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# On the generalised real Section Conjecture

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**Abstract**

In this thesis, we prove the generalised pro-2 real Section Conjecture for what we call *equivariantly triangulable* varieties over $\mathbb{R}$. Examples include all smooth varieties as well as all (possibly singular) affine/projective varieties. Building on this, we derive the generalised real Section Conjecture in the geometrically étale simply connected case.

**Zusammenfassung**

In dieser Arbeit beweisen wir die verallgemeinerte reelle pro-2 Schnittvermutung für sogenannte *äquivariant triangulierbare* Varietäten über $\mathbb{R}$. Dies beinhaltet alle glatten Varietäten, sowie alle (möglicherweise singuläre) affinen/projektiven Varietäten. Darauf aufbauend zeigen wir die verallgemeinerte reelle Schnittvermutung im geometrisch étale einfach zusammenhängenden Fall.

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# Contents

# 0 Introduction

In this thesis, we study the *generalised Section Conjecture* over the real numbers $\mathbb{R}$, utilising the so-called *Sullivan Conjecture* from homotopy theory. To adequately contextualise the present manuscript, we begin with a brief overview of anabelian geometry. To this end, let us fix the following situation:

**Setup.** Let $k$ be a field with separable closure $\bar{k} \supset k$, and let $X$ be a geometrically connected quasi-compact and quasi-separated scheme over $k$. Write $G_k := \mathrm{Gal}(\bar{k}/k)$ for the absolute Galois group of $k$, $X_{\bar{k}}$ for the base change of $X$ to $\bar{k}$, and let $\bar{x} \in X_{\bar{k}}$ be a geometric point.

The basic idea of anabelian geometry is to study the geometry and arithmetic of $X/k$ in terms of the so-called *fundamental exact sequence* of étale fundamental groups

$$(\pi_1^{\text{ét}}(X/k)) \qquad 1 \longrightarrow \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \longrightarrow \pi_1^{\text{ét}}(X, \bar{x}) \longrightarrow G_k \longrightarrow 1,$$

which one has in the above situation (see [SGA 1, Exposé IX, Théorème 6.1]). The following section makes this idea more precise:

**Grothendieck's anabelian geometry.** In a letter to Faltings [14] from 1983, Grothendieck formulated his "yoga of anabelian geometry". To keep this overview reasonably short, let us restrict to the characteristic 0 case (as Grothendieck actually also did in his letter). Thus, let $k/\mathbb{Q}$ be an arithmetically rich ground field, usually a finitely generated field extension of $\mathbb{Q}$.

In his letter, Grothendieck postulated the existence of so-called *anabelian schemes* (over $k$). Here, the feature that decides whether a scheme $X$ shows "anabelian characteristics" is the extent to which it is determined by means of its fundamental exact sequence ($\pi_1^{\text{ét}}(X/k)$). If the geometry of $X/k$ is, in its entirety, determined by its fundamental exact sequence, $X$ is said to be *anabelian*. Grothendieck did not make this intuition into a formal definition and instead postulated three conjectures, the *main conjectures of anabelian geometry*, that anabelian schemes ought to satisfy.

He only formulated these conjectures for *anabelian curves*, which, as we will later see when encountering the étale homotopy type, was probably a wise decision. In order to give a precise formulation of Grothendieck's anabelian conjectures, let us introduce the following notations:

**0.1 Notation** (outer homomorphisms)**.** Let $\mathcal{E}\colon 1 \longrightarrow \bar{\pi} \longrightarrow \pi \longrightarrow G \longrightarrow 1$ be a short exact sequence of profinite groups and let $\pi' \to G$ be another homomorphism of profinite groups.

(a) The set of *outer homomorphisms* from $\pi' \to \pi$ over $G$ is given by

$$\mathrm{Hom}_G^{\mathrm{out}}(\pi', \pi) := \mathrm{Hom}_G(\pi', \pi)_{\bar{\pi}},$$

where $\bar{\pi}$ acts via conjugation.

(b) Similarly, the set of *outer isomorphisms* from $\pi' \to \pi$ over $G$ is given by

$$\mathrm{Isom}_G^{\mathrm{out}}(\pi', \pi) := \mathrm{Isom}_G(\pi', \pi)_{\bar{\pi}}.$$

(c) Finally, in the case that $\pi' = G \to G$ is the identity map, we write

$$\mathcal{S}(\mathcal{E}) := \mathcal{S}(\pi \twoheadrightarrow G) := \mathrm{Hom}_G^{\mathrm{out}}(G, \pi)$$

for the set of $\bar{\pi}$-conjugacy classes of sections of $\pi \twoheadrightarrow G$.

(d) In the case of the fundamental exact sequence, we consequently write $\mathcal{S}(\pi_1^{\text{ét}}(X/k))$ for the set of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$-conjugacy classes of sections of $\pi_1^{\text{ét}}(X, \bar{x}) \twoheadrightarrow G_k$.

The étale fundamental group is a pointed invariant and therefore only functorial with respect to basepoint-preserving morphisms. Passing to outer homomorphisms remedies this technical nuisance:

**0.2 Construction.** Let $Y$ and $X$ be qcqs geometrically connected schemes over $k$ and $\bar{y} \to Y$ as well as $\bar{x} \to X$ geometric points.

(1) Given a morphism of schemes $f: Y \to X$ over $k$, the functoriality of $\pi_1^{\text{ét}}(-)$ induces a map

$$f_*: \pi_1^{\text{ét}}(Y, \bar{y}) \to \pi_1^{\text{ét}}(X, f(\bar{y})).$$

The choice of an étale path $\gamma: f(\bar{y}) \to \bar{x}$ furthermore yields a conjugation isomorphism

$$\gamma(-)\gamma^{-1}: \pi_1^{\text{ét}}(X, f(\bar{y})) \to \pi_1^{\text{ét}}(X, \bar{x}).$$

The composite $\gamma f_* \gamma^{-1}$ thus determines a map $\pi_1^{\text{ét}}(Y, \bar{y}) \to \pi_1^{\text{ét}}(X, \bar{x})$.

(2) It turns out that varying the étale path $\gamma: f(\bar{y}) \to \bar{x}$ conjugates $\gamma f_* \gamma^{-1}$ precisely by $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$. We therefore get a well-defined map

$$\mathrm{Hom}_k(Y, X) \to \mathrm{Hom}_{G_k}^{\text{out}}(\pi_1^{\text{ét}}(Y, \bar{y}), \pi_1^{\text{ét}}(X, \bar{x})), \quad f \mapsto [\gamma f_* \gamma^{-1}]$$

for any choice of geometric points $\bar{y} \to Y$ and $\bar{x} \to X$.

With this observation in place, we are ready to state Grothendieck's anabelian conjectures. First of all, he conjectured that the isomorphism class of an anabelian curve should be entirely determined by its étale fundamental group:

**0.3 Isomorphism Conjecture.** Let $X$ and $Y$ be two connected anabelian curves over some finitely generated field extension $k/\mathbb{Q}$. Then the canonical map

$$\mathrm{Isom}_k(Y, X) \to \mathrm{Isom}_{G_k}^{\text{out}}(\pi_1^{\text{ét}}(Y), \pi_1^{\text{ét}}(X))$$

is a bijection.

