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On the Hofer–Zehnder Capacity of Twisted Tangent Bundles

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Abstract

In this thesis, we deal with the Hofer–Zehnder capacity of disc subbundles of twisted tangent bundles. While in the literature for most cases only the finiteness of this capacity is shown, we use symmetries to determine exact values of the capacity. We therefore restrict ourselves to a class of homogeneous Kähler manifolds, called Hermitian symmetric spaces. For these, we construct a symplectomorphism that identifies the twisted tangent bundle, or at least a neighborhood of the zero section that we can specify explicitly, and the Hermitian tangent bundle. The advantage of the Hermitian tangent bundle is that the fibers are symplectic. This makes it easier to study holomorphic curves, which we use to obtain an upper bound on the Hofer–Zehnder capacity. We get the lower bound by specifying a Hamiltonian that generates a circle action. The oscillation of such a Hamiltonian always yields a lower bound.
We also clarify the relationship between the twisted, respectively Hermitian, symplectic structure to the hyperkähler structure in a neighborhood of the zero section of the tangent bundle of a Hermitian symmetric space.

For various reasons, it is much harder to determine the Hofer–Zehnder capacity for standard tangential bundles than it is for the twisted case. Nevertheless, we were able to compute the Hofer–Zehnder capacity for the disc subbundle of the standard tangent bundle of the complex projective space $\mathbb{C}\mathrm{P}^n$ and the real projective space $\mathbb{R}\mathrm{P}^n$. To obtain the lower bound it is for the former sufficient to consider the kinetic Hamiltonian, i.e. geodesic flow, while in the second case, geodesic billiards must be used. For the upper bound one uses the symmetries of the spaces to show that the disc subbundle of the tangent bundles compactify to the product of two complex projective spaces $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n$ in the first case and the complex projective space $\mathbb{C}\mathrm{P}^n$ in the second case. In these compact symplectic manifolds one can again study holomorphic spheres in order to construct upper bounds. In fact, we also show in the twisted case that the disc subbundle of the tangent bundle of the complex projective space compactifies to the product, but now with differently weighted factors.

Furthermore, this thesis includes the computation of the Hofer–Zehnder capacity of Hermitian symmetric spaces of compact type. This exploits the fact that Hermitian symmetric spaces can be represented as coadjoint orbits. In this representation it is relatively easy to specify a Hamiltonian which generates a semi-free circle action and which attains its minimum at an isolated point. The oscillation of such a Hamiltonian provides both lower and upper bounds and thus determines the capacity.

Zusammenfassung

In dieser Arbeit befassen wir uns mit der Hofer–Zehnder Kapazität von Scheiben-Unterbündeln getwisteter Einheits-Tangentialbündel. Während in der Literatur in den meisten Fällen nur die Endlichkeit dieser Kapazität gezeigt wird, nutzen wir Symmetrien aus um genaue Werte der Kapazität zu bestimmen. Wir beschränken uns daher auf eine Klasse von homogenen Kähler Mannigfaltigkeiten, die Hermitschen symmetrischen Räume. Für diese konstruieren wir einen Symplektomorphismus zwischen dem getwisteten Tangentialbündel, oder zumindest einer Umgebung des Nullschnitts die wir genau angeben können, und dem Hermitschen Tangentialbündel. Der Vorteil des Hermitschen Tangentialbündel ist, dass die Fasern symplektisch sind. Dies macht es einfacher holomorphe Kurven zu studieren, welche wir nutzen um eine obere Schranke für die Hofer–Zehnder Kapazität zu bekommen. Die untere Schranke erhalten wir, in dem wir einen Hamiltonian angeben, der eine S^1-Wirkung generiert. Die Oszillation eines solchen Hamiltonians liefert stets eine untere Schranke.
Wir klären außerdem die Beziehung der getwisteten und der Hermitschen symplektischen Struktur zu der hyperkähler Struktur in einer Umgebung des Nullschnitts des Tangentialbündles eines Hermitschen symmetrischen Raums.

Die Berechnung der Hofer–Zehnder Kapazität von standard Tangentialbündeln ist aus verschiedenen Gründen deutlich schwieriger, als für den getwisteten Fall. In dieser Arbeit konnte dennoch die Hofer–Zehnder Kapazität des Scheiben-Unterbündels des standard Tangentialbündels des komplex-projektiven Raums $\mathbb{C}\mathrm{P}^n$ und des reel-projektiven Raums $\mathbb{R}\mathrm{P}^n$ berechnet werden. Für ersteren genügt es als untere Schranke den kinetischen Hamiltonian, also geodätischen Fluss, zu betrachten, während im zweiten Fall geodätische Billards genutzt werden müssen. Für die obere Schranke nutzt man in beiden Fällen die Symmetrien der Räume um zu zeigen, dass die Scheiben-Unterbündel des Tangentialbündel zum Produkt zweier komplex-projektiver Räume $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n$ im ersten Fall und zum komplex-projektiven Raum $\mathbb{C}\mathrm{P}^n$ im zweiten Fall kompaktifizieren. In diesen kompakten Räumen kann man dann wieder holomorphe Sphären studieren um obere Schranken zu konstruieren. Tatsächlich zeigen wir auch im getwisteten Fall, dass das Scheiben-Unterbündel des Tangentialbündel des komplex projektiven Raums zum Produkt kompaktifiziert, jetzt allerdings mit unterschiedlich gewichteten Faktoren.

Desweiteren beinhaltet diese Arbeit die Berechnung der Hofer–Zehnder Kapazität kompakter Hermitscher symmetrischer Räume von kompaktem Typ. Diese nutzt aus, dass Hermitsche symmetrische Räume als koadjungierte Orbits dargestellt werden können. In dieser Darstellung lässt sich relativ einfach ein Hamiltonian angeben der eine halb freie S^1-Wirkung generiert und der sein Minimum an einem isolierten Punkt annimmt. Die Oszillation eines solchen Hamiltonians liefert sowohl untere als auch obere Schranke und bestimmt somit die Kapazität.

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Contents

1. Introduction

(Hofer–Zehnder) capacity:

Symplectic capacities are symplectic invariants that measure the size of a symplectic manifold (W, ω). They arose as obstructions for symplectic embeddings, first discovered by Gromov in 1985 with his famous non-squeezing theorem [22]. Two examples for symplectic capacities are the Gromov width c_G and the Hofer–Zehnder capacity c_{HZ}. We postpone their precise definitions to section 3.1, but roughly speaking the first is determined by the largest open ball $(B_R(0), \omega_0)$ one can symplectically embed into (W, ω) and the second is the number c so that every Hamiltonian with oscillation bigger than c has a fast[1] non-constant periodic orbit. The Gromov width takes a special position among capacities, as it is the smallest. In particular it yields a lower bound to all other symplectic capacities. It is therefore an interesting question to investigate whether other capacities agree with the Gromov width and when they differ. For example Gromov [22] resp. Hofer–Viterbo [26] determined

$$c_G(\mathbb{C}\mathrm{P}^n, \omega_{FS}) = \pi \quad \text{resp.} \quad c_{HZ}(\mathbb{C}\mathrm{P}^n, \omega_{FS}) = \pi,$$

while Gromov [22] (also [41, Ex. 12.1.6]) resp. Lu [37] found

$$c_G(\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n, \omega_{FS} \oplus \omega_{FS}) = \pi \quad \text{resp.} \quad c_{HZ}(\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n, \omega_{FS} \oplus \omega_{FS}) = 2\pi,$$

where ω_{FS} denotes the Fubini-Study form normalized such that on the generator of $H_2(\mathbb{C}\mathrm{P}^n, \mathbb{Z})$ it takes value π. Note here that capacities are in general very hard to compute and mainly known in examples, but knowledge of their values yield important information about the symplectic manifold. For example even finiteness of the Hofer–Zehnder capacity has drastic implications on the Hamiltonian dynamics possible on (W, ω). Indeed as proven by Hofer, Zehnder [27, Ch. 4.4, Thm.2] (and strengthened by Macarini, Schlenk [38]) finiteness implies existence of periodic orbits on almost all compact energy hypersurfaces.

A standard class of examples for symplectic manifolds are cotangent bundles $(T^*M, \mathrm{d}\lambda)$, where

$$\lambda : T^*M \to \mathbb{R}; \ (q, p) \mapsto p(q)$$

is the canonical one form. This class of symplectic manifolds is of special interest in physics as it describes the phase space of a constraint system with configuration space M. Fixing a Riemann metric g on M we can identify cotangent bundle and tangent bundle. We write $D_\rho M := \{(x, v) \in TM | \ g_x(v, v) < \rho^2\}$ for the open disc-subbundle of radius $\rho \in \mathbb{R}_{>0}$. Using the fact that the Hamiltonian vector field of the kinetic energy function

$$E : TM \to \mathbb{R} \quad (x, v) \mapsto \frac{1}{2} g_x(v, v)$$

[1] Fast means of period $T < 1$.

generates the geodesic flow, one can show that a lower bound for the Hofer–Zehnder capacity of $(D_1 M, \mathrm{d}\lambda)$ is the length of the shortest geodesic. An important result by Weber [49] shows that for closed M and a non-zero class $\nu \in \pi_1(M)$, the Hofer-Zehnder capacity relative to the zero section $c^{\nu}_{HZ}(D_1 M, M, \mathrm{d}\lambda)$[2] is given by the length of the shortest geodesic in the class $\nu \in \pi_1(M)$. For non-aspherical, homogeneous spaces the result was extended to the class of contractible loops ($\nu = 0$) by Benedetti and Kang [5]. There is no reason to expect these theorems to hold for the non-relative capacity, as $c_{HZ}(D_1 M, M, \mathrm{d}\lambda)$ is computed using Hamiltonians constant along the zero section. Indeed for the flat torus $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$ the Hofer–Zehnder capacity is twice the length of the shortest geodesic [27, Ch.4.4, Prop. 4].

There are some further results concerning finiteness of the Hofer–Zehnder capacity. For example the Hofer–Zehnder capacity of a standard disc-bundle over a closed manifold is finite, if the base manifold carries a circle action with non-contractible orbits [29] or if M is rationally inessential [18] or if M if the Hurewicz map $\pi_2(M) \to H_2(M, \mathbb{Z})$ is non zero [2]. All these results rely on the observation by Irie [29, Cor. 3.5] that vanishing of symplectic homology implies finiteness of the Hofer–Zehnder capacity. Viterbo's isomorphism identifies symplectic homology with the homology of the free loop space. Albers, Frauenfelder and Oancea showed in [2] that the homology of the free loop spaces vanishes if one uses coefficients twisted by a not weakly exact, closed two form $\sigma \in \Omega^2(M)$. Groman and Merry [21] establish an isomorphism between this homology of the free loop space with twisted coefficients and symplectic homology of the twisted tangent bundle $(TM, \mathrm{d}\lambda - \pi^*\sigma)$ where $\pi : TM \to M$ denotes the foot point projection. Thus combined with Albers–Frauenfelder–Oancea and Irie they prove finiteness of the Hofer–Zehnder capacity of all disc-bundles with symplectic structure twisted by a not weakly exact two form,

$$c^0_{HZ}(D_\rho M, \mathrm{d}\lambda - \pi^*\sigma) < \infty.$$

Another strategy of bounding the Hofer–Zehnder (or any other) capacity is to study embeddings into manifolds where the capacity is known. For example if M sits as a Lagrangian in a symplectic manifold (W, ω) of finite Hofer–Zehnder capacity, then by Lagrangian neighborhood theorem also a small disc bundle $(D_\varepsilon M; \mathrm{d}\lambda)$ sits inside (W, ω). As scaling of fiber only scales the symplectic form finiteness of the capacity follows for all radii. Similarly for all symplectic submanifolds a neighborhood looks like symplectic normal bundle. Lu [36] proved, that all symplectic vector bundles (over symplectic manifolds) have finite Hofer–Zehnder capacity. In particular a neighborhood of the zero section of the symplectically twisted tangent bundle $(TM, \mathrm{d}\lambda - \pi^*\sigma)$ has finite Hofer–Zehnder capacity, but the twist destroys the scaling property, thus not even the finiteness result continues to hold for all radii.

Results of this thesis

The idea of this thesis is to use symmetries of the base manifold to construct such embeddings and normal forms explicitly. This will give explicit bounds for the Hofer–Zehnder capacity. We will consider the case where (M, g, σ) is a (locally) Hermitian symmetric space. These are homogenous Kähler manifolds, i.e. g is a Riemannian metric and there

[2] For some class $\nu \in \pi_1(M)$ one defines c^{ν}_{HZ} by considering only periodic orbits in class ν. In particular $c_{HZ} \leq c^{\nu}_{HZ}$ for all $\nu \in \pi_1(M)$. The precise definition is in Section 3.1.

exists an integrable complex structure j such that $g(j\cdot,\cdot) = \sigma(\cdot,\cdot)$. Further one can identify $M = G/K$ for some Lie group G and a compact subgroup K. Both the metric g and the symplectic form σ are invariant under G. In particular the induced action of G on TM is preserves the twisted symplectic structure

$$\omega_s := \mathrm{d}\lambda - s\pi^*\sigma,$$

where we included an additional parameter $s > 0$. We will see that the action is not only symplectic, but Hamiltonian.

Irreducible Hermitian symmetric spaces come in two types, compact and non-compact. Hermitian symmetric spaces of compact resp. non-compact type have non-negative resp. non-positive holomorphic bisectional curvature. Their tangent bundles therefore carry the structure of negative resp. positive vector bundles. Negative vector bundles globally admit a symplectic structure with symplectic fibers, that restricted to the zero section agrees with the symplectic structure of the base. We denote this symplectic structure by

$$\mathrm{d}\tau/2 - s\pi^*\sigma.$$

For positive vector bundles this closed two form is only non-degenerate on the neighborhood of the zero section

$$U_{2s}M := \{(x,v) \in TM| \ |g_x(jR_{jv,v}w,w)| < 2|s|\|w\|^2 \quad \forall w \in T_xM\},$$

where the holomorphic bisectional curvature is uniformly bounded from below by $2|s|$. Observe that a priori this symplectic structure is fairly different from the twisted symplectic structure as the fibers are Lagrangian for ω_s. By the symplectic neighborhood theorem there must be a small neighborhood of the zero section, where the two symplectic forms $\mathrm{d}\lambda - s\pi^*\sigma$ and $\mathrm{d}\tau/2 - s\pi^*\sigma$ can be identified. Our main theorem states that this symplectomorphism actually exists globally.

Theorem A. *Let M be an irreducible Hermitian symmetric space of compact type, then there exists an G-equivariant symplectomorphism*

$$\Psi : (TM, \mathrm{d}\lambda - s\pi^*\sigma) \to (TM, \mathrm{d}\tau/2 - s\pi^*\sigma).$$

If M is of non-compact type, then there exists an equivariant symplectomorphism

$$\Psi : (U_{s^2}M, \mathrm{d}\lambda - s\pi^*\sigma) \to (U_{2s}M, \mathrm{d}\tau/2 - s\pi^*\sigma).$$

Further the symplectomorphisms intertwine the moment maps of the Hamiltonian G-actions.

This symplectomorphism can be used to compute the Hofer-Zehnder capacity for suitable subsets of the the twisted tangent bundle (TM, ω_s). Indeed Lu [36] shows, that the Hofer-Zehnder capacity of disc-subbundle of symplectic vector bundles is given by the symplectic area of the fiber class. Now our symplectomorphism does not map disc-bundles to disc-bundles. We can therefore only give bounds for the capacity of $(D_\rho M, \omega_s)$.

Theorem B. *Let (M, g, σ) be isometrically covered by an irreducible Hermitian symmetric space of rank r, then*

$$\frac{2\pi}{\kappa}\left(\sqrt{s^2 + \kappa\rho^2/r} - s\right) \leq c_{HZ}(D_\rho M, \omega_s) \leq c^0_{HZ}(D_\rho M, \omega_s) \leq \frac{2\pi r}{\kappa}(\sqrt{s^2 + \kappa\rho^2} - s)$$

for any constants $s, \rho > 0$ satisfying $s^2 + \kappa\rho^2 > 0$. Here, κ denotes the maximal resp. minimal holomorphic sectional curvature.

Observe that in the rank one case upper and lower bound agree and thus determine the precise value

$$c_{HZ}(D_\rho M, \omega_s) = c^0_{HZ}(D_\rho M, \omega_s) = \frac{2\pi}{\kappa}(\sqrt{s^2 + \kappa\rho^2} - s)$$

as long as $s^2 + \kappa\rho^2 > 0$.

As discovered independently by Feix [16] and Kaledin [31] there also exists a hyperkähler structure in a neighborhood of the zero section of any real-analytic Kähler manifold (M, g, j). A hyperkähler structure on a manifold consists of three symplectic forms compatible with the same metric, such that the corresponding complex structures satisfy the quaternionic commutator relations. In the case of a tangent bundle two of the three symplectic forms are the canonical symplectic structure $\mathrm{d}\lambda$ and its pullback via the complex structure $j^*\mathrm{d}\lambda$. For the last symplectic form denoted by ω_I, the fibers and the zero section of the tangent bundle are symplectic. More precisely restricted to the zero section it coincides with the Kähler form σ of M. We show that there is also a symplectomorphism identifying $-\omega_I$ with $\mathrm{d}\lambda - \pi^*\sigma$.

Theorem C. *If M is a Hermitian symmetric space of compact type there are G-equivariant symplectomorphisms identifying*

$$(TM, \mathrm{d}\lambda - \pi^*\sigma) \cong (TM, -\omega_I) \cong (TM, \mathrm{d}\tau/2 - \pi^*\sigma).$$

If M is a Hermitian symmetric space of non-compact type, then

$$(U_{s^2}M, \mathrm{d}\lambda - \pi^*\sigma) \cong (U_{s^2}M, -\omega_I) \cong (U_{2s}M, \mathrm{d}\tau/2 - \pi^*\sigma).$$

In total we relate the following three symplectic structures on the tangent bundle of a locally Hermitian symmetric space:

1. The symplectically twisted symplectic structure $(TM, \mathrm{d}\lambda - \pi^*\sigma)$ below the Mané critical value.

2. The symplectic structure $(TM, \mathrm{d}\tau/2 - \pi^*\sigma)$ induced by the Hermitian curvature in a neighborhood of the zero section determined by the holomorphic bisectional curvature.

3. The hyperkähler structure on a neighborhood of the zero section discovered by Feix and Kaledin.

Non of these symplectic identifications work if the magnetic twist vanishes. A first step into the direction of the untwisted case might be the last chapter, where we consider the space $\mathbb{C}\mathrm{P}^n$, but potentially one can generalize this to other Hermitian symmetric spaces. We normalize the Fubini–Study metric so that all geodesics have length 2π. This implies that the Fubini–Study form takes the value 4π on the generator $[\mathbb{C}\mathrm{P}^1] \in H_2(\mathbb{C}\mathrm{P}^n, \mathbb{Z})$.

Theorem D. *There is a symplectomorphism, which is equivariant with respect to the Hamiltonian* $\mathrm{SU}(n+1)$*-actions,*

$$F : (D_\rho \mathbb{C}\mathrm{P}^n, \omega_s) \to (\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n \setminus \bar{\Delta}, R_1\sigma \ominus R_2\sigma),$$

where $\bar{\Delta} \subset \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n$ *denotes the anti-diagonal divisor*

$$\bar{\Delta} = \{(p,q) \in \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n \mid \mathrm{dist}(p,q) \textit{ maximal}\}$$

and R_1, R_2 *are determined by*

$$s = R_2 - R_1, \quad \rho = 2\sqrt{R_1 R_2}.$$

Further

$$\overline{(D_1 \mathbb{C}\mathrm{P}^n, \mathrm{d}\lambda - s\pi^*\sigma)} \cong (\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n \setminus \bar{\Delta}, R_1\sigma \ominus R_2\sigma),$$

where $\overline{(D_1 \mathbb{C}\mathrm{P}^n, \mathrm{d}\lambda - s\pi^*\sigma)}$ *denotes the symplectic compactification of the disc-bundle using a Lerman cut with respect to the Hamiltonian circle action given by the reparametrized magnetic geodesic flow.*

Indeed this symplectomorphism also holds for $s = 0$ and we can use it to determine the Hofer–Zehnder capacity.

Corollary A. *Equip* $\mathbb{C}\mathrm{P}^n$ *with the Fubini–Study metric, then*

$$c_{HZ}(D_1 \mathbb{C}\mathrm{P}^n, \mathrm{d}\lambda) = l,$$

where l *denotes the length of the geodesics.*

The case $s = 0$ of the above symplectomorphism can be interpreted as compact complexification of $\mathbb{C}\mathrm{P}^n$. On the other hand there is also a real form of $\mathbb{C}\mathrm{P}^n$ namely $\mathbb{R}\mathrm{P}^n$. In [1] Adaloglou showed a symplectic version of the fact that the disc subbundle of $T\mathbb{R}\mathrm{P}^2$ compactifies to $\mathbb{C}\mathrm{P}^2$. We generalize this to all dimensions.

Theorem E. *There is an* $SO(n+1)$*-equivariant symplectomorphism*

$$F : (D_{1/2} \mathbb{R}\mathrm{P}^n, \mathrm{d}\lambda) \to (\mathbb{C}\mathrm{P}^n \setminus Q^{n-1}, \sigma),$$

where Q^{n-1} *denotes the quadric*

$$Q^{n-1} := \{[z_0 : \ldots : z_n] \mid z_0^2 + \ldots + z_n^2 = 0\} \subset \mathbb{C}\mathrm{P}^n.$$

Further

$$\overline{(D_{1/2} \mathbb{R}\mathrm{P}^n, \mathrm{d}\lambda)} \cong (\mathbb{C}\mathrm{P}^n, \sigma),$$

where $\overline{(D_{1/2} \mathbb{R}\mathrm{P}^n, \mathrm{d}\lambda)}$ *denotes the symplectic compactification of the disc-bundle using a Lerman cut with respect to the Hamiltonian circle action given by geodesic flow.*

As corollary we obtain bounds on the Hofer–Zehnder capacity of the standard unit disc bundle over $\mathbb{R}\mathrm{P}^n$.

Theorem F. *Equip* $\mathbb{R}\mathrm{P}^n$ *with the constant curvature metric, then*

$$c_{HZ}(D_1 \mathbb{R}\mathrm{P}^n, \mathrm{d}\lambda) = 2l,$$

where l *denotes the length of the geodesics.*

Corollary A and Theorem F are particularly interesting when comparing them to the results from Weber [49, Thm A] and Benedetti and Kang [5, Cor. 2.8], where they showed that the Hofer–Zehnder capacity relative to the zero section is the length of the shortest geodesic. Observe that this means that for $\mathbb{C}\mathrm{P}^n$ relative and absolute agree, while for $\mathbb{R}\mathrm{P}^n$ they differ by a factor of two.

In some situations one does not need to exploit all the symmetries of a manifold. Already the existence of a Hamiltonian circle action yields quite some information about a symplectic manifold. For symplectic manifolds with Hamiltonian circle action there are fairly good techniques to bound the Gromov width and the Hofer–Zehnder capacity. This was already recognised by Hwang and Suh [28], where they compute Gromov-width and Hofer–Zehnder capacity of closed Fano symplectic manifolds with a semifree Hamiltonian circle action with isolated maximum. Indeed the oscillation of the moment map $\nu : M \to \mathbb{R}$ of the circle action yields a lower bound for the Hofer–Zehnder capacity. If in addition the maximum (or minimum) is isolated, the difference $\max(\nu) - \mathrm{smax}(\nu)$ bounds the Gromov width from below. Here $\mathrm{smax}(\nu)$ denotes the value of ν at the second highest critical set. On the other hand Tolman and McDuff studied the theory of pseudoholomorphic curves in symplectic manifolds with Hamiltonian circle actions intensively. They proved a localization principle which allows to count only S^1-invariant pseudoholomorphic curves when computing Gromov–Witten invariants. This makes it easier (or possible) to compute them in some cases we are considering. One can then use the idea of Hofer–Viterbo [26] that was generalized by Lu [37] to bound the Gromov width and the Hofer–Zehnder capacity from above by the symplectic area of a homology class $A \in H_2(M, \mathbb{Z})$ that admits a non-vanishing 1-point \2-point Gromov–Witten invariant. In this way we compute the Hofer–Zehnder capacity of all (irreducible) Hermitian symmetric spaces of compact type.

Theorem G. *Let (M, σ) be an irreducible Hermitian symmetric space normalized so that $\sigma(A) = 4\pi$ for the generator $A \in H_2(M, \mathbb{Z})$. Then*

$$c_{HZ}(M, \sigma) = 4\pi r.$$

Similarly we can also determine the Gromov width. This was already done by Loi, Mossa and Zuddas [35], but our prove only uses a Hamiltonian circle action and no additional structure.

Outline of this thesis

Chapter 2 introduces notations and concepts needed for the other chapters. We start by recalling the geometry of tangent bundles. In particular the precise definitions of all symplectic structures appearing in Theorems B and C are given. Then we present a short introduction to magnetic systems and the definition of the Mañé's critical value. Lastly, we introduce the manifolds we shall work with, Hermitian symmetric spaces. Particularly the representation of Hermitian symmetric spaces as coadjoint orbits will be extremely helpful later on.

Chapter 3 introduces the notion of symplectic capacities and especially the Hofer–Zehnder capacity. In order to find upper bounds to this capacity we will need to work with pseudoholomorphic curves. Thus we recap very roughly this theory in order to state a theorem by Hofer–Viterbo [26] and a generalization by Lu [37] that yield an upper bound to the Hofer–Zehnder capacity in terms of non-vanishing Gromov–Witten invariants.

Chapter 4 contains the proofs of Theorem G and a special case of Theorem B. Both proofs use the existence of a Hamiltonian circle action. Indeed as recalled in the first part of Chapter 4 a lower bound of the Hofer–Zehnder capacity of such manifolds is often given by the oscillation of the moment map. Further we state results by McDuff and Tolman [42] that help computing Gromov-Witten in this setup. In section 4.2 we prove Theorem B for surfaces of constant curvature using the fact that the magnetic geodesic flow is totally periodic. In particular we do not need to use the symplectomorphism in Theorem A. Sadly this fairly elegant argument is very four dimensional, thus can not be used for the general case. Section 4.3 contains the proof of Theorem G. The Hamiltonian circle action is obtained from the presentation of Hermitian symmetric spaces as coadjoint orbits.

Chapter 5 contains the proofs of Theorem A, B and C. The first section is merely an introduction, but also the proof of Theorem A in the euclidean case. The following three sections contain different proofs of Theorem A increasing in generality. The first proof is for surfaces of constant curvature and uses that in dimension four there exists a global frame of $T(TM \setminus 0_{TM})$. The second proof is for spaces of constant holomorphic sectional curvature (rank 1 Hermitian symmetric spaces) and uses that in this case one can explicitly compute the differential of Ψ using Jacobi fields. The last (of the three) section contains the full proof of Theorem A. It uses strongly the interpretation of Hermitian symmetric spaces as coadjoint orbits. In section 5.5 we deduce Theorem B from A. Theorem C is then analogously proven in section 5.6.

Chapter 6 contains the proofs of Theorem E and Theorem D. Both proofs use the presentation of the complex projective space as coadjoint orbit. The symplectomorphisms are mostly determined by setting the moment maps of the Hamiltonian $U(n+1)$- resp. $SO(n+1)$-actions equal. From Theorems E and D the Corollary A and Theorem F are deduced.

Chapter 7 contains an outlook and presents some partial results that are not fully proven yet. In the beginning of the chapter we present a table with precise values of the Hofer–Zehnder capacity of disc sub bundles of twisted tangent bundles that where determined in this thesis or where known before. The rest of the chapter gives an outlook on how to complete the table of values for the Hofer–Zehnder capacity further.

2. Preliminaries I: Differential geometry

2.1. Geometry of tangent bundles

In this section we recall the elements of [23] that are relevant for our work. Some calculations are adapted as all calculations in [23] were done for the Levi-Civita but work analogously for other connections. For general connections another source is [14]. Let (M,σ) be a symplectic manifold and denote by $\pi : TM \to M$ the tangent bundle. The kernel of the differential $\mathrm{d}\pi : TTM \to TM$ defines a distribution in $\mathcal{V} \subset TTM$, called *vertical distribution*. Choose a compatible almost complex j on M and denote the associated metric by g. The almost complex structure is in general not integrable still it turns the tangent bundle into a complex vector bundle. If we pick as connection the following modification

$$\tilde{\nabla}_X Y := \nabla_X Y - \frac{1}{2} j(\nabla_X j)Y$$

of the Levi-Civita connection ∇, we turn the tangent bundle into a Hermitian vector bundle. Indeed $\tilde{\nabla} g = 0$ and $\tilde{\nabla} j = 0$, but the new connection $\tilde{\nabla}$ is not torsion free in general. Its torsion is precisely the Nijenhuis-tensor. This connection determines a complement $\mathcal{H} \subset TTM$ called *horizontal distribution*, i.e.

$$TTM = \mathcal{H} \oplus \mathcal{V}.$$

Both $\mathcal{V}_{(x,v)}$ and $\mathcal{H}_{(x,v)}$ are as vector spaces isomorphic to T_xM. In particular we can lift vectors in T_xM and also vector fields on TM horizontally and vertically.

Definition 2.1.1 (Horizontal & vertical lift)**.** *Let $(x,v) \in TM$. Any tangent vector $w \in T_xM$ can be lifted horizontally (vertically) to a vector in $\mathcal{H}_{(x,v)}$ ($\mathcal{V}_{(x,v)}$). Explicitly the horizontal lift is defined by*

$$w \mapsto w^{\mathcal{H}} := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\gamma(t), P_\gamma v)$$

where $\gamma : (-\varepsilon, \varepsilon) \subset \mathbb{R} \to M$ is a smooth curve satisfying $\gamma(0) = x$ and $\dot{\gamma}(0) = w$ and P_γ denotes parallel transport along γ. The vertical lift is defined by

$$w \mapsto w^{\mathcal{V}} := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (x, v + tw).$$

The following proposition gives the commutator relations for vertical and horizontal lifts of vector fields.

Proposition 2.1.2 ([14], Lemma 2)**.**
Let X and Y be vector fields on M, then their lifts satisfy the following commutator relations

(i) $[X^{\mathcal{V}}, Y^{\mathcal{V}}] = 0$

(ii) $[X^{\mathcal{H}}, Y^{\mathcal{V}}] = (\tilde{\nabla}_X Y)^{\mathcal{V}}$

(iii) $[X^{\mathcal{H}}, Y^{\mathcal{H}}] = [X, Y]^{\mathcal{H}} - (R(X, Y)v)^{\mathcal{V}}$

Here R denotes the Riemannian curvature tensor of the Hermitian connection $\tilde{\nabla}$.

We can define four vector fields

$$\begin{aligned} X_{(x,v)} &:= v^{\mathcal{H}} \\ H_{(x,v)} &:= (jv)^{\mathcal{H}} \\ Y_{(x,v)} &:= v^{\mathcal{V}} \\ V_{(x,v)} &:= (jv)^{\mathcal{V}} \end{aligned}$$

Observe that these are not lifts of vector fields on the base, but locally can be written as linear combinations of those. For example

$$X_{(x,v)} = v^i \partial_i^{\mathcal{H}},$$

for some local frame $(\partial_1, \ldots, \partial_{2n})$ of TM. Using Proposition 2.1.2 one can compute their commutators.

Proposition 2.1.3.

$$[V, X] = H, \qquad [V, H] = -X, \qquad [V, Y] = 0, \qquad [Y, X] = X, \qquad [Y, H] = H,$$

and

$$[X, H]_{(x,v)} = (R(v, jv)v)^{\mathcal{V}}.$$

These vector fields are non-zero and linearly independent outside the zero-section. To obtain a dual description on the cotangent bundle in terms of dual one forms we need to choose a metric on TM. For this we pick the Sasaki-metric $\hat{g}$. This is the metric that takes the form $g \oplus g$ with respect to the splitting $\mathcal{H} \oplus \mathcal{V}$ in horizontal and vertical. It turns out that the one-forms dual to X, Y, H, V are no strangers.

Lemma 2.1.4. *The one forms that are via the Sasaki-metric dual to X, H, Y, V respectively are*

- *λ the (metric pullback) of the canonical 1-form on T^*M,*
- *$j^*\lambda =$ the pullback of λ via $j : TM \to TM$,*
- *$\mathrm{d}E$ where $E(x, v) = \frac{1}{2} g_x(v, v)$ is the kinetic energy.*
- *$\mathrm{d}^c E = \mathrm{d}E \circ I$ where $I = j \ominus j$ is an almost complex structure on TM.*

Proof. Clearly

$$\iota_X \hat{g}(\cdot) = \hat{g}(X, \cdot) = g(v, \mathrm{d}\pi \cdot) = \lambda(\cdot).$$

Similarly

$$\iota_H \hat{g}(\cdot) = \hat{g}(H, \cdot) = g(jv, \mathrm{d}\pi \cdot) = g(jv, \mathrm{d}\pi \mathrm{d}j \cdot) = \lambda(\mathrm{d}j \cdot) = j^*\lambda(\cdot),$$

where we used that $\mathrm{d}\pi\mathrm{d}j = \mathrm{d}\pi$ as j is a bundle map lifting the identity. The third one is maybe the most tricky. First observe that $\mathrm{d}E$ vanishes on $\mathcal{H}$ as for every $w \in T_xM$ we find

$$\mathrm{d}E(w^{\mathcal{H}})_{(x,v)} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} E(\gamma(t), P_\gamma v(t)) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g_{\gamma(t)}(P_\gamma v(t), P_\gamma v(t)) = 0$$

because $\tilde{\nabla} g = 0$. Further

$$\mathrm{d}E(w^{\mathcal{V}}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} E(x, v+tw) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g_x(v+tw, v+tw) = g_x(v,w)$$

and we conclude $\hat{g}(Y,\cdot) = \mathrm{d}E$. Finally

$$\iota_V \hat{g}(\cdot) = \hat{g}(V,\cdot) = g(jv, \mathcal{P}_{\mathcal{V}}\cdot) = -g(v, j\mathcal{P}_{\mathcal{V}}\cdot) = g(v, \mathcal{P}_{\mathcal{V}} I\cdot) = \mathrm{d}E \circ I(\cdot),$$

where $\mathcal{P}_{\mathcal{V}}$ denotes projection to the vertical subspace. □

We shall call the dual of V the *angular form*, as it is dual to the vector field that generates rotation $e^{jt} : T_xM \to T_xM$ in the fibers, and denote it by $\tau := \iota_V \hat{g}$. Using the commutator relations in Proposition 2.1.2 we can compute the exterior derivatives of λ and τ.

Proposition 2.1.5. *We write the 2-forms in matrix representation with respect to the splitting of $TTM = \mathcal{H} \oplus \mathcal{V}$. So the upper left entry eats two horizontal vectors, the upper right a horizontal and a vertical and so on. In this representation the exterior derivatives of λ and τ are given as*

$$\mathrm{d}\lambda = \begin{pmatrix} g(v, T(\cdot,\cdot)) & -g \\ g & 0 \end{pmatrix}, \quad \mathrm{d}\tau = \begin{pmatrix} g(jv, R(\cdot,\cdot)v) & 0 \\ 0 & 2\sigma \end{pmatrix}$$

Here T and R denote respectively torsion and curvature of $\tilde{\nabla}$.

Proof. We first prove two identities that will be useful. Let A, B be any vector fields on M, then

$$A^{\mathcal{V}}(g(v,B))(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g_x(v + tA_x, B_x) = g_x(A_x, B_x).$$

Further

$$A^{\mathcal{H}}(g(v,B))(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g_{x(t)}(P_x v(t), B_{x(t)}) = g_x(v, (\tilde{\nabla}_A B)_x),$$

where $x(t) \in M$ is an integral curve of A, i.e. $\dot{x} = A$ and $P_x v(t)$ denotes the parallel transport of $v \in T_{x(0)}M$ along $x(t)$ with respect to the Hermitian connection $\tilde{\nabla}$. In total we find

$$A^{\mathcal{V}}(g(v,B)) = g(A,B), \quad A^{\mathcal{H}}(g(v,B)) = g(v, \tilde{\nabla}_A B).$$

Similarly also

$$A^{\mathcal{V}}(g(jv,B)) = g(jA,B), \quad A^{\mathcal{H}}(g(jv,B)) = g(jv, \tilde{\nabla}_A B)$$

holds. We can now compute $\mathrm{d}\lambda$ using Proposition 2.1.2 and the formula for the differential of a 1-form

$$\mathrm{d}\lambda(\hat{A}, \hat{B}) = \hat{A}\lambda(\hat{B})) - \hat{B}(\lambda(\hat{A})) - \lambda([\hat{A}, \hat{B}])$$

for any vector fields $\hat{A}, \hat{B}$ on TM. Now

$$
\begin{aligned}
\mathrm{d}\lambda(A^{\mathcal{V}}, B^{\mathcal{V}}) &= A^{\mathcal{V}}(\lambda(B^{\mathcal{V}})) - B^{\mathcal{V}}(\lambda(A^{\mathcal{V}})) - \lambda([A^{\mathcal{V}}, B^{\mathcal{V}}]) = 0 \\
\mathrm{d}\lambda(A^{\mathcal{V}}, B^{\mathcal{H}}) &= A^{\mathcal{V}}(\lambda(B^{\mathcal{H}})) - B^{\mathcal{H}}(\lambda(A^{\mathcal{V}})) - \lambda([A^{\mathcal{V}}, B^{\mathcal{H}}]) = A^{\mathcal{V}}(g(v, B)) = g(A, B) \\
\mathrm{d}\lambda(A^{\mathcal{H}}, B^{\mathcal{H}}) &= A^{\mathcal{H}}(\lambda(B^{\mathcal{H}})) - B^{\mathcal{H}}(\lambda(A^{\mathcal{H}})) - \lambda([A^{\mathcal{H}}, B^{\mathcal{H}}]) \\
&= A^{\mathcal{H}}(g(v, B)) - B^{\mathcal{H}}(g(v, A)) - \lambda([A, B]^{\mathcal{H}}) \\
&= g(v, \tilde{\nabla}_A B) - g(v, \tilde{\nabla}_B A) - g(v, [A, B]) = g(v, T(A, B)).
\end{aligned}
$$

Similarly we can also compute $\mathrm{d}\tau$

$$
\begin{aligned}
\mathrm{d}\tau(A^{\mathcal{V}}, B^{\mathcal{V}}) &= A^{\mathcal{V}}(\tau(B^{\mathcal{V}})) - B^{\mathcal{V}}(\tau(A^{\mathcal{V}})) - \tau([A^{\mathcal{V}}, B^{\mathcal{V}}]) \\
&= A^{\mathcal{V}}(g(jv, B)) - B^{\mathcal{V}}(g(jv, A)) = g(jA, B) - g(jB, A) = 2\sigma(A, B) \\
\mathrm{d}\tau(A^{\mathcal{V}}, B^{\mathcal{H}}) &= A^{\mathcal{V}}(\tau(B^{\mathcal{H}})) - B^{\mathcal{H}}(\tau(A^{\mathcal{V}})) - \tau([A^{\mathcal{V}}, B^{\mathcal{H}}]) \\
&= -B^{\mathcal{H}}(g(jv, A)) + \tau((\tilde{\nabla}_B A)^{\mathcal{V}}) = -g(jv, \tilde{\nabla}_B A) + g(jv, \tilde{\nabla}_B A) = 0 \\
\mathrm{d}\tau(A^{\mathcal{H}}, B^{\mathcal{H}}) &= A^{\mathcal{H}}(\tau(B^{\mathcal{H}})) - B^{\mathcal{H}}(\tau(A^{\mathcal{H}})) - \tau([A^{\mathcal{H}}, B^{\mathcal{H}}]) \\
&= \tau((R(A, B)v)^{\mathcal{V}}) = g(jv, R(A, B)v).
\end{aligned}
$$

□

Observe that $\mathrm{d}\tau$ is not symplectic as it is degenerate on the zero-section. But we can add the pullback of σ to make it non-degenerate at least in a neighborhood of the zero-section.

Lemma 2.1.6. *For any real number $s > 0$ the closed two-form*

$$\mathrm{d}\tau/2 - s\pi^*\sigma \tag{2.1}$$

is non-degenerate and thus symplectic in the neighborhood of the zero-section

$$U_{2s}M := \{(x, v) \in TM \mid g(jv, R(w, jw)v) \leq 2s\sigma(w, jw) = 2s\|w\|^2 \quad \forall w \in T_xM\} \tag{2.2}$$

determined by the holomorphic bisectional curvature.

2.1.1. Hyperkähler structure

We will now have a closer look at the tangent bundle of Kähler manifolds. Denote by $\eta := j^*\lambda$. In view of Proposition 2.1.5, we see that on the tangent bundle of a Kähler manifolds two symplectic structures naturally arise, namely

$$\mathrm{d}\lambda \equiv \begin{pmatrix} 0 & -g \\ g & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{d}\eta \equiv \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}$$

the blocks of the matrices represent the splitting into horizontal and vertical coordinates. The torsion term vanishes for an integrable almost complex structure. One reasonable question is, do they belong to a hyperkähler structure?

Definition 2.1.7 ([11] Hyperkähler Structure)**.**
A hyperkähler manifold is a Riemannian manifold (N, G) endowed with three complex structures I, J and K compatible with G, i.e. the forms $\omega_I(\cdot,\cdot) := G(I\cdot,\cdot)$, $\omega_J(\cdot,\cdot) := G(J\cdot,\cdot)$ and $\omega_K(\cdot,\cdot) := G(K\cdot,\cdot)$ are closed and thus symplectic. Further I, J and K, considered as endomorphisms of the real tangent bundle, satisfy the relation $I \circ J = -J \circ I = K$.

Actually the integrability condition on the almost complex structures I, J and K in the definition is redundant by the following proposition.

Proposition 2.1.8 ([25], Thm. 2)**.** *A Riemannian manifold (N, G) with two almost complex structures satisfying $I \circ J = -J \circ I =: K$ is hyperkähler if and only if the corresponding forms ω_I, ω_J and ω_K are closed.*

Any hyperkähler (M, I, J, K, G) manifold also admits a holomorphic symplectic structure $\omega_c := \omega_J + i\omega_K \in \Omega^{2,0}(M, \mathbb{C})$ with respect to the complex structure I. The converse is not true in general.

Assume now that $\mathrm{d}\lambda$ and $\mathrm{d}\eta$ belong to a hyperkähler structure. Then there must be a metric G on TM and two complex structures J and K such that

$$\mathrm{d}\lambda(\cdot, J\cdot) = G(\cdot,\cdot) = \mathrm{d}\eta(\cdot, K\cdot).$$

It follows that

$$\mathrm{d}\lambda(\cdot, JK\cdot) = -\mathrm{d}\eta(\cdot,\cdot)$$

and thus the third complex structure is implicitly determined to be

$$I = JK \equiv \begin{pmatrix} -j & 0 \\ 0 & j \end{pmatrix}.$$

This is indeed a complex structure! The proof of integrability of I is given in appendix A.1. It regards the tangent bundle with a canonic holomorphic symplectic structure

$$\omega_c := \mathrm{d}\lambda + i\mathrm{d}\eta \in \Omega^{2,0}(TM, \mathbb{C}).$$

But if there is also a hyperkähler structure what could G be? A first guess might be the Sasaki-metric, but sadly with this choice it turns out that $\omega_I = G(I\cdot,\cdot)$ is not closed unless the manifold is flat (see appendix A.2). Nevertheless we can ask if there is a different metric on the tangent bundle turning it into a hyperkähler manifold. Actually the answer to this question is yes (for real-analytic Kähler manifolds), at least in a neighborhood of the zero section as shown by B. Feix [16] and D. Kaledin [31] independently.

Theorem 2.1.9. *([16] Thm. A) Let X be a real-analytic Kähler manifold. Then there exists a hyperkähler metric in a neighbourhood of the zero section of the cotangent bundle which is compatible with the canonical holomorphic-symplectic structure.*
Furthermore, the S^1-action $(x, v) \to (x, e^{jt}v)$ rotating the fibers is isometric and the restriction of the hyperkähler metric to the zero section induces the original Kähler metric.

We shall explore this theorem for Hermitian symmetric spaces in chapter 5 and recover the hyperkähler structure explicitly.

2.2. Magnetic systems

In this section we will introduce magnetic systems. All definitions and more details can for example be found in [4]. Let (M, g) be a Riemannian manifold. Additionally pick a closed two form $\sigma \in \Omega^2(M)$. We will refer to the triple (M, g, σ) as *magnetic system*. Since σ is closed we can use it to define a twisted symplectic structure on the tangent bundle

$$\omega_\sigma := \mathrm{d}\lambda - \pi^*\sigma.$$

This indeed defines a symplectic form as ω_σ is closed

$$\mathrm{d}\omega_\sigma = \mathrm{d}^2\lambda - \mathrm{d}\pi^*\sigma = -\pi^*\mathrm{d}\sigma = 0$$

and non-degenerate because it takes the form

$$\omega_\sigma = \begin{pmatrix} \sigma & -g \\ g & 0 \end{pmatrix}$$

with respect to the splitting $TTM = \mathcal{H} \oplus \mathcal{V}$, where we used the Levi-Civita connection to define the horizontal distribution.

Definition 2.2.1 (Lorentz force)**.**
The Lorentz force is the bundle map $F : TM \to TM$ defined via

$$g_x(F_x(v), w) = \sigma_x(v, w).$$

The Lorentz force determines the Hamiltonian vector field for the kinetic energy with respect to the twisted symplectic form.

Lemma 2.2.2. *([6] Lemma 6.1) Let $E(x, v) := \frac{1}{2}g_x(v, v)$ denote the kinetic energy. The Hamiltonian vector field X_E is given by*

$$(X_E)_{(x,v)} = v^{\mathcal{H}} + F(v)^{\mathcal{V}}.$$

If $\sigma = 0$, the Lorentz force vanishes and $X_E = v^{\mathcal{H}}$, i.e. X_E generates the geodesic flow. For $\sigma \neq 0$ the flow is referred to as magnetic geodesic flow.

In the special case where σ is exact (i.e. $\sigma = \mathrm{d}\theta$ for some 1-form $\theta \in \Omega^1(M)$) we can shift the zero-section to see that the twisted symplectic form is equivalent to the standard symplectic form. Denote $A \in \mathfrak{X}(M)$ the vector field dual to θ defined via $g(A, \cdot) = \theta(\cdot)$. Consider the map

$$L_A : TM \to TM;\ (x, v) \mapsto (x, v + A_x).$$

Then

$$L_A^*(\lambda)_{(x,v)} = \lambda_{(x,v+A_x)}(\mathrm{d}L_A\cdot) = g_x(v+A_x, \mathrm{d}\pi\mathrm{d}L_A\cdot) = g_x(v, \mathrm{d}\pi\cdot)+g_x(A_x, \mathrm{d}\pi\cdot) = \lambda_{(x,v)}+\pi^*\theta$$

where we used $\mathrm{d}\pi\mathrm{d}L_A = \mathrm{d}\pi$ as L_A is a bundle map. We see that this map maps the twisted symplectic structure to the standard one. Further the kinetic Hamiltonian transforms as

$$E(L_A(x, v)) = \frac{1}{2}|v + A|^2 \equiv \frac{1}{2}|v|^2 + A \cdot v + V(x)$$

which has the form of an electro-magnetic Hamiltonian for a charge moving in a magnetic field $B \equiv \mathrm{rot}A$. This is why it is called a magnetic system!

