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Hodge Theoretic Aspects of Extended Mirror Symmetry

Under supervision of

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Abstract

We formulate extended mirror symmetry of Calabi-Yau threefolds with D-branes as an equivalence between variations of mixed Hodge structure under the mirror map. After an introduction to Hodge theoretic closed string mirror symmetry, we review the relation between D-branes, normal functions and extensions by algebraic cycles on the side of the B-model. We define an extension of the A-model variation of mixed Hodge structure whose flat connection is derived from an enhancement of the quantum product by holomorphic disks ending on Lagrangian submanifolds. Our construction is based on the Solomon-Tukachinsky axioms for open Gromov-Witten invariants together with the open WDVV equations and matches the predictions from extended mirror symmetry. For the particular case of homology spheres, we define an extension of Iritani's Gamma-integral local system and propose an extended version of the Gamma conjecture. We demonstrate the validity of the conjecture for the standard pair of branes in case of the quintic and prove a corresponding extended Mirror Theorem. Using the extended holomorphic anomaly equations, we explore novel invariants from one-loop amplitudes for cycles of van Geemen-type, whose A-model geometry is at present unknown.

Zusammenfassung

Wir formulieren die erweiterte Spiegelsymmetrie von Calabi-Yau 3-Mannigfaltigkeiten mit D-branen als ein Äquivalenz zwischen Variationen von gemischter Hodge-Struktur unter der Spiegelabbildung. Nach einer Einführung in die Hodge-theoretische Spiegelsymmetrie für geschlossene Strings, arbeiten wir die Verbindungen zwischen D-branen, Normalfunktionen und Erweiterungen durch algebraische Zykel auf Seiten des B-Modells auf. Wir definieren eine Erweiterung der A-Modell Variation von gemischter Hodge-Struktur, deren flacher Zusammenhang aus einer Ergänzung des Quantenproduktes um holomorphe Disks abgeleitet ist, die auf Lagrange-Untermannigfaltigkeiten enden. Unsere Konstruktion beruht auf den Solomon-Tukachinsky Axiomen für offene Gromov-Witten Invarianten zusammen mit den offenen WDVV Gleichungen und deckt sich mit den Vorhersagen der erweiterten Spiegelsymmetrie. Für den Spezialfall von Homologiesphären definieren wir eine Erweiterung von Iritanis Gamma-integralem lokalen System und schlagen eine erweiterte Variante der Gamma-Vermutung vor. Wir zeigen die Gültigkeit dieser Vermutung für das Standardpaar von Branen im Falle der Quintik und beweisen einen entsprechenden erweiterten Spiegelsatz. Indem wir die erweiterten holomorphen Anomaliegleichungen verwenden, untersuchen wir neuartige Invarianten von Schleifen-Amplituden für Zykel vom van Geemen-Typ, deren A-Modell Geometrie zu diesem Zeitpunkt unbekannt ist.

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Contents

1. Introduction

Initiated by the predictions of enumerative invariants for the quintic in [CdlOGP], mirror symmetry has been a source of plentiful novel mathematics related to the geometry of string compactifications. In its original form, it asserts a relationship between "mirror pairs" of Calabi-Yau threefolds X and Y, whose utilization as extra dimensions in string theory leads to the same effective physics in four dimensions. In terms of the topological phases of string theory [Wit1], it can be understood as an equivalence between the A-model on X and the B-model on Y, implying the characteristic relationship of Hodge numbers in Figure 1.1 and the encoding of highly non-trivial enumerative information of X in the Hodge theory of Y.

In particular, the Gromov-Witten invariants of X can be extracted from the periods that determine the variation of Hodge structure (VHS) associated to the middle dimensional cohomology of the mirror family $\mathcal{Y} \to \Delta^*$. This objective crucially involves the asymptotic behavior of this VHS near a point of "maximal degeneracy" in the moduli space, that can be locally described as a variation of mixed Hodge structure (VMHS) of a special (Hodge-Tate) type [Del4]. A Hodge theoretic framework for the mirror phenomenon is then based on the observation that this object admits an A-model interpretation in terms of the H^2-module structure on the even dimensional cohomology $H^{\text{even}}(X)$ defined by Gromov-Witten invariants leading to a quantum product [Mor2, Mor3] . As observed in [CF1, CF2], the appearance of variations of MHS of this kind in the A-model can be viewed as a general consequence of the cup product being deformed by a quantum potential subject to a WDVV-type of equation. While this quantum product captures the holomorphic components of the B-model periods, a certain (Gamma-)integral structure is necessary to link the Hodge asymptotics with intrinsic topological quantities of the A-model [KKP, Iri]. An equivalence of the respective variations then defines the notion of Hodge theoretic mirror pairs and is the consequence of Mirror Theorems [Giv, LLY], which imply the enumerative predictions of mirror symmetry in case of the quintic.

Open strings and D-branes become relevant in the homological mirror symmetry program [Kon1], in which mirror pairs of Calabi-Yau threefolds are characterized by an equivalence of the D-brane A_∞-categories associated to the A- and B-model. In the A-model, the derived Fukaya category $\mathrm{Fuk}(X)$ is defined via Floer theory, with objects given by suitably decorated Lagrangian submanifolds

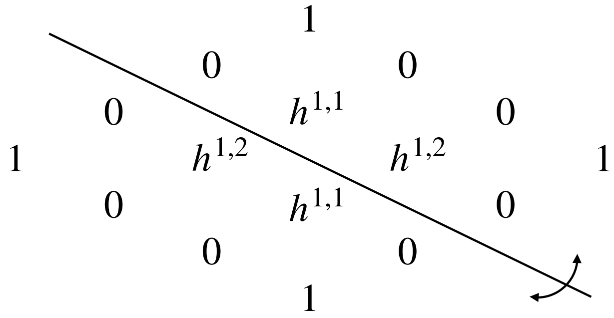

Figure 1.1.: Hodge diamond of a Calabi-Yau threefold. Given a mirror pair, the Hodge numbers are mirrored according to $h^{3-p,q}(X) = h^{p,q}(Y)$, exchanging deformation parameters for symplectic and complex structures. In terms of the Hodge diamond, this corresponds to a mirroring along the depicted axis.

of X, while on the B-side, the derived category of coherent sheaves $D^b(Y)$ is purely classical. Homological mirror symmetry then amounts to the statement

$$\mathrm{Fuk}(X) \cong D^b(Y), \tag{1.1}$$

which is by now proven for the quintic [She] and known to imply Hodge-theoretic closed string mirror symmetry with its enumerative predictions [GPS].

While the enumerative geometry of Lagrangian submanifolds implicitly appears in the definition of the Fukaya category [Fuk2], an explicit calculation of open Gromov-Witten invariants via mirror symmetry has originally been achieved only in certain (non-compact) toric situations [AV, AKV]. The first implementation of the traditional mirror principle in the presence of D-branes in compact Calabi-Yau threefolds, which are most relevant to phenomenological questions, has been put forward in [Wal1, PSW]. In analogy to the case of closed strings, the invariants associated to a given Lagrangian arise in a Hodge-theoretic quantity attached to the mirror object in the derived category, when expanded around a point of maximal degeneracy. Physically, this involves an equivalence between the corresponding A- and B-brane superpotentials, which are the basic observables of a given brane vacuum. While this thesis is exclusively concerned with the geometric aspects of brane superpotentials, their additional relevance in the standard scenarios for the construction of realistic string vacua, as for example [KKLT], should be noted. In this context, they appear as non-perturbative instanton corrections to the scalar supergravity potential governing the effective physics in four-dimensional spacetime [Wit4]. The physical situation which is relevant for this work is depicted in Figure 1.2.

On the B-side, the superpotential is given by the holomorphic Chern-Simons invariant. In many situations, it can be calculated from an algebraic cycle $\mathcal{C} \to \Delta^*$

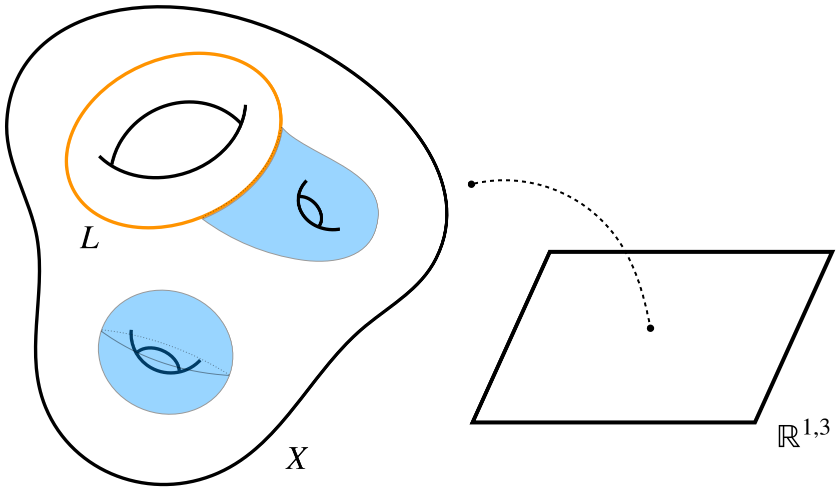

Figure 1.2.: Worldsheet instantons and string compactifications. The spacetime superpotential of a type IIA string compactification on X receives non-perturbative corrections from closed string worldsheet instantons. The presence of a space filling D6-brane wrapping a supersymmetric (special Lagrangian) cycle L additionally leads to instanton contributions from open strings that determine the brane vacuum. Aided by (extended) mirror symmetry, both objects are computable from the classical geometry of the mirror Calabi-Yau $\mathcal{Y}$.

that represents the algebraic Chern class of the B-brane. More precisely, the algebraic cycle determines a Hodge theoretic normal function that generally classifies extensions of variations of MHS [MW]. Until now, this extension of the VMHS in the B-model has not been interpreted geometrically in the A-model, partly due to the unavailability of sufficiently general axioms for open Gromov-Witten invariants which were only recently found in a series of papers by Solomon-Tukachinsky [ST1, ST2, ST3]. The purpose of this work is to fill this gap by providing an A-model description for the extended VMHS of the B-model. Guided by the lessons of the closed string, the central ingredients of our construction are extensions of the quantum product, the WDVV equations, and the Gamma-integral local system.

Parts of Chapters 3-6 of this dissertation are uploaded to the arXiv (see [HW]) and submitted to a journal.

Overview

This dissertation is structured in the following way. As many constructions and assumptions made on the side of the A-model are motivated by the physics of brane superpotentials, we begin with a review of all the relevant physical concepts in Chapter 2. This includes the general notion of a string vacuum and the topological phases of superstring theory, for which we describe the physical aspects underlying the mirror phenomenon. Subsequently, we discuss the sector of the open topological string, which will be the main focus of this work. We give a geometric account of A- and B-branes together with their associated superpotentials in the relevant regime, whose mathematical interpretation as Hodge theoretic (truncated) normal functions will be the topic of later chapters. The basis for the prediction of open Gromov-Witten invariants given in [Wal1], for which this thesis aims to provide a better understanding, is physically motivated and revisited in a Hodge theoretic language in Chapter 6.

In order to establish the mathematical background, the topic of Chapter 3 is the formulation of standard closed string mirror symmetry in terms of Hodge theory, following [DK2, dSJKP]. Starting with general definitions, we discuss the appearance of Hodge-Tate variations of MHS arising in both the A- and the B-model. On the B-side, we study the degenerating behavior of the VHS associated to the middle-dimensional cohomology of a family of threefolds $\mathcal{Y} \to \Delta^*$ around a point of maximal degeneracy. After collecting the relevant definitions and axioms of Gromov-Witten theory, we then describe the reconstruction of the B-model objects based on this data coming from the mirror A-model. While focusing on the one-parameter case, we emphasize the role of the WDVV equations in a multi-parameter situation and further describe the relevance of the Gamma class in defining the correct integral local system. We conclude the chapter with a description of Hodge theoretic mirror pairs and discuss the standard example of [CdlOGP] in this context.

Chapter 4 is concerned with the extension of B-model data by algebraic cycles following [MW, SVW2]. In this geometric situation, we describe their associated normal functions in terms of the Abel-Jacobi map and explain how the B-brane superpotential arises in this setting as a well-defined mathematical object. We then put a special emphasis on the asymptotic behavior of normal functions around a point of maximal degeneracy and derive the general structure of the B-brane superpotential based on monodromy considerations.

The A-model analog of this superpotential is discussed in Chapter 5. We begin with an overview of related topics in Lagrangian Floer theory pertaining to the Fukaya category of Calabi-Yau threefolds, providing the context to formulate open Gromov-Witten theory. After establishing topological properties of Lagrangians that lead to appropriate brane vacua, we characterize the superpotential taking into account the Solomon-Tukachinsky axioms, before addressing various classical contributions related to gauge bundle data on the brane.

Having identified all relevant structures on the side of the A-model, we are able to turn to the main objective of this thesis in Chapter 6. We define the notion of extended mirror pair based on an equivalence of extension classes and then provide a recipe to reconstruct the relevant normal function intrinsically in the A-model. The construction is based on an extension of the quantum product by holomorphic disks, subject to the open WDVV equations [Alc, ST3, CZ], and a characterization of an extended version of the Gamma-integral local system in certain examples. In particular, we are able to formulate an extended version of the Gamma conjecture for a special class of Lagrangians submanifolds. By studying the limiting behavior of the normal function in the A-model, we identify a type of algebraic object that abstractly captures properties of the A-model that mirror Hodge asymptotics in the presence of an algebraic cycle. Putting everything together, we conclude with the formulation of an extended Mirror Theorem for the main example of [Wal1], based on the equivalence of extensions of variations of MHS arising on the two sides of the duality. This theorem includes an A-model interpretation of the Abel-Jacobi limit as the Chern-Simons invariant of the mirror Lagrangian, whose explicit calculation is reviewed in Appendix B.

The construction of the extended Gamma-integral local system does not immediately generalize to the arithmetically interesting algebraic cycles akin to the van Geemen lines described in [Wal4, JW]. In these examples, where the A-model geometry is so far unknown, we produce novel invariants based on the calculation of one-loop amplitudes in Chapter 7. This includes the analysis of solutions to the extended holomorphic anomaly equations [Wal3] in three case studies that can hopefully contribute to the project of identifying potential mirror A-branes.

The main contribution of this thesis is the construction of an extended A-model VMHS that mirrors the Hodge theory of the corresponding B-model, as demonstrated for the quintic in Theorem 6.12. In all examples for which the topology of the Lagrangian submanifold is understood, this includes an intrinsic topological interpretation of the Abel-Jacobi limit, as described in Conjecture 6.8. When it comes to a Hodge theoretic understanding, these results elevate extended mirror symmetry to a more equal footing with its closed string counterpart in many regards, while also leading towards novel research directions, proposals for which we collected in Chapter 8.

2. String and Brane Vacua

In this chapter we present a review of the physics background that is relevant for the topics presented in this thesis. Besides providing additional context to the mathematical theory that will be developed, it is supposed to serve two purposes: We want to briefly introduce all physical concepts that also appear in the purely mathematical discussion and motivate various constructions of the coming chapters. The following treatment is kept on a very qualitative level without focus on detailed calculations, all of which can be found in a vast literature (see below). In Section 2.1 we begin with a motivation for the study of Calabi-Yau threefolds in string theory, before turning to its topological phases and the phenomenon of mirror symmetry in Section 2.2. We end with a discussion of D-branes in Section 2.3, whose role in the context of mirror symmetry will be most import in what follows.

2.1. Calabi-Yau Manifolds as String Backgrounds

We will partially follow the reviews [NV, Von] and also [Hor, Asp, CK2]. The propagation of closed bosonic strings in some spacetime can be described by a *sigma model* in which the dynamical fields are given by maps

$$\phi : \Sigma \longrightarrow M \tag{2.1}$$

from a closed two-dimensional surface Σ, together with a prescribed metric h, into a *target space* or *string background* M. It is viewed as the worldsheet sweeped out by the motion of closed strings in M which, at this stage, is taken to be a general Lorentzian manifold of any dimension d. The case in which Σ has a boundary corresponds to the open string, which will be discussed in Section 2.3. The coordinate fields of ϕ are naturally governed by a sigma model action that couples their dynamics to the geometry of M, defining a two-dimensional field theory on the worldsheet. Due to the symmetries of the sigma model action, this field theory classically only depends on the conformal class of the metric h, so that we can think of (Σ, h) as a compact Riemann surface.

Applying the procedure of quantization to these fields produces a Hilbert space of quantum states, consisting of a tower of string excitations including a massless level in which we are mostly interested. The interactions between string states can be described in terms of a path integral

$$\left\langle \prod_{i=1}^{n} V_i \right\rangle_X = \sum_{\text{genus}} \int [\mathscr{D}\phi \mathscr{D}h] \prod_{i=1}^{n} V_i \, e^{iS(\phi,h)}, \tag{2.2}$$

where the insertion V_i of a state is geometrically related to a puncture on the relevant Riemann surface. The integral contains a sum over all possible genera and an integral over the moduli space of conformal structures, coupling the sigma model to two-dimensional quantum gravity. The path integral is weighted by the classical action and endowed with an appropriately normalized measure, generalizing Feynman's ordinary path integral to string-like trajectories.

The massless states are of a special significance as they form the sigma model's geometric background and therefore interact with various geometric quantities associated to the target space. Famously, one of these excitations corresponds to a symmetric traceless tensor that can be interpreted as deformations of the spacetime metric G away from being flat and can be physically viewed as a graviton. The corresponding antisymmetric excitation of the closed string is called the *Kalb-Ramond field* which we will think of as a closed 2-form $B \in H^2(X;\mathbb{R})$. From the point of view of the sigma model, it naturally becomes part of the modular parameters of the theory, as we will see below.

Bosonic string theory can only be viewed as a toy model, as it cannot produce spacetime fermions and suffers infrared divergences associated to unnatural tachyon excitations. In order to make contact with observed nature it is therefore necessary to decorate the worldsheet theory of the bosonic sigma model with fermionic degrees of freedom in a way that remedies both of the above mentioned issues. The resulting superstring theory can be described by a sigma model based on a supergeometric extension of Riemann surfaces [Wit5]. In this way, the worldsheet of the superstring is endowed with a fermionic sector and a supersymmetry transformation that exchanges bosonic and fermionic degrees of freedom. The left and right moving generators $Q, \overline{Q}$ of this transformation satisfy anti-commutation relations of the form

$$\left\{Q, \overline{Q}\right\} \sim P, \quad \{Q, Q\} = 0, \quad \left\{\overline{Q}, \overline{Q}\right\} = 0, \tag{2.3}$$

where P is the four-momentum generating translations in spacetime. They are part of the superconformal algebra that replaces the classical symmetry group in case of the superstring. In the end, there are various ways in which bosonic and fermionic sectors can be combined on the worldsheet theory, leading to five perturbatively well defined superstring theories without the deficiences of the bosonic string. In what follows, we will be mostly concerned with the *type IIA/B* string.

In any case, the superconformal symmetry of the classical theory needs to survive the quantization process in order for the quantum theory to be well defined. The potential presence of associated anomalies in both the worldsheet theory and the loop expansion from the spacetime perspective produce nontrivial constraints on the allowed background geometry M. It can be seen that the worldsheet theory in flat space is constrained in a way that the central extension of the quantum super Virasoro algebra, comprising of generators of the superconformal symmetry, must vanish. This forces a constraint on the number of bosonic and fermionic degrees of freedom and the dimension of spacetime to the value $d = 10$. In a more general geometric background described by the sigma model, conformal invariance is equivalent to the vanishing of the beta function of the spacetime metric, order by order in perturbation theory. As it turns out, the beta function is at one-loop level proportional to the Ricci tensor of M

$$\beta_{\mu\nu}(G) \sim R_{\mu\nu} + \text{ higher loop corrections}, \tag{2.4}$$

which means that M is required to be Ricci-flat in order to define a proper solution of superstring theory. While it is known that the beta function does not receive quantum corrections at two- and three-loop level, the first non-zero contribution arises at four-loop and is a rational multiple of the zeta value $\zeta(3)$ [GvdVZ], which will have important appearances at various other places in this thesis. We will refer to a target space that is compatible with a quantum superstring theory to a given loop-level as a *string vacuum*.

In order to make contact with physics observed in 3+1 extended spacetime dimensions it is necessary to tailor realistic string vacua in a way that is able to reproduce the expected phenomenology. One way to do so is to consider solutions in which spacetime is topologically of the form

$$M = \mathbb{R}^{1,3} \times X \tag{2.5}$$

where $\mathbb{R}^{1,3}$ denotes ordinary, 3+1 dimensional Minkowski spacetime and X is a compact space of real dimension six. The phenomenological advantage of such a *string compactification* is that it is possible to study a regime in which the volume of X is assumed to be small in comparison to relevant scales in the extended dimensions.

The details of the geometry of X then determine the field content and dynamics of the four-dimensional effective theory that arises in $\mathbb{R}^{1,3}$ at larger scales. There are various phenomenologically favourable configurations that are potentially able to solve problems in particle physics and cosmology. One important tool in this endeavor is spacetime supersymmetry, which is therefore often supposed to be unbroken at least on some energy scale. Geometrically, this requirement translates to the most prominent class of string vacua, given by the following type of background.

Definition 2.1. A d-dimensional *Calabi-Yau manifold*[1] X is a compact, simply connected Kähler manifold with trivial canonical bundle

$$k_X = \bigwedge^d T_X^* \cong \mathcal{O}_X. \tag{2.6}$$

Equivalently, X admits a nowhere vanishing holomorphic d-form Ω which corresponds to a section of (2.6).

By Yau's theorem [Yau], X then famously admits a Kähler metric for which the Ricci curvature vanishes, making Calabi-Yau threefolds viable solutions to superstring theory, at least to the three-loop level of the sigma model perturbative expansion. The relevant Ricci flat metric is in general very hard to determine explicitly, but the corresponding volume form can be expressed in terms of the holomorphic d-form via

$$\mathrm{vol}_X = \Omega \wedge \overline{\Omega}. \tag{2.7}$$

As the choice of Ω is only unique up to a scalar multiple, the volume form is equally only well-defined up to scaling. A further important topological consequence of (2.6) is that the integral first Chern class of X must vanish,

$$c_1(X) = c_1(T_X) = 0 \in H^2(X;\mathbb{Z}). \tag{2.8}$$

Our main class of examples of Calabi-Yau manifolds are given by the following type of hypersurfaces.

Example 2.2. The canonical bundle of complex projective space is given by $k_{\mathbb{P}^N} = \mathcal{O}_{\mathbb{P}^N}(-N-1)$. For a smooth hypersurface $i : X \hookrightarrow \mathbb{P}^N$ of degree d the canonical bundle can be computed by the adjunction formula yielding

$$k_X = i^* k_{\mathbb{P}^N} \otimes \mathcal{O}_X(d) \cong \mathcal{O}_X(-N-1+d), \tag{2.9}$$

such that every smooth hypersurface of degree $N+1$ in $\mathbb{P}^N$ is Calabi-Yau. In particular, each quintic $X \subset \mathbb{P}^4$ is a Calabi-Yau threefold.

This thesis will be mainly concerned with Hodge theoretic properties of Calabi-Yau threefolds, where precise definitions will be given in Chapter 3. For the purpose of exposition it will suffice to study symmetries of the Hodge numbers and their corresponding effects on the geometry and physics of string backgrounds. One way to view the Hodge numbers is as dimensions of sheaf cohomology groups

$$h^{p,q}(X) = \dim H^q(X; \bigwedge^p T_X^*) \tag{2.10}$$

associated to various powers of the cotangent bundle. In case of a Calabi-Yau threefold, Definition 2.1 immediately implies that $h^{1,0} = 0$ and $h^{3,0} = 1$. Furthermore, the Hodge numbers satisfy the well known symmetry relations

$$h^{p,q} = h^{q,p} = h^{3-p,3-q} = h^{3-q,3-p}, \tag{2.11}$$

[1] In case of Calabi-Yau 3-manifolds, we will often simply use the term *threefold*.

which then already constrain most of their values. In particular, the only non-trivial and independent Hodge numbers are $h^{1,1}$ and $h^{1,2}$, which have a direct geometric interpretation as deformation parameters of the symplectic and complex structures that underly the Kähler manifold. Physically, these deformation parameters also arise as massless fields in the effective theory on $\mathbb{R}^{1,3}$.

Recall that the symplectic structure is determined by the Kähler class, a choice of two-form $J \in H^2(X;\mathbb{R})$. From the point of view of the sigma model the Kähler class arises from the massless closed string excitation associated to the spacetime metric as a consequence of the Kähler condition. As described above, the Kalb-Ramond B-field defines another two-form $B \in H^2(X;\mathbb{R})$ that equally has to be viewed as a defining property of the string vacuum. The Hodge number $h^{1,1}$ is then associated to the parameters space

$$H^1(X;\bigwedge\nolimits^1 T_X^*) \cong H^2(X;\mathbb{C}), \tag{2.12}$$

which records the variation of the *complexified Kähler class* of X, given by the form $\omega = B + iJ \in H^2(X;\mathbb{C})$, taking both objects into account. Turning to the complex structure, the relevant sheaf cohomology group becomes

$$H^2(X;\bigwedge\nolimits^1 T_X^*) \cong H^1(X;\mathcal{O}_X \otimes T_X) \cong H^1(X;T_X), \tag{2.13}$$

by using Serre duality and the Calabi-Yau condition. The right hand side of (2.13) can be viewed as the tangent space to the moduli space of complex structures via the Kodaira-Spencer map, such that $h^{1,2}$ is counting the number of complex structure parameters on X.

2.2. Topological Strings and the Mirror Phenomenon

As described in the previous section, the full superstring theory defined on a Calabi-Yau threefold interacts with both the symplectic and the complex geometric aspects of its background. However, there is a way in which certain topological phases can be extracted from the full spectrum which are only dependent on one type of parameter. These versions of *topological string theory* [Wit1] can be viewed as a toy model for the study of various types of string dynamics and dualities, but also capture supersymmetric observables of the full superstring theory. Their construction is based on a projection of states onto a spectrum that is nontrivially annihilated by a fermionic nilpotent symmetry, in very close analogy to the BRST procedure. This symmetry transformation corresponds in this context to a certain combination of the supercharges that are nilpotent due to the supersymmetry algebra satisfying (2.3).

In case of the closed (type II) string there is a $\mathcal{N} = (2,2)$ supersymmetry algebra generated by two fermionic supercharges $Q_\pm$ together with their complex conjugates $\overline{Q}_\pm$. In the supersymmetric sigma model, the bosonic sector determines the coordinate functions of the string, while fermionic degrees of freedom can be understood as complex differential forms. The action of the supercharges on both sectors can be determined explicitly, and it turns out that special combinations of the generators have a well known geometric meaning. These projections of the sigma model's spectrum are referred to as *topological twists* of the sigma model. In one version, called the *(topological) A-model*, the supercharges add up to the de Rham differential

$$Q_+ + \overline{Q}_- = d : \Omega^k(X) \longrightarrow \Omega^{k+1}(X) \tag{2.14}$$

such that its "physical" states correspond to classes in the de Rham cohomology $H^k_{\mathrm{dR}}(X)$ of the target space. On the other hand, the combination

$$Q_+ + \overline{Q}_+ = \overline{\partial} : \Omega^{p,q}(X) \longrightarrow \Omega^{p,q+1}(X) \tag{2.15}$$

reproduces the Dolbeault operator, acting anti-holomorphically on complex differential forms with a (p,q)-grading according to the number of its holomorphic and anti-holomorphic components. The spectrum of the resulting *(topological) B-model* therefore consist of classes in the Dolbeault cohomology $H^{p,q}_{\overline{\partial}}(X)$ associated to X. In both manifestations of the twisted theory, the dynamics of the remaining field content loses its explicit dependence on the metric, and the theories are in this sense topological.

The dynamics of the A- and B-model is determined by correlations functions defined in terms of a version of the path integral (2.2). In stark contrast to the general situation in quantum field theory, the path integral in presence of supersymmetry often simplifies significantly due to a phenomenon called *supersymmetric localization*. It refers to the fact the only non-vanishing contributions to the path integral arise from the fixed point locus of the fermionic symmetry generated by the supercharges Q. In case of the A-model, the Q-invariant field configurations translate to the maps $\phi : \Sigma \to X$ that are holomorphic, i.e. $\overline{\partial}\phi = 0$. Such fields are physically interpreted as worldsheet *instantons*, as they lack any time-dependence from the point of view of the Minkowskian part in (2.5), see also Figure 1.2. Turning to the B-model, the path integral localizes to field configurations that satisfy $d\phi = 0$, i.e. the sigma model consisting of constant maps. This means that the B-model essentially reduces to an ordinary quantum field theory on X. For our purposes, the most important type of interaction is the tree-level data captured by the *three-point correlation function* associated to the moduli for the complex and symplectic structure described earlier. Given a fixed background defined by a Calabi-Yau threefold (X, Ω, ω) with prescribed complex structure and complexified Kähler class, we start with the A-model and first consider elements $\omega_1, \omega_2, \omega_3 \in H^2(X;\mathbb{C})$. A path integral calculation shows that the three-point correlation function is given by (see e.g. [CK2])

$$\langle \omega_1, \omega_2, \omega_3 \rangle_X = \int_X \omega_1 \wedge \omega_2 \wedge \omega_3 + \sum_{\beta \neq 0 \in H_2(X;\mathbb{Z})} \widetilde{N}_\beta \int_\beta \omega_1 \int_\beta \omega_2 \int_\beta \omega_3 \; e^{2\pi i \int_\beta \omega}. \quad (2.16)$$

While the first term of (2.16) is classical, this function crucially contains a highly non-trivial sum of quantum contributions related to worldsheet instantons. Namely, the numbers $\widetilde{N}_\beta$ are defined as the count of possible instanton configurations associated to a given homology class $\beta \in H_2(X;\mathbb{Z})$ in a way that has to be made mathematically precise (see Section 3.3). Given $\theta_1, \theta_2, \theta_3 \in H^1(X, T_X)$ on the side of the B-model, which we interpret as vector fields on the moduli space of complex structures on X, the three-point correlation function is of the form

$$\langle \theta_1, \theta_2, \theta_3 \rangle_X = \int_X \Omega \wedge \left(\nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \Omega \right), \quad (2.17)$$

where ∇ denotes the Gauß-Manin connection. The details of how this function arises in the B-model will be given in Section 3.2. At this point it is important to note that (2.17) is essentially determined by the periods of the holomorphic three-form Ω, and therefore obtainable from standard calculations in complex algebraic geometry, contrary to the case of (2.16).

The point at which the mirror phenomenon enters into the story of topological strings is the definition of the generators $Q_\pm, \overline{Q}_\pm$ of the $\mathcal{N} = (2,2)$ supersymmetry algebra. These are only well defined up to a choice of sign where, from a physical point of view, both versions should lead to equally valid string vacua. However, reversing the sign cannot generally translate into a symmetry of a given Calabi-Yau background: This would exchange the notion of A- and B-model on X, which is generally not possible as the related cohomology groups do not have the same dimension. A remedy for this inconsistency lies in the proposal that for each string vacuum given by a Calabi-Yau threefold X, there exists another *mirror dual* Calabi-Yau threefold Y on which the sigma model yields a dual physical theory, up to the sign change described above. This duality implies that the A-model on X should be determined by the B-model on Y and vice versa. One consequence is therefore that the symplectic moduli of X correspond to the complex moduli of Y, i.e.

$$H^1(X; \textstyle\bigwedge^1 T_X^*) \cong H^2(Y; \textstyle\bigwedge^1 T_Y^*). \quad (2.18)$$

An investigation of the full spectrum produces similar equalities in all degrees such that the duality involves a mirroring of Hodge numbers

$$h^{3-p,q}(X) = h^{p,q}(Y) \quad (2.19)$$

that can be expressed by a symmetry of the Hodge diamond as depicted in Figure 1.1. In Definition 3.7 we will introduce a map between the relevant moduli spaces, defined in the neighborhood of special boundary points, of which (2.18) represents a linearization.

Perhaps the most amazing prediction of mirror symmetry between X and Y is the equality of three-point correlation functions

$$\langle \omega_1, \omega_2, \omega_3 \rangle_X = \langle \theta_1, \theta_2, \theta_3 \rangle_Y \,. \tag{2.20}$$

As stressed before, the A-model correlation function contains highly nontrivial quantum corrections associated to X, while the B-model correlation function is essentially a classical object. This means that, given a mirror dual pair of Calabi-Yau manifolds, the computation of highly non-trivial enumerative information in the A-model can be outsourced to the B-model, where the corresponding task is comparatively simple. As we will see in Section 3.6, the equality (2.20) does only hold in a certain *large volume limit*, for which the volume of X is large in comparison to the relevant quantum effects. Indeed, (2.16) is only a meaningful expression in a small neighborhood around such a large radius limit point (cf. Remark 3.14). The most well known example of a mirror pair in the above sense was first constructed in [GP] and the enumerative invariants contained in (2.20) were predicted in [CdlOGP].

So far we have presented mirror symmetry as a duality or phenomenon that arises in the geometry of string compactifications. This involves various objects that are not well defined mathematically, like the path integral in quantum field theory or the notion of superconformal field theory. The study of mirror symmetry in the mathematical context therefore involves organizing aspects of these physical theories into well-defined mathematical structures for which the phenomenon can be formalized. There are various such ways in which parts of the duality can be captured, including geometric approaches in the toric context [Bat], an equivalence between D-brane categories (see Section 2.3) and, for our purposes most importantly, a Hodge theoretic formulation. The Hodge theoretic approach is most closely related to the enumerative predictions described above and the relevant theory will be systematically developed in Chapter 3.

2.3. A- and B-Branes

So far, we focused on the sector of the closed string, whose massless excitations lead to Calabi-Yau threefolds as string backgrounds. However, this is only half of the story and misses the open string sector, which has an equally important role to play in both physical and mathematical applications. From a qualitative point of view, its description in terms of a sigma model

$$\phi : (\Sigma, \partial\Sigma) \longrightarrow X \tag{2.21}$$

is quite analogous to the closed case, with the exception that we have to consider two-dimensional surfaces with boundaries in the definition of the worldsheet. The key novelty is therefore the necessity to specify appropriate boundary conditions on the end points of the string, which determine whether they are allowed to move beyond specific subspaces or not. The subspaces onto which strings can be constrained in this way are called *D-branes* and are, through their interaction with open strings and potential backreactions with the geometric background, regarded as dynamical objects in open string theory. The superstring theory in which open superstrings and D-branes are included through an orientifold action is referred to as *type I* theory.

Again, massless excitations have a geometric meaning also for the open string, where they are related to the subspace on which the D-brane is localized. Those excitations parallel to the brane transform as vectors on the worldvolume of the brane and therefore have an interpretation as gauge fields in quantum field theory. The coupling to open strings then means that a single D-brane always comes equipped with the structure of a U(1)-bundle that defines the corresponding gauge theory[2]. Orthogonal excitations can be physically interpreted as the Goldstone bosons associated to the breakdown of Poincaré invariance and are related to geometric deformation parameters of the brane, as discussed in more detail in Appendix A. In superstring theory, D-branes additionally couple to certain higher form gauge potentials called *Ramond-Ramond (RR) fields*, such that a brane carries both a mass (proportional to its volume) and a natural charge associated to this coupling.

Turning to the topological phases of superstring theory, the truncation of the physical spectrum by the topologically twisted supersymmetry algebra also has an effect on the admissible boundary conditions of the open string. In the simplest cases, and in the large volume regime, topological branes correspond to smooth submanifolds with additional structure, where the requirement of compatibility with A- and B-type twisted supersymmetry poses restrictions on the number of dimension the brane is allowed to have. It turns out that in the A-model the dimension has to be odd, while there are only even dimensional branes in the B-model. A further important dynamical property of D-branes is the fact that the stable, energy minimizing configurations saturate the *Bogomol'nyi-Prasad-Sommerfeld (BPS) bound*, generally satisfied in supersymmetric quantum field theory. It says that the mass of a supersymmetric state

[2] We will mostly focus on the case of single branes. In case of a stack of N branes, the gauge group enhances to U(N).

is always bounded from below by a certain central charge of the supersymmetry algebra, which in case of D-branes corresponds to the RR charge. Given a p-dimensional brane N this means that

$$\int_N \mathrm{vol}_N \sim M \geq Z, \tag{2.22}$$

where the left hand side corresponds to the mass of the brane, determined by the volume induced from the metric on X and the brane tension, and Z is a topological charge. The energetically stable configurations, called *BPS branes*, are therefore specified by the property to be volume minimizing in their respective homology class. Namely, the saturation of the BPS bound for a p-dimensional brane mathematically translates into the notion of the brane being *calibrated* with respect to an appropriate p-form on X.

On a Calabi-Yau threefold, the type of calibrated submanifold depends on the dimension and corresponds to well known objects in symplectic and complex algebraic geometry [Joy]. In the A-model, BPS branes are given by special Lagrangian submanifolds $i : L \hookrightarrow X$, $\dim(L) = 3$, defined by the conditions

$$i^*\omega = 0, \qquad i^* \operatorname{Im}(\Omega) = 0. \tag{2.23}$$

Special Lagrangians in Calabi-Yau threefolds are calibrated by $\operatorname{Re}(\Omega)$. In the B-model, suitably normalized powers of the Kähler class always define a calibration, for which the calibrated submanifolds are simply the complex submanifolds $C \subset X$.

Due to the coupling to the open string sector, the topological branes in both twisted theories are only specified when a (holomorphic) U(1)-bundle, describing the associated gauge theory, is chosen. The basic observable of a given brane configuration then is the spacetime *superpotential*, which is closely related to certain versions of the Chern-Simons functional [Wit3], and depends on both open and closed string moduli. Given a pair $\mathcal{L} = (L, \mathcal{E})$ of a special Lagrangian submanifold, together with a U(1)-bundle in the A-model, the superpotential is given by[3]

$$\mathcal{W}_A = \int_L \operatorname{Tr}(A\, dA) + \text{instanton corrections}, \tag{2.24}$$

where $A \in \Omega^1(L, \mathfrak{u}(1))$ is the connection 1-form of the gauge bundle. The open string moduli are geometrically realized by the choice of connection 1-form, while the closed string Kähler modulus appears in the weighting of the instanton corrections as before. In analogy to the case of closed strings, these instantons can be described by a count of holomorphic maps from Riemann surfaces with

[3] This is the case only for a globally trivial bundle, a more general formulation of the Chern-Simons invariant is given in Definition 5.9.

boundary on L and a weighting that takes into account the topological data of the image. In general, the critical values of the superpotential are in correspondence with the vacua of physical theories living on the brane, posing further conditions on the type of geometry that is allowed in this context. Indeed, a classical constraint is the requirement for the bundle $\mathcal{E}$ to be flat, while quantum corrections produce two further conditions associated to quantum anomalies of the ghost number grading and tadpole cancellation. Both quantum properties of the brane have an immediate interpretation in the context of Floer theory and the Fukaya category, and are further discussed in Section 5.1. We will usually refer to the pair $\mathcal{L}$ as an *A-brane* on X, and the structure of the A-brane superpotential (2.24) can be best understood in a regime of two limits. For one, we again consider a large volume limit that controls the effect of instanton corrections in a way that makes the expression meaningful, as in the case of closed string instantons. Furthermore, for weak string coupling the dynamics of open and closed strings separate, such that the restriction onto the critical locus depends only on closed string moduli. In these limits, the superpotential takes the form [TV, CKLT, PSW]

$$\mathcal{W}_A\Big|_{\partial\mathcal{W}_A=0} = \frac{1}{2}\int_{\Gamma_4}\omega\cup\omega + \int_{\Gamma_2}\omega + \widetilde{c} + \sum_{\beta\neq 0\in H_2(X,L;\mathbb{Z})}\widetilde{n}_\beta\, e^{2\pi i\int_\beta\omega}, \tag{2.25}$$

where Γ_4, Γ_2 are smooth 4- and 2-chains with boundary on L and $\widetilde{n}_\beta$ denotes, roughly speaking, the number of holomorphic disks ending on L. In Section 5.2 we focus on a situation in which both the constant $\widetilde{c}$ and the numbers $\widetilde{n}_\beta$ can be made precise and discuss the various terms appearing in (2.25). Turning to the B-model, we note that the *B-brane* given by an algebraic curve $C\subset Y$ can also be understood in terms of a holomorphic vector bundle $E\to Y$, when C is the image of the algebraic second Chern class

$$E\longmapsto c_2^{\mathrm{alg}}(E)=C\in\mathrm{CH}^2(Y) \tag{2.26}$$

in the corresponding Chow group. The superpotential is in this case given by the holomorphic Chern-Simons functional

$$\mathcal{W}_B=\int_X\Omega\wedge\mathrm{Tr}\left(A\,\overline{\partial}A\right), \tag{2.27}$$

where now $A\in\Omega^{0,1}(X;\mathrm{End}(E))$. As in the case of the three-point correlation functions in the B-model, the expression (2.27) does not receive quantum corrections. In order to study the vacuum structure of the theory, we again reduce it to the critical locus, which is known to lead to the membrane integral [AV, KKLM]

$$\mathcal{W}_B\Big|_{\partial\mathcal{W}_B=0}=\int_\Gamma\Omega, \tag{2.28}$$

when C is homologically trivial in X and where Γ is a smooth 3-chain with boundary given by C. Again, (2.28) does only depend on closed string moduli in terms of the choice of holomorphic 3-form on X. This specific form of the superpotential can be interpreted in terms of a *(truncated) normal function*, that is more precisely described in Chapter 4.

