\mathfrak{m}_i}}/(X_1^\rho,\dots,X_n^\rho) \\ \begin{bmatrix} \omega \\ X^\rho \end{bmatrix} &\longmapsto \left(\begin{bmatrix} \omega_1 \\ X^\rho \end{bmatrix}, \dots, \begin{bmatrix} \omega_r \\ X^\rho \end{bmatrix} \right) \end{aligned}$$

where, as in Lemma 1.57, $\left[\begin{smallmatrix} \omega \\ X^\rho \end{smallmatrix}\right]$ denotes the image of ω under the canonical map

$$\widehat{M_{\mathfrak{m}}}/(X_1^\rho,\dots,X_n^\rho) \longrightarrow \varinjlim_{j} \widehat{M_{\mathfrak{m}}}/(X_1^j,\dots,X_n^j),$$

and ω_i denotes the image of ω in $\widehat{M_{\mathfrak{m}_i}}$.

(iii) *The diagram*

$$\begin{tikzcd} \bigoplus_{i=1}^{s} H^n_{z_i}(\omega_Z) \arrow[r, "\gamma", "\sim"'] \arrow[dr, "\sum \mathrm{res}_{z_i}"'] & H^n_0(\pi_*\omega_Z) \arrow[r, "H^n_0(\sigma)"] & H^n_0(\omega_{\mathbb{D}^n}) \arrow[dl, "\mathrm{res}_0"] \\ & K & \end{tikzcd} \tag{4.6}$$

commutes, where $\sigma = \sigma_\pi$ is defined as in Definition 4.9 below.

Proof. (i) The isomorphism γ is the map from [Bey97a, Lemma 4.2.9 (a)], where the only comment is that the assertion regarding this map is well known from local cohomology. Therefore, we note that γ is obtained as the composite

$$\bigoplus_{i=1}^{s} H^n_{\mathfrak{m}_i}(N) \xrightarrow{\sim} H^n_{\bigcap_{i=1}^s \mathfrak{m}_i}(N) = H^n_{\sqrt{\varphi(\mathfrak{m})R}}(N) = H^n_{\varphi(\mathfrak{m})R}(N) = H^n_{\mathfrak{m}}(N)$$

with $N := \Gamma(Z,\mathcal{F})$ and where the first arrow is the Mayer-Vietoris map from Corollary 1.61, the first equality is due to $\bigcap_{i=1}^s \mathfrak{m}_i = \sqrt{\varphi(\mathfrak{m})R}$ (which is true since $\mathfrak{m}_1,\dots,\mathfrak{m}_s$ are precisely the maximal ideals containing $\varphi(\mathfrak{m})R$ and R is a Jacobson ring), the second equality is due to Remark 1.55, and the final equality is due to independence of base (Proposition 1.58 (ii)).

(ii) This is true by the arguments in [Bey97a, Proof of Lemma 4.2.9 (c)].

(iii) This assertion is precisely [Bey97a, Lemma 4.2.9 (c)].

∎

4.2 The map from local cohomology into compactly supported cohomology

Let $Z = \mathrm{Sp}(R)$ be a connected smooth affinoid space of dimension d, let $\mathring{W} \subseteq Z$ be a special affinoid wide-open space. In [Bey97a, Lemma 4.2.6], a canonical injection

$$H_z^d(\omega_Z) \longrightarrow H_c^d(\mathring{W}, \omega_Z)$$

is constructed for every point $z \in \mathring{W}$:

Lemma 4.6 ([Bey97a, Lemma 4.2.6])**.** *Let $Z = \mathrm{Sp}(R)$ be a connected smooth affinoid space of dimension d and $\mathring{W} \subseteq Z$ a special affinoid wide-open space. Let $z \in \mathring{W}$ be a point and $\mathcal{F}$ a coherent sheaf on Z. Then there exists a canonical map*

$$H_z^d(\mathcal{F}) \longrightarrow H_c^d(\mathring{W}, \mathcal{F}) \tag{4.7}$$

that is functorial in $\mathcal{F}$.

Proof. For our purposes, we needn't know the explicit construction of (4.7). But for the sake of completeness we still give an outline of the construction carried out by Beyer in the proof of [Bey97b, Lemma 5.20], for $Z = \mathbb{D}^d, \mathring{W} = \mathring{\mathbb{D}}^d$ and $z = 0$. First note that there is a canonical map $H_{\{0\}}^d(\mathring{\mathbb{D}}^d, \mathcal{F}) \longrightarrow H_c^d(\mathring{\mathbb{D}}^d, \mathcal{F})$. Indeed, $\mathring{\mathbb{D}}^d \setminus \{0\}$ is admissible open in $\mathring{\mathbb{D}}^d$ (because it is even Zariski open), so one can define $\Gamma_{\{0\}}(\mathring{\mathbb{D}}^d, \mathcal{F})$ analogously to (1.4) in Definition 1.29. Then, taking any open affinoid $X \subseteq \mathring{\mathbb{D}}^d$ which contains 0, the inclusion $\Gamma_{\{0\}}(\mathring{\mathbb{D}}^d, \mathcal{F}) \subseteq \Gamma_X(\mathring{\mathbb{D}}^d, \mathcal{F})$ induces the desired map. We will construct a map

$$H_0^d(\mathcal{F}) \longrightarrow H_{\{0\}}^d \mathring{\mathbb{D}}^d, \mathcal{F}),$$

from which (4.7) is obtained by composing with the canonical map $H_{\{0\}}^d(\mathring{\mathbb{D}}^d, \mathcal{F}) \longrightarrow H_c^d(\mathring{\mathbb{D}}^d, \mathcal{F})$. First shorten the notation by recalling that $R = K\langle \xi_1, \dots, \xi_d \rangle$ and setting $M = \Gamma(Z, \mathcal{F})$. Then Proposition 1.65 and Remark 1.49 tell us that we need to construct a map

$$H_0^d(\mathcal{F}) = H^{d-1}(\mathrm{Spec}\, R \setminus \{0\}, \widetilde{M}) \dashrightarrow H^{d-1}(\mathring{\mathbb{D}}^d \setminus \{0\}, \widetilde{M}) = H_{\{0\}}^d(\mathring{\mathbb{D}}^d, \mathcal{F}),\ d \geq 2$$

resp. a map

$$H_0^1(\mathcal{F}) = \frac{H^0(\mathrm{Spec}\, R \setminus \{0\}, \widetilde{M})}{H^0(\mathrm{Spec}\, R, \widetilde{M})} \dashrightarrow \frac{H^0(\mathring{\mathbb{D}}^1 \setminus \{0\}, \widetilde{M})}{H^0(\mathring{\mathbb{D}}^1, \widetilde{M})} = H_{\{0\}}^1(\mathring{\mathbb{D}}^1, \mathcal{F}).$$

We will do this by interpreting the sheaf cohomology groups as Čech cohomology groups with respect to certain open coverings. Choose a zero sequence $(\varepsilon_m)_{m \in \mathbb{N}}$ of numbers in $|K^\times|, \varepsilon_m = |c_m|$ and let

$$U_{i,m} := \{x \in \mathring{\mathbb{D}}^d \colon \varepsilon_m < |x_i|\}.$$

All finite intersections of the $U_{i,m}$ are quasi-Stein spaces, hence $\mathfrak{U}_m := (U_{i,m})_i$ is a Leray covering of the space $\mathring{\mathbb{D}}^d \setminus \mathbb{D}^d(\varepsilon_m) = \{x \in \mathring{\mathbb{D}}^n \colon \varepsilon_m < |x_i| \text{ for some } i\}$ and $\mathfrak{U} := (U_{i,m})_{i,m}$ is a Leray covering of $\mathring{\mathbb{D}}^d - \{0\}$. On the other hand, the

$$V_i := \operatorname{Spec} A_{\xi_i}$$

yield a covering $\mathfrak{V} := (V_i)_i$ of $\operatorname{Spec} R \setminus \{0\}$ by affine open subschemes (hence a Leray covering). Thus $H^{d-1}(\mathring{\mathbb{D}}^d \setminus \{0\}, \widetilde{M}) = H^{d-1}(\mathfrak{U}, \widetilde{M})$ is a subquotient of

$$\check{C}^{d-1}(\mathfrak{U}, \widetilde{M}) = \varprojlim_m \check{C}^{d-1}(\mathfrak{U}_m, \widetilde{M}) = \varprojlim_m \Gamma(\bigcap_{i=1}^d U_{i,m}, \widetilde{M}) = \varprojlim_m (R\langle c_m\xi_1^{-1}, \dots, c_m\xi_d^{-1}\rangle \otimes_R M)$$

and $H^{d-1}(\operatorname{Spec} R \setminus \{z\}, \widetilde{M}) = H^{d-1}(\mathfrak{V}, \widetilde{M})$ is a subquotient of

$$\check{C}^{d-1}(\mathfrak{V}, \widetilde{M}) = \Gamma(\bigcap_{i=1}^d V_i, \widetilde{M}) = R[\xi_1^{-1}, \dots, \xi_d^{-1}] \otimes_R M.$$

Then the obvious map

$$R[t_1^{-1}, \dots, t_d^{-1}] \otimes_R M \to \varprojlim_m (R\langle c_m\xi_1^{-1}, \dots, c_m\xi_d^{-1}\rangle \otimes_R M)$$

induces the desired maps. □

Lemma 4.7. *Let Z be a connected smooth affinoid space of dimension d. Let $\mathring{W} \subseteq Z$ be a special affinoid wide-open space. Then there exist points $x_1, \dots, x_m$ in $\mathring{W}$ such that the image of the map*

$$\bigoplus_{i=1}^m H^d_{x_i}(\omega_Z) \longrightarrow H^d_c(\mathring{W}, \omega_Z)$$

is dense in $H^d_c(\mathring{W}, \omega_Z)$.
More precisely: Let

$$\pi \colon Z = \operatorname{Sp} R \longrightarrow \mathbb{D}^d = \operatorname{Sp} T_d$$

be any finite surjective morphism defining $\mathring{W}$. Let $\{z_1, \dots, z_r\} = \pi^{-1}(0) \cap \mathring{W}$ and $\{z_1, \dots, z_r, z_{r+1}, \dots z_s\} = \pi^{-1}(0)$. Set

$$\mathring{Z} = \pi^{-1}(\mathring{\mathbb{D}}^d).$$

Then the image of the canonical map

$$\bigoplus_{i=1}^s H^d_{z_i}(\omega_Z) \longrightarrow H^d_c(\mathring{Z}, \omega_Z)$$

is dense in $H^d_c(\mathring{Z}, \omega_Z)$ and the image of the following map induced by restriction

$$\bigoplus_{i=1}^r H^d_{z_i}(\omega_Z) \longrightarrow H^d_c(\mathring{W}, \omega_Z)$$

is dense in $H^d_c(\mathring{W}, \omega_Z)$.

Proof. This is [Bey97a, Lemma 4.2.9 (d)]. □

4.3 Beyer's trace map

An immediate consequence of Lemma 4.7 is that there can exist at most one continuous map $t\colon H^d_c(\mathring{W}, \omega_Z) \longrightarrow K$ with the property that, for every $x \in \mathring{W}$, the diagram

$$\begin{tikzcd} H^d_x(\omega_Z) \arrow[rr] \arrow[dr, "\mathrm{res}_x"'] & & H^d_c(\mathring{W}, \omega_Z) \arrow[dl, "t"] \\ & K & \end{tikzcd}$$

commutes. In fact, there does exist such a map, namely the trace map:

Proposition 4.8 (Characterising property of the trace map for special affinoid wide-open spaces)**.** *Let Z be a connected smooth affinoid space of dimension d and let $\mathring{W} \subseteq Z$ be a special affinoid wide-open space. Then there exists a unique continuous morphism $t\colon H^d_c(\mathring{W}, \omega_Z) \longrightarrow K$ with the property that, for every $x \in \mathring{W}$, the diagram*

$$\begin{tikzcd} H^d_x(\omega_Z) \arrow[rr] \arrow[dr, "\mathrm{res}_x"'] & & H^d_c(\mathring{W}, \omega_Z) \arrow[dl, "t"] \\ & K & \end{tikzcd}$$

commutes. This map t is called the trace map*. An explicit construction of t is given below in Construction 4.10.*

Proof. In [Bey97a, Proposition 4.2.10 (a)], it is proved that the trace map t_π from Construction 4.10 below (constructed by using any choice of a finite surjective separable morhism $\pi\colon Z \longrightarrow \mathbb{D}^d$) satisfies the desired universal property. □

The trace map will be constructed with help of the *residue map*

$$\mathrm{res}\colon H^n_c(\mathring{\mathbb{D}}^n, \omega_{\mathring{\mathbb{D}}^n}) \longrightarrow K \tag{4.8}$$

from [Bey97a, Definition 2.1.1]. By the same arguments as for Lemma 1.53, one sees that each cohomology class in $H^n_c(\mathring{\mathbb{D}}^n, \omega_{\mathring{\mathbb{D}}^n})$ is represented by an n-form

$$f = (\sum_{\alpha<0} a_\alpha X^\alpha)\, dX_1 \wedge \ldots \wedge dX_n.$$

Then res acts as

$$\mathrm{res}(f) = a_{(-1,\ldots,-1)}.$$

By [Bey97a, Proposition 2.1.3], this definition of res is independent of the choice of coordinates $(X_1, \ldots, X_n)$ on $\mathbb{D}^n$.

Another important ingredient in the construction of the trace map is the following map from [Bey97a, page 234]:

Definition 4.9 (The map σ)**.** *Let Z be a connected smooth affinoid space of dimension n. Let $\mathring{W} \subseteq Z$ be a special affinoid wide-open space with associated finite surjective separable morphism*

$$\pi \colon Z = \operatorname{Sp} R \longrightarrow \mathbb{D}^n = \operatorname{Sp} T_n.$$

We denote the trace map of the finite field extension

$$E = Q(T_n) \hookrightarrow L = Q(R), \tag{4.9}$$

by $\mathrm{Tr}_{L/E}$. It induces a map on the n-forms

$$\begin{aligned} \Omega^n_{L/K} = \Omega^n_{E/K} \otimes_E L &\xrightarrow{\sigma} \Omega^n_{E/K} \\ \omega \otimes b &\longmapsto \mathrm{Tr}_{L/E}(b) \cdot \omega, \end{aligned} \tag{4.10}$$

where $\Omega^n_{L/K} = \Omega^n_{E/K} \otimes_E L$ holds because L/E is separable. Moreover, we have $\Omega^n_{L/K} = Q(R) \otimes_R \Omega^n_{R/K}$ and $\sigma(\Omega^n_{R/K}) \subseteq \Omega^n_{T_n/K}$, so σ restricts to a map

$$\sigma \colon \Omega^n_{R/K} \longrightarrow \Omega^n_{T_n/K},$$

which induces a map

$$\sigma \colon \pi_* \omega_Z \longrightarrow \omega_{\mathbb{D}^n}.$$

We write $\sigma = \sigma_\pi$ when we want to stress the dependence on π.

Now we reproduce [Bey97a, Definition 4.2.4]:

Construction 4.10 (Trace map for special affinoid wide-open spaces)**.** *Let Z be a connected smooth affinoid space of dimension n. Let $\mathring{W} \subseteq Z$ be a special affinoid wide-open space with associated finite surjective separable morphism*

$$\pi \colon Z \longrightarrow \mathbb{D}^n.$$

Let $\mathring{Z} := \pi^{-1}(\mathring{\mathbb{D}}^n)$. The trace map

$$t = t_\pi \colon H^n_c(\mathring{W}, \omega_Z) \longrightarrow K$$

is defined as the following composite map (which is in fact independent of the choice of the finite surjective separable morphism π since it satisfies Proposition 4.8):

$$H^n_c(\mathring{W}, \omega_Z) \hookrightarrow H^n_c(\mathring{Z}, \omega_Z) \xrightarrow{\sim} H^n_c(\mathring{\mathbb{D}}^n, \pi_*\omega_Z) \xrightarrow{H^n_c(\sigma)} H^n_c(\mathring{\mathbb{D}}^n, \omega_{\mathring{\mathbb{D}}^n}) \xrightarrow{\mathrm{res}} K$$

where the first two maps are as in (3.5) and the third is induced by the σ from Definition 4.9.

The definition of the trace map can be extended to smooth Stein spaces, since they admit admissible covers consisting of special affinoid wide-open spaces:

Lemma 4.11. *Let S be a connected smooth Stein space of dimension n. Then S has a cover $\{\mathring{W}_i\}_{i\in\mathbb{N}}$ consisting of admissible open, special affinoid wide-open subsets $\mathring{W}_i$ such that*

- *the ambient affinoid space $W_i \supseteq \mathring{W}_i$ in the sense of Definition 3.1 can be chosen to be connected and contained in S,*

- *$\mathring{W}_i \subseteq \mathring{W}_{i+1}$, and*

- *$\mathring{W}_i$ is smooth of dimension n.*

Proof. Since it is a connected Stein space, S has an admissible open cover $\{W_i\}_{i\in\mathbb{N}}$ by connected affinoid subsets W_i satisfying $W_i \Subset W_{i+1}$ (see Definition 1.33 above). We claim that the image of W_i under the reduction map $p_{i+1}\colon W_{i+1} \longrightarrow \widetilde{W_{i+1}}$ is a finite subset of $\widetilde{W_{i+1}}$. Then we can define $\mathring{W}_{i+1}$ as the preimage of this finite set under p_{i+1} and it is immediate that this $\mathring{W}_{i+1}$ satisfies the desired conditions. To prove the claim, choose an affinoid generating system $f_1, \dots, f_m$ of $\mathcal{O}(W_{i+1})$ over K such that

$$W_i \subseteq \{w \in W_{i+1} \colon |f_j(z)| < 1 \text{ for all } j = 1, \dots, m\}.$$

Then

$$p_{i+1}(W_i) \subseteq \{y \in \widetilde{W_{i+1}} \colon \widetilde{f}_j(y) = 0 \text{ for all } j = 1, \dots, m\}$$

by Proposition 1.5. Thus it suffices to show that the set of all $y \in \widetilde{W_{i+1}}$ satisfying $\widetilde{f}_j(y) = 0$ for all $j = 1, \dots, m$ is finite. For $R := \mathcal{O}(W_{i+1})$, the morphism $\varphi\colon K\langle\xi_1, \dots \xi_m\rangle \longrightarrow R$ mapping ξ_j to f_j is surjective and hence finite, so $\widetilde{\varphi}\colon k[\xi_1, \dots \xi_m] \longrightarrow \widetilde{R}$ is finite by Theorem 1.10. Setting $\widetilde{\pi} := \mathrm{Sp}(\widetilde{\varphi})$, we have that $\widetilde{\pi}^{-1}(0)$ is finite, so there exist only finitely many maximal ideals $\mathfrak{m}$ in $\widetilde{R}$ such that $\widetilde{\varphi}^{-1}(\mathfrak{m}) = (\widetilde{\xi}_1, \dots, \widetilde{\xi}_m)$. In other words, there exist only finitely many maximal ideals $\mathfrak{m}$ in $\widetilde{R}$ such that $\widetilde{f}_j \in \mathfrak{m}$, which means that $\{y \in \widetilde{W_{i+1}} \colon \widetilde{f}_j(y) = 0 \text{ for all } j\}$ is finite. □

Definition 4.12 (Trace map for Stein spaces)**.** *Let S be a connected smooth Stein space of dimension n. In the notation of Lemma 4.11, we have the trace morphisms*

$$t_i\colon H_c^n(\mathring{W}_i, \omega_S) \longrightarrow K.$$

Since the diagrams

$$\begin{tikzcd} H_c^n(\mathring{W}_i, \omega_S) \arrow[r] \arrow[dr, "t_i"'] & H_c^n(\mathring{W}_{i+1}, \omega_S) \arrow[d, "t_{i+1}"] \\ & K \end{tikzcd} \tag{4.11}$$

commute (by [Bey97a, Corollary 4.2.12]), the t_i *induce a map*

$$t\colon \varinjlim_i H^n_c(\mathring{W}_i, \omega_S) = H^n_c(S, \omega_S) \longrightarrow K. \tag{4.12}$$

Beyer shows in [Bey97b, Proof of Satz 7.1] that this t *satisfies Theorem 5.3 below. In particular, it follows by standard universal abstract nonsense (cf. [Har77, Proposition III.7.2]) that, up to a unique automorphism of* ω_S*,* t *is independent of the choice of the covering* $\{\mathring{W}_i\}_{i\in\mathbb{N}}$*.*

Chapter 5

The relative trace map and the main theorem

5.1 Statement of the main theorem

Recall from [BLR95, Definition 2.1] that a morphism $f\colon X \longrightarrow Y$ of rigid spaces is called *smooth of relative dimension* d at a point $x \in X$ if there exists an open neighbourhood U of x and a closed immersion $j\colon U \hookrightarrow \mathbb{D}^n_Y$ over Y such that the sheaf of ideals defining $j(U)$ as a closed subvariety of $\mathbb{D}^n_Y$ is generated by $(n-d)$ sections $g_{d+1}, \ldots, g_n$ and the differentials $dg_{d+1}, \ldots, dg_n$ are linearly independent in $\Omega^1_{\mathbb{D}^n_Y/Y} \otimes k(x)$. The morphism f is called *étale* at x if it is smooth of relative dimension 0. As [BLR95, Proposition 2.3] recalls, the canonical sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow f^*\Omega^1_{Y/K} \longrightarrow \Omega^1_{X/K} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0 \tag{5.1}$$

is exact and locally split if f is smooth. Moreover, [FV12, Proposition 8.1.1] and its proof yield the following characterisation of étale morphisms between affinoid spaces:

Proposition 5.1. *For a morphism* $\mathrm{Sp}(B) \longrightarrow \mathrm{Sp}(A)$*, the following are equivalent:*

(i) f *is étale.*

(ii) $A \longrightarrow B$ *is flat and* $\Omega^1_{B/A} = 0$*.*

(iii) *There exists a presentation*

$$B = A\langle x_1, \ldots, x_n\rangle/(g_1, \ldots, g_n)$$

such that the image of $\det(\partial g_i/\partial x_j)$ *is a unit in* B*.*

To a finite étale morphism of smooth d-dimensional rigid spaces over K, we can attach a relative trace map as follows:

Definition 5.2 (A relative trace map)**.** *Let $f\colon X \longrightarrow Y$ be a finite étale morphism of smooth d-dimensional rigid spaces. Since f pulls back affinoids to affinoids (because f is finite), any coherent $f_*\mathcal{O}_X$-module $\mathcal{M}$ can naturally be viewed as a coherent $\mathcal{O}_X$-module $\widetilde{\mathcal{M}}$ such that*

$$f_*\widetilde{\mathcal{M}} = \mathcal{M}$$

(i.e. $\widetilde{\mathcal{M}}(\alpha^{-1}(V)) = \mathcal{M}(V)$ for all open affinoids $V \subseteq Y$) and $(-)^\sim$ is an equivalence of categories (cf. [EGAI, Proposition I.9.2.5]). Since f is étale, we have $\Omega^1_{X/Y} = 0$ so taking the d-th exterior power in the exact sequence (5.1) yields a natural isomorphism

$$(f_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \omega_Y)^\sim \xrightarrow{\sim} \omega_X, \tag{5.2}$$

where ω_Y denotes the sheaf of holomorphic d-forms on Y. Since f is finite flat, the natural map

$$\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \omega_Y \xrightarrow{\sim} \mathcal{H}om_Y(f_*\mathcal{O}_X, \omega_Y) \tag{5.3}$$

is an isomorphism. Finally, since f is finite flat, we have the usual trace pairing (cf. Section 5.3)

$$f_*\mathcal{O}_X \longrightarrow \mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{O}_Y). \tag{5.4}$$

The relative trace map *is now defined to be the composite map*

$$\begin{aligned} f_*\omega_X &\overset{(5.2)}{\cong} f_*(f_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \omega_Y)^\sim \\ &\xrightarrow{(5.4)} f_*(\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \omega_Y)^\sim \\ &\overset{(5.3)}{\cong} f_*\mathcal{H}om_Y(f_*\mathcal{O}_X, \omega_Y)^\sim \\ &= \mathcal{H}om_Y(f_*\mathcal{O}_X, \omega_Y) \xrightarrow{g \longmapsto g(1)} \omega_Y \end{aligned}$$

and is denoted by t_f.

See Section 5.3 for a more down-to-earth description of the relative trace map in terms of affinoids.
On the other hand, the (absolute) trace map from Section 4.3 satisfies [Bey97b, Satz 7.1]:

Theorem 5.3 (Serre duality for smooth rigid Stein spaces)**.** *Let X be a smooth rigid K-space of dimension d and let $\omega_X = \bigwedge^d \Omega^1_{X/K}$ be the sheaf of holomorphic d-forms on X. Let $H^*_c(X, -)$ denote the cohomology with compact support. If X is Stein, then there is a canonical trace morphism*

$$t\colon H^d_c(X, \omega_X) \longrightarrow K$$

which has the following property: If $\mathcal{F}$ is a coherent sheaf on X, then the composite of the trace map t with the canonical pairing

$$H_c^{d-i}(X,\mathcal{F}) \times \operatorname{Ext}^i_X(\mathcal{F},\omega_X) \longrightarrow H_c^d(X,\omega_X)$$

induces an isomorphism of topological K-vector spaces

$$H_c^{d-i}(X,\mathcal{F})^\vee \xrightarrow{\sim} \operatorname{Ext}^i_X(\mathcal{F},\omega_X) \tag{5.5}$$

for all $i \geq 0$.