2.2.1. Mañé's critical value

In the non-twisted case the geodesic flow on different energy hyper surfaces is conjugated, this is not the case anymore for magnetic systems. Indeed there are values where the dynamics on the energy hyper surface changes dramatically. We shall in examples see that this usually happens at the so called Mañé's critical value. Our main source for this section is [12].

Definition 2.2.3. *Consider (TM, ω_σ) for some closed Riemannian manifold (M, g) and a closed two form $\sigma \in \Omega^2(M)$. Denote $\hat{M}$ the universal cover M. Define*

$$c(M, g, \sigma) := \inf_{\theta} \sup_{x \in \hat{M}} \hat{E}(x, {}^g\theta_x),$$

where $\hat{E}$ is the lift of E, the infimum is taken over primitives of $\hat{\sigma}$ and ${}^g\theta$ denotes the metric dual of θ. If $\hat{\sigma}$ is not exact, then $c(M, g, \sigma) := \infty$ by convention.

Remark 2.2.4. *One can use different coverings and different Hamiltonians to define other Mañé critical values, but in this thesis we will restrict to the the universal cover and the kinetic Hamiltonian as in the definition above.*

The Mañé's critical value can also be defined in terms of the Lagrangian $\hat{L}$ the Legendre dual of $\hat{E}$,

$$\hat{L}(x, v) = \frac{1}{2}|v|^2 - \theta_x(v).$$

On an absolute continuous curve $\gamma : [0, T] \to \hat{M}$ define the action of $\hat{L}$ as

$$A_{\hat{L}}(\gamma) = \int_0^T \hat{L}(\gamma(t), \dot{\gamma}(t))\mathrm{d}t.$$

Proposition 2.2.5 ([13])**.** *The Mañé critical value satisfies*

$$c(M, g, \sigma) = \inf\{k \in \mathbb{R} : A_{\hat{L}+k}(\gamma) \geq 0 \text{ for any absolutely continuous closed curve } \gamma\}.$$

Denote by S_k the energy hyper surface $\{E = k\}$. It is said to be of *virtual contact type*, if $\hat{\sigma}|_{\hat{S}_k} = \mathrm{d}\alpha$ for a contact form α satisfying

$$\sup_{x \in \hat{S}_k} |\alpha_x| \leq C < \infty \quad \text{and} \quad \inf_{x \in \hat{S}_k} |\alpha_x| \geq \varepsilon > 0,$$

where R is a vector field generating the kernel of $\mathrm{d}\lambda$.

Lemma 2.2.6 (Lemma 5.1 [12])**.** *For any $k > c$, the hyper surface S_k is virtually contact.*

In particular if the hyper surface is not virtually contact, then we obtain a lower bound for the Mañé critical value, i.e. $k \leq c$.

2.3. Coadjoint orbits

Coadjoint orbits are a pleasant class of examples for homogeneous symplectic manifolds. We will later see that also Hermitian symmetric spaces can be realized as coadjoint orbits, but for now let G be a finite dimensional, real, semisimple Lie group and denote by $\mathfrak{g}$ its Lie algebra. All proofs and details of this section can be found in Kirillov's book [33, Ch. 1].

Adjoint representation:
We denote by C_g the conjugation by $g \in G$, i.e.

$$C_g : G \to G; \quad h \mapsto ghg^{-1}.$$

Then the *adjoint representation* of G on $\mathfrak{g}$ is given by

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}), \quad g \mapsto \mathrm{Ad}_g := (\mathrm{d}C_g)_e.$$

We further denote the induced adjoint representation of the Lie algebra by

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}); \quad X \mapsto \mathrm{ad}_X := (\mathrm{dAd})_e(X).$$

In fact,

$$\mathrm{ad}_X(Y) = [X, Y] \in \mathfrak{g}.$$

Killing form:
The *Killing form* is defined as the symmetric bilinear form

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}; \quad (X, Y) \mapsto \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y).$$

It is invariant under Lie algebra automorphisms, in particular under the adjoint action, and thus ad_{X} is skew-symmetric with respect to B for any $X \in \mathfrak{g}$. Moreover, B is in our cases non-degenerate as $\mathfrak{g}$ is semisimple.

Coadjoint representation:
The *coadjoint representation* of G on the dual Lie algebra $\mathfrak{g}^*$ is denoted by

$$\mathrm{Ad}^* : G \to \mathrm{Gl}(\mathfrak{g}^*); \; g \mapsto \mathrm{Ad}^*_g.$$

It is the dual of the adjoint representation and thus implicitly defined via

$$\langle \mathrm{Ad}^*_g F, X \rangle = \langle F, \mathrm{Ad}_{g^{-1}} X \rangle, \quad \forall F \in \mathfrak{g}^* \; \forall X \in \mathfrak{g},$$

where $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ denotes the natural pairing. We use non-degeneracy of the Killing form to identify $\mathfrak{g}^* \cong \mathfrak{g}$. It intertwines adjoint and coadjoint action as

$$\langle \mathrm{Ad}^*_g F, Y \rangle := \langle F, \mathrm{Ad}_{g^{-1}} Y \rangle = B(X_F, \mathrm{Ad}_{g^{-1}} Y) = B(\mathrm{Ad}_g X_F, Y),$$

for any $Y \in \mathfrak{g}$ and $F \in \mathfrak{g}^*$ with dual $X_F \in \mathfrak{g}$. Hence, we will from now on use adjoint and coadjoint descriptions interchangeably.

Coadjoint orbits:

The orbit $O_p := \mathrm{Ad}^*_G(p) \subset \mathfrak{g}^*$ of a point $p \in \mathfrak{g}^*$ under the coadjoint action is called *coadjoint orbit*. It can be identified with the homogeneous space G/G_p ,where G_p is the stabilizer of p, via the isomorphism

$$\mathrm{Ad}^*_g(p) \mapsto g \cdot G_p.$$

For an element of the Lie-algebra $a \in \mathfrak{g}$, we define the induced vector field at $x \in O_p$ as

$$(a)^\#_x := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}^*_{\exp ta} x.$$

As coadjoint action is transitive on O_p, we can represent every vector in $T_x O_p$ in this way by an element of $\mathfrak{g}$. Using the identification of adjoint and coadjoint orbit via the Killing form, we obtain the following identifications

$$(a)^\#_x = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}_{\exp ta} x = [a, x] \quad \& \quad T_x O_p \cong [\mathfrak{g}, x].$$

Kirillov-Kostant-Souriau form:
Coadjoint orbits carry a natural G-invariant symplectic structure, called *KKS-form*[1]. At $p \in O_p$ it is for $a, b \in \mathfrak{g}$ given by

$$\sigma_p(a^\#_p, b^\#_p) := -\langle p, [a, b]\rangle,$$

where the natural pairing $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is extended equivariantly to a symplectic form on O_p. Well-definedness of this definition and non-degeneracy follow from the fact that the kernel of $\langle p, [\cdot, \cdot]\rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is precisely $\mathfrak{g}_p$, the Lie-algebra of G_p, and invariance under G_p. Closedness follows from the Jacobi identity. Using the Killing form one can push the symplectic structure to the adjoint orbit, also denoted by σ, e.g.

$$\sigma_x(a^\#_x, b^\#_x) := -B(x, [a, b]) \quad \forall x \in O_p \quad \forall a, b \in \mathfrak{g}.$$

Moment map:
For a symplectic group action $\Psi : G \to \mathrm{Symp}(N, \sigma);\ g \mapsto \Psi_g$ on a general symplectic manifold (N, ω), a map $\mu : N \to \mathfrak{g}^*$ is called *moment map* if

$$\mathrm{d}\langle \mu, a\rangle = \iota_{a^\#}\omega \quad \forall a \in \mathfrak{g}.$$

We additionally require μ to be equivariant with respect to the coadjoint action, i.e.

$$\mu(\Psi_g(x)) = \mathrm{Ad}^*_g(\mu(x)) \quad \forall x \in O_p,\ \forall g \in G.$$

A symplectic action that admits a moment map is called *Hamiltonian*.

Indeed the obvious symplectic action of G on O_p is Hamiltonian and the moment map is given by the inclusion

$$\mu : O_p \hookrightarrow \mathfrak{g} \cong \mathfrak{g}^*.$$

[1]KKS stands for Kirillov–Kostant–Souriau.

Proof. Since the pairing is Ad_G invariant we have

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} B(\mathrm{Ad}_{\exp t\xi_1} x, \mathrm{Ad}_{\exp t\xi_1}\xi_2) = B([\xi_1, x], \xi_2) + B(x, [\xi_1, \xi_2]) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} B(\mu(\mathrm{Ad}_{\exp t\xi_1} x), \xi_2) - \sigma_x([x, \xi_1], [x, \xi_2]), \end{aligned}$$

for all $x \in O_p$ and $\xi_1, \xi_2 \in \mathfrak{g}$. From here we immediately see

$$\mathrm{d}\langle \mu, \xi \rangle = \iota_{\xi^\#}\sigma \quad \forall \xi \in \mathfrak{g},$$

since the (co-)adjoint action is transitive on the orbit. □

2.4. Hermitian symmetric spaces

In this section we collect some basic material of Hermitian symmetric spaces. The sources are mainly [24] and [51]. Our focus lies on the description of Hermitian symmetric spaces as coadjoint orbits.

2.4.1. Symmetric spaces

Definition 2.4.1. *A Riemannian symmetric space is a connected Riemannian manifold (M, g) with the property that the geodesic reflection at any point is an isometry of M. Explicitly this means for any point $p \in M$ there is an isometry $s_p : M \to M$ that satisfies*

$$s_p(p) = p \quad \& \quad (\mathrm{d}s_p)_p = -\mathrm{id}.$$

Symmetric spaces are complete as we can use the geodesic symmetry to extend any geodesic segment to infinite length. Because M is connected, Hopf-Rinow's theorem implies that any two points p and q in M can be joined by a geodesic $\gamma : \mathbb{R} \to M$ satisfying $\gamma(0) = p$ and $\gamma(t) = q$ for some $t \in \mathbb{R}$. It follows that $s_{\gamma(t/2)}(p) = q$ thus the connected component of the identity of the isometry group $G = \mathrm{Is}^0(M)$ acts transitively. We can realize M as homogeneous space $M = G/K$ by picking a base point $o \in M$ and calling the stabilizer $\mathrm{Stab}_G(o) =: K$.

Orthogonal symmetric Lie algebras [24, Ch. V.1]:
The involutive isometry s_o induces an involution on G via $g \mapsto s_o \circ g \circ s_o$ and its differential is a so called *Cartan involution* $\theta : \mathfrak{g} \to \mathfrak{g}$. As an involution, it has eigenvalues ± 1 and we split $\mathfrak{g}$ into the eigenspaces of θ, i.e. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $\theta|_\mathfrak{k} = 1$, $\theta|_\mathfrak{p} = -1$. Moreover, θ is a Lie algebra automorphism leaving the Killing-form B invariant. The decomposition is thus orthogonal with respect to B. Further, being a Lie algebra automorphism implies the commutator relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \quad \text{and} \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{2.3}$$

The Lie subalgebra $\mathfrak{k}$ can be identified with the Lie algebra of K, thus is compact. Further the differential of $\pi : G \to M$ has kernel $\mathfrak{k}$ and thus induces an identification

$$\mathrm{d}\pi_e|_\mathfrak{p} : \mathfrak{p} \xrightarrow{\sim} T_oM.$$

The pair $(\mathfrak{g}, \theta)$ is what is called an *orthogonal symmetric Lie algebra* (short OSLA).

Definition 2.4.2. *An orthogonal symmetric Lie algebra (OSLA) is a pair* $(\mathfrak{g}, \theta)$ *where*

- $\mathfrak{g}$ *is a real Lie algebra,*
- $\theta \in \mathrm{End}(\mathfrak{g})$ *such that* $\theta^2 = \mathrm{id}$ *but* $\theta \neq \mathrm{id}$*,*
- $\mathfrak{k} := E_1(\theta)$[2] *is compact* [3]*.*

Irreducibility [24, Ch. VIII.5]:
A symmetric space M is called *irreducible* if it does not split as a product $M = M_1 \times M_2$ of symmetric spaces M_1, M_2. Irreducibility can also be seen algebraically.

Definition 2.4.3. *An OSLA* $(\mathfrak{g}, \theta)$ *is irreducible if*

- $\mathfrak{g}$ *is semisimple,*
- $\mathfrak{k}$ *contains no ideal of* $\mathfrak{g}$*,*
- *the Lie algebra* $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ *acts irreducibly on* $\mathfrak{p}$*.*

Indeed if M is irreducible then so is the associated OSLA.

Duality – compact vs. non-compact type [24, Ch. V.2]:
Irreducible Hermitian symmetric spaces fall into two types (*compact* and *non-compact*) according to their OSLA's.

Definition 2.4.4. *An irreducible OSLA is called of compact type if the Killing form is negative definite. It is called of non-compact type if the Killing form restricted to* $\mathfrak{p} \times \mathfrak{p}$ *is positive definite.*

Indeed, one can show that every irreducible OSLA is either of compact or non-compact type. These types are dual in the following sense. We consider the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ and the natural complexification $\theta^{\mathbb{C}}$ of θ and define

$$(\mathfrak{g}^{\vee}, \theta^{\vee}) := (\mathfrak{k} \oplus i\mathfrak{p}, \theta^{\mathbb{C}}|_{\mathfrak{k}+i\mathfrak{p}}).$$

If $(\mathfrak{g}, \theta)$ is an OSLA of compact type then $(\mathfrak{g}^{\vee}, \theta^{\vee})$ is an OSLA of non-compact type and the other way around [24, Prop. 2.1, Ch. V].

Euclidean type:
In principle there is a third type of symmetric spaces, called *Euclidean type*. This type occurs if $\mathfrak{p}$ is an abelian ideal of $\mathfrak{g}$. Indeed all symmetric spaces of Euclidean type can be isometrically identified with an Euclidean space. Furthermore, every symmetric space M can be decomposed as a product [24, Prop. 4.2, Ch. V]

$$M = M_0 \times M_- \times M_+,$$

where M_0 is a Euclidean space and M_+, M_- are symmetric spaces of the compact and non-compact type, respectively.

[2] The eigenspace of θ for the eigenvalue 1.
[3] A Lie algebra is called compact if its Killing form is negative definite.

Uniqueness of invariant metric:
Irreducibility also ensures that there is (up to scalar multiple) a unique G-invariant metric, i.e. any invariant metric is induced by $\pm\lambda^2 B|_{\mathfrak{p}\times\mathfrak{p}}$ for some real constant $\lambda \neq 0$ and the sign chosen such that $\pm B|_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite. In particular the G-invariant metric g we started with is of this form. To see this assume there was another one. This would define another K-invariant scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{p}$. Now we define a symmetric operator $S : \mathfrak{p} \to \mathfrak{p}$ implicitly via

$$\langle\cdot,\cdot\rangle = B|_{\mathfrak{p}}(S\cdot,\cdot).$$

As both scalar products are K-invariant S must commute with all elements of K. Now S is symmetric and therefore diagonalizable over $\mathbb{R}$. All eigenspaces are K-invariant subspaces, but the action of K on $\mathfrak{p}$ is irreducible so S must be of the form $\lambda^2\cdot\mathrm{id}$ for some $\lambda \neq 0$.

Curvature of symmetric spaces [24, Ch. IV.4]:
As the sectional curvature is invariant under isometries it is enough to determine sectional curvature at the base point. At the base point o we can exploit the fact that $T_oM \cong \mathfrak{p}$. We can thus express the curvature tensor R of M at o in terms of the curvature tensor of G ([24, Thm. 4.2, Ch. IV]), i.e.

$$R(a,b)c(o) = -[[a,b],c] \quad \forall a,b,c \in \mathfrak{p} \cong T_oM. \tag{2.4}$$

Recall that the Riemannian metric g on M is induced by $+\lambda^2 B$ resp. $-\lambda^2 B$ for M of compact resp. non-compact type and some real constant $\lambda \neq 0$. As

$$g_o(a, R(a,b)b) = \mp\lambda^2 B(a,[[a,b],b]) = \pm\lambda^2 B([a,b],[a,b]) \quad \forall a,b \in \mathfrak{p}$$

and $B|_{\mathfrak{k}\times\mathfrak{k}} > 0$, it follows that if M is of compact resp. non-compact type it has non-negative resp. non-positive sectional curvature. The sectional curvature of spaces of Euclidean type vanishes identically [24, Thm. 3.1, Ch. V.3].

Totally geodesic subspaces [24, Ch. IV.7]:
A submanifold $N \subset M$ is *geodesic* at $p \in N$ if for all $v \in T_pN \subset T_pM$ the M-geodesic with tangent v is contained in N. The submanifold is *totally geodesic* if it is geodesic at every point in N. A totally geodesic submanifold of a symmetric space is itself a symmetric space as the geodesic symmetries restrict to the submanifold. The algebraic characterization of totally geodesic submanifolds of symmetric spaces leads to the notion of *Lie triple systems.*

Definition 2.4.5. *Let $\mathfrak{g}$ be a Lie algebra. A Lie triple system is a vector space $\mathfrak{n} \subset \mathfrak{g}$ such that*

$$[\mathfrak{n},[\mathfrak{n},\mathfrak{n}]] \subset \mathfrak{n}.$$

Indeed, $N \subset M$ is a totally geodesic submanifold containing o if $\mathfrak{n} := \mathrm{d}\pi^{-1}(T_oN) \subset \mathfrak{p}$ is a Lie triple system. Conversely, if $\mathfrak{n} \subset \mathfrak{p}$ is a Lie triple system, then $N := \exp_o \mathrm{d}\pi(\mathfrak{n})$ is a totally geodesic submanifold [24, Thm. 7.2, Ch. IV].

Maximal flat subspaces [24, Ch. V.6] :
A Riemannian manifold F is *flat* if all sectional curvatures vanish identically. A *maximal flat* $F \subset M$ is a flat submanifold that is not contained in a flat submanifold of higher dimension. Via the exponential map, maximal flats F correspond one-to-one to maximal abelian subalgebras $\mathfrak{a} \subset \mathfrak{p}$ [24, Thm. 6.1, Ch. V].

Theorem 2.4.6 ([24] Thm. 6.2, Ch. V)**.** *Let M be a symmetric space of compact/ non-compact type and $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathfrak{p}$ maximal abelian subalgebras, then there exists a $k \in K$ such that*

$$\mathrm{Ad}_k(\mathfrak{a}_1) = \mathfrak{a}_2.$$

The Theorem yields a well-defined notion of the rank of a symmetric space.

Definition 2.4.7. *The rank of M is the dimension of maximal flats.*

Locally symmetric spaces:
We call a Riemannian manifold (M, g) *locally symmetric* if it is isometrically covered by a symmetric space. Locally symmetric spaces can also be characterized by the following theorem.

Theorem 2.4.8 ([24] Thm. 1.1, Ch. IV)**.** *A Riemann manifold (M, g) is locally symmetric if and only if the Riemannian curvature tensor is parallel, i.e. $\nabla R = 0$ where ∇ denotes the Levi-Civita connection.*

2.4.2. Hermitian symmetric spaces

Until now we did not discuss any relations between symmetric spaces and symplectic manifolds. We will see that Hermitian symmetric spaces are precisely at the intersection, i.e. they are symmetric spaces with an G-invariant symplectic form.

Definition 2.4.9. *A Hermitian symmetric space is a connected complex manifold with Hermitian structure (M, g, j), such that the geodesic reflection at any point is a holomorphic isometry of M. Explicitly this means for any point $p \in M$ there is a holomorphic isometry $s_p : M \to M$ that satisfies*

$$s_p(p) = p \quad \& \quad (\mathrm{d}s_p)_p = -\mathrm{id}.$$

All Hermitian symmetric spaces are symmetric spaces, so everything discussed in the previous section continues to hold. Nevertheless the class of Hermitian symmetric spaces is much smaller than the class of symmetric space. Indeed irreducible Hermitian symmetric spaces can be characterized by the following theorem.

Theorem 2.4.10 ([24] Thm. 6.1. Ch. VIII)**.** *(i) The compact irreducible Hermitian symmetric spaces are exactly the manifolds G/K where G is a connected compact simple Lie group with center $\{e\}$ and K has nondiscrete center and is a maximal connected proper subgroup of G.*

(ii) The noncompact irreducible Hermitian symmetric spaces are exactly the manifolds $G^\vee/K$ where $G^\vee$ is a connected noncompact simple Lie group with center $\{e\}$ and K has nondiscrete center and is a maximal compact subgroup of $G^\vee$.

The center of K can be described more accurately in both cases compact and noncompact.

Proposition 2.4.11 ([24] Thm. 6.1. Ch. VIII)**.** *The center $C(K)$ of the group K in Theorem 2.4.10 (i) and (ii) is analytically isomorphic to the circle group.*

Uniqueness of invariant complex structure:
By Prop. 2.1.3 we can identify $C(K) \cong S^1$ and so the Lie algebra of $C(K)$ is identified with $i\mathbb{R}$. Denote $Z \in \mathfrak{g}$ the element that corresponds to i under this identification. Now $A := \mathrm{ad}_Z$ is an antisymmetric endomorphism of $\mathfrak{p}$ which is also Ad_K-invariant. Thus A^2 is symmetric, negative definite and Ad_K-invariant. By the same argument (similar to Schur's lemma) as in the proof of uniqueness of the invariant metric, this means $\mathrm{ad}_Z^2 = -\lambda^2 \mathrm{id}_{\mathfrak{p}}$ for some $\lambda \in \mathbb{R}$. Finally, $\lambda = 1$ because $e^{2\pi t A}$ has eigenvalue $e^{\pm 2\pi t i \lambda}$ and for $t = 1$ this eigenvalue must be equal to 1 since $e^{2\pi A}$ is the identity.

Thus $j_o = \mathrm{ad}_Z$ defines a complex structure on $T_oM \cong \mathfrak{p}$. Observe that j_o is K-invariant as for all $k \in \mathfrak{k}$ and $v \in T_oM \cong \mathfrak{p}$ we have

$$\mathrm{ad}_k(j_o(v)) = \mathrm{ad}_k([Z, v]) = -[Z, [v, k]] - [v, [k, Z]] = [Z, \mathrm{ad}_k v] = j_o(\mathrm{ad}_k(v)).$$

Therefore we can extend j_o equivariantly to a G-invariant almost complex structure j on M. Now the invariant metric g and the invariant almost complex structure j determine an invariant Hermitian metric h on $TM^{\mathbb{C}}$. Analogously to the proof of uniqueness of the Riemannian metric g one shows that G-invariant hermitian metric h on $TM^{\mathbb{C}}$ is also unique up to scalar multiple. To do so observe that j promotes the adjoint representation to a complex irreducible representation on $\mathfrak{p}^{\mathbb{C}}$. In particular one can directly apply Schur's lemma to obtain uniqueness. As g and h are unique up to scalar multiple, the complex structure j is unique up to sign. In particular the almost complex structure j must up to sign coincide with the complex structure we started with. Therefore j must be integrable. Actually one can also independently show that j must be integrable. We do this by computing Nijenhuis-tensor N_j at o. This is enough as Nijenhuis-tensor is invariant under j-biholomorphisms. For all $a, b \in \mathfrak{p} \cong T_oM$ we have

$$\begin{aligned}
N_j(a,b)(o) &:= \Big([a^\#, b^\#] + j([j(a^\#), b^\#] + [a^\#, j(b^\#)]) - [ja^\#, jb^\#]\Big)(o) \\
&= ([a,b] + [Z, [[Z,a], b] + [a, [Z,b]]] - [[Z,a],[Z,b]])^\# (o) \\
&= ([a,b] - [[Z,a],[Z,b]])^\# (o) \\
&= ([a,b] + [Z, [b, [Z,a]]] + [b, [[Z,a], Z]])^\# (o) \\
&= ([a,b] + [b,a])^\# (o) = 0,
\end{aligned}$$

where the third and fifth equality use $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and Z in the center of $\mathfrak{k}$ and the fourth uses the Jacobi-identity. This proves that there are no almost hermitian symmetric spaces.

Intermezzo – Root systems:
Let $\mathfrak{g}^{\mathbb{C}}$ be a semi-simple complex Lie algebra. A *Cartan subalgebra* $\mathfrak{h}^{\mathbb{C}}$ is a maximal abelian subalgebra such that, for each $h \in \mathfrak{h}^{\mathbb{C}}$, ad_h is diagonalizable. In particular, the operators ad_h can be diagonalized simultaneously. This leads to the definition of roots and root spaces.

Definition 2.4.12. *A root of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ is a non-zero linear form $\alpha : \mathfrak{h}^{\mathbb{C}} \to \mathbb{C}$ such that the corresponding root space*

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}^{\mathbb{C}} \mid \mathrm{ad}_h X = \alpha(h) X \ \forall\ h \in \mathfrak{h}^{\mathbb{C}}\}$$

is nonzero. Denote the set of roots by Δ.

The root spaces are simultaneous eigenspaces of ad_h, for all $h \in \mathfrak{h}^{\mathbb{C}}$, and we get a decomposition of $\mathfrak{g}^{\mathbb{C}}$ into the direct sum of root spaces

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

The subspace $\mathfrak{g}_0$ is the centralizer of $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$. As for example explained in [45, Ch. 3, Thm. 3], we have that

$$\mathfrak{g}_0 = \mathfrak{h}^{\mathbb{C}}.$$

Note that using the Jacobi-identity $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, in particular $\mathfrak{h}_\alpha := [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}^{\mathbb{C}}$ for all $\alpha, \beta \in \Delta$. Indeed, one can show ([45, Ch. VI, Thm. 2]) that for all roots $\alpha \in \Delta$ the spaces $\mathfrak{g}_\alpha$ and $\mathfrak{h}_\alpha$ are one-dimensional and that there exists a unique element $H_\alpha \in \mathfrak{h}_\alpha$ determined by $\alpha(H_\alpha) = 2$. It is then easy to see that for each non-zero element $X_\alpha \in \mathfrak{g}_\alpha$ there exists an element Y_α in $\mathfrak{g}_{-\alpha}$ such that

$$[H_\alpha, X_\alpha] = 2X_\alpha, \quad [H_\alpha, Y_\alpha] = -2Y_\alpha \text{ and } [X_\alpha, Y_\alpha] = H_\alpha.$$

These elements generate a copy of $\mathfrak{sl}(2, \mathbb{C})$ that we shall denote by $\mathfrak{g}[\alpha]$.

Polyspheres/ Polydiscs:
We go back to our set up before the intermezzo. Denote by $\mathfrak{g}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$. It decomposes as

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-,$$

where $\mathfrak{k}^{\mathbb{C}}$ is the complexification of $\mathfrak{k}$ and $\mathfrak{p}_\pm$ are the $\pm i$-eigenspaces of the complex linear extension of $j = \mathrm{ad}_Z$. Assume without loss of generality that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the compact real form and denote by $\mathfrak{g}^\vee = \mathfrak{k} \oplus i\mathfrak{p}$ its non-compact dual. As described in [24, Ch. VIII.7] the maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{k}$ complexifies to a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$. We denote by Δ the set of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. Since $[\mathfrak{h}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}] \subset \mathfrak{k}^{\mathbb{C}}$ and $[\mathfrak{h}^{\mathbb{C}}, \mathfrak{p}^{\mathbb{C}}] \subset \mathfrak{p}^{\mathbb{C}}$ the root space $\mathfrak{g}_\alpha$ is either contained in $\mathfrak{k}^{\mathbb{C}}$ or $\mathfrak{p}^{\mathbb{C}}$. The roots are called *compact* or *non-compact*, respectively. In particular,

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha, \qquad \mathfrak{p}^{\mathbb{C}} = \bigoplus_{\beta} \mathfrak{g}_\beta$$

where α runs over compact roots and β runs over all non-compact roots. Furthermore one can partition into positive and negative non-compact roots according to the sign of $-i\beta(Z)$. Indeed, this is compatible with the decomposition $\mathfrak{p}^{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ and we can write

$$\mathfrak{p}_+ = \bigoplus_{\beta} \mathfrak{g}_\beta, \qquad \mathfrak{p}_- = \bigoplus_{\beta} \mathfrak{g}_{-\beta},$$

where β runs over all positive non-compact roots. Two roots $\alpha, \beta \in \Delta$ are called *strongly orthogonal* if $\alpha \pm \beta \notin \Delta$, which implies $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$. By [24, Prop. 7.4, Ch. VIII] there exist strongly orthogonal positive non-compact roots $\gamma_1, \ldots, \gamma_r$. Thus, the subspace

$$\bigoplus_{i=1}^{r} \mathfrak{g}^{\mathbb{C}}[\gamma_i] \subset \mathfrak{g}^{\mathbb{C}}$$

is isomorphic to $\mathfrak{sl}(2,\mathbb{C})^r$. The intersection with $\mathfrak{g}$ resp. $\mathfrak{g}^\vee$ yield subalgebras of $\mathfrak{g}$ isomorphic to the compact real form $\mathfrak{su}(2)^r$ resp. the dual non-compact real form $\mathfrak{sl}(2,\mathbb{R})^r$. Intersecting these with $\mathfrak{p}$ resp. $i\mathfrak{p}$ yield Lie-triple systems in $\mathfrak{g}$ resp. $\mathfrak{g}^\vee$ and thus realize totally geodesically embedded polyspheres resp. polydiscs of G/K resp. $G^\vee/K$. Indeed Hermitian symmetric space corresponding to $\mathfrak{su}(2)$ is $(\mathbb{CP}^1)^r$, while the Hermitian symmetric space corresponding to $\mathfrak{sl}(2,\mathbb{R})$ is $(\mathbb{CH}^1)^r$. This polysphere resp. polydisc obtained be integrating the copy $\mathfrak{su}(2)^r$ resp. $\mathfrak{sl}(2,\mathbb{R})^r$ can translated by the adjoint action of G to see that there is a polysphere resp. polydisc through every point. In total we obtain the polysphere resp. polydisc theorem.

Theorem 2.4.13 (Polysphere/ polydisc theorem [52], p. 280)**.** *Let M be an irreducible Hermitian symmetric space of rank r. For any point $q = (x, v) \in TM$, there exists a point $p = (x_0, v_0) \in T\Sigma^r$ and a holomorphic totally geodesic embedding*

$$\iota_{p,q} : \Sigma^r = \Sigma \times \ldots \times \Sigma \hookrightarrow M$$

such that

$$\iota_{p,q}(x_0) = x \quad \text{and} \quad (\mathrm{d}\iota_{p,q})_{x_0} v_0 = v.$$

Here, $\Sigma = \mathbb{CP}^1$ in the compact case and $\Sigma = \mathbb{CH}^1$ in the non-compact case.

Remark 2.4.14. *These embeddings are equivariant in a double sense. We denote by H either* $\mathrm{SU}(2)$ *in the compact case or* $\mathrm{SL}(2,\mathbb{R})$ *in the non-compact case.*

Translation and reparametrization: *For all $g \in G$ and $h \in H^r$ the following diagram commutes*

$$\begin{array}{ccc} \Sigma^r & \xhookrightarrow{\iota_{p,q}} & M \\ \big\downarrow h & & \big\downarrow g \\ \Sigma^r & \xhookrightarrow{\iota_{hp,gq}} & M \end{array}.$$

Here the action $h : \Sigma \to \Sigma$ should be interpreted as reparametrization, while the arrow $g : M \to M$ translates a polysphere resp. polydisc through $q = (x, v)$ to a polysphere resp. polydisc through $gq = (g(x), \mathrm{d}g(x))$. We say a polysphere resp. polydisc goes through $q = (x, v)$ if it goes through x and v is tangent to it.

H^r-equivariance of $\iota_{p,q}$: *As discussed above Theorem 2.4.13 every embedding $\iota_{p,q}$ comes from a Lie algebra monomorphism $k : \mathfrak{h}^r \hookrightarrow \mathfrak{g}$. This can be integrated to a monomorphism of Lie groups we denote by $R : H^r \hookrightarrow G$. Then $\iota_{p,q}$ is also equivariant with respect to R, i.e. for all $h \in H^r$ the following diagram commutes*

$$\begin{array}{ccc} \Sigma^r & \xhookrightarrow{\iota_{p,q}} & M \\ \big\downarrow h & & \big\downarrow R(h) \\ \Sigma^r & \xhookrightarrow{\iota_{p,q}} & M \end{array}.$$

Remark 2.4.15. *From now on we will not need root systems again. One nice thing about the proofs in this thesis is, that they only use the polysphere/ polydisc theorem and no root systems explicitly. They were only included to convince the reader that the polysphere/ polydisc theorem holds, but one could equally well just use this theorem as a black box.*

Foliation of the tangent bundle of a Hermitian symmetric space:
The polysphere resp. polydisc theorem tells us that for every point $(x,v) \in TM$ there is a polysphere resp. polydisc through x with v tangent to it. We want to investigate now if these $T\Sigma^r$, where $\Sigma = \mathbb{C}\mathrm{P}^1$ resp. $\Sigma = \mathbb{C}\mathrm{H}^1$, form a foliation. It is not hard to see that through some points (for example points on the zero section) go more than one $T\Sigma^r$. So our foliation will be singular, but we can characterize an open dense set of points, where the foliation is not singular.

An element $v \in \mathfrak{p}$ is called *regular* if its centralizer

$$Z_{\mathfrak{p}}(v) := \{w \in \mathfrak{p} \mid [v,w] = 0\}$$

has dimension as small as possible. The smallest dimension possible is equal to r the rank of M. This is because $Z_{\mathfrak{p}}$ can be identified with the union of all maximal abelian subalgebras $\mathfrak{a} \subset \mathfrak{p}$ containing v, i.e.

$$Z_{\mathfrak{p}}(v) = \cup \mathfrak{a}.$$

The inclusion $\cup\mathfrak{a} \subset Z_{\mathfrak{p}}(v)$ is immediate. On the other hand any element $w \in Z_{\mathfrak{p}}(v)$ satisfies $[v,w] = 0$ and can thus be extended to a maximal abelian subspace $\mathfrak{a}$ containing v and w. In particular regular vectors lie in a unique maximal abelian subspace explicitly given by

$$\mathfrak{a}_v = Z_{\mathfrak{p}}(v).$$

One can show that the set of regular vectors is open and dense [45, Prop. 1, Ch.III.2]. We call a point $(x,v) \in TM$ *regular* if $\mathrm{Ad}_g v \in \mathfrak{p}$ is regular for $g \in G$ such that $\mathrm{Ad}_g(x) = o$. The set of regular points is denoted by $T^{\mathrm{reg}}M$. Observe that picking r vectors, each tangent to a factor in the polysphere resp. polydisc through x, we obtain a maximal abelian subspace of $\mathrm{Ad}_{g^{-1}}\mathfrak{p}$. Thus there is up to reparametrization only a unique polysphere resp. polydisc through a regular point $(x,v) \in T^{\mathrm{reg}}M$. As all maximal flats are conjugate (see Thm. 2.4.6) the same holds true for their complexifications the polyspheres/ polydiscs.

Theorem 2.4.16. *Every polysphere/ polydisc through o can be mapped to any other polysphere/ polydisc through o by an element of K.*

In view of Remark 2.4.14 we obtain a smooth foliation of $T^{\mathrm{reg}}M$ by $T^{\mathrm{reg}}\Sigma^r$. In particular, we can define locally on a neighborhood $U \subset T^{\mathrm{reg}}M$ of a regular point (x,v) the projections

$$\pi_i : U \to T\Sigma$$

on the i-th factor of the product $T\Sigma^r$. In addition the following quantities are locally well-defined and smooth

$$v_i := \pi_i(v), \quad r_i := |v_i|, \quad X_i := (v_i)^{\mathcal{H}}, \quad H_i := (jv_i)^{\mathcal{H}}, \quad Y_i := (v_i)^{\mathcal{V}}, \quad V_i := (jv_i)^{\mathcal{V}}.$$

There are two distributions on $T^{\mathrm{reg}}M$ that we will need later, denote

$$\Upsilon := \mathrm{span}_{\mathbb{R}}\{Y_1, \ldots, Y_r\} \subset TT^{\mathrm{reg}}M \quad \text{and} \quad \mathcal{D} := \{a^{\#} | a \in \mathfrak{g}\} \subset TT^{\mathrm{reg}}M.$$

At $p = (o,v) \in T^{\mathrm{reg}}M$ for any $v \in T_o^{\mathrm{reg}}M$ we can identify the following sub spaces

$$\Upsilon_p = \mathrm{span}_{\mathbb{R}}\{Y_1, \ldots, Y_r\}|_{(o,v)} \cong (\mathfrak{a}_v)^{\mathcal{V}} \quad \text{and} \quad \mathcal{D}_p = \mathfrak{p}^{\mathcal{H}} \oplus [\mathfrak{k}, v]^{\mathcal{V}}.$$

Lemma 2.4.17. *If $v \in \mathfrak{p}$ is regular, then*

$$\mathfrak{a}_v^{\perp} = [\mathfrak{k}, v],$$

where $\perp$ denotes the orthogonal with respect to the Killing form B.

Proof. Take $k \in \mathfrak{k}$, then for all $w \in \mathfrak{a}_v$ we have

$$B([k, v], w) = -B(k, [w, v]) = 0$$

as $[w, v] = 0$. Thus $[k, v] \in \mathfrak{a}_v^{\perp}$. It remains to be shown that $\mathfrak{a}_v^{\perp} \subset [\mathfrak{k}, v]$. We show instead $[\mathfrak{k}, v]^{\perp} \subset \mathfrak{a}_v$. For this take $w \in [\mathfrak{k}, v]^{\perp}$, then

$$B([k, v], w) = 0 \quad \forall k \in \mathfrak{k} \Rightarrow \quad B(k, [v, w]) = 0 \quad \forall k \in \mathfrak{k} \Rightarrow \quad [v, w] \in \mathfrak{k}^{\perp}.$$

On the other hand, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Hence, $[v, w] = 0$ and therefore $w \in Z_{\mathfrak{p}}(v)$. As v is regular we have $Z_{\mathfrak{p}}(v) = \mathfrak{a}_v$ and the claim follows. □

Corollary 2.4.18. *At every regular $p \in TM$ we have*

$$T_pTM = \mathcal{D}_p \oplus \Upsilon_p.$$

The next question is, what happens at the non-regular points? Indeed, the foliation becomes singular at non-regular points. Through every non-regular point go more than one polysphere/ polydisc. This is expressed by the fact that the distributions Υ and $\mathcal{D}$ become lower dimensional on singular points.

2.4.3. Hermitian symmetric spaces as coadjoint orbits

As Corollary of Theorem 2.4.10 and Proposition 2.4.11 we can finally deduce the realization of Hermitian symmetric spaces as coadjoint orbits. The corollary is known to the experts, but as we could not find a reference the proof is included here.

Corollary 2.4.19. *Every Hermitian symmetric space can be realized as (co-)adjoint orbit.*

Proof. We prove the compact case, the noncompact case follows by duality. As $C(K)$ is analytically isomorphic to the circle group there exists an element $z \in C(K)$ different from the unit. Now K is a sub group of the centralizer $C_G(z)$ of z in G. As the center of G is trivial, we have $C_G(z) \neq G$ so K coincides with the identity component of $C_G(z)$ by maximality of K. Denote by Z a generator of $C(K)$. Clearly, on the one hand, $K \subset \mathrm{Stab}_G(Z)$ and on the other hand, $\mathrm{Stab}_G(Z) \subset C_G(z)$. Thus

$$\mathrm{Stab}_G(Z) = K,$$

as stabilizers of simple groups are connected and therefore we may identify G/K with the (co-)adjoint orbit O_Z of G at $Z \in \mathfrak{g}$. □

Kähler structure:
The last question that needs to be answered is, whether the KKS symplectic structure σ of $O_Z \cong M$ complements the hermitian structure (g, j) to a Kähler structure. As σ, g and j are G-invariant it is enough to check compatibility at $Z \in O_Z \subset \mathfrak{g}$, indeed for all $a, b \in \mathfrak{p} \cong T_Z O_Z$

$$g_Z(j_Z a, b) = -B([Z, a], b) = -B(Z, [a, b]) = \sigma_Z(a, b).$$

By uniqueness of g and j we obtain the following theorem.

Theorem 2.4.20. *The G-invariant triple (g, j, σ), defined on $\mathfrak{p} \cong T_Z M$ as*

$$g_Z(\cdot,\cdot) := -B(\cdot,\cdot), \quad j_Z(\cdot) := [Z,\cdot] \quad \text{and} \quad \sigma_Z(\cdot,\cdot) := -B(Z,[\cdot,\cdot])$$

is compatible and equivariantly extends to the up to scalar multiple unique invariant Kähler structure of $M \cong O_Z$.

The following Lemma will be useful for future calculations.

Lemma 2.4.21. *At any point $x \in M \cong O_Z \subset \mathfrak{g}$ the Kähler structure is given by*

$$g_x(v,w) := -B(v,w), \quad j_x(v) := [x,v] \quad \text{and} \quad \sigma_x(v,w) := -B(x,[v,w]),$$

for all $v, w \in T_x M \cong [x,\mathfrak{g}] \subset \mathfrak{g}$.

Proof. We prove the formulas for the metric and the complex structure, the formula for the symplectic form follows. Clearly there exists an element $g \in G$ such that $x = \mathrm{Ad}_g(Z)$. Using this we find for the metric

$$g_x(v,w) := g_Z(\mathrm{Ad}_{g^{-1}}v, \mathrm{Ad}_{g^{-1}}w) = -B(\mathrm{Ad}_{g^{-1}}v, \mathrm{Ad}_{g^{-1}}w) = -B(v,w)$$

as the Killing form is Ad_G-invariant. For the complex structure we similarly find

$$j_x(v) := \mathrm{Ad}_g\left(j_Z\left(\mathrm{Ad}_{g^{-1}}v\right)\right) = \mathrm{Ad}_g[Z, \mathrm{Ad}_{g^{-1}}v] = [\mathrm{Ad}_g Z, v] = [x,v].$$

□

De Rham cohomolgy:
We will quickly determine the second de Rham cohomology of hermitian symmetric spaces of compact type. We did not find a proof in the literature, so we present what we learned from discussions with Maria Beatrice Pozzetti.

Proposition 2.4.22. *Let (M,g) be an irreducible hermitian symmetric space of compact type and denote $\sigma \in \Omega^2(M)$ the corresponding invariant Kähler form. Then the second de Rham cohomology group is generated by $[\sigma]$, i.e.*

$$H^2_{\mathrm{dR}}(M,\mathbb{R}) \cong \mathbb{R}.$$

Proof. Denote by $\Omega^2_G(M)$ the set of G-invariant 2-forms on M. Every $\nu \in \Omega^2_G(M)$ is closed. This can be seen as follows. For any point $p \in M$ denote by s_p the geodesic symmetry, then on the one hand

$$s_p^*\nu = (-1)^2\nu = \nu$$

on the other hand

$$s_p^*\mathrm{d}\nu = (-1)^3\mathrm{d}\nu = -\mathrm{d}\nu,$$

using that G acts transitively and that $s_p \circ g \circ s_p^{-1} \in G$ for all $g \in G$ implies that $s_p^*\nu$ is also G-invariant. In total this means

$$\mathrm{d}\nu = -\mathrm{d}\nu = 0.$$

Further every de Rham cohomology class $\alpha \in H^2_{\mathrm{dR}}(M)$ can be represented by an invariant form. Let μ be a G bi-invariant probability measure on G. We define the G-average of a 2-form $\eta \in \Omega^2(M)$ with respect to μ as

$$\bar{\nu}_p(v,w) := \int_G (g^*\nu)_p(v,w)\ \mathrm{d}\mu(g) \quad \text{for all } v,w \in T_pM.$$

Then for any closed 2-dimensional submanifold $\Sigma \subset M$ we have

$$\begin{aligned}\bar{\nu}(\Sigma) &= \int_\Sigma \bar{\nu} = \int_\Sigma \left(\int_G (g^*\nu)_p(\partial_s\Sigma, \partial_t\Sigma)\mathrm{d}\mu(g)\right)\mathrm{d}s\mathrm{d}t \\ &= \int_G \left(\int_\Sigma (g^*\nu)_p(\partial_s\Sigma, \partial_t\Sigma)\mathrm{d}s\mathrm{d}t\right)\mathrm{d}\mu(g) = \int_G \nu(g(\Sigma))\mathrm{d}\mu(g) \\ &= \int_G \nu(\Sigma)\mathrm{d}\mu(g) = \nu(\Sigma),\end{aligned}$$

where we think of $\Sigma(s,t)$ as a parametrization of $\Sigma \subset M$. All that is left to do is to show that there is up to scalar multiple only one invariant 2-form. Take some $\nu \in \Omega^2_G(M)$, then there exists a K-invariant symmetric operator $A : \mathfrak{p} \to \mathfrak{p}$ satisfying $\nu(A\cdot,\cdot) = \sigma(\cdot,\cdot)$. Then, by the same argument as in the proof of uniqueness of the invariant metric g, A must be a multiple of the identity, because the representation of K on $\mathfrak{p}$ is irreducible. □

Moment maps:
We will later in this thesis study the induced action of G on the tangent bundle of $M = O_Z$. Actually, the action can also be seen as the restriction of the diagonal adjoint action of G on $\mathfrak{g} \times \mathfrak{g}$ to

$$TM = \left\{(x,v) \in \mathfrak{g} \times \mathfrak{g} \mid x = \mathrm{Ad}_g(Z),\ v \in \mathrm{ann}(x)^\perp\right\},$$

where $\mathrm{ann}(x) = \{\eta \in \mathfrak{g} \mid [\eta, x] = 0\}$. In view of this, we see that evaluated at a point $(x,v) \in TM$ the induced vector field $a^\#$ takes the form

$$a^\#_{(x,v)} = ([a,x],[a,v]) \in \mathfrak{g} \times \mathfrak{g}.$$

This representation of $a^\#$ will be useful for what comes. By construction the 1-forms λ, η and τ are invariant under isometries and consequently G-invariant. It therefore makes sense to ask whether there exist moment maps for $\mathrm{d}\lambda, \mathrm{d}\eta$ and $\mathrm{d}\tau$.

Theorem 2.4.23. *The G-action on TM is Hamiltonian with respect to the three symplectic[4] forms $\mathrm{d}\lambda, \mathrm{d}\eta$ and $\mathrm{d}\tau$. The moment maps are respectively given by*

$$\mu_\lambda(x,v) = [x,v], \quad \mu_\eta(x,v) = v, \quad \mu_\tau(x,v) = -[[x,v],v] \quad \forall (x,v) \in TM \subset \mathfrak{g} \times \mathfrak{g},$$

using the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form B.