As is the case for potentials in general, only differences of their values constitute proper physical observables. It is therefore important to stress that also the brane superpotentials in both topological phases have to be viewed as relative invariants. This means that the expressions (2.25) and (2.28) have to be interpreted as measuring an observable that is only meaningful when a reference brane vacuum is chosen. In most situations this implies that we have to consider Lagrangian or algebraic cycles with various homologically equivalent connected components, and that the relative chains have to be viewed as interpolating between the associated vacua. The relative nature of the superpotential includes the possibility to compare two vacua in which the only difference is the topological type of the flat bundle. This is a crucial ingredient in the correct understanding of the superpotential associated to the real quintic, as studied in Section 6.5. In those situations where a canonical reference vacuum is chosen among a collection of A- or B-branes, the symbol $\mathcal{W}$ will be used to denote the corresponding superpotential. The difference of two superpotentials measured in relation to the same vacuum

$$\mathcal{T}_{ij} := \mathcal{W}_j - \mathcal{W}_i \tag{2.29}$$

is often understood as the *BPS domain wall tension* between the two brane vacua. Applying the mirror principle to a pair of mirror dual Calabi-Yau threefolds, X and Y, suggests that for appropriately chosen mirror symmetric brane configurations there should be an equality of domain wall tensions

$$\mathcal{T}_A^{(X)} = \mathcal{T}_B^{(Y)} \tag{2.30}$$

under the mirror map. Such an equality was first successfully established in case of the quintic by Walcher in [Wal1]. The analogy to the equality of three point functions (2.20) lies in the fact that the superpotential encodes the two-point correlation function on the disk, associated to the relevant brane vacua. Together, these objects determine the tree-level structure of the underlying open-closed physical theory and are expected to satisfy

$$\langle \omega_1, \omega_2 \rangle_{(X,L)} = \langle \theta_1, \theta_2 \rangle_{(Y,C)} \tag{2.31}$$

as a consequence of (2.30). As we will see, these two-point correlation functions on the disk in both topologically twisted theories also have a well defined meaning in the Hodge theoretic context. Their precise definitions in the relevant limit are given in Section 4.3 and Section 6.1.

We close this chapter with a brief account of more general A- and B-branes and their role in a categorical approach to mirror symmetry. The A-branes on X described above constitute the simplest objects of a decorated version of the Fukaya category, which will be described in more detail in Section 5.1. Due to convergence issues regarding the instanton corrections in (2.24), the Fukaya category is currently only understood in a small neighborhood around large volume. Its counterpart on the B-model side is given by the derived category of coherent sheaves $D^b\,\mathrm{Coh}(Y)$ on Y, of which holomorphic vector bundles with their associated complex submanifolds are elementary examples. Both categories are endowed with an A_∞ structures and the equivalence of categories[4]

$$\mathrm{Fuk}(X) \cong D^b\,\mathrm{Coh}(Y) \tag{2.32}$$

in the large volume regime is the statement of *homological mirror symmetry* [Kon1]. While still a conjecture in the context of general Calabi-Yau threefolds, homological mirror symmetry is by now proven for quintic threefolds of the type described in Example 2.2 [She]. For our purposes, there is no need to go into more details regarding the explicit realization of the equivalence. However, it can be viewed as providing examples of brane configurations for which equalities of the type (2.30) hold, leading to the notion of extended mirror pair introduced in Section 6.1.

[4] This involves a certain derived version of the Fukaya category.

3. Closed String Mirror Symmetry

Having provided the physical context in which mirror symmetry arises as a duality of Calabi-Yau threefolds, we now focus on the Hodge theoretic approach based on which many aspects on the phenomenon can be made precise. In this chapter we begin with a treatment of those structures that are physically related to the closed string described in Section 2.1 and Section 2.2. This includes the enumerative predictions of [CdlOGP] contained in the equality of three-point correlation functions (2.20). We start with a collection of relevant definitions in (mixed) Hodge theory in Section 3.1, before describing how they arise in the B-model in Section 3.2. A mathematical description of the instanton corrections in (2.16) is given in Section 3.3. How the related structures can be organized Hodge theoretically is discussed in Section 3.4 and Section 3.5, and we end with the definition of Hodge theoretic mirror pairs in Section 3.6.

3.1. Mixed Hodge Theory

A mathematical framework that systematically captures the enumerative aspects of the mirror phenomenon is based on Hodge theory. While traditionally studied on the B-model side, the key insight in the present setting is that the corresponding algebraic structure also arises in the A-model. We therefore start with a mostly algebraic collection of the relevant definitions, together with brief explanations based on the geometric realization in the context of complex algebraic geometry, see e.g. [PS, CMSP, CK2], and also [DK2] for the case of threefolds.

Definition 3.1. Let $H_{\mathbb{Z}}$ be an abelian group and denote by $H = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ its complexification. We call H a *pure Hodge Structure (HS) of weight* n, if it admits a *Hodge decomposition*

$$H = \bigoplus_{p+q=n} H^{p,q} \tag{3.1}$$

into complex subspaces subject to the condition $\overline{H^{p,q}} = H^{q,p}$, with respect to the complex conjugation determined by the complexification. An element of $H^{p,q}$ is said to be of *Hodge type* (p,q). Equivalently[1], H is endowed with a decreasing *Hodge filtration* $F^\bullet$

$$H = F^0 \supset F^1 \supset \cdots \supset F^{n-1} \supset F^n \supset F^{n+1} = \{0\}, \tag{3.2}$$

[1]By setting $H^{p,q} = F^p \cap \overline{F^q}$ and $F^p = \bigoplus_{i \geq p} H^{i,n-i}$.

satisfying $H = F^p \oplus \overline{F^{n-p+1}}$ for all p. A *polarization* is a non-degenerate bilinear form

$$Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \longrightarrow \mathbb{Z}, \tag{3.3}$$

linearly extended to H, that is $(-1)^n$-symmetric and satisfies the *Hodge-Riemann bilinear relations*

$$\begin{aligned} Q(\alpha,\beta) = 0 \quad & \text{for } \alpha \in H^{p,q}, \beta \in H^{p',q'} \text{ with } (p,q) \neq (q',p') \\ i^{p-q}\, Q(\alpha,\overline{\alpha}) > 0 \quad & \text{for } \alpha \neq 0 \in H^{p,q}. \end{aligned} \tag{3.4}$$

A morphism of (polarized) Hodge structures is a map between abelian groups which is compatible with the Hodge filtration.

The geometric standard case in which Hodge structures of weight n naturally arise is the integral cohomology $H^n(Y;\mathbb{Z})$ of compact Kähler manifolds Y in degree n. Here, the Hodge Theorem guarantees that the de Rham cohomology with complex coefficients always admits a decomposition into Dolbeault cohomology groups

$$H^n_{\mathrm{dR}}(Y;\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}_{\overline{\partial}}(Y). \tag{3.5}$$

When $Y \hookrightarrow \mathbb{P}^N$ is projective, the notion of polarization is closely related to the choice of an integral Kähler 2-form. Namely, by the Kodaira embedding theorem, the pullback of the 2-form $[\omega_{\mathrm{FS}}]$ associated to the unique Fubini-Study metric on $\mathbb{P}^N$ defines an integral Kähler class $\omega \in H^2(Y;\mathbb{Z})$. The pairing

$$Q(\alpha,\beta) = (-1)^{\frac{n(n-1)}{2}} \int_Y \alpha \wedge \beta \wedge \omega^{\dim(Y)-n}, \quad \alpha,\beta \in H^3(Y;\mathbb{C}), \tag{3.6}$$

then defines a polarization on the part of the cohomology that is generated from ω in spirit of the Lefschetz hyperplane theorem, i.e. on primitive cohomology. While a sum of expressions like (3.6) can be used to generally define a polarization on the full cohomology, this thesis will be mostly concerned with a situation in which all cohomology classes are generated via cup product with ω from a single non-primitive class. When the modular parameters of the geometric configuration are varied, as discussed below, the Hodge-Riemann bilinear relations (3.4) are necessary to constrain moduli in a way that makes polarized Hodge structures classifiable by their *periods*, i.e. integrals of closed forms over homology classes.

Before turning to this more dynamical situation, we consider the general case in which the Hodge decomposition theorem fails. The standard example is a non-smooth variety where the strata of a singularity locus can be associated Hodge structures of different weights that cannot be viewed as independent from one another. The correct way in which this can be algebraically summarized is due to Deligne [Del2].

Definition 3.2. Let $H_{\mathbb{Z}}$ be an abelian group together with a decreasing Hodge filtration $F^{\bullet}$ on H and an increasing *weight filtration* $W_{\bullet}$

$$\{0\} = W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_{m-1} \subset W_m = H_{\mathbb{Q}} \tag{3.7}$$

on $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. If the graded quotients

$$\mathrm{Gr}_k^W H = {}^{W_k}\!\diagup_{W_{k-1}} \otimes_{\mathbb{Z}} \mathbb{C} \tag{3.8}$$

define a pure HS of weight k with induced Hodge Filtration

$$\mathrm{Gr}_k^W F^p = {}^{(F^p \cap W_k + W_{k-1})}\!\diagup_{W_{k-1}} \otimes_{\mathbb{Z}} \mathbb{C} \tag{3.9}$$

for all k, the resulting object is called a *mixed Hodge structure (MHS)*.

Even though the standard Hodge decomposition fails in the context of a general MHS, there still is a unique decomposition

$$H = \bigoplus_{p,q \in \mathbb{Z}} H^{p,q}, \quad H^{p,q} = F^p \cap W_{p+q} \cap \left(\overline{F^p} + \sum_{j \geq 0} \left(\overline{F^{q-j-1}} \cap W_{p+q-j-2} \right) \right) \tag{3.10}$$

called the *Deligne-Bigrading* of H. It turns out that this bigrading is related to both the Hodge and weight filtrations in a way that most closely resembles the situation for a pure HS. Namely, it is uniquely determined by the properties

$$\begin{gathered} F^k = \bigoplus_{p,q \in \mathbb{Z}, p \geq k} H^{p,q}, \qquad W_l = \bigoplus_{p,q \in \mathbb{Z}, p+q \leq l} H^{p,q} \\ H^{p,q} \equiv \overline{H^{q,p}} \mod \bigoplus_{a<p, b<q} H^{a,b}. \end{gathered} \tag{3.11}$$

We will refer to the elements in the $H^{p,q}$-component of the MHS defined via (3.10) as of *Hodge-Deligne type* (p, q).

Example 3.3. A simple but important example is the pure HS $\mathbb{Z}(-1)$ of weight 2 given by the abelian group $(2\pi i)^{-1}\mathbb{Z} \subset \mathbb{C}$ for which the complexification is given by $\mathbb{Z}(-1) \otimes_{\mathbb{Z}} \mathbb{C} = H^{1,1}$. Here, the exponent of the $2\pi i$-prefactor is meant to record the weight, such that the n-fold product

$$\mathbb{Z}(-n) = \mathbb{Z}(-1) \otimes \cdots \otimes \mathbb{Z}(-1) \tag{3.12}$$

is a pure HS of weight $2n$ and Hodge-Deligne type (n, n) with $\mathbb{Z}(-n) \otimes_{\mathbb{Z}} \mathbb{C} = H^{n,n}$. A MHS satisfying $\mathrm{Gr}_{2i}^W = \mathbb{Z}(-i)^{\oplus d_i}$ and $\mathrm{Gr}_{2i+1}^W = \{0\}$ for all i is referred to as of *Hodge-Tate* type. We will think of it as consisting of building blocks of the form (3.12) in each weight.

In a geometric setting it is natural to consider not only a single variety but an appropriate family $\pi : \mathcal{Y} \to B$ over some parameter space. The analog of the abelian group $H^n(Y;\mathbb{Z})$ here corresponds to the higher direct images $R^n\pi_*\mathbb{Z}$ defining a locally constant sheaf on B, with stalks given by the integral cohomology of the fibers. How the information encoded in the (M)HS associated to each fiber varies with modular parameters is then captured by the following object.

Definition 3.4. Let $\mathcal{H}_\mathbb{Z}$ be an integral local system on a complex manifold B. We call the holomorphic vector bundle $\mathcal{H} = \mathcal{H}_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{O}_B$ a *variation of Hodge structure (VHS) of weight n over B*, if it admits a decreasing Hodge filtration $\mathcal{F}^\bullet \subset \mathcal{H}$ by holomorphic subbundles that induces a pure HS of weight n on each fiber and such that the *Gauß-Manin connection* $\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_B$, specified by $\nabla(\mathcal{H}_\mathbb{Z}) = 0$, satisfies the condition

$$\nabla\left(\mathcal{F}^p\right) \subset \mathcal{F}^{p-1} \otimes \Omega^1_B, \tag{3.13}$$

called *Griffiths transversality*. Similarly, a *variation of mixed Hodge structure (VMHS)* is specified by an additional increasing weight filtration $\mathcal{W}_\bullet \subset \mathcal{H}_\mathbb{Q}$ on $\mathcal{H}_\mathbb{Q} = \mathcal{H}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$ such that each fiber of $\mathrm{Gr}^{\mathcal{W}}_k\mathcal{H}$ defines a pure HS of weight k. A polarization of a variation of (M)HS is understood fiberwise and a morphism is a map between local systems that is compatible with all relevant additional structures.

Concretely, and in the specific case of a smooth projectic family $\mathcal{Y} \to B$, the Gauß-Manin connection can be locally described as

$$\nabla(\sigma \otimes f) = \sigma \otimes df, \qquad \sigma \in \mathcal{H}_\mathbb{Z}, f \in \mathcal{O}_B. \tag{3.14}$$

Roughly speaking, the Griffiths transversality condition reflects the fact that when differentiating a vertically closed differential form on the total space with p holomorphic components, the increase in holomorphic degree can only take place in a horizontal direction, i.e. along the base.

3.2. Maximal Degenerations of Calabi-Yau Threefolds

Following [SVW2, DK2], let $\pi : \mathcal{Y} \to \Delta^*$ be a smooth family of projective Calabi-Yau threefolds over a punctured disk Δ^* with local coordinate z, for which each fiber $\pi^{-1}(z) = Y_z$ is torsion free, simply connected and has Hodge numbers

$$h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1. \tag{3.15}$$

We think of the base as a small neighborhood around a boundary point of the compactified moduli space $\widehat{\mathcal{M}}(Y)$ of complex structures on a reference fiber Y

and further assume that there is a semi-stable continuation to the disk Δ

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook] \arrow[d] & \overline{\mathcal{Y}} \arrow[d] & Y_0 \arrow[l, hook'] \arrow[d] \\ \Delta^* \arrow[r, hook] & \Delta & \{0\}, \arrow[l, hook'] \end{tikzcd} \tag{3.16}$$

where the singular fiber Y_0 has at most normal crossing singularities.

While most of the story is true for a larger class of Calabi-Yau threefolds, we will be mainly concerned with the examples arising as complete intersections in some (weighted) projective space (see e.g. Example 2.2). These can be classified into 14 distinct types that are distinguished by their associated Hodge theory [DM2] and their periods are governed by a Picard-Fuchs differential equation of a generalized hypergeometric type, to be discussed below. Our running example will be the mirror quintic [GP], while we keep the notation flexible enough to also cover its closest relatives, see e.g Chapter 7. Consider the family of varieties

$$Y_\psi = \left\{ (x_1 : \cdots : x_5) \in \mathbb{P}^4 \,\middle|\, W_\psi = \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1x_2x_3x_4x_5 = 0 \right\} \subset \mathbb{P}^4. \tag{3.17}$$

It is invariant under the Greene-Plesser group

$$G = \left\{ (a_1, \ldots, a_5) \in \left(\mathbb{Z}/5\mathbb{Z}\right)^5 \,\middle|\, \sum_i a_i \equiv 0 \bmod 5 \right\} \Big/ \left(\mathbb{Z}/5\mathbb{Z}\right) \tag{3.18}$$

of symmetries leaving W invariant. Given an element $a \in G$, its action on Y_z is defined by multiplication of homogeneous coordinates with fifth roots of unity, i.e.

$$a \cdot (x_1 : \cdots : x_5) := (\mu^{a_1} x_1 : \cdots : \mu^{a_5} x_5), \qquad \mu = \exp(2\pi i/5). \tag{3.19}$$

While Y_ψ is generically smooth, the quotient Y_ψ/G has a singular locus consisting of the divisors $D_{ij} = \{x_i = x_j = 0\}/G$, that is inherited from the fixed points of the G-action on $\mathbb{P}^4$. By [Mor2], there is a smooth resolution

$$\widetilde{Y_\psi/G} \longrightarrow Y_\psi/G \tag{3.20}$$

of the quotient which is both compatible with the action of the Greene-Plesser group and the Calabi-Yau condition, see also [GGK1]. Slightly abusing notation, we denote by $\mathcal{Y}$ the resulting family, with fibers obtained by smoothly resolving these singularities, and having the Hodge numbers (3.15).

Due to the invariance of the mirror quintic family under rescalings of homogeneous coordinates by fifth roots of unity $\mathcal{Y}_\psi \cong \mathcal{Y}_{\mu\psi}$, it is natural to choose a local coordinate on the moduli space that incorporates this invariance. The usual choice is to set $z = (5\psi)^{-5}$ as the parametrization of the complex structure parameter. Topologically, the complex moduli space is now given by $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, i.e. a thrice punctured sphere in which each marking corresponds to a special singular point. At the Landau-Ginzburg point $z = \infty$ ($\psi = 0$) the symmetry group of the variety

$$Y_\infty = \left\{ (x_1 : \cdots : x_5) \in \mathbb{P}^4 \,\middle|\, W_0 = \sum_{i=1}^{5} x_i^5 = 0 \right\} \subset \mathbb{P}^4 \tag{3.21}$$

enhances because the condition in (3.18) becomes immaterial, leading to an orbifold singularity of the moduli space. The special point

$$z = z_C = 1 - 5^5 z \qquad (\psi^5 = 1) \tag{3.22}$$

is called the conifold point, because here the fiber of the variety defined by (3.17) acquires singularities whose neighborhood looks like a cone over $S^2 \times S^3$. The conifold locus has an important role to play in the general context of mirror symmetry, as the different ways in which the singularity can be resolved leads to description of a topology-change that can be physically interpreted as a geometric transition between different string vacua [GMS]. A topological description of this phenomenon can be found in [Ban]. For the purpose of this thesis the most import singularity corresponds to the puncture at $z = 0$ ($\psi = \infty$), usually called the *large complex structure limit*. At this point, the variety naively degenerates to a union of five intersecting $\mathbb{P}^3$'s defined by

$$Y_0 = \left\{ (x_1 : \cdots : x_5) \in \mathbb{P}^4 \,\middle|\, W_\infty = x_1 x_2 x_3 x_4 x_5 = 0 \right\} \subset \mathbb{P}^4. \tag{3.23}$$

This type of degeneration is considered "maximal" either on the basis of maximally decreasing volume determined by (2.7), see [KS], or due to considerations of a monodromy classification [Mor1, Del4] that will be discussed further below. As described in [GGK1], the singular fiber in the large complex structure limit admits a semi stable reduction, such that a small neighborhood of this singular point is our standard example of the family $\mathcal{Y} \to \Delta^*$ described in (3.16). The full moduli space of the mirror quintic is sketched in Figure 3.1.

As Y_z varies across Δ^*, its middle-dimensional integral cohomology fits into a $\mathbb{Z}$-local system given by the higher direct image $\mathcal{H}^3_\mathbb{Z} = R^3\pi_*\mathbb{Z}$ whose fibers are given by $(\mathcal{H}^3_\mathbb{Z})_z = H^3(Y_z; \mathbb{Z})$. Over $\mathbb{C}$, these cohomology groups admit a Hodge decomposition

$$H^3 := H^3(Y_z; \mathbb{C}) = H^{3,0}(Y_z) \oplus H^{2,1}(Y_z) \oplus H^{1,2}(Y_z) \oplus H^{0,3}(Y_z) \tag{3.24}$$

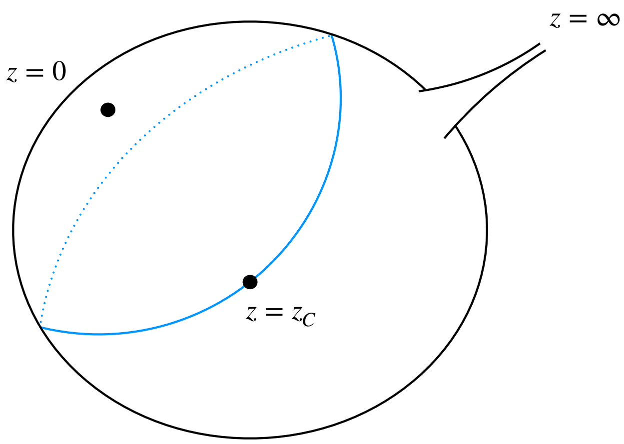

Figure 3.1.: Sketch of the B-model moduli space. Topologically, it is given by $\mathbb{P}^1$ with 3 distinguished points corresponding to the large complex structure ($z = 0$), conifold ($z = z_C$) and Landau-Ginzburg ($z = \infty$) points. The orbifold singularity at $z = \infty$ is depicted. The blue line indicates the radius of convergence of the periods expanded around $z = 0$ (see below).

for which there is a decreasing Hodge filtration

$$F^p = \bigoplus_{i \geq p} H^{i,3-i}(Y_z) \tag{3.25}$$

with $F^3 \subset F^2 \subset F^1 \subset F^0 = H^3$. The Hodge filtration varies holomorphically across Δ^*, such that $\mathcal{F}^p := F^p \otimes \mathcal{O}_{\Delta^*}$ defines a corresponding filtration of holomorphic subbundles on $\mathcal{H}^3 := \mathcal{H}^3_{\mathbb{Z}} \otimes \mathcal{O}_{\Delta^*}$. A natural antisymmetric polarization form $Q(\cdot,\cdot) : \mathcal{H}^3_{\mathbb{Z}} \otimes \mathcal{H}^3_{\mathbb{Z}} \to \mathbb{Z}$ is induced from the Poincaré duality pairing on integral cohomology and can be extended linearly to $\mathcal{H}^3$. The local system $\mathcal{H}^3_{\mathbb{C}} = \mathcal{H}^3_{\mathbb{Z}} \otimes \mathbb{C}$ uniquely determines the flat Gauß-Manin connection $\nabla : \mathcal{H}^3 \to \mathcal{H}^3 \otimes \Omega^1_{\Delta^*}$ with $\nabla(\mathcal{H}^3_{\mathbb{C}}) = 0$ that satisfies the Griffiths transversality condition $\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_{\Delta^*}$. This collection of algebraic data $(\mathcal{H}^3_{\mathbb{Z}}, \mathcal{H}^3, \mathcal{F}^\bullet, \nabla, Q)$ defines a polarized, integral VHS of weight 3 over Δ^*.

For a small loop $\gamma(t)$ around the singular point $0 \in \Delta$ based at some $z \in \Delta^*$, we can lift any class $g_z \in H^3(Y_z;\mathbb{Z})$ to a flat section $g(t) \in H^3(Y_{\gamma(t)};\mathbb{Z})$ over $[0,1]$ with $g(0) = g$. The monodromy operator $M : \mathcal{H}^3_{\mathbb{Z}} \to \mathcal{H}^3_{\mathbb{Z}}$ is then defined by $M(g) = g(1)$ in each fiber and guaranteed to be quasi-unipotent by the monodromy theorem [Lan]: Given a VHS over Δ^*, there is always an integer $m \geq 0$ such that

$$(M^m - id)^{n+1} = 0, \tag{3.26}$$

where n is (at most) the Hodge theoretic weight. Assuming that M is unipotent, i.e. $m=1$, the monodromy logarithm $N := \log(M) : \mathcal{H}^3_{\mathbb{Q}} \to \mathcal{H}^3_{\mathbb{Q}}$, defined by

$$N = \log(M) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (M - id)^k \,, \tag{3.27}$$

is a nilpotent operator on $\mathcal{H}^3_{\mathbb{Q}} = \mathcal{H}^3_{\mathbb{Z}} \otimes \mathbb{Q}$. Note that the sum (3.27) is always finite due to the unipotency of the monodromy. It induces the unique increasing monodromy weight filtration $W_\bullet := W_\bullet(N)$, satisfying the properties

$$\begin{gathered} W_{-1} = \{0\} \subset W_0 \subset \cdots \subset W_6 = \mathcal{H}^3_{\mathbb{Q}}, \\ N(W_k) \subset W_{k-2}, \qquad N^k : \mathrm{Gr}^W_{3+k} \xrightarrow{\sim} \mathrm{Gr}^W_{3-k}, \end{gathered} \tag{3.28}$$

where $\mathrm{Gr}^W_k = W_k/W_{k-1}$ for $k = 0, ..., 3$, from which explicit expressions for the form of its pieces can be derived [Gri, CK2]. In our situation, they are explicitly given by

$$\begin{aligned} W_0 &= \mathrm{im}(N^3) \\ W_1 &= \mathrm{im}(N^2) \cap \ker(N) \\ W_2 &= \mathrm{im}(N) \cap \ker(N) + \mathrm{im}(N^2) \cap \ker(N^2) \\ W_3 &= \ker(N) + \mathrm{im}(N) \cap \ker(N^2) + \mathrm{im}(N^2) \cap \ker(N^3) \\ W_4 &= \ker(N^2) + \mathrm{im}(N) \cap \ker(N^3) \\ W_5 &= \ker(N^3). \end{aligned} \tag{3.29}$$

The monodromy logarithm can also be employed to define the untwisted local system with corresponding connection

$$\widetilde{\mathcal{H}}^3_{\mathbb{Z}} := \exp\left(-\frac{\log(z)}{2\pi i} N\right) \mathcal{H}^3_{\mathbb{Z}}, \qquad \nabla^c := \nabla + \frac{N}{2\pi i} \frac{dz}{z}, \tag{3.30}$$

from which Deligne's canonical extension $\widetilde{\mathcal{H}}^3 := \widetilde{\mathcal{H}}^3_{\mathbb{Z}} \otimes \mathcal{O}_\Delta \to \Delta$ over the puncture can be constructed [Del4].

Lemma 3.5. The sections $\widetilde{g} \in \widetilde{\mathcal{H}}^3_{\mathbb{Z}}$ of the canonical extension are single-valued, i.e. they have trivial monodromy.

Proof. Monodromy around the puncture in the coordinate z is accomplished by $z \mapsto z\exp(2\pi i)$ such that $\log(z) \mapsto \log(z) + 2\pi i$. The action on a general element of $\widetilde{\mathcal{H}}^3_{\mathbb{Z}}$ is then given by

$$\begin{aligned} M(\widetilde{g}) = M\left(\exp\left(-\frac{\log(z)}{2\pi i}N\right) g\right) &= \exp\left(-\frac{\log(z)}{2\pi i}N\right) \exp(-N) M\, g \\ &= \exp\left(-\frac{\log(z)}{2\pi i}N\right) M^{-1} M\, g = \widetilde{g}, \end{aligned} \tag{3.31}$$

yielding the desired invariance under monodromy. □

As a consequence of the nilpotent orbit theorem [Sch], the Hodge filtration also extends to holomorphic subbundles $\widetilde{\mathcal{F}}^p \subset \widetilde{\mathcal{H}}^3 = \widetilde{\mathcal{H}}^3_{\mathbb{Z}} \otimes \mathcal{O}_\Delta$, with limiting filtration $F_0^\bullet \subset \widetilde{\mathcal{H}}^3_0$. This canonical extension of the Hodge theoretic data allows to assign a meaningful limit to the VHS at the puncture that carries information about the degeneration of the fiber Y_0 over $0 \in \Delta$. For this purpose, we untwist a multivalued basis $\{g_i\}$ of $\mathcal{H}^3_{\mathbb{Z}}$ via (3.30) and consider the local system $\widetilde{\mathcal{H}}^3{}_{\mathbb{Z},0}$ generated by $\widetilde{g}_i(0)$. The collection of data $(\widetilde{\mathcal{H}}^3_{\mathbb{Z},0}, \widetilde{\mathcal{H}}^3_0, F_0^\bullet, W_\bullet)$ defines a MHS called the *limiting mixed Hodge structure* (LMHS) of $\mathcal{H}^3$.

In the given situation, the LMHS can be classified in terms of the monodromy behavior around the puncture [GGK1], as depicted in Figure 3.2. While the upper left panel shows the nonsingular situation with trivial monodromy, all the further degenerations have an interesting geometric realization: The upper right panel corresponds to the degeneration at the aforementioned conifold locus $z = z_C$, in which cycles represented by 3-spheres collapse to a point in the B-model. The monodromy structure depicted in the lower left panel arises in a degeneration where the Calabi-Yau threefold collapses to a union of Fano varieties, referred to as Tyurin degeneration [DHT]. Of particular interest in the context of (Hodge theoretic) mirror symmetry is the case of a *maximal degeneration* depicted in the lower right panel. From this point of view, it is characterized by *maximally unipotent monodromy (MUM)* with $(M - id)^4 = 0$ and $(M - id)^3 \neq 0$ at $z = 0$.

At a MUM point, the LMHS is Hodge-Tate of the form

$$\mathbb{Z}(-3) \xrightarrow{N} \mathbb{Z}(-2) \xrightarrow{N} \mathbb{Z}(-1) \xrightarrow{N} \mathbb{Z}(0) \tag{3.32}$$

with $W_{2i} = \ker N^{i+1}$ and $\mathrm{Gr}^W_{2i} \cong \mathbb{Z}(-i)$ for $i = 0, ..., 3$ while otherwise $\{0\}$. In a neighborhood around the puncture, the limiting Hodge filtration can be extended to a filtration that is constant with respect to (3.30), leading to a VHS

$$\mathcal{H}^3_{\text{nilp}} := (\mathcal{H}^3_{\mathbb{Z}}, \mathcal{H}^3, \exp(-\frac{\log(z)}{2\pi i} N)\, F_0^\bullet, \nabla, Q), \tag{3.33}$$

called the *nilpotent orbit*, which approximates the original VHS asymptotically. Griffiths transversality follows from the property (3.59) of the Gauß-Manin connection discussed further below, which here implies $N(F_0^p) \subset F_0^{p-1}$. Together with the weight filtration $\mathcal{W}_\bullet = W_\bullet \otimes \mathcal{O}_{\Delta^*}$ the nilpotent orbit defines a Hodge-Tate VMHS on Δ^* with LMHS (3.32). It can be understood as the most trivial VMHS with a prescribed LMHS and its structure will become more apparent in our main example (3.52). Indeed, following Deligne [Del4], the filtration $\mathcal{W}_\bullet$ pairs up also with the original Hodge filtration such that the data $(\mathcal{H}^3_{\mathbb{Z}}, \mathcal{H}^3, \mathcal{F}^\bullet, \mathcal{W}_\bullet, \nabla, Q)$ defines a polarized, integral VMHS of Hodge-Tate type

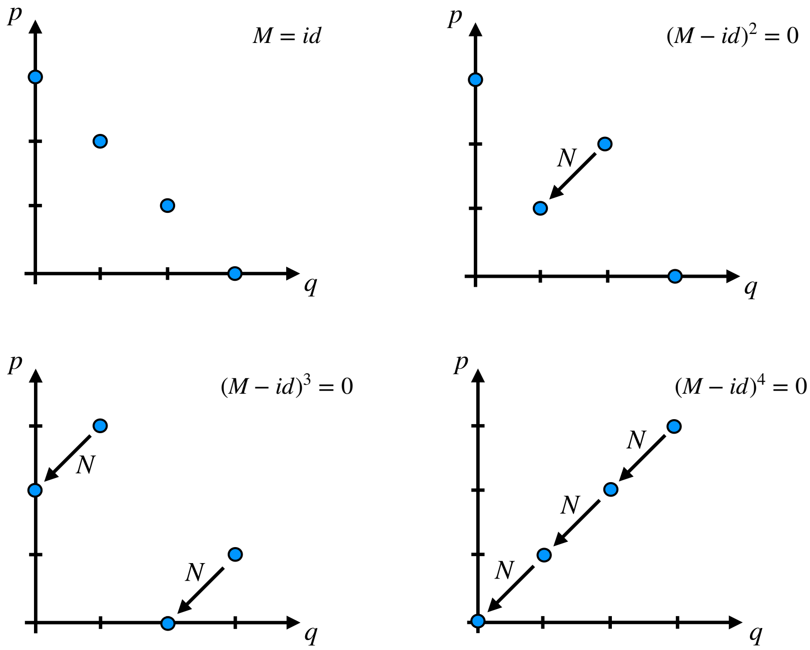

Figure 3.2.: Monodromy classification of limiting mixed Hodge structures. There are four possibilities in which a geometric degeneration can be reflected in the structure of the monodromy. The bullets at position (p, q) indicate that there is a non-zero component of corresponding Hodge-Deligne type. The arrows depict the action of the monodromy logarithm N.

in a neighborhood Δ^* of the MUM-point. Its periods consist of a ∇^c-constant part encoded in the nilpotent orbit, as well as holomorphic components vanishing at $0 \in \Delta$ [CK1]. This VMHS can be viewed as describing the degeneration of the original variation of pure HS to the LMHS (3.32) in terms of its periods, on which we will focus now.

We denote by $\Omega \in \Gamma(\mathcal{F}^3, \Delta^*)$ a choice of nonvanishing holomorphic 3-form on $\mathcal{Y}$. For any multivalued flat section g of $\mathcal{H}^3_{\mathbb{C}}$ the periods $Q(g, \Omega)$ are holomorphic functions on Δ^* satisfying a Picard-Fuchs differential equation of generalized hypergeometric type [DM2]. In a neighborhood of the MUM-point, it can be expressed in terms of a canonical logarithmic vector field via

$$D_{\mathrm{PF}}(\theta)\, Q(g, \Omega) = 0, \qquad \theta := \frac{d}{d\log(z)} = z\frac{d}{dz}. \tag{3.34}$$

While the periods generally diverge when approaching the puncture, the *limiting periods* $Q(\tilde{g}(0), \Omega)$ defined in terms of the untwisted local system are well defined and determine the LMHS at $z = 0$. In the examples of one-parameter families arising from complete intersections in weighted projective space [*op.cit.*], the

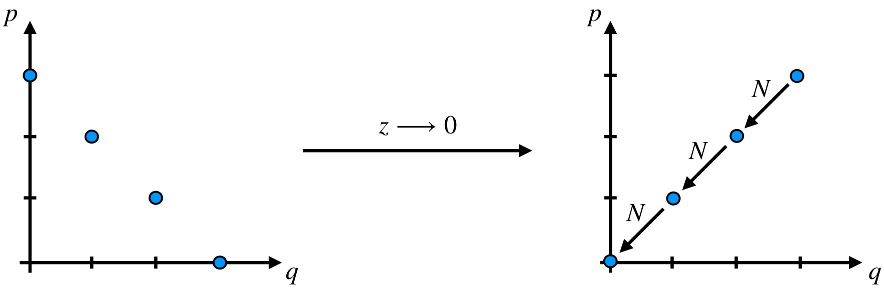

Figure 3.3.: LMHS associated to $\mathcal{H}^3$ at a point of maximal degeneration. The process of the degeneration can be summarized in the asymptotic behavior of the periods of the VMHS arising from a deformation of the nilpotent orbit.

Picard-Fuchs operator is of the generalized hypergeometric type

$$D_{\mathrm{PF}}\,(-) = (2\pi i)^2 \left[\theta^4 - z\prod_{k=1}^{4}(\theta + r_k)\right](-), \qquad r_k \in \mathbb{Q}. \tag{3.35}$$

It can always be brought into the form

$$D_{\mathrm{PF}}\,(-) = (2\pi i)^2 \left[\theta^4 + E_3(z)\,\theta^3 + E_2(z)\,\theta^2 + E_1(z)\,\theta\right](-), \tag{3.36}$$

where each E_i is analytic at $z = 0$ but has a singularity at the conifold point $z_C = 0$. We note that the first term is always given by

$$E_3(z) = -\frac{2\alpha z}{1-\alpha z}, \qquad z_C = 1 - \alpha z. \tag{3.37}$$

The singularity of the Picard-Fuchs operator at the conifold locus reflects the fact that the periods generally have a radius of convergence touching the corresponding point in the moduli space, as depicted in Figure 3.1.

Example 3.6. Turning to our running example, we consider the mirror quintic family and give an explicit description of the periods when expanded in a neighborhood of the boundary point with maximally unipotent monodromy. The periods can be explicitly calculated by integrating the holomorphic volume form over the vanishing cycle [CdlOGP], from which also the form of the Picard-Fuchs operator can be derived. For the standard choice of holomorphic 3-form

$$\Omega_z = \left(\frac{5}{2\pi i}\right)^3 \mathrm{Res}_{\,W=0}\frac{\sum_{i=1}^{5}(-1)^i x_i\, dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_5}{W}, \tag{3.38}$$

where W denotes the defining polynomial (3.17) in the variable z, the Picard-Fuchs operator is given by [DM2]

$$D_{\mathrm{PF}}\,(-) = (2\pi i)^2 \left[\theta^4 - 5z\prod_{k=1}^{4}(5\theta + k)\right](-). \tag{3.39}$$

Taking this operator to be given, the periods can be easily determined from a power series Ansatz and the Frobenius method. Namely, the unique fundamental solution of the Picard-Fuchs equation is found to be given by

$$\varpi_0 = \sum_{n=0}^{\infty} \frac{\Gamma(5n+1)}{\Gamma(n+1)^5}\, z^n = 1 + 120\, z + 113400\, z^2 + 168168000\, z^3 + \cdots\,, \tag{3.40}$$

which we will refer to as the *fundamental period* of the mirror quintic. A full system of solutions can now be derived via the Frobenius method, starting with the following hypergeometric series

$$\varpi(z,h) := \sum_{n=0}^{\infty} \frac{\Gamma(5(n+h)+1)}{\Gamma(n+h+1)^5}\, z^{n+h}, \tag{3.41}$$

with $\varpi_0 = \varpi(z,0)$. Because $D_{\mathrm{PF}}\,\varpi(z,s) = h^4 z^h + \mathcal{O}(h^5)$ and ∂_h commutes with D_{PF}, the functions

$$\varpi_i := (\partial_h)^i \Big|_{h=0} \varpi(z,h), \qquad i = 0,1,2,3, \tag{3.42}$$

are all solutions to the Picard-Fuchs equation. Writing

$$\varphi_i(z) = \sum_{n=0}^{\infty} (\partial_h)^i \Big|_{h=0} \frac{\Gamma(5(n+h)+1)}{\Gamma(n+h+1)^5}\, z^n, \tag{3.43}$$

these additional periods are iteratively given by the formula

$$\varpi_i(z) = \sum_{k=0}^{i} \binom{i}{k} \varphi_k(z) \log(z)^{i-k}. \tag{3.44}$$

Starting with the first iteration of this process

$$\varpi_1 = \varpi_0 \log(z) + \varphi_1 = \varpi_0 \log(z) + 770\, z + 810225\, z^2 + \frac{3745679000}{3}\, z^3 + \cdots\,, \tag{3.45}$$

the periods exhibit a diverging behavior for $i = 1,2,3$.

Following [SVW2, DK2, CK2, Mor1], we start from the periods ϖ_0 and ϖ_1, where we can find integral flat generators $g_i \in W_{2i} \cap \mathcal{H}^3_{\mathbb{Z}}$ together with a Hodge basis $\{e_j\} := \{e_3, e_2, e_1, e_0\}$ where $e_j \in \mathcal{F}^j$, which lead to a structure that can be reinterpreted from an A-model point of view. Starting with the holomorphic 3-form $e_3 = \Omega$ and the generator g_0 Poincaré dual to the minimal integral vanishing cycle, the function $\varpi_0 = Q(g_0, e_3)$ is up to a constant the unique fundamental period which is analytic at $z = 0$. Assuming the monodromy is *small* [Mor2], there exists a further generator g_1 with period $\varpi_1 = Q(g_1, e_3)$ that transforms according to $M(g_1) = g_1 + g_0$ such that

$$q(z) := \exp(2\pi i t(z)), \qquad t(z) = \frac{\varpi_1(z)}{\varpi_0(z)}, \tag{3.46}$$

is a well-defined function that serves as the unique *canonical coordinate* on Δ^*.

Definition 3.7. The map $m : q(z) \mapsto z(q)$ relating the coordinate z to the canonical coordinate q is called the *mirror map*.

Example 3.8. For the mirror quintic, the canonical coordinate can be computed from the periods (3.40) and (3.45) and is given by the power series

$$q(z) = z + 770\, z^2 + 1014275\, z^3 + \cdots\,. \tag{3.47}$$

The mirror map

$$z(q) = q - 770\, q^2 + 171525\, q^3 + \cdots \tag{3.48}$$

can be obtained by computing the inversion of the power series (3.47).

We choose (3.46) as the canonical coordinate and set $e_3 := \Omega/\varpi_0$, essentially normalizing the fundamental period. As in [SVW2, dSJKP], we can complete $\{g_0, g_1\}$ to an integral flat basis $\{g_i\} := \{g_3, g_2, g_1, g_0\}$ with periods $\varpi_i = Q(g_i, e_3)$ in which the full monodromy and its logarithm are represented by the matrices

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & \kappa & 1 & 0 \\ -\frac{a+2\kappa}{12} & \kappa & 1 & 1 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \frac{\kappa}{2} & \kappa & 0 & 0 \\ -\frac{a}{12} & \frac{\kappa}{2} & 1 & 0 \end{pmatrix}, \qquad \kappa, a \in \mathbb{Z}. \tag{3.49}$$

Furthermore, we find that the matrix

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.50}$$

represents the polarization form in both bases $\{g_i\}$ and $\{e_j\}$.