Here $H_c^{d-i}(X,\mathcal{F})^\vee$ denotes the space of continuous linear forms on $H_c^{d-i}(X,\mathcal{F})$, equipped with the strong dual topology. Moreover, $\operatorname{Ext}^i_X(\mathcal{F},\omega_X)$ is equipped with the canonical topology for global sections of a coherent sheaf (see Definition 1.41), as $\operatorname{Ext}^i_X(\mathcal{F},\omega_X) = H^0(X,\mathcal{E}xt^i_X(\mathcal{F},\omega_X))$ due to the spectral sequence for the derived functor of the composition being degenerate. Indeed, the spectral sequence degenerates since X is quasi-Stein and $\mathcal{E}xt^i_X(\mathcal{F},\omega_X)$ is a coherent $\mathcal{O}_X$-module for all i (cf. [Chi90, Proposition 3.3 and also the discussion preceding Lemma 3.7]).

Remark 5.4. *In the setting of Theorem 5.3, it is equivalent to assert that (5.5) is an isomorphism for $i = 0$ (then it is automatically an isomorphism for all $i > 0$).*

Proof. This assertion is a by-product of several of Beyer's proofs on his way to the proof of Theorem 5.3, we simply gather his arguments here. Assume (5.5) is an isomorphism for $i = 0$.

In a first step, one proves the assertion for $X = \mathring{\mathbb{D}}^d$. One does this by considering the contravariant δ-functors

$$T^i := H_c^{d-i}(\mathring{\mathbb{D}}^d, -)^\vee$$

and

$$S^i := \operatorname{Ext}^i_{\mathring{\mathbb{D}}^d}(-,\omega_{\mathbb{D}^d})$$

on the category of coherent $\mathcal{O}_{\mathbb{D}^d}$-modules. Since S^i is a universal δ-functor and we by assumption have an isomorphism between S^0 and T^0, it suffices to show that T^i is also universal. For this, it suffices to prove that T^i is coeffaceable, i.e. that for every coherent $\mathcal{O}_{\mathbb{D}^d}$-module $\mathcal{F}$ and every surjection $u\colon O^r_{\mathbb{D}^d} \twoheadrightarrow \mathcal{F}$, we have $T^i(u) = 0$ for all $i > 0$. But $T^i(u) = 0$ because $T^i(u)$ is a map from $H_c^{d-i}(\mathring{\mathbb{D}}^d,\mathcal{F})^\vee$ to $H_c^{d-i}(\mathring{\mathbb{D}}^d,\mathcal{O}^r_{\mathbb{D}^d})^\vee$ and one can show that $H_c^{d-i}(\mathring{\mathbb{D}}^d,\mathcal{O}^r_{\mathbb{D}^d})^\vee = 0$ for $i > 0$. Indeed, it suffices to show this for $r = 1$, where it is done by an explicit calculation of Čech cohomology [Bey97a, Corollary 1.2.4].

In a second step, one proves the assertion for $X = \mathring{Z}$ an affinoid wide-open space inside an affinoid space Z. The strategy is the same as in the first step: One considers the contravariant δ-functors

$$T^i := H_c^{d-i}(\mathring{Z}, -)^\vee$$

and

$$S^i := \operatorname{Ext}^i_{\mathring{Z}}(-, \omega_Z)$$

on the category of coherent $\mathcal{O}_Z$-modules and shows that T^i is coeffaceable. In the same way as in the first step above, the coeffaceability of T^i will follow from the fact that $H^{d-i}_c(\mathring{Z}, \mathcal{O}_Z)^\vee = 0$ ($i > 0$). To prove the vanishing of $H^{d-i}_c(\mathring{Z}, \mathcal{O}_Z)^\vee$, choose a closed immersion $\iota\colon Z \hookrightarrow \mathbb{D}^n$ as in Definition 3.6 and observe that

$$H^{d-i}_c(\mathring{Z}, \mathcal{O}_Z)^\vee = H^{d-i}_c(\mathring{\mathbb{D}}^n, \iota_*\mathcal{O}_Z)^\vee = \operatorname{Ext}^{n-(d-i)}_{\mathring{\mathbb{D}}^d}(\iota_*\mathcal{O}_Z, \omega_{\mathbb{D}^d}),$$

the first equality being due to Proposition 1.51 and the second due to the first step above. Finally, [Bey97a, Lemma 4.1.6] proves that $\operatorname{Ext}^{n-(d-i)}_{\mathring{\mathbb{D}}^d}(\iota_*\mathcal{O}_Z, \omega_{\mathbb{D}^d})$ vanishes for $i > 0$ by showing the vanishing locally for each point $x \in \mathbb{D}^n$, in the following way: For $x \in \mathbb{D}^n \setminus \iota(Z)$, the local Ext-groups vanish in all exponents since the first argument is $\iota_*\mathcal{O}_{Z,x} = 0$. For $x \in \iota(Z)$, one uses the Auslander-Buchsbaum formula for projective dimension: setting $R = \mathcal{O}_{\mathbb{D}^n,x}$ and $A = \iota_*\mathcal{O}_{Z,x}$, we have a ring epimorphism $R \twoheadrightarrow A$. Since R is regular, the finitely generated R-module A has finite projective dimension, which by the Auslander Buchsbaum formula [Eis13, Theorem 19.9] then amounts to

$$\operatorname{pd}_R(A) = \operatorname{depth}(\mathfrak{m}_R, R) - \operatorname{depth}(\mathfrak{m}_R, A),$$

whence it suffices to show that $\operatorname{depth}(\mathfrak{m}_R, R) - \operatorname{depth}(\mathfrak{m}_R, A) = n - d$. To prove this last equality, note that by definition we have $\operatorname{depth}(\mathfrak{m}_R, R) = \operatorname{depth}(\mathfrak{m}_R)$ and the latter is equal to $\operatorname{codim}(\mathfrak{m}_R)$ since R is regular and in particular Cohen-Macaulay. Furthermore, by definition we have $\operatorname{codim}(\mathfrak{m}_R) = \dim R_{\mathfrak{m}_R} = \dim R = n$. Next, note that $\mathfrak{m}_R$ maps surjectively onto $\mathfrak{m}_A$ so in particular the action of $\mathfrak{m}_R$ on A is the same as the action of $\mathfrak{m}_A$ on A, i.e. we have $\operatorname{depth}(\mathfrak{m}_R, A) = \operatorname{depth}(\mathfrak{m}_A, A)$ and the latter is equal to $\dim A = d$ by the same argument as for R -- since A is regular (due to Z being smooth at x) and hence in particular Cohen-Macaulay.

In the third and final step, one observes that we have

$$H^{d-i}_c(X, \mathcal{F})^\vee = (\varinjlim_j H^{d-i}_c(\mathring{U}_j, \mathcal{F}))^\vee = \varprojlim_j H^{d-i}_c(\mathring{U}_j, \mathcal{F})^\vee$$

for any covering $\{\mathring{U}_j\}_j$ by affinoid wide-opens (which exists by Lemma 4.11 and Proposition 3.10), and by the second step above

$$\varprojlim_j H^{d-i}_c(\mathring{U}_j, \mathcal{F})^\vee = \varprojlim_j \operatorname{Ext}^i_{\mathring{U}_j}(\mathcal{F}, \omega_X) = \operatorname{Ext}^i_X(\mathcal{F}, \omega_X)$$

for all $i \geq 0$. $\square$

Our ultimate goal in this chapter is to prove the following compatibility of the relative trace map with Beyer's absolute trace map:

Theorem 5.5. *Let $\alpha\colon X \longrightarrow Y$ be a finite étale morphism of smooth connected d-dimensional Stein spaces over K. Then the diagram*

$$\begin{tikzcd} & H^d_c(Y,\omega_Y) \arrow[dr, "t_Y"] & \\ H^d_c(Y,\alpha_*\omega_X) \arrow[ur, "H^d_c(Y,t_\alpha)"] \arrow[dr, "\cong"'] & & K. \\ & H^d_c(X,\omega_X) \arrow[ur, "t_X"'] & \end{tikzcd}$$

commutes, where $t_\alpha\colon \alpha_*\omega_X \longrightarrow \omega_Y$ is the relative trace map.

The proof of this theorem is the content of Section 5.4 below. Note that, in the statement of Theorem 5.5, we have used that the map $H^d_c(Y,\alpha_*\omega_X) \longrightarrow H^d_c(X,\omega_X)$ is an isomorphism, which is due to Proposition 1.51.

Notation 5.6. We denote the composite map

$$H^d_c(X,\omega_X) \xrightarrow{\sim} H^d_c(Y,\alpha_*\omega_X) \xrightarrow{H^d_c(Y,t_\alpha)} H^d_c(Y,\omega_Y)$$

by q_α.

Thus we have to show that

$$\begin{tikzcd} H^d_c(Y,\omega_Y) \arrow[dr, "t_Y"] & \\ & K \\ H^d_c(X,\omega_X) \arrow[uu, "q_\alpha"] \arrow[ur, "t_X"'] & \end{tikzcd} \tag{5.6}$$

commutes. This notation is fixed for the remainder of Chapter 5.

5.2 Compatible coverings by special affinoid wide-opens

We will need the following "relative version" of Lemma 4.11:

Lemma 5.7. *There exist admissible covers $\{\mathring{U}_i\}_{i\in\mathbb{N}}$ of X and $\{\mathring{V}_i\}_{i\in\mathbb{N}}$ of Y by special affinoid wide-opens as in Lemma 4.11, such that $\alpha(\mathring{U}_i) \subset \mathring{V}_i$ and the map $\alpha\colon U_i \longrightarrow V_i$ on the ambient affinoid spaces is finite étale. More precisely, we have a commutative diagram*

$$\begin{tikzcd} \mathring{V}_i \arrow[r, "\subseteq"] & V_i \arrow[r, two heads, "\pi_{V_i}"] & \mathbb{D}^d \\ \mathring{U}_i \arrow[u, two heads] \arrow[r, "\subseteq"] & U_i \arrow[u, two heads, "\alpha"] \arrow[ur, "\pi_{U_i}"'] & \end{tikzcd}$$

for each i, where U_i and V_i are connected smooth affinoids, α is finite étale, π_{V_i} and π_{U_i} are finite and separable and "define" $\mathring{V}_i$ resp. $\mathring{U}_i$ (in the sense of Proposition 3.5 (iii)), and all arrows with two heads are surjective.

Proof. We obtain the existence of $\{\mathring{V}_i\}_{i\in\mathbb{N}}$ by Lemma 4.11. Moreover, we have a connected und smooth affinoid space $V_i = \mathrm{Sp}(A_i)$ with $\mathring{V}_i \subseteq V_i \subseteq Y$, such that $\mathring{V}_i$ is the preimage of finitely many points under the reduction map p_{V_i}, say

$$\mathring{V}_i = p_{V_i}^{-1}(\{\widetilde{v}_1, \ldots, \widetilde{v}_r\}).$$

The preimage of an affinoid space under a finite morphism is again affinoid, so $U_i' := \alpha^{-1}(V_i)$ is affinoid. Moreover, the restriction $\alpha\colon U_i' \longrightarrow V_i$ is also finite étale. Replacing U_i' with one[1] of its Zariski-connected components U_i (which is an affinoid subdomain in U_i', say $U_i = \mathrm{Sp}(B_i)$), the restriction $\alpha\colon U_i \longrightarrow V_i$ is again finite étale. Here we use that U_i is "clopen" in U_i' (Remark 1.14 (iii)): the restriction is again étale since it arises by composition with the open immersion $U_i \hookrightarrow U_i'$ (which is étale) and, on the other hand, it is again finite since it arises by composition with the closed immersion $U_i \hookrightarrow U_i'$ (which is finite). The restriction $\alpha\colon U_i \longrightarrow V_i$ being finite étale, the associated ring morphism $A_i \longrightarrow B_i$ is finite flat. We may assume that $U_i \neq \emptyset$. Indeed, since $X \neq \emptyset$, we have $U_i \neq \emptyset$ for all $i \gg 0$, so we may re-index and forget the small i. Next we argue that $A_i \longrightarrow B_i$ is injective. This is true since any flat ring morphism $R \longrightarrow S$ with R a domain and $S \neq 0$ is necessarily injective, and A_i is a domain since V_i is connected (see Remark 1.15). So $A_i \longrightarrow B_i$ is an injective integral morphism, which implies that the map $\mathrm{Spec}(B_i) \longrightarrow \mathrm{Spec}(A_i)$ is surjective. Thus

$$\alpha\colon U_i \longrightarrow V_i$$

is surjective as well. Note that α being finite implies that $\widetilde{\alpha}$ is also finite (by Theorem 1.10) and that α being surjective implies that $\widetilde{\alpha}$ is also surjective (by Remark 1.4).

Since $\widetilde{\alpha}$ is finite, $\Omega := \widetilde{\alpha}^{-1}(\{\widetilde{v}_1, \ldots, \widetilde{v}_r\})$ is a finite subset of $\widetilde{U}_i$ and hence

$$\mathring{U}_i := p_{U_i}^{-1}(\Omega)$$

is special affinoid wide-open in U_i. Then we have $\alpha(\mathring{U}_i) \subseteq \mathring{V}_i$ because $p_{V_i} \circ \alpha = \widetilde{\alpha} \circ p_{U_i}$ by functoriality of the reduction. In fact, we claim that we have $\mathring{U}_i = \alpha^{-1}(\mathring{V}_i)$, so the restriction $\mathring{U}_i \longrightarrow \mathring{V}_i$ of α is again surjective. To see this, let $u \in \alpha^{-1}(\mathring{V}_i)$, which by definition of $\mathring{V}_i$ means that

$$\{\widetilde{v}_1, \ldots, \widetilde{v}_r\} \ni p_{V_i}(\alpha(u)) = \widetilde{\alpha}(p_{U_i}(u)),$$

so $p_{U_i}(u) \in \widetilde{\alpha}^{-1}(\{\widetilde{v}_1, \ldots, \widetilde{v}_r\}) = \Omega$, i.e. $u \in {p_{U_i}}^{-1}(\Omega) = \mathring{U}_i$ which proves our claim. In particular it follows that $\{\mathring{U}_i\}_{i\in\mathbb{N}}$ is the preimage of the admissible open cover

[1] In general, an affinoid space Z has finitely many Zariski-connected components, since affinoid algebras are Noetherian. The Zariski-connected components are affinoid subdomains of Z and define an admissible affinoid covering of Z, as is mentioned in the discussion following 5.3/Definition 9 in [Bos14].

$\{\mathring{V}_i\}_{i\in\mathbb{N}}$ of Y under $X \longrightarrow Y$, hence it is itself an admissible open cover of X. Now take a finite surjective map $\widetilde{\pi}_{V_i}\colon \widetilde{V}_i \longrightarrow \mathbb{A}^d$ with $\widetilde{\pi}_{V_i}(\widetilde{v}_j) = 0$ for all $j = 1, \ldots, r$ and a separable lift $\pi_{V_i}\colon V_i \longrightarrow \mathbb{D}^d$ (see Remark 3.3). Define

$$\widetilde{\pi}_{U_i} := \widetilde{\pi}_{V_i} \circ \widetilde{\alpha}.$$

Then $\widetilde{\pi}_{U_i}(\Omega) = \widetilde{\pi}_{V_i}(\widetilde{\alpha}(\Omega)) \subseteq \widetilde{\pi}_{V_i}(\{\widetilde{v}_1, \ldots, \widetilde{v}_r\}) = \{0\}$. Moreover, $\widetilde{\pi}_{U_i}$ is finite surjective since $\widetilde{\pi}_{V_i}$ and $\widetilde{\alpha}$ are both finite and surjective. Therefore, any separable lift of $\widetilde{\pi}_{U_i}$ satisfies the conditions in the assertion, by Lemma 3.2. In particular, the separable lift

$$\pi_{U_i} := \pi_{V_i} \circ \alpha$$

satisfies the desired conditions. Here we used that α is separable, which is true because it is unramified at all closed points of U and hence necessarily generically unramified (cf. the proof of Lemma 1.25). $\square$

5.3 More on the relative trace map

Consider the diagram from Lemma 5.7 for a fixed i, but omit the index i in the notation. Thus we have that $\alpha\colon \mathring{U} \longrightarrow \mathring{V}$ is the restriction of a surjective finite étale morhism

$$\alpha\colon U = \mathrm{Sp}(B) \longrightarrow V = \mathrm{Sp}(A).$$

The associated ring morphism $A \longrightarrow B$ is, in particular, finite and flat. By Lemma 5.7 and Remark 1.7, we have the commutative diagram

$$\begin{tikzcd} V \arrow[r, two heads, "\pi_V"] & \mathbb{D}^d \\ U \arrow[u, two heads, "\alpha"] \arrow[ur, two heads, "\pi_U"'] & \end{tikzcd} \qquad \text{associated to} \qquad \begin{tikzcd} A \arrow[d, hook] & K\langle \xi_1, \ldots, \xi_d\rangle \arrow[l, hook'] \arrow[dl, hook'] \\ B & \end{tikzcd} \tag{5.7}$$

where all morphisms are finite and A and B are integral domains.
Since A is Noetherian, the finite flat ring morphism $A \longrightarrow B$ makes B into a finitely presented flat A-module, i.e. a finitely generated projective A-module. Hence the natural map

$$\begin{aligned} \mathrm{can}\colon B^* \otimes_A B &\xrightarrow{\sim} \mathrm{End}_A(B) \\ f \otimes b &\longmapsto [x \longmapsto f(x) \cdot b] \end{aligned}$$

is an isomorphism, where $B^* := \mathrm{Hom}_A(B, A)$. The trace $\mathrm{Tr}_{B/A}$ is now defined as the composite

$$\begin{tikzcd} B \arrow[r] \arrow[rrr, bend right, "\mathrm{Tr}_{B/A}"'] & \mathrm{End}_A(B) \arrow[r, "\mathrm{can}^{-1}"] & B^* \otimes_A B \arrow[r] & A \end{tikzcd}$$

where the first map sends $b \in B$ to the endomorphism given by multiplication by b, and the last map is given by "evaluation". If B is free of finite rank over A, then $\mathrm{Tr}_{B/A}$ coincides with the usual trace map from linear algebra.
On the other hand, since α is finite étale, we have

$$\Omega^d_{B/K} \cong \Omega^d_{A/K} \otimes_A B$$

according to (5.2) in Definition 5.2.

Definition 5.8 (The map τ). *We define a trace map τ at the level of differentials, by putting*

$$\begin{aligned} \Omega^d_{B/K} = \Omega^d_{A/K} \otimes_A B &\xrightarrow{\tau} \Omega^d_{A/K} \\ \omega \otimes b &\longmapsto \mathrm{Tr}_{B/A}(b) \cdot \omega. \end{aligned} \tag{5.8}$$

This corresponds to a map

$$\mathfrak{T} \colon \alpha_* \omega_U \longrightarrow \omega_V.$$

Remark 5.9. *The map $\mathfrak{T}$ is equal to the restriction of the relative trace map t_α.*

Proof. The relative trace map t_α restricts to $t_\alpha \colon \alpha_*\omega_U \longrightarrow \omega_V$, where it is associated to a homomorphism of modules $\Omega^d_{B/K} \longrightarrow \Omega^d_{A/K}$. To see that the homomorphism in question coincides with the natural homomorphism τ, note that U and V being affinoid allows us to work with modules instead of sheaves in Definition 5.2, whence $t_\alpha \colon \alpha_*\omega_U \longrightarrow \omega_V$ is associated to the composite

$$\Omega^d_{B/K} \xrightarrow[(5.2)]{\sim} B \otimes_A \Omega^d_{A/K} \xrightarrow[(5.4)]{} B^* \otimes_A \Omega^d_{A/K} \xrightarrow[(5.3)]{\sim} \mathrm{Hom}_A(B, \Omega^d_{A/K}) \xrightarrow{f \longmapsto f(1)} \Omega^d_{A/K}$$

which is precisely the homomorphism τ. Here we used that (5.4) at the level of modules is induced by the isomorphism $B \longrightarrow B^*, b \longmapsto \mathrm{Tr}_{B/A}(b \cdot (-))$. $\square$

We will need the following compatibility of τ with $\sigma_U := \sigma_{\pi_U}$ and $\sigma_V := \sigma_{\pi_V}$ from Definition 4.9:

Lemma 5.10. *The diagram*

$$\begin{tikzcd} \Omega^d_{B/K} \arrow[r, "\sigma_U"] \arrow[d, "\tau"'] & \Omega^d_{K\langle \xi_1, \dots, \xi_d \rangle/K} \\ \Omega^d_{A/K} \arrow[ur, "\sigma_V"'] & \end{tikzcd}$$

commutes. It corresponds to the commutative diagram

$$\begin{tikzcd} \pi_{U*}\omega_U = \omega_U \arrow[r, "\sigma_U"] \arrow[d, "\pi_{V*}(\mathfrak{T})"'] & \omega_{\mathbb{D}^d} \\ \pi_{V*}\omega_V. \arrow[ur, "\sigma_V"'] & \end{tikzcd}$$

Proof. In the following, we write T_d for $K\langle \xi_1, \ldots, \xi_d \rangle$ and $Q(B)^* = \mathrm{Hom}_{Q(A)}(Q(B), Q(A))$. Due to the formulae (4.10) and (5.8), we see that it suffices to show that

$$\mathrm{Tr}_{Q(B)/Q(T_d)} = \mathrm{Tr}_{Q(A)/Q(T_d)} \circ \mathrm{Tr}_{B/A}\,. \tag{5.9}$$

We will show that the restriction of $\mathrm{Tr}_{Q(B)/Q(A)}$ to B coincides with $\mathrm{Tr}_{B/A}$, i.e. that the diagram

$$\begin{tikzcd}
Q(B) \arrow[r] & \mathrm{End}_{Q(A)}(Q(B)) \arrow[r, "\sim"] & Q(B)^* \otimes_{Q(A)} Q(B) \arrow[r] & Q(A) \\
B \arrow[u] \arrow[r] & \mathrm{End}_A(B) \arrow[r, "\sim"] & B^* \otimes_A B \arrow[r] & A \arrow[u]
\end{tikzcd}$$

commutes, whence the desired equality (5.9) follows by the transitivity of the trace in towers of field extensions. Lemma 5.11 below tells us that $Q(B) = B \otimes_A Q(A)$, so in particular extension of scalars yields a map

$$\begin{aligned} \mathrm{End}_A(B) &\longrightarrow \mathrm{End}_{Q(A)}(Q(B)) \\ \phi &\longmapsto \phi_Q := \phi \otimes \mathrm{id}_{Q(A)} \end{aligned}$$

and similarly

$$\begin{aligned} B^* &\longrightarrow Q(B)^* \\ f &\longmapsto f_Q := f \otimes \mathrm{id}_{Q(A)}. \end{aligned}$$

Thus we can expand the above diagram to

$$\begin{tikzcd}
Q(B) \arrow[r] & \mathrm{End}_{Q(A)}(Q(B)) \arrow[r, "\sim"] & Q(B)^* \otimes_{Q(A)} Q(B) \arrow[r] & Q(A) \\
B \arrow[u] \arrow[r] & \mathrm{End}_A(B) \arrow[u] \arrow[r, "\sim"] & B^* \otimes_A B \arrow[u] \arrow[r] & A. \arrow[u]
\end{tikzcd}$$

Now it easy easy to see that each of the three squares in the diagram commute. To see, for instance, that the middle square commutes, note that the isomorphism $Q(B) \cong B \otimes_A Q(A)$ means that any element of $Q(B)$ can be written as $\frac{x}{a}$ with $x \in B$ and $a \in A$, and any ϕ_Q then acts on it as $\phi_Q(\frac{x}{a}) = \frac{\phi(x)}{a}$. The commutativity of the middle square amounts to showing that, if $\phi(x) = \sum_i f_i(x) b_i$ for all $x \in B$, then $\phi_Q(\frac{x}{a}) = \sum_i f_{iQ}(\frac{x}{a})\frac{b_i}{1}$ for all $x \in B$ and all $a \in A$. But $\phi_Q(\frac{x}{a}) = \frac{\phi(x)}{a}$ and $\sum_i f_{iQ}(\frac{x}{a})\frac{b_i}{1} = \sum_i \frac{f_i(x)}{a}\frac{b_i}{1} = \frac{\sum_i f_i(x) b_i}{a} = \frac{\phi(x)}{a}$ as well, so we are done. □

Lemma 5.11. *If R' is an overring of an integral domain R such that each $r' \in R'$ is integral over R and such that no element of $R \setminus \{0\}$ is a zero divisor in R', then the localisation $R'_{R\setminus\{0\}}$ of R' at the multiplicative subset $R \setminus \{0\}$ coincides with the total ring of fractions $Q(R')$.*

Proof. This is proved in the discussion after [BGR84, 3.1.3/Proposition 3]. □

5.4 Proof of the main theorem

5.4.1 Reduction to the case of special affinoid wide-open spaces

From Lemma 5.7 we obtain the diagram

$$\begin{tikzcd} \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d, equal] \\ H_c^d(\mathring{U}_i, \omega_X) \arrow[r, "q_\alpha"] \arrow[d] & H_c^d(\mathring{V}_i, \omega_Y) \arrow[r, "t_i"] \arrow[d] & K \arrow[d, equal] \\ H_c^d(\mathring{U}_{i+1}, \omega_X) \arrow[r, "q_\alpha"] \arrow[d] & H_c^d(\mathring{V}_{i+1}, \omega_Y) \arrow[r, "t_{i+1}"] \arrow[d] & K \arrow[d, equal] \\ \cdots & \cdots & \cdots \end{tikzcd} \tag{5.10}$$

where all squares on the right-hand side commute by [Bey97a, Corollary 4.2.12] and all squares on the left-hand side also commute, so taking $\varinjlim_i$ produces

$$H_c^d(X, \omega_X) \xrightarrow{q_\alpha} H_c^d(Y, \omega_Y) \xrightarrow{t_Y} K. \tag{5.11}$$

On the other hand, if we can show that the composite map

$$H_c^d(\mathring{U}_i, \omega_X) \xrightarrow{q_\alpha} H_c^d(\mathring{V}_i, \omega_Y) \xrightarrow{t_i = t_{\mathring{V}_i}} K$$

coincides with

$$H_c^d(\mathring{U}_i, \omega_X) \xrightarrow{t_{\mathring{U}_i}} K,$$

then taking $\varinjlim_i$ in (5.10) produces t_X. Hence (5.11) would coincide with t_X and Theorem 5.5 would be proved. Thus we are reduced to showing:

Theorem 5.12. *The diagram*

$$\begin{tikzcd} H_c^d(\mathring{V}_i, \omega_Y) \arrow[dr, "t_{\mathring{V}_i}"] & \\ & K \\ H_c^d(\mathring{U}_i, \omega_X) \arrow[uu, "q_\alpha"] \arrow[ur, "t_{\mathring{U}_i}"'] & \end{tikzcd} \tag{5.12}$$

commutes for all i.