Proof. The maps are clearly equivariant, as commutators are. Further, for any $a \in \mathfrak{g}$, $(x,v) \in TM$ we have

$$\begin{aligned}\mathrm{d}(B(\mu_\lambda(x,v),a)) &= \mathrm{d}(B([x,v],a)) = \mathrm{d}(B(v,[x,a])) \\ &= \mathrm{d}(g(v, \mathrm{d}\pi(a^\#)) = \mathrm{d}(\hat{g}(X, a^\#)) = \mathrm{d}(\lambda(a^\#)) = \iota_{a^\#}\mathrm{d}\lambda\end{aligned}$$

[4] Actually, $\mathrm{d}\tau$ is only symplectic outside the zero-section. Still its moment map is globally defined.

as λ is invariant under the flow of $a^{\#}$ and as a consequence $\mathcal{L}_{a^{\#}}\lambda = 0$. Analogously we find

$$\mathrm{d}(B(\mu_\eta(x,v),a)) = \mathrm{d}(B(v,a)) = \mathrm{d}(B([x,v],[x,a])) = \mathrm{d}(g(H,a^{\#})) = \mathrm{d}(\eta(a^{\#})) = \iota_{a^{\#}}\mathrm{d}\eta$$

and

$$\begin{aligned}\mathrm{d}(B(\mu_\tau(x,v),a)) &= \mathrm{d}(B([[x,v],v],a)) = -\mathrm{d}(B([x,v],[v,a])) \\ &= -\mathrm{d}(g(V,a^{\#})) = -\mathrm{d}(\tau(a^{\#})) = -\iota_{a^{\#}}\mathrm{d}\tau.\end{aligned}$$

□

Polyspheres resp. polydiscs as suborbits:
We want to give an explicit description of the polyspheres resp. polydiscs in Theorem 2.4.13 as suborbits. First we fix some notation. We denote by Σ_i the i-th factor of Σ^r. Every factor can be realized as an adjoint orbit in $\mathfrak{h}$. Here $\mathfrak{h}$ denotes either $\mathfrak{su}(2)$ in the compact case or $\mathfrak{sl}(2,\mathbb{R})$ in the non-compact case. Denote Z_i the up to sign unique element in the center of $\mathfrak{h}$ such that $\mathrm{ad}_{Z_i}^2 = -\mathrm{id}$. Then $\Sigma_i \cong O_{Z_i}$ and the standard Kähler structure coincides with the Kähler structure obtain as in Theorem 2.4.20 up to multiple.

From the discussion above Theorem 2.4.13 we know that every polysphere resp. polydisc

$$\iota_{p,q} : \Sigma^r \hookrightarrow M$$

comes from integrating a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{su}(2)^r$ resp. $\mathfrak{sl}(2,\mathbb{R})^r$. In particular for every embedding $\iota_{p,q}$ there is an injective Lie algebra homomorphism

$$k_{p,q} : \mathfrak{h}^r \hookrightarrow \mathfrak{g}$$

such that $(\mathrm{d}\iota_{p,q})_{x_0} = k_{p,q}|_{T_{x_0}\Sigma^r}$, where $p = (x_0, v_0) \in T\Sigma^r$. By equivariance of the embedding (see Remark 2.4.14, translation and reparametrization), we may restrict to $p = (Z_0 := \sum_i Z_i, v_0)$ for some $v_0 \in T_{Z_0}\Sigma$ and $q = (Z, v)$ for some $v \in T_Z M$. We abbreviate $\iota := \iota_{p,q}$ and $k := k_{p,q}$.

Proposition 2.4.24. *The affine linear map*

$$K : \mathfrak{h}^r \to \mathfrak{g};\ \xi \mapsto k(\xi) + Z - k(Z_0)$$

extends $\iota : \Sigma^r \to M$ equivariantly with respect to the adjoint action of $H^r \subset G$. This means the following diagrams commute

(Extension)
$$\begin{array}{ccc} \Sigma^r & \xrightarrow{\ \iota\ } & M \\ \downarrow & & \downarrow \\ \mathfrak{h}^r & \xrightarrow{\ K\ } & \mathfrak{g} \end{array}, \tag{2.5}$$

where the vertical arrows are the inclusions as coadjoint orbits and

(Equivariance)
$$\begin{array}{ccc} \mathfrak{h}^r & \xrightarrow{\ K\ } & \mathfrak{g} \\ {\scriptstyle \mathrm{Ad}_h}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_{R(h)}} \\ \mathfrak{h}^r & \xrightarrow{\ K\ } & \mathfrak{g} \end{array} \tag{2.6}$$

for all $h \in H^r$, where $R : H^r \hookrightarrow G$ denotes as is Remark 2.4.14 the monomorphism of Lie groups that integrates k, i.e. $(\mathrm{d}R)_e = k$.

Proof. To prove the Lemma we first need to show that $Z - k(Z_0)$ is invariant under $R(H^r)$, i.e. we need to show that for any $\xi \in \mathfrak{h}^r$ the following commutator vanishes

$$[k(\xi), Z - k(Z_0)] = 0. \tag{2.7}$$

Denote the Cartan decomposition of $\mathfrak{h}^r$ as $\mathfrak{h}^r = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. The map k respects the Cartan decomposition, i.e. $k(\mathfrak{k}_0) \subset \mathfrak{k}$ and $k(\mathfrak{p}_0) \subset \mathfrak{p}$. To prove Eq. (2.7) we look at two cases $\xi \in \mathfrak{k}_0$ and $\xi \in \mathfrak{p}_0$.

Case $\xi \in \mathfrak{k}_0$: We see that

$$[k(\xi), Z - k(Z_0)] = [k(\xi), Z] - [k(\xi), k(Z_0)] = k[\xi, Z_0] = 0,$$

where the second equality uses $k(\xi) \in \mathfrak{k}$ and Z in the center of K and that k is a Lie algebra homomorphism. The last equality uses that Z_0 is in the center of $\mathfrak{h}^r$.

Case $\xi \in \mathfrak{p}_0$: As $\xi \in \mathfrak{p}_0 \cong T_{Z_0}\Sigma^r$ we can use $k(\xi) = \mathrm{d}\iota_{Z_0}(\xi)$, it follows that

$$[k(\xi), Z - k(Z_0)] = [\mathrm{d}\iota_{Z_0}(\xi), Z] - k([\xi, Z_0]) = j_Z \mathrm{d}\iota_{Z_0}(\xi) - \mathrm{d}\iota_{Z_0}(j_{Z_0}\xi) = 0,$$

where we used again that k is a Lie algebra homomorphism, that $j_Z = \mathrm{ad}_Z$ and $j_{Z_0} = \mathrm{ad}_{Z_0}$ and in the last equation that ι is holomorphic.

Further $k : \mathfrak{h}^r \hookrightarrow \mathfrak{g}$ is H^r-equivariant as $k = (\mathrm{d}R)_e$ and R satisfies

$$R(\mathrm{Ad}_h(\tilde{h})) = R(h\tilde{h}h^{-1}) = R(h)R(\tilde{h})R(h)^{-1} = \mathrm{Ad}_{R(h)}R(\bar{h}) \quad \forall\, h, \tilde{h} \in H^r,$$

as it is a Lie group homomorphism.

Equivariance of k and invariance of $Z - k(Z_0)$ imply equivariance of K, i.e. diagram (2.6). Last we need to check if K extends ι. As K and ι are equivariant and H^r acts transitively on Σ^r it is enough to check this at one point $p \in \Sigma^r$. We choose $p = Z_0$ and find

$$K(Z_0) = k(Z_0) + Z - k(Z_0) = Z = \iota(Z_0).$$

This proves diagram (2.5) and thus finishes the proof of the Proposition. □

3. Preliminaries II: Symplectic topology

3.1. Hofer–Zehnder capacity

The Hofer–Zehnder capacity is a certain symplectic invariant. The term capacity refers to a class of symplectic invariants that measure the size of a symplectic manifold. A first observation is that every symplectomorphism preserves the volume form. Thus volume is a global symplectic invariant, but the famous non-squeezing theorem [22] proved by Gromov in 1985 shows that volume is not a very precise measurement of the size of a symplectic manifold.

Theorem 3.1.1 (Gromov's non-squeezing theorem).
Let $B(r)$ and $Z(R)$ be the symplectic submanifolds of $(\mathbb{R}^{2n}, \omega_0)$ given by

$$B(r) = \left\{(x, y) \in \mathbb{R}^{2n} \middle| \; |x|^2 + |y|^2 < r^2\right\}, \qquad Z(R) = \left\{(x, y) \in \mathbb{R}^{2n} \middle| \; x_1^2 + y_1^2 < R^2\right\}.$$

Then we can find a symplectic embedding

$$(B(r), \omega_0) \hookrightarrow (Z(R), \omega_0)$$

if and only if $r \leq R$.

□

This means for $R < r$ we can not find a symplectic embedding, while finding a volume preserving embedding is not a problem at all. We will now introduce symplectic invariants that in dimension greater than two differ from volume, called symplectic capacities. The main reference for this section is the book by H. Hofer and E. Zehnder [27].

Definition 3.1.2 (Symplectic capacity).
Denote by $\mathcal{S}_{2n}$ the set of symplectic manifolds of dimension 2n. A symplectic capacity is a map

$$\mathcal{S}_{2n} \to \mathbb{R} \cup \{\infty\}; \quad (M, \omega) \mapsto c(M, \omega)$$

that satisfies

(1) ***Monotonicity:*** $c(M, w) \leq c(N, \tau)$
if there exists a symplectic embedding $\varphi : (M, \omega) \to (N, \tau)$.

(2) ***Conformality:*** $c(M, a\omega) = |a| c(M, \omega)$
for all $a \in \mathbb{R}, \; a \neq 0$.

(3) ***Nontriviality:*** $c(B(1), \omega_0) = \pi = c(Z(1), \omega_0)$
where $B(1)$ and $Z(1)$ are the symplectic ball and the symplectic cylinder.

In two dimensions the total area $c(M,\omega) := |\int_M \omega|$ is a capacity. In higher dimensions the symplectic invariant $(\mathrm{Vol})^{1/n}$ is no capacity, as the cylinder has infinite volume. The first examples of symplectic capacities are the Gromov width c_G and the cylindrical capacity c_Z.

Definition 3.1.3.

$$c_G(M,\omega) := \sup\left\{\pi r^2 \middle| \ \exists \ \textit{symplectic embedding } \phi : (B(r),\omega_0) \to (M,\omega)\right\}$$

$$c_Z(M,\omega) := \inf\left\{\pi r^2 \middle| \ \exists \ \textit{symplectic embedding } \phi : (M,\omega) \to (Z(r),\omega_0)\right\}$$

That these actually define capacities follows directly from the non-squeezing. Further any other symplectic capacity c satisfies

$$c_G(M,\omega) \leq c(M,\omega) \leq c_Z(M,\omega).$$

In particular this implies that if Gromov width and cylindrical capacity agree, all capacities are the same. We shall now introduce a capacity that measures size in terms of the possible Hamiltonian dynamics. For this denote by $\mathcal{H}(M,\omega)$ the set of smooth functions H on M, that satisfy

(1) there is a compact set $K \subset M$ such that $K \subset M \setminus \partial M$ and

$$H(M \setminus K) = m(H) \quad \text{(a constant)},$$

(2) there is an open set $U \subset K$ such that

$$H(U) = 0,$$

(3) $0 \leq H(x) \leq m(H)$ for all $x \in M$.

A function $H \in \mathcal{H}(M,\omega)$ will be called admissible if all periodic solutions to $\dot{x} = X_H(x)$ are either constant or have period $T > 1$. We write $\mathcal{H}_a(M,\omega) \subset \mathcal{H}(M,\omega)$ for the subset of admissible functions.

Definition 3.1.4 (Hofer–Zehnder capacity)**.**
The Hofer–Zehnder capacity is given by

$$c_{HZ}(M,\omega) = \sup\left\{m(H) \middle| \ H \in \mathcal{H}_a(M,\omega)\right\}.$$

As the name tells us, the Hofer-Zehnder capacity is indeed a symplectic capacity. The proof can be found in [27] chapter 3. We see that finding a lower bound to this capacity might be directly constructed by finding an admissible Hamiltonian. On the other hand finding upper bounds is much harder and indeed often impossible. For example Usher [46] shows that even closed symplectic manifolds often have infinite Hofer–Zehnder capacity. However a finite Hofer–Zehnder capacity actually implies something usually referred to as almost existence theorem.

Theorem 3.1.5. *([27] Ch. 4; Thm. 1) Let $H : M \to \mathbb{R}$ a smooth function and $\lambda \in \mathbb{R}$ a regular value. Fix $\varepsilon > 0$ such that all $t \in (\lambda - \epsilon, \lambda + \epsilon)$ are also regular. If $H^{-1}(\lambda) \subset M$ is compact and bounds a symplectic submanifold of finite Hofer-Zehnder capacity, then*

$$\mu\left(\{t \in (\lambda - \epsilon, \lambda + \epsilon) | \text{ there exists a periodic orbit } \gamma \text{ s.t. } H(\gamma) = t\}\right) = 2\epsilon,$$

where μ denotes the Lebesgue measure.

Actually Macarini and Schlenk showed that the almost existence still holds if only the thickening of the energy hyper-surface has finite Hofer–Zehnder capacity [38].

In some cases we will also look at a relative version of the Hofer–Zehnder capacity (defined by V. Ginzburg and B. Gürel in [20]).

Definition 3.1.6 (relative)**.** *For a subset $Z \subset M$ that doesn't touch the boundary, i.e. $\mathrm{cl}(Z) \cap \partial M = \emptyset$, we denote by $\mathcal{H}(M, Z, \omega)$ the set of smooth functions satisfying*

a) $H|_{M\setminus K} = m(H)$ *and* $H|_U = 0$,

b) $0 \leq H(x) \leq m(H)$ *for all* $x \in M$,

for an open neighborhood $U \supset Z$ and a compact set $K \supset U$. A function $H \in \mathcal{H}(M, Z, \omega)$ will be called admissible if all periodic solutions to $\dot{x} = X_H(x)$ are either constant or have period $T > 1$. We write $\mathcal{H}_a(M, Z, \omega) \subset \mathcal{H}(M, \omega)$ for the subset of admissible functions. The relative Hofer–Zehnder capacity is then defined as

$$c_{\mathrm{HZ}}(M, Z, \omega) := \sup\{\max H \mid H \in \mathcal{H}_a(M, Z, \omega)\}.$$

Remark 3.1.7. *Observe that clearly $c_{\mathrm{HZ}}(M, Z, \omega) \leq c_{\mathrm{HZ}}(M, \omega)$ for any $Z \subset M$.*

As for the Hofer–Zehnder capacity there is an almost existence result in case of finite relative Hofer–Zehnder capacity [20, Thm. 2.14]. It says that if $c_{\mathrm{HZ}}(M, Z, \omega) < \infty$ and $H : M \to \mathbb{R}$ is a proper smooth function with $H|_Z = \min H$, then almost all compact regular energy levels carry periodic orbits.

Another variation of the Hofer–Zehnder capacity is the π_1-sensitive Hofer–Zehnder capacity

Definition 3.1.8 (π_1-sensitive)**.** *Fix a class $\nu \in \pi_1(M)$, then*

$$c^{\nu}_{HZ}(M, \omega) := \sup\{\max H \mid H \in \mathcal{H}^{\nu}_a(M, \omega)\},$$

where $\mathcal{H}^{\nu}_a(M, \omega)$ is the subset of $\mathcal{H}(M, \omega)$ such that all periodic orbits in the homology class ν are either constant (if $\nu = 0$) or have period $T > 1$.

3.2. Pseudo-holomorphic curves

One approach to finding upper bounds to the Hofer–Zehnder capacity is an observation by Hofer–Viterbo [26] that existence of pseudo-holomorphic curves can imply very generally existence of periodic orbits. This idea was then generalized by Lu [37] and phrased in terms of non-vanishing Gromov–Witten invariants. In this section we will very roughly introduce the theory of pseudo-holomorphic curves to state the theorems by Hofer–Viterbo [26] and Lu [37] that yield explicit upper bounds for c_{HZ}. For the introduction of the general theory of pseudo-holomorphic curves we closely follow McDuff–Salamon [43]. Another reference is Wendl [50].

Almost complex structures:
An *almost complex structure* J on a smooth manifold M is a smooth bundle map

$$J : TM \to TM$$

such that $J^2 = -\mathrm{id}_{TM}$. It is called ω-compatible if $g_J(\cdot,\cdot) := \omega(\cdot, J\cdot)$. is a Riemannian metric. We shall denote

$$\mathcal{J}(M,\omega) = \{\omega\text{-compatible almost complex structures on } M\}$$

this is a topological spaces with respect to C^∞_{loc}-topology, e.g. we consider a sequence J_k to converge if all its derivatives converge uniformly on compact subsets of TM.

Pseudoholomorphic curves:
Fix a closed, connected Riemann surface $(\Sigma, j, \mathrm{dvol}_\Sigma)$. A map $u : \Sigma \to M$ is called *pseudoholomorphic* (or J-holomorphic) if its differential is complex linear, i.e.

$$\mathrm{d}u \circ j = J \circ \mathrm{d}u.$$

Denote by

$$\mathcal{M}(A,\Sigma;J) := \{u \in C^\infty(\Sigma, M)|\ J \circ \mathrm{d}u = \mathrm{d}u \circ j,\ [u] = A\}$$

the moduli space of holomorphic curves in a fixed homology class $A \in H_2(M;\mathbb{Z})$. Further let $\mathcal{B} \subset C^\infty(\Sigma, M)$ denote the space of all smooth maps $u : \Sigma \to M$ that represent the homology class $A \in H_2(M.\mathbb{Z})$. It will be useful to interpret the set of solutions to this equation as zero-set of a section

$$\mathcal{S} : \mathcal{B} \to \mathcal{E};\ u \mapsto (u, \bar\partial_J(u))$$

in an infinite dimensional vector bundle $\mathcal{E} \to \mathcal{B}$. The fiber over $u \in \mathcal{B}$ is the space

$$\mathcal{E}_u = \Omega^{0,1}(\Sigma, u^*TM)$$

of smooth J-antilinear 1-forms on Σ with values in u^*TM and

$$\bar\partial_J(u) := \frac{1}{2}\left(\mathrm{d}u + J \circ \mathrm{d}u \circ j\right) \in \mathcal{E}_u.$$

Energy identity:
The *energy* of a smooth map $u : \Sigma \to M$ is defined as

$$E(u) := \frac{1}{2}\int_\Sigma |\mathrm{d}u|_J^2 \mathrm{dvol}_\Sigma,$$

where the norm of the real linear operator $L := \mathrm{d}u_z : T_z\Sigma \to T_{u(z)}M$ is given by

$$|L|_J := |\zeta|^{-1}\sqrt{|L(\zeta)|_J^2 + |L(j\zeta)|_J^2}$$

for $0 \neq \zeta \in T_z\Sigma$. If J is compatible with ω and u is J-holomorphic, then

$$E(u) = \int_\Sigma u^*\omega.$$

In particular Stoke's theorem tells us that if $[u] = 0 \in H_2(M, \mathbb{R})$, then $E(u) = 0$ and therefore u must be constant.

Unique continuation:
Assume that $u, v : \Sigma \to M$ are both J-holomorphic and agree at some point $z \in \Sigma$ to infinite order, then $u \equiv v$ (Cor. 2.3.3 [40]). For holomorphic maps (so integrable complex structures) this is an immediate consequence of the fact that holomorphic functions are analytic.

Somewhere injective curves:
A J-holomorphic curve $u : \Sigma \to M$ is said to be *somewhere injective* if there is a point $z \in \Sigma$ at which $\mathrm{d}u_z : T_z\Sigma \to T_{u(z)}M$ is injective and $u^{-1}(u(z)) = \{z\}$. It is said to be multiply covered if there exists a closed Riemann surface (Σ', j'), a J-holomorphic curve $u' : \Sigma' \to M$ and a branched covering $\varphi : \Sigma \to \Sigma'$ such that

$$u = u' \circ \varphi \qquad \deg(\varphi) > 1.$$

A curve is called simple if it is not multiply covered. Indeed every simple curve is somewhere injective and the set of non-injective points is at most countable and can only accumulate at the critical points of u (Prop. 2.5.1 [40]). We will denote the subspace of simple curves by

$$\mathcal{M}^*(A, \Sigma; J) := \{u \in \mathcal{M}(A, \Sigma; J) |\ u \text{ simple}\}.$$

Conversely, for closed connected domains, somewhere injective implies simple, thus in our set up the notions are equivalent [50, Prop. 2.6].

Transversality:
We want to see that for generic almost complex structures the spaces $\mathcal{M}^*(A, \Sigma; J)$ are smooth finite dimensional manifolds. For this we need to show that $\mathcal{S}$ is transverse to the zero-section. The vertical differential is the composition

$$D_u : T_u\mathcal{B} = \Omega^0(\Sigma, u^*TM) \xrightarrow{\mathrm{d}\mathcal{S}_u} T_{(u,0)}\mathcal{E} = T_u\mathcal{B} \oplus \mathcal{E}_u \xrightarrow{\pi_u} \mathcal{E}_u = \Omega^{0,1}(\Sigma, u^*TM)$$

of the differential $\mathrm{d}\mathcal{S}_u$ and the projection π_u. Observe that the section $\mathcal{S}$ is transverse to the zero-section if and only if D_u is surjective for every $u \in \mathcal{M}^*(A, \Sigma; J)$.

Regular almost complex structures:
Fix a closed Riemann surface $(\Sigma, j, \mathrm{dvol}_\Sigma)$ and a homology class $A \in H_2(M, \mathbb{Z})$. An almost complex structure J on M is called *regular* if D_u is onto for every $u \in \mathcal{M}^*(A, \Sigma; J)$ [40, Def. 3.1.5]. Denote the set of regular, compatible almost complex structures by $\mathcal{J}_{\mathrm{reg}}(A, \Sigma)$. Indeed a generic almost complex structure is regular [40, Thm. 3.1.6].
In some situation we will need to work with an explicit (almost) complex structure and not a generic one. The *splitting principle* yields a criterion when (integrable) complex structures are regular. It uses the fact that any holomorphic vector bundle over $\mathbb{CP}^1$ can be decomposed into holomorphic line bundles.

Lemma 3.2.1 (Lem. 3.3.1 [40]). *Let (M, ω, j) be a Kähler manifold and $u : \mathbb{C}\mathrm{P}^1 \to M$ a holomorphic sphere. Decompose u^*TM into holomorphic line bundles, i.e.*

$$u^*TM \cong L_1 \oplus \ldots \oplus L_n.$$

If every summand L_i has Chern number $c_1(L_i) \geq -1$, then D_u is onto.

We shall now discuss an example in detail as we will need this example later.

Example 3.2.2. *We look at $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n$ with complex structure $j \ominus j$, where j is the standard complex structure of $\mathbb{C}\mathrm{P}^n$. As homology class $A \in H_2(\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n, \mathbb{Z})$ we take the generator of any of the two factors. Such a holomorphic sphere is of the form*

$$u : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n; z \mapsto (v_1(z), v_2(z)),$$

where $v_1, v_2 : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ are holomorphic spheres in $\mathbb{C}\mathrm{P}^n$. Let us without loss of generality assume $[u] = [\mathbb{C}\mathrm{P}^1 \times pt.] \in H_2(\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n, \mathbb{Z})$, then

$$\deg(v_1) = 1 \text{ and } \deg(v_2) = 0.$$

It follows that

$$u^*T(\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n) = v_1^*T\mathbb{C}\mathrm{P}^n \oplus \mathbb{C}^n.$$

In particular half of the L_i's will be trivial and thus have Chern number zero. A holomorphic sphere $v_1 : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ of degree one takes in homogeneous coordinates the form

$$v_1([z : w]) = [a_0 z + b_0 w : \ldots : a_n z + b_n w].$$

The group of biholomorphisms of $\mathbb{C}\mathrm{P}^n$ can map any two distinct points to arbitrary two other distinct points. We may therefore assume that $v_1([1 : 0]) = [1 : 0 : \ldots : 0]$ and $v_1([0 : 1]) = [0 : 1 : 0 : \ldots : 0]$. This implies

$$v([z, w]) = [z : w : 0 : \ldots : 0].$$

Clearly $v_1^*T\mathbb{C}\mathrm{P}^n \cong T\mathbb{C}\mathrm{P}^1 \oplus v_1^*N\mathbb{C}\mathrm{P}^1$. We therefore need to decompose the normal bundle of $v(\mathbb{C}\mathrm{P}^1)$ into line bundles. The idea is to find suitable holomorphic sections. For this observe that $v_1(\mathbb{C}\mathrm{P}^1)$ is covered by the two coordinate patches U_z and U_w defined by $z \neq 0$ resp. $w \neq 0$. In the coordinates on U_w we can define $n-1$ holomorphic sections in the normal bundle as follows

$$\xi_i|_{U_w} : (\frac{z}{w}, 0, \ldots, 0) \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\frac{z}{w}, 0, \ldots, t, \ldots, 0).$$

It looks like this section doesn't have any zeros, but changing coordinates to the coordinates on U_z reveals a zero at $[0, 1, 0, \ldots, 0]$. Indeed applying the coordinate change

$$(a_1, \ldots, a_n) \mapsto (a_1^{-1}, a_1^{-1}a_2, \ldots, a_1^{-1}a_n)$$

yields

$$\xi_i|_{U_z} : (\frac{w}{z}, 0, \ldots, 0) \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\frac{w}{z}, 0, \ldots, \frac{w}{z}t, \ldots, 0)$$

and thus the section has a simple zero at $[0 : 1 : 0 \ldots : 0]$. As all the sections are linearly independent, they determine a decomposition of the normal bundle into line bundles L_i. The first Chern number can be determined by the sum of zero's of a holomorphic section counted with multiplicity. As explained above the ξ_i's have exactly one simple zero, thus $c_1(L_i) = 1 \geq -1$ and therefore satisfy Lemma 3.2.1. This shows that $j \ominus j$ is regular for the generators of $H_2(\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n, \mathbb{Z})$.

Fredholm property:
One can extend the previous definition of D_u to the case that $u : \Sigma \to M$ is not smooth but in Sobolev-class $W^{k,p}$ for some integer $k \geq 1$ and $p > 2$. Actually if J is smooth any $u \in W^{k,p}(\Sigma, M)$ that satisfies $\bar{\partial}_J(u) = 0$ will still be smooth by an elliptic bootstrapping argument. The spaces of class $W^{k,p}$ are Banach spaces and it can then be shown ([40, Ch. 3.1 & Thm. C.1.10]) that the vertical differential is not only bounded but even a Fredholm operator with index given by

$$\mathrm{ind} D_u = n(2 - 2g) + 2c_1(u^*TM).$$

Here $2n$ is the dimension of M, g is the genus of Σ and c_1 denotes the first Chern number, i.e. the first Chern class evaluated on the fundamental class.

Moduli space:
It follows now from the implicit function theorem for Banach spaces [40, Thm. A.3.3] that $\mathcal{M}^*(A, \Sigma; J)$ is a smooth finite dimensional manifold. Observe that by surjectivity of D_u the Fredholm index of D_u is the same as the dimension of the kernel and thus the dimension of the moduli space.

Theorem 3.2.3 ([40] Thm. 3.1.6.)**.** *If J is regular the space $\mathcal{M}^*(A, \Sigma; J)$ is a smooth manifold of dimension*

$$\dim \mathcal{M}^*(A, \Sigma; J) = n(2 - 2g) + 2c_1(A).$$

It carries a natural orientation.

Further the oriented bordism class of the moduli space $\mathcal{M}^*(A, \Sigma; J)$ does not depend on the choice of the regular almost complex structure J [40, Thm. 3.1.8]. Even though this is rather useless if we cannot establish some sort of compactness of the moduli space.

Pointwise constraints:
Fix a finite sequence of pairwise distinct points $\mathbf{w} = (w_1, \ldots, w_m) \in \Sigma^m$ and define the *evaluation map*

$$\mathrm{ev}_{\mathbf{w}} : \mathcal{M}^*(A, \Sigma; J) \to M^m; \quad u \mapsto (u(w_1), \ldots, u(w_m)).$$

For a smooth submanifold $X \subset M^m$ consider the constrained moduli space

$$\mathcal{M}^*(A, \Sigma; \mathbf{w}, X; J) := \{u \in \mathcal{M}^*(A, \Sigma; J) | \ \mathrm{ev}_{\mathbf{w}}(u) \in X\}.$$

By [40, Thm. 3.4.1] for generic almost complex structure it is a smooth orientable manifold of dimension

$$\dim \mathcal{M}^*(A, \Sigma; \mathbf{w}, X; J) = \dim \mathcal{M}^*(A, \Sigma; J) - \mathrm{codim} X.$$

An almost complex structure is regular for the constrained problem if it is regular in the unconstrained sense and the evaluation map is transverse to X.

Bubbling:
In this part we shall specialize to the case $\Sigma = \mathbb{C}\mathrm{P}^1$ to simplify some arguments, but

bubbling also works for all other pseudoholomorphic curves [40, ch. 4.2]. A first step towards compactness is done by investigating limits of sequences of pseudoholomorphic curves $u^\nu : \mathbb{C}\mathrm{P}^1 \to M$. Indeed by [40, Thm. 4.1.1.], if

$$\sup_\nu \|\mathrm{d}u^\nu\|_{L^\infty} < \infty,$$

then u^ν has a sub sequence which converges uniformly with all derivatives to a pseudoholomorphic curve $u : \mathbb{C}\mathrm{P}^1 \to M$. Thus compactness can only fail if there exists a sequence $z^\nu \in \mathbb{C}\mathrm{P}^1$ such that

$$|\mathrm{d}u^\nu(z^\nu)| := c^\nu \xrightarrow{\nu\to\infty} \infty.$$

We may assume that $|\mathrm{d}u^\nu(z)|$ attains its maximum at z_ν. As $\mathbb{C}\mathrm{P}^1$ is compact we can restrict to a sub sequence of z^ν also denoted by z^ν that converges to a point $z_0 \in \Sigma$. Holomorphically identify $\mathbb{C}\mathrm{P}^1 = \mathbb{C}\cup\{\infty\}$ such that $z_0 \equiv 0$ and define $v^\nu := u^\nu \circ \psi^\nu : \mathbb{C} \to M$ where $\psi^\nu(z) := z^\nu + z/c^\nu$ is a Möbius transformation (with fixpoint ∞). Now

$$|\mathrm{d}v^\nu(0)| = 1 \;\;\&\;\; |\mathrm{d}v^\nu(z)| \le 1,$$

thus v^ν converges to a non-constant pseudoholomorphic map $v : \mathbb{C} \to M$. As the energy is conformally invariant, the map

$$\mathbb{C} \setminus \{0\} \to M;\;\; z \to v(1/z)$$

has finite energy and by the removal of singularities [40, Thm. 4.1.2.] can be extended to a pseudoholomorphic sphere. One says the sphere $v : S^2 \to M$ *bubbles off.*
Bubbling can only happen at finitely many points $\Gamma \subset S^2$, if one takes these points away the sequence $u^\nu|_{S^2\setminus\Gamma}$ of punctures J-holomorphic sphere, has a subsequence that converges locally uniformly to a punctured J-holomorphic sphere $u : \mathbb{C}\mathrm{P}^1 \setminus \Gamma \to M$. As u has finite energy all punctures are removable. This is called *convergence modulo bubbling.*

Compactness:
Clearly we can not expect $\mathcal{M}(A, \mathbb{C}\mathrm{P}^1; J)$ to be compact, as the non-compact group of $\mathrm{PSL}(2, \mathbb{C})$ acts on it by reparametrization through Möbius transformations. In order to have any hope of compactness one therefore needs to fix this reparametrization symmetry. This can for example be done by pointwise constraints or taking a quotient by the group of reparametrizations, as we shall discuss later. As discussed in the previous section apart from the reparametrization the only other obstruction to compactness is bubbling. In some situations, for example if the class A is minimal in some sense, bubbling can not occur. In these situations our moduli space is indeed compact and therefore defines an oriented bordism class $[\mathcal{M}]$.

3.2.1. Hofer–Viterbo's Theorem

In this section we will discuss a first method to bound the Gromov-width and the Hofer–Zehnder capacity from above, using pseudoholomorphic spheres.

To find an upper bound of the Hofer–Zehnder capacity, we will use an idea of H. Hofer and C. Viterbo [26] that shows existence of periodic orbits in the presence of holomorphic spheres. They studied the moduli space $\mathcal{M}$ of holomorphic spheres $u : S^2 \to M$ satisfying

$$\begin{aligned} u(0) &\in \Sigma_0, \\ u(\infty) &\in \Sigma_\infty, \\ [u] &= A \\ \frac{1}{2}\omega(A) &= \int_{|z|\leq 1} u^*\omega, \end{aligned} \tag{3.1}$$

for two closed disjoint submanifolds Σ_0 and Σ_∞ and some $A \in H_2(M,\mathbb{Z})$. Observe that these conditions fix the reparametrization symmetry up to an S^1 (rotation of the complex plane). Further they assume A to be *minimal*, i.e.

$$\omega(A) = \inf\{\omega(B) \mid B \in [S^2, M],\ \omega(B) > 0\},$$

which ensures that bubbling cannot occur. Thus their moduli space is compact and defines an oriented cobordism class. Since they didn't fix the S^1-symmetry their moduli space inherits a free S^1-action and one considers cobordisms of compact, oriented S^1-manifolds.

Theorem 3.2.4 (Hofer–Viterbo '92)**.** *Let (M,ω) be a closed symplectic manifold, A a minimal free homotopy class in $[S^2, M]$, J a regular ω-compatible almost complex structure and Σ_0, Σ_∞ two disjoint nonempty closed submanifolds of M. Suppose $H : M \to \mathbb{R}$ is a Hamiltonian such that*

$$H|_{\mathcal{U}(\Sigma_0)} = \min(H) \quad \textit{and} \quad H|_{\mathcal{U}(\Sigma_\infty)} = \max(H),$$

then if the bordism class $[\mathcal{M}] \neq 0$ the Hamiltonian system $\dot{x} = X_H(x)$ possesses a non constant contractible T-periodic solution with

$$0 < T(\max(H) - \min(H)) < \omega(A).$$

In particular

$$c_{HZ}(M \setminus \Sigma_\infty, \Sigma_0, \omega) \leq \omega(A)$$

and if Σ_0 is a one point set we can conclude

$$c^0_{HZ}(M \setminus \Sigma_\infty, \omega) \leq \omega(A).$$

The idea of the proof of this theorem is, that turning on a Hamiltonian perturbation of the Cauchy–Riemann equation yields the parametrized moduli space

$$\mathcal{C} = \left\{(\lambda, u) \in [0,\infty) \times C^\infty(S^2, M) \mid \text{satisfying (3.1) and (3.2)}\right\}$$

where (3.2) is the perturbed Cauchy-Riemann equation [1]

$$\partial_s u + J(\partial_t u - \lambda X_H(u)) = 0 \tag{3.2}$$

in holomorphic coordinates $(s,t) \in \mathbb{R} \times S^1 \cong S^2 \setminus \{0,\infty\}$. The parametrized moduli space is a smooth manifold with boundary. Observe that the boundary at $\lambda = 0$ is the moduli

[1] Also called Floer equation, but in this setup it will be interpreted as a perturbation of the Cauchy–Riemann equation.

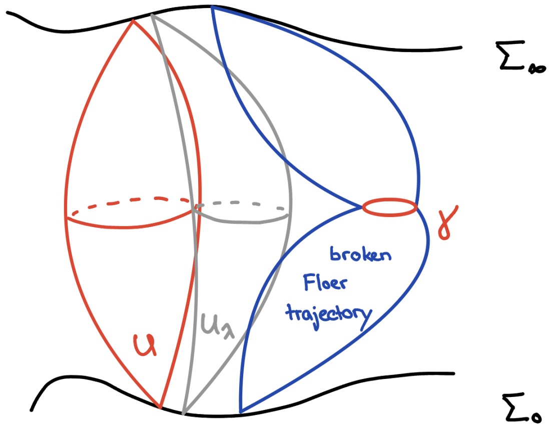


Figure 3.1.: *In red we see a pseudoholomorphic sphere u connecting the two disjoint submanifolds Σ_0 and Σ_∞. The gray curve represents a solution to the perturbed Cauchy–Riemann equation for a small enough real parameter λ. Finally the blue curve depicts a broken Floer trajectory. It is clear from the picture that the orbit γ, at which the Floer trajectory breaks, can not be constant by minimality of A. Further γ must be contractible as it can be contracted along the broken Floer trajectory.*

space of holomorphic curves $\mathcal{M}$. Roughly, Hofer and Viterbo argue that if the parametrized moduli space was compact then $[\mathcal{M}] = 0$. The argument is not as direct as one might think because $\mathcal{C}$ is not the bordism to the empty set as it may not be a manifold. One needs perturb the Fredholm section that has $\mathcal{C}$ as zero-set slightly, to obtain transversality. Conversely this means $[\mathcal{M}] \neq 0$ implies that $\mathcal{C}$ is non-compact. Now positivity of energy

$$0 \leq E(u_\lambda) = \omega(u) - \lambda(\max(H) - \min(H)), \quad \forall u_\lambda \in \mathcal{C}$$

implies that there must be another boundary consisting of broken Floer trajectories of λH for some $\lambda \leq \omega(A)/(\max(H) - \min(H))$. As Floer trajectories break at 1-periodic orbits, this yields existence of a 1-periodic orbit for λH. As A is minimal the orbit can not be constant thus H has a non-constant contractible orbit of period less than $\omega(A)/(\max(H) - \min(H))$.

Very similarly one can bound the Gromov-width, essentially copying the proof of Gromov's non-squeezing theorem [22] considering a parametrized moduli space where the parameter deforms the almost complex structure.

Theorem 3.2.5 (Gromov)**.** *Let (M, ω) be a closed symplectic manifold, A a minimal free homotopy class in $[S^2, M]$, J a regular ω-compatible almost complex structure and $\Sigma_0 = \{p_0\}, \Sigma_\infty$ two disjoint nonempty closed submanifolds of M. If the bordism class $[\mathcal{M}(\Sigma_0, \Sigma_\infty, A, J)] \neq 0$, then*

$$c_G(M, \omega) \leq \omega(A).$$

The idea of proof is the following. Assume there was a symplectic embedding $f : B_R(0) \hookrightarrow M$. Composing with a Hamiltonian diffeomorphism we may assume $f(0) = p_0$. Let J_1 be a regular almost complex structure that pulls back to the standard complex structure

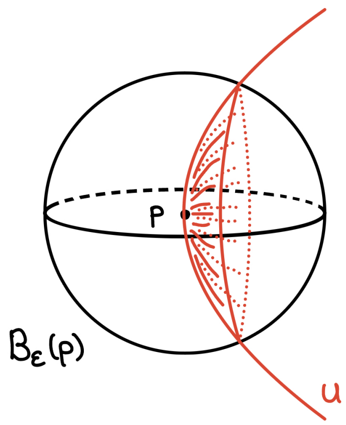


Figure 3.2.: *The figure sketches the intersection of a pseudoholomorphic curve through p with a symplectic ball of radius $\varepsilon > 0$ centered at p. Gromov's monotonicity states that the area of this intersection is at least $\pi\varepsilon^2$.*

on $B_R(0)$, i.e. $f^* J_1|_{\mathrm{Im}(f)} = i$. As the space of almost complex is contractible one finds a path of almost complex structures J_t connecting $J = J_0$ and J_1. The parametrized moduli space

$$\mathcal{C}(\Sigma_0, \Sigma_\infty, A, \{J_t\}) := \Big\{(t, u) \in [0,1] \times C^\infty(S^2, M) | u \in \mathcal{M}(\Sigma_0, \Sigma_\infty, A, J_t)\Big\}$$

is a compact, oriented S^1-manifold with boundary $\mathcal{M}(\Sigma_0, \Sigma_\infty, A, J)$ and $\mathcal{M}(\Sigma_0, \Sigma_\infty, A, J_1)$, i.e. a cobordism. As $[\mathcal{M}(\Sigma_0, \Sigma_\infty, A, J)] \neq 0$ by assumption this implies $[\mathcal{M}(\Sigma_0, \Sigma_\infty, A, J_1)] \neq 0$. In particular we have J_1-holomorphic spheres through p_0. These pull back to holomorphic discs through 0 in $B_R(0)$. Now Gromov's monotonicity implies

$$\pi R^2 \leq \omega(A).$$

3.2.2. Gromov-Witten invariants

The condition of A being minimal in Theorems 3.2.4 and 3.2.5 is very strong. In order to remove it one needs to deal with the bubbling phenomenon, as it obstructs compactness of the moduli spaces and thus prevents defining the cobordism class $[\mathcal{M}]$. The idea is to compactify the moduli space by adding limits of sequences of pseudoholomorphic curves. This procedure is called Gromov-compactification. The limits are no longer holomorphic curves in the previous sense, but so called nodal curves.

Nodal curves:
A *nodal* J-holomorphic curve with m marked points representing the homology class $A \in H_2(M, \mathbb{Z})$ is a tuple

$$(\Sigma, j, u, (\zeta_1, \dots, \zeta_m), \Delta)$$

consisting of

- a closed Riemann surface (Σ, j) with connected components $\Sigma_1, \dots, \Sigma_p$,

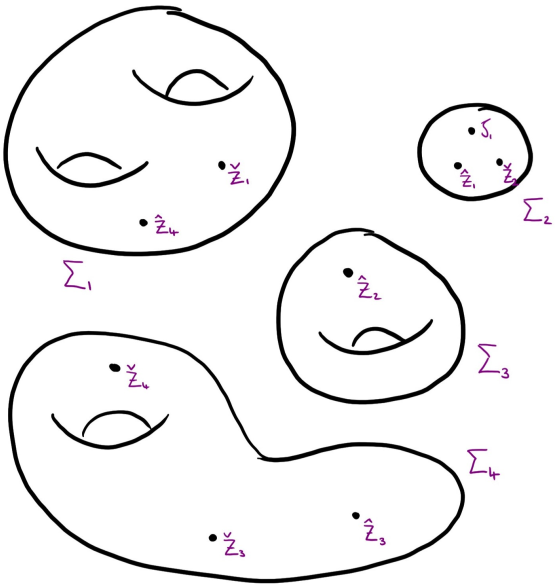


Figure 3.3.: *The domain of a nodal curve consisting of four connected components, one marked point and four nodes.*

- a J-holomorphic curve $u : (\Sigma, j) \to (M, J)$, such that $[u] := \sum_{i=1}^{p} u_*[\Sigma_i] = A \in H_2(M)$,
- an ordered set of distinct points $(\zeta_1, \ldots, \zeta_m) \in \Sigma^m$,
- a set of nodes $\Delta = \{\{\hat{z}_1, \check{z}_1\}, \ldots, \{\hat{z}_r, \check{z}_r\}\}$ such that $\hat{z}_1, \check{z}_1, \ldots, \hat{z}_r, \check{z}_r, \zeta_1, \ldots, \zeta_m \in \Sigma$ are distinct and
$$u(\hat{z}_i) = u(\check{z}_i) \quad \text{for all} \quad i = 1, \ldots, r.$$

We say such a nodal curve is of *arithmetic genus* g if by replacing all nodal points $\hat{z}_i, \check{z}_i$ by circles $\hat{C}_i, \check{C}_i$ and gluing together the circles $\hat{C}_i$ and $\check{C}_i$ we obtain a closed connected surface $\hat{\Sigma}$ of genus g.

Stable curves:
A nodal curve is called *stable* if, after removing all the marked points $\zeta_1, \ldots, \zeta_m$ and nodal points Δ from Σ to produce a punctured surface $\dot{\Sigma}$, every connected component of $\dot{\Sigma}$ on which u is constant has negative Euler characteristic. This means constant spheres need to have at least three marked points. We call those spheres where u is constant *ghost bubbles*. The stability condition is necessary to make the isotropy subgroup of every stable map finite. Define the space of stable nodal curves as

$$\overline{\mathcal{M}}_{g,m}(A; J) := \{[(\Sigma, j, u, (\zeta_1, \ldots, \zeta_m), \Delta)]\}/\sim$$

where
$$[(\Sigma, j, u, (\zeta_1, \ldots, \zeta_m), \Delta)] \sim [(\Sigma', j', u', (\zeta'_1, \ldots, \zeta'_m), \Delta']$$
if there exists a biholomorphic map $\varphi : (\Sigma, j) \to (\Sigma', j')$ such that $u = u' \circ \varphi$ and $\varphi(\zeta_i) = \zeta'_i,\ \varphi(\hat{z}_i) = \hat{z}'_i,\ \varphi(\check{z}_i) = \check{z}_i$.

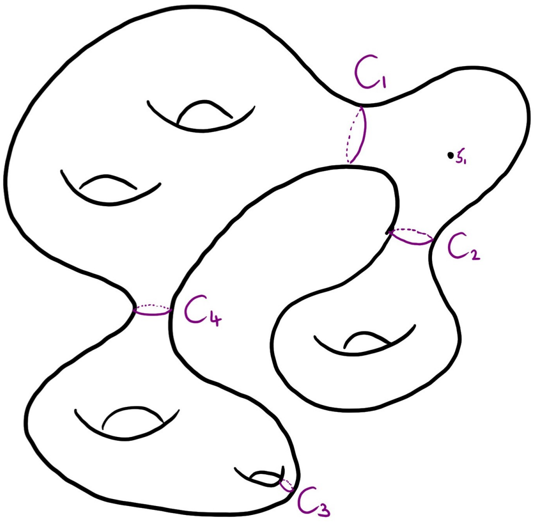


Figure 3.4.: *The figure shows how the arithmetic genus of the nodal curve shown in figure 3.3 can be determined by constructing the surface $\hat{\Sigma}$. This surface has arithmetic genus 5.*

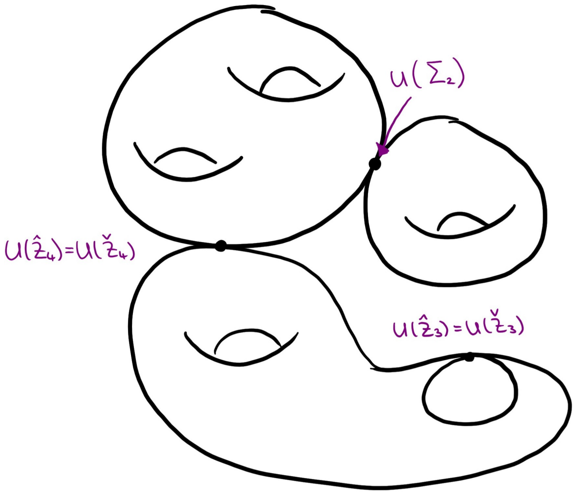


Figure 3.5.: *The figure illustrates the image of the nodal curve in figure 3.3. One can see that Σ_2 is a ghost bubble as u is constant on Σ_2. The nodal curve is still stable as Σ_2 has three marked points or nodal points.*

Dual graph representation:
Nodal surfaces can be represented by labelled graphs Γ. Let $(\Sigma, (\zeta_1, \ldots, \zeta_m), \Delta)$ be a nodal surface, then the vertices in Γ correspond to the components Σ_i of Σ and are labeled by their genus g_i. For each pair of nodes, we add an edge, for each marked point $\zeta_j \in \Sigma_i$ we attach a half edge to the vertex representing Σ_i and label it with the index j. For an example see figure 3.6.