Introducing the logarithmic vector field that corresponds to the canonical coordinate

$$\delta := 2\pi i \frac{d}{d\log(q)} = 2\pi i\, q \frac{d}{dq} = 2\pi i\, \frac{q}{z}\frac{dz}{dq}\, z\frac{d}{dz} = 2\pi i\, \frac{q}{z}\frac{dz}{dq}\,\theta, \tag{3.51}$$

we denote by $\nabla(\delta) := \nabla_t$ its contraction with the Gauß-Manin connection. The flat basis $\{g_i\}$ is then related to the Hodge basis $\{e_j\}$ by (see also [CdlOGP] and [CK2, Proposition 5.6.1])

$$\begin{aligned} g_0 = e_0, \quad g_1 = e_1 + \log q\, e_0, \quad g_2 = e_2 + \delta^2\mathcal{F}\, e_1 + \delta\mathcal{F}\, e_0 \\ g_3 = e_3 - \log q\, e_2 + \left(\delta\mathcal{F} - \log q\, \delta^2\mathcal{F}\right) e_1 + (2\mathcal{F} - \log q\, \delta\mathcal{F})\, e_0, \end{aligned} \tag{3.52}$$

where the periods are determined by the prepotential

$$\mathcal{F} = \frac{1}{(2\pi i)^3}\frac{\kappa}{6}\log(q)^3 + \frac{1}{(2\pi i)^2}\frac{\kappa}{4}\log(q)^2 - \frac{1}{(2\pi i)}\frac{a}{24}\log(q) + f(q). \tag{3.53}$$

Here, $f(q)$ is a single valued function, which extends holomorphically over the puncture

$$f(q) = \frac{\widetilde{b}\,\zeta(3)}{(2\pi i)^3} + \frac{1}{(2\pi i)^3}\sum_{d=1}^{\infty}\widetilde{N}_d\, q^d, \qquad \widetilde{b} \in \mathbb{Q}. \tag{3.54}$$

When the family $\mathcal{Y}$ is defined over $\mathbb{Q}$, the coefficients $\widetilde{N}_d \in \mathbb{Q}$ of the q-series expansion are generally rational numbers. The explicit form of the basis (3.52) and the prepotential (3.53) depends on the choice of the integral basis $\{g_i\}$ an its underlying monodromy. It can be determined by iteratively constructing the Hodge filtration of the graded quotients of Hodge-Tate type, as for example in [Del4, GGK1, SVW2]. The existence of a prepotential capturing the periods can be viewed as a consequence of the Hodge-Riemann bilinear relations (3.4), which constrain their structure in the given situation.

By untwisting the local system (3.52) according to (3.30), we get a basis $\{\widetilde{g}_i\}$ of the canonical extension which encodes the limiting asymptotics of the periods $Q(g_i, e_j)$. Roughly speaking, it is obtained by dropping the logarithmic terms from the basis $\{g_i\}$. The *limiting period matrix* is then defined by

$$\Pi_{q=0} := Q(\widetilde{g}_i(0), e_j)_{i,j=0,\ldots,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{a}{24} & -\frac{\kappa}{2} & 1 & 0 \\ \frac{b\,\zeta(3)}{(2\pi i)^3} & \frac{a}{24} & 0 & 1 \end{pmatrix} \tag{3.55}$$

and determines the LMHS at $q = 0$. On the other hand, the nilpotent orbit to which we come back later, is obtained by only keeping the logarithmic terms in (3.52). It is therefore determined only by the structure of the monodromy.

Remark 3.9. The entries of the limiting period matrix can be viewed as certain extension classes arising in the composition series (3.32) [GGK1]. The most interesting extension class lies in

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-3), \mathbb{Z}(0)) = \mathbb{C}/\mathbb{Z}(3), \tag{3.56}$$

and corresponds to the constant term of (3.54), whose theoretical origin is explained in [GGK2]. It determines the limit of the period ϖ_3

$$\lim_{q\to 0}\varpi_3 := Q(\widetilde{g}_3(0), e_3) = \frac{b\,\zeta(3)}{(2\pi i)^3}, \qquad b = 2\widetilde{b}, \tag{3.57}$$

in the LMHS. From a physical perspective, it can be interpreted as a four-loop correction to the sigma model metric on the corresponding A-model Calabi-Yau background [CdlOGP, GvdVZ]. As we will see, the rational multiple b in (3.57) corresponds to a topological invariant (the Euler number) of the mirror manifold.

In terms of the logarithmic vector field (3.51), the Gauß-Manin connection ∇_t in the basis $\{e_j\}$ is fully determined by the prepotential via

$$\nabla_t = d + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\mathfrak{C} & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \otimes \frac{dq}{2\pi i\, q}, \qquad \mathfrak{C} = \delta^3 \mathcal{F} = Q(\nabla_t^3\, e_3, e_3). \tag{3.58}$$

The function $\mathfrak{C}$ is called the *Yukawa coupling* of $\mathcal{H}^3$ and (3.58) serves as the mathematical definition of the three-point correlation function (2.17) of the closed B-model discussed in Chapter 2.

It is a general feature of unipotent monodromy in this context that the monodromy logarithm encodes the singularity structure of the Gauß-Manin connection in a certain sense. Namely, the monodromy logarithm at $q = 0$, i.e. in the basis $\{e_j\}$, can be computed from the residues of the Gauß-Manin connection at the singularity via [Del1]

$$N_{q=0} = -2\pi i\, \mathrm{Res}_{q=0} \nabla. \tag{3.59}$$

In the situation at hand this means that

$$N_{q=0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.60}$$

An explicit formula for the Yukawa coupling can then be determined from a certain differential equation (cf. [CK2]).

Proposition 3.10. Let $\mathcal{Y} \to \Delta^*$ be a one-parameter family of complete intersection Calabi-Yau threefolds in a small neighborhood around a point for which the associated weight 3 VHS has maximally unipotent monodromy. Then the Yukawa coupling in the canonical coordinate is given by

$$\mathfrak{C} = \frac{\kappa}{(1-\alpha z)\, \varpi_0^2} \left(\frac{q}{z} \frac{dz}{dq} \right)^3. \tag{3.61}$$

Proof. We begin with the unnormalized coupling $Q(\nabla_\theta^3 \Omega, \Omega)$ in the coordinate z and compute its derivative

$$z\frac{d}{dz} Q\left(\nabla_\theta^3 \Omega, \Omega\right) = Q\left(\nabla_\theta^4 \Omega, \Omega\right) + Q\left(\nabla_t^3 \Omega, \nabla_t \Omega\right) = \frac{1}{2} Q\left(\nabla_t^4 \Omega, \Omega\right). \quad (3.62)$$

Here, the second equality follows from Griffiths transversality: Differentiating the equation $Q(\nabla_\theta^2 \Omega, \Omega) = 0$ twice yields

$$Q\left(\nabla_\theta^3 \Omega, \nabla_\theta \Omega\right) = -\frac{1}{2} Q\left(\nabla_\theta^4 \Omega, \Omega\right). \quad (3.63)$$

By virtue of the Picard-Fuchs equation (3.36), the unnormalized holomorphic 3-form Ω satisfies the relation

$$\nabla_\theta^4 \Omega = -E_3 \nabla_\theta^3 \Omega - E_2 \nabla_\theta^2 \Omega - E_1 \nabla_\theta \Omega. \quad (3.64)$$

Plugging $\nabla_\theta^4 \Omega$ into (3.62) and using (3.37) produces the differential equation

$$z\frac{d}{dz} Q\left(\nabla_\theta^3 \Omega, \Omega\right) = \frac{\alpha z}{1-\alpha z} Q\left(\nabla_\theta^3 \Omega, \Omega\right). \quad (3.65)$$

with a solution that can be determined up to a constant factor

$$Q\left(\nabla_\theta^3 \Omega, \Omega\right) = \frac{c}{1-\alpha z}, \qquad c \in \mathbb{C}. \quad (3.66)$$

We now turn to the normalized Yukawa coupling in the canonical coordinate. In case of the normalized 3-form e_3, computing the derivative similar to (3.62) produces additional terms that all vanish by Griffiths transversality. Furthermore, when switching to the canonical coordinate q we have to take into account the chain rule (3.51). The corresponding solution is given by

$$\mathfrak{C} = \frac{c}{(1-\alpha z)\, \varpi_0^2} \left(\frac{q}{z}\frac{dz}{dq}\right)^3, \qquad c \in \mathbb{C}, \quad (3.67)$$

which determines the Yukawa coupling up to the same constant. As $\mathfrak{C} = c + \mathcal{O}(q)$, it suffices to compute $\mathfrak{C}(0) = c$ to determine this constant. For this purpose we consider the monodromy logarithm and its relation to the Gauß-Manin connection (3.60). From this we can deduce that

$$N_{q=0}\, e_2 = \kappa\, e_1 = \mathfrak{C}(0)\, e_1, \quad (3.68)$$

such that $\mathfrak{C}(0) = \kappa$. □

Example 3.11. In case of the mirror quintic, the constants $\kappa = 5$, $a = 50$ and $b = -200$ can be determined from monodromy considerations[2], such that the prepotential is given by

$$\mathcal{F} = \frac{1}{(2\pi i)^3}\frac{5}{6}\log(q)^3 + \frac{1}{(2\pi i)^2}\frac{5}{4}\log(q)^2 - \frac{1}{(2\pi i)}\frac{25}{12}\log(q) + f(q). \tag{3.69}$$

Via Proposition 3.10, we can compute the Yukawa coupling including its holomorphic part

$$\mathfrak{C} = \delta^3\mathcal{F} = \kappa + \delta^3 f(q) = 5 + 2875\,q + 4876875\,q^2 + 8564575000\,q^3 + \cdots\,, \tag{3.70}$$

using the periods and the mirror map.

3.3. Gromov-Witten Invariants and Quantum Cohomology

We are now ready to turn to the A-model, where the aim is to reinterpreted the Hodge-theoretic objects of the preceding section in terms of the symplectic geometry of the mirror manifold. For this purpose, let X be a simply connected and projective Calabi-Yau threefold and denote by $\omega = B + iJ = t[H]$ the complexified Kähler class with $t \in \mathfrak{H}$ in the upper half plane. This means, we assume for simplicity that $h^{1,1} = 1$. For the B-model, we have described how the Yukawa coupling arises in the maximal degeneration of a family of threefolds and, as pointed out in Chapter 2, one feature of the physical mirror phenomenon is the equivalence between the three-point functions in the topologically twisted theories. In order to make the three-point function in the A-model (2.16) precise, we first have to give a geometric description of the instanton corrections $\widetilde{N}_\beta$ arising therein, see e.g. [CK2].

Roughly speaking, in their role as coefficients of the topological string amplitude they should geometrically correspond to the number of holomorphic genus 0 curves $C \subset X$ that represent a given homology class $\beta \in H_2(X;\mathbb{Z})$. In an enumerative problem like this it is necessary to specify a number of incidence relations, e.g. intersection points with cycles, that make the relevant moduli space zero dimensional. With this objective in mind, we consider cycles Z_i in X and are generally interested in the numbers

$$\#\left\{C \subset X \,\middle|\, \mathrm{genus}(C) = g,\ [C] = \beta,\ C \cap Z_i \neq \emptyset \text{ for all } i\right\}. \tag{3.71}$$

A way of making this count of curves well defined is due to Kontsevich [Kon2] and involves the notion of *stable maps* and their moduli space. Instead of a

[2] For κ and a this can be seen from the basis (3.52), while b requires a continuation to the conifold locus [CdlOGP].

curve $C \subset X$, we consider an abstractly defined curve together with a holomorphic map $f : C \to X$ that satisfies $f_*[C] = \beta$. Note that such an approach quite strongly resembles the sigma model (2.1) discussed in Chapter 2. When we want to specify incidence relations, this can be achieved by requiring that a set of distinct points $p_i \in C$ is mapped to cycles $f(p_i) \in Z_i$ for $Z_i \in H_*(X;\mathbb{C})$. Naively, the numbers (3.71) are obtained by integrating an appropriate form over a moduli space of isomorphism classes of such curves that needs to be sufficiently well-behaved. In particular, the moduli space has to admit a compactification, leading to the notion of stability which we discuss next.

In the ordinary case of marked curves $(C, p_1, \ldots, p_n)$ without reference to a map to some ambient space, it is necessary to make various assumptions in order for a compact moduli space to exist. Crucially, it is necessary to include not only smooth but certain kinds of singular curves into the discussion, which can be interpreted as limiting configurations of degenerating curves. Here it suffices to only focus on singularities that can be locally described as an ordinary double point. The notion of *stability* is now related to the fact that the moduli problem is only solvable (in a way to be made precise), when each reducible component of a curve only has finitely many automorphisms. This can be enforced by requiring that (i) an irreducible component $D \subset C$ with $D \cong \mathbb{P}^1$ has at least 3 marked points and (ii) if C has only one irreducible component of genus 1, there is at least one marked point. A marked point refers in this context either to one of the incidence relations $(p_1, \ldots, p_n)$ or the nodal singularities. In this way it is ensured that all continuous[3] automorphisms of a Riemann sphere or an elliptic curve are fixed by the specification of the markings. The moduli problem can be assessed by first defining a contravariant functor

$$\mathfrak{F} : \mathbf{Sch}_{\mathbb{C}} \longrightarrow \mathbf{Sets}, \qquad B \longmapsto \{\mathcal{C}_B \to B\} \tag{3.72}$$

sending a scheme over $\mathbb{C}$ to a family of pointed curves over B. A solution of the moduli problem is usually thought of as a representation

$$\mathfrak{F} \xrightarrow{\sim} \mathrm{Hom}(-, \widehat{\mathcal{M}}_{g,n}) \tag{3.73}$$

of this functor by a, preferably simple, parameter space. In the given case of curves it turns out that the above representation cannot be achieved by a scheme. This problem is generally related to the fact that many curves still admit finite automorphism groups. The correct object that incorporates such automorphisms is a *Deligne-Mumford stack*, which we will think of as the algebraic version of a compact orbifold with finite group action. As it turns out [DM1], in the particular moduli problem for curves the moduli stack $\widehat{\mathcal{M}}_{g,n}$ is a smooth Deligne-Mumford stack (or smooth orbifold) of dimension $3g - 3 + n$

[3] By this we mean those automorphisms that are isotopic to the identity.

when $n + 2g \geq 3$.

Turning again to maps $f : C \to X$, a similar strategy can be employed. The correct stability conditions are slight modifications of the ones presented above and are given by the requirement that (i) an irreducible component $D \subset C$ with $D \cong \mathbb{P}^1$ and $f(D) = \{pt.\}$ has at least 3 marked points and (ii) if C has genus $g = 1$ and $f(C) = \{pt.\}$, then there are $n > 0$ marked points. The corresponding moduli problem again has a solution in the sense that the functor of families can be represented by a Deligne-Mumford stack $\widehat{\mathcal{M}}_{g,n}(X, \beta)$ [Kon2], however in the case of stable maps it is not smooth in general. Before discussing this issue we come back to our initial task in the simplest situation.

A general feature of the moduli space $\widehat{\mathcal{M}}_{g,n}(X, \beta)$ of stable maps is the existence of *evaluation maps*

$$\mathrm{ev}_i : \widehat{\mathcal{M}}_{g,n}(X, \beta) \longrightarrow X, \qquad (f, C, p_1, \ldots, p_n) \longmapsto f(p_i), \tag{3.74}$$

sending a given stable map to the image of one of the marked points in X. Assuming $\widehat{\mathcal{M}}_{g,n}(X, \beta)$ to be a smooth Deligne-Mumford stack, or smooth compact orbifold, it can be shown that its compex dimension is given by [Kon2, CK2]

$$\dim_{\mathbb{C}} \widehat{\mathcal{M}}_{g,n}(X, \beta) = (1 - g)(\dim_{\mathbb{C}} X - 3) - \int_\beta c_1(X) + n, \tag{3.75}$$

which simplifies significantly in case of a Calabi-Yau threefold. In this particular case, a numerical invariant can be extracted when incidence relations are specified in a way that fixes n complex or $2n$ real parameters. To do so, consider cohomology classes $\gamma_i = PD(Z_i) \in H^{\mathrm{even}}(X; \mathbb{C})$ Poincaré dual to the cycles which aid as such incidence relations. Assuming $\sum_i \deg(\gamma_i) = 2n$, the count (3.71) can then be formalized by the *Gromov-Witten invariant*

$$GW_{g,\beta}(\gamma_1, \ldots, \gamma_n) = \int_{\widehat{\mathcal{M}}_{g,n}(X,\beta)} \bigcup_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i), \tag{3.76}$$

in which we integrate an appropriate $2n$-form over the real $2n$-dimensional moduli space of stable maps.

In the more general case, this naive approach will fail as the moduli space might have strata on which the dimension deviates from the expected value (3.75). Fortunately, there is a remedy for this issue based on the idea that there is proper substitute for the fundamental class also when $\widehat{\mathcal{M}}_{g,n}(X, \beta)$ is not smooth, which can be employed to define the invariants quite analogous to (3.76). As the construction of the relevant *virtual fundamental class* is rather involved, we refer the reader to the exposition in [CK2] and content ourselves with the fact that the more sophisticated version of (3.76) is always defined, with the rough intuition sketched in the beginning of this section persisting. There are both algebro-geometric and symplectic approaches to the general problem, both of which produce consistent properties or axioms for the resulting invariants, attributed to Kontsevich-Manin [KM]. For our purposes it will suffice to rely on these axioms, which we now formulate for the general situation.

Axioms 3.12. Given cohomology classes $\gamma_i \in H^*(X;\mathbb{C})$ with $i = 1, ..., n$, the invariants $GW_{g,\beta}(\gamma_1, ..., \gamma_n)$ of X satisfy the axioms

- *(Degree)* $GW_{g,\beta}(\gamma_1, ..., \gamma_n) = 0$ unless

$$2(1-g)(\dim X - 3) - 2\int_\beta c_1(X) + 2n = \sum_{i=1}^{n} \deg(\gamma_i). \tag{3.77}$$

- *(Equivariance)* For $\dagger = \deg(\gamma_i)\cdot\deg(\gamma_{i+1})$ it is

$$GW_{g,\beta}(\gamma_1, \ldots, \gamma_i, \gamma_{i+1}, \ldots, \gamma_n) = (-1)^\dagger\, GW_{g,\beta}(\gamma_1, \ldots, \gamma_{i+1}, \gamma_i, \ldots, \gamma_n). \tag{3.78}$$

- *(Unit)* An insertion of the unit $[X] \in H^0(X;\mathbb{C})$ yields

$$GW_{g,\beta}([X], \gamma_1, ..., \gamma_{n-1}) = 0. \tag{3.79}$$

- *(Zero)* The restriction to $g = 0$ and $\beta = 0$ yields

$$GW_{0,0}(\gamma_1, ..., \gamma_n) = \begin{cases} \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3 & \text{if} \quad n = 3 \\ 0 & \text{otherwise.} \end{cases} \tag{3.80}$$

- *(Divisor)* For $\deg(\gamma_n) = 2$ it is

$$GW_{g,\beta}(\gamma_1, ..., \gamma_n) = \left(\int_\beta \gamma_n\right)\cdot GW_{g,\beta}(\gamma_1, ..., \gamma_{n-1}). \tag{3.81}$$

- *(Splitting)* The invariants associated[4] to curves coming from the divisor

$$\widehat{\mathcal{M}}_{g_1,n_1+1} \times \widehat{\mathcal{M}}_{g_2,n_2+1} \to \widehat{\mathcal{M}}_{g,n}, \qquad g_1 + g_2 = g,\ n_1 + n_2 = n, \tag{3.82}$$

  are given by

$$\sum_{\beta_1+\beta_2=\beta}\sum_{i,j} Q^{ij}\, GW_{g_1,\beta_1}(\gamma_1, \ldots, \gamma_{n_1}, e_i)\cdot GW_{g_1,\beta_1}(e_j, \gamma_{n_1+1}, \ldots, \gamma_n), \tag{3.83}$$

  where $\{e_i\}$ denotes a homogeneous basis for which the Poincaré pairing is represented by a matrix with entries Q_{ij} and $Q^{ij} = (Q_{ij})^{-1}$.

[4] We refer the reader to [CK2] for an explicit construction.

- *(Reduction)* The invariants associated to curves coming from the divisor

$$\widehat{\mathcal{M}}_{g-1,n+2} \longrightarrow \widehat{\mathcal{M}}_{g,n}, \tag{3.84}$$

are given by

$$\sum_{i,j} Q^{ij}\, GW_{g-1,\beta}(\gamma_1, \ldots, \gamma_n, e_i, e_j). \tag{3.85}$$

- *(Invariance)* The numbers $GW_{g,\beta}(\gamma_1, \ldots, \gamma_n)$ are constant under deformations of the Kähler class ω.

Remark 3.13. We have formulated the Kontsevich-Manin axioms for Gromov-Witten invariants in terms of the invariants itself, rather than their associated classes, as it is often done. The reason is that this will make the comparison to their open cousins more apparent, where an immediate focus on the invariants is more convenient. We dropped the effectivity axiom from our list, stating that the invariants vanish except $\int_\beta \omega \geq 0$, because this is always the case when we focus on holomorphic maps.

For now and until Chapter 7 we will focus our attention on the tree-level case, i.e. $g = 0$, and denote $GW_{0,\beta}(\gamma_1, \ldots, \gamma_n) = GW_\beta(\gamma_1, \ldots, \gamma_n)$. We will also use the notation GW_β for those invariants without insertions. To these tree-level invariants we can associate their generating function, called the *Gromov-Witten Potential*, which can be physically interpreted as the genus 0 partition function for the closed A-type topological string. We define it by

$$\Phi = \frac{1}{6}\int_X \omega \cup \omega \cup \omega + \Phi_h = \frac{\kappa}{6} t^3 + \Phi_h, \tag{3.86}$$

where[5]

$$\kappa = \int_X H \cup H \cup H, \qquad [H] \in H^2(X; \mathbb{C}), \tag{3.87}$$

is the classical triple intersection number in terms of the hyperplane class and

$$\Phi_h = \frac{1}{(2\pi i)^3} \sum_{\beta \in H_2(X;\mathbb{Z})\setminus\{0\}} GW_\beta\, q^\beta, \qquad q^\beta = \exp(2\pi i \int_\beta \omega), \tag{3.88}$$

is the quantum part of the potential. We will assume this quantum part to converge and think of it as a holomorphic function on Δ^*. The conventional prefactor of $1/(2\pi i)^3$ will become especially clear in a later comparison to the B-model.

Remark 3.14. It is conjectured [CK2], that the power series (3.88) is convergent for sufficiently small q. From a physical point of view, such an assumption is plausible as q measures the impact of quantum effects on the physical theory,

[5] It is no coincidence that the letter κ is repurposed at this point.

such that a restriction to a sufficiently small neighborhood around $q = 0$ should be expected to have a well-defined physical meaning. At the moment there is no proof of this statement in the mathematical literature. As a consequence, this problem is often dealt with by thinking of Φ_h as a formal power series in the variable q. Geometrically, this means that one is forced to restrict oneself to an infinitesimal or formal neighborhood around $q = 0$. For our purposes the details of how to make this precise are immaterial, as we will not be concerned with questions of convergence in this thesis.

In case of a Calabi-Yau threefold X, the properties of Gromov-Witten invariants have a special significance pertaining to the cohomology of X. Namely, they can be used to define a deformation of the ordinary cup product on $H^*(X;\mathbb{C})$ of the form

$$* = \cup + \mathcal{O}(q), \tag{3.89}$$

that can be interpreted as leading to a "quantum" version of the intersection product. While the intersection product requires honest intersections at single points, its quantum version will compute intersections up to the presence of a holomorphic sphere, or closed string instanton.

Definition 3.15. Let (X, ω) be a Calabi-Yau threefold with complexified Kähler class ω. For $\gamma_1 \in H^k(X;\mathbb{C})$ and $\gamma_2 \in H^l(X;\mathbb{C})$ the operation

$$\gamma_1 * \gamma_2 = \sum_{\beta \in H_2(X;\mathbb{Z})} \sum_{i,j} GW_\beta(\gamma_1, \gamma_2, e_i)\, Q^{ij}\, e_j\, q^\beta \in H^{k+l}(X;\mathbb{C}) \tag{3.90}$$

is called the *(small) quantum product.* The graded ring $(H^*(X;\mathbb{C}), *)$ is called the *(small) quantum cohomology* of X.

Remark 3.16. There is also a *big* version of quantum cohomology in which not only invariants with $n = 3$ but all invariants are taken into account. Even though the corresponding quantum product contains considerably more information, we are exclusively interested in its smaller cousin as it already captures all the data that is relevant for our Hodge theoretic applications.

Note that due to the zero axiom, the classical contribution to the quantum product

$$\gamma_1 * \gamma_2 \Big|_{q=0} = \sum_{i,j} GW_0(\gamma_1, \gamma_2, e_i)\, Q^{ij}\, e_j = \sum_{i,j} \left(\int_X \gamma_1 \cup \gamma_2 \cup e_i \right) Q^{ij}\, e_j \tag{3.91}$$

is indeed representing the ordinary cup product. In the next section we will see how this deformation of the cup product can be algebraically summarized in terms of a VMHS. Of particular enumerative importance will be the associativity of the quantum product, which reflects certain equalities of the numbers discussed in the splitting axiom. In what follows, we will interpret this condition as the flatness of a certain connection on even degree cohomology.

3.4. A-model Variations of Hodge Structure

Following Morrison [Mor3], closed string mirror symmetry can then be established by a reconstruction of the Hodge theoretic structures discussed in Section 3.2 from objects that are inherent to the associated A-model geometry, see also [DK2]. These include the Gromov-Witten invariants and the quantum cup product introduced in Section 3.3. We fix X to be a simply connected and projective Calabi-Yau threefold with Hodge numbers

$$h^{0,0} = h^{1,1} = h^{2,2} = h^{3,3} = 1 \tag{3.92}$$

mirror to (3.15). The standard example on the A-model side is given by the Fermat quintic

$$X = \left\{ (x_1 : \cdots : x_5) \in \mathbb{P}^4 \,\middle|\, x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0 \right\} \subset \mathbb{P}^4, \tag{3.93}$$

see [CdlOGP, GP]. On the even degree cohomology

$$H^{\mathrm{even}} := H^{\mathrm{even}}(X;\mathbb{C}) = H^{0,0}(X) \oplus H^{1,1}(X) \oplus H^{2,2}(X) \oplus H^{3,3}(X) \tag{3.94}$$

we define the decreasing A-model Hodge filtration

$$F^p = \bigoplus_{i \leq 3-p} H^{i,i}(X) \subset H^{\mathrm{even}} \tag{3.95}$$

with $F^3 \subset F^2 \subset F^1 \subset F^0 = H^{\mathrm{even}}$, inspired by the characteristic symmetry of Hodge numbers depicted in Figure 1.1. For $t \in \mathfrak{H}$ in the upper half plane, we again denote by $\omega = B + iJ = t[H]$ the complexified Kähler class and consider a region of the Kähler moduli space $\widehat{\mathcal{K}}(X)$ given by a punctured disk Δ^* parametrized by $q = e^{2\pi i t}$ around the boundary point $q = 0$. As the ordinary Kähler form J measures the volume of X in a classical sense, the value $q = 0$ is often considered as the *large volume limit* of the string vacuum. This limit point will correspond to the large complex structure limit in the B-model via the mirror map, hence the description of the local coordinate by the same symbol.

In order to construct an object that mirrors the VMHS coming from $\mathcal{Y}$ we define the vector bundle $\mathcal{H}^{\mathrm{even}} := H^{\mathrm{even}} \otimes \mathcal{O}_{\Delta^*}$ to which we can extend the filtration by $\mathcal{F}^p := F^p \otimes \mathcal{O}_{\Delta^*}$. The antisymmetric polarization form $Q(\cdot,\cdot) : \mathcal{H}^{\mathrm{even}} \otimes \mathcal{H}^{\mathrm{even}} \to \mathbb{C}$ is again induced from the cup-product on H^{even} and given by

$$Q(\alpha,\beta) = (-1)^i \int_X \alpha \cup \beta, \quad \alpha \in H^{i,i}(X), \quad \beta \in H^{3-i,3-i}(X). \tag{3.96}$$

In the basis $\{e_j\} := \{e_3, e_2, e_1, e_0\}$ with $e_j \in H^{6-2j}(X;\mathbb{C})$ given by

$$e_3 = [X], \quad e_2 = [H], \quad e_1 = -[\ell], \quad e_0 = [p], \tag{3.97}$$

the polarization form is represented by the matrix (3.50), whose entries we denote by $Q_{ij} = Q(e_i, e_j)$. We use physicists' notation for coefficients of the inverse matrix, writing them in terms of upper indices $Q^{ij} = (Q_{ij})^{-1}$.

So far, the construction only contains classical information encoded in the cup product. However, it is the quantum information contained in the Gromov-Witten invariants and the quantum product that actually varies with the Kähler parameter t. It is therefore natural to introduce the *Dubrovin (or A-model, or Quantum) connection* that measures the energy of quantum effects accordingly. The Dubrovin connection on $\mathcal{H}^{\text{even}}$ is defined in terms of the small quantum product (cf. Definition 3.15) $e_2 * (-) : H^{\text{even}}(X) \to H^{\text{even}}(X)$ with the hyperplane class $e_2 = [H]$, i.e.

$$\nabla_t(e_k) = e_2 * e_k = \sum_{l,m} \sum_{\beta \in H_2(X;\mathbb{Z})} GW_\beta(e_2, e_k, e_l)\, q^\beta\, Q^{lm}\, e_m, \tag{3.98}$$

whose energy zero contribution $e_2 * e_k|_{q=0} = e_2 \cup e_k \in H^{\text{even}}(X)$ corresponds to the ordinary cup product. On the bundle $\mathcal{H}^{\text{even}}$ with Hodge basis $\{e_j\}$ the action of the Dubrovin connection is then given by

$$\nabla_t := d + \Big([H]*\Big) \otimes dt = d + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\Phi''' & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \otimes dt. \tag{3.99}$$

It records the variation of the small quantum product with respect to the Kähler parameter and the analog of the Yukawa coupling (3.58) is here identified as the closed A-model three-point correlation function (2.16)

$$\Phi''' = \sum_\beta GW_\beta(H,H,H)\, q^\beta = \int_X H * H * H = Q(\nabla_t^3 e_3, e_3), \tag{3.100}$$

by means of the Divisor axiom of Axioms 3.12. Any $\gamma \in H^{2k}(X)$ for $k \leq n-p$ is a section of $\mathcal{F}^p$ and Griffiths transversality $\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_{\Delta^*}$ follows from the fact that $[H]*\gamma \in H^{2k+2}(X)$ is a section of $\mathcal{F}^{p-1}$. In the one-parameter case, flatness of ∇ is an immediate consequence of associativity and commutativity of the small quantum product. In the notation of Doran-Kerr [DK2], the complex local system $\mathcal{H}^{\text{even}}_{\mathbb{C}} = \ker(\nabla)$ can be systematically constructed in terms of Φ and the basis $\{e_j\}$ by setting

$$\begin{aligned} \tilde\sigma(e_0) := e_0, \quad \tilde\sigma(e_1) := e_1, \quad \tilde\sigma(e_2) &:= e_2 + \Phi_h''\, e_1 + \Phi_h'\, e_0, \\ \tilde\sigma(e_3) &:= e_3 + \Phi_h'\, e_1 + 2\Phi_h\, e_0 \end{aligned} \tag{3.101}$$

and then defining the *quantum deformed classes* by

$$g_i = \sigma(e_i) := \tilde\sigma(e^{-\omega} \cup e_i). \tag{3.102}$$

This leads to a flat basis which corresponds to a complex solution of the quantum differential equation defined by the A-model connection [*op.cit.*]. We will refer to the functions $Q(g_i, e_j)$ as the *A-model periods* of $\mathcal{H}^{\text{even}}$. While their B-model analogs have an interpretation in terms of honest period integrals, the A-model periods are simply understood as the coefficients appearing in the local system (3.101). The quantum deformed classes can then be physically interpreted as cycles attaining a quantum deformed volume measured by $Q(g_i, e_j)$.

Remark 3.17. In the one-parameter setting, flatness of the Dubrovin connection is an automatic consequence of the algebraic properties of the operation $[H] * (-)$. In the case $h^{1,1}(X) = n$ of several Kähler parameters, we denote by

$$\omega = \sum_{i=1}^{r} t_i \, [H_i] \longmapsto q = (q_1, \ldots, q_r) = \left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_r}\right) \in (\Delta^*)^r, \quad t_i \in \mathfrak{H}, \tag{3.103}$$

the Kähler class and corresponding coordinates of the Kähler moduli space around a small punctured polydisk around the large volume limit point. The curvature of the Dubrovin connection is now given by

$$R_\nabla(\partial_i, \partial_j)\Big([H_k]\Big) = [H_i] * \Big([H_j] * [H_k]\Big) - [H_j] * \Big([H_i] * [H_k]\Big), \tag{3.104}$$

and vanishes only when associativity of the small quantum product is persisting also in this more general situation. While commutativity is a given on H^{even}, associativity leads to constraints on the Gromov-Witten invariants discussed in the splitting axiom of Axioms 3.12. This constraint can be formulated as a system of partial differential equations satisfied by the Gromov-Witten potential

$$\sum_{a,b} \partial_a \partial_i \partial_j \, \Phi \cdot Q^{ab} \cdot \partial_b \partial_k \partial_l \, \Phi = \sum_{a,b} \partial_a \partial_i \partial_k \, \Phi \cdot Q^{ab} \cdot \partial_b \partial_j \partial_l \, \Phi, \quad \text{for all } i, j, k, l. \tag{3.105}$$

These are known as the *Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations* and can be understood as an equivalence of Gromov-Witten invariants associated to different boundary strata of the moduli space of stable maps with four marked points [Wit2, DVV1, DVV2]. A nice geometric proof of this property of Gromov-Witten invariants in the simplest case of $\widehat{\mathcal{M}}_{g,n}(X, \beta)$ being smooth can be found in [AMTV].

In order to study the asymptotics of the A-model periods $Q(g_i, e_j)$ we turn again to the monodromy $M : \mathcal{H}^{\text{even}}_{\mathbb{C}} \to \mathcal{H}^{\text{even}}_{\mathbb{C}}$ of the local system around the puncture. By (3.59), its logarithm at $q = 0$ can be computed from the residue $\operatorname{Res}_{q=0}(\nabla)$, which is seen to coincide with the energy zero contribution to the quantum product [CK2] (cf. (3.91))

$$2\pi i \operatorname{Res}_{q=0}(\nabla) = \left(\sum_{l,m} GW_0(e_j, e_k, e_l) \, Q^{lm} e_m \right) = \left(\sum_{l,m} \left(\int_X e_j \cup e_k \cup e_l \right) Q^{lm} e_m \right). \tag{3.106}$$

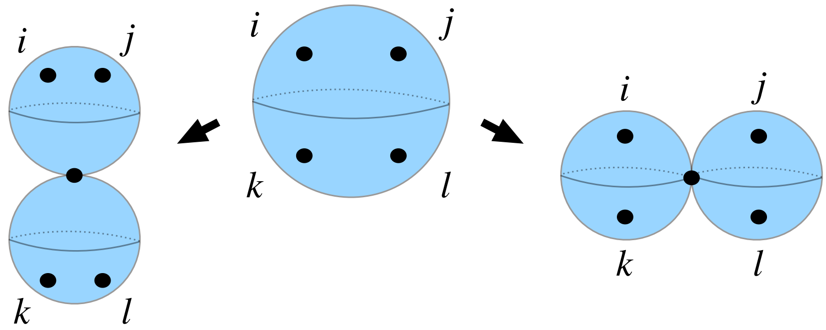

Figure 3.4.: Geometric interpretation of the WDVV equations. The collision of marked points leads to boundary strata of the moduli space consisting of nodal curves. Gromov-Witten invariants associated to the depicted degenerations coincide.

The monodromy logarithm $N_{q=0} = -2\pi i \operatorname{Res}_{q=0}(\nabla)$ is therefore given by the cup product $-[H] \cup (-) : H^{\text{even}}(X) \to H^{\text{even}}(X)$, which in the basis $\{e_j\}$ is represented by the matrix

$$N_{q=0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.107}$$

where κ is the classical intersection number. For example in case of the Fermat quintic (3.93), it is $\kappa = 5$. The weight filtration $W_\bullet := W(N)_\bullet$ associated to this monodromy is at $0 \in \Delta$ given by

$$W_k = \bigoplus_{i \geq 3-k/2} H^{2i}, \tag{3.108}$$

invoking the Hard Lefschetz theorem $[H]^k \cup (-) : H^{3-k} \xrightarrow{\sim} H^{3+k}$. We regard the graded pieces $\operatorname{Gr}_k^W H^{\text{even}} = H^{2n-k}$ as having a pure Hodge structure of weight k and denote by $F_0^\bullet = \widetilde{\mathcal{F}}_0^\bullet$ the limit of the Hodge filtration $\widetilde{\mathcal{F}}^\bullet = F^\bullet \otimes \mathcal{O}_\Delta$ that can be trivially extended to the full disk. By untwisting the local system according to (3.30), this defines a MHS $(\widetilde{\mathcal{H}}_{\mathbb{Z},0}^{\text{even}}, \widetilde{\mathcal{H}}_0^{\text{even}}, F_0^\bullet, W_\bullet)$ of Hodge-Tate type at $0 \in \Delta$. Similar to the construction in the B-model, we can extend it to a nilpotent orbit

$$\mathcal{H}_{\text{nilp}}^{\text{even}} := (\mathcal{H}_{\mathbb{C}}^{\text{even}}, \mathcal{H}^{\text{even}}, e^{-tN} F_0^\bullet, \nabla, Q) \tag{3.109}$$

in a small neighborhood around the puncture, with Griffiths transversality following from $N(F_0^p) \subset F_0^{p-1}$ and LMHS given by (3.32). Paired up with the filtration $\mathcal{W}_\bullet = W_\bullet \otimes \mathcal{O}_{\Delta^*}$, this defines a Hodge-Tate VMHS that is constant with respect to the corresponding untwisted connection (3.30) and fully determined by the H^2-module structure on H^{even} induced by the cup product. Again, also the original Hodge filtration leads to a polarized, complex VMHS $(\mathcal{H}^{\mathrm{even}}_{\mathbb{C}}, \mathcal{H}^{\mathrm{even}}, \mathcal{F}^\bullet, \mathcal{W}_\bullet, \nabla, Q)$ of Hodge-Tate type on Δ^*, in which the additional holomorphic components of the periods are derived from the quantum part of the Gromov-Witten potential (3.88), see also [CK2, Theorem 8.5.11]. Its structure is similar to (3.52), except for the constant terms (i.e. the integral structure) which will be discussed further below.

The fact that a VMHS of Hodge-Tate type arises in this context can be viewed as a general consequence of the WDVV equations for a deformation of the H^2-module structure on even degree cohomology to a quantum product [CF1, CF2, FP]. While (big) quantum cohomology is often studied in the realm of Frobenius manifolds [Man], only part of this structure is required for this Hodge theoretic application.

Definition 3.18. Let $V = \bigoplus_{i=0}^{2n} V_i$ with $n \in \mathbb{N}$ and $V_0 = \langle 1 \rangle$ be a graded finite dimensional $\mathbb{C}$-vector space together with a symmetric non-degenerate bilinear form $Q : V \times V \to \mathbb{C}$ that pairs V_{2i} with $V_{2(n-i)}$. Given a graded module structure

$$* : \mathrm{Sym}(V_2) \times V \longrightarrow V, \tag{3.110}$$

the collection $(V, Q, *)$ is called a *graded V_2-Frobenius module of weight n*, if

$$Q(v, w_1 * w_2) = Q(w_1, v * w_2) \tag{3.111}$$

for all $v \in V_2$ and $w_1, w_2 \in V$, i.e. the module structure $*$ is compatible with the bilinear form Q.

For simplicity, we will focus on the case $n = 3$, $\dim V_{2i} = 1$ for all i and else zero, which captures the situation of Frobenius modules arising from Calabi-Yau threefolds. A feature of Frobenius modules is that, analogous to the case of a Frobenius manifold, the module structure can be encoded in a (classical) potential function $F_c : V \to \mathbb{C}$. Namely, given a graded basis $\{e_j\}$ of V with $e_j \in V_{2(3-j)}$ and associated linear coordinates t_j on V, the module structure is recovered via

$$e_i * e_j = \sum_{k,l} \frac{\partial^3 F_c}{\partial t_i \partial t_j \partial t_k} Q^{kl} e_l, \qquad e_i \in V_2,\ e_j, \dots e_l \in V. \tag{3.112}$$

Here, the potential function is given by [CF1]

$$F_c = t_1 t_2 t_3 + \frac{1}{6} Q(e_2 * e_2, e_2)\, t_2^3. \tag{3.113}$$

In case of our standard examples of Calabi-Yau threefolds arising as complete intersections, the classical potential is therefore uniquely determined by the classical intersection number $\kappa = Q(e_2 * e_2, e_2)$. The following notion can be viewed as an abstraction of the quantum product as deformation of the standard H^2-module structure on even degree cohomology.

Definition 3.19. Let $(V, Q, *)$ be a V_2-Frobenius module of weight $n = 3$ and classical potential F_c. Setting $r = \dim V_2$, we consider a graded basis $\{e_j\}$ with associated coordinates t_j and define $q_i = \exp(2\pi i t_i)$ for $e_i \in V_2$. A *quantum potential* is a function $F : V \to \mathbb{C}$ of the form $F = F_c + F_q$, with F_q being a convergent power series in the q_i vanishing at $q_i = 0$ for all i. We further require that it satisfies the WDVV equations

$$\sum_{a,b} \partial_a \partial_i \partial_j \, F \cdot Q^{ab} \cdot \partial_b \partial_k \partial_l \, F = \sum_{a,b} \partial_a \partial_i \partial_k \, F \cdot Q^{ab} \cdot \partial_b \partial_j \partial_l \, F, \quad \text{for all } i, j, k, l, \tag{3.114}$$

where $e_i, e_j, e_k, e_l \in V_2$ and $e_a, e_b \in V$. Then $(V, Q, *_q)$ is called a *quantum deformed Frobenius module* if the operation

$$e_i \, *_q \, e_j = \sum_{k,l} \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} Q^{kl} e_l, \qquad e_i \in V_2, \; e_j, e_k, e_l \in V, \tag{3.115}$$

defines a Frobenius module for each $q = (q_1, \dots q_r)$.

In the Calabi-Yau case, the quantum part of the potential corresponds to the quantum part of the Gromov-Witten potential, such that

$$F = t_1 t_2 t_3 + \frac{1}{6} Q(e_2 * e_2, e_2) \, t_2^3 + \frac{1}{(2\pi i)^3} \sum_{d=1}^{\infty} \widetilde{N}_d \, q^d, \tag{3.116}$$

where $q = (q_1, \dots q_r)$ as in Remark 3.17, encodes a Frobenius module defined by the small quantum product. Note that the first term in (3.116) has to be added in order to ensure compatibility with the unit in V, which here corresponds to the coordinate t_3 by our choice of basis, see (3.97). One of the main results of [CF1, CF2] is that there is a 1:1 correspondence between the weight 3 quantum deformed Frobenius modules and variations of MHS that arise in a maximal degeneration of pure Hodge structures of weight 3 at a MUM boundary point. How to establish this correspondence starting from the A-model side was described at the beginning of this section, while the inverse direction essentially corresponds to the transformation into the canonical coordinate described in Definition 3.7. We defer the reader to [*op.cit.*] for a more detailed discussion, and also a more general treatment of the multiparameter case and different weights. How the correspondence interacts with mirror symmetry will be described in Section 3.6. Before going there, we have to discuss how to fix the integral local system of the A-model VMHS.