To prove Theorem 5.12, we need to prepare the necessary ingredients, the last of which will be Lemma 5.16 below.

5.4.2 The relative trace map at the level of local cohomology

For notational convenience from now on, we drop the index i, so that we are in the same setting as in Section 5.3: $\alpha\colon \mathring{U} \longrightarrow \mathring{V}$ is the restriction of a surjective finite étale morhism

$$\alpha\colon U = \mathrm{Sp}(B) \longrightarrow V = \mathrm{Sp}(A).$$

The associated ring morphism $A \longrightarrow B$ is, in particular, finite and flat. We have the commutative diagram

$$\begin{tikzcd} V \arrow[r, two heads, "\pi_V"] & \mathbb{D}^d \\ U \arrow[u, two heads, "\alpha"] \arrow[ur, "\pi_U"'] & \end{tikzcd} \qquad \text{associated to} \qquad \begin{tikzcd} A \arrow[d, hook] & K\langle \xi_1, \ldots, \xi_d \rangle \arrow[l, hook'] \arrow[dl, hook'] \\ B & \end{tikzcd}$$

where all morphisms are finite. Let

$$\{x_1, \ldots, x_r\} = \pi_U^{-1}(0) \cap \mathring{U} \quad \text{and} \quad \{x_1, \ldots, x_r, x_{r+1}, \ldots, x_s\} = \pi_U^{-1}(0).$$

Note that $\{\alpha(x_1), \ldots, \alpha(x_r)\} \subseteq \pi_V^{-1}(0) \cap \mathring{V}$, since $\pi_U = \pi_V \circ \alpha$ by Lemma 5.7 and we have $\{\alpha(x_1), \ldots, \alpha(x_s)\} = \pi_V^{-1}(0)$ since $\pi_U = \pi_V \circ \alpha$ and α is surjective. Denote the cardinality of $\{\alpha(x_1), \ldots, \alpha(x_s)\}$ by s' and let

$$\{y_1, \ldots, y_{s'}\} = \{\alpha(x_1), \ldots, \alpha(x_s)\} = \pi_V^{-1}(0)$$

with $y_i \neq y_j$ for $i \neq j$. We have $s' \leq s$ and it may happen that $s' < s$. Define r' in the same way for $\{\alpha(x_1), \ldots, \alpha(x_r)\}$. We bring local cohomology into the game: Using the maps from Section 4.2, expand the diagram (5.12) to obtain

$$\begin{tikzcd} \bigoplus_{i=1}^{r'} H^d_{y_i}(\omega_V) \arrow[r] \arrow[rrd, bend left=40, "\sum \mathrm{res}_{y_i}"] & H^d_c(\mathring{V}, \omega_V) \arrow[rd, "t_{\mathring{V}}"] \arrow[d, phantom, shift left=30, "?"] & \\ & & K \\ \bigoplus_{i=1}^{r} H^d_{x_i}(\omega_U) \arrow[r, "\text{dense}"] \arrow[rru, bend right=40, "\sum \mathrm{res}_{x_i}"'] & H^d_c(\mathring{U}, \omega_U) \arrow[uu, "q_\alpha"] \arrow[ru, "t_{\mathring{U}}"'] & \end{tikzcd} \tag{5.13}$$

where the lower horizontal map has dense image by Lemma 4.7. The two outer "slices"

$$\begin{tikzcd}
\bigoplus_{i=1}^{r'} H^d_{y_i}(\omega_V) \arrow[r] \arrow[dr, bend left=60, "\sum \mathrm{res}_{y_i}"] & H^d_c(\mathring{V}, \omega_V) \arrow[d, "t_{\mathring{V}}"] \\
& K
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& & K \\
\bigoplus_{i=1}^{r} H^d_{x_i}(\omega_U) \arrow[r] \arrow[urr, bend right=60, "\sum \mathrm{res}_{x_i}"'] & H^d_c(\mathring{U}, \omega_U) \arrow[ur, "t_{\mathring{U}}"] &
\end{tikzcd}$$

in (5.13) commute by Proposition 4.8. We have placed a question mark in the triangle in (5.13) for psychological reasons - as a reminder that we need to show that the triangle commutes.

Next, we use the map $\tau\colon \Omega^d_{B/K} \longrightarrow \Omega^d_{A/K}$ from Section 5.3 to obtain an induced map

$$H^d_x(\tau)\colon H^d_x(\omega_U) \longrightarrow H^d_{\alpha(x)}(\omega_V) \tag{5.14}$$

for every point $x \in U$, via Definition 5.13 below. (Recall that we use the notation

$$H^d_x(\omega_U) = H^d_{\mathfrak{m}_x}(\Omega^d_{B/K})$$

etc.)

Definition 5.13 (Induced maps on local cohomology in general)**.** *Let R and S be rings, $\varphi\colon R \longrightarrow S$ a ring morphism, $\mathfrak{b} \subseteq S$ an ideal, M an R-module and N an S-module. Let $\rho\colon N \longrightarrow M$ be an R-linear map. Then we define*

$$H^j_{\mathfrak{b}}(\rho)\colon H^j_{\mathfrak{b}}(N) \longrightarrow H^j_{\varphi^{-1}(\mathfrak{b})}(M)$$

as follows. The inclusion $\varphi(\varphi^{-1}(\mathfrak{b}))S \subseteq \mathfrak{b}$ implies $\Gamma_{\mathfrak{b}}(N) \subseteq \Gamma_{\varphi(\varphi^{-1}(\mathfrak{b}))S}(N)$, whence we get a natural map

$$H^j_{\mathfrak{b}}(N) \longrightarrow H^j_{\varphi(\varphi^{-1}(\mathfrak{b}))S}(N). \tag{5.15}$$

Moreover,

$$H^j_{\varphi(\varphi^{-1}(\mathfrak{b}))S}(N) \cong H^j_{\varphi^{-1}(\mathfrak{b})}(N) \tag{5.16}$$

by independence of base (Proposition 1.58 (ii)). Finally, $\Gamma_{\varphi^{-1}(\mathfrak{b})}$ is a functor on R-modules and hence gives rise to

$$H^j_{\varphi^{-1}(\mathfrak{b})}(N) \longrightarrow H^j_{\varphi^{-1}(\mathfrak{b})}(M). \tag{5.17}$$

The desired map $H^j_{\mathfrak{b}}(\rho)$ ist the composite of these three maps.

Choosing $R = A, S = B, \mathfrak{b} = \mathfrak{m}_x, M = \Omega^d_{A/K}, N = \Omega^d_{B/K}$ and $\rho = \tau$ in Definition 5.13 yields $H^d_x(\tau)$.
For each i we have the map

$$\bigoplus_{x \in \alpha^{-1}(y_i)} H^d_x(\omega_U) \xrightarrow{\sum_x H^d_x(\tau)} H^d_{y_i}(\omega_V).$$

Lemma 5.14 (Explicit description of $\sum_x H^d_x(\tau)$)**.** *Let $M = \Omega^d_{A/K}, N = \Omega^d_{B/K}$ and consider the trace map*

$$\tau \colon N \longrightarrow M$$

from Definition 5.8. Let $\mathfrak{m}_y \subseteq A$ be a maximal ideal that pulls back to $\mathfrak{m}$ in T_d, where $\mathfrak{m}$ denotes the ideal corresponding to the point $0 \in \mathbb{D}^d$. For every $\mathfrak{m}_x \subseteq B$ that pulls back to $\mathfrak{m}_y$, taking completions in the ring diagram (5.7) yields the diagram

$$\begin{tikzcd} A^{\wedge \mathfrak{m}_y} \arrow[d] & T_d{}^{\wedge \mathfrak{m}} \arrow[l] \arrow[dl] \\ B^{\wedge \mathfrak{m}_x} & \end{tikzcd} \tag{5.18}$$

where all morphisms are finite due to Proposition 1.67. Let $X_1, \ldots, X_d$ be a system of parameters for $T_d{}^{\wedge \mathfrak{m}}$ - so that the images in $A^{\wedge \mathfrak{m}_y}$ resp. $B^{\wedge \mathfrak{m}_x}$ are also a system of parameters due to Corollary 1.68. Then the map

$$\bigoplus_{x \in \alpha^{-1}(y)} H^d_x(N) \xrightarrow{\sum_x H^d_x(\tau)} H^d_y(M)$$

identifies, via Proposition 1.59 in conjunction with Lemma 1.57, with the map

$$\varinjlim_{\rho} N^{\wedge \mathfrak{m}_y}/(X_1^\rho, \ldots, X_n^\rho) \xrightarrow{\tau^{\wedge \mathfrak{m}_y}} \varinjlim_{\rho} M^{\wedge \mathfrak{m}_y}/(X_1^\rho, \ldots, X_n^\rho). \tag{5.19}$$

Proof. Note that

$$\widehat{\mathfrak{m}_y}\widehat{B} = (\mathfrak{m}_y\widehat{A})\widehat{B} = \mathfrak{m}_y(\widehat{A}\widehat{B}) = \mathfrak{m}_y\widehat{B} = \mathfrak{m}_y(B\widehat{B}) = (\mathfrak{m}_y B)\widehat{B} = \widehat{\mathfrak{m}_y B}$$

and

$$N^{\wedge \mathfrak{m}_y B} = \varprojlim_n N/(\mathfrak{m}_y B)^n N = \varprojlim_n N/\mathfrak{m}_y^n N = N^{\wedge \mathfrak{m}_y}$$

since we have $(\mathfrak{m}_y B)N = \mathfrak{m}_y(BN) = \mathfrak{m}_y N$ and by similar logic also $(\mathfrak{m}_y B)^n N = \mathfrak{m}_y^n N$. The identities $\widehat{\mathfrak{m}_y B} = \widehat{\mathfrak{m}_y}\widehat{B}$ and $N^{\wedge \mathfrak{m}_y B} = N^{\wedge \mathfrak{m}_y}$ explain the first identity in the bottow row of the diagram (5.20) below.

By going through Definition 5.13, we see that the map $\sum_x H^d_x(\tau)$ is the composite of the arrows in the top row of the diagram

$$\begin{tikzcd}
\bigoplus_{x\in\alpha^{-1}(y)} H^d_{\mathfrak{m}_x}(N) \arrow[r, "\cong", "\sum_x (5.15)"'] & H_{\mathfrak{m}_y B}(N) \arrow[r, equal, "(5.16)"'] \arrow[d, equal] & H_{\mathfrak{m}_y}(N) \arrow[r, "\tau", "(5.17)"'] & H_{\mathfrak{m}_y}(M) \arrow[d, equal] \\
& H_{\widehat{\mathfrak{m}_y B}}(N^{\wedge \mathfrak{m}_y B}) \arrow[r, equal, "(5.16)"'] & H_{\widehat{\mathfrak{m}_y}}(N^{\wedge \mathfrak{m}_y}) \arrow[r, "\tau^{\wedge \mathfrak{m}_y}", "(5.17)"'] & H_{\widehat{\mathfrak{m}_y}}(M^{\wedge \mathfrak{m}_y})
\end{tikzcd} \tag{5.20}$$

where the first map in the top row

$$\bigoplus_{x\in\alpha^{-1}(y)} H^d_{\mathfrak{m}_x}(N) \xrightarrow{\sim} H_{\mathfrak{m}_y B}(N)$$

is an isomorphism by the same Mayer-Vietoris argument as in our proof of Lemma 4.5 (i). On the other hand, the bottom row in the diagram produces the map (5.19), hence we are done. $\square$

Taking the direct sum over all $i = 1, \dots, r'$ we obtain the map

$$\bigoplus_{i=1}^{r} H^d_{x_i}(\omega_U) = \bigoplus_{i=1}^{r'} \bigoplus_{x\in\alpha^{-1}(y_i)} H^d_x(\omega_U) \longrightarrow \bigoplus_{i=1}^{r'} H^d_{y_i}(\omega_V)$$

which we denote by $\oplus_i H^d_{x_i}(\tau)$, abusing the notation. This yields the diagram

$$\begin{tikzcd}
\bigoplus_{i=1}^{r'} H^d_{y_i}(\omega_V) \arrow[r] \arrow[rrd, bend left, "\sum \mathrm{res}_{y_i}"] & H^d_c(\mathring{V}, \omega_V) \arrow[rd, "t_{\mathring{V}}"] & \\
\arrow[r, phantom, "?"] & \arrow[r, phantom, "?"] & K \\
\bigoplus_{i=1}^{r} H^d_{x_i}(\omega_U) \arrow[r, "\text{dense}"] \arrow[uu, "\oplus_i H^d_{x_i}(\tau)"] \arrow[rru, bend right, "\sum \mathrm{res}_{x_i}"'] & H^d_c(\mathring{U}, \omega_U) \arrow[uu, "q_\alpha"] \arrow[ru, "t_{\mathring{U}}"'] &
\end{tikzcd} \tag{5.21}$$

with outer semicircle

$$\begin{tikzcd}
\bigoplus_{i=1}^{r} H^d_{x_i}(\omega_U) \arrow[rr, "\oplus_i H^d_{x_i}(\tau)"] \arrow[rd, bend right, "\sum \mathrm{res}_{x_i}"'] & \arrow[d, phantom, "?"] & \bigoplus_{i=1}^{r'} H^d_{y_i}(\omega_V) \arrow[ld, bend left, "\sum \mathrm{res}_{y_i}"] \\
& K. &
\end{tikzcd} \tag{5.22}$$

Lemma 5.15. *The outer semicircle (5.22) commutes.*

Proof. Using the commutative diagram (4.6) from Lemma 4.5 (iii), we can re-write the diagram (5.22) as

$$\begin{tikzcd}
\bigoplus_{i=1}^{r} H^d_{x_i}(\omega_U) \arrow[rr, "\oplus_i H^d_{x_i}(\tau)"] \arrow[d] & & \bigoplus_{i=1}^{r'} H^d_{y_i}(\omega_V) \arrow[d] \\
\bigoplus_{i=1}^{s} H^d_{x_i}(\omega_U) \arrow[d, "\gamma_U"', "\cong"] & & \bigoplus_{i=1}^{s'} H^d_{y_i}(\omega_V) \arrow[d, "\cong"', "\gamma_V"] \\
H^d_0(\pi_{U*}\omega_U) \arrow[d, "H^d_0(\sigma_U)"'] & & H^d_0(\pi_{V*}\omega_V) \arrow[d, "H^d_0(\sigma_V)"] \\
H^d_0(\omega_{\mathbb{D}^d}) \arrow[dr, "\mathrm{res}_0"'] & & H^d_0(\omega_{\mathbb{D}^d}) \arrow[dl, "\mathrm{res}_0"] \\
& K. &
\end{tikzcd}$$

To show that this diagram commutes, we expand it so that it has three parts:

$$\begin{tikzcd}
\bigoplus_{i=1}^{r} H^d_{x_i}(\omega_U) \arrow[rr, "\oplus_i H^d_{x_i}(\tau)"] \arrow[d] & (1) & \bigoplus_{i=1}^{r'} H^d_{y_i}(\omega_V) \arrow[d] \\
\bigoplus_{i=1}^{s} H^d_{x_i}(\omega_U) \arrow[rr, "\oplus_i H^d_{x_i}(\tau)"] \arrow[d, "\gamma_U"', "\cong"] & (2) & \bigoplus_{i=1}^{s'} H^d_{y_i}(\omega_V) \arrow[d, "\cong"', "\gamma_V"] \\
H^d_0(\pi_{U*}\omega_U) \arrow[rr, "H^d_0(\pi_{V*}(\mathfrak{T}))"'] \arrow[d, "H^d_0(\sigma_U)"'] & & H^d_0(\pi_{V*}\omega_V) \arrow[d, "H^d_0(\sigma_V)"] \\
H^d_0(\omega_{\mathbb{D}^d}) \arrow[dr, "\mathrm{res}_0"'] & (3) & H^d_0(\omega_{\mathbb{D}^d}) \arrow[dl, "\mathrm{res}_0"] \\
& K. &
\end{tikzcd}$$

Then (1) commutes for obvious reasons and (3) commutes by Lemma 5.10. Thus it remains to prove that (2) commutes. Letting $M = \Omega^d_{A/K}, N = \Omega^d_{B/K}$ and $\mathfrak{m}$ denote the maximal ideal corresponding to $0 \in \mathbb{D}^d$, the diagram (2) is equivalent to a diagram of the form

$$\begin{tikzcd} \varinjlim_\rho \bigoplus_{i=1}^{s} \widehat{N_{\mathfrak{m}_{x_i}}}/(X_1^\rho,\dots,X_n^\rho) \arrow[r] & \varinjlim_\rho \bigoplus_{i=1}^{s'} \widehat{M_{\mathfrak{m}_{y_i}}}/(X_1^\rho,\dots,X_n^\rho) \\ \varinjlim_\rho \widehat{N_{\mathfrak{m}}}/(X_1^\rho,\dots,X_n^\rho) \arrow[r] \arrow[u, "\cong"', "\tilde{\gamma}_U^{-1}"] & \varinjlim_\rho \widehat{M_{\mathfrak{m}}}/(X_1^\rho,\dots,X_n^\rho) \arrow[u, "\cong"', "\tilde{\gamma}_V^{-1}"] \end{tikzcd} \tag{5.23}$$

by Lemma 4.5 (ii). Moreover, for each y_i we have an isomorphism of B-modules

$$B^{\wedge \mathfrak{m}_{y_i} B} \cong \bigoplus_{x\in\alpha^{-1}(y_i)} \widehat{B_{\mathfrak{m}_x}}$$

by (1.27) from the proof of Proposition 1.67, whence

$$\widehat{N_{\mathfrak{m}_{y_i}}} = N \otimes_B B^{\wedge \mathfrak{m}_{y_i} B} = \bigoplus_{x\in\alpha^{-1}(y_i)} N \otimes_B \widehat{B_{\mathfrak{m}_x}} = \bigoplus_{x\in\alpha^{-1}(y_i)} \widehat{N_{\mathfrak{m}_x}}.$$

Thus

$$\varinjlim_\rho \bigoplus_{i=1}^{s} \widehat{N_{\mathfrak{m}_{x_i}}}/(X_1^\rho,\dots,X_n^\rho) \cong \varinjlim_\rho \bigoplus_{i=1}^{s'} \widehat{N_{\mathfrak{m}_{y_i}}}/(X_1^\rho,\dots,X_n^\rho)$$

and the diagram (5.23) becomes

$$\begin{tikzcd} \varinjlim_\rho \bigoplus_{i=1}^{s'} \widehat{N_{\mathfrak{m}_{y_i}}}/(X_1^\rho,\dots,X_n^\rho) \arrow[r, "\oplus_i \tau^{\wedge \mathfrak{m}_{y_i}}"] & \varinjlim_\rho \bigoplus_{i=1}^{s'} \widehat{M_{\mathfrak{m}_{y_i}}}/(X_1^\rho,\dots,X_n^\rho) \\ \varinjlim_\rho \widehat{N_{\mathfrak{m}}}/(X_1^\rho,\dots,X_n^\rho) \arrow[r, "\tau^{\wedge \mathfrak{m}}"] \arrow[u, "\cong"'] & \varinjlim_\rho \widehat{M_{\mathfrak{m}}}/(X_1^\rho,\dots,X_n^\rho) \arrow[u, "\cong"'] \end{tikzcd}$$

with maps according to Lemma 5.14. This last diagram obviously commutes, hence we are done. ☐

Lemma 5.16. *The square*

$$\begin{tikzcd} \bigoplus_{i=1}^{r'} H^d_{y_i}(\omega_V) \arrow[r] & H^d_c(\mathring{V},\omega_V) \\ \bigoplus_{i=1}^{r} H^d_{x_i}(\omega_U) \arrow[r] \arrow[u, "\oplus_i H^d_{x_i}(\tau)"] & H^d_c(\mathring{U},\omega_U) \arrow[u, "q_\alpha"'] \end{tikzcd}$$

from Diagram (5.21) commutes.

Proof. We expand the diagram so that it has three parts:

$$\begin{tikzcd}
\bigoplus_{i=1}^{r'} H^d_{y_i}(\omega_V) \arrow[r] & \bigoplus_{i=1}^{s'} H^d_{y_i}(\omega_V) \arrow[r] & H^d_c(\mathring{\mathcal{V}}, \omega_V) \arrow[r] & H^d_c(\mathring{V}, \omega_V) \\
\bigoplus_{i=1}^{r} H^d_{x_i}(\omega_U) \arrow[r] \arrow[u, "\oplus_i H^d_{x_i}(\tau)"] \arrow[ur, phantom, "(1)"] & \bigoplus_{i=1}^{s} H^d_{x_i}(\omega_U) \arrow[r] \arrow[u] \arrow[ur, phantom, "(2)"] & H^d_c(\mathring{\mathcal{U}}, \omega_U) \arrow[r] \arrow[u, "q_\alpha"] \arrow[ur, phantom, "(3)"] & H^d_c(\mathring{U}, \omega_U) \arrow[u, "q_\alpha"]
\end{tikzcd}$$

where $\mathring{\mathcal{V}} := \pi_V^{-1}(\mathring{\mathbb{D}}^d)$ and $\mathring{\mathcal{U}} := \pi_V^{-1}(\mathring{\mathbb{D}}^d)$. Then (1) and (3) commute for obvious reasons, so it remains to prove that (2) commutes. Using Lemma 4.5 (i) and Remark 3.11, we can re-write (2) as

$$\begin{tikzcd}
H^d_0(\pi_{V*}\omega_V) \arrow[r] & H^d_c(\mathring{\mathbb{D}}^d, \pi_{V*}\omega_V) \\
H^d_0(\pi_{U*}\omega_U) \arrow[r] \arrow[u, "H^d_0(\pi_{V*}(\mathfrak{T}))"] & H^d_c(\mathring{\mathbb{D}}^d, \pi_{U*}\omega_U). \arrow[u, "H^d_c(\pi_{V*}(\mathfrak{T}))"']
\end{tikzcd}$$

This diagram commutes by the functoriality of the horizontal maps, so we are done. □

5.4.3 Conclusion of the proof

Proof of Theorem 5.12. We again consider the diagram (5.21):

$$\begin{tikzcd}
\bigoplus_{i=1}^{r} H^d_{y_i}(\omega_V) \arrow[r] \arrow[rrd, bend left, "\sum \mathrm{res}_{y_i}"] & H^d_c(\mathring{V}, \omega_V) \arrow[rd, "t_{\mathring{V}}"] & \\
 & & K. \\
\bigoplus_{i=1}^{r} H^d_{x_i}(\omega_U) \arrow[r, "\text{dense}"] \arrow[uu, "\oplus_i H^d_{x_i}(\tau)"] \arrow[rru, bend right, "\sum \mathrm{res}_{x_i}"'] & H^d_c(\mathring{U}, \omega_U) \arrow[uu, "q_\alpha"] \arrow[ru, "t_{\mathring{U}}"'] \arrow[u, phantom, "?", shift left=5ex] &
\end{tikzcd}$$

The objective is to show that the triangle with the question mark commutes. Since the lower horizontal map (call it η) has dense image, it suffices to show that

$$t_{\mathring{U}} \circ \eta = t_{\mathring{V}} \circ q_\alpha \circ \eta. \tag{5.24}$$

By the commutativity of the lower "slice" in the diagram, we have $t_{\mathring{U}} \circ \eta = \sum \mathrm{res}_{x_i}$. On the other hand, $t_{\mathring{V}} \circ q_\alpha \circ \eta$ is equal to the composite of $\sum \mathrm{res}_{y_i}$ with $\oplus_i H^d_{x_i}(\tau)$ by Lemma 5.16 and the commutativity of the upper "slice". But the composite of $\sum \mathrm{res}_{y_i}$ with $\oplus_i H^d_{x_i}(\tau)$ also coincides with $\sum \mathrm{res}_{x_i}$ by Lemma 5.15. Hence both sides of (5.24) are equal to $\sum \mathrm{res}_{x_i}$, which completes the proof of Theorem 5.12 and thus the proof of Theorem 5.5. $\square$

5.5 Some consequences

Let $\alpha\colon X \longrightarrow Y$ be a finite étale morphism of smooth d-dimensional Stein spaces over K and let $\mathcal{G}$ be a coherent sheaf on Y. Denote the adjunction morphism $\mathcal{G} \longrightarrow \alpha_*\alpha^*\mathcal{G}$ by ξ_α and let $(-)^\vee = \mathrm{Hom}^{cont}_K(-, K)$ denote the continuous dual. An easy consequence of Theorem 5.5 is:

Proposition 5.17. *The diagram*

$$\begin{tikzcd}
H^{d-i}_c(X, \alpha^*\mathcal{G})^\vee \arrow[r, "\sim"] \arrow[ddd] & \mathrm{Ext}^i_X(\alpha^*\mathcal{G}, \omega_X) \arrow[d, "f \longmapsto \alpha_*(f)"] \\
 & \mathrm{Ext}^i_Y(\alpha_*\alpha^*\mathcal{G}, \alpha_*\omega_X) \arrow[d, "\mathrm{Ext}^i_Y(\alpha_*\alpha^*\mathcal{G}, t_\alpha)"] \\
 & \mathrm{Ext}^i_Y(\alpha_*\alpha^*\mathcal{G}, \omega_Y) \arrow[d, "\mathrm{Ext}^i_Y(\xi_\alpha, \omega_Y)"] \\
H^{d-i}_c(Y, \mathcal{G})^\vee \arrow[r, "\sim"] & \mathrm{Ext}^i_Y(\mathcal{G}, \omega_Y)
\end{tikzcd}$$

commutes for all $i \geq 0$, where the horizontal isomorphisms come from the Serre duality pairing.