Even more surprising, he postulated that all (dominant) morphisms of anabelian curves could be recovered from étale fundamental groups:

**0.4 Homomorphism Conjecture.** Let $X$ and $Y$ be two connected anabelian curves over some finitely generated field extension $k/\mathbb{Q}$. Then the canonical map

$$\mathrm{Hom}_k^{\text{dom}}(Y, X) \to \mathrm{Hom}_{G_k}^{\text{op,out}}(\pi_1^{\text{ét}}(Y), \pi_1^{\text{ét}}(X))$$

is a bijection. Here, $\mathrm{Hom}_k^{\text{dom}}$ denotes the set of *dominant* homomorphisms and $\mathrm{Hom}_{G_k}^{\text{op,out}}$ the set of *open* outer group homomorphisms.

**0.5 Remark.** In a later remark, Grothendieck even conjectured that one could relax the assumptions on $Y$ in the homomorphism conjecture to $Y$ being any smooth variety. This has been shown to be true by Mochizuki, see Remark 0.8.

In the case of $Y = \mathrm{Spec}(k)$, we introduce the following name for the map of Construction 0.2:

**0.6 Definition** (Kummer map)**.** The map $\kappa_{X/k} : X(k) \to \mathcal{S}(\pi_1^{\text{ét}}(X/k))$ given by

$$a \mapsto s_a := [\gamma a_* \gamma^{-1}]$$

is called *Kummer map*.

In his final anabelian conjecture, Grothendieck envisioned that one should even be able to detect rational points in an anabelian fashion:

**0.7 Section Conjecture.** Let $X$ be a connected and proper anabelian curve over some finitely generated field extension $k/\mathbb{Q}$. Then the Kummer map

$$\kappa_{X/k} : X(k) \to \mathrm{Hom}_{G_k}^{\text{out}}(G_k, \pi_1^{\text{ét}}(X)) = \mathcal{S}(\pi_1^{\text{ét}}(X/k))$$

is a bijection.

There is one significant question left, namely: Which curves are actually supposed to satisfy the above three conjectures? This is where Grothendieck turned his anabelian philosophy into a precise programme: He conjectured that *hyperbolic curves* are anabelian.

**0.8 Remark** (state of the art)**.**

(a) There is a lot ought to be said, but ultimately left unsaid, about anabelian geometry in this introduction: Anabelian geometry in *positive characteristic*, *birational anabelian geometry* and many other things. Instead, we refer the reader to Pop's excellent "lectures on anabelian phenomena in geometry and arithmetic" [34] for a more comprehensive overview of the different subdisciplines of anabelian geometry, as well as the state of the art (of 2011) on these.

(b) For hyperbolic curves, two of the three main conjectures have been solved:

   (1) The Isomorphism Conjecture has been resolved by Tamagawa's doctoral thesis [45] in the affine case and Mochizuki's article [29] in the proper case.

   (2) Moreover, Mochizuki resolved the Homomorphism Conjecture in [30].

(c) The Section Conjecture, however, is still open to this day despite substantial efforts to resolve it.

To be more precise, Grothendieck already knew how to prove injectivity of the Kummer map. The surjectivity of $\kappa_{X/k} : X(k) \to \mathcal{S}(\pi_1^{\text{ét}}(X/k))$ is the truly hard part. There is, however, one surprising case where the Section Conjecture has seen a satisfactory solution: When working over the base field $k = \mathbb{R}$.

**The real Section Conjecture.** Over the reals, injectivity of $\kappa_{X/\mathbb{R}}$ ceases to hold. Nevertheless, two real points $a, b$ of $X$ determine the same conjugacy-class of sections if and only if they lie in the same connected component of the real analytification $X(\mathbb{R})$ of $X$. Moreover, $\kappa_{X/\mathbb{R}}$ indeed is surjective as has first been proven by Mochizuki:

**0.9 Theorem** (Mochizuki, 2003; [31, Theorem 3.13-3.15])**.** *Let $X/\mathbb{R}$ be a smooth, geometrically connected curve of genus $g \geq 1$. Then the Kummer map*

$$\kappa_{X/\mathbb{R}}\colon\ \pi_0 X(\mathbb{R}) \to \mathcal{S}(\pi_1^{\text{ét}}(X/\mathbb{R}))$$

*is a bijection of finite sets. More generally, the same holds for any geometrically connected algebraic étale* $\mathrm{K}(\pi,1)$*-space* $X/\mathbb{R}$.

**0.10 Remark.**

(a) Note that, in contrast to Section Conjecture 0.7, $X$ is not assumed to be proper in Theorem 0.9. This is a pecularity of working over the real numbers. Usually, the naive Section Conjecture fails in the non-proper case because of so-called *cuspidal sections* (which are the sections induced by rational points of the boundary of $X$ in its smooth completion $\bar{X}$). Over $\mathbb{R}$, any such boundary point can be obtained as a limit of a sequence of real points lying in (the same connected component of) $X$. Therefore, the cuspidal section attached to this boundary point is already detected by a connected component of $X(\mathbb{R})$.

(b) Over general $k$, the correct formulation of the Section Conjecture in the non-proper case takes cuspidal sections into account.

Wickelgren moreover refined the above theorem in [46] by proving that, in fact, the *geometrically 2-step nilpotent (pro-2) quotient* of $\pi_1^{\text{ét}}(X/\mathbb{R})$ suffices to determine $\pi_0 X(\mathbb{R})$.

**A 2-step nilpotent version.** In order to give a precise statement of Wickelgren's result, we first need to introduce some further notation.

**0.11 Notation.** Let $\mathcal{E}\colon 1 \longrightarrow \bar{\pi} \longrightarrow \pi \longrightarrow G \longrightarrow 1$ be a short exact sequence of profinite groups and let $\bar{\pi} \twoheadrightarrow \bar{\pi}'$ be a fixed quotient of $\bar{\pi}$.

(1) Pushing out the above exact sequence along $\bar{\pi} \twoheadrightarrow \bar{\pi}'$ yields a new short exact sequence

$$(\mathcal{E}') \qquad 1 \longrightarrow \bar{\pi} \longrightarrow \pi *_{\bar{\pi}} \bar{\pi}' \longrightarrow G \longrightarrow 1$$

together with a map

$$\begin{array}{ccccccccccc} \mathcal{E} & & 1 & \longrightarrow & \bar{\pi} & \longrightarrow & \pi & \longrightarrow & G & \longrightarrow & 1 \\ \downarrow & & & & \downarrow & \ulcorner & \downarrow & & \downarrow{=} & & \\ \mathcal{E}' & & 1 & \longrightarrow & \bar{\pi}' & \longrightarrow & \pi *_{\bar{\pi}} \bar{\pi}' & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

of short exact sequences.

(2) In particular, we get an induced map $\mathcal{S}(\mathcal{E}) \to \mathcal{S}(\mathcal{E}')$.

**0.12 Definition** (descending central series)**.** The *descending central series* $C_\bullet\Gamma$ of a profinite group $\Gamma$ is defined inductively by

$$C_{-1}\Gamma = \Gamma \quad \text{and} \quad C_{-(n+1)}\Gamma = [\Gamma, C_{-n}\Gamma],$$

for $n \geq 2$. Here, $[U, V]$ denotes the closed subgroup generated by the commutators

$$[u, v] = uvu^{-1}v^{-1}$$

for $u \in U$ and $v \in V$.

**0.13 Notation.** Let $\mathcal{E}: 1 \longrightarrow \bar{\pi} \longrightarrow \pi \longrightarrow G \longrightarrow 1$ be a short exact sequence of profinite groups.

(1) We write $C_{\geq -n}(\mathcal{E})$ for the short exact sequence obtained by pushing out $\mathcal{E}$ along the quotient map $\bar{\pi} \twoheadrightarrow \bar{\pi}/C_{-(n+1)}\bar{\pi}$ and call it the *n-step nilpotent quotient of* $\mathcal{E}$.