3.5. The Gamma-Integral Local System

So far the construction of this VMHS in the A-model only involved the specification of a complex local system as a solution to the quantum differential equation defined by the Dubrovin connection. A subtle problem arises in the definition of the correct integral local system, which is reflected by the fact that the limiting A-model periods $Q(\tilde{g}_i, e_j)$ do not reproduce the limiting period matrix of the B-model, including the crucial zeta value $\zeta(3)$. A direct construction of this integral local system $\mathcal{H}_{\mathbb{Z}}^{\text{even}}$ from a pure A-model perspective is due to Iritani [Iri] and Katzarkov-Kontsevich-Pantev [KKP], while we follow [DK2, dSJKP]. It starts with a basis $\{\xi_i\}$ of the algebraic K-theory $K_0^{\text{alg}}(X)$ which is symplectic with respect to the Mukai pairing

$$\langle \xi, \xi' \rangle_M = \int_X ch(\xi^\vee \otimes \xi') \cup Td(X), \quad Td(X) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \cdots, \tag{3.117}$$

where $Td(X)$ denotes the Todd class of X. From this, the integral quantum deformed classes (cf. (3.102))

$$g_i := \sigma\left(\hat{\Gamma}(X) \cup ch(\xi_i)\right), \qquad ch(\xi_i) \in H^{\text{even}}, \tag{3.118}$$

are defined in terms of the Chern characters of the basis elements and the *Gamma class*

$$\hat{\Gamma}(X) := \exp\left(\sum_{k\geq 2} \frac{(-1)^k (k-1)!}{(2\pi i)^k} \zeta(k)\, ch_k(X)\right) \in H^{\text{even}} \tag{3.119}$$

of X. The Gamma class is often viewed as a certain "square-root" of $Td(X)$, because the pairing between the correctly quantum deformed classes reproduces the Mukai pairing. This means that we have

$$Q\left(g(\xi), g(\xi')\right) = \langle \xi, \xi' \rangle_M, \tag{3.120}$$

writing $g(\xi) = \sigma\left(\hat{\Gamma}(X) \cup ch(\xi)\right)$, as proven in [Iri].

For a Calabi-Yau threefold as above, with Chern classes $c_1 = 0$, $c_2 = a\,[\ell]$ and $c_3 = b\,[p]$, the Todd class is simply $Td(X) = 1 + \frac{a}{12}[\ell]$ and the Gamma class becomes

$$\hat{\Gamma}(X) = 1 - \frac{\zeta(2)}{(2\pi i)^2} c_2 + \frac{\zeta(3)}{(2\pi i)^3}(c_1 c_2 - c_3) = [X] + \frac{a}{24}[\ell] - \frac{b\,\zeta(3)}{(2\pi i)^3}[p]. \tag{3.121}$$

Starting with the right basis of the algebraic K-theory of X, a flat basis of the form (3.52) can be derived purely from data coming from the A-model topology. Closely following [DK2, dSJKP], we consider the coherent sheaves

$$\xi_3 = \mathcal{O}_X, \quad \xi_2 = \mathcal{O}_H - \frac{a + 2\kappa}{12}\mathcal{O}_p, \quad \xi_1 = -\mathcal{O}_\ell - \mathcal{O}_p, \quad \xi_0 = \mathcal{O}_p, \tag{3.122}$$

constructed from the skyscraper sheaves associated to even degree cohomology elements as in (3.97). They are chosen in such a way, that their Mukai pairing is symplectic and represented by the matrix (3.50). As described for example in [BKV], the Chern characters of (3.122) can be computed by resolving the coherent sheaves by vector bundles via $\mathcal{O}_D = \mathcal{O}_X - \mathcal{O}(-D)$, and then using $ch(\mathcal{O}(D)) = \exp(D)$ for any divisor. Applying this procedure leads to the Chern characters

$$\begin{aligned} ch(\xi_3) = [X], \quad ch(\xi_2) = [H] + \frac{\kappa}{2}\,[\ell] - \frac{a}{12}\,[p] \\ ch(\xi_1) = -[\ell], \quad ch(\xi_0) = [p]. \end{aligned} \tag{3.123}$$

For example, as $\xi_3 = \mathcal{O}_X$ we then have

$$\langle \xi_3, \xi_2 \rangle_M = \int_X ch(\xi_2) \cup Td(X) = \int_X \left([H] + \frac{\kappa}{2}\,[\ell] \right) = 0 \tag{3.124}$$

for degree reasons, while it is trivially $\langle \xi_3, \xi_1 \rangle = 1$. From here, it is easy to construct the Gamma-integral local system according to (3.118). Leaving aside quantum corrections encoded in the holomorphic part of the A-model periods, the combination $g_i^{\text{nilp}} = \hat{\Gamma}(X) \cup e^{-\omega} \cup ch(\xi_i)$ gives

$$\begin{aligned} g_3^{\text{nilp}} &= [X] - t\,[H] + \left(\frac{\kappa}{2} t^2 + \frac{a}{24} \right) [\ell] + \left(-\frac{\kappa}{6} t^3 - \frac{a}{24} t + \frac{b\,\zeta(3)}{(2\pi i)^3} \right) [p] \\ g_2^{\text{nilp}} &= [H] + \left(-\kappa t - \frac{\kappa}{2} \right) [\ell] + \left(\frac{\kappa}{2} t^2 + \frac{\kappa}{2} t - \frac{a}{24} \right) [p] \\ g_1^{\text{nilp}} &= -[\ell] + t\,[p], \qquad g_0^{\text{nilp}} = [p], \end{aligned} \tag{3.125}$$

and corresponds to the desired nilpotent orbit. Based on (3.125), it can be seen that this basis indeed satisfies the property (3.120).

The large volume monodromy and its logarithm in the basis $\{g_i\} := \{g_3, g_2, g_1, g_0\}$ are given by (3.49) and the corresponding limiting periods correctly reproduce the asymptotic behavior of the B-model, where the previously missing values are from an A-model perspective provided by the Gamma class. In particular, the corresponding LMHS, that is also visible in (3.125), is of the form (3.32).

Definition 3.20. The collection of algebraic data $(\mathcal{H}^{\text{even}}_{\mathbb{Z}}, \mathcal{H}^{\text{even}}, \mathcal{F}^\bullet, \mathcal{W}_\bullet, \nabla, Q)$, defining an integral, polarized VMHS over the punctured disk Δ^*, is called the *A-model VMHS* associated to (X, ω).

Having established the construction of the A-model VMHS, we are finally in a position to bridge the gap towards the B-model in the following section, and give a precise definition for a mirror pair that was physically described in Section 2.2.

3.6. Hodge Theoretic Mirror Pairs

The two variations of MHS associated to A-model and B-model geometries around suitably chosen limit points in the respective moduli spaces can be used to formulate closed string mirror symmetry in a Hodge-theoretic framework.

Definition 3.21. A pair of Calabi-Yau threefold families (X, ω) and $\mathcal{Y}$ is called a *(Hodge theoretic) mirror pair* if the polarized variations of MHS on $\mathcal{H}^{\text{even}}(X)$ and $\mathcal{H}^3(\mathcal{Y})$ are isomorphic

$$\begin{tikzcd} \mathcal{H}^{\text{even}}(X) \arrow[r, "\sim"] \arrow[d] & \mathcal{H}^3(\mathcal{Y}) \arrow[d] \\ \Delta_q^* \arrow[r, "m"] & \Delta_z^*, \end{tikzcd} \tag{3.126}$$

where Δ_q^* and Δ_z^* are small neighborhoods around MUM-type boundary points of $\widehat{\mathcal{K}}(X)$ and $\widehat{\mathcal{M}}(Y)$ related by the mirror map $m : q(z) \mapsto z(q)$.

Given a mirror pair, the periods of Ω in the B-model attain an A-model interpretation as quantum corrected volumes of (D-branes wrapping) the integral cycles Poincaré dual to g_i, e.g.

$$\delta\mathcal{F} = \frac{\kappa}{2}t^2 + \frac{\kappa}{2}t - \frac{a}{24} + \frac{1}{(2\pi i)^2}\sum_{d=1}^{\infty} d\widetilde{N}_d\, q^d = Q(g_2, e_3). \tag{3.127}$$

The choice of integral local system corresponds to the specification of the limiting asymptotics of both A- and B-model periods. Given a mirror pair, their equality establishes a connection between topological data of X with certain extension data of the Hodge theory of $\mathcal{Y}$, cf. Remark 3.9. The quantum differential equations coming from the Gauß-Manin and Dubrovin connections are identified, leading to an equality of the respective A-model and B-model three-point functions under the mirror map. From this, the Yukawa coupling of $\mathcal{Y}$

$$\mathfrak{C}(q(z)) = \kappa + \sum_{d=1}^{\infty} d^3\widetilde{N}_d\, q^d = \Phi'''(q), \tag{3.128}$$

computes the Gromov-Witten invariants of the mirror X with $\widetilde{N}_d = GW_\beta$ for $\beta = d[\ell]$. While in general $\widetilde{N}_d \in \mathbb{Q}$ for families defined over the rationals, an integral expansion of the holomorphic part of the Gromov-Witten potential is given by a Gopakumar-Vafa multicover formula [GV, KSV]

$$\Phi_h = \frac{1}{(2\pi i)^3}\sum_{d=1}^{\infty}\widetilde{N}_d\, q^d = \frac{1}{(2\pi i)^3}\sum_{d=1}^{\infty} N_d\,\mathrm{Li}_3(q^d) := \frac{1}{(2\pi i)^3}\sum_{d,k=1}^{\infty} N_d\,\frac{q^{dk}}{k^3}, \quad N_d \in \mathbb{Z}, \tag{3.129}$$

where the instanton numbers N_d are closely related to the number of degree d curves lying in X, see e.g. [CK2].

Example 3.22. The most well-known example of a mirror pair was orignally studied in [GP, CdlOGP] and consists of the Fermat quintic X (3.93) on the A-model side and the mirror quintic family $\mathcal{Y}$, obtained by resolving the quotient of (3.17), on the B-model side. The prediction of [*op.cit.*] consists of the equality of the superconformal field theories underlying the A-model on X and the B-model on $\mathcal{Y}$, which in particular implies the equality of three-point functions under the mirror map, as discussed in Section 2.2. Starting in the B-model, the Picard-Fuchs operator of $\mathcal{Y}$ is given by (3.39) (see Example 3.6) and the Yukawa coupling can be computed via Proposition 3.10 leading to

$$\mathfrak{C} = \frac{5}{(1-5^5 z)\,\varpi_0^2}\left(\frac{q}{z}\frac{dz}{dq}\right)^3 = 5 + 2875\,q + 4876875\,q^2 + 8564575000\,q^3 + \cdots . \tag{3.130}$$

That the classical parts of the three-point functions coincide, amounts to computing the triple-intersection number in the A-model via Bézout's theorem

$$\kappa = \int_X H \cup H \cup H = 5. \tag{3.131}$$

That the Yukawa coupling of $\mathcal{Y}$ indeed produces the Gromov-Witten invariants of X in all degrees in terms of the Gromov-Witten potential

$$\mathfrak{C} = \Phi''', \quad (2\pi i)^3\,\Phi_h = 2875\,q + \frac{4876875}{8}\,q^2 + \frac{8564575000}{27}\,q^3 + \cdots , \tag{3.132}$$

is the statement of the Mirror Theorems [Kon2, Giv, LLY] proving the prediction of [CdlOGP] via a localization formula. The equality of three-point function is the first essential ingredient in showing that (X, ω) and $\mathcal{Y}$ define a mirror pair in the sense of Definition 3.21, as described in [CK2]. Namely, after mapping the sections $[X]$ and $[p]$ of $\mathcal{H}^{\mathrm{even}}(X)$ to the sections e_3 and g_0 of $\mathcal{H}^3(\mathcal{Y})$, the variations of complex variations of MHS are isomorphic under the mirror map if the differential equations defined by the Gauß-Manin and Dubrovin connections are identified [Theorem 8.6.4][*op.cit.*]. In the canonical basis defined by $\{e_j\}$ the latter are completely determined by $\mathfrak{C}$ and Φ''' respectively, from which the statement follows. The second ingredient is related to the underlying integral local systems $\mathcal{H}^{\mathrm{even}}_{\mathbb{Z}}$ and $\mathcal{H}^3_{\mathbb{Z}}$. In the B-model, it is determined by the limiting periods, which are for the choice of basis in Section 3.2 given by

$$Q(\tilde{g}_3(0), e_2) = \frac{25}{12}, \qquad Q(\tilde{g}_3(0), e_3) = -\frac{25i\,\zeta(3)}{\pi^3}. \tag{3.133}$$

As described in Section 3.5, the specification of an integral local system of this kind in the A-model can be achieved utilizing the Gamma class (3.121). Indeed, the constants $a = 50$ and $b = -200$ of (3.55) arise in this context as the second Chern class and Euler class of X. In particular, the LMHS of the B-model geometry is determined by topological quantities associated to the A-model.

Remark 3.23. In Section 2.3 we briefly touched on the notion of homological mirror symmetry, relating the A- and B-brane categories of X and $\mathcal{Y}$. This categorical statement, proven for the quintic in [She], turns out to be a more general statement than the Hodge theoretic Mirror Theorem described in Example 3.22. The reason is that the variations of MHS arising in the latter, can be derived from the respective brane categories, such that Hodge theoretic mirror symmetry can be viewed as a consequence of homological mirror symmetry in this context [GPS].

The central role of a multicover formula of the type (3.129) in relating string amplitudes to integral enumerative invariants is not unique to the Yukawa coupling. The overarching concept, of which we will encounter various manifestations later on, is the following [SVW1].

Definition 3.24. Given $s \in \mathbb{N}$, a formal power series

$$V(q) = \sum_{d=1}^{\infty} a_d\, q^d \in q\,\mathbb{Q}[\![q]\!] \tag{3.134}$$

with rational coefficients and without constant term is called an *s-function* if it can be written as a linear combination of s-Logarithms

$$V(q) = \sum_{d=1}^{\infty} b_d\, \mathrm{Li}_s(q^d), \qquad \mathrm{Li}_s(q) = \sum_{k=1}^{\infty} \frac{q^k}{k^s}, \tag{3.135}$$

with integral coefficients $b_d \in \mathbb{Z}$.

Remark 3.25. Definition 3.24 is equivalent to a certain property of the coefficients a_d in relation to the Frobenius endomorphism $\mathrm{Fr}_p(q) = q^p$. Namely, $V(q)$ is an s-function if and only if the combinations

$$\frac{1}{p^s}\, a_{d/p} - a_d \in \mathbb{Z}_p \tag{3.136}$$

are p-integral[6] for all p and d, with $a_{d/p} = 0$ if $p \nmid d$ (see [*op.cit.*]). This point of view then allows for a generalization of the concept to formal power series $V(q) \in q\,K[\![q]\!]$ with coefficients $a_d \in K/\mathbb{Q}$ lying in an algebraic number field. In this situation, we have to view the coefficients of $V(q)$ as p-adic numbers and replace the p-adic integers $\mathbb{Z}_p$ in (3.136) with the localization $\mathcal{O}_\mathfrak{p}$ of the ring of algebraic integers at a prime ideal $\mathfrak{p} \subset \mathcal{O}$ lying over p, away from the branch locus of $\mathrm{Spec}(\mathcal{O}) \to \mathrm{Spec}(\mathbb{Z})$. We refer the reader to [KSV, SVW2, Mü2] for a treatment of the arithmetic details.

[6] This means that there are no factors of p in the denominator, such that the elements define p-adic integers.

In the language of s-functions, the Yukawa coupling (3.128) is a 3-function. Its analog for the open string, to be discussed in Section 4.3, will be a 2-function and the possibility of a non-trivial field extension of $\mathbb{Q}$ entering the story is related to the field of definition of the relevant algebraic cycle. The resulting arithmetically twisted multicover formula in the simplest examples will be discussed in Section 6.1 and in particular Example 6.2. The open and closed topological string one-loop amplitudes, to be discussed in Chapter 7, will be seen to be examples for 1-functions.

4. B-Branes and Normal Functions

In this chapter we describe how the concept of D-branes can be embedded into the Hodge theoretic framework developed in Chapter 3, while focussing exclusively on the B-model. As in Section 2.3 we are mostly interested in B-type topological branes that can be understood as complex submanifolds, or algebraic cycles, of the given string vacuum. An algebraic consequence of the presence of algebraic cycles is the production of extensions of variations of MHS, which we discuss in Section 4.1. The classification of extensions in terms of the Abel-Jacobi map is the topic of Section 4.2, where we also encounter the B-brane superpotential (2.28). We close with an analysis of the asymptotics of the Abel-Jacobi map in Section 4.3 in a neighborhood of a MUM-type boundary point, revealing the open string analog of the Yukawa coupling that will play a crucial role in what follows.

4.1. Extensions from Algebraic Cycles

Following [MW, SVW2], we continue with the type of family $\pi : \mathcal{Y} \to \Delta^*$ described in Section 3.2 and consider for each fiber Y_z an algebraic curve $i : C_z \hookrightarrow Y_z$ varying with $z \in \Delta^*$ such that $\pi \circ i : \mathcal{C} \to \Delta^*$ is a smooth family which admits a semi-stable continuation over Δ. In what follows, we will denote by $\mathrm{CH}^2(\mathcal{Y})_{\mathrm{hom}}$ the subgroup of the Chow group consisting of homologically trivial algebraic cycles. When the family of cycles $\mathcal{C} \in \mathrm{CH}^2(\mathcal{Y})_{\mathrm{hom}}$ is homologically trivial, meaning that $i_*([C_z]) = 0 \in H_2(Y_z;\mathbb{Z})$ for each $z \in \Delta^*$, we can associate to the pair $(\mathcal{Y}, \mathcal{C})$ a VMHS that can be realized as an extension

$$0 \longrightarrow \mathcal{H}^3 \longrightarrow \hat{\mathcal{H}}^3 \longrightarrow \mathcal{I} \longrightarrow 0, \qquad \hat{\mathcal{H}}^3 \in \mathrm{Ext}_{\mathrm{VMHS}}(\mathcal{I}, \mathcal{H}^3), \tag{4.1}$$

of the pure VHS $\mathcal{I}$ of weight 4 by $\mathcal{H}^3$. In case of a Kähler manifold it is important to note that there can be no irreducible cycle represented by a complex submanifold which is in itself homologically equivalent to zero. The reason is that the Kähler form defines a calibration on each fiber Y_z with respect to which complex submanifolds are calibrated. In particular this implies that compact complex submanifolds of a Kähler manifold are homologically volume minimizing (see also Section 2.3). The homologically trivial cycle described above is therefore always consisting of several connected components that are homologically equivalent to one another. For now we assume for simplicity that $H^0(C_z, \mathbb{Z}) = \mathbb{Z}^2$, meaning that C_z consists of two homologically equivalent components. The extension is fiberwise coming from the exact sequence

$$0 \longrightarrow H^3(Y_z;\mathbb{C}) \longrightarrow H^3(Y_z \setminus C_z;\mathbb{C}) \longrightarrow \ker\Big(H^0(C_z;\mathbb{C}) \xrightarrow{i_!} H^4(Y_z;\mathbb{C})\Big) \longrightarrow 0, \tag{4.2}$$

where $i_! := PD \circ i_* \circ PD$ denotes the Gysin map. Under our assumptions, the kernel above is one-dimensional and generated by the difference of classes associated to the two connected components of C_z. The extension of local systems

$$0 \longrightarrow \mathcal{H}^3_{\mathbb{Z}} \longrightarrow \hat{\mathcal{H}}^3_{\mathbb{Z}} \longrightarrow \mathcal{I}_{\mathbb{Z}} \longrightarrow 0 \tag{4.3}$$

arises geometrically from the short exact sequence

$$0 \to H_3(Y_z;\mathbb{Z}) \to H_3(Y_z, C_z;\mathbb{Z}) \to \ker\Big(H_2(C_z;\mathbb{Z}) \to H_2(Y_z;\mathbb{Z})\Big) \to 0 \tag{4.4}$$

by Poincaré duality where, after a suitable Tate-twist, $\mathcal{I}_{\mathbb{Z},z} = \mathbb{Z}(-2)$ is Hodge-Tate of pure Hodge type (2,2). The extended Hodge filtration $\hat{F}^p$ on the cohomology of the complement is coming from the grading of the sheaf[1] $\Omega^\bullet_{\mathcal{Y}}\langle\log \mathcal{C}\rangle$ of rational holomorphic forms and satisfies

$$\hat{F}^2 \big/ F^2 = \mathbb{Z}(-2)^d, \qquad \hat{F}^p \big/ \hat{F}^{p+1} = F^p \big/ F^{p+1}. \tag{4.5}$$

We define the Hodge filtration $\hat{\mathcal{F}}^p = \hat{F}^p \otimes \mathcal{O}_{\Delta^*}$ of holomorphic subbundles on $\hat{\mathcal{H}}^3 = \hat{\mathcal{H}}^3_{\mathbb{Z}} \otimes \mathcal{O}_{\Delta^*}$ and the Gauß-Manin connection canonically extends to $\hat{\mathcal{H}}^3$ by specifying the horizontal sections according to $\nabla(\hat{\mathcal{H}}^3_{\mathbb{C}}) = 0$. The only non-vanishing components of the weight filtration $\mathcal{W}_\bullet$ on $\hat{\mathcal{H}}^3_{\mathbb{Q}}$ are set to $\mathcal{W}_3 = \mathcal{H}^3_{\mathbb{Q}}$ and $\mathcal{W}_4 = \hat{\mathcal{H}}^3_{\mathbb{Q}}$, such that the data $(\hat{\mathcal{H}}^3_{\mathbb{Z}}, \hat{\mathcal{H}}^3, \hat{\mathcal{F}}^\bullet, \mathcal{W}_\bullet, \nabla, Q)$ defines an integral, polarized VMHS that fits as an extension in the exact sequence (4.1).

By a theorem of Carlson [Car], extensions of variations of MHS of this kind are classified by the *intermediate Jacobian fibration*

$$\mathrm{Ext}^1_{\mathrm{VMHS}}(\mathcal{I}_{\mathbb{Z}}, \mathcal{H}^3) \cong \mathcal{J}^2(\mathcal{H}^3) := \mathcal{H}^3 \big/ \big(\mathcal{F}^2 + \mathcal{H}^3_{\mathbb{Z}}\big), \tag{4.6}$$

in terms of a *normal function* ν of $\mathcal{H}^3$, given by a holomorphic section ν of $\mathcal{J}^2(\mathcal{H}^3)$ that satisfies the horizontality condition $\nabla\tilde{\nu} \in \mathcal{F}^1 \otimes \Omega_{\Delta^*}$, where $\tilde{\nu}$ is any lift of ν to $\mathcal{H}^3$. The section which determines a given extension can be explicitly described by taking both an integral lift h and an $\hat{\mathcal{F}}^2$-lift f of the generator of $\mathcal{I}_{\mathbb{Z}}$ in $\hat{\mathcal{H}}^3$, such that the normal function is defined as the class of the difference $[h - f] \in \mathcal{J}^2(\mathcal{H}^3)$. Because $\nabla^2 = 0$ and modulo the ambiguity

[1] Details of the construction can be found for example in [PS], see also [Ker].

of the $\hat{\mathcal{F}}^2$-lift f, the derivatives $\nabla\tilde{\nu}$ characterize the $\mathbb{C}$-VMHS locally by the *Griffiths-Green-Voisin infinitesimal invariant*

$$\left[\nabla\tilde{\nu}\right] \in \frac{\ker\left(\nabla : \mathcal{F}^1 \otimes \Omega^1_{\Delta^*} \to \mathcal{F}^0 \otimes \Omega^2_{\Delta^*}\right)}{\mathrm{im}\left(\nabla : \mathcal{F}^2 \to \mathcal{F}^1 \otimes \Omega^1_{\Delta^*}\right)}, \tag{4.7}$$

of $\mathcal{H}^3$ [Voi]. Its vanishing implies that the normal function is locally constant, meaning that locally there always exists a lift $\tilde{\nu} \in \mathcal{H}^3$ satisfying $\nabla\tilde{\nu} = 0$. In case of an extension coming from an algebraic cycle, vanishing of the infinitesimal invariant is geometrically reflected by the cycle being locally constant, i.e. independent of the modular parameter z.

4.2. The Abel-Jacobi map and B-Brane Superpotentials

In the geometric situation of a homologically trivial algebraic cycle, Poincaré duality implies an isomorphism [Voi]

$$\mathcal{J}^2(\mathcal{H}^3) \cong {\left(\mathcal{F}^2\right)^\vee} \big/ {\left(\mathcal{H}^3_{\mathbb{Z}}\right)^\vee}, \tag{4.8}$$

which allows to characterize the extension by a functional $\nu_{\mathcal{C}} : \mathcal{F}^2 \to \mathcal{O}_{\Delta^*}$ defined modulo integral periods. To define its action in each fiber Y_z, we follow [Ker] and choose any relative 3-chain $\Gamma_z \in H_3(Y_z, C_z; \mathbb{Z})$ with boundary $\partial\Gamma_z = C_z$ and define the integral lift as a current

$$h = (2\pi i)^2 \delta_{\Gamma_z} \in H^3\left(Y_z \setminus C_z; \mathbb{Z}(2)\right) \cong H_3\left(Y_z, C_z; \mathbb{Z}(-1)\right). \tag{4.9}$$

The $\hat{\mathcal{F}}^2$-lift is given by another current $f \in \hat{F}^2$ with $df = (2\pi i)^2 \delta_{C_z}$. Their difference $h - f$ is a class in $H^3(Y_z)$, well-defined modulo $F^2H^3(Y_z) + H^3(Y_z; \mathbb{Z}(2))$ and the equivalence class in the intermediate Jacobian

$$[h - f] \in J^2(H^3) = \mathcal{J}^2(\mathcal{H}^3)_z \tag{4.10}$$

gives rise to a functional defined up to integral periods by

$$\nu_{\mathcal{C}}(\eta)_z := \int_{Y_z} (h - f) \wedge \eta = (2\pi i)^2 \int_{\Gamma_z} \eta, \qquad \eta \in F^2H^3(Y_z), \tag{4.11}$$

where the second term vanishes by type considerations because $f \wedge \eta \in F^4 = \{0\}$.

Definition 4.1. The *Abel-Jacobi map*[2]

$$AJ : \mathrm{CH}^2(\mathcal{Y})_{\mathrm{hom}} \longrightarrow \mathcal{J}^2(\mathcal{H}^3), \qquad \mathcal{C} \longmapsto \nu_{\mathcal{C}}, \tag{4.12}$$

is defined by the assignment of the functional (4.11) to a homologically trivial algebraic cycle $\mathcal{C}$.

[2] We focus on the case of middle dimensional cohomology of Calabi-Yau threefolds.

Acting in each fiber on $F^2H^3(Y_z) = H^{2,1}(Y_z) \oplus H^{3,0}(Y_z)$, the Abel-Jacobi map has two components arising from its restriction to each summand. We will refer to these components as the *truncated normal functions* coming from an algebraic cycle. Of central importance in what follows will be the truncated normal function associated to the holomorphic 3-form

$$\mathcal{W}_B(z) = Q(AJ(\mathcal{C}), e_3) = (2\pi i)^2 \int_{\Gamma_z} \Omega_z, \tag{4.13}$$

because it can be physically interpreted as the B-brane superpotential (2.28) associated to the open string vacuum determined by the cycle, as described in Section 2.3. Its asymptotic behavior around a MUM-type boundary point will be the main focus of Section 4.3, because it is the open string analog of the prepotential (3.53) and contains the full information of the VMHS in the limit. As described by Morrison-Walcher in [MW], the superpotential satisfies an inhomogeneous version of the Picard-Fuchs equation

$$D_{\mathrm{PF}}\, \mathcal{W}_B = \mathcal{J}, \tag{4.14}$$

where D_{PF} is the ordinary Picard-Fuchs operator of $\mathcal{Y}$ and the inhomogeneity $\mathcal{J}$ is additive with respect to the corresponding cycles.

In the more general situation of a cycle consisting of any number of connected components, we take the following point of view to define the Abel-Jacobi map. It depends on the choice of a homologically equivalent cycle $\mathcal{C}_0$ that serves as a fixed reference with $\mathcal{C} - \mathcal{C}_0 \in \mathrm{CH}^2(\mathcal{Y})_{\mathrm{hom}}$. The normal function is then fiberwise determined by integrating $\eta \in F^2H^3(Y_z)$ over a relative 3-chain Γ_z with boundary $\partial\Gamma_z = C_z - C_{0,z}$. In general we will therefore consider cycles

$$\mathcal{C} = \bigcup_{k=0}^{d} \mathcal{C}_k, \qquad C_z = \bigcup_{k=0}^{d} C_{z,k}, \tag{4.15}$$

assuming that all irreducible components are homologically equivalent to each other, i.e. $[C_{z,k}] - [C_{z,k'}] = 0 \in H_2(Y_z, \mathbb{Z})$, for each pair of indices k and k'. We choose $\mathcal{C}_0 \subset \mathcal{C}$ as a fixed and globally defined reference cycle with respect to which homological equivalence is measured and denote by $AJ(\mathcal{C}_k) = \nu_{\mathcal{C}_k}$ the normal function described above, slightly abusing notation. Suitable choices for $\mathcal{C}_0$ are the coordinate lines on $\mathcal{Y}$, see e.g. [JW]. This determines an extension similar to (4.1), where $\mathcal{I}_{\mathbb{Z},z} = \mathbb{Z}(-2)^{\oplus d}$, and we interpret the truncated normal functions $\mathcal{W}_k$ as the B-brane superpotential associated to the open string vacuum determined by $\mathcal{C}_k$ relative to the reference $\mathcal{C}_0$. For any two homologically equivalent cycles $\mathcal{C}_k, \mathcal{C}_{k'} \subset \mathcal{C}$ the differences

$$\mathcal{T}_{k,k'} := \mathcal{W}_k - \mathcal{W}_{k'}, \qquad D_{\mathrm{PF}}\, \mathcal{T}_{k,k'} = \mathcal{J}_k - \mathcal{J}_{k'}, \tag{4.16}$$

have an interpretation as BPS domain wall tensions between the open string vacua associated to $\mathcal{C}_k$ and $\mathcal{C}_{k'}$. This construction is in line with the relative nature of the B-brane superpotential in the physical context, as sketched in Section 2.3.

4.3. Asymptotics and Abel-Jacobi Limits

When the monodromy of the extended local system $\hat{M} : \hat{\mathcal{H}}^3_{\mathbb{Z}} \to \hat{\mathcal{H}}^3_{\mathbb{Z}}$ is unipotent, its logarithm $\hat{N} = \log(\hat{M})$ is a nilpotent operator that preserves the filtration $\mathcal{W}_\bullet$ of $\hat{\mathcal{H}}^3$. In such a situation there exists a relative monodromy weight filtration $\hat{\mathcal{W}}_\bullet := \hat{\mathcal{W}}_\bullet(\hat{N}, \mathcal{W}_\bullet)$ on $\hat{\mathcal{H}}^3_{\mathbb{Q}}$ such that $\hat{N}(\hat{\mathcal{W}}_k) \subset \hat{\mathcal{W}}_{k-2}$ and $\hat{\mathcal{W}}_\bullet$ induces on each $\mathrm{Gr}_k^{\mathcal{W}}$ the weight filtration of the map induced by $\hat{N}$ [Del3], see also [BZ]. With respect to this relative filtration and due to its mapping into $H^4(Y_z;\mathbb{C})$ in (4.2), $\mathcal{I}_{\mathbb{Z}}$ is of weight 4 and the extension can be described in terms of the canonical coordinate around a MUM-point and the Hodge basis $\{e_j\}$. In what follows, we will first assume that the rank d of $\mathcal{I}_{\mathbb{Z}}$ is 1. For any lift

$$\tilde{\nu} = h - f = \mathcal{S}\, e_3 + \mathcal{U}\, e_2 + \mathcal{V}\, e_1 + \mathcal{W}\, e_0 \in \mathcal{H}^3 \tag{4.17}$$

with $\mathcal{S}, \mathcal{U}, \mathcal{V}, \mathcal{W}$ holomorphic on Δ^*, the validity of the horizontality condition $Q(\nabla_t \tilde{\nu}, e_3) = 0$ corresponds to the vanishing of the coefficient of e_0 in $\nabla_t \tilde{\nu}$. This derivative is given by

$$\nabla_t \tilde{\nu} = \nabla_t (h - f) = \delta \mathcal{S}\, e_3 + (\mathcal{S} + \delta \mathcal{U})\, e_2 + \Big(\mathcal{U} \delta^3 \mathcal{F} + \delta \mathcal{V}\Big)\, e_1 + (-\mathcal{V} + \delta \mathcal{W})\, e_0, \tag{4.18}$$

with the canonical logarithmic vector field $\delta = 2\pi i\, q\, d/dq$ introduced in (3.51). It is now evident that the horizontality condition is here independent of the $\mathcal{F}^2$-part of the $\hat{\mathcal{F}}^2$-lift f, such that maximal degeneracy provides a canonical lift[3] of the normal function given by setting $\mathcal{S} = \mathcal{U} = 0$ [SVW2], i.e.

$$\tilde{\nu} = h - f = \mathcal{V}\, e_1 + \mathcal{W}\, e_0, \tag{4.19}$$

where $\mathcal{V}$ and $\mathcal{W}$ are the truncated normal functions of $\tilde{\nu}$. From now on, we will denote by $\tilde{\nu}$ this canonical lift of the normal function, slightly abusing notation. Vanishing of the coefficient of e_0 in $\nabla_t \tilde{\nu}$ then implies $\mathcal{V} = \delta \mathcal{W}$. The integral lift becomes part of the extended basis $\{g_i, h\}$ of $\hat{\mathcal{H}}^3_{\mathbb{Z}}$ and satisfies by definition $\nabla(h) = 0$, which further leads to

$$\nabla_t f = -\theta \mathcal{V}\, e_1 = -\theta^2 \mathcal{W}\, e_1, \tag{4.20}$$

such that the extension of the Gauß-Manin connection is fully determined by the B-brane superpotential $\mathcal{W}$. The open string analog of the Yukawa coupling

[3] This point of view is also taken in [Usu].

is therefore given by a certain representative of the infinitesimal invariant (4.7) in the canonical coordinate. Namely, the derivative of the canonical lift $\nabla\widetilde{\nu}$ is fully determined by its $\mathcal{F}^1$ component, which motivates the definition [SVW2]

$$\mathfrak{D} := -Q(\nabla_t \widetilde{\nu}, \nabla_t e_3) = Q(\nabla_t^2 \widetilde{\nu}, e_3) = \theta^2 \mathcal{W}, \tag{4.21}$$

where the second equality follows from differentiating the horizontality condition. In particular, the full information of the extension can be recovered from the superpotential and its inhomogeneous Picard-Fuchs equation in the given situation.

In general the monodromy is not guaranteed to be unipotent, as the cycle might branch when encircling the puncture, exchanging the connected components of $\mathcal{C}$. This implies that $\hat{M}$ has finite order r when restricted to $\mathcal{I}_{\mathbb{Z}}$, where we assume for simplicity that all arising orbits are of the same order. To ensure the existence of a globally well defined cycle, we have to pass to an r-fold cover $\hat{z} = z^{1/r} \mapsto z$, branched at $0 \in \Delta$, and consider the pullback

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[dr, bend right] & \hat{\mathcal{Y}} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ & \Delta^*_{\hat{z}} \arrow[r] & \Delta^*_z . \end{tikzcd} \tag{4.22}$$

On the cover, the extended monodromy logarithm is then given by $\hat{N} = \log(\hat{M}^r)$, and after pullback along (4.22), the degree of the local system $\hat{\mathcal{I}}_{\mathbb{Z}}$ is d/r. The extended monodromy logarithm is now generally of the form [GGK2]

$$\hat{N} = \begin{pmatrix} rN & L \\ 0 & 0 \end{pmatrix}, \qquad L \in \mathrm{Hom}_{\mathbb{Q}}(\hat{\mathcal{I}}_{\mathbb{Q}}, \mathcal{H}^3_{\mathbb{Q}}). \tag{4.23}$$

While we can find a splitting of the sequence (4.1) over $\mathbb{Q}$, such that $L = 0$ and $\hat{\mathcal{I}}_{\mathbb{Q}}$ has weight 3, the integral local system $\hat{\mathcal{I}}_{\mathbb{Z}}$ has weight 4 in general. In the integral basis $\{g_i, h_k\}$, the logarithm $\hat{N}$ is therefore represented by

$$\hat{N} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -r & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{r\kappa}{2} & r\kappa & 0 & 0 & \lambda_1 & \cdots & \lambda_d \\ -\frac{ra}{12} & \frac{r\kappa}{2} & r & 0 & s_1 & \cdots & s_d \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \frac{\lambda_k}{r}, \frac{\lambda_k}{2} + s_k \in \mathbb{Z} \text{ for all } k, \tag{4.24}$$

where the extending generator h_k is mapped to $\hat{N}(h_k) = \lambda_k\, g_1 + s_k\, g_0 \in \mathcal{W}_2$ and the integrality constraints ensure that the monodromy on the cover is integral.

Over $\mathbb{Q}$ we can find a basis such that $\hat{N}(h_k) = s_k\, g_0 \in \mathcal{W}_0$ and $h_k \in \hat{\mathcal{W}}_3$, but this change of coordinates cannot be performed while retaining integrality of the generators in general [SVW2]. The graded pieces of the filtration $\hat{\mathcal{W}}_\bullet$ then fit into the diagram (cf. [Usu])

$$\begin{tikzcd}
 & \mathrm{Gr}_6^{\mathcal{W}} \arrow[r, "\sim"] \arrow[d, "rN"'] & \mathrm{Gr}_6^{\hat{\mathcal{W}}} \arrow[d, "\hat{N}"] & & \\
0 \arrow[r] & \mathrm{Gr}_4^{\mathcal{W}} \arrow[r] \arrow[d, "rN"'] & \mathrm{Gr}_4^{\hat{\mathcal{W}}} \arrow[r] \arrow[d, "\hat{N}"] & \hat{\mathcal{I}}_{\mathbb{Z}} \arrow[r] \arrow[d, "L"] & 0 \\
0 \arrow[r] & \mathrm{Gr}_2^{\mathcal{W}} \arrow[r] \arrow[d, "rN"'] & \mathrm{Gr}_2^{\hat{\mathcal{W}}} \arrow[r] \arrow[d, "\hat{N}"] & H \arrow[r] & 0 \\
 & \mathrm{Gr}_0^{\mathcal{W}} \arrow[r, "\sim"] & \mathrm{Gr}_0^{\hat{\mathcal{W}}}, & &
\end{tikzcd} \tag{4.25}$$

where horizontal lines are exact sequences and H corresponds to an r-torsion group given by the quotient

$$H := {}^{\mathbb{Z}\langle g_0\rangle}\big/_{r\mathbb{Z}\langle g_0\rangle}, \tag{4.26}$$

reflecting the branching behavior of the cycle.

Example 4.2. Returning to the mirror quintic family $\mathcal{Y}$, defined by resolving the quotient of (3.17), the motivating example for the discussion above is the algebraic cycle defined by

$$C_{\pm,\psi} = \left\{ x_1 + x_2 = 0, x_3 + x_4 = 0, x_5^2 = \pm\sqrt{5\psi}\, x_1 x_3 = 0 \right\} \subset Y_\psi. \tag{4.27}$$

Its image under the Greene-Plesser group is referred to as the *Deligne conics* in [Wal4]. In view of the physical context of the superpotential in which we always consider a regime in which the brane vacuum decouples from open string deformations, we note that the Deligne conics have no open string moduli, meaning that they admit no holomorphic deformations inside of a fixed Y_ψ. A more detailed account of this property can be found in Appendix A. As the components of $C_{\pm,\psi}$ are exchanged under monodromy around $z = 0$, the algebraic curve $\mathcal{C}_\pm \subset \hat{\mathcal{Y}}$ is only globally well defined after passing to a double cover as described above. By [MW], the combination $[C_+ - C_-] = 0 \in H_2(Y_z)$ and similarly $\mathcal{C}_+ - \mathcal{C}_- \in \mathrm{CH}^2_{\mathrm{hom}}(\mathcal{Y})$, such that the Deligne conics determine a normal function of the VHS $\mathcal{H}^3(\mathcal{Y})$. For the standard choice of holomorphic 3-form (3.38), its associated truncated normal function, or B-brane superpotential, satisfies the inhomogeneous Picard-Fuchs equation

$$D_{\mathrm{PF}}\mathcal{W}_B = (2\pi i)^4 \left[\theta^4 - 5z \prod_{k=1}^{4} (5\theta + k)\right] \left(\int_\Gamma \Omega\right) = \mathcal{J}_{\mathrm{Del}} := -\frac{15}{4}\sqrt{z}, \tag{4.28}$$

where $\partial\Gamma_z = [\mathcal{C}_{+,z}] - [\mathcal{C}_{-,z}]$ and the square root is an indicator for the branching behavior of the conics.

The extended Gauß-Manin connection in the basis $\{e_j, f_k\}$ and the canonical coordinate q is now given by

$$\nabla_t = d + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -\mathfrak{C} & 0 & 0 & -\mathfrak{D}_1 & \cdots & -\mathfrak{D}_d \\ 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \otimes \frac{dq}{2\pi i\, q}, \qquad \mathfrak{D}_k = \delta^2 \mathcal{W}_k = Q(\nabla_t^2 \widetilde{\nu}, e_3) \tag{4.29}$$

and determined by the Yukawa coupling $\mathfrak{C}$ together with the infinitesimal invariants $\mathfrak{D}_k$ coming from all superpotentials. It describes a VMHS of Hodge-Tate type on Δ^* which we think of as recording the degeneration of the VMHS given by the extension in the neighborhood of the singular point. Its components and their respective Hodge-Deligne types are depicted in Figure 4.1.