Genus zero:

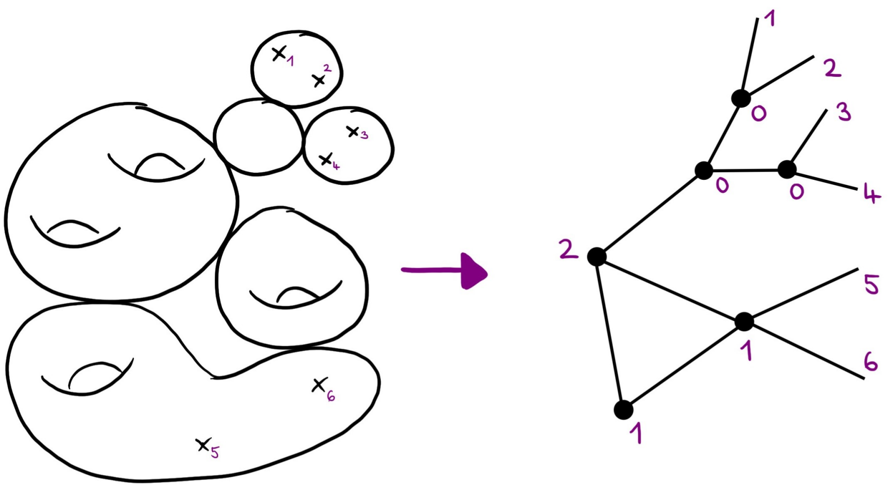


Figure 3.6.: *Dual graph representation of a stable nodal Riemann surface.*

We will from now on restrict to the genus zero case, as on the one hand it simplifies the following arguments and on the other hand is the only case we will need. Observe that for a nodal curve

$$(\Sigma, j, u, (\zeta_1, \ldots, \zeta_m), \Delta) \in \overline{\mathcal{M}}_{0,m}(A; J)$$

all connected components of Σ must be spheres S^2_α and the dual graph must be a tree T. On a tree we can make sense of the notion of intervals. For $\alpha, \beta \in T$ the interval $[\alpha, \beta] \subset T$ is the set of vertices lying on the unique chain connecting α and β. We can then for $\alpha, \beta \in T$ define the sub tree

$$T_{\alpha\beta} := \{\gamma |\ \beta \in [\alpha, \gamma]\}.$$

An example is shown in figure 3.7. Every vertex α of the tree T corresponds to a pseudo-

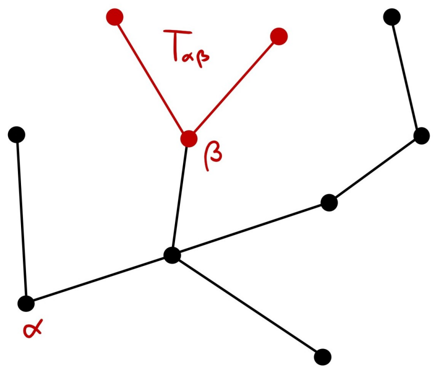


Figure 3.7.: *The red part of the tree is $T_{\alpha\beta}$.*

holomorphic sphere $u_\alpha : S^2_\alpha \to M$. Every edge $\alpha\beta$ of T corresponds to a node. Denote the nodal points by $z_{\alpha\beta} \in S^2_\alpha$ and $z_{\beta\alpha} \in S^2_\beta$. The energy of a nodal curve $u \in \overline{\mathcal{M}}_{g,m}(A; J)$ is defined as

$$E(u) := \sum_{\alpha \in T} E(u_\alpha)$$

and we denote

$$m_{\alpha\beta}(u) := \sum_{\gamma \in T_{\alpha\beta}} E(u_\gamma).$$

Gromov-convergence:
We can now say what convergence of a sequence of holomorphic spheres to a nodal curve means. Let $J \in \mathcal{J}(M,\omega)$ and

$$(u,z) = (\{u_\alpha\}_{\alpha\in T}, \{z_{\alpha\beta}\}, \xi_1, \dots, \xi_m) \in \overline{\mathcal{M}}_{0,m}(A;J)$$

be a stable nodal map with dual graph T. A sequence of pseudoholomorphic spheres $u^\nu : S^2 \to M$ with m distinct marked points $\xi_1^\nu, \dots, \xi_m^\nu \in S^2$ is said to *Gromov converge* to (u,z) if there exists a collection of Möbius transformations $\{\phi_\alpha^\nu\}_{\alpha\in T}^{\nu\in\mathbb{N}}$ such that the following holds.

(Map): For every $\alpha \in T$ the sequence $u_\alpha^\nu := u^\nu \circ \phi_\alpha^\nu : S^2 \to M$ converges to u_α uniformly on every compact subset of $S_\alpha^2 \setminus Z_\alpha$, where Z_α is the set of nodal points in S_α^2.

(Energy): For every two vertices $\alpha, \beta \in T$ that are joined by en edge in T,

$$m_{\alpha\beta}(u) = \lim_{\epsilon\to 0} \lim_{\nu\to\infty} E(u_\alpha^\nu, B_\epsilon(z_{\alpha\beta}))$$

where $E(u_\alpha^\nu, B_\epsilon(z_{\alpha\beta}))$ denotes the energy of the J-holomorphic curve $u_\alpha^\nu|_{B_\epsilon(z_{\alpha\beta})}$.

(Rescaling): If $\alpha, \beta \in T$ are joined by an edge in T, then $\phi_{\alpha\beta}^\nu := (\phi_\alpha^\nu)^{-1} \circ \phi_\beta^\nu$ converges to $z_{\alpha\beta}$ uniformly on every compact subset of $S^2 \setminus \{z_{\beta\alpha}\}$.

(Marked Point): $\xi_i = \lim_{\nu\to\infty} (\phi_{\alpha_i}^\nu)^{-1}(\xi_i^\nu)$ for $i = 1, \dots, m$.

Gromov-compactness:
We can now state Gromov's compactness Theorem.

Theorem 3.2.6 (Thm. 5.3.1. [40])**.**
Let (M,ω) be a compact symplectic manifold and $J \in \mathcal{J}(M,\omega)$. Then any sequence of non-constant holomorphic curves $(u^\nu, \xi^\nu) \in \mathcal{M}(A;J) \times (S^2)^m$ has a subsequence that Gromov converges to a stable nodal curve in $\overline{\mathcal{M}}_{0,m}(A;J)$. The limit is up to equivalence unique.

The idea is that the only obstruction to the existence of a convergent subsequence is bubbling. If one removes the points where bubbling occurs, the holomorphic curves converge. The bubbles connect and thus form nodes on the limit curve. The total energy is preserved under the bubbling process and since every bubble takes by [40, Prop. 4.1.4] at least $\hbar > 0$ of the energy of u there can only be finitely many bubbles.

Pseudo-cycles [40, chpt. 6.5]:
Consider a smooth m-dimensional manifold X. An arbitrary subset $B \subset X$ is said to be of dimension at most d if it is contained in the image of a map $g : W \to X$ which is defined on a manifold W whose components have dimension less than or equal to d. A d-dimensional *pseudocycle* in X is a smooth map

$$f : V \to X$$

defined on an oriented d-dimensional manifold V such that $f(V)$ has a compact closure and

$$\dim \Omega_f \leq \dim V - 2, \quad \Omega_f := \{ \lim f(x_n) | \text{sequences } x_n \text{ with no limit points in } V \}.$$

A pseudocycle determines a compact subset $\overline{f(V)} \subset X$ with boundary of maximal dimension less than $d-2$. Thus roughly speaking $\overline{f(V)}$ is close to compact submanifolds without boundary and can therefore be used to determine a (weak representative) of a homology class. One can define a notion of cobordisms, transversality and intersections similarly as one does for compact submanifolds without boundary.

Gromov–Witten invariants:
Gromov–Witten invariants are defined as suitable intersection product of pseudocycles using the fact that the evaluation map defines a pseudocycle.

Theorem 3.2.7 (Thm. 6.6.1 [40])**.** *Let (M, ω) be a closed semipositive symplectic 2n-manifold and let $J \in \mathcal{J}(M, \omega)$ regular*[2]*. Let $A \in H_2(M; \mathbb{Z})$ be a spherical homology class that is not a nontrivial integer multiple of a spherical homology class B with Chern number $c_1(B) = 0$. Then the evaluation map*

$$\mathrm{ev} : \mathcal{M}^*_{0,k}(A; J) \to M^k$$

is a pseudocycle of dimension $2n + 2c_1(A) + 2k - 6$. Its cobordism class is independent of J.

Based on this Theorem one can finally define Gromov–Witten invariants.

Theorem 3.2.8 (Thm. 6.6.1 [40])**.** *Let (M, ω) be a closed semipositive symplectic $2n$-manifold, m be a non-negative integer, and $A \in H_2(M; \mathbb{Z})$ be a spherical homology class that is not a nontrivial integer multiple of a spherical homology class B with Chern number $c_1(B) = 0$. Then the homomorphism*

$$\mathrm{GW}^M_{A,k} : H^*(M)^{\otimes k} \to \mathbb{Z}; \quad \mathrm{GW}^M_{A,k}(a_1, \ldots, a_k) := f \cdot \mathrm{ev}_J,$$

is independent of the regular almost complex structure J and the pseudocycle f used to define it. Here $f : U \to M^k$ k is a pseudocycle Poincaré dual to $\pi_1^* a_1 \cup \ldots \cup \pi_k^* a_m$, and $\pi_i : M^k \to M$ denotes the projection onto the i-th factor.

3.2.3. Lu's Theorem

We can now state a more general version of Hofer–Viterbo's theorem 3.2.4 that was proven by Lu. He removes the condition of $A \in H_2(M, \mathbb{Z})$ being minimal, that ensures that the moduli space is compact, by working with the Gromov compactification and Gromov–Witten invariants. His theorem is stated in terms of so called pseudo symplectic capacities of Hofer–Zehnder type.

[2] For nodal curves the definition of regular is a bit more complex. One needs regularity for all components and something called edge transversality. For the precise definition see [41, Def. 6.2.1].

Definition 3.2.9. *For a connected symplectic manifold (M, ω) of dimension at least 4 and two nonzero homology classes $\alpha_0, \alpha_\infty \in H_*(M, \mathbb{Q})$, we call a smooth function $H : M \to \mathbb{R}$ $(\alpha_0, \alpha_\infty)$-admissible (resp. $(\alpha_0, \alpha_\infty)^\circ$-admissible) if there exist two compact submanifolds P and Q of M with connected smooth boundaries and of codimension zero such that the following condition groups (1)(2)(3)(4)(5)(6) (resp. (1)(2)(3)(4)(5)(6)') hold:*

(1) $P \subset \mathrm{Int}(Q)$ *and* $Q \subset \mathrm{Int}(M)$*;*

(2) $H|_P = 0$ *and* $H|_{M \setminus \mathrm{Int}(Q)} = \max H$*;*

(3) $0 \leq H \leq \max H$*;*

(4) There exist cycle representatives of α_0 and α_∞, still denoted by α_0, α_∞, such that $\mathrm{supp}(\alpha_0) \subset \mathrm{Int}(P)$ *and* $\mathrm{supp}(\alpha_\infty) \subset M \setminus Q$*;*

(5) There are no critical values in $(0, \varepsilon) \cup (\max H - \varepsilon, \max H)$ for a small $\varepsilon = \varepsilon(H) > 0$;

(6) The Hamiltonian system $\dot{x} = X_H(x)$ on M has no non-constant fast periodic solutions;

(6)' The Hamiltonian system $\dot{x} = X_H(x)$ on M has no non-constant contractible fast periodic solutions.

We respectively denote by

$$\mathcal{H}_{\mathrm{ad}}(M, \omega; \alpha_0, \alpha_\infty) \quad \text{and} \quad \mathcal{H}^\circ_{\mathrm{ad}}(M, \omega; \alpha_0, \alpha_\infty)$$

the set of all $(\alpha_0, \alpha_\infty)$-admissible and $(\alpha_0, \alpha_\infty)^\circ$-admissible functions. Define the pseudo symplectic capacities of Hofer–Zehnder type

$$C^{(2)}_{HZ}(M, \omega; \alpha_0, \alpha_\infty) := \sup\{\max H |\ H \in \mathcal{H}_{\mathrm{ad}}(M, \omega; \alpha_0, \alpha_\infty)\}$$

and

$$C^{(2\circ)}_{HZ}(M, \omega; \alpha_0, \alpha_\infty) := \sup\{\max H |\ H \in \mathcal{H}^\circ_{\mathrm{ad}}(M, \omega; \alpha_0, \alpha_\infty)\}.$$

Lu proved that these pseudo symplectic capacities can be bounded from above when suitable Gromov–Witten invariants do not vanish.

Theorem 3.2.10 (Thm. 1.10,[37])**.** *Let (M, ω) be a closed symplectic manifold of dimension $\dim M \geq 4$ and fix two non-zero homology classes $\alpha_0, \alpha_\infty \in H_2(M, \mathbb{Q})$. Suppose there exists a homology class $A \in H_2(M; \mathbb{Z})$ for which the Gromov-Witten invariant*[3]

$$\mathrm{GW}^{(M,\omega)}_{A,g,k+2}(\alpha_0, \alpha_\infty, \beta_1, \ldots, \beta_m) \neq 0$$

for some homology classes $\beta_1, \ldots, \beta_m \in H_*(M; \mathbb{Q})$ and an integer $m \geq 1$. Then

$$C^{(2)}_{HZ}(M, \omega; \alpha_0, \alpha_\infty) \leq \omega(A)$$

and if A is a spherical class, i.e. $g = 0$ then also

$$C^{(2\circ)}_{HZ}(M, \omega; \alpha_0, \alpha_\infty) \leq \omega(A).$$

[3] In principle Gromov–Witten invariants can be defined for arbitrary genus g. This theorem holds for arbitrary genus, thus the g appears as index.

As these pseudo capacities are a bit uncommon we rephrased this theorem in terms of the relative Hofer–Zehnder capacity. Indeed, the following Corollary is weaker than Lu's theorem and follows more or less directly from it.

Corollary 3.2.11. *Let (M,ω) be a closed symplectic manifold of dimension $\dim M \geq 4$ and fix two disjoint closed*[4] *connected submanifolds $\Sigma_0, \Sigma_\infty \subset M$ of codimension at least two. Denote by $[\Sigma_0], [\Sigma_\infty] \in H_2(M,\mathbb{Q})$ the induced homology classes. Suppose there exists a homology class $A \in H_2(M;\mathbb{Z})$ for which the Gromov-Witten invariant*

$$\mathrm{GW}^{(M,\omega)}_{A,g,k+2}([\Sigma_0],[\Sigma_\infty],\beta_1,\ldots,\beta_m) \neq 0$$

for some homology classes $\beta_1,\ldots,\beta_m \in H_*(M;\mathbb{Q})$ and an integer $m \geq 1$. Then the relative Hofer–Zehnder capacity

$$c_{HZ}(M \setminus \Sigma_\infty, \Sigma_0;\omega) \leq \omega(A)$$

and if A is a spherical class, i.e. $g=0$ then also

$$c^\circ_{HZ}(M \setminus \Sigma_\infty, \Sigma_0;\omega) \leq \omega(A).$$

Proof. We need to show that

$$\mathcal{H}_{\mathrm{ad}}(M \setminus \Sigma_\infty, \Sigma_0;\omega) \subset \mathcal{H}_{\mathrm{ad}}(M,\omega;[\Sigma_0],[\Sigma_\infty])$$

because then

$$c_{HZ}(M \setminus \Sigma_\infty, \Sigma_0,\omega) \leq C^{(2)}_{HZ}(M,\omega;[\Sigma_0],[\Sigma_\infty]).$$

Indeed as Σ_0, Σ_∞ are connected and of codimension at least two, the boundary of any small closed disc sub bundle $D_\varepsilon\Sigma_0, D_\varepsilon\Sigma_\infty$ of the normal bundle is connected and we can set $P = D_\varepsilon\Sigma_0$ and $Q = M\setminus\mathrm{Int}(D_\varepsilon\Sigma_\infty)$. For $\varepsilon > 0$ small enough condition (1) is satisfied as Σ_0 and Σ_∞ are disjoint and compact. Now let $H \in \mathcal{H}_{\mathrm{ad}}(M \setminus \Sigma_\infty, \Sigma_0;\omega)$, then H vanishes on an open neighborhood of Σ_0 and constantly attains its maximum an a neighborhood of Σ_∞. In particular for $\varepsilon > 0$ small enough H also satisfies condition (2). Conditions (3) and (6) hold true per definition and (4) per construction. Condition (5) follows as M is compact, thus critical values can not accumulate. The π_1-sensitive claim follows analogously. □

Remark 3.2.12. *As finitely many points can be moved by Hamiltonian diffeomorphisms, i.e.* $\mathrm{Ham}(M,\omega)$ *is k-transitive, it follows that 2-point invariants yield upper bounds to the Hofer–Zehnder capacity of M.*

One can also find upper bounds for the Gromov-width using Lu's Theorem. This also demonstrates that Lu's theorem is quite a bit stronger then the Corollary.

Corollary 3.2.13. *Let (M,ω) be a closed symplectic manifold of dimension $\dim M = 2n \geq 4$. Suppose there exists a homology class $A \in H_2(M;\mathbb{Z})$ and a homology class $\alpha_\infty \in H_d(M;\mathbb{Q})$, $d \leq 2n-2$ for which the Gromov-Witten invariant*

$$\mathrm{GW}^{(M,\omega)}_{A,g,k+2}([pt.],\alpha_\infty,\beta_1,\ldots,\beta_m) \neq 0$$

for some homology classes $\beta_1,\ldots,\beta_m \in H_*(M;\mathbb{Q})$ and an integer $m \geq 1$. Then

$$c_G(M,\omega) \leq \omega(A).$$

[4] compact with no boundary!

Proof. Denote $\Sigma_\infty := \{p_0\} \subset M$ and $\Sigma_\infty \subset M$ a representative of α_∞, i.e. $[\Sigma_\infty] = \alpha_\infty$. We may pick $p_0 \notin \Sigma_\infty$. Suppose there was a symplectic embedding

$$\varphi : (B_R^{2n}(0), \omega_0) \hookrightarrow (M, \omega).$$

Composing with a Hamiltonian diffeomorphism we can assume $\varphi(0) = p_0$. As $p_0 \notin \Sigma_\infty$ there is an isotopy $\Psi_t : M \to M$ fixing p_0 and pushing Σ_∞ out of the embedded ball, i.e.

$$\Psi_t(p_0) = p_0 \quad \forall t \quad \& \quad \Psi_1(\Sigma_\infty) \subset M \setminus \varphi(B_R^{2n}(0)).$$

Explicitly Ψ_t can be realized by pushing forward the radial scaling of the ball and extending it suitably on M. We conclude that

$$\varphi : (B_R^{2n}(0), \omega_0) \hookrightarrow (M \setminus \Psi_1(\Sigma_\infty), \omega)$$

and therefore it follows that

$$\begin{aligned}\pi R^2 &\leq c_G(M \setminus \Psi_1(\Sigma_\infty), \omega) \leq c_{HZ}(M \setminus \Psi_1(\Sigma_\infty), \omega) \\ &\leq C_{HZ}^{(2)}(M, [\Psi_1(\Sigma_\infty)], [\Sigma_0]) \leq C_{HZ}^{(2)}(M, [\Sigma_\infty], [\Sigma_0]) = \omega(A).\end{aligned}$$

The second inequality follows as the Gromov width is the smallest capacity, the third inequality follows from the previous corollary, the equality uses that the pseudo-capacity only depends on homology classes and the last inequality is Lu's theorem. □

4. Hamiltonian S^1-manifolds

A Hamiltonian S^1-manifold is a symplectic manifold (M, ω) that admits a Hamiltonian $H : M \to \mathbb{R}$ such that the associated Hamiltonian flow is 1-periodic. On these manifolds we have fairly good tools to understand the Gromov-width and the Hofer–Zehnder capacity. For example a lower bound for the Hofer–Zehnder capacity can often immediately be given in terms of the Hamiltonian H.

Lemma 4.0.1. *Let (M, ω) be a compact symplectic manifold that admits a non-trivial Hamiltonian circle action with moment map $H : M \to \mathbb{R}$. Further assume that H attains its minimum on the interior and, if M has a boundary, its maximum constantly on the boundary of M. Then*

$$c_{HZ}(M, \omega) \geq \operatorname{osc}(H) = \max H - \min H.$$

Proof. We need to modify the Hamiltonian H generating the circle action slightly so that it becomes admissible. This can be done with the help of a function $f : [a, b] \to [0, \infty)$ satisfying

$$\begin{aligned} &0 \leq f'(x) < 1, \\ &f(x) = 0 \quad \text{near} \quad a, \\ &f(x) = b - a - \varepsilon \quad \text{near} \quad b \end{aligned}$$

with $a = \min H$ and $b = \max H$. Then all solutions to the Hamiltonian system with Hamiltonian $\tilde{H} = f \circ H$ have period

$$T = \frac{1}{f'(E)} > 1.$$

Thus $\tilde{H}$ is admissible and we find the estimate

$$c_{HZ}(M, \omega) \geq \operatorname{osc}(\tilde{H}) = \operatorname{osc}(H) - \varepsilon, \qquad \forall \varepsilon > 0$$

and the claim follows.

□

In some cases also a lower bound of the Gromov-width can be obtained.

Proposition 4.0.2 ([32] Prop. 2.8)**.** *In addition to the assumptions of Lemma 4.0.1 assume that the Hamiltonian circle action is semi free and that the minimum is isolated, then*

$$c_G(M, \omega) \geq \operatorname{smin}(H) - \min H,$$

where $\operatorname{smin}(H)$ *denotes the second lowest critical value.*

On the other hand we are also in a good position to expect that some 1-point and 2-point Gromov-Witten invariants in suitable homology classes do not vanish, as we have pseudoholomorphic curves going through every point. Indeed we may assume our compatible almost complex structure J to be S^1-invariant[1]. Now as described in [41, Ex. 5.1.5] the S^1-orbit of a gradient flow line of H is a J-holomorphic sphere u connecting critical points $c_\pm$ of H and $\omega(u) = H(c_+) - H(c_-)$. Any non-critical point lies on a gradient flow line and at every critical point we have either incoming or outgoing flow lines. Thus we see that these gradient spheres go through every point. It is still highly non-trivial to show that some Gromov–Witten invariant does not vanish as there might be many (nodal or broken) pseudoholomorphic spheres through every point, so that they cancel in the count.

4.1. Localization and Gromov–Witten invariants

The first important result towards explicit computations of Gromov–Witten invariants in the context of Hamiltonian S^1-manifolds is the localization principle proved by McDuff and Tolman in [42, sec. 4.2].

Proposition 4.1.1 (Prop. 4.10 [42]). *Let (M,ω) be a closed symplectic manifold with a Hamiltonian S^1-action. Fix a S^1-invariant, regular, compatible almost complex structure J and a S^1-invariant pseudo-cycle $\alpha : Z \to M^k$ that represents $a_1 \times \ldots \times a_k \in H^2(M^k, \mathbb{Z})$. Then a connected component of $\mathcal{M}_{0,k}(M, J, A; Z)$ makes no contribution to $\mathrm{GW}^M(a_1, \ldots, a_k; A)$ unless it contains an S^1-invariant element.*

In practise this means when counting pseudoholomorphic we may restrict to counting S^1-invariant curves. McDuff [39] found using this localization, a condition for when the GW-invariant $\mathrm{GW}^M_{0,1}(pt.; A)$ does not vanish.

Proposition 4.1.2 ([39]; Prop. 4.3). *Suppose that (M,ω) is a Hamiltonian S^1-manifold whose maximal and minimal fixed point sets are divisors. Suppose further that at least one of the following conditions holds:*

(i) the action is semi-free, or

(ii) there is an ω-tame almost complex structure J on $D_{\max}$ such that the non-constant J-holomorphic spheres in $D_{\max}$ do not go through every point.

Then $\mathrm{GW}^M_{0,1}(pt.; A) \neq 0$, where $A \in H_2(M, \mathbb{Z})$ is the homology class represented by the S^1-orbit of a gradient flow line connecting minimum and maximum.

Combining this Proposition with Lu's Theorem or rather its Corollary 3.2.11 we obtain the following corollary.

Corollary 4.1.3. *Let (M,ω) be a closed symplectic manifold and let $H : M \to \mathbb{R}$ be a Hamiltonian that induces a semi-free circle action. Denote by $H_{\max}$ the critical set where H attains its maximum and by $H_{\min}$ the critical set where H attains its minimum. Further assume that $H_{\min}$ is a divisor, then*

$$c_{HZ}(M \setminus H_{\max}, \omega) = c^0_{HZ}(M \setminus H_{\max}, \omega) = \max H - \min H.$$

[1] If J is not invariant, we can always average the corresponding metric to be S^1-invariant and then redefine J.

If $H_{\min}$ is no divisor we can still calculate the relative Hofer-Zehnder capacity

$$c_{HZ}(M \setminus H_{\max}, H_{\min}, \omega) = \max H - \min H.$$

Proof. As H is a moment map for a semi-free circle action it is a Morse-Bott function [41, Lem. 5.5.6]. Thus its critical manifolds are symplectic submanifolds. We can blow up along $H_{\max}$ to obtain the blow-up manifold $(\tilde{M}, \tilde{\omega}_\lambda)$ (where λ parametrizes the symplectic size of the blow-up) and a divisor $D_{\max} \subset \tilde{M}$. We can choose an almost-complex structure invariant under the Hamiltonian circle action, thus the Hamiltonian circle action survives the blow up and we denote its moment map by $\tilde{H} : \tilde{M} \to \mathbb{R}$. For the blow-up we needed to cut out a neighborhood of $H_{\max}$ and therefore

$$\operatorname{osc}(\tilde{H}) = \operatorname{osc}(H) - \mathcal{O}(\lambda),$$

where $\mathcal{O}(\lambda)$ is a term that goes to zero for $\lambda \to 0$. As $\tilde{H}$ is again Morse-Bott and takes its maximum on $D_{\max}$ (at least for λ small enough) and its minimum on $H_{\min}$ we know that there must be a gradient flow line of $X_{\tilde{H}}$ from $D_{\max}$ to H_{min}. Denote by A the homology class of the S^1-orbit of this flow line. Then by McDuff 4.1.2 we know that $\mathrm{GW}^M_{0,1}(pt.; A) \neq 0$ and thus by Lu 3.2.11

$$c^0_{HZ}(\tilde{M} \setminus D_{\max}, \tilde{\omega}_\lambda) \leq \tilde{\omega}_\lambda(\alpha) = \operatorname{osc}(\tilde{H}) = \operatorname{osc}(H) - \mathcal{O}(\lambda).$$

On the other hand by construction of the blow-up we know that $(\tilde{M} \setminus D_{\max}, \tilde{\omega}_\lambda)$ is symplectomorphic to $(M \setminus \mathcal{N}_\lambda(H_{\max}), \omega)$. Thus

$$c^0_{HZ}(M \setminus \mathcal{N}_\lambda(H_{\max}), \omega) \leq \operatorname{osc}(H) - \mathcal{O}(\lambda)$$

and inner regularity of the Hofer–Zehnder capacity assures that in the limit $\lambda \to 0$

$$c^0_{HZ}(M \setminus H_{\max}, \omega) \leq \operatorname{osc}(H).$$

Since we can modify H close to $H_{\min}$ and $H_{\max}$ so that it becomes admissible $\operatorname{osc}(H)$ bounds $c_{HZ}(M \setminus H_{\max}, \omega)$ from below and we obtain

$$\operatorname{osc}(H) \leq c_{HZ}(M \setminus H_{\max}, \omega) \leq c^0_{HZ}(M \setminus H_{\max}, \omega) \leq \operatorname{osc}(H).$$

If $H_{\min}$ is no divisor we also need to blow up along $H_{\min}$ and therefore we only compute the Hofer–Zehnder capacity for the class of admissible Hamiltonians vanishing on $H_{\min}$. This yields the relative Hofer–Zehnder capacity. □

We can use this Corollary to immediately compute our first (relative) Hofer–Zehnder capacity.

Example 4.1.4. *Let us consider the standard tangent bundle $(TM, \mathrm{d}\lambda)$ of some Zoll manifold (M, g)[2]. Denote by l the length of the geodesics. If we remove the zero section the Hamiltonian*

$$H : TM \setminus 0_{TM} \to \mathbb{R}; \ (x, v) \mapsto l|v|$$

[2] A metric on M is called Zoll if all geodesics are closed and of same length.

is smooth and generates a free Hamiltonian circle action. We can now do a Lerman-cut at $H = \varepsilon$ and $H = l$. The resulting manifold $(\overline{D_1M \setminus 0_{TM}}, \mathrm{d}\lambda)$ is closed with a semi-free Hamiltonian circle action. We can now apply Cor. 4.1.3 to obtain

$$c_{HZ}(D_1M, M, \mathrm{d}\lambda) = l,$$

which confirms the results of [49, Prop. 4.3] and [5, Cor. 2.8] in the special case of Zoll manifolds.

The same idea of computing Hofer–Zehnder capacities using Lu's theorem in the context of Hamiltonian circle actions was also used by Hwang and Suh [28] for closed Fano[3] symplectic manifolds with semi-free Hamiltonian circle action.

Theorem 4.1.5 (Thm. 1.1. [28])**.** *Let (M, ω) be a closed Fano symplectic manifold with a semifree Hamiltonian circle action. The Gromov width and the Hofer–Zehnder capacity are estimated as*

(a) $c_G(M, \omega) \leq \max(H) - \min(H) \leq c_{HZ}(M, \omega)$.

(b) Further if H_{min} is a point, then

$$c_G(M, \omega) = \operatorname{smin}(H) - \min(H), \quad c_{HZ}(M, \omega) = \max(H) - \min(H).$$

One nice observation from this theorem is that it is compatible with taking products. If Fano symplectic manifolds (M_1, ω_1), (M_2, ω_2) with Hamiltonian circle actions generated by H_1, H_2 satisfy the prerequisites of Theorem 4.1.5 (b), then so does $(M_1 \times M_2, a\omega_1 \oplus b\omega_2)$ with Hamiltonian $aH_1 \circ \pi_1 + bH_2 \circ \pi_2$, where π_1, π_2 are the projections on the first resp. second factor. In particular

$$c_G(M_1 \times M_2, a\omega_1 \oplus b\omega_2) = \min\{|a| c_G(M_1, \omega_1), |b| c_G(M_2, \omega_2)\}$$

and

$$c_{HZ}(M_1 \times M_2, a\omega_1 \oplus b\omega_2) = |a| c_{HZ}(M_1, \omega_1) + |b| c_{HZ}(M_2, \omega_2)$$

while for arbitrary symplectic manifolds only

$$c_G(M_1 \times M_2, a\omega_1 \oplus b\omega_2) \geq \min\{|a| c_G(M_1, \omega_1), |b| c_G(M_2, \omega_2)\}$$

and

$$c_{HZ}(M_1 \times M_2, a\omega_1 \oplus b\omega_2) \geq |a| c_{HZ}(M_1, \omega_1) + |b| c_{HZ}(M_2, \omega_2)$$

holds.

Corollary 4.1.6. *Let (M, ω) be a closed Fano symplectic manifold with a semifree Hamiltonian circle action and H_{min} a point, then the Hofer–Zehnder capacity of any compact neighborhood of the zero-section in $(T^*M, \mathrm{d}\lambda)$ is bounded.*

[3] They call (M, ω) Fano if there exists a compatible S^1-invariant almost complex structure such that all non-constant pseudo holomorphic spheres have positive Chern number. In particular monotone implies Fano.

Proof. The zero-section is a Lagrangian diffeomorphic to M. Also the diagonal in $(M \times M, \omega \ominus \omega)$ is such a Lagrangian. By the previous considerations $c_{HZ}(M \times M, \omega \ominus \omega)$ is finite. This implies by Lagrangian neighborhood theorem that the Hofer–Zehnder capacity of some neighborhood of the zero-section must be finite. Scaling the fibers of the disc-bundle only scales the symplectic form and thus the capacity. We can therefore shrink any compact subset of T^*M to fit in the neighborhood of the zero-section. □

This corollary is a special case of the main theorem in [2] by Albers, Frauenfelder and Oancea. Indeed for all such Fano symplectic manifolds the Hurewicz map

$$\pi_2(M) \to H_2(M; \mathbb{Z})$$

is nonzero, because all gradient spheres represent non-zero elements in $H_2(M; \mathbb{Z})$.

4.2. Magnetic geodesic flow on constant curvature surfaces

In this section we shall see that the magnetic geodesic flow for the symplectically twisted form on surfaces of constant sectional curvature gives rise to a Hamiltonian circle action. So let Σ be a geodesically complete, 2-dimensional, smooth manifold and g a Riemannian metric on Σ of constant curvature κ. Further denote by σ the corresponding area form and by j the compatible complex structure. The circle action we are going to construct will be a reparametrization of the magnetic geodesic flow and have contractible orbits. Hence, we may work on the universal cover, which can be identified with $\mathbb{CP}^1, \mathbb{C}^1, \mathbb{CH}^1$ depending on the sign of the curvature. We want to study the Hamiltonian flow for the kinetic energy

$$E : T\Sigma \to \mathbb{R}, \quad (x, v) \mapsto \frac{1}{2}|v|_x^2,$$

with respect to the twisted symplectic structure $\mathrm{d}\lambda - s\pi^*\sigma$. Using Lemma 2.2.2 we can determine the Hamiltonian vector field

$$(X_E)_{(x,v)} = (v)^{\mathcal{H}} + s(jv)^{\mathcal{V}} = X + sV.$$

We need to look for periodic solutions $\gamma(t) = (x(t), v(t))$ to

$$\dot{\gamma} = X_E = X + sV. \tag{4.1}$$

Applying $\mathrm{d}\pi$ to this equation yields

$$\dot{x} = \mathrm{d}\pi\dot{\gamma} \overset{(4.1)}{=} \mathrm{d}\pi(X + sV) = \mathrm{d}\pi\mathcal{L}^H(v) = v.$$

Thus our solutions must be of the form $\gamma(t) = (x(t), \dot{x}(t))$. On the other hand applying the projection $\mathcal{P} : TT\Sigma \to \mathcal{V}$ on the vertical bundle yields

$$\nabla_{\dot{x}}\dot{x} = \mathcal{P}(\dot{\gamma}) \overset{(4.1)}{=} \mathcal{P}(X + sV) = sjv.$$

This means the projection to M of solutions are curves of geodesic curvature $\kappa_g = |\frac{s}{v}|$. If R denotes the radius (with respect to the Riemannian metric g) of a geodesic circle we know using normal polar coordinates that its circumference C and the geodesic curvature κ_g are

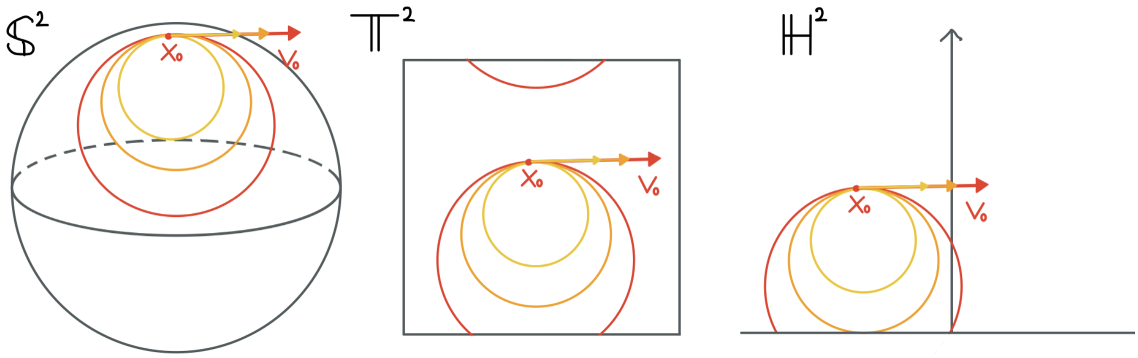


Figure 4.1.: *The picture shows families of geodesic circles. The vectors v_0 of different length indicate that the corresponding magnetic geodesic is a parametrized geodesic circle of geodesic curvature $\kappa_g = s/|v_0|$.*

$$C = \frac{2\pi}{\sqrt{\kappa}} \sin(\sqrt{\kappa}R) = \frac{2\pi\sqrt{\kappa}^{-1}\tan(\sqrt{\kappa}R)}{\sqrt{1+(\tan(\sqrt{\kappa}R))^2}}, \tag{4.2}$$

$$\kappa_g = \frac{\sqrt{\kappa}}{\tan(\sqrt{\kappa}R)}. \tag{4.3}$$

Here we use the convention $\sqrt{-1} = i$ and the formulas $-i\sin(ix) = \sinh(x)$, $\cos(ix) = \cosh(x)$. Observe that in the hyperbolic case the geodesic curvature of geodesic circles can not be less than $\sqrt{|\kappa|}$. Indeed curves of geodesic curvature less than $\sqrt{|\kappa|}$ do not close up. We therefore restrict to the regime of strong magnetic field, i.e. $s^2 + \kappa|v|^2 > 0$. See Figure 4.1 for a visualisation. Inserting κ_g into (4.2) yields

$$C = \frac{2\pi}{\kappa_g\sqrt{1+\kappa/\kappa_g^2}} = \frac{2\pi|v|}{\sqrt{s^2+\kappa|v|^2}},$$

where in the last step we inserted $\kappa_g = s/|v|$. Now, we conclude that the period is given by

$$T = \frac{C}{|v|} = \frac{2\pi}{\sqrt{s^2+\kappa|v|^2}}.$$

In particular the reparametrization $H = h \circ E$ with

$$h : \mathbb{R}_{\geq 0} \to \mathbb{R}; \quad h(E) = \frac{2\pi}{\kappa}\left(\sqrt{s^2+2\kappa E} - |s|\right)$$

induces a Hamiltonian S^1-action (of period $T = 1$). We can now conclude the following proposition.

Proposition 4.2.1. *Let (Σ, g, j) a Riemann surface of constant sectional curvature κ. Then for constants $s \in \mathbb{R} \setminus \{0\}$ and $\rho > 0$ satisfying $s^2 + \kappa\rho^2 > 0$ the Hamiltonian*

$$H : D_\rho M \to \mathbb{R}; \; (x, v) \mapsto \frac{2\pi}{\kappa}\left(\sqrt{s^2+\kappa|v|^2} - |s|\right)$$

generates a Hamiltonian circle action on the disc-subbundle $(D_\rho M, \omega_s)$ of the magnetically twisted tangent bundle.

Observe that all computations survive the limit $\kappa \to 0$. For example the Hamiltonian H is in the case $\kappa = 0$ given by $H = E/|s|$.

If we solve (4.3) for the radius R we obtain

$$R = \frac{1}{\sqrt{\kappa}} \arctan(\sqrt{\kappa}|v/s|).$$

Based on this we define a function b that will become more important later on.

Definition 4.2.2 (Radial function of geodesic circles)**.** *We define the function*

$$b : TM \to \mathbb{R}_{\geq 0};\ \ (x, v) \mapsto \frac{\arctan(\frac{\sqrt{\kappa}|v|}{|s|})}{\sqrt{\kappa}|v|}.$$

Geometrically $R = |v|b(|v|)$ is the radius of the geodesic circles that are the trajectories of X_H. Observe that b is smooth and even in $|v|$ and thus b is also smooth on the zero-section.

4.2.1. Hofer–Zehnder capacity for magnetic surfaces

We just found that these twisted tangent bundles are Hamiltonian S^1-manifolds. Now we can use this to determine their Hofer–Zehnder capacity.

Theorem 4.2.3. *Let (Σ, g_κ) be a closed connected orientable Riemannian surface with constant curvature κ. Denote by σ_κ the corresponding area form, by λ the canonical one-form on $T\Sigma$ and define the disc bundle $D_\rho\Sigma := \{(x, v) \in T\Sigma \mid |v| < \rho\}$ of radius ρ with respect to g_κ. Then, whenever $s^2 + \kappa\rho^2 > 0$ for some $s \in \mathbb{R} \setminus 0$, we have*

$$c^0_{\mathrm{HZ}}(D_\rho\Sigma, \mathrm{d}\lambda - s\pi^*\sigma_\kappa) = \frac{2\pi}{\kappa}\left(\sqrt{s^2 + \kappa\rho^2} - |s|\right).$$

Proof. Observe that for $\rho > 0$ satisfying $s^2 + \kappa\rho^2 > 0$ the Hamiltonian

$$H : D_{\rho+\varepsilon}\Sigma \to \mathbb{R};\ \ H(x, v) = \frac{2\pi}{\kappa}\left(\sqrt{s^2 + \kappa|v|^2} - |s|\right)$$

is well-defined, smooth and generates a semi-free S^1-action. We can use the Lerman-cut construction at the regular energy surface $\{|v| = \rho\}$ to compactify $(D_\rho\Sigma, \omega_s)$. The compactification $(\overline{D_\rho\Sigma}, \overline{\omega_s})$ is a closed symplectic 4-manifold with semi-free Hamiltonian circle action. The critical set where the Hamiltonian attains its minimum corresponds to the zero-section and is thus of codimension two. Now we are precisely in the setup of Corollary 4.1.3 and its application yields

$$c^0_{HZ}(\overline{D_\rho\Sigma}) \setminus \bar{H}_{\max}, \bar{\omega}_s) = \frac{2\pi}{\kappa}\left(\sqrt{s^2 + \kappa\rho^2} - |s|\right)$$

The critical set $\bar{H}_{\max}$ where the Hamiltonian $\bar{H} : \overline{D_\rho M} \to \mathbb{R}$ attains its maximum is precisely the set we glued in to compactify $D_\rho\Sigma$, this means

$$(\overline{D_\rho\Sigma} \setminus \bar{H}_{\max}, \bar{\omega}_s) \cong (D_\rho\Sigma, \omega_s)$$

and thus the claim follows. □

The theorem covers three types of surfaces: spheres ($\kappa > 0$), tori ($\kappa = 0$) and higher genus surfaces ($\kappa < 0$). The assumption $s^2 + \kappa\rho^2 > 0$ does not put any additional constraint on the sphere, for the torus it tells us that the magnetic field does not vanish, i.e. $s \neq 0$, and for higher genus surfaces it tells us to look at strong magnetic fields, i.e. $|s| > \sqrt{-\kappa}\rho$.

Remark 4.2.4. *While $c_{\mathrm{HZ}} \leq c^0_{\mathrm{HZ}}$ always holds, we found in these three cases that actually*

$$c^0_{\mathrm{HZ}}(D_\rho\Sigma, \mathrm{d}\lambda - s\pi^*\sigma_\kappa) = c_{\mathrm{HZ}}(D_\rho\Sigma, \mathrm{d}\lambda - s\pi^*\sigma_\kappa),$$

as the lower bound was obtained from a Hamiltonian with no fast periodic orbits, neither contractible nor not contractible.

4.3. Capacities of Hermitian symmetric spaces

In this section we will compute the Gromov width and the Hofer–Zehnder capacity of irreducible Hermitian symmetric spaces of compact type. The Gromov width was already determined by Loi, Mossa and Zuddas [35], but the value for the Hofer–Zehnder capacity at least to the knowledge of the author was still unknown. We will prove respectively give an alternative proof of the following two theorems.

Theorem 4.3.1 (Thm. G)**.** *Let (M, g) be an irreducible Hermitian symmetric space of compact type. Denote by r the rank of M and normalize σ such that $\sigma(A) = 4\pi$ for A the homology class of any factor in a poly-sphere. Then the Hofer–Zehnder capacity is given by*

$$c_{HZ}(M, \sigma) = 4\pi r.$$

For the special case of complex Grassmanians Theorem 4.3.1 was proven by [28, Ex. 4.1].

Theorem 4.3.2 ([35] Thm. 1)**.** *Let (M, g) be an irreducible Hermitian symmetric space of compact type. Normalize σ such that $\sigma(A) = 4\pi$ for A the homology class of any factor in a poly-sphere. Then the Gromov width is given by*

$$c_G(M, \sigma) = 4\pi.$$

We shall see that all (irreducible) Hermitian symmetric spaces satisfy the prerequisites of Theorem 4.1.5 by Hwang and Suh. Indeed Hermitian symmetric spaces are monotone and thus Fano as shown in [10, Ch. 5, §16]. Further as stated in the following Lemma the representation of M as adjoint orbit $O_Z \subset \mathfrak{g}$ almost immediately yields a Hamiltonian circle action.

Lemma 4.3.3. *Let M be an irreducible Hermitian symmetric space. The Hamiltonian function*

$$\nu : M \cong O_Z \to \mathbb{R}, \;\; x \mapsto 2\pi B(Z, x)$$

generates a circle action. Here $B(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ denotes the Killing form.

Proof. Let us compute the Hamiltonian vector field,

$$\mathrm{d}\nu_x(\cdot) = 2\pi B(Z, [x, \cdot]) = -2\pi B([Z, \cdot], x) = 2\pi\iota_{Z^\#}\sigma.$$

We conclude $X_\nu = 2\pi Z^\#$, which clearly generates a circle action, as the group generated by Z is isomorphic to S^1. We shall see later that the prefactor is there to ensure that the period of the circle action is one (see figure 4.2). □

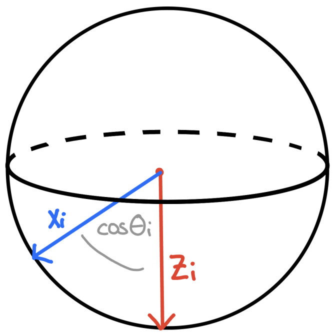


Figure 4.2.: *In order to satisfy* $\mathrm{ad}_{Z_i}^2 = -\mathrm{id}$, *the norm of* Z_i *needs to be one. We see that* (Z_i, x_i) *is equal to the height function, which generates a circle action of period* 2π. *In particular* ν *generates a circle action of period one.*

In order for ν to fulfill the prerequisites of Theorem 4.1.5 we need to show that ν has an isolated minimum. Clearly Z is a critical point of ν. We claim that the point Z is the isolated minimum of ν.

Lemma 4.3.4. *The Hessian of* ν *at* $p = Z$ *is positive definite, thus* $p = Z$ *is isolated and the global minimum.*

Proof. Take $a, b \in \mathfrak{p} \cong T_Z M$, then

$$\mathrm{Hess}_Z(\nu)(a_Z^{\#}, b_Z^{\#}) = a^{\#}\left(b^{\#}(\nu)\right)|_Z = a^{\#}\left(\mathrm{d}\nu(b^{\#})\right)|_Z = a^{\#}\left(2\pi B(Z,[b,p])\right)|_{p=Z}$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} 2\pi B(Z,[b,\mathrm{Ad}_{e^{ta}}(p)])\right)|_{p=Z} = 2\pi B(Z,[b,[a,Z]]) = -2\pi B(a,b).$$

We conclude that

$$\mathrm{Hess}_Z = -2\pi B(\cdot,\cdot)|_{\mathfrak{p}\times\mathfrak{p}}.$$

In particular the Hessian is positive definite as the Killing-form restricted to $\mathfrak{p}$ is negative definite in the compact case. This shows that ν is a local minimum. In general any Hamiltonian generating a circle action is a Morse-Bott function, its critical submanifolds are symplectic and their indices and coindices are even [41, Lem. 5.5.7]. By [41, Lem. 5.5.5] all level sets of such functions are connected, thus $p = Z$ is the global minimum. ☐

This shows that our moment map ν satisfies the prerequisites of Theorem 4.1.5.

Lemma 4.3.5. *The Hamiltonian* ν *satisfies*

$$\max(\nu) - \min(\nu) = 4\pi r, \quad \mathrm{smin} - \min(\nu) = 4\pi,$$

where $\mathrm{smin}(\nu)$ *denotes the second lowest value of* ν *at a critical point.*

Proof. Observe that for any $x \in M$ there exists a poly-sphere through x and Z. From Lemma 2.4.24 we know that the poly-spheres are sub orbits and stay in an affine copy of $(\mathfrak{su}(2))^r$. Thus we can decompose $Z = \sum_i Z_i + c$ and $x = \sum_i x_i + c$, where c the vector orthogonal to the affine subspace (see Figure 4.3). Recall that we picked $Z \in \mathfrak{g}$ the unique generator of the center of K such that $[Z,[Z,v]] = -v$ for all $v \in \mathfrak{p}$. In particular it follows

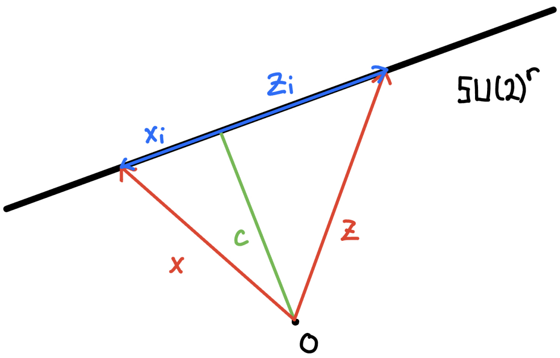


Figure 4.3.: *The figure schematically shows how the poly-spheres sits in an affine copy of* $\mathfrak{su}(2)^r$.

that $|Z_i| = 1$. Indeed, this is equivalent to the above condition as $[Z_i, [Z_i, v]] = -|Z_i|^2 v$ for the case $\mathfrak{su}(2)$. See figure 4.2 for a visualization. We compute

$$(Z, x) = \sum_{i=1}^{r} (Z_i, x_i) + |c|^2 = \sum_{i=1}^{r} \cos(\theta_i) + |c|^2,$$

where we used that $|Z_i| = |x_i| = 1$. Therefore we conclude

$$\max(\nu) - \min(\nu) = 4\pi r \quad \text{and} \quad \text{smin}(\nu) - \min(\nu) = 4\pi.$$

□

These three Lemmas together with Theorem 4.1.5 prove Theorem 4.3.1 and 4.3.2.