(2) In the case $n = 1$ we simply write $\mathcal{E}^{\mathrm{ab}} := C_{\geq -1}\mathcal{E}$ and refer to it as the *abelianisation* of $\mathcal{E}$.

(3) Given a prime $\ell$, we write $\mathcal{E}^{\wedge \ell}$ for the short exact sequence obtained by pushing out $\mathcal{E}$ along the maximal pro-$\ell$ quotient map $\bar{\pi} \twoheadrightarrow \bar{\pi}^{\wedge \ell}$ (see §1.6 for $\Sigma = \{\ell\}$).

We can now provide a concise statement of Wickelgren's result:

**0.14 Theorem** (Wickelgren's 2-step nilpotent real Section Conjecture; [46, Theorem 1.1])**.** *Let $X/\mathbb{R}$ be a smooth projective geometrically connected curve of genus $g \geq 1$. Then the map*

$$\pi_0 X(\mathbb{R}) \to \mathrm{im}(\mathcal{S}(C_{\geq -2}(\pi_1^{\text{ét}}(X/\mathbb{R})^{\wedge 2})) \to \mathcal{S}(\pi_1^{\text{ét}}(X/\mathbb{R})^{\mathrm{ab},\wedge 2})$$

*induced by the Kummer map is a bijection.*

As already mentioned, Grothendieck only formulated his main conjectures for anabelian curves. Since our results apply to more general varieties, let us also briefly discuss higher-dimensional anabelian geometry.

**What about higher-dimensional varieties?** In dimension 1, Grothendieck postulated that hyperbolic curves should be anabelian. In higher dimensions, the anabelian programme becomes much more speculative. Not only is it unknown whether the anabelian conjectures hold, but there is not even a consensus about which higher-dimensional varieties should be anabelian to begin with.

Grothendieck postulated that at least successive smooth fibrations of anabelian curves should be anabelian. In their seminal paper [40], Schmidt and Stix made substantial progress toward this conjecture by proving (among other important results) that successive fibrations of hyperbolic curves satisfy the Isomorphism Conjecture. More precisely:

**0.15 Theorem** (Schmidt-Stix; [40, Corollary 1.6])**.** *Let $Y$ and $X$ be strongly hyperbolic Artin neighborhoods (see [40, Def. 6.1]) over a finitely generated field extension $k$ of $\mathbb{Q}$. Then the natural map*

$$\mathrm{Isom}_k(Y, X) \to \mathrm{Isom}^{\mathrm{out}}_{G_k}(\pi_1^{\text{ét}}(Y), \pi_1^{\text{ét}}(X))$$

*is a bijection.*

As a corollary, they deduced the following striking result, which was also predicted by Grothendieck in his letter:

**0.16 Theorem** (Schmidt-Stix; [40, Corollary 1.7])**.** *Let $X$ be a smooth and geometrically connected variety over a finitely generated field extension $k$ of $\mathbb{Q}$. Then every point of $x$ has a basis of Zariski-neighborhoods consisting of anabelian varieties (in the sense that they satisfy the Isomorphism Conjecture).*

It is important to note that, although the above results are all formulated in terms of étale fundamental groups, the methods used to deduce these theorems heavily rely on Artin and Mazur's *étale homotopy theory* from [AM], which we will now discuss.

**The étale homotopy type.** The aim of étale homotopy theory is to attach an entire *homotopy type* $\Pi^{\text{ét}}(X)$, called the *étale homotopy type*, to a scheme $X$.

The following constitutes a minimal set of properties that any sensible notion of étale homotopy type ought to satisfy:

**0.17 Desideratum.** The étale homotopy type $\Pi^{\text{ét}}(X)$ should simultaneously refine the étale fundamental group as well as the étale cohomology of $X$:

(a) There is a natural identification of profinite groups

$$\pi_1(\Pi^{\text{ét}}(X), \bar{x}) = \pi_1^{\text{ét}}(X, \bar{x})$$

for any geometric point $\bar{x} \to X$.

(b) There is a natural identification

$$\mathrm{H}^*(\Pi^{\text{ét}}(X); A) = \mathrm{H}^*_{\text{ét}}(X; A)$$

for any finite abelian group $A$.

This first requirement already shows that $\Pi^{\text{ét}}(X)$ cannot be represented by an ordinary topological space: The fundamental group $\pi_1(T)$ of any such $T$ is a plain discrete group, but $\pi_1^{\text{ét}}(X)$ carries itself a profinite topology. Instead, just how profinite groups are represented by formal cofiltered limits of finite groups, $\Pi^{\text{ét}}(X)$ should be modelled as a *formal cofiltered (homotopy) limit* of (suitably finite) topological spaces, i.e. a notion of *profinite homotopy type*.

The first construction of $\Pi^{\text{ét}}(X)$ along these lines was given by Artin and Mazur in [AM]. They constructed an étale homotopy type $\Pi^{\text{ét}}_{\text{AM}}(X)$ satisfying the requirements formulated in 0.17 as an object of the category Pro(ho(**sSet**)) — the pro-category (see §1.1) of the homotopy category of simplicial sets.

A major drawback of Artin and Mazur's construction is that it requires first passing to the homotopy category and then forming the pro-category. However, in order to employ techniques from abstract homotopy theory (in the sense of Quillen) to analyse $\Pi^{\text{ét}}(X)$, one would much rather do things the other way around: First pass to the pro-category, and then to a homotopy category. It has taken considerable effort from many people, notably Friedlander (see, e.g., [13]) and Isaksen (see [11]), to make this happen. One reason for this is that it is technically very demanding to set up a well-behaved homotopy theory of formal cofiltered limits, i.e. equip pro-categories with model structures.

**0.18.** In this thesis, we exclusively work with the $\infty$-categorical incarnation $\Pi^{\text{ét}}_\infty(X)$ defined via Lurie's shape theory as can be found, for example, in [20], where also the connection to $\Pi^{\text{ét}}_{\text{AM}}(X)$ is discussed. We give a concise introduction to $\Pi^{\text{ét}}_\infty(X)$ in (§2.1).

The étale homotopy type lets us define *higher étale homotopy groups*.

**0.19 Definition** (higher étale homotopy groups)**.** Let $X$ be a qcqs scheme, $\bar{x}$ a geometric point of $X$ and $n \geq 1$. The $n$-th *étale homotopy group* of $X$ at $x$ is given by

$$\pi_n^{\text{ét}}(X, \bar{x}) := \pi_n(\Pi^{\text{ét}}(X), \bar{x}).$$

If $n \geq 2$, it is a profinite abelian group.

It turns out that for curves, these are not so interesting. They are étale $\mathrm{K}(\pi, 1)$-schemes (see Theorem 2.3.12) in the following sense:

**0.20 Definition** (étale $\mathrm{K}(\pi,1)$)**.** A qcqs scheme $X$ is said to be an *étale* $\mathrm{K}(\pi,1)$ if, for any choice of geometric point $\bar{x}$ of $X$ and any $n \geq 2$, the étale homotopy group $\pi_n^{\text{ét}}(X,\bar{x})$ vanishes.

Note that all the anabelian main conjectures rely solely on $\pi_1^{\text{ét}}$. Therefore, it is unreasonable to expect any of them to hold for schemes that are not étale $\mathrm{K}(\pi,1)$. Instead, one has to consider variants of the above conjectures utilising the full étale homotopy type in this case. In the following section, we present the appropriate variant of the Section Conjecture.