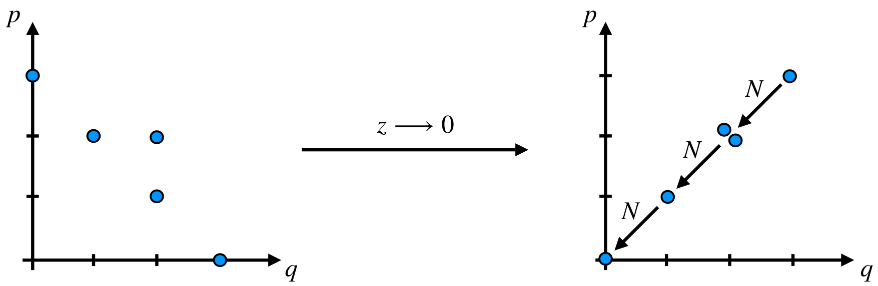

Figure 4.1.: LMHS associated to $\hat{\mathcal{H}}^3$ at a point of maximal degeneration. In contrast to the situation of Figure 3.3, we begin with a VMHS whose weight filtration comes from the extension by an algebraic cycle. The process of its degeneration is determined by both the asymptotic behavior of the periods and the normal function.

Remark 4.3. The VMHS $\hat{\mathcal{H}}^3$ can also be viewed from a dual perspective[4] by considering relative cohomology $H^3(Y_z, C_z)$ instead of the cohomology of the complement $H^3(Y_z \setminus C_z)$. In this case, the extension is of the form

$$0 \longrightarrow \mathcal{I}^\vee \longrightarrow \left(\hat{\mathcal{H}}^3\right)^\vee \longrightarrow \mathcal{H}^3 \longrightarrow 0, \qquad \left(\hat{\mathcal{H}}^3\right)^\vee \in \mathrm{Ext}_{\mathrm{VMHS}}(\mathcal{H}^3, \mathcal{I}^\vee), \tag{4.30}$$

[4]The author thanks Felipe Espreafico for helpful discussions around this point of view.

where $\mathcal{I}^\vee$ is a pure VHS of weight 2. It arises geometrically from the short exact sequence

$$0 \longrightarrow \operatorname{coker}\left(H^2(Y_z) \to H^2(C_z)\right) \longrightarrow H^3(Y_z, C_z) \longrightarrow H^3(Y_z) \longrightarrow 0 \tag{4.31}$$

coming from the long exact sequence of the pair (Y_z, C_z). The extended Hodge filtration is specified by

$$\hat{F}^1\big/F^1 = \mathbb{Z}(-1)^d, \qquad \hat{F}^p\big/\hat{F}^{p+1} = F^p\big/F^{p+1}, \tag{4.32}$$

and the weight filtration satisfies $\mathcal{W}_3 = (\hat{\mathcal{H}}^3)^\vee$ and $\mathcal{W}_2 = \mathcal{I}^\vee$. The two variations of MHS are dual in the sense that the respective extending generators are dual with respect to the duality pairing

$$H^3(Y_z \setminus C_z) \times H^3(Y_z, C_z) \longrightarrow \mathbb{C}, \qquad (\alpha, \beta) \longmapsto \int_{PD(\alpha)} \beta. \tag{4.33}$$

We denote the extending generator of $(\hat{\mathcal{H}}^3)^\vee$ by $f_k^\vee$. The monodromy weight filtration associated to a MUM-point and the corresponding LMHS can be analyzed similar to the presentation in the preceding section. The Hodge-Deligne types of the degeneration are depicted in Figure 4.2.

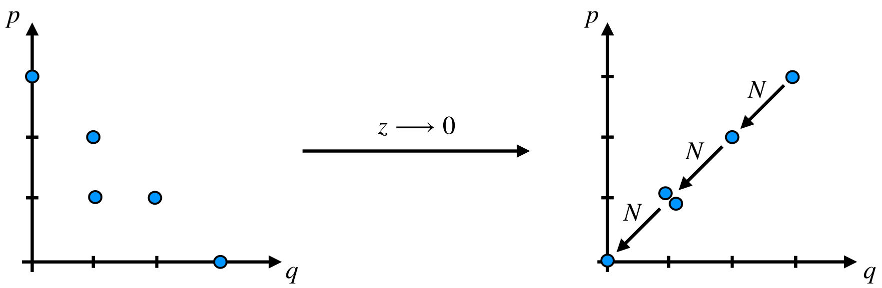

Figure 4.2.: LMHS associated to $(\hat{\mathcal{H}}^3)^\vee$ at a point of maximal degeneration. The dual picture can be viewed as arising from mirroring Figure 4.1 at the "center of mass" of the pure HS $\mathcal{H}^3$.

In the basis $\{e_j, f_k^\vee\}$ the Gauß-Manin connection in the canonical coordinate of the degeneration limit then takes the form [Esp]

$$\nabla_t = d + \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -\mathfrak{C} & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -\mathfrak{D}_1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\mathfrak{D}_d & 0 & 0 & \cdots & 0 \end{pmatrix} \otimes \frac{dq}{2\pi i\, q}, \tag{4.34}$$

in which the nontrivial entries correspond to the Yukawa coupling and infinitesimal invariants associated to all extending generators as in (4.29). A similar approach is also employed in the study of $\mathcal{N}=1$ special geometry of [LMW1, LMW2].

On the r-fold cover we pass to the local coordinate $\hat{q}=q^{1/r}$ and the structure (4.24) of the extended monodromy logarithm $\hat{N}$ implies that the superpotential is generally of the form

$$\mathcal{W}_k=\frac{1}{(2\pi i r)^2}\frac{\lambda_k}{2}\log(q)^2+\frac{1}{2\pi i r}s_k\log(q)+w_k(q), \tag{4.35}$$

with a single valued function

$$w_k(q)=\tilde{c}_k+\frac{1}{(2\pi i)^2}\sum_{d=1}^{\infty}\widetilde{n}_d\, q^{d/r} \tag{4.36}$$

extending holomorphically over the puncture. Here, the constant $\tilde{c}_k$ and the coefficients $\widetilde{n}_d$ of the q-series are constrained by the arithmetic of the situation. In the standard case, considered above, the family $\mathcal{Y}$ has a model over $\mathbb{Q}$, and the coefficients $\widetilde{N}_d$ in (3.54) are rational numbers. Even so, the cycle $\mathcal{C}$ will generically only be defined over an algebraic number field $K/\mathbb{Q}$, $\tilde{n}_d\in K$ (or possibly a subextension thereof), and the constant $\widetilde{c}_k$ is related to special values of L-functions associated to K. We refer to [GGK2, Wal4, SVW2] for general expositions of this phenomenon, and to [Wal4, LW, JW, JMW] for illustration in various examples. Concretely, and analogous to the situation of Remark 2.1, the constant term of the expansion is related to the limit of the Abel-Jacobi map associated to $\mathcal{C}$ in the canonical coordinate. As shown in [GGK2], this limit takes values in the *Néron-Model*

$$AJ(C_{\hat{q}})\xrightarrow{\hat{q}\to 0}AJ(C_0)\in G\ltimes {}^{\mathbb{C}}\!/_{\mathbb{Z}(2)}, \tag{4.37}$$

with a finite group G generated by cycles limiting to distinct components in the singular fiber Y_0 and $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-2),\mathbb{Z}(0))\cong\mathbb{C}/\mathbb{Z}(2)$ arising as the image of a regulator map acting on the degenerated cycle C_0. To fix ideas, in the simplest non-trivial case that $K/\mathbb{Q}$ is a quadratic number field, the Abel-Jacobi limit is expected to be of the general form

$$\lim_{q\to 0}Q(h_k,e_3):=Q(\widetilde{h}_k(0),e_3)=\frac{c_k\sqrt{\Delta}\,L(2,\chi)}{(2\pi i)^2},\qquad c_k\in\mathbb{Q}, \tag{4.38}$$

in which Δ denotes the discriminant of the field extension and $L(2,\chi)$ is the value of the Dirichlet L-function

$$L(2,\chi)=\sum_{k=1}^{\infty}\frac{\chi(k)}{k^2} \tag{4.39}$$

associated to the non-trivial quadratic character $\chi(k) = \left(\frac{\Delta}{k}\right)$. In the particular situation of a cycle defined over $\mathbb{Q}$, the Abel-Jacobi limit will therefore always be a rational multiple of $\zeta(2)/(2\pi i)^2$, similar to Remark 3.9, which in this case is itself rational. When the extension is defined by the superpotential (4.35), the finite group G contains an r-torsion subgroup coming from the covering, and the components of the Abel-Jacobi limit

$$Q(\widetilde{h}_k(0), e_2) = \frac{s_k}{r}, \qquad Q(\widetilde{h}_k(0), e_3) = \widetilde{c}_k, \tag{4.40}$$

can be expressed in terms of the untwisted local system which determines the LMHS.

5. A-Brane Superpotentials

After having established the role of the B-brane superpotential and its relation to Hodge theoretic normal functions in Chapter 4, we now turn to its A-model counterpart. Its embedding into the context of the A-model VMHS will be subsequently discussed in Chapter 6. The central quantum ingredient to the A-brane superpotential is a count of disk invariants whose appearance in the context of Lagrangian Floer theory will be explained in Section 5.1. The nature of the resulting enumerative invariants as open Gromov-Witten invariants and how they arise in specific geometric configurations is described in Section 5.2. The specification of an A-brane vaccum requires the choice of certain bundle data on top of a Lagrangian submanifold. The classical contributions to the superpotential that arise from these choices are outlined in Section 5.3.

5.1. Lagrangian Floer Theory and the Fukaya Category

In a neighborhood of a large volume limit point, the D-brane geometry of the A-model is encoded in the Fukaya category of X. As discussed in Chapter 2, the relevant objects in the present context are BPS branes given by compact, special Lagrangian submanifolds $i : L \hookrightarrow X$ with respect to the Kähler class and holomorphic 3-form on X. Morphisms between these objects are related to a special version of Lagrangian intersection Floer cohomology and the A_∞-algebra associated to Lagrangian submanifolds [FOOO1]. This section provides an overview of the various concepts that arise in this context, without going into too much detail when it comes to explicit constructions. The main aim is to motivate the notion of superpotential used by Fukaya-Oh-Ohta-Ono and its relevance for questions in symplectic geometry, before we discuss its relation to extended mirror symmetry and the physical superpotential. Fortunately, the focus on physically relevant configurations implies that various objects, like the moduli space of disks to be discussed below, simplify substantially which allows to keep the treatment close to classical objects in differential geometry.

A morphism between two such objects L_0, L_1 is defined by the corresponding Floer cochain complex spanned by their intersection points

$$\mathrm{Hom}(L_0, L_1) = CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \mathbb{C}\langle p \rangle. \tag{5.1}$$

We will always assume that any pair of Lagrangians is transversally intersecting such that (5.1) is well defined. A more general configuration of Lagrangians can, under our assumptions, always be made transveral by a systematic pertubation via hamiltonian diffeomorphisms, as described in [FOOO1]. Compositions between morphism groups fit into a hierarchy of structure maps

$$\mathfrak{m}_k : \mathrm{Hom}(L_0, L_1) \otimes \cdots \otimes \mathrm{Hom}(L_{k-1}, L_k) \longrightarrow \mathrm{Hom}(L_k, L_0) \tag{5.2}$$

for $k = 0, 1, 2, \ldots$, that are constructed from the count of holomorphic disks with boundary on $L = \bigcup_{i=0}^{k} L_i$, as depicted in Figure 5.1. Similarly to the sphere count[1] described in Section 3.3, the disk invariants are defined via the compactification $\widehat{\mathcal{M}}_{k+1}(X, L, \beta)$ of the moduli space of holomorphic genus zero open stable maps $u : (D, \partial D) \to (X, L)$ with degree $\beta \in H_2(X, L; \mathbb{Z})$ and one boundary component with $k+1$ marked points $p_0, \ldots p_k$. For now, we only consider the genus zero case and suppress the corresponding index. One way to describe this moduli space is as the quotient of the moduli space of stable maps $C \to X$ with respect to an anti-holomorphic involution acting pointwise on each C [*op.cit.*]. This means that the relative object $\widehat{\mathcal{M}}_{k+1}(X, L, \beta)$ generally inherits the subtleties pertaining to the stacky nature of its absolute counterpart described in Section 3.3. We similarly focus on the simplest example, in which the moduli space can be thought of as a smooth orbifold with boundary and corners of dimension

$$\dim_{\mathbb{R}} \widehat{\mathcal{M}}_{k+1}(X, L, \beta) = \dim_{\mathbb{C}} X - 3 - \mu(\beta) + k + 1. \tag{5.3}$$

In a situation in which the Lagrangian arises as the fixed point set of an anti-holomorphic involution (see Assumptions 5.5 below) the *Maslov index* $\mu(\beta)$ can be defined as

$$\mu(\beta) = \int_{\widehat{\beta}} c_1(X), \tag{5.4}$$

where $\widehat{\beta} \in H_2(X; \mathbb{Z})$ denotes the homology class of the double cover of the disk. In particular, the moduli problem again significantly simplifies in the situation of such a Lagrangian in Calabi-Yau threefolds: As $\dim_{\mathbb{C}} X = 3$ and $\mu(\beta) = 0$ for all $\beta \in H_2(X, L; \mathbb{Z})$, it suffices to specify $k+1$ boundary constraints as incidence relations in order to make the disk count well defined. A precise treatment of the moduli space and its relation to the product in Floer theory involves an analysis that can be thought of as a generalization of Morse theory, see for example [Aur] for an introductory discussion.

We denote by $\mathcal{M}_{k+1}(X, L, \beta, p_0, \ldots p_k)$ the moduli space in which $k+1$ boundary constraints are specified. When the relevant analysis is taken care of, the moduli

[1] In order to avoid the introduction of too much new notation, we recycle some of the symbols used in Section 3.3. This should not lead to confusion, as the current section exclusively deals with the open string case.

space is seen to be zero-dimensional [*op.cit.*] and the product operation is defined by the formula

$$\mathfrak{m}_k(p_1,\ldots,p_k) = \sum_{\beta\in H_2(X,L;\mathbb{Z})} \sum_{p_0\in L_k\cap L_0} \#\mathcal{M}_{k+1}(X,L,\beta,p_0,\ldots p_k)\, q^\beta\, p_0. \tag{5.5}$$

In analogy to (3.88), the number of holomorphic disks $\#\mathcal{M}_{k+1}(X,L,\beta,p_0,\ldots p_k)$ is weighted by the energy of the relevant homology class with

$$q^\beta = \exp(2\pi i \int_\beta \omega), \tag{5.6}$$

such that (5.5) has to be generally viewed as a formal power series in q. In [FOOO1], convergence issues are taken care of by considering coefficients in a certain Novikov field, but based on physical principles we will assume that all such power series are convergent in a small neighborhood around the large volume limit $q = 0$ to which we restrict our attention. The level of understanding of these convergence issues is similar to Remark 3.14. An example of the product operation in the Fukaya category is shown in Figure 5.1.

The composition of two morphisms defined by the product operation $\mathfrak{m}_2$ fails to be associative in a way that is controlled by all higher products $\mathfrak{m}_k$ for $k > 0$.

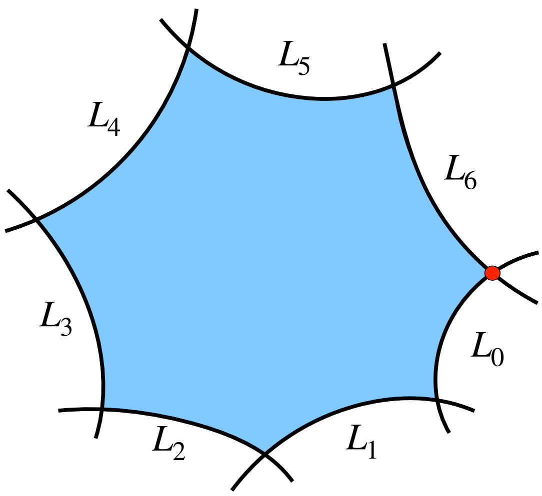

Figure 5.1.: A product operation in the Fukaya category. A circular configuration of transversally intersecting Lagrangians is shown. The product (5.5) defined by $\mathfrak{m}_k$ is determined by the number of disks that bound the configuration. It is constructed in a way that the k intersection points in $L_i \cap L_{i+1}$, $i = 0, ..., k-1$ enter the product as inputs, while the distinguished point in $L_k \cap L_0$ is considered as output (colored in red).

Namely, the structure maps satisfy the A_∞-relations

$$\sum_{k_1+k_2=k+1} (-1)^* \, \mathfrak{m}_{k_1}(p_1,\dots,p_i,\mathfrak{m}_{k_2}(p_{i+1},\dots,p_{i+k_2}),p_{i+k_2+1},\dots p_k) = 0, \tag{5.7}$$

making $\mathrm{Fuk}(X)$ into an A_∞-category[2]. Algebraically, the relation (5.7) reflects the stratification behavior of the boundary $\partial\mathcal{M}_{k+1}(X,L,\beta)$, where each stratum arises from a disk degeneration in which two punctures collide in the limit. The sign depends on the degrees of all punctures and can be chosen in such a way that all such boundary contributions add up to zero.

As motivated in Chapter 2 the geometric object that corresponds to an A-brane is not only specified by the Lagrangian condition, but also requires a choice of flat U(1)-bundle $\mathcal{E} \to L$, i.e. we have to consider pairs $\mathcal{L} = (L,\mathcal{E})$. Following [Fuk1], the correct modification of the Floer cochain groups that take this additional data into account is given by

$$\mathrm{Hom}(\mathcal{L}_0,\mathcal{L}_1) = CF(\mathcal{L}_0,\mathcal{L}_1) = \bigoplus_{p\in L_0\cap L_1} \mathrm{Hom}(\mathcal{E}_0\big|_p,\mathcal{E}_1\big|_p). \tag{5.8}$$

Given a similar configuration of Lagrangians as before, each decorated with a rank 1 local system, let $\gamma_i \in \mathrm{Hom}(\mathcal{E}_i|_{p_i},\mathcal{E}_i|_{p_i+1})$ be the isomorphism defined by parallel transport of the local system along L_i with $i = 0,\dots k$. For each intersection point $p_i \in L_i\cap L_{i+1}$, denote by $f_i \in \mathrm{Hom}(\mathcal{E}_i|_{p_i},\mathcal{E}_{i+1}|_{p_i})$ an element of the corresponding Floer cochain complex for $i = 1,\dots,k$. Then the composition

$$\mathrm{hol}_{f_1,\dots,f_k} = \gamma_k \circ f_k \circ \cdots \circ \gamma_1 \circ f_1 \circ \gamma_0 \tag{5.9}$$

can be viewed as the holonomy of the local system when going around the circular configuration of Lagrangians [Aur]. The product operation has to be adjusted by this holonomy factor and is defined by

$$\mathfrak{m}_k(f_1,\dots,f_k) = \sum_{\beta\in H_2(X,L;\mathbb{Z})} \sum_{p_0\in L_k\cap L_0} \#\mathcal{M}_{k+1}(X,L,\beta,p_0,\dots p_k)\, q^\beta \,\mathrm{hol}_{f_1,\dots,f_k}\, p_0. \tag{5.10}$$

The resulting operators define the A_∞ structure maps of this decorated version of the Fukaya category.

The identity morphism for an object L in $\mathrm{Fuk}(X)$ corresponds to the Floer cochain complex $CF(L,L)$ associated to a single Lagrangian. It is defined by using a Hamiltonian diffeomorphism $\varphi : X \to X$ such that $L\cap\varphi(L)$ has finitely many transversal intersection points and $CF(L,L) := CF(L,\varphi(L))$ is well defined. In this particular case, there is another description of the relevant

[2] The resulting object is sometimes also called a *curved* A_∞-category, as at the current stage we do allow for the tadpole anomalies discussed below.

cochain complex that is much closer to ordinary de Rham cohomology than to Morse Cohomology. Namely, an intersection point of given degree can be associated to a smooth differential form, Poincaré dual to a cycle which it intersects [ST1]. Here, we assume that the marked points $u(p_i)$ lie on cycles in L, Poincaré dual to smooth forms $\alpha_i \in \Omega^*(L)$, and we have evaluation maps $\mathrm{ev}_i : \widehat{\mathcal{M}}_{k+1}(X, L, \beta) \to L$, sending the stable map u to $u(p_i)$. In this approach, a proper $\mathbb{Z}$-grading of the complex can only be achieved in special situations, including special Lagrangians in Calabi-Yau threefolds[3], see for example [Aur]. The Floer cochain complex of L can here be thought of as a deformation of the de Rham complex $\Omega^\bullet(L)$ in which the differential and the cup product

$$\mathfrak{m}_1 = d + \mathcal{O}(q)\,, \qquad \mathfrak{m}_2 = \cup + \mathcal{O}(q) \tag{5.11}$$

are deformed by the holomorphic disks in $\mathcal{M}_{k+1}(X, L, \beta)$. For each homology class β we define the operators

$$\mathfrak{m}_{k,\beta} : \Omega^\bullet(L)^{\otimes k} \to \Omega^\bullet(L)[2-k], \quad \mathfrak{m}_{k,\beta}(\alpha_1, \ldots, \alpha_k) := (\mathrm{ev}_0)_* \left(\bigcup_{i=1}^{k} (\mathrm{ev}_i)^* \alpha_i \right), \tag{5.12}$$

with the exceptional case $k = 1$, $\beta = 0$ being set to $\mathfrak{m}_{1,0} = d$. In the geometrically well-behaved examples [ST1], the push-forward map $(\mathrm{ev}_0)_*$ can be understood as integration of a differential form along the fibers of $\widehat{\mathcal{M}}_{k+1}(X, L, \beta) \to L$. The A_∞-structure maps are then defined by

$$\mathfrak{m}_k := \sum_{\beta \in H_2(X,L;\mathbb{Z})} \mathfrak{m}_{k,\beta}\, q^\beta, \tag{5.13}$$

where, as always, we assume all power series to be convergent in a small neighborhood of the large volume limit point. In the case in which the moduli spaces can be thought of as smooth orbifolds with corners, the degree $2-k$ of the structure maps is an immediate consequence of the relative dimension that enters into the push-forward map in (5.12), namely

$$\dim_{\mathbb{R}} \mathcal{M}_{k+1}(X, L, \beta) - \dim_{\mathbb{R}} L = k - 2, \tag{5.14}$$

using (5.3). The condition for the moduli space to be orientable, such that the integration along fibers is well defined, is related to L being relatively spin, which is always ensured in the given situation [FOOO1].

The condition $\mathfrak{m}_1^2 = 0$ for $\mathfrak{m}_1$ to define a differential is generally obstructed when there is a non-zero "tadpole anomaly" $\mathfrak{m}_0 \neq 0 \in \Omega^2(L)$, coming from the disk amplitude with one boundary marked point. This can be seen from the special manifestation of the A_∞ relations in degree 2,

$$\mathfrak{m}_1^2(\alpha) = \mathfrak{m}_2(\mathfrak{m}_0, \alpha) - \mathfrak{m}_2(\alpha, \mathfrak{m}_0), \qquad \alpha \in \Omega^\bullet(L). \tag{5.15}$$

[3] The reason is that in this case the Maslov index vanishes for all relative homology classes.

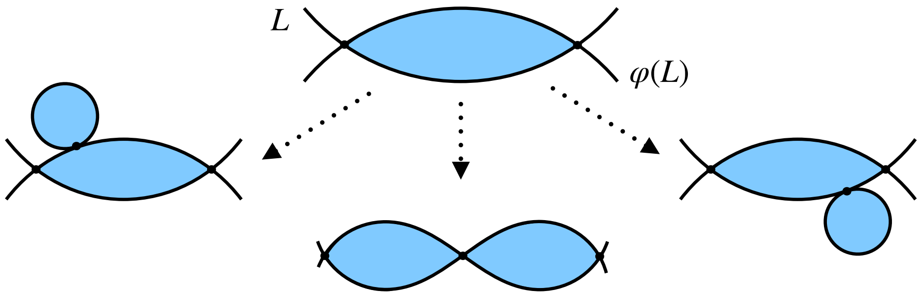

Figure 5.2.: Geometric origin of the tadpole anomaly. Holomorphic disks that enter into the definition of the Floer differential can degenerate in various ways. The broken disk in the middle depicts $\mathfrak{m}_1^2$, whose vanishing is obstructed due to the A_∞-relations. As described above, the obstruction is related to the presence of $\mathfrak{m}_0$, counting disks with exactly one puncture on the boundary. Physically, $\mathfrak{m}_0$ can be regarded as a tadpole in which an open string state is created from the brane vacuum.

Again, this example of the A_∞ relations can be interpreted from a geometric point of view as a cancellation of certain boundary terms of the moduli space. In this case, the relevant boundary is $\partial\widehat{\mathcal{M}}_2(X, L, \beta)$ and the corresponding disk degenerations are depicted in Figure 5.2.

In the specific situation of Calabi-Yau threefolds, the correction for this anomaly amounts to deforming the A_∞ structure according to

$$\mathfrak{m}_k^b(\alpha_1, \dots, \alpha_k) := \sum_{l_0,\dots,l_k} \mathfrak{m}_{k+l_0+\dots+l_k}(\underbrace{b,\dots,b}_{l_0}, \alpha_1, \underbrace{b,\dots,b}_{l_1}, \dots, \underbrace{b,\dots,b}_{l_{k-1}}, \alpha_k, \underbrace{b,\dots,b}_{l_k}), \tag{5.16}$$

by a 1-form $b \in \Omega^1(L)$ that satisfies the A_∞-Maurer-Cartan equation

$$\mathfrak{m}_0^b = \mathfrak{m}_0 + \mathfrak{m}_1(b) + \mathfrak{m}_2(b, b) + \mathfrak{m}_3(b, b, b) + \dots = 0. \tag{5.17}$$

Given the existence of such a Maurer-Cartan element, or *bounding cochain* [FOOO1], it can be shown that the deformed differential then satisfies $(\mathfrak{m}_1^b)^2 = 0$ such that *Floer cohomology*

$$HF^n(L) := H^n(\Omega^\bullet(L), \mathfrak{m}_1^b) = \frac{\ker\left(\mathfrak{m}_1^b : \Omega^n(L) \to \Omega^{n+1}(L)\right)}{\operatorname{im}\left(\mathfrak{m}_1^b : \Omega^{n-1}(L) \to \Omega^n(L)\right)} \tag{5.18}$$

is defined. In this case, L is considered an unobstructed element of the Fukaya category $\mathrm{Fuk}(X)$. However, for the purpose of describing the structure of the superpotential, we are only interested in the space of bounding cochains that enter into the definition of Floer cohomology, rather than the invariant itself.

Namely, the space of bounding cochains can be identified with the critical locus of a functional that is also called *superpotential*[4] in [FOOO1, Fuk2]. This notion of superpotential has also been studied in the general context of A_∞-structures associated to D-brane categories in both the A- and B-model [Laz, Tom]. It is given by

$$\Psi(b) = \sum_{k=0}^{\infty} \frac{1}{k+1} Q_L\left(\mathfrak{m}_k(b,\dots,b),b\right) + \mathfrak{m}_{-1}, \quad \mathfrak{m}_{-1} = \sum_{\beta\in H_2(X,L;\mathbb{Z})} \mathfrak{m}_{-1,\beta}\, q^{\beta}, \tag{5.19}$$

where the constant term with

$$\mathfrak{m}_{-1,\beta} = \#\widehat{\mathcal{M}}_0(X,L,\beta), \qquad \dim \widehat{\mathcal{M}}_0(X,L,\beta) = 0, \tag{5.20}$$

corresponds to a count of disks without boundary constraints. This count is well defined for Calabi-Yau threefolds as the associated moduli space has dimension zero. The pairing $Q_L : CF^\bullet(L) \times CF^\bullet(L) \to \mathbb{C}$ is induced from the Poincaré duality pairing on ordinary cohomology. Given a basis $\{x_i\}$ of $H^1(L;\mathbb{C})$, the pairing satisfies a cyclical symmetry[5]

$$Q_L\left(\mathfrak{m}_k(x_0,\dots,x_{k-1}),x_k\right) = (-1)^k\, Q_L\left(\mathfrak{m}_k(x_k,x_0,\dots,x_{k-2}),x_{k-1}\right), \tag{5.21}$$

where we note that all relevant expressions are non-zero due to the degree of the structure maps (5.13). The relevance of the Fukaya-Oh-Ohta-Ono superpotential is twofold. For one, its critical locus indeed corresponds to solutions of the A_∞-Maurer-Cartan equation [FOOO1, Proposition 3.6.50].

Proposition 5.1. Let $\{x_i\}$ be a basis of $H^1(L;\mathbb{C})$ such that any bounding cochain is of the form $b = \sum_i b_i x_i$. Then the bounding cochain is in the critical locus, i.e. $\partial_i\Psi(b) = 0$ for all i, if and only if $\mathfrak{m}_0^b = 0$.

Proof. Computing the derivative of $\Psi(b)$ with respect to the ith coordinate yields

$$\partial_i\Psi(b) = \pm\sum_{k=0}^{\infty} Q_L\left(\mathfrak{m}_k(b,\dots,b),x_i\right), \tag{5.22}$$

after using the cyclical symmetry (5.21) to transport each x_i to the rightmost position. Setting $\partial_i\Psi(b) = 0$ for all i is then equivalent to

$$\sum_{k=0}^{\infty} Q_L\left(\mathfrak{m}_k(b,\dots,b),x_i\right) = Q_L(\mathfrak{m}_0^b,x_i) = 0 \tag{5.23}$$

[4]Whenever confusion is likely we will refer to this notion of superpotential as the Fukaya-Oh-Ohta-Ono superpotential. It is closely related to, but not identical, to its physical namesake.

[5]Here we ignore abovementioned transversality issues, see [FOOO1, Conjecture 3.6.48].

for all i, such that $\mathfrak{m}_0^b = 0$ follows from the non-degeneracy of the Poincaré pairing. $\square$

The second important property of Ψ is related to its enumerative content, as the main result of [Fuk2] is that the inclusion of $\mathfrak{m}_{-1}$ makes Ψ a numerical invariant on each critical point. The way in which enumerative information is packaged in the Fukaya-Oh-Ohta-Ono superpotential is described in [Theorem 7.1][*op.cit.*], where it is shown that there is a resummation in which Ψ takes the form

$$\Psi(b) = \sum_{\beta \in H_2(X,L;\mathbb{Z})} \mathfrak{m}_{-1,\beta}\, q^\beta\, y^{\partial\beta}, \quad y^{\partial\beta} = \exp\left(2\pi i \int_{\partial\beta} b\right). \tag{5.24}$$

Here, the enumerative invariants $\mathfrak{m}_{-1,\beta}$ are weighted not only by the energy of the holomorphic disk, but also by an appropriate factor that depends on the bounding cochain. As noted in [FOOO1], the property (5.22) persists also when terms are added to $\Psi(b)$ that do not depend on the open string deformation parameters x_i. Physically, the Fukaya-Oh-Ohta-Ono superpotential should therefore be viewed as the quantum part of a function that also contains certain classical terms that play a role for the corresponding brane vaccum in string theory. One contribution of this thesis is a specification and A-model interpretation of these classical terms, based on the physical properties of A-brane superpotentials and a comparison with the B-model geometry on the mirror side. The enumerative invariants contained in $\Psi(b)$ can in special situations be extracted by an extended version of the mirror principle, provided certain contributions by holomorphic spheres are excluded. These problematic contributions are attributed to a certain wall-crossing phenomenon in [Fuk2] and render the disk count ill-defined in general (cf. Remark 5.2). As hypothesized in [*op.cit.*], the calculation of disk invariants presented in [Wal1] then relies on an investigation of brane configurations in which various choices of decorations by flat bundles leads to a cancellation of the anomalous terms. The way in which this has to be interpreted from the point of view of Hodge theory is discussed in Chapter 6 and in particular Section 6.5.

In view of the possibility to decorate objects in $\mathrm{Fuk}(X)$ by flat bundles leading to the structure maps (5.10) weighted by holonomy factors, the Fukaya-Oh-Ohta-Ono superpotential has a close analog which is equally well known in physics [Vaf, KKLM]. This involves the weighting of the disk count not only by energy, but also by an correspondingly exponentiated factor of holonomy, leading to a complexification of the full moduli space of the brane by Wilson lines when open string deformation parameters are present. In most of what follows we will focus on Lagrangians with very special properties (see Assumptions 5.5), whose superpotentials considerably simplify. Namely, we will be able to ignore the contributions by bounding cochains and the complexification will at most contribute a factor by a root of unity. The details of this approach are discussed in the following sections.

5.2. A-Brane Vacua and Open Gromov-Witten Invariants

In [ST1, ST2, ST3], Solomon-Tukachinsky further generalized the numerical invariant defined by the Fukaya-Oh-Ohta-Ono superpotential $\Psi(b)$ by using the bulk-deformed A_∞-algebra associated to Lagrangian submanifolds [Fuk2, FOOO1]. This involves a moduli space $\widehat{\mathcal{M}}_{k+1,l}(X,L,\beta)$ which takes not only boundary constraints into consideration, but also allows for incidence relations in the bulk of the disks, similar to the case of ordinary stable maps. The dimension of this moduli space is given by

$$\dim_{\mathbb{R}} \widehat{\mathcal{M}}_{k+1,l}(X,L,\beta) = \dim_{\mathbb{C}} X - 3 - \mu(\beta) + k + 1 + 2l \tag{5.25}$$

and it admits canonical evaluation maps

$$\begin{tikzcd}[row sep=small] & L \\ \widehat{\mathcal{M}}_{k+1,l}(X,L,\beta) \arrow[ur, "\mathrm{ev}_i^\partial"] \arrow[dr, "\mathrm{ev}_j"'] & \\ & X, \end{tikzcd} \tag{5.26}$$

that can be used to define a generalization of the A_∞-structure maps $\mathfrak{m}_k$ as follows. The corresponding operators

$$\mathfrak{q}_{k,l,\beta} : \Omega^\bullet(L)^{\otimes k} \otimes \Omega^\bullet(X)^{\otimes l} \longrightarrow \Omega^\bullet(L)[k+2l-2], \tag{5.27}$$

for each relative homology class $\beta \in H_2(X,L;\mathbb{Z})$, are given by the straightforward enhancement

$$\mathfrak{q}_{k,l,\beta}(\alpha_1,\cdots\alpha_k,\gamma_1,\cdots,\gamma_l) := \left(\mathrm{ev}_0^\partial\right)_* \left(\bigcup_{i=1}^k \left(\mathrm{ev}_i^\partial\right)^* \alpha_i \cup \bigcup_{j=1}^l \left(\mathrm{ev}_j\right)^* \gamma_j\right) \tag{5.28}$$

of (5.12) and we similarly set

$$\mathfrak{q}_{k,l} := \sum_{\beta\in H_2(X,L;\mathbb{Z})} \mathfrak{q}_{k,l,\beta}\, q^\beta. \tag{5.29}$$

Doing so allows to make contact with the closed string sector and replacing $\mathfrak{m}$ with $\mathfrak{q}$ in (5.19) leads to enumerative invariants similar to (5.24) that incorporate interactions between the two sectors. A careful analysis of the properties of the resulting superpotential, as performed in [ST2], reveals axioms for these *open Gromov-Witten invariants* with both boundary and bulk constraints. Here, bulk constraints are specified by intersections of the disks' interior with cycles in X, while boundary constraints are always point-like and closely related to the choice of bounding cochain.

Remark 5.2. When a holomorphic disk without boundary constraints has spherical degree $\beta = i_*\widetilde{\beta} \in \mathrm{im}\,(H_2(X) \to H_2(X,L))$, it can degenerate in a way in which the boundary collapses to a point in L. These contributions spoil invariance of Ψ but can be corrected when L is homologically trivial by further including sphere invariants with one puncture lying on a 4-chain Γ_4 with $\partial\Gamma_4 = L$ [PSW]. A formalization of this procedure results in *enhanced* open Gromov-Witten invariants [ST3, Proposition 4.19] which in general depend on the choice of Γ_4. In addition to the bounding cochain, it becomes part of the data making the disk count well defined.

In order to systematically address the anomaly described in Remark 5.2, we make use of a general property of 3-cycles in compact symplectic 6-manifolds. It can be interpreted as a guarantee for the existence of a canonical reference cycle. The following is based on [Ban, Proposition 3.2] and [Fre, Theorem 1.4].

Proposition 5.3. Let (X,ω) be a simply connected, compact and symplectic threefold. Then any three-dimensional homology class $\alpha \in H_3(X,\mathbb{Z})$ can be represented by an immersed Lagrangian sphere $S^3 \subset X$.

Proof. The argument is following a version of the h-principle known as the Gromov-Lees theorem [Gro, Lee]. Given a smooth immersion $f : L \to X$, such that $Df : TL \longrightarrow TX$ is a fiberwise Lagrangian monomorphism, there exists a Lagrangian immersion in the same homotopy class if the cohomological obstruction $[f^*\omega] \in H^2(L)$ vanishes. As X is simply connected, the Hurewicz map

$$\pi_3(X) \longrightarrow H_3(X;\mathbb{Z}), \qquad [f : S^3 \to X] \longmapsto f_*[S^3] \tag{5.30}$$

is onto such that there always exists a continuous map $f : S^3 \to X$ with $f_*[S^3] = \alpha$. We can smoothly approximate f and make it into an immersion with transversal self-intersections by a general position argument, denoting the resulting map by the same symbol. In order to study the properties of the differential Df we consider the normal bundle of $f(S^3)$. We choose local trivializations over neighborhoods around two poles that intersect along an equator homeomorphic to S^2, such that the transition functions correspond to continuous maps $S^2 \to \mathrm{GL}(3,\mathbb{R})$. They are therefore classified by the group $\pi_2(\mathrm{GL}(3,\mathbb{R}))$ which vanishes by Bott-periodicity, meaning that the normal bundle is trivial. As the normal bundle can be identified with the cotangent bundle T^*S^3 via the symplectic form, we have a global splitting $f^*(TX) \cong TS^3 \oplus T^*S^3$ which implies that $Df_p : T_pS^3 \to T_{f(p)}X$ defines a Lagrangian subspace for each $p \in S^3$. The statement then follows from the vanishing of the cohomological obstruction $H^2(S^3) = 0$. □

Even with their existence assured, the explicit construction of these Lagrangian spheres can be very involved in practice, see for example [She], where they play a crucial role in the proof of homological mirror symmetry for the quintic. This is part of a notorious problem of finding examples of Lagrangian submanifolds in compact threefolds, in which the (extended) mirror principle could become an important tool: In a situation in which a given A-model geometry is hard to understand, it can be instructive to study its enumerative invariants that can be extracted from the mirror B-model in a relatively simple way.

There is one systematic way to construct special Lagrangians in compact Calabi-Yau threefolds that is completely understood. While occasionally commenting on more general situations, this construction will be our main focus and will contain all running examples.

Lemma 5.4. Let X be a projective Calabi-Yau manifold of complex dimension n together with an antiholomorphic involution $\iota : X \to X$ satisfying $\iota^2 = id$ and $\iota_* \mathcal{J} = -\mathcal{J}$, where $\mathcal{J} : T_X \to T_X$ is the complex structure. Then the fixed point set

$$\mathrm{Fix}(\iota) = \{x \in X \mid \iota(x) = x\}, \tag{5.31}$$

if non-empty, defines a real n-dimensional special Lagrangian submanifold of X.

Proof. Given a holomorphic chart (U, ϕ) of X around a fixed point of ι, the map $\phi \circ \iota \circ \phi^{-1}$ defines an antiholomorphic involution on $\mathbb{C}^n$. Its fixed point set is diffeomorphic to $\mathbb{R}^n$ and it follows that $\mathrm{Fix}(\iota)$ is an n-dimensional real submanifold of X, which we assume to be non-empty. As X is Kähler, the symplectic form is related to the Kähler metric $\omega(\cdot,\cdot) = g(\mathcal{J}\cdot,\cdot)$, which can be expressed in local coordinates via

$$\omega = i \sum_{i,j} g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j. \tag{5.32}$$

We can always find a local coordinate system in which the involution acts by $\iota(z) = \bar{z}$ and it follows that

$$\iota^*\omega = i \sum_{i,j} g_{j\bar{i}}\, d\bar{z}^i \wedge dz^j = -i \sum_{i,j} g_{j\bar{i}}\, dz^j \wedge d\bar{z}^i = -\omega, \tag{5.33}$$

where we are using the fact that the Kähler metric is Hermitian. In a similar fashion the involution acts on the holomorphic 3-form. We can always find a local splitting $\Omega = \mathrm{Re}(\Omega) + i\,\mathrm{Im}(\Omega)$ into real and imaginary parts for which

$$\iota^*\Omega = \mathrm{Re}(\Omega) - i\,\mathrm{Im}(\Omega) = \overline{\Omega}. \tag{5.34}$$

Now let $i : \mathrm{Fix}(\iota) \hookrightarrow X$ denote the inclusion. When restricting on the fixed point set, it follows from (5.33) and (5.34) that

$$i^*\omega = i^*\iota^*\omega = -i^*\omega, \qquad i^*\mathrm{Im}(\Omega) = i^*\iota^*\mathrm{Im}(\Omega) = -i^*\mathrm{Im}(\Omega), \tag{5.35}$$

such that $i^*\omega = 0$ and $i^*\mathrm{Im}(\Omega) = 0$, meaning that $\mathrm{Fix}(\iota)$ is special Lagrangian. $\square$

Assumptions 5.5. Drawing from these known examples of Lagrangian submanifolds in compact Calabi-Yau threefolds described in Lemma 5.4, we assume that $L = \{x \in X \,|\, \iota(x) = x\}$ is the fixed point set of an anti-holomorphic involution $\iota : X \to X$. As described around (5.4), this implies that every class $\beta \in H_2(X, L; \mathbb{Z})$ has vanishing Maslov index $\mu(\beta) = 0$. We further assume that L is a rational homology sphere, i.e.

$$H_k(L; \mathbb{Q}) = H_k(S^3, \mathbb{Q}) \tag{5.36}$$

for all k. To make contact with the examples of [KW], we will generally allow r-torsion in $H_1(L; \mathbb{Z}) \cong H^2(L; \mathbb{Z})$. In these cases, $H^1(L; \mathbb{Z}) = 0$ follows from the universal coefficient theorem (see also [Liu])

$$0 \to \mathrm{Ext}^1\Big(H_0(L; \mathbb{Z}), \mathbb{Z}\Big) \cong 0 \longrightarrow H^1(L; \mathbb{Z}) \longrightarrow 0 \cong \mathrm{Hom}\Big(H_1(L; \mathbb{Z}), \mathbb{Z}\Big) \to 0. \tag{5.37}$$

In particular, L is classically rigid as a special Lagrangian because $H^1(L; \mathbb{R}) = 0$ due to McLean's theorem [McL], see also Appendix A. The latter condition physically corresponds to the absence of open string moduli of the A-brane. If L is homologically trivial, there exists a relative 4-chain $[\Gamma] \in H_4(X, L)$ with boundary $\partial[\Gamma] = [L]$. Whenever L is not homologically trivial, we use Proposition 5.3 and include a homologically equivalent (immersed) Lagrangian sphere $L_0 \cong S^3$ as a canonical fixed reference and consider $L \cup L_0$ in order to correct for the failure mode addressed in Remark 5.2. In this case there exists a relative 4-chain $[\Gamma] \in H_4(X, L \cup L_0)$ with boundary $\partial[\Gamma] = [L] - [L_0]$. Lastly, we will assume that L remains Lagrangian under small deformations of the Kähler class in Δ^*.

