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On the Dynamics and Information Theory of Quantum and Statistical Field Theories on Subregions

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Abstract

This thesis is concerned with the study of classical statistical and relativistic quantum field theories on spatial subregions. Restricting a state of a field theory to a spatial subregion immediately yields two questions which we address in this work: the information theory of such reduced states and their time evolution.

With regard to the information theory of reduced states, we consider states of Gaussian statistical field theories. Such theories occur in the description of classical thermodynamic systems close to a second-order phase transition or as the high-temperature limit of quantum field theories. Due to the infinite number of degrees of freedom in such theories, we employ the concept of relative entropies and demonstrate that they are well-defined for systems with a continuum of degrees of freedom. In particular, we investigate the relative entropy between field theories in a finite volume with different masses and boundary conditions. We demonstrate how the relative entropy depends crucially on the dimension of Euclidean space. Furthermore, we show that the mutual information between two disjoint regions in $\mathbb{R}^d$ is finite if and only if the two regions are separated by a finite distance. We argue that this result is due to the Markov property of such theories.

Regarding the time evolution of states restricted to spatial subregions, we turn our attention to states of relativistic quantum field theories. Starting from a lattice regularized scalar field theory, we demonstrate how the derivative term in the action linearly couples the interior and the exterior degrees of freedom, leading to non-unitary open system dynamics for the reduced theory. Furthermore, we incorporate initial state correlations by considering local excitations of global thermal states. We show that, due to the local structure of the theory, the non-unitary contributions to the time evolution of the reduced state are entirely contained in an effective boundary action.

Zusammenfassung

Diese Dissertation beschäftigt sich mit klassischen statistischen und relativistischen Quantenfeldtheorien auf räumlichen Unterregionen. Die Beschränkung feldtheoretischer Zustände auf räumliche Unterregionen wirft unmittelbar zwei Fragen auf, die in dieser Arbeit behandelt werden: die Informationstheorie solcher reduzierten Zustände und deren Zeitentwicklung.

Bezüglich der Informationstheorie solcher reduzierten Zustände betrachten wir Zustände Gaußscher statistischer Feldtheorien. Solche Theorien treten bei der Beschreibung klassischer thermodynamischer Systeme in der Nähe eines Phasenübergangs zweiter Ordnung sowie im Hochtemperaturlimes von Quantenfeldtheorien auf. Aufgrund der unendlichen Anzahl von Freiheitsgraden in solchen Theorien verwenden wir das Konzept der relativen Entropie und zeigen, dass diese auch für Systeme mit einem Kontinuum an Freiheitsgraden wohldefiniert ist. Insbesondere untersuchen wir die relative Entropie zwischen Feldtheorien in einem endlichen Volumen mit verschiedenen Massen und Randbedingungen. Wir zeigen, dass die relative Entropie maßgeblich von der Euklidschen Raumdimension abhängt. Weiterhin zeigen wir, dass die gegenseitige Information (auch Transinformation genannt) zwischen zwei disjunkten Gebieten in $\mathbb{R}^d$ dann und nur dann endlich ist, wenn die beiden Gebiete durch eine endliche Distanz voneinander getrennt sind. Wir argumentieren, dass diese Beobachtung auf die Markow-Eigenschaft solcher Theorien zurückzuführen ist.

Hinsichtlich der Zeitentwicklung von Zuständen, die auf räumliche Teilgebiete beschränkt sind, konzentrieren wir uns auf Zustände relativistischer Quantenfeldtheorien. Ausgehend von einer gitterregularisierten skalaren Feldtheorie zeigen wir, dass der Ableitungsterm in der Wirkung die Freiheitsgrade im Inneren und Äußeren linear koppelt, was zu einer nicht-unitären Zeitentwicklung der reduzierten Theorie führt. Weiterhin integrieren wir Korrelationen im Anfangszustand, indem wir lokale Anregungen globaler thermischer Zustände betrachten. Wir zeigen, dass aufgrund der lokalen Struktur der Theorie die nicht-unitären Beiträge zur Zeitentwicklung vollständig in einer effektiven Randwirkung enthalten sind.

Contents

0 Prologue

The aim of my doctoral studies, as stated in the dissertation agreement, was to gain new insights into the entanglement properties of quantum field theories. This thesis is the summary of an attempt to achieve this goal and a collection of results that ultimately emerged from this endeavour. To provide context for what is to come, it is important to note that most, if not all, of the motivation for this work stems from this original intent.

States of *relativistic* quantum systems are generically entangled for space-like separated regions of spacetime [1]. Thus, the study of the relation between entanglement and causality, i.e., the entanglement properties of quantum field theories, involves the restriction of states to subregions of spacetime and the information theory of such states [2, 3]. The present work follows this line of thought.

When we first started looking at entanglement in quantum field theories, it quickly became clear that restricting a field theory to a spatial subregion is an interesting concept in itself. This led to two key questions: What can we learn about the information theory of a state of a field theory restricted to a subregion of space? And how does a state of a field theory restricted to a spatial subregion evolve in time? This work is an attempt to answer these questions.

Instead of starting with quantum field theories, we will begin our investigation by examining classical statistical field theories. These theories are intriguing in connection with quantum field theories as well as on their own, see Section 2. The key characteristic of such systems is that – similar to quantum fields – they possess a continuum of degrees of freedom and have a local structure. As we will demonstrate, these properties lead to a rich information theory.

We then proceed to study the local time evolution of a state in a relativistic quantum field theory. The non-unitary "open system" dynamics of states restricted to subregions of space are caused by both the (infinitesimal) nearest-neighbour interactions induced by the local structure of relativistic quantum field theories and the generic entanglement in states of such theories. We demonstrate that the dissipation and noise contributions to the dynamics of the reduced state are entirely contained in a boundary term due to the local structure of such theories. Additionally, we argue that such dynamics may be responsible for local thermalization in closed relativistic quantum systems.

Structure of the Thesis. Part I introduces the three main concepts used in this work: Information theory, statistical field theory and quantum field theory. Chapter 1 defines all

necessary objects from information theory, with special emphasis on relative entropy and mutual information. In Chapters 2 and 3, the basics of statistical field theory and quantum field theory, respectively, are briefly introduced. Parts II and III form the core of this work. In Part II, we apply the concepts of information theory to Gaussian statistical field theory. Chapter 4 focuses on the study of the relative entropy between different states of a free massive scalar field in a finite volume, while Chapter 5 examines the mutual information between two regions of Euclidean space. In Part III, we study the local time evolution of states in relativistic quantum field theories. In particular, we introduce a lattice model for a free massive scalar field in Chapter 6 and we derive the local time evolution of states in this model in Chapter 7. We summarize our results and discuss their implications as well as possible future directions for research in Part IV. Part V provides some technical details and proofs needed in the main body of this thesis.

Notations and Conventions. Elements of $\mathbb{R}^d$ with $d \geq 2$ are denoted using boldface, e.g., $\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{p}, \boldsymbol{q}, \ldots \in \mathbb{R}^d$. Throughout this thesis, we use the physicist's convention for the inner product $\langle ., . \rangle$ of a complex Hilbert space $\mathcal{H}$, i.e., $\langle ., . \rangle$ is linear in the second argument and conjugate linear in the first. The Fourier transform $\hat{f}$ of a function f and the inverse Fourier transform are, respectively, defined as

$$\hat{f}(\boldsymbol{p}) = \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\, \mathrm{d}^d x\,, \qquad f(\boldsymbol{x}) = \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{p})\, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\, \frac{\mathrm{d}^d p}{(2\pi)^d}\,.$$

The time derivative of a function f is denoted by a dot, i.e., $\dot{f} \coloneqq \partial f / \partial t$. We always work in natural units, i.e., $\hbar = c = k_{\mathrm{B}} = 1$.

Part I

Introduction

This Part provides a concise introduction to information theory as well as statistical and quantum field theories. Chapter 1 motivates relative entropy as *the* fundamental entropic quantity and highlights its applicability to systems with an infinite number of degrees of freedom. As a special case of relative entropy, we discuss the mutual information as a general measure of the dependence of two random variables. The mathematical theory of information originated from Shannon's seminal paper in 1948 [4]. The interested reader is referred to [5] for a comprehensive introduction to information theory. In 1951, Kullback and Leibler [6] introduced the relative entropy as a measure of the dissimilarity between two probability distributions. The relative entropy is a central concept in information theory and has found numerous applications in statistics, machine learning, and physics.

Chapter 2 introduces the fundamentals of statistical field theory. Statistical fields arise as an effective description of classical statistical systems close to a second order phase transition [7–11]. They can also be viewed as Euclidean versions of relativistic quantum field theories [12–15] or as the infinite temperature limit of a quantum field theory [16]. In all these cases, statistical fields are classical probability theories which are typically defined via functional integrals. The Gaussian model, used as an approximation to non-linear (interacting) theories, is the simplest example of a statistical field theory and the one considered this thesis.

Finally, Chapter 3 provides a brief introduction to quantum field theory using the example of a free scalar field. Quantum field theory is the theoretical framework that unifies quantum mechanics and special relativity and is used in the description of elementary particles and their interactions as well as condensed matter systems. The information theoretic properties of quantum fields, particularly in connection with entanglement, are of great interest in contemporary research [2, 3, 17–40]. Standard references for quantum field theory include [41–43]. For a more mathematically oriented exposition of this subject, the reader is referred to, e.g., [44–46].

1 Information Theory

Disclaimer. This Chapter closely follows [47, Ch. 3], which was largely written by me. Parts are taken verbatim from the aforementioned reference.

In this Chapter we introduce the relative entropy and discuss its properties. In particular, we focus on the relative entropy between two Gaussian probability measures, which correspond to free field theories. Besides its properties as a divergence, we will also discuss its statistical interpretation. In particular, we argue that the relative entropy can be thought of as a measure of the distinguishability of two probability measures. From a mathematical point of view, the relative entropy allows for a meaningful generalization of the concept of an entropy to probability measures on infinite dimensional spaces. We conclude this Chapter with a discussion of a special type of relative entropy, the mutual information.

The most well-known notion of entropy is Shannon's entropy, which is defined for discrete random variables as the expectation value of the information content of a realization.

Definition 1.1 (Information content, [48])**.** Let X be a random variable with alphabet $\mathcal{X}$ and probability mass function $p_X(x) \coloneqq \mathbb{P}(X = x)$. The *information content* or *surprisal* of a realization $x \in \mathcal{X}$ of the random variable X is defined as

$$i_X(x) \coloneqq -\log p_X(x) = -\log \mathbb{P}(X = x)\,. \tag{1.1}$$

Definition 1.2 (Shannon's entropy, [4])**.** The *entropy* of a random variable X with alphabet $\mathcal{X}$ is defined as

$$S(X) \coloneqq \mathbb{E}[i_X] = -\sum_{x\in\mathcal{X}} p_X(x) \log p_X(x)\,. \tag{1.2}$$

As emphasized in [49] (and already mentioned by Shannon himself in [4]), all entropies are relative entropies between a probability measure and another measure. In case of Shannon's entropy, the reference measure is the counting measure on the alphabet $\mathcal{X}$. For the often used differential entropy, the reference measure is the Lebesgue measure on $\mathbb{R}^d$. However, the differential entropy of probability distributions on $\mathbb{R}^n$ is neither non-negative nor invariant under a change of variables. An additional problem arises in the case of probability measures on infinite dimensional spaces, describing field theories in the continuum, where it is not clear how to make sense of the "functional" entropy

$$S[p] = -\int p[\varphi] \log p[\varphi]\; \mathcal{D}\varphi\,. \tag{1.3}$$

In practice, such a functional entropy generically suffers from UV-divergences, as can be seen from a simple one-loop calculation. The choice of another probability measure as reference measure leads to what is usually referred to as a relative entropy.

Definition 1.3 (Relative entropy, [6])**.** Let $(\mathscr{X}, \Sigma, \mu_i)$, $i = 1, 2$, be two probability spaces and suppose that μ_1 is absolutely continuous with respect to μ_2 (see Appendix C.2). The *relative entropy* or *Kullback-Leibler divergence* of μ_1 with respect to μ_2 is defined as

$$S(\mu_1 \| \mu_2) \coloneqq \int_{\mathscr{X}} \frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_2}(x) \log\left[\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_2}(x)\right] \mathrm{d}\mu_2(x) = \int_{\mathscr{X}} \log\left[\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_2}(x)\right] \mathrm{d}\mu_1(x)\,, \tag{1.4}$$

where $\mathrm{d}\mu_1/\mathrm{d}\mu_2$ denotes the Radon-Nikodym derivative of μ_1 with respect to μ_2.

The relative entropy is an information theoretic measure of the distinguishability of the two probability measures μ_1 and μ_2. It is non-negative and vanishes precisely when $\mu_1 = \mu_2$ [50, Thm. 3.1]. It is, however, not a metric as it is not symmetric and does not satisfy the triangle inequality. Rather, it is an example of a (directed) statistical divergence, a concept that is central in the field of information geometry [51–53]. The relative entropy is invariant under parameter transformations and additive for independent random variables [50, Ch. 2]. Finally, it satisfies a monotonicity property in the form of the data processing inequality [54, Thm. 9]. Since it has all the desired properties of an entropy while being well-defined for general probability measures[1], we regard relative entropy as *the* fundamental entropic quantity.

This work studies the case where μ_1 and μ_2 are Gaussian measures. By the Feldman-Hájek Theorem [57, 58] (cf. Theorem C.7), two Gaussian measures are either equivalent or mutually singular. If two Gaussian measures μ_1 and μ_2 are equivalent, $\log[\mathrm{d}\mu_1/\mathrm{d}\mu_2]$ is integrable with respect to μ_1 and the relative entropy is finite. Moreover, $\mathrm{d}\mu_1/\mathrm{d}\mu_2 > 0$ μ_2-a.e. [59, Sec. 3.2], i.e., the Radon-Nikodym derivative between two such measures is supported (almost) everywhere.

If μ_1 and μ_2 are mutually singular, then there exists an event $A \in \Sigma$ that is impossible with respect to μ_1, i.e., $\mu_1(A) = 0$, but certain with respect to μ_2, i.e., $\mu_2(A) = 1$. In this case, μ_1 and μ_2 are, in a sense, maximally different and one usually defines the relative entropy between them to be $+\infty$. Therefore, for the case of two Gaussian measures μ_1 and μ_2 on a common measurable space $(\mathscr{X}, \Sigma)$, the relative entropy is defined as[2]

$$S(\mu_1 \| \mu_2) = \begin{cases} \displaystyle\int_{\mathscr{X}} \frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_2}(x) \log\left[\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_2}(x)\right] \mathrm{d}\mu_2(x) & \text{if } \mu_1 \sim \mu_2 \\ +\infty & \text{if } \mu_1 \perp \mu_2 \end{cases} . \tag{1.5}$$

[1] In fact, a relative entropy can even be defined for states on von Neumann algebras in terms of Tomita-Takesaki modular theory [55, 56].

[2] For details on the notation for equivalence and mutual singularity of measures, see Appendix C.2.

One often interprets the second argument of the relative entropy as a model (or reference, or approximation) for the true theory given in the first argument. In particular, the relative entropy $S(\mu_1\|\mu_2)$ quantifies the average excess information content from using μ_2 as a model when the true theory is given by μ_1. The relative entropy is zero (and hence the theories are indistinguishable) precisely when $\mu_1 = \mu_2$. For $\mu_1 \neq \mu_2$, the distinguishability is positive and increases monotonically as the two theories become "more different". In the extreme case where the two theories are mutually singular, i.e., when the model predicts zero probability for an event that is certain with respect to the true distribution, the two theories can be perfectly distinguished from one another, and we set $S(\mu_1\|\mu_2) = +\infty$.

A special case of the relative entropy is the mutual information.

Definition 1.4 (Mutual information, [4, 60, 61])**.** Let X and Y be random variables with joint law μ_{XY} and marginal laws μ_X and μ_Y, respectively, such that μ_{XY} is absolutely continuous with respect to $\mu_X \otimes \mu_Y$. The *mutual information* of X and Y is defined as

$$\begin{aligned} I(X:Y) &\coloneqq S(\mu_{XY} \,\|\, \mu_X \otimes \mu_Y) \\ &= \int_{\mathscr{X}\times\mathscr{Y}} \frac{\mathrm{d}\mu_{XY}}{\mathrm{d}\mu_X\,\mathrm{d}\mu_Y}(x,y)\, \log\left[\frac{\mathrm{d}\mu_{XY}}{\mathrm{d}\mu_X\,\mathrm{d}\mu_Y}(x,y)\right] \mathrm{d}\mu_X(x)\,\mathrm{d}\mu_Y(y)\,. \end{aligned} \tag{1.6}$$

For the particular case where we consider probability distributions on a finite set, we can write the mutual information in terms of Shannon entropies as

$$I(X:Y) = S(X) + S(Y) - S(XY)\,. \tag{1.7}$$

This provides an interpretation of the mutual information as the average amount of information shared by the random variables X and Y. In particular, the mutual information is zero precisely when X and Y are independent, and we say that the mutual information is a measure of the (in-)dependence of two random variables.

For later use, we consider the case where X and Y are centred random variables following a Gaussian distribution over $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then, the joint distribution p_{XY} is a multi-variate normal distribution with $(n+m)\times(n+m)$-dimensional covariance matrix

$$\Sigma = \begin{pmatrix} \Sigma_X & \Sigma_{XY} \\ \Sigma_{XY}^{\mathsf{T}} & \Sigma_Y \end{pmatrix}, \tag{1.8}$$

where Σ_{XY} is a $(n\times m)$-matrix, called the cross-covariance matrix of the joint distribution p_{XY}. Using the definition of the mutual information, we see that the mutual information is the relative entropy between the "true" distribution containing all cross-correlations of the random variables X and Y and the "model" which coincides with the true distribution except that it contains no cross-correlations. Using the expression for the relative entropy

between multivariate Gaussian distributions (cf. [62, Ch. 1]) the mutual information can then be written as

$$I(X:Y) = \frac{1}{2}\log\left[\frac{\det\Sigma_X \, \det\Sigma_Y}{\det\Sigma}\right] . \tag{1.9}$$

2 Statistical Field Theories

Disclaimer. This Chapter closely follows [47, Ch. 2], which was largely written by me. Parts are taken verbatim from the aforementioned reference.

In this Chapter we describe how to realize classical statistical field theories via functional integrals. Statistical field theories arise, for example, in the description of second order phase transitions of systems such as uniaxial and isotropic (anti-)ferromagnets like the Ising model, fluids, superfluids and superconductors [63]. Such a model is defined at a fundamental microscopic length scale Λ^{-1}, where Λ is some high momentum (or ultraviolet) cut-off scale. In the case of a spin model, for example, this microscopic scale is the lattice constant of the atomic lattice. The interactions of the microscopic degrees of freedom are assumed to be short-range. The length scale L at which one conducts experiments is called the macroscopic scale and typically one can assume $\Lambda^{-1} \ll L$.

In the vicinity of a critical point, the correlation length ξ is much greater than the microscopic scale and even diverges as the system approaches the critical temperature. When the system is close to a critical point and the correlation length is much larger than the microscopic scale but still significantly smaller than the macroscopic scale, we introduce a mesoscale $\lambda \lesssim \xi$ such that the system exhibits a scale hierarchy $\Lambda^{-1} \ll \lambda \ll L$. At the mesoscopic scale the system is approximately homogeneous and fluctuations at scales between the microscopic and mesoscopic scales are small. It is then reasonable to average out the these small scale fluctuations. One then postulates that, close to a critical point, the thermodynamic partition sum of the system can be approximated by a formal integration over all configurations $\eta(x)$ of the order parameter, fluctuating at scales between λ and Λ. More explicitly [7–9],

$$Z \approx \int \mathcal{D}\eta \, \mathrm{e}^{-\beta H_\lambda[\eta]} \,, \tag{2.1}$$

where H_λ is an effective Hamiltonian.

If we assume the effective Hamiltonian to be quadratic in the order parameter η, we arrive at the *Gaussian model*, whose information-theoretic properties we study in Part II. The Gaussian model is the first correction to Landau's mean field theory [11, Ch. XIV]. It takes fluctuations of the order parameter into account but assumes that the fluctuations follow a normal distribution around some mean value. It is well known that the validity of the Gaussian approximation depends crucially on the dimension of space. For Ising-type systems, the space dimension above which mean field theory and the Gaussian approximation can reli-

ably describe critical phenomena, called the upper critical dimension d_c, is four. The origin of the specific value of the upper critical dimension can be explained via the renormalization group [64, 65]. For $d > d_c$, all higher order couplings become irrelevant and the theory is trivial, while for $d < d_c$ there are relevant higher order couplings which need to be taken into account nonperturbatively close to the critical point. In the edge case $d = d_c$, higher order couplings can be shown to be marginally irrelevant. While for $d > d_c$ the Gaussian approximation yields a reasonable description of the system everywhere in the phase diagram, for the physically relevant spatial dimensions $d \leq 3$ the Gaussian approximation breaks down around the critical point. However, at sufficiently high temperatures the Gaussian contributions dominate and the Gaussian model is valid.

Another important area of application of functional integrals are Euclidean quantum field theories, i.e., relativistic quantum field theories analytically continued to imaginary time. Euclidean quantum field theories have the same structure as theories in classical statistical mechanics [15]. In particular, they can be seen as probability theories defined by a Gibbs-type measure on some infinite dimensional space of "field configurations". Under certain conditions [13, 14, 66–68], one can recover the relativistic theory from these probability theories. This provides a connection between functional probability measures and relativistic quantum field theories. The Gaussian field theory considered in this work, which corresponds to a single massive scalar boson field without any interactions, is one of the simplest quantum field theories one can consider. Nevertheless, a sound understanding of the Gaussian theory is necessary for the treatment of interacting theories. For example, at least for weak interactions, one can treat the interacting theory as a perturbation of the free theory, i.e., the theory without interactions [44, 69–71]. The free theory thus provides a starting point for the study of more complicated theories. Furthermore, certain aspects of interacting theories are already captured in the corresponding free theory, in particular the (infinitesimal) nearest-neighbour interaction induced by the kinetic part of the action. Finally, we note that what corresponds to a mass in the relativistic context has then the significance of an inverse correlation length, $m = \xi^{-1}$.

Lastly, we mention that a classical statistical field theory may arise as the infinite temperature limit of a quantum field theory [16]. Recall that a quantum field over d-dimensional space in a canonical thermal state at inverse temperature $\beta = T^{-1}$ may be defined via a Euclidean path integral over the cylinder $\mathbb{S}_\beta \times \mathbb{R}^d$, where $\mathbb{S}_\beta$ is the circle of circumference β [68, 72–74]. The infinite temperature limit corresponds to the limit $\beta \to 0$, in which the circle fibre vanishes. The time-zero correlation function of two field operators in a free massive scalar field theory, which at finite temperature $T > 0$ is given by [72, Eq. 3.19]

$$\langle \hat{\Phi}(\boldsymbol{x}) \hat{\Phi}(\boldsymbol{y}) \rangle_\beta = \frac{1}{2} \int_{\mathbb{R}^d} \frac{1}{\sqrt{\boldsymbol{p}^2 + m^2}} \coth\left(\frac{\beta}{2} \sqrt{\boldsymbol{p}^2 + m^2} \right) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \frac{\mathrm{d}^d p}{(2\pi)^d} \,, \tag{2.2}$$

behaves for large T like

$$\langle\hat{\Phi}(\boldsymbol{x})\hat{\Phi}(\boldsymbol{y})\rangle_\beta \sim T\int_{\mathbb{R}^d}\frac{1}{\boldsymbol{p}^2+m^2}\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\,\frac{\mathrm{d}^d p}{(2\pi)^d}\;, \tag{2.3}$$

which is, up to a factor of T, the correlation function of a classical statistical field theory on d-dimensional Euclidean space, cf. the remainder of this Chapter. This is of course in accordance with the expectation that for large temperatures the thermal fluctuations dominate the quantum fluctuations.

We recall that in the functional integral formalism, a free scalar (statistical or Euclidean quantum) field theory is defined by a Gaussian probability measure. Formally, this measure is given by the expression

$$\mathrm{d}\mu = \frac{1}{Z}\,\exp[-S_\mathrm{E}[\varphi]]\,\mathcal{D}\varphi\;, \tag{2.4}$$

where Z is a normalization constant, $\mathcal{D}\varphi$ is a formal Lebesgue measure on the space of field configurations and $S_\mathrm{E}[\varphi]$ is the Euclidean action functional given by

$$S_\mathrm{E}[\varphi] = \frac{1}{2}\int_\Omega \varphi(x)\big(-\triangle+m^2\big)\varphi(x)\;\mathrm{d}^d x\;, \tag{2.5}$$

where $\triangle \coloneqq \sum_{i=1}^d \partial^2/\partial x_i^2$ is the Laplace operator and m is the mass of the field. It is instructive to first consider this field theory regularized on a finite lattice[1] $\mathscr{L}$ [75, Sec. 1.3]. Assuming that this lattice has N lattice sites, the Gaussian measure defining the theory is just a Gaussian measure on $\mathbb{R}^N$ given by

$$\mathrm{d}\mu_\Gamma(\boldsymbol{\varphi}) = \frac{1}{\sqrt{\det(2\pi C)}}\;\exp\left[-\frac{1}{2}\,\boldsymbol{\varphi}^\mathsf{T} C^{-1}\boldsymbol{\varphi}\right]\mathrm{d}^N\varphi\;, \tag{2.6}$$

In this case, the field configurations are given by real N-component vectors $\boldsymbol{\varphi}$. The inverse covariance matrix in (2.6), sometimes called the precision matrix, is given by

$$C^{-1} \coloneqq -\triangle_\mathscr{L} + m^2\mathbb{1}_N\;, \tag{2.7}$$

where $\mathbb{1}_N$ is the $N\times N$ unit matrix and $-\triangle_\mathscr{L}$ is the lattice Laplacian, see, e.g., [75, Sec. 1.3]. Since we work on a finite lattice, some boundary conditions have to be imposed in the definition of $-\triangle_\mathscr{L}$.

We can think of the continuum theory as the limit[2] of the lattice theory when the lattice constant ε is sent to zero, the number of lattice sites N is sent to infinity and the system size $N\varepsilon^d$ is kept fixed. In this limit, the lattice Laplacian $-\triangle_\Gamma + m^2\mathbb{1}_N$ becomes the usual "continuum" Laplacian $-\triangle+m^2$ satisfying some boundary conditions. Furthermore, we

[1] See also Chapter 6.
[2] For a detailed discussion see, e.g., [76, 77].

may expect the field configurations φ to become functions[3] $\varphi(x)$ defined on the Euclidean spacetime region Ω. Assuming a vanishing mean field configuration, the Gaussian measure (and thus the field theory) is completely determined by a covariance operator $\hat{C} = (-\triangle + m^2)^{-1}$, where suitable boundary conditions are to be imposed in the precise definition of the differential operator $-\triangle + m^2$.

Since the theory is determined by (the inverse of) a differential operator, its properties depend non-trivially on d, the dimension of Euclidean spacetime. In particular, the dimension d determines the asymptotic behaviour of the spectrum of the covariance operator, i.e., the ultraviolet properties of the covariance operator. For example, in $d = 1$, the theory is UV-finite, since the corresponding correlation function is continuous. This crucial dependence of the UV properties of the covariance operator on the Euclidean spacetime dimension also has consequences for the relative entropy, see Chapter 4.

It was shown by Nelson [13, 14] that the free scalar field has the Markov property and is thus a Markov random field [79]. Intuitively, the Markov property of the free scalar field stems from the fact that the Euclidean action functional only contains "nearest-neighbour" interactions introduced by the Laplacian. In the lattice-regularized theory, the nearest-neighbour property becomes manifest (see also Part III), and the theory is essentially an Ising ferromagnet, see also [70, Sec. IV] and [80, Sec. IX.1].

[3] As discussed in Section C.1, we actually realize the field theory as a Gaussian measure on a space of generalized functions or distributions. See also [71, Sec. §I.2] and [78].

3 Quantum Field Theories

In this Chapter, we describe the fundamentals of relativistic quantum field theories using the prototypical example of a massive, non-interacting scalar (i.e., bosonic) field. We begin with the algebraic approach to quantum field theory before making the connection to the Euclidean path integral formalism. As in statistical field theories, the free field provides an explicitly solvable model that serves as a starting point for the study of interacting theories and is thus of fundamental importance. Furthermore, the study of local time evolution discussed in Part III will show that the kinetic part of a field theory, containing the (infinitesimal) nearest neighbour interactions between spatial points, i.e., the non-interacting part, is responsible for the coupling of "system" and "environment" degrees of freedom and thus for the emergence of a non-trivial local time evolution.

As argued in, e.g., [81, Sec. 1.3], a physical system is defined by its physical properties (also called *observables*) and the set of states in which the system can be prepared in. Mathematically, the set of observables can naturally be given the structure of a (real) Jordan Banach algebra, where the notion of a norm arises from the fact that the scale of every real experimental apparatus is necessarily bounded. A *state* of a physical system associates to each observable of a physical system a real number, called the *expected value* of this observable. In an experiment, this number is (approximately) obtained by performing replicated measurements of the observable of interest in identically prepared states, and by taking the average over the results of measurements [81]. Mathematically, a state is represented as a positive and normalized continuous functional on the algebra of observables.

For technical reasons, it is convenient to extend the above discussion to *complex* algebras containing the algebra of observables as a subalgebra, which yields the description of physical systems in terms of C^*- and von Neumann algebras [45, 81–83]. This algebraic description is applicable to both classical and quantum systems; Classical systems are described by commutative algebras, while quantum systems are described by non-commutative algebras. One of the most important examples of a non-commutative algebra is the Weyl algebra, which we shall briefly discuss in the following.

We know from text book quantum mechanics that the position and momentum operators of a particle satisfy the canonical commutation relations (CCRs)

$$[\hat{x}_i, \hat{p}_j] = \mathrm{i}\delta_{ij}\mathbb{1} \ . \tag{3.1}$$

The CCRs imply that the position and momentum operators cannot simultaneously have finite norm [81, Sec. 3.1]. This technical difficulty can be overcome by considering the unitary operators generated by the self-adjoint position and momentum operators instead. In addition, this approach is convenient when one considers systems with an infinite amount of degrees of freedom, such as quantum field theories.

Let[1] $(K_{\mathbb{R}}, \sigma)$ be a symplectic space, i.e., a real vector space $K_{\mathbb{R}}$ equipped with a non-degenerate symplectic form $\sigma : K_{\mathbb{R}} \times K_{\mathbb{R}} \to \mathbb{R}$. If $K_{\mathbb{R}}$ is either infinite or even dimensional, we call $(K_{\mathbb{R}}, \sigma)$ a classical phase space.

Definition 3.1 (Weyl algebra)**.** Let $(K_{\mathbb{R}}, \sigma)$ be a symplectic space with non-degenerate symplectic form σ. The C^*-algebra $\mathfrak{W}(K_{\mathbb{R}}, \sigma)$ generated by the set of (abstract) Weyl operators $\{W(f)\}_{f \in K_{\mathbb{R}}}$ satisfying the Weyl relations

$$W(f)W(g) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\sigma(f,g)}\, W(f+g)\;, \qquad (W(f))^* = W(-f)\;, \tag{3.2}$$

$f, g \in K_{\mathbb{R}}$, is called the Weyl algebra of $(K_{\mathbb{R}}, \sigma)$.

For any non-degenerate symplectic space $(K_{\mathbb{R}}, \sigma)$, the corresponding Weyl algebra exists and is unique up to *-isomorphisms [84, Thm. 2.1]. If the underlying symplectic space is not important for the discussion, we denote the associated Weyl algebra simply by $\mathfrak{W}$. The Weyl algebra is unital with $\mathbb{1} = W(0)$ and, by construction, every $W(f)$ is unitary. The Weyl algebra $\mathfrak{W}(K_{\mathbb{R}}, \sigma)$ is precisely the completion of the linear span of Weyl operators in the (unique) C^*-norm. Since span $\{W(f)\}_{f \in K_{\mathbb{R}}}$ is norm dense in $\mathfrak{W}(K_{\mathbb{R}}, \sigma)$, linearity and boundedness of algebraic states implies that every state ω on $\mathfrak{W}(K_{\mathbb{R}}, \sigma)$ is uniquely fixed by its values on the Weyl operators.

Before we discuss representations of the Weyl algebra, we introduce a useful class of states on it. As already stated, a state ω on $\mathfrak{W}$ is completely specified by its values on Weyl operators. The state on all of $\mathfrak{W}$ is then obtained by continuous extension.

Definition 3.2 (Gaussian state, [3, Sec. 2.2.1])**.** Let $(K_{\mathbb{R}}, \sigma)$ be a symplectic space and $b : K_{\mathbb{R}} \times K_{\mathbb{R}} \to \mathbb{R}$ be a positive symmetric bilinear form such that

$$\frac{1}{2}|\sigma(f,g)| \leq \sqrt{b(f,f)\, b(g,g)}\;, \qquad f, g \in K_{\mathbb{R}}\;. \tag{3.3}$$

The state ω_b on the Weyl algebra $\mathfrak{W}(K_{\mathbb{R}}, \sigma)$ defined by

$$\omega_b(W(f)) = \mathrm{e}^{-\frac{1}{2}b(f,f)}\;, \qquad f \in K_{\mathbb{R}}\;, \tag{3.4}$$

is called the Gaussian (or quasifree) state associated with b.

The Gelfand-Naimark-Segal (GNS) representation [82] of a Gaussian state can always be described in terms of a Fock space [3, Sec. 2.2.1].

[1] We follow the notation of [3].

Finally, we give some comments on representations of the Weyl algebra. The fact that the Weyl algebra $\mathfrak{W}$ is simple [85, Thm. 3.7(iv)] ensures that every (non-trivial) representation of $\mathfrak{W}$ is faithful [82]. Let $(\mathcal{H}, \pi)$ be a representation of $\mathfrak{W}$. Since $\mathfrak{W}$ is simple, π is an isometric *-isomorphism onto $\pi[\mathfrak{W}] \subset \mathfrak{B}(\mathcal{H})$ and the image algebra $\pi[\mathfrak{W}]$ is a C^*-subalgebra of $\mathfrak{B}(\mathcal{H})$. Since every representation is an isomorphism, we may study the Weyl algebra $\mathfrak{W}$ as a C^*-algebra in any (convenient) representation. The von Neumann algebra of $\mathfrak{W}$ generated by a representation $(\mathcal{H}, \pi)$ is given by $(\pi[\mathfrak{W}])''$, i.e., the double commutant of the image algebra, and it is denoted by $\mathcal{W}_\pi$. If $(\mathcal{H}, \pi)$ is an irreducible representation, the fact that $\mathfrak{W}$ is simple implies that $\pi[\mathfrak{W}] \subsetneq \mathfrak{B}(\mathcal{H})$. In particular, $\pi[\mathfrak{W}]$ does not contain compact operators. Furthermore, for any irreducible representation $(\mathcal{H}, \pi)$, $\mathcal{W}_\pi = \mathfrak{B}(\mathcal{H})$ [86, Thm. 32.6].

A representation of the Weyl algebra $\mathfrak{W}(K_\mathbb{R}, \sigma)$ is called regular if the operators $\pi(W(f))$ are strongly continuous in f. The following important Theorem states that all regular representations of the Weyl algebra of systems with finitely many degrees of freedom are unitarily equivalent.

Theorem 3.1 (Stone-von Neumann, [81, Thm. 3.2.2])**.** *Let $K_\mathbb{R}$ be finite dimensional. Then, all regular representations of the Weyl algebra $\mathfrak{W}(K_\mathbb{R}, \sigma)$ are unitarily equivalent.*

Importantly, the Stone-von Neumann uniqueness Theorem does *not* hold for systems with infinitely many degrees of freedom, such as quantum field theories. Physically, the existence of unitarily inequivalent representations in systems with infinitely many degrees of freedom is related to phenomena in thermodynamic systems, like spontaneous symmetry breaking and superselection sectors [45]. Another manifestation of the existence of unitarily inequivalent representations is Haag's Theorem [87], which states that the vacuum representation of a free and that of an interacting theory are not unitarily equivalent.

In the remainder of this Section, we introduce the vacuum representation of a non-interacting scalar Bose field. This representation is canonically given by a Fock space representation and may be interpreted as (the closure of) all finite particle states of the theory. We first introduce the unique Gaussian vacuum state of the free scalar field.

Definition 3.3 (Vacuum Representation, [3, Sec. 2.4.1] & [88, 89])**.** Let $K_\mathbb{R} = C_0^\infty(\mathbb{R}^d, \mathbb{R}) \times C_0^\infty(\mathbb{R}^d, \mathbb{R})$ be the real vector space of Cauchy data for the Klein-Gordon equation and equip it with the non-degenerate symplectic form σ given by

$$\sigma(f, g) := \int_{\mathbb{R}^d} (f_1(x)g_2(x) - f_2(x)g_1(x)) \, \mathrm{d}^d x \,, \qquad f, g \in K_\mathbb{R} \,. \tag{3.5}$$

Define the bilinear form $b_{\text{vac}} : K_\mathbb{R} \times K_\mathbb{R} \to \mathbb{R}$ by

$$b_{\text{vac}}(f, g) := \frac{1}{2}\Big(\big\langle f_1, D^{-1/2} g_1 \big\rangle + \big\langle f_2, D^{1/2} g_2 \big\rangle\Big) \,, \tag{3.6}$$

where D is the unique self-adjoint extension of $(-\triangle + m^2)|_{C_0^\infty}$ as an operator on $L^2(\mathbb{R}^d)$, see Appendix B. The Gaussian state ω_{vac} associated with b_{vac} is called the vacuum state of the

free scalar field. The GNS representation of ω_{vac} is called the vacuum representation of the free scalar field.