**The generalised Section Conjecture.** Within the realm of anabelian geometry, the present manuscript constitutes a further step in the study of anabelian phenomena of higher-dimensional varieties as in the work of Schmidt and Stix [40]. Concretely, we study the *generalised real Section Conjecture*. In order to give a precise statement of our main results, we first briefly explain how one formulates the generalised Section Conjecture using the étale homotopy type:

**0.21.** Similarly to the fundamental exact sequence, we proved that there is a *fundamental fibre sequence*

$$\Pi_\infty^{\text{ét}}(X_{\bar{k}}) \to \Pi_\infty^{\text{ét}}(X) \to \mathrm{B}G_k$$

of étale homotopy types in joint work with Peter J. Haine and Sebastian Wolf, see [15, Corollary 0.5] or Theorem 2.1.16.

This suggests to replace the set $\mathcal{S}(\pi_1^{\text{ét}}(X/k))$ of Grothendieck's Section Conjecture with the set of homotopy classes of sections of the above fibration $\Pi_\infty^{\text{ét}}(X) \to \mathrm{B}G_k$:

**Definition** (2.2.1)**.** The set of *étale sections* of $X/k$ is given by

$$\mathcal{S}^{\text{ét}}(X/k) := \pi_0 \operatorname{map}_{\mathrm{B}G_k}(\mathrm{B}G_k, \Pi_\infty^{\text{ét}}(X)).$$

With an appropriate replacement for $\mathcal{S}(\pi_1^{\text{ét}}(X/k))$ in place, the *generalised Section Conjecture* becomes:

**Conjecture** (generalised Section Conjecture, 2.2.3)**.** The canonical map

$$X(k) \to \mathcal{S}^{\text{ét}}(X/k), \quad a \mapsto [a_*]$$

is a bijection.

By performing a homotopy-theoretic analogue of the construction in 0.11 on the fundamental fibre sequence 0.21, we are furthermore able to define a pro-$\ell$ version $\mathcal{S}_\ell^{\text{ét}}(X/k)$ of $\mathcal{S}^{\text{ét}}(X/k)$ for every prime $\ell$, resulting in:

**Conjecture** (generalised pro-$\ell$ Section Conjecture, 2.3.9)**.** The composition

$$X(k) \to \mathcal{S}^{\text{ét}}(X/k) \to \mathcal{S}_\ell^{\text{ét}}(X/k)$$

is a bijection.

## 0.1 Overview of results

In order to give a precise statement of our main results, let us quickly introduce the following notation:

**0.22 Definition** (equivariantly triangulable, 3.3.1)**.** A scheme $X/\mathbb{R}$ is called *equivariantly triangulable* if its complex analytification $X(\mathbb{C})$ admits the structure of a finite-dimensional $\mathbb{Z}/2$-CW complex.

**0.23 Remark.** In Proposition 3.3.11, we show that $X/\mathbb{R}$ is equivariantly triangulable in the following cases:

(1) $X$ is smooth over $\mathbb{R}$.

(2) $X$ is affine and of finite type over $\mathbb{R}$.

(3) $X$ is projective over $\mathbb{R}$.

However, we expect this to hold in much greater generality, namely whenever $X/\mathbb{R}$ is separated and of finite type (see Remark 3.3.12).

Our first main result is, in first approximation, a higher-dimensional generalisation of Wickelgren's pro-2 real Section Conjecture:

**Theorem A** (3.3.6)**.** Let $X$ be any equivariantly triangulable qcqs scheme of finite type over $\mathbb{R}$. Then $X/\mathbb{R}$ satisfies the *generalised pro-2 Section Conjecture*: The canonical map

$$\pi_0 X(\mathbb{R}) \to \mathcal{S}_2^{\text{ét}}(X/\mathbb{R})$$

is a bijection.

**0.24 Remark.** In the case of a hyperbolic curve $X/\mathbb{R}$, Theorem A recovers the classical pro-2 real Section Conjecture (see Corollary 2.3.13). Our Theorem A thus in particular provides yet another proof of the classical real Section Conjecture for hyperbolic curves, see Corollary 3.3.16.

Leveraging on the fact that simply connected homotopy types are nilpotent, we deduce that a surprisingly large class of varieties over $\mathbb{R}$ satisfies the full generalised Section Conjecture:

**Theorem B** (3.3.9)**.** Any equivariantly triangulable and geometrically étale simply connected qcqs scheme $X$ of finite type over $\mathbb{R}$ satisfies the *generalised Section Conjecture*: The canonical map

$$\pi_0 X(\mathbb{R}) \to \mathcal{S}^{\text{ét}}(X/\mathbb{R})$$

is a bijection of sets.

**0.25 Remark.** Theorem A and B showcase the following novel phenomena in anabelian geometry, which are worth highlighting:

(a) Theorems A and B are the first anabelian results that do not require smoothness of the varieties in question.

(b) In existing results of anabelian geometry, all the geometric information is extracted from $\pi_1^{\text{ét}}$. In this sense, Theorem B is *orthogonal* to all other existing results in anabelian geometry.

Our proof is very homotopy-theoretic in nature (see §0.2 for an outline of the proof) and is based upon the (proven, see [28], [25] and [7]) *Sullivan Conjecture*. In order to make the Sullivan Conjecture applicable to the real Section Conjecture, we furthermore proved some results in profinite homotopy theory. As these results might be of interest on their own, we are now going to briefly present them as well.

**0.26.** Fix some nonempty set of primes $\Sigma$.

Firstly, we slightly expanded upon Lurie's [SAG, Appendix E] by working out some basic facts surrounding the notion of *nilpotency* in ($\Sigma$-)profinite homotopy theory. Here, our main theorem is a homotopy-theoretic analogue of the well-known fact that nilpotent profinite groups are the product over their $p$-Sylow subgroups:

**Theorem C** (1.6.22)**.** The following are equivalent for a connected $\Sigma$-profinite anima $K$:

(1) $K$ is nilpotent (in the sense of Definition 1.6.17).

(2) The canonical map

$$K \to \prod_{\ell \in \Sigma} K_\ell^\wedge$$

is an equivalence of ($\Sigma$-)profinite anima.

Here, $(-)_\ell^\wedge$ denotes Lurie's *$\ell$-profinite completion functor* (see §(1.4)).

As explained in (§0.2), the key ingredient we prove in order to deduce Theorem B from Theorem A is the following:

**Theorem D** (3.2.4)**.** Let $G$ be a finite $p$-group and $K$ a connected nilpotent profinite anima with $G$-action. Assume that $K^{\mathrm{h}G} \neq \varnothing$. Then the canonical map

$$(K^{\mathrm{h}G})_p^\wedge \to (K_p^\wedge)^{\mathrm{h}G}$$

is an equivalence of $p$-profinite anima.

Theorem D is a straightforward consequence of Theorem C and our final result, a computation in non-abelian profinite group cohomology. This latter computation is probably well-known to experts, but we were unable to find a reference for it:

**Theorem E** (3.1.1)**.** Let $\Gamma$ be a profinite group acting on another profinite group $N$. If the supernatural orders of $\Gamma$ and $N$ are coprime, then $\mathrm{H}^1(\Gamma, N)$ vanishes.