Example 5.6. An anti-holomorphic involution $\iota : \mathbb{P}^4 \to \mathbb{P}^4$ with $\iota(x_i) = \overline{x}_i$ restricted to the Fermat quintic (3.93) has a fixed point locus

$$L = \Big\{(y_1 : \cdots : y_5) \in \mathbb{RP}^4 \,\Big|\, y_1^5 + y_2^5 + y_3^5 + y_4^5 + y_5^5 = 0\Big\} \subset X, \tag{5.38}$$

which is special Lagrangian with respect to the Kähler class ω and holomorphic 3-form Ω on X. Solving (5.38) for any y_i over $\mathbb{R}$ topologically identifies this real quintic with a copy of real projective space $L \cong \mathbb{RP}^3$, whose fundamental class defines an element $[L] \in H_3(X, \mathbb{Z})$ in homology.

Under Assumptions 5.5, the constant term of the Fukaya-Oh-Ohta superpotential satisfies [FOOO2, Fuk2]

$$\mathfrak{m}_{-1,\beta} = \mathfrak{m}_{-1,\iota_*\beta}, \tag{5.39}$$

leading to the equality $\Psi(b) = \Psi(-b)$, which is best visible from the point of view of (5.24). As a consequence, the bounding cochain $b = 0$ is always a critical point of $\Psi(b)$, leading to disk invariants without boundary constraints. In the bulk deformed situation considered in [ST2], this invariant essentially corresponds to the value of

$$\mathfrak{q}_{-1,l}(\gamma_1,\ldots,\gamma_l) = \sum_{\beta\in H_2(X,L;\mathbb{Z})} \mathfrak{q}_{-1,l,\beta}(\gamma_1,\ldots,\gamma_l)\, q^{\beta}. \tag{5.40}$$

This is in line with the axioms for enhanced open Gromov-Witten invariants proposed in [ST3] (cf. Remark 5.2) implying that all disk invariants with boundary constraints vanish for $\beta \neq 0$ in case of Lagrangians with $\mu(\beta) = 0$ in Calabi-Yau threefolds. We therefore consider a version of the axioms for enhanced open Gromov-Witten invariants adapted to our particular situation in which only bulk constraints are present[6].

Axioms 5.7. Given cohomology classes $\gamma_i \in H^*(X;\mathbb{C})$ with $i = 1, ..., n$, the invariants $OGW_\beta(\gamma_1, ..., \gamma_n)$ of the pair (X, L) satisfy the axioms

- *(Degree)* $OGW_\beta(\gamma_1, ..., \gamma_n) = 0$ unless

$$\dim_{\mathbb{C}} X - 3 - \mu(\beta) + 2n = \sum_{i=1}^{n} \deg(\gamma_i). \tag{5.41}$$

- *(Equivariance)* For $\dagger = \deg(\gamma_i)\cdot\deg(\gamma_{i+1})$ it is

$$OGW_\beta(\gamma_1,\ldots,\gamma_i,\gamma_{i+1},\ldots,\gamma_n) = (-1)^{\dagger} OGW_\beta(\gamma_1,\ldots,\gamma_{i+1},\gamma_i,\ldots,\gamma_n). \tag{5.42}$$

- *(Unit)* An insertion of the unit $[X] \in H^0(X;\mathbb{C})$ yields[7]

$$OGW_\beta([X],\gamma_1,...,\gamma_{n-1}) = \begin{cases} \int_{\Gamma_4}\gamma_1 & [L]=0,\ \beta=0 \text{ and } n=2 \\ 0 & \text{otherwise.} \end{cases} \tag{5.43}$$

- *(Zero)* The restriction to $\beta = 0$ yields

$$OGW_0(\gamma_1,...,\gamma_n) = \begin{cases} \int_{\Gamma_4}\gamma_1\cup\gamma_2 & [L]=0 \text{ and } n=2 \\ 0 & \text{otherwise.} \end{cases} \tag{5.44}$$

- *(Divisor)* For $\deg(\gamma_n) = 2$ it is

$$OGW_\beta(\gamma_1,...,\gamma_n) = \left(\int_\beta \gamma_n\right)\cdot OGW_\beta(\gamma_1,...,\gamma_{n-1}). \tag{5.45}$$

[6] The author thanks Sara Tukachinsky for various explanations regarding open Gromov-Witten theory.

[7] Note that all of the relevant integrals are well defined because there is always a lift to relative cohomology under Assumptions 5.5, see also (5.48).

- *(Invariance)* The numbers $OGW_\beta(\gamma_1, \ldots, \gamma_n)$ are constant under deformations of the Kähler class ω for which L remains Lagrangian.

Remark 5.8. Similar to the case of ordinary Gromov-Witten invariants, and as discussed in Remark 3.13, there is no version of the effectivity axiom as we consider all disks to have non-negative energy $\int_\beta \omega \geq 0$. The open string analog of the splitting axiom is not visible in the situation of $b = 0$, as we do not allow for boundary constraints that would arise from certain types of disk degenerations. In a more general context, the construction of open Gromov-Witten invariants of [*op.cit.*] crucially allows for disks to separate such that they intersect up to a cycle Poincaré dual to the bounding cochain, which can be viewed as a substitution for the splitting axiom in the open string case. Lastly, we remark that we only consider disks of genus $g = 0$, such that we do not have a version of the reduction axiom. The reason is that higher genus open Gromov-Witten invariants are even harder to define from the point of view of Floer theory, cf. Chapter 7.

We again denote by OGW_β the invariants without constraints, i.e. $n = 0$. Based on the divisor axiom we consider the generating function of open Gromov-Witten invariants

$$\Psi = \frac{1}{2}\int_{\Gamma_4} \omega \cup \omega + \Psi_h = \frac{\lambda}{2} t^2 + \Psi_h, \tag{5.46}$$

where $\lambda = \int_{\Gamma_4} H \cup H$ corresponds to the energy zero contribution by the zero axiom. In analogy to (3.88) we define the quantum part by

$$\Psi_h = \frac{1}{(2\pi i)^2} \sum_{\beta \in H_2(X,L;\mathbb{Z})\setminus\{0\}} OGW_\beta \, q^\beta, \tag{5.47}$$

which we think of as a holomorphic function on Δ^*. The function Ψ can be interpreted as the A-brane superpotential, or BPS domain wall tension with quantum corrections by disk instantons, between two open string vacua determined by L and L_0. Note that by the exact sequence

$$\cdots \longrightarrow H^1(L;\mathbb{C}) \cong 0 \longrightarrow H^2(X,L;\mathbb{C}) \longrightarrow H^2(X;\mathbb{C}) \longrightarrow 0 \cong H^2(L;\mathbb{C}) \longrightarrow \cdots \tag{5.48}$$

and under Assumptions 5.5, there always exists a canonical lift of the Kähler class to relative cohomology such that the first term of (5.46) is well defined. The fact that Hamiltonian isotopies of L do not affect the A-brane superpotential corresponds to the invariance of the Abel-Jacobi map under rational equivalence in the B-model. Altering the choice of Γ in the definition of the enhanced open Gromov-Witten invariants will change the A-brane superpotential only by the closed string period $Q(g_2, e_3) = \Phi'$ (cf. (3.101)).

5.3. Classical Contributions

To characterize the Lagrangian submanifold as an A-brane it is necessary to also specify a flat U(1)-bundle $\mathcal{E}$ on L, leading to further classical contributions to the superpotential of the pair $\mathcal{L} = (L, \mathcal{E})$ that arises from different choices of flat bundle $\mathcal{E}$ and $\mathcal{E}_0$ on the same Lagrangian L, see Section 2.3. In addition, the A_∞-structure maps $\mathfrak{m}_k$ have to be weighted by the holonomy $\mathrm{hol}_{\partial\beta}(\mathcal{E})$ of the corresponding flat connection [Fuk1] (cf. (5.10)) which, under Assumptions 5.5, changes the holomorphic part of the superpotential by a rth root of unity as a constant prefactor. The flat bundles are topologically classified by their first Chern class in the r-torsion group $H^2(L;\mathbb{Z})$ and the analog of homological equivalence is the fact that they cannot be distinguished after their embedding into the simply connected ambient space X, i.e.

$$i_! : H^2(L;\mathbb{Z}) \longrightarrow H^5(X;\mathbb{Z}), \qquad i_!\Big(c_1(\mathcal{E}) - c_1(\mathcal{E}_0)\Big) = 0. \tag{5.49}$$

Writing $s[\gamma] = [c_1(\mathcal{E})] - [c_1(\mathcal{E}_0)] \in H^2(L;\mathbb{Z})$, the additional contribution to the domain wall tension comes from a relative homology class $[\Gamma_2] \in H_2(X, L;\mathbb{Z})$ with boundary Poincaré dual to $s[\gamma]$. The Puppe sequence

$$0 \longrightarrow \pi_1(X, L) \longrightarrow \pi_0(L) \xrightarrow{\sim} \pi_0(X) \longrightarrow 0 \tag{5.50}$$

implies $\pi_1(X, L) = 0$ such that $\pi_2(X, L) \cong H_2(X, L;\mathbb{Z})$ follows from the relative Hurewicz theorem. Therefore, the 2-chain Γ_2 admits a representation in relative homology by a disk D with $[\Gamma_2] = s[D]$. From the long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & H_2(X;\mathbb{Z}) \arrow[r] \arrow[d, "PD"] & H_2(X, L;\mathbb{Z}) \arrow[r] \arrow[d, "PD"] & H_1(L;\mathbb{Z}) \arrow[r] \arrow[d, "PD"] & 0 \\ \cdots \arrow[r] & H^4(X;\mathbb{Z}) \arrow[r] & H^4(X \setminus L;\mathbb{Z}) \arrow[r] & H^2(L;\mathbb{Z}) \arrow[r] & 0 \end{tikzcd} \tag{5.51}$$

we see that the class $r[D] \in H_2(X, L;\mathbb{Z})$ can be lifted to $[\ell] \in H_2(X;\mathbb{Z})$ such that the associated currents satisfy $[\delta_\ell] = r[\delta_D] \in H^4(X \setminus L;\mathbb{Z})$. The contribution to the domain wall tension is then given by

$$\int_{\Gamma_2} \omega = \frac{s}{r} \int_\ell \omega = \frac{s}{r} t, \qquad s \in \mathbb{Z}, \tag{5.52}$$

and well defined due to the exact sequence (5.48).

Under monodromy $t \mapsto t+1$ around the puncture, the domain wall tension is not invariant but changes up to integral periods that can be physically interpreted as a change in the integrally quantized flux of Ramond-Ramond fields which naturally couple to the A-brane in addition to the U(1)-gauge field [Wal1]. In addition, the gauge theory leads to a Chern-Simons term $cs(\mathcal{E})$ that arises as a constant three-loop contribution to the tension, which in full is up to closed string periods given by the function [TV, CKLT, PSW]

$$\mathcal{W}_A(q) = \frac{1}{2}\int_{\Gamma_4} \omega \cup \omega + \int_{\Gamma_2} \omega + cs(\mathcal{E}) + \Psi_h. \tag{5.53}$$

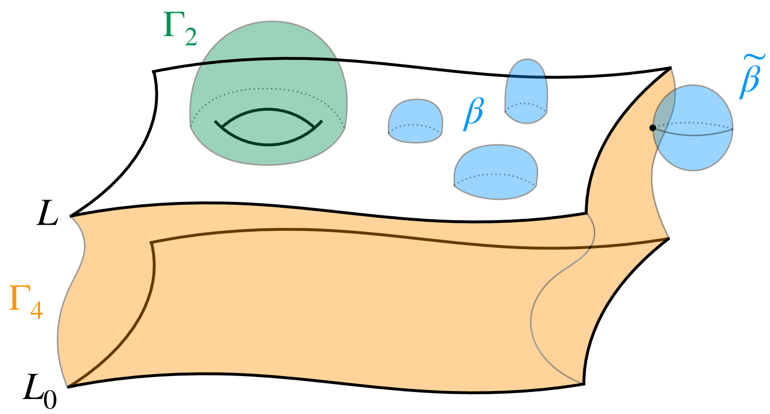

Figure 5.3.: Contributions to the A-brane superpotential. The domain wall tension between two A-brane vacua classically consists of the volume of a 4-chain Γ_4 interpolating between the Lagrangians and a 2-chain Γ_2 interpolating between choices of flat bundles. These are corrected by open and closed string instantons, together with the Chern-Simons invariant associated to the gauge bundles.

When contributions coming from the bundle data are taken into account, large volume monodromies are in general not integral. This anomaly can be compensated by introducing several A-branes whose superpotentials are exchanged when $t \mapsto t+1$, such that the full domain wall spectrum is invariant under integral monodromy. This is the A-model analog of an algebraic cycle branching when encircling the puncture in the B-model (4.25) and forces the superpotential to take the general form (4.35). The integrality constraint $\lambda/r \in \mathbb{Z}$ of (4.24) arises in A-model terms from the integral version

$$\longrightarrow H^1(L;\mathbb{Z}) \cong 0 \longrightarrow H^2(X,L;\mathbb{Z}) \longrightarrow H^2(X;\mathbb{Z}) \longrightarrow {}^{\mathbb{Z}}\!/\!{}_{r\mathbb{Z}} \cong H^2(L;\mathbb{Z}) \longrightarrow \tag{5.54}$$

of the short exact sequence (5.48). The condition $\lambda/2+s \in \mathbb{Z}$ can be physically viewed as a consequence of the mixing of different discrete open string moduli, as discussed in [KW].

In case of a trivial U(1)-bundle $\mathcal{E}$, the three-loop contribution to the superpotential given by the Chern-Simons invariant is of the form

$$cs(\mathcal{E}) = -\frac{1}{(2\pi i)^2}\int_L \mathrm{Tr}\,(A \wedge dA)\,, \tag{5.55}$$

where $A \in \Omega^1(L, \mathfrak{u}(1))$ is the globally well-defined connection 1-form associated to the bundle. In general, we consider the following construction of the Chern-Simons invariant [DW].

Definition 5.9. Whenever L can be thought of as the boundary of an oriented 4-manifold Γ_4 onto which the flat bundle admits an extension $\widetilde{\mathcal{E}}$ with connection 1-form $\widetilde{A}$ and curvature $F_{\widetilde{A}}$, we call

$$cs(\mathcal{E}) = -\frac{1}{(2\pi i)^2} \int_{\Gamma_4} \mathrm{Tr}\left(F_{\widetilde{A}} \wedge F_{\widetilde{A}}\right) = -\int_{\Gamma_4} ch_2(\widetilde{\mathcal{E}}) \tag{5.56}$$

the *Chern-Simons invariant of* $\mathcal{E}$, where $ch_2(\widetilde{\mathcal{E}}) = \frac{1}{2}c_1(\widetilde{\mathcal{E}})^2 - c_2(\widetilde{\mathcal{E}})$.

Note that the two formulas for $cs(\mathcal{E})$ agree in the globally trivial case by Stokes' theorem. The existence of the relevant 4-manifold is generally obstructed by the bordism theory of the ambient space X and the classifying space of the bundle[8]. We note that for our purposes in Chapter 6 it will suffice to have a smooth map $g : \Gamma_4 \to X$ in order to make sense of various integral expressions via pullback along g.

Proposition 5.10. Let $L, L_0 \subset X$ be homologous (immersed) three-dimensional submanifolds in a threefold X. Then an interpolating 4-chain can be represented by a smooth oriented bordism $\Gamma_4 \in \Omega_4^{\mathrm{SO}}(X)$. Furthermore, any U(1)-bundle on $L \cup L_0$ admits an extension onto Γ_4.

Proof. In order to assess whether there is a bordism $[\Gamma_4, g] \in \Omega^{\mathrm{SO}}(X)$ fitting into the diagram

$$\begin{tikzcd}
L \arrow[dr, "\text{incl.}"'] \arrow[drr, bend left] & & \\
& \Gamma_4 \arrow[r, dotted, "g"] & X, \\
L_0 \arrow[ur, "\text{incl.}"] \arrow[urr, bend right] & &
\end{tikzcd} \tag{5.57}$$

we first consider the Atiyah-Hirzebruch spectral sequence

$$H_p(X; \Omega_q^{\mathrm{SO}}) \Longrightarrow \Omega_{p+q}^{\mathrm{SO}}(X). \tag{5.58}$$

Because the bordism group $\Omega_k^{\mathrm{SO}} = \Omega_k^{\mathrm{SO}}(pt)$ is trivial in degrees $k = 1, 2, 3$, while $\Omega_0^{\mathrm{SO}} = \mathbb{Z}$, the spectral sequence shows that the bordism version of the Hurewicz map

$$\Omega_3^{\mathrm{SO}}(X) \xrightarrow{\sim} H_3(X; \mathbb{Z}), \qquad [f : M \to X] \longmapsto f_*[M], \tag{5.59}$$

is an isomorphism [DK1]. Therefore, homological equality $[L] = [L_0]$ implies the existence of an oriented bordism together with a continuous map $g : \Gamma_4 \to X$

[8]The author is grateful to Markus Banagl for a helpful conversation about these topics.

that can be smoothly approximated. The possibility to find an extension of the bundles on L and L_0 can be addressed similarly by considering the bordism theory of the classifying space $\mathrm{BU}(1) \cong \mathbb{P}^\infty$. The obstruction to the existence of a bordism which simultaneously interpolates between L and L_0, together with the corresponding bundle data, lives in the bordism group $\Omega_3^{\mathrm{SO}}(X \times \mathbb{P}^\infty; \mathbb{Z})$. The Hurewicz isomorphism again reduces the problem to homology, which can be computed with the Künneth formula

$$H_3(X \times \mathbb{P}^\infty; \mathbb{Z}) \cong \bigoplus_{i+j=3} H_i(X; \mathbb{Z}) \otimes H_j(\mathbb{P}^\infty; \mathbb{Z}) = H_3(X; \mathbb{Z}), \tag{5.60}$$

using $H_{\mathrm{odd}}(\mathbb{P}^\infty; \mathbb{Z}) = 0$ and $H_1(X, \mathbb{Z}) = 0$. □

Remark 5.11. The explicit computation of the Chern-Simons invariant reviewed in Appendix B is based on index theory, where it is necessary to specify a spin structure on Γ_4 in addition to an orientation. Obstructions to the existence of a spin structure can be analyzed similar to Proposition 5.10, where the bordism groups $\Omega_k^{\mathrm{Spin}}(X)$ are up to torsion identical to $\Omega_k^{\mathrm{SO}}(X)$, namely [Mil]

$$\Omega_0^{\mathrm{Spin}} = \mathbb{Z}, \quad \Omega_1^{\mathrm{Spin}} = \mathbb{Z}/2\mathbb{Z}, \quad \Omega_2^{\mathrm{Spin}} = \mathbb{Z}/2\mathbb{Z}, \quad \Omega_3^{\mathrm{Spin}} = 0. \tag{5.61}$$

Here, the additional constraint in dimension 1 and 2 corresponds to a choice of anti-periodic spin structure on the circle and products thereof. As both X and $\mathbb{P}^\infty$ are torsion-free, an analogous argument shows that we can always find a suitable bordism admitting a spin structure.

6. Extended Mirror Symmetry

In this chapter we are finally able to tackle the central task of providing a Hodge theoretic context in which extended mirror symmetry is on a more equal footing with its closed string counterpart described in Chapter 3. We begin with a mathematical definition of the notion of extended mirror pair in Section 6.1, which is based on a comparison of the relevant A- and B-brane superpotentials. A characterization of the extension class in the A-model and its relation to the open WDVV equations is discussed in Section 6.2, where we define the extended A-model VMHS. As for closed strings, this construction does not immediately produce the correct asymptotic behavior of the normal function, such that a modification by a relative version of the Gamma class is required. Its definition and role in an extended Gamma-integral local system is given in Section 6.3. The structure arising in the A-model can be interpreted as a certain extension of Frobenius modules, a point of view which is explained in Section 6.4. Concluding with the main example of a mirror pair for the quintic in Section 6.5, we formulate the statements of [Wal1] in this new language which culminates in an extended version of the standard Mirror Theorem.

6.1. Extended Mirror Pairs

Starting from a mirror pair with isomorphic variations $\mathcal{H}^{\mathrm{even}}(X) \cong \mathcal{H}^3(Y)$ of MHS, the extension in the B-model can be given an A-model interpretation by composing the B-brane superpotential $\mathcal{W}_B$ with the mirror map. As described in the previous section, the resulting function can be understood as A-brane superpotential $\mathcal{W}_A$ of an associated A-model geometry and defines an extension

$$0 \longrightarrow \mathcal{H}^{\mathrm{even}} \longrightarrow \hat{\mathcal{H}}^{\mathrm{even}} \longrightarrow \mathcal{I} \longrightarrow 0, \qquad \hat{\mathcal{H}}^{\mathrm{even}} \in \mathrm{Ext}^1_{\mathrm{VMHS}}\left(\mathcal{I}, \mathcal{H}^{\mathrm{even}}\right), \tag{6.1}$$

of the pure VHS $\mathcal{I}$ of weight 4 by the A-model VMHS $\mathcal{H}^{\mathrm{even}}$. This identification motivates the following definition in analogy to the absolute case of Section 3.6.

Definition 6.1. Let (X, ω) and $\mathcal{Y}$ be a mirror pair of Calabi-Yau threefold families, together with a collection of algebraic cycles $\mathcal{C} \subset \mathcal{Y}$ and A-branes $\mathcal{L}$ in X. We call $(X, \mathcal{L})$ and $(\mathcal{Y}, \mathcal{C})$ an *extended (Hodge theoretic) mirror pair*, if the associated extensions of variations of MHS $\hat{\mathcal{H}}^{\mathrm{even}}(X)$ and $\hat{\mathcal{H}}^3(\mathcal{Y})$ are isomorphic over small neighborhoods around MUM-type boundary points related by the mirror map.

From the discussion in Section 4.3, Definition 6.1 is equivalent to the identifications of the A- and B-brane superpotentials under the mirror map

$$\mathcal{W}_A(q) = \mathcal{W}_B(z(q)), \tag{6.2}$$

as the truncated normal functions $\mathcal{W}_{A/B} = Q(\widetilde{\nu}, e_3)$ with respect to the canonical lift (4.19) fully determine the extension in a situation of maximal degeneracy. Recall that we canonically normalize e_3 in the B-model, such that $\varpi_0(z) = 1$. Given an extended mirror pair with $\mathcal{I}$ of rank 1, the superpotential $\mathcal{W}_B$ can therefore be understood as computing the quantum corrected volume of (a D-brane wrapping) the relative 4-chain Γ_4 interpolating between the A-brane vacua (cf. (3.127))

$$\mathcal{W}_A = \frac{\lambda}{2r^2}t^2 + \frac{s}{r}t + \widetilde{c} + \frac{\xi}{(2\pi i)^2}\sum_{d=1}^{\infty}\widetilde{n}_d\, q^{d/r} = Q(h_k, e_3), \qquad \xi^r = 1, \tag{6.3}$$

with asymptotic behavior specified by the limiting period matrix

$$\hat{\Pi}_{q=0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{a}{24} & -\frac{\kappa}{2} & 1 & 0 & \frac{s}{r} \\ \frac{b\,\zeta(3)}{(2\pi i)^3} & \frac{a}{24} & 0 & 1 & \widetilde{c} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.4}$$

In particular, the perturbative three-loop correction to the A-brane superpotential, related to the Abel-Jacobi limit in the B-model, is expected to be given by the Chern-Simons invariant $cs(\mathcal{E})$ in the known examples. The identification of flat connections implies

$$\mathfrak{D}(q(z)) = \lambda + \xi\sum_{d=1}^{\infty}\frac{d^2\widetilde{n}_d}{r^2}\,q^{d/r} = \Psi''(q), \tag{6.5}$$

providing an A-model interpretation of the infinitesimal invariant (4.21) as the disk two-point correlation function of the mirror A-brane. By the Solomon-Tukachinsky Axioms 5.7, this corresponds to the generating function of open Gromov-Witten invariants with two bulk insertions for $\widetilde{n}_d = OGW_\beta$ and $\beta = d[\ell]$. When L arises as the fixed point locus of an anti-holomorphic involution, it is conjectured that the mirror algebraic cycle is generally defined over $\mathbb{Q}$, such that the open Gromov-Witten invariants $\widetilde{n}_d \in \mathbb{Q}$ are rational numbers. In these cases, there exists an Ooguri-Vafa multicover formula analogous to (3.129) in which the quantum part of the superpotential is expressed as an integral linear combination of dilogarithms (assuming $r = 1$ for simplicity)

$$\Psi_h = \frac{1}{(2\pi i)^2} \sum_{d=1}^{\infty} \widetilde{n}_d \, q^d = \frac{1}{(2\pi i)^2} \sum_{d=1}^{\infty} n_d \, \mathrm{Li}_2(q^d) := \frac{1}{(2\pi i)^2} \sum_{d,k=1}^{\infty} n_d \, \frac{q^{dk}}{k^2}, \quad n_d \in \mathbb{Z}, \tag{6.6}$$

i.e. Ψ_h is a 2-function in the sense of Definition 3.24. However, as discussed in Section 3.2, there are examples in which the B-model predicts the invariants to lie in the algebraic number field $\widetilde{n}_d \in K/\mathbb{Q}$ that underlies the cycle in the B-model. The analog of the multicover expansion (6.6) then corresponds to the characterization of Ψ_h as a 2-function with coefficients in K as described in Remark 3.25. Concretely, when $K = \mathbb{Q}(\sqrt{\Delta})$ is a quadratic number field (cf. (4.38)), the superpotential admits an expansion of the form (assuming $r = 1$)

$$\Psi_h = \frac{1}{(2\pi i)^2} \sum_{d=1}^{\infty} \widetilde{n}_d \, q^d = \frac{1}{(2\pi i)^2} \sum_{d,k=1}^{\infty} (n_d^{(0)} + \sqrt{\Delta}\chi(k) n_d^{(1)}) \frac{q^{dk}}{k^2}, \tag{6.7}$$

where $\chi(k) = \left(\frac{\Delta}{k}\right)$ is the non-trivial quadratic character, and $n_d^{(0)}, n_d^{(1)} \in \mathbb{Z}$. A more general discussion of abelian number fields can be found in [Mü1]. An enumerative A-model interpretation of these non-rational invariants remains to be found.

Example 6.2. An example in which non-trivial arithmetic plays a role in the B-model is the algebraic cycle defined by

$$\begin{aligned} x_1 + \mu x_2 + \mu^2 x_3 = 0, \quad & a(x_1 + x_2 + x_3) = 3x_4, \quad b(x1 + x_2 + x_3) = 3x_5 \\ & ab\psi = 6, \qquad a^5 + b^5 = 27, \end{aligned} \tag{6.8}$$

where μ is a third root of unity, $\mu^3 = 1$. For each a, b subject to (6.8), these equations define a degree 1 curve in (3.17) and their image in the Greene-Plesser quotient $\mathcal{Y}$ is known as the *van Geemen lines* (see e.g. [Wal4]). To this family we can associate a normal function that is independent of a and b, as the Abel-Jacobi map is invariant under algebraic equivalence. Denoting by $\mathcal{C}_\mu$ the curve for a given root of unity, let Γ_{μ,μ^2} be the smooth 3-chain with boundary given by $\partial[\Gamma_{\mu,\mu^2,z}] = [\mathcal{C}_{\mu^2,z}] - [\mathcal{C}_{\mu,z}]$ in each fiber. For the standard choice of holomorphic 3-form (3.38), the resulting truncated normal function satisfies an inhomogeneous Picard-Fuchs equation

$$D_{\mathrm{PF}} \mathcal{W}_B = (2\pi i)^4 \left[\theta^4 - 5z \prod_{k=1}^{4} (5\theta + k)\right] \left(\int_{\Gamma_{\mu,\mu^2}} \Omega\right) = \mathcal{J}_{\mathrm{vG}} \tag{6.9}$$

with inhomogeneity (expanded around a small neighborhood around $z = 0$)

$$\mathcal{J}_{\mathrm{vG}} = \sqrt{-3} \cdot \frac{32}{45} \cdot \frac{\frac{63}{\psi^5} + \frac{1824}{\psi^{10}} - \frac{512}{\psi^{15}}}{\left(1 - \frac{128}{3\psi^5}\right)^{5/2}} = \sqrt{-3}\left(140000\, z + \frac{178000000000}{3} z^2 + \cdots\right). \tag{6.10}$$

The associated expansion of the A-brane superpotential is given by

$$\frac{(2\pi i)^2}{\sqrt{-3}}\,\Psi_h = 140000\,q + \frac{11148100000}{3}\,q^2 + \frac{5015947794500000}{27}\,q^3 + \cdots \tag{6.11}$$

and satisfies an arithmetically twisted Ooguri-Vafa integrality (6.7) specified by $\Delta = -3$. In line with (4.38), the constant contribution to the superpotential

$$\widetilde{c} = \lim_{q\to 0} Q(h, e_3) = Q(\widetilde{h}(0), e_3) = -\frac{390\sqrt{-3}}{(2\pi i)^2}\,L(2,\chi), \tag{6.12}$$

in terms of the Dirichlet L-function (4.39) with quadratic character, corresponds to the Abel-Jacobi limit of the van Geemen lines at $q = 0$, as described in [LW], and further studied in [JMW].

The A-model geometry mirror to the van Geemen lines is currently unknown, but expected to not satisfy Assumptions 5.5. In particular, the Abel-Jacobi limit is unlikely to be related to the Chern-Simons invariant alone, cf. the outlook in Chapter 8.

6.2. An A-model Origin of the Extension Class

We consider a collection of Lagrangian submanifolds $i : L = \bigcup_k L_k \hookrightarrow X$ with $k = 0, ..., d$ that satisfy Assumptions 5.5 and are homologically equivalent as cycles in X, meaning that $[L_k] - [L_{k'}] = 0 \in H_3(X)$ for each pair of indices k and k'. The long exact homology sequence of the pair (X, L) produces the sequences

$$\begin{aligned} 0 \longrightarrow H_6(X) \longrightarrow H_6(X,L) \longrightarrow 0& \\ 0 \longrightarrow H_4(X) \longrightarrow H_4(X,L) \longrightarrow H_3(L) \longrightarrow H_3(X) \longrightarrow \cdots& \\ 0 \longrightarrow H_2(X) \longrightarrow H_2(X,L) \longrightarrow 0& \\ \cdots \longrightarrow H_0(L) \longrightarrow H_0(X) \longrightarrow H_0(X,L) \longrightarrow 0,& \end{aligned} \tag{6.13}$$

and defines an extension of the even cohomology in pure weight 4 (cf. [DK2])

$$\begin{tikzcd} 0 \arrow[r] & H_4(X) \arrow[r] \arrow[d, "PD"] & H_4(X,L) \arrow[r] \arrow[d, "PD"] & \ker\Big(H_3(L) \xrightarrow{i_*} H_3(X)\Big) \arrow[r] \arrow[d, "PD"] & 0 \\ 0 \arrow[r] & H^2(X) \arrow[r] & H^2(X\setminus L) \arrow[r] & \ker\Big(H^0(L) \xrightarrow{i_!} H^3(X)\Big) \arrow[r] & 0, \end{tikzcd} \tag{6.14}$$

where the kernel of $i_! : H^0(L) \to H^3(X)$ is of rank d with generators Poincaré dual to $[L_k] - [L_0] \in H_3(X)$. Their lifts to $H^2(X \setminus L)$ are the currents

$$[\delta_{\Gamma_4^k}] \in H^2(X\setminus L) \cong H_4(X,L), \tag{6.15}$$

defined via the integral relation

$$\int_X \delta_{\Gamma_4^k} \cup \eta = \int_{\Gamma_4^k} \eta, \qquad \eta \in H^4(X, L), \tag{6.16}$$

where $[\Gamma_4^k] \in H_4(X, L)$ is a 4-chain with boundary $\partial[\Gamma_4^k] = [L_k] - [L_0]$.

Remark 6.3. While cup products as in (6.16) can be made precise in all of our examples, we will usually view them as being defined in terms of certain integral expressions. Namely, in most down-to-earth terms we set

$$\delta_{\Gamma_4^k} \cup \eta := \left(\int_{\Gamma_4^k} \eta\right) \cdot [p], \qquad [p] \in H^6(X). \tag{6.17}$$

We note that the relevant integrals are always well defined using pullback along $g : \Gamma_4 \to X$ (cf. Proposition 5.10) and the fact that the relevant classes can always be liftet to relative cohomology via a sequence similar to (5.48).

From this, we can define an extension of $\mathcal{H}^{\text{even}}$ which on the level of graded vector bundles takes the form

$$\hat{H}^{\text{even}} := H^{\text{even}}(X) \oplus \mathbb{C}(-2)^{\oplus d}\langle\{\delta_{\Gamma_4^1}, \ldots, \delta_{\Gamma_4^d}\}\rangle, \quad \hat{\mathcal{H}}^{\text{even}} := \hat{H}^{\text{even}} \otimes \mathcal{O}_{\Delta^*}. \tag{6.18}$$

An extension of the A-model Hodge filtration to $\hat{\mathcal{H}}^{\text{even}}$ is given by

$$\hat{F}^p = \bigoplus_{i \leq 3-p} \hat{H}^{2i} \subset \hat{H}^{\text{even}}, \qquad \hat{\mathcal{F}}^p := \hat{F}^p \otimes \mathcal{O}_{\Delta^*}, \tag{6.19}$$

which is seen to satisfy the conditions

$$\hat{F}^2 \big/ F^2 = \mathbb{C}(-2)^{\oplus d}, \qquad \hat{F}^p \big/ \hat{F}^{p+1} = F^p \big/ F^{p+1}. \tag{6.20}$$

We think of the currents (6.16) as $\hat{F}^2$-lifts $f_k = [\delta_{\Gamma_4^k}] \in \hat{F}^2$ compatible with the Hodge filtration on $\hat{H}^{\text{even}}$.

The H^2-module structure on H^{even} induced by the cup product naturally extends to $\hat{H}^{\text{even}}$ via

$$e_2 \cup f_k = [H] \cup [\delta_{\Gamma_4^k}] = \left(\int_X \delta_{\Gamma_4^k} \cup H \cup H\right)[\ell] = \left(\int_{\Gamma_4^k} H \cup H\right)[\ell] \tag{6.21}$$

and we seek a deformation of this module defined in terms of the open Gromov-Witten potential which matches the extension in the B-model. The flat connection on $\hat{\mathcal{H}}^{\text{even}}$ is therefore given by a family of algebraic structures

$$[H] \circledast (-) : \hat{H}^{\text{even}} \to \hat{H}^{\text{even}} \tag{6.22}$$

that is an augmentation of the small quantum product by holomorphic disks. On H^{even} it reduces to $e_2 * (-)$, while its action on the extending generators is defined by the rule

$$\nabla_t(f_k) = e_2 \circledast f_k := \sum_{l,m} \sum_{\beta \in H_2(X,L;\mathbb{Z})} OGW^k_\beta(e_2, e_l)\, q^\beta\, Q^{lm}\, e_m = \Psi''_k\, [\ell], \tag{6.23}$$

where we denote by Ψ_k and OGW^k_β the superpotential and open Gromov-Witten invariants of L_k. Similar to the case of closed strings, the zero energy contribution to this extended quantum product corresponds to $e_2 \circledast f_k|_{q=0} = e_2 \cup f_k$. The extension of the Dubrovin connection is determined by both the open and closed Gromov-Witten potentials and satisfies Griffiths transversality $\nabla_t(f_k) \in \hat{\mathcal{F}}^1$ due to the degree axiom. Its flatness in the one-parameter case is an immediate consequence of the algebraic properties of the operation $[H] \circledast (-)$. In the basis $\{e_j, f_k\}$ it is represented by the matrix

$$\nabla_t = d + \Big([H] \circledast (-)\Big) \otimes dt := d + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -\Phi''' & 0 & 0 & -\Psi''_1 & \cdots & -\Psi''_d \\ 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \otimes dt, \tag{6.24}$$

with $\nabla_t f_k = -\Psi''_k\, e_1$, c.f. (4.20). In analogy to the absolute case we construct a ∇-flat $\mathbb{C}$-local system $\hat{\mathcal{H}}^{\text{even}}_{\mathbb{C}}$ by first setting

$$\tilde{\sigma}(f_k) := f_k + \Psi'_{k,h}\, e_1 + \Psi_{k,h}\, e_0 \tag{6.25}$$

in terms of the holomorphic part $\Psi_{k,h}$ of the generating function of open Gromov-Witten invariants of L_k, and then defining the *quantum deformed current* as

$$h_k = \sigma(f_k) := \tilde{\sigma}(e^{-\omega} \cup f_k) = e^{-\omega}\, \delta_{\Gamma^k_4} + \Psi'_{k,h}\, e_1 + \Psi_{k,h}\, e_0. \tag{6.26}$$

Lemma 6.4. The quantum deformed current h_k is a flat section with respect to the extended Dubrovin connection, i.e. $\nabla(h_k) = 0$.

Proof. In view of Remark 6.3, we write

$$h_k = f_k + t\left(\int_{\Gamma_4} H \cup H\right) e_1 + \frac{t^2}{2}\left(\int_{\Gamma_4} H \cup H\right) e_0 + \Psi'_{k,h}\, e_1 + \Psi_{k,h}\, e_0, \tag{6.27}$$

such that flatness is seen to be an immediate consequence of $\nabla_t f_k = -\Psi''_k\, e_1$. $\square$

The local system is extended to $\hat{\mathcal{H}}^{\text{even}}_{\mathbb{C}}$ by $\nabla(h_k) = 0$ and we regard $\hat{\mathcal{H}}^{\text{even}}_{\mathbb{C}}$ as a complex solution to the quantum differential equation coming from $[H] \circledast (-)$.

In the multiparameter setup of Remark 3.17, vanishing of the A-model curvature poses a further algebraic constraint on the extended quantum product coming from

$$R_\nabla(\partial_i, \partial_j)\Big([\delta_{\Gamma_4^k}]\Big) = [H_i] \circledast \Big([H_j] \circledast [\delta_{\Gamma_4^k}]\Big) - [H_j] \circledast \Big([H_i] \circledast [\delta_{\Gamma_4^k}]\Big). \tag{6.28}$$

It can be interpreted as an equality of open Gromov-Witten invariants coming from certain boundary strata of the disk moduli space that correspond to disks with boundary on L_k, three marked points in the interior and no marked points on the boundary. In terms of the Gromov-Witten potential and superpotential associated to L_k, this can be expressed by the following condition.

Proposition 6.5. Flatness of the extended Dubrovin connection (6.24) is equivalent to the concurrent validity of both the ordinary WDVV equations (3.105), and the system of partial differential equations

$$\sum_{a,b} \partial_a\partial_i\partial_j\,\Phi \cdot Q^{ab} \cdot \partial_b\partial_l\,\Psi_k = \sum_{a,b} \partial_a\partial_i\,\Psi_k \cdot Q^{ab} \cdot \partial_b\partial_j\partial_l\,\Phi, \quad \text{for all } i, j, k, l, \tag{6.29}$$

which can be identified as the open WDVV equations [ST3, Alc, CZ] under the Assumptions 5.5.

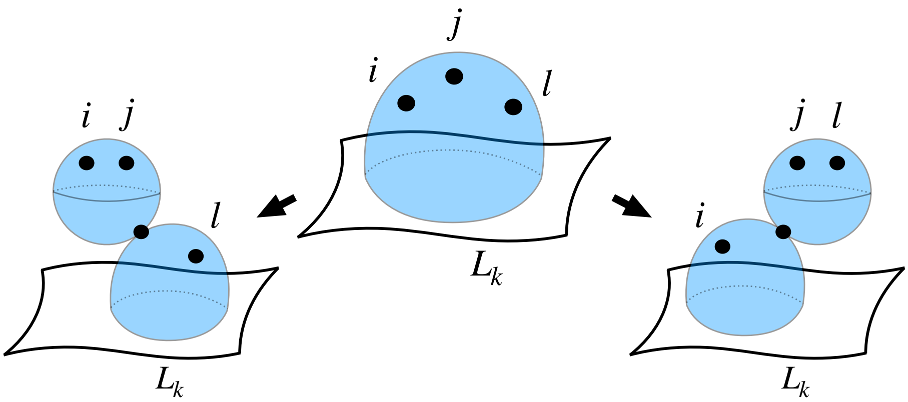

Figure 6.1.: Geometric interpretation of the open WDVV equations.