The vacuum representation of the free scalar field can be realized as a Schrödinger representation, which is also known as the "Q"-space representation [13, 14, 44, 69–71, 80, 88, 89]. More precisely, the Weyl algebra can be represented as an algebra of bounded operators acting on the complex Hilbert space $L^2(Q, \mu)$, where Q is some infinite dimensional space of field configurations, e.g., $\mathcal{S}'_\beta(\mathbb{R}^d)$, see Appendix C.1, and μ is a centred Gaussian measure with covariance operator $\hat{C} = (2\hat{\omega})^{-1}$. Here, $\hat{\omega}$ is the integral operator corresponding to the relativistic dispersion relation $\omega(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2}$. In this representation, the field operator acts as a multiplication operator and the conjugate momentum field operator acts as a suitably defined differential operator [88, 89]. The fields in the "Q"-space representation are contained in the Euclidean functional integral formalism introduced in Chapter 2 as time-zero fields, making the connection between the relativistic free scalar quantum field and the classical Euclidean statistical field theory manifest [44].

Part II

Information Theory of Gaussian Statistical Field Theory

This Part applies the concepts of information theory to Gaussian statistical field theory. In Chapter 4, we study the relative entropy between different states of a Gaussian scalar field theory, parametrized by different masses and boundary conditions, in a finite volume. We observe that the equivalence of two Gaussian measures describing these states – and thus the finiteness of the relative entropy – depends crucially on the dimension of space. In particular, we find that two distinct finite-volume states are mutually singular at and above the upper critical dimension $d = 4$ of Ising-like systems.

In Chapter 5, we discuss the mutual information of two scalar fields (interpreted as random processes) in two disjoint regions of space. We show that a necessary and sufficient condition for the finiteness of this mutual information (in all dimensions) is that there is a finite distance between the two regions. We then demonstrate (via the specific example of a one-dimensional field theory), that the mutual information shows expected behaviour. Finally, we argue that these results are due to the Markov property of the scalar field.

Disclaimer. This part of the thesis is based on [47], of which I am a co-author. Parts of this publication, namely those which were contributed by me, are partially taken verbatim from [47] in the following.

Interest in information-theoretical aspects of physical theories has grown rapidly in recent decades. Originally introduced to physics via the concept of entropy by Clausius and Boltzmann, information theory now plays a fundamental role in a wide variety of subfields of physics. Examples include classical statistical physics in and out of equilibrium [90–94], quantum mechanics, especially in connection with quantum computing, [95–100], the black hole information paradox [101] and quantum field theory, whose entanglement properties remain an area of active research [17–25, 29, 31, 102–108].

For the specific case of continuum quantum field theories, the computation (and even the definition) of entropies is typically complicated due to the difficulties arising from the infinite number of degrees of freedom. A suitable notion of entropy for continuum field theories is the relative entropy (or Kullback-Leibler divergence) [6], which quantifies the (information-theoretic) distinguishability of two states. Colloquially, since the relative entropy compares two states of the theory, the ultraviolet divergences present in both states cancel each other out, giving finite results even in the continuum, see e.g. [109, 110]. Relative entropies in relativistic quantum field theories can be rigorously defined using Tomita-Takesaki modular theory [2, 3, 45, 55, 56].

In this part of the thesis, we are concerned with the properties of relative entropy in the context of *classical* Gaussian statistical field theory[2]. Physically, this model can be interpreted either as a Gaussian approximation of a classical statistical system close to a second order phase transition or as the Euclidean version of the relativistic quantum field theory of a single massive scalar boson or as the infinite temperature limit of such a theory. In the functional integral formalism, this model is defined by a functional Gaussian probability measure and is thus a problem of classical probability theory. It is also the starting point for the construction of interacting theories [44, 69–71]. Recent works dealing with the information theory of Euclidean field theories include [47, 112, 113].

A statistical field theory shares an important property with quantum field theories, namely a continuum of degrees of freedom. Therefore, we need to use relative entropies to study the information-theoretic properties of this model. Besides physical considerations, Gaussian field theories are also interesting from a purely probability-theoretic point of view. In contrast to "ordinary" Gaussian measures on $\mathbb{R}^n$, which are specified by multivariate normal distributions, Gaussian measures on infinite dimensional spaces (which are needed in field theory because the number of degrees of freedom is infinite) are much more subtle to handle. In particular, the relative entropy between two Gaussian measures on infinite dimensional spaces can be infinite even if both measures are non-degenerate. One of the main challenges of this work will be to determine the conditions under which the relative entropy between two Gaussian field theories is finite.

Since we are working with relative entropies, we need to choose *two* probability measures that we want to compare with each other. As we are considering only free theories (i.e., Gaussian measures), our options for choosing these measures are limited. Different states

[2] In the mathematical literature, this model is known as the Gaussian free field, see [78, 111].

(or theories) with physical meaning can be obtained by choosing different boundary conditions, different masses, or a combination of both. In the main part of this paper we derive conditions under which the relative entropy between two Gaussian measures, corresponding to different choices of the above possibilities, is finite. We will see that the finiteness of the relative entropy between two field theories with different masses depends critically on the dimension d of Euclidean spacetime. In particular, we find that $d = 4$ is a critical dimension for the relative entropy between theories with different masses and the same (classical) boundary conditions[3], in the sense that it is infinite at and above this dimension. Furthermore, we demonstrate that the relative entropy between two field theories over a bounded region with different boundary conditions can, in general, be infinite in all dimensions. For the special case of the relative entropy between two field theories with Robin boundary conditions, we find that it is finite precisely when $d < 3$.

Another quantity we will consider is the mutual information between two disjoint regions of Euclidean spacetime. Interpreted as the average amount of information shared between the fields in each of the different regions, the mutual information provides an insight into the mutual dependence of these fields. Since mutual information takes into account non-linear relationships between random variables, it is generically more general than other correlation quantifiers that only consider linear relationships such as covariance. We will show that the properties of the mutual information between two disjoint regions are dictated by the Markov property of the scalar field [13, 14]. In particular, the mutual information is finite if and only if the regions are separated by a finite distance. Moreover, by explicitly calculating the mutual information between two intervals in $d = 1$, we show that only the degrees of freedom at the boundaries of the regions contribute to the mutual information, reinforcing the notion that the Markov property plays a crucial role in the study of mutual information [114]. It is worth noting that recent developments in the area of analogue quantum field simulators [115] may provide an opportunity to experimentally study the results regarding mutual information obtained in this work, see also the discussion in Part IV.

[3] In the remainder of this work, we call Dirichlet, Neumann, Robin and periodic boundary conditions "classical" boundary conditions, see Appendix B.

4 Relative Entropy in Gaussian Statistical Field Theory

This Chapter studies the relative entropy between different states of a statistical field theory. In Section 4.1, we consider two free scalar field theories on a bounded domain with different mass parameters m_1 and m_2 but the same classical boundary conditions. We show that the equivalence of two such field theories depends non-trivially on d, the dimension of Euclidean space. For Dirichlet, Neumann and periodic boundary conditions on a cubic region, we then calculate the relative entropy of theories with different masses.

In Section 4.2, we discuss the properties of the relative entropy between two field theories on a bounded domain with different boundary conditions. While, in general, two field theories with different boundary conditions can be mutually singular in all Euclidean spacetime dimensions, we show that the relative entropy between two Robin field theories is finite if $d \leq 2$. We then calculate the relative entropy between two such theories in one Euclidean spacetime dimension.

4.1 Field Theories with Different Masses

Let Ω be some bounded open subset of $\mathbb{R}^d$ with boundary $\partial\Omega$. Consider centred Gaussian measures of the form $\mu_i = \mathcal{N}(0, \hat{G}_i)$ (for definitions and notation, see Appendix C.1), where $\hat{G}_i = (-\triangle_{\mathrm{X}} + m_i^2)^{-1}$, $m_i > 0$, is an operator on $L^2(\Omega)$. Here, $-\triangle_{\mathrm{X}}$ is some self-adjoint extension of $-\triangle|_{C_0^\infty(\Omega)}$ with "X"-boundary conditions (see Appendix B). For two such field theories, there is a simple necessary and sufficient condition for equivalence in terms of the covariance operators.

Theorem 4.1. *Let $m_1, m_2 > 0$ be such that $m_1 \neq m_2$. The centred Gaussian measures $\mu_1 = \mathcal{N}(0, \hat{G}_1)$ and $\mu_2 = \mathcal{N}(0, \hat{G}_2)$, where $\hat{G}_i = (-\triangle_{\mathrm{X}} + m_i^2)^{-1}$, $i \in \{1, 2\}$, are equivalent if and only if $\hat{G}_2$ (and hence also $\hat{G}_1$) is a Hilbert-Schmidt operator on $L^2(\Omega)$.*

Proof. Obviously $\hat{G}_1^{-1}$ and $\hat{G}_2^{-1}$ have the same form domain, namely that of $-\triangle_{\mathrm{X}}$. As $\hat{G}_1^{-1}$ and $\hat{G}_2^{-1}$ are bounded from below by a positive number, by Theorem A.2, the ranges of the square roots of their inverses thus coincide, i.e., $\hat{G}_1^{1/2}[L^2(\Omega)] = \hat{G}_2^{1/2}[L^2(\Omega)]$. Let $\{\lambda_n\}_{n=1}^\infty$

be the sequence of eigenvalues of $-\triangle_{\mathrm{X}}$, enumerated in non-decreasing order, and define $\hat{B} \coloneqq \hat{G}_1^{-1/2}\hat{G}_2^{1/2}$. Then,

$$\|\hat{B}\hat{B}^* - I\|_{\mathrm{HS}}^2 = \sum_{n=1}^{\infty}\left(\frac{\lambda_n + m_1^2}{\lambda_n + m_2^2} - 1\right)^2 = \sum_{n=1}^{\infty}\left(\frac{m_1^2 - m_2^2}{\lambda_n + m_2^2}\right)^2 = (m_1^2 - m_2^2)^2\,\|\hat{G}_2\|_{\mathrm{HS}}^2\,. \tag{4.1}$$

Therefore, by Theorem C.10, $\mu_1 \sim \mu_2$ precisely when $\hat{G}_2 \in \mathrm{HS}(L^2(\Omega))$. ■

If the covariance operators $\hat{G}_1$ and $\hat{G}_2$ are Hilbert-Schmidt, by (C.29), the relative entropy between μ_1 and μ_2 takes the simple form

$$D_{\mathrm{KL}}(\mu_1\|\mu_2) = \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{m_1^2 - m_2^2}{\lambda_n + m_2^2} - \log\left(\frac{m_1^2 - m_2^2}{\lambda_n + m_2^2} + 1\right)\right]. \tag{4.2}$$

We can see that the above series is indeed convergent precisely when $\hat{G}_2$ is Hilbert-Schmidt. More specifically, define $x_n \coloneqq (m_1^2 - m_2^2)/(\lambda_n + m_2^2)$. As $n \to \infty$, $x_n \to 0$ and we can expand the logarithm around $x_n = 0$ for large n, i.e., $\log(x_n + 1) = x_n - \frac{1}{2}x_n^2 + \mathcal{O}(x_n^3)$. So for large n, the summands in the above series behave like $\frac{1}{2}x_n^2 + \mathcal{O}(x_n^3)$. We see that the additional term from the regularized Fredholm determinant (cf. Appendix C.3) exactly cancels the problematic term x_n, which would lead to a convergent series if and only if $\hat{G}_2$ was also of trace class. Below, when we consider the infinite volume relative entropy density, it will become clear that this additional term from the regularized Fredholm determinant plays a role similar to a mass counterterm.

We do not yet know when the covariance operators are Hilbert-Schmidt. Recall that the covariance operators considered here are the inverses of the differential operator $-\triangle+m^2$. Thus, we expect the asymptotic behaviour of the eigenvalues of $\hat{G}_1$ and $\hat{G}_2$ to depend on d, the dimension of Euclidean space. For the case where $-\triangle_{\mathrm{X}}$ is the Dirichlet Laplacian $-\triangle_{\mathrm{D}}$, this is made precise by Weyl's law, see [116, 117], [118, Sec. XIII.15] and [119, Sec. VI.4]. More precisely, let $\Omega \subset \mathbb{R}^d$ be a bounded domain with piecewise smooth boundary. Let $\{\lambda_n\}_{n=1}^{\infty}$ be the sequence of eigenvalues of the Dirichlet Laplacian $-\triangle_{\mathrm{D}}$, enumerated in non-decreasing order. Then, as n tends to infinity, the eigenvalues satisfy the asymptotic behaviour

$$\lambda_n \sim \text{const.} \times n^{2/d}\,, \tag{4.3}$$

where $\sim$ denotes asymptotic equivalence. Weyl's law generalizes to Neumann, Robin and periodic boundary conditions if the boundary of the region Ω is sufficiently regular [120], as well as to the case of free boundary conditions [121, 122].

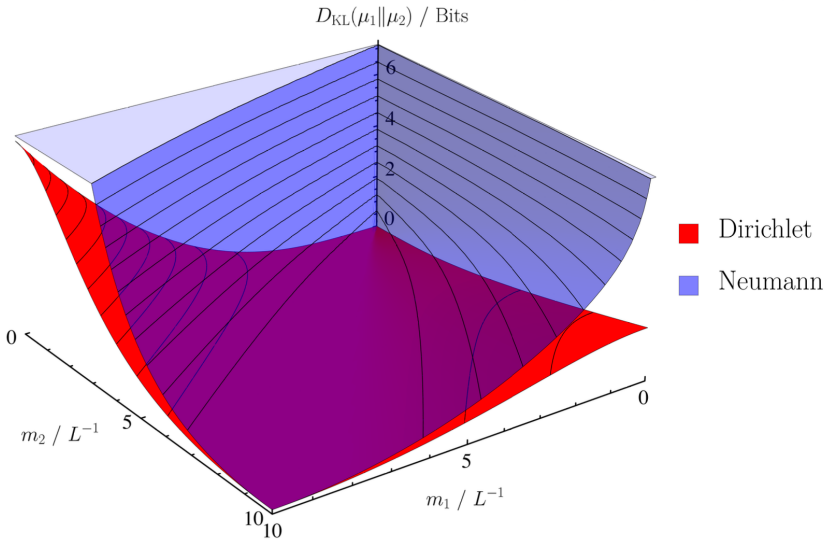


Figure 4.1: The relative entropy between two field theories with masses m_1 and m_2, respectively, on an interval of length L. We consider Dirichlet and Neumann boundary conditions and plot the relative entropy in bits against the masses in units of the inverse interval length L^{-1}. Note that in the limit $m_i \to 0$ the relative entropy is finite for Dirichlet boundary conditions but diverges for Neumann boundary conditions. This is due to the zero mode of the Laplacian that is present when we choose Neumann boundary conditions.

Weyl's law implies that the covariance operators $\hat{G}_{\mathrm{X}}$, $\mathrm{X} \in \{\mathrm{D}, \mathrm{N}, \mathrm{P}, \sigma, \mathrm{F}\}$, are Hilbert-Schmidt[1] in Euclidean spacetime dimensions $d < 4$. If $d = 1$, they are also of trace class. For $d \geq 4$, two "X"-boundary condition field theories with different masses are therefore mutually singular. In particular, for $d = 4$, the Hilbert-Schmidt norm diverges logarithmically. We recall that $d = 4$ is exactly the upper critical dimension of a scalar field theory. It is no coincidence that the relative entropy is infinite at and above the upper critical dimension, as can be seen from the discussion of the relative entropy density at the end of this Section.

In the following, we restrict ourselves to the case where Ω is an open d-cube of edge length L. In this case, the Dirichlet, Neumann and periodic covariance operators ($\hat{G}_{\mathrm{D}}$, $\hat{G}_{\mathrm{N}}$ and $\hat{G}_{\mathrm{P}}$, respectively) are Hilbert-Schmidt precisely when $d < 4$. We start with the case $d = 1$, i.e., we consider a field theory on an interval of length L. The eigenvalues of the Dirichlet Laplacian $-\triangle_{\mathrm{D}}$ on an interval of length L are given by $(n\pi)^2/L^2$, $n \in \mathbb{N}$. Using the series representation of the log-gamma function [124] together with standard techniques for evaluating Matsubara sums of free massive propagators [125], we obtain the following closed-form expression for

[1] In particular, this means that the corresponding Green's function is square integrable in the sense that [123, Thm. VI.23]

$$\int_\Omega \int_\Omega (G_{\mathrm{X}}(\boldsymbol{x}, \boldsymbol{y}))^2 \, \mathrm{d}^d x \, \mathrm{d}^d y < +\infty \,, \tag{4.4}$$

for $d < 4$, where $G_{\mathrm{F}} = G$.

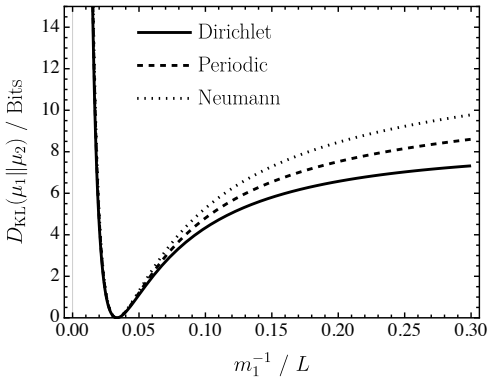


Figure 4.2: The relative entropy between two field theories with masses m_1 and $m_2 = 30L^{-1}$, respectively, on an interval of length L. We consider Dirichlet, periodic and Neumann boundary conditions and plot the relative entropy in units of bits against the inverse mass (or correlation length) m_1^{-1} in units of the interval length L. We observe the ordering $D_{\mathrm{KL}}^{\mathrm{D}} \leq D_{\mathrm{KL}}^{\mathrm{P}} \leq D_{\mathrm{KL}}^{\mathrm{N}}$, with equality only for coinciding masses.

the relative entropy between two Dirichlet field theories with different masses on an interval of length L,

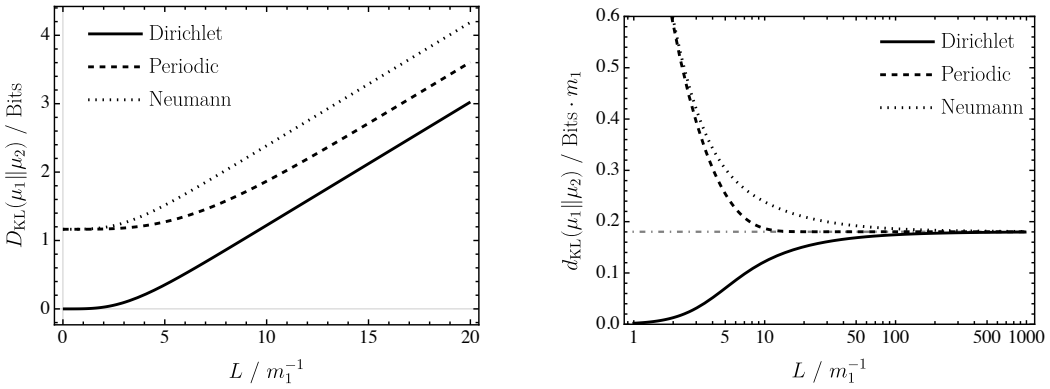
$$D_{\mathrm{KL}}^{\mathrm{D}}(\mu_1\|\mu_2) = \frac{1}{4}\left[\left(1 - \frac{m_1^2}{m_2^2}\right) + \frac{L(m_1^2 - m_2^2)}{m_2 \tanh(Lm_2)} - 2\log\left(\frac{m_2 \sinh(Lm_1)}{m_1 \sinh(Lm_2)}\right)\right] . \tag{4.5}$$

We now discuss the properties of this quantity. For a fixed system size L, the relative entropy (4.5) increases as we increase the absolute value of the mass difference $|m_1 - m_2|$ and is zero precisely when $m_1 = m_2$. This is consistent with our interpretation of the relative entropy as a measure of distinguishability: The greater the difference in masses, the more "different" the corresponding field theories are, and the better we can distinguish between them. If the masses are the same, then the theories are the same (remember that we have chosen the same boundary conditions for both fields) and there is no way to distinguish between them. So the relative entropy should be zero in this case.

The expression in (4.5) can easily be adapted to other typical boundary conditions. The eigenvalues of the Neumann Laplacian $-\triangle_{\mathrm{N}}$ on an interval of length L are given by $(n\pi)^2/L^2$, $n \in \mathbb{N}_0$. Recalling the series representation of the relative entropy (4.2), the Neumann relative entropy then reads

$$D_{\mathrm{KL}}^{\mathrm{N}}(\mu_1\|\mu_2) = \frac{1}{2}\left[\frac{m_1^2}{m_2^2} - \log\left(\frac{m_1^2}{m_2^2}\right) - 1\right] + D_{\mathrm{KL}}^{\mathrm{D}}(\mu_1\|\mu_2) . \tag{4.6}$$

The function $f(x) = x - \log x - 1$ is non-negative for positive x and zero precisely at $x = 1$. Thus, for $m_1 \neq m_2$, the Neumann relative entropy is strictly larger than the Dirichlet relative entropy, cf. Fig. 4.1 and 4.2.



(a) (b)

Figure 4.3: (**a**) The relative entropy between two field theories with masses m_1 and $m_2 = \frac{1}{2}m_1$, respectively, on an interval of length L for three different boundary conditions. The relative entropy in units of bits is plotted against the system size L in units of the correlation length or inverse mass m_1^{-1}. We see that as soon as the system size is larger than the largest correlation length (in this case, $m_2^{-1} = 2m_1^{-1}$), the relative entropy scales linearly with L. If L is smaller than both correlation lengths, the relative entropy is close to zero for Dirichlet boundary conditions and attains a constant value for Neumann and periodic boundary conditions. (**b**) The relative entropy density as a function of the system size L. We see that in the infinite volume limit $L \to +\infty$, the relative entropy density converges for all three boundary conditions to the limit given in (4.8), represented in the Figure as a grey dash-dotted line.

The eigenvalues of the Laplacian $-\triangle_{\mathrm{P}}$ with periodic boundary conditions on an interval of length L are given by $(2n\pi)^2/L^2$, $n \in \mathbb{N}$, each of multiplicity two, together with the smallest eigenvalue 0 of multiplicity one. The relative entropy for periodic boundary conditions then reads

$$D_{\mathrm{KL}}^{\mathrm{P}}(\mu_1\|\mu_2) = \frac{1}{2}\left[\frac{m_1^2}{m_2^2} - \log\left(\frac{m_1^2}{m_2^2}\right) - 1\right] + 2D_{\mathrm{KL}}^{\mathrm{D}}(\mu_1\|\mu_2)\Big|_{L\to L/2} . \tag{4.7}$$

Here, $D_{\mathrm{KL}}^{\mathrm{D}}(\mu_1\|\mu_2)|_{L\to L/2}$ denotes the Dirichlet relative entropy given in (4.5) but with L replaced by $L/2$. As shown in Fig. 4.2, we observe the ordering $D_{\mathrm{KL}}^{\mathrm{D}} \leq D_{\mathrm{KL}}^{\mathrm{P}} \leq D_{\mathrm{KL}}^{\mathrm{N}}$, where equality holds only for equal masses.

Instead of changing the masses for some fixed system size, we can also keep the masses fixed and vary L. For large L, (4.5) scales linearly in L, which is reminiscent of the extensive behaviour of an entropy. Thus, the distinguishability is large when both length scales set by the inverse masses are small compared to the size of the system. As shown in Fig. 4.3a, this is true for all three boundary conditions considered here. Conversely, if both inverse masses are large compared to the size of the system, distinguishability is low. As can be seen from (4.6) and (4.7), the additional L-independent term from the zero eigenvalue of $-\triangle_{\mathrm{N}}$ and $-\triangle_{\mathrm{P}}$ causes the Neumann and periodic relative entropies to attain a constant value in the limit $L \to 0$, while the Dirichlet relative entropy vanishes as the system size approaches zero. This difference in the behaviour is due to the absence of a zero mode of the Dirichlet

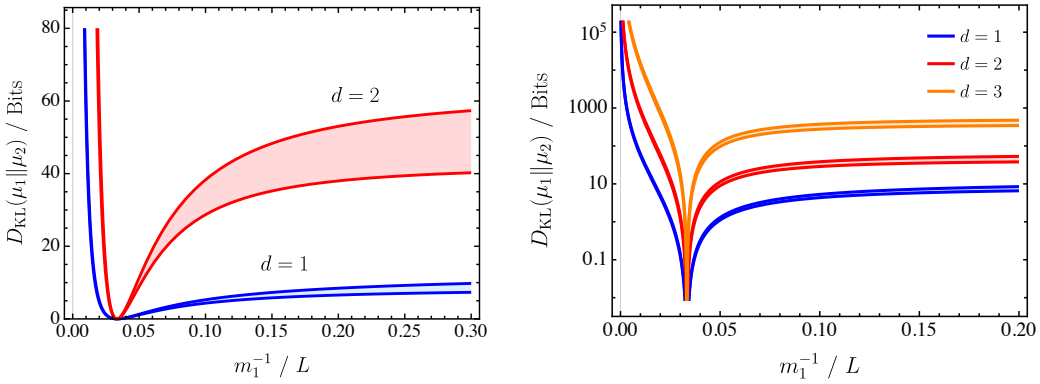


Figure 4.4: (**a**) The relative entropy between two field theories with masses m_1 and $m_2 = 30L^{-1}$ on a d-cube of edge length L in dimensions $d = 1$ (blue) and $d = 2$ (red). We plot the relative entropy in units of bits against the correlation length or inverse mass m_1^{-1} in units of the edge length L. The lower (upper) bound of each shaded region represents the Dirichlet (Neumann) relative entropy. We observe that for a fixed edge length L the relative entropy is larger in higher dimensions. (**b**) Logarithmic plot of the relative entropy to include the case $d = 3$ (orange). We use the same parameters as in (a).

Laplacian. More precisely, the limit $L \to 0$ corresponds to the limit $m_1 \to 0$ and the value of the relative entropy in this limit is determined by the zero mode of the Laplacian.

We conclude the discussion of the case $d = 1$ by considering a relative entropy density, cf. Fig. 4.3b. We define the relative entropy density as $d^{\mathrm{X}}_{\mathrm{KL}}(\mu_1\|\mu_2) := L^{-1}D^{\mathrm{X}}_{\mathrm{KL}}(\mu_1\|\mu_2)$, where $\mathrm{X} \in \{\mathrm{D}, \mathrm{P}, \mathrm{N}\}$. In the infinite volume limit, i.e., in the limit $L \to +\infty$, we observe that the relative entropy density converges and the limit is independent of the boundary conditions considered here. More precisely, for all $\mathrm{X} \in \{\mathrm{D}, \mathrm{P}, \mathrm{N}\}$, the infinite volume limit of the relative entropy density is given by

$$\lim_{L\to+\infty} d^{\mathrm{X}}_{\mathrm{KL}}(\mu_1\|\mu_2) = \frac{(m_1 - m_2)^2}{4m_2} \,. \tag{4.8}$$

We now study field theories with different mass parameters in $d \geq 2$. Recall that we choose Ω to be an open d-cube of edge length L. In this case, the eigenvalues of the Laplacians $-\triangle_{\mathrm{D}}$, $-\triangle_{\mathrm{N}}$ and $-\triangle_{\mathrm{P}}$ in $d \geq 2$ can straight-forwardly obtained from the case $d = 1$ discussed above. Unlike for $d = 1$, we are not able to obtain a closed-form expression for the Dirichlet relative entropy as we did in (4.5). Instead, we have to approximate the series (4.2) numerically. For the relative entropy between two fields with different masses on a cubic region in dimensions $d \leq 3$, we observe that the relative entropy in dimensions $d = 2$ and $d = 3$ shows the same qualitative behaviour as in $d = 1$, cf. Fig. 4.4. In particular, the Neumann relative entropy is strictly greater (except at coinciding masses) than the Dirichlet relative entropy. Furthermore, we see that for a fixed edge length L the relative entropy is larger in higher dimensions.

We continue with a calculation of the infinite volume relative entropy density. Let Ω again be the open d-cube of edge length L in $\mathbb{R}^d$, $d < 4$. Using (4.2), we can write the Dirichlet relative entropy between two fields with masses m_1 and m_2, respectively, over Ω as

$$D^{\mathrm{D}}_{\mathrm{KL}}(\mu_1\|\mu_2) = \frac{1}{2}\sum_{\boldsymbol{n}\in\mathbb{N}^d}\left[\frac{m_1^2-m_2^2}{\frac{(|\boldsymbol{n}|\pi)^2}{L^2}+m_2^2} - \log\left(\frac{m_1^2-m_2^2}{\frac{(|\boldsymbol{n}|\pi)^2}{L^2}+m_2^2}+1\right)\right], \tag{4.9}$$

where $\boldsymbol{n} = (n_1,\dots,n_d)$ and $|\boldsymbol{n}|^2 = n_1^2+\ldots+n_d^2$. Upon defining $\frac{\Delta k}{2\pi} \coloneqq L^{-1}$ and replacing the summation over $\mathbb{N}^d$ by a summation over $\mathbb{Z}^d$, the Dirichlet relative entropy density $d^{\mathrm{D}}_{\mathrm{KL}}(\mu_1\|\mu_2) = L^{-d}D^{\mathrm{D}}_{\mathrm{KL}}(\mu_1\|\mu_2)$ reads

$$d^{\mathrm{D}}_{\mathrm{KL}}(\mu_1\|\mu_2) = \frac{1}{2^{d+1}}\sum_{\boldsymbol{n}\in\mathbb{Z}^d}\frac{(\Delta k)^d}{(2\pi)^d}\left[\frac{m_1^2-m_2^2}{\frac{(\Delta k|\boldsymbol{n}|)^2}{4}+m_2^2} - \log\left(\frac{m_1^2-m_2^2}{\frac{(\Delta k|\boldsymbol{n}|)^2}{4}+m_2^2}+1\right)\right] + \dots, \tag{4.10}$$

where the ellipsis denotes terms that take into account summands where one or more $n_i = 0$. These terms vanish in the limit $L \to +\infty$, so we will omit them in the remainder of the calculation. Since the infinite volume limit of the relative entropy density is independent of the choice of boundary conditions, we define $d_{\mathrm{KL}}(\mu_1\|\mu_2) = \lim_{L\to+\infty} d^{\mathrm{D}}_{\mathrm{KL}}(\mu_1\|\mu_2)$ to be *the* (infinite volume) relative entropy density. Since the function appearing in the sum (4.10) is Riemann integrable, we can take the limit $L \to +\infty$, which corresponds to $\Delta k \to 0$, and arrive at the convergent improper integral

$$d_{\mathrm{KL}}(\mu_1\|\mu_2) = \frac{1}{2}\int_{\mathbb{R}^d}\frac{\mathrm{d}^d p}{(2\pi)^d}\left[\frac{m_1^2-m_2^2}{|\boldsymbol{p}|^2+m_2^2} - \log\left(\frac{m_1^2-m_2^2}{|\boldsymbol{p}|^2+m_2^2}+1\right)\right], \tag{4.11}$$

where we made the substitution $p_i \to 2p_i$ to get rid of the factor $\frac{1}{4}$ in the denominator. Using d-dimensional spherical coordinates, the relative entropy density can be written as

$$d_{\mathrm{KL}}(\mu_1\|\mu_2) = \frac{\pi^{-\frac{d}{2}}}{2^d\,\Gamma(d/2)}\int_0^\infty q^{d-1}\left[\frac{m_1^2-m_2^2}{q^2+m_2^2} - \log\left(\frac{m_1^2-m_2^2}{q^2+m_2^2}+1\right)\right]\mathrm{d}q\,. \tag{4.12}$$

We can evaluate this integral by performing an integration by parts. More precisely, for $d < 4$,

$$d_{\mathrm{KL}}(\mu_1\|\mu_2) = \frac{\pi^{1-\frac{d}{2}}}{d\,2^{d+1}\Gamma(d/2)}\,\frac{dm_1^2m_2^{d-2}+2(m_2^d-m_1^d)-dm_2^d}{\sin\left(\frac{d\pi}{2}\right)}\,. \tag{4.13}$$

For integer dimensions relevant in this work, the infinite volume relative entropy density reads

$$d = 1: \qquad d_{\mathrm{KL}}(\mu_1\|\mu_2) = \frac{(m_1 - m_2)^2}{4m_2}\,, \tag{4.14}$$

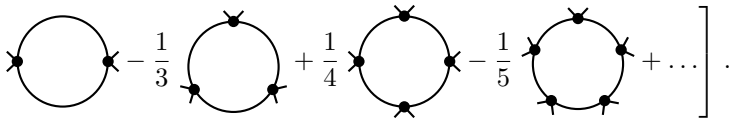
$$d = 2: \qquad d_{\mathrm{KL}}(\mu_1\|\mu_2) = \frac{1}{8\pi}\left(m_2^2 - m_1^2 + m_1^2 \log\frac{m_1^2}{m_2^2}\right), \tag{4.15}$$

$$d = 3: \qquad d_{\mathrm{KL}}(\mu_1\|\mu_2) = \frac{(2m_1 + m_2)(m_1 - m_2)^2}{24\pi}\,. \tag{4.16}$$

Note that the result for $d = 1$ coincides with result obtained previously.

Finally, we discuss the dependence of the relative entropy density on the dimension of Euclidean spacetime. We define $g \coloneqq m_1^2 - m_2^2$ and interpret it as a coupling constant. For $m_1^2 < 2m_2^2$, we can expand the logarithm in (4.11), yielding the expansion

$$\begin{aligned} d_{\mathrm{KL}}(\mu_1\|\mu_2) &= \frac{1}{2}\int_{\mathbb{R}^d} \frac{\mathrm{d}^d p}{(2\pi)^d}\left[\frac{1}{2}\frac{g^2}{(|\boldsymbol{p}|^2 + m_2^2)^2} - \frac{1}{3}\frac{g^3}{(|\boldsymbol{p}|^2 + m_2^2)^3} + \frac{1}{4}\frac{g^4}{(|\boldsymbol{p}|^2 + m_2^2)^4} - \dots\right] \\ &= \frac{1}{2}\left[\frac{1}{2}\;-\;\frac{1}{3}\;+\;\frac{1}{4}\;-\;\frac{1}{5}\;+\dots\right]. \end{aligned} \tag{4.17}$$



In the diagrammatic expression in the second line, each vertex contributes a factor g and all external momenta are set to zero. Note that the leading (order g^2) diagram,  , is divergent for $d \geq 4$, reflecting the crucial dependence of the relative entropy on the Euclidean spacetime dimension.

We note that (4.17) is structurally equivalent to the renormalized one loop effective potential of a scalar field theory with quartic self-interaction in two or three Euclidean spacetime dimensions, see, e.g., [126, Sec. 5.3.3]. In particular, the *regularized* Fredholm determinant in the expression for the relative entropy between two Gaussian measures, (C.29), provides precisely the mass counterterm that cancels the divergent diagram  . However, for $d \geq 4$, the diagram  is also divergent and we would need an additional counterterm which the regularized Fredholm determinant does not provide. This is the origin of the divergence of the relative entropy (density) in dimensions $d \geq 4$.

Finally, we comment briefly on the limit where one of the fields becomes massless. For Dirichlet boundary conditions in $d < 4$, the limit $m_i \to 0$ gives a finite result, while for Neumann and periodic boundary conditions this limit does not exist, which is due to the zero mode of the Neumann and periodic Laplacians. For the relative entropy densities given in (4.14) to (4.16) we observe that the limit $m_2 \to 0$ exists only for $d = 3$ or, more precisely, for $2 < d < 4$, as can be seen from (4.13). This is of course due to the well-known infrared divergence of massless theories in $d \leq 2$. Note that the limit $m_1 \to 0$ yields a finite result for all $d < 4$.

4.2 Field Theories with Different Boundary Conditions

In this Section, we study the relative entropy between two fields over a bounded region with different boundary conditions. For simplicity, we assume that both fields have the same mass, and note that different masses can be incorporated using the results of Section 4.1. In general, two field theories with different boundary conditions can be mutually singular even if they have the same mass parameter. As a specific example of two such fields, we consider the relative entropy between a Dirichlet and a Neumann field. We then show that the relative entropy between two Robin fields (which includes the case where one field is a Neumann field) is finite only in Euclidean spacetime dimensions $d < 3$.

Recall from Theorem C.10 that a necessary condition for the equivalence of two centred Gaussian measures $\mu_1 = \mathcal{N}(0, \hat{C}_1)$ and $\mu_2 = \mathcal{N}(0, \hat{C}_2)$ (and hence for the finiteness of the relative entropy between them) is that the form domains of the precision operators $\hat{C}_i^{-1}$ coincide. For the case where $\mu_1 = \mu_\mathrm{D}$ is the Dirichlet field and $\mu_2 = \mu_\mathrm{N}$ is the Neumann field, this boils down to whether the form domains of $-\triangle_\mathrm{D}$ and $-\triangle_\mathrm{N}$ are equal. Let Ω be a bounded Lipschitz region in $\mathbb{R}^d$. The form domains of the Dirichlet and Neumann Laplacians are given by[2] $\mathcal{Q}(-\triangle_\mathrm{D}) = H_0^{+1}(\Omega)$ and $\mathcal{Q}(-\triangle_\mathrm{N}) = H^{+1}(\Omega)$, cf. [118, Ch. XIII]. It can be shown that $H_0^{+1}(\Omega) \subset H^{+1}(\Omega)$ [118, p. 253]. In general, however, $H_0^{+1}(\Omega)$ is a proper subset of $H^{+1}(\Omega)$. For example, if Ω is C^0, then $H_0^{+1}(\Omega) \subsetneq H^{+1}(\Omega)$ [127, Cor. 3.29(vii)]. For the specific case $d = 1$ and $\Omega = (a, b)$, a bounded open interval in $\mathbb{R}$, $H^{+1}((a, b))$ consists of all absolutely continuous functions on (a, b) whose distributional derivatives are in $L^2((a, b))$, while $H_0^{+1}((a, b))$ consists precisely of those functions in $H^{+1}((a, b))$ whose continuous extensions to $[a, b]$ vanish at the endpoints of the interval. Thus, a non-zero constant function is in $H^{+1}((a, b))$ but not in $H_0^{+1}((a, b))$.