## 0.2 Outline of the proofs of Theorems A and B

Our proofs of Theorems A & B proceed in several steps, which we are now going to outline:

**Step 1: Reinterpretation in terms of homotopy fixed points.** We employ Quick's strategy from [37] to reinterpret the Section Conjecture as a problem comparing fixed points to homotopy fixed points. In our setup, this reinterpretation is based on the following Theorem of Lurie, see [SAG, E.6.5.1]:

**Ingredient 1.** Let $G$ be a profinite group.

Denote by $\mathbf{Ani}_\pi^\wedge(G)$ the $\infty$-category of profinite anima with continuous $G$-action and by $(\mathbf{Ani}_\pi^\wedge)_{/\mathrm{B}G}$ the $\infty$-category of profinite anima over $\mathrm{B}G$. Then the construction

$$K \quad \mapsto \quad (K/\!\!/G \to * /\!\!/G = \mathrm{B}G)$$

carrying a profinite $G$-anima $K$ to its homotopy quotient $K/\!\!/G$ furnishes an equivalence of $\infty$-categories $\mathbf{Ani}_\pi^\wedge(G) \to (\mathbf{Ani}_\pi^\wedge)_{/\mathrm{B}G}$. The inverse of this equivalence is given by sending a map $L \to \mathrm{B}G$ to its fibre equipped with the natural $G$-action.

**0.27.** The fundamental fibre sequence 0.21

$$\Pi_\infty^{\text{ét}}(X_{\bar{k}}) \to \Pi_\infty^{\text{ét}}(X) \to \mathrm{B}G_k$$

implies that, under the equivalence of Ingredient 1, the map $\Pi_\infty^{\text{ét}}(X) \to \mathrm{B}G_k$ corresponds to $\Pi_\infty^{\text{ét}}(X_{\bar{k}})$ equipped with its natural $G_k$-action.

As, under the equivalence of Ingredient 1, the mapping anima $\mathrm{map}_{\mathrm{B}G}(\mathrm{B}G, K/\!\!/G)$ gets furthermore identified with the materialisation $|K^{\mathrm{h}G}|$ (see §1.3.1) of the homotopy fixed point anima $K^{\mathrm{h}G}$ (see §1.5), we obtain:

**0.28.** There is a canonical bijection

$$\mathcal{S}^{\text{ét}}(X/k) = \pi_0\, \Pi_\infty^{\text{ét}}(X_{\bar{k}})^{\mathrm{h}G_k}.$$

Ingredient 1 therefore lets us reformulate the Section Conjecture as a comparison

$$X(k) = X(\bar{k})^{G_k} \;\rightsquigarrow\; \pi_0\, \Pi_\infty^{\text{ét}}(X_{\bar{k}})^{\mathrm{h}G_k} = \mathcal{S}^{\text{ét}}(X/k)$$

of fixed points with homotopy fixed points. In [44], Sullivan proposed his conjecture about exactly this kind of comparison.

**Step 2: Applying the Sullivan Conjecture.** In our proof, we employ the following version of the Sullivan Conjecture, again due to Lurie:

**Ingredient 2** (Sullivan Conjecture, see 1.7.1)**.**

Let $p$ be a prime number, $G$ a finite $p$-group and $K$ a finite-dimensional $G$-CW complex. Then the composite of the canonical maps

$$(K^G)_p^\wedge \to (K^{\mathrm{h}G})_p^\wedge \to (K_p^\wedge)^{\mathrm{h}G}$$

is an equivalence of $p$-profinite anima.

Here, given a space $L$ with $G$-action, $L^G$ denotes the *fixed point space* and $L^{\mathrm{h}G}$ the *homotopy fixed point anima* of $L$ with respect to the given $G$-action (§1.5).

Now we apply Ingredient 2 to the complex analytification $X(\mathbb{C})$.

**0.29.** If $X$ is equivariantly triangulable, the Sullivan Conjecture from above is applicable to $X(\mathbb{C})$ and, since $X(\mathbb{R}) = X(\mathbb{C})^{\mathbb{Z}/2}$, we obtain a chain of equivalences

$$\begin{aligned} X(\mathbb{R})_2^\wedge &\simeq (X(\mathbb{C})^{\mathbb{Z}/2})_2^\wedge \\ &\simeq (X(\mathbb{C})_2^\wedge)^{\mathrm{h}\mathbb{Z}/2} \qquad \text{by the Sullivan Conjecture.} \end{aligned}$$

Our objective is to eventually extend the above chain of equivalences in such a way that Theorem A is what remains when applying $\pi_0(-)$ everywhere.

Since $\mathcal{S}^{\text{ét}}(X/\mathbb{R}) = \pi_0\, \Pi_\infty^{\text{ét}}(X_\mathbb{C})^{\mathrm{h}G_k}$ is defined purely in terms of the étale homotopy type $\Pi_\infty^{\text{ét}}(X_\mathbb{C})$, we need a bridge from the topological to the algebraic world:

**Step 3: The generalised Riemann existence theorem.** In order to continue, we have to relate

$$X(\mathbb{C}) \longleftrightarrow \Pi_\infty^{\text{ét}}(X_{\mathbb{C}}).$$

To this end, recall that the Riemann existence theorem asserts that there is a canonical isomorphism of profinite groups

$$\pi_1^{\text{ét}}(X_{\mathbb{C}}) \quad \cong \quad \pi_1(X(\mathbb{C}))^\wedge,$$

where $(-)^\wedge$ denotes profinite completion.

Artin and Mazur refined this isomorphism to an equivalence of homotopy types:

**Ingredient 3** (generalised Riemann existence, see 3.3.3)**.** Let $X$ be a scheme of finite type over $\mathbb{R}$. Then the profinite completion (see 1.2.14) of $X(\mathbb{C})$, denoted by $X(\mathbb{C})^\wedge_\pi$, coincides with the étale homotopy type $\Pi_\infty^{\text{ét}}(X_{\mathbb{C}})$.

Ingredient 3 is the crucial input enabling us to relate the Sullivan Conjecture with the Section Conjecture.

**0.30.** Indeed, the generalised Riemann existence theorem supplies equivalences

$$\begin{aligned} X(\mathbb{C})^\wedge_2 &\simeq (X(\mathbb{C})^\wedge_\pi)^\wedge_2 \\ &\simeq \Pi_\infty^{\text{ét}}(X_{\mathbb{C}})^\wedge_2, \end{aligned}$$

compatible with the respective $\mathbb{Z}/2$-actions, and therefore lets us prolong the chain of equivalences of 0.29 to

$$\begin{aligned} X(\mathbb{R})^\wedge_2 &\simeq (X(\mathbb{C})^{\mathbb{Z}/2})^\wedge_2 \\ &\simeq (X(\mathbb{C})^\wedge_2)^{\mathrm{h}\mathbb{Z}/2} && \text{by the Sullivan Conjecture} \\ &\simeq (\Pi_\infty^{\text{ét}}(X_{\mathbb{C}})^\wedge_2)^{\mathrm{h}\mathbb{Z}/2} && \text{by the generalised Riemann existence theorem.} \end{aligned}$$

The set of connected components of $(\Pi_\infty^{\text{ét}}(X_{\mathbb{C}})^\wedge_2)^{\mathrm{h}\mathbb{Z}/2}$ is precisely $\mathcal{S}_2^{\text{ét}}(X/\mathbb{R})$. So, since $(-)^\wedge_2$ preserves connected components, Theorem A is what remains of the equivalence

$$X(\mathbb{R})^\wedge_2 \simeq (\Pi_\infty^{\text{ét}}(X_{\mathbb{C}})^\wedge_2)^{\mathrm{h}\mathbb{Z}/2}$$

on $\pi_0$.