Proof. For the convenience of the reader, we repeat the argument from the proof of [SV23, Lemma 4.2.8]. First consider the diagram

$$\begin{tikzcd}
H_c^{d-i}(X,\alpha^*\mathcal{G}) & \times & \mathrm{Ext}^i_X(\alpha^*\mathcal{G},\omega_X) \arrow[r] \arrow[d, "f\longrightarrow\alpha_*(f)"] & H^d_c(X,\omega_X) & {} \arrow[d, phantom, "(1)"] \\
H_c^{d-i}(Y,\alpha_*\alpha^*\mathcal{G}) \arrow[u, "\cong"] \arrow[d, equal] & \times & \mathrm{Ext}^i_Y(\alpha_*\alpha^*\mathcal{G},\alpha_*\omega_X) \arrow[r] \arrow[d, "\mathrm{Ext}^i_Y(\alpha_*\alpha^*\mathcal{G},t_\alpha)"] & H^d_c(Y,\alpha_*\omega_X) \arrow[u, "\cong"] \arrow[d, "H^d_c(Y,t_\alpha)"] & {} \arrow[d, phantom, "(2)"] \\
H_c^{d-i}(Y,\alpha_*\alpha^*\mathcal{G}) & \times & \mathrm{Ext}^i_Y(\alpha_*\alpha^*\mathcal{G},\omega_Y) \arrow[r] \arrow[d, "\mathrm{Ext}^i_Y(\xi_\alpha,\omega_Y)"] & H^d_c(Y,\omega_Y) \arrow[d, equal] & {} \arrow[d, phantom, "(3)"] \\
H_c^{d-i}(Y,\mathcal{G}) \arrow[u, "H_c^{d-i}(Y,\xi_\alpha)"] & \times & \mathrm{Ext}^i_Y(\mathcal{G},\omega_Y) \arrow[r] & H^d_c(Y,\omega_Y) & {}
\end{tikzcd}$$

The lower part (3) commutes due to the naturality of the Yoneda-Cartier pairing in the coherent sheaf. The commutativity of the Yoneda-Cartier pairings in the upper two parts (1) and (2) is a consequence of functoriality properties. Expanding this diagram as follows

$$\begin{tikzcd}
H_c^{d-i}(X,\alpha^*\mathcal{G}) & \times & \mathrm{Ext}^i_X(\alpha^*\mathcal{G},\omega_X) \arrow[r] \arrow[d, "f\longrightarrow\alpha_*(f)"] & H^d_c(X,\omega_X) \arrow[r, "t_X"] & K \arrow[dd, equal] \\
H_c^{d-i}(Y,\alpha_*\alpha^*\mathcal{G}) \arrow[u, "\cong"] \arrow[d, equal] & \times & \mathrm{Ext}^i_Y(\alpha_*\alpha^*\mathcal{G},\alpha_*\omega_X) \arrow[r] \arrow[d, "\mathrm{Ext}^i_Y(\alpha_*\alpha^*\mathcal{G},t_\alpha)"] & H^d_c(Y,\alpha_*\omega_X) \arrow[u, "\cong"] \arrow[d, "H^d_c(Y,t_\alpha)"] & \\
H_c^{d-i}(Y,\alpha_*\alpha^*\mathcal{G}) & \times & \mathrm{Ext}^i_Y(\alpha_*\alpha^*\mathcal{G},\omega_Y) \arrow[r] \arrow[d, "\mathrm{Ext}^i_Y(\xi_\alpha,\omega_Y)"] & H^d_c(Y,\omega_Y) \arrow[d, equal] \arrow[r, "t_Y"] & K \arrow[d, equal] \\
H_c^{d-i}(Y,\mathcal{G}) \arrow[u, "H_c^{d-i}(Y,\xi_\alpha)"] & \times & \mathrm{Ext}^i_Y(\mathcal{G},\omega_Y) \arrow[r] & H^d_c(Y,\omega_Y) \arrow[r, "t_Y"] & K
\end{tikzcd}$$

and observing that the new part also commutes (due to Theorem 5.5), we obtain the commutativity of the diagram in the assertion. □

In the special case $\mathcal{G} = \mathcal{O}_Y$ we have $\alpha^*\mathcal{O}_Y = \mathcal{O}_X$ and hence $\mathrm{Hom}_X(\alpha^*\mathcal{O}_Y,\omega_X) = \mathrm{Hom}_X(\mathcal{O}_X,\omega_X) = \omega_X(X)$, so that the commutativity of the above diagram in particular yields:

Corollary 5.18. *The diagram*

$$\begin{tikzcd}
H^d_c(X,\mathcal{O}_X)^\vee \arrow[r, "\sim"] \arrow[d] & \omega_X(X) \arrow[d, "t_\alpha(Y)"] \\
H^d_c(Y,\mathcal{O}_Y)^\vee \arrow[r, "\sim"] & \omega_Y(Y)
\end{tikzcd}$$

commutes.

Note that the domain of $t_\alpha(Y)$ is indeed $\alpha_*\omega_X(Y) = \omega_X(X)$.

Chapter 6

Compatibility of the trace map with (completed) base change

In this chapter, we consider the following setting: Let K' be a complete field extension of K and, for any (separated) rigid space X over K, let

$$X' := X \widehat{\otimes}_K K'$$

denote the base change of X to K' as in [BGR84, §9.3.6]. If R is a K-affinoid algebra, we accordingly use the notation

$$R' := R \widehat{\otimes}_K K'.$$

Remark 6.1. *We recall a few well-known facts about base change, all of which are also recalled in [Con99, page 475]:*
One defines $\mathrm{Sp}(R) \widehat{\otimes}_K K' := \mathrm{Sp}(R \widehat{\otimes}_K K')$ *for any K-affinoid algebra R, and in order to globalize this to more general rigid spaces X, one uses that the overlap of any two open affinoids in X is quasi-compact (i.e. one uses the quasi-separatedness of X).*
If K' is infinite over K, then the base change functor can't be realised as a fiber froduct and we don't have a morphism $X' \longrightarrow X$. Nevertheless, the base change functor is always compatible with the formation of fiber products and takes closed immersions to closed immersions.
Furthermore, even though we in general don't have a morphism $X' \longrightarrow X$, there is an exact "pullback" functor

$$\mathcal{F} \rightsquigarrow \mathcal{F}'$$

from coherent sheaves on X to coherent sheaves on X', defined in the obvious way over affinoids: If $\mathcal{F}$ is a coherent sheaf on $X = \mathrm{Sp}(R)$ and M the finitely generated R-module associated to $\mathcal{F}$, then $\mathcal{F}'$ is the coherent sheaf on X' that is associated to the finitely generated R'-module

$$M' := K' \widehat{\otimes}_K M = R' \widehat{\otimes}_R M = R' \otimes_R M.$$

where the first equality is obtained by the associativity of the completed tensor product [BGR84, 2.1.7/Proposition 7], and the second equality is due to Lemma 6.7 (ii) below.

6.1 A brief recollection on completed tensor products

For any locally convex K-vector space V, the *Hausdorff* (or *separated*) *completion* of V is the unique (up to a unique topological isomorphism) complete Hausdorff locally convex K-vector space $\widehat{V}$ together with a continuous map $V \longrightarrow \widehat{V}$ that is universal for continuous K-linear maps from V to complete Hausdorff locally convex K-vector spaces. The map $V \longrightarrow \widehat{V}$ has dense image in $\widehat{V}$ and induces a topological isomorphism between $V/\overline{\{0\}}$ and the image, see [Sch13, Proposition 7.5].

When discussing the tensor product $V \otimes_K W$ of locally convex K-vector spaces V and W, we always mean the *projective tensor product*, i.e. we consider $V \otimes_K W$ equipped with the *projective tensor product topology* [PS10, Definition 10.3.2]. This is the strongest locally convex topology on $V \otimes_K W$ for which the bilinear map $V \times W \longrightarrow V \otimes_K W$ is continuous. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are families of seminorms inducing the topology of V and W respectively, then the projective topology on $V \otimes_K W$ is induced by the family $\{p \otimes q \colon p \in \mathcal{P}_1, q \in \mathcal{P}_2\}$, where $p \otimes q$ acts on $z \in V \otimes_K W$ as

$$(p \otimes q)(z) := \inf\{\max_{1 \le i \le n} p(x_i)q(y_i) \colon n \in \mathbb{N}, z = \sum_{i=1}^{n} x_i \otimes y_i\}.$$

In particular, it follows immediately that the canonical isomorphism $K \otimes_K V \xrightarrow{\sim} V$ is topological.

The *completed tensor product* $V \mathbin{\widehat{\otimes}}_K W$ is the Hausdorff completion of the projective tensor product $V \otimes_K W$. In particular, we have a canonical topological isomorphism

$$K \mathbin{\widehat{\otimes}}_K V \xrightarrow{\sim} \widehat{V}. \tag{6.1}$$

Due to lack of an adequate reference, we quickly prove the following:

Remark 6.2. (i) *Hausdorff completion commutes with arbitrary direct sums of locally convex vector spaces (hence in particular with finite direct products).*

(ii) *The projective tensor product commutes with arbitrary locally convex direct sums (hence in particular with finite direct products).*

Proof. (i) Let $(V_i)_{i \in I}$ be a family of locally convex vector spaces. Since each $\widehat{V_i}$ is Hausdorff and complete, the sum $\bigoplus_i \widehat{V_i}$ is Hausdorff and complete by [Sch13, Corollary 5.4 and Lemma 7.8], and we have a natural map $\bigoplus_i V_i \longrightarrow \bigoplus_i \widehat{V_i}$.

Combining the universal property of the locally convex direct sum and the universal property of each $\widehat{V}_i$, it is straightforward to see that $\bigoplus_i \widehat{V}_i$ satisfies the universal property of $(\bigoplus_i V_i)^\wedge$.
The "hence in particular" part of the assertion now follows from the fact that, if the set I is finite, then the identity map $\bigoplus_i V_i \xrightarrow{\sim} \prod_i V_i$ is a topological isomorphism by [Sch13, Lemma 5.2 ii.].

(ii) Let V be a locally convex vector space and $(W_i)_{i\in I}$ a family of locally convex vector spaces. We need to argue that the isomorphism

$$V \otimes_K \bigoplus_i W_i \xrightarrow{\sim} \bigoplus_i V \otimes_K W_i \tag{6.2}$$

is topological. Let ι_j denote the canonical map $W_j \longrightarrow \bigoplus_i W_i$. By functoriality of the projective tensor product, the maps $\mathrm{id}_V \otimes \iota_j$ are continuous, hence the universal property of $\bigoplus_i V \otimes_K W_i$ yields a continuous map $\bigoplus_i V \otimes_K W_i \longrightarrow V \otimes_K \bigoplus_i W_i$, which is the inverse of the map (6.2). The continuity of the map $\bigoplus_i V \otimes_K W_i \longrightarrow V \otimes_K \bigoplus_i W_i$ means that the projective tensor product topology on $V \otimes_K \bigoplus_i W_i$ is coarser than the locally convex direct sum topology (transported onto $V \otimes_K \bigoplus_i W_i$ via the bijection (6.2)). On the other hand, the locally convex direct sum topology on $V \otimes_K \bigoplus_i W_i$ is the coarsest one for which the maps $\mathrm{id}_V \otimes \iota_j$ are all continuous. Since these are continuous for the projective tensor product topology, we conclude that the projective tensor product topology on $V \otimes_K \bigoplus_i W_i$ is finer than the locally convex direct sum topology, completing the proof of the assertion.

□

Corollary 6.3. (i) *The completed tensor product commutes with arbitrary direct sums of locally convex vector spaces (hence in particular with finite direct products).*

(ii) *If K' is a finite extension field of K and V is Hausdorff and complete, then the natural map $K' \otimes_K V \xrightarrow{\sim} K' \widehat{\otimes}_K V$ is an isomorphism.*

Proof. Assertion (i) is obtained by combining both assertions of Remark 6.2. On the other hand, since K' is topologically isomorphic to a finite direct product of copies of K, assertion (ii) follows immediately from Assertion (i) and (6.1) above, bearing in mind that $V = \widehat{V}$. □

Remark 6.4. *In general, we have*

$$H^j_c(X, \mathcal{F}) \,\widehat{\otimes}_K\, K' \ncong H^j_c(X', \mathcal{F}').$$

(even when K' is finite over K), due to the fact that the left-hand side is Hausdorff whereas the right-hand side can be a non-Hausdorff space, cf. [Bos21, Remark 1.11].

However, for a special affinoid wide-open space (resp. a Stein space) S over K, we discuss comparison maps

$$H_c^j(S,\mathcal{F}) \mathbin{\widehat{\otimes}}_K K' \longrightarrow H_c^j(S',\mathcal{F}')^{\wedge}$$

in Section 6.2 (resp. Section 6.4).

6.2 Comparison maps for base change of compactly supported cohomology

It is well-known that the class of smooth morphisms of rigid spaces is stable under base change. Moreover, the dimension of a rigid space is invariant under base change by [Con99, Lemma 2.1.5]. The following remark recalls that the class of (quasi-)Stein spaces is also stable under base change.

Remark 6.5. *Let S be a quasi-Stein Space. Then the base change S' of S to K' is quasi-Stein as well. If S is moreover Stein, then so is S'.*

Proof. Let $\{D_i\}_{i\in\mathbb{N}}$ be a cover of S as in Definition 1.32 and let D_i' denote the base change of D_i to K'. Then $\{D_i'\}_{i\in\mathbb{N}}$ is an admissible affinoid cover of S' and the maps

$$\mathcal{O}_X(D_{i+1}') = \mathcal{O}_X(D_{i+1}) \mathbin{\widehat{\otimes}}_K K' \longrightarrow \mathcal{O}_X(D_i) \mathbin{\widehat{\otimes}}_K K' = \mathcal{O}_X(D_i')$$

have dense image, so S' is quasi-Stein. If S is moreover Stein, i.e. $D_i \Subset D_{i+1}$ then also $D_i' \Subset D_{i+1}'$ because the characterisation $(*)$ in Definition 1.33 of relative compactness is obviously stable under base change. Hence S' is Stein as well. □

Proposition 6.6. *Let Z be a connected smooth affinoid space, $\mathcal{F}$ a coherent sheaf on Z. Let $\mathring{W} \subseteq Z$ be a special affinoid wide-open space with associated finite surjective morphism*

$$\pi\colon Z \longrightarrow \mathbb{D}_K^n$$

whose restriction

$$\varpi\colon \mathring{W} \longrightarrow \mathring{\mathbb{D}}_K^n$$

we denote by ϖ. Let $\varepsilon \in (0,1)$ and set

$$S = \mathring{W}, \quad V = \varpi^{-1}(\mathbb{D}_K^n(\varepsilon)) \;\text{ and }\; X = S \setminus V.$$

Then:

(i) *We have $V' = \varpi'^{-1}(\mathbb{D}_{K'}^n(\varepsilon))$ and*

$$X' = S' \setminus V'. \tag{6.3}$$

(ii) *There is a natural map*

$$H^j_V(S,\mathcal{F}) \otimes_K K' \longrightarrow H^j_{V'}(S',\mathcal{F}') \tag{6.4}$$

for all $j \geq 0$.

(iii) *Taking* $\varinjlim_\varepsilon$ *in (6.4) yields a natural map*

$$H^j_c(S,\mathcal{F}) \otimes_K K' \longrightarrow H^j_c(S',\mathcal{F}') \tag{6.5}$$

which induces a map on completions

$$H^j_c(S,\mathcal{F}) \mathbin{\widehat{\otimes}}_K K' \longrightarrow H^j_c(S',\mathcal{F}')^\wedge. \tag{6.6}$$

(iv) *If* K' *is moreover finite over* K*, then all three maps (6.4), (6.5) and (6.6) are isomorphisms.*

Proof. (i) Since extension of scalars is compatible with the formation of fiber products, it is in particular compatible with taking preimages under morphisms. Applying this to ϖ, we have

$$V' = (\varpi^{-1}(\mathbb{D}^n_K(\varepsilon)))' = \varpi'^{-1}(\mathbb{D}^n_{K'}(\varepsilon))$$

and

$$X' = (\varpi^{-1}(\mathring{\mathbb{D}}^n_K \setminus \mathbb{D}^n_K(\varepsilon)))' = \varpi'^{-1}((\mathring{\mathbb{D}}^n_K \setminus \mathbb{D}^n_K(\varepsilon))').$$

Next, we claim that

$$(\mathring{\mathbb{D}}^n_K \setminus \mathbb{D}^n_K(\varepsilon))' = \mathring{\mathbb{D}}^n_{K'} \setminus \mathbb{D}^n_{K'}(\varepsilon). \tag{6.7}$$

To see this, note that

$$\mathring{\mathbb{D}}^n_K \setminus \mathbb{D}^n_K(\varepsilon) = \bigcup_{i,\delta} U_{K,i,\delta} \quad \text{with} \quad U_{K,i,\delta} := \{x \in \mathring{\mathbb{D}}^n_K \colon \delta \leq |x_i| \leq 1-\delta\}$$

where i runs through $1,\dots,n$ and δ runs through a zero sequence. Due to how the base change functor is defined, $(\mathring{\mathbb{D}}^n_K \setminus \mathbb{D}^n_K(\varepsilon))'$ is obtained by gluing the $(U_{K,i,\delta})'$. But $(U_{K,i,\delta})' = U_{K',i,\delta}$, whence (6.7) follows. Altogether, we have

$$X' = \varpi'^{-1}(\mathring{\mathbb{D}}^n_{K'} \setminus \mathbb{D}^n_{K'}(\varepsilon)) = S' \setminus \varpi'^{-1}(\mathbb{D}^n_{K'}(\varepsilon)) = S' \setminus V',$$

which proves (6.3).

(ii) By definition, we have $H^0_V(S,\mathcal{F}) = \ker(H^0(S,\mathcal{F}) \longrightarrow H^0(X,\mathcal{F}))$ and $H^0_{V'}(S',\mathcal{F}') = \ker(H^0(S',\mathcal{F}') \longrightarrow H^0(X',\mathcal{F}'))$, where the latter equality uses (6.3). For any rigid space Y and any coherent sheaf $\mathcal{G}$ ony Y, we have the natural map

$$H^0(Y,\mathcal{G}) \otimes_K K' \longrightarrow H^0(Y,\mathcal{G}) \mathbin{\widehat{\otimes}}_K K' = H^0(Y',\mathcal{G}')$$

where the last equality is due to the definition of $\mathcal{G}'$. This yields the vertical maps in the commutative diagram

$$\begin{tikzcd} H^0(S',\mathcal{F}') \arrow[r] & H^0(X',\mathcal{F}') \\ H^0(S,\mathcal{F})\otimes_K K' \arrow[r] \arrow[u] & H^0(X,\mathcal{F})\otimes_K K'. \arrow[u] \end{tikzcd} \tag{6.8}$$

By restricting the left vertical map to the kernel of the lower horizontal map (which coincides with $H^0_V(S,\mathcal{F})\otimes_K K'$ due to the flatness of $K \longrightarrow K'$) and observing that this map then lands in the kernel of the upper horizontal map, we obtain the desired map (6.4) for $j=0$. For $j \geq 1$ we can apply (6.3) and Remark 1.49 to see that it is equivalent to give a natural map

$$H^{j-1}(X,\mathcal{F})\otimes_K K' \longrightarrow H^{j-1}(X',\mathcal{F}'). \tag{6.9}$$

To construct the map (6.9), we imitate the proof of [Stacks, Tag 02KH], which calls for a finite Leray covering of X. To see that there exists a finite Leray covering of X, first note that ϖ is a finite morphism. Indeed, since $\mathring{W}$ is a union of connected components of $\pi^{-1}(\mathring{\mathbb{D}}^n_K)$ and hence a clopen subspace, the inclusion $\mathring{W} \hookrightarrow \pi^{-1}(\mathring{\mathbb{D}}^n_K)$ is a closed immersion and in particular a finite map, whence its composite with the finite map $\pi^{-1}(\mathring{\mathbb{D}}^n_K) \longrightarrow \mathring{\mathbb{D}}^n_K$ is also finite, i.e. $\varpi\colon \mathring{W} \longrightarrow \mathring{\mathbb{D}}^n_K$ is finite. Now, if we let $\mathfrak{W}$ be the finite Leray cover of $\mathring{\mathbb{D}}^n_K\backslash\mathbb{D}^n_K(\varepsilon)$ defined in (1.12), then $\mathfrak{U} := \varpi^{-1}\mathfrak{W}$ is a finite cover of X which is a Leray cover due to Remark 1.46. Next, the definition of $\mathcal{F}'$ and the fact that the completed tensor product commutes with finite direct products yields the following relation between Čech complexes

$$\check{C}^{j-1}(\mathfrak{U},\mathcal{F})\,\widehat{\otimes}_K K' = \check{C}^{j-1}(\mathfrak{U}',\mathcal{F}'). \tag{6.10}$$

By precomposing with the map

$$\check{C}^{j-1}(\mathfrak{U},\mathcal{F})\otimes_K K' \longrightarrow \check{C}^{j-1}(\mathfrak{U},\mathcal{F})\,\widehat{\otimes}_K K' \tag{6.11}$$

we obtain a natural map

$$\check{C}^{j-1}(\mathfrak{U},\mathcal{F})\otimes_K K' \longrightarrow \check{C}^{j-1}(\mathfrak{U}',\mathcal{F}'). \tag{6.12}$$

The cohomology of the left-hand side in (6.12) is $H^{j-1}(\mathfrak{U},\mathcal{F})\otimes_K K'$ (because $K \longrightarrow K'$ is flat), so taking cohomology in (6.12) yields the desired natural map

$$H^{j-1}(\mathfrak{U},\mathcal{F})\otimes_K K' \longrightarrow H^{j-1}(\mathfrak{U}',\mathcal{F}'). \tag{6.13}$$

(iii) We only need to show that the V (resp. the V'), for varying $0<\varepsilon<1$, form a cofinal subfamily of the family of all finite unions of admissible affinoids in $\mathring{\mathbb{D}}^n_K$ (resp. in $\mathring{\mathbb{D}}^n_{K'}$). But this follows from Remark 1.52 via Lemma 1.50, bearing in mind that $V' = \varpi'^{-1}(\mathbb{D}^n_{K'}(\varepsilon))$ by (i).

(iv) Now assume that K' is finite over K. Obviously, it suffices to show that the map (6.4) is an isomorphism, or, equivalently, that (6.9) is an isomorphism, or, equivalently, that (6.13) is an isomorphism. For this, note that the map (6.11) is an isomorphism due to Corollary 6.3 (ii) since $\check{C}^{j-1}(\mathfrak{U}, \mathcal{F})$ is a complex of Fréchet spaces (which are in particular Hausdorff and complete). Hence the map (6.12) is also an isomorphism, which, by passage to cohomology, yields that (6.13) is now an isomorphism. This settles the assertion for $j \geq 1$. The case $j = 0$ follows from the fact that both vertical maps in the diagram (6.8) are now isomorphisms due to Corollary 6.3 (ii).

□

6.3 Base-changing separable Noether normalisation

We prove a number of preparatory results that we will need in order to obtain base change results for the trace map.

Lemma 6.7 ([Con99, Lemma 1.1.5])**.** *Let R be an affinoid algebra over K and (as always) $R' = K' \widehat{\otimes}_K R$. Then:*

(i) *The natural map $R \longrightarrow R'$ is faithfully flat.*

(ii) *For any finite R-module M, the natural map*

$$R' \otimes_R M \xrightarrow{\sim} R' \widehat{\otimes}_R M \tag{6.14}$$

is a topological isomorphism.

(iii) *For any maximal ideal $\mathfrak{m} \subseteq R$, there exist prime ideals in R' over $\mathfrak{m}$. These prime ideals are all maximal.*

Corollary 6.8. *If $R \longrightarrow S$ is a finite (resp. finite injective) morphism of affinoid K-algebras, then the base change $R' = K' \widehat{\otimes}_K R \longrightarrow K' \widehat{\otimes}_K S = S'$ is also finite (resp. finite injective).*

Proof. Note that the map $R' = K' \widehat{\otimes}_K R \longrightarrow K' \widehat{\otimes}_K S = S'$ is identified with the map $R' \widehat{\otimes}_R R \longrightarrow R' \widehat{\otimes}_R S$ by the associativity of the completed tensor product [BGR84, 2.1.7/Proposition 7]. Next, (6.14) in Lemma 6.7 tells us that this latter map arises by viewing $R \longrightarrow S$ as a map between finite R-modules and applying $R' \otimes_R (-)$ to it. But $R' \otimes_R (-)$ preserves finiteness (and injectivity too, since $R \longrightarrow R'$ is flat by Lemma 6.7 (i)), so we are done. □

The main result of this section is the stability of separable Noether normalisation morphisms under base change:

Lemma 6.9. *Let $Z = \mathrm{Sp}(R)$ be a connected smooth affinoid space and*

$$\varphi\colon T_d \hookrightarrow R$$

a finite injective separable morphism. Then the morphism

$$\varphi'\colon T_d(K') \hookrightarrow R'$$

obtained by base change to K' is finite, injective and separable.