The above example shows that a Neumann field and a Dirichlet field are mutually singular in all Euclidean spacetime dimensions. We now study the relative entropy between a Neumann and a Robin field. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Recall from Chapter B that the Neumann and Robin forms are defined by

$$\mathfrak{q}_\mathrm{N}(f, g) = \int_\Omega \left(\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + m^2 f g\right) \mathrm{d}^d x \,, \tag{4.18}$$

$$\mathfrak{q}_\sigma(f, g) = \int_\Omega \left(\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + m^2 f g\right) \mathrm{d}^d x + \int_{\partial\Omega} \sigma f g \, \mathrm{d}S \,, \tag{4.19}$$

where σ is a positive[3] continuous function on $\partial\Omega$. The Robin form can be written as $\mathfrak{q}_\sigma = \mathfrak{q}_\mathrm{N} + \mathfrak{b}_\sigma$, where $\mathfrak{b}_\sigma$ is the form

$$\mathfrak{b}_\sigma(f, g) = \int_{\partial\Omega} \sigma(\boldsymbol{x}) f(\boldsymbol{x}) g(\boldsymbol{x}) \, \mathrm{d}S(\boldsymbol{x}) \,. \tag{4.20}$$

[2] For definitions, see Appendix A.2.

[3] If we choose $\sigma \geq 0$, then $\mathfrak{q}_\sigma$ is strictly positive and we avoid complications from negative eigenvalues of $-\triangle_\sigma$, see [128, Sec. 1.1.20] and the discussion of the one-dimensional case below.

It is shown in [128, p. 34] that the boundary form $\mathfrak{b}_\sigma$ is infinitesimally relatively form bounded with respect to the Neumann form $\mathfrak{q}_\mathrm{N}$. This means that the Neumann and Robin form domains coincide (cf. [129, Thm. VI.1.33]), and in particular $\mathcal{Q}(-\triangle_\mathrm{N}) = \mathcal{Q}(-\triangle_\sigma) = H^{+1}(\Omega)$. Clearly $\mathfrak{b}_\sigma$ is symmetric, densely defined and positive. However, it is not closable[4].

As usual, we use the notation $\hat{G}_\mathrm{N} = (-\triangle_\mathrm{N} + m^2)^{-1}$ and $\hat{G}_\sigma = (-\triangle_\sigma + m^2)^{-1}$. Unlike the case of two fields with different masses but the same (local) boundary conditions, studied in Section 4.1, the covariance operators $\hat{G}_\mathrm{N}$ and $\hat{G}_\sigma$ have no common eigenbasis. Therefore, we cannot use the simple condition for $\hat{B}\hat{B}^* - I$ to be a Hilbert-Schmidt operator on $L^2(\Omega)$, which we used in Theorem 4.1. From another point of view, the additional boundary term in the Robin form (4.19) prevents us from simply adding and subtracting the precision operators $\hat{G}_\mathrm{N}^{-1}$ and $\hat{G}_\sigma^{-1}$. However, following an idea from [130], we can add and subtract their corresponding bilinear forms. Using results from Appendix C, we see that two field theories over Ω with Neumann and Robin boundary conditions, respectively, are equivalent precisely when

$$\hat{B}^*\hat{B} - I = \left(\hat{G}_\sigma^{-1/2}\hat{G}_\mathrm{N}^{1/2}\right)^*\left(\hat{G}_\sigma^{-1/2}\hat{G}_\mathrm{N}^{1/2}\right) - I \tag{4.21}$$

is a Hilbert-Schmidt operator on $L^2(\Omega)$. This is equivalent to the requirement that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\widetilde{\Delta\mathfrak{q}}(e_n, f_m)\right)^2 < +\infty\,, \tag{4.22}$$

where $\{e_n\}_{n=1}^\infty$ and $\{f_m\}_{m=1}^\infty$ are any two orthonormal bases in $L^2(\Omega)$ and $\widetilde{\Delta\mathfrak{q}}$ is the bounded bilinear form on $L^2(\Omega)$ defined by

$$\widetilde{\Delta\mathfrak{q}}(f,g) \coloneqq \mathfrak{b}_\sigma\left(\hat{G}_\mathrm{N}^{1/2}f, \hat{G}_\mathrm{N}^{1/2}g\right) = \langle f, (\hat{B}^*\hat{B} - I)\, g\rangle_{L^2(\Omega)}\,. \tag{4.23}$$

Let $R_\mathrm{N}(\boldsymbol{x},\boldsymbol{y})$ be the distributional kernel of $\hat{G}_\mathrm{N}^{1/2}$. We conclude from (4.23) that the operator $\hat{B}^*\hat{B} - I$ has a kernel Ψ_σ given by

$$\Psi_\sigma(\boldsymbol{x},\boldsymbol{y}) = \int_{\partial\Omega}\sigma(\boldsymbol{z})\, R_\mathrm{N}(\boldsymbol{x},\boldsymbol{z})R_\mathrm{N}(\boldsymbol{z},\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{z})\,. \tag{4.24}$$

Since $\hat{B}^*\hat{B} - I$ is an operator on $L^2(\Omega)$, it is Hilbert-Schmidt if and only if it has a kernel that is square integrable. A quick calculation shows that this requirement can be written as

$$\iint_\Omega |\Psi_\sigma(\boldsymbol{x},\boldsymbol{y})|^2\,\mathrm{d}^d x\,\mathrm{d}^d y = \iint_{\partial\Omega}\sigma(\boldsymbol{z})\sigma(\boldsymbol{z}')|G_\mathrm{N}(\boldsymbol{z},\boldsymbol{z}')|^2\,\mathrm{d}S(\boldsymbol{z})\,\mathrm{d}S(\boldsymbol{z}') < +\infty\,, \tag{4.25}$$

i.e., a Neumann and a Robin field are equivalent (in the sense of measures) precisely when the Green's functions are square integrable on the boundary. This is certainly the case in

[4] To see this, note that $\mathfrak{b}_\sigma$ is a generalization of the form considered in [129, Ex. VI.1.26], and we can use the same arguments as there.

$d = 1$, since then the Green's functions are continuous on the diagonal and the integral over the boundary becomes a sum over a finite number of boundary points.

For $d > 1$, we assume that $\partial\Omega$ is sufficiently regular, e.g. C^1. We can then decide whether the Green's function is square integrable by checking whether the singularity at $\boldsymbol{z} = \boldsymbol{z}'$ is square integrable. Since the behaviour of the Green's function for small distances $|\boldsymbol{x} - \boldsymbol{y}|$ is a UV property, it should not depend on the choice of boundary conditions, and we can use (B.6), the small distance behaviour of the fundamental solution, to estimate the singularity (see also [80, Lem. III.2]). In particular, for $|\boldsymbol{x} - \boldsymbol{y}| \ll m^{-1}$,

$$G_{\mathrm{N}}(\boldsymbol{x}, \boldsymbol{y}) \sim \begin{cases} \log \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} & \text{for } d = 2 \\ \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^{d-2}} & \text{for } d > 2 \end{cases} . \tag{4.26}$$

Fix $\boldsymbol{z}' \in \partial\Omega$. Choose the coordinate system in $\mathbb{R}^d$ such that $\boldsymbol{z}' = \mathbf{0}$ and $\partial\Omega$ is flat near $\boldsymbol{z}'$ and lying in the plane $y_d = 0$[5]. Then, the integral of $(G_{\mathrm{N}})^2$ near $\boldsymbol{z} = \boldsymbol{z}' = \mathbf{0}$, e.g., over a ball $\mathbb{B}_\varepsilon$ with radius $\varepsilon \ll m^{-1}$, is essentially given by

$$\int_{\mathbb{B}_\varepsilon} \frac{\mathrm{d}^{d-1} z}{|\boldsymbol{z}|^{2d-4}} \sim \int_0^\varepsilon \frac{\mathrm{d}r}{r^{d-2}} \tag{4.27}$$

This integral is finite for $d < 3$ and divergent otherwise. In particular, the integral diverges logarithmically for $d = 3$.

We conclude that a Neumann and a Robin field with equal masses over a bounded region Ω are equivalent in the sense of measures (and thus the relative entropy between them is finite) only in dimensions $d < 3$. Note that this result easily generalizes to the case of two Robin fields (in particular, a Neumann field is a Robin field for the special choice $\sigma \equiv 0$). Note that the case where one of the fields is a Dirichlet field can be formally incorporated by choosing $\sigma(\boldsymbol{x}) \equiv \sigma_* > 0$ and then taking the limit $\sigma_* \to +\infty$. Then the condition in (4.25) is violated in all dimensions, which is consistent with our conclusion from the discussion of the Dirichlet and Neumann form domains above.

We now give a concrete calculation of the relative entropy between two Robin fields in $d = 1$. As already mentioned (and also discussed in [69]), the relative entropy between two such theories is finite. Consider two field theories on an interval with equal masses but different Robin boundary conditions of the form

$$\frac{\partial f}{\partial n} = -\sigma f , \qquad \sigma \geq 0 . \tag{4.28}$$

In particular, this class of boundary conditions includes Neumann ($\sigma = 0$), free ($\sigma = m$) and Dirichlet ($\sigma = +\infty$) boundary conditions.

[5] If $\partial\Omega$ is not flat near $\boldsymbol{z}'$, we can "flatten out" $\partial\Omega$ in the vicinity of $\boldsymbol{z}'$ using a continuously differentiable map, cf. [131, Sec. 5.4].

In the following, let $\Omega = (0, L)$. The Radon-Nikodym derivative of the field theory μ_0 with free boundary conditions with respect to a field theory μ_σ with boundary conditions (4.28) can be shown to be (cf. [69, Thm. II.31], [80, p. 263])

$$\frac{\mathrm{d}\mu_0}{\mathrm{d}\mu_\sigma}(\varphi) = \mathrm{e}^{\frac{\sigma-m}{2}(\varphi(0)^2+\varphi(L)^2)} \int \mathrm{e}^{-\frac{\sigma-m}{2}(\varphi(0)^2+\varphi(L)^2)} \, \mathrm{d}\mu_0(\varphi) \, . \tag{4.29}$$

The normalization constant on the right-hand side can be computed, using the change of variables formula [132, Eq. (0.1)], to wit

$$\begin{aligned} \int \mathrm{e}^{-\frac{\sigma-m}{2}(\varphi(0)^2+\varphi(L)^2)} \, \mathrm{d}\mu_0(\varphi) &= \frac{1}{2\pi\sqrt{\det\Sigma}} \int_{\mathbb{R}^2} \mathrm{e}^{-\frac{\sigma-m}{2}\|\boldsymbol{x}\|^2} \, \mathrm{e}^{-\frac{1}{2}\boldsymbol{x}^\mathsf{T}\Sigma^{-1}\boldsymbol{x}} \, \mathrm{d}^2x \\ &= \frac{2m}{\sqrt{(\sigma+m)^2 - \mathrm{e}^{-2mL}(\sigma-m)^2}} \, , \end{aligned} \tag{4.30}$$

where Σ and $\widetilde{\Sigma}$ are 2×2 matrices given by

$$\Sigma = \begin{pmatrix} G(0,0;m) & G(0,L;m) \\ G(L,0;m) & G(L,L;m) \end{pmatrix} = \frac{1}{2m} \begin{pmatrix} 1 & \mathrm{e}^{-mL} \\ \mathrm{e}^{-mL} & 1 \end{pmatrix} , \tag{4.31}$$

and $\widetilde{\Sigma} = \left(\Sigma^{-1} + (\sigma - m)\mathbb{1}_2\right)^{-1}$.

Given the density in (4.29), it is straightforward to calculate the relative entropy between μ_0 and μ_σ,

$$\begin{aligned} D_{\mathrm{KL}}(\mu_0 \| \mu_\sigma) &= \int \log\left[\frac{2m}{\sqrt{(\sigma+m)^2 - \mathrm{e}^{-2mL}(\sigma-m)^2}} \, \mathrm{e}^{\frac{\sigma-m}{2}(\varphi(0)^2+\varphi(L)^2)} \right] \mathrm{d}\mu_0(\varphi) \\ &= \frac{1}{2}\left[\log\left(\frac{4m^2}{(\sigma+m)^2 - \mathrm{e}^{-2mL}(\sigma-m)^2} \right) + \frac{\sigma}{m} - 1 \right] . \end{aligned} \tag{4.32}$$

We plot $D_{\mathrm{KL}}(\mu_0 \| \mu_\sigma)$ in Fig. 4.5 for $L = 1$. We see that $D_{\mathrm{KL}}(\mu_0 \| \mu_\sigma)$ vanishes precisely when $\sigma = m$. It diverges when we take the limit $\sigma \to +\infty$, which is again consistent with our conclusion from the discussion of the Dirichlet and Neumann form domains. Note that the expression in (4.32) diverges in the limit

$$\sigma \to -m \tanh\left(\frac{mL}{2}\right) < 0 \tag{4.33}$$

from above. This is due to the fact that for this value of σ the operator $-\triangle_\sigma + m^2$ gets a zero eigenvalue. More precisely, for $\sigma < 0$, the Robin Laplacian $-\triangle_\sigma$ has at least one negative eigenvalue, with the smallest eigenvalue $\lambda_1(\sigma) = -\mu^2$ satisfying $\lambda(\sigma) < -|\sigma|^2$, where μ is the smallest positive solution of [128, Sec. 1.1.20]

$$\tanh(\mu L) - \frac{2|\sigma|\mu}{\mu^2 + \sigma^2} = 0 \, . \tag{4.34}$$

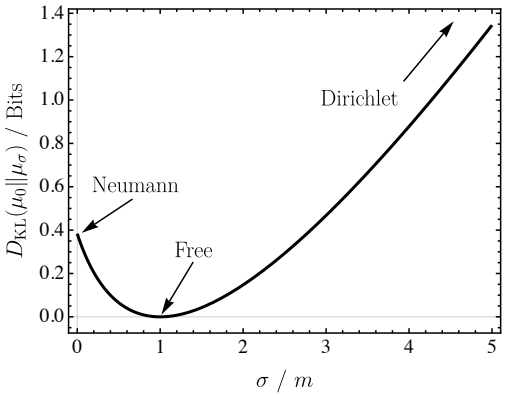


Figure 4.5: The relative entropy between a one-dimensional field theory with free boundary conditions and a field theory with σ-boundary conditions given in (4.28), both with mass m, on an interval $\Omega = (0, 1)$. As indicated by the arrows, the special values $\sigma = 0$, $\sigma = m$ and $\sigma = +\infty$ correspond to Neumann, free and Dirichlet boundary conditions, respectively. The relative entropy scales linearly in σ for large σ, indicating that a σ-boundary condition field is mutually singular to a Dirichlet field.

If σ is set to the value in (4.33), then $\mu = m$ is the smallest solution to the above equation and therefore the operator $-\triangle_\sigma + m^2$ is no longer strictly positive for $\sigma \leq -m \tanh\left(\frac{mL}{2}\right)$. This explains the divergence of (4.32) for too small values of σ.

5 Mutual Information in Gaussian Statistical Field Theory

In this Chapter, we discuss a special kind of relative entropy, the mutual information between two disjoint regions Ω_A and Ω_B. We show that the mutual information between two bounded regions is always finite if the these regions are separated by a finite distance. Furthermore, we give an example where the mutual information is infinite if the regions touch, i.e., if the separation distance is zero. We argue that the reason for this behaviour is the Markov property of a scalar Euclidean field theory [13, 14, 69, 70, 80].

In the following, let Ω_A and Ω_B be disjoint open C^0 subsets of $\mathbb{R}^d$[1]. Furthermore, let $L^2(\Omega_A \cup \Omega_B)$ be the Hilbert space of square-integrable functions on $\Omega_A \cup \Omega_B$. Then, $L^2(\Omega_A \cup \Omega_B) \simeq L^2(\Omega_A) \oplus L^2(\Omega_B)$ where the isomorphism is given by the map $f \mapsto (f_A, f_B)$, where f_A (f_B) is the restriction of f to Ω_A (Ω_B). We define the mutual information between the regions Ω_A and Ω_B to be the relative entropy

$$I(\Omega_A : \Omega_B) \coloneqq S(\mu_{AB} \| \mu_A \otimes \mu_B) \, . \tag{5.1}$$

In the above expression, μ_{AB} is the free scalar field theory of mass $m > 0$ (and vanishing mean) over the region $\Omega_A \cup \Omega_B$ with free boundary conditions. Similarly, μ_A and μ_B are the free scalar field theories with the same mass m (and vanishing means) over the regions Ω_A and Ω_B, respectively, and $\mu_A \otimes \mu_B$ is the product measure.

These field theories can be interpreted as follows. The measure μ_{AB} describes a theory reduced to the region $\Omega_A \cup \Omega_B$, containing correlations between all spacetime points in $\Omega_A \cup \Omega_B$. In contrast, the field theory $\mu_A \otimes \mu_B$ only describes correlations between spacetime points within each regions, but it does not contain any cross-correlations between the regions. More precisely, if $\boldsymbol{x}_A \in \Omega_A$ and $\boldsymbol{x}_B \in \Omega_B$, then $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ are correlated in the theory μ_{AB} but uncorrelated in the theory $\mu_A \otimes \mu_B$. In other words, the field φ_A in Ω_A and the field φ_B in Ω_B are independent Gaussian random variables in the theory $\mu_A \otimes \mu_B$. The mutual information $I(\Omega_A : \Omega_B)$ can thus be interpreted as the distinguishability of a theory containing all cross-correlations and a theory containing no cross-correlations at all. We proceed by discussing the covariance operators of these two field theories.

[1] We say an open subset $\Omega \subset \mathbb{R}^d$ is C^0 (i.e., continuous) if its boundary $\partial\Omega$ can locally be represented by the graph of a continuous function from $\mathbb{R}^{d-1}$ to $\mathbb{R}$, see [127, Sec. 1.1] or [133, Def. 1.2.1.1]. This is a rather mild regularity assumption and holds for most "usual" regions like spheres and rectangles. Note that the union of the touching rectangles considered at the end of this section is *not* a C^0 subset.

The covariance operator of the centred Gaussian measure μ_{AB}, denoted $\hat{G}_0$, is given by

$$(\hat{G}_0 f)(\boldsymbol{x}) = \int_{\Omega_A \cup \Omega_B} G(\boldsymbol{x}, \boldsymbol{y}; m) f(\boldsymbol{y}) \, \mathrm{d}^n y \,, \qquad f \in L^2(\Omega_A \cup \Omega_B) \,, \tag{5.2}$$

where G is the fundamental solution of $-\triangle + m^2$ given in (B.2). Thus, we can interpret μ_{AB} as the free scalar field over $\Omega_A \cup \Omega_B$ with free boundary conditions (see Appendix B). Recalling that $L^2(\Omega_A \cup \Omega_B) \simeq L^2(\Omega_A) \oplus L^2(\Omega_B)$, we can represent $\hat{G}_0$ as an operator on $L^2(\Omega_A) \oplus L^2(\Omega_B)$ by the matrix

$$\hat{G}_0 = \begin{pmatrix} \hat{G}_A & \hat{G}_{AB} \\ \hat{G}_{AB}^* & \hat{G}_B \end{pmatrix} , \tag{5.3}$$

where the operators $\hat{G}_A : L^2(\Omega_A) \to L^2(\Omega_A)$, $\hat{G}_B : L^2(\Omega_B) \to L^2(\Omega_B)$ and $\hat{G}_{AB} : L^2(\Omega_B) \to L^2(\Omega_A)$ are defined as

$$f_A \mapsto (\hat{G}_A f_A)(\boldsymbol{x}) \;\; = \int_{\Omega_A} G(\boldsymbol{x}, \boldsymbol{y}; m) f_A(\boldsymbol{y}) \, \mathrm{d}^n y \,, \qquad \boldsymbol{x} \in \Omega_A \,, \tag{5.4}$$

$$f_B \mapsto (\hat{G}_B f_B)(\boldsymbol{x}) \;\; = \int_{\Omega_B} G(\boldsymbol{x}, \boldsymbol{y}; m) f_B(\boldsymbol{y}) \, \mathrm{d}^n y \,, \qquad \boldsymbol{x} \in \Omega_B \,, \tag{5.5}$$

$$f_B \mapsto (\hat{G}_{AB} f_B)(\boldsymbol{x}) = \int_{\Omega_B} G(\boldsymbol{x}, \boldsymbol{y}; m) f_B(\boldsymbol{y}) \, \mathrm{d}^n y \,, \qquad \boldsymbol{x} \in \Omega_A \,. \tag{5.6}$$

The corresponding covariance reads

$$\begin{aligned} \mathrm{Cov}(f, g) &= \langle f, \hat{G}_0 g \rangle_{L^2(\Omega_A \cup \Omega_B)} \\ &= \langle f_A, \hat{G}_A g_A \rangle_{L^2(\Omega_A)} + \langle f_B, \hat{G}_B g_B \rangle_{L^2(\Omega_B)} + \langle f_A, \hat{G}_{AB} g_B \rangle_{L^2(\Omega_A)} \\ &\quad + \langle f_B, \hat{G}_{AB}^* g_A \rangle_{L^2(\Omega_B)} \,. \end{aligned} \tag{5.7}$$

From the above expression we can see that the operator $\hat{G}_{AB}$, together with its adjoint $\hat{G}_{AB}^*$, describes correlations across the two regions Ω_A and Ω_B. Therefore, we call $\hat{G}_{AB}$ the cross-covariance operator of μ_{AB} [134, 135].

For the product measure $\mu_A \otimes \mu_B$, the covariance operator $\hat{G}_\otimes$ is given by

$$(\hat{G}_\otimes f)(\boldsymbol{x}) = \chi_{\Omega_A}(\boldsymbol{x}) \int_{\Omega_A} G(\boldsymbol{x}, \boldsymbol{y}; m) f_A(\boldsymbol{y}) \, \mathrm{d}^n y + \chi_{\Omega_B}(\boldsymbol{x}) \int_{\Omega_B} G(\boldsymbol{x}, \boldsymbol{y}; m) f_B(\boldsymbol{y}) \, \mathrm{d}^n y \,, \tag{5.8}$$

where χ_{Ω_i} is the indicator function on Ω_i. We can represent $\hat{G}_\otimes$ as an operator on $L^2(\Omega_A) \oplus L^2(\Omega_B)$ by the matrix

$$\hat{G}_\otimes = \begin{pmatrix} \hat{G}_A & 0 \\ 0 & \hat{G}_B \end{pmatrix} . \tag{5.9}$$

By construction $\mu_A \otimes \mu_B$ contains no cross-correlations between the regions Ω_A and Ω_B.

From the definition of the mutual information in (5.1) and the discussion in Chapter 1, we see that $I(\Omega_A : \Omega_B)$ is finite precisely when $\mu_{AB} \sim \mu_A \otimes \mu_B$. In order to find conditions when this is the case, we first need some auxiliary results from the theory of Sobolev spaces, most of which are obtained in [69, 80].

Recall that D is the unique self-adjoint extension of $(-\triangle + m^2)|_{C_0^\infty(\mathbb{R}^d)}$. The Hilbert-Sobolev space $H^{\pm 1}(\mathbb{R}^d)$ of order ± 1 is the closure of $\mathscr{S}(\mathbb{R}^d)$ in the inner product[2] $\langle f, g\rangle_{\pm 1} = \langle f, D^{\pm 1} g\rangle_{L^2}$. We recall the scale of Hilbert spaces

$$H^{+1}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset H^{-1}(\mathbb{R}^d) \,. \tag{5.10}$$

In particular, every $\varphi \in H^{-1}(\mathbb{R}^d)$ can be interpreted as a tempered distribution, i.e., an element in $\mathcal{S}^*$. For any closed subset K of $\mathbb{R}^d$, the space of distributions with support in K,

$$H_K^{-1} = \big\{\varphi \in H^{-1}(\mathbb{R}^d) \;:\; \operatorname{supp}\varphi \subseteq K\big\} \tag{5.11}$$

is a closed subspace of $H^{-1}(\mathbb{R}^d)$. We denote by e_K the orthogonal projection from $H^{-1}(\mathbb{R}^d)$ onto H_K^{-1}.

Let $\Omega \subset \mathbb{R}^d$ be an open subset, $H^{-1}(\Omega)$ the Hilbert-Sobolev space as defined in Appendix A.2 and define $p_\Omega := I - e_{\mathbb{R}^d\setminus\Omega}$, the projection onto $(H^{-1}_{\mathbb{R}^d\setminus\Omega})^\perp$. This is essentially a projection to distributions with support in Ω.

Lemma 5.1 [69, Lem. II.24]**.** *Let $-\triangle_{\mathrm{D}}$ be the Dirichlet Laplacian on $L^2(\Omega)$. For every $f \in H^{-1}(\Omega) \supset L^2(\Omega)$,*

$$(-\triangle_{\mathrm{D}} + m^2)^{-1} f = \hat{G}_{\mathrm{D}} f = D^{-1} p_\Omega f \,. \tag{5.12}$$

Upon recalling that the inner product in $H^{-1}(\Omega)$ is given by $\langle f, g\rangle_{H^{-1}(\Omega)} = \langle p_\Omega f, p_\Omega g\rangle_{-1}$ and using [127, Cor. 3.29(ii)], we infer that $H^{-1}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the inner product

$$\langle f, g\rangle_{H^{-1}(\Omega)} = \langle f, \hat{G}_{\mathrm{D}} g\rangle_{L^2(\Omega)} \,, \qquad f, g \in C_0^\infty(\Omega) \,. \tag{5.13}$$

The proof that $\mu_{AB} \sim \mu_A \otimes \mu_B$ if the regions are separated relies on the following

Lemma 5.2 [69, Lem. II.35], [71, Thm. III.16]**.** *Let Ω_1 and Ω_2 be open subsets in $\mathbb{R}^d$ and denote their closures by Λ_1 and Λ_2, respectively. Suppose that Ω_1 is bounded and Ω_1 and Ω_2 are separated by a finite distance. Then, the operator $\alpha = e_{\Lambda_1} e_{\Lambda_2} e_{\Lambda_1}$ is of trace class on $H^{-1}(\mathbb{R}^d)$ and $\|\alpha\| < 1$. Moreover, if f is an eigenvector of α such that the corresponding eigenvalue is nonzero, then $f \in H^{-1}(\partial\Omega_1)$.*

The following Lemma is useful for showing that the form domains of two precision operators coincide.

[2] Note that these inner products differ from those used in the definition of Sobolev spaces in Appendix A.2 due to the general mass term. Nevertheless, the norms induced by these inner products and those used in Appendix A.2 are equivalent.

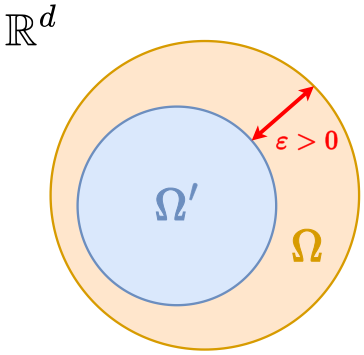


Figure 5.1: Sketch of regions considered in Theorem 5.4. The bounded open sets Ω and Ω' are chosen such that $\overline{\Omega'}$ is a subset of Ω and $\mathrm{dist}(\Omega', \partial\Omega) > 0$. Two field theories with free and Dirichlet boundary conditions on $\partial\Omega$, respectively, will, in general, not be equivalent. However, they are equivalent if we restrict them to the smaller region Ω'.

Lemma 5.3 [136, Prop. B.1]**.** *Let T_1 and T_2 be two bounded self-adjoint operators on a separable Hilbert space $\mathcal{H}$ with norm $\|.\|$. The ranges of T_1 and T_2 coincide, i.e., $T_1[\mathcal{H}] = T_2[\mathcal{H}]$, if and only if there exist constants $\gamma, \Gamma > 0$, $\gamma \leq \Gamma$, such that $\gamma\|T_1 f\| \leq \|T_2 f\| \leq \Gamma\|T_1 f\|$ for all $f \in \mathcal{H}$.*

The following Theorem states that while a Dirichlet field theory and a field theory with free boundary conditions over a bounded region Ω are generally mutually singular, if we restrict both field theories to a smaller subregion Ω', then they are in fact equivalent, see also Fig. 5.1.

Theorem 5.4 [69, Thm. II.34]**.** *Let Ω and Ω' be bounded open regions in $\mathbb{R}^d$ such that $\Omega' \subset \Omega$ and $\mathrm{dist}(\Omega', \partial\Omega) > 0$. Let $\mu = \mathcal{N}(0, \hat{G}_0)$ and $\nu = \mathcal{N}(0, \hat{G}_{\mathrm{D}}^{\Omega})$ be centred Gaussian measures over Ω', where $\hat{G}_{\mathrm{D}}^{\Omega}$ is the integral operator on $L^2(\Omega')$ whose integral kernel is the restriction to $\Omega' \times \Omega'$ of the Dirichlet Green's function on $\Omega \times \Omega$. Then the measures μ and ν are equivalent, which implies that the relative entropy between them is finite.*

Remark. Note that this result also holds when we consider boundary conditions other than Dirichlet [80, Thm. II.2]. Specifically, it also holds for Neumann boundary conditions [80, Thm. III.6].

Proof. Let $\Lambda^{\mathrm{ext}} = \mathbb{R}^d \setminus \Omega$. For all $f, g \in L^2(\Omega')$, $\langle f, \hat{G}_0 g\rangle_{L^2(\Omega')} = \langle f, g\rangle_{-1}$, and, by Lemma 5.1, $\langle f, \hat{G}_{\mathrm{D}}^{\Omega} g\rangle = \langle f, p_\Omega g\rangle_{-1}$. Denote the closure $\overline{\Omega'}$ by Λ'. Using the self-adjointness of $e_{\Lambda'}$ with respect to $\langle ., .\rangle_{-1}$ and the fact that $e_{\Lambda'}$ acts as the identity on $H^{-1}_{\Lambda'} = \widetilde{H}^{-1}(\Omega')$, we can furthermore write

$$\langle f, p_\Omega g\rangle_{-1} = \langle f, Pg\rangle_{-1} \ , \qquad f, g \in H^{-1}_{\Lambda'} \supset L^2(\Omega) \ , \tag{5.14}$$

where we defined $P \coloneqq I - \alpha$ and $\alpha \coloneqq e_{\Lambda'} e_{\Lambda^{\mathrm{ext}}} e_{\Lambda'}$. By Lemma 5.2, $\alpha \in \mathrm{HS}(H^{-1}_{\Lambda'})$ (see also [69, p. 169] and [71, p. 217]) and $\|\alpha\| < 1$. In particular, this means that $P = I - \alpha$ is

strictly positive and boundedly invertible. Furthermore, P is self-adjoint as an operator on $H^{-1}(\mathbb{R}^d)$, and it has a unique positive square root $P^{1/2}$.

By Lemma B.1, $\langle f, \hat{G}_{\mathrm{D}}^{\Omega} f\rangle_{L^2(\Omega')} \leq \langle f, \hat{G}_0 f\rangle_{L^2(\Omega')}$ for all $f \in L^2(\Omega')$. Furthermore,

$$\langle f, \hat{G}_0 f\rangle_{L^2(\Omega')} = \langle f, f\rangle_{-1} \leq \|P^{-1}\| \, \langle f, Pf\rangle_{-1} = \|P^{-1}\| \, \langle f, \hat{G}_{\mathrm{D}}^{\Omega} f\rangle_{L^2(\Omega')} \ . \tag{5.15}$$

Thus, by Lemma 5.3, $\hat{G}_0^{1/2}[L^2(\Omega')] = (\hat{G}_{\mathrm{D}}^{\Omega})^{1/2}[L^2(\Omega')]$.

We define $\hat{B} \coloneqq \hat{G}_0^{-1/2}(\hat{G}_{\mathrm{D}}^{\Omega})^{1/2}$ on $L^2(\Omega')$. Let $\{\phi_n\}_{n=1}^{\infty}$ be an eigenbasis of $\hat{G}_0$ in $L^2(\Omega')$. Then, $\{\psi_n\}_{n=1}^{\infty}$ where $\psi_n \coloneqq \hat{G}_0^{-1/2}\phi_n$ is an orthonormal basis in $H_{\Lambda'}^{-1} = \widetilde{H}^{-1}(\Omega')$. Then, as $\alpha \in \mathrm{HS}(H_{\Lambda'}^{-1})$,

$$\|\hat{B}\hat{B}^* - I\|_{\mathrm{HS}(L^2(\Omega'))}^2 = \sum_{n=1}^{\infty} \|(P-I)\psi_n\|_{-1}^2 = \|\alpha\|_{\mathrm{HS}\left(H_{\Lambda'}^{-1}\right)}^2 < +\infty \ . \tag{5.16}$$

Therefore, by Theorem C.10, $\mathcal{N}(0, \hat{G}_0) \sim \mathcal{N}(0, \hat{G}_{\mathrm{D}}^{\Omega})$. ■

We will use Theorem 5.4 to show that the mutual information between separated regions is finite. Another result that will turn out to be useful is that a Dirichlet field over $\Omega_A \cup \Omega_B$ (with Ω_A and Ω_B separated) factorizes into a product measure. We first need the following

Proposition 5.5 [118, Prop. XIII.3]**.** *Let Ω_A and Ω_B be disjoint open subsets of $\mathbb{R}^d$. Then, the Dirichlet Laplacian $-\triangle_{\mathrm{D}}^{\Omega_A \cup \Omega_B}$ on $L^2(\Omega_A \cup \Omega_B)$ can be represented as an operator on $L^2(\Omega_A) \oplus L^2(\Omega_B)$ as*

$$-\triangle_{\mathrm{D}}^{\Omega_A \cup \Omega_B} = -\triangle_{\mathrm{D}}^{\Omega_A} \oplus -\triangle_{\mathrm{D}}^{\Omega_B} \ , \tag{5.17}$$

where $-\triangle_{\mathrm{D}}^{\Omega_i}$ is the Dirichlet Laplacian on $L^2(\Omega_i)$, $i \in \{A, B\}$.

From the above Proposition, we see that the Dirichlet covariance operator on $L^2(\Omega_A \cup \Omega_B)$ can be represented as an operator on $L^2(\Omega_A) \oplus L^2(\Omega_B)$ as

$$\hat{G}_{\mathrm{D}}^{\Omega_A \cup \Omega_B} = \hat{G}_{\mathrm{D}}^{\Omega_A} \oplus \hat{G}_{\mathrm{D}}^{\Omega_B} \ . \tag{5.18}$$

Notice that a Dirichlet field theory over $\Omega_A \cup \Omega_B$ therefore contains no cross-correlations between the regions Ω_A and Ω_B. Intuitively, we can think of Dirichlet boundary conditions as separating a bounded open region from the interior of its complement. The boundary $\partial\Omega$ acts as a "wall" between Ω and $\mathrm{int}(\mathbb{R}^d \setminus \Omega)$ as the fields are fixed to be zero on $\partial\Omega$. In other words, Dirichlet boundary conditions on $\partial\Omega$ cause the field in Ω to decouple from the field in $\mathrm{int}(\mathbb{R}^d \setminus \Omega)$ [69, p. 120]. Thus, there is no correlation between a point in Ω_A and a point in Ω_B as the field in Ω_A cannot influence the field in Ω_B and vice versa. This should be compared to free boundary conditions. As seen from (B.28), free boundary conditions are non-local and in particular, the value of the normal derivative at a point on the boundary is given by a surface integral over $\partial(\Omega_A \cup \Omega_B)$. For example, if Ω_A and Ω_B are separated by a finite distance, and we consider $-\triangle + m^2$ on $\Omega_A \cup \Omega_B$ with free boundary conditions, then

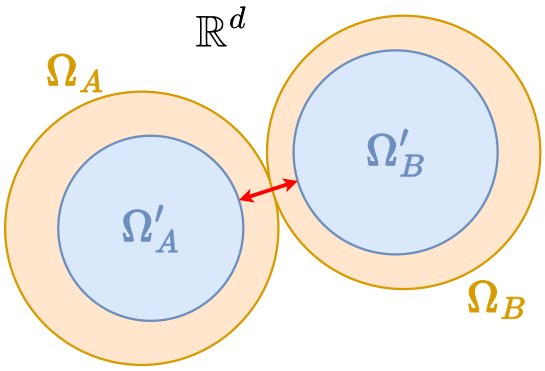


Figure 5.2: Sketch of regions considered in Theorem 5.6. The open subsets Ω'_A and Ω'_B are separated by a finite distance, indicated by the red double arrow. Thus, there exist two disjoint (possibly touching) open sets Ω_A and Ω_B such that $\operatorname{dist}(\Omega'_A, \partial\Omega_A) > 0$ and $\operatorname{dist}(\Omega'_B, \partial\Omega_B) > 0$. For such Ω'_A and Ω'_B, the field theory μ_{AB} with free boundary conditions over $\Omega'_A \cup \Omega'_B$, containing all cross-correlations between Ω'_A and Ω'_B, is equivalent to the theory $\mu_A \otimes \mu_B$. Therefore, the mutual information $I(\Omega'_A : \Omega'_B)$ for two such regions is finite.

the value of $\partial f/\partial n$ at a point $\boldsymbol{x} \in \partial\Omega_A$ depends on the value of f on the *whole* boundary $\partial(\Omega_A \cup \Omega_B)$. This is due to the fact that the kernel k in (B.28) is supported everywhere on the boundary, see also [80]. Therefore, for a field theory with free boundary conditions, the two regions Ω_A and Ω_B "communicate" via the non-local boundary conditions. Hence, the inverse of $-\triangle+m^2$ with free boundary conditions, $\hat{G}_0$, cannot be written as $\hat{G}_A \oplus \hat{G}_B$ as is the case for Dirichlet boundary conditions. Rather, $\hat{G}_0$ also contains components describing cross-correlations, see (5.3).