**0.31.** With this final description, there is only one obstruction to proving the full Section Conjecture left: The 2-profinite completion $(-)^\wedge_2$ appearing in $(\Pi_\infty^{\text{ét}}(X_{\mathbb{C}})^\wedge_2)^{\mathrm{h}\mathbb{Z}/2}$ but not in the set of étale sections $\mathcal{S}^{\text{ét}}(X/\mathbb{R}) = \pi_0\, \Pi_\infty^{\text{ét}}(X_{\mathbb{C}})^{\mathrm{h}\mathbb{Z}/2}$.

This is where our Theorem D comes into play:

**Step 4: "A + D $\implies$ B".** In order to finally deduce the real Section Conjecture for $X$, we would like the following to hold:

**Ingredient 4.** The canonical map

$$(\Pi_\infty^{\text{ét}}(X_{\mathbb{C}})^{\mathrm{h}\mathbb{Z}/2})^\wedge_2 \to (\Pi_\infty^{\text{ét}}(X_{\mathbb{C}})^\wedge_2)^{\mathrm{h}\mathbb{Z}/2}$$

is an equivalence of 2-profinite anima.

**0.32.** Assuming Ingredient 4, we can yet again prolong the chain of equivalences of (0.30) to

$$\begin{aligned} X(\mathbb{R})^\wedge_2 &\simeq (X(\mathbb{C})^{\mathbb{Z}/2})^\wedge_2 \\ &\simeq (X(\mathbb{C})^\wedge_2)^{\mathrm{h}\mathbb{Z}/2} && \text{by the Sullivan Conjecture} \\ &\simeq (\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_2)^{\mathrm{h}\mathbb{Z}/2} && \text{by the Riemann existence theorem} \\ &\simeq (\Pi^{\text{ét}}_\infty(X_\mathbb{C})^{\mathrm{h}\mathbb{Z}/2})^\wedge_2 && \text{by Ingredient 4,} \end{aligned}$$

and, since $(-)^\wedge_2$ preserves connected components, conclude that the real Section Conjecture holds:

$$\begin{aligned} \pi_0 X(\mathbb{R}) &= \pi_0 X(\mathbb{R})^\wedge_2 \\ &= \pi_0(\Pi^{\text{ét}}_\infty(X_\mathbb{C})^{\mathrm{h}\mathbb{Z}/2})^\wedge_2 \\ &= \pi_0 \, \Pi^{\text{ét}}_\infty(X_\mathbb{C})^{\mathrm{h}\mathbb{Z}/2} \\ &= \mathcal{S}^{\text{ét}}(X/\mathbb{R}). \end{aligned}$$

Unfortunately, Ingredient 4 does not hold in general. In fact, it already fails in the case of ordinary homotopy theory:

**0.33.** In his seminal article [44], Sullivan's conjecture originally asked for the map

$$(K^G)^\wedge_p \to (K^{\mathrm{h}G})^\wedge_p$$

to be an equivalence. In [7], Carlsson showed that this conjecture does not hold in general: Instead, as in our Ingredient 2, one should look at the composition

$$(K^G)^\wedge_p \to (K^{\mathrm{h}G})^\wedge_p \to (K^\wedge_p)^{\mathrm{h}G},$$

which, using Bousfield-Kan's $p$-completion, he shows to be an equivalence. So the discrepancy of Sullivans original conjecture and what Carlsson showed lies precisely in a comparison of the form

$$(K^{\mathrm{h}G})^\wedge_p \to (K^\wedge_p)^{\mathrm{h}G} \tag{†}$$

as Ingredient 4 is asking for in the case $p = 2$ and $K = \Pi^{\text{ét}}_\infty(X_\mathbb{C})$.

Nevertheless, Carlsson demonstrates that (†) is indeed an equivalence, provided that $K$ is *nilpotent*. Our Theorem D, a profinite analogue of this fact, shows that the above Ingredient 4 holds, provided that $\Pi^{\text{ét}}_\infty(X_\mathbb{C})$ is nilpotent. Since this is particularly the case whenever $X/\mathbb{R}$ is geometrically étale simply connected, we arrive at Theorem B by means of 0.32.

## 0.3 Related work

Since much of the related work has already been discussed in the introduction, we will be very brief here. Within the realm of anabelian geometry, our main Theorems A and B are most closely related to the work of Schmidt and Stix [40], since we also consistently employ the étale homotopy type to address higher-dimensional phenomena, along with the existing body of results on the real Section Conjecture. It should be noted that among the various proofs of the classical real Section Conjecture available in the literature — most notably Mochizuki's [31], which contains the first such proof, as well as those by Stix [42], Vistoli and Bresciani [5], Pál [32] and Wickelgren's nilpotent version [46] — the methods of the present paper are most similar to those of Pál and Wickelgren: Pál's proof is certainly influenced by ideas surrounding the Sullivan Conjecture, and Wickelgren actually derives her result by applying the Sullivan Conjecture.

We should also mention Quick's article [37] again: We follow his strategy to reinterpret the Section Conjecture in terms of homotopy fixed points. This reinterpretation is crucial for our use of the Sullivan Conjecture.

### 0.4 Linear overview

Section 1 is dedicated to profinite homotopy theory: We briefly recall some facts about Serre's mod $\mathcal{C}$ homotopy theory and pro-objects in $\infty$-category theory (§1.1) and then take some time to set up the notions and results of ($\Sigma$-)profinite homotopy theory we need in (§1.2 - §1.5). Here, it should be noted that the contents of (§1.2 - §1.5) are essentially a subset of [SAG, Appendix E]. It is in (§1.6) that we slightly expand on SAG: We discuss *nilpotency* in the context of profinite homotopy theory, resulting in a proof of Theorem D of (§0.1).

Section 2 discusses various forms of the generalised Section Conjecture: We briefly touch upon our choice of model of the étale homotopy type $\Pi_\infty^{\text{ét}}(X)$ via Lurie's shape theory in (§2.1). Then, we formulate the generalised Section Conjecture in (§2.2) and show that it recovers Grothendieck's Section Conjecture for étale $\mathrm{K}(\pi, 1)$ schemes. In (§2.3), we combine (§1.6) with (§2.2): We introduce the notion of *generalised $\Sigma$-nilpotent étale sections* for any choice of set of primes $\Sigma$, again compare it with the classical notion of geometrically ($\Sigma$-)nilpotent sections for hyperbolic curves, and use this to formulate a generalised pro-$\ell$ Section Conjecture.

Section 3 is where we prove Theorems A, B, D and E: Theorem E is proven in (§3.1) and subsequently combined with Theorem C to deduce Theorem D in (§3.2). Finally, we deduce Theorems A and B in (§3.3).

### Notation and Conventions

(1) We freely make use of the language of $\infty$-categories as developed by Lurie in his trilogy [HTT], [HA] and [SAG].

(2) By abuse of notation, we will consider ordinary categories (i.e. 1-categories) as $\infty$-categories via the nerve construction $\mathrm{N}(-)$, and usually suppress it from the notation.

(3) We follow Scholze and Clausen's suggestion to replace the term "space" by "anima". We write **Ani** for what Lurie calls "$\infty$-category of spaces" and refer to it as "$\infty$-category of anima".