Proof. The morphism $\varphi'\colon T_d(K') \hookrightarrow R'$ obtained by base change to K' is finite injective by Corollary 6.8. Since R is an integral domain, R has pure dimension d, so R' also has pure dimension d by [Bos70, Lemma 2.5]. Since R' is moreover reduced, Lemma 1.27 tells us that φ' is torsion-free, so we have an induced morphism

$$Q(T_d(K')) \hookrightarrow Q(R').$$

We need to prove that this morphism is étale. Since $Q(R)$ is a separable field extension of $Q(T_d)$ by assumption, the morphism

$$Q(T_d) \hookrightarrow Q(R)$$

is étale. Tensoring with $Q(R)$, it follows by [Stacks, Tag 00U2 part (5)] that the structure morphism

$$Q(T_d') \hookrightarrow Q(T_d') \otimes_{Q(T_d)} Q(R)$$

is then étale too, where $T_d' := T_d(K')$. We will show that

$$Q(T_d') \otimes_{Q(T_d)} Q(R) \cong Q(R'), \tag{6.15}$$

whence $Q(R')$ is étale over $Q(T_d')$ and we are done.

As in the Proof of Corollary 6.8, we see that the map $T_d' \hookrightarrow R'$ arises by applying $T_d' \otimes_{T_d} (-)$ to the map $T_d \hookrightarrow R$, i.e. we have

$$R' = T_d' \otimes_{T_d} R.$$

With the multiplicative subset $S := T_d \setminus \{0\} \subseteq T_d$, we then have

$$\begin{aligned} S^{-1}R' &= S^{-1}(T_d') \otimes_{S^{-1}T_d} S^{-1}R \\ &= S^{-1}(T_d') \otimes_{Q(T_d)} Q(R) \end{aligned}$$

where the last equality holds because $S^{-1}T_d = Q(T_d)$ by definition and $S^{-1}R = Q(R)$ by Lemma 5.11. On the other hand, we also wish to apply Lemma 5.11 to the finite ring extension $T_d' \hookrightarrow R'$. This is possible since no element of $T_d' \setminus \{0\}$ is a zero divisor in R'. Indeed, this is due to R' having pure dimension d

and Lemma 1.27. Hence Lemma 5.11 tells us that $Q(R') = T^{-1}R'$ with the multiplicative set $T := T'_d \setminus \{0\}$. But obviously $T^{-1}R' = T^{-1}(S^{-1}R')$. Altogether, we have

$$\begin{aligned} Q(R') = T^{-1}(S^{-1}R') &= Q(T'_d) \otimes_{T'_d} S^{-1}R' \\ &= Q(T'_d) \otimes_{T'_d} (S^{-1}(T'_d) \otimes_{Q(T_d)} Q(R)) \\ &= (Q(T'_d) \otimes_{T'_d} S^{-1}(T'_d)) \otimes_{Q(T_d)} Q(R) \\ &= S^{-1}Q(T'_d) \otimes_{Q(T_d)} Q(R) \\ &= Q(T'_d) \otimes_{Q(T_d)} Q(R), \end{aligned}$$

where the last equality holds because $S^{-1}Q(T'_d) = Q(T'_d)$. This proves (6.15) and completes the proof of the lemma. □

Consider again the setting of Lemma 6.9. Since Z' is smooth, the local rings $\mathcal{O}_{Z',z'}$ are integral domains for all $z' \in Z'$. Then the proof of Remark 1.15 tells us that the irreducible components of Z' are pairwise disjoint and hence they coincide with the connected components of Z'. In other words, letting $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ denote the minimal prime ideals in R', we have

$$R' = \prod_{i=1}^{s} R'/\mathfrak{p}_i.$$

Since R is an integral domain, it has pure dimension d, so R' also has pure dimension d by [Bos70, Lemma 2.5]. Since R' is moreover reduced, Lemma 1.27 (or rather its proof) tells us that composing φ' with the projection $R' \twoheadrightarrow R'/\mathfrak{p}_i$ yields a finite *injective* map

$$\varphi'_i \colon T_d(K') \hookrightarrow R'/\mathfrak{p}_i.$$

for all i. Hence we have an induced finite field extension $Q(T_d(K')) \hookrightarrow Q(R'/\mathfrak{p}_i)$ for each i, and we wish to explain how Lemma 6.9 implies that these extensions are separable. The localisation of a reduced ring at a minimal prime ideal is a field by [Stacks, Tag 00EU], hence $R'_{\mathfrak{p}_i}$ is a field and the natural map $R'_{\mathfrak{p}_i} \xrightarrow{\sim} Q(R'/\mathfrak{p}_i)$ is an isomorphism. Moreover, $\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_s$ is the set of zero divisors in R' by [Stacks, Tag 00EW], so there are natural maps $Q(R') \longrightarrow R_{\mathfrak{p}_i}$ (since any non-zero-divisor is contained in $R \setminus \mathfrak{p}_i$), hence a natural map $Q(R') \longrightarrow R_{\mathfrak{p}_1} \times \ldots \times R_{\mathfrak{p}_s}$ which is in fact an isomorphism by [Stacks, Tag 02LX], so we have

$$Q(R') \xrightarrow{\sim} R_{\mathfrak{p}_1} \times \ldots \times R_{\mathfrak{p}_s} \xrightarrow{\sim} Q(R'/\mathfrak{p}_1) \times \ldots \times Q(R'/\mathfrak{p}_s). \tag{6.16}$$

In general, if F is a field and $A = A_1 \times \ldots \times A_n$ is a finite product of F-algebras, then A is étale over F if and only if each A_i is étale over F, see [Stacks, Tag 00U2 part (11)]. Therefore, Lemma 6.9 tells us (equivalently) that the finite field extensions induced by φ'_i

$$Q(T_d(K')) \hookrightarrow Q(R'/\mathfrak{p}_i)$$

are separable for all i.

6.4 Base change results for the trace map

Lemma 6.10 ([Con99, Lemma 3.1.1]). *Let X be a quasi-separated rigid space over K, $\iota\colon U \hookrightarrow X$ a Zariski-open set and $Y \hookrightarrow X$ a closed immersion whose underlying space in X is the complement of U. Then the base change ι' of ι to K' is an open immersion and the analytic set $Y' \hookrightarrow X'$ is the complement of U' in X'.*

Proposition 6.11. *Let $Z = \mathrm{Sp}(R)$ be a connected smooth affinoid space, $\mathring{W} \subseteq Z$ be a special affinoid wide-open space and $\pi\colon Z \longrightarrow \mathbb{D}^n_K$ an associated finite surjective separable morphism. Consider the morphism*

$$\pi'\colon Z' \longrightarrow \mathbb{D}^n_{K'}$$

obtained by base change to K'. Then:

(i) *π' is finite, surjective and separable.*

(ii) *Letting $Z'_1, \dots, Z'_s$ denote the connected components of Z', then the restriction $\pi'_i\colon Z'_i \longrightarrow \mathbb{D}^n_{K'}$ of π to Z'_i is finite, surjective and separable for each i.*

(iii) *$\mathring{W}' \subseteq Z'$ is a finite union of connected components of ${\pi'}^{-1}(\mathring{\mathbb{D}}^n_{K'})$.*

Proof. Assertion (i) follows from Lemma 6.9. Assertion (ii) follows from the discussion at the end of Section 6.3. It remains to prove (iii), i.e. that $\mathring{W}'$ embeds into Z' as a finite union of connected components of ${\pi'}^{-1}(\mathring{\mathbb{D}}^n_{K'})$. Setting $\mathring{Z} := \pi^{-1}(\mathring{\mathbb{D}}^n_K)$ and $(Z')^{\circ} := {\pi'}^{-1}(\mathring{\mathbb{D}}^n_{K'})$, we have $(\mathring{Z})' = (Z')^{\circ}$ since extension of scalars is compatible with taking preimages under morphisms. The space $\mathring{W}$ is a finite union of connected components of $\mathring{Z}$. If U is a connected component in $\mathring{Z}$ (so in particular it is clopen in $\mathring{Z}$), then we can apply Lemma 6.10 to the inclusion $U \hookrightarrow \mathring{Z}$ to deduce that $U' \longrightarrow (Z')^{\circ}$ is an open immersion too. On the other hand, since base change takes closed immersions to closed immersions, we see that $U' \longrightarrow (Z')^{\circ}$ is also a closed immersion, so U' is clopen in $(Z')^{\circ}$. Then Remark 1.14 (iii) tells us that U' is a union of connected components of $(Z')^{\circ}$, whence the assertion follows. □

Due to Proposition 6.11, we can construct a map

$$\sigma = \sigma_{\pi'}\colon \Omega^n_{R'/K'} \longrightarrow \Omega^n_{T'_n/K'}$$

by invoking the same formula as in Definition 4.9, but now using the trace $\mathrm{Tr}_{L'/E'}$ of the finite étale morphism

$$E' := Q(T'_n) \hookrightarrow L' := Q(R')$$

induced by $\pi'\colon Z' \longrightarrow \mathbb{D}^n_{K'}$.

Remark 6.12. *In the decomposition $L' = L'_1 \times \ldots \times L'_s$ according to (6.16), each L'_i is finite separable over E', so we also have the map*

$$\Omega^n_{L'/K'} = \bigoplus_{i=1}^{s} \Omega^n_{L'_i/K'} \xrightarrow{\sum_i \sigma_i} \Omega^n_{E'/K'}. \tag{6.17}$$

which is readily seen to coincide with σ.

As before, σ restricts to a map $\Omega^n_{R'/K'} \longrightarrow \Omega^n_{T'_n/K'}$ or, equivalently, a map

$$\sigma\colon \pi'_*\omega_{Z'} \longrightarrow \omega_{\mathbb{D}^n_{K'}}.$$

We write $\sigma = \sigma_{\pi'}$ when we want to stress the dependence on π'.

Remark 6.13. *Let R be an affinoid algebra over K. Then*

$$\Omega^n_{R'/K'} = \Omega^n_{R/K} \mathbin{\widehat{\otimes}}_K K'$$

for all n.

Proof. As in the proof of Corollary 6.8, we see that

$$\Omega^n_{R/K} \mathbin{\widehat{\otimes}}_K K' = \Omega^n_{R/K} \mathbin{\widehat{\otimes}}_R R' = \Omega^n_{R/K} \otimes_R R',$$

so it remains to see that $\Omega^n_{R/K} \otimes_R R' = \Omega^n_{R'/K'}$. For $n = 1$ this is the content of [Ber+67, Satz 2.5.2], whence it follows for all n since the tensor product commutes with exterior powers. ☐

Lemma 6.14. *The diagram*

$$\begin{tikzcd} \Omega^n_{R/K} \mathbin{\widehat{\otimes}}_K K' \arrow[r, "(\sigma_\pi)'"] \arrow[d, equal] & \Omega^n_{T'_d/K'} \\ \Omega^n_{R'/K'} \arrow[ur, "\sigma_{\pi'}"'] & \end{tikzcd}$$

commutes. In other words,

$$\begin{tikzcd} (\pi_*\omega_Z)' \arrow[r, "(\sigma_\pi)'"] \arrow[d, equal] & \Omega^n_{T'_d/K'} \\ \pi'_*\omega_{Z'} \arrow[ur, "\sigma_{\pi'}"'] & \end{tikzcd}$$

commutes.

Proof. We write $L = Q(R), E = Q(T_d)$ and $L' = Q(R'), E' = Q(T_d')$ for brevity. We can view $(\sigma_\pi)'$ as the composite[1]

$$\Omega^n_{R/K}\,\widehat{\otimes}_K\,K' \to \Omega^n_{L/K}\,\widehat{\otimes}_K\,K' = (L\otimes_E\Omega^n_{E/K})\,\widehat{\otimes}_K\,K' \to L\otimes_E\Omega^n_{E'/K'} \xrightarrow{\mathrm{Tr}_{L/E}\otimes\mathrm{id}} \Omega^n_{E'/K'}$$

and $\sigma_{\pi'}$ as the composite

$$\Omega^n_{R'/K'} \longrightarrow \Omega^n_{L'/K'} = L'\otimes_{E'}\Omega^n_{E'/K'} \xrightarrow{\mathrm{Tr}_{L'/E'}\otimes\mathrm{id}} \Omega^n_{E'/K'}$$

whence we need to show that the diagram

$$\begin{tikzcd}
\Omega^n_{R/K}\,\widehat{\otimes}_K\,K' \arrow[r] \arrow[d, equal] & \Omega^n_{L/K}\,\widehat{\otimes}_K\,K' \arrow[r, equal] & (L\otimes_E\Omega^n_{E/K})\,\widehat{\otimes}_K\,K' \arrow[r] & L\otimes_E\Omega^n_{E'/K'} \arrow[r, "\mathrm{Tr}_{L/E}\otimes\mathrm{id}"] & \Omega^n_{E'/K'} \\
\Omega^n_{R'/K'} \arrow[r] & \Omega^n_{L'/K'} \arrow[r, equal] & L'\otimes_{E'}\Omega^n_{E'/K'} \arrow[r, "\mathrm{Tr}_{L'/E'}\otimes\mathrm{id}"] & \Omega^n_{E'/K'} \arrow[ur, equal] &
\end{tikzcd}$$

commutes. Under the identification

$$L' = L\otimes_E E'$$

from the proof of Lemma 6.9, we certainly have $\mathrm{Tr}_{L'/E'} = \mathrm{Tr}_{L/E}\otimes\mathrm{id}_{E'}$. Thus we can replace the last arrow in the lower line of the above diagram with $L\otimes_E E'\otimes_{E'}\Omega^n_{E'/K'} \xrightarrow{\mathrm{Tr}_{L/E}\otimes\mathrm{id}_{E'}\otimes\mathrm{id}} \Omega^n_{E'/K'}$, whence we obtain the diagram

$$\begin{tikzcd}
\Omega^n_{R/K}\,\widehat{\otimes}_K\,K' \arrow[r] \arrow[d, equal] & \Omega^n_{L/K}\,\widehat{\otimes}_K\,K' \arrow[r, equal] & (L\otimes_E\Omega^n_{E/K})\,\widehat{\otimes}_K\,K' \arrow[r] & L\otimes_E\Omega^n_{E'/K'} \arrow[r, "\mathrm{Tr}_{L/E}\otimes\mathrm{id}"] & \Omega^n_{E'/K'} \arrow[d, equal] \\
\Omega^n_{R'/K'} \arrow[r] & \Omega^n_{L'/K'} \arrow[r, equal] & L\otimes_E E'\otimes_{E'}\Omega^n_{E'/K'} \arrow[rr, "\mathrm{Tr}_{L/E}\otimes\mathrm{id}_{E'}\otimes\mathrm{id}"] & & \Omega^n_{E'/K'}
\end{tikzcd}$$

which now obviously commutes, so we are done. $\square$

In the setting of Proposition 6.11, we can use $\sigma_{\pi'}$ to define the trace map

$$t_{\mathring{W}'} = t_{\pi'}\colon H^n_c(\mathring{W}', \omega_{Z'}) \longrightarrow K'$$

exactly as in Construction 4.10 (even though $\mathring{W}'$ is not necessarily connected). Letting $Z_1', \ldots, Z_s'$ denote the connected components of Z', $\mathring{Z}_i' := \pi_i'^{-1}(\mathring{\mathbb{D}}_{K'})$ and $\mathring{W}_i'$

[1] One obtains the third map in the composite by taking the completion of the map $(L\otimes_E\Omega^n_{E/K})\otimes_K K' \longrightarrow L\otimes_E\Omega^n_{E'/K'}$ and observing that $L\,\widehat{\otimes}_E\,\Omega^n_{E'/K'} = L\otimes_E\Omega^n_{E'/K'}$ since both factors in the tensor product are finite over E.

denote the union of those connected components of $\mathring{Z}'$ that are contained in $\mathring{Z}'_i$, we have the commutative diagram

$$\begin{tikzcd}
H^n_c(\mathring{W}', \omega_{Z'}) \arrow[r] \arrow[d, equal] & H^n_c(\mathring{Z}', \omega_{Z'}) \arrow[r, "\sim"] & H^n_c(\mathring{\mathbb{D}}^n_{K'}, \pi'_*\omega_{Z'}) \arrow[r, "H^n_c(\sigma_{\pi'})"] \arrow[d, equal] & H^n_c(\mathring{\mathbb{D}}^n_{K'}, \omega_{\mathring{\mathbb{D}}^n_{K'}}) \\
\bigoplus_{i=1}^{s} H^n_c(\mathring{W}'_i, \omega_{Z'}) \arrow[r] & \bigoplus_{i=1}^{s} H^n_c(\mathring{Z}'_i, \omega_{Z'}) \arrow[r, "\sim"] & \bigoplus_{i=1}^{s} H^n_c(\mathring{\mathbb{D}}^n_{K'}, \pi'_{i*}\omega_{Z'}) \arrow[ur, bend right, "H^n_c(\sigma_{\pi'_i})"'] &
\end{tikzcd}$$

which tells us that $t_{\pi'} = \sum_i t_{\pi'_i}$ and in particular shows that $t_{\pi'}$ does not depend on π' (since we know that each $t_{\pi'_i}$ does not depend on π'_i).

We are now ready to prove the main results of this chapter, which concern the compatibility of the trace map with (completed) base change, first for special affinoid wide-opens and then for Stein spaces in general:

Proposition 6.15. *Let $Z = \mathrm{Sp}(R)$ be a connected smooth affinoid space of dimension n, $\mathring{W} \subseteq Z$ a special affinoid wide-open space. Then the diagram*

$$\begin{tikzcd}
H^n_c(\mathring{W}', \omega_{Z'}) \arrow[dr, "t_{\mathring{W}'}"] & \\
& K' \\
K' \otimes_K H^n_c(\mathring{W}, \omega_Z) \arrow[uu] \arrow[ur, "id_{K'} \otimes t_{\mathring{W}}"'] &
\end{tikzcd} \tag{6.18}$$

commutes. Therefore, taking completions, we find that

$$\begin{tikzcd}
H^n_c(\mathring{W}', \omega_{Z'})^\wedge \arrow[dr, "t_{\mathring{W}'}"] & \\
& K' \\
K' \,\widehat{\otimes}_K\, H^n_c(\mathring{W}, \omega_Z) \arrow[uu] \arrow[ur, "(t_{\mathring{W}})'"'] &
\end{tikzcd}$$

commutes.

Proof. First of all, in the special case $\mathring{W} = \mathring{\mathbb{D}}^n_K, Z = \mathbb{D}^n_K$ with coordinates $X = (X_1, \ldots, X_n)$, we have $H^n_c(\mathring{\mathbb{D}}^n, \omega_{\mathbb{D}^n}) \cong K\langle X^{-1}\rangle^\dagger \cdot \frac{dX}{X}$. In particular, $H^n_c(\mathring{\mathbb{D}}^n_{K'}, \omega_{\mathbb{D}^n_{K'}})$

is Hausdorff and complete and the diagram (6.18) in the assertion is just the diagram

$$\begin{tikzcd} K'\langle X^{-1}\rangle^\dagger \cdot \frac{dX}{X} \arrow[dr, "(\mathrm{res}_{K'})"] & \\ & K' \\ K' \otimes_K (K\langle X^{-1}\rangle^\dagger \cdot \frac{dX}{X}) \arrow[uu] \arrow[ur, "\mathrm{id}_{K'}\otimes \mathrm{res}_K"'] & \end{tikzcd}$$

which obviously commutes. For a general special affinoid wide-open space $\mathring{W} \subseteq Z$ and an associated finite surjective separable morphism $\pi\colon Z \longrightarrow \mathbb{D}^n_K$, we have to show that the outer contour of the diagram

$$\begin{tikzcd} H^n_c(\mathring{W}', \omega_{Z'}) \arrow[r] & H^n_c(\mathring{Z}', \omega_{Z'}) \arrow[r, "\sim"] & H^n_c(\mathring{\mathbb{D}}^n_{K'}, \pi'_*\omega_{Z'}) \arrow[r, "\sigma_{\pi'}"] & H^n_c(\mathring{\mathbb{D}}^n_{K'}, \omega_{\mathring{\mathbb{D}}^n_{K'}}) \arrow[dr, "\mathrm{res}_{K'}"] & \\ & & & & K' \\ H^n_c(\mathring{W}, \omega_Z)_{K'} \arrow[r] \arrow[uu] & H^n_c(\mathring{Z}, \omega_Z)_{K'} \arrow[r, "\sim"] \arrow[uu] & H^n_c(\mathring{\mathbb{D}}^n_K, \pi_*\omega_Z)_{K'} \arrow[r, "(\sigma_\pi)'"] \arrow[uu] & H^n_c(\mathring{\mathbb{D}}^n_K, \omega_{\mathring{\mathbb{D}}^n_K})_{K'} \arrow[uu] \arrow[ur, "(\mathrm{res}_K)'"'] & \end{tikzcd}$$

commutes, where we have written $(-)_{K'}$ instead of $K' \otimes_K (-)$ in the lower line, for ease of notation. The left-hand square certainly commutes, since the first vertical map (from the left) is just the restriction of the second map to a direct summand. Next, we claim that the middle square commutes. To see this, we first consider any given $\varepsilon \in (0,1)$ and let $\mathfrak{W}$ be the finite Leray cover of $\mathring{\mathbb{D}}^n_K \setminus \mathbb{D}^n_K(\varepsilon)$ defined in (1.12). Then we recall from (the proofs of) Proposition 6.6, Proposition 1.51 and Proposition 1.44 that the middle square is obtained by taking $\varinjlim_\varepsilon$ of the diagrams

$$\begin{tikzcd} H^{n-1}(\pi'^{-1}\mathfrak{W}', \omega_{Z'}) \arrow[r, equal] & H^{n-1}(\mathfrak{W}', \pi'_*\omega_{Z'}) \\ H^{n-1}(\pi^{-1}\mathfrak{W}, \omega_Z) \otimes_K K' \arrow[r, equal] \arrow[u] & H^{n-1}(\mathfrak{W}, \pi_*\omega_Z) \otimes_K K' \arrow[u] \end{tikzcd}$$

which obviously commute, whence our claim follows. Similarly, the commutativity of the right-hand square above comes down to the commutativity of

$$\begin{tikzcd} \check{C}^{n-1}(\mathfrak{W}', \pi'_*\omega_{Z'}) \arrow[r, "\sigma_{\pi'}"] & \check{C}^{n-1}(\mathfrak{W}', \omega_{\mathring{\mathbb{D}}^n_{K'}}) \\ \check{C}^{n-1}(\mathfrak{W}, \pi_*\omega_Z) \otimes_K K' \arrow[r, "\sigma_\pi \otimes \mathrm{id}"] \arrow[u] & \check{C}^{n-1}(\mathfrak{W}, \omega_{\mathring{\mathbb{D}}^n_K}) \otimes_K K' \arrow[u] \end{tikzcd}$$

which follows from Lemma 6.14. Finally, the right-hand triangle commutes by what we have already discussed. The assertion follows. □

Theorem 6.16. *Let X be a connected smooth Stein space of dimension n. Choosing an admissible open cover $\{\mathring{W}_i\}_{i\in\mathbb{N}}$ of X consisting of special affinoid wide-open subsets (as in Lemma 4.11) and taking the limit over all the maps $K' \otimes_K H_c^n(\mathring{W}_i, \omega_X) \to H_c^n(\mathring{W}_i', \omega_{X'})$ yields a map*

$$K' \otimes_K H_c^n(X, \omega_X) \to H_c^n(X', \omega_{X'}) \tag{6.19}$$

which is in fact canonical and makes the diagram

$$\begin{tikzcd}
H_c^n(X', \omega_{X'}) \arrow[dr, "t_{X'}"] & \\
& K' \\
K' \otimes_K H_c^n(X, \omega_X) \arrow[uu] \arrow[ur, "id\otimes t_X"'] &
\end{tikzcd}$$

commute. Taking completions, we find that

$$\begin{tikzcd}
H_c^n(X', \omega_{X'})^{\wedge} \arrow[dr, "t_{X'}"] & \\
& K' \\
K' \widehat{\otimes}_K H_c^n(X, \omega_X) \arrow[uu] \arrow[ur, "(t_X)'"'] &
\end{tikzcd}$$

commutes.