We can now show that μ_{AB} and $\mu_A \otimes \mu_B$ are equivalent if Ω_A and Ω_B are separated by a finite distance.

Theorem 5.6. *Let Ω'_A and Ω'_B be disjoint bounded open subsets of $\mathbb{R}^d$ such that $\operatorname{dist}(\Omega'_A, \Omega'_B) > 0$. Then, the measures μ_{AB} and $\mu_A \otimes \mu_B$ are equivalent.*

Proof. As $\operatorname{dist}(\Omega'_A, \Omega'_B) > 0$, there exist disjoint bounded open subsets Ω_A and Ω_B of $\mathbb{R}^d$ such that $\operatorname{dist}(\Omega'_A, \partial\Omega_A) > 0$ and $\operatorname{dist}(\Omega'_B, \partial\Omega_B) > 0$, cf. Figure 5.2. Let μ^{D}_{AB} the field theory over $\Omega'_A \cup \Omega'_B$ that is the restriction of the Dirichlet theory over $\Omega_A \cup \Omega_B$ in the sense described in Theorem 5.4. By Theorem 5.4, $\mu_{AB} \sim \mu^{\mathrm{D}}_{AB}$.

By Proposition 5.5, $\hat{G}^{\Omega_A \cup \Omega_B}_{\mathrm{D}} = \hat{G}^{\Omega_A}_{\mathrm{D}} \oplus \hat{G}^{\Omega_B}_{\mathrm{D}}$ and hence $\mu^{\mathrm{D}}_{AB} = \mu^{\mathrm{D}}_A \otimes \mu^{\mathrm{D}}_B$. By employing Theorem 5.4 again, we see that $\mu_A \sim \mu^{\mathrm{D}}_A$ and $\mu_B \sim \mu^{\mathrm{D}}_B$, which implies that $\mu_A \otimes \mu_B \sim \mu^{\mathrm{D}}_A \otimes \mu^{\mathrm{D}}_B$. Thus $\mu_{AB} \sim \mu^{\mathrm{D}}_{AB} = \mu^{\mathrm{D}}_A \otimes \mu^{\mathrm{D}}_B \sim \mu_A \otimes \mu_B$ and therefore, by transitivity, $\mu_{AB} \sim \mu_A \otimes \mu_B$. ∎

From the above Theorem we see that $\operatorname{dist}(\Omega_A, \Omega_B) > 0$ is a sufficient condition for the mutual information $I(\Omega_A : \Omega_B)$ to be finite. We now present an example where two regions Ω_A and Ω_B touch, i.e., where the separation distance is zero, and where the corresponding measures are mutually singular.

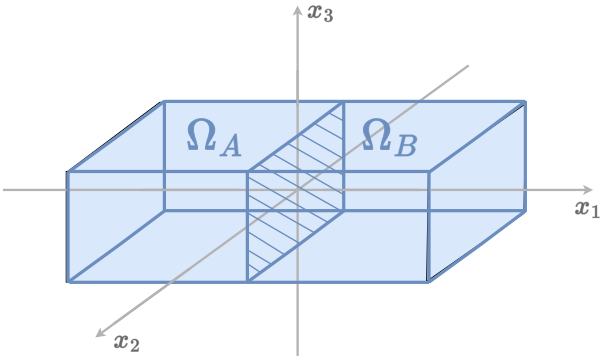


Figure 5.3: Sketch of touching rectangular regions in $d = 3$. The two open rectangles Ω_A and Ω_B touch in the $x_1 = 0$ plane. Notice that for this configuration $\partial(\Omega_A \cup \Omega_B) \neq \partial(\overline{\Omega_A \cup \Omega_B})$ as the rectangle in the $x_1 = 0$ plane (hatched region) is a subset of $\partial(\Omega_A \cup \Omega_B)$ but not of $\partial(\overline{\Omega_A \cup \Omega_B})$. As discussed in the main text, one can think of Ω_A and Ω_B "sharing" this part of the boundary. Due to the Markov property of free scalar fields, the information on this surface is equal to the information in the whole region and thus the mutual information $I(\Omega_A : \Omega_B)$ is infinite.

Let $\Omega_A, \Omega_B \subset \mathbb{R}^d$ be touching open rectangles as sketched in Fig. 5.3 for $d = 3$. We denote by Ω_{AB} the interior of $\overline{\Omega_A \cup \Omega_B}$. Notice that $\partial(\Omega_A \cup \Omega_B) \neq \partial\Omega_{AB}$. Let μ_{AB} be the scalar field theory of mass $m > 0$ with free boundary conditions over $\Omega_A \cup \Omega_B$. Recall that the reproducing kernel Hilbert space (RKHS) (cf. Appendix C.2) of μ_{AB} is defined as the closure of $C_0^\infty(\Omega_A \cup \Omega_B)$ in $\|.\|_{-1}$-norm, which is just the Hilbert-Sobolev space $\widetilde{H}^{-1}(\Omega_A \cup \Omega_B)$, for definitions see [137, Ch. 3] and [127] as well as Appendix A.2 and Tab. C.1. For the specific open sets Ω_A and Ω_B considered here, namely the touching open rectangles, we furthermore have

$$\widetilde{H}^{-1}(\Omega_{AB}) = H^{-1}(\overline{\Omega_A \cup \Omega_B}) = \widetilde{H}^{-1}(\Omega_A \cup \Omega_B) \,. \tag{5.19}$$

The first equality follows from [137, Thm. 3.29(ii)] and the fact that $\overline{\Omega_{AB}} = \overline{\Omega_A \cup \Omega_B}$ and the second equality follows from [127, Lem. 3.17(v)] using [127, Lem. 3.10(vii)]. The equality $\widetilde{H}^{-1}(\Omega_{AB}) = \widetilde{H}^{-1}(\Omega_A \cup \Omega_B)$ implies that we may equivalently think of μ_{AB} as the field theory with open boundary conditions over Ω_{AB} rather than over $\Omega_A \cup \Omega_B$. Informally speaking, we conclude that it does not matter whether or not we include $\Gamma = \Omega_{AB} \setminus (\Omega_A \cup \Omega_B)$ (the hatched region in Fig. 5.3) in the definition of the field theory.

The Cameron-Martin space of μ_{AB}, $\mathsf{H}_{\mu_{AB}}$, is given by $\hat{G}_0[\widetilde{H}^{-1}(\Omega_{AB})]$, cf. [138, Thm. 3.2.3]. By noticing that we can write $\hat{G}_0 = r_{\Omega_{AB}} \circ D^{-1}$, where $r_{\Omega_{AB}}$ denotes the restriction of a function in $H^{+1}(\mathbb{R}^d)$ to Ω_{AB}, [127, Lem. 3.2 & Eq. (19)] imply that $\mathsf{H}_{\mu_{AB}} = H^{+1}(\Omega_{AB})$, see also [127, Thm. 3.12(iii)]. As $\mathsf{H}_{\mu_{AB}}$ and the form domain of $\hat{G}_0^{-1}$ coincide as sets (cf. Lemma C.5), we have $\mathcal{Q}(\hat{G}_0^{-1}) = H^{+1}(\Omega_{AB})$.

On the other hand, using an analogous argument as above, the form domain of the precision operator $\hat{G}_\otimes^{-1} = \hat{G}_A^{-1} \oplus \hat{G}_B^{-1}$ of the product measure $\mu_A \otimes \mu_B$ is given by

$$\mathcal{Q}(\hat{G}_\otimes^{-1}) = \mathcal{Q}(\hat{G}_A^{-1}) \oplus \mathcal{Q}(\hat{G}_B^{-1}) = H^{+1}(\Omega_A) \oplus H^{+1}(\Omega_B) = H^{+1}(\Omega_A \cup \Omega_B) \,, \tag{5.20}$$

where we used [118, p. 268]. Suppose for a moment that Ω_A and Ω_B do not touch, i.e., there is a finite distance $\varepsilon > 0$ between these rectangles. Then, $\Omega_{AB} = \Omega_A \cup \Omega_B$ and $\mathcal{Q}(\hat{G}_0^{-1}) = \mathcal{Q}(\hat{G}_\otimes^{-1})$, which is a necessary condition for the equivalence of μ_{AB} and $\mu_A \otimes \mu_B$, cf. Theorem C.10. This is of course in accordance with the result obtained in Theorem 5.6, which implies that $\mathcal{Q}(\hat{G}_0^{-1}) = \mathcal{Q}(\hat{G}_\otimes^{-1})$ if Ω_A and Ω_B are separated. If Ω_A and Ω_B do touch (in the sense described above), however, then $\Omega_{AB} \neq \Omega_A \cup \Omega_B$ and we need to check whether $H^{+1}(\Omega_{AB})$ and $H^{+1}(\Omega_A \cup \Omega_B)$ coincide as sets.

We recall that for Ω an open subset of $\mathbb{R}^d$, every $f \in H^{+1}(\Omega)$ is absolutely continuous on almost all straight lines that are parallel to the coordinate axes, cf. [139, Thm. 1.1.3/1 & p. 7]. Consider the function that is zero on Ω_A and one on Ω_B. This function is in $H^1(\Omega_A \cup \Omega_B)$ but not in $H^{+1}(\Omega_{AB})$, as it is not continuous along the x_1-direction. Therefore, $H^{+1}(\Omega_{AB}) \neq H^{+1}(\Omega_A \cup \Omega_B)$ and, by Theorem C.10, $\mu_{AB} \perp \mu_A \otimes \mu_B$. We emphasise that this result also holds in $d = 1$. In particular, let $a, b, c \in \mathbb{R}$ such that $a < b < c$ and let $\Omega_A = (a, b)$, $\Omega_B = (b, c)$ and $\Omega_{AB} = (a, c)$. The Hilbert-Sobolev spaces $H^{+1}((a, c))$ and $H^{+1}((a, b) \cup (b, c))$ do not coincide as sets, see also [140, p. 343]. More precisely, a function that is absolutely continuous on (a, c) except at the point b is an element of $H^{+1}(\Omega_A \cup \Omega_B)$ but not of $H^{+1}(\Omega_{AB})$. Therefore, for $\Omega_A = (a, b)$ and $\Omega_B = (b, c)$ we have $I(\Omega_A : \Omega_B) = +\infty$.

One may be tempted to explain the divergence of the mutual information between two touching regions by the singularity of the fundamental solution G at coinciding Euclidean spacetime points. More precisely, if the regions touch, the correlations between points close to the touching surface are arbitrarily large and this may seem like the cause of the divergence of the mutual information. However, in $d = 1$, the fundamental solution is continuous and bounded, cf. (B.3). Yet, as demonstrated above, the mutual information between two touching open intervals is infinite. Therefore, the analytic properties of the fundamental solution around the diagonal does not seem to be the cause of the divergence of the mutual information. We note that a similar observation has been made in [141] in the context of entanglement entropy of relativistic quantum field theories. In particular, there the authors show that no matter how regular the Green's function is on the diagonal, the entanglement entropy in the corresponding quantum field theory is always UV divergent.

We argue that the divergence of the mutual information is due to the Markov property of the free Euclidean scalar field rather than due to the analytic properties of the Green's function in the vicinity of coinciding Euclidean spacetime points. We will show this with the case $d = 1$, where we can, at least formally, calculate the mutual information explicitly. Let $\Omega \subset \mathbb{R}^d$ be open and denote by φ the field with free boundary conditions. This random variable can be decomposed as [80, p. 242]

$$\varphi(f) = \varphi_{\mathrm{D}}(f) + \varphi_\partial(e_{\partial\Omega} f) \,, \tag{5.21}$$

where φ_{D} is the Dirichlet field over Ω and φ_∂ is the field on the boundary $\partial\Omega$ with covariance operator $\hat{G}_0$, called the boundary field. The expression (5.21) has to be understood as the

independent sum of the Gaussian random variables φ_{D} and φ_{∂}, i.e., the law of φ is a product measure of the form $\mu_{\mathrm{D}} \otimes \mu_{\partial}$, see [80, Sec. III.2] and [71, Prop. I.7]. Let Ω_A and Ω_B be open subsets of $\mathbb{R}^d$ separated by a finite distance. We denote by φ^i_{D} and $\varphi_{\partial i}$ the Dirichlet and boundary field over Ω_i and $\partial\Omega_i$, respectively, $i \in \{A, B\}$, and by $\varphi^{AB}_{\mathrm{D}}$ and $\varphi_{\partial AB}$ the Dirichlet and boundary field over $\Omega_A \cup \Omega_B$ and $\partial(\Omega_A \cup \Omega_B)$. Recall from Chapter 1 that we can write the mutual information between the two separated regions Ω_A and Ω_B as

$$I(\Omega_A : \Omega_B) = S(\varphi_A) + S(\varphi_B) - S(\varphi_{AB}) \,, \tag{5.22}$$

where $S(X)$ denotes the entropy of the random variable X. Note that this expression is purely formal as the entropies appearing on the right-hand side are not defined when the random variable X is a statistical field φ. Upon splitting the random variables φ_i according to (5.21) and noting that the entropy is additive for independent random variables, we can formally write

$$I(\Omega_A : \Omega_B) = S(\varphi^A_{\mathrm{D}}) + S(\varphi_{\partial A}) + S(\varphi^B_{\mathrm{D}}) + S(\varphi_{\partial B}) - S(\varphi^{AB}_{\mathrm{D}}) - S(\varphi_{\partial AB}) \,. \tag{5.23}$$

However, the Dirichlet field $\varphi^{AB}_{\mathrm{D}}$ factorises also into two independent Gaussian variables, see Proposition 5.5 and the subsequent discussion. In particular, $\varphi^{AB}_{\mathrm{D}} = \varphi^A_{\mathrm{D}} + \varphi^B_{\mathrm{D}}$. Thus, the contributions from the Dirichlet fields cancel in (5.23) and we conclude that for separated regions Ω_A and Ω_B

$$I(\Omega_A : \Omega_B) = S(\varphi_{\partial A}) + S(\varphi_{\partial B}) - S(\varphi_{\partial AB}) = I(\partial\Omega_A : \partial\Omega_B) \,. \tag{5.24}$$

We see that only the boundaries of the two regions Ω_A and Ω_B matter for the mutual information. This is a direct consequence of the Markov property of scalar Euclidean field theories [114].

We now explicitly consider the case $d = 1$. Let $a, b, c, d \in \mathbb{R}$ such that $a < b < c < d$ and define $\Omega_A = (a, b)$ and $\Omega_B = (c, d)$. In this case, the laws of the random variables $\varphi_{\partial A}$, $\varphi_{\partial B}$ and $\varphi_{\partial AB}$ are ordinary multivariate Gaussian distributions with vanishing mean and covariance matrices

$$\Sigma_A = \frac{1}{2m} \begin{pmatrix} 1 & \mathrm{e}^{-m|a-b|} \\ \mathrm{e}^{-m|a-b|} & 1 \end{pmatrix} \,, \qquad \Sigma_B = \frac{1}{2m} \begin{pmatrix} 1 & \mathrm{e}^{-m|c-d|} \\ \mathrm{e}^{-m|c-d|} & 1 \end{pmatrix} \tag{5.25}$$

and

$$\Sigma_{AB} = \frac{1}{2m} \begin{pmatrix} 1 & \mathrm{e}^{-m|a-b|} & \mathrm{e}^{-m|a-c|} & \mathrm{e}^{-m|a-d|} \\ \mathrm{e}^{-m|a-b|} & 1 & \mathrm{e}^{-m|b-c|} & \mathrm{e}^{-m|b-d|} \\ \mathrm{e}^{-m|a-c|} & \mathrm{e}^{-m|b-c|} & 1 & \mathrm{e}^{-m|c-d|} \\ \mathrm{e}^{-m|a-d|} & \mathrm{e}^{-m|b-d|} & \mathrm{e}^{-m|c-d|} & 1 \end{pmatrix} \,, \tag{5.26}$$

respectively. Here, we used the expression of the fundamental solution in $d = 1$, (B.3). Using the expression for the relative entropy between multivariate Gaussian distributions, the mutual information between Ω_A and Ω_B can be written as

$$I((a,b):(c,d)) = \frac{1}{2}\log\left[\frac{\det\Sigma_A \, \det\Sigma_B}{\det\Sigma_{AB}}\right] . \tag{5.27}$$

The determinants of the covariance matrices can be computed explicitly, and we arrive at

$$I((a,b):(c,d)) = -\frac{1}{2}\log\left(1 - \mathrm{e}^{-2m(c-b)}\right) . \tag{5.28}$$

We observe that only the boundary points b and c, i.e., the boundary points "facing each other", appear in the expression for the mutual information. This once again reflects the Markov property of the scalar field[3]. It also means that the mutual information between two intervals is independent of the length of each of the intervals and only depends on the separation distance $\delta \coloneqq c - b$ between the two intervals. For small δ, the mutual information behaves like $-\frac{1}{2}\log(2m\delta)$ and we see that the mutual information diverges as $\delta \to 0^+$, which is consistent with our previous result that the mutual information has to be set to $+\infty$ when the two intervals touch. Finally, we note that, for fixed separation distance δ, the mutual information decreases as the mass m is increased. This is consistent with the interpretation of the mass as the inverse of the correlation length. In particular, as we decrease the correlation length, the mutual information between two finitely separated regions decreases.

We note that (5.28) can be generalized to other boundary conditions. More precisely, let $L > 0$ be such that $0 \leq a < b < c < d \leq L$. Then, the Dirichlet and Neumann Green's functions for the interval $\Omega = (0, L)$ are given by [69]

$$\begin{aligned} G_{\mathrm{D}}(x,y) &= \frac{1}{m\sinh(mL)} \begin{cases} \sinh(m(L-x))\sinh(my) & x \geq y\,, \\ \sinh(mx)\sinh(m(L-y)) & x < y\,, \end{cases} \\ G_{\mathrm{N}}(x,y) &= \frac{1}{m\sinh(mL)} \begin{cases} \cosh(m(L-x))\cosh(my) & x \geq y\,, \\ \cosh(mx)\cosh(m(L-y)) & x < y\,, \end{cases} \end{aligned} \tag{5.29}$$

Then, by an analogous calculation as above, the mutual information between two intervals (a, b) and (c, d) with $0 \leq a < b < c < d \leq L$ and with Dirichlet and Neumann boundary conditions on $\partial\Omega = \{0, L\}$, respectively, can be written as

$$\begin{aligned} I_{\mathrm{D}}((a,b):(c,d)) &= -\frac{1}{2}\log\left(\frac{\left(\mathrm{e}^{2mb} - \mathrm{e}^{2mL}\right)\left(\mathrm{e}^{2mc} - 1\right)}{\mathrm{e}^{2mb} - \mathrm{e}^{2mc}}\right) , \\ I_{\mathrm{N}}((a,b):(c,d)) &= -\frac{1}{2}\log\left(\frac{\left(\mathrm{e}^{2mL} - 1\right)\left(\mathrm{e}^{2mb} - \mathrm{e}^{2mc}\right)}{(\mathrm{e}^{2mL} + \mathrm{e}^{2mb})(1 + \mathrm{e}^{2mc})}\right) . \end{aligned} \tag{5.30}$$

[3]Note that in $d = 1$ we can interpret a scalar field as a continuous time Markov process and the Markov property implies that "the future depends on the past only through the present" [142].

Notice that once again the mutual information only depends on the boundary points b and c and not on the length of the intervals. Furthermore, the mutual information diverges as the two intervals touch, i.e., as $(c-b) \to 0^+$.

The need for a finite separation distance between Ω_A and Ω_B for finite values of the mutual information is consistent with results obtained in the context of quantum mutual information in relativistic quantum field theories, see, e.g., [26, 27, 30].

Part III

Local Dynamics of a Relativistic Quantum Field

This Part concerns the time evolution of a reduced state of a relativistic quantum field theory on a spatial subregion of $\mathbb{R}^d$. In Chapter 6, we introduce a lattice model of a scalar field theory. This is done for the purpose of facilitating a clearer understanding of the splitting of the action into interior and exterior degrees of freedom, which is more transparent in the lattice model than in the continuum theory. Furthermore, we verify the consistency of this splitting by considering a reduced state of a Euclidean field theory and check whether the results obtained are consistent with those presented in Part II of this thesis.

In Chapter 7, we examine the local dynamics of a relativistic quantum field theory on a spatial subregion. Our analysis reveals that the time evolution of the reduced state of the field theory is non-unitary. This non-unitarity can be attributed to two factors. Firstly, the interior degrees of freedom (referred to as the "system") are linearly coupled to the exterior degrees of freedom (referred to as the "environment") via the Laplacian in the action. Secondly, since we are working in a relativistic quantum field theory, we should consider correlated initial states. Initially, we ignore such initial state correlations and assume a factorizing initial state. However, later on, we incorporate initial state correlations by considering local excitations of thermal states. It is demonstrated that the non-unitary contributions to the dynamics of the reduced state are entirely contained in an effective boundary action, which is due to the local nature of the theory. Finally, a stochastic partial differential equation for the field expectation values in the interior region is derived. This equation of motion is shown to be a Klein-Gordon equation with stochastic boundary conditions.

Disclaimer. At the time of this writing, a publication on the topics in this Part is in preparation. At the time this thesis is published, the publication may be available and may contain significant overlap with the content of this Part. However, the content of this Part is based on my own research and has been formulated by me.

The time evolution of an isolated quantum system is unitary and therefore its entropy is constant in time. For example, a pure state of a closed quantum system remains pure during its time evolution. Nevertheless, an isolated many-body quantum system in a non-equilibrium state can thermalize under such unitary dynamics [143–145]. This is especially the case if one considers only the expectation values of observables supported on a (small) subsystem [146, 147]. A driving principle for the thermalization of many-body quantum systems is thought to be the generation of entanglement between subsystems [147–149], which can also be measured experimentally [150].

Thermalization caused by entanglement generation may also be important in high-energy experiments [108, 151, 152]. Moreover, a detailed understanding of the "local" subsystem dynamics and the related concepts of entanglement generation and entropy increase may be crucial for understanding the relationship between quantum field theory and relativistic fluid dynamics [94]. Relativistic fluid dynamics [153–155] provide a powerful phenomenological description of the dynamics of quantum fields, for example in heavy ion collisions [156–158]. One attempt to understand the relationship between quantum field theory and relativistic fluid dynamics is based on the concept of local thermal equilibrium. It is assumed that spatial subsystems, which are small compared to the scale on which typical experiments are conducted, are *open* quantum systems, i.e., they are not isolated from their environment. Therefore, such a local time evolution is *non-unitary*, since information can be exchanged with the environment. In contrast to the unitary time evolution of closed systems, entropy is no longer constant under non-unitary time evolution, and entropy can increase locally. This is thought to be one of the reason for the emergence of local thermal equilibrium and thus the applicability of relativistic fluid dynamics to quantum field theories.

In this Part of the thesis, we aim to gain insight into the emergence of local thermalization of relativistic quantum field theories by studying the structure of the local dynamics in a relativistic field theory. As a toy model, we consider a massive scalar field on $d+1$ dimensional Minkowski spacetime. We are interested in the dynamics of the field theory in a spatial subregion Ω. Defining the field within this region as the "system" and the field outside the region as the "environment", the differential operator in the action of the field theory induces a linear system-environment coupling similar to the Caldeira-Leggett model [159–161]. This coupling leads to open dynamics of the system, i.e. the time evolution of the reduced state of the system is not unitary, since information can dissipate between the interior and exterior regions. As described above, such a non-unitary time evolution is a necessary condition for an increase of entropy and thus for local thermalization.

Besides the presence of a system-environment interaction, the initial state of a field theory plays a crucial role in determining the system's dynamics. Therefore, it is important to consider the implications of the initial state when performing calculations. Although an uncorrelated initial state is typically assumed for practical reasons, this assumption is difficult to justify in the case of relativistic field theories. Thermal states, including the vacuum, are correlated across space. Even when classical correlations are absent (like in the vacuum

state), it is well-known that states of relativistic quantum field theories are highly (and in some sense even maximally) entangled [2, 3]. As a result, our analysis must account for the possibility of initial state correlations, which further complicates the problem.

The main objective of this Part is to derive the effective dynamics of a local field theory in a model where the environment integrals are exactly solvable. In particular, we assume the initial state of the environment to be Gaussian and the dynamics of the environment to be linear. Furthermore, we derive a stochastic equation of motion for the field expectation values. Due to the local nature of the theory, the stochastic parts of the dynamics are encoded in the spatial boundary conditions of the differential equation. We show that the boundary conditions derived from the effective dynamics are reminiscent of the free boundary conditions already encountered in Part II.

6 Lattice Model

In this Chapter we consider a regularized model of a massive scalar field theory. More precisely, we discretize space into a regular lattice and compactify space into a torus, yielding a system with finitely many degrees of freedom with significantly fewer subtleties than the continuum theory. In particular, the splitting of the model into a "system" (the degrees of freedom inside some region Λ) and an "environment" (the degrees of freedom in the complement of Λ) is more transparent if we consider a lattice theory.

Following [75, Sec. 1.3.1], we denote by $\varepsilon > 0$ the lattice spacing, by $L \gg \varepsilon$ the size of the lattice and require $L/(2\varepsilon) \in \mathbb{N}$. Then, we define $\mathscr{L}^d_{\varepsilon,L} = \varepsilon\mathbb{Z}^d/L\mathbb{Z}^d$ to be our (finite) spatial d-dimensional lattice with periodic boundary conditions. Furthermore, for some $\Omega \subsetneq \mathbb{R}^d/L\mathbb{R}^d$, let $\Lambda_1 = \Omega \cap \mathscr{L}^d_{\varepsilon,L}$ be the corresponding sublattice. Following [70, Ch. IV], we define a norm on $\mathbb{Z}^d$ via $\|n\|_{\mathbb{Z}} = \sum_{i=1}^d |n_i|$ and define the interior of the lattice Λ as

$$\Lambda_1^\circ = \{x = \varepsilon n \in \Lambda_1 \ : \ m\varepsilon \in \Lambda_1 \ \text{ if } \ \|n-m\|_{\mathbb{Z}} = 1\} \, . \tag{6.1}$$

Finally, we define $\partial\Lambda = \Lambda_1 \setminus \Lambda_1^\circ$, $\Lambda_2 = \mathscr{L}^d_{\varepsilon,L} \setminus \Lambda_1$ and $\overline{\Lambda_2} = \Lambda_2 \cup \partial\Lambda = \mathscr{L}^d_{\varepsilon,L} \setminus \Lambda_1^\circ$ to be the boundary of Λ_1, the complement (or exterior to) Λ_1, and the "closure" of the complement of Λ_1, respectively, cf. Fig. 6.1.

Denote by $\ell^2_{\mathbb{R}}(\mathscr{L}^d_{\varepsilon,L}, \varepsilon^d)$ the real Hilbert space of real-valued functions on the lattice $\mathscr{L}^d_{\varepsilon,L}$ with inner product $\langle ., . \rangle_\varepsilon$ given by

$$\langle f, g \rangle_\varepsilon = \varepsilon^d \sum_{x \in \mathscr{L}^d_{\varepsilon,L}} f_x \, g_x \, . \tag{6.2}$$

We define the quadratic form $\mathfrak{q}_\varepsilon$ on $\ell^2_{\mathbb{R}}(\mathscr{L}^d_{\varepsilon,L}, \varepsilon^d)$ as

$$\mathfrak{q}_\varepsilon(f) \coloneqq \varepsilon^d \sum_{x \in \mathscr{L}^d_{\varepsilon,L}} (\mathbf{d}_\varepsilon f_x)^2 \, , \tag{6.3}$$

where $\mathbf{d}_\varepsilon$ denotes the lattice gradient defined as [75, Eq. 1.64]

$$(\mathbf{d}_\varepsilon f)_{x,k} = \frac{1}{\varepsilon}(f_{x+\varepsilon\hat{e}_k} - f_x) \, , \qquad k \in \{1, \dots, d\} \, , \tag{6.4}$$

where $\hat{e}_k$ is the k-th unit vector in $\mathbb{Z}^d$.

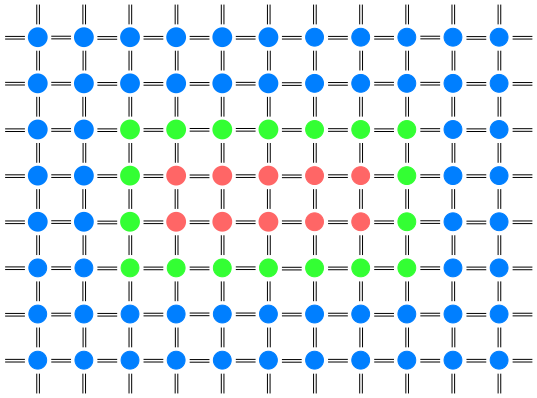


Figure 6.1: Example of the regions Λ_1, Λ_1°, Λ_2, $\overline{\Lambda_2}$ and $\partial\Lambda$ on a two-dimensional lattice. The region Λ_1 consists of the union of all red and green lattice sites, while its interior Λ_1° is made up of all red sites only, cf. (6.1). The exterior region Λ_2 consists of all blue sites, while its closure $\overline{\Lambda_2}$ also includes the green sites. The boundary $\partial\Lambda$ is the set of all green lattice sites.

The self-adjoint operator associated with the form $\mathfrak{q}_\varepsilon$ is the *lattice Laplacian* $-\triangle_\varepsilon := \mathbf{d}_\varepsilon^* \mathbf{d}_\varepsilon$, where the adjoint gradient $\mathbf{d}_\varepsilon^*$ is defined as the backwards difference operator acting as [75, Eq. 1.71]

$$(\mathbf{d}_\varepsilon^* f)_{x,k} = \frac{1}{\varepsilon}(f_{x-\varepsilon\hat{e}_k} - f_x) \,, \qquad k \in \{1, \dots, d\} \,. \tag{6.5}$$

The lattice Laplacian acts on functions in $\ell^2_{\mathbb{R}}(\mathscr{L}^d_{\varepsilon,L}, \varepsilon^d)$ as

$$(-\triangle_\varepsilon f)_x = \frac{1}{\varepsilon^2} \sum_{k=1}^{d} (2f_x - f_{x+\varepsilon\hat{e}_k} - f_{x-\varepsilon\hat{e}_k}) \,. \tag{6.6}$$

Moreover, $-\triangle_\varepsilon$ can be written as a $|\mathscr{L}^d_{\varepsilon,L}| \times |\mathscr{L}^d_{\varepsilon,L}|$ matrix $-\hat{\triangle}_\varepsilon$, interpreted as a discrete integral kernel, with entries

$$(-\hat{\triangle}_\varepsilon)_{xy} = \frac{1}{\varepsilon^2} \sum_{k=1}^{d} \big(2\delta^\varepsilon_{xy} - \delta^\varepsilon_{x+\varepsilon\hat{e}_k,y} - \delta^\varepsilon_{x-\varepsilon\hat{e}_k,y}\big) \,, \tag{6.7}$$

where $\delta^\varepsilon_{xy} := \varepsilon^{-d}\delta_{xy}$ is the lattice Dirac-δ. In particular, we have

$$(-\triangle_\varepsilon f)_x = \varepsilon^d \sum_{y \in \mathscr{L}^d_{\varepsilon,L}} (-\hat{\triangle}_\varepsilon)_{xy} \, f_y \,. \tag{6.8}$$

In order to illustrate the difference between $-\triangle_\varepsilon$ and $-\hat{\triangle}_\varepsilon$, we consider the limit $\varepsilon \to 0$, which formally yields

$$-\triangle_\varepsilon \xrightarrow{\varepsilon\to 0} -\triangle \,, \qquad -\hat{\triangle}_\varepsilon \xrightarrow{\varepsilon\to 0} -\triangle\, \delta^{(d)}(x-y) \,, \tag{6.9}$$

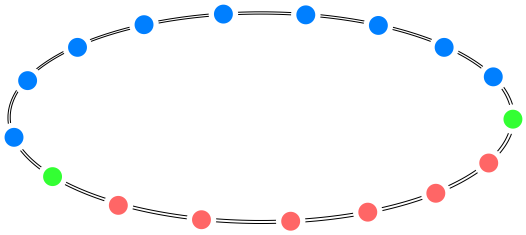


Figure 6.2: Sketch of a one-dimensional lattice with periodic boundary conditions. The color-coding is as in Fig. 6.1. In particular, the interior region Λ_1 (union of green and red lattice sites) corresponds to a single interval $\Omega \subsetneq \mathbb{R}/L\mathbb{R}$ with N_1 lattice sites, while the exterior region Λ_2 (blue sites) corresponds to the complement of Λ_1 with N_2 lattice sites. The boundary $\partial\Lambda$ consists of the green lattice sites, and we denote the endpoints of the interval by a and b, respectively, i.e., $\partial\Lambda \coloneqq \{a, b\}$.

where $-\triangle$ is the continuum Laplacian and $\delta^{(d)}(x-y)$ is the d-dimensional Dirac-δ distribution. Therefore, we can indeed interpret $-\hat{\triangle}_\varepsilon$ as a discrete integral kernel with respect to the inner product of $\ell^2_{\mathbb{R}}(\mathscr{L}^d_{\varepsilon,L}, \varepsilon^d)$, while $-\triangle_\varepsilon$ is the corresponding operator acting on functions.

Let $\Lambda_1 \subsetneq \mathscr{L}^d_{\varepsilon,L}$ be a sublattice with boundary $\partial\Lambda$ and complement Λ_2. For every $f \in \ell^2_{\mathbb{R}}(\mathscr{L}^d_{\varepsilon,L}, \varepsilon^d)$, we define f_1, f_2 and $\underline{f_2}$ to be the restrictions of f to Λ_1, Λ_2 and $\overline{\Lambda_2}$, respectively. We now wish to find a decomposition of the quadratic form $\mathfrak{q}_\varepsilon$ in three parts, one for the interior, one for the exterior and one coupling the interior and the exterior via their common boundary.

It is instructive to start with a one-dimensional model. Let $\mathscr{L}^1_{\varepsilon,L}$ be a one-dimensional periodic lattice with $N \in \mathbb{N}$ lattice sites. Furthermore, let Λ_1 be the sublattice corresponding to an interval $\Omega \subsetneq \mathbb{R}/L\mathbb{R}$ with N_1 lattice sites and Λ_2 be the complement of Λ_1 with N_2 lattice sites, cf. Fig. 6.2. Then, the lattice Laplacian $-\hat{\triangle}_\varepsilon$ on $\ell^2_{\mathbb{R}}(\mathscr{L}^1_{\varepsilon,L}, \varepsilon)$ can be written as an $N \times N$ matrix given by

$$-\hat{\triangle}_\varepsilon = \frac{1}{\varepsilon}\frac{1}{\varepsilon^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \tag{6.10}$$

The corresponding quadratic form $\mathfrak{q}_\varepsilon$ is given by

$$\mathfrak{q}_\varepsilon(f) = \varepsilon^2 \sum_{x,y \in \mathscr{L}^1_{\varepsilon,L}} f_x(-\hat{\triangle}_\varepsilon)_{xy} f_y \,. \tag{6.11}$$

Next, we define the $N_1 \times N_1$ matrix $-\hat{\triangle}_\varepsilon^{\,1}$, the $N_2 \times N_2$ matrix $-\hat{\triangle}_\varepsilon^{\,2}$ and the $(N_2+2) \times (N_2+2)$ matrix $-\hat{\triangle}_\varepsilon^{\,2,\star}$ as

$$-\hat{\triangle}_\varepsilon^{\,1} = \frac{1}{\varepsilon}\frac{1}{\varepsilon^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \tag{6.12}$$

$$-\hat{\triangle}_\varepsilon^{\,2} = \frac{1}{\varepsilon}\frac{1}{\varepsilon^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \tag{6.13}$$

$$-\hat{\triangle}_\varepsilon^{\,2,\star} = \frac{1}{\varepsilon}\frac{1}{\varepsilon^2}\begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \end{pmatrix}. \tag{6.14}$$

By a straightforward calculation, we find that the quadratic form $\mathfrak{q}_\varepsilon$ can be written as

$$\mathfrak{q}_\varepsilon(f) = \mathfrak{q}_\varepsilon^1(f_1) + \mathfrak{q}_\varepsilon^{2,\star}(\underline{f_2}) \,, \tag{6.15}$$

where $\mathfrak{q}_\varepsilon^1$ and $\mathfrak{q}_\varepsilon^{2,\star}$ are the quadratic forms associated with the lattice Laplacians $-\triangle_\varepsilon^1$ and $-\triangle_\varepsilon^{2,\star}$, respectively, i.e.,

$$\mathfrak{q}_\varepsilon^1(f_1) = \varepsilon^2 \sum_{x,y\in\Lambda_1} f_{1,x}\,(-\hat{\triangle}_\varepsilon^{\,1})_{xy}\, f_{1,y} \,, \qquad \mathfrak{q}_\varepsilon^{2,\star}(\underline{f_2}) = \varepsilon^2 \sum_{x,y\in\overline{\Lambda_2}} \underline{f_{2,x}}\,(-\hat{\triangle}_\varepsilon^{\,2,\star})_{xy}\, \underline{f_{2,y}} \,. \tag{6.16}$$

Let $\partial\Lambda \coloneqq \{a, b\}$ be the boundary of the interior lattice Λ_1 and define $\partial\Lambda' \coloneqq \{a + \varepsilon\hat{n}, b + \varepsilon\hat{n}\}$, where $\hat{n}$ is the unit outward (with respect to Λ_1) normal vector to the boundary $\partial\Lambda$. Notice that the Laplacian $-\triangle_\varepsilon^1$ describes *homogeneous* Dirichlet boundary conditions, while the Laplacian $-\triangle_\varepsilon^{2,\star}$ describes *inhomogeneous* Dirichlet boundary conditions given by the value

of f_1 on the boundary $\partial\Lambda \coloneqq \{a, b\}$. More precisely, following, e.g., [162, Sec. 2.3.1], let us introduce the N_2-component vector b_f as

$$b_f^{\mathsf{T}} = \frac{1}{\varepsilon^2}(f_a, 0, 0, \ldots, 0, 0, f_b) \; . \tag{6.17}$$

Then, by direct calculation, it can be shown that

$$-\triangle_\varepsilon^{2,\star} \underline{f_2}\big|_{\Lambda_2} = -\triangle_\varepsilon^2 f_2 - b_f \; . \tag{6.18}$$

Similarly, the quadratic form $\mathfrak{q}_\varepsilon^{2,\star}$ can be written as

$$\mathfrak{q}_\varepsilon^{2,\star}(\underline{f_2}) = \mathfrak{q}_\varepsilon^2(f_2) - 2\,\langle f_2, b_f\rangle_\varepsilon \; , \tag{6.19}$$

where $\mathfrak{q}_\varepsilon^2$ is the quadratic form associated with the Laplacian $-\triangle_\varepsilon^2$, i.e., the quadratic form in Λ_2 with homogeneous Dirichlet boundary conditions on $\partial\Lambda$.