## 1 Some ($\Sigma$-)profinite homotopy theory

This section develops all the purely homotopy-theoretical machinery and results we use. We start with a brief background section (§1.1) containing a short recollection of Serre's mod $\mathcal{C}$ homotopy theory and the notion of pro-objects in the language of $\infty$-categories. In (§1.2), we introduce the notion of *$\Sigma$-profinite anima* for a chosen set of primes $\Sigma$ and discuss some of their formal properties. Next, we specialise to the case $\Sigma = \pi$ of all primes in (§1.3): Here, we adapt well-known concepts of homotopy theory, most importantly *homotopy groups* and *cohomology groups*, to the profinite setting. In (§1.4), we go to the other extreme and specialise to $\Sigma = \{p\}$ the set consisting of a single prime. The resulting *p-profinite homotopy theory* is especially easy to control, as it turns out to be entirely determined by $\mathbb{F}_p$-cohomology. We also introduce the *p-profinite completion* functor $(-)^\wedge_p$ and discuss some of its pleasant properties. We then briefly discuss the notion of *homotopy fixed points* in (§1.5). In (§1.6), we adapt the notion of "nilpotent anima" to the setting of profinite homotopy theory: Among other things, we show that Bousfield-Kan $p$-completion $\mathbb{F}_{p^\infty}(-)$ agrees with $p$-profinite completion $(-)^\wedge_p$ of *π-finite* anima and use that to deduce Theorem D from the classical arithmetic fracture square. Finally, in (§1.7), we state the variant of the Sullivan Conjecture, due to Lurie, that we use.

## 1.1 Background

### Serre's mod $\mathcal{C}$ theory

In this section, we quickly recall the basics of Serre's mod $\mathcal{C}$ theory as originally laid out in [41]. The main result relevant to the present paper is Theorem 1.1.3, a version of the classical Hurewicz theorem relative to a fixed class of abelian groups.

**1.1.1 Definition** (Hurewicz class)**.** Let $\mathcal{C}$ be a non-empty collection of abelian groups.

(a) $\mathcal{C}$ is a *class*, if for any exact sequence $A \longrightarrow B \longrightarrow C$ of abelian groups $A, C \in \mathcal{C}$ implies $B \in \mathcal{C}$.

(b) $\mathcal{C}$ is a *Hurewicz class* if it is a class satisfying the the following additional properties:

- (⊗) If $A, B \in \mathcal{C}$, then also $A \otimes_{\mathbb{Z}} B$ and $\mathrm{Tor}_1^{\mathbb{Z}}(A, B) \in \mathcal{C}$.
- (H) If $A \in \mathcal{C}$, then also $\mathrm{H}_i(A; \mathbb{Z}) \in \mathcal{C}$ for $i > 0$. Here, $\mathrm{H}_*(A; \mathbb{Z})$ denotes the group homology of $A$ acting trivially on $\mathbb{Z}$.

**1.1.2 Definition.** Let $\mathcal{C}$ be a class and $\varphi : A \to B$ a homomorphism of abelian groups.

(a) $\varphi$ is a $\mathcal{C}$-*monomorphism*, if $\ker(\varphi) \in \mathcal{C}$.

(b) $\varphi$ is a $\mathcal{C}$-*epimorphism*, if $\mathrm{coker}(\varphi) \in \mathcal{C}$.

(c) $\varphi$ is a $\mathcal{C}$-*isomorphism* if it is both a $\mathcal{C}$-monomorphism and a $\mathcal{C}$-epimorphism.

**1.1.3 Theorem** (Serres' Hurewicz mod $\mathcal{C}$, [41, Théorème 1])**.** *Let $\mathcal{C}$ be a Hurewicz class of abelian groups, $n$ a positive integer and $K$ a topological space with $\pi_0 K = \pi_1 K = *$. Suppose $\pi_i K \in \mathcal{C}$ for $i < n$. Then, for $0 < i < n$, also $\mathrm{H}_i(K; \mathbb{Z}) \in \mathcal{C}$ and $\pi_n K \to \mathrm{H}_n(K; \mathbb{Z})$ is a $\mathcal{C}$-isomorphism.*

In this thesis, we make use of the following Hurewicz classes:

**1.1.4 Example.** Let $p$ be a prime number and $\varnothing \neq \Sigma$ a set of prime numbers. The following are Hurewicz classes:

(1) The collection $\mathcal{C} = \{0\}$. Serre's Hurewicz mod $\mathcal{C}$ recovers the ordinary Hurewicz theorem in this case.

(2) The collection $\mathcal{C}$ of finitely generated abelian groups.

(3) The collection $\mathcal{C}$ of finite abelian $p$-groups.

(4) The collection $\mathcal{C}$ of uniquely $p$-divisible abelian groups.

(5) The collection $\mathcal{C}$ of $\Sigma$-finite abelian groups (Definition 1.2.3).

### Categories of pro-objects

In this section, we will collect some basic facts concerning pro-categories. We will only discuss the $\infty$-categorical version as this also subsumes the 1-categorical concept, i.e.

$$\mathrm{Pro}(\mathrm{N}(\mathcal{C})) \simeq \mathrm{N}(\mathrm{Pro}(\mathcal{C})),$$

for any ordinary category $\mathcal{C}$ (see Remark 1.1.15).

**1.1.5 Recollection** (equivalence of $\infty$-categories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be $\infty$-categories and $F: \mathcal{C} \to \mathcal{D}$ a functor.

(a) Recall that a functor $G: \mathcal{D} \to \mathcal{C}$ is said to be *homotopy inverse to* $F$ if $G \circ F$ and $F \circ G$ are isomorphic to the identity functors $\mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{D}}$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ and $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$, respectively.

(b) $F$ is said to be an *equivalence of $\infty$-categories* if it admits a homotopy inverse.

(c) $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exists some equivalence of $\infty$-categories $F: \mathcal{C} \to \mathcal{D}$.

**1.1.6 Notation.** Given two $\infty$-categories $\mathcal{C}$ and $\mathcal{D}$, we write $\mathrm{Fun}^{\mathrm{cofilt}}(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for the full subcategory spanned by those functors $F: \mathcal{C} \to \mathcal{D}$ that preserve cofiltered limits.

We define the pro-category of an $\infty$-category $\mathcal{C}$ via a universal property:

**1.1.7 Definition** (pro-category)**.** Let $\mathcal{C}$ be an $\infty$-category. A functor $j: \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ *exhibits* $\mathrm{Pro}(\mathcal{C})$ *as pro-category of* $\mathcal{C}$ if the following hold:

(a) The $\infty$-category $\mathrm{Pro}(\mathcal{C})$ admits cofiltered limits.

(b) Given any $\infty$-category $\mathcal{E}$ admitting cofiltered limits, precomposition with $j$ induces an equivalence

$$j^*: \mathrm{Fun}^{\mathrm{cofilt}}(\mathrm{Pro}(\mathcal{C}), \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

of $\infty$-categories.

**1.1.8 Remark.** If $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ and $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})'$ exhibit $\mathrm{Pro}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})'$ as pro-categories of $\mathcal{C}$ respectively, then the above definition gives rise to a canonical equivalence $\mathrm{Pro}(\mathcal{C}) \simeq \mathrm{Pro}(\mathcal{C})'$. Therefore, if there exists a functor $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ exhibiting $\mathrm{Pro}(\mathcal{C})$ as a pro-category of $\mathcal{C}$, it is essentially unique.

**1.1.9 Notation.** Let $\mathcal{C}$ be an $\infty$-category.

(a) We write $j: \mathcal{C} \to \mathrm{Fun}(\mathcal{C}, \mathbf{Ani})^{\mathrm{op}}, c \mapsto h^c = \mathrm{map}_{\mathcal{C}}(c, -)$ for the Yoneda embedding.