5. Symplectically twisted tangent bundle as Hermitian vector bundle

The goal of this section is to relate the following symplectic structures of the tangent bundle of locally Hermitian symmetric spaces M.

1. The symplectically twisted symplectic structure $(TM, \mathrm{d}\lambda - \pi^*\sigma)$ below the Mané critical value.

2. The symplectic structure $(TM, \mathrm{d}\tau/2 - \pi^*\sigma)$ in a neighborhood of the zero-section determined by the holomorphic bisectional curvature.

3. The symplectic form ω_I belonging to the hyperkähler structure that exists in a neighborhood of the zero-section.

The identification of the above structures evolved in three steps, generalizing the class of manifolds. The first construction only works for constant curvature surfaces as it plays with the fact that in dimension four the vector fields X, H, Y, V form a global frame of $T(TM \setminus 0_{TM})$. The second construction only works for spaces with constant holomorphic sectional curvature and uses that the differential of our symplectomorphism can be expressed in terms of Jacobi fields, which are particularly easy to determine in the case of constant holomorphic sectional curvature. The last and most general construction uses the global symmetries of Hermitian symmetric spaces. Intertwining the moment maps of the group actions already determines most of the map. Further observe that locally near the zero-section all three structures must be equivalent by the symplectic neighborhood theorem. This section is concerned with making things work globally or at least in an explicit neighborhood of the zero-section. We start the section with an observation for the easiest case, a flat manifold.

Remark 5.0.1. *All our constructions work equally well on quotients by finite subgroups of the isometry group. We shall therefore always work on the universal cover. For example flat manifolds are covered by Euclidean space. Only when speaking about capacities we assume our base manifold to be closed. Further we fix the real parameter $s > 0$. The calculations should also go through for $s < 0$, but changing the sign $s \to -s$ one also needs to change $j \to -j$ and $\tau \to -\tau$, so it gets a little tedious to trace.*

5.1. Observation for symplectic vector spaces

As a first example let us look at a symplectic vector space (V, σ). Choose a compatible complex structure j such that $\sigma(\cdot, j\cdot) =: g$ defines a metric. The tangent bundle TV will be identified with $V \times V$. We can think of the factors as horizontal and vertical. Now we want to compare three symplectic structures on $TV = V \times V$ namely $\mathrm{d}\tau/2 - s\pi^*\sigma$, $\mathrm{d}\lambda - s\pi^*\sigma$

and in the case $s = 1$ also $-\omega_I$. The formulas in Proposition (2.1.5) in the introduction yield in the flat case

$$d\tau/2 - s\pi^*\sigma := \begin{pmatrix} -s\sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

where the matrix is decomposed with respect to the identification $TV = V \times V$. We immediately see that $d\tau - \pi^*\sigma = -\omega_I$ in the flat case. The symplectically twisted structure is given by

$$d\lambda - s\pi^*\sigma := \begin{pmatrix} -s\sigma & g \\ -g & 0 \end{pmatrix}.$$

For both symplectic structures the kinetic Hamiltonian $E(x,v) = \frac{1}{2}g(v,v)$ generates a circle action. In the Hermitian case the orbits are simply $(x, e^{jt}v)$. For the symplectically twisted case the orbits are of the form $(\gamma(t), \dot\gamma(t))$, where γ parametrizes a circle of radius $|\dot\gamma|/s$ in the affine plane $\gamma(0) + \mathrm{span}\{\dot\gamma(0), j\dot\gamma(0)\}$. We claim that the map that intertwines

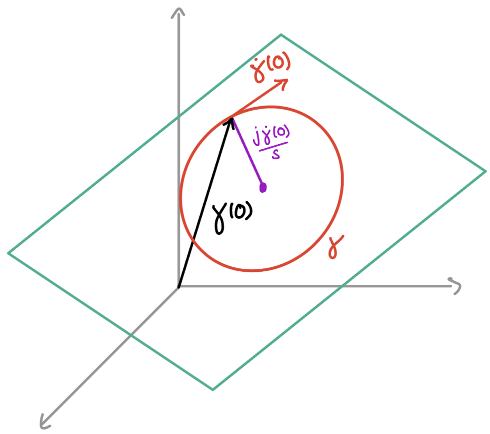


Figure 5.1.: *The Hamiltonian circle action for the symplectically twisted tangent bundle.*

these circle actions

$$\Psi : \left(TV, \begin{pmatrix} -s\sigma & 0 \\ 0 & \sigma \end{pmatrix}\right) \to \left(TV, \begin{pmatrix} -s\sigma & g \\ -g & 0 \end{pmatrix}\right); \quad \begin{pmatrix} x \\ v \end{pmatrix} \mapsto \begin{pmatrix} x + jv/s \\ v/\sqrt{s} \end{pmatrix} = \begin{pmatrix} 1 & j/s \\ 0 & 1/\sqrt{s} \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

is a symplectomorphism. The proof is one line of matrix multiplication

$$\begin{pmatrix} 1 & 0 \\ -j/s & 1/\sqrt{s} \end{pmatrix} \begin{pmatrix} -s\sigma & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} 1 & j/s \\ 0 & 1/\sqrt{s} \end{pmatrix} = \begin{pmatrix} -s\sigma & g \\ -g & 0 \end{pmatrix}.$$

We conclude that all three symplectic structures are equivalent in the flat case. The idea of this chapter is to generalize the above symplectomorphism to some non-flat Kähler manifolds. This will be done by replacing the linear maps by flows of the vector fields $Y := (v)^{\mathcal{V}}$ and $H := (jv)^{\mathcal{H}}$. The linear case can also be rephrased in this language. Simply look at

$$\Phi^Y_a(x,v) = (x, e^a v), \quad \Phi^H_b(x,v) = (x + bjv, v).$$

Now we identify

$$\Psi = \Phi^Y_{\ln(\sqrt{s}^{-1})} \circ \Phi^H_{s^{-1}} : TV \to TV.$$

5.2. Constant curvature surfaces

The results of this sections are also contained in an arXiv article of the author [7]. Riemannian surfaces (Σ, g) of constant curvature κ are our main examples, all further generalizations will be reduced to this case by using the polysphere/ polydisc theorem 2.4.13. The case of surfaces is special, as it is the only case where the vector fields X, H, Y and V determine a frame of TTM outside the zero-section. This makes it possible to construct the symplectomorphism in a more elegant and direct way than in the higher dimensional cases. Observe that the commutation relations Prop. 2.1.3 for constant curvature simplify to

$$\begin{aligned} [Y,X] &= X, \qquad [Y,H] = H, \qquad [Y,V] = 0, \\ [V,X] &= H, \qquad [V,H] = -X, \qquad [X,H] = 2E\kappa V. \end{aligned}$$

These translate into the following formulas for the exterior derivatives of the (metric) dual frame $(\lambda, \eta, \mathrm{d}E, \tau)$ of 1-forms

$$\begin{aligned} \mathrm{d}\lambda &= \frac{1}{2E}(\mathrm{d}E \wedge \lambda + \tau \wedge \eta) \\ \mathrm{d}\eta &= \frac{1}{2E}(\lambda \wedge \tau + \mathrm{d}E \wedge \eta) \\ \mathrm{d}\tau &= \kappa\eta \wedge \lambda + \frac{1}{E}\mathrm{d}E \wedge \tau. \end{aligned}$$

Observe that in particular

$$\mathrm{d}\left(\frac{\tau}{2E}\right) = -\frac{k}{2E}\lambda \wedge \eta = -\kappa\pi^*\sigma$$

and thus

$$\mathrm{d}\lambda - s\pi^*\sigma = \mathrm{d}\left(\lambda + \frac{s}{2E\kappa}\tau\right) \quad \text{and} \quad \mathrm{d}\tau/2 - s\pi^*\sigma = \frac{1}{2}\mathrm{d}\left(1 + \frac{s}{2\kappa E}\right)\tau$$

outside the zero section. As indicated in the previous section we will use the flows of our vector fields H and Y to construct the symplectomorphism. This computation was done for curvature $\kappa = 1$ in [6, Thm. A.1.], we only adopt it to general constant curvature. Let ξ be some 1-form on $T\Sigma$. We want to compute $\mathrm{d}\Phi_t^H(\xi)$. For the sake of this we make use of the fact that $\lambda, \eta, \mathrm{d}E, \tau$ yields a global dual frame outside the zero section, thus $(\Phi_t^H)^*(\xi) = x\lambda + y\eta + z\tau + w\mathrm{d}E$ for some coefficients x, y, z, w depending on t. Thus

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[(\Phi_{-t}^H)^*(\Phi_t^H)^*\lambda\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[(\Phi_{-t}^H)^*(x\lambda + y\eta + z\tau + w\mathrm{d}E)\right] \\ &= \dot{x}\lambda + \dot{y}\eta + \dot{z}\tau + \dot{w}\mathrm{d}E - x\mathcal{L}_H\lambda - y\mathcal{L}_H\eta - z\mathcal{L}_H\tau - w\mathcal{L}_H\mathrm{d}E \\ &= \dot{x}\lambda + \dot{y}\eta + \dot{z}\tau + \dot{w}\mathrm{d}E - x\iota_H\mathrm{d}\lambda - y\iota_H\mathrm{d}\eta - y\mathrm{d}(\iota_H\eta) - z\iota_H\mathrm{d}\tau \\ &= \dot{x}\lambda + \dot{y}\eta + \dot{z}\tau + \dot{w}\mathrm{d}E + x\tau + y\mathrm{d}E - 2y\mathrm{d}E - 2E\kappa z\lambda. \end{aligned}$$

This yields the following system of ODE's,

$$\dot{x} = 2E\kappa z, \quad \dot{y} = 0, \quad \dot{z} = -x \quad \dot{w} = y.$$

Denote the initial data by $x_0, y_0, z_0, w_0 \in \mathbb{R}$, then the system is solved by

$$x(t) = x_0 \cos(\sqrt{2E\kappa}t) + z_0\sqrt{2\kappa E}\sin(\sqrt{2E\kappa}t), \quad y(t) = y_0,$$
$$z(t) = z_0 \cos(\sqrt{2E\kappa}t) - \frac{x_0}{\sqrt{2E\kappa}}\sin(\sqrt{2E\kappa}t), \quad w(t) = w_0 + y_0 t.$$

Plugging in the initial conditions for ξ equal to $\lambda, \eta, \tau, \mathrm{d}E$ we obtain

$$(\Phi_t^H)^*\lambda = \cos(\sqrt{2E\kappa}t)\lambda - \frac{1}{\sqrt{2E\kappa}}\sin(\sqrt{2E\kappa}t)\tau,$$
$$(\mathrm{d}\Phi_t^H)^*\eta = \eta, \quad (\mathrm{d}\Phi_t^H)^*\mathrm{d}E = \mathrm{d}E,$$
$$(\mathrm{d}\Phi_t^H)^*\tau = \cos(\sqrt{2E\kappa}t)\tau + \sqrt{2E\kappa}\sin(\sqrt{2E\kappa}t)\lambda.$$

Analogously one gets ODE's determining the pullback via the flow of Y,

$$\dot{x} = x, \quad \dot{y} = y, \quad \dot{z} = 2z \quad \dot{w} = 2w$$

with solutions

$$x(t) = x_0 e^t, \quad y(t) = y_0 e^t, \quad z(t) = z_0 e^{2t}, \quad w(t) = w_0 e^{2t}.$$

Thus

$$(\Phi_t^Y)^*\lambda = e^t\lambda, \quad (\Phi_t^Y)^*\eta = e^t\eta, \quad (\Phi_t^Y)^*\tau = e^{2t}\tau, \quad (\Phi_t^Y)^*\mathrm{d}E = e^{2t}\mathrm{d}E.$$

For the symplectomorphism we will not only need the flows of H, Y for some constant times, but rather times depending on $r := |v|$. We make the ansatz

$$\varphi := \Phi_{a(r)}^Y \circ \Phi_{b(r)}^H : \; D_\rho\Sigma \to D_{e^a\rho}\Sigma$$

for some smooth even functions $a, b : \mathbb{R} \to \mathbb{R}$. The differential of φ is

$$\mathrm{d}\varphi = \frac{\dot{a}}{b} Y \otimes \mathrm{d}r - a\dot{b}\left(\mathrm{d}\Phi_a^Y H\right) \otimes \mathrm{d}r + \mathrm{d}\Phi_a^Y \mathrm{d}\Phi_b^H$$

and $\mathrm{d}\Phi_a^Y H = e^a H$, thus τ vanishes on the first two summands. Now we can compute

$$\varphi^*\left(\frac{1}{2} + \frac{s}{\kappa r^2}\right)\tau = \left(\frac{1}{2} + \frac{s}{\kappa e^{2a} r^2}\right)e^{2a}(\cos(\sqrt{\kappa}rb)\tau + \sqrt{\kappa}r\sin(\sqrt{\kappa}rb)\lambda) \overset{!}{=} \alpha + \frac{s}{r^2\kappa}\tau.$$

We conclude that the functions a, b must fulfill the following equations

$$\frac{1}{\sqrt{\kappa}r} = \left(\frac{1}{2} + \frac{s}{\kappa e^{2a} r^2}\right)e^{2a}\sin(\sqrt{\kappa}rb),$$
$$\frac{s}{r^2\kappa} = \left(\frac{1}{2} + \frac{s}{\kappa e^{2a} r^2}\right)e^{2a}\cos(\sqrt{\kappa}rb).$$

They are solved by

$$b(r) = \frac{\arctan(\sqrt{\kappa}r/s)}{\sqrt{\kappa}r}, \tag{5.1}$$
$$a(r) = \frac{1}{2}\ln\left(\frac{2}{\kappa r^2}\left(\sqrt{s^2 + \kappa r^2} - s\right)\right). \tag{5.2}$$

Indeed these are well-defined and smooth as long as $\kappa r^2 + s^2 > 0$ and $s > 0$. Observe that φ is a symplectomorphism onto its image as the function $re^{a(r)}$ is monotone in r. In total we have proven the following theorem.

Theorem 5.2.1. *Let (Σ, g) be a surface with constant curvature κ. Then if $s > 0$, $s^2 + \kappa\rho^2 > 0$, there is a symplectomorphism*

$$\varphi : (D_\rho\Sigma, \lambda - s\pi^*\sigma) \to \left(D_{e^{a(\rho)}\rho}\Sigma, \mathrm{d}\tau/2 - s\pi^*\sigma\right)$$

intertwining magnetic geodesic flow and fiberwise rotation. In particular for non-negative curvature $\kappa \geq 0$ the symplectomorphism extends to the full tangent bundle.

Taking a closer look we may even realize that we already know the functions a, b. Indeed

$$E \circ \varphi = \frac{1}{2}e^{2a}r^2 = \frac{1}{\kappa}\left(\sqrt{s^2 + \kappa r^2} - s\right)$$

is the Hamiltonian generating the circle action on the symplectically twisted tangent bundle and $b(r)r$ gives the radius of the geodesic circles that are the projected magnetic geodesics (see Def. 4.2.2). So the map φ is indeed just the straight forward generalization of the flat case. The flow of H is sending a point on the geodesic circle to the circle center, indeed if we denote by $\gamma_{(x,v)}(t) = \exp_x(tv)$ the geodesic starting at x in the direction v the flow of H is given by

$$\Phi_t^H(x, v) = (\gamma_{(x,jv)}(t), (P_{\gamma_{(x,jv)}}v)(t)).$$

So we find

$$\varphi(x, v) = (\gamma_{(x,jv)}(b), e^a(P_{\gamma_{(x,jv)}}v)(b)).$$

Remark 5.2.2. *Observe that in all formulas we can take the limit $\kappa \to 0$ and recover the formulas for vector spaces.*

In the case of negative curvature the symplectomorphism is only valid for the regime $s^2 + \kappa\rho^2 > 0$. We can use the same ansatz to show another symplectomorphism in the regime $s^2 + \kappa\rho^2 < 0$. As this is only relevant for negative curvature we shall set $\kappa = -1$.

Theorem 5.2.3. *Let (Σ, g) be a surface with constant curvature $\kappa = -1$. For $a < b$ we denoted by $D_a^b\Sigma := \{(x, v) \in T\Sigma \mid a < |v|\}$ the annulus subbundle. Then if $s < \rho$, there is a symplectomorphism*

$$\tilde{\varphi} : (D_s^\rho\Sigma, \lambda - s\pi^*\sigma) \to \left(D_0^{\sqrt{\rho^2 - s^2}}\Sigma, \mathrm{d}\lambda\right)$$

intertwining magnetic geodesic and geodesic flow.

Proof. We use the ansatz $\tilde{\varphi} = \Phi^Y_{\tilde{a}(r)} \circ \Phi^H_{\tilde{b}(r)}$ for some functions $\tilde{a}, \tilde{b}$. Then

$$\tilde{\varphi}^*\lambda = e^{\tilde{a}}\left(\cosh(r\tilde{b}))\lambda + \sinh(r\tilde{b})/r\tau\right) \stackrel{!}{=} \alpha - \frac{s}{r}\tau.$$

Thus $\tilde{a}$ and $\tilde{b}$ must fulfill

$$-\frac{s}{r} = \sinh(r\tilde{b})e^{\tilde{a}}, \qquad 1 = \cosh(r\tilde{b})e^{\tilde{a}},$$

which is solved by

$$\tilde{a}(r) = \frac{1}{2}\ln\left(\frac{r^2 - s^2}{r^2}\right), \qquad \tilde{b}(r) = \frac{1}{r}\operatorname{arctanh}\left(-\frac{s}{r}\right).$$

These are smooth as $s < \rho$, further $re^{\tilde{a}(r)}$ is monotone in r and therefore $\tilde{\varphi}$ is a diffeomorphism. □

This symplectomorphism can be used to compute the relative capacity.

Corollary 5.2.4 (Ex. 4.7;[19]). *If $s^2 + \kappa\rho^2 < 0$ the Hofer–Zehnder capacity of $(D_\rho\Sigma, \omega_s)$ relative to $D_{\frac{|s|}{\sqrt{-\kappa}}}\Sigma$ is given by*

$$c^\nu_{\mathrm{HZ}}\left(D_\rho\Sigma, D_{\frac{|s|}{\sqrt{-\kappa}}}\Sigma, \omega_s\right) = l_\nu\sqrt{-(\kappa\rho^2 + s^2)},$$

where l_ν denotes the shortest length of a closed geodesic in the free-homotopy class ν of loops in Σ.

Proof. J. Weber determined in [49, Thm. 4.3] the value of the BPS-capacity of unit disc-bundles with canonical symplectic structure relative to the zero section to be

$$c^\nu_{BPS}(D_1\Sigma, \Sigma, \omega_0) = l_\nu. \tag{5.3}$$

Ginzburg and Gürel showed in [20] section 2.2 that the relative Hofer–Zehnder capacity coincides with the BPS-capacity for standard bundles as defined in [49]. Combining this with the symplectomorphism above yields the corollary. □

The value was already given in Ginzburg [19, Ex. 4.7], we wrote down the proof as we could not find the details in the literature.

5.3. Spaces of constant holomorphic sectional curvature

In this section we will generalize our result from the previous section to higher dimensions. As it turns out the spaces that are suitable higher dimensional analogues of constant curvature surfaces are spaces of constant holomorphic sectional curvature.

Definition 5.3.1. *The holomorphic sectional curvature of a Kähler manifold (M, g, j) is defined to be*

$$\mathrm{Hol} : TM \to \mathbb{R};\ (x, v) \mapsto \frac{g(v, R(v, jv)jv)}{|v|^4},$$

where R denotes the Riemannian curvature tensor of M.

Let now (M, g, j) be a Kähler manifold of constant holomorphic sectional curvature κ. Further denote $\sigma(\cdot,\cdot) := g(j\cdot,\cdot)$ the associated Kähler form on M. Manifolds of constant holomorphic curvature can be characterized by the following theorem.

Theorem 5.3.2 ([34], Thm. 7.9). *The universal cover of a space of constant holomorphic sectional curvature κ is holomorphically isometric to the model space of holomorphic sectional curvature κ, i.e. complex projective space $\mathbb{C}\mathrm{P}^n$, complex space $\mathbb{C}^n$ or complex hyperbolic space $\mathbb{C}\mathrm{H}^n$.*

Since we will focus on contractible periodic solutions to Hamiltonian systems we can (for now) pretend that M is one of the model spaces. The model spaces have one striking property. Take a point $(x, v) \in TM$ outside the zero-section, then there is an (up to isometry) unique totally geodesic copy of $\mathbb{C}\mathrm{P}^1$, $\mathbb{C}^1$ or $\mathbb{C}\mathrm{H}^1$ through x with v tangent to it. This means that some geometric and dynamical properties reduce to the study of these in one complex dimension. We can still not just copy the two dimensional proof as it strongly relied on the fact that the vector fields X, H, Y, V form a frame of $TT\Sigma$ outside the zero-section. Instead we will compute the differential of φ explicitly using Jacobi-fields. This means we need to solve Jacobi-equations and this is possible since the curvature tensor for spaces of constant holomorphic sectional curvature is relatively simple. Indeed as shown in [34, Prop. 7.6] for a space of constant holomorphic sectional curvature κ the curvature tensor is given by

$$R(W,Y)Z = \frac{\kappa}{4}[g(W,Z)Y - g(Y,Z)W - g(W,jZ)jY + g(Y,jZ)jW - 2g(W,jY)jZ]. \quad (5.4)$$

Using this formula the last commutator relation of Prop. 2.1.3 again simplifies to

$$[X,H] = (R(v,jv)v)^{\mathcal{V}} = 2\kappa EV$$

and the exterior derivative of τ is

$$\mathrm{d}\tau = -2\omega^{\mathcal{V}} + \frac{\kappa}{2}[\lambda \wedge \eta + 2E\pi^*\omega].$$

Theorem 5.3.3. *Let (M, g, j) be a Kähler manifold of constant holomorphic sectional curvature κ and let $\rho > 0$, $s > 0$ satisfy $s^2 + \kappa\rho^2 > 0$. Then*

$$\varphi := \Phi^Y_a \circ \Phi^H_b : (D_\rho M, \mathrm{d}\lambda - s\pi^*\sigma) \to (D_{e^a\rho}M, \mathrm{d}\tau/2 - s\pi^*\sigma)$$

is a symplectomorphism. Further the Hamiltonian circle action on $(D_\rho M, \omega_s)$ is intertwined with the fiberwise rotation $e^{2\pi jt} : T_pM \to T_pM$.

The rest of this section contains the proof of this theorem. The strategy is very straight forward. We compute the differential explicitly using Jacobi fields. Then we plug the differential into the symplectic form to show that the map is symplectic.

Take a point $(x, v) \in D_\rho M$ and consider the curve

$$\gamma(t) = \exp_x(-tb(|v|)jv)$$

We abbreviate $\phi := \Phi^H_{-b} = (\Phi^H_b)^{-1}$ and recall that the flow of H for time b is given by sending x to $\gamma(1)$ and parallel transporting v along the curve, i.e.

$$\phi(x,v) = (\gamma(1), (P_\gamma v)(1)).$$

Since γ is a geodesic and j is parallel (i.e. $\nabla j = 0$ for Kähler manifolds) we can rewrite the parallel transport to find

$$\phi(x,v) = \Big(\gamma(1), b(|v|)^{-1} j\dot\gamma(1)\Big).$$

To determine the pullback $\phi^*\omega_s$ we need to compute the differential $\mathrm{d}\phi$. We start with the vertical directions. Consider a path $(x, v(s)) \subset T_xM$[1] with $v(0) = v$ and $\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}v(s) = w$

[1] Attention! We ran out of parameters. This s is a time like parameter and has nothing to do with the magnetic field strength in ω_s.

for some $w \in T_v(T_xM) = T_xM$. Then

$$\begin{aligned}
\mathrm{d}\phi(0,w) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\phi(x,v(s)) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\Big(\gamma_s(1), b(|v|)^{-1}j\dot{\gamma}_s(1)\Big) \\
&= \Big(\partial_s\gamma_s(1), \partial_s(b(|v_s|)^{-1})j\dot{\gamma}_s(1) + b(|v|)^{-1}\nabla_{\partial_s}(j\dot{\gamma}_s(1))\Big)\Big|_{s=0} \\
&= \Big(\partial_s\gamma_s(1), \partial_s(b(|v_s|)^{-1})j\dot{\gamma}_s(1) + b(|v|)^{-1}j\nabla_{\partial_t}(\partial_s\gamma_s(1))\Big)\Big|_{s=0}.
\end{aligned}$$

Observe that γ_s is a family of geodesics thus $\partial_s\gamma_s$ evaluated at $s=0$ is the Jacobi field X_w determined by

$$X_w(0) = 0 \text{ and } \dot{X}_w(0) = -j[(\partial_s b(|v_s|))v + b(|v|)w]. \tag{5.5}$$

This means we can rewrite

$$\mathrm{d}\phi(0,w) = \Big(X_w(1), \partial_s(b(|v|)^{-1})j\dot{\gamma}(1) + b(|v|)^{-1}j\dot{X}_w(1)\Big).$$

Lemma 5.3.4. *The Jacobi fields X_w along γ determined by the initial conditions (5.5) are given by*

$$X_v(t) = t\left(\frac{b'r+b}{b}\right)\dot{\gamma}(t), \quad X_{jv}(t) = \frac{\sin(\sqrt{\kappa}brt)}{\sqrt{\kappa}br}j\dot{\gamma}(t)$$

and for w orthogonal to v and jv we have

$$X_w(t) = \frac{-2}{\sqrt{\kappa}r}\sin\left(\frac{\sqrt{\kappa}}{2}brt\right)jw(t),$$

where $w(t)$ denotes the parallel transport of w along $\gamma(t)$.

Proof. Any Jacobi field X must satisfy the Jacobi equation

$$\ddot{X} = R(\dot{\gamma}, X)\dot{\gamma}.$$

Plugging $W = Z = \dot{\gamma}$ and $Y = X$ into (5.4) yields

$$\ddot{X} = R(\dot{\gamma}, X)\dot{\gamma} = \frac{\kappa}{4}[g(X,\dot{\gamma})\dot{\gamma} - g(\dot{\gamma},\dot{\gamma})X - 3g(X, j\dot{\gamma})j\dot{\gamma}].$$

For the calculations to come we will abriviate $r := |v|$ and $b = b(r)$. Lets now first assume $w = v$, then we can set $v(s) = e^s v$. Thus

$$X_v(0) = 0 \;\;\&\;\; \dot{X}_v(0) = -j(b'r+b)v.$$

The solution of the Jacobi equation for this initial condition is

$$X_v(t) = t\left(\frac{b'r+b}{b}\right)\dot{\gamma}(t).$$

This can easily be verified by plugging $X_v(t)$ into the Jacobi equation and using the simplified form of the curvature tensor. Assume secondly that $w = jv$, then we can set $v(s) = e^{sj}v$, thus

$$X_{jv}(0) = 0 \;\;\&\;\; \dot{X}_{jv}(0) = bv.$$

The solution to the Jacobi equation for this initial data is

$$X_{jv}(t) = \frac{\sin(\sqrt{\kappa}brt)}{\sqrt{\kappa}br} j\dot{\gamma}(t).$$

Finally assume w orthogonal to v and jv. Further choose v_s such that $|v_s| = |v|$ then

$$X_w(0) = 0 \ \ \& \ \ \dot{X}_w(0) = -bjw.$$

The solution to the Jacobi equation for this initial data is

$$X_w(t) = \frac{-2}{\sqrt{\kappa}r} \sin\left(\frac{\sqrt{\kappa}}{2}brt\right) jw(t),$$

where $w(t)$ denotes the parallel transport of w along $\gamma(t)$. □

Next we compute the horizontal directions of the differential. For this take a curve $x(s) \in M$ with $x(0) = x$ and $\dot{x}(0) = w$. Further let $v(s)$ be the parallel vector field along $x(s)$ with $v(0) = v$. Similar to before we look at the 1-parameter family of geodesics

$$\gamma_s(t) := \exp_{x(s)}(-tbjv(s)).$$

Now the differential in this horizontal direction is

$$\mathrm{d}\phi(w,0)) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \phi(x(s), v(s)) = (\partial_s\gamma_s(1), b^{-1}j\nabla_{\partial_t}\partial_s\gamma_s(1))|_{s=0}.$$

We can rewrite this using that $\partial_s\gamma_s$ is the Jacobi field $\tilde{X}_w$ uniquely determined by

$$\tilde{X}_w(0) = w \quad \text{and} \quad \dot{\tilde{X}}_w(0) = 0 \tag{5.6}$$

to see that

$$\mathrm{d}\phi(w,0)) = (\tilde{X}_w(1), b^{-1}j\dot{\tilde{X}}_w(1)).$$

Analogous to the vertical case one checks the following Lemma.

Lemma 5.3.5. *The Jacobi fields $\tilde{X}_w$ along γ determined by the initial conditions (5.6) are given by*

$$\tilde{X}_v(t) = \cos(\sqrt{\kappa}brt)b^{-1}j\dot{\gamma}(t), \quad \tilde{X}_{jv}(t) = -b^{-1}\dot{\gamma}(t)$$

and for w orthogonal to v and jv we have

$$\tilde{X}_w(t) = \cos\left(\frac{\sqrt{\kappa}br}{2}t\right) w(t),$$

where $w(t)$ is the parallel transport of w along γ.

We can now finally compute the pullback $\phi^*\omega_s$.

Lemma 5.3.6. *The pullback of ω_s via ϕ is given by*

$$\phi^*\omega_s = \begin{pmatrix} -h'(E)^{-1}\sigma^{\|} & 0 & 0 & 0 \\ 0 & -e^{-2a(r)}\sigma^{\perp} & 0 & 0 \\ 0 & 0 & h'(E)\sigma^{\|} & 0 \\ 0 & 0 & 0 & e^{2a(r)}\sigma^{\perp} \end{pmatrix},$$

where $\sigma^{\|}$ is σ restricted to the subspace spanned by v and jv and $\sigma^{\perp}$ is σ restricted to the orthogonal complement.

Remark 5.3.7. *The interested reader may check Calabi's paper [11], to find that this is precisely the formula he gives for $-\omega_I$, the symplectic structure compatible with the complex structure $-I := -(j \ominus j)$. Indeed the hyperkähler structure of the tangent bundle of spaces of constant holomorphic curvature is one of the very first hyperkähler structures discovered.*

Proof. First we compute the vertical part using Lemma 5.3.4,

$$\begin{aligned}
\phi^*\omega_s((0,v),(0,jv)) &= \omega_s\left(\left(X_v(1), -\frac{b'r}{b^2}j\dot\gamma(1) + b^{-1}j\dot X_v(1)\right), \left(X_{jv}(1), \frac{1}{b}\dot X_{jv}(1)\right)\right) \\
&= \omega_s\left(\left(\frac{b'r+b}{b}\dot\gamma(1), \frac{1}{b}j\dot\gamma(1)\right), \left(\frac{h'(E)}{b}j\dot\gamma(1), -\frac{sh'(E)}{b}\dot\gamma(1)\right)\right) \\
&= \frac{h'(E)}{b^2}g(j\dot\gamma(1), j\dot\gamma(1)) + \frac{sh'(E)(b'r+b)}{b^2}g(\dot\gamma(1),\dot\gamma(1)) \\
&- s\frac{h'(E)(b'r+b)}{b^2}g(j\dot\gamma(1), j\dot\gamma(1)) \\
&= h'(E)|v|^2 > 0,
\end{aligned}$$

where we used that $\sin(\sqrt{\kappa}br)/\sqrt{\kappa} = rh'(E) = r/\sqrt{s^2+\kappa r^2}$ and $\cos(\sqrt{\kappa}br) = sh'(E)$. Next we repeat the calculation for w orthogonal to v and jv

$$\begin{aligned}
\phi^*\omega_s((0,w),(0,jw)) &= \omega_s((X_w(1), b^{-1}j\dot X_w(1)), (X_{jw}(1), b^{-1}\dot X_{jw}(1)) \\
&= g(b^{-1}j\dot X_w(1), X_{jw}(1)) - g(X_w(1), b^{-1}j\dot X_{jw}(1)) - sg(jX_w(1), X_{jw}(1)) \\
&= \frac{4}{r\sqrt{\kappa}}\cos(\sqrt{\kappa}br/2)\sin(\sqrt{\kappa}br/2)|w|^2 - \frac{4s}{\kappa r^2}\sin(\sqrt{\kappa}br/2)^2|w|^2 \\
&= \frac{2|w|^2}{r\sqrt{\kappa}}\sin(\sqrt{\kappa}br) - \frac{2s|w|^2}{\kappa r^2}(1-\cos(\sqrt{\kappa}br)) \\
&= \frac{2}{\kappa r^2}(\sqrt{s^2+\kappa r^2} - s)|w|^2 = e^{2a(r)}|w|^2 > 0.
\end{aligned}$$

Observe that all other terms vanish as w, jw were assumed to be orthogonal to v and jv.

Next we compute $\phi^*\omega_s$ in the horizontal entries using Lemma 5.3.6

$$\begin{aligned}
\phi^*\omega_s((v,0),(jv,0)) &= \omega_s(\left(\tilde X_v(1), b^{-1}j\dot{\tilde X}_v(1)\right), \left(\tilde X_{jv}(1), b^{-1}\dot{\tilde X}_{jv}(1)\right)) \\
&= \omega_s(\left(\frac{sh'(E)}{b}j\dot\gamma(1), \frac{\kappa r^2 h'(E)}{b}\dot\gamma(1)\right), \left(-\frac{1}{b}\dot\gamma(1), 0\right)) \\
&= \frac{-\kappa r^2 h'(E)}{b^2}g(\dot\gamma(1),\dot\gamma(1)) - s\frac{sh'(E)}{b^2}g(\dot\gamma(1),\dot\gamma(1)) \\
&= -\sqrt{s^2+\kappa r^2}|v|^2,
\end{aligned}$$

and

$$\begin{aligned}
\phi^*\omega_s((w,0),(jw,0)) &= \omega_s((\tilde{X}_w(1), b^{-1}j\dot{\tilde{X}}_w(1)), (\tilde{X}_{jw}(1), b^{-1}\dot{\tilde{X}}_{jw}(1)) \\
&= g(b^{-1}j\dot{\tilde{X}}_w(1), \tilde{X}_{jw}(1)) - g(\tilde{X}_w(1), b^{-1}j\dot{\tilde{X}}_{jw}(1)) - sg(j\tilde{X}_w(1), \tilde{X}_{jw}(1)) \\
&= -\sqrt{\kappa}r\cos(\sqrt{\kappa}br/2)\sin(\sqrt{\kappa}br/2)|w|^2 - s\cos(\sqrt{\kappa}br/2)^2|w|^2 \\
&= \frac{-\sqrt{\kappa}r|w|^2}{2}\sin(\sqrt{\kappa}br) - \frac{s|w|^2}{2}(1+\cos(\sqrt{\kappa}br)) \\
&= \frac{1}{2}(-\kappa r^2 h'(E) - s - s^2h'(E))|w|^2 \\
&= -\frac{1}{2}(\sqrt{s^2+\kappa r^2} + s)|w|^2 \\
&= -e^{-2a(r)}|w|^2.
\end{aligned}$$

Observe that restricted to the zero section this yields $-s\pi^*\sigma$. To finish the computation of the pullback we need to look at mixed terms

$$\phi^*\omega((w_1,0),(0,w_2)).$$

These vanish clearly whenever $w_1 \perp w_2$, so let us look at the three remaining cases. First,

$$\begin{aligned}
\phi^*\omega_s((v,0),(0,v)) &= \omega_s(\left(\tilde{X}_v(1), b^{-1}j\dot{\tilde{X}}_v(1)\right), \left(X_v(1), -\frac{b'r}{b^2}j\dot{\gamma}(1) + b^{-1}j\dot{X}_v(1)\right)) \\
&= \omega_s(\left(\frac{sh'(E)}{b}j\dot{\gamma}(1), \frac{\kappa r^2h'(E)}{b}\dot{\gamma}(1)\right), \left(\frac{b'r+b}{b}\dot{\gamma}(1), \frac{1}{b}j\dot{\gamma}(1)\right)) \\
&= \frac{\kappa r^2 h'(E)}{b}\frac{b'r+b}{b}g(\dot{\gamma}(1),\dot{\gamma}(1)) - \frac{sh'(E)}{b^2}g(j\dot{\gamma}(1), j\dot{\gamma}(1)) \\
&+ s\frac{sh'(E)}{b}\frac{b'r+b}{b}g(\dot{\gamma}(1),\dot{\gamma}(1)) \\
&= ((s^2+\kappa r^2)sh'(E)^2 - s)h'(E)r^2 \\
&= 0,
\end{aligned}$$

where we used $b'r + b = sh'(E)^2$. Next we look at

$$\begin{aligned}
\phi^*\omega_s((jv,0),(0,jv)) &= \omega_s(\left(\tilde{X}_{jv}(1), b^{-1}j\dot{\tilde{X}}_{jv}(1)\right), \left(X_{jv}(1), b^{-1}j\dot{X}_{jv}(1)\right)) \\
&= \omega_s(\left(-\frac{1}{b}\dot{\gamma}(1), 0\right), \left(\frac{h'(E)}{b}j\dot{\gamma}(1), -\frac{sh'(E)}{b}\dot{\gamma}(1)\right)) \\
&= -\frac{sh'(E)}{b^2}g(\dot{\gamma}(1),\dot{\gamma}(1)) + s\frac{h'(E)}{b^2}g(j\dot{\gamma}(1), j\dot{\gamma}(1)) \\
&= 0.
\end{aligned}$$

Finally we compute for a w orthogonal to v and jv

$$\begin{aligned}
\phi^*\omega_s((w,0),(0,w)) &= \omega_s(\Big(\tilde{X}_w(1), b^{-1}j\dot{\tilde{X}}_w(1)\Big), \Big(X_w(1), b^{-1}j\dot{X}_w(1)\Big)) \\
&= g(b^{-1}j\dot{\tilde{X}}_w(1), X_w(1)) - g(\tilde{X}_w(1), b^{-1}j\dot{X}_w(1)) - sg(j\tilde{X}_w(1), X_w(1)) \\
&= (\sin(\sqrt{\kappa}br/2)^2 - \cos(\sqrt{\kappa}br/2)^2 + \frac{2s}{\sqrt{\kappa}r}\sin(\sqrt{\kappa}br/2)\cos(\sqrt{\kappa}br/2))|w|^2 \\
&= (-\cos(\sqrt{\kappa}br) + \frac{s}{r}\sin(\sqrt{\kappa}br))|w|^2 \\
&= (-sh'(E) + sh'(E))|w|^2 \\
&= 0.
\end{aligned}$$

□

Observe that $\phi^*\omega_s$ restricted to the fiber is fairly close to the standard symplectic structure $\mathrm{d}\lambda_v$ on $\mathbb{C}^n$. Indeed if we consider the scaled Liouville primitive $e^{2a(r)}\lambda_v$, then

$$\mathrm{d}(e^{2a(r)}\lambda_v) = 2a'(r)e^{2a(r)}\mathrm{d}r \wedge \lambda_v + e^{2a(r)}\mathrm{d}\lambda_v.$$

Therefore

$$\mathrm{d}(e^{2a(r)}\lambda_v)(v,jv) = (a'(r)r+1)e^{2a(r)}r^2 = h'(E)|v|^2$$

and for any w orthogonal to v and jv we have

$$\mathrm{d}(e^{2a(r)}\lambda_v)(w,jw) = e^{2a(r)}|w|^2.$$

Thus $\mathrm{d}(e^{2a(r)}\lambda)$ coincides with $\phi^*\omega_s$ restricted to the fibers. If we now pullback by $m = (\Phi^Y_{a(r)})^{-1}$ we will obtain the standard symplectic structure on the fibers. A straight forward calculation shows that $(\Phi^Y_{a(r)})^{-1} = \Phi^Y_{\bar{a}(r)}$ with

$$\bar{a}(r) = \frac{1}{2}\ln\left(\frac{1}{\kappa r^2}(\left(\frac{\kappa}{2}r^2 + s\right)^2 - s^2)\right).$$

We compute

$$h'(E(e^{a(r)}r))^{-1} = \sqrt{s^2 + e^{2a(r)}\kappa r^2} = \frac{\kappa}{2}r^2 + s$$

and thus the block form after scaling takes the form

$$m^*\phi^*\omega_s = \begin{pmatrix} (\frac{\kappa}{2}r^2+s)\sigma^{\parallel} & 0 & 0 & 0 \\ 0 & (\frac{\kappa}{4}r^2+s)\sigma^{\perp} & 0 & 0 \\ 0 & 0 & \sigma^{\parallel} & 0 \\ 0 & 0 & 0 & \sigma^{\perp} \end{pmatrix}.$$

This symplectic form is not a stranger, actually comparing it to $\mathrm{d}\tau$ in (5.10) yields

$$(\varphi^{-1})^*\omega_s = m^*\phi^*\omega_s = \mathrm{d}\tau/2 - s\pi^*\sigma.$$

This finishes the proof of Theorem 5.3.3.

5.4. Locally Hermitian symmetric spaces

This section contains the final generalization of the symplectomorphism identifying the two symplectic structures on the tangent bundle TM of a locally Hermitian symmetric spaces M. Actually it is fine to assume that M is simply connected, thus globally Hermitian symmetric as all our constructions are equivariant and thus survive the quotient by a discrete group action. Further we may assume M to be irreducible, because all Hermitian symmetric spaces are products of those and Euclidean spaces. Compared to the previous sections, we change perspective a little. We already observed that the tangent bundle of a Hermitian symmetric space admits a huge group action. This group action is Hamiltonian with respect to both symplectic structures (see Thm. 2.4.23). Instead of computing the differential of the diffeomorphism explicitly, we will now use the moment maps to see that the diffeomorphism is symplectic in the spirit of the following Proposition.

Proposition 5.4.1. *Assume we have a Hamiltonian G-action on* (N_i, ω_i)*. If* $\varphi : N_1 \to N_2$ *is an equivariant smooth bijection such that*

$$\begin{array}{ccc} N_1 & \xrightarrow{\quad\varphi\quad} & N_2 \\ & {\scriptstyle \mu_1}\searrow \quad \swarrow{\scriptstyle \mu_2} & \\ & \mathfrak{g} & \end{array}$$

commutes and the distribution $\mathcal{D} \subset TN_1$ *tangent to the* G*-orbits admits a complement* Υ *that is isotropic for both symplectic forms* ω_1 *and* $\varphi^*\omega_2$*, then* φ *is actually a symplectomorphism i.e.* $\varphi^*\omega_2 = \omega_1$*.*

Proof. We need to show that

$$\varphi^*\omega_2(v, w) = \omega_1(v, w) \quad \forall v, w \in TN_1.$$

By non-degeneracy of ω_1 and ω_2 it follows then, that also $\mathrm{d}\varphi$ must be non-degenerate hence a diffeomorphism.
We can decompose the tangent bundle $TN_1 = \mathcal{D} \oplus \Upsilon$. Recall that $\mathcal{D}_x = \{(a_1^\#)_x := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} e^{ta}(x) \mid a \in \mathfrak{g}\}$. In particular $\mathrm{d}\varphi a_1^\# = a_2^\#$ by equivariance. So if either $v = a_1^\# \in \mathcal{D}$ or $w = a_1^\# \in \mathcal{D}$ we get the equality as follows

$$\omega_1(a_1^\#, \cdot) = \mathrm{d}(\mu_1, a) = \mathrm{d}(\mu_2 \circ \varphi, a) = \omega_2(a_2^\#, \mathrm{d}\varphi\ \cdot) = \omega_2(\mathrm{d}\varphi\ a_1^\#, \mathrm{d}\varphi\ \cdot) = \varphi^*\omega_2(a_1^\#, \cdot).$$

The second equality holds due to the moment map triangle and the fourth due to equivariance of φ. If $v, w \in \Upsilon$ we get

$$\omega_1(v, w) = 0 = \omega_2(\mathrm{d}\varphi(v), \mathrm{d}\varphi(w))$$

as Υ is isotropic with respect to both symplectic forms by assumption. □

In the previous cases we used for the construction of the equivariant diffeomorphism the fact that the magnetic geodesic flow is totally periodic. Sadly on general Hermitian symmetric spaces this is not true anymore. Still the magnetic geodesic flow is (at least for low energy levels) very nice, indeed it is quasi- periodic. This can be seen in terms of the polysphere resp. polydisc theorem. As the polyspheres resp. polydiscs are totally geodesic complex submanifolds the magnetic geodesic flow restricts to the magnetic geodesic flow on products of constant curvature surfaces. Thus it is quasi-periodic. One can even still find a Hamiltonian H in a neighborhood of the zero-section generating a circle action. Just reparametrizing the kinetic Hamiltonian is not sufficient anymore. Inspired by [8], we shall work with spectral functions for the self-adjoint operator

$$jR_{jv,v} : T_xM \to T_xM; \quad w \mapsto jR(jv,v)w$$

for some $v \in T_xM$. Let

$$U_\varpi M := \{(x,v) \in TM| \ |g_x(jR_{jv,v}w,w)| < \varpi\|w\|^2 \ \ \forall w \in T_xM\}$$

denote the neighborhood of the zero-section with the absolute value of the holomorphic-bisectional curvature bounded by $\varpi/\|v\|^2$.

Lemma 5.4.2. *For the polysphere resp. polydisc i.e.* $\Sigma \in \{\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{H}^1\}$ *we have* $(D_\rho\Sigma)^r = U_{|\kappa|\rho^2}\Sigma^r$*, where* κ *denotes the curvature of* Σ*.*

Proof. We have the following chain of equivalences

$$\begin{aligned}(x,v) \in (D_\rho\Sigma)^r &\Leftrightarrow \|v_i\|^2 < \rho^2 \ \ \forall i \\ &\Leftrightarrow |g_x(jR_{jv,v}w,w)| = |\kappa| \sum_i \|v_i\|^2\|w_i\|^2 \leq |\kappa|\rho^2\|w\|^2 \ \ \forall w \in T_x\Sigma^r \\ &\Leftrightarrow (x,v) \in U_{|\kappa|\rho^2}\Sigma^r.\end{aligned}$$

□

Remark 5.4.3. *For general irreducible Hermitian symmetric spaces we define* κ *to be the maximal resp. minimal holomorphic sectional curvature. This is the same as the curvature of any of the factors of an embedded polysphere resp. polydisc, which can be seen as follows. Take any* $(x,v) \in TM$*, then*

$$\frac{g_x(jR(jv,v)v,v)}{|v|^4} = \sum_{i=1}^r \frac{\kappa|v_i|^4}{|v|^4} \quad \begin{cases} \leq \kappa & \text{for } \kappa \text{ positive} \\ \geq \kappa & \text{for } \kappa \text{ negative}\end{cases},$$

where the splitting is with respect to any polysphere resp. polydisc through (x,v)*. Indeed equality is obtained when* v *points along a factor of the polysphere resp. polydisc.*

On a polysphere resp. polydisc the sum

$$H(x,v) = \frac{2\pi}{\kappa}\sum_{i=1}^r \left(\sqrt{s^2 + \kappa|v_i|^2} - s\right)$$

where v_i denotes the projection of v to the i-th factor, generates a circle action. The operator $jR_{jv,v}$ has eigenvalues $\kappa|v|^2$, we want to use this and spectral functions of $jR_{jv,v}$ to construct a Hamiltonian that generates a circle action on TM.