In summary, we have found that the quadratic form $\mathfrak{q}_\varepsilon$ can be split as

$$\mathfrak{q}_\varepsilon(f) = \mathfrak{q}_\varepsilon^1(f_1) + \mathfrak{q}_\varepsilon^2(f_2) - 2\,\langle f_2, b_f\rangle_\varepsilon \; , \tag{6.20}$$

where $\mathfrak{q}_\varepsilon^1$ and $\mathfrak{q}_\varepsilon^2$ are the quadratic forms associated with the Laplacians $-\triangle_\varepsilon^1$ and $-\triangle_\varepsilon^2$, respectively, and b_f is the vector of boundary values of f_1. Notice that $\mathfrak{q}_\varepsilon^1$ and $\mathfrak{q}_\varepsilon^2$ only contain degrees of freedom in Λ_1 and Λ_2, respectively, while the last term linearly couples the two regions across boundary $\partial\Lambda$. This result generalizes to higher dimensions in a straightforward way.

6.1 Example: Euclidean Field Theory

In order to further elucidate the above splitting of the quadratic form $\mathfrak{q}_\varepsilon$, we now consider a Euclidean field theory on the lattice. Let $\mathscr{L}_{\varepsilon,L}^1$ be a one-dimensional periodic lattice and Λ_1 and Λ_2 be two sublattices corresponding to intervals as defined above, cf. Fig. 6.2. Then, the classical field theory on the lattice is described by the quadratic form S_E, called the Euclidean action, given by

$$S_\mathrm{E}(f) = \frac{\varepsilon^2}{2} \sum_{x,y\in\mathscr{L}_{\varepsilon,L}^1} f_x \,(-\hat{\triangle}_\varepsilon + m^2\delta^\varepsilon)_{xy}\, f_y \; , \tag{6.21}$$

where $m > 0$ is a mass parameter of the theory.

Since the mass term $m^2\delta^\varepsilon$ does not couple neighbouring lattice sites, we can, analogues to the case of the form $\mathfrak{q}_\varepsilon$, split the Euclidean action S_E into three parts, one for the interior,

one for the exterior and one coupling the interior and the exterior via a common boundary. More precisely, we have

$$S_{\mathrm{E}}(f) = S_{\mathrm{E}}^1(f_1) + S_{\mathrm{E}}^2(f_2) - \langle f_2, b_f \rangle_\varepsilon \ , \tag{6.22}$$

where S_{E}^1 and S_{E}^2 are the Euclidean actions associated with the Laplacians $-\triangle_\varepsilon^1$ and $-\triangle_\varepsilon^2$, respectively, and b_f is the vector of boundary values of f_1 as introduced in (6.17). Notice that from the point of view of the field in region Λ_2, $-\langle f_2, b_f \rangle_\varepsilon$ is a source term and the boundary values of the field in Λ_1 act as an external source supported exclusively on the boundary.

The Euclidean action S_{E} defines a classical statistical (lattice) field theory via a centred Gaussian measure μ on $\mathbb{R}^N$ given by

$$\mathrm{d}\mu(f) \coloneqq \frac{1}{Z} \mathrm{e}^{-S_{\mathrm{E}}(f)} \, \mathrm{d}^N f \ , \tag{6.23}$$

where Z is a normalization constant and $\mathrm{d}^N f$ is the N-dimensional Lebesgue measure. The measure μ is completely characterized by its covariance operator $\hat{G}$ given by

$$\hat{G} = \varepsilon^{-2}(-\hat{\triangle}_\varepsilon + m^2 \delta^\varepsilon)^{-1} \ , \qquad G_{xy} \coloneqq (\hat{G})_{xy} = \varepsilon^{-2} \int_{\mathbb{R}^N} f_x f_y \, \mathrm{d}\mu(f) \ , \tag{6.24}$$

where we rescaled the covariance operator by a factor of ε^{-2} to interpret it as an integral operator with respect to the inner product $\langle ., . \rangle_\varepsilon$.

Upon splitting the action into three parts, the probability density with respect to $\mathrm{d}^N f = \mathrm{d}^{N_1} f_1 \, \mathrm{d}^{N_2} f_2$ factorizes as

$$\mathrm{d}\mu(f) = \mathrm{d}\mu(f_1, f_2) = \frac{1}{Z} \exp\left[-\left(S_{\mathrm{E}}^1(f_1) + S_{\mathrm{E}}^2(f_2) - \langle f_2, b_f \rangle_\varepsilon\right)\right] \mathrm{d}^{N_1} f_1 \, \mathrm{d}^{N_2} f_2 \ , \tag{6.25}$$

We obtain a reduced theory for the region Λ_1 by considering the marginal μ_1 given by integrating out the degrees of freedom in Λ_2, i.e.,

$$\begin{aligned} \mathrm{d}\mu_1(f_1) &= \int_{\mathbb{R}^{N_2}} \mathrm{d}\mu(f_1, f_2) \\ &= \frac{1}{Z} \left(\int_{\mathbb{R}^{N_2}} \exp\left[-\left(S_{\mathrm{E}}^2(f_2) - \langle f_2, b_f \rangle_\varepsilon\right)\right] \mathrm{d}^{N_2} f_2 \right) \mathrm{e}^{-S_{\mathrm{E}}^1(f_1)} \, \mathrm{d}^{N_1} f_1 \\ &= \frac{1}{Z} \sqrt{\det\left(2\pi \hat{G}_{\mathrm{D},2}\right)} \, \mathrm{e}^{-\widetilde{S_{\mathrm{E}}^1}(f_1)} \, \mathrm{d}^{N_1} f_1 \ , \end{aligned} \tag{6.26}$$

where $\hat{G}_{\mathrm{D},2}$ is the covariance operator[1] associated with the Euclidean action S_{E}^2, i.e.,

$$\hat{G}_{\mathrm{D},2} \coloneqq \varepsilon^{-2} \left(-\hat{\triangle}_\varepsilon^{\,2} + m^2 \delta^\varepsilon\right)^{-1} \ , \tag{6.27}$$

[1] The index D indicates the Dirichlet boundary conditions described by the Euclidean action S_{E}^2.

and $\widetilde{S^1_{\mathrm{E}}}$ is a quadratic form given by

$$\widetilde{S^1_{\mathrm{E}}}(f_1) = S^1_{\mathrm{E}}(f_1) - \frac{\varepsilon^2}{2} b_f^{\mathsf{T}} \hat{G}_{\mathrm{D},2}\, b_f = S^1_{\mathrm{E}}(f_1) - \frac{1}{2} \sum_{x,y \in \partial\Lambda} f_{1,x}\, \frac{(\hat{G}_{\mathrm{D},2})_{x+\varepsilon\hat{n},y+\varepsilon\hat{n}}}{\varepsilon^2}\, f_{1,y}\ , \tag{6.28}$$

where $\hat{n}$ is again the unit outward normal vector with respect to Λ_1. Since $\hat{G}_{\mathrm{D},2}$ is a discrete Dirichlet Green's function, it is natural to extend it to a $(N_2+2)\times(N_2+2)$ matrix, which we again denote by $\hat{G}_{\mathrm{D},2}$, whose entries are zero if $x \in \partial\Lambda \vee y \in \partial\Lambda$. Thus, we see that

$$\begin{aligned} \frac{(\hat{G}_{\mathrm{D},2})_{x+\varepsilon\hat{n},y+\varepsilon\hat{n}}}{\varepsilon^2} &= \frac{1}{\varepsilon^2}\Big((\hat{G}_{\mathrm{D},2})_{x+\varepsilon\hat{n},y+\varepsilon\hat{n}} - (\hat{G}_{\mathrm{D},2})_{x+\varepsilon\hat{n},y} - (\hat{G}_{\mathrm{D},2})_{x,y+\varepsilon\hat{n}} + (\hat{G}_{\mathrm{D},2})_{xy} \Big) \\ &=: (\partial^{\varepsilon}_{n_x} \partial^{\varepsilon}_{n_y} \hat{G}_{\mathrm{D},2})_{xy} \end{aligned} \tag{6.29}$$

for $x, y \in \partial\Lambda$, where we introduced the discrete normal derivative $\partial^{\varepsilon}_n := \hat{n} \cdot \mathbf{d}_{\varepsilon}$.

Thus, we finally obtain

$$\begin{aligned} \widetilde{S^1_{\mathrm{E}}}(f_1) &= S^1_{\mathrm{E}}(f_1) - \frac{1}{2} \sum_{x,y \in \partial\Lambda} f_{1,x}\, (\partial^{\varepsilon}_{n_x} \partial^{\varepsilon}_{n_y} \hat{G}_{\mathrm{D},2})_{xy}\, f_{1,y} \\ &= \frac{\varepsilon^2}{2} \sum_{x,y \in \Lambda_1} f_{1,x}\, (-\hat{\triangle}_{\varepsilon}^{1,m} + m^2 \delta^{\varepsilon})_{xy}\, f_{1,y}\ , \end{aligned} \tag{6.30}$$

where $-\triangle_{\varepsilon}^{1,m}$ is the Laplacian on Λ_1 with non-local and mass dependent boundary conditions such that the above equality holds. More precisely, $-\hat{\triangle}_{\varepsilon}^{1,m}$ is defined as

$$-\hat{\triangle}_{\varepsilon}^{1,m} := -\hat{\triangle}_{\varepsilon}^{1} - B^m_{\partial\Lambda}\ , \tag{6.31}$$

where $B^m_{\partial\Lambda}$ is a (mass dependent) matrix "concentrated on the boundary" [70, Sec. IV.2], see also [70, Thm. IV.7]. In particular, we have

$$(B^m_{\partial\Lambda})_{xy} = \frac{1}{\varepsilon^2} \begin{cases} (\partial^{\varepsilon}_{n_x} \partial^{\varepsilon}_{n_y} \hat{G}_{\mathrm{D},2})_{xy} & \text{if } x, y \in \partial\Lambda\ , \\ 0 & \text{otherwise}\ . \end{cases} \tag{6.32}$$

The mass dependence of the boundary term $B^m_{\partial\Lambda}$ is a consequence of the mass dependence of the Dirichlet Green's function $\hat{G}_{\mathrm{D},2}$. In d dimensions, the above expression generalizes to[2]

$$\widetilde{S^1_{\mathrm{E}}}(f_1) = \frac{\varepsilon^{2d}}{2} \sum_{x,y \in \Lambda_1} f_{1,x}\, (-\hat{\triangle}_{\varepsilon}^{1} + m^2 \delta^{\varepsilon})_{xy}\, f_{1,y} - \frac{\varepsilon^{2(d-1)}}{2} \sum_{x,y \in \partial\Lambda} f_{1,x}\, (\partial^{\varepsilon}_{n_x} \partial^{\varepsilon}_{n_y} \hat{G}_{\mathrm{D},2})_{xy}\, f_{1,y}\ . \tag{6.33}$$

We observe that the effect of integrating out the exterior lattice Λ_2 is to induce non-local and mass dependent boundary conditions on the interior lattice Λ_1 such that the covariance

[2] We assume that the region Λ_1 is chosen such that the normal unit vector $\hat{n}$ is well-defined for all boundary points.

matrix associated with the reduced theory is given by the restriction of the global covariance matrix to the interior lattice Λ_1. Of course, this is just a manifestation of the fact that the reduced theory describes, by construction, the same physics in the region Λ_1 as the full theory on the entire lattice $\mathscr{L}^d_{\varepsilon,L}$. That the effect of the exterior degrees of freedom is completely described by a boundary term is a consequence of the Markov property of the free scalar field, see the discussions in [47, 69, 70, 80].

6.2 Quantum Lattice System

We now turn to quantum lattice systems representing lattice regularized scalar quantum field theories[3]. Let $\mathscr{L}^d_{\varepsilon,L}$ be a d-dimensional periodic lattice as defined in the previous Section. We denote by $\hat{\Phi}_x$ and $\hat{\Pi}_x$ the unbounded field and conjugated momentum field operators, respectively, at the lattice site $x \in \mathscr{L}^d_{\varepsilon,L}$, satisfying the canonical commutation relations

$$\left[\hat{\Phi}_x, \hat{\Phi}_y\right] = \left[\hat{\Pi}_x, \hat{\Pi}_y\right] = 0 \,, \qquad \left[\hat{\Phi}_x, \hat{\Pi}_y\right] = \mathrm{i}\delta^{\varepsilon}_{xy}\mathbb{1} \,, \qquad x, y \in \mathscr{L}^d_{\varepsilon,L} \,. \tag{6.34}$$

The dynamics of the system are governed by the Hamiltonian of an anharmonic lattice defined as

$$\hat{H} := \frac{\varepsilon^d}{2} \sum_{x \in \mathscr{L}^d_{\varepsilon,L}} \left[\hat{\Pi}_x^2 + \varepsilon^d \sum_{y \in \mathscr{L}^d_{\varepsilon,L}} \left(\hat{\Phi}_x \left(-\hat{\triangle}_\varepsilon + m^2 \delta^\varepsilon \right)_{xy} \hat{\Phi}_y \right) + V(\hat{\Phi}_x) \right] , \tag{6.35}$$

where V is some suitable potential bounded from below.

We define the Schrödinger representation of the lattice theory to be the representation of the Weyl algebra on the complex representation Hilbert space $\mathcal{H} = \bigotimes_{x \in \mathscr{L}^d_{\varepsilon,L}} L^2(\mathbb{R}, \mathrm{d}\varphi_x)$. In the Schrödinger representation, the Hamiltonian acts as the operator

$$\hat{H} := \frac{\varepsilon^d}{2} \sum_{x \in \mathscr{L}^d_{\varepsilon,L}} \left[-\frac{\partial^2}{\partial \varphi_x^2} + \varepsilon^d \sum_{y \in \mathscr{L}^d_{\varepsilon,L}} \left(\varphi_x \left(-\triangle_\varepsilon + m^2 \delta^\varepsilon \right)_{xy} \varphi_y \right) + V(\varphi_x) \right] , \tag{6.36}$$

where $-\partial^2/\partial\varphi_x^2$ denotes the unique self-adjoint realization of the Laplacian on $L^2(\mathbb{R}, \mathrm{d}\varphi_x)$, see Appendix B.

[3] For details on such systems, see, e.g., [82, 83, 163, 164].

Let $\Lambda_1 \subsetneq \mathscr{L}^d_{\varepsilon,L}$ be a sublattice with boundary $\partial\Lambda$ and complement Λ_2. Except for the term containing the lattice Laplacian, the Hamiltonian in (6.36) is local in the lattice indices, and we may split it, following the discussion above, as $\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_{12}$, where

$$\begin{aligned}
\hat{H}_1 &= \frac{\varepsilon^d}{2} \sum_{x\in\Lambda_1} \left[-\frac{\partial^2}{\partial\varphi_x^2} + \varepsilon^d \sum_{y\in\Lambda_1} \left(\varphi_x \left(-\hat{\triangle}^1_\varepsilon + m^2\delta^\varepsilon \right)_{xy} \varphi_y \right) + V(\varphi_x) \right] , \\
\hat{H}_2 &= \frac{\varepsilon^d}{2} \sum_{x\in\Lambda_2} \left[-\frac{\partial^2}{\partial\varphi_x^2} + \varepsilon^d \sum_{y\in\Lambda_2} \left(\varphi_x \left(-\hat{\triangle}^2_\varepsilon + m^2\delta^\varepsilon \right)_{xy} \varphi_y \right) + V(\varphi_x) \right] , \\
\hat{H}_{12} &= -\varepsilon^{d-2} \sum_{x\in\partial\Lambda} \varphi_{x+\varepsilon\hat{n}}\, \varphi_x \ ,
\end{aligned} \tag{6.37}$$

where $\hat{n}$ denotes again the outward normal vector (with respect to Λ_1) on the boundary $\partial\Lambda$ and $-\triangle^1_\varepsilon$ and $-\triangle^2_\varepsilon$ are the lattice Laplacians on Λ_1 and Λ_2, respectively, describing homogeneous Dirichlet boundary conditions. Notice that the boundary potential $\hat{H}_{12}$ linearly couples the fields in Λ_1 and Λ_2 across $\partial\Lambda$.

7 Local Dynamics of Relativistic Quantum Field Theories

In this Chapter, we study the local dynamics of a relativistic quantum field theory using the concrete example of a free, massive scalar field. To do this, we employ the Feynman-Vernon influence functional approach described in Appendix D. Even though states in relativistic quantum field theories are generically entangled [2, 3], we will first consider a product ansatz for the initial state, i.e., a state that can be written as a tensor product of states in the two regions Λ_1 and Λ_2. This will allow us to elucidate the influence of the environment on the reduced dynamics of the system. After that, we incorporate initial state correlations by considering an initial state that is a (local) perturbation of a thermal state. Finally, we derive a stochastic equation of motion for the expectation value of the field in the interior region.

For definiteness, we consider a lattice model as introduced in Chapter 6. We denote the field in Λ_1 by φ and the field in Λ_2 by ϕ, respectively. The dynamics of the theory are assumed to be governed by the Hamiltonian of a free massive scalar field on a lattice given by (6.35) with $V \equiv 0$. In particular, this Hamiltonian can be split as $\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_{12}$, where $\hat{H}_1$ and $\hat{H}_2$ are the Hamiltonians associated with the regions Λ_1 and Λ_2, respectively, and $\hat{H}_{12}$ couples the two regions across the boundary $\partial\Lambda$, see (6.37).

The dynamics described by this Hamiltonian can also be described by the classical action S given by[1]

$$S[f] = \frac{\varepsilon^d}{2} \int_0^t \mathrm{d}\tau \sum_{x \in \mathscr{L}^d_{\varepsilon,L}} \left[\dot{f}_x^2 - \varepsilon^d \sum_{y \in \mathscr{L}^d_{\varepsilon,L}} \left(f_x \left(-\hat{\triangle}_\varepsilon + m^2 \delta^\varepsilon \right)_{xy} f_y \right) \right] , \tag{7.1}$$

where $\dot{f}_x \coloneqq \partial f_x / \partial \tau$. This action again splits as

$$S[\varphi, \phi] = S_1[\varphi] + S_2[\phi] + S_{12}[\varphi, \phi] , \tag{7.2}$$

[1] In this Part of the thesis, we use the same notation for the classical action as for the entropy in Part II. The context should make it clear which one is meant.

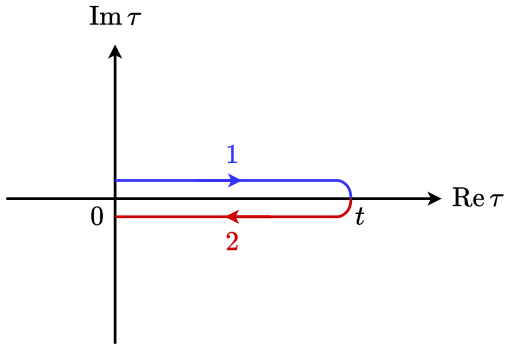


Figure 7.1: Schwinger-Keldysh contour $\mathscr{C}$ for the influence functional. The contour starts at $\tau = 0$, extends to $\tau = t$ and ends again at $\tau = 0$. The contour is composed of two branches. Branch 1 is the forward branch and branch 2 is the backward branch as indicated by the arrows in the sketch. For illustration purposes, the two branches are deformed into the complex plane.

where S_1 and S_2 are the action functionals associated with the regions Λ_1 and Λ_2, respectively, given by

$$S_i[f] = \frac{\varepsilon^d}{2} \int_0^t \mathrm{d}\tau \sum_{x\in\Lambda_i} \left[\dot{f}_x^2 - \varepsilon^d \sum_{y\in\Lambda_i} \left(f_x \left(-\hat{\triangle}_\varepsilon^{\,i} + m^2 \delta^\varepsilon \right)_{xy} f_y \right) \right] \tag{7.3}$$

for $i \in \{1, 2\}$ and S_{12} couples the fields at the boundary, i.e.,

$$S_{12}[\varphi, \phi] = -\varepsilon^{d-2} \int_0^t \mathrm{d}\tau \sum_{x\in\partial\Lambda} \phi_{x+\varepsilon\hat{n}}\, \varphi_x \ . \tag{7.4}$$

7.1 Factorizing Initial State

We start by assuming that the initial state of the system factorizes with respect to the splitting into regions Λ_1 and Λ_2. More precisely, we assume that the initial state of the system is described by a density operator of the form $\hat{\rho}(0) = \hat{\rho}_1(0) \otimes \hat{\rho}_2(0)$, where $\hat{\rho}_1(0)$ and $\hat{\rho}_2(0)$ are the initial states of the regions Λ_1 and Λ_2, respectively. In the Schrödinger representation, such a product state can be written in terms of density kernels as

$$\rho(0; \varphi^1, \phi^1, \varphi^2, \phi^2) = \rho_1(0; \varphi^1, \varphi^2)\, \rho_2(0; \phi^1, \phi^2) \ . \tag{7.5}$$

In this case, the influence functional is given by [165, Sec. 3.2]

$$\mathfrak{F}_{\mathrm{IF}}[\varphi^1, \varphi^2] = \mathrm{e}^{\mathrm{i}S_{\mathrm{IF}}[\varphi^1, \varphi^2, t]} = \int_{\mathscr{C}} \mathcal{D}\phi\ \mathrm{e}^{\mathrm{i}(S_2[\phi] + S_{12}[\varphi, \phi])} \rho_2(0; \phi_0^1, \phi_0^2) \ , \tag{7.6}$$

where $\mathscr{C}$ is Schwinger-Keldysh contour, see Fig. 7.1.

Consider now, for simplicity, Gaussian initial conditions in the environment. In this case, the Gaussian integrals on the expression for the influence functional can be performed exactly, and we obtain for the influence action [165, Sec. 3.2.2]

$$S_{\mathrm{IF}}[\varphi^1,\varphi^2,t] = \frac{\mathrm{i}}{2}\int_0^t \mathrm{d}\tau\,\mathrm{d}\tau'\,\varepsilon^{2(d-1)}\sum_{x,y\in\partial\Lambda}\varphi^a_x(\tau)\,(\mathcal{K}_{xy})_{ab}(\tau,\tau')\,\varphi^b_y(\tau')\;, \tag{7.7}$$

where summation over repeated "time-path" indices $a,b\in\{1,2\}$ is implied and $\mathcal{K}$ is the path ordered propagator given by

$$\mathcal{K}_{xy}(\tau,\tau') \coloneqq \varepsilon^{-2}\begin{pmatrix} \langle \mathcal{T}\phi_{x+\varepsilon\hat{n}}(\tau)\,\phi_{y+\varepsilon\hat{n}}(\tau')\rangle & -\langle \phi_{y+\varepsilon\hat{n}}(\tau')\,\phi_{x+\varepsilon\hat{n}}(\tau)\rangle \\ -\langle \phi_{x+\varepsilon\hat{n}}(\tau)\,\phi_{y+\varepsilon\hat{n}}(\tau')\rangle & \langle \widetilde{\mathcal{T}}\phi_{x+\varepsilon\hat{n}}(\tau)\,\phi_{y+\varepsilon\hat{n}}(\tau')\rangle \end{pmatrix}. \tag{7.8}$$

Here, $\mathcal{T}$ denotes time-ordering (the latest time to the left), $\widetilde{\mathcal{T}}$ denotes anti time-ordering (the latest time to the right) and the expectation values $\langle\cdot\rangle$ are meant with respect to the path integral of the region Λ_2 *without* the coupling to the field in Λ_1, i.e., with respect to the field theory in Λ_2 with spatial Dirichlet boundary conditions. More precisely, the expectation values are given by

$$\langle\mathcal{O}(t')\rangle = \int_{\mathscr{C}}\mathcal{D}\phi\;\mathrm{e}^{\mathrm{i}S_2[\phi]}\,\mathcal{O}(t')\,\rho_2(0;\phi^1_0,\phi^2_0)\;. \tag{7.9}$$

We observe from (7.7) that the influence action is a boundary action. This is a consequence of the fact that the Laplacian linearly couples the interior and exterior field at the boundary and thus the boundary field acts as a source term for the exterior theory. Furthermore, we see from (7.8) that the path ordered propagator scales with a factor ε^{-2}, and it is thus not clear how this expression behaves in the continuum limit.

In order to illuminate on this issue (and to prepare for the discussion in Section 7.2), we now assume that the exterior initial state $\hat{\rho}_2(0)$ is a thermal state with respect to the exterior Hamiltonian $\hat{H}_2$, i.e., $\hat{\rho}_2(0)=\hat{\sigma}_2(\beta)\coloneqq Z_2^{-1}\mathrm{e}^{-\beta\hat{H}_2}$. Physically, this corresponds to a system where the interior and exterior fields are decoupled by imposing Dirichlet boundary conditions on the boundary and letting the exterior equilibrate to a thermal state of inverse temperature β. In this case, the expectation values in (7.8) are taken with respect to the thermal state $\hat{\sigma}_2(\beta)$. The expression for the influence functional in this case reads

$$\mathfrak{F}_{\mathrm{IF}}[\varphi^1,\varphi^2] = \int_{\mathscr{C}_\beta}\mathcal{D}\phi\;\exp\big[\mathrm{i}S_2[\phi^1]+\mathrm{i}S_{12}[\varphi^1,\phi^1]-\mathrm{i}S_2[\phi^2]-\mathrm{i}S_{12}[\varphi^2,\phi^2]-S^2_{\mathrm{E}}[\phi^\beta]\big]\;, \tag{7.10}$$

where we used the expression of thermal states in terms of imaginary time path integrals, see Section 7.2.

We introduce the following 2-point functions,

$$G^{11}_{xy}(\tau,\tau') \coloneqq \langle \mathcal{T}\phi_x(\tau)\,\phi_y(\tau')\rangle_\beta \ , \qquad G^{12}_{xy}(\tau,\tau') \coloneqq \langle \phi_y(\tau')\,\phi_x(\tau)\rangle_\beta \ , \tag{7.11}$$

$$G^{21}_{xy}(\tau,\tau') \coloneqq \langle \phi_x(\tau)\,\phi_y(\tau')\rangle_\beta \ , \qquad G^{22}_{xy}(\tau,\tau') \coloneqq \langle \widetilde{\mathcal{T}}\phi_x(\tau)\,\phi_y(\tau')\rangle_\beta \ , \tag{7.12}$$

for $x, y \in \Lambda_2$, where $\langle . \rangle_\beta$ denotes the expectation value with respect to the state $\hat{\sigma}_2(\beta)$. Notice that these correlators are thermal 2-point functions for a theory with homogeneous spatial Dirichlet boundary conditions, and we can again naturally extend them to $\overline{\Lambda_2}$ by setting $G^{ij}_{xy}(\tau,\tau') = 0$ when at least one spatial index lies on the boundary $\partial\Lambda$.

With these definitions, the path ordered propagator $\mathcal{K}$ can be written as

$$\mathcal{K}_{xy}(\tau,\tau') \coloneqq \begin{pmatrix} g^{11}_{xy}(\tau,\tau') & -g^{12}_{xy}(\tau,\tau') \\ -g^{21}_{xy}(\tau,\tau') & g^{22}_{xy}(\tau,\tau') \end{pmatrix} \tag{7.13}$$

where we introduced the notation

$$g^{ij}_{xy}(\tau,\tau') \coloneqq (\partial^\varepsilon_{n_x}\partial^\varepsilon_{n_y} G^{ij})_{xy}(\tau,\tau') \tag{7.14}$$

and ∂^ε_n is again the discrete normal derivative, cf. Chapter 6.

Upon introducing the notation $\mathrm{x} = (\tau, x)$ and $\mathrm{y} = (\tau', y)$, we can write the continuum version of influence action as

$$S_{\mathrm{IF}}[\varphi^1,\varphi^2,t] = \frac{\mathrm{i}}{2}\int_0^t \mathrm{d}\tau'\,\mathrm{d}\tau \int_{\partial\Omega} \mathrm{d}S(x)\,\mathrm{d}S(y)\ \varphi^a(\mathrm{x})\,\mathcal{K}_{ab}(\mathrm{x},\mathrm{y})\,\varphi^b(\mathrm{y}) \ , \tag{7.15}$$

where the kernel $\mathcal{K}$ is given by

$$\mathcal{K}(\mathrm{x},\mathrm{y}) \coloneqq \begin{pmatrix} g^{11}(\mathrm{x},\mathrm{y}) & -g^{12}(\mathrm{x},\mathrm{y}) \\ -g^{21}(\mathrm{x},\mathrm{y}) & g^{22}(\mathrm{x},\mathrm{y}) \end{pmatrix} \tag{7.16}$$

and

$$g^{ij}(\mathrm{x},\mathrm{y}) \coloneqq \left(\frac{\partial^2}{\partial n_x \partial n_y} G^{ij}\right)(\mathrm{x},\mathrm{y}) \ . \tag{7.17}$$

7.2 Initial State Correlations

In this Section we generalize the result from the previous Section to the case where the initial state of the system is not a product state, i.e., it contains correlations between the system and the environment [166–168]. More precisely, following, e.g., [168], we consider an initial state of the form

$$\rho_{\beta,\lambda}(0;\varphi^1,\phi^1,\varphi^2,\phi^2) = \int [\mathrm{d}\overline{\varphi}^1][\mathrm{d}\overline{\varphi}^2]\ \lambda(\varphi^1,\overline{\varphi}^1,\varphi^2,\overline{\varphi}^2)\,\rho_\beta(0;\overline{\varphi}^1,\phi^1,\overline{\varphi}^2,\phi^2) \ , \tag{7.18}$$

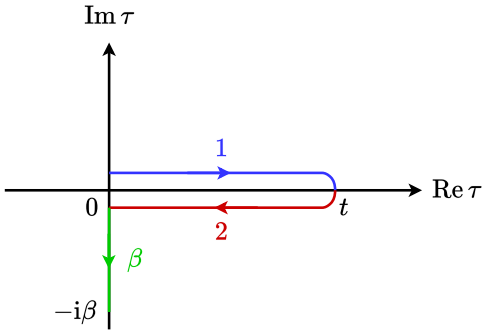


Figure 7.2: Schwinger-Keldysh contour $\mathscr{C}_\beta$ for the influence functional in the presence of a thermal state. In addition to the forward and backward branches, the contour also contains a branch β over the imaginary time interval from 0 to $-\mathrm{i}\beta$, accounting for the thermal part of the initial state. The contour is closed, i.e., the field at imaginary time $-\mathrm{i}\beta$ on the β branch is identified with the field at time 0 on the forward branch. For illustration purposes, the real time branches are deformed into the complex plane.

where we defined

$$[\mathrm{d}\overline{\varphi}^i] \coloneqq \prod_{x\in\Lambda_1} \mathrm{d}\overline{\varphi}^i_x \,. \tag{7.19}$$

Here, ρ_β is the density kernel of the canonical thermal state at inverse temperature β and λ, called the preparation function in [168], parametrizes a deviation from the thermal state within the region Λ_1. We assume that the initial thermal state ρ_β is Gaussian (which amounts to the requirement that the Hamiltonian is quadratic, i.e., $V \equiv 0$ in (6.35)).

The global thermal density kernel can be written as a path integral over field configurations on the imaginary time interval $-\mathrm{i}[0,\beta]$. More precisely, we have

$$\rho_\beta(0;\overline{\varphi}^1,\phi^1,\overline{\varphi}^2,\phi^2) = \frac{1}{Z_\beta}\int_{\overline{\varphi}^2,\phi^2}^{\overline{\varphi}^1,\phi^1} \mathcal{D}\widetilde{\varphi}\,\mathcal{D}\widetilde{\phi}\,\mathrm{e}^{-S_\mathrm{E}[\widetilde{\varphi},\widetilde{\phi}]}\,, \tag{7.20}$$

where the Euclidean action S_E is given by

$$S_\mathrm{E}[f] = \frac{\varepsilon^d}{2}\int_0^\beta \mathrm{d}\tau \sum_{x\in\mathscr{L}^d_{\varepsilon,L}} \left[(\partial_\tau f_x)^2(\tau) + \varepsilon^d \sum_{y\in\mathscr{L}^d_{\varepsilon,L}} f_x(\tau)\,(-\hat{\triangle}_\varepsilon + m^2\delta^\varepsilon)_{xy}\,f_y(\tau)\right]. \tag{7.21}$$

With this setup, it was shown in [168, Sec. III] that the reduced density kernel at time t can be obtained by evolving the initial preparation function, i.e.,

$$\begin{aligned}&\rho_{\mathrm{S},\beta,\lambda}(t;\varphi^1,\varphi^2) = \\ &\frac{1}{Z_\beta}\int^{\varphi^1,\varphi^2} \mathcal{D}\widetilde{\varphi}^1\mathcal{D}\widetilde{\varphi}^2\mathcal{D}\widetilde{\varphi}^\beta\,\mathrm{e}^{\mathrm{i}(S_1[\widetilde{\varphi}^1]-S_1[\widetilde{\varphi}^2])-S^1_\mathrm{E}[\widetilde{\varphi}^\beta]+\mathrm{i}S^\beta_\mathrm{IF}[\widetilde{\varphi}^1,\widetilde{\varphi}^2,\widetilde{\varphi}^\beta,t]}\,\lambda(\varphi^1_0,\varphi^{\beta,1}_0,\varphi^2_0,\varphi^{\beta,2}_0)\,,\end{aligned} \tag{7.22}$$

where the *generalized* influence action S^{β}_{IF} is given by the generalized Feynman-Vernon influence functional $\mathfrak{F}_{\mathrm{IF}}$ as

$$\begin{aligned}\mathfrak{F}^{\beta}_{\mathrm{IF}}[\varphi^1,\varphi^2,\varphi^\beta] &= \mathrm{e}^{\mathrm{i}S^{\beta}_{\mathrm{IF}}[\varphi^1,\varphi^2,\varphi^\beta,t]} \\ &= \int_{\mathscr{C}_\beta} \mathcal{D}\phi\; \mathrm{e}^{\mathrm{i}(S_2[\phi^1]+S_{12}[\varphi^1,\phi^1]-S_2[\phi^2]-S_{12}[\varphi^2,\phi^2])-S^2_{\mathrm{E}}[\phi^\beta]-S^{12}_{\mathrm{E}}[\varphi^\beta,\phi^\beta]}\,,\end{aligned} \tag{7.23}$$

where $\mathscr{C}_\beta$ is the closed Schwinger-Keldysh contour over two connected time paths as well as over an imaginary time contour accounting for the thermal part of the initial state, cf. Fig. 7.2.

The above expression for the generalized influence functional should be compared to the influence functional in (7.10) for the case of a product state with thermal environment. The only difference is the additional term S^{12}_{E} in the exponent, which acts as a boundary source term on the thermal branch. Therefore, we conclude that the initial state correlations are incorporated by a linear system-environment coupling across the spatial boundary on the thermal branch and the expression for the influence functional may be seen as a Gaussian integral over the contour $\mathscr{C}_\beta$ with a (both real and imaginary) time dependent source term on the boundary.