Proof. First of all, we note that the isomorphism

$$K' \otimes_K \varinjlim_i H_c^n(\mathring{W}_i, \omega_X) \cong \varinjlim_i K' \otimes_K H_c^n(\mathring{W}_i, \omega_X)$$

is topological. Indeed, this is proved verbatim as in Remark 6.2 (ii). Now, the map (6.19) is by definition the unique map that makes the left-hand rectangle in

$$\begin{tikzcd}
H_c^n(X', \omega_{X'}) \arrow[rr, equal] & & \varinjlim_i H_c^n(\mathring{W}_i', \omega_{X'}) \arrow[dr] & \\
& & & K' \\
K' \otimes_K H_c^n(X, \omega_X) \arrow[uu, dotted] \arrow[r, equal] & K' \otimes_K \varinjlim_i H_c^n(\mathring{W}_i, \omega_X) \arrow[r, equal] & \varinjlim_i K' \otimes_K H_c^n(\mathring{W}_i, \omega_X) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

commute, and we have to show that the resulting outer contour in the above diagram commutes (cf. Definition 4.12). But this is immediate, since the right-hand triangle commutes by Proposition 6.15. It only remains to prove that the map (6.19) is canonical, i.e. independent of the choice of the cover $\{\mathring{W}_i\}_{i\in\mathbb{N}}$. Given another cover $\{\mathring{V}_i\}_{i\in\mathbb{N}}$ of X as in the assertion, we can assume that $\mathring{V}_i \subseteq \mathring{W}_i$ (otherwise replace $\mathring{V}_i$ by $\mathring{V}_i \cap \mathring{W}_i$ which is again special affinoid wide-open by [Bey97a, Lemma 5.1.3]). Then we need to show that the diagram

$$\begin{tikzcd}
\varinjlim_i H^n_c(\mathring{V}'_i, \omega_{X'}) \arrow[r, equal] & H^n_c(X', \omega_{X'}) \arrow[r, equal] & \varinjlim_i H^n_c(\mathring{W}'_i, \omega_{X'}) \\
\varinjlim_i K' \otimes_K H^n_c(\mathring{V}_i, \omega_X) \arrow[u] \arrow[rr] & & \varinjlim_i K' \otimes_K H^n_c(\mathring{W}_i, \omega_X) \arrow[u]
\end{tikzcd}$$

commutes. Granting that the map $\mathring{V}'_i \longrightarrow \mathring{W}'_i$ is an open immersion, we have induced maps $H^n_c(\mathring{V}'_i, \omega_{X'}) \longrightarrow H^n_c(\mathring{W}'_i, \omega_{X'})$ whose limit over all i makes the outer contour and the upper semicircle in the extended diagram

$$\begin{tikzcd}
\varinjlim_i H^n_c(\mathring{V}'_i, \omega_{X'}) \arrow[r, equal] \arrow[rr, bend left] & H^n_c(X', \omega_{X'}) \arrow[r, equal] & \varinjlim_i H^n_c(\mathring{W}'_i, \omega_{X'}) \\
\varinjlim_i K' \otimes_K H^n_c(\mathring{V}_i, \omega_X) \arrow[u] \arrow[rr] & & \varinjlim_i K' \otimes_K H^n_c(\mathring{W}_i, \omega_X) \arrow[u]
\end{tikzcd}$$

commute, whence the desired commutativity of the rectangle follows. It remains to check that $\mathring{V}'_i \longrightarrow \mathring{W}'_i$ is an open immersion. Dropping the index i for notational convenience, it suffices to show that $\mathring{W}' \longrightarrow X'$ is an open immersion (since then $V' \subseteq X'$ as an admissible open too, and V' lands in the admissible open $W' \subseteq X'$). Adopting our standard notation $\mathring{W} \subseteq Z$ and $\pi\colon Z \longrightarrow \mathbb{D}^n_K$, the desired conclusion follows since $\mathring{W}'$ is an admissible open in ${\pi'}^{-1}(\mathbb{D}^n_{K'})$ by Proposition 6.11 (iii), ${\pi'}^{-1}(\mathring{\mathbb{D}}^n_{K'})$ is certainly an admissible open in Z', and Z' is an admissible open in X' by the explicit construction of X'. □

Next, we prove that the Yoneda-Cartier pairing is also compatible with base change:

Proposition 6.17. *Let X be a smooth rigid Stein K-space of dimension d and let $\mathcal{F}$ be a coherent sheaf on X. Then the diagram*

$$\begin{tikzcd}
H^{d-i}_c(X', \mathcal{F}') & \times & \mathrm{Ext}^i_{X'}(\mathcal{F}', \omega_{X'}) \arrow[r] & H^d_c(X', \omega_{X'}) \\
H^{d-i}_c(X, \mathcal{F}) \arrow[u] & \times & \mathrm{Ext}^i_X(\mathcal{F}, \omega_X) \arrow[u] \arrow[r] & H^d_c(X, \omega_X) \arrow[u]
\end{tikzcd}$$

commutes for all $i \geq 0$.

Proof. We will prove the following equivalent formulation of the theorem: Letting $\alpha_{i,\mathcal{F}} \colon H_c^{d-i}(X,\mathcal{F}) \longrightarrow H_c^{d-i}(X',\mathcal{F}')$ denote the comparison map for base change, the diagram

$$\begin{tikzcd}
\operatorname{Ext}^i_{X'}(\mathcal{F}',\omega_{X'}) \arrow[r] & \operatorname{Hom}_{K'}(H_c^{d-i}(X',\mathcal{F}'), H_c^d(X',\omega_{X'})) \arrow[dr, "(-)\circ\alpha_{i,\mathcal{F}}"] & \\
& & \operatorname{Hom}_K(H_c^{d-i}(X,\mathcal{F}), H_c^d(X',\omega_{X'})) \\
\operatorname{Ext}^i_X(\mathcal{F},\omega_X) \arrow[uu] \arrow[r] & \operatorname{Hom}_K(H_c^{d-i}(X,\mathcal{F}), H_c^d(X,\omega_X)) \arrow[ur, "\alpha_{0,\omega_X}\circ(-)"'] &
\end{tikzcd}$$

commutes for all $i \geq 0$. We can view the content of this diagram as having two maps from the δ-functor $\operatorname{Ext}^i_X(-,\omega_X)$ to the δ-functor $\operatorname{Hom}_K(H_c^{d-i}(X,-), H_c^d(X',\omega_{X'}))$, and we want to prove that these maps coincide. Note that the latter is indeed a δ-functor, since it is the composite $\operatorname{Hom}_K(-, H_c^d(X',\omega_{X'})) \circ H_c^{d-i}(X,-)$ of the δ-functor $H_c^{d-i}(X,-)$ with the exact functor $\operatorname{Hom}_K(-, H_c^d(X',\omega_{X'}))$. Now, since $\operatorname{Ext}^i_X(-,\omega_X)$ is a universal δ-functor, it suffices to show that the mentioned maps coincide for $i = 0$, i.e. that the above diagram commutes for $i = 0$. Given a $\gamma \in \operatorname{Hom}_X(\mathcal{F},\omega_X)$, if we denote its image in $\operatorname{Hom}_{X'}(\mathcal{F}',\omega_{X'})$ by γ' -- so, for $U \subseteq X$ affinoid, γ' over U' is $\mathcal{F}'(U') = \mathcal{F}(U) \mathbin{\widehat{\otimes}}_K K' \xrightarrow{\gamma \widehat{\otimes} \mathrm{id}} \omega_X(U) \mathbin{\widehat{\otimes}}_K K' = \omega_{X'}(U')$ -- then the commutativity of the above diagram for $i = 0$ amounts to the commutativity of

$$\begin{tikzcd}
H_c^d(X',\mathcal{F}') \arrow[r, "H_c^d(X',\gamma')"] & H_c^d(X',\omega_{X'}) \\
H_c^d(X,\mathcal{F}) \arrow[u, "\alpha_{0,\mathcal{F}}"] \arrow[r, "H_c^d(X,\gamma)"] & H_c^d(X,\omega_X) \arrow[u, "\alpha_{0,\omega_X}"']
\end{tikzcd}$$

which holds true for all $\gamma \in \operatorname{Hom}_X(\mathcal{F},\omega_X)$ since it is evident from the construction of our base-change-comparison maps α that they are functorial in this sense. Thus the proposition is proved. $\square$

Now we can summarise the content of Theorem 6.16 and Proposition 6.17 in the following:

Corollary 6.18. *Let X be a smooth rigid Stein K-space of dimension d. Then the Serre duality pairing from Theorem 5.3 is compatible with base change, in the following sense: For every coherent sheaf $\mathcal{F}$ on X, the diagram*

$$\begin{tikzcd}
H_c^{d-i}(X',\mathcal{F}') & \times & \operatorname{Ext}^i_{X'}(\mathcal{F}',\omega_{X'}) \arrow[r] & H_c^d(X',\omega_{X'}) \arrow[r, "t_{X'}"] & K' \\
H_c^{d-i}(X,\mathcal{F}) \arrow[u] & \times & \operatorname{Ext}^i_X(\mathcal{F},\omega_X) \arrow[u] \arrow[r] & H_c^d(X,\omega_X) \arrow[u] \arrow[r, "t_X"] & K \arrow[u]
\end{tikzcd}$$

commutes for all $i \geq 0$.

Chapter 7

Comparison with other recent results

In their preprint [AL20], Abe and Lazda consider partially proper morphisms $f\colon X \longrightarrow Y$ of analytic adic spaces, the proper pushforward $f_!$ (defined by taking those sections of f_* whose support is proper over Y) and its derived functor $\boldsymbol{R}f_!$, which in particular yields a notion of (relative) compactly supported cohomology via

$$H^q_c(X/Y, -) := \Gamma(X, \boldsymbol{R}^q f_!(-)). \tag{7.1}$$

They prove that $\boldsymbol{R}f_!$ is compatible with composition (see (7.2) in §7.1 below). Moreover, for a special class of smooth partially proper morphisms, namely those that are smooth and "partially proper in the sense of Kiehl" (see Definitions 7.1 and 7.3 below), they construct an $\mathcal{O}_Y$-linear "trace map" in the derived category of $\mathcal{O}_Y$-modules that is local on the base Y and compatible with composition (see, for instance, Proposition 7.10 below). It has the following explicit description when $Y = \operatorname{Spa}(A, A^+)$ is affinoid and

$$f\colon X = \mathring{\mathbb{D}}^d_Y \longrightarrow Y$$

is the relative d-dimensional open unit disc over Y: choosing coordinates $z = (z_1, \ldots, z_d)$ yields the the following commutative diagram

$$\begin{tikzcd}
H^d_c(\mathring{\mathbb{D}}^d_Y/Y, \omega_{X/Y}) \arrow[rr, "\sim"] \arrow[dr, "\mathrm{Tr}_{X/Y}"'] & & A\langle z^{-1}\rangle^\dagger \frac{dz}{z} \arrow[dl, "\sum_{i\le 0} a_i z^i \frac{dz}{z} \mapsto a_0"] \\
& A &
\end{tikzcd}$$

where $A\langle z^{-1}\rangle^\dagger$ denotes the ring of overconvergent series in z^{-1} (cf. our counterparts: Lemma 1.53 and the residue map (4.8) in Section 4.3).

Recall Huber's functor r from the category of rigid analytic varieties over $\operatorname{Sp}(K)$ to the category of adic spaces over $\operatorname{Spa}(K, \mathfrak{o}_K)$, which is characterised by the following properties [Hub96, (1.1.11)]:

a) If $X = \mathrm{Sp}(A)$ is an affinoid rigid variety, then $r(X) = \mathrm{Spa}(A, \mathring{A})$. Moreover, if $f\colon \mathrm{Sp}(B) \longrightarrow \mathrm{Sp}(A)$ is a morphism of affinoid rigid varieties induced by a K-algebra morphism $\varphi\colon A \longrightarrow B$, then $f\colon \mathrm{Spa}(B, \mathring{B}) \longrightarrow \mathrm{Spa}(A, \mathring{A})$ is the morphism of adic spaces which is induced by the morphism $\varphi\colon (A, \mathring{A}) \longrightarrow (B, \mathring{B})$.

b) If $f\colon X \hookrightarrow Y$ is an open embedding of rigid varieties, then $r(f)\colon r(X) \hookrightarrow r(Y)$ is an open embedding of adic spaces.

c) A family $(X_i)_{i\in I}$ of admissible open subsets of a rigid variety X is an admissible open covering of X if and only if $r(X) = \bigcup_{i\in I} r(X_i)$.

d) r is fully faithful.

For every rigid analytic variety X over $\mathrm{Sp}(K)$, the adic space $r(X)$ is locally of finite type over $\mathrm{Spa}(K, \mathfrak{o}_K)$. The functor r induces an equivalence from the category of quasi-separated rigid analytic varieties over $\mathrm{Sp}(K)$ to the category of adic spaces over $\mathrm{Spa}(K, \mathfrak{o}_K)$ whose structure morphisms are quasi-separated and locally of finite type. If $\boldsymbol{r}$ denotes the morphism from the site $\boldsymbol{r(X)}$ of $r(X)$ to the site $\boldsymbol{X}$ of X given by

$$\begin{aligned} \boldsymbol{r(X)} &\longrightarrow \boldsymbol{X} \\ r(U) &\longleftarrow\!\shortmid U, \end{aligned}$$

then the induced morphism between the corresponding toposes $\boldsymbol{r(X)}^{\sim}$ and $\boldsymbol{X}^{\sim}$ (recall that this morphism consists of the direct image functor $\boldsymbol{r}_*\colon \boldsymbol{r(X)}^{\sim} \longrightarrow \boldsymbol{X}^{\sim}$, the inverse image functor $\boldsymbol{r}^*\colon \boldsymbol{X}^{\sim} \longrightarrow \boldsymbol{r(X)}^{\sim}$ and an adjunction between them) is an equivalence of toposes.

In this chapter, we will show how our setting embeds into the setting of Abe and Lazda via Huber's functor r. Then we compare the two notions of compactly supported cohomology, by proving in Lemma 7.8 that our compactly supported cohomology $H_c^j(X, \mathcal{F})$ of a rigid Stein space X coincides with the cohomology $H_c^j(r(X)/\mathrm{Spa}(K, \mathfrak{o}_K), \boldsymbol{r}^*\mathcal{F})$ obtained from [AL20] as in (7.1) above. Then, in Subsection 7.3.1, we prove that Abe and Lazda's trace map $\mathrm{Tr}_{X/Y}$ can recover Beyer's trace map when $Y = \mathrm{Spa}(K, \mathfrak{o}_K)$. In Subsection 7.3.2, we examine their trace map $\mathrm{Tr}_{X/Y}$ in the case when X/Y is finite étale and discuss the following question (Question 7.12):

Let X and Y be adic spaces coming from rigid analytic varieties via Huber's functor r, and let $\alpha\colon X \longrightarrow Y$ be a finite étale morphism. Does $\mathrm{Tr}_{X/Y}$ coincide with the classical trace map $\alpha_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y$?

We do not know the answer to this question in general, but we give a conjectural outline how one might obtain a positive answer in the special case when $X = Y$ and α is the identity. Finally, in Subsection 7.3.3, we prove that under the assumption of an affirmative answer to Question 7.12, one can relate Abe and Lazda's trace map $\mathrm{Tr}_{X/Y}$ to our relative trace map t_α and recover our compatibility result (Theorem 5.5) from their result on the compatibility of $\mathrm{Tr}_{X/Y}$ with respect to composition.

We adopt the standing hypotheses of [AL20, §2] for the rest of this chapter, i.e. we assume that all adic spaces are analytic and satisfy one of the standard conditions [Hub96, (1.1.1)] that guarantee that they are sheafy.

7.1 The definition of $f_!$ and comparison with our $H_c^\bullet(X,-)$

Definition 7.1 ([Hub96, Remark 1.3.19 i)]). *A separated morphism $f\colon X \longrightarrow Y$ of rigid spaces is called* Kiehl-partially-proper *if there exist an admissible open covering $Y = \bigcup_{i\in I} Y_i$ by affinoid open subspaces Y_i and, for each i, an admissible open covering $f^{-1}(Y_i) = \bigcup_{j\in J_i} X_{ij}$ by affinoid open subspaces X_{ij} that can be "properly enlarged over Y_i within $f^{-1}(Y_i)$", meaning that there exists an affinoid open subspace X'_{ij} of $f^{-1}(Y_i)$ such that*

$$X_{ij} \Subset_{Y_i} X'_{ij}$$

(cf. Definition 1.33) for each i and each j.

If we additionally require the morphism to be quasi-compact, we recover Kiehl's definition of a proper morphism, i.e. a morphism of rigid spaces is proper if and only if it is quasi-compact and Kiehl-partially-proper.

Example 7.2. The structure morphism $X \longrightarrow \mathrm{Sp}(K)$ of a rigid Stein space X is Kiehl-partially-proper.

If $Y = \mathrm{Spa}(R, R^+)$ is a Tate affinoid adic space, then we define the open unit disk $\mathring{\mathbb{D}}_Y^r$ resp. the affine space $\mathbb{A}_Y^{r,\mathrm{an}}$ to be the union over $n \in \mathbb{N}$ of "closed disks of radius $|\varpi|^{1/n}$ resp. $|\varpi|^{-n}$" for a topologically nilpotent unit $\varpi \in R$, as in [AL20, §4.1]. When $Y = \mathrm{Spa}(K, \mathfrak{o}_K)$, this collides with our notation for disks in rigid geometry, which is not a problem since it will always be clear from the context whether the disk is being regarded as a rigid variety or as the associated adic space.

Under Huber's functor r, the condition in Definition 7.1 obviously translates to condition (i) in the following Definition 7.3 (see also Remark 7.6 further below):

Definition 7.3 ([AL20, Definition 4.1.4 and Proposition 4.1.3]). *A morphism $f\colon X \longrightarrow Y$ of adic spaces is called* Kiehl-partially-proper *if it is separated, taut and satisfies one of the following equivalent conditions:*

(i) *Locally on X and Y, there exist open covers $\{X_i\}_{i\in I}$ and $\{X'_i\}_{i\in I}$ of X with $X_i \subseteq X'_i$, integers $N_i \geq 1$, and closed immersions $X'_i \hookrightarrow \mathbb{D}_Y^{N_i}$ over Y such that X_i lands in the open disk $\mathring{\mathbb{D}}_Y^{N_i}$.*

(ii) *Locally on X and Y, there exists an open cover $\{X_i\}_{i\in I}$ of X, integers $N_i \geq 1$, and closed immersions $X_i \hookrightarrow \mathring{\mathbb{D}}_Y^{N_i}$ over Y.*

Remark 7.4 ([AL20, Remark 4.1.5]). *Let f be a morphism of adic spaces.*

(i) *If f is Kiehl-partially-proper, then it is partially proper (i.e. separated, locally of $^+$weakly finite type and universally specialising).*

(ii) *If X and Y are quasi-separated adic spaces locally of finite type over a discretely valued, height one affinoid field, then any partially proper map $X \longrightarrow Y$ is Kiehl-partially-proper by [Hub96, Remark 1.3.19].*

(iii) *For any Y, the morphisms $\mathring{\mathbb{D}}^r_Y \longrightarrow Y$ and $\mathbb{A}^{r,\mathrm{an}}_Y \longrightarrow Y$ are Kiehl-partially-proper.*

Definition 7.5 (Proper pushforward according to [AL20, Definition 3.1.1])**.** *Let $f \colon X \longrightarrow Y$ be a morphism of adic spaces that is separated and locally of $^+$weakly finite type, $\mathcal{F}$ a sheaf on X, $V \subseteq Y$ an open subset and $s \in \Gamma(f^{-1}(V), \mathcal{F})$ a section. Then*

$$\mathrm{Supp}(s) := \{x \in f^{-1}(V) \colon s_x \neq 0\}$$

is a closed subset of the adic space $f^{-1}(V)$ and hence in particular a "germ" (in the sense of [AL20, Definition 2.2.1]). Therefore, one can ask whether the map of germs $\mathrm{Supp}(s) \longrightarrow V$ is proper [AL20, the discussion preceding Definition 2.2.3]. The proper pushforward

$$f_!\mathcal{F} \subseteq f_*\mathcal{F}$$

is defined to be the subsheaf consisting of sections $s \in \Gamma(V, f_*\mathcal{F}) = \Gamma(f^{-1}(V), \mathcal{F})$ that have proper support over V. The functor $H^0(Y, f_!(-))$ will also be denoted by $H^0_c(X/Y, -)$ or $\Gamma_c(X/Y, -)$.

If f is partially proper, then the support of $s \in \Gamma(V, f_*\mathcal{F})$ is proper over V if and only if it is quasi-compact over V. If f is proper, then

$$f_! = f_*$$

holds. By [AL20, Proposition 3.3.1], the proper pushforward is left-exact and commutes with composition. For partially proper morphisms, the compatibility of the proper pushforward with respect to composition extends to the total derived functor of the proper pushforward, i.e. if

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

are partially proper morphisms of adic spaces, then there is a canonical isomorphism

$$\boldsymbol{R}(g \circ f)_! \cong \boldsymbol{R}g_! \circ \boldsymbol{R}f_! \tag{7.2}$$

of functors $\boldsymbol{D}^+(X) \longrightarrow \boldsymbol{D}^+(Y)$, according to [AL20, Corollary 3.4.7].

Remark 7.6. *If $f \colon X \longrightarrow Y$ is Kiehl-partially-proper morphism of rigid varieties, then $r(f) \colon r(X) \longrightarrow r(Y)$ is a Kiehl-partially-proper morphism of adic spaces.*

Proof. In view of the definitions 7.1 and 7.3, we only need to argue that $r(f)$ is separated and taut. But, $r(f)$ is in fact partially proper by [Hub96, Remark 1.3.19 iii)] and hence in particular taut (by [Hub96, Lemma 5.1.4 ii)]) and separated. □

Assuming that the ground field is discretely valued, Abe and Lazda prove the following comparison result[1] between their compactly supported cohomology and the one from [van92]:

Lemma 7.7 ([AL20, Lemma 3.2.2]). *Let* $f\colon X \longrightarrow Y$ *be a partially proper, locally of finite type morphism of adic spaces, with* Y *affinoid and of finite type over a discretely valued affinoid field. Then* $H_c^0(X/Y, -)$ *coincides with the following subfunctor of* $H^0(X, -)$*, obtained by transporting the notion of compactly supported cohomology from [van92] to the adic context:*

$$H^0_{c,\mathrm{vdP}}(X/Y, -) := \varinjlim_{U'} H^0_{\overline{U'}}(X, -)$$

where the U' *run over all finite unions of affinoids* U_i *that can be properly enlarged over* Y *within* X *(i.e., for which there exist affinoids* $V_i \subseteq X$ *such that* $U_i \Subset_Y V_i$*) and* $H^0_{\overline{U}}$ *denotes sections with support in the closed subset* $\overline{U} \subseteq X$*.*

The only part in the proof of [AL20, Lemma 3.2.2] that relies on the fact that the ground field is discretely valued is the use of [Hub96, Remark 1.3.19 iii) (b) $\Longrightarrow$ (a)] to deduce that f being partially proper implies that it is Kiehl-partially-proper and hence that the collection of those open affinoids $U \subseteq X$ that can be properly enlarged over Y within X forms a basis for the topology of X. We are interested in the case where f is the structure morphism of a Stein space. Since f is then in particular Kiehl-partially-proper, we obtain the following result without having to assume that the ground field is discretely valued: For any rigid Stein space X over K and any coherent sheaf $\mathcal{F}$ on X, we have

$$H^j_{c,\mathrm{vdP}}(r(X)/\operatorname{Spa}(K, \mathfrak{o}_K), \boldsymbol{r}^*\mathcal{F}) = H^j_c(r(X)/\operatorname{Spa}(K, \mathfrak{o}_K), \boldsymbol{r}^*\mathcal{F})$$

for all $j \geq 0$.
Moreover, using the equivalence of the underlying toposes of X and $r(X)$, we can transport our definition of $H^0_c(X, -)$ to the adic context, where it becomes

$$\varinjlim_{U} H^0_{\overline{U}}(r(X), -)$$

where U runs over all finite unions U of affinoids, $\overline{U}$ denotes the closure of U in $r(X)$ and $H^0_{\overline{U}}$ denotes sections with support in the closed subset $\overline{U} \subseteq r(X)$. Since X is Stein, every open affinoid in X can be properly enlarged within X, whence we obtain

$$H^j_c(X, \mathcal{F}) = H^j_{c,\mathrm{vdP}}(r(X)/\operatorname{Spa}(K, \mathfrak{o}_K), \boldsymbol{r}^*\mathcal{F})$$

for all $j \geq 0$. Summarising the above, we obtain:

[1] We note that [AL20, Remark 3.2.3] gives an example of how this comparison result is false in greater generality, when one drops the assumptions on f.

Lemma 7.8. *Let X be a rigid Stein space over K and $\mathcal{F}$ a coherent sheaf on X. Then*

$$H_c^j(X, \mathcal{F}) = H_{c,\mathrm{vdP}}^j(r(X)/\operatorname{Spa}(K, \mathfrak{o}_K), \boldsymbol{r}^*\mathcal{F}) = H_c^j(r(X)/\operatorname{Spa}(K, \mathfrak{o}_K), \boldsymbol{r}^*\mathcal{F})$$

for all $j \geq 0$.

7.2 Abe and Lazda's trace map

In this section, we summarise the construction of the trace map from [AL20, §5]. Two building blocks are needed, which are reviewed in §7.2.1 and §7.2.2 below.

7.2.1 The trace map for the relative open disk

Let $X = \mathring{\mathbb{D}}_Y^d$ be the relative open unit disk over a Tate affinoid adic space $Y = \operatorname{Spa}(R, R^+)$ and let

$$p\colon \mathring{\mathbb{D}}_Y^d \longrightarrow Y$$

be the structure morphism. Choosing a topologically nilpotent unit $\varpi \in R$ and coordinates $z_1, \ldots, z_d$ on X, we have the following description of $H_c^q(\mathring{\mathbb{D}}_Y^d/Y, \omega_{\mathring{\mathbb{D}}_Y^n/Y})$ as in [AL20, §5.1]:

$$H_c^q(\mathring{\mathbb{D}}_Y^d/Y, \omega_{\mathring{\mathbb{D}}_Y^n/Y}) = \begin{cases} R\langle z_1^{-1}, \ldots, z_d^{-1}\rangle^\dagger \cdot \dfrac{dz_1}{z_1} \wedge \ldots \wedge \dfrac{dz_d}{z_d} & \text{if } q = d \\ 0 & \text{if } q \neq d \end{cases}$$

where $R\langle z_1^{-1}, \ldots, z_d^{-1}\rangle^\dagger$ denotes the ring of series of the form

$$\sum_{i_1,\ldots,i_d \leq 0} r_{i_1,\ldots,i_d} z_1^{i_1} \cdots z_d^{i_d}, \quad r_{i_1,\ldots,i_d} \in R,$$

for which there exists $n \geq 1$ such that $r_{i_1,\ldots,i_d}^n \varpi^{i_1+\ldots+i_d} \to 0$ as $(i_1, \ldots, i_d) \to -\infty$. Switching to multi-index notation, in particular

$$\frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_d}{z_d} = \frac{dz_1 \wedge \ldots \wedge dz_d}{z_1 \cdots z_d} =: \frac{dz}{z},$$

the trace map is defined as

$$\begin{aligned} \operatorname{Tr}_{z_1,\ldots,z_d}\colon H_c^d(\mathring{\mathbb{D}}_Y^d/Y, \omega_{\mathring{\mathbb{D}}_Y^n/Y}) &\longrightarrow H^0(Y, \mathcal{O}_Y) \\ \sum_{i \leq 0} r_i z^i \cdot \frac{dz}{z} &\longmapsto r_0 \end{aligned}$$

which globalises to

$$\operatorname{Tr}_{z_1,\ldots,z_d}\colon \boldsymbol{R}^d p_! \omega_{\mathring{\mathbb{D}}_Y^n/Y} \longrightarrow \mathcal{O}_Y$$

for any adic space Y. It is independent of the choice of coordinates $z_1, \dots, z_d$ by [AL20, Corollary 5.3.3]. This trace map generalises the residue map

$$\mathrm{res}\colon H^d_c(\mathring{\mathbb{D}}^d, \omega_{\mathring{\mathbb{D}}^d}) \longrightarrow K$$

from Section 4.3. (Indeed, res projects onto the (-1)-th coefficient with respect to the basis dz, which coincides with the 0-th coefficient with respect to dz/z.)