Proposition 5.4.4. *If* $s^2 + \kappa\rho^2 > 0$*, then the Hamiltonian*

$$H : U_{|\kappa|\rho^2}M \to \mathbb{R}; \ (x,v) \mapsto g_x(h(jR_{jv,v})v,v), \quad \text{where} \ \ h(y) := \frac{2\pi}{y}\left(\sqrt{s^2+y} - s\right)$$

is well-defined, differentiable and generates a circle action.

Proof. It is well defined, because the condition $s^2+\kappa\rho^2 > 0$ makes sure that the eigenvalues of $jR_{jv,v}$ are bounded from below by $-s^2$. Further h is smooth, thus the same holds for H. Next we compute $\mathrm{d}H$ using the chain rule. The horizontal part must vanish, as g, j and R are parallel, thus

$$\mathrm{d}H = (0, g(\tilde{h}(jR_{jv,v})v, \cdot)) = (\tilde{h}(jR_{jv,v})v, s\tilde{h}(jR_{jv,v})jv)\begin{pmatrix} -s\sigma & g \\ g & 0 \end{pmatrix},$$

where $\tilde{h}(y) = h'(y)y + h(y)$. It follows that

$$X_H = (\tilde{h}(jR_{jv,v})v)^{\mathcal{H}} + s(\tilde{h}(jR_{jv,v})jv)^{\mathcal{V}}.$$

Observe that in view of Eq. 2.4 the operator $jR_{jv,v} : T_xM \to T_xM$ restricts to the tangent space of the polysphere resp. polydisc Σ^r through x tangent to v. As the polyspheres resp. polydiscs are totally geodesic $(\tilde{h}(jR_{jv,v})v)^{\mathcal{H}} \in TT\Sigma^r$. The embedding is also complex thus $(\tilde{h}(jR_{jv,v})jv)^{\mathcal{V}} \in TT\Sigma^r$. In total we see that X_H is tangent to $T\Sigma^r$. Observe that this means that the diagrams of the form

$$\begin{array}{ccc} (D_\rho\Sigma)^r & \xrightarrow{\mathrm{d}\iota} & U_{|\kappa|\rho^2}M \\ & \searrow^{\sum H\circ\pi_i} \quad \swarrow_{H} & \\ & \mathfrak{g} & \end{array}$$

commute. On the polysphere resp. polydisc $jR_{jv,v}$ is diagonal with respect to the product structure, i.e.

$$jR_{jv,v} = \begin{pmatrix} \kappa|v_1|^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \kappa|v_r|^2 \end{pmatrix}$$

where v_i is the projection of v to the ith factor. Now it is easy to see that on the polysphere resp. polydisc the Hamiltonian is given by

$$H(x, v) = \frac{2\pi}{\kappa}\sum_{i=1}^{r}\left(\sqrt{s^2+\kappa|v_i|^2} - s\right).$$

In particular it generates the diagonal Hamiltonian circle action on the product. □

In the cases of negative constant holomorphic sectional curvature κ the neighborhood of the zero-section on which the Hermitian symplectic form is well-defined was a disc-bundle of constant radius $\sqrt{-2s/\kappa}$. We need to replace the disc bundle with $U_{2s}M$ (see (2.2)). We can now state our main theorem.

Theorem 5.4.5 (Thm. A)**.** *Let M be an irreducible Hermitian symmetric space of compact type, then there exists an equivariant symplectomorphism*

$$\Psi : (TM, \mathrm{d}\lambda - s\pi^*\sigma) \to (TM, \mathrm{d}\tau/2 - s\pi^*\sigma).$$

If M is of non-compact type the symplectomorphism exists only on a neighborhood of the zero-section, namely

$$\Psi : (U_{s^2}M, \mathrm{d}\lambda - s\pi^*\sigma) \to (U_{2s}M, \mathrm{d}\tau/2 - s\pi^*\sigma).$$

Further the symplectomorphisms intertwines in both cases the moment maps of the Hamiltonian G-actions and the Hamiltonians H and $2\pi E$ $(E(x,v) = \frac{1}{2}g_x(v,v))$.

The rest of this section contains the proof of this Theorem and is divided into four steps.

(1) Find a candidate Ψ that intertwines the circle actions.

(2) Show that Ψ is an equivariant diffeomorphism.

(3) Show that the moment triangle commutes.

  (a) Reduce the general case to the case of polyspheres resp. polydiscs.

  (b) Show that the moment triangle commutes for polyspheres resp. polydiscs.

(4) Show that $\Upsilon = \operatorname{span}_{\mathbb{R}}\{Y_1, \dots, Y_r\}$ (see Cor. 2.4.18) is an isotropic complement of $\mathcal{D}$ for both $\mathrm{d}\lambda - s\pi^*\sigma$ and $\Psi^*\left(\mathrm{d}\tau/2 - s\pi^*\sigma\right)$ and use Proposition 5.4.1 to conclude the proof.

Step (1):
Similar to the definition of H we modify our original symplectomorphism φ using spectral functions of $jR_{jv,v}$. Recall from Eq. (5.1) that

$$\varphi(x, v) = \left(\exp_x(b(\kappa r^2)jv), e^{a(\kappa r^2)}P_\gamma v(1)\right),$$

with functions[2]

$$b(y) = \frac{\arctan(\sqrt{y}/s)}{\sqrt{y}} \quad \text{and} \quad a(y) = \frac{1}{2}\ln\left(\frac{2}{y}(\sqrt{s^2 + y} - s)\right).$$

In analogy we define $\Psi := m \circ \phi$[3], where

$$\phi(x, v) := (\exp_x(b(jR_{jv,v})jv), P_\gamma v(1)) \quad \text{and} \quad m(x, v) = (x, e^{a(jR_{jv,v})}v).$$

Observe that Ψ is smooth as a and b are. Further it is defined whenever the eigenvalues of $jR_{jv,v}$ are bounded from below by $-s^2$. For compact type Hermitian symmetric spaces this holds on the whole tangent bundle TM, for non-compact type Hermitian symmetric spaces this holds on $U_{s^2}M \subset TM$. Let us check that Ψ indeed intertwines the Hamiltonian circle actions.

Lemma 5.4.6. *The map Ψ intertwines H and $2\pi E$, i.e.*

$$H = 2\pi E \circ \Psi.$$

Proof. We compute

$$\begin{aligned} 2\pi E(\Psi((x, v)) &= \pi g_x(e^{a(jR_{jv,v})}v, e^{a(jR_{jv,v})}v) = \pi g_x(e^{2a(R_{jv,v})}v, v) \\ &= g_x(h(R_{jv,v})v, v) = H(x, v). \end{aligned}$$

□

Step (2):
We want to show that Ψ is an equivariant diffeomorphism. We start with equivariance.

[2] By slight abuse of notation we changed the argument of the functions from r to $y \equiv \kappa r^2$.
[3] Compared to the previous section we exchanged $m \leftrightarrow m^{-1}$ and $\phi \leftrightarrow \phi^{-1}$. Sorry!

Lemma 5.4.7. *The map Ψ is equivariant under the action of the isometry group.*

Proof. All objects, i.e. metric, curvature, exponential map are invariant under the action of isometries thus also Ψ is. Explicitly let $I : M \to M$ be an isometry, then

$$\begin{aligned}\Psi(\mathrm{d}I(x,v)) &= \Big(\exp_{I(x)}(b(jR_{j\mathrm{d}I_x v,\mathrm{d}I_x v})j\mathrm{d}I_x v), e^{a(jR_{j\mathrm{d}I_x v,\mathrm{d}I_x v})}P_\gamma \mathrm{d}I_x v(1)\Big)\\ &= \Big(\exp_{I(x)}(\mathrm{d}I_x b(jR_{jv,v})jv), \mathrm{d}I_x e^{a(jR_{jv,v})}P_\gamma v(1)\Big)\\ &= \Big(I(\exp_x(b(jR_{jv,v})jv), \mathrm{d}I_x e^{a(jR_{jv,v})}P_\gamma v(1)\Big)\\ &= \mathrm{d}I(\Psi(x,v)).\end{aligned}$$

□

Lemma 5.4.8. *The map Ψ is a diffeomorphism and Ψ^{-1} is defined on TM resp. $U_{2s}M$ in the compact resp. non-compact case.*

Proof. Also in analogy to the constant curvature case one can explicitly give an inverse $\Psi^{-1} = \Phi^{-1} \circ m^{-1}$ where

$$\Phi^{-1}(x,v) = (\exp_x(-b(jR_{jv,v})jv), P_\gamma v(1))$$

and

$$m^{-1}(x,v) = (x, e^{\bar{a}(jR_{jv,v})v}v),$$

with

$$\bar{a}(y) = \frac{1}{2}\ln\left(\frac{1}{y}\left(\left(\frac{y}{2}+s\right)^2 - s^2\right)\right).$$

The inverse is well defined whenever the eigenvalues of $jR_{jv,v}$ are bounded from below by $-4s$, i.e. on TM resp. $U_{4s}M$ in the compact resp. non-compact case. In particular it is well-defined on the image of Ψ namely TM resp. $U_{2s}M$. The inverse is smooth as $\bar{a}$ and b are smooth. □

Step (3):
We need to assure that Ψ intertwines the moment maps, i.e.

$$\mu_\lambda - s\mu_\sigma = (\mu_\tau/2 - s\mu_\sigma) \circ \Psi.$$

This is difficult to see if one considers the full hermitian symmetric space, but relatively easy to prove for polyspheres resp. polydiscs. We will show now that we can actually reduce the general case to polyspheres resp. polydiscs.

Lemma 5.4.9. *The diagrams*

$$\begin{array}{ccc}(T\mathbb{C}\mathrm{P}^1)^r & \xrightarrow{\mathrm{d}\iota} & TM\\ {\scriptstyle\varphi\times\ldots\times\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\Psi}\\ (T\mathbb{C}\mathrm{P}^1)^r & \xrightarrow{\mathrm{d}\iota} & TM\end{array} \tag{5.7}$$

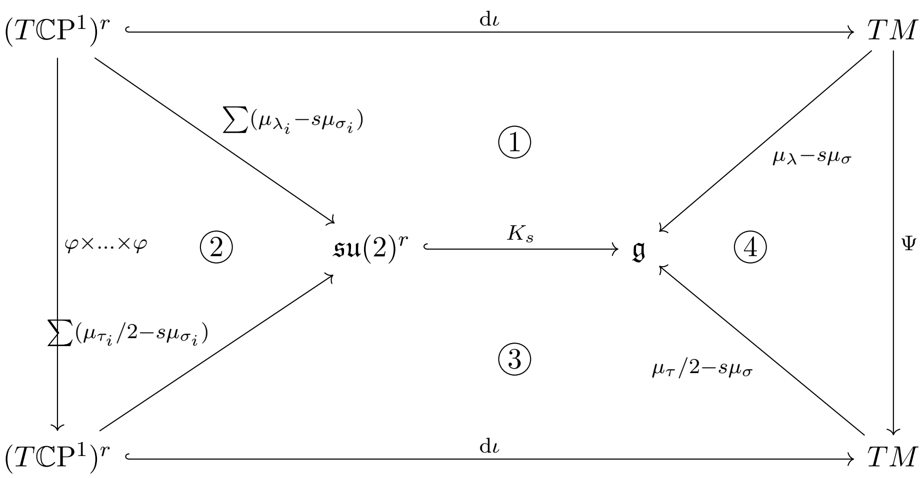
in the compact case and

$$\begin{array}{ccc} (D_{s/\sqrt{-\kappa}}\mathbb{C}\mathrm{H}^1)^r & \xrightarrow{\mathrm{d}\iota} & U_{s^2}M \\ {\scriptstyle \varphi\times\ldots\times\varphi}\big\downarrow & & \big\downarrow{\scriptstyle \Psi} \\ (D_{\sqrt{-2s/\kappa}}\mathbb{C}\mathrm{H}^1)^r & \xrightarrow{\mathrm{d}\iota} & U_{2s}M \end{array} \tag{5.8}$$

in the non-compact case commute. Here φ denotes the map realizing the symplectomorphism for surfaces as constructed in the proof of Theorem 5.2.1.

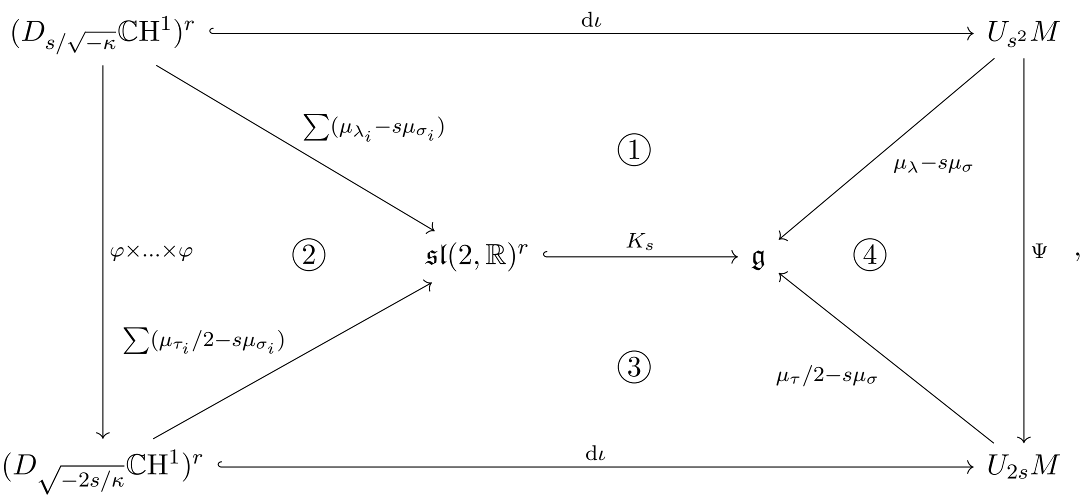
Proof. The self adjoint endomorphism $jR_{jv,v}$ restricts to the tangent spaces of the polyspheres/disc. As the embedding is complex totally geodesic it follows that Ψ also restricts to the copies of $T\Sigma^r$. Further $jR_{jv,v}$ is diagonal with respect to the splitting of $T\Sigma^r$ as product $T\Sigma \times \ldots \times T\Sigma$ and therefore the diagram holds. □

Next we include the moment maps into the diagrams (5.7) and (5.8). For the compact case we look at



(5.9)

and for the non-compact case we look at



where $K_s : \mathfrak{h}^r \hookrightarrow \mathfrak{g}$ is an affine embedding we will specify below, μ_λ & μ_τ are the moment maps defined in the introduction 2.4.23 and $\mu_\sigma(x, v) := x$. To see that these are the correct moment maps combine Theorem 2.4.23 and the fact that the inclusion $M \cong O_Z \hookrightarrow \mathfrak{g}$ is a moment map for σ. The idea is that commutativity of ④ follows from commutativity of ① − ③. As all maps are equivariant and the embeddings are well behaved under G as explained in Remark 2.4.14, we may assume $\iota = \iota_{p,q}$ for $p = (Z, v)$ and $q = (Z_0 = \sum_{i=1}^r Z_i, v_0)$ for some elements $v \in T_Z M$ resp. $v_0 \in T_{Z_i}\Sigma_i$. Here Z_i denotes the up to sign unique element in the center of $\mathfrak{h}_i$ such that $O_{Z_i} \cong \Sigma$ and the Kähler structure from Theorem 2.4.20 coincides with the standard Kähler structure. The index indicates the factor in the product.

We are now exactly in the setup of Proposition 2.4.24. We define K_s to be

$$K_s : \mathfrak{h}^r \hookrightarrow \mathfrak{g};\ h \mapsto k(h) - s(Z - k(Z_0)).$$

The affine embedding K_s is a variation of the affine embedding K in Proposition 2.4.24.

Lemma 5.4.10. *The sub diagrams ① and ③ commute with this choice of affine embedding.*

Proof. We start with ①. Take $(y, w) \in (T\mathbb{C}\mathrm{P}^1)^r$ resp. $(y, w) \in (D_{s/\sqrt{-\kappa}}\mathbb{C}\mathrm{H}^1)^r$, then

$$\begin{aligned} K_s(\mu_\lambda(y, w) - s\mu_\sigma(y, w)) &= k(\mu_\lambda(y, w) - s\mu_\sigma(y, w)) - s(Z + k(Z_0)) \\ &= k([y, w]) - sK(y) \overset{*}{=} [k(y), k(w)] - sK(y) \\ &\overset{**}{=} [K(y), k(w)] - sK(y) \overset{***}{=} [\iota(y), \mathrm{d}\iota_y(w)] - s\iota(y) \\ &= \mu_\lambda(\mathrm{d}\iota(y, w)) - s\mu_\sigma(\mathrm{d}\iota(y, w)). \end{aligned}$$

The first two and the last equations are just plugging in definitions. Equation $*$ uses that k is a Lie algebra homomorphism, $**$ uses that $[Z - k(Z_0), k(w)] = 0$ by (2.7) and $***$ uses that K extends ι, $K|_{\Sigma^r} = \iota$ (see Prop. 2.4.24).

Similarly we can compute ③. Again take $(y, w) \in (T\mathbb{C}\mathrm{P}^1)^r$ resp. $(y, w) \in (D_{\sqrt{-2s/\kappa}}\mathbb{C}\mathrm{H}^1)^r$, then

$$\begin{aligned} K_s((\mu_\tau(y, w)/2 - s\mu_\sigma(y, w)) &= k(\mu_\tau(y, w)/2 - s\mu_\sigma(y, w)) - s(Z + k(Z_0)) \\ &= k([y, [y, w]]/2) - sK(y) \\ &\overset{*}{=} [k(y), [k(y), k(w)]]/2 - sK(y) \\ &\overset{**}{=} [K(y), [K(y), k(w)]]/2 - sK(y) \\ &\overset{***}{=} [\iota(y), [\iota(y), \mathrm{d}\iota_y w]]/2 - s\iota(y) \\ &= \mu_\tau(\mathrm{d}\iota(y, w))/2 - s\mu_\sigma(\mathrm{d}\iota(y, w)). \end{aligned}$$

The first two and the last equations are just plugging in definitions. Equation $*$ uses that k is a Lie algebra homomorphism, $**$ uses that $[Z - k(Z_0), k(w)] = 0$ by (2.7) and $***$ uses that K extends ι, $K|_{\Sigma^r} = \iota$ (see Prop. 2.4.24). □

Observe that if we can now show that the moment map triangle commutes in the two dimensional case, commutativity of ② and thus commutativity of ④ follows.

Step (3b):
We reduced the problem to the 2-dimensional case, thus $\Sigma \in \{\mathbb{CP}^1, \mathbb{CH}^1\}$. In this case the isometries act transitively on the unit-sphere subbundle of $T\Sigma$, by equivariance it is therefore enough to show that

$$\mu_2(\varphi(Z_0, rv_0)) = \mu_1(Z_0, rv_0) \tag{5.10}$$

for some fixed $(Z_0, v_0) \in T\Sigma$ and arbitrary $r \geq 0$[4]. Here we denoted $\mu_1 := \mu_\lambda - s\mu_\sigma$ and $\mu_2 := \frac{1}{2}\mu_\tau - s\mu_\sigma$ the moment maps with respect to $\mathrm{d}\lambda - s\pi^*\sigma$ and $\mathrm{d}\tau/2 - \pi^*\sigma$ respectively. The geodesic starting at $x \in M \subset \mathfrak{g}$ in direction $v \in T_xM \subset \mathfrak{g}$ is given by

$$\gamma(t) = e^{tjv} x e^{-tjv} \tag{5.11}$$

as then

$$\dot{\gamma}(0) = [jv, x] = -j^2 v = v.$$

All that is left to show is that equation (5.10) is satisfied for a suitable choice of (x_0, v_0). We will show this case by case realizing the surfaces explicitly as (co-)adjoint orbits.

The case $M = \mathbb{CP}^1$ **:** Here $G = \mathrm{SU}(2)$ and the Lie-algebra is

$$\mathfrak{su}(2) = \left\langle a_1 := \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ a_2 := \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ a_3 := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

The generators satisfy

$$[a_1, a_2] = -a_3,\ [a_3, a_1] = -a_2,\ [a_3, a_2] = a_1.$$

We can identify $\mathbb{CP}^1$ as coadjoint orbit of a_3, i.e. $\mathbb{CP}^1 \cong \mathcal{O}_{a_3}$. The identification is such that the Fubini-Study Kähler structure coincides with the KKS-Kähler structure. We need to show that the composition $\varphi = \Phi_a^Y \circ \Phi_b^H$ is the map that intertwines the moment maps. Here $\kappa = 1$ thus

$$a(r^2) = \frac{1}{2}\ln\left(\frac{2}{r^2}(\sqrt{s^2 + r^2} - s)\right),\quad b(r^2) = \frac{1}{r}\arctan\left(\frac{r}{s}\right).$$

In view of Eq. (5.11) we can compute the flow of $H := (jv)^{\mathcal{H}}$ as follows

$$\Phi_b^H(a_3, ra_1) = (e^{bra_1} a_3 e^{-bra_1}, ra_1),$$

where

$$e^{bra_1} = \begin{pmatrix} e^{ibr/2} & 0 \\ 0 & e^{-ibr/2} \end{pmatrix}$$

thus

$$e^{bra_1} a_3 e^{-bra_1} = \frac{1}{2}\begin{pmatrix} 0 & e^{ibr} \\ -e^{-ibr} & 0 \end{pmatrix}.$$

[4]Attention! This r has nothing to do with the rank, it is the norm of v_0.

We use Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ and the identities

$$\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}, \quad \sin(\tan^{-1}(x)) = \frac{x}{\sqrt{1+x^2}}$$

in order to find the following expression

$$\begin{pmatrix} 0 & e^{ibr} \\ -e^{-ibr} & 0 \end{pmatrix} = \cos\left(\tan^{-1}\left(\frac{r}{s}\right)\right) a_3 + \sin\left(\tan^{-1}\left(\frac{r}{s}\right)\right) a_2 = \frac{1}{\sqrt{r^2+s^2}}(sa_3 + ra_2).$$

So in particular we have that:

$$\begin{aligned} \mu_2(\Phi_a^Y \circ \Phi_b^H(a_3, ra_1)) &= \mu_2(\frac{1}{\sqrt{r^2+s^2}}(sa_3+ra_2), e^{a(r)}ra_1) \\ &= \frac{e^{2a(r)}r^2}{2\sqrt{r^2+s^2}}[[sa_3+ra_2, a_1], a_1] - \frac{s}{\sqrt{r^2+s^2}}(sa_3+ra_2) \\ &= \frac{\sqrt{s^2+r^2}-s}{\sqrt{s^2+r^2}}(-sa_3-ra_2) - \frac{s}{\sqrt{r^2+s^2}}(sa_3+ra_2) \\ &= -sa_3 - ra_2 = r[a_3, a_1] - sa_3 \\ &= \mu_1(a_3, ra_1). \end{aligned}$$

Which finishes the compact case.

The case $M = \mathbb{C}\mathrm{H}^1$ **:** Here $G = \mathrm{SU}(1,1)$ and the Lie-algebra is

$$\mathfrak{su}(1,1) = \left\langle a_1 := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ a_2 := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ a_3 := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

The generators satisfy

$$[a_1, a_2] = a_3, \ [a_3, a_1] = -a_2, \ [a_3, a_2] = a_1.$$

We can identify $\mathbb{C}\mathrm{H}^1$ as coadjoint orbit of a_3, i.e. $\mathbb{C}\mathrm{H}^1 \cong \mathcal{O}_{a_3}$. The identification is such that the standard Kähler structure coincides with the KKS-Kähler structure. We need to show that the composition $\varphi = \Phi_a^Y \circ \Phi_b^H$ is the map that intertwines the moment maps. Here $\kappa = -1$ thus

$$a(-r^2) = \frac{1}{2}\ln\left(\frac{-2}{r^2}(\sqrt{s^2-r^2}-s)\right), \quad b(-r^2) = \frac{1}{r}\mathrm{arctanh}\left(\frac{r}{s}\right).$$

We compute

$$\Phi_b^H(a_3, ra_1) = (e^{bra_1}a_3e^{-bra_1}, ra_1),$$

where

$$e^{bra_1} = \begin{pmatrix} e^{br/2} & 0 \\ 0 & e^{-br/2} \end{pmatrix}$$

thus

$$e^{bra_1}a_3e^{-bra_1} = \frac{1}{2}\begin{pmatrix} 0 & e^{br} \\ -e^{-br} & 0 \end{pmatrix}.$$

We use the identities $e^x = \cosh(x) + \sinh(x)$ and

$$-\cosh(\tanh^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}, \quad \sinh(\tanh^{-1}(x)) = \frac{x}{\sqrt{1-x^2}}$$

in order to find the following expression

$$\begin{pmatrix} 0 & e^{br} \\ -e^{-br} & 0 \end{pmatrix} = \cosh\left(\tanh^{-1}\left(\frac{r}{s}\right)\right) a_3 + \sinh\left(\tanh^{-1}\left(\frac{r}{s}\right)\right) a_2 = \frac{1}{\sqrt{s^2-r^2}}(sa_3 + ra_2).$$

So in particular we have that:

$$\begin{aligned}
\mu_2(\Phi_a^Y \circ \Phi_b^H(a_3, ra_1)) &= \mu_2(\frac{1}{\sqrt{s^2-r^2}}(sa_3 + ra_2), e^{a(r)} ra_1) \\
&= \frac{e^{2a(r)} r^2}{2\sqrt{s^2-r^2}}[[sa_3 + ra_2, a_1], a_1] - \frac{s}{\sqrt{s^2-r^2}}(sa_3 + ra_2) \\
&= \frac{s - \sqrt{s^2-r^2}}{\sqrt{s^2-r^2}}(sa_3 + ra_2) - \frac{s}{\sqrt{s^2-r^2}}(sa_3 + ra_2) \\
&= -sa_3 - ra_2 = r[a_3, a_1] - sa_3 \\
&= \mu_1(a_3, ra_1),
\end{aligned}$$

which finishes the proof of the non-compact case.

The case $M = \mathbb{C}^1$ **:** As a fun fact we do the same calculation for the flat case, even though clearly showing directly that $\varphi = \Phi_a^Y \circ \Phi_b^H$ is a symplectomorphism is much less stressful. Here $G = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$ and the Lie-algebra is

$$\mathfrak{g} = \left\langle a_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ a_2 := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ a_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$$

The generators satisfy

$$[a_1, a_2] = 0, \ [a_3, a_1] = -a_2, \ [a_3, a_2] = a_1.$$

We can identify $\mathbb{C}^1$ as coadjoint orbit of a_3, i.e. $\mathbb{C}^1 \cong \mathcal{O}_{a_3}$. The identification is such that the standard Kähler structure coincides with the KKS-Kähler structure. We need to show that the composition $\varphi = \Phi_a^Y \circ \Phi_b^H$ is the map that intertwines the moment maps. Here $\kappa = 0$ thus

$$a = \frac{1}{2}\ln\left(\frac{1}{s}\right), \quad b = \frac{1}{s}.$$

We compute

$$\Phi_b^H(a_3, ra_1) = (e^{bra_1} a_3 e^{-bra_1}, ra_1),$$

where

$$e^{bra_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r/s & 0 & 1 \end{pmatrix},$$

thus

$$e^{bra_1}a_3e^{-bra_1} = \begin{pmatrix} 0 & 0 & 0 \\ -r/s & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = -\frac{r}{s}a_2 + a_3.$$

In particular we have that:

$$\begin{aligned} \mu_2(\Phi_a^Y \circ \Phi_b^H(a_3, ra_1)) &= \mu_2(-\frac{r}{s}a_2 + a_3, \frac{r}{\sqrt{s}}a_1) \\ &= \frac{r^2}{s}[[-\frac{r}{s}a_2 + a_3, a_1], a_1] + ra_2 - sa_3 \\ &= 0 + ra_2 - sa_3 \\ &= -r[a_3, a_1] - sa_3 \\ &= \mu_1(a_3, ra_1). \end{aligned}$$

By the previous computations we have proven the following Lemma:

Lemma 5.4.11. *The moment map triangles*

$$\begin{array}{ccc} TM & \xrightarrow{\Psi} & TM \\ & {\scriptstyle\mu_1}\searrow \quad \swarrow{\scriptstyle\mu_2} & \\ & \mathfrak{g} & \end{array}$$

resp.

$$\begin{array}{ccc} U_{s^2}M & \xrightarrow{\Psi} & U_{2s}M \\ & {\scriptstyle\mu_1}\searrow \quad \swarrow{\scriptstyle\mu_2} & \\ & \mathfrak{g} & \end{array}$$

commute.

Step (4):
The last condition we need to check in order to apply 5.4.1 (and thus finish the proof of Thm. 5.4.5) is the existence of a complement of $\mathcal{D} := \{a^\# \mid a \in \mathfrak{g}\} \subset TTM$ that is isotropic with respect to both symplectic forms $\omega_1 := \mathrm{d}\lambda - s\pi^*\sigma$ and $\Psi^*\omega_2 := \Psi^*(\mathrm{d}\tau/2 - s\pi^*\sigma)$. Recall that by Corollary 2.4.18 on the open dense set of regular points $\Upsilon = \mathrm{span}\{Y_1, \ldots, Y_r\}$ is a complement of $\mathcal{D}$ and observe that Υ is clearly isotropic for ω_1 as it is contained in the vertical distribution.

Lemma 5.4.12. *Υ is isotropic for $\Psi^*\omega_2$.*

Proof. In the view of the foliation of TM by tangent spaces of polspheres resp. polydiscs $T\Sigma^r$ (end of section 2.4.2), near a regular point it makes sense to look at the vector fields Y_i. We need to compute $\mathrm{d}\Psi(Y_i)$. As Ψ splits with respect to the product we conclude that $\mathrm{d}\Psi(Y_i) \in T\Sigma_i$. Further also ω_2 splits with respect to the product $T\Sigma_1 \times \ldots \times T\Sigma_r$ and therefore $T\Sigma_i$ and $T\Sigma_2$ are ω_2-orthogonal for $i \neq j$. It follows that

$$\Psi^*\omega_2(Y_i, Y_j) = \omega_2(\mathrm{d}\Psi Y_i, \mathrm{d}\Psi Y_j) = 0 \quad \forall i, j \in \{1, \ldots, r\}.$$

□

Using Proposition 5.4.1 we find that $\mathrm{d}\Psi$ is symplectic on the open, dense set of regular points, but as Ψ is smooth this already implies that $\mathrm{d}\Psi$ is symplectic everywhere. Thus this Lemma finishes the proof of Theorem 5.4.5.

5.5. Hofer–Zehnder capacity

The Hofer–Zehnder capacity of disc sub bundles of Hermitian vector bundles was computed by Lu [36, Thm. 1.3] to be the area of the fibers. We showed in the last section that Ψ intertwines H and $2\pi E$. In particular Ψ identifies the sub level sets

$$\Psi\left(H^{-1}(\pi R^2)\right) = E^{-1}\left(R^2/2\right) = D_R M.$$

We conclude using [36, Thm. 1.3] that

$$c_{HZ}(H^{-1}(\pi R^2), \omega_s) = c^0_{HZ}(H^{-1}(\pi R^2), \omega_s) = c^0_{HZ}(D_R M, \mathrm{d}\tau/2 - s\pi^*\sigma) = \pi R^2.$$

The bounds for $c_{HZ}(D_\rho M, \omega_s)$ are now obtained by asking what is the largest sub level set of H that lies in $D_\rho M$ and what is the smallest sub level set of H that contains $D_\rho M$.

Theorem 5.5.1 (Thm. B)**.** *Let (M, g, σ) be isometrically covered by an irreducible Hermitian symmetric space of rank r, then*

$$\frac{2\pi}{\kappa}\left(\sqrt{s^2 + \kappa\rho^2/r} - s\right) \le c_{HZ}(D_\rho M, \omega_s) \le c^0_{HZ}(D_\rho M, \omega_s) \le \frac{2\pi r}{\kappa}(\sqrt{s^2 + \kappa\rho^2} - s)$$

for any constants $s, \rho > 0$ satisfying $s^2 + \kappa\rho^2 > 0$. Here, κ denotes the maximal resp. minimal holomorphic sectional curvature.

Proof. For the lower bound we need to find a suitable sub level set of H that is contained in $D_\rho M$. Assume $(x, v) \in \left\{H \le \frac{2\pi}{\kappa}\left(\sqrt{s^2 + \kappa\rho^2/r} - s\right)\right\}$, then we can use that v splits into $\sum_i v_i$ along a polysphere resp. polydisc and we obtain the following chain of inequalities

$$\begin{aligned}
& H(x, v) = g_x(h(jR_{jv,v})v, v) \le \frac{2\pi}{\kappa}\left(\sqrt{s^2 + \kappa\rho^2/r} - s\right) \\
\Rightarrow\ & \sum_i \frac{2\pi}{\kappa}\left(\sqrt{s^2 + \kappa|v_i|^2} - s\right) \le \frac{2\pi}{\kappa}\left(\sqrt{s^2 + \kappa\rho^2/r} - s\right) \\
\Rightarrow\ & \frac{1}{\kappa}\left(\sqrt{s^2 + \kappa|v_i|^2} - s\right) \le \frac{1}{\kappa}\left(\sqrt{s^2 + \kappa\rho^2/r} - s\right) \qquad \forall\ i \\
\Rightarrow\ & |v_i|^2 \le \rho^2/r \qquad \forall\ i \\
\Rightarrow\ & |v|^2 \le \rho^2.
\end{aligned}$$

This implies that the sub level set $\left\{H \le \frac{2\pi}{\kappa}\left(\sqrt{s^2 + \kappa\rho^2/r} - s\right)\right\}$ is in $D_\rho M$ and thus the lower bound holds. For the upper bound of the capacity we have to find a sub level set of H that contains $D_\rho M$. For this let $(x, v) \in D_\rho M$, then again considering the splitting of v into $\sum_i v_i$ along a polysphere resp. polydisc we find

$$|v|^2 \le \rho^2 \Rightarrow \quad |v_i| \le \rho^2 \quad \forall i.$$

Then by monotonicity of the square root it follows that

$$\begin{aligned}
& \frac{1}{\kappa}\left(\sqrt{s^2 + \kappa|v_i|^2} - s\right) \le \frac{1}{\kappa}\left(\sqrt{s^2 + \kappa\rho^2} - s\right) \quad \forall i \\
\Rightarrow\ & H(x, v) = g_x(h(jR_{jv,v})v, v) = \sum_i \frac{2\pi}{\kappa}\left(\sqrt{s^2 + \kappa|v_i|^2} - s\right) \le \frac{2\pi r}{\kappa}\left(\sqrt{s^2 + \kappa\rho^2} - s\right).
\end{aligned}$$

This implies that $D_\rho M \subset H^{-1}\left(\frac{2\pi r}{\kappa}\left(\sqrt{s^2 + \kappa\rho^2} - s\right)\right)$ and the upper bound follows, which finishes the proof. □

Remark 5.5.2. *In the preparation of this thesis the author spent quite some time to avoid using this theorem by Lu for the upper bound. For the non-compact case this is indeed possible. We can compactify the disc bundle* $(D_\rho M, \omega_s)$ *using a Lerman cut with respect to the Hamiltonian circle action induced by* H. *The resulting manifold* $\overline{D_\rho M}$ *is topologically a fiber bundle*

$$\mathbb{C}\mathrm{P}^n \hookrightarrow \overline{D_\rho M} \twoheadrightarrow M.$$

In the case that the universal cover of M *is of non-compact type,* M *is aspherical. In particular the fiber class* $[\mathbb{C}\mathrm{P}^1] \in H_2(\overline{D_\rho M}, \mathbb{Z})$ *is minimal in the Hofer–Viterbo sense. We can apply their theorem 3.2.4 to see that* $\omega([\mathbb{C}\mathrm{P}^1])$ *is the upper bound. If the universal cover of* M *is of compact type,* M *will certainly not be aspherical.*

We will see in Theorem 6.1.1 that

$$(\overline{D_\rho \mathbb{C}\mathrm{P}^n}, \bar{\omega}_s) \cong (\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n, R_1\sigma \ominus R_2\sigma)$$

for constants R_1, R_2 *determined by* s, ρ. *In particular* $(\overline{D_\rho \mathbb{C}\mathrm{P}^n}, \bar{\omega}_s)$ *might not even be monotone. An alternative for the upper bound could come from filtered symplectic homology. This issues are addressed in a little more detail in the outlook 7.4, but we did not make this approach work.*

In the case of irreducible Hermitian symmetric spaces M of compact type we can also say something about the Hofer–Zehnder capacity of non-constant magnetic systems. Let $\nu \in \Omega^2(M)$ be an arbitrary closed 2-form. By Proposition 2.4.22 we know that $[\nu] = [s\sigma]$ in $H^2(M, \mathbb{R})$ for some $s \in \mathbb{R}$ as $H^2(M, \mathbb{R})$ is generated by σ. Thus we find an exact form $\mathrm{d}\theta$ such that $\nu + \mathrm{d}\theta = s\sigma$. But this means $(T^*M, \mathrm{d}\lambda + \pi^*\nu)$ is symplectomorphic to $(T^*M, \mathrm{d}\lambda + s\pi^*\sigma)$ via the map that shifts the zero-section

$$(x, p) \mapsto (x, p + \theta(x)).$$

Clearly this map does not map disc bundles to disc bundles but we can still get some inclusions depending on $\theta \in \Omega^1(M)$. Denote

$$\theta_{\max} = \max_x |\theta(x)|,$$

then

$$(D_\rho M, \omega_\nu) \hookrightarrow (D_{\rho + \theta_{\max}} M, \omega_s).$$

and if $\theta_{\max} < \lambda$ we also find

$$(D_{\rho - \theta_{\max}} M, \omega_s) \hookrightarrow (D_\rho M, \omega_\nu).$$

We can therefore deduce upper and sometimes lower bounds of the HZ-capacity for arbitrary magnetic systems. There is a freedom of choosing the primitive θ that we can use to optimize the bound. Simply replace $\theta_{\max}$ by

$$\inf_\theta \max_x |\theta(x)|.$$

We proved the following Corollary of Theorem 5.5.1.

Corollary 5.5.3. *Let (M, σ) be a rank r Hermitian symmetric space of compact type. Pick an arbitrary $\nu \in \Omega_2(M)$ representing a non-zero class in $H^2(M, \mathbb{R})$. Then*

$$c^0_{HZ}(D_\rho M, \omega_\nu) \leq \frac{2\pi r}{\kappa}\left(\sqrt{s^2 + \kappa\tilde{\rho}^2} - |s|\right),$$

where

$$\tilde{\rho} := \rho + \inf_\theta \max_x |\theta(x)|$$

and the infimum is taken over all 1-forms $\theta \in \Omega^1(M)$ satisfying $\nu + \mathrm{d}\theta = s\sigma$.

5.6. Hyperkähler structure

The hyperkähler structure of cotangent bundles of Hermitian symmetric spaces was described explicitly by Biquard and Gauduchon [8]. The case of constant holomorphic sectional curvature[5] is one of the very first hyperkähler structures ever described by Calabi [11].

Theorem 5.6.1 ([8])**.** *Let M be a Hermitian symmetric space, then there is a unique G-invariant hyperkähler metric on TM resp. $U_{s^2}M$ in the compact resp. non-compact type case, such that the Kähler form compatible with $I = j \ominus j$ is given by*

$$\omega_I = \pi^*\sigma + \mathrm{d}\mathrm{d}^c\nu,$$

with

$$\nu((x, v)) = g_x(F(jR_{jv,v})v, v), \quad F(y) = \frac{1}{y}\left(\sqrt{1+y} - 1 - \ln\frac{1+\sqrt{1+y}}{2}\right).$$

If M is of non-compact type the hyperkähler metric is incomplete.

We will now prove that as in the constant holomorphic sectional curvature case, this symplectic form $-\omega_I$ sits somewhere 'between' $\mathrm{d}\lambda - \pi^*\sigma$ and $\mathrm{d}\tau/2 - \pi^*\sigma$.

Theorem 5.6.2 (Thm. C)**.** *If M is a Hermitian symmetric space of compact type there are equivariant symplectomorphisms identifying*

$$(TM, \mathrm{d}\lambda - \pi^*\sigma) \cong (TM, -\omega_I) \cong (TM, \mathrm{d}\tau/2 - \pi^*\sigma).$$

If M is a Hermitian symmetric space of non-compact type, then there are equivariant symplectomorphisms identifying

$$(U_{s^2}M, \mathrm{d}\lambda - \pi^*\sigma) \cong (U_{s^2}M, -\omega_I) \cong (U_{2s}M, \mathrm{d}\tau/2 - \pi^*\sigma).$$

The proof goes mostly analogous to the proof of Theorem 5.4.5. Replacing $\Psi = m \circ \phi$ with ϕ for the first and with m for the second symplectomorphism. To copy the proof of Theorem 5.4.5 in this situation we need a moment map with respect to ω_I. We can compute

$$\mathrm{d}^c\nu = j^*\mathrm{d}\nu = -g(j\tilde{F}(jR_{jv,v})v, P_{\mathcal{V}}\,\cdot), \quad \tilde{F}(y) = F'(y)y + F(y),$$

where $P_{\mathcal{V}}$ denotes the projection $TTM \to \mathcal{V}$ on the vertical distribution. We can now easily deduce a moment map for ω_I.

[5] Constant holomorphic sectional curvature is the same as rank one Hermitian symmetric space.

Theorem 5.6.3. *The symplectic action of G on (TM, ω_I) resp. $(U_{s^2}M, \omega_I)$ is Hamiltonian with moment map*

$$\mu_I(x,v) = -[v, j\tilde{F}(jR_{jv,v})v] + x$$

Proof. The map μ_I is equivariant as commutators and $jR_{jv,v}$ are. We check by a direct computation, that

$$\begin{aligned}\iota_{a^\#}(\mathrm{d}\mathrm{d}^c\nu) &= -\mathrm{d}(\hat{g}((j\tilde{F}(jR_{jv,v})v)^{\mathcal{V}}, a^\#)) = -\mathrm{d}(g(j\tilde{F}(jR_{jv,v})v, P_{\mathcal{V}}(a^\#))) \\ &= \mathrm{d}(B(j\tilde{F}(jR_{jv,v})v, [a,v])) = \mathrm{d}(B([v, j\tilde{F}(jR_{jv,v})v], a)),\end{aligned}$$

where in the first equation we used that $\mathcal{L}_{a^\#}\mathrm{d}^c\nu = 0$. Now recall from section 2.3 the inclusion map is a moment map with respect to the symplectic form σ, thus

$$\iota_{a^\#}\omega_I = \iota_{a^\#}\mathrm{d}\mathrm{d}^c\nu + \iota_{a^\#}\pi^*\sigma = \mathrm{d}B(-[v, j\tilde{F}(jR_{jv,v})v] + x, a) = \mathrm{d}(\mu_I, a)$$

follows. □

Next we want to include ϕ resp. m in the diagram analogous to (5.9). For this we need to ensure that the subdiagram ③ resp. ① commutes for $-\mu_I$. By equivariance under H^r and Remark 2.4.14 we may again assume that $\iota = \iota_{p,q}$ with $p = (Z, v)$ and $q = (Z_0, v_0)$ for some $v \in T_Z M$ and some $v_0 \in T_{Z_0}\Sigma^r$.

Lemma 5.6.4. *The diagram*

$$\begin{array}{ccc} (T\mathbb{C}\mathrm{P}^1)^r & \xrightarrow{\mathrm{d}\iota_{p,q}} & TM \\ {\scriptstyle -\sum_i \mu_{I_i}}\downarrow & & \downarrow{\scriptstyle -\mu_I} \\ \mathfrak{su}(2)^r & \xrightarrow{k - Z + k(Z_0)} & \mathfrak{g} \end{array} \tag{5.12}$$

and respectively

$$\begin{array}{ccc} (D_{s/\sqrt{-\kappa}}\mathbb{C}\mathrm{H}^1)^r & \xrightarrow{\mathrm{d}\iota_{p,q}} & TM \\ {\scriptstyle -\sum_i \mu_{I_i}}\downarrow & & \downarrow{\scriptstyle -\mu_I} \\ \mathfrak{su}(1,1)^r & \xrightarrow{k - Z + k(Z_0)} & \mathfrak{g} \end{array} \tag{5.13}$$

commutes.

Proof. By equivariance it is enough to show commutativity at the point $q = (Z_0, v_0)$. There we have

$$\begin{aligned} k(-\mu_I(Z_0, v_0)) - Z + k(Z_0) &= k([v_0, j\tilde{F}(jR_{jv_0,v_0})v_0] - Z_0) - Z + k(Z_0) \\ &= -[\mathrm{d}\iota_{Z_0}v_0, \mathrm{d}\iota_{Z_0}\tilde{F}(jR_{jv_0,v_0})v_0] - \iota(Z_0) \\ &= -\mu_I(\mathrm{d}\iota(Z_0, v_0)), \end{aligned}$$

which finishes the proof. □

In the case of constant curvature surfaces all eigenvalues of $jR_{jv,v}$ are identically $y = \kappa r^2$. We compute

$$\begin{aligned}\tilde{F}(y) &:= F'(y)y + F(y) \\ &= \left(\frac{1}{y^2}\ln\left(\frac{1+\sqrt{1+y}}{2}\right) - \frac{1}{2y(\sqrt{1+y}+1)}\right)y - F(y) \\ &= \frac{1}{\sqrt{1+y}+1}.\end{aligned}$$

Thus μ_I is in the case of surfaces given by

$$\mu_I(x,v) = \frac{\kappa r^2}{\sqrt{1+\kappa r^2}+1}x + x = \left(\frac{\kappa r^2 + 1 + \sqrt{1+\kappa r^2}}{\sqrt{1+\kappa r^2}+1}\right)x = \sqrt{1+\kappa r^2}x.$$

We want to show that $-\phi^*\omega_I = \mathrm{d}\lambda - s\pi^*\sigma$ and $m^*(\mathrm{d}\tau/2 - \pi^*\sigma) = -\omega_I$. For this we need to prove that in the case of constant curvature surfaces we have

$$\mu_\lambda - \mu_\sigma = -\mu_I \circ \phi \quad \text{and} \quad -\mu_I = (\mu_\tau/2 - \mu_\sigma) \circ m.$$

We can verify this by the following calculations using the notation at the end of section 5.4,

$$\begin{aligned}-\mu_I(\phi(a_3, ra_1)) &= -\mu_I\left(\frac{1}{\sqrt{1+\kappa r^2}}(a_3 + ra_2), ra_1\right) = -(a_3 + ra_2) \\ &= [a_3, ra_1] - a_3 = \mu_1(a_3, ra_1).\end{aligned}$$

And similarly

$$\begin{aligned}\mu_2(m(a_3, ra_1)) &= \mu_2(a_3, e^{a(r)}ra_1) = \frac{1}{2}e^{2a}r^2[[a_3, a_1], a_1] - a_3 \\ &= \left(-\frac{1}{2}e^{2a}r^2\kappa r^2 - 1\right)a_3 = -\sqrt{1+\kappa r^2}a_3 \\ &= -\mu_I(a_3, ra_1).\end{aligned}$$

The last step of copying the proof of Theorem 5.4.5 is to make sure that Υ is isotropic for $\omega_I, \phi^*\omega_I$ and $m^*(\mathrm{d}\tau/2 - \pi^*\sigma)$. But this again follows because all symplectic structures split with respect to the product $T\Sigma \times \ldots \times T\Sigma$. This finishes the proof of Theorem 5.6.2.