Just like in the last Section, this integral can be solved explicitly and yields the following expression for the influence action

$$S^{\beta}_{\mathrm{IF}}[\varphi^1,\varphi^2,\varphi^\beta,t] = \frac{\mathrm{i}}{2}\int_{I^a}\int_{I^b} \mathrm{d}\tau\,\mathrm{d}\tau'\;\varepsilon^{2(d-1)}\sum_{x,y\in\partial\Lambda}\varphi^a_x(\tau)\,(\mathcal{K}^{\beta}_{xy})_{ab}(\tau,\tau')\,\varphi^b_y(\tau')\,, \tag{7.24}$$

where summation over time-path indices $a,b\in\{1,2,\beta\}$ is assumed and the integration interval I^a is given by $I^1=I^2=(0,t)$ and $I^\beta=(0,\beta)$. The kernel $\mathcal{K}^\beta$ is given by

$$\mathcal{K}^{\beta}_{xy}(\tau,\tau') \coloneqq \begin{pmatrix} g^{11}_{xy}(\tau,\tau') & -g^{12}_{xy}(\tau,\tau') & \mathrm{i}g^{1\beta}_{xy}(\tau,\tau') \\ -g^{21}_{xy}(\tau,\tau') & g^{22}_{xy}(\tau,\tau') & -\mathrm{i}g^{2\beta}_{xy}(\tau,\tau') \\ \mathrm{i}g^{\beta 1}_{xy}(\tau,\tau') & -\mathrm{i}g^{\beta 2}_{xy}(\tau,\tau') & -g^{\beta\beta}_{xy}(\tau,\tau') \end{pmatrix}, \tag{7.25}$$

where again a lower case g indicates a double (discrete) normal derivative in the spatial indices, cf. (7.14), and we introduced the new 2-point functions

$$\begin{aligned} G^{\beta\beta}_{xy}(t,t') &\coloneqq \langle\phi_x(-\mathrm{i}t)\,\phi_y(-\mathrm{i}t')\rangle_\beta = \langle\phi_y(-\mathrm{i}t')\,\phi_x(-\mathrm{i}t)\rangle_\beta\,, \\ G^{i\beta}_{xy}(t,t') &\coloneqq \langle\phi_y(-\mathrm{i}t')\,\phi_x(t)\rangle_\beta\,, \\ G^{\beta i}_{xy}(t,t') &\coloneqq \langle\phi_x(-\mathrm{i}t)\,\phi_y(t')\rangle_\beta\,, \end{aligned} \tag{7.26}$$

for $x, y \in \Lambda_2$ and $i \in \{1, 2\}$. Here, the expectation values are again the thermal expectation values with respect to the exterior state $\hat{\sigma}_2(\beta)$ as indicated by the subscript β. Notice that $G^{\beta 1} = G^{\beta 2}$ and $G^{1\beta} = G^{2\beta}$. In continuum notation, the influence action can be written as

$$S_{\mathrm{IF}}^{\beta}[\varphi^1, \varphi^2, \varphi^\beta, t] = \frac{\mathrm{i}}{2} \int_{I^a} \int_{I^b} \mathrm{d}\tau \, \mathrm{d}\tau' \int_{\partial\Omega} \mathrm{d}S(x) \, \mathrm{d}S(y) \; \varphi^a(\mathrm{x}) \, \mathcal{K}_{ab}^{\beta}(\mathrm{x}, \mathrm{y}) \, \varphi^b(\mathrm{y}) \; , \tag{7.27}$$

where the kernel $\mathcal{K}^\beta$ is given by

$$\mathcal{K}^{\beta}(\mathrm{x}, \mathrm{y}) = \begin{pmatrix} g^{11}(\mathrm{x}, \mathrm{y}) & -g^{12}(\mathrm{x}, \mathrm{y}) & \mathrm{i}g^{1\beta}(\mathrm{x}, \mathrm{y}) \\ -g^{21}(\mathrm{x}, \mathrm{y}) & g^{22}(\mathrm{x}, \mathrm{y}) & -\mathrm{i}g^{2\beta}(\mathrm{x}, \mathrm{y}) \\ \mathrm{i}g^{\beta 1}(\mathrm{x}, \mathrm{y}) & -\mathrm{i}g^{\beta 2}(\mathrm{x}, \mathrm{y}) & -g^{\beta\beta}(\mathrm{x}, \mathrm{y}) \end{pmatrix} . \tag{7.28}$$

We note that the generalized influence action S_{IF}^{β} can be written as

$$S_{\mathrm{IF}}^{\beta}[\varphi^1, \varphi^2, \varphi^\beta, t] = S_{\mathrm{IF}}[\varphi^1, \varphi^2, t] + S_{\mathrm{IF}}^{\mathrm{corr.}}[\varphi^1, \varphi^2, \varphi^\beta, t] - \mathrm{i}\Delta S_{\mathrm{E}}^{1}[\varphi^\beta] \; , \tag{7.29}$$

where the first term is the influence action for the product ansatz (see Section 7.1), the second term accounts for the initial correlations between the system and the environment, and the third term contains the non-local boundary conditions for the Euclidean action of the region Λ_1, cf. Section 6.1. More precisely, the third term is given by

$$\Delta S_{\mathrm{E}}^{1}[\varphi^\beta] = \frac{1}{2} \int_0^\beta \mathrm{d}\tau \, \mathrm{d}\tau' \; \varepsilon^{2(d-1)} \sum_{x,y \in \partial\Lambda} \varphi_x^\beta(\tau) \, g_{xy}^{\beta\beta}(\tau, \tau') \, \varphi_y^\beta(\tau') \; , \tag{7.30}$$

and in particular we have

$$\widetilde{S_{\mathrm{E}}^{1}}[\varphi^\beta] = S_{\mathrm{E}}^{1}[\varphi^\beta] - \Delta S_{\mathrm{E}}^{1}[\varphi^\beta] \; , \tag{7.31}$$

see (6.30) and the related discussion in Section 6.1.

Plugging these results back into the expression for the reduced density kernel, we obtain

$$\rho_{\mathrm{S},\beta,\lambda}(t; \varphi^1, \varphi^2) = \int^{\varphi^1, \varphi^2} \mathcal{D}\widetilde{\varphi}^1 \mathcal{D}\widetilde{\varphi}^2 \mathcal{D}\widetilde{\varphi}^\beta \; \mathrm{e}^{\mathrm{i}S_{\mathrm{eff}}[\widetilde{\varphi}^1, \widetilde{\varphi}^2, \widetilde{\varphi}^\beta, t] - \widetilde{S_{\mathrm{E}}^{1}}[\widetilde{\varphi}^\beta]} \; \lambda(\varphi_0^1, \varphi_0^{\beta,1}, \varphi_0^2, \varphi_0^{\beta,2}) \; , \tag{7.32}$$

where we defined the effective action S_{eff} as

$$S_{\mathrm{eff}}[\varphi^1, \varphi^2, \varphi^\beta, t] \coloneqq S_1[\varphi^1] - S_1[\varphi^2] + S_{\mathrm{IF}}[\varphi^1, \varphi^2, t] + S_{\mathrm{IF}}^{\mathrm{corr.}}[\varphi^1, \varphi^2, \varphi^\beta, t] \; . \tag{7.33}$$

Notice that the limit $t \to 0^+$ reproduces the correct initial condition for the system.

We remark that the effect of initial correlations between the system and the environment is again encoded in boundary terms, yielding non-local boundary conditions for the differential operator in the action.

7.3 Dynamics of Field Expectation Values

In this Section, we derive a stochastic equation of motion for the expectation values of the field in the interior region. We start by considering the continuum effective action S_{eff}

$$S_{\text{eff}}[\varphi^1, \varphi^2, t] \coloneqq S_1[\varphi^1] - S_1[\varphi^2] + S_{\text{IF}}[\varphi^1, \varphi^2, t] \,, \tag{7.34}$$

where S_{IF} is the influence action for factorizing initial conditions and a thermal environment as given in (7.15). We then derive the equation of motion for the expectation values of the field in the interior region by taking the functional derivative of the effective action with respect to the field.

Following [165], we introduce the difference field $\Psi \coloneqq \varphi^1 - \varphi^2$ and the "centre of mass" field $\Xi \coloneqq \frac{1}{2}(\varphi^1 + \varphi^2)$. Using the difference and centre of mass fields, the influence action S_{IF} can be written as [165, Sec. 3.2.2]

$$S_{\text{IF}}[\Psi, \Xi, t] = \int_0^t \mathrm{d}\tau \, \mathrm{d}\tau' \int_{\partial\Omega} \mathrm{d}S(x) \, \mathrm{d}S(y) \left[\Psi(\mathrm{x}) \, \mathbf{D}(\mathrm{x}, \mathrm{y}) \, \Xi(\mathrm{y}) + \frac{\mathrm{i}}{2} \Psi(\mathrm{x}) \, \mathbf{N}(\mathrm{x}, \mathrm{y}) \, \Psi(\mathrm{y}) \right] , \tag{7.35}$$

where $\mathbf{D}$ and $\mathbf{N}$ are the dissipation and noise kernels, respectively, given by

$$\mathbf{D}(\mathrm{x}, \mathrm{y}) = \mathrm{i}\Theta(\tau - \tau') \, \frac{\partial^2}{\partial n_x \partial n_y} \big(G^{21}(\mathrm{x}, \mathrm{y}) - G^{12}(\mathrm{x}, \mathrm{y}) \big) \,, \tag{7.36}$$

$$\mathbf{N}(\mathrm{x}, \mathrm{y}) = \frac{1}{2} \frac{\partial^2}{\partial n_x \partial n_y} \big(G^{21}(\mathrm{x}, \mathrm{y}) + G^{12}(\mathrm{x}, \mathrm{y}) \big) \,. \tag{7.37}$$

In the following, we will denote by $\hat{\mathbf{D}}$ and $\hat{\mathbf{N}}$ the integral operators with kernels $\mathbf{D}(\mathrm{x}, \mathrm{y})$ and $\mathbf{N}(\mathrm{x}, \mathrm{y})$, respectively.

When writing the expression $S_1[\varphi^1] - S_1[\varphi^2]$ in terms of the fields Ψ and Ξ, we arrive at the expression

$$S_1[\varphi^1] - S_1[\varphi^2] = \int_0^t \mathrm{d}\tau \int_\Omega \mathrm{d}^d x \, \left[\dot{\Psi}\dot{\Xi} - \boldsymbol{\nabla}\Psi \cdot \boldsymbol{\nabla}\Xi - m^2 \Psi \Xi \right] . \tag{7.38}$$

For the effective action $S_{\text{eff.}}$ defined in (7.34), we obtain

$$\begin{aligned} S_{\text{eff.}}[\Psi, \Xi, t] = &\int_0^t \mathrm{d}\tau \int_\Omega \mathrm{d}^d x \, \left[\dot{\Psi}(\mathrm{x}) \, \dot{\Xi}(\mathrm{x}) - \boldsymbol{\nabla}\Psi(\mathrm{x}) \cdot \boldsymbol{\nabla}\Xi(\mathrm{x}) - m^2 \Psi(\mathrm{x}) \Xi(\mathrm{x}) \right] \\ &+ \int_0^t \mathrm{d}\tau \, \mathrm{d}\tau' \int_{\partial\Omega} \mathrm{d}S(x) \, \mathrm{d}S(y) \left[\Psi(\mathrm{x}) \, \mathbf{D}(\mathrm{x}, \mathrm{y}) \, \Xi(\mathrm{y}) + \frac{\mathrm{i}}{2} \Psi(\mathrm{x}) \, \mathbf{N}(\mathrm{x}, \mathrm{y}) \, \Psi(\mathrm{y}) \right] . \end{aligned} \tag{7.39}$$

We furthermore can write the factor containing the noise kernel as the Fourier transform of a Gaussian measure (see also Appendix C), i.e. (we use the notation $\int_{t,\partial\Omega} = \int_0^t \mathrm{d}\tau \int_{\partial\Omega} \mathrm{d}S(x)$ and similarly for $\int_{t,\Omega}$),

$$\exp\left[-\frac{1}{2}\iint_{t,\partial\Omega} \Psi(\mathrm{x})\,\mathbf{N}(\mathrm{x},\mathrm{y})\,\Psi(\mathrm{y})\right] = \int \mathrm{d}\mathcal{P}(\xi)\ \exp\left[\mathrm{i}\int_{t,\partial\Omega} \xi(\mathrm{x})\Psi(\mathrm{x})\right], \tag{7.40}$$

where $\mathcal{P}$ is a centred Gaussian measure with covariance given by

$$\mathbb{E}_{\mathcal{P}}[\xi(\mathrm{x})\xi(\mathrm{y})] = \mathbf{N}(\mathrm{x},\mathrm{y})\ . \tag{7.41}$$

Thus, we can write

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}S_{\text{eff.}}[\Psi,\Xi,t]} = \\ &\int \mathrm{d}\mathcal{P}(\xi)\ \exp\left[\mathrm{i}\left(\int_{t,\Omega} \dot\Psi\,\dot\Xi - \boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}\Xi - m^2\Psi\Xi\right) + \mathrm{i}\left(\int_{t,\partial\Omega} \xi\Psi + \Psi\hat{\mathbf{D}}\Xi\right)\right]. \end{aligned} \tag{7.42}$$

We may now write the effective action as containing the Gaussian random variable ξ as

$$S^{\xi}_{\text{eff.}}[\Psi,\Xi,t] = \int_{t,\Omega} \dot\Psi\,\dot\Xi - \boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}\Xi - m^2\Psi\Xi + \int_{t,\partial\Omega} \xi\Psi + \Psi\hat{\mathbf{D}}\Xi\ . \tag{7.43}$$

For the above expression to make sense, we need to keep in mind that ultimately we need to average with respect to the measure $\mathcal{P}$.

The stochastic equation of motion for the expectation value of the interior field is obtained by extremizing the effective action with respect to the difference field Ψ, i.e. (see [165, Sec. 5.1.3] and [169, App. A]),

$$\left.\frac{\delta S^{\xi}_{\text{eff.}}[\Psi,\Xi,t]}{\delta\Psi}\right|_{\Psi=0} = 0\ . \tag{7.44}$$

For the case at hand and upon denoting the expectation value of the field φ by v, this yields the following stochastic boundary value problem for the Klein-Gordon equation,

$$\left(\frac{\partial^2}{\partial\tau^2} - \triangle + m^2\right) v(\mathrm{x}) = 0\ , \qquad \mathrm{x}\in(0,t]\times\Omega\ , \tag{7.45}$$

$$\frac{\partial v}{\partial n_x}(\mathrm{x}) + (\hat{\mathbf{D}}v)(\mathrm{x}) = -\xi(\mathrm{x})\ , \qquad \mathrm{x}\in[0,t]\times\partial\Omega\ , \tag{7.46}$$

$$\begin{cases} v(0,x) = f(x) \\ \dot v(0,x) = g(x) \end{cases}, \qquad x\in\Omega\ . \tag{7.47}$$

We observe that the stochastic part of the dynamics is encoded in the spatial boundary conditions for the field. The noise kernel $\mathbf{N}$ induces a stochastic "force" term $-\xi$ in (7.46), while the dissipation kernel $\mathbf{D}$ causes the boundary conditions to be non-local both in space

and time. More precisely, fix some $\mathrm{x} = (t', x) \in (0, t) \times \partial\Omega$. Then, the term $(\hat{\mathbf{D}}v)(\mathrm{x})$ in (7.46) is given by

$$\begin{aligned}(\hat{\mathbf{D}}v)(\mathrm{x}) &= \int_0^t \mathrm{d}\tau \int_{\partial\Omega} \mathrm{d}S(y)\, \mathbf{D}(\mathrm{x},\mathrm{y})\, v(\mathrm{y}) \\ &= \mathrm{i} \int_0^{t'} \mathrm{d}\tau \int_{\partial\Omega} \mathrm{d}S(y)\, \frac{\partial^2}{\partial n_x \partial n_y} \big(G^{21}(\mathrm{x},\mathrm{y}) - G^{12}(\mathrm{x},\mathrm{y})\big)\, v(\mathrm{y})\ .\end{aligned} \tag{7.48}$$

Notice that the Heaviside function in the expression for the dissipation kernel $\mathbf{D}$ (see (7.36)) restricts the time integral to the interval $[0, t']$. We see that the normal derivative of the field expectation value v at a point $\mathrm{x} = (t', x)$ of the spacetime boundary $\mathbb{R}_+ \times \partial\Omega$ depends on the value of the field expectation value at all points of the spatial boundary $\partial\Omega$ at all times up to t'. This generalizes the result from Section 6.1, where the normal derivative depends on the value of the field on the whole boundary $\partial\Omega$ of some Euclidean region Ω. The dependence of the normal derivative on the value of the field expectation in the whole past indicates that the time evolution described by (7.45) – (7.47) is non-Markovian.

Part IV

Conclusion

Disclaimer. The conclusion and outlook to Part II of this thesis are taken from [47]. Parts of this conclusion, namely those which were contributed by me, are partially taken verbatim from the aforementioned reference.

In this thesis, we have studied the information theoretic properties of classical statistical field theories and the time evolution of states of relativistic quantum field theories restricted to a spatial subregion.

In the context of information theory in statistical field theories, we considered the relative entropy between theories with different masses or boundary conditions. Our findings revealed a crucial dependence on the dimension of Euclidean spacetime in the case of different masses. In particular, we demonstrated that the relative entropy is finite in dimensions $d < 4$ and infinite in higher dimensions. Furthermore, we demonstrated that in dimensions $d < 4$, the relative entropy behaves in a manner consistent with its interpretation in terms of the distinguishability of two theories.

In the case of fields with different boundary conditions, we demonstrated that the relative entropy between a Dirichlet and a Neumann field (or, more generally, a Robin field) is always infinite. The relative entropy between two Robin fields is finite if the dimension d of Euclidean spacetime is less than three. The relative entropy between a Robin field and a field with free boundary conditions in one spatial dimension was explicitly calculated.

Furthermore, we discussed the mutual information between two disjoint regions in Euclidean space. Our findings show that the mutual information between two such regions is finite if and only if the regions are separated by a finite distance. We argue that this is due to the Markov property of the scalar field, which implies that only the degrees of freedom at the boundary of the region contribute to the mutual information. This idea is supported by an explicit calculation of the mutual information in $d = 1$.

An intriguing observation pertains to the significance of boundary terms in the context of information-theoretic aspects of field theories. To illustrate, the mutual information between degrees of freedom in non-overlapping regions would vanish if local boundary conditions (such as Dirichlet or Neumann) were selected at every point on the boundary. It is only through less restrictive choices, such as free boundary conditions, that information can be shared between regions.

For the local time evolution of states in relativistic quantum field theories, we have shown that the time evolution is non-unitary and that the reduced state evolves in time like an open quantum system. In particular, the dynamical coupling between the system (the degrees of freedom within some spatial region) and the environment (the degrees of freedom in the complement of that region) occurs via the differential operator in the classical action. Due to this local structure, the coupling between the system and the environment is linear (and thus reminiscent of the Caldeira-Leggett model [159–161]) and only across the boundary of the region. Therefore, the effects of the system-environment coupling on the time evolution of the reduced density operator are entirely contained in an effective boundary action.

In addition to the linear system-environment coupling, it is necessary to consider initial state correlations. This is due to the fact that states in relativistic quantum field theories are generically entangled across spacetime regions. We incorporated initial state correlations by considering (global) thermal states with local excitations. Since the excitations are purely

contained within the interior region, the initial state correlations are only due to the global thermal state, which can be taken care of using a Euclidean path integral representation. It is noteworthy that the incorporation of initial state correlations results in the emergence of an effective boundary term in the action, exhibiting a structural similarity to the one induced by the system-environment coupling. In particular, no bulk terms are induced by the initial state correlations considered here.

It is worth noting that the boundary terms encoding the non-unitary contributions to the time evolution of the reduced state are similar to the free boundary conditions considered in Part II in the context of information theory in classical field theories. More specifically, the dissipation and noise kernels are both given by a double normal derivative of an exterior Dirichlet 2-point function. In particular, in Section 6.1 we considered a Euclidean field theory and demonstrated that the splitting of the action yields a boundary action that precisely describes the free boundary conditions already encountered in Part II. This indicates that the aforementioned structure of the boundary action, namely a double normal derivative of some exterior Dirichlet Green's function, is a generic phenomenon when considering the restriction of field theoretic states to spatial subregions.

Finally, we derived a stochastic equation of motion for the time evolution within a spatial region. Structurally, this equation is just the usual Klein-Gordon equation but with stochastic spatial boundary conditions. More precisely, these boundary conditions are induced by the dissipation and noise kernels previously shown to be entirely supported on the boundary. We argue that such a stochastic partial differential equation is interesting on its own right. It would be of interest to investigate the properties of solutions to such an equation, in particular the emergence of non-trivial boundary effects due to the stochastic boundary conditions.

We now propose a number of avenues for further investigation. With regard to the information theory of statistical fields, we suggest further investigations of more general choices of boundary conditions. Given the significant role that boundary conditions play in determining information-theoretic properties, it would be highly interesting to investigate in more detail how different states are fixed through such boundary terms, including those beyond the classical boundary conditions that we have investigated in this work.

An obvious extension of the study of relative entropy in classical field theory presented here is to allow for more general choices of theories. For example, one can consider space-dependent masses, which would lead to the theory of Schrödinger operators and is closely related to the possible inclusion of gauge fields and a space-dependent Riemannian metric. Finally, the generalization to interacting theories, i.e. non-Gaussian theories, immediately comes to mind when studying this manuscript. In a functional formulation of Euclidean field theory this is possible, but requires further regularization (in the ultraviolet) and renormalization. For the information-theoretic properties and the propagation of information between regions, the quadratic sector of the theory seems to play the most important role, and we therefore believe that many of the insights gained here will persist, at least on the conceptual and qualitative level, also for interacting theories.

Beyond classical field theory, relative entropies can also be applied to *quantum* field theories. As mentioned in the introduction, this has already been done successfully in the framework of algebraic quantum field theory [2, 3, 32–36, 38, 45, 55, 56, 170]. It would now be interesting to investigate how the functional integral approach can be combined with the algebraic formalism. Functional methods have great practical advantages over the algebraic approach in the sense that they allow one to treat (phenomenologically interesting) interacting theories non-perturbatively, although at the cost of mathematical rigour. In this sense, the present work can also be seen as a step towards a functional treatment of information theory in continuum quantum field theories.

Finally, recent developments in the field of analogue quantum field simulators have opened up new avenues for the experimental investigation of information-theoretic properties of quantum field theories. In particular, as described in [115], ultracold Bose gases can be prepared in thermal states with respect to a massive Klein-Gordon Hamiltonian. Once the full quantum covariance matrix of this state has been reconstructed, the mutual information between subintervals can be extracted. It is reasonable to assume that if this system is at high temperatures, the thermal fluctuations will dominate the quantum fluctuations, allowing the field theory to be effectively described by a classical field theory. This would allow for the experimental investigation of the information-theoretic properties of classical field theories, as discussed in this thesis. It would be of interest to investigate the transition from the quantum to the classical regime and the associated changes in the information-theoretic properties beyond the high temperature regime. In order to do this, results for the *quantum* mutual information in the same setup would be required.

With regard to the local time evolution, it would be beneficial to investigate whether and in what sense the non-unitary time evolution of the reduced state causes local thermalization and thus a first-principle explanation of the emergence of an effective fluid dynamic description of quantum field theories. It is conceivable that in order to truly observe thermalization, one must go beyond the non-interacting toy model considered in this work. Nevertheless, the model presented here provides a foundation for further investigations, as the system-environment coupling occurs via the free part of the action even in interacting theories.

Finally, it is necessary to define precisely what is meant by local thermalization. Interestingly, this may be an information-theoretic question, which would lead back to the previous part of this thesis and, in particular, to the notation of a relative entropy. As argued in, for example, [94] and [171], a reasonable notion of subsystem thermalization is the vanishing of the quantum relative entropy $S(\hat{\rho}_\Lambda \| \hat{\sigma}^\beta_\Lambda)$, where $\hat{\rho}_\Lambda$ is the reduced state of a subsystem Λ and $\hat{\sigma}^\beta_\Lambda$ is the *reduced* thermal state of the same subsystem of some inverse temperature β. It would be of interest to ascertain whether this concept of thermalization can be related to the local time evolution of the reduced state within the context of the model presented here.

Part V

Supplementary Material

In the last part of this thesis, we collect some supplementary results needed in the main body of this work. Chapter A provides a brief introduction to selected topics in functional analysis. In particular, we discuss the connection between (unbounded) self-adjoint operators and semi-bounded quadratic forms *à la* Kato [129] and give a short summary of Hilbert-Sobolev spaces.

In Chapter B, we use the results from the theory of quadratic forms introduced in Section A.1 to rigorously define the Laplace operator with boundary conditions on a bounded domain $\Omega \subset \mathbb{R}^d$. More specifically, we discuss the classical choices of Dirichlet, Neumann, Robin and periodic boundary conditions and furthermore introduce, following [80], so-called free boundary conditions.

Chapter C provides a brief introduction to Gaussian measure theory. We show how Minlos' Theorem [172] can be used to rigorously define Gaussian measures on spaces of generalized functions or distributions. Furthermore, we provide conditions for equivalence of Gaussian measures in Section C.2. Finally, we derive a useful formula for the relative entropy between two Gaussian measures of the type used in Part II of this thesis.

Finally, Chapter D provides a brief introduction to the Schwinger-Keldysh and Feynman-Vernon influence functional formalisms used to describe the time evolution of open quantum systems. The techniques outlined in this Chapter are used in Part III of this thesis to study the local dynamics of a relativistic quantum field.

Disclaimer. Appendices A, B, and C are taken from [47] of which I am a co-author. Parts of these appendices, namely those which were contributed by me, are partially taken verbatim from [47] in the following.

A Functional Analysis

This Appendix provides a brief overview of the functional analysis concepts used in this work. References on the subject include [118, 123, 129, 173].

A.1 Operators on Hilbert Space

In the following, let $\mathcal{H}$ be a real or complex separable Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. An operator $T \in \mathscr{B}(\mathcal{H})$ is called positive if $\langle f, Tf \rangle \geq 0$ for all $f \in \mathcal{H}$. Note that a bounded positive operator is necessarily self-adjoint if $\mathcal{H}$ is a Hilbert space over $\mathbb{C}$ but that this is not true for real Hilbert spaces. An operator $T \in \mathscr{B}(\mathcal{H})$ is called strictly positive if it is positive and $\langle f, Tf \rangle = 0$ precisely when $f = 0$. We call a symmetric operator T with domain $\mathfrak{D}(T) \subset \mathcal{H}$ bounded from below (or semi-bounded) if there exists $c \in \mathbb{R}$ such that $\langle f, Tf \rangle \geq c\|f\|^2$ for all $f \in \mathfrak{D}(T)$.

Proposition A.1. *Let $T : \mathfrak{D}(T) \to \mathcal{H}$ be a self-adjoint operator bounded from below by some $c > 0$. Then, T is a bijection and its inverse T^{-1} is a strictly positive bounded operator on $\mathcal{H}$ with $\|T^{-1}\| \leq c^{-1}$.*

Proof. Since T is bounded from below by a positive number, we have $\|T^{1/2} f\| \geq 0$ for all $f \in \mathfrak{D}(T)$ with equality precisely when $f = 0$. Therefore, $T^{1/2}$ is injective and consequently so is T. Since $\mathrm{ran}(T)^{\perp} = \ker(T^*)$ for any densely defined T [173, Prop. 1.6], injectivity implies that the range of T is dense in $\mathcal{H}$. Furthermore, by Cauchy-Schwarz, $\|Tf\| \geq c\|f\|$ for all $f \in \mathfrak{D}(T)$. Fix $\phi \in \mathcal{H}$. Then, there exists $\{\phi_n\}_{n=1}^{\infty} \subset \mathrm{ran}(T)$ such that $\phi_n \to \phi$ in $\mathcal{H}$. Let $\{f_n\}_{n=1}^{\infty} \subset \mathfrak{D}(T)$ be such that $\phi_n = Tf_n$ for all $n \in \mathbb{N}$. By the semi-boundedness of T, $\{f_n\}_{n=1}^{\infty}$ is Cauchy, and we denote its limit by f. By assumption T is self-adjoint and thus its graph $\Gamma(T)$ is complete in $\mathcal{H} \oplus \mathcal{H}$. In particular, $\lim_{n\to\infty}(f_n, Tf_n) = (f, Tf) \in \Gamma(T)$, which implies that $\phi = Tf$. Therefore, $\mathrm{ran}(T) = \mathcal{H}$.

By Hellinger-Toeplitz [123, p. 84], T^{-1} is bounded. Fix $\psi \in \mathcal{H}$, which can be written as $\psi = T\varphi$ for some $\varphi \in \mathcal{H}$. Then, $\langle \psi, T^{-1}\psi \rangle = \langle T\varphi, \varphi \rangle \geq c\|\varphi\|^2 \geq 0$, where, by the injectivity of T, equality holds if and only if $\psi = 0$. Therefore, T^{-1} is strictly positive. Lastly,

$$\|T^{-1}\| = \sup_{\psi \neq 0} \frac{\|T^{-1}\psi\|}{\|\psi\|} = \sup_{\varphi \neq 0} \frac{\|\varphi\|}{\|T\varphi\|} = \sup_{\|\varphi\|=1} \left(\|T\varphi\|\right)^{-1} \leq c^{-1} \,. \tag{A.1}$$

■

In the following, we assume $\mathcal{H}$ to be a complex Hilbert space. Let $\mathfrak{q} : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ be a (not necessarily bounded) sesquilinear form, $\mathcal{D}$ a dense linear subspace of $\mathcal{H}$. Then, $\mathcal{D}$ is called the form domain of $\mathfrak{q}$, and we write $\mathcal{Q}(\mathfrak{q}) = \mathcal{D}$. The sesquilinear form $\mathfrak{q}$ defines a quadratic form on $\mathcal{Q}(\mathfrak{q})$ in the obvious way. Conversely, a quadratic form on a complex Hilbert space defines a sesquilinear form via polarization. We call a sesquilinear form $\mathfrak{q}$ symmetric (or Hermitian) if $\mathfrak{q}(f, g) = \overline{\mathfrak{q}(g, f)}$ for all $f, g \in \mathcal{Q}(\mathfrak{q})$. A symmetric form $\mathfrak{q}$ is called bounded from below if there exists $c \in \mathbb{R}$ such that $\mathfrak{q}(f) \geq c\|f\|^2$ for all $f \in \mathcal{Q}(\mathfrak{q})$, in which case we simply write $\mathfrak{q} \geq c$. The largest such number c is called the lower bound of $\mathfrak{q}$. A symmetric form $\mathfrak{q}$ is called positive if $\mathfrak{q} \geq 0$.

A form $\mathfrak{q}$, bounded from below by $c \in \mathbb{R}$, is called closed if the form domain $\mathcal{Q}(\mathfrak{q})$ is complete with respect to the norm

$$|||f|||^2 = \mathfrak{q}(f) + (1 - c)\|f\|^2 \,. \tag{A.2}$$

If $\mathfrak{q}$ is closed and $S \subset \mathcal{Q}(\mathfrak{q})$ is $|||.|||$-dense in $\mathcal{Q}(\mathfrak{q})$, we call S a form core of $\mathfrak{q}$. A form $\mathfrak{q}$ is called closable if it has a closed extension.

Theorem A.2 (Kato's representation Theorem)**.** *Let $\mathfrak{q}$ be a densely defined, closed, symmetric form bounded from below. Then, there exists a unique densely defined, self-adjoint operator T, bounded from below with the same lower bound as $\mathfrak{q}$, such that*

- $\mathfrak{D}(T) \subset \mathcal{Q}(\mathfrak{q})$ *and* $\mathfrak{q}(f, g) = \langle Tf, g\rangle$ *for all* $f \in \mathfrak{D}(T)$ *and* $g \in \mathcal{Q}(\mathfrak{q})$,
- $\mathfrak{D}(T)$ *is a form core of* $\mathfrak{q}$,
- *if* $f \in \mathcal{Q}(\mathfrak{q})$, $h \in \mathcal{H}$ *and* $\mathfrak{q}(f, g) = \langle h, g\rangle$ *for every* g *belonging to a core of* $\mathfrak{q}$, *then* $f \in \mathfrak{D}(T)$ *and* $Tf = h$.

If $\mathfrak{q}$ is, in addition, positive, then $\mathcal{Q}(T) = \mathfrak{D}(T^{1/2})$ and $\mathfrak{q}(f, g) = \langle T^{1/2}f, T^{1/2}g\rangle$ for all $f, g \in \mathcal{Q}(\mathfrak{q})$.

Proofs of this Theorem can be found in [123, Thm. VIII.15] and [129, Thm. VI.2.1, Thm. VI.2.6 & Thm. VI.2.23]. We call T the operator associated with the form $\mathfrak{q}$ and call $\mathcal{Q}(T) \coloneqq \mathcal{Q}(\mathfrak{q})$ the form domain of the operator T.

A.2 Hilbert-Sobolev Spaces

In the following, let $s \in \mathbb{R}$, $d \in \mathbb{N}$ and $m > 0$. We denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of functions of rapid decrease equipped with its usual Fréchet topology [174].

Definition A.1. The *Bessel potential of order s and mass m* $\mathcal{J}^s : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is defined as the continuous linear operator given by

$$(\mathcal{J}^s f)(x) = \int_{\mathbb{R}^d} (|\boldsymbol{p}|^2 + m^2)^{\frac{s}{2}} \hat{f}(\boldsymbol{p})\, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\, \frac{\mathrm{d}^d p}{(2\pi)^d} \,, \tag{A.3}$$

where $\hat{f}$ denotes the Fourier transform of f.

Definition A.2 (Hilbert-Sobolev space)**.** The Hilbert-Sobolev space $H^s(\mathbb{R}^d)$ of order s on $\mathbb{R}^d$ is defined as the Hilbert space completion of the space of Schwartz functions $\mathscr{S}(\mathbb{R}^d)$ with respect to the inner product

$$\langle f, g\rangle_s \coloneqq \langle \mathcal{J}^s f, \mathcal{J}^s g\rangle_{L^2(\mathbb{R}^d)} \ . \tag{A.4}$$

We have the following chain of continuous inclusions of Hilbert spaces

$$\ldots \hookrightarrow H^{+2}(\mathbb{R}^d) \hookrightarrow H^{+1}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow H^{-1}(\mathbb{R}^d) \hookrightarrow H^{-2}(\mathbb{R}^d) \hookrightarrow \ldots \ , \tag{A.5}$$

where each inclusion has dense image and $H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$.

Let $K \subset \mathbb{R}^d$ be closed. The linear space

$$H^s_K = \left\{\varphi \in H^s(\mathbb{R}^d) \ : \ \operatorname{supp}\varphi \subseteq K\right\} , \tag{A.6}$$

is a closed subspace of $H^s(\mathbb{R}^d)$. We denote by e_K the orthogonal projection from $H^s(\mathbb{R}^d)$ onto H^s_K. Let $\Omega \subset \mathbb{R}^d$ be open. The space $H^s(\Omega)$ is defined as the linear space of restrictions to Ω (in the sense of distributions) of elements in $H^s(\mathbb{R}^d)$. It is a Hilbert space when equipped with the inner product

$$\langle f, g\rangle_{H^s(\Omega)} \coloneqq \langle p_\Omega F, p_\Omega G\rangle_s \ , \tag{A.7}$$

where $F, G \in H^s(\mathbb{R}^d)$ such that $f = F|_\Omega$ and $g = G|_\Omega$ and $p_\Omega \coloneqq I - e_{\mathbb{R}^d\setminus\Omega}$. Equivalently, $H^s(\Omega)$ is the Hilbert space completion of $C^\infty(\overline{\Omega})$ with respect to the inner product (A.7). Finally, we define, for $\Omega \subset \mathbb{R}^d$ open, the space $\widetilde{H}^s(\Omega)$ as the closure of $C^\infty_0(\Omega)$ in $H^s(\mathbb{R}^d)$ and the space $H^s_0(\Omega)$ as the closure of $C^\infty_0(\Omega)$ in $H^s(\Omega)$. The spaces $\widetilde{H}^s(\Omega)$ and $H^s_0(\Omega)$ are Hilbert spaces when equipped with the inner products of $H^s(\mathbb{R}^d)$ and $H^s(\Omega)$, respectively.

B Boundary Conditions of the Laplace Operator

Throughout this Section, we denote by $\triangle := \sum_{i=1}^{d} \partial^2/\partial x_i^2$ the Laplace operator on $\mathbb{R}^d$ and let $m > 0$ be a positive mass parameter. Furthermore, we denote the operator $f \mapsto (-\triangle+m^2)f$ defined on $C_0^\infty(\mathbb{R}^d)$ by $(-\triangle+m^2)|_{C_0^\infty}$. It is a densely defined, strictly positive and symmetric operator on $L^2(\mathbb{R}^d)$ bounded from below by m^2. Since it is symmetric, it is closable [123, p. 255]. Moreover, it can be shown that $(-\triangle+m^2)|_{C_0^\infty}$ is essentially self-adjoint as an operator on $L^2(\mathbb{R}^d)$ [175, Thm. 3.5.3]. Therefore, $(-\triangle+m^2)|_{C_0^\infty}$ (as an operator on $L^2(\mathbb{R}^d)$) has one and only one self-adjoint extension [123, p. 256], namely its closure

$$D := \overline{(-\triangle+m^2)|_{C_0^\infty}}^{L^2(\mathbb{R}^d)} . \tag{B.1}$$

The domain of D is the completion of $C_0^\infty(\mathbb{R}^d)$ in the norm $|||f||| = \|f\|_{L^2} + \|Df\|_{L^2}$, see [123, Prob. VIII.15], i.e., $\mathfrak{D}(D) = H^{+2}(\mathbb{R}^d)$ (see Appendix A.2). Since D is the closure of a strictly positive operator, it is also strictly positive and furthermore it is also bounded from below by m^2, see [173, §14].