Remark 7.9 ([AL20, Remark 5.1.3])**.** *One can replace $p\colon \mathring{\mathbb{D}}^d_Y \longrightarrow Y$ everywhere by the relative analytic affine space $p\colon \mathbb{A}^{d,\mathrm{an}}_Y \longrightarrow Y$ and construct the trace map in a similar way. This generalises the trace map*

$$\begin{aligned} H^d_c(\mathbb{A}^{d,\mathrm{an}}_Y, \omega_{\mathbb{A}^{d,\mathrm{an}}_Y/Y}) &\longrightarrow \mathcal{O}(Y) \\ \sum_{i\le 0} r_i z^i \cdot \frac{dz}{z} &\longmapsto r_0 \end{aligned}$$

from [van92, 2.4], where it is proved that each $f \in H^d_c(\mathbb{A}^{d,\mathrm{an}}_Y, \omega_{\mathbb{A}^{d,\mathrm{an}}_Y/Y})$ can uniquely be written as $f = \sum_{i\le 0} r_i z^i \cdot \frac{dz}{z}$ (in multi-index notation) with coefficients $r_i \in \mathcal{O}(Y)$ and for which there exists some $R \in |K^\times|$ such that f converges on $\{(z_1, \dots, z_d, y)\colon \text{all } |z_i| > R\}$.

7.2.2 Duality for regular closed immersions

As in [AL20, Definition 5.2.3], we say that a closed immersion $u\colon X \hookrightarrow D$ of adic spaces is *regular of codimension* c if the associated ideal sheaf $\mathcal{J}_X$ on D is locally generated by a regular sequence of c elements. Then $u^*(\mathcal{J}_X/\mathcal{J}_X^2)$ is locally free of rank c, and its dual $u^*(\mathcal{J}_X/\mathcal{J}_X^2)^\vee$ is called *the normal sheaf* to X in D. The top exterior power of the normal sheaf is denoted by $\mathfrak{n}_{X/D}$. (We mention that exterior powers commute with dual spaces by [Lan02, Proposition XIX.1.5].) By the duality [AL20, Lemma 5.2.4] for regular closed immersions, we have, for any perfect complex $\mathcal{F}$ of $\mathcal{O}_X$-modules, a canonical isomorphism

$$\mathrm{Tr}_u\colon u_*\boldsymbol{R}\mathcal{H}om_X(\mathcal{F}, \mathfrak{n}_{X/D}) \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om_D(u_*\mathcal{F}, \mathcal{O}_D)[c] \tag{7.3}$$

in $\boldsymbol{D}(\mathcal{O}_D)$, natural in $\mathcal{F}$ and compatible with composition. In §7.2.3, we will review the construction of the trace map for morphisms $f\colon X \longrightarrow Y$ where Y is a finite-dimensional adic space and f is smooth of relative dimension d and admits a factorisation over a closed immersion u into a open unit disk over Y:

$$\begin{tikzcd} X \arrow[rr, hook, "u"] \arrow[dr, "f"'] & & \mathring{\mathbb{D}}^N_Y \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where p is the natural projection. In this situation, u is regular of codimension $N-d$. In fact, the sequence

$$0 \longrightarrow u^*(\mathcal{J}_X/\mathcal{J}_X^2) \longrightarrow u^*\Omega^1_{\mathring{\mathbb{D}}^N_Y/Y} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0 \tag{7.4}$$

is exact (cf. [BLR95, Proposition 2.5]), so taking top exterior powers yields an isomorphism $u^*\omega_{\mathring{\mathbb{D}}^N_Y/Y} \cong \mathfrak{n}^\vee_{X/\mathring{\mathbb{D}}^N_Y} \otimes \omega_{X/Y}$ and then tensoring with $\mathfrak{n}_{X/\mathring{\mathbb{D}}^N_Y}$ finally yields

$$\mathfrak{n}_{X/\mathring{\mathbb{D}}^N_Y} \otimes u^*\omega_{\mathring{\mathbb{D}}^N_Y/Y} \cong \omega_{X/Y}. \tag{7.5}$$

Plugging in $\mathcal{F} = \mathcal{O}_X$ in (7.3), tensoring both sides with $\omega_{\mathring{\mathbb{D}}^N_Y/Y}$ and then simplifying with (7.5) and the projection formula [AL20, Corollary 3.8.2], we obtain an isomorphism

$$\mathrm{Tr}_u\colon u_*\omega_{X/Y} \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om_{\mathring{\mathbb{D}}^N_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^N_Y/Y})[N-d] \tag{7.6}$$

as follows:

$$\begin{tikzcd}
u_*(\mathfrak{n}_{X/\mathring{\mathbb{D}}^N_Y}) \otimes \omega_{\mathring{\mathbb{D}}^N_Y/Y} \arrow[r, "(7.3)", "\cong"'] \arrow[d, equal, "\text{proj. formula}"] & \boldsymbol{R}\mathcal{H}om_{\mathring{\mathbb{D}}^N_Y}(u_*\mathcal{O}_X, \mathcal{O}_{\mathring{\mathbb{D}}^N_Y})[N-d] \otimes \omega_{\mathring{\mathbb{D}}^N_Y/Y} \arrow[d, equal] \\
u_*(\mathfrak{n}_{X/\mathring{\mathbb{D}}^N_Y} \otimes u^*\omega_{\mathring{\mathbb{D}}^N_Y/Y}) \arrow[d, equal, "(7.5)"] & \boldsymbol{R}\mathcal{H}om_{\mathring{\mathbb{D}}^N_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^N_Y/Y})[N-d] \\
u_*\omega_{X/Y} \arrow[ur] &
\end{tikzcd} \tag{7.7}$$

Both complexes in (7.6) only have cohomology in degree 0, where we obtain an isomorphism of sheaves

$$u_*\omega_{X/Y} \xrightarrow{\sim} \mathcal{E}xt^{N-d}_{\mathring{\mathbb{D}}^N_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^N_Y/Y}). \tag{7.8}$$

7.2.3 The trace map for closed subspaces of the open disk

As in §7.2.2, let Y be a finite-dimensional adic space and $f\colon X \longrightarrow Y$ smooth of relative dimension d admitting a factorisation over a closed immersion u into a open unit disk over Y:

$$\begin{tikzcd}
X \arrow[rr, hook, "u"] \arrow[dr, "f"'] & & \mathring{\mathbb{D}}^N_Y \arrow[dl, "p"] \\
& Y. &
\end{tikzcd}$$

Consider the isomorphism

$$\mathrm{Tr}_u\colon \boldsymbol{R}u_!\omega_{X/Y} \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om_{\mathring{\mathbb{D}}^N_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^N_Y/Y})[N-d]$$

from (7.6), where we have rewritten the left-hand side using $u_* = \boldsymbol{R}u_* = \boldsymbol{R}u_!$ (the first equality being due to the exactness of u_* and the second due to the equality $u_* = u_!$). If we apply $\boldsymbol{R}^d p_!$ to the above isomorphism, we obtain

$$\begin{array}{ccc} \boldsymbol{R}^d f_! \omega_{X/Y} & \xrightarrow{\sim} & \boldsymbol{R}^d p_! \boldsymbol{R}\mathcal{H}om_{\mathring{\mathbb{D}}^N_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^N_Y/Y})[N-d] \\ & & \| \\ & & \boldsymbol{R}^N p_! \boldsymbol{R}\mathcal{H}om_{\mathring{\mathbb{D}}^N_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^N_Y/Y}). \end{array}$$

Composing with the map induced by $\mathcal{O}_{\mathring{\mathbb{D}}^N_Y} \longrightarrow u_*\mathcal{O}_X$ then yields

$$\boldsymbol{R}^d f_! \omega_{X/Y} \longrightarrow \boldsymbol{R}^N p_! \omega_{\mathring{\mathbb{D}}^N_Y/Y}.$$

Finally, composing with $\mathrm{Tr}_{z_1,\dots,z_N}$ from §7.2.1 for a choice of coordinates on $\mathring{\mathbb{D}}^N_Y$ produces the trace map

$$\mathrm{Tr}_{X/Y} \colon \boldsymbol{R}^d f_! \omega_{X/Y} \longrightarrow \mathcal{O}_Y.$$

Then [AL20, Proposition 5.3.1] tells us that $\mathrm{Tr}_{X/Y}$ satisfies:

Proposition 7.10. *Suppose that Y is a finite-dimensional adic space, and that $f \colon X \longrightarrow Y$ is a smooth morphism of relative dimension d, factoring through a closed immersion into an open unit polydisc over Y. Then:*

(i) $\mathrm{Tr}_{X/Y}$ *does not depend on the choice of embedding $u \colon X \hookrightarrow \mathring{\mathbb{D}}^N_Y$ over Y.*

(ii) *Suppose that $g \colon Y \longrightarrow Z$ is a smooth morphism of relative dimension e, factoring through a closed immersion into some relative open disc $\mathring{\mathbb{D}}^M_Z$. In short, we have the commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f \quad (\dim d)"] \arrow[dr, "(\dim d+e) \quad g\circ f"'] & & Y \arrow[dl, "g \quad (\dim e)"] \\ & Z. & \end{tikzcd}$$

Then, via the identification resulting from

$$\omega_{X/Z} = \omega_{X/Y} \otimes f^*\omega_{Y/Z} \tag{7.9}$$

and the projection formula [AL20, Corollary 3.8.2], the diagram

$$\begin{tikzcd} \boldsymbol{R}^{d+e}(g\circ f)_! \omega_{X/Z} \arrow[rr, "\boldsymbol{R}g_!(\mathrm{Tr}_{X/Y})"] \arrow[dr, "\mathrm{Tr}_{X/Z}"'] & & \boldsymbol{R}^e g_! \omega_{Y/Z} \arrow[dl, "\mathrm{Tr}_{Y/Z}"] \\ & \mathcal{O}_Z & \end{tikzcd}$$

commutes.

7.2.4 The trace map for Kiehl-partially-proper morphisms

Let Y be a finite-dimensional adic space, and $f\colon X \longrightarrow Y$ a smooth morphism of relative dimension d which is Kiehl-partially-proper. Then Definition 7.3 (ii) says that, locally on Y, there exists a cover of X by open subspaces U_i such that the restriction $U_i \longrightarrow Y$ is as in §7.2.3, so we have the trace maps

$$\mathrm{Tr}_{U_i/Y}\colon \boldsymbol{R}^d f_! \omega_{U_i/Y} \longrightarrow \mathcal{O}_Y$$

which, as Abe and Lazda prove, glue to a map

$$\mathrm{Tr}_{X/Y}\colon \boldsymbol{R}^d f_! \omega_{X/Y} \longrightarrow \mathcal{O}_Y$$

that satisfies properties analogous to Proposition 7.10 above.

7.3 Comparison of results

7.3.1 Recovering Beyer's trace map from $\mathrm{Tr}_{r(X)/\,\mathrm{Spa}(K,\mathfrak{o}_K)}$

Let X be a connected d-dimensional smooth rigid Stein space over K and let

$$f\colon X \longrightarrow \mathrm{Sp}(K)$$

be the structure morphism. Then the structure morphism $r(f)\colon r(X) \longrightarrow \mathrm{Spa}(K,\mathfrak{o}_K)$ is smooth of relative dimension d and Kiehl-partially-proper, so we have the trace map $\mathrm{Tr}_{r(X)/\,\mathrm{Spa}(K,\mathfrak{o}_K)}$ according to §7.2.4. Evaluating this trace map on global sections and transporting it onto $H_c^d(X,\omega_X)$ via Lemma 7.8, we obtain

$$\mathrm{Tr}_X\colon H_c^d(X,\omega_X) \longrightarrow K.$$

Lemma 7.11. *The map* Tr_X *coincides with Beyer's trace map* t *from Definition 4.12.*

Proof. The map Tr_X is constructed locally on X by using closed immersions

$$X \supseteq U \hookrightarrow \mathring{\mathbb{D}}_K^N. \tag{7.10}$$

Since X is Stein, [Lüt73, Theorem 4.25] tells us that there exists a global closed immersion

$$u\colon X \hookrightarrow \mathbb{A}_K^{N,\mathrm{an}}. \tag{7.11}$$

By Remark 7.9, we can replace (7.10) by (7.11) in the construction of Tr_X in §7.2.3, whence we see that Tr_X fits into the commutative diagram

$$\begin{tikzcd}
\boldsymbol{R}^d f_!\omega_X(\mathrm{Sp}(K)) \arrow[r, equal] \arrow[dddr, "\mathrm{Tr}_X"'] & H^d_c(X,\omega_X) \arrow[r, equal, "\text{Prop. 1.51}"] & H^d_c(\mathbb{A}_K^{N,\mathrm{an}}, u_*\omega_X) \arrow[d, equal, "\text{same as (7.8)}"] \\
 & & H^d_c(\mathbb{A}_K^{N,\mathrm{an}}, \mathcal{E}xt^{N-d}_{\mathbb{A}_K^{N,\mathrm{an}}}(u_*\mathcal{O}_X, \omega_{\mathbb{A}_K^{N,\mathrm{an}}})) \arrow[d, equal, "\text{(degenerate) spectral sequence}"] \\
 & & \mathrm{Ext}^N_c(u_*\mathcal{O}_X, \omega_{\mathbb{A}_K^{N,\mathrm{an}}}) \arrow[d, "\text{restrict along } \mathcal{O}_{\mathbb{A}_K^{N,\mathrm{an}}} \to u_*\mathcal{O}_X"] \\
 & K & H^N_c(\mathbb{A}_K^{N,\mathrm{an}}, \omega_{\mathbb{A}_K^{N,\mathrm{an}}}) = \mathrm{Ext}^N_c(\mathcal{O}_{\mathbb{A}_K^{N,\mathrm{an}}}, \omega_{\mathbb{A}_K^{N,\mathrm{an}}}) \arrow[l, "\mathrm{Tr}_{z_1,\dots,z_N}"']
\end{tikzcd}$$

where Ext^q_c on $\mathbb{A}_K^{N,\mathrm{an}}$ denotes the q-th derived functor of

$$\mathrm{Ext}^0_c(\mathcal{F},\mathcal{G}) := H^0_c(\mathbb{A}_K^{N,\mathrm{an}}, \mathcal{H}om_{\mathbb{A}_K^{N,\mathrm{an}}}(\mathcal{F},\mathcal{G}))$$

as in [van92, 2.2]. Now we consider the composite

$$\mathrm{Ext}^N_c(u_*\mathcal{O}_X, \omega_{\mathbb{A}_K^{N,\mathrm{an}}}) \xrightarrow{\beta} H^0(\mathbb{A}_K^{N,\mathrm{an}}, u_*\mathcal{O}_X)^\vee \xrightarrow{\gamma} K$$

in the notation of [van92, 3.4], i.e. β denotes the Serre duality pairing from [van92, Proposition 2.6 (3.)] for the sheaf $u_*\mathcal{O}_X$ and γ denotes evaluation at 1. We will also write $\beta = \beta(u_*\mathcal{O}_X)$ and $\gamma = \gamma(u_*\mathcal{O}_X)$ to keep track of the sheaf if other sheaves come into play below. We claim that $\gamma \circ \beta$ coincides with the composite

$$\mathrm{Ext}^N_c(u_*\mathcal{O}_X, \omega_{\mathbb{A}_K^{N,\mathrm{an}}}) \to \mathrm{Ext}^N_c(\mathcal{O}_X, \omega_{\mathbb{A}_K^{N,\mathrm{an}}}) = H^N_c(\mathbb{A}_K^{N,\mathrm{an}}, \omega_{\mathbb{A}_K^{N,\mathrm{an}}}) \xrightarrow{\mathrm{Tr}_{z_1,\dots,z_N}} K.$$

Due to the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^N_c(u_*\mathcal{O}_X, \omega_{\mathbb{A}_K^{N,\mathrm{an}}}) \arrow[r, "\beta(u_*\mathcal{O}_X)"] \arrow[d] & H^0(\mathbb{A}_K^{N,\mathrm{an}}, u_*\mathcal{O}_X)^\vee \arrow[r, "\gamma(u_*\mathcal{O}_X)"] \arrow[d] & K \arrow[d, equal] \\
\mathrm{Ext}^N_c(\mathcal{O}_X, \omega_{\mathbb{A}_K^{N,\mathrm{an}}}) \arrow[r, "\beta(\mathcal{O}_X)"] & H^0(\mathbb{A}_K^{N,\mathrm{an}}, \mathcal{O}_X)^\vee \arrow[r, "\gamma(\mathcal{O}_X)"] & K
\end{tikzcd}$$

it suffices to show that

$$\begin{tikzcd}
\mathrm{Ext}^N_c(\mathcal{O}_X, \omega_{\mathbb{A}_K^{N,\mathrm{an}}}) \arrow[r, "\beta(\mathcal{O}_X)"] \arrow[d, equal] & H^0(\mathbb{A}_K^{N,\mathrm{an}}, \mathcal{O}_X)^\vee \arrow[r, "\gamma(\mathcal{O}_X)"] & K \\
H^N_c(\mathbb{A}_K^{N,\mathrm{an}}, \omega_{\mathbb{A}_K^{N,\mathrm{an}}}) \arrow[urr, bend right, "\mathrm{Tr}_{z_1,\dots,z_N}"'] & &
\end{tikzcd}$$

commutes. But this is true by the definition of $\beta(\mathcal{O}_X)$, cf. [van92, proof of Lemma 2.5]. Hence we see that $\gamma \circ \beta$ expands the above commutative diagram to

$$\begin{tikzcd}
\boldsymbol{R}^d f_! \omega_X(\mathrm{Sp}(K)) \arrow[r, equal] \arrow[ddddr, "\mathrm{Tr}_X"'] & H^d_c(X, \omega_X) \arrow[r, equal, "\text{Prop. 1.51}"] & H^d_c(\mathbb{A}^{N,\mathrm{an}}_K, u_*\omega_X) \arrow[d, equal, "\text{same as (7.8)}"] & \\
& & H^d_c(\mathbb{A}^{N,\mathrm{an}}_K, \mathcal{E}xt^{N-d}_{\mathbb{A}^{N,\mathrm{an}}_K}(u_*\mathcal{O}_X, \omega_{\mathbb{A}^{N,\mathrm{an}}_K})) \arrow[d, equal, "\text{(degenerate) spectral sequence}"] & \\
& & \mathrm{Ext}^N_c(u_*\mathcal{O}_X, \omega_{\mathbb{A}^{N,\mathrm{an}}_K}) \arrow[d, "\text{restrict along } \mathcal{O}_{\mathbb{A}^{N,\mathrm{an}}_K} \longrightarrow u_*\mathcal{O}_X"] \arrow[ddl, "\gamma\circ\beta"'] & \\
& & & \\
& K & H^N_c(\mathbb{A}^{N,\mathrm{an}}_K, \omega_{\mathbb{A}^{N,\mathrm{an}}_K}) = \mathrm{Ext}^N_c(\mathcal{O}_{\mathbb{A}^{N,\mathrm{an}}_K}, \omega_{\mathbb{A}^{N,\mathrm{an}}_K}) \arrow[l, "\mathrm{Tr}_{z_1,\dots,z_N}"] &
\end{tikzcd}$$

where everything commutes. Now the upper trapezoid in the above diagram is exactly how van der Put's trace map is defined in [van92, 3.4], so Tr_X coincides with van der Put's trace map. By [van92, Proposition 3.6], van der Put's trace map satisfies the same Serre Duality[2] for Stein spaces (Theorem 5.3 above) as Beyer's trace map, hence they coincide by standard universal abstract nonsense. This completes the proof of the lemma. □

7.3.2 $\mathrm{Tr}_{X/Y}$ when X/Y is finite étale

The purpose of this subsection is to discuss the following question:

Question 7.12. *Let X and Y be adic spaces coming from rigid analytic varieties via Huber's functor r, and let $\alpha\colon X \longrightarrow Y$ be a finite étale morphism. Does $\mathrm{Tr}_{X/Y}$ coincide with the classical trace map*

$$\alpha_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y?$$

We will give a conjectural outline how one might obtain a positive answer in a special case, culminating in Remark 7.16 below. In the following, we take inspiration from the proof of [Con00, Lemma 2.8.2]. We begin by noting that the question is local on the base, whence we may assume that X and Y are affinoid and $X \longrightarrow Y$ is associated to a finite ring map $A \longrightarrow B$ for which there exists a presentation

$$B = A\langle x_1, \dots, x_n\rangle/(f_1, \dots, f_n) \tag{7.12}$$

such that the image of $\det(\partial f_i/\partial x_j)$ is a unit in B (see Proposition 5.1). Assume that the images of the x_i in B are topologically nilpotent (i.e. they have supremum

[2]In fact, [van92, Proposition 3.6] only shows duality for $i = 0$, but this suffices by Remark 5.4.

norm less than 1). In particular, the closed immersion corresponding to the top line of the diagram

$$\begin{tikzcd}
A\langle x_1,\dots,x_n\rangle/(f_1,\dots,f_n) & & A\langle x_1,\dots,x_n\rangle \arrow[ll] \\
& A \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

takes X into $\mathring{\mathbb{D}}^n_Y$ so we see that α admits a factorisation

$$\begin{tikzcd}
X \arrow[rr, hook, "u"] \arrow[dr, "\alpha"'] & & \mathring{\mathbb{D}}^n_Y \arrow[dl, "p"] \\
& Y. &
\end{tikzcd}$$

We have $\omega_{X/Y} = \mathcal{O}_X$ since α is étale. We introduce more notation and let

$$\xi\colon \mathcal{O}_X \xrightarrow{\sim} \mathfrak{n}_{X/\mathring{\mathbb{D}}^n_Y} \otimes u^*\omega_{\mathring{\mathbb{D}}^n_Y/Y}$$

denote the isomorphism from (7.5) and let

$$\eta\colon \mathcal{E}xt^n_{\mathring{\mathbb{D}}^n_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^n_Y/Y}) \xrightarrow{\sim} u_*(\mathfrak{n}_{X/\mathring{\mathbb{D}}^n_Y} \otimes u^*\omega_{\mathring{\mathbb{D}}^n_Y/Y})$$

denote the isomorphism obtained by walking through the upper rectangular part of the diagram (7.7) from right to left. Then, by design, the composite $\eta^{-1} \circ u_*(\xi)$ coincides with the map Tr_u from (7.6). We will describe the maps η and ξ explicitly further below in (7.16) and Lemma 7.13, respectively. To prove Proposition 7.12, we need to show that the composition

$$\begin{tikzcd}[column sep=small]
\alpha_*\mathcal{O}_X \arrow[r, equal] & p_!u_*\mathcal{O}_X \arrow[r, "\xi", "\cong"'] & p_!u_*(\mathfrak{n}_{X/\mathring{\mathbb{D}}^n_Y} \otimes u^*\omega_{\mathring{\mathbb{D}}^n_Y/Y}) \arrow[r, "\eta^{-1}", "\cong"'] & p_!\,\mathcal{E}xt^n_{\mathring{\mathbb{D}}^n_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^n_Y/Y}) \arrow[d, equal, "\text{spectral sequence}"] \\
& & & \boldsymbol{R}^n p_! \boldsymbol{R}\mathcal{H}om_{\mathring{\mathbb{D}}^n_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^n_Y/Y}) \arrow[d, "\text{via } \mathcal{O}_{\mathring{\mathbb{D}}^n_Y} \to u_*\mathcal{O}_X"] \\
& & \mathcal{O}_Y & \boldsymbol{R}^n p_!\omega_{\mathring{\mathbb{D}}^n_Y/Y} \arrow[l, "\mathrm{Tr}_{x_1,\dots,x_n}"']
\end{tikzcd}$$

coincides with the classical trace map $\alpha_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y$. To further shorten the notation, we denote the composite of the two vertical arrows in the above diagram by

$$\beta\colon p_!\,\mathcal{E}xt^n_{\mathring{\mathbb{D}}^n_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^n_Y/Y}) \longrightarrow \boldsymbol{R}^n p_!\omega_{\mathring{\mathbb{D}}^n_Y/Y}.$$

As recalled in Definition 5.2, we can naturally view any coherent $u_*\mathcal{O}_X$-module $\mathcal{F}$ as a coherent $\mathcal{O}_X$-module $\widetilde{\mathcal{F}}$ such that $u_*\widetilde{\mathcal{F}} = \mathcal{F}$. Thus we can also view η as $\eta = u_*\widetilde{\eta}$ with

$$\widetilde{\eta}\colon \mathcal{E}xt^n_{\mathring{\mathbb{D}}^n_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^n_Y/Y})^\sim \xrightarrow{\sim} \mathfrak{n}_{X/\mathring{\mathbb{D}}^n_Y} \otimes \omega_{\mathring{\mathbb{D}}^n_Y/Y}.$$