5.7. Mañé critical value

This section contains an attempt in determining the Mañé critical value of Hermitian symmetric spaces (M, g, σ) of non-compact type. The Mañé critical value of Hermitian symmetric spaces of compact type is always infinite as σ is not weakly exact. For the non-compact case we give a lower bound, but did not finish the construction of a primitive to obtain an upper bound.

Conjecture for the upper bound:
Recall from Section 2.2.1 that the Mañé critical value is defined by

$$c(M, g, \sigma) := \frac{1}{2} \inf_{\theta} \sup_{x \in \hat{M}} \|{}^g\theta_x\|^2,$$

where $\hat{M}$ denotes the universal cover of M, the infimum is taken over primitives of σ and ${}^g\theta$ denotes the metric dual of θ. Thus in order to bound the Mañé critical value from above we need to find a primitive of σ. It is well known that the Kähler form σ is not only exact but even admits a Kähler potential. Usually the Kähler potential is determined using Bergman kernels see for example Wienhard [51, Ch. 3.4].

The idea is to determine a K-invariant Kähler potential in terms of the description as coadjoint orbit. The corresponding K-invariant primitive of σ should, restricted to any totally geodesic polydisc, yield the direct sum of the standard contact structure on the hyperbolic disc. Thus the Mañé critical value of (M, g, σ) should be bounded from above by r-times the Mañé critical value of $\mathbb{C}\mathrm{H}^1$, which was computed by [13, section 5.2].

We will first give a detailed description of the two dimensional case. Identify $\mathfrak{su}(1,1)$ with $\mathbb{R}^{2,1}$ via the map

$$\mathfrak{su}(1,1) \to \mathbb{R}^{2,1}; \ \frac{1}{2}\begin{pmatrix} x_1 & x_2 + x_3 \\ x_2 - x_3 & -x_1 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The Killing form is mapped to the billinear form

$$(\vec{x}, \vec{y}) = x_1 y_1 - x_2 y_2 - x_3 y_3$$

and the Lie bracket to

$$\vec{x} \tilde{\times} \vec{y} = \begin{pmatrix} x_3 y_2 - x_2 y_3 \\ x_1 y_3 - x_3 y_1 \\ -x_2 y_1 + x_1 y_2 \end{pmatrix}.$$

The coadjoint orbit O_Z for $Z = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the upper half of the two-sheeted hyperboloid, parametrized by

$$\vec{x}(\varphi, \theta) = \begin{pmatrix} \sin(\varphi)\sinh(\theta) \\ \cos(\varphi)\sinh(\theta) \\ \cosh(\theta) \end{pmatrix},$$

for $\varphi \in [0, 2\pi)$ and $\theta \in [0, \infty)$. The invariant Kähler structure is in these coordinates given by

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sinh(\theta)^2 \mathrm{d}\varphi^2, \quad \sigma = \sinh(\theta)\mathrm{d}\theta \wedge \mathrm{d}\varphi$$

and

$$j = \begin{pmatrix} 0 & \sinh(\theta) \\ -\sinh(\theta)^{-1} & 0 \end{pmatrix}.$$

In order to construct a Kähler potential we define the function

$$\nu : O_Z \cong \mathbb{C}\mathrm{H}^1 \to \mathbb{R}; \ x \mapsto (Z, x).$$

Now, the following Lemma holds.

Lemma 5.7.1. *The function* $\ln(1+\nu)$ *is a Kähler potential for* σ *in particular the one-form*

$$\frac{1}{1+\nu}\mathrm{d}^c\nu$$

is a primitive of σ*.*

Proof. In coordinates we have $\nu(x) = \cosh(\theta)$, thus applying d^c to ν yields:

$$\begin{aligned}\frac{1}{1+\nu}\mathrm{d}^c\nu &= \frac{1}{1+\cosh(\theta)}(\mathrm{d}\cosh(\theta)) \circ j = \frac{\sinh(\theta)}{1+\cosh(\theta)}\mathrm{d}\theta \circ j \\ &= \frac{\sinh(\theta)^2}{1+\cosh(\theta)}\mathrm{d}\varphi = \frac{\sinh(\theta)^2(1-\cosh(\theta))}{1-\cosh(\theta)^2}\mathrm{d}\varphi \\ &= (\cosh(\theta)-1)\mathrm{d}\varphi.\end{aligned}$$

Now it is easy to see that

$$\mathrm{d}\left(\frac{1}{1+\nu}\mathrm{d}^c\nu\right) = \mathrm{d}\left((\cosh(\theta)-1)\mathrm{d}\varphi\right) = \sinh(\theta)\mathrm{d}\theta \wedge \mathrm{d}\varphi = \sigma.$$

□

This was a little preparation for the general case. We consider the following map

$$A : M = O_Z \to \mathrm{Sym}(\mathfrak{p});\ p \mapsto A_p, \quad \text{where } A_p(v) = [Z, [p, v]].$$

Indeed $A_p(v) \in \mathfrak{p}$ as $\mathfrak{p} = [Z, \mathfrak{g}]$, further A_p is symmetric as

$$\begin{aligned}(w, A_p(v)) &= (w, [Z, [p, v]]) = ([p, [Z, w]], v) = \\ &- ([Z, [w, p]], v) - ([w, [p, Z]], v) \\ &= (A_p(w), v) + (\underbrace{[p, Z]}_{\in\mathfrak{p}}, \underbrace{[w, v]}_{\in\mathfrak{k}}) = (A_p(w), v).\end{aligned}$$

Furthermore in the case of $M = \mathbb{C}\mathrm{H}^1$ we have:

Lemma 5.7.2. *For* $M = \mathbb{C}\mathrm{H}^1$ *we can identify*

$$A_p(v) = -(Z, p)v = -\nu(p)v.$$

Proof. We may assume $Z = e_3$ and $p = p_1e_1 + p_3e_3$, where e_1, e_2, e_3 denote the standard orthonormal basis of $\mathbb{R}^{2,1}$. Now a small computation shows

$$A_p(e_1) = -(Z, p)e_1 \quad \text{and} \quad A_p(e_2) = -(Z, p)e_2.$$

□

The next conjecture states how the Kähler potential might be expressed in terms of A.

Conjecture 5.7.3. *Let* (M, σ) *be a hermitian symmetric space of non-compact type, then the function* $\mathrm{tr}(\ln(\mathbb{1}_{2n} - A))$ *is a Kähler potential for* σ*.*

The conjecture seems reasonable because of the following observation. For every point p there is a polydisc through Z and p. As shown in Lemma 2.4.24 every polydisc lies in an affine copy of $\mathfrak{su}(1,1)^r$. Therefore we can decompose $Z = \sum Z_i + c$ and $p = \sum p_i + c$, where c commutes with $\mathfrak{su}(1,1)^r$, i.e. $[c,h] = 0$ for all $h \in \mathfrak{su}(1,1)^r$. Assume $v \in \mathfrak{p} \cap \mathfrak{su}(1,1)^r$, then we can also split $v = \sum_i v_i$ with respect to the product. As c commutes with $\mathfrak{su}(1,1)^r$ we find

$$A_p(v) = [Z,[p,v]] = \sum_i [Z_i,[p_i,v_i]] = -\sum_i (Z_i,p_i)v_i,$$

where the last equation follows from the previous Lemma 5.7.2. This means that restricted to the polydisc our conjecture holds.

Lower bound:
We will now follow the proof of Lemma 6.11 in [12] to find a lower bound. Consider the family of closed curves $\gamma_R : [0,T] \to M$ inside a polydisc of the form $(\gamma_1, \dots, \gamma_r)$, where $\gamma_i : [0,T] \to \mathbb{C}\mathrm{H}^1$ parametrizes a geodesic circle of radius R with speed $|\dot\gamma_i| = \sqrt{2k/r}$. We know that the primitive of σ on M pulls back to the primitive $\sum_i \frac{\mathrm{d}x_i}{y_i}$ on the polydisc, where we identified $\mathbb{C}\mathcal{H}^1$ with the hyperbolic half plane. Now we can compute

$$\begin{aligned}
\mathcal{A}_{L+k}(\gamma) &= \sum_{i=1}^r \int_0^T \left(\frac{1}{2y_i^2}(\dot x_i^2 + y_i^2) + \frac{\dot x_i}{y_i} + \frac{k}{r} \right) \mathrm{d}t \\
&= \sum_{i=0}^r \left(\int_0^T \frac{2k}{r}\mathrm{d}t - \int_{D_R} \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2} \right) \\
&= r \left(\sqrt{\frac{2k}{r}} l - A \right),
\end{aligned}$$

where we used that $T = l\sqrt{r/2k}$ and l denotes the hyperbolic circumference and A the hyperbolic area of a geodesic disc D_R of Radius R. We plug in $l = 2\pi \sinh(R)$ and $A = 2\pi(\cosh(R) - 1)$ to find

$$k < \frac{r}{2} \quad \Rightarrow \quad \mathcal{A}_{L+k}(\gamma_R) \to -\infty \text{ for } R \to \infty.$$

By the alternative description of the Mañé critical value via the action functional (see Section 2.2.1) we have $r/2 \geq c(M,g,\sigma)$ and together with Conjecture 5.7.3 we obtain the following conjecture.

Conjecture 5.7.4. *The Mañé critical value of the magnetic system (M,g,σ) is $\frac{r}{2}$.*

6. Symmetries of magnetic $\mathbb{C}\mathrm{P}^n$

In this chapter we will first prove a magnetic version of the fact that $D_1\mathbb{C}\mathrm{P}^n$ sits in the product $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n$ as complement of the anti-diagonal divisor. The symplectomorphism we build also works in the case of vanishing magnetic field. This situation can be interpreted as complexification of $\mathbb{C}\mathrm{P}^n$. The second part of this chapter deals with the (open dense) symplectic embedding of the disc sub bundle $(D_{1/2}\mathbb{R}\mathrm{P}^n, \mathrm{d}\lambda)$ into $(\mathbb{C}\mathrm{P}^n, \sigma)$. Observe that $\mathbb{R}\mathrm{P}^n$ is fixed by an anti-holomorphic involution of $\mathbb{C}\mathrm{P}^n$, thus it can be seen as a real form of $\mathbb{C}\mathrm{P}^n$. We will use these symplectomorphisms to determine the Hofer–Zehnder capacity of $(D_1\mathbb{C}\mathrm{P}^n, \mathrm{d}\lambda)$ and $(D_1\mathbb{R}\mathrm{P}^n, \mathrm{d}\lambda)$.

6.1. (Magnetic) complexification

Let us start with a purely topological observation. We consider the open disc-subbundle of the tangent bundle of the complex projective space, i.e. $D\mathbb{C}\mathrm{P}^n \subset T\mathbb{C}\mathrm{P}^n$. The boundary of $D\mathbb{C}\mathrm{P}^n$ is a S^{2n-1}-bundle. If we collapse this boundary fiberwise (using the Hopf-fibration) we obtain a compactification $\overline{(D\mathbb{C}\mathrm{P}^n)}$ of the disc-bundle.[1] The compactification is a $\mathbb{C}\mathrm{P}^n$-bundle and topologically the same as the projectivization $\mathbb{P}(T\mathbb{C}\mathrm{P}^n \oplus \underline{\mathbb{C}})$ of the Whitney sum of the tangent bundle with a trivial complex line. Using the Euler-sequence we find that this is isomorphic to $\mathbb{P}(L^{\oplus n+1})$ where L is the tautological line bundle. Since projectivization does not see tensoring by a line bundle it can be identified with $\mathbb{P}(\underline{\mathbb{C}}^{n+1})$ which is clearly $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n$. In total we obtain the following line of identifications

$$\overline{(D\mathbb{C}\mathrm{P}^n)} \cong \mathbb{P}(T\mathbb{C}\mathrm{P}^n \oplus \underline{\mathbb{C}}) \cong \mathbb{P}(L^{\oplus n+1}) \cong \mathbb{P}(L \otimes \underline{\mathbb{C}}^{\oplus n+1}) \cong \mathbb{P}(\underline{\mathbb{C}}^{n+1}) \cong \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n.$$

The aim of this section is to prove the following analogous statement in the symplectic category. We will similarly to Section 5.4 construct the symplectomorphism using the $\mathrm{SU}(n+1)$-symmetries of $(D_1\mathbb{C}\mathrm{P}^n, \mathrm{d}\lambda - s\pi^*\sigma)$ and $(\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n, R_1\sigma \ominus R_2\sigma)$. Recall from Section 2.3 that we can realize $\mathbb{C}\mathrm{P}^n$ as adjoint orbit of $\mathrm{SU}(n+1)$. In particular the inclusion $\mathbb{C}\mathrm{P}^n \hookrightarrow \mathfrak{su}(n+1)$ is a moment map with respect to the invariant Kähler form (for $\mathbb{C}\mathrm{P}^n$ usually called Fubini-Study form). It follows that the diagonal action on the product $(\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n, R_1\sigma \ominus R_2\sigma)$ is Hamiltonian. Further by Theorem 2.4.23 the induced action on the tangent bundle is also Hamiltonian.

Theorem 6.1.1 (Thm. D)**.** *There is a symplectomorphism, which is equivariant with respect to the Hamiltonian* $\mathrm{SU}(n+1)$ *actions,*

$$F : (D_\rho\mathbb{C}\mathrm{P}^n, \omega_s) \to (\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n \setminus \bar{\Delta}, R_1\sigma \ominus R_2\sigma),$$

where $\bar{\Delta} \subset \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n$ *denotes the anti-diagonal divisor*

$$\bar{\Delta} = \{(p,q) \in \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n \mid \mathrm{dist}(p,q) \ \textit{maximal}\}$$

[1] In the symplectic category this will correspond to a Lerman cut.

and R_1, R_2 are determined by

$$s = R_2 - R_1, \quad \rho = 2\sqrt{R_1 R_2}.$$

Further

$$\overline{(D_1\mathbb{C}\mathrm{P}^n, \mathrm{d}\lambda - s\pi^*\sigma)} \cong (\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n, R_1\sigma \ominus R_2\sigma),$$

where $\overline{(D_1\mathbb{C}\mathrm{P}^n, \mathrm{d}\lambda - s\pi^*\sigma)}$ denotes the symplectic compactification of the disc-bundle using a Lerman cut with respect to the Hamiltonian circle action given by the reparametrized magnetic geodesic flow.

This theorem can be used to compute the Hofer–Zehnder capacity of the twisted disc-bundle. Observe that this symplectomorphism also works in the case of vanishing magnetic field i.e. $s = 0$. We are therefore in addition able to compute the Hofer–Zehnder capacity of the untwisted tangent bundle.

Corollary 6.1.2 (Cor. A)**.** *For any $s \in \mathbb{R}$ the Hofer–Zehnder capacity of $(D_\rho\mathbb{C}\mathrm{P}^n, \omega_s)$ is given by*

$$c_{\mathrm{HZ}}(D_\rho\mathbb{C}\mathrm{P}^n, \omega_s) = c^0_{\mathrm{HZ}}(D_\rho\mathbb{C}\mathrm{P}^n, \omega_s) = 2\pi\left(\sqrt{s^2 + \rho^2} - |s|\right).$$

6.1.1. The complex projective space as adjoint orbit

In order to prove Theorem 6.1.1 we will consider $\mathbb{C}\mathrm{P}^n$ as adjoint orbit in $\mathfrak{su}(n+1)$. We follow [15] for the presentation of $\mathbb{C}\mathrm{P}^n$ as adjoint orbit. The complex projective space can be identified with the homogeneous space

$$\mathbb{C}\mathrm{P}^n = {}^{\mathrm{SU}(n+1)}\!\big/_{\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))},$$

where $\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$ denotes the matrices in $\mathrm{U}(n) \times \mathrm{U}(1)$ of determinant one. Fix

$$Z := \frac{1}{n+1}\begin{pmatrix} -i_n & \\ & ni \end{pmatrix} \in \mathfrak{su}(n+1).$$

The stabilizer of Z under the adjoint action is precisely $\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$, thus $\mathbb{C}\mathrm{P}^n$ can be realized as the adjoint orbit $O_Z \subset \mathfrak{su}(n+1)$. We can relate this description of $\mathbb{C}\mathrm{P}^n$ with the standard description of complex lines in $\mathbb{C}^{n+1}$ by identifying Z with the complex line corresponding to the last coordinate axis of $\mathbb{C}^{n+1}$ and extending this identification equivariantly. The stabilizer group $\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$ is isomorphic to $\mathrm{U}(n)$ via the map

$$\mathrm{U}(n) \to \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1)); \ A \mapsto \begin{pmatrix} A & \\ & \det A^{-1} \end{pmatrix}.$$

On Lie-algebra level the inclusion looks like

$$\mathfrak{u}(n) \hookrightarrow \mathfrak{su}(n+1); \ B \mapsto \begin{pmatrix} B & \\ & -\mathrm{tr}B \end{pmatrix}.$$

Thus the Lie-algebra splits as

$$\mathfrak{su}(n+1) = \mathfrak{p} \oplus \mathfrak{u}(n),$$

where

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0_n & -\bar{z} \\ z^t & 0 \end{pmatrix} \,\middle|\, z \in \mathbb{C}^n \right\}$$

is the subspace corresponding to the tangent space of $\mathbb{C}\mathrm{P}^n$ at Z. Observe that for any

$$v = \begin{pmatrix} 0_n & -\bar{z} \\ z^t & 0 \end{pmatrix} \in T_Z\mathbb{C}\mathrm{P}^n = \mathfrak{p}$$

we have that

$$j_Z v := [Z, v] = \frac{1}{n+1}\left(\begin{pmatrix} -i_n & \\ & ni \end{pmatrix} \begin{pmatrix} 0_n & -\bar{z} \\ z^t & 0 \end{pmatrix} - \begin{pmatrix} 0_n & -\bar{z} \\ z^t & 0 \end{pmatrix} \begin{pmatrix} -i_n & \\ & ni \end{pmatrix} \right) = \begin{pmatrix} 0_n & -(\overline{iz}) \\ (iz)^t & 0 \end{pmatrix}$$

recovers the standard complex structure on $\mathbb{C}\mathrm{P}^n$. Further the negative of the Killing form

$$(X, Y) := -2\mathrm{tr}(X \cdot Y)$$

yields an Ad-invariant scalar product on $\mathfrak{su}(n+1)$. Restricted to $\mathfrak{p}$ it is compatible with j_Z and equivariantly extended it yields the standard invariant Kähler structure of $\mathbb{C}\mathrm{P}^n$.

6.1.2. Proof of Theorem D

We want to symplectically identify the twisted disc-bundle $(D_\rho\mathbb{C}\mathrm{P}^n, \omega_s)$ with the split symplectic manifold $(\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n \setminus \bar{\Delta}, R_1\sigma \ominus R_2\sigma)$. Recall that the $\mathrm{SU}(n+1)$-action on the twisted tangent bundle is Hamiltonian with moment map

$$\mu_{D_\rho\mathbb{C}\mathrm{P}^n}(x, v) = [x, v] - sx,$$

where we interpret $\mathbb{C}\mathrm{P}^n$ as coadjoint orbit in $\mathfrak{su}(n+1)$. Also the diagonal $\mathrm{SU}(n+1)$ action on $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n$ is Hamiltonian with moment map

$$\mu_{\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n}(N, S) = R_1 N - R_2 S.$$

This action restricts to an action on $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n \setminus \bar{\Delta}$ as it leaves the anti-diagonal divisor $\bar{\Delta} \subset \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n$ invariant. We can now prove Theorem 6.1.1. Observe that scaling the fibers

$$\Phi_Y^{-\ln(R_1)} : \ (D_\rho\mathbb{C}\mathrm{P}^n, \mathrm{d}\lambda - s\pi^*\sigma) \to \left(D_{\rho/R_1}\mathbb{C}\mathrm{P}^n, R_1\left(\mathrm{d}\lambda - \frac{s}{R_1}\pi^*\sigma \right) \right)$$

is an equivariant symplectomorphism. Therefore, we can restrict to the case $R_1 = 1$ and $R_2 = R$. Recall that the vector field $X_{(x,v)} := v^{\mathcal{H}}$ generates geodesic flow, i.e. $\Phi_X^t : D_\rho\mathbb{C}\mathrm{P}^n \to D_\rho\mathbb{C}\mathrm{P}^n$ is given by

$$\Phi_X^t(x, v) = (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t)),$$

for $\gamma_{(x,v)}(t) := \exp_x(tv)$ the geodesic starting at x in direction v. If $c : (-\rho, \rho) \to (0, \infty)$ is a smooth even function, we can define the smooth map

$$\varphi_c : D_\rho\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n; \ (x, v) \mapsto \gamma_{(x,v)}(c(|v|)),$$

which is the geodesic flow for time $c(|v|)$ projected to $\mathbb{C}P^n$. Our claim is now that (for suitable smooth even functions c_1, c_2) the map

$$F : (D_{2\sqrt{R}}\mathbb{C}P^n, \omega_{R-1}) \to (\mathbb{C}P^n \times \mathbb{C}P^n \setminus \bar{\Delta}, \sigma \ominus R\sigma); \quad (x, v) \mapsto (\varphi_{-c_1}(x, j_x v), \varphi_{c_2}(x, j_x v))$$

is a symplectomorphism. For the proof we want to use Prop. 5.4.1. Thus we need to show that the moment map triangle commutes. From the definition of F we see that

$$F(x, v) = (\gamma_{(x,j_x v)}(-c_1), \gamma_{(x,j_x v)}(c_2)),$$

and we will determine $c_1,\ c_2$ imposing the relation of the moment maps

$$\mu_{D_\rho \mathbb{C}P^n}(x, v) = \mu_{\mathbb{C}P^n\times\mathbb{C}P^n}(F(x, v)).$$

Per construction the moment maps and the map F are equivariant. Hence it is enough to check the moment map triangle at some point

$$x = Z = \frac{1}{n+1}\begin{pmatrix} -i_{n-1} & & \\ & -i & 0 \\ & 0 & ni \end{pmatrix}, \quad v = \frac{r}{2}\begin{pmatrix} 0_{n-1} & & \\ & 0 & -i \\ & -i & 0 \end{pmatrix} \in T_Z\mathbb{C}P^n, \ r = |v| > 0,$$

as the group action is transitive on the sphere subbundle of $T\mathbb{C}P^n$. We compute

$$\mu_{D_\rho \mathbb{C}P^n}(x, v) = [x, v] - (R-1)x = \frac{r}{2}\begin{pmatrix} 0_n & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix} - \frac{R-1}{n+1}\begin{pmatrix} -i_{n-1} & & \\ & -i & 0 \\ & 0 & ni \end{pmatrix}.$$

The geodesic through x in direction $j_x v$ is given by

$$\gamma_{(x,j_x v)}(t) = \mathrm{Ad}_{\exp(-tv)}x = \frac{1}{n+1}\begin{pmatrix} -i_{n-1} & & \\ & -i(\cos(rt)^2 - n\sin(rt)^2) & (n+1)\sin(rt)\cos(rt) \\ & -(n+1)\sin(rt)\cos(rt) & i(n\cos(rt)^2 - \sin(rt)^2) \end{pmatrix}.$$

It parametrizes an affine circle in $\mathfrak{su}(n+1)$ since

$$V(t) := \frac{1}{2}\left(\gamma_{(x,j_x v)}(t) - \gamma_{(x,j_x v)}(t + \frac{\pi}{2r})\right) = \frac{1}{2}\begin{pmatrix} 0_{n-1} & & \\ & -i\cos(2rt) & \sin(2rt) \\ & -\sin(2rt) & i\cos(2rt) \end{pmatrix}$$

satisfies $|V|^2 = -2\mathrm{tr}(V^2) = 1$. The circle is centered at

$$y := \frac{1}{2}\left(\gamma_{(x,j_x v)}(t) + \gamma_{(x,j_x v)}(t + \frac{\pi}{2r})\right) = \frac{1}{n+1}\begin{pmatrix} -i_{n-1} & 0 & 0 \\ 0 & \frac{n-1}{2}i & 0 \\ 0 & 0 & \frac{n-1}{2}i \end{pmatrix}.$$

Observe that $\gamma_{(x,j_x v)}(t) = V(t) + y$.

Remark 6.1.3. *The image of* $\mathrm{span}(v, jv)$ *under the exponential map yields an embedded affine sphere. This is the special case of Proposition 2.4.24 for the complex projective space, a compact Hermitian symmetric space of rank one. As the circle has radius one, the sphere has area 4π. The Fubini–Study is therefore normalized to have area 4π evaluated on the generator of $H_2(\mathbb{C}P^n, \mathbb{Z})$.*

Now the second moment map is given by

$$\mu_{\mathbb{C}P^n\times\mathbb{C}P^n}(F(x,v)) = \gamma_{(x,j_xv)}(-c_1) - R\gamma_{(x,j_xv)}(c_2) = (1-R)y + V(-c_1) - RV(c_2).$$

Comparing the coefficients of the two matrices yields two equations for c_1 and c_2:

$$\begin{aligned} \sin(2c_1r) + R\sin(2c_2r) &= r, \\ \cos(2c_1r) - R\cos(2c_2r) &= 1-R. \end{aligned} \tag{6.1}$$

The moment map triangle commutes if and only if these two equations hold.

Lemma 6.1.4. *A solution to (6.1) exists if and only if*

$$r := |v| \leq 2\sqrt{R} =: \rho.$$

Proof. The proof is elementary geometric and shown in figure 6.1. □

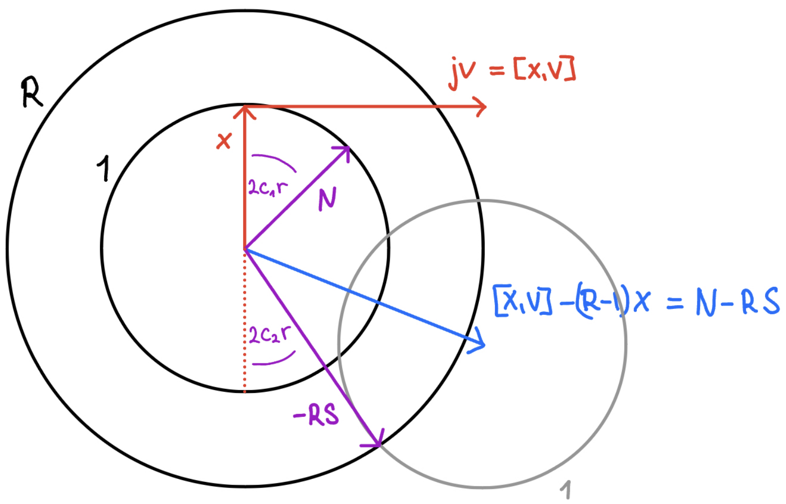


Figure 6.1.: *The geodesic $\gamma_{(x,jv)}$ parametrizes a round circle. We draw this circle twice (in black), one of radius 1 and one of radius R. The moment map $jv-(R-1)x$ must be equal to $N-RS$ for two vectors N,S of length 1. If $0<r:=|v|<2\sqrt{R}$, the grey circle of radius 1 intersects the outer black circle exactly twice. The vector $-RS$ is the first intersection point counting in the direction determined by jv from $N-RS$. If $r=0$ there is only one intersection, it determines $-RS$.*

Lemma 6.1.5. *The functions*

$$c_1, c_2 : \ (-\rho,\rho) \to (0,\infty)$$

implicitly defined via the equation (6.1) are smooth and even.

Proof. Applying the implicit function theorem directly to the equations (6.1) yields smoothness of c_1, c_2 whenever $r\neq 0$. Further, they are even, as the defining equations (6.1) are invariant under $r\to -r$. For the case $r\to 0$ we rewrite (6.1) in terms of the function

$$G:(-\rho,\rho)\times(0,\infty)^2\to\mathbb{R}^2; \qquad (r,c_1,c_2)\to\begin{pmatrix} 2\tau(2c_1r)c_1 + 2R\tau(2c_2r)c_2 - 1 \\ 4\sigma(2c_1r)c_1^2 - 4R\sigma(2c_2r)c_2^2 \end{pmatrix},$$

where $\tau, \sigma : \mathbb{R} \to \mathbb{R}$ are the smooth functions given by

$$\tau(x) = \frac{\sin(x)}{x}, \qquad \sigma(x) = \frac{1-\cos(x)}{x^2}.$$

The equations (6.1) are now equivalent to $G(r, c_1, c_2) = 0$, taking the derivative in c_1, c_2 yields

$$\mathrm{d}_{c_1,c_2} G = \begin{pmatrix} 4r\tau'(2c_1r)c_1 + 2\tau(2c_1r) & 4Rr\tau'(2c_2r)c_2 + 2R\tau(2c_2r) \\ 8r\sigma'(2c_1r)c_1^2 + 8\sigma(2c_1r)c_1 & -8Rr\sigma'(2c_2r)c_2^2 - 8R\sigma(2c_2r)c_2 \end{pmatrix}.$$

Observe that taking the limit $r \to 0$ in (6.1) yields

$$c_1(0) + Rc_2(0) = 1/2, \qquad c_1(0)^2 - Rc_2(0)^2 = 0$$

thus

$$2c_1(0) = (1+\sqrt{R})^{-1}, \qquad 2c_2(0) = (\sqrt{R}+R)^{-1}.$$

Further, $\tau(0) = 1 = 2\sigma(0)$, $\tau'(0) = 0 = \sigma'(0)$ and therefore

$$\mathrm{d}_{c_1,c_2} G|_{r=0} = \begin{pmatrix} 2 & 2R \\ 4c_1(0) & -4Rc_2(0) \end{pmatrix}.$$

We see that

$$\det\left(\mathrm{d}G_{(c_1,c_2)}|_{r=0}\right) = -8R(c_1(0) + c_2(0)) \neq 0$$

and it follows by the implicit function theorem that c_1, c_2 are smooth at $r = 0$. □

This finishes the construction of the smooth map $F : D_\rho\mathbb{C}P^n \to \mathbb{C}P^n \times \mathbb{C}P^n$. But is it a diffeomorphism? We can explicitly give an inverse. A point $(N, S) \in \mathbb{C}P^n \times \mathbb{C}P^n \setminus \bar{\Delta}$ defines a unique shortest geodesic γ connecting from N to S. As shown in Figure 6.2 one can geometrically determine $c_1(r)r$ and $c_2(r)r$ and thus deduce r from the first of the relations (6.1). Denote by $v \in T_N\mathbb{C}P^n$ the vector of length r pointing in the direction of γ. Now

$$F^{-1}(N, S) = (\exp_N(c_1(r)v), Pv) = \Phi_X^{c_1}(N, v)$$

is well defined and smooth as geodesic flow and c_1 are smooth. Here Pv denotes the parallel transport along the geodesic starting at N in direction v. Thus F is bijective and smooth intertwining the moment maps.

Further the codimension of the orbits of $SU(n+1)$ is one. Thus any complement of $\mathcal{D}$ is 1-dimensional and thus isotropic for both symplectic forms ω_s and $F^*(\sigma \ominus R\sigma)$. It follows from Prop. 5.4.1 that F is a symplectomorphism. Observe that as F is symplectic its differential must be invertible. A smooth, invertible map with invertible differential is a diffeomorphism.

Remark 6.1.6. *A closed almost complex manifold (W, J) is called (compact) complexification of a real manifold M if there exists an antiholomorphic involution*

$$I : W \to W; \quad \mathrm{d}I \circ J = -J \circ \mathrm{d}I$$

such that M is isomorphic to the fixed point set of I. The symplectomorphism constructed above realizes the complexification of $\mathbb{C}P^n$ in the case $s = 0$. The antiholomorphic involution on $(\mathbb{C}P^n \times \mathbb{C}P^n, j \ominus j)$ is given by

$$I : \mathbb{C}P^n \times \mathbb{C}P^n \to \mathbb{C}P^n \times \mathbb{C}P^n; \quad (N, S) \mapsto (S, N).$$

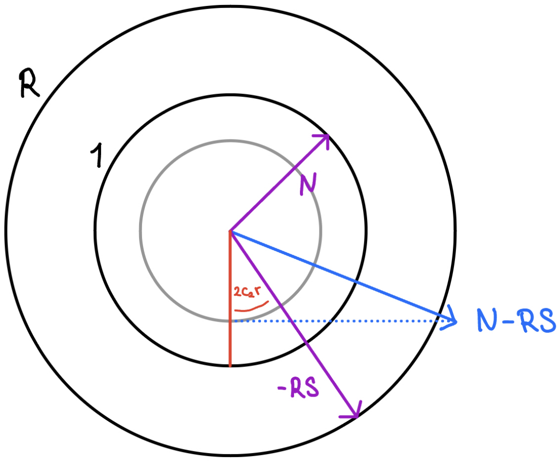


Figure 6.2.: *The vector $N - RS$ is given and we need to decompose it into two orthogonal vectors jv and $-(R-1)x$. The second vector must be of length $R-1$, thus possible candidates come from tangent lines to the grey circle of radius $R-1$ starting at $N - RS$. If $N \neq S$ there are precisely two of those. We pick the first one rotating $N - RS$ in the direction of γ. The angle between N resp. $-S$ and the red line is $2c_1r$ resp. $2c_2r$. If $N = S$ then $2c_1r = 2c_2r = 0$.*

6.1.3. Hofer–Zehnder capacity

The symplectomorphism F can also be used to compute the Hofer–Zehnder capacity of $(D_\rho\mathbb{C}\mathrm{P}^n, \omega_s)$, in particular for the case $s = 0$, which was not included previously.
Recall from section 4.2 that the Hamiltonian

$$H : D_\rho\mathbb{C}\mathrm{P}^n \to \mathbb{R}; \quad (x, v) \mapsto 2\pi\left(\sqrt{s^2 + |v|^2} - |s|\right)$$

is the moment map of a circle action on $(D_\rho\mathbb{C}\mathrm{P}^n, \omega_s)$. Thus $H \circ F^{-1}$ must generate a Hamiltonian circle action on $(\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n \setminus \Delta, R_1\sigma \ominus R_2\sigma)$. Denote $(x, v) := F^{-1}(N, S)$. By construction $\mu_1(x, v) = \mu_2(N, S)$ and therefore

$$\begin{aligned} |\mu_1(x,v)|^2 = |\mu_2(N,S)|^2 \;&\Leftrightarrow\; |v|^2 + (R_2 - R_1)^2 = R_1^2 + R_2^2 - 2R_1R_2(N,S) \\ &\Leftrightarrow\; |v|^2 = 2R_1R_2(1 - (N,S)). \end{aligned}$$

Thus

$$H(F^{-1}(N,S)) = 2\pi\left(|R_1N - R_2S|^2 - |R_2 - R_1|\right).$$

Using a Lerman cut on both sides of the symplectomorphism, we obtain an identification of

$$\left(\overline{(D_\rho\mathbb{C}\mathrm{P}^n)}, \bar{\omega}_s\right) \cong (\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n, R_1\sigma \ominus R_2\sigma),$$

as the symplectomorphism is equivariant with respect to the Hamiltonian circle actions on both sides.

Theorem 6.1.7 (Cor. A)**.** *The Hofer–Zehnder capacity of the (magnetically twisted) disc sub bundle of the tangent bundle is given by*

$$c_{\mathrm{HZ}}(D_\rho\mathbb{C}\mathrm{P}^n, \omega_s) = c^0_{\mathrm{HZ}}(D_\rho\mathbb{C}\mathrm{P}^n, \omega_s) = 2\pi\left(\sqrt{s^2 + \rho^2} - |s|\right).$$

Proof. For the upper bound we want to use Lu's theorem 3.2.11. Therefore we need to compute a non-vanishing Gromov-Witten invariant. Recall from example 3.2.2 that the standard complex structure $j \ominus j$ is regular for the split symplectic structure of $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n$ and the homology classes $[pt. \times \mathbb{C}\mathrm{P}^1], [\mathbb{C}\mathrm{P}^1 \times pt.] \in H_2(\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n, \mathbb{Z})$. So take $A \in \{[pt. \times \mathbb{C}\mathrm{P}^1], [\mathbb{C}\mathrm{P}^1 \times pt.]\} \subset H_2(\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n, \mathbb{Z})$. We have a unique holomorphic sphere through a generic point intersecting the diagonal as there is exactly one holomorphic sphere connecting two distinct points in $\mathbb{C}\mathrm{P}^n$. Further the intersection number $[A] \cdot [\bar{\Delta}] = 1$ in $H_*(\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n, \mathbb{Z})$, as $[\mathbb{C}\mathrm{P}^1] \cdot [\mathbb{C}\mathrm{P}^{n-1}] = 1$ in $H_*(\mathbb{C}\mathrm{P}^n, \mathbb{Z})$. In total we find

$$\mathrm{GW}_A(pt., \Delta, \bar{\Delta}) = \mathrm{GW}_A(pt., \Delta) \left(A \cdot [\bar{\Delta}] \right) = 1.$$

Applying Lu's theorem 3.2.11 yields

$$c_{HZ}(\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n \setminus \bar{\Delta}, R_1\sigma \ominus R_2\sigma) \leq \sigma(\mathbb{C}\mathrm{P}^1) \cdot \min\{R_1, R_2\} = 4\pi \cdot \frac{1}{2} \left(\sqrt{s^2 + \rho^2} - |s| \right).$$

By theorem 6.1.1 the upper bound follows. This is also a lower bound by Lemma 4.0.1 since it is the oscillation of the Hamiltonian generating the circle action. □

Remark 6.1.8. *This also implies that*

$$c_{HZ}(D_1\mathbb{C}\mathrm{P}^n, \mathrm{d}\lambda) = 2\pi.$$

We can compare this to the computation of the Hofer–Zehnder capacity relative to the zero-section in Example 4.1.4, where we saw that

$$c_{HZ}(D_1\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n, \mathrm{d}\lambda) = l = 2\pi.$$

This is because the Fubini-Study form is normalized such that $\sigma(\mathbb{C}\mathrm{P}^1) = 4\pi = l^2/\pi$. We see that indeed relative and absolute Hofer-Zehnder capacity agree.

6.2. Real form

It is a well known fact that the complement of the quadric $\mathbb{C}\mathrm{P}^n \setminus Q^{n-1}$ is diffeomorphic to $T\mathbb{R}\mathrm{P}^n$, where in homogeneous coordinates the quadric is given by

$$Q^{n-1} := \{[z_0 : \ldots : z_n] \mid z_0^2 + \ldots + z_n^2 = 0\} \subset \mathbb{C}\mathrm{P}^n.$$

The symplectic version of this statement in dimension $n = 2$ was proven by Adaloglou [1]. In this section we give a (different) proof for all dimensions. We will as in the previous section use the fact that there is a symmetry group essentially determining the symplectomorphism. We equip $\mathbb{R}\mathrm{P}^n$ with the constant curvature metric. Thus the isometry group is $\mathrm{SO}(n+1)$ and it induces a symplectic action on $(D_1\mathbb{R}\mathrm{P}^n)$ in the usual way. Further $\mathrm{SO}(n+1)$ can be identified as subgroup of $\mathrm{SU}(n+1)$ restricting to real coefficients. Hence, $\mathrm{SO}(n+1)$ acts also symplectically on $(\mathbb{C}\mathrm{P}^n, \sigma)$, where σ denotes the Fubini-Study form.

Theorem 6.2.1 (Thm. E)**.** *There is an $SO(n+1)$-equivariant symplectomorphism*

$$F : (D_{1/2}\mathbb{R}\mathrm{P}^n, \mathrm{d}\lambda) \to (\mathbb{C}\mathrm{P}^n \setminus Q^{n-1}, \sigma).$$

Further

$$\overline{(D_{1/2}\mathbb{R}\mathrm{P}^n, \mathrm{d}\lambda)} \cong (\mathbb{C}\mathrm{P}^n, \sigma).$$

where $\overline{(D_{1/2}\mathbb{R}\mathrm{P}^n, \mathrm{d}\lambda)}$ denotes the symplectic compactification of the disc-bundle using a Lerman cut with respect to the Hamiltonian circle action given by geodesic flow.

Using the description of $\mathbb{C}\mathrm{P}^n$ as adjoint orbit $O_Z \subset \mathfrak{su}(n+1)$ explained in Section 6.1.1, we can identify $\mathbb{R}\mathrm{P}^n$ as an $\mathrm{SO}(n+1) \subset \mathrm{SU}(n+1)$ sub orbit

$$\mathbb{R}\mathrm{P}^n \cong \{\mathrm{Ad}_g(Z)| \ \ g \in \mathrm{SO}(n+1)\} \subset O_Z \cong \mathbb{C}\mathrm{P}^n.$$

It is fixed under the involution

$$I : O_Z \to O_Z; \ p \mapsto p^T,$$

as can be seen by direct computation

$$(gZg^{-1})^T = \bar{g}Z^T g^T = gZg^{-1} \ \ \forall g \in \mathrm{SO}(n+1).$$

The involution I is antiholomorphic, because for all $v \in \mathfrak{p}$ we find

$$\mathrm{d}I_Z(j_Z(v)) = [Z,v]^T = (Zv - vZ)^T = -(Zv^T - v^T Z) = -[Z, v^T] = -j_Z(\mathrm{d}I_Z(v)).$$

6.2.1. Proof of Theorem E

Analogously to the previous sections we will prove Theorem 6.2.1 by intertwining moment maps on both sides. Therefore we first need to find out what the moment maps on both sides are.

Lemma 6.2.2. *The* $\mathrm{SO}(n+1)$*-action on* $T\mathbb{R}\mathrm{P}^n$ *is Hamiltonian. The moment map is given by*

$$\mu_1 : T\mathbb{R}\mathrm{P}^n \mapsto \mathfrak{so}(n+1); \ (x,v) \mapsto [x,v].$$

Proof. The map μ_1 is the restriction of the moment map for $(T\mathbb{C}\mathrm{P}^n, \mathrm{d}\lambda)$. We therefore only need to check that the map actually takes values in $\mathfrak{so}(n+1)$. From the description as suborbit it is only clear that $[x,v] \in \mathfrak{su}(n+1)$. The action of $\mathrm{SO(n+1)}$ on $T\mathbb{R}\mathrm{P}^n$ is transitive on the sphere subbundle, thus it is fine to check this at some point

$$x = \frac{1}{n+1}\begin{pmatrix} -i_{n-1} & & \\ & -i & 0 \\ & 0 & ni \end{pmatrix} \in \mathbb{R}\mathrm{P}^n, \quad v = \frac{r}{2}\begin{pmatrix} 0_{n-1} & & \\ & 0 & -i \\ & -i & 0 \end{pmatrix} \in T_x\mathbb{R}\mathrm{P}^n \cong [x, \mathfrak{so}(n+1)].$$

Observe that $x = Z$ and $r = |v|$. Indeed for this choice we see

$$\overline{[x,v]} = [\bar{x}, \bar{v}] = [-x, -v] = [x,v]$$

and therefore $[x,v] \in \mathfrak{so}(n+1)$. □

Lemma 6.2.3. *The* $\mathrm{SO}(n+1)$*-action on* $\mathbb{C}\mathrm{P}^n$ *is Hamiltonian. The moment map is given by*

$$\mu_2 : \mathbb{C}\mathrm{P}^n \mapsto \mathfrak{so}(n+1); \ p \mapsto \mathrm{Re}(p).$$

Proof. Equivariance of the map is clear as conjugation with a real matrix commutes with taking the real part. The rest of the proof is the following straight forward computation.

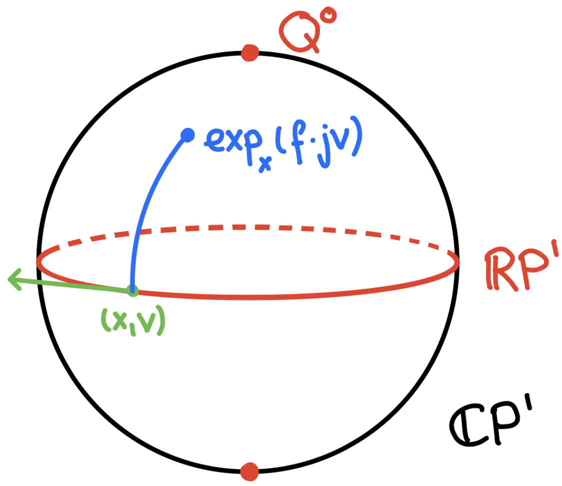


Figure 6.3.: *The figure shows a sketch of the map $F : D_{1/2}\mathbb{R}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$. In dimension $n = 1$ the quadric is given by the two antipodal points furthest away from $\mathbb{R}\mathrm{P}^1$. In particular the image of F coincides with $\mathbb{C}\mathrm{P}^1 \setminus Q^0$.*

Any tangent vector $v \in T_x\mathbb{R}\mathrm{P}^n$ is of the form $v = \xi_x^\#$ for some $\xi \in \mathfrak{so}(n+1)$, thus for every $a \in \mathfrak{so}(n+1)$ we find

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\mathrm{Ad}_{e^{t\xi}}\mathrm{Re}(x), \mathrm{Ad}_{e^{t\xi}}a) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\mathrm{Ad}_{e^{t\xi}}\mathrm{Re}(x), a) + (\mathrm{Re}(x), [\xi, a]) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\mathrm{Re}(\mathrm{Ad}_{e^{t\xi}}x), a) + (x, [\xi, a]) \\ &= \mathrm{d}(\mathrm{Re}, a)_x(\xi^\#) + \omega_x(\xi^\#, a^\#) \\ &= \mathrm{d}(\mathrm{Re}, a)_x(v) + \omega_x(v, a^\#). \end{aligned}$$

For the third equality we used that for two real matrices a, b such that $a, ib \in \mathfrak{su}(n+1)$ we know that

$$(a, ib) = -2\mathrm{tr}(a \cdot ib) = -2i\mathrm{tr}(a \cdot b) \in \mathbb{R} \quad \Rightarrow \quad (a, ib) = 0.$$

□

We now make the following ansatz for the symplectomorphism realizing Theorem 6.2.1,

$$F : D_{1/2}\mathbb{R}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n; \quad (x, v) \mapsto \exp_x(-f(|v|)j_x v) = e^{f(|v|)v} x e^{-f(|v|)v},$$

for some function $f : [0, 1/2) \to \mathbb{R}$. This can be visualized nicely in dimension $n = 1$ as shown in figure 6.3. For the above choice of (x, v) we find

$$e^{f(r)v} = \begin{pmatrix} \mathbb{1}_{n-1} & & \\ & \cos(fr) & i\sin(fr) \\ & i\sin(fr) & \cos(fr) \end{pmatrix}.$$

It follows that

$$e^{f(r)v}xe^{-f(r)v} = \frac{1}{n+1}\begin{pmatrix} -i_{n-1} & & \\ & i(n\sin(fr)^2-\cos(fr)^2) & -\frac{1}{2}(n+1)\sin(2fr) \\ & \frac{1}{2}(n+1)\sin(2fr) & i(n\cos(fr)^2-\sin(fr)^2) \end{pmatrix}$$

and thus

$$\mu_2(F(x,v)) = \frac{1}{2}\begin{pmatrix} 0_{n-1} & & \\ & 0 & -\sin(2fr) \\ & \sin(2fr) & 0 \end{pmatrix}.$$

On the other hand

$$\mu_1(x,v) = [x,v] = \begin{pmatrix} 0_{n-1} & & \\ & 0 & -r \\ & r & 0 \end{pmatrix}.$$

Comparing the matrix entries we find that f must satisfy

$$\sin(2fr) = 2r, \tag{6.2}$$

which is invertible for $r \in [0,1/2)$ so that

$$f(r) = \frac{\sin^{-1}(2r)}{2r}.$$

As f extends to a smooth even function on $(-1/2,1/2)$ it follows that F is a smooth map intertwining the moment maps.