Since D is bounded from below by $m^2 > 0$, it is a bijection from its domain $H^{+2}(\mathbb{R}^d)$ onto $L^2(\mathbb{R}^d)$, cf. Proposition A.1. Furthermore, its inverse D^{-1} is a strictly positive bounded self-adjoint pseudo-differential operator with $\|D^{-1}\| \leq m^{-2}$, called the Green's operator, acting on functions $f \in L^2(\mathbb{R}^d)$ as $f \mapsto G * f$, where $*$ denotes convolution and G is a symmetric integral kernel of positive type, called the fundamental solution of D, given by [75, Sec. 1.5]

$$G(\boldsymbol{x}, \boldsymbol{y}; m) = \mathcal{F}^{-1}[(|\boldsymbol{p}|^2 + m^2)^{-1}](\boldsymbol{x} - \boldsymbol{y}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{m}{|\boldsymbol{x} - \boldsymbol{y}|}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(m|\boldsymbol{x} - \boldsymbol{y}|) , \tag{B.2}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Here, $K_\alpha(z)$ is the modified Bessel function of the second kind [176]. For Euclidean spacetime dimensions $d \leq 3$, the fundamental solution can be written as

$$d = 1: \qquad G(x, y; m) = \frac{1}{2m} \mathrm{e}^{-m|x-y|} , \tag{B.3}$$

$$d = 2: \qquad G(\boldsymbol{x}, \boldsymbol{y}; m) = \frac{1}{2\pi} K_0(m|\boldsymbol{x} - \boldsymbol{y}|) , \tag{B.4}$$

$$d = 3: \qquad G(\boldsymbol{x}, \boldsymbol{y}; m) = \frac{1}{4\pi} \frac{\mathrm{e}^{-m|\boldsymbol{x}-\boldsymbol{y}|}}{|\boldsymbol{x} - \boldsymbol{y}|} . \tag{B.5}$$

The small distance behaviour of the fundamental solution, i.e., for $\varepsilon \coloneqq |\boldsymbol{x} - \boldsymbol{y}| \ll m^{-1}$, is given by [75, Lem. 1.10]

$$G(\varepsilon; m) = \frac{\text{const.}}{m^{2-d}} \begin{cases} \log \varepsilon^{-1} & \text{for } d = 2 \\ \varepsilon^{-(d-2)} & \text{for } d \geq 3 \end{cases} . \tag{B.6}$$

In order to define the Laplace operator as a self-adjoint operator on $L^2(\Omega)$, $\Omega \subsetneq \mathbb{R}^d$ open, we need to impose suitable boundary conditions. Unlike for the "global" case $\Omega = \mathbb{R}^d$, there are uncountable infinitely many self-adjoint extensions for this "local" case. Different self-adjoint extensions correspond to different boundary conditions. We first introduce boundary conditions via the variational principle, see [177, 178]. The connection to self-adjoint extensions of $(-\triangle + m^2)|_{C_0^\infty}$ is then made via quadratic forms, see [129] and Section A.1.

Let $\Omega \subset \mathbb{R}^d$ be open, bounded and with sufficiently smooth boundary $\partial\Omega$. For every $f \in C^\infty(\overline{\Omega})$, we define the Euclidean action functional

$$\begin{aligned} &S_{\mathrm{E}}[f; b] = S_{\mathrm{E}}^{\Omega}[f] + S_{\mathrm{E}}^{\partial\Omega}[f; b] \\ &= \frac{1}{2} \int_\Omega \Big[(\boldsymbol{\nabla} f(\boldsymbol{x}))^2 + m^2 (f(\boldsymbol{x}))^2 \Big] \mathrm{d}^d x + \frac{1}{2} \iint_{\partial\Omega} b(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{x}) f(\boldsymbol{y}) \, \mathrm{d}S(\boldsymbol{x}) \, \mathrm{d}S(\boldsymbol{y}) , \end{aligned} \tag{B.7}$$

as well as its corresponding bilinear form

$$\begin{aligned} \mathfrak{q}_b(f, g) = &\int_\Omega \big[(\boldsymbol{\nabla} f(\boldsymbol{x})) \cdot (\boldsymbol{\nabla} g(\boldsymbol{x})) + m^2 f(\boldsymbol{x}) g(\boldsymbol{x}) \big] \mathrm{d}^d x \\ &+ \iint_{\partial\Omega} b(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{x}) g(\boldsymbol{y}) \, \mathrm{d}S(\boldsymbol{x}) \, \mathrm{d}S(\boldsymbol{y}) , \end{aligned} \tag{B.8}$$

where b is a symmetric (possibly formal) distributional kernel. This kernel specifies the boundary conditions on $\partial\Omega$. For example, if we want to describe local boundary conditions, b takes the form

$$b(\boldsymbol{x}, \boldsymbol{y}) = \delta^{(d-1)}(\boldsymbol{x} - \boldsymbol{y}) \, \widetilde{b}(\boldsymbol{y}) , \tag{B.9}$$

where $\widetilde{b}$ is a (possibly also generalized) function on $\partial\Omega$.

By varying the action, we obtain

$$\begin{aligned}\delta S_{\mathrm{E}} = &\int_\Omega \big((-\triangle+m^2)f(\boldsymbol{x})\big)\delta f(\boldsymbol{x})\,\mathrm{d}^d x \\ &+ \int_{\partial\Omega} \frac{\partial f}{\partial n}(\boldsymbol{x})\,\delta f(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) + \iint_{\partial\Omega} b(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{x})\,\delta f(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\;,\end{aligned} \tag{B.10}$$

where $\partial/\partial n \coloneqq \boldsymbol{n}\cdot\boldsymbol{\nabla}$ and $\boldsymbol{n}$ is the outward pointing unit vector on $\partial\Omega$. The principle of stationary action, $\delta S_{\mathrm{E}} = 0$, then yields the following set of equations,

$$(-\triangle+m^2)f(\boldsymbol{x}) = 0\;, \qquad \boldsymbol{x}\in\Omega\;, \tag{B.11}$$

$$\frac{\partial f}{\partial n}(\boldsymbol{x}) + \int_{\partial\Omega} b(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}) = 0\;, \qquad \boldsymbol{x}\in\partial\Omega\;. \tag{B.12}$$

We thus see that the action (B.7) indeed yields the homogeneous boundary value problem described by the kernel b.

In the following, we make the connection between actions of the form (B.7) and self-adjoint extensions of $(-\triangle+m^2)|_{C_0^\infty}$. We start with the classical choices of Dirichlet, Neumann and Robin boundary conditions. All these boundary conditions are local in the sense that the kernel b is of the form (B.9). Thus, in this case we only need to specify the function $\widetilde{b}$. In addition, we study periodic boundary conditions, which can be written as a periodic sum of local boundary conditions of the form (B.9).

The first boundary conditions we consider are of Dirichlet type, which corresponds to $f\equiv 0$ on $\partial\Omega$. Formally, we can incorporate Dirichlet boundary conditions by choosing $\widetilde{b}\equiv+\infty$. To make the discussion more rigorous, instead of employing an infinite boundary term, we change the domain of the Dirichlet action to a subspace of functions which directly satisfy Dirichlet boundary conditions[1].

Definition B.1 (Dirichlet Laplacian)**.** The Dirichlet form is the quadratic form $\mathfrak{q}_{\mathrm{D}}$ on $C_0^\infty(\Omega)\times C_0^\infty(\Omega)$ is defined via

$$\mathfrak{q}_{\mathrm{D}}(f,g) = \int_\Omega \big(\boldsymbol{\nabla}f\cdot\boldsymbol{\nabla}g + m^2 fg\big)\,\mathrm{d}^d x\;. \tag{B.13}$$

The unique positive self-adjoint operator $-\triangle_{\mathrm{D}}+m^2$ associated to $\mathfrak{q}_{\mathrm{D}}$ is determined via

$$\overline{\mathfrak{q}_{\mathrm{D}}}(f,g) = \langle f, (-\triangle_{\mathrm{D}}+m^2)g\rangle_{L^2(\Omega)}\;, \qquad f\in\mathcal{Q}(-\triangle_{\mathrm{D}})\;,\;\; g\in\mathfrak{D}(-\triangle_{\mathrm{D}})\;. \tag{B.14}$$

The operator $-\triangle_{\mathrm{D}}$ is called the Dirichlet Laplacian.

[1] Following the terminology of [177], we call such boundary conditions essential. If the boundary conditions are such that the kernel b can be chosen in a way that the corresponding action is well-defined, we call them natural. Examples of natural boundary conditions are Neumann and local Robin boundary conditions discuss ed below.

The form $\mathfrak{q}_\mathrm{D}$ is densely defined, symmetric and bounded from below by $m^2 > 0$. Furthermore, as the gradient $\boldsymbol{\nabla}$ defined on $C_0^\infty(\Omega)$ is closable as an operator on $L^2(\Omega)$ [128, p. 14], the Dirichlet form is closable [173, Prop. 2.2].

Next we discuss Robin and – as a special case – Neumann boundary conditions, which correspond to $\widetilde{b} = \sigma$, where σ is a continuous function on $\partial\Omega$.

Definition B.2 (Neumann and Robin Laplacian)**.** The Robin form corresponding to the boundary function $\sigma \in C^0(\partial\Omega)$ is the quadratic form $\mathfrak{q}_\sigma$ on $C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$ is defined via

$$\mathfrak{q}_\sigma(f,g) = \int_\Omega \left(\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + m^2 f g\right) \mathrm{d}^d x \,. \tag{B.15}$$

The unique positive self-adjoint operator $-\triangle_\sigma + m^2$ associated to $\mathfrak{q}_\sigma$ is determined via

$$\overline{\mathfrak{q}_\sigma}(f,g) = \langle f, (-\triangle_\sigma + m^2) g\rangle_{L^2(\Omega)} \,, \qquad f \in \mathcal{Q}(-\triangle_\sigma) \,, \quad g \in \mathfrak{D}(-\triangle_\sigma) \,. \tag{B.16}$$

The operator $-\triangle_\sigma$ is called the Robin Laplacian associated to the function σ. The special choice $\sigma \equiv 0$ corresponds to the Neumann Laplacian $-\triangle_\mathrm{N}$ [118, Sec. XIII.15].

Note that the boundary form $\int_{\partial\Omega} \sigma f g \,\mathrm{d}S$ is, for all $\sigma \in C^0(\partial\Omega)$, infinitesimally relatively form bounded with respect to the Neumann form $\mathfrak{q}_\mathrm{N}$ [128, p. 34].

As a last example of "classical" boundary conditions, we introduce periodic boundary conditions, which are another example of essential boundary conditions. Let $\Omega = (0,L)^d$ be an open d-cube of edge length L. We fix periodic boundary conditions by the formal choice

$$b(\boldsymbol{x},\boldsymbol{y}) = \lim_{\widetilde{b}\to+\infty} \widetilde{b} \left[\sum_{r_j = \pm 1} \delta^{(d-1)}(\boldsymbol{x}-\boldsymbol{y}) - \delta^{(d-1)}(\boldsymbol{x}-\boldsymbol{y}+L\boldsymbol{r}) \right] , \tag{B.17}$$

for which (B.12) becomes

$$\frac{\partial f}{\partial n}(\boldsymbol{x}) + \lim_{\widetilde{b}\to+\infty} \widetilde{b}[f(\boldsymbol{x}) - f(\boldsymbol{x}+L\boldsymbol{r})] = 0 \,, \tag{B.18}$$

where $\boldsymbol{x} + L\boldsymbol{r}$ is the boundary point opposite to $\boldsymbol{x}$. In the (formal) limit $\widetilde{b} \to +\infty$, this equation yields the familiar periodic boundary conditions,

$$f(\boldsymbol{x}) - f(\boldsymbol{x}+L\boldsymbol{r}) = 0 \,, \tag{B.19}$$

$$\frac{\partial f}{\partial n}(\boldsymbol{x}) + \frac{\partial f}{\partial n}(\boldsymbol{x}+L\boldsymbol{r}) = 0 \,. \tag{B.20}$$

Alternatively, just like for Dirichlet boundary conditions, we can incorporate periodic boundary conditions by choosing the domain of functions suitably.

Definition B.3 (Periodic Laplacian)**.** The quadratic form corresponding to periodic boundary conditions, denoted by $\mathfrak{q}_{\mathrm{P}}$, is the quadratic form on $\mathfrak{D}(P_0) \times \mathfrak{D}(P_0)$ defined via

$$\mathfrak{q}_{\mathrm{P}}(f,g) = \int_\Omega \left(\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + m^2 f g\right) \mathrm{d}^d x \,, \tag{B.21}$$

where[2]

$$\mathfrak{D}(P_0) = \left\{ f \in C^\infty(\overline{\Omega}) \,:\, f|_{x_i=-L/2} = f|_{x_i=+L/2} \,, \quad i = 1,\dots,d \right\} \subset C^\infty(\overline{\Omega}) \,. \tag{B.22}$$

The self-adjoint operator associated with the closures of the periodic form is denoted by $-\triangle_{\mathrm{P}} + m^2$, where $-\triangle_{\mathrm{P}}$ is the Laplacian corresponding to periodic boundary conditions, or simply the periodic Laplacian.

Finally, we consider what is called in the literature free boundary conditions [44, 69, 80]. Unlike for the classical boundary conditions considered so far, where we directly prescribed boundary conditions in the definition of the Laplacian, we can alternatively fix some bounded operator and check whether it is the inverse of a self-adjoint extension of $(-\triangle + m^2)|_{C_0^\infty(\Omega)}$. In particular, let Ω be a bounded open subset of $\mathbb{R}^d$. Recall the definition of the fundamental solution G given in (B.2). We define the operator $\hat{G}_{\mathrm{F}}$ on $L^2(\Omega)$ by

$$(\hat{G}_{\mathrm{F}} f)(\boldsymbol{x}) = \int_\Omega G(\boldsymbol{x}, \boldsymbol{y}; m) f(\boldsymbol{y}) \, \mathrm{d}^d y \,, \qquad f \in L^2(\Omega) \,. \tag{B.23}$$

From the properties of D^{-1}, we can see that $\hat{G}_{\mathrm{F}}$ is a strictly positive, bounded and self-adjoint operator on $L^2(\Omega)$ with $\|\hat{G}_{\mathrm{F}}\| \leq m^{-2}$. The operator $\hat{G}_{\mathrm{F}}$ is a kind of volume potential [179], more specifically the Bessel potential [180–182].

This definition demands a physical motivation. Suppose we are given a free scalar field of mass m on Euclidean spacetime $\mathbb{R}^d$. From this we want to obtain a *reduced* theory, i.e., a theory that fully describes the physics within a bounded region Ω but contains no information about the physics in the exterior $\mathbb{R}^d \setminus \Omega$. Heuristically, we can obtain such a *local* theory from the *global* theory by integrating out the degrees of freedom in the exterior region, i.e., marginalizing the global probability distribution. Then, all expectation values of observable supported in Ω can be computed with either the global or the local theory[3]. Since the theory under consideration here is Gaussian with vanishing mean, we can construct such a reduced (or marginalized) theory simply by requiring that it describes the same correlations in the region Ω as the global theory. This motivates the following discussion.

It is shown in [80, Thm. II.6] that the inverse of $\hat{G}_{\mathrm{F}}$ is a self-adjoint extension of $(-\triangle + m^2)|_{C_0^\infty(\Omega)}$. We now derive the boundary conditions that are satisfied by functions in the

[2] For a motivation of the notation, see [128, Sec. §2].

[3] This is similar to the concepts of partial trace and reduced density operator in quantum information theory [95, Sec. 2.4.3].

domain of this operator. This question is treated in [80, Sec. II.2] (see also [183]). Fix $f \in \mathfrak{D}(\hat{G}_{\mathrm{F}}^{-1}) \cap C^2(\Omega) \cap C^1(\overline{\Omega})$. Using Green's second identity, we have, for all $\boldsymbol{x} \in \Omega$,

$$f(\boldsymbol{x}) = (\hat{G}_{\mathrm{F}}\hat{G}_{\mathrm{F}}^{-1}f)(\boldsymbol{x}) = f(\boldsymbol{x}) + \int_{\partial\Omega}\left[\frac{\partial G(\boldsymbol{x},\boldsymbol{y};m)}{\partial n_y}f(\boldsymbol{y}) - \frac{\partial f(\boldsymbol{y})}{\partial n_y}G(\boldsymbol{x},\boldsymbol{y};m)\right]\mathrm{d}S(\boldsymbol{y}) \ , \quad \text{(B.24)}$$

which implies

$$\int_{\partial\Omega}\frac{\partial G(\boldsymbol{x},\boldsymbol{y};m)}{\partial n_y}f(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}) = \int_{\partial\Omega}\frac{\partial f(\boldsymbol{y})}{\partial n_y}G(\boldsymbol{x},\boldsymbol{y};m)\,\mathrm{d}S(\boldsymbol{y}) \ , \qquad \boldsymbol{x} \in \Omega \ . \quad \text{(B.25)}$$

It can be shown[4] [80, Sec. II.2] that in the limit $\Omega \ni \boldsymbol{x}' \to \boldsymbol{x} \in \partial\Omega$ the left-hand side of the above equation can be written as

$$\int_{\partial\Omega}\frac{\partial G(\boldsymbol{x},\boldsymbol{y};m)}{\partial n_y}f(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}) = \iint_{\partial\Omega}G(\boldsymbol{x},\boldsymbol{y};m)k(\boldsymbol{y},\boldsymbol{z};m)f(\boldsymbol{z})\,\mathrm{d}S(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{z}) \quad \text{(B.26)}$$

for all $\boldsymbol{x} \in \partial\Omega$. Here, the kernel k is given by

$$k(\boldsymbol{x},\boldsymbol{y};m) = \frac{\partial^2}{\partial n_x \partial n_y}G_{\mathrm{D}}^{\text{ext.}}(\boldsymbol{x},\boldsymbol{y};m) \ , \quad \text{(B.27)}$$

where $G_{\mathrm{D}}^{\text{ext.}}$ is the Green's function of the Dirichlet problem in the exterior domain $\Omega^{\text{ext.}} = \mathbb{R}^d \setminus \overline{\Omega}$. The condition (B.25) thus yields the boundary condition

$$\frac{\partial f}{\partial n}(\boldsymbol{x}) = \int_{\partial\Omega}k(\boldsymbol{x},\boldsymbol{y};m)f(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}) \ , \qquad \boldsymbol{x} \in \partial\Omega \ . \quad \text{(B.28)}$$

Following [80], we call these boundary conditions *free* boundary conditions. We make two important observations. First, the boundary conditions in (B.28) are non-local in the sense that they cannot be written in the form (B.9), i.e., the normal derivative at a point on the boundary depends on the value of the function at each point of the boundary. As discussed in Section 5, this is necessary for two disjoint open regions Ω_A and Ω_B to "communicate" with each other. Secondly, the boundary condition (more precisely, the self-adjoint extension) depends on the mass as indicated by the mass dependence of the kernel $k(.,.;m)$ in (B.28).

The operators $-\triangle_{\mathrm{X}} + m^2$, $\mathrm{X} \in \{\mathrm{D},\mathrm{N},\mathrm{P},\sigma\}$, as well as $\hat{G}_{\mathrm{F}}^{-1}$ introduced above are strictly positive, and we denote their inverses by $\hat{G}_{\mathrm{D}}$, $\hat{G}_{\mathrm{N}}$, $\hat{G}_{\mathrm{P}}$, $\hat{G}_{\sigma}$ and $\hat{G}_{\mathrm{F}}$, respectively. These operators are bounded (and even compact if Ω is bounded) operators on $L^2(\Omega)$. Furthermore, these operators are continuous linear maps from $\mathcal{D}(\Omega)$ to $\mathcal{D}'_\beta(\Omega)$. Therefore, by the Schwartz Kernel Theorem [174, Thm. 51.7], they possess distributional integral kernels, which are of course just the Green's functions of the operator $-\triangle+m^2$ corresponding to some choice of boundary condition. In Part II of this thesis, the operators $\hat{G}_{\mathrm{X}}$ serve as the covariance operators of free massive scalar field theories over a bounded region Ω.

[4] The reader may also compare this with the theory of surface layer potentials, see, e.g., [137, 179, 184].

We finish this Section with an operator inequality. Let T_1 and T_2 be two bounded operators on a separable Hilbert space $\mathcal{H}$. If $T_2 - T_1$ is a positive operator, we write $T_1 \leq T_2$.

Lemma B.1 [80, Sec. III.1], see also [44, Sec. 7.7]**.** *Let Ω be a bounded open subset of $\mathbb{R}^d$ with boundary $\partial\Omega$. Then, the following operator inequality holds.*

$$\hat{G}_{\mathrm{D}} \leq \hat{G}_{\mathrm{F}} \leq \hat{G}_{\mathrm{N}} \,. \tag{B.29}$$

If Ω is a rectangular domain, we have the additional inequality

$$\hat{G}_{\mathrm{D}} \leq \hat{G}_{\mathrm{P}} \leq \hat{G}_{\mathrm{N}} \,. \tag{B.30}$$

C Gaussian Measure Theory

In this Chapter we review the theory of Gaussian measures on locally convex topological vector spaces (LCTVSs) and their properties. The motivation for this Chapter is to provide the necessary background for a rigorous treatment of Gaussian functional integrals describing non-interacting (bosonic) statistical field theories. Standard references for the construction of Gaussian measures describing such field theories include [13, 14, 44, 69–71, 80, 185, 186]. For a treatment of measures, in particular Gaussian measures, on infinite dimensional spaces, see, e.g., [132, 138, 187, 188].

C.1 Gaussian Functional Integrals

In the following, let $\mathscr{X}$ be a LCTVS over $\mathbb{R}$ and $\mathscr{X}^*$ its topological dual. Furthermore, denote by $\mathcal{E}(\mathscr{X})$ the cylindrical σ-algebra generated by the dual $\mathscr{X}^*$ [138, App. A].

Definition C.1 (Gaussian measures, [138, Def. 2.2.1])**.** A probability measure μ on the measurable space $(\mathscr{X}, \mathcal{E}(\mathscr{X}))$ is called *Gaussian* if every $f \in \mathscr{X}^*$ is a Gaussian random variable, i.e., if its push-forward measure $f_*\mu$ is a Gaussian measure on $\mathbb{R}$. It is called a *centred* Gaussian measure if all $f_*\mu$, $f \in \mathscr{X}^*$, have vanishing mean.

In the language of random processes [189], a probability measure on $(\mathscr{X}, \mathcal{E}(\mathscr{X}))$ is Gaussian if the random process on $(\mathscr{X}, \mathcal{E}(\mathscr{X}), \mu)$ indexed by $\mathscr{X}^*$, $\{\varphi_f\}_{f\in\mathscr{X}^*}$, where $\varphi_f \coloneqq f(\varphi)$, is Gaussian [132].

The Gaussian measures consider in this work are Radon measures on LCTVSs [138, Def. A.3.10].

Definition C.2 (Mean and Covariance, [138, Def. 2.2.7], see also [138, Thm. 3.2.3])**.** Let μ be a Radon Gaussian measure on a LCTVS $\mathscr{X}$. The *mean of* μ, written a_μ, is an element of $\mathscr{X}$ and is defined by

$$a_\mu(f) \coloneqq \mathbb{E}[f] = \int_{\mathscr{X}} f(\varphi)\, \mathrm{d}\mu(\varphi)\,, \qquad f \in \mathscr{X}^*\,. \tag{C.1}$$

The *covariance operator of* μ, denoted R_μ, is a linear map $R_\mu : \mathscr{X}^* \to \mathscr{X}$, defined by

$$\begin{aligned} R_\mu(f)(g) &\coloneqq \mathbb{E}[(f - a_\mu(f))(g - a_\mu(g))] \\ &= \int_{\mathscr{X}} (f(\varphi) - a_\mu(f))(g(\varphi) - a_\mu(g))\, \mathrm{d}\mu(\varphi)\,, \end{aligned} \tag{C.2}$$

for all $f, g \in \mathscr{X}^*$. The covariance operator R_μ induces a symmetric bilinear form Cov on $\mathscr{X}^* \times \mathscr{X}^*$ via $\mathrm{Cov}(f,g) \coloneqq R_\mu(f)(g)$ for all $f, g \in \mathscr{X}^*$. The corresponding quadratic form, called the *covariance of* μ, is positive, i.e., $\mathrm{Cov}(f,f) \geq 0$ for all $f \in \mathscr{X}^*$.

Let μ be a measure on $(\mathscr{X}, \mathcal{E}(\mathscr{X}))$. The Fourier transform of μ, denoted $\hat{\mu}$, is a map $\hat{\mu} : \mathscr{X}^* \to \mathbb{C}$ defined by [138, Def. A.3.17]

$$\hat{\mu}(f) = \int_X \exp[\mathrm{i} f(\varphi)] \, \mathrm{d}\mu(\varphi) \,, \qquad f \in \mathscr{X}^* \,. \tag{C.3}$$

Theorem C.1 [138, Thm. 2.2.4]**.** *A Radon measure μ on a locally convex space X is Gaussian precisely when its Fourier transform is given by*

$$\hat{\mu}(f) = \exp\left[-\frac{1}{2} B(f,f) + \mathrm{i} m(f)\right] . \tag{C.4}$$

Here, m is a linear functional on $\mathscr{X}^*$ and B is a symmetric bilinear form on $\mathscr{X}^* \times \mathscr{X}^*$ such that the corresponding quadratic form is positive, i.e., $B(f,f) \geq 0$ for all $f \in \mathscr{X}^*$.

Remark. Following Definition C.2, we identify B with the covariance Cov and m with the mean a_μ, see also [132, Prop. 2.2] and [138, Thm. 2.2.4].

Definition C.3 (Characteristic functional, [190, Def. 1.5.1])**.** Let $\mathfrak{X}$ be a nuclear space[1]. A function $\mathcal{C} : \mathfrak{X} \to \mathbb{C}$ is called a characteristic functional if *(i)* $\mathcal{C}$ is continuous, *(ii)* $\mathcal{C}$ is of positive type and *(iii)* $\mathcal{C}$ is normalized, i.e., $\mathcal{C}(0) = 1$.

Lemma C.2. *Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle ., . \rangle$ and $\mathfrak{X}$ a nuclear space such that the inclusion $\iota : \mathfrak{X} \hookrightarrow \mathcal{H}$ is continuous. Suppose $m \in \mathfrak{X}^*$ and $T \in \mathscr{B}(\mathcal{H})$ is self-adjoint and positive. Then, the functional $\mathcal{C} : \mathfrak{X} \to \mathbb{C}$ defined by*

$$\mathcal{C}(f) = \exp\left[-\frac{1}{2} \langle T(\iota f), \iota f \rangle + \mathrm{i} m(f)\right] , \qquad f \in \mathfrak{X} \,, \tag{C.5}$$

is a characteristic functional.

Proof. Clearly, $\mathcal{C}$ is normalized and continuous. Fix $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and $f_1, \dots, f_n \in \mathfrak{X}$. Then,

$$\sum_{i,j=1}^{n} \overline{\alpha_i} \alpha_j \, \mathcal{C}(f_i - f_j) = \sum_{i,j=1}^{n} \overline{\beta_i} \beta_j \, \exp[A_{ij}] \,, \tag{C.6}$$

[1] For a definition of nuclear spaces, see [174].

where $\beta_i \coloneqq \alpha_i \overline{\mathcal{C}(f_i)}$ and $A_{ij} \coloneqq \langle f_i, T f_j \rangle$ (we suppress the inclusion ι for the remainder of the proof). Let $\mathcal{H}_{\mathbb{C}}$ be the complexification of $\mathcal{H}$ and denote by $\langle ., . \rangle_{\mathbb{C}}$ the canonical inner product in $\mathcal{H}_{\mathbb{C}}$ [191]. Furthermore, let $T_{\mathbb{C}}$ be the complexification of T. Then, for all $z \in \mathbb{C}^n$,

$$\sum_{i,j=1}^{n} \overline{z_i} z_j A_{ij} = \left\langle \sum_{i=1}^{n} z_i f_i, T_{\mathbb{C}} \left(\sum_{j=1}^{n} z_j f_j \right) \right\rangle_{\mathbb{C}} \geq 0 \,, \tag{C.7}$$

where we used that T is positive and thus so is $T_{\mathbb{C}}$. This means that A is a positive matrix and, by the Schur product Theorem [192, Thm. VII], so is the matrix $\exp[A_{ij}]$. Therefore, $\mathcal{C}$ is of positive type. ∎

Theorem C.3 (Minlos, [172], [193, Thm. 4.3 & Thm. 4.4])**.** *Let $\mathfrak{X}$ be a nuclear space. Every characteristic functional on $\mathfrak{X}$ is the Fourier transform of a Radon probability measure on $\mathfrak{X}'_\beta$, the topological dual of $\mathfrak{X}$ equipped with the strong dual topology.*

Corollary C.4 (Centred Gaussian measure on space of distributions)**.** *Let $\mathfrak{X} = \mathcal{D}(\Omega)$, $\Omega \subset \mathbb{R}^d$ open and choose $\mathcal{H} = L^2(\Omega)$. Furthermore, let $\hat{C} \in \mathscr{B}(L^2(\Omega))$ be self-adjoint and positive. Then, there exists a centred Radon Gaussian measure on $\mathcal{D}'_\beta(\Omega)$ with covariance*

$$\mathrm{Cov}(f, f) = \langle \hat{C}(\iota f), \iota f \rangle_{L^2(\Omega)} \,, \qquad f \in \mathcal{D}(\Omega) \,. \tag{C.8}$$

We denote this Gaussian measure by $\mathcal{N}(0, \hat{C})$.

Given a centred Radon Gaussian measure μ on a LCTVS $\mathscr{X}$, we define the Hilbert space $L^2(\mu) \coloneqq L^2(X, \mu)$ in the usual way.

Definition C.4 (Reproducing kernel Hilbert space, [132, Ch. 5], [138, Sec. 2.2])**.** Let μ be a Radon Gaussian measure on a LCTVS $\mathscr{X}$. The closure of the set $\mathscr{X}^*$ with respect to the norm of $L^2(\mu)$ and equipped with the $L^2(\mu)$-inner product is called the *reproducing kernel Hilbert space (RKHS)* of the measure μ and is denoted by $\mathscr{X}^*_\mu$. The elements of $\mathscr{X}^*_\mu$ are called the μ-measurable linear functionals.

Definition C.5 (Cameron-Martin space, [194, Def. 3.24])**.** Let μ be a Radon Gaussian measure on a LCTVS $\mathscr{X}$. Consider the set

$$\mathring{\mathsf{H}}_\mu \coloneqq \{\varphi \in \mathscr{X} \,:\, \exists\, \hat{\varphi} \in \mathscr{X}^* \text{ such that } \varphi = R_\mu(\hat{\varphi})\} = R_\mu[\mathscr{X}^*] \subset \mathscr{X} \,. \tag{C.9}$$

We define on $\mathring{\mathsf{H}}_\mu \times \mathring{\mathsf{H}}_\mu$ the inner product $\langle\!\langle \varphi, \psi \rangle\!\rangle \coloneqq R_\mu(\hat{\varphi})(\hat{\psi})$ and norm $\|\varphi\|^2_\mu = \langle\!\langle \varphi, \varphi \rangle\!\rangle$. The completion of $\mathring{\mathsf{H}}_\mu$ with respect to $\|.\|_\mu$, equipped with the inner product $\langle\!\langle ., . \rangle\!\rangle$, is called the *Cameron-Martin space (CMS)* of μ and is denoted by H_μ. One can show that $\mathsf{H}_\mu = R_\mu[X^*_\mu]$ [138, Thm. 3.2.3].

At this point it is instructive to consider explicit examples of reproducing kernel Hilbert spaces and Cameron-Martin spaces encountered in Part II of this thesis. Specifically, consider the centred Gaussian measures $\mu = \mathcal{N}(0, \hat{G}_{\mathrm{X}})$, where $\mathrm{X} \in \{\mathrm{F}, \mathrm{D}\}$, for definitions see

Boundary condition	RKHS	CMS
Free	$\widetilde{H}^{-1}(\Omega)$	$H^{+1}(\Omega)$
Dirichlet	$H_0^{-1}(\Omega) = H^{-1}(\Omega)$	$H_0^{+1}(\Omega)$

Table C.1: Reproducing kernel Hilbert space (RKHS) and Cameron-Martin space (CMS) of a free scalar field theory over a bounded region $\Omega \subset \mathbb{R}^d$ with free and Dirichlet boundary conditions.

Appendix B. In other words, we study the RKHS and CMS of a free scalar field theory of mass $m > 0$ with free or Dirichlet boundary conditions.

We start with free boundary conditions. By definition, the RKHS of $\mu_{\mathrm{F}} = \mathcal{N}(0, \hat{G}_{\mathrm{F}})$, $\mathfrak{X}_{\mu_{\mathrm{F}}}$, is the closure of $C_0^\infty(\Omega)$ in the norm $\|\hat{G}_{\mathrm{F}}^{1/2}.\|_{L^2(\Omega)}$. But for every $f \in C_0^\infty(\Omega)$,

$$\|\hat{G}_{\mathrm{F}}^{1/2} f\|_{L^2(\Omega)} = \|D^{-1/2} f\|_{L^2(\mathbb{R}^d)} \,, \tag{C.10}$$

where D is the unique self-adjoint extension of $(-\triangle + m^2)|_{C_0^\infty}$. But the norm $\|D^{-1/2}.\|_{L^2(\mathbb{R}^d)}$ is equivalent to the norm of the Hilbert-Sobolev space $H^{-1}(\mathbb{R}^d)$. Therefore, the RKHS of μ_{F} is the closure of $C_0^\infty(\Omega)$ in $\|.\|_{-1}$, which is just the Hilbert-Sobolev space $\widetilde{H}^{-1}(\Omega)$, see Appendix A.2 for details. The CMS of μ_{F} is given by $\mathsf{H}_{\mu_{\mathrm{F}}} = \hat{G}_{\mathrm{F}}[\widetilde{H}^{-1}(\Omega)]$, cf. [138, Thm. 3.2.3]. By noticing that we can write $\hat{G}_{\mathrm{F}} = r_\Omega \circ D^{-1}$, where r_Ω denotes the restriction of a function in $H^{+1}(\mathbb{R}^d)$ to Ω, [127, Lem. 3.2 & Eq. (19)] imply that $\mathsf{H}_{\mu_{\mathrm{F}}} = H^{+1}(\Omega)$, see also [127, Thm. 3.12(iii)].

Next, we consider a Dirichlet field $\mu_{\mathrm{D}} = \mathcal{N}(0, \hat{G}_{\mathrm{D}})$. Again, by definition, the RKHS of μ_{D} is the closure of $C_0^\infty(\Omega)$ in the norm $\|\hat{G}_{\mathrm{D}}^{1/2}.\|_{L^2(\Omega)}$. Using [69, Cor. II.25] (cf. also the discussion in Chapter 5), this norm is equivalent to the norm in $H^{-1}(\Omega)$ and thus the RKHS of μ_{D} is given by $\mathfrak{X}_{\mu_{\mathrm{D}}} = H_0^{-1}(\Omega) = H^{-1}(\Omega)$, where the second equality follows from [127, Cor. 3.29(ii)]. The CMS of μ_{D} is given by $\mathsf{H}_{\mu_{\mathrm{D}}} = \hat{G}_{\mathrm{D}}[H^{-1}(\Omega)] = H_0^{+1}(\Omega)$, see also [131, Ch. 6]. We summarize the results of the last two paragraphs in Table C.1.

Notice that $H_0^{+1}(\Omega)$ is precisely the form domain of the Dirichlet Laplacian $-\triangle_{\mathrm{D}} + m^2$, see, e.g., [118, Sec. XIII.15]. It is not a coincidence that the form domain of the precision operator of a Gaussian measure $\mathcal{N}(0, \hat{C})$ coincides with its CMS, as can be seen from the following

Lemma C.5. *For a Gaussian measure $\mu = \mathcal{N}(0, \hat{C})$ as defined in Corollary C.4, H_μ coincides with $\mathcal{Q}(\hat{C}^{-1}) = \mathfrak{D}(\hat{C}^{1/2}) = \hat{C}^{1/2}[\mathcal{H}]$ as a set.*

Proof. Fix $\varphi \in \hat{C}^{1/2}[\mathcal{H}]$. Then, there exists $h \in \mathcal{H}$ such that $\varphi = \hat{C}^{1/2} h$. By assumption, $\mathfrak{X}$ is dense in $\mathcal{H}$, $\hat{C}^{1/2}$ is continuous and $\hat{C}^{1/2}[\mathcal{H}]$ is dense in $\mathcal{H}$. Thus $\hat{C}^{1/2}[\mathfrak{X}]$ is also dense in $\mathcal{H}$. Hence, there exists a sequence $\{h_n\}_{n=1}^\infty$ in $\hat{C}^{1/2}[\mathfrak{X}]$ such that $h_n \to h$ in $\mathcal{H}$. Then,

$$\lim_{n\to\infty} \|\varphi - \hat{C}^{1/2} h_n\|_\mu = \lim_{n\to\infty} \|\hat{C}^{-1/2}(\varphi - \hat{C}^{1/2} h_n)\|_{\mathcal{H}} = \lim_{n\to\infty} \|h - h_n\|_{\mathcal{H}} = 0 \,. \tag{C.11}$$

This means there exists a sequence in $\hat{C}[\mathfrak{X}]$ that converges to φ in $\|.\|_\mu$ and hence $\varphi \in \mathsf{H}_\mu$. Therefore, $\hat{C}^{1/2}[\mathcal{H}] \subset \mathsf{H}_\mu$.