The difference of lifts $h_k - f_k$ is d-closed and in each fiber well defined modulo $\mathcal{F}^2 + \mathcal{H}^{\text{even}}_{\mathbb{Z}}$, where $\mathcal{H}^{\text{even}}_{\mathbb{Z}}$ denotes Iritani's Gamma-integral local system. In fact, what we have described so far is best understood as the canonical lift akin to (4.19) in the B-model. The extension is then classified by a normal function $\nu_{L_k} = [h_k - f_k] \in \mathcal{J}^2(\mathcal{H}^{\text{even}})$ given by a section of the intermediate Jacobian fibration

$$\mathcal{J}^2(\mathcal{H}^{\text{even}}) = \mathcal{H}^{\text{even}} \big/ \left(\mathcal{F}^2 + \mathcal{H}^{\text{even}}_{\mathbb{Z}}\right) \cong \left(\mathcal{F}^2\right)^\vee \big/ (\mathcal{H}^{\text{even}}_{\mathbb{Z}})^\vee, \tag{6.30}$$

where the second equality is an immediate consequence of Poincaré duality. It is given by a functional defined up to integral periods that acts on each fiber by

$$\nu_{L_k}(\eta)_t := \int_X (h_k - f_k) \cup \eta = \int_X h_k \cup \eta, \qquad \eta \in F^2 H^{\text{even}}(X), \tag{6.31}$$

where the second term vanishes for dimensionality reasons as $f \cup \eta$ is at most a 4-form.

Proposition 6.6. Let $L = \bigcup_k L_k \subset X$ be a collection of Lagrangians as described above. Then the sections $\nu_{L_k} \in \mathcal{J}^2(\mathcal{H}^{\text{even}})$, defined by the functionals (6.31), are normal functions of $\mathcal{H}^{\text{even}}$.

Proof. As described above, the difference $\widetilde{\nu}_{L_k} = h_k - f_k$ can be viewed as the canonical lift described in Section 4.3 such that the only property that remains to be shown is the horizontality condition

$$Q(\nabla_t \widetilde{\nu}_{L_k}, e_3) = \int_X \nabla_t (h_k - f_k) \cup [X] = 0. \tag{6.32}$$

This can be seen by an explicit calculation that is very similar to the discussion around (4.19) in the B-model. Namely, the truncated normal functions are given by

$$\mathcal{V}_k = Q(\widetilde{\nu}_{L_k}, e_2) = \int_{\Gamma_4^k} \omega \cup [H] + \Psi'_{h,k}, \quad \mathcal{W}_k = Q(\widetilde{\nu}_{L_k}, e_3) = \frac{1}{2}\int_{\Gamma_4^k} \omega \cup \omega + \Psi_{h,k}, \tag{6.33}$$

and reproduce the expected structure $\mathcal{V}_k = \mathcal{W}'_k$ of the A-brane superpotential. This last condition is equivalent to the vanishing of the coefficient of $e_0 = [p]$ in $\nabla_t \widetilde{\nu}_{L_k}$, from which horizontality of the normal function follows. $\square$

As discussed around (6.5), the infinitesimal invariants correspond to the two-point correlation functions on the disk

$$\Psi''_k = \sum_{\beta \in H_2(X,L;\mathbb{Z})} \mathrm{OGW}^k_\beta(H,H)\, q^\beta = \int_X H \circledast H \circledast \delta_{\Gamma_4^k} = Q(\nabla_t^2 \widetilde{\nu}_{L_k}, e_3), \tag{6.34}$$

in a way that is quite analogous to the absolute case (3.100).

6.3. Extending the Gamma-Integral Local System

At this stage, our approach has two deficiencies: The additional classical contributions to the A-brane superpotential (5.53) coming from the flat bundle are related to the choice of a specific integral local system $\hat{\mathcal{H}}^{\text{even}}_{\mathbb{Z}}$ underlying the extension. Analogous to the closed string case presented in Chapter 3, they are expected to arise in this context as mirrors of the Abel-Jacobi limit and determine the LMHS of the extension as a constant of integration of the quantum differential equation coming from $[H] \circledast (-)$. Their quantum interpretation in pure A-model terms presumably requires an extension of Iritani's $\hat{\Gamma}$-construction which takes the bundle data into account and correctly specifies the asymptotics of A-model periods including the constant term $Q(\widetilde{h}(0), e_3) = \widetilde{c}$ of (4.36) associated to the number field that underlies the mirror object. We conjecture that a systematic construction of a relative Gamma class and the corresponding quantum Abel-Jacobi map

$$AJ : \mathcal{L} \longmapsto \nu_{\mathcal{L}} = [h - f] \in \mathcal{J}^2(\mathcal{H}^{\text{even}}) \tag{6.35}$$

will generally reproduce the limiting periods of the mirror B-model. Secondly, as already noted in [DK2], the short exact sequence (6.14) cannot explain an extension class between two flat bundles on the same Lagrangian. We view this as a symptom of the cohomology not fully representing the A-brane geometry and resort to the fact that the corresponding truncated normal function can often be derived after including an auxiliary reference (see Section 6.5). The Lagrangian sphere discussed in Assumptions 5.5 is a canonical choice for this reference and can be viewed as analog of the coordinate lines in the respective B-model computations.

Guided by known examples and the physics underlying A-brane superpotentials, we give a conjectural expression for a relative Gamma class which reproduces the right integral structure for all established cases with a specific A-model geometry.

Definition 6.7. Consider an A-brane $\mathcal{L} = (L, \mathcal{E})$ that satisfies Assumptions 5.5. According to Proposition 5.10, let $\widetilde{\mathcal{E}}$ be an extension of the flat bundle to the bounding 4-chain Γ_4 with $\partial\Gamma_4 = L$. Then we identify the Chern character

$$\hat{\Gamma}(\mathcal{L}) := ch(\widetilde{\mathcal{E}}) = 1 + ch_1(\widetilde{\mathcal{E}}) + ch_2(\widetilde{\mathcal{E}}) = 1 + c_1(\widetilde{\mathcal{E}}) + \frac{1}{2}\left(c_1(\widetilde{\mathcal{E}})^2 - 2c_2(\widetilde{\mathcal{E}})\right) \tag{6.36}$$

as the *relative Gamma class* of $\mathcal{L}$.

Conjecture 6.8. Let $(X, \mathcal{L})$ and $(\mathcal{Y}, \mathcal{C})$ be an extended mirror pair for which $\mathcal{L}$ satisfies Assumptions 5.5. Then the truncated normal function associated to $\mathcal{C}$ satisfies[1]

$$Q(AJ(\mathcal{C}_k), e_3) = (2\pi i)^2 \int_{\Gamma_{k,z}} \Omega_z = \int_X \widetilde{\sigma}\left(e^{-\omega} \cup \hat{\Gamma}(\mathcal{L}_k) \cup f_k\right), \tag{6.37}$$

i.e. the classes

$$h_k = \sigma\left(\hat{\Gamma}(\mathcal{L}_k) \cup f_k\right) = \widetilde{\sigma}\left(e^{-\omega} \cup \hat{\Gamma}(\mathcal{L}_k) \cup f_k\right) \tag{6.38}$$

are the generators of the extended integral local system in the A-model.

[1] Here, $\Gamma_{k,z}$ is the smooth 3-chain with boundary $\partial[\Gamma_{k,z}] = [C_{k,z}]$ discussed in Section 4.2.

Conjecture 6.8 is motivated by known examples (see Theorem 6.12) and in general by the following observation (see also Remark 6.3).

Proposition 6.9. Consider an A-brane $\mathcal{L} = (L, \mathcal{E})$ in X that satisfies Assumptions 5.5 and let $\hat{\mathcal{H}}^{\text{even}}$ be an extension of the A-model VMHS for which the integral generator h is defined via (6.38). Then the truncated normal function $Q(\nu_{\mathcal{L}}, e_3)$ associated to the extension class $\nu_{\mathcal{L}} := [h - f] \in \mathcal{J}^2(\mathcal{H}^{\text{even}})$ is up to integral periods given by the A-brane superpotential $\mathcal{W}_A$, including all contributions of the flat bundle $\mathcal{E}$. In particular, the extension is fully determined by $\mathcal{W}_A$ and its derivative, i.e.

$$\widetilde{\nu}_{\mathcal{L}} = \left(\int_{\Gamma_4} \omega \cup H + \int_D H + \Psi'_h \right) e_1 + \left(\frac{1}{2} \int_{\Gamma_4} \omega \cup \omega + \int_D \omega + cs(\mathcal{E}) + \Psi_h \right) e_0 \tag{6.39}$$

is the canonical lift of the normal function.

Proof. A computation similar to the proof of Proposition 6.6 shows that the relative Gamma class $\hat{\Gamma}(\mathcal{L})$ immediately yields the additional contribution of the Chern-Simons invariant (Definition 5.9) to the truncated normal function, as the Chern character $ch(\widetilde{\mathcal{E}}) = 1 + ch_1(\widetilde{\mathcal{E}}) + ch_2(\widetilde{\mathcal{E}})$ of the extended bundle $\widetilde{\mathcal{E}}$ has an expression

$$ch(\widetilde{\mathcal{E}}) = \operatorname{tr} \exp \left(\frac{-F_{\widetilde{A}}}{2\pi i} \right) = 1 - \frac{1}{2\pi i} \operatorname{tr}(F_{\widetilde{A}}) + \frac{1}{(2\pi i)^2} \frac{\operatorname{tr}(F_{\widetilde{A}} \wedge F_{\widetilde{A}})}{2} \tag{6.40}$$

via Chern-Weil theory. Furthermore, the second term of (6.40) gives a 2-chain integral

$$\int_\Delta \omega, \qquad \Delta = PD\left(c_1(\widetilde{\mathcal{E}})\right) \in H_2(\Gamma_4, L; \mathbb{Z}) \cong H^2(\Gamma_4; \mathbb{Z}). \tag{6.41}$$

The homology class Δ does not always admit a representation by a disk, as Γ_4 need not be simply connected. However, because the diagram

$$\begin{tikzcd} H^2(\Gamma_4; \mathbb{Z}) \arrow[r, "\text{incl.}^*"] \arrow[d, "PD"] & H^2(L) \arrow[d, "PD"] \\ H_2(\Gamma_4, L, \mathbb{Z}) \arrow[r, "\partial"] & H_1(L) \end{tikzcd} \tag{6.42}$$

commutes, Δ bounds the 1-cycle representing $c_1(\mathcal{E})$, such that

$$\int_\Delta \omega = \int_{\Gamma_2} \omega \quad \text{mod integral periods,} \tag{6.43}$$

where $\Gamma_2 \in H_2(X, L; \mathbb{Z})$ is the class of a disk discussed around (5.52). Putting everything together yields the expression (6.39). ∎

In those cases were Assumptions 5.5 are met, we interpret the extension $\hat{\mathcal{H}}^{\text{even}}$ with underlying extended Gamma-integral structure $\hat{\mathcal{H}}^{\text{even}}_{\mathbb{Z}}$ as the *extended A-model VMHS* that produces the relevant extension class intrinsically in the A-model. Notably, this includes examples in which the Lagrangian is a spherical 3-manifold whose Chern-Simons invariant is always rational [Nis], see also Appendix B for a discussion of lens spaces. This is in line with the expected rational Abel-Jacobi limit for a mirror algebraic cycle that is defined over $\mathbb{Q}$, cf. Section 4.3. Further speculations regarding K-theory and the Mukai pairing can be found in the outlook (Chapter 8).

6.4. Asymptotics and Frob Modules

To make the monodromy $\hat{M} : \hat{\mathcal{H}}^{\text{even}}_{\mathbb{Z}} \to \hat{\mathcal{H}}^{\text{even}}_{\mathbb{Z}}$ around $q = 0$ unipotent, we pass to a r-fold cover with local coordinate $\hat{q} = q^{1/r}$ and extended monodromy logarithm $\hat{N} = \log(\hat{M}^r)$. At the puncture, $\hat{N}$ is given by $-2\pi i \operatorname{Res}_{\hat{q}=0}(\nabla)$ and we consider

$$\nabla_{\hat{q}}\left([\delta_{\Gamma_4^k}]\right) = \frac{r}{2\pi i q} \sum_{\beta \in H_2(X,L;\mathbb{Z})} \mathrm{OGW}_\beta(H,H)\, q^\beta\, [\ell]. \tag{6.44}$$

As whenever $\beta \neq 0$ all holomorphic disks have positive area $\int_\beta \omega > 0$, we have $q^\beta \to 0$ as $q \to 0$. The residue is then given by

$$\operatorname{Res}_{\hat{q}=0} \nabla_{\hat{q}}\left([\delta_{\Gamma_4^k}]\right) = \frac{r}{2\pi i}\, \mathrm{OGW}_0(H,H)\, [\ell] = \frac{r}{2\pi i}\left(\int_{\Gamma_4^k} H \cup H\right)[\ell], \tag{6.45}$$

which is $r/(2\pi i)$ times the only non-vanishing entry of the matrix representing the cup product $e_2 \cup f_k$. Along similar lines (see [CK2]) we can compute the remaining entries of the residue matrix such that the monodromy logarithm of the extended A-model VMHS at $0 \in \Delta_{\hat{q}}$ is given by $-r[H]\cup(-) : \hat{H}^{\text{even}} \to \hat{H}^{\text{even}}$, which in the basis $\{e_i, f_k\}$ is represented by the matrix

$$\hat{N}_{\hat{q}=0} = r \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \kappa & 0 & 0 & \lambda_1 & \cdots & \lambda_d \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{6.46}$$

Here, r corresponds to the order of the torsion group $H^2(L;\mathbb{Z})$. The associated relative weight filtration $\hat{W}_\bullet = \hat{W}_\bullet(\hat{N})$ leads to a filtration $\hat{\mathcal{W}}_\bullet = \hat{W} \otimes \mathcal{O}_{\Delta^*}$ of

$\hat{\mathcal{H}}^{\text{even}}$ such that the graded pieces fit into the diagram with horizontal exact sequences

$$\begin{tikzcd}
 & \mathcal{H}^0 \arrow[r, "\sim"] \arrow[d, "rN"'] & \hat{\mathcal{H}}^0 \arrow[d, "\hat{N}"] & & \\
0 \arrow[r] & \mathcal{H}^2 \arrow[r] \arrow[d, "rN"'] & \hat{\mathcal{H}}^2 \arrow[r] \arrow[d, "\hat{N}"] & \mathcal{I}_{\mathbb{Z}} \arrow[r] \arrow[d, "L"] & 0 \\
0 \arrow[r] & \mathcal{H}^4 \arrow[r] \arrow[d, "rN"'] & \hat{\mathcal{H}}^4 \arrow[r] \arrow[d, "\hat{N}"] & H \arrow[r] & 0 \\
 & \mathcal{H}^6 \arrow[r, "\sim"] & \hat{\mathcal{H}}^6, & &
\end{tikzcd} \tag{6.47}$$

cf. (6.13). There is an immediate resemblance with (4.25), where we in particular identify the torsion group H in (4.25) with $H^2(L;\mathbb{Z})$ and the degree of the branched covering $\hat{\mathcal{Y}} \to \Delta^*_{\hat{z}}$ with its order.

The nilpotent orbit associated to the extension $\hat{\mathcal{H}}^{\text{even}}$ is fully determined by the H^2-module structure on $\hat{H}^{\text{even}}$ defined by the cup product (6.21). Its deformation to an extended quantum product subject to the open WDVV equations can be viewed as an extended version of a Frobenius module in the sense of [CF2], which we discuss further below. In view of Remark 4.3, also the A-model construction admits a dual perspective. Starting from the cohomology of the pair (X, L),

$$\begin{aligned}
0 \longrightarrow H^0(X,L) \longrightarrow H^0(X) \longrightarrow H^0(L) \longrightarrow \cdots \\
0 \longrightarrow H^2(X,L) \longrightarrow H^2(X) \longrightarrow 0 \\
\cdots \longrightarrow H^3(X) \longrightarrow H^3(L) \longrightarrow H^4(X,L) \longrightarrow H^4(X) \longrightarrow 0 \\
0 \longrightarrow H^6(X,L) \longrightarrow H^6(X) \longrightarrow 0,
\end{aligned} \tag{6.48}$$

it is based on the short exact sequence

$$0 \longrightarrow \operatorname{coker}\Big(H^3(X) \to H^3(L)\Big) \longrightarrow H^4(X,L) \longrightarrow H^4(X) \longrightarrow 0. \tag{6.49}$$

Defining A-model Hodge and weight filtrations according to Remark 4.3 leads to a VMHS $(\hat{\mathcal{H}}^{\text{even}})^\vee$ which is dual to $\hat{\mathcal{H}}^{\text{even}}$ via

$$H^2(X \setminus L) \times H^4(X,L) \longrightarrow \mathbb{C}, \qquad (\alpha, \beta) \longmapsto \int_{PD(\alpha)} \beta. \tag{6.50}$$

We denote by $f_k^\vee$ the extending generators and specify the corresponding extended quantum product by

$$e_2 \check{\circledast} e_2 := \sum_{\beta \in H_2(X;\mathbb{Z})} GW_\beta(e_2, e_2)\, q^\beta\, e_1 + \sum_{\beta \in H_2(X,L;\mathbb{Z})} OGW_\beta^k(e_2, e_2)\, q^\beta\, f_k^\vee, \tag{6.51}$$

such that the variations of MHS $(\hat{\mathcal{H}}^{\text{even}})^\vee$ and $(\hat{\mathcal{H}}^3)^\vee$ match by construction . In particular, the extended Dubrovin connection is now given by

$$\nabla_t = d + \Big([H] \check{\circledast} (-)\Big) \otimes dt = d + \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -\Phi''' & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -\Psi_1'' & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\Psi_d'' & 0 & 0 & \cdots & 0 \end{pmatrix} \otimes \frac{dq}{2\pi i\, q}, \tag{6.52}$$

cf. (4.34). The extended quantum product (6.51) on $H^{\text{even}}(X, L)$ arising in this way as a mirror of the Gauß-Manin connection coincides with a special case of the much more general relative quantum product $\beth(e_2, e_2)$ constructed in [ST3]. The product $\beth$ is defined for any Lagrangian submanifold and its associativity is equivalent to the validity of a more general set of open WDVV equations including boundary punctures in terms of bounding cochains (see Section 5.2). In would be interesting to further study this general construction in the present context, in particular in view of including boundary punctures and potential mirror geometries in the B-model.

From this point of view the VMHS $(\hat{\mathcal{H}}^{\text{even}})^\vee$ can also be related to a structure put forward in [Alc]. In the closed string context, associativity of the quantum product in terms of the WDVV equations and the corresponding existence of the Gromov-Witten potential makes the even degree cohomology into a Frobenius manifold. While part of this structure is lost in the open string situation, an adapted object was introduced in [*op.cit.*] that generally captures the geometry of the open WDVV equations.

Definition 6.10. Let M be a manifold with a flat connection $\nabla : T_M \to T_M \otimes \Omega^1_M$ and a symmetric bilinear form $\circledast : T_M \otimes T_M \to T_M$ on its tangent bundle. Then M is called a *Frob manifold* if $\circledast$ is associative and arises locally from a vector field $\mathcal{G}$ called *vector potential*, i.e.

$$X \circledast Y = [X, [Y, \mathcal{G}]] = \nabla_X \nabla_Y \mathcal{G}, \tag{6.53}$$

for all ∇-flat vector fields X and Y.

A Frob manifold can be thought of as a Frobenius manifold in which the requirement of a metric compatible with the bilinear form is dropped. As a consequence, not all of the structure constants as in (3.115) can be integrated into a single potential function, leading to the notion of a vector that carries the product information. As shown in [*op.cit.*], an open version of the WDVV equations arise as associativity constraints in a situation where the Frob manifold is given by an extension

$$0 \longrightarrow I \longrightarrow M \longrightarrow N \longrightarrow 0, \tag{6.54}$$

where all morphisms should be compatible with the Frob manifold data. As is the case for Frobenius manifolds, not the entirety of the multiplicative structure is relevant for questions in Hodge theory and the extended A-model VMHS, where we are only interested in the H^2-module structure induced from the extended quantum product. For this reason we want to introduce the following object, that should be viewed as an extension of a Frobenius module as in Definition 3.18 and Definition 3.19.

Definition 6.11. Let $V = \bigoplus_{i=0}^{2n} V_i$ with $n \in \mathbb{N}$ and $V_0 = \langle 1 \rangle$ be a graded finite dimensional $\mathbb{C}$-vector space together with a graded module structure

$$\circledast : \mathrm{Sym}(V_2) \times V \longrightarrow V. \tag{6.55}$$

We call $(V, \circledast)$ a *graded V_2-Frob module of weight n* if the module structure arises from a classical vector potential: Given a graded basis $\{e_j\}$ of V with associated linear coordinates t_j, there is a vector potential with components $G_c^k : V \to \mathbb{C}$ such that

$$e_i \circledast e_j = \sum_k \frac{\partial^2 G_c^k}{\partial t_i \partial t_j} e_k, \qquad e_i \in V_2,\ e_j, e_k \in V. \tag{6.56}$$

We then consider the specific situation in which $(V, \circledast)$ is of weight $n = 3$ and given by an extension (6.54) of a Frobenius module in the sense of Definition 3.18 with I of rank d, where the extending generators are not contained in V_2. Set $\dim V_2 = r$ and define $q_i = \exp(2\pi i t_i)$ when $e_i \in V_2$. A *quantum vector potential* is a vector with components $G^k : V \to \mathbb{C}$, each of the form $G^k = G_c^k + G_q^k$ where G_q^k is a convergent power series in the q_i vanishing at $q_i = 0$ for all i. Note that when considering an extension of a Frobenius module, there are always components of the form $\partial_i F$, $i = 1, \ldots, r$, that can be integrated to a single potential. In addition to the WDVV equations (3.114) of this potential, we require the vector potential to satisfy the open WDVV equations

$$\sum_{a,b} \partial_a \partial_i \partial_j \, F \cdot Q^{ab} \cdot \partial_b \partial_l \, G^k = \sum_{a,b} \partial_a \partial_i \, G^k \cdot Q^{ab} \cdot \partial_b \partial_j \partial_l \, F, \quad \text{for all } i, j, k, l, \tag{6.57}$$

where $e_i, e_j, e_k, e_l \in V_2$ and $e_a, e_b \in V$. We then call $(V, \circledast_q)$ a *quantum deformed Frob module* if the operation

$$e_i \circledast_q e_j = \sum_k \frac{\partial^2 G^k}{\partial t_i \partial t_j} e_k, \qquad e_i \in V_2,\ e_j, e_k \in V, \tag{6.58}$$

defines a Frob module for each $q = (q_1, \ldots q_r)$.

The obvious example of a quantum deformed Frob module that we have in mind is related to the short exact sequence

$$0 \longrightarrow \operatorname{coker}\left(H^3(X) \to H^3(L)\right) \longrightarrow H^{\text{even}}(X, L) \longrightarrow H^{\text{even}}(X) \longrightarrow 0, \quad (6.59)$$

which defines a rank d extension of a quantum deformed Frobenius module, where $L = \bigcup_k L_k \subset X$ as above. Here, the open WDVV equations (6.29) encode associativity of the product (6.51) and the associated vector potential is given by

$$G = (\partial_1 F, \ldots, \partial_r F, \Psi_1, \ldots, \Psi_d)\,, \quad (6.60)$$

with $h^{1,1} = r$ and $\text{rk}(I) = d$, and where F is given by the modified Gromov-Witten potential (3.116). Note that part of the vector potential can actually be integrated to a single quantum potential: A compatible metric exists for those components that are associated to the closed string sector, where $H^{\text{even}}(X)$ is an ordinary Frobenius module. Only the components coming from the extension have a live on their own, which is a known feature of $\mathcal{N} = 1$ special geometry as described in [LMW1, LMW2].

Given an extended mirror pair according to Definition 6.1, we find a correspondence between weight 3, rank d quantum deformed Frob modules in the A-model with extensions of variations of MHS coming from a homologically trivial algebraic cycle as described in Remark 4.3. We comment on potential future investigations in Chapter 8.

6.5. The Example of the Real Quintic

Now that we have established all of the relevant machinery in both the A- and the B-model, we are in a position to rephrase the main example of [Wal1] in this new language. It concerns a pair of cycles lying in the Fermat quintic (X, ω) and mirror quintic family $\mathcal{Y} \to \Delta^*$ discussed in Example 3.22 that is shown to be an extended mirror pair in the sense of Definition 6.1. On the A-side, we consider the real quintic $L = \mathbb{RP}^3$ that arises as the fixed point locus of an anti-holomorphic involution on X, as described in Example 5.6. Flat U(1)-bundles $\mathcal{E}$ on L are up to isomorphism classified by their integral first Chern class $c_1(\mathcal{E}) \in H^2(L;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ such that there are two possible A-brane configurations which we will denote by $\mathcal{L}_+ = (L, \mathcal{E}_+)$ and $\mathcal{L}_- = (L, \mathcal{E}_-)$. Based on constraints imposed by the consistency of A-brane charges [*op.cit.*], the most general superpotentials associated to $\mathcal{L}_\pm$ are up to closed string periods, i.e. mod $t\mathbb{Z} + \mathbb{Z}$, given by

$$\mathcal{W}_+ = \frac{t^2}{4} + \Psi_h, \qquad \mathcal{W}_- = \frac{t^2}{4} - \frac{t}{2} + \frac{1}{4} - \Psi_h, \qquad \Psi_h = \frac{1}{(2\pi i)^2} \sum_{d=1}^{\infty} \widetilde{n}_d\, q^{d/2}, \quad (6.61)$$

with $\widetilde{n}_d = OGW_\beta$ for $\beta = d\,[\ell]$. We interpret these superpotentials as the domain wall tensions of $\mathcal{L}_\pm$ in relation to a fixed Lagrangian sphere $L_0 \cong S^3 \subset X$ in the same homology class (cf. Proposition 5.3), where the classical currents (6.16) account for the terms proportional to t^2. Thinking of the superpotentials as truncated normal functions $\mathcal{W}_\pm = Q(\widetilde{\nu}_\pm, e_3)$, the class $\nu_\pm = [\widetilde{\nu}_\pm] \in \mathcal{J}^2(\mathcal{H}^{\text{even}})$ defines an extension of the A-model VMHS associated to (X, ω). Note that because the superpotentials are measured with respect to the same Lagrangian L_0 up to homology, the corresponding $\hat{\mathcal{F}}^2$-lifts $f_\pm$ coincide, while the quantum deformed currents $h_\pm$ differ due to different choices of flat bundle on L, see also Figure 6.2.

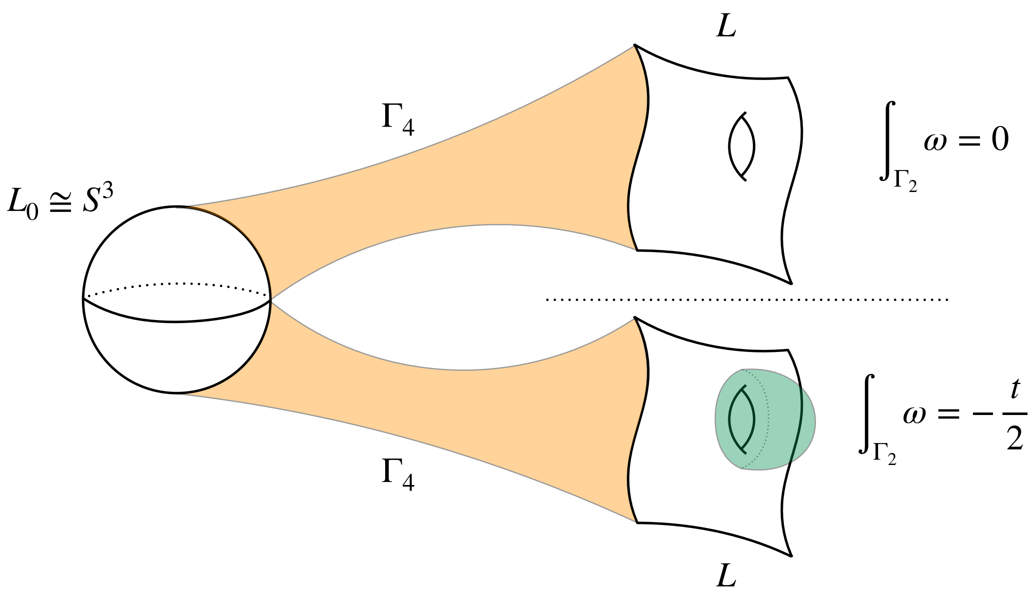

Figure 6.2.: Domain wall tension coming from the real quintic. The superpotentials $\mathcal{W}_\pm$ arise from the two choices of flat bundle on L, when measured in reference to a fixed Lagrangian sphere L_0. In case of the non-trivial Chern class, the tension receives additional contributions from the bundle data that specifies the A-brane.

Under monodromy $\hat{M}$ of the corresponding extended local system, the extending generators $h_\pm$ are exchanged because

$$\mathcal{W}_+(t+1) = \mathcal{W}_-(t) + t, \qquad \mathcal{W}_-(t+1) = \mathcal{W}_+(t), \tag{6.62}$$

such that we have to consider a pullback along the double cover $\hat{q} = q^{1/2} \mapsto q$ and the monodromy logarithm $\hat{N} = \log(\hat{M}^2)$. In the basis $\{g_i, h_\pm\}$ the respective matrices are given by

$$\hat{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 1 & 0 \\ -5 & 5 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \hat{N} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 5 & 10 & 0 & 0 & 1 & 1 \\ -\frac{25}{3} & 5 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.63}$$

Taking the difference $\mathcal{T}_A = \mathcal{W}_+ - \mathcal{W}_-$ yields the domain wall tension between the vacua specified by the bundles $\mathcal{E}_+$ and $\mathcal{E}_-$ on the same Lagrangian. It is up to integral periods of the form

$$\mathcal{T}_A = \frac{t}{2} - \frac{1}{4} + 2\Psi_h = \frac{t}{2} - \left(\frac{1}{4} + \frac{1}{2\pi^2}\sum_{d=1}^{\infty} \widetilde{n}_d\, q^{d/2}\right) \quad \mathrm{mod}\ t\mathbb{Z} + \mathbb{Z}, \tag{6.64}$$

and likewise defines an extension of $\mathcal{H}^{\mathrm{even}}$ on the double cover with canonical lift of the normal function $\widetilde{\nu} = \mathcal{T}_A'\, e_1 + \mathcal{T}_A\, e_0$. Under monodromy $t \mapsto t+1$ the domain wall tension behaves as

$$\mathcal{T}_A(t+1) = -\mathcal{T}_A(t) + t, \tag{6.65}$$

and the matrix

$$\hat{N} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 5 & 10 & 0 & 0 & 0 \\ -\frac{25}{3} & 5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.66}$$

represents the corresponding monodromy logarithm of the extended local system.

On the side of the B-model we consider the Deligne conics $\mathcal{C}_\pm \to \Delta^*$ defined in Example 4.2. Recall that these algebraic cycles are only globally well defined after passing to the double cover $\hat{\mathcal{Y}}$ defined over $\hat{z} = z^{1/2} \mapsto z$, as the components of $C_{\pm,\psi}$ are exchanged under monodromy around $z = 0$. Choosing in each fiber a smooth 3-chain Γ_z with boundary $\partial\Gamma_z = [C_{+,z}] - [C_{-,z}]$, the B-brane superpotential is given by the integral

$$\mathcal{T}_B = \int_{\Gamma_z} \Omega_z. \tag{6.67}$$

The statement of extended mirror symmetry for these two cycles can now be formulated as the following Mirror Theorem for the quintic.

Theorem 6.12. The cycles given by both choices of flat bundle on the real quintic $\mathcal{L}_\pm$ in X and the Deligne conics $\mathcal{C}_\pm \subset \hat{\mathcal{Y}}$ determine an extended mirror pair in the sense of Definition 6.1. In particular, the integral local system in the A-model is reproduced by the relative Gamma class.

Proof. The B-brane superpotential $\mathcal{T}_B$ satisfies an inhomogeneous Picard-Fuchs equation with inhomogeneity [MW] (see also Example 4.2)

$$\mathcal{J}_{\mathrm{Del}} = -\frac{15}{4}\sqrt{z}, \tag{6.68}$$

for the standard choice of holomorphic 3-form (3.38) on $\hat{\mathcal{Y}}$ and its particular solution $\tau(z)$ computes the open Gromov-Witten potential $\Psi_h(q)$ of the real quintic. The particular solution corresponds to a multiple of the hypergeometric series (3.41) evaluated at $h = 1/2$,

$$\tau(z) = -\frac{1}{4}\sum_{n=0}^{\infty}\frac{\Gamma(5(n+1/2)+1)}{\Gamma(n+1/2+1)^5}\, z^{n+1/2}, \tag{6.69}$$

and by the localization calculation of [PSW] the open Gromov-Witten potential is obtained by canonical normalization and composition with the mirror map, i.e.

$$(2\pi i)^2\, \Psi_h(q) = \frac{\tau(z(q))}{\varpi_0(z(q))}. \tag{6.70}$$

This identification establishes an equivalence of the A- and B-brane superpotentials up to solutions of the homogeneous Picard-Fuchs equation. Employed as truncated normal functions in terms of the canonical lift (4.19), this induces an isomorphism of the underlying complex variations of MHS. Equality of integral local systems corresponds to an identification of the classical terms in the superpotentials which determine the LMHS. In the A-model, the physical prediction for their value is given in (6.64). After analytic continuation of $\mathcal{T}_B$ to the Landau-Ginzburg point, the requirement of integral monodromy provides the LMHS of the B-model [Wal1]

$$\mathcal{T}_B(z) = \frac{\varpi_1(z)}{2} - \left(\frac{\varpi_0(z)}{4} + \frac{1}{2\pi^2}\,\tau(z)\right), \qquad \mathcal{T}_A(q) = \frac{\mathcal{T}_B(z(q))}{\varpi_0(z(q))}, \tag{6.71}$$

which reproduces (6.64) on the A-side. In a fashion intrinsic to the A-model, this integral structure can be also derived from the relative Gamma classes associated to a suitable configuration of A-branes. We consider $\mathcal{L}_\pm = (L \cup L_0, \mathcal{E}_\pm)$, where we include a reference 3-sphere L_0 using Proposition 5.3 to make the configuration homologically trivial. The corresponding integral local system is now specified by $\hat{\Gamma}(\mathcal{L}_\pm)$ via (6.38). By Proposition 6.9 we have canonical lifts of the respective normal functions

$$\begin{aligned}\tilde{\nu}_+ &= \left(\int_{\Gamma_4}\omega\cup H + \Psi_h'\right)e_1 + \left(\frac{1}{2}\int_{\Gamma_4}\omega\cup\omega + \Psi_h\right)e_0,\\ \tilde{\nu}_- &= \left(\int_{\Gamma_4}\omega\cup H - \int_D H - \Psi_h'\right)e_1 + \left(\frac{1}{2}\int_{\Gamma_4}\omega\cup\omega - \int_D \omega + cs(\mathcal{E}_-) - \Psi_h\right)e_0,\end{aligned} \tag{6.72}$$

where for a globally trivial bundle both the first Chern class and the Chern-Simons invariant vanish (mod $\mathbb{Z}$). In particular, there are no contributions from bundle data on the reference 3-sphere L_0. By the index calculation presented in Appendix B, the Chern-Simons invariant of the non-trivial bundle on L is given by $cs(\mathcal{E}_-) = 1/4$ and the 2-chain contribution is given according to (5.52). Computing the difference $\mathcal{T}_\pm = \mathcal{W}_+ - \mathcal{W}_-$ of the superpotentials $\mathcal{W}_\pm = Q(\widetilde{\nu}_\pm, e_3)$ leads to a cancellation of the terms proportional to t^2 and disk instantons associated to L_0, leaving the expression (6.64). $\square$

The flat connection ∇_t in the canonical coordinate is determined by the infinitesimal invariant

$$\mathfrak{D}(z(q)) = 2\Psi_h'' = 15\,q^{1/2} + 6900\,q^{3/2} + 13603140\,q^{5/2} + \cdots\,, \tag{6.73}$$

which we identify as twice the two-point correlation function on the disk by the divisor axiom, see Axioms 5.7. Here, the open Gromov-Witten potential of L is given by

$$(2\pi i)^2\,\Psi_h = 30\,q^{1/2} + \frac{4600}{3}\,q^{3/2} + \frac{5441256}{5}\,q^{5/2} + \cdots\,. \tag{6.74}$$

The functional which mirrors the Abel-Jacobi map in the A-model is fully determined by its action on $\mathcal{F}^2$ with limiting A-model periods

$$Q(\widetilde{h}(0), e_2) = -\frac{1}{2}, \qquad Q(\widetilde{h}(0), e_3) = \frac{1}{4}, \tag{6.75}$$

where the first component is related to the double cover and we can think of the second component as

$$cs(\mathcal{E}_-) = \frac{1}{4} = -\frac{6\,\zeta(2)}{(2\pi i)^2} \in \mathbb{C}/\mathbb{Z}(2). \tag{6.76}$$

From the point of view of the B-model, this Abel-Jacobi limit was found by monodromy considerations around the conifold point in [Wal1, MW]. It would be interesting to also verify its value based on the methods of [GGK2] applied to the degenerated Deligne conics and the associated regulator image.

A version of Theorem 6.12 can also be formulated for the further examples of extended mirror symmetry for one-parameter families discussed in [KW]. These include the sextic, octic and decimic hypersurfaces of the Doran-Morgan classification [DM2], together with cycles analogous to the Deligne conics in Example 4.2. In all of these cases, the mirror A-brane satisfies Assumptions 5.5 and the superpotential corresponds to a domain wall tension between two choices of flat bundle on a Lagrangian that is topologically $\mathbb{RP}^3$, similar to the real quintic.

7. One-Loop Amplitudes from van Geemen Lines

Algebraic cycles like the van Geemen lines, whose definition underlies an algebraic number field rather than $\mathbb{Q}$, are perhaps the most interesting subjects of future investigation. At the time of writing, there is no example of such a cycle for which the mirror A-model geometry is fully understood. It should generally be expected that in these cases Assumptions 5.5 are not satisfied such that both the q-series expansion of the normal function as well as its limiting value carry arithmetic features whose A-model interpretation is yet to be unveiled. Topological string theory provides a further tool to generate geometric data of the A-model based on B-model calculations that is physically related to open string loop amplitudes. This chapter is meant to demonstrate how this can be achieved in arithmetically interesting examples, where this new enumerative information could help assessing future candidate mirrors. We begin with a general overview of the computational technique developed in [BCOV, Wal3, Wal2] in Section 7.1. Afterwards, we discuss the example of the van Geemen lines (see Example 6.2) and two of its analogs inside of the mirror octic, found in [JW], in Section 7.2 and Section 7.3.

7.1. Loop Amplitudes and Tadpole Cancellation

In this context, the main object of study is the topological string partition function, in which we allow for a general genus g and any number h of boundary components [Wal3]. Similar to its tree-level counterpart, it is of the form

$$\mathcal{F}^{(g,h)} = \sum_{\beta \neq 0} OGW_\beta^{(g,h)}\, q^\beta, \tag{7.1}$$

where the coefficients $OGW_\beta^{(g,h)}$ denote the corresponding open Gromov-Witten invariants of degree $\beta \in H_2(X, L; \mathbb{Z})$. As before, we also allow for situations in which the canonical coordinate expresses a branching behavior where the formal power series is weighted by an additional factor of $1/r$. The precise definition of these more general Gromov-Witten invariants involves a moduli space $\widehat{\mathcal{M}}^{(g,h)}(X, L, \beta)$ very similar to the one considered in Section 5.1, except for the fact that we now allow for additional topologies of the open stable maps $u : (\Sigma, \partial\Sigma) \to (X, L)$.

However, not much is known about the geometry of $\widehat{\mathcal{M}}^{(g,h)}(X, L, \beta)$ for a general pair (g, h), such that there is currently no formulation of axioms analogous to Axioms 5.7. In terms of the virtual fundamental class, a naive definition of these more general open Gromov-Witten invariants is

$$OGW_{\beta}^{(g,h)} = \int_{\widehat{\mathcal{M}}^{(g,h)}(X,L,\beta)} 1, \tag{7.2}$$

when we assume that the moduli space can be described as a smooth orbifold with boundary and corners. However, the corresponding boundary strata are in general much more complicated than in the case of disks, producing various anomalies (including the ones discussed in Remark 5.2) that hinder a general understanding of the enumerative meaning of (7.2). The real case, as studied in [GZ1, GZ2, GZ3], again provides an exception where the definition can be made precise. Similar to the discussion of disks in Chapter 5, the problematic behavior at the boundary is in these cases controlled by the action of an anti-holomorphic involution.

The partition function (7.1) described so far is an object intrinsic to the A-model that is, as always, only well-defined in a formal neighborhood around the large volume limit. By the mirror principle, this q-series can also be viewed from a B-model point of view in which it arises as the large complex structure limit in the canonical coordinate of a function that is more generally defined on the moduli space of complex structure of the mirror Calabi-Yau $\mathcal{Y}$, see e.g. [Liu] for a mathematical survey. While this B-model partition function depends on the complex structure, it does not preserve the distinction of local holomorphic and antiholomorphic coordinates, which can be viewed as a consequence of the non-holomorphy of the Hodge decomposition (3.1). In case of the closed string, the famous result of [BCOV] is that this deviation from being holomorphic is captured by certain contributions of degenerating surfaces which allows to systematically compute the amplitude in terms of the recursive *holomorphic anomaly equations*. An extension of this procedure to the open string sector including D-branes was achieved in [Wal3], where the basic philosophy of the anomaly arising form the boundary $\partial\widehat{\mathcal{M}}^{(g,h)}(X, L, \beta)$ persists. The resulting *extended holomorphic anomaly equations* allow for a recursive computation of the numbers $OGW_{\beta}^{(g,h)}$, starting in case of a single D-brane with the Yukawa coupling $\mathfrak{C}$ and the infinitesimal invariant $\mathfrak{D}$ that act as interaction vertices in certain Feynman rules that describe the decomposition of the worldsheet.