Lemma 6.2.4. *The image of F is contained in $\mathbb{C}\mathrm{P}^n \setminus Q^{n-1}$, i.e. F is well-defined.*

Proof. We need to show that $F(x,v) \in \mathbb{C}\mathrm{P}^n \setminus Q^{n-1}$ for all $(x,v) \in D_{1/2}\mathbb{R}\mathrm{P}^n$. As $\mathrm{SO}(n+1)$ acts transitively on the unit sphere bundle we may assume that $x = [1 : 0 : \ldots : 0]$ in homogeneous coordinates and v tangent to the copy of $\mathbb{R}\mathrm{P}^1 \subset \{[z : w : 0 : \ldots : 0] \in \mathbb{C}\mathrm{P}^n\}$. We reduced our claim to dimension $n = 1$. As shown in figure 6.3 in this case it is obvious that $(x,v) \notin Q^0$. □

Lemma 6.2.5. *$F : D_{1/2}\mathbb{R}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n \setminus Q^{n-1}$ is bijective.*

Proof. We can explicitly give an inverse. A point $p \in \mathbb{C}\mathrm{P}^n \setminus Q^{n-1}$ and its conjugate $I(p)$ are joined by a unique unit speed geodesic $\gamma : [0,l] \to \mathbb{C}\mathrm{P}^n$. This can be seen as follows. If $p \notin \mathbb{R}\mathrm{P}^n$, p and $I(p)$ lie in a unique copy of $\mathbb{C}\mathrm{P}^1$. Now we look at figure 6.3 to see that there is a unique shortest geodesic joining p and $I(p)$ if and only if $p \neq \mathbb{Q}^{n-1}$. In view of equation (6.2) we have

$$\mathrm{length}(\gamma) := l = \sin(2f(|v|)|v|) = 2|v|$$

thus

$$F^{-1}(p) = (\gamma(l/2), \frac{l}{2}\dot{\gamma}(l/2))$$

defines the inverse. □

We see that F is an equivariant smooth bijection that intertwines the moment maps. Again any complement of $\mathcal{D}$ is one-dimensional and thus isotropic for any symplectic form. Thus F fulfills the prerequisites of Prop. 5.4.1 and therefore is the symplectomorphism realizing the symplectic identification in Theorem 6.2.1.

6.2.2. Hofer–Zehnder capacity

The embedding of $(D_1\mathbb{R}\mathrm{P}^n, \mathrm{d}\lambda)$ into $(\mathbb{C}\mathrm{P}^n, 2\sigma)$ yields an upper bound for the Hofer–Zehnder capacity.

Corollary 6.2.6. *Equip $\mathbb{R}\mathrm{P}^n$ with the constant curvature metric, then the Hofer–Zehnder capacity of the unit disc sub bundle of $T\mathbb{R}\mathrm{P}^n$ is bounded from above by*

$$c_{HZ}(D_1\mathbb{R}\mathrm{P}^n, \mathrm{d}\lambda) \leq 2l,$$

where l denotes the length of the geodesics.

Proof. The upper bound follows immediately from the symplectic embedding of $(D_1\mathbb{R}\mathrm{P}^n, \mathrm{d}\lambda)$ into $(\mathbb{C}\mathrm{P}^n, 2\sigma)$ and the fact that $c_{HZ}(\mathbb{C}\mathrm{P}^n, 2\sigma) = 8\pi$. Our normalization of the Fubini-Study form is $2\sigma([\mathbb{C}\mathrm{P}^1]) = 8\pi = \pi \cdot 2^2$ and thus $l = 2\pi \cdot 2 = 4\pi$. □

Recall from Example 4.1.4 that the relative Hofer–Zehnder capacity for manifolds with Zoll metric is given by the length of the geodesics. In particular we obtain a lower bound for the total capacity

$$4\pi = l \leq c_{HZ}(D_1\mathbb{R}\mathrm{P}^n).$$

We claim that this lower bound can be improved.

Theorem 6.2.7 (Thm. F)**.** *Equip $\mathbb{R}\mathrm{P}^n$ with the constant curvature metric, then the Hofer–Zehnder capacity of the unit disc sub bundle of $T\mathbb{R}\mathrm{P}^n$ is given by*

$$c_{HZ}(D_1\mathbb{R}\mathrm{P}^n, \mathrm{d}\lambda) = 2l,$$

where l denotes the length of the geodesics.

Proof. We already showed that the capacity is bounded from above by $2l$. All that is left to do is to find an admissible Hamiltonian with oscillation $2l$. The idea is to look at the dynamics for a family of Hamiltonians

$$H_\varepsilon(x, v) = \sqrt{\|v\|_x^2 + V_\varepsilon(x)}$$

with potentials $V_\varepsilon : \mathbb{R}\mathrm{P}^n \to \mathbb{R}$ that approximate billiard dynamics on a hemisphere as $\varepsilon \to 0$. We may lift the potential to S^n and assume that $V_\varepsilon(-x) = V_\varepsilon(x)$ so that the potential descends to a function V_ε on $\mathbb{R}\mathrm{P}^n$. We fix a point $N \in S^n$. The isotropy subgroup of $SO(n+1)$ that fixes N is isomorphic to $SO(n)$ and leaves the equatorial copy of S^{n-1} invariant. We want the family V_ε to be rotation invariant i.e. $V_\varepsilon(R(x)) = V_\varepsilon(x)$ for all $R \in \mathrm{SO(n)}$. This implies that $V_\varepsilon(x)$ only depends on the distance of x from N. In particular the gradient $\nabla V_\varepsilon(x)$ points along the geodesic connecting x and N. The Hamiltonian vector field for H_ε with respect to $\mathrm{d}\lambda$ is given by

$$(X_{H_\varepsilon})_{(x,v)} = \frac{1}{H_\varepsilon}\left(X_{(x,v)} - (\nabla V_\varepsilon(x))^{\mathcal{V}}\right),$$

were X is the vector field generating geodesic flow. In particular $(X_{H_\varepsilon})_{(x,v)} \in TTS^2$ for the totally geodesically embedded copy of S^2 through x and N tangent to v. We reduced the problem to the situation in dimension $n = 2$. Now we introduce coordinates on this copy of S^2 to describe the dynamics explicitly. We use $\theta \in [0, \pi]$ and $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$. Where N corresponds to $\theta = 0$. Set

$$V_\varepsilon(x) = \begin{cases} 1, & \text{for } \theta(x) \in [\pi/2 - \varepsilon/2, \pi/2] \\ 0, & \text{for } \theta(x) \in [0, \pi/2 - \varepsilon] \end{cases}$$

and interpolate smoothly for $\theta(x) \in [\pi/2 - \varepsilon, \pi/2 - \varepsilon/2]$ so that $V_\varepsilon \geq 0$ and $V'_\varepsilon \geq 0$. Now extend V_ϵ according to the required symmetries, i.e. $V_\varepsilon(x) = V_\varepsilon(-x)$ and $V_\varepsilon(Rx) = V_\varepsilon(x)$ for all $R \in \mathrm{SO}(n)$. In this coordinates the Hamiltonian reads

$$H_\varepsilon(x, v) = \sqrt{\dot{\theta}^2 + \sin(\theta)^2 \dot{\varphi}^2 + V_\epsilon(\theta)},$$

where by abuse of notation we write $\dot{\varphi}, \dot{\theta}$ instead of v_φ, v_θ and $V_\varepsilon(\theta)$ instead of $V_\epsilon(x)$ as the potential only depends on θ. Observe that H_ε does not depend on φ, this implies immediately that $\sin(\theta)^2 \dot{\varphi}$ is preserved. We conclude that the maximal angular velocity $\dot{\varphi}_{\max}$ corresponds to the minimal radius $\theta_{\min}$. We want to find upper bounds for the periods of all orbits for the Hamiltonian H_ε as ε tends to zero. Therefore we take an arbitrary orbit $\gamma(t) = (\theta(t), \varphi(t), \dot{\theta}(t), \dot{\varphi}(t))$ with energy $H_\varepsilon(\gamma) < 1$. We split into two cases: $\theta_{\min} \geq \frac{\pi}{2} - \sqrt{\varepsilon}$ and $\theta_{\min} \leq \frac{\pi}{2} - \sqrt{\varepsilon}$.

Case $\theta_{\min} \geq \frac{\pi}{2} - \sqrt{\varepsilon}$: At $\theta(t) = \theta_{\min}$ we have $\dot{\theta} = 0$, thus

$$1 > H_\varepsilon = \sqrt{\sin(\theta_{\min})^2 \dot{\varphi}_{\max}^2 + V_\epsilon(\theta_{\min})} \geq \sin(\theta_{\min}) \dot{\varphi}_{\max} \Rightarrow \dot{\varphi}_{\max} \leq \frac{1}{\sin(\theta_{\min})}.$$

Clearly the period T must satisfy

$$T > \frac{2\pi}{\dot{\varphi}_{\max}} \geq 2\pi \sin(\theta_{\min}) \geq 2\pi \cos(\sqrt{\varepsilon}) \xrightarrow{\varepsilon \longrightarrow 0} 2\pi.$$

Case $\theta_{\min} \leq \frac{\pi}{2} - \sqrt{\varepsilon}$: For small ε certainly $\varepsilon < \sqrt{\varepsilon}$, so $\theta_{\min} \leq \frac{\pi}{2} - \sqrt{\varepsilon}$ implies $V(\theta_{\min}) = 0$. In particular segments of the orbit are geodesics parametrized by unit speed. There are at least two of these segments as by symmetry of the potential the orbit must leave the region where $V_\varepsilon \neq 0$ with the same angle it enters the region. The length of these segments is $\pi - 2l_\varepsilon$, where l_ε is the length of the geodesic segment contained in the region $\pi/2 - \varepsilon \leq \theta \leq \pi/2$ where the potential V_ε does not vanish as depicted in Figure 6.4. Now we can bound the period from below as follows

$$T > 2\pi - 4l_\varepsilon \geq 2\pi(1 - \sin(l_\varepsilon)) = 2\pi \left(1 - \frac{\sin(\varepsilon)}{\sin(\pi/2 - \theta_{\min})}\right) \geq 2\pi \left(1 - \frac{\sin(\varepsilon)}{\sin(\sqrt{\varepsilon})}\right) \xrightarrow{\varepsilon \longrightarrow 0} 2\pi.$$

The bound on $\sin(l_\varepsilon)$ is obtained using the spherical law of sines

$$\frac{\sin(l_\varepsilon)}{\sin(\pi/2)} = \frac{\sin(\varepsilon)}{\sin(\pi/2 - \theta_{\min})}$$

as explained in Figure 6.4.

In total we conclude that for small enough $\varepsilon > 0$ the period is bounded by

$$T > 2\pi(1 - \sqrt{\varepsilon})$$

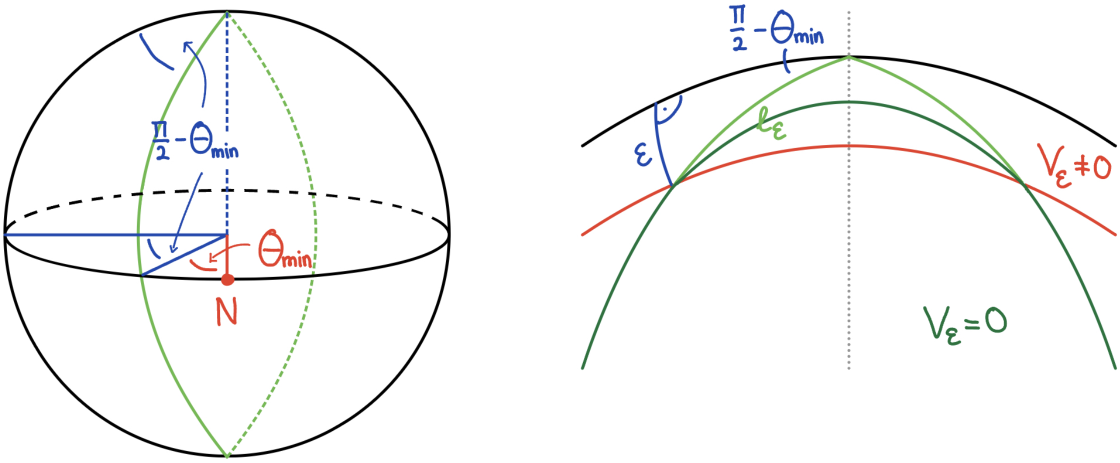


Figure 6.4.: *On the left we see in green undashed a geodesic segment on the hemisphere centered at N. Indeed it hits the equator with angle $\frac{\pi}{2} - \theta_{\min}$. On the right we see in dark green a sketch of how the trajectory might continue when it enters the region where the potential does not vanish. By conservation of angular momentum the picture must be symmetric with respect to the grey dotted line. In particular the trajectory leaves the region with the same angle entering it. Further we see that the segment of the geodesic l_ε is the hypotenuse of a right spherical triangle. The length of the side opposite to $\frac{\pi}{2} - \theta_{\min}$ is ε and thus we can determine l_ε using the spherical law of sines.*

and therefore the orbits for the scaled Hamiltonian $(1-\sqrt{\varepsilon})2\pi H_\varepsilon$ have periods $T \geq 1$. Now we can, analogously to the construction in the proof of Lemma 4.0.1, use a function $f : [0,1] \to [0,\infty)$ changing the oscillation arbitrary small so that $f \circ H_\varepsilon$ is admissible. In particular

$$c_{HZ}(D_1\mathbb{R}\mathrm{P}^n) \geq (1-\sqrt{\varepsilon})2\pi \mathrm{osc}(H_\varepsilon) = (1-\sqrt{\varepsilon})2\pi \quad \forall \varepsilon > 0,$$

which finishes the proof as $2\pi = 2l$ for the normalization we worked with. □

7. Conclusion and outlook

In this section we collect what we learned and knew already about the Hofer–Zehnder capacity of disc sub bundles $(D_\rho M, \omega_s)$ of (symplectically twisted) tangent bundles. We used symmetries to constructed explicit symplectic embeddings, which made it possible to replace finiteness results of the Hofer–Zehnder capacity by precise bounds and values. The following table gives an overview of these values. For simplicity we set $\rho = 1$ and $\kappa = \pm 1$ or $\kappa = 0$. The left column denotes the universal cover of M. The length of the shortest geodesic of M is denoted by l. Observe that κ fixes the normalization of l, for example if $\kappa = 1$ the geodesics on $\mathbb{C}\mathrm{P}^n$ have length 2π. There are some remarks we want to make looking at the table.

Relative vs. absolute:
For symplectically twisted tangent bundles ($s \neq 0$) the Hofer–Zehnder capacity relative to the zero section $c_{HZ}(D_1 M, M, \omega_s)$ coincides (at least in the cases we studied) with the absolute one. This can be easily seen as the Hamiltonian we used for the lower bound vanishes identically on the zero-section. On the other hand for the standard tangent bundle ($s = 0$) of $\mathbb{R}\mathrm{P}^n$ or flat manifolds the absolute is twice the relative capacity. It is reasonable to expect this also from the point of view of pseudoholomorphic curves, as the zero-section is Lagrangian it cuts the holomorphic cylinder that parametrizes the tangent bundle of a closed geodesic into two halves. From this point of view it is surprising and seems rather coincidental that in some cases as for $\mathbb{C}\mathrm{P}^n$ relative and absolute capacity coincide. In principle it should be possible to adapt the Hofer-Viterbo argument using holomorphic curves with boundary mapped to Lagrangian submanifolds in order to compute the Hofer–Zehnder capacity relative to Lagrangians. This phenomenon seems very related to Biran's Lagrangian barriers in the sense of [9]. Recently Brendel and Schlenk demonstrated this impressively computing the Gromov width of $\mathbb{C}\mathrm{P}^2$ with certain singular Lagrangians, called pinwheels, removed. If one could establish a version of Hofer–Viterbo's Theorem for holomorphic discs with boundary mapped to the Lagrangian, one can prove the analogous statements for the Hofer–Zehnder capacity. Actually in the case of the Lagrangian $\mathbb{R}\mathrm{P}^2 \subset \mathbb{C}\mathrm{P}^2$ one gets away without using holomorphic discs.

Example 7.0.1. *As an example of Biran's decomposition of Kähler manifolds [9, Thm. 1.A.] one gets that $(\mathbb{C}\mathrm{P}^n \backslash \mathbb{R}\mathrm{P}^n, \sigma)$ is symplectomorph to the unit disc subbundle of Boothby–Wang bundle over the quadric (Q^{n-1}, σ). In particular after a Lerman cut at the boundary of the disc-bundle we can apply Corollary 4.1.3 to find that*

$$c_{HZ}(\mathbb{C}\mathrm{P}^n \setminus \mathbb{R}\mathrm{P}^n, \sigma) = \frac{1}{2} c_{HZ}(\mathbb{C}\mathrm{P}^n, \sigma).$$

This is because the area of the fibers of the Boothby–Wang bundle over Q^n have (per construction) half the area of the holomorphic sphere that represent the generator of $H_2(\mathbb{C}\mathrm{P}^n, \mathbb{Z})$. In this case one does not need to use holomorphic discs, because $\mathbb{R}\mathrm{P}^n$ has a Zoll metric, *we can therefore compactify the complement of* $\mathbb{R}\mathrm{P}^n$ *by a Lerman cut and thus compactify the holomorphic discs to holomorphic spheres.*

$\hat{M}$	s	κ	$\leq c_{HZ}$	$c_{HZ} \leq$	$c_{HZ}(D_1M, M)$	c^0_{HZ}
$\mathbb{C}\mathrm{P}^n$	>0	1	$2\pi(\sqrt{s^2+1}-s)$	same	same	same
	$=0$	1	$l=2\pi$	same	same	same
$\mathbb{C}\mathrm{H}^n$	≥ 1	-1	$2\pi(s-\sqrt{s^2-1})$	same	same	same
	<1	-1	$l\sqrt{\rho^2-s^2})$ for $n=1$	?	?	∞
	$=0$	-1	l	?	l	∞
$\mathbb{C}^n$	>0	0	π/s	same	same	same
	$=0$	0	$2l$	same	l	∞
HSSCT	>0	1	$2\pi(\sqrt{s^2+1/r}-s)$	$2\pi r(\sqrt{s^2+1}-s)$	same	same
	$=0$	1	l	$<\infty$	l	?
HSSNT	>1	-1	$2\pi(s-\sqrt{s^2-1/r})$	$2\pi r(s-\sqrt{s^2-1})$	same	same
	≤ 1	-1	?	?	?	∞
	$=0$	-1	l	?	l	∞
$\mathbb{R}\mathrm{P}^n$	$=0$	1	$2l=8\pi$	same	$l=4\pi$	∞
$\mathbb{R}^n$	$=0$	0	$2l$	same	l	∞
$M_{\mathbb{R}}$	$=0$	± 1	l	?	l	?
S^{2n+1}	$=0$	1	$l=2\pi$	?	$l=2\pi$	?
BW	$=0$	± 1	l	?	l	?

Fiberwise convex subsets of symplectic vector bundles:
The inaccuracy for the bounds of the Hofer–Zehnder capacity of Hermitian symmetric spaces comes from the fact that the disc-bundle can be identified with a fiberwise convex neighborhood of the zero-section of a symplectic vector bundle and we only know the value

for disc sub bundles. Looking at the upper bound one gets the impression that eventually only the size of the fibers determines the value of the capacity. This could indicate that the true value of the Hofer–Zehnder capacity is the minimal action of a characteristic on the boundary of the fibers, i.e.

$$c_{HZ}(D_1M,\omega_s) = \frac{2\pi}{\kappa}(\sqrt{s^2-\kappa} - s)$$

for $s^2 > -\kappa$, independent of the rank r. An attempt to prove this could go as follows. Let $H : UM \to \mathbb{R}$ be an admissible Hamiltonian and UM some fiberwise convex neighborhood of the zero-section. Denote $(x_0, v_0) \in UM$ a point where H attains its minimum. Then

$$H|_{U_{x_0}M} : (U_{x_0}M, \omega_0) \to \mathbb{R}$$

is admissible and has the same oscillation as H. Further $U_{x_0}M \subset \mathbb{R}^n$ is convex so if the oscillation of H is larger than the minimal action of a characteristic on the boundary, $H|_{U_{x_0}M}$ has a fast periodic orbit. This however doesn't imply that H has a fast periodic orbit, as $X_{H|_{U_{x_0}M}} \neq X_H$. So the question one needs to answer is the following.

Question 7.0.2. *Is it possible to modify H near $U_{x_0}M$ without changing the oscillation of H too much such that $\mathrm{d}\pi(X_H)_{(x_0,v)} = 0$ for all $v \in U_{x_0}M$?*

Flat manifolds:
The values for flat manifolds T^n (Tori) were not derived in this thesis, but will appear independently in joint work with Gabriele Benedetti and Kai Zehmisch. Roughly speaking the lower bound comes from playing billiard (see also section 7.2). The upper bound comes from holomorphic cylinders and the symplectic identification of the tangent bundle of a torus $(TT^n, \mathrm{d}\lambda)$ with the product of complex cylinders $(Z \times \ldots \times Z, l_1\sigma_0 \oplus \ldots \oplus l_n\sigma_0)$, where σ_0 denotes the standard symplectic form on Z and l_i are the lengths of the geodesics that generate $\pi_1(T^n)$.

Claim 7.0.3. *Let (T^n, g) be a flat torus. Denote by l the length of the shortest geodesic, then*

$$c_{HZ}(D_1T^n, \mathrm{d}\lambda) = 2l.$$

Weak and vanishing magnetic strength s:
The difficulty with weak or vanishing magnetic term is that there is no Hamiltonian circle action we can use for lower and upper bound. We saw in the example of $\mathbb{C}\mathrm{P}^n$ that we can overcome this problem by considering the embedding of $D_1\mathbb{C}\mathrm{P}^n$ into the product $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n$. In principle this should be possible for all Hermitian symmetric spaces. This includes the important example of constant curvature surfaces. Note that for this class of examples even finiteness of the Hofer–Zehnder capacity is not known. Using the Lagrangian embedding

$$M \hookrightarrow (M \times M, \sigma \ominus \sigma)$$

to find upper bounds is explained a little more in section 7.1.

π_1-sensitive capacity:

For simply connected manifolds the π_1-sensitive capacity must clearly agree with the full capacity. If the magnetic field is strong they also agree as the lower bound is constructed from a Hamiltonian circle action and the upper bound comes from holomorphic spheres, thus orbits must be contractible. In the non-simply connected scenario with vanishing or weak magnetic field everything can happen. For example closed hyperbolic manifolds have infinite π_1-sensitive capacity, as there are no contractible geodesics.

Extending the class of manifolds:
Most of the considerations worked explicitly because M is a complex manifold. The only example of a real manifold we were considering is $\mathbb{R}\mathrm{P}^n$ the real form of $\mathbb{C}\mathrm{P}^n$. Surely it should be possible to consider also real forms of the other Hermitian symmetric spaces $M_{\mathbb{R}}$. This is discussed in some more detail in section 7.1.
Another possible direction will be to consider the total space of Boothby-Wang bundles over Hermitian symmetric spaces. These are homogenous manifolds for the same group. An important example is S^{2n+1}. Even though $(D_1S^n, \mathrm{d}\lambda)$ is a very classic example of a symplectic manifold its Hofer–Zehnder capacity is unknown. Some more details are in section 7.3

7.1. Complexifications & real forms

By the symplectic or Lagrangian neighborhood theorem for small enough ρ the twisted disc-bundle embeds into the product

$$(D_\rho M, \omega_s) \hookrightarrow (M \times M, R_1\sigma \ominus R_2\sigma),$$

where $s = R_1 - R_2$. Intertwining the Hamiltonian group action on both sides it should be possible to at least extend this to a finite explicit value of ρ. The embedding should be constructed replacing the functions c_1, c_2 in the proof of Theorem D by spectral functions on the operator $jR_{jv,v}$. To obtain an optimal embedding one will need to replace the disc-bundle by a suitable convex neighborhood of the zero-section.

The analog strategy should also work for real forms of Hermitian symmetric spaces. Indeed also on other symmetric spaces there exist antiholomorphic involutions $I : M \to M$ (see for example Jeffrey [30]). Denote by $M_{\mathbb{R}}$ the fixed point set of I also called real form of M. By the results of Jeffrey [30] these are real (thus Lagrangian) totally geodesic subspaces that are also symmetric spaces. Again the Lagrangian neighborhood theorem asserts that for small ρ there must be a symplectic embedding

$$(D_\rho M_{\mathbb{R}}, \mathrm{d}\lambda) \hookrightarrow (M, \sigma).$$

Using the Hamiltonian group action it should again be possible to construct this embedding explicitly and therefore find explicit bounds on ρ. Further we know the precise value of the Gromov width and the Hofer–Zehnder capacity of (M, σ) and therefore we should obtain explicit bounds on the Gromov width and the Hofer–Zehnder capacity of $(D_1 M_{\mathbb{R}}, \mathrm{d}\lambda)$. Again it seems likely that the function f must be replaced by a spectral function on $jR_{jv,v}$. If it is possible to obtain an optimal embedding one probably also needs to replace the disc bundle by a more appropriate neighborhood of the zero-section.

Example 7.1.1. *As an example we reinterpret [27, Ch.4.4, Prop. 4]. We look at* $M = T^n := (S^1)^n = (\mathbb{R}/\mathbb{Z})^n$ *with flat metric normalized so that the minimal length of geodesics is one. Its tangent bundle trivializes* $TT^n \cong T^n \times \mathbb{R}^n$*, therefore it is not hard to see that we can symplectically identify*

$$(TT^n, \mathrm{d}\lambda) \cong (Z^n, \omega_0 \oplus \ldots \oplus \omega_0)$$

where $Z = S^1 \times \mathbb{R}$ *denotes the cylinder and* ω_0 *denotes the standard symplectic structure on* Z*. We can compactify the cylinders to 2-tori quotienting by a* $\mathbb{Z}$*-action. In particular we obtain the symplectic embedding*

$$(D_{1/2}T^n, \mathrm{d}\lambda) \hookrightarrow (T^2 \times \ldots \times T^2, \omega_0 \oplus \ldots \oplus \omega_0).$$

Equip $(T^2 \times \ldots \times T^2, \omega_0 \oplus \ldots \oplus \omega_0)$ *with the compatible complex structure* $j \oplus \ldots \oplus j$*. In particular there are holomorphic curves through every point parameterizing each factor. One would like to conclude that therefore*

$$c_{HZ}(D_1 T^n, \mathrm{d}\lambda) \leq 2 = 2l.$$

but it is not so clear whether these holomorphic curves are stable as the complex structure is not regular. The bound however is valid by the much simpler argument of [27, Ch.4.4, Prop. 4]. That this bound is exact is also shown in [27, Ch.4.4, Prop. 4], but one could also obtain it using billiard dynamics as we will describe briefly in the next section.

7.2. Billiard dynamics for lower bounds

From discussions with Gabriele Benedetti and Kai Zehmisch on the Hofer–Zehnder capacity of the unit disc subbundle the idea came up that one should be able to improve the lower bound obtained from geodesic flow using geodesic billiards. Recall that if l denotes the length of the shortest geodesic of a Riemannian manifold (M, g), then the Hamiltonian flow for the kinetic energy $E(x, v) = \frac{1}{2}|v|^2$ is the geodesic flow. In particular the periodic orbits project to closed geodesics γ and their period is

$$T = \text{length}(\gamma)/|v|.$$

It follows that all closed non-constant orbits of $H(x, v) = l|v|$ have periods bigger or equal to 1. Modifying H near the zero-section and near the boundary yields an admissible Hamiltonian and thus

$$l \leq c_{HZ}(D_1 M, \mathrm{d}\lambda).$$

In order to improve this bound one can try to add a potential, i.e. consider a Hamiltonian of the form $H(x, v) = \sqrt{|v|^2 + V(x)}$. Ideally we would like to add a potential of the form

$$V(x) = \begin{cases} 0, & \text{for } x \in B_{l/2}(p) \\ 1, & \text{for } x \in M \setminus B_{l/2}(p) \end{cases},$$

where $B_{l/2}(p)$ denotes the geodesic ball centered at some point p of radius half the length of the shortest geodesic. The sub level set $\{H < 1\}$ is contained in $D_\rho M$ and the dynamics of H restricted to this sub level set is simple. Points in the complement of $TB_{l/2}(p)$ are not in the sub level set, while orbits starting at points inside the geodesic ball follow geodesic billiard trajectories. If there are no contractible geodesics, the shortest periodic billiard orbit should be the radial two bounce orbit through p and thus have length $2l$ and period $2l$.[1] In particular if we could modify H so that it becomes admissible without changing the periods and the oscillations to much, we would obtain

$$2l \leq c_{HZ}(D_1 M, \mathrm{d}\lambda).$$

Surely there are many things that could go wrong and one would need to set up a nice family of admissible potentials $V_\varepsilon : M \to \mathbb{R}$ approximating V so that all periodic orbits converge to periodic billiard trajectories, while controlling also the periods. Eventually one can use the approximation schemes developed in [3] and also [48] to realize this. It is in general also not at all clear whether two bounce orbits minimize the length. On the sphere this is not the case as $B_{l/2}(p)$ contains closed geodesics, so zero bounce orbits! On the other hand for the real projective space see Theorem 6.2.7 this approach works.

7.3. Boothby–Wang bundles

This is an outlook for joint work with Levin Maier and Steffen Schmidt. We want to study the Boothby–Wang bundles over Hermitian symmetric spaces. Denote $M \cong G/K$ an irreducible Hermitian symmetric space, i.e. G is a simple Lie group with trivial center and K is the maximal compact proper sub group. Recall that the center $C(K)$ of K is analytically isomorphic to the circle group S^1 and we denoted its generator by Z. We could identify M with the adjoint orbit $O_Z \subset \mathfrak{g}$ at Z. Observe that $\bar{K} := K/C(K)$ is a group as the center of a group is normal. Denote $\bar{\mathfrak{k}}$ the Lie-algebra, it is not to hard to see that we can identify

$$\bar{\mathfrak{k}} \cong \langle Z \rangle^\perp \subset \mathfrak{k},$$

where the orthogonal is with respect to the Killing form. As $\bar{K}$ is compact and connected, its exponential map $\bar{\mathfrak{k}} \to \bar{K}$ is surjective, in particular we may see $\bar{K}$ as a subgroup of K (or equivalently K as central extension of $\bar{K}$). Now it makes sense to look at the homogeneous space $\bar{M} := G/\bar{K}$. Observe that the projection

$$\pi : \bar{M} \to M; \ g \cdot [\bar{K}] \mapsto g \cdot [K]$$

provides $\bar{M}$ with the structure of a principle S^1-bundle over M. We claim it is not just a S^1-bundle, but the Boothby-Wang (or pre-quantization) bundle for the invariant symplectic form $\sigma \in \Omega_2(M)$. Recall that σ was the same as the KKS-symplectic structure when we describe M as adjoint orbit O_Z. Indeed the manifold $\bar{M}$ carries a contact form α defined as

$$\alpha_p(a_p^\#) = (\pi(p), a).$$

We claim that $\mathrm{d}\alpha = -\pi^*\sigma$. The definition is G-invariant therefore it is fine to check the claim at a point p over Z, i.e. $\pi(p) = Z$. Observe that α is well-defined as Z is orthogonal to $\bar{\mathfrak{k}}$. Now we compute its differential

$$\mathrm{d}\alpha_p(a^\#, b^\#) = a^\#(\alpha(b^\#)) - b^\#(\alpha(a^\#)) - \alpha([a^\#, b^\#]) = -(Z, [a, b]) = -\pi^*\omega(a^\#, b^\#)$$

[1] This is not so clear a priory! Eventually one can assure this with some curvature condition.

and see that indeed the claim holds. How could this description help computing $c_{HZ}(\bar{M}, \mathrm{d}\lambda)$? Let us have a look at the most prominent example, namely the Hopf-fibration

$$S^{2n+1} \cong \mathrm{SU}(n+1)/\mathrm{SU}(n) \to \mathbb{C}\mathrm{P}^n \cong \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1)).$$

Identify $(\mathbb{C}\mathrm{P}^{n+1}, 2\sigma)$ with the Lerman cut of the standard ball $\overline{(B_2(0), \omega_0)}$ at radius 2. We can now embed the odd dimensional sphere of radius 1 into the product $\mathbb{C}\mathrm{P}^{n+1} \times \mathbb{C}\mathrm{P}^n$ via the map

$$\varphi : S^{2n+1} \hookrightarrow \mathbb{C}\mathrm{P}^{n+1} \times \mathbb{C}\mathrm{P}^n; x \mapsto (x, \pi(x)).$$

We claim that this embedding is Lagrangian, i.e.

$$\varphi^*(2\sigma \ominus \sigma) = 0.$$

It follows immediately that the Hofer–Zehnder capacity of $(D_1 S^{2n+1}, \mathrm{d}\lambda)$ is finite[2]. The hope would be that using the symmetries one can again determine an explicit embedding and thus explicit bounds of the capacity.

Remark 7.3.1. *In general if $(L, \mathrm{d}(r^2\alpha))$ denotes the symplectization of the Boothby–Wang bundle $\pi : \bar{M} \to M$ over a symplectic manifold (M, σ) and $(\bar{L}, \omega)$ denotes the Lerman cut of L at radius 2, then the embedding*

$$\bar{M} \hookrightarrow \bar{L} \times M; x \mapsto (x, \pi(x))$$

is Lagrangian with respect to the symplectic structure $\omega \ominus \sigma$. If we knew that $c_{HZ}(\bar{L} \times M, \omega \ominus \sigma)$ was finite the same would hold for $(D_1\bar{M}, \mathrm{d}\lambda)$.

7.4. Bounds from symplectic cohomology

A different method for bounding the Hofer–Zehnder capacity avoiding Gromov–Witten invariants might come from symplectic cohomology. The author spend quite some time trying to make this approach work, in this section we will present the partial results in this direction.

If symplectic cohomology of a Liouville domain (W, α) vanishes it follows that the symplectic cohomology capacity

$$c_{SH}(W, \mathrm{d}\alpha) := \inf\{a > 0 \mid \iota_a([M]) = 0\}$$

is finite, where $\iota_a : H_*(W, \partial W) \to SH^*_{<a}(W)$. This capacity is an upper bound for the Hofer–Zehnder capacity [29].

Observe that for the case we are interested in (symplectically twisted tangent bundles over Hermitian symmetric spaces of compact type) symplectic homology vanishes by results from Groman and Merry [21], relying on the fact that the invariant symplectic structure σ is not weakly exact. In the case of the symplectically twisted tangent bundle over a Hermitian symmetric space of compact type, vanishing of symplectic cohomology can be seen more explicitly using the symplectomorphism

$$(TM, \mathrm{d}\lambda - \pi^*\sigma) \cong (TM, \mathrm{d}\tau/2 - \pi^*\sigma),$$

[2] S^n is rationally inessential, thus finiteness also follows from [18].

constructed in this thesis. Non-negative curvature of M ensures that $(TM, \mathrm{d}\tau/2-\pi^*\sigma)$ is a negative vector bundle as studied by Ritter [44]. In particular a maximum principle holds and symplectic homology of $(DM, \mathrm{d}\tau/2-\pi^*\sigma)$ can be defined using radial Hamiltonians quadratic in $r = |v|$ of non-integer slope in the complement of DM.

The first Chern class $c_1(TM, \mathrm{d}\lambda - \pi^*\sigma) = c_1(TM, \mathrm{d}\tau/2 - \pi^*\sigma)$ vanishes as the foot point projection $TM \to M$ determines a Lagrangian foliation. This means the index of a 1-periodic orbit is well-defined, independent of the capping.

Vanishing of symplectic homology now follows from an argument that also appears in [17] and [47]. For this we look at the Hamiltonian $H_a(x, v) := a\pi|v|^2 + b$ for some numbers $a \in \mathbb{R}\setminus\mathbb{Z}$ and $b \in \mathbb{R}$. As $\pi|v|^2$ generates a circle action on $(TM, \mathrm{d}\tau/2 - s\pi^*\sigma)$ this Hamiltonian has no non-constant 1-periodic orbits and all points on the zero-section are critical, so constant 1-periodic orbits. We can use an auxiliary Morse-function f on M to modify H_a near the zero section so that the new Hamiltonian $\tilde{H}_a$ has only non-degenerate periodic orbits. All 1-periodic orbits of $\tilde{H}_a$ are constant and correspond to the critical points of f. The Conley–Zehnder index is now easily determined from the Robin–Salamon index and the Morse-index. The Hamiltonian circle action generated by $\pi|v|^2$ is fiberwise rotation $(x, v) \mapsto (x, e^{2\pi jt}v)$. Thus the linearized flow at a point $(x, 0)$ on the zero-section is given by the path of symplectic matrices

$$\Psi(t) = \exp(2\pi jaA),$$

for $A = 0_{2n} \oplus 1_{2n}$ represented with respect to the splitting $T_{(x,0)}(TM) \cong \mathcal{H}_{(x,0)} \oplus \mathcal{V}_{(x,0)}$. From there it is easy to see that the corresponding Robin–Salamon index is precisely

$$\mathrm{RS}(\gamma) = (2k + 1)n$$

for k the largest integer smaller than a. The Conley–Zehnder index is now

$$\mathrm{CZ}(\gamma) = \mathrm{RS}(\gamma) + n - \mathrm{M}(\gamma) = 2n(k + 1) - \mathrm{M}(\gamma) \geq 2nk,$$

where $\mathrm{M}(\gamma)$ denotes the Morse-index. It follows that $FH^*(\tilde{H}_a)$ is supported in degree larger than $2nk$ and taking the direct limit $(k \to \infty)$ we see that symplectic cohomology vanishes.

Further the maximum principle prevents Floer trajectories of $\tilde{H}_a$ from leaving the zero-section thus Floer-cohomology coincides with quantum homology shifted in degree, i.e.

$$FH^*(\tilde{H}_a) \cong QH_{*-2kn}(M).$$

In order to use the vanishing of symplectic cohomology for some quantitative bound of the Hofer–Zehnder capacity we would need to find a filtration of symplectic homology that we can use. Considering action filtration directly is difficult, as (M, σ) is not aspherical.
An idea was to work with radial Hamiltonians and a universal choice for the capping, even though it might be possible to define a Floer complex using only orbits with this universal choice of capping, it is not clear at all if and how this 'smaller' complex relates to the full complex.

A. Appendix

This Appendix contains two computations. The first shows that the almost complex structure $j \ominus j$ on the tangent bundle of a Kähler manifold (M, g, j) is always integrable. The second shows that the Sasaki-metric is not the hyperkähler metric compatible with $\mathrm{d}\lambda$ and $\mathrm{d}\eta$ unless the Kähler manifold (M, g, j) is flat.

A.1. Canonical holomorphic symplectic structure

To check that $I := j \ominus j$ is integrable we compute the Nijenhuis-tensor, i.e. we need to show that

$$N_I(\hat{A}, \hat{B}) = [\hat{A}, \hat{B}] + I\left([I\hat{A}, B] + [\hat{A}, I\hat{B}]\right) - [I\hat{A}, I\hat{B}] = 0$$

for all vector field $\hat{A}, \hat{B}$ on TM. We show this going through all possible combinations of vertical and horizontal lifts of vector fields A, B on M. Observe that

$$I(A^{\mathcal{V}}) = (-jA)^{\mathcal{V}} \quad \text{and} \quad I(A^{\mathcal{H}}) = (jA)^{\mathcal{H}}.$$

Therefore we find

$$N_I(A^{\mathcal{V}}, B^{\mathcal{V}}) = [A^{\mathcal{V}}, B^{\mathcal{V}}] + I\left([(-jA)^{\mathcal{V}}, B^{\mathcal{V}}] + [A^{\mathcal{V}}, (-jB)^{\mathcal{V}}]\right) - [(-jA)^{\mathcal{V}}, (-jB)^{\mathcal{V}}] = 0,$$

using (i) of Proposition 2.1.2. Next we compute

$$\begin{aligned} N_I(A^{\mathcal{H}}, B^{\mathcal{V}}) &= [A^{\mathcal{H}}, B^{\mathcal{V}}] + I\left([(-jA)^{\mathcal{H}}, B^{\mathcal{V}}] + [A^{\mathcal{H}}, (-jB)^{\mathcal{V}}]\right) - [(-jA)^{\mathcal{H}}, (-jB)^{\mathcal{V}}] \\ &= (\nabla_A B)^{\mathcal{V}} + (j\nabla_{jA} B)^{\mathcal{V}} + (j\nabla_A jB)^{\mathcal{V}} - (\nabla_{jA} jB)^{\mathcal{V}} = 0, \end{aligned}$$

using (ii) of Proposition 2.1.2 and the fact that for Kähler manifolds $\nabla_{jA} B = \nabla_A jB = j\nabla_A B$ as can be easily verified using holomorphic coordinates. Last we look at two horizontal entries and find

$$\begin{aligned} N_I(A^{\mathcal{H}}, B^{\mathcal{H}}) &= [A^{\mathcal{H}}, B^{\mathcal{H}}] + I\left([(-jA)^{\mathcal{H}}, B^{\mathcal{H}}] + [A^{\mathcal{H}}, (-jB)^{\mathcal{H}}]\right) - [(-jA)^{\mathcal{H}}, (-jB)^{\mathcal{H}}] \\ &= N_j(A, B)^{\mathcal{H}} - (R(A, B)v)^{\mathcal{V}} - (jR(jA, B)v)^{\mathcal{V}} - (jR(A, jB)v)^{\mathcal{V}} \\ &\quad + R(jA, jB)v)^{\mathcal{V}} = 0, \end{aligned}$$

using that j is integrable and that R is j-linear as (M, g, j) is Kähler.

A.2. Not a hyperkähler metric

We want to see why the Sasaki-metric $G = g \ominus g$ is not the hyperkähler metric compatible with $\mathrm{d}\lambda$ and $\mathrm{d}\eta$ unless the Kähler manifold (M, g, j) is flat. We do this by showing that

$$\mathrm{d}\omega_I \neq 0 \quad \text{for } \omega_I(\cdot, \cdot) := G(I\cdot, \cdot).$$

Recall that the exterior derivative of a 2-form can be computed with the identity

$$\begin{aligned} \mathrm{d}\omega_I(\hat{A},\hat{B},\hat{C}) &= \hat{A}(\omega_I(\hat{B},\hat{C})) - \hat{B}(\omega_I(\hat{A},\hat{C})) + \hat{C}(\omega_I(\hat{A},\hat{B})) \\ &\quad - \omega_I([\hat{A},\hat{B}],\hat{C}) + \omega_I([\hat{A},\hat{C}],\hat{B}) - \omega_I([\hat{B},\hat{C}],\hat{A}) \end{aligned}$$

for arbitrary vector fields $\hat{A},\hat{B},\hat{C}$ on TM. Using this we find

$$\begin{aligned} \mathrm{d}\omega_I(A^{\mathcal{V}},B^{\mathcal{V}},C^{\mathcal{V}}) &= A^{\mathcal{V}}(\omega_I(B^{\mathcal{V}},C^{\mathcal{V}}) - B^{\mathcal{V}}(A^{\mathcal{V}},C^{\mathcal{V}})) + C^{\mathcal{V}}(\omega_I(A^{\mathcal{V}},C^{\mathcal{V}})) \\ &= A^{\mathcal{V}}(\sigma(B,C)) - B^{\mathcal{V}}(\sigma(A,C)) + C^{\mathcal{V}}(\sigma(A,B)) = 0, \\ \mathrm{d}\omega_I(A^{\mathcal{V}},B^{\mathcal{V}},C^{\mathcal{H}}) &= C^{\mathcal{H}}(\omega_I(A^{\mathcal{V}},B^{\mathcal{V}})) + \omega_I([A^{\mathcal{V}},C^{\mathcal{H}}],B^{\mathcal{V}}) - \omega_I([B^{\mathcal{V}},C^{\mathcal{H}}],A^{\mathcal{V}}) \\ &= C^{\mathcal{H}}(\sigma(A,B)) - \omega_I((\nabla_C A)^{\mathcal{V}},B^{\mathcal{V}}) + \omega_I((\nabla_C B)^{\mathcal{V}},A^{\mathcal{V}}) \\ &= \sigma(\nabla_C A,B) + \sigma(A,\nabla_C B) - \sigma(\nabla_C A,B) + \sigma(\nabla_C B,A) = 0, \\ \mathrm{d}\omega_I(A^{\mathcal{V}},B^{\mathcal{H}},C^{\mathcal{H}}) &= A^{\mathcal{V}}(\omega_I(B^{\mathcal{H}},C^{\mathcal{H}}) - \omega_I([B^{\mathcal{H}},C^{\mathcal{H}}],A^{\mathcal{V}}) \\ &= -A^{\mathcal{V}}(\sigma(B,C) + \omega_I((R(B,C)v)^{\mathcal{V}},A^{\mathcal{V}}) = \sigma(R(B,C)v,A), \\ \mathrm{d}\omega_I(A^{\mathcal{H}},B^{\mathcal{H}},C^{\mathcal{H}}) &= A^{\mathcal{H}}(\omega_I(B^{\mathcal{H}},C^{\mathcal{H}})) - B^{\mathcal{H}}(\omega_I(A^{\mathcal{H}},C^{\mathcal{H}})) + C^{\mathcal{H}}(\omega_I(A^{\mathcal{H}},B^{\mathcal{H}})) \\ &\quad - \omega_I([A^{\mathcal{H}},B^{\mathcal{H}}],C^{\mathcal{H}}) + \omega_I([A^{\mathcal{H}},C^{\mathcal{H}}],B^{\mathcal{H}}) - \omega_I([B^{\mathcal{H}},C^{\mathcal{H}}],A^{\mathcal{H}}) \\ &= A^{\mathcal{H}}(\sigma(B,C)) - B^{\mathcal{H}}(\sigma(A,C)) + C^{\mathcal{H}}(\sigma(A,B)) \\ &\quad - \omega_I([A,B]^{\mathcal{H}},C^{\mathcal{H}}) + \omega_I([A,C]^{\mathcal{H}},B^{\mathcal{H}}) - \omega_I([B,C]^{\mathcal{H}},A^{\mathcal{H}}) \\ &= \sigma(\nabla_A B,C) - \sigma(\nabla_B A,C) + \sigma(\nabla_C A,B) \\ &\quad + \sigma(B,\nabla_A C) - \sigma(A,\nabla_B C) + \sigma(A,\nabla_C B) \\ &\quad - \sigma([A,B],C) + \sigma([A,C],B) - \sigma([B,C],A) \\ &= \sigma(A,T(C,B)) + \sigma(B,T(A,C)) + \sigma(C,T(B,A)) = 0. \end{aligned}$$

We used $\nabla\sigma = 0$ and integrability of j for the last identity as then torsion vanishes. Note that the red coloured term indeed only vanishes if (M,g) is flat!

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