Moreover, we have

$$\alpha_* \circ (-)^\sim = p_!$$

because the identity $\alpha_* = p_! \circ u_*$ implies $\alpha_* \circ (-)^\sim = p_! \circ u_* \circ (-)^\sim = p_! \circ \mathrm{id} = p_!$. Hence the above diagram can be viewed as

$$\begin{tikzcd}
\alpha_*\mathcal{O}_X \arrow[r, "\xi", "\cong"'] & \alpha_*(\mathfrak{n}_{X/\mathring{\mathbb{D}}^n_Y} \otimes u^*\omega_{\mathring{\mathbb{D}}^n_Y/Y}) \arrow[r, "\widetilde{\eta}^{-1}", "\cong"'] & \alpha_*(\mathcal{E}xt^n_{\mathring{\mathbb{D}}^n_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^n_Y/Y})^\sim) \arrow[d, "\beta"] \\
& \mathcal{O}_Y & \boldsymbol{R}^n p_! \omega_{\mathring{\mathbb{D}}^n_Y/Y} \arrow[l, "\mathrm{Tr}_{x_1,\dots,x_n}"']
\end{tikzcd}$$

In the following, we will also write η instead of $\widetilde{\eta}$ for ease of notation. Since Y is affinoid, we can compute at the level of global sections. Hence, setting

$$R = A\langle x_1, \dots, x_n\rangle, \quad J = (f_1, \dots, f_n) \quad \text{and} \quad M = \Omega^n_{R/A},$$

the goal would be to show that the diagram

$$\begin{tikzcd}
B = R/J \arrow[r, "\xi", "\cong"'] \arrow[dr, "\mathrm{Tr}_{B/A}"'] & (\bigwedge^n J/J^2)^\vee \otimes_{R/J} M/JM \arrow[r, "\eta^{-1}", "\cong"'] & \mathrm{Ext}^n_R(R/J, M) \arrow[d, "\beta"] \\
& A & H^n_c(\mathring{\mathbb{D}}^n_Y/Y, \omega_{\mathring{\mathbb{D}}^n_Y/Y}) \arrow[l, "a_0 \leftarrow\!\shortmid \sum_{i \le 0} a_i z^i \frac{dz}{z}"']
\end{tikzcd} \tag{7.13}$$

commutes, where $\mathrm{Tr}_{B/A}$ is the trace of the finite flat ring map $A \longrightarrow B$ as in Section 5.3. To justify that $\mathrm{Ext}^n_R(R/J, M)$ should indeed sit in the top right of the above diagram, note that

$$\begin{aligned}
\alpha_*(\mathcal{E}xt^n_{\mathring{\mathbb{D}}^n_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^n_Y/Y})^\sim)(Y) &= \mathcal{E}xt^n_{\mathring{\mathbb{D}}^n_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^n_Y/Y})^\sim(X) \\
= \mathcal{E}xt^n_{\mathring{\mathbb{D}}^n_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^n_Y/Y})^\sim(u^{-1}(\mathring{\mathbb{D}}^n_Y)) &= \mathcal{E}xt^n_{\mathring{\mathbb{D}}^n_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^n_Y/Y})(\mathring{\mathbb{D}}^n_Y) \\
= \mathrm{Ext}^n_{\mathring{\mathbb{D}}^n_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^n_Y/Y}) &= \mathrm{Ext}^n_R(R/J, M)
\end{aligned}$$

where the fourth equality $H^0(\mathring{\mathbb{D}}^n_Y, \mathcal{E}xt^n_{\mathring{\mathbb{D}}^n_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^n_Y/Y})) = \mathrm{Ext}^n_{\mathring{\mathbb{D}}^n_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^n_Y/Y})$ comes from the spectral sequence for the derived functor of the composition being degenerate. Indeed, this spectral sequence is degenerate since $\mathring{\mathbb{D}}^n_Y$ is quasi-Stein and $\mathcal{E}xt^i_{\mathring{\mathbb{D}}^n_Y}(u_*\mathcal{O}_X, \omega_{\mathring{\mathbb{D}}^n_Y/Y})$ is a coherent $\mathcal{O}_{\mathring{\mathbb{D}}^n_Y}$-module for all i (see [Chi90, Proposition 3.3]). To study whether (7.13) commutes, we first need to describe the relevant maps explicitly.

The map ξ

Due to the smoothness of α, we can assume that $(f_1, \dots, f_n)$ is a regular sequence and J/J^2 is a free R/J-module of rank n with basis given by the residue classes $\overline{f_i}$ of the f_i.

Lemma 7.13. *The map ξ has the following explicit description*

$$\begin{aligned}\xi\colon B = R/J &\xrightarrow{\sim} (\bigwedge^n J/J^2)^\vee \otimes_{R/J} M/JM \\ 1 &\longmapsto [\overline{f_1} \wedge \ldots \wedge \overline{f_n}]^\vee \otimes u^*(df_1 \wedge \ldots \wedge df_n)\end{aligned}$$

where, for $m \in M$, $u^*(m) := m \otimes 1 \in M \otimes_R R/J = M/JM$.

Proof. Since X/Y is étale, the exact sequence (7.4) is just the isomorphism

$$\delta : J/J^2 \xrightarrow{\sim} \Omega^1_{R/A} \otimes_R R/J$$

where, for any $b \in J$, we have $\delta(\bar{b}) = db \otimes 1$. Taking the n-th exterior power of the isomorphism δ and then tensoring it with $(\bigwedge^n J/J^2)^\vee$ yields the map described in the assertion of the lemma. But it also yields ξ by the definition of ξ, cf. (7.5). □

The map ξ is canonical by construction, so it is independent of the choice of elements $f_1, \ldots, f_n \in J$ yielding a presentation as in (7.12). This is indeed evident in the explicit description of ξ, for if $h_1, \ldots, h_n \in J$ is another choice of such elements, then any choice of a matrix $G = (g_{ij})$ with coefficients $g_{ij} \in R$ satisfying $\overline{h_i} = \sum_j \overline{g_{ij}}\,\overline{f_j}$ yields the relations $[\overline{h_1} \wedge \ldots \wedge \overline{h_n}]^\vee = \overline{\det(G)}^{-1} \cdot [\overline{f_1} \wedge \ldots \wedge \overline{f_n}]^\vee$ and $u^*(dh_1 \wedge \ldots \wedge dh_n) = \overline{\det(G)} \cdot u^*(df_1 \wedge \ldots \wedge df_n)$.

The map η

Next, the description of the map $\eta\colon \operatorname{Ext}^n_R(R/J, M) \xrightarrow{\sim} (\bigwedge^n J/J^2)^\vee \otimes_{R/J} M/JM$ will be entirely analogous to the one in [Chi90, (4.14)]. For this, we first need to discuss Koszul complexes. If S is a ring and $t_1, \ldots, t_d$ an ordered sequence of d elements, the Koszul complex $K_\bullet(\boldsymbol{t})$ has the term

$$K_{-p}(\boldsymbol{t}) := \bigwedge^p S^d$$

in degree $-p$ for $0 \le p \le d$ and boundary operator $d_{-p}\colon K_{-p}(\boldsymbol{t}) \longrightarrow K_{-(p-1)}(\boldsymbol{t})$ determined by $d_{-p}(e_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum_{j=1}^p (-1)^{j+1} t_{i_j} e_{i_1} \wedge \ldots \wedge \widehat{e_{i_j}} \wedge \ldots \wedge e_{i_p}$, where $e_1, \ldots, e_d \in S^d$ denote the standard basis vectors. Then, given any S-module N, we are interested in the complex

$$K^\bullet(\boldsymbol{t}; N) := \operatorname{Hom}_S(K_{-\bullet}(\boldsymbol{t}), N)$$

which is concentrated in degrees from 0 to d and whose boundary maps are given by

$$d^p(\gamma)(e_{i_1} \wedge \ldots \wedge e_{i_{p+1}}) = \sum_{j=1}^{p+1} (-1)^{j+1} \cdot t_{i_j} \cdot \gamma(e_{i_1} \wedge \ldots \wedge \widehat{e_{i_j}} \wedge \ldots \wedge e_{i_{p+1}}).$$

If $t_1, \ldots, t_d$ is a regular sequence, then $K_\bullet(\boldsymbol{t})$ provides a free resolution of S/I, where $I := (t_1, \ldots, t_d)$. In particular, the complex $K^\bullet(\boldsymbol{t}; N)$ then computes the Ext-groups $\mathrm{Ext}^p_S(S/I, N)$, i.e. we have isomorphisms

$$\psi^p_{t,N} \colon \mathrm{Ext}^p_S(S/I, N) \xrightarrow{\sim} H^p(K^\bullet(\boldsymbol{t}; N)). \tag{7.14}$$

Since the d-th cohomology group of $K^\bullet(\boldsymbol{t}; N)$ is N/IN (cf. the discussion subsequent to (1.19)), composing this identification with $\psi^d_{t,N}$ yields an isomorphism

$$\varphi_{t,N} \colon \mathrm{Ext}^d_S(S/I, N) \xrightarrow{\sim} N/IN \tag{7.15}$$

in degree d, dependent on the choice of $\boldsymbol{t}$. This dependence on $\boldsymbol{t}$ can be made explicit as follows: If $u_1, \ldots, u_d \in I$ is another choice of such elements and $G = (g_{ij})$ is a $(d \times d)$-matrix with coefficients $g_{ij} \in S$ satisfying $\overline{u_i} = \sum_j \overline{g_{ij}}\,\overline{t_j}$, then the exterior powers of the matrix G yield an isomorphism between the complexes $K^\bullet(\boldsymbol{t}; N)$ and $K^\bullet(\boldsymbol{u}; N)$. In particular, we see that $\varphi_{u,N} = \overline{\det(G)} \cdot \varphi_{t,N}$. Therefore, the relation $[\overline{u_1} \wedge \ldots \wedge \overline{u_d}]^\vee = \overline{\det(G)}^{-1} \cdot [\overline{t_1} \wedge \ldots \wedge \overline{t_d}]^\vee$ then implies that the isomorphism

$$\begin{aligned} \eta_{I,N} \colon \mathrm{Ext}^d_S(S/I, N) &\xrightarrow{\sim} \Big(\bigwedge^d I/I^2\Big)^\vee \otimes_{S/I} N/IN \\ y &\longmapsto [\overline{t_1} \wedge \ldots \wedge \overline{t_d}]^\vee \otimes \varphi_{t,N}(y) \end{aligned}$$

is canonical. Our η is the following instance of this isomorphism:

$$\eta = \eta_{J,M}. \tag{7.16}$$

The map β

It remains to describe the map β. For $m \in \mathbb{N}$, we consider the open cover $\mathfrak{U}_{\varepsilon_m} = \{U_{i,\varepsilon_m}\}_{i=1,\ldots,n}$ of $\mathring{\mathbb{D}}^n_Y \setminus \mathbb{D}^n_Y(\varepsilon_m)$, where $U_{i,\varepsilon_m} = \{(z_1, \ldots, z_n, y) \colon |z_i| > \varepsilon_m\}$ and $\varepsilon_m \in |K^\times|$ satisfies $\varepsilon_m \to 1$ as $m \to \infty$. Then, for any coherent sheaf $\mathcal{F}$ on $\mathring{\mathbb{D}}^n_Y$, we have $H^{n-1}(\mathfrak{U}_{\varepsilon_m}, \mathcal{F}) = H^{n-1}(\mathring{\mathbb{D}}^n_Y \setminus \mathbb{D}^n_Y(\varepsilon_m), \mathcal{F}) = H^n_{\mathbb{D}^n_Y(\varepsilon_m)}(\mathring{\mathbb{D}}^n_Y, \mathcal{F})$ for $n \geq 2$ and the $\varinjlim_m$ of these computes $H^n_c(\mathring{\mathbb{D}}^n_Y/Y, \mathcal{F})$, whereas for $n = 1$ we have $H^1_{\mathbb{D}^1_Y(\varepsilon_m)}(\mathring{\mathbb{D}}^1_Y, \mathcal{F}) = \mathrm{Coker}(H^0(\mathring{\mathbb{D}}^1_Y, \mathcal{F}) \longrightarrow H^0(\mathring{\mathbb{D}}^1_Y \setminus \mathbb{D}^1_Y(\varepsilon_m), \mathcal{F}))$. Now we *would like to* define a map of complexes

$$\theta^\bullet_N \colon K^\bullet(\boldsymbol{f}; N) \longrightarrow \varinjlim_m \check{C}^{\bullet-1}(\mathfrak{U}_{\varepsilon_m}, \mathcal{F}) \tag{7.17}$$

(where $\mathcal{F}$ corresponds to the R-module N over $\mathbb{D}^n_Y$) as follows: Given a $\gamma \in K^p(\boldsymbol{f}; N) = \mathrm{Hom}_R(K_{-p}(\boldsymbol{f}), N)$, choose any $m \gg 0$ and send γ to the Čech $(p-1)$-cochain whose $(i_1, \ldots, i_p)$-coordinate (for $i_1 < \ldots < i_p$) is

$$\frac{\gamma(e_{i_1} \wedge \ldots \wedge e_{i_p})}{f_{i_1} \cdots f_{i_p}}.$$

In the following, we often simply write θ instead of $\theta_N^\bullet$, unless we really need to keep track of N or the degree. For θ to be well-defined, we would need each power series f_{i_j} to be invertible on $\mathring{\mathbb{D}}_Y^n \setminus \mathbb{D}_Y^n(\varepsilon_m)$ for $m \gg 0$. However, this is not true in general in $n \geq 2$ variables, for example: $f(X,Y) := X - Y$ has the sequence of zeros (p^n+1, p^n+1) which approaches $(1,1)$ if $K = \mathbb{Q}_p$. Nevertheless, in the special case of Remark 7.16 below, this condition is satisfied and θ is well-defined.

Lemma 7.14. *Whenever it is well-defined, θ is indeed a map of complexes.*

Proof. We will prove that

$$\begin{tikzcd}
\mathrm{Hom}_R(K_{-p}(\boldsymbol{f}), N) \arrow[r, "d^p"] \arrow[d, "\theta^p"'] & \mathrm{Hom}_R(K_{-(p+1)}(\boldsymbol{f}), N) \arrow[d, "\theta^{p+1}"] \\
\check{C}^{p-1}(\mathfrak{U}_{\varepsilon_m}, \mathcal{F}) \arrow[r, "\partial^{p-1}"] & \check{C}^p(\mathfrak{U}_{\varepsilon_m}, \mathcal{F})
\end{tikzcd}$$

commutes. Starting in the top left of the above diagram with a $\gamma \in \mathrm{Hom}_R(K_{-p}(\boldsymbol{f}), N)$ and first going right and then down yields the Čech p-cochain whose $(i_1, \ldots, i_{p+1})$-coordinate is

$$\frac{d^p(\gamma)(e_{i_1} \wedge \ldots \wedge e_{i_{p+1}})}{f_{i_1} \cdots f_{i_{p+1}}} = \frac{\sum_{j=1}^{p+1} (-1)^{j+1} \cdot f_{i_j} \cdot \gamma(e_{i_1} \wedge \ldots \wedge \widehat{e_{i_j}} \wedge \ldots \wedge e_{i_{p+1}})}{f_{i_1} \cdots f_{i_{p+1}}}.$$

On the other hand, first going down and then right yields the Čech p-cochain whose $(i_1, \ldots, i_{p+1})$-coordinate is

$$\begin{aligned}
&\sum_{j=1}^{p+1} (-1)^{j+1} \cdot (\text{the } (i_1, \ldots, \widehat{i_j}, \ldots, i_{p+1})\text{-coordinate of } \theta^p(\gamma)) \\
=&\sum_{j=1}^{p+1} (-1)^{j+1} \cdot \frac{\gamma(e_{i_1} \wedge \ldots \wedge \widehat{e_{i_j}} \wedge \ldots \wedge e_{i_{p+1}})}{f_{i_1} \cdots \widehat{f_{i_j}} \cdots f_{i_{p+1}}}
\end{aligned}$$

which obviously coincides with the above, proving that θ is a map of complexes. $\square$

For the sake of further discussion, assume that the power series f_i are invertible on $\mathring{\mathbb{D}}_Y^n \setminus \mathring{\mathbb{D}}_Y^n(\varepsilon_m)$ for $m \gg 0$ (so in particular θ is well-defined). Then there is a canonical map of functors

$$\mathrm{Ext}^0_R(R/J, -) \longrightarrow H^0_c(\mathring{\mathbb{D}}_Y^n / Y, -). \tag{7.18}$$

Indeed, the left-hand side, when evaluated at a module N, consists of elements of N that are annihilated by $J = (f_1, \ldots, f_n)$. Such elements, when regarded as global sections of the corresponding sheaf, have compact support (due to the assumption on the f_i), i.e. they live in the right-hand side of (7.18). Now we want to describe β with the help of θ, but first we need to introduce more notation. For $p \geq 2$, let

$$\overline{\theta^p_N} \colon H^p(K^\bullet(\boldsymbol{f}; N)) \longrightarrow \varinjlim_m H^{p-1}(\mathfrak{U}_{\varepsilon_m}, \mathcal{F}) = H^p_c(\mathring{\mathbb{D}}_Y^n / Y, \mathcal{F})$$

denote the map induced by θ in cohomology in degree p and let

$$\overline{\theta^1_N}\colon H^1(K^\bullet(\boldsymbol{f};N)) \longrightarrow \frac{\varinjlim_m H^0(\mathfrak{U}_{\varepsilon_m},\mathcal{F})}{H^0(\mathring{\mathbb{D}}^n_Y,\mathcal{F})} = H^1_c(\mathring{\mathbb{D}}^n_Y/Y,\mathcal{F})$$

denote the composite of the canonical projection modulo $H^0(\mathring{\mathbb{D}}^n_Y,\mathcal{F})$ with the map induced by θ in cohomology in degree 1. Moreover, let

$$\overline{\theta^0_N}\colon H^0(K^\bullet(\boldsymbol{f};N)) = \mathrm{Hom}_R(R/J,N) \longrightarrow H^0_c(\mathring{\mathbb{D}}^n_Y/Y,\mathcal{F})$$

be defined as in (7.18). Finally, recall the isomorphisms

$$\psi^p_{f,(-)}\colon \mathrm{Ext}^p_R(R/J,-) \xrightarrow{\sim} H^p(K^\bullet(\boldsymbol{f};-))$$

as in (7.14).

Conjecture 7.15. *Assume that the power series f_i are invertible on $\mathring{\mathbb{D}}^n_Y \setminus \mathring{\mathbb{D}}^n_Y(\varepsilon_m)$ for $m \gg 0$ (so in particular θ is well-defined). Then:*

(i) *The maps*

$$\overline{\theta^\bullet_{(-)}} \circ \psi^\bullet_{f,(-)}\colon \mathrm{Ext}^\bullet_R(R/J,-) \longrightarrow H^\bullet_c(\mathring{\mathbb{D}}^n_Y/Y,-)$$

build a morphism of δ-functors. Note that $\overline{\theta^0_{(-)}} \circ \psi^0_{f,(-)}$ by definition coincides with (7.18).

(ii) *β is induced by the unique δ-functorial map*

$$\mathrm{Ext}^\bullet_R(R/J,-) \longrightarrow H^\bullet_c(\mathring{\mathbb{D}}^n_Y/Y,-)$$

which is the canonical map (7.18) in degree 0.

If Conjecture 7.15 were true, an immediate consequence would be that

$$\beta = \overline{\theta^n_M} \circ \psi^n_{f,M} \tag{7.19}$$

as maps $\mathrm{Ext}^n_R(R/J,M) \longrightarrow H^n_c(\mathbb{A}^n_Y/Y,\omega_{\mathbb{A}^n_Y/Y})$.

Remark 7.16. *Consider the special case where $X = Y, \alpha = id$ and $f_i = x_i$ and assume that (7.19) holds. Then the diagram (7.13) commutes.*

Proof. Starting with $1 \in B = A$ in the top left of the diagram (7.13) and using the explicit descriptions of ξ and η, we first observe that $\eta^{-1}(\xi(1))$ is the element $y \in \mathrm{Ext}^n_R(R/J,M)$ which satisfies $\varphi_{f,M}(y) = u^*(dx_1 \wedge \ldots \wedge dx_n)$. Then $\psi^n_{f,M}(y) \in H^n(K^\bullet(\boldsymbol{f};M))$ is represented by the element $\gamma \in \mathrm{Hom}_R(\bigwedge^n R^n, M)$ which is determined by $\gamma(e_1 \wedge \ldots \wedge e_n) = dx_1 \wedge \ldots \wedge dx_n$. Thus we see that $\beta(y) = \overline{\theta^n_M}(\psi^n_{f,M}(y)) = \overline{\theta^n_M}(\overline{\gamma}) =$ is the class of

$$\frac{dx_1 \wedge \ldots \wedge dx_n}{x_1 \cdots x_n}.$$

Finally, applying $\mathrm{Tr}_{x_1,\ldots,x_n}$ to this obviously produces 1, which coincides with $\mathrm{Tr}_{A/A}(1)$. $\square$

7.3.3 Recovering compatibility with respect to composition

Theorem 7.17. *Let $\alpha\colon X \longrightarrow Y$ be a finite étale morphism of smooth connected d-dimensional Stein spaces over K, so we have the commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "\alpha \quad (\dim 0)"] \arrow[dr, "(\dim d) \quad g\circ\alpha"'] & & Y \arrow[dl, "g \quad (\dim d)"] \\ & \mathrm{Sp}(K) & \end{tikzcd}$$

of smooth morphisms. Assume that Question 7.12 has an affirmative answer. Then, starting with the commutativity of the diagram

$$\begin{tikzcd} \boldsymbol{R}^d(g\circ\alpha)_!\omega_X \arrow[rr, "\boldsymbol{R}g_!(\mathrm{Tr}_{X/Y})"] \arrow[dr, "\mathrm{Tr}_X"'] & & \boldsymbol{R}^d g_!\omega_Y \arrow[dl, "\mathrm{Tr}_Y"] \\ & \mathcal{O}_{\mathrm{Sp}(K)} & \end{tikzcd} \tag{7.20}$$

(which is due to Proposition 7.10 (ii)), evaluating on global sections and applying the identifications from Lemma 7.8 and Lemma 7.11 recovers the commutativity of the diagram

$$\begin{tikzcd} H^d_c(X,\omega_X) \arrow[r, equal] \arrow[dr, "t_X"'] & H^d_c(Y,\alpha_*\omega_X) \arrow[r, "H^d_c(Y,t_\alpha)"] & H^d_c(Y,\omega_Y) \arrow[dl, "t_Y"] \\ & K. & \end{tikzcd}$$

Proof. First of all, the identity (7.9) amounts in our case to

$$\omega_X = \alpha^*\omega_Y \tag{7.21}$$

since α is étale. Moreover, since α is finite and thus in particular proper, we have $\alpha_! = \alpha_*$. We also have $\boldsymbol{R}\alpha_* = \alpha_*$ since α is finite. Thus in our case, the projection formula [AL20, Corollary 3.8.2] yields the isomorphism

$$\alpha_*(\alpha^*\omega_Y) = \omega_Y \otimes \alpha_*\mathcal{O}_X. \tag{7.22}$$

Combining (7.21) and (7.22), we note that the evaluation of (7.20) on global sections yields that

$$\begin{tikzcd} H^d_c(X,\omega_X) \arrow[r, equal] \arrow[drr, "\mathrm{Tr}_X"'] & H^d_c(Y,\alpha_*\omega_X) \arrow[r, equal] & H^d_c(Y,\omega_Y\otimes\alpha_*\mathcal{O}_X) \arrow[r, "H^d_c(Y,\mathrm{id}\otimes\mathrm{Tr}_{X/Y})"] & H^d_c(Y,\omega_Y\otimes\mathcal{O}_Y) \arrow[d, equal] \\ & & & H^d_c(Y,\omega_Y) \arrow[dl, "\mathrm{Tr}_Y"] \\ & & K & \end{tikzcd}$$

commutes. But we can expand this diagram to

$$\begin{tikzcd}
H^d_c(X,\omega_X) \arrow[r, equal] \arrow[ddrr, "\mathrm{Tr}_X = t_X"'] & H^d_c(Y,\alpha_*\omega_X) \arrow[r, equal] \arrow[drr, "H^d_c(Y,t_\alpha)"] & H^d_c(Y,\omega_Y \otimes \alpha_*\mathcal{O}_X) \arrow[r, "H^d_c(Y,\mathrm{id}\otimes\mathrm{Tr}_{X/Y})"] & H^d_c(Y,\omega_Y \otimes \mathcal{O}_Y) \arrow[d, equal] \\
 & & & H^d_c(Y,\omega_Y) \arrow[dl, "\mathrm{Tr}_Y = t_Y"] \\
 & & K &
\end{tikzcd}$$

and our goal is to deduce the commutativity of the lower trapezoid. Given the commutativity of the outer contour of the diagram, it suffices to show the commutativity of the upper triangle. But this is due to the commutativity of

$$\begin{tikzcd}
\alpha_*\omega_X \arrow[r, equal] \arrow[drr, "t_\alpha"'] & \omega_Y \otimes \alpha_*\mathcal{O}_X \arrow[r, "\mathrm{id}\otimes\mathrm{Tr}_{X/Y}"] & \omega_Y \otimes \mathcal{O}_Y \arrow[d, equal] \\
 & & \omega_Y
\end{tikzcd}$$

which holds because $\mathrm{Tr}_{X/Y} \colon \alpha_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y$ coincides with the usual trace map

$$\alpha_*\mathcal{O}_X \xrightarrow[(5.4)]{} \mathcal{H}om_Y(\alpha_*\mathcal{O}_X, \mathcal{O}_Y) \xrightarrow{f \longmapsto f(1)} \mathcal{O}_Y$$

by our assumption that Question 7.12 has an affirmative answer. $\square$

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