The annulus amplitude $\mathcal{A} = \mathcal{F}^{(0,2)}$ arises physically as the one-loop amplitude of an open string, in which we assume that both boundary conditions are specified by a single D-brane. It is one of the simplest partition functions in this context, where the corresponding anomaly equation is, when going back to the large complex structure limit, given by[1] [Wal2]

$$\partial_t \mathcal{A} = \partial_t \mathcal{F}^{(0,2)} = \mathfrak{D}^2\, \mathfrak{C}^{-1} + \partial_t f^{(0,2)}. \tag{7.3}$$

Here, the holomorphic function $f^{(0,2)} : \Delta^* \to \mathbb{C}$ is the *holomorphic ambiguity* that can be viewed as a "constant of integration" of the respective differential equation. The idea behind this approach is that any amplitude can be computed by an equation similar to (7.3) up to a finite number of such ambiguities, which can often be fixed by studying their behavior at singular points in the moduli space. One physical requirement for the amplitude is related to an interpretation as counting integral BPS invariants in terms of an Ooguri-Vafa multicover formula, where the one-loop function in particular is expected to be a 1-function in the sense of Definition 3.24. The 1-function property can be systematically assessed by verifying the condition on prime numbers (3.136) for various choices of $f^{(0,2)}$, for example using `Mathematica` [Wol]. The validity of the recursion relation furthermore depends on the decoupling of continuous open string deformation parameters, which is always an assumption that already enters into the definition of the superpotential (2.28) from which the infinitesimal invariant is derived.

It was further noted in [Wal2] that in order to achieve Ooguri-Vafa integrality and a satisfactory BPS interpretation of loop amplitudes in general, it is necessary to also include the count of unoriented topological string worldsheets at any order in string perturbation theory. This requirement can be understood as the analog of tadpole cancellation[2] for topological strings, where in case of the annulus amplitude it suffices to bring the contribution of Klein bottles and Möbius strips into the mix. As described in [*op.cit.*], the contributions of the Möbius strip decouple in case of brane vacua separated by discrete Wilson lines, which we also assume in what follows. Although this is a sensible assumption from the physical point of view, where we always restrict to a limit without continuous open string deformation parameters, an explicit verification of this condition would require a better understanding of the relevant A-model geometry. In case of the one-parameter models, the Klein bottle contributions are seen to be generally of the form [*op.cit.*]

$$\mathcal{K} \sim \frac{1}{2\pi i} \log\left[\left(\frac{q}{z}\frac{dz}{dq}\right)(1-\alpha z)^{-\frac{1}{4}}\right], \tag{7.4}$$

where $1-\alpha z$ describes the conifold locus in moduli space. When computing one-loop amplitudes we will therefore proceed as follows: We first solve the extended

[1] Note that due to a different normalization of the infinitesimal invariant, we have a prefactor of the first term that differs from the existing literature.

[2] This phenomenon is distinct from the tadpole anomaly in the context of Floer theory discussed in Section 5.1.

holomorphic anomaly equation (7.3) and choose the holomorphic ambiguity in a way that makes $\mathcal{A}$ into a 1-function. As it turns out in the examples at hand, this Ooguri-Vafa integrality can always be achieved for the annulus amplitude itself. We then find that the combination

$$\mathcal{A}+\mathcal{K} = \frac{1}{2\pi i}\sum_{d=1}^{\infty}\widetilde{n}_d^{(0,2)}\, q^d = \frac{1}{2\pi i}\sum_{d=1}^{\infty} n_d^{(0,2)}\,\mathrm{Li}_1(q^d) = \sum_{d,k=1}^{\infty} n_d^{(0,2)}\,\frac{q^{dk}}{k} \tag{7.5}$$

turns out to be a 1-function with $\widetilde{n}_d^{(0,2)} \in \mathbb{Q}$ whose integral coefficients can be extracted iteratively via the formula

$$n_d^{(0,2)} = \widetilde{n}_d^{(0,2)} - \sum_{k|d}\frac{n_{d/k}^{(0,2)}}{k} \in \mathbb{Z}. \tag{7.6}$$

We interpret the integers $n_d^{(0,2)} \in \mathbb{Z}$ as BPS invariants of the D-brane configuration and list the low degree examples in Appendix C. Although this approach is analogous to [Wal2], the fact that the mirror of van Geemen-type algebraic cycles is not expected to be real means that the corresponding enumerative invariants are strictly speaking not well-defined. We view our success in finding integral invariants also in these cases as further evidence for a broader applicability of this general strategy.

7.2. One-Loop Invariants of the van Geemen Lines

Starting with the mirror quintic, we consider the van Geemen lines with superpotential (6.11) from which we derive their infinitesimal invariant

$$\mathfrak{D} = \partial_t^2\mathcal{T}_A = \sqrt{-3}\left(140000\,q + \frac{44592400000}{3}\,q^2 + \frac{5015947794500000}{3}\,q^3 + \cdots\right). \tag{7.7}$$

Integrating the extended holomorphic anomaly equation for the annulus amplitude yields

$$\int\left(\frac{\mathfrak{D}^2}{\mathfrak{C}}\right)\frac{dq}{2\pi i\, q} = \frac{1}{2\pi i}\left(-5880000000\,q^2 - \frac{2490412400000000}{3}\,q^3 - \cdots\right), \tag{7.8}$$

which does not satisfy the desired Ooguri-Vafa integrality without correctly specifying the holomorphic ambiguity. Based on examples from [Wal2], it is expected that the holomorphic ambiguity exhibits a singular behavior at the singularity locus of the inhomogeneity (6.10), and we find experimentally that

$$\begin{aligned} f^{(0,2)} &= -\frac{1}{2\pi i}\frac{27}{8}\log\left(1-5^5\frac{128}{3}\,q\right) \\ &= \frac{1}{2\pi i}\left(450000\,q + 29653500000\,q^2 + \frac{7861631558750000q^3}{3} + \cdots\right) \end{aligned} \tag{7.9}$$

is the minimal choice which does the job. This means that we interpret the sum

$$\begin{aligned}\mathcal{A} &= \int \left(\frac{\mathfrak{D}^2}{\mathfrak{C}}\right) \frac{dq}{2\pi i\, q} + f^{(0,2)} \\ &= \frac{1}{2\pi i}\left(450000\, q + 23773500000\, q^2 + 1790406386250000\, q^3 + \cdots\right)\end{aligned} \tag{7.10}$$

as the relevant annulus amplitude. While its coefficients are generally rational, it satisfies the 1-function property that we checked up to degree $d \sim 50$. We note that one way to verify Ooguri-Vafa integrality in this case is to assess whether the coefficients of the exponentiated power series

$$\exp\left[2\pi i(\mathcal{A}+f^{(0,2)})\right] = 1+450000\, q+125023500000\, q^2+27675981386250000\, q^3+\cdots \tag{7.11}$$

are integral, see [SVW2, Propositon 7]. As described above, tadpole cancellation requires the inclusion of the Klein bottle amplitude, which in case of the mirror quintic and in our conventions is given by

$$\begin{aligned}\mathcal{K} &= -\frac{1}{2\pi i}\, 4 \log\left[\left(\frac{q}{z}\frac{dz}{dq}\right)\left(1-5^5 z\right)^{-\frac{1}{4}}\right] \\ &= \frac{1}{2\pi i}\left(-45\, q - \frac{582725}{2}\, q^2 - 590044750 q^3 - \cdots\right).\end{aligned} \tag{7.12}$$

The sum of the annulus and Klein bottle amplitudes

$$\mathcal{A}+\mathcal{K} = \frac{1}{2\pi i}\left(449955\, q + \frac{47546417275}{2}\, q^2 + 1790405796205250\, q^3 + \cdots\right) \tag{7.13}$$

again has rational coefficients $\widetilde{n}_d^{(0,2)} \in \mathbb{Q}$ but admits a resummation of the type (7.5) for which $n_d^{(0,2)} \in \mathbb{Z}$ are expected to have a proper interpretation as counting BPS states. The results in low degrees for both the annulus amplitude and the proper BPS counts are given in Table C.1 and Table C.2. As the overall irrational prefactor gets resolved, one could speculate that these invariants admit a standard enumerative interpretation, as is the case for cycles defined over $\mathbb{Q}$.

7.3. One-Loop Invariants of lines on the Mirror Octic

Turning to the next examples, we consider the one-parameter family of varieties defined by

$$Y_z = \left\{W = x_1^8 + x_2^8 + x_3^8 + x_4^8 + 4x_5^2 - \psi\, x_1 x_2 x_3 x_4 x_5 = 0\right\} \subset \mathbb{WP}^4_{1,1,1,1,4} \tag{7.14}$$

in weighted projective space. We choose $z = (4\psi)^{-8}$ as local coordinate and form the quotient with respect to the group of symmetries leaving the polynomial W invariant. This procedure yields a one-parameter family of Calabi-Yau threefolds $\mathcal{Y} \to \Delta^*$ with Hodge numbers of mirror quintic type (3.15), called the mirror octic, see e.g. [KW]. The mirror octic family together with the special fiber at $\psi = 0$

$$X = \left\{x_1^8 + x_2^8 + x_3^8 + x_4^8 + 4x_5^2 = 0\right\} \subset \mathbb{WP}^4_{1,1,1,1,4}, \tag{7.15}$$

thought of as a symplectic manifold, form a mirror pair whose treatment is very similar to the standard example of [CdlOGP] as both complex and symplectic structure moduli are one dimensional. The Picard-Fuchs operator is given by

$$D_{\mathrm{PF}}\left(-\right) = (2\pi i)^4 \left[\theta^4 - 2^{16} z \left(\theta + \frac{1}{8}\right)\left(\theta + \frac{3}{8}\right)\left(\theta + \frac{5}{8}\right)\left(\theta + \frac{7}{8}\right)\right]\left(-\right) \tag{7.16}$$

and leads to a Yukawa-coupling

$$\mathfrak{C} = \frac{\kappa}{(1-\alpha z)\,\varpi_0^2}\left(\frac{q}{z}\frac{dz}{dq}\right)^3 = 2 + 29504\,q + 1030708800\,q^2 + 38440454795264\,q^3 + \cdots\,, \tag{7.17}$$

where the conifold locus is specified by $\alpha = 2^{16}$ and the classical contribution corresponds to the triple-intersection number

$$\kappa = \frac{1}{\prod_{i=1}^n \nu_i}\int_X H \cup H \cup H = \frac{8}{4} = 2 \tag{7.18}$$

that has to be correctly weighted by $(\nu_0, \nu_1, \nu_2, \nu_3, \nu_4) = (1,1,1,1,4)$, cf. [KT]. There are two families of lines in the mirror octic that can be viewed as analogous to the van Geemen lines [JW]. The first is governed by an inhomogeneous Picard-Fuchs equation with inhomogeneity

$$\mathcal{J}_1(z) = \sqrt{-7}\cdot\frac{147}{16}\cdot\frac{-823543 + 184534\,\psi^8 + 129\,\psi^{16}}{\psi^4(\psi^8 - 2401)^{5/2}}, \tag{7.19}$$

from which we can derive the infinitesimal invariant

$$\mathfrak{D} = \sqrt{-7}\left(77672448\,q + 9459686780092416\,q^2 + \cdots\right). \tag{7.20}$$

Using the extended holomorphic anomaly equation (7.3) we find for the annulus amplitude

$$\mathcal{A} = \frac{1}{2\pi i}\left(-10557766062047232\,q^2 - 1714329236777071559049216\,q^3 + \cdots\right), \tag{7.21}$$

which turns out to be a 1-function in itself, setting $f^{(0,2)} = 0$. The corresponding enumerative invariants are collected in Table C.3. Again, a satisfactory BPS interpretation is only expected after including Klein bottle contributions given by the formula (7.12), which in case of the mirror octic amounts to

$$\mathcal{K} = \frac{1}{2\pi i}\left(-2304\,q - 183464704\,q^2 - 6942319837184\,q^3 - \cdots\right). \tag{7.22}$$

The combined amplitude

$$\mathcal{A} + \mathcal{K} = \frac{1}{2\pi i}\left(-2304\, q - 10557766245511936\, q^2 - \cdots\right) \tag{7.23}$$

then admits a multicover resummation whose low degree coefficients can be found in Table C.4. As the resulting invariants turn out to be negative, we collected their absolute values for the purpose of readability. This is likely a consequence of a missing factor of normalization, which can only be fixed by a more direct comparison with the (currently unknown) A-model geometry.

The second van Geemen-like inhomogeneity takes the form

$$\mathcal{J}_2(z) = \sqrt{-3}\cdot\frac{3}{16}\cdot\frac{\psi(8+\psi^2)}{(\psi^2-7)^{5/2}} \tag{7.24}$$

and leads to the infinitesimal invariant

$$\mathfrak{D} = \sqrt{-3}\left(48\, q^{1/4} + 4896\, q^{1/2} + 483840\, q^{3/4} + 48733440\, q + \cdots\right). \tag{7.25}$$

We note that unlike the previous examples, the superpotential associated to this cycle exhibits a branching behavior, such that we have to consider an appropriate covering $q^{1/4} \mapsto q$ in the large volume limit. The annulus amplitude can again be defined with vanishing holomorphic ambiguity and is given by

$$\mathcal{A} = \frac{1}{2\pi i}\left(-6912\, q^{1/2} - 940032\, q^{3/4} - 105629184\, q - \cdots\right), \tag{7.26}$$

where the associated (negative) invariants are given in Table C.5. The BPS invariants of the combined amplitude

$$\mathcal{A} + \mathcal{K} = \frac{1}{2\pi i}\left(-6912\, q^{1/2} - 940032\, q^{3/4} - 105631488\, q - \cdots\right) \tag{7.27}$$

can be found in Table C.6. As a consequence of the covering, the growth of the invariants is slower than for the previous examples. It is also worth noting that the contributions by non-orientable surfaces in this case only arise in degrees that are multiples of four.

We stress that the computations presented in this section have to be viewed as preliminary case studies, whose final assessments depend upon a further comparison with A-model investigations. For example, even though we demonstrated that the extended holomorphic anomaly equations can in principle be used to produce integral invariants, their explicit form might require a further normalization that is not obtainable from a B-model perspective at this point. Also, the lack of a precise definition of higher genus open Gromov-Witten invariants, and in particular of the associated integral BPS numbers, necessitates further consistency checks between the A- and B-model geometries. We therefore suggest to revisit this subject as soon as canditate mirrors of the van Geemem lines and its cousins are available. An outlook on further examples is given in Chapter 8.

8. Summary and Outlook

The aim of this work was to derive implications of the extended mirror principle [Wal1] for the geometry and topology of A-branes, in order to develop a better Hodge theoretic understanding of the phenomenon built upon the A-model VMHS [Mor3]. As a first step, we reviewed the physical concepts and the relevant Hodge theory behind closed string mirror symmetry for one-parameter Calabi-Yau threefolds, with a particular emphasis on the quantum structures that underly the topological A-model.

In regard to B-branes, we collected what is known about their relation to Hodge theoretic normal functions, when the corresponding extension class is produced by the presence of an algebraic cycle [MW]. We described how the B-brane superpotential, in terms of the infinitesimal invariant of a certain canonical lift, fully determines the extension in a degeneration limit with maximally unipotent monodromy. The expansion of the superpotential around this singularity was characterized using monodromy considerations, which identifies a certain constant term as the limiting value of the relevant Abel-Jacobi map [GGK2].

Prior studies established that the geometric derivation of the normal function could not be imitated directly in the A-model, because the real locus of an antiholomorphic involution is typically not homologically trivial [DK2]. For this reason, we proceeded to carefully analyze the structure of A-brane superpotentials associated to Lagrangians with favorable topological properties. In particular, we explained the relevance and existence of a homologically equivalent Lagrangian sphere, with respect to which the superpotential has to be measured in order to handle a certain tadpole anomaly arising in open Gromov-Witten theory. In combination with the Solomon-Tukachinsky axioms [ST1, ST2], this enabled us to formulate an open Gromov-Witten potential in very close analogy to its absolute counterpart. We explained how the gauge bundle carried by an A-brane furthermore contributes additional classical terms, which are especially important in understanding the main example of the real quintic.

Equipped with this more systematic understanding of the geometric data that determines the superpotential, we were able to describe a construction based on which the extension class can be produced independently in the A-model. The crucial insight consisted of the fact that including a Lagrangian sphere as a fixed reference cycle allows to geometrically define the expected normal function in a fashion similar to the B-model. A flat connection acting on the extension was constructed from an extension of the quantum cup product by holomorphic disks, leading to an identification of the infinitesimal invariant with the open string two-point function and producing the open WDVV equations [Alc, ST3, CZ] as associativity constraints. While this normal function of the A-model VMHS correctly captures the topological contributions to the superpotential, a modification is required to also introduce the classical terms pertaining to gauge bundle data. The fact that these correspond to a certain constant of integration from the point of view of the quantum differential equation defined by the extended quantum product, led us to suspect that they should arise from a relative version of the Gamma class [Iri, KKP]. Indeed, based on the physical expectation that the Abel-Jacobi limit should correspond to the Chern-Simons invariant in certain examples (satisfying Assumptions 5.5), we were able to construct an extension of the Gamma integral local system build from the bundle data, leading to Conjecture 6.8. We demonstrated the validity of the conjecture for the main example of the extended mirror principle [Wal1], by formulating a Mirror Theorem (see Theorem 6.12) that rephrases the duality as an isomorphism between extensions of variations of MHS.

Our construction of the relative Gamma class does not immediately apply beyond Assumptions 5.5, which excludes algebraic cycles like the van Geemen lines at this point. We demonstrated how the extended holomorphic anomaly equations of [Wal3] can be employed to compute invariants for the mirror A-model also in these cases, which might help in further studying these arithmetically interesting examples in the future.

Our work produced new insights regarding future research directions. An inexhausitive list is presented in the following.

Beyond Spheres

Cycles with an underlying algebraic number field are conjectured to be mirrored by Lagrangians that do not satisfy Assumptions 5.5, and it is unclear how to adapt the construction of the relative Gamma class to these examples whose geometry is at present unknown. The crucial missing piece is the analog of the Abel-Jacobi limit that in these cases bears interesting arithmetic features. While its B-model origin is explained in [JMW], the index calculation of Appendix B might hint at a realization in the A-model. Namely, the η-invariant is known to have a close relation to special values of L-functions that generally arise in corrections to the signature theorem for manifolds with boundary [ADS]. It would be interesting to investigate potential relations with the Abel-Jacobi limit in arithmetically interesting cases, like the van Geemen lines (Example 6.2) by studying the index theory of candidate mirrors. In case of hyperbolic submanifolds, the index of the Dirac operator (B.8) might receive further corrections corresponding to a nontrivial space of harmonic spinors.

Extending the Mukai Pairing

The way in which we defined the relative Gamma class is not quite on equal footing with the construction for the closed string described in Section 3.5, because it does not include the underlying K-theory and potential extensions of the Mukai-pairing. Closing this gap likely involves a careful analysis of the algebraic K-theory of the complement $X \setminus L$ of the Lagrangian brane and would be necessary to give a precise definition of the A-model version of the Abel-Jacobi map sketched in (6.35). One difficulty consists of the fact that not all properties of the branes are captured by topology alone. For example in case of the real quintic, one should therefore rather consider the subcategory

$$\begin{tikzcd} \mathcal{L}_+ \arrow[loop left, "CF(\mathcal{L}_+,\mathcal{L}_+)"] \arrow[r, bend left, "CF(\mathcal{L}_+,\mathcal{L}_-)"] & \mathcal{L}_- \arrow[l, bend left, "CF(\mathcal{L}_-,\mathcal{L}_+)"] \arrow[loop right, "CF(\mathcal{L}_-,\mathcal{L}_-)"] \end{tikzcd} \tag{8.1}$$

of the full Fukaya category, where $\mathcal{L}_\pm = (L, \mathcal{E}_\pm)$ are Lagrangians with flat bundles as described in Section 6.5. A further interesting question in a similar direction is whether homological mirror symmetry can be seen to imply extended Hodge theoretic mirror symmetry in the sense of Definition 6.1, using the methods of [GPS], see also Remark 3.23.

Frob Modules and open WDVV equations

For extended mirror pairs, we identified quantum deformed Frob modules (cf. Definition 6.11), defined by the H^2-module structure coming from the extended quantum product, as mirroring the asymptotics of variations of MHS extended by an algebraic cycle. We expect that it is possible to proof a more general correspondence between these structures, using the methods of [CF1]. Furthermore, it would be interesting to study how quantum deformed Frob modules more broadly relate to Hodge theoretic objects in the B-model. While it is reasonable to expect there to be a general correspondence between Frob modules of any rank with variations of MHS of the type discussed in Remark 4.3, one should expect the case of higher weight to carry additional structure, similar to higher weight Frobenius modules [CF2, FP]. As the higher weight situation is geometrically not directly associated with Calabi-Yau threefolds, boundary punctures in disk invariants do play a role in this context. We expect the relevant associativity equations to be the more general open WDVV equations discussed in [Alc, ST3, CZ]. For example, boundary punctures arise in this general treatment in the associativity constraints coming from multiplying extending generators. These are absent in our treatment due to the decoupling of the open and closed string sectors described in Section 2.3. We also expect there to be some correspondence with the structure induced from the relative quantum product מ of [ST3].

Loop-Amplitudes and Number Fields

Beside the van Geemen lines and the cycles inside of the mirror octic discussed in Section 7.3, there are further families of algebraic cycles described for the quintic [Wal4] and decimic hypersurfaces [JW] with an underlying algebraic number field. We attempted to perform a calculation similar to the one presented in Chapter 7 for all of them, but did not succeed in finding the choice of holomorphic ambiguity that makes the annulus amplitude into a 1-function. One possible reason is that in all of these examples the parameters of the family are constrained by polynomials which require an asymptotic solution at the MUM-point, and a corresponding expansion of the associated inhomogeneity. A better understanding of the resulting singularity structure might be necessary to find the right boundary conditions for the holomorphic ambiguity. Another difficulty is related to the fact that in those examples the annulus amplitude[1], for example

$$\mathcal{A} = -28814401800\, q + 463194713736000\sqrt{114}\, q^{3/2} - \cdots\,, \tag{8.2}$$

is not guaranteed to have only rational coefficients (here they lie in $\mathbb{Q}(\sqrt{114})$), leading to a significant increase in computational demand for the calculation. An improved understanding of the one-loop amplitudes could also be acquired by studying potential extensions of their mathematically precise definition in terms of Ray-Singer torsions described in [FLY].

[1] This example is computed from a cycle in the mirror decimic with inhomogeneity [JW, Equation 2.32].

A. Rigidity of A- and B-branes

The main examples of branes that are considered in this thesis are rigid in the sense that they admit no open string deformation parameters. One reason is that in the A-model there are no examples in which a Lagrangian in a Calabi-Yau threefold can be deformed independently of bulk Kähler transformations (cf. Assumptions 5.5).

In the A-model, rigidity reduces to a purely topological condition that is captured by McLean's theorem [McL].

Theorem A.1. Let X be a Calabi-Yau manifold and $L \subset X$ be a compact special Lagrangian submanifold. Then the moduli space of special Lagrangian deformations of L inside of X is of dimension $\dim H^1(L;\mathbb{R})$.

On general grounds, infinitesimal deformations of L are given by sections $H^0(L;N_{L|X})$ of the normal bundle. Roughly speaking, McLean's theorem states that those deformations that are identified with harmonic one-forms via the symplectic form respect the special Lagrangian condition. In most known examples, the A-model geometry satisfies Assumptions 5.5 and rigidity follows from the fact that the Lagrangian is a rational homology sphere. In the particular case of the real quintic and its analogs [KW], it is a consequence of $H^1(\mathbb{RP}^3) = 0$.

The Deligne conics Example 4.2 enter into Theorem 6.12 as the mirror B-model configuration, where based on physical principles a similar rigidity is expected. While this is in line with the Clemens conjecture, which states that any curve in a quintic should be generically isolated, there are examples (like the van Geemen lines) in which they vary in families. This appendix provides a brief computation based on the strategy of [Kat] that shows that the Deligne conics indeed cannot be independently deformed from the bulk complex structure parameters of the mirror quintic.

Given a Calabi-Yau threefold Y on the B-model side, open string excitations orthogonal to a cycle C again correspond to sections of the normal bundle $N_{C|Y}$. These sections in $H^0(Y;N_{C|Y})$ geometrically correspond to infinitesimal deformations of C inside of Y, leaving the complex structure of Y fixed. It might be the case that a given infinitesimal deformation cannot be integrated to a finite one, where the corresponding obstruction is measured by higher sheaf cohomology groups. Here, the normal bundle is defined via

$$0 \longrightarrow T_C \longrightarrow T_Y\big|_C \longrightarrow N_{C|Y} \longrightarrow 0. \tag{A.1}$$

In case of $C \cong \mathbb{P}^1$ the first Chern class of the tangent bundle T_C corresponds to the Euler characteristic of $\mathbb{P}^1$, i.e. $c_1(T_C) = 2$. The normal bundle $N_{C|Y}$ has degree two and by the additivity of Chern classes and the Calabi-Yau condition $c_1(T_X) = 0$ it follows that $N_{C|Y}$ is generally of the form

$$N_{C|Y} = \mathcal{O}(-1+n) \oplus \mathcal{O}(-1-n), \tag{A.2}$$

such that $c_1(N_{C|Y}) = -2$. As the bundle $\mathcal{O}(k)$ has $k+1$ holomorphic sections, (A.1) means that the curve admits n infinitesimal deformation parameters. According to the Clemens conjecture, a generic curve is expected to have a normal bundle with $n = 0$, meaning that the associated moduli space is zero-dimensional and the curve is isolated.

We consider the conics $C_{\pm,\psi}$ defined by (4.27) inside of the variety

$$Y = \Big\{W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi\, x_1x_2x_3x_4x_5 = 0\Big\} \subset \mathbb{P}^4. \tag{A.3}$$

In view of the deformation problem it suffices to focus on (A.3), as the quotient by the Greene-Plesser group does not induce additional moduli. Following [Kat], we rewrite the polynomial in terms of the defining equations of the Deligne conics, i.e.

$$W = \left(x_5^2 \mp \sqrt{5\psi}\, x_1x_3\right) f + (x_1 + x_2)\, g_{1,2} + (x_3 + x_4)\, g_{3,4}, \tag{A.4}$$

where f is a cubic and $g_{1,2}$, $g_{3,4}$ are quartics. It is a general consequence of Hilbert's Nullstellensatz that such a decomposition can always be found. Explicitly, the relevant polynomials are given by

$$\begin{aligned} f = x_5^3 \pm \sqrt{5\psi}\, x_1x_3x_5, \quad g_{1,2} &= \sum_{k=0}^{4} (-1)^k x_1^k x_2^{4-k} + \frac{5}{2}\psi\, x_1x_3x_5(x_3 - x_4), \\ g_{3,4} &= \sum_{k=0}^{4} (-1)^k x_3^k x_4^{4-k} + \frac{5}{2}\psi\, x_1x_3x_5(x_1 - x_2). \end{aligned} \tag{A.5}$$

From [Kat] we know that the corresponding cycle is rigid, whenever there are no non-trivial relations among the polynomials f, $g_{1,2}$ and $g_{3,4}$. Consider the polynomial ring $R = \mathbb{C}[x_1, \ldots, x_5]$ defined by the homogeneous coordinates on $\mathbb{P}^4$ and let $I = \langle f, g_{1,2}, g_{3,4}\rangle \subset R$ be the ideal generated by the polynomials. Polynomial relations among the generators of I as well as higher syzygies are encoded in a free resolution

$$\cdots \xrightarrow{s_{i+2}} F_{i+1} \xrightarrow{s_{i+1}} F_i \xrightarrow{s_i} \cdots \xrightarrow{s_2} F_1 \xrightarrow{s_1} F_0 \xrightarrow{s_0} I, \tag{A.6}$$

where each F_i is a finitely generated R-module and $s_{i+1} \circ s_i = 0$ defines a differential. An algorithm which determines a minimal free resolution via Gröbner basis methods is implemented in the software `Macaulay2` [GS] and produces the resolution

$$0 \xrightarrow{s_3} R^1 \xrightarrow{s_2} R^3 \xrightarrow{s_1} R^3 \xrightarrow{s_0} I. \tag{A.7}$$

The map $s_0 : R^3 \longrightarrow I$ is simply the matrix given by the generators f, $g_{1,2}$ and $g_{3,4}$. Their relations are encoded in $s_1 : R^3 \to R^3$, which is given by the matrix

$$\begin{pmatrix} -g_{1,2} & f & 0 \\ -g_{3,4} & 0 & f \\ 0 & -g_{3,4} & g_{1,2} \end{pmatrix}. \tag{A.8}$$

The differential condition $s_1 \circ s_0 = 0$ corresponds to

$$\begin{pmatrix} -g_{1,2} & f & 0 \\ -g_{3,4} & 0 & f \\ 0 & -g_{3,4} & g_{1,2} \end{pmatrix} \begin{pmatrix} f \\ g_{1,2} \\ g_{3,4} \end{pmatrix} = \begin{pmatrix} -g_{1,2}\, f + f\, g_{1,2} \\ -g_{3,4}\, f + f\, g_{3,4} \\ -g_{3,4}\, g_{1,2} + g_{1,2}\, g_{3,4} \end{pmatrix} = 0 \tag{A.9}$$

and shows that the only relations among these polynomials are trivial in the sense that they are automatically satisfied as a consequence of commutativity. The normal bundle to the Deligne conics in Y is therefore given by (A.2) with $n = 0$ such that

$$H^0(C; N_{C/Y}) = H^0\left(\mathcal{O}(-1) \oplus \mathcal{O}(-1)\right) = 0, \tag{A.10}$$

meaning that the conics do not admit infinitesimal deformation parameters.

B. Chern-Simons Invariants of Lens Spaces

We review the computation of Chern-Simons invariants of flat U(1)-bundles on three-dimensional lens spaces given in [Nis]. These spaces are given by quotients $L = S^3/G$, where $G = \mathbb{Z}/n\mathbb{Z}$ is a cyclical group, and the flat U(1)-bundles $\mathcal{E}_k$ on L, $k = 1, \dots, n$, are classified by their first Chern class

$$c_1(\mathcal{E}_k) \in H^2(L;\mathbb{Z}) \cong {}^{\mathbb{Z}}\!/\!_{n\mathbb{Z}}. \tag{B.1}$$

Following Definition 5.9, let Γ_4 be a 4-manifold with boundary $\partial\Gamma_4 = L$. By Proposition 5.10 and Remark 5.11 we can choose Γ_4 such that it admits a spin structure and every bundle $\mathcal{E}_k$ can be extended to a bundle $\widetilde{\mathcal{E}}_k$ on Γ_4. As Γ_4 is even dimensional, the spin bundle is of the form $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$ and there are twisted chiral Dirac operators

$$\begin{tikzcd} C^\infty(\Gamma_4, \mathcal{E}_k \otimes \mathcal{S}_+) \arrow[r, bend left, "D_k^+"] & C^\infty(\Gamma_4, \mathcal{E}_k^* \otimes \mathcal{S}_-). \arrow[l, bend left, "D_k^-"] \end{tikzcd} \tag{B.2}$$

The Chern-Simons invariant $cs(\mathcal{E})$ mod $\mathbb{Z}$ can be extracted from the Fredholm index associated to the operator D_k^+, using the Atiyah-Patodi-Singer index theorem [APS]. Recall that in case of a compact 4-manifold M with spin structure and flat bundle $\mathcal{E}$, the index theorem gives a formula for the Fredholm index of the twisted chiral Dirac operator D^+ as an integral over characteristic classes on M

$$\mathrm{ind}(D^+) = \int_M ch(\mathcal{E})\, \hat{A}(M), \tag{B.3}$$

where $\hat{A}(M)$ denotes the $\hat{A}$-genus. In the given case it can be expressed by

$$\hat{A}(M) = 1 - \frac{1}{24} p_1(M), \qquad p_1(M) \in H^4(M;\mathbb{Z}), \tag{B.4}$$

in terms of the first Pontryagin class of the tangent bundle of M. In case of a 4-manifold with boundary, the usual $\hat{A}$-genus is corrected by boundary contributions which are related to the space of harmonic spinors and the η-invariant of the Dirac operator restricted to L [*op.cit.*]. As the positively curved lens spaces admit no harmonic spinors by a vanishing theorem of Lichnerowicz [Lic], the boundary term essentially corresponds to the η-invariant. It is generally defined in terms of the eta series

$$\eta(s) = \sum_{\lambda \neq 0} \frac{\operatorname{sgn}(\lambda)}{|\lambda|^s}, \tag{B.5}$$

where λ denotes the eigenvalues of the Dirac operator. The *eta invariant* is defined as the special value $\eta(0)$. While an explicit calculation of the eta invariant can be very involved and generally involves analytic continuation, in the context at hand there is a description purely in terms of the representation theory of the fundamental group. Namely, it can be calculated from the representations of the fundamental group $\pi_1(S^3/G) = G$ in the gauge group

$$\rho_k : G \longrightarrow \mathrm{U}(1), \qquad 1 \longmapsto \exp\left(\frac{2\pi i k}{n}\right) \tag{B.6}$$

and the corresponding embedding in SU(2)

$$\sigma_k : G \longrightarrow \mathrm{SU}(2), \qquad 1 \longmapsto \begin{pmatrix} \exp\left(\frac{2\pi i k}{n}\right) & 0 \\ 0 & \exp\left(-\frac{2\pi i k}{n}\right) \end{pmatrix}, \tag{B.7}$$

which can be associated to each flat bundle $\mathcal{E}_k$. We denote by $\chi_k : G \to \mathbb{C}$ and $\psi_k : G \to \mathbb{C}$ the respective characters. The Fredholm index of D_k^+ is now given by the Kronheimer-Nakajima formula [KN]

$$\begin{aligned} \operatorname{ind}(D_k^+) &= \int_{\Gamma_4} ch(\widetilde{\mathcal{E}}_k)\, \widehat{A}(\Gamma_4) - \frac{\eta(D_k^+|_L)}{2} \\ &= -cs(\mathcal{E}_k) - \frac{1}{24}\int_{\Gamma_4} p_1(\Gamma_4) - \frac{1}{|G|}\sum_{g\neq e} \frac{\chi_k(g)}{\psi_k(g) - 2} \qquad \in \mathbb{Z}, \end{aligned} \tag{B.8}$$

where again $\widehat{A}(\Gamma_4) = 1 - \frac{1}{24} p_1(\Gamma_4)$ and we used Definition 5.9. For the globally trivial flat connection $\widetilde{\mathcal{E}}_0$ we have $cs(\mathcal{E}_0) = 0$ and $\chi_0(g) = 0$ for every $g \neq e \in G$. Noting that $\operatorname{ind}(D_k^+) - \operatorname{ind}(D_0^+) \in \mathbb{Z}$, identifies the Chern-Simons invariant up to integers with the expression

$$cs(\mathcal{E}_k) = \frac{1}{|G|}\sum_{g\neq e}\frac{\chi_k(g)}{\psi_k(g)-2} = \frac{1}{n}\sum_{l=1}^{n-1} \frac{\exp\left(\frac{2\pi i k l}{n}\right) - 1}{\exp\left(\frac{2\pi i l}{n}\right) + \exp\left(\frac{-2\pi i l}{n}\right) - 2} \quad \operatorname{mod} \mathbb{Z}. \tag{B.9}$$

While it is now easy to explicitly evaluate the Chern-Simons invariant mod $\mathbb{Z}$, there is a residue calculation by which this expression can be substantially simplified [Nis]. We denote by $\mu_l = \exp(2\pi i l/n)$ the non-trivial roots of unity with $l = 1, \ldots, n-1$ and consider the meromorphic function

$$f_k(z) = \frac{z^k - 1}{(z-1)^2(z^n - 1)} \tag{B.10}$$

on $\mathbb{P}^1$ with poles at μ_l for each l. From

$$\mathrm{Res}_{z=1} f_k(z) = \frac{k^2}{2n} - \frac{k}{2}, \qquad \mathrm{Res}_{z=\mu_l} f_k(z) = \frac{1}{n} \frac{\mu_l^k - 1}{\mu_l + \mu_l^{-1} - 2}, \tag{B.11}$$

the residue theorem gives

$$cs(\mathcal{E}_k) = \sum_{l=1}^{n-1} \mathrm{Res}_{z=\mu_l} f_k(z) = -\mathrm{Res}_{z=1} f_k(z) = -\frac{k^2}{2n} + \frac{k}{2} \mod \mathbb{Z}, \tag{B.12}$$

where we chose in integration contour of a sufficiently large radius that can be contracted in $\mathbb{P}^1$. In the particular case of the real quintic considered in Theorem 6.12, the Chern-Simons invariants of the bundles $\mathcal{E}_\pm$ are therefore given by

$$cs(\mathcal{E}_+) = 0, \qquad cs(\mathcal{E}_-) = \frac{1}{4}. \tag{B.13}$$

C. One-Loop Invariants for Cycles of van Geemen-type

d	Annulus invariant
1	450000
2	23773275000
3	1790406386100000
4	157530934723763250000
5	15139055443028646937410000
6	1540089817611589084662813150000
7	163029163159364675037035977110150000
8	17772125517155222253293432782661415000000
9	1981616814874464036487469614902857258565000000
10	224944212508279700687674585930876820793357401595000
11	25908706334188064501645668553106040120500364200408150000
12	3020292465269009710631261730042826767452054933172473928900000

Table C.1.: Annulus invariants for van Geemen lines. The table depicts integral invariants for the van Geemen lines in the mirror quintic with inhomogeneity (6.10). They are obtained from the extended holomorphic anomaly equation for the annulus amplitude, as described in Chapter 7.

d	one-loop BPS invariant $n_d^{(0,2)}$
1	449955
2	23772983660
3	1790405796055265
4	157530933661020547400
5	15139055441136207256023089
6	1540089817608204537427570779960
7	163029163159358576752736946561620460
8	17772125517155211182491689478757233167000
9	1981616814874464016251412276822502155708670750
10	224944212508279700650460886682337646351225750144428
11	25908706334188064501576869007764916105042871515577505435
12	3020292465269009710631133942033864603435520830764435130938760

Table C.2.: One-loop BPS invariants for van Geemen lines. The table depicts integral invariants for the van Geemen lines in the mirror quintic with inhomogeneity (6.10). They are obtained from the extended holomorphic anomaly equation for the annulus amplitude, together with non-orientable contributions, as described in Chapter 7.

d	Annulus invariant
1	0
2	10557766062047232
3	1714329236777071559049216
4	248576380862536666243259726561280
5	35438330778327715998516711049248248954880
6	5059047452269582001119102756478593734671566110720
7	726308381964591879055617937791541589476890442732155174912
8	104944004754637010754837870497062978799491671793258655364962844672
9	15256596300456643023673821115480876066666283949440296163036869630120427520
10	2230375252837817537985315467273540067180440548379548755871630735490804880066478080
11	327686862616785753903522137627099143462308733377964353963274363171428041632568118148595712
12	48357042047270387451903621581591509062663235272451176169068226684840139178058042887096244918812672

Table C.3.: Annulus invariants for van Geemen-type lines. The table depicts integral invariants for the van Geemen-type lines in the mirror octic with inhomogeneity (7.19). They are obtained from the extended holomorphic anomaly equation for the annulus amplitude, as described in Chapter 7.

d	one-loop BPS invariant $-n_d^{(0,2)}$
1	2304
2	10557766245510784
3	1714329236784013878885632
4	248576380862536921081938624145792
5	35438330778327716007959092503079754578688
6	5059047452269582001119457290462371759902336774272
7	726308381964591879055617951265369340849048109489053536512
8	104944004754637010754837870497580339800155481223516528740743090560
9	15256596300456643023673821115480896104521664258371178485836150332264142848
10	2230375252837817537985315467273540067962202719827365836669076310658349316121844608
11	327686862616785753903522137627099143462339422853450571799346530093307386868387247326971136
12	48357042047270387451903621581591509062663236483645538231579500120207823938472029597654580760881024

Table C.4.: One-loop BPS invariants for van Geemen-type lines. The table depicts integral invariants for the van Geemen-type lines in the mirror octic with inhomogeneity (7.19). They are obtained from the extended holomorphic anomaly equation for the annulus amplitude, together with non-orientable contributions, as described in Chapter 7.

d	Annulus invariant
1	0
2	6912
3	940032
4	105625728
5	11299405824
6	1190560230144
7	124945591468032
8	13116289936022784
9	1379614459875047424
10	145491878436969500928
11	15386158358510645329920
12	1631613771604970173215360
13	173478387324471102664998912
14	18490277345840216879312892672
15	1975315723596705506113943488512
16	211471178237844288917751141949440
17	22683918279826037646217285551783936
18	2437659428057500570605654329223613440
19	262396070372666060201611930650073325568
20	28288995592279571924821670816282430069120
21	3054253906132364039414947065170045291501568
22	330199333030782329189853396938684939718651648
23	35742985406317466563464388413304988519122993152
24	3873582888899995421446800054463239250913834025216

Table C.5.: Annulus invariants for van Geemen-type lines. The table depicts integral invariants for the van Geemen-type lines in the mirror octic with inhomogeneity (7.24). They are obtained from the extended holomorphic anomaly equation for the annulus amplitude, as described in Chapter 7.

d	one-loop BPS invariant $-n_d^{(0,2)}$
1	0
2	6912
3	940032
4	105628032
5	11299405824
6	1190560230144
7	124945591468032
8	13116290119486336
9	1379614459875047424
10	145491878436969500928
11	15386158358510645329920
12	1631613771611912493051776
13	173478387324471102664998912
14	18490277345840216879312892672
15	1975315723596705506113943488512
16	211471178237844543756430039533952
17	22683918279826037646217285551783936
18	2437659428057500570605654329223613440
19	262396070372666060201611930650073325568
20	28288995592279571934264052270113935692928
21	3054253906132364039414947065170045291501568
22	330199333030782329189853396938684939718651648
23	35742985406317466563464388413304988519122993152
24	3873582888899995421447154588447017276144604688768

Table C.6.: One-loop BPS invariants for van Geemen-type lines. The table depicts integral invariants for the van Geemen-type lines in the mirror octic with inhomogeneity (7.24). They are obtained from the extended holomorphic anomaly equation for the annulus amplitude, together with non-orientable contributions, as described in Chapter 7.

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