Conversely, fix $\varphi \in \mathsf{H}_\mu$. Then, there exists $\{\varphi_n\}_{n=1}^\infty$ in $\hat{C}[\mathfrak{X}]$ such that $\varphi_n \to \varphi$ with respect to $\|.\|_\mu$. From the definition of $\|.\|_\mu$, we see that $\{\varphi_n\}_{n=1}^\infty$ being Cauchy in H_μ implies that $\{\hat{C}^{-1/2}\varphi_n\}_{n=1}^\infty$ is Cauchy in $\mathcal{H}$. As $\mathcal{H}$ is complete, there exists $h \in \mathcal{H}$ such that $\hat{C}^{-1/2}\varphi_n \to h$ in $\mathcal{H}$. The sequence $\{\varphi_n\}_{n=1}^\infty = \hat{C}^{1/2}[\{\hat{C}^{-1/2}\varphi_n\}_{n=1}^\infty]$ is the image of a converging sequence under a continuous map and is hence convergent. Furthermore, its limit is $\hat{C}^{1/2}h =: \psi \in \hat{C}^{1/2}[\mathcal{H}]$. Therefore, $\{\varphi_n\}_{n=1}^\infty$ converges to an element in $\hat{C}^{1/2}[\mathcal{H}]$ with respect to $\mathcal{H}$-norm. Now $\|\varphi_n - \psi\|_\mu = \|\hat{C}^{-1/2}(\varphi_n - \psi)\|_\mathcal{H} \to 0$ as $n \to \infty$. By the uniqueness of the limit, $\varphi = \psi \in \hat{C}^{1/2}[\mathcal{H}]$. Therefore, $\mathsf{H}_\mu \subset \hat{C}^{1/2}[\mathcal{H}]$. ■

C.2 Equivalence of Gaussian Measures

Definition C.6 (Absolute continuity, equivalence and mutual singularity, [59, Def. 3.2.1])**.** Let μ and ν be two measures on a measurable space $(\mathscr{X}, \Sigma)$.

- The measure ν is called *absolutely continuous* with respect to μ, written $\nu \ll \mu$, if $\nu(A) = 0$ for every $A \in \Sigma$ with $\mu(A) = 0$. If $\nu \ll \mu$ and $\mu \ll \nu$, then the measures μ and ν are called *equivalent*, and we write $\mu \sim \nu$.

- The measure ν is called *singular* with respect to μ, written $\nu \perp \mu$, if there exists a set $A \in \Sigma$ such that $\mu(A) = 0$ and $\nu(\mathscr{X} \setminus A) = 0$. Clearly $\nu \perp \mu$ implies $\mu \perp \nu$. Therefore, we call two such measures *mutually singular*.

Throughout the remainder of this Section we make two assumptions. First, we assume we have a double $(\mathfrak{X}, \mathcal{H})$, where $\mathcal{H}$ is a real Hilbert space with inner product $\langle ., . \rangle_\mathcal{H}$ and norm $\|.\|_\mathcal{H}$ and $\mathfrak{X} \subset \mathcal{H}$ is a nuclear space such that the inclusion $\iota : \mathfrak{X} \hookrightarrow \mathcal{H}$ is continuous with dense image. Secondly, we assume $\hat{C} : \mathcal{H} \to \mathcal{H}$ is a self-adjoint, strictly positive (hence injective) and compact operator. Furthermore, its inverse $\hat{C}^{-1}$ is an unbounded, densely defined, self-adjoint operator that is bounded from below by some $c > 0$.

Proposition C.6. *Let $\hat{C}_1$ and $\hat{C}_2$ be two self-adjoint, strictly positive and compact operators on a Hilbert space $\mathcal{H}$. Suppose $\hat{C}_1^{1/2}[\mathcal{H}] = \hat{C}_2^{1/2}[\mathcal{H}]$. Then, the operator $\hat{B} \coloneqq \hat{C}_1^{-1/2}\hat{C}_2^{1/2}$ is bounded and boundedly invertible.*

Proof. Clearly $\hat{B}$ is injective. Suppose $(h_n, \hat{B}h_n) \to (h, g)$ in $\mathcal{H} \oplus \mathcal{H}$. We can use the continuity of $\hat{C}_2^{1/2}$ together with the self-adjointness of $\hat{C}_1^{-1/2}$ to see that

$$\langle f, g \rangle_\mathcal{H} = \lim_{n\to\infty} \langle f, \hat{B}h_n \rangle_\mathcal{H} = \langle \hat{C}_1^{-1/2} f, \hat{C}_2^{1/2} h \rangle_\mathcal{H} = \langle f, \hat{B}h \rangle_\mathcal{H} \tag{C.12}$$

for all $f \in \hat{C}_1^{1/2}[\mathcal{H}]$. By assumption, $\hat{C}_1^{1/2}[\mathcal{H}]$ is dense in $\mathcal{H}$. Together with the continuity of the inner product this implies that $\langle f, g \rangle_\mathcal{H} = \langle f, \hat{B}h \rangle_\mathcal{H}$ for all $f \in \mathcal{H}$. Hence, $g = \hat{B}h$

and the graph of $\hat{B}$ is closed. Therefore, by the closed graph Theorem [123, Thm. III.12], $\hat{B}$ is bounded. By the symmetry of the above reasoning in the indices 1 and 2, the inverse $\hat{B}^{-1} = \hat{C}_2^{-1/2}\hat{C}_1^{1/2}$ is also bounded. ■

Theorem C.7 (Feldman-Hájek, [57, 58], [138, Thm. 2.7.2])**.** *Let $\mathscr{X}$ be a locally convex space and μ and ν two Gaussian measures on $\mathscr{X}$. Then μ and ν are either equivalent or mutually singular.*

Theorem C.8 [138, Thm. 6.4.6]**.** *Two centred Radon Gaussian measures μ and ν on a locally convex space $\mathscr{X}$ are equivalent precisely when H_μ and H_ν coincide as sets and there exists an invertible operator A on H_μ such that $AA^* - I \in \mathrm{HS}(\mathsf{H}_\mu)$ and $\|h\|_\nu = \|A^{-1}h\|_\mu$ for all $h \in \mathsf{H}_\mu$.*

Lemma C.9. *Two centred Gaussian measures $\mu = \mathcal{N}(0, \hat{C}_\mu)$ and $\nu = \mathcal{N}(0, \hat{C}_\nu)$ are equivalent precisely when $\hat{C}_\mu^{1/2}[\mathcal{H}] = \hat{C}_\nu^{1/2}[\mathcal{H}]$ and $\hat{B}\hat{B}^* - I$ is a Hilbert-Schmidt operator on $\mathcal{H}$, where $\hat{B} := \hat{C}_\mu^{-1/2}\hat{C}_\nu^{1/2}$.*

Remark 1. Here, $\hat{B}^*$ is the continuous extension of the operator $\hat{C}_\nu^{1/2}\hat{C}_\mu^{-1/2}$ defined on $\hat{C}_\mu^{1/2}[\mathcal{H}]$.

Remark 2. This condition should be compared to the necessary and sufficient condition given in the Feldman–Hájek Theorem [57, 58] for the case of Gaussian measures on a Hilbert space, see also [136, Sec. 2.3.2].

Proof. Suppose $\hat{C}_\mu^{1/2}[\mathcal{H}] = \hat{C}_\nu^{1/2}[\mathcal{H}]$ and $\hat{B}\hat{B}^* - I$ is a Hilbert-Schmidt operator on $\mathcal{H}$. By Lemma C.5, $\hat{C}_\mu^{1/2}[\mathcal{H}] = \hat{C}_\nu^{1/2}[\mathcal{H}]$ implies that H_μ and H_ν coincide as sets. We define the operator $A : \mathsf{H}_\mu \to \mathsf{H}_\mu$ via $h \mapsto \hat{C}_\nu^{1/2}\hat{C}_\mu^{-1/2}h$. Clearly, $\|\hat{C}_\nu^{-1/2}h\|_\mathcal{H} = \|\hat{C}_\mu^{-1/2}(\hat{B}^*)^{-1}h\|_\mathcal{H}$ for all $h \in \hat{C}_\mu^{1/2}[\mathcal{H}]$, which implies that $\|h\|_\nu = \|A^{-1}h\|_\mu$ for all $h \in \mathsf{H}_\mu$. The equivalence of the norms $\|.\|_\mu$ and $\|.\|_\nu$ (cf. [136, Prop. B.1]) implies that A^{-1} is bounded. More precisely, there exists $\gamma > 0$ such that $\|h\|_\nu \leq \gamma\|h\|_\mu$ for all $h \in \hat{C}_\mu^{1/2}[\mathcal{H}] = \hat{C}_\nu^{1/2}[\mathcal{H}]$ and thus

$$\|A^{-1}\| = \sup_{\|h\|_\mu=1} \|A^{-1}h\|_\mu = \sup_{\|h\|_\mu=1} \|h\|_\nu \leq \sup_{\|h\|_\mu=1} \gamma\|h\|_\mu = \gamma\,. \tag{C.13}$$

We still need to find the adjoint of A. Recalling that $\hat{C}_\mu$ and $\hat{C}_\nu$ are self-adjoint on $\mathcal{H}$ and using the definition of the inner product on H_μ, we see that

$$\langle\!\langle f, Ag\rangle\!\rangle = \langle\!\langle \hat{C}_\mu^{1/2}\hat{C}_\nu^{1/2}\hat{C}_\mu^{-1}f, g\rangle\!\rangle \tag{C.14}$$

for all $f, g \in H_\mu$. Hence, $A^* = \hat{C}_\mu^{1/2}\hat{C}_\nu^{1/2}\hat{C}_\mu^{-1}$. Let $\{\phi_n\}_{n=1}^\infty$ be an orthonormal basis in $\mathcal{H}$ contained in $\hat{C}_\mu^{1/2}[\mathcal{H}]$, e.g., the eigenbasis of $\hat{C}_\mu$. Then $\{\hat{C}_\mu^{1/2}\phi_n\}_{n=1}^\infty$ is an orthonormal basis in H_μ. Then,

$$\|\hat{B}\hat{B}^* - I\|_{\mathrm{HS}(\mathcal{H})}^2 = \sum_{n=1}^\infty \|(\hat{C}_\nu\hat{C}_\mu^{-1} - I)\,\hat{C}_\mu^{1/2}\phi_n\|_\mu^2 = \|AA^* - I\|_{\mathrm{HS}(\mathsf{H}_\mu)}^2\,, \tag{C.15}$$

which implies that $AA^* - I \in \mathrm{HS}(\mathsf{H}_\mu)$. Therefore, by Theorem C.8, $\mu \sim \nu$.

Conversely, suppose $\mu \sim \nu$. Then, H_μ and H_ν coincide as sets, which implies that $\hat{C}_\mu^{1/2}[\mathcal{H}] = \hat{C}_\nu^{1/2}[\mathcal{H}]$. Furthermore, there exists an invertible operator A on H_μ such that $AA^* - I \in \mathrm{HS}(\mathsf{H}_\mu)$ and $\|h\|_\nu = \|A^{-1}h\|_\mu$. By the definitions of $\|.\|_\mu$ and $\|.\|_\nu$, the last property can be written as $\|A^{-1}h\|_\mu = \|\hat{C}_\mu^{1/2}\hat{C}_\nu^{-1/2}h\|_\mu$ for all $h \in \hat{C}_\mu^{1/2}[\mathcal{H}]$. Thus, A^{-1} coincides with $U\hat{C}_\mu^{1/2}\hat{C}_\nu^{-1/2}$, where U is an orthogonal transformation on H_μ. Furthermore, $A = \hat{C}_\nu^{1/2}\hat{C}_\mu^{-1/2}U^*$ and, by the same calculation as above, $A^* = U\hat{C}_\mu^{1/2}\hat{C}_\nu^{1/2}\hat{C}_\mu^{-1}$. Thus, $AA^* = \hat{C}_\nu\hat{C}_\mu^{-1}$, which, by (C.15), implies that $\hat{B}\hat{B}^* - I$ is a Hilbert-Schmidt operator on $\mathcal{H}$. ■

By the above Lemma, a necessary condition for the equivalence of μ and ν is that $\hat{B}\hat{B}^* - I$ is a Hilbert-Schmidt operator on $\mathcal{H}$. If $\hat{C}_\mu^{1/2}[\mathcal{H}] = \hat{C}_\nu^{1/2}[\mathcal{H}]$, then, by [138, Lem. 6.3.1(ii)], $\hat{B}\hat{B}^* - I$ is a Hilbert-Schmidt operator on $\mathcal{H}$ if and only if $\hat{B}^*\hat{B} - I$ is. We summarize these results in the following

Theorem C.10. *Two centred Gaussian measures $\mu = \mathcal{N}(0, \hat{C}_\mu)$ and $\nu = \mathcal{N}(0, \hat{C}_\nu)$ are equivalent precisely when $\hat{C}_\mu^{1/2}[\mathcal{H}] = \hat{C}_\nu^{1/2}[\mathcal{H}]$ and one of the following equivalent conditions holds:*

1. *$\hat{B}\hat{B}^* - I$ is a Hilbert-Schmidt operator on $\mathcal{H}$,*
2. *$\hat{B}^*\hat{B} - I$ is a Hilbert-Schmidt operator on $\mathcal{H}$,*

where $\hat{B} := \hat{C}_\mu^{-1/2}\hat{C}_\nu^{1/2}$. Of course, this statement also holds if we interchange μ and ν.

C.3 A Formula for the Relative Entropy

In the following, let $\mu = \mathcal{N}(0, \hat{C}_\mu)$ and $\nu = \mathcal{N}(0, \hat{C}_\nu)$ be two equivalent centred Gaussian measures as defined in Section C.1. We recall that in this case the relative entropy is given by

$$S(\mu\|\nu) = \int_{\mathfrak{X}_\beta^*} \log\left[\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(\varphi)\right] \mathrm{d}\mu(\varphi) \,, \tag{C.16}$$

where $\mathrm{d}\mu/\mathrm{d}\nu$ is the Radon-Nikodym derivative of μ with respect to ν.

Theorem C.11 [138, Col. 6.4.11]**.** *Let $\mu = \mathcal{N}(0, \hat{C}_\mu)$ and $\nu = \mathcal{N}(0, \hat{C}_\nu)$ be two equivalent centred Gaussian measures. Then, the Radon-Nikodym density of μ with respect to ν is given by*

$$\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(\varphi) = \exp\left[\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha_n - 1}{\alpha_n}(\eta_n(\varphi))^2 - \log\alpha_n\right)\right] , \tag{C.17}$$

where $\{\alpha_n\}_{n=1}^\infty$ is the sequence of eigenvalues of $\hat{B}\hat{B}^*$, where $\hat{B} := \hat{C}_\nu^{-1/2}\hat{C}_\mu^{1/2}$, with corresponding eigenvectors $\{\phi_n\}_{n=1}^\infty$ in $\mathcal{H}$ and η_n is the inclusion of $\hat{C}_\nu^{-1/2}\phi_n$ into $L^2(\mathfrak{X}_\beta^*, \nu) =: L^2(\nu)$.

Remark. Notice that this Theorem is essentially [138, Col. 6.4.11], which gives the analogous result for Gaussian measures on Hilbert spaces. For convenience, we shall reproduce the proof with minimal adaption to the present case, following the aforementioned reference as well as [195, 196].

Proof. Define $\hat{S} \coloneqq \hat{B}\hat{B}^*$. Since $\mu \sim \nu$ implies that $\hat{S} - I$ is Hilbert-Schmidt on $\mathcal{H}$, there exists, by the Hilbert-Schmidt Theorem [123, Thm. VI.16], an orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$ in $\mathcal{H}$, such that $(\hat{S} - I)\phi_n = (\alpha_n - 1)\phi_n$ for all $n \in \mathbb{N}$, where the α_n are the eigenvalues of $\hat{S}$. Furthermore, the series $\sum_{n=1}^{\infty}(\alpha_n - 1)^2$ converges. The sequence $\{\hat{C}_\nu^{-1/2}\phi_n\}_{n=1}^{\infty}$ is an orthonormal basis in $\mathfrak{X}_\nu$, the RKHS of ν. We recall that $\mathfrak{X}_\nu \subset L^2(\nu)$ and we denote by η_n the inclusion of $\hat{C}_\nu^{-1/2}\phi_n$ in $L^2(\nu)$.

Define

$$s_N(\varphi) \coloneqq \sum_{n=1}^{N}\left(\frac{\alpha_n - 1}{\alpha_n}(\eta_n(\varphi))^2 - \log\alpha_n\right) \in L^2(\nu) \,. \tag{C.18}$$

Recall that, by Wick's theorem [71, Prop. I.2], for all $f, g \in L^2(\nu)$,

$$\langle f^2, g^2\rangle_{L^2(\nu)} = \|f\|^2_{L^2(\nu)}\|g\|^2_{L^2(\nu)} + 2\langle f, g\rangle^2_{L^2(\nu)} \,, \tag{C.19}$$

where the squares have to be understood as the pointwise products of the linear functionals f and g on $\mathfrak{X}^*$. In particular, for two orthonormal basis vectors η_n and η_m, $\langle \eta_n^2, \eta_m^2\rangle_{L^2(\nu)} = 1 + 2\delta_{nm}$. From this, we see that [196, Lem. 13]

$$\frac{1}{2}\langle(\eta_n^2 - 1), (\eta_m^2 - 1)\rangle_{L^2(\nu)} = \delta_{nm} \,. \tag{C.20}$$

Therefore, $\{\frac{1}{\sqrt{2}}(\eta_n^2 - 1)\}_{n=1}^{\infty}$ is an orthonormal system in $L^2(\nu)$. Following [196, Lem. 14], we rewrite s_N as

$$s_N = \sum_{n=1}^{N}\left(\frac{\sqrt{2}(\alpha_n - 1)}{\alpha_n}\frac{1}{\sqrt{2}}(\eta_n^2 - 1) + \frac{\alpha_n - 1}{\alpha_n} - \log\alpha_n\right) = r_N + t_N \,, \tag{C.21}$$

where

$$r_N \coloneqq \sum_{n=1}^{N}\frac{\sqrt{2}(\alpha_n - 1)}{\alpha_n}\frac{1}{\sqrt{2}}(\eta_n^2 - 1) \,, \qquad t_N \coloneqq \sum_{n=1}^{N}\left(\frac{\alpha_n - 1}{\alpha_n} - \log\alpha_n\right) \,. \tag{C.22}$$

As $L^2(\nu)$ is complete, we only need to show that $\{r_N\}_{N=1}^{\infty}$ and $\{t_N\}_{N=1}^{\infty}$ are Cauchy. We start with $\{t_N\}_{N=1}^{\infty}$. Without loss of generality, assume $N > M$. Then,

$$\|t_N - t_M\|_{L^2(\nu)} \leq \sum_{n=M}^{N}\left|\frac{\alpha_n - 1}{\alpha_n} - \log\alpha_n\right| \,. \tag{C.23}$$

Hence, convergence of $\sum_{n=1}^{\infty}|(\alpha_n - 1)\alpha_n^{-1} - \log\alpha_n|$ in $\mathbb{R}$ implies convergence of $\{t_N\}_{N=1}^{\infty}$ in $L^2(\nu)$. To see that this is indeed the case, recall that $\sum_{n=1}^{\infty}(\alpha_n - 1)^2 < +\infty$ as $\hat{S} - I \in \mathrm{HS}(\mathcal{H})$. This implies that $\sum_{n=1}^{\infty}|(\alpha_n - 1)\alpha_n^{-1} - \log\alpha_n|$ converges, as shown in [195, p. 336f]. Therefore, $t \coloneqq \lim_{N\to\infty} t_N \in L^2(\nu)$ exists. Following an analogous argument, convergence of $\sum_{n=1}^{\infty}(1 - \alpha_n^{-1})^2$ in $\mathbb{R}$ implies convergence of $\{r_N\}_{N=1}^{\infty}$ in $L^2(\nu)$. The convergence of

$\sum_{n=1}^{\infty}(1-\alpha_n^{-1})^2$ follows from the convergence of $\sum_{n=1}^{\infty}(1-\alpha_n)^2$ [195, p. 336]. Therefore, $s \coloneqq \lim_{N\to\infty} s_N \in L^2(\nu)$ exists. We can rewrite $s(\varphi)$ as

$$s(\varphi) = \sum_{n=1}^{\infty}\left(1-\frac{1}{\alpha_n}\right)\left[(\eta_n(\varphi))^2-1\right] - \underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{\alpha_n}-\log\frac{1}{\alpha_n}-1\right)}_{\in\mathbb{R}} . \tag{C.24}$$

Therefore, by [138, Cor. 6.4.10], $\rho(\varphi) \coloneqq \exp[\frac{1}{2}s(\varphi)]$ is in $L^1(\nu)$ and $\|\rho\|_{L^1(\nu)}^{-1}\rho\cdot\nu$ is a Gaussian measure.

Finally, we show that $\mathrm{d}\mu/\mathrm{d}\nu = \rho$, i.e., that $\mu = \rho\cdot\nu$. First, note that

$$\langle\eta_n,\eta_m\rangle_{L^2(\mu)} = \langle\hat{C}_\mu^{1/2}\hat{C}_\nu^{-1/2}\phi_n, \hat{C}_\mu^{1/2}\hat{C}_\nu^{-1/2}\phi_m\rangle_{\mathcal{H}} = \langle\hat{S}\phi_n,\phi_m\rangle_{\mathcal{H}} = \alpha_n\delta_{nm} . \tag{C.25}$$

Fix $\xi\in\mathfrak{X}_\nu\subset L^2(\nu)$. Such a ξ can be written as $\xi=\sum_{n=1}^{\infty}c_n\eta_n$ with $\sum_{n=1}^{\infty}c_n^2<+\infty$. As $\{\eta_n\}_{n=1}^{\infty}$ is an orthonormal basis in $\mathfrak{X}_\nu$, every η_n is a standard Gaussian random variable and thus, using the change of variables formula, we see that

$$\int_{\mathfrak{X}_\beta^*}\exp\left[\frac{\alpha_n-1}{2\alpha_n}\eta_n^2+\mathrm{i}c_n\eta_n\right]\mathrm{d}\nu(\varphi) = \sqrt{\alpha_n}\,\exp\left[-\frac{1}{2}c_n^2\alpha_n\right] . \tag{C.26}$$

In particular, we therefore have

$$\int_{\mathfrak{X}_\beta^*}\mathrm{e}^{\mathrm{i}\xi(\varphi)}\rho(\varphi)\,\mathrm{d}\nu(\varphi) = \exp\left[-\frac{1}{2}\sum_{n=1}^{\infty}c_n^2\alpha_n\right] . \tag{C.27}$$

An analogous calculation, using (C.25), yields

$$\int_{\mathfrak{X}_\beta^*}\mathrm{e}^{\mathrm{i}\xi(\varphi)}\,\mathrm{d}\mu(\varphi) = \exp\left[-\frac{1}{2}\sum_{n=1}^{\infty}c_n^2\alpha_n\right] . \tag{C.28}$$

Thus, $\hat{\mu}=\widehat{\rho\cdot\nu}$ and, as a Gaussian measure is uniquely determined by its Fourier transform [59, Lem. 7.13.5], $\mu=\rho\cdot\nu$. We conclude that the Radon-Nikodym density of μ with respect to ν is given by (C.17). ∎

Corollary C.12. *Let $\mu=\mathcal{N}(0,\hat{C}_\mu)$ and $\nu=\mathcal{N}(0,\hat{C}_\nu)$ be equivalent. Then, the relative entropy of μ with respect to ν is given by*

$$S(\mu\|\nu) = -\frac{1}{2}\log\det{}_2(\hat{S}) = \frac{1}{2}\sum_{n=1}^{\infty}(\alpha_n-\log\alpha_n-1) , \tag{C.29}$$

where $\det_2$ denotes the regularized Fredholm determinant [197, Sec. 6], see also [198].

D Schwinger-Keldysh and Feynman-Vernon Formalism

In this Chapter we introduce the Schwinger-Keldysh formalism [199, 200], see also [201, 202], which is a powerful tool to study real time properties of quantum systems out of equilibrium. We start by introducing the path integral formalism for quantum mechanics, and then we show how to extend it to the Schwinger-Keldysh formalism. Using the Schwinger-Keldysh formalism, we subsequently derive the Feynman-Vernon influence functional formalism [203], which allows us to study the time evolution of *open* quantum systems, i.e., quantum systems in contact with an environment system. This Chapter mainly follows [165, Ch. 3]. The exposition of this topic is, compared to the rest of this thesis, more heuristic, which is mainly due to the complications stemming from Feynman's real time path integral formalism. Nevertheless, we will provide rigorous results where possible.

Consider a system of $N \in \mathbb{N}$ quantum mechanical particles in d spatial dimensions. The Schrödinger representation of this system is given by the representation Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{dN})$ and the canonical commutation relations are implemented by representing position and momentum operators as multiplication and differential operators, respectively. Suppose $\hat{H}$ is the Hamiltonian of the system. Throughout this Chapter, we assume the Hamiltonian to be time *in*dependent. Then, $\hat{H}$ is the generator of a one-parameter unitary group describing the time evolution of the system. The time evolution of a state $\psi \in \mathcal{H}$ is given by

$$\psi(t_2) = \hat{U}(t_2 - t_1)\,\psi(t_1)\ , \tag{D.1}$$

where $\hat{U}(t_2 - t_1) = \exp[-\mathrm{i}\hat{H}(t_2 - t_1)]$. The infinitesimal form of this time evolution is given by the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\psi(t) = \hat{H}\,\psi(t)\ . \tag{D.2}$$

More generally, for a state represented by a density operator $\hat{\rho}$, i.e., a positive operator on $\mathcal{H}$ with unit trace, the time evolution is given by

$$\hat{\rho}(t_2) = \hat{U}(t_2 - t_1)\,\hat{\rho}(t_1)\,\hat{U}^\dagger(t_2 - t_1)\ , \tag{D.3}$$

the infinitesimal form of which is the Liouville-von Neumann equation

$$\mathrm{i}\frac{\partial}{\partial t}\hat{\rho}(t) = [\hat{H}, \hat{\rho}(t)] \,. \tag{D.4}$$

As an operator on an L^2-space, we can associate a (possibly distributional) integral kernel $U(t_2 - t_1; x, x')$ to the time-evolution operator $\hat{U}(t_2 - t_1)$, which is (proportional to) the fundamental solution of the Schrödinger equation, i.e.,

$$\left(\mathrm{i}\frac{\partial}{\partial t} - \hat{H}\right) U(t_2 - t_1; x, x') = \mathrm{i}\delta(x - x')\delta(t_2 - t_1) \,. \tag{D.5}$$

For example, if $\hat{H} = \hat{H}_0 = \hat{p}^2/(2m)$ is the free Hamiltonian of a single particle, then the kernel of the time evolution operator is given by [204, Eq. 3.3]

$$U(t_2 - t_1; x, x') = \left(\frac{m}{2\pi\mathrm{i}(t_2 - t_1)}\right)^{1/2} \exp\left[\frac{\mathrm{i}m(x - x')^2}{2(t_2 - t_1)}\right] . \tag{D.6}$$

More generally, the kernel of the time evolution operator can *formally*[1] be written as a Feynman path integral over all paths $\varphi : [t_1, t_2] \to \mathbb{R}^d$ connecting the points x and x' at times t_1 and t_2, respectively [207]. The path integral representation of the time evolution operator is given by

$$U(t_2 - t_1; x, x') = \int_{x(t_1)=x}^{x(t_2)=x'} \mathcal{D}x(t)\, \mathrm{e}^{\mathrm{i}S[x(t)]} =: \int_{x}^{x'} \mathcal{D}x(t)\, \mathrm{e}^{\mathrm{i}S[x(t)]} \,, \tag{D.7}$$

where S is the classical action functional of the system, and we assumed the Hamiltonian $\hat{H}$ to be of the form $\hat{H} = \hat{p}^2/(2m) + V(\hat{x})$, where V is the potential energy of the system. The classical action S corresponding to this Hamiltonian is given by [73, Ch. 2]

$$S[x(t)] = \int_{t_1}^{t_2} \mathrm{d}t \left[\frac{m}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - V(x(t))\right] . \tag{D.8}$$

Given a (formal) expression of the time evolution kernel, the wave function ψ at time t_2 can be written as

$$\psi(t_2, x) = \int \mathrm{d}x'\; U(t_2 - t_1; x, x')\, \psi(t_1, x') \,. \tag{D.9}$$

For notational brevity, we will set $t_1 = 0$ and $t_2 = t$ for the remainder of this Section.

Just like the time evolution operator, the density operator $\hat{\rho}(t)$ can also be represented by a kernel, which we denote by $\rho(t; x, x')$. The expectation of an observable $\hat{\mathcal{O}}$ is then given by

$$\langle\hat{\mathcal{O}}\rangle_t = \mathrm{Tr}\Big\{\hat{\rho}(t)\,\hat{\mathcal{O}}\Big\} = \int \mathrm{d}x\,\mathrm{d}x'\; \rho(t; x, x')\, \mathcal{O}(x, x') \,, \tag{D.10}$$

[1] For a rigorous discussion of Feynman's path over histories approach, see, e.g., [205] or [206] for a rigorous discussion of the Feynman-Vernon approach.

where $\mathcal{O}(x, x')$ is the kernel of the operator $\hat{\mathcal{O}}$. Starting from (D.3) and using (D.7), the time evolution of the kernel of the density operator can be written as

$$\begin{aligned}\rho(t; x, x') &= \int \mathrm{d}x_1\, \mathrm{d}x_2\; U(t; x, x_1)\, \rho(0; x_1, x_2)\, U^\dagger(t; x_2, x') \\ &= \int^{x(t)=x, x'(t)=x'} \mathcal{D}x(\tau)\, \mathcal{D}x'(\tau)\; \mathrm{e}^{\mathrm{i}(S[x(\tau)]-S[x'(\tau)])}\, \rho(0; x(0), x'(0))\ .\end{aligned} \tag{D.11}$$

Notice that the above path integral contains summation over *two* histories. As a consequence, expectation values of observables can be written as

$$\langle \hat{\mathcal{O}} \rangle_{t'} = \int^{x(t)=x'(t)} \mathcal{D}x(\tau)\, \mathcal{D}x'(\tau)\; \mathrm{e}^{\mathrm{i}(S[x(\tau)]-S[x'(\tau)])}\, \rho(0; x(0), x'(0))\, \mathcal{O}(x(t'), x'(t'))\ . \tag{D.12}$$

The above expression may be interpreted as integral over histories on a *closed* time path $\mathscr{C}$ with one branch forward in time from 0 to t and one branch backward in time from t to 0, cf. Fig. 7.1. Therefore, the above approach is called the *Schwinger-Keldysh closed time path* formalism.

Suppose now that the system consists of several degrees of freedom, and that we have a physical motivation for splitting the system into a "system" and an "environment". We may then integrate out the environment variables x_E, which amounts to averaging over all possible environment configurations. This yields the reduced density operator of the system, whose Schrödinger kernel is given by

$$\rho_\mathrm{S}(t; x, x') = \int \mathrm{d}x_\mathrm{E}\; \rho(t; x, x_\mathrm{E}, x', x_\mathrm{E})\ , \tag{D.13}$$

which is essentially taking the partial trace of the density operator over the environment degrees of freedom, cf. [95].

Furthermore, we may then write the Hamiltonian of the system as $\hat{H} = \hat{H}_\mathrm{S} + \hat{H}_\mathrm{E} + \hat{H}_\mathrm{SE}$, where $\hat{H}_\mathrm{S}$, $\hat{H}_\mathrm{E}$ and $\hat{H}_\mathrm{SE}$ are the Hamiltonians of the system, the environment and the interaction between system and environment, respectively. Upon assuming a product initial state, i.e., $\hat{\rho} = \hat{\rho}_\mathrm{S} \otimes \hat{\rho}_\mathrm{E}$, the time evolution of the density operator of the system is then given by

$$\begin{aligned}\rho_\mathrm{S}(t; x, x') &= \int \mathrm{d}x_\mathrm{E}\; \rho(t; x, x_\mathrm{E}, x', x_\mathrm{E}) \\ &= \int^{x(t)=x, x'(t)=x'} \mathcal{D}x(\tau)\, \mathcal{D}x'(\tau)\; \mathrm{e}^{\mathrm{i}(S[x]-S[x']+S_\mathrm{IF}[x,x',t])}\, \rho_\mathrm{S}(0; x(0), x'(0))\ ,\end{aligned} \tag{D.14}$$

where S_{IF} is the *influence action*, given in terms of the *Feynman-Vernon influence functional* $\mathfrak{F}_{\mathrm{IF}}$ as

$$\begin{aligned}\mathfrak{F}_{\mathrm{IF}}[x,x',t] &= \mathrm{e}^{\mathrm{i}S_{\mathrm{IF}}[x,x',t]} \\ &= \int_{\mathscr{C}} \mathcal{D}x_{\mathrm{E}}\, \mathrm{e}^{\mathrm{i}(S_{\mathrm{E}}[x^1_{\mathrm{E}}]-S_{\mathrm{E}}[x^2_{\mathrm{E}}]+S_{\mathrm{SE}}[x^1,x^1_{\mathrm{E}}]-S_{\mathrm{SE}}[x^2,x^2_{\mathrm{E}}])}\rho_2(0;x^1_{\mathrm{E}}(0),x^2_{\mathrm{E}}(0))\ .\end{aligned} \tag{D.15}$$

Notice that the influence action is complex and in particular we have $S_{\mathrm{IF}}[x,x',t] = -\overline{S_{\mathrm{IF}}[x',x,t]}$ [165, Sec. 3.2.1]. Therefore, the influence action describes the non-unitary evolution of the system due to the interaction with the environment.

Glossary

$S(f)$	Classical action of a field configuration f
$\partial\Omega$	Boundary of a set $\Omega \subset \mathbb{R}^d$
$\overline{\Omega}$	Closure of a set $\Omega \subset \mathbb{R}^d$
$\delta^{(d)}(\boldsymbol{x} - \boldsymbol{y})$	Dirac delta distribution in d dimensions
$\mathbf{D}$, $\mathbf{N}$	Dissipation and noise kernels, respectively
$\mathfrak{D}(T)$, $\mathcal{Q}(T)$	Domain and form domain, respectively, of an operator T
$S_{\text{eff.}}$	Effective action
$S(X)$	Shannon's entropy of a random variable X
$S(\mu_1 \| \mu_2)$	Relative entropy (or Kullback-Leibler divergence) of a probability measure μ_1 with respect to another probability measure μ_2
$S_{\text{E}}[\varphi]$	Euclidean action of a field configuration φ
$\mathbb{R}^d$	d-dimensional Euclidean space
$\mathbb{E}[X]$	Expectation value of a random variable X
$\mathfrak{F}_{\text{IF}}$	Feynman-Vernon influence functional
S_{IF}	Influence action
$\hat{\Phi}$, $\hat{\Pi}$	Field operator and conjugate momentum field operator, respectively
$\mathcal{D}\varphi$	Formal functional integral measure
$\Gamma(T)$	Graph of an operator T
$\hat{H}$	Hamilton operator
$L^p(X, \mu)$	Space of p-integrable functions on a measure space (X, Σ, μ)
$H^s(\mathbb{R}^d)$	Hilbert-Sobolev space of order $s \in \mathbb{R}$ on $\mathbb{R}^d$
H^s_K, $H^s(\Omega)$, $\widetilde{H}^s(\Omega)$, $H^s_0(\Omega)$	Local Hilbert-Sobolev spaces, see Section A.2
$i_X(x)$	Information content (or surprisal) of a realization x of a random variable X
$\ker(T)$, $\operatorname{ran}(T)$	Kernel and range, respectively, of an operator T
$\triangle$	(Negative definite) Laplace operator
$\triangle_\varepsilon$	(Negative definite) lattice Laplace operator
$\hat{\triangle}_\varepsilon$	Discrete integral kernel of lattice Laplace operator
$\mathscr{L}^d_{\varepsilon,L}$	d-dimensional regular lattice of size L with lattice spacing ε
$K_\alpha(z)$	Modified Bessel function of the second kind
$I(X : Y)$	Mutual information of two random variables X and Y

$\boldsymbol{\nabla}$	Nabla operator or gradient
$\mathbf{d}_\varepsilon$, $\mathbf{d}_\varepsilon^*$	Lattice gradient and adjoint lattice gradient, respectively
$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$	The set of natural, integer, real and complex numbers, respectively
$\mathscr{B}(\mathcal{H})$, $\mathscr{T}(\mathcal{H})$, $\mathrm{HS}(\mathcal{H})$	Set of bounded, trace-class and Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$, respectively
$\mathbb{P}(x)$	Probability of an event x
$\mathcal{K}$	Path ordered propagator
$\mathscr{C}$, $\mathscr{C}_\beta$	Schwinger-Keldysh contours
ℓ^2	Hilbert space of square summable sequences
ω	Algebraic quantum state
$C^\infty(\overline{\Omega})$	Set of restrictions of smooth functions on $\mathbb{R}^d$ to the closure of a set $\Omega \subseteq \mathbb{R}^d$
$C_0^\infty(\Omega)$, $\mathscr{S}(\mathbb{R}^d)$	Space of compactly supported smooth functions (test functions) on an open set $\Omega \subseteq \mathbb{R}^d$ and Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^d$, respectively
$\mathcal{D}(\Omega)$, $\mathcal{S}(\mathbb{R}^d)$	Test function spaces equipped with their natural topologies
$\hat{U}$	Time evolution operator
$\mathfrak{W}(K_\mathbb{R}, \sigma)$	Weyl algebra of symplectic space $(K_\mathbb{R}, \sigma)$

List of Publications

This thesis is based on the following publications:

- **Relative Entropy and Mutual Information in Gaussian Statistical Field Theory**
  Stefan Floerchinger, Markus Schröfl
  e-Print: 2307.15548 [cond-mat.stat-mech]

- **Local Time Evolution of Relativistic Quantum Fields**
  Stefan Floerchinger, Markus Schröfl
  In preparation.

During the course of my Ph.D., the following publications were written but are not included in this thesis:

- **Relative entropic uncertainty relation for scalar quantum fields**
  Stefan Floerchinger, Tobias Haas, Markus Schröfl
  e-Print: 2107.07824 [quant-ph]
  doi: 10.21468/SciPostPhys.12.3.089
  Published in: SciPost Phys. 12 (2022) 3, 089

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—Thomas Bernhard