(b) A *prorepresentation* of a functor $F: \mathcal{C} \to \mathbf{Ani}$ is a cofiltered diagram $\mathcal{I} \to \mathcal{C}, i \mapsto d_i$ together with an equivalence

$$F \simeq \lim_{i \in \mathcal{I}} h^{d_i},$$

where the limit is taken in $\mathrm{Fun}(\mathcal{C}, \mathbf{Ani})^{\mathrm{op}}$.

(c) We say that a functor $F: \mathcal{C} \to \mathbf{Ani}$ is *prorepresentable* if there exists a propresentation of it.

(d) We write $\mathrm{Fun}^{\mathrm{pro}}(\mathcal{C}, \mathbf{Ani})$ for the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathbf{Ani})$ spanned by the prorepresentable functors.

By the dual of [HTT, 5.3.5.4], pro-categories always exist:

**1.1.10 Theorem** (existence of pro-categories, Lurie)**.** *Let $\mathcal{C}$ be an $\infty$-category.*

(1) *The restriction of the Yoneda embedding*

$$j\colon \mathcal{C} \to \mathrm{Fun}^{\mathrm{pro}}(\mathcal{C}, \mathbf{Ani})^{\mathrm{op}}, \quad c \mapsto h^c = \mathrm{map}_{\mathcal{C}}(c, -)$$

*exhibits* $\mathrm{Fun}^{\mathrm{pro}}(\mathcal{C}, \mathbf{Ani})^{\mathrm{op}}$ *as a pro-category of* $\mathcal{C}$*.*

(2) *If $\mathcal{C}$ admits finite limits, then a functor is prorepresentable if and only if it preserves finite limits. Therefore,*

$$j\colon \mathcal{C} \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathbf{Ani})^{\mathrm{op}}, \quad c \mapsto h^c = \mathrm{map}_{\mathcal{C}}(c, -)$$

*exhibits* $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathbf{Ani})^{\mathrm{op}}$ *as a pro-category of $\mathcal{C}$. Here,* $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathbf{Ani})$ *denotes the full subcategory of* $\mathrm{Fun}(\mathcal{C}, \mathbf{Ani})$ *spanned by the left exact functors.*

**1.1.11 Remark.** Let $\mathcal{C}$ be an $\infty$-category. Note that, by Theorem 1.1.10, Remark 1.1.8 and the Yoneda lemma, any functor $j\colon \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ exhibiting $\mathrm{Pro}(\mathcal{C})$ as a pro-category of $\mathcal{C}$ is necessarily fully faithful.

**1.1.12 Notation** («lim»)**.** Let $\mathcal{C}$ be an $\infty$-category, $j\colon \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ a pro-category of $\mathcal{C}$ and $c \in \mathrm{Pro}(\mathcal{C})$ a pro-object in $\mathcal{C}$. By Theorem 1.1.10, there exists a cofiltered diagram $\mathcal{I} \to \mathcal{C}, i \mapsto c_i$ and an equivalence

$$c \simeq \lim_{i \in \mathcal{I}} j(c_i).$$

By abuse of notation, we will from now on simply write $c \simeq «\lim»_i\, c_i$ in this case.

We will make extensive use of the following observation:

**1.1.13 Theorem** (existence of pro-adjoints)**.** *Let $R\colon \mathcal{D} \to \mathcal{C}$ be a functor between $\infty$-categories with finite limits. If $R$ preserves finite limits, it admits a pro-left adjoint $L\colon$ $\mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{D})$ that, with respect to the identifications of Theorem 1.1.10,*

$$\mathrm{Pro}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathbf{Ani})^{\mathrm{op}} \quad \textit{and} \quad \mathrm{Pro}(\mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}, \mathbf{Ani})^{\mathrm{op}},$$

*is given by precomposition with $R$.*

*Proof.* The assumptions imply that

$$L\colon \mathcal{C} \to \mathrm{Pro}(\mathcal{D}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}, \mathbf{Ani})^{\mathrm{op}}, \quad c \mapsto [d \mapsto \mathrm{map}_{\mathcal{C}}(c, R(d))]$$

is well-defined. The fact that its extension $L\colon$ $\mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{D})$ is left adjoint to $R\colon$ $\mathrm{Pro}(\mathcal{D}) \to \mathrm{Pro}(\mathcal{C})$ follows immediately from the Yoneda lemma. □

We will make frequent use of the following important result about diagrams in pro-categories:

**1.1.14 Proposition.** *Let $\mathcal{C}$ be an $\infty$-category and $\mathcal{I}$ a finite partially ordered set. Then precomposition along $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ induces an equivalence of $\infty$-categories*

$$\mathrm{Pro}(\mathrm{Fun}(\mathcal{I}, \mathcal{C})) \to \mathrm{Fun}(\mathcal{I}, \mathrm{Pro}(\mathcal{C})).$$

*In other words, any diagram $\mathcal{I} \to \mathrm{Pro}(\mathcal{C})$ can be obtained, in an essentially unique way, as a cofiltered limit of diagrams $\mathcal{I} \to \mathcal{C}$.*

*Proof.* Using the natural identifications

$$\mathrm{Pro}(-) \simeq \mathrm{Ind}(-^{\mathrm{op}})^{\mathrm{op}} \quad \text{and} \quad \mathrm{Fun}(-,-)^{\mathrm{op}} \simeq \mathrm{Fun}(-^{\mathrm{op}}, -^{\mathrm{op}}),$$

this follows from dualizing [HTT, Prop. 5.3.5.15]:

$$\begin{aligned}
\mathrm{Pro}(\mathrm{Fun}(\mathcal{I}, \mathcal{C})) &\simeq \mathrm{Ind}(\mathrm{Fun}(\mathcal{I}, \mathcal{C})^{\mathrm{op}})^{\mathrm{op}} \\
&\simeq \mathrm{Ind}(\mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}}))^{\mathrm{op}} \\
&\simeq \mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}))^{\mathrm{op}} \quad \text{(by [HTT, Prop. 5.3.15])} \\
&\simeq \mathrm{Fun}(\mathcal{I}, \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}) \\
&\simeq \mathrm{Fun}(\mathcal{I}, \mathrm{Pro}(\mathcal{C})).
\end{aligned}$$

□

**1.1.15 Remark.** If $\mathcal{C}$ is an ordinary category with finite limits, there similarly is an identification

$$\mathrm{Pro}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}},$$

see (the opposite of) [24, Prop. 6.1.7]. This can be leveraged to see that

$$\mathrm{Pro}(\mathrm{N}(\mathcal{C})) \simeq \mathrm{N}(\mathrm{Pro}(\mathcal{C}))$$

in this case: According to [HTT, 5.5.6.16], any left-exact functor preserves ($k$-)truncatedness of objects and morphisms. This shows that any left exact functor $\mathrm{N}(\mathcal{C}) \to \mathbf{Ani}$ factorises over the inclusion $\mathrm{N}(\mathbf{Set}) \subset \mathbf{Ani}$ implying that

$$\begin{aligned}
\mathrm{Pro}(\mathrm{N}(\mathcal{C})) &= \mathrm{Fun}^{\mathrm{lex}}(\mathrm{N}(\mathcal{C}), \mathbf{Ani})^{\mathrm{op}} \\
&\simeq \mathrm{Fun}^{\mathrm{lex}}(\mathrm{N}(\mathcal{C}), \mathrm{N}(\mathbf{Set}))^{\mathrm{op}} \\
&\simeq \mathrm{N}(\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}}) \\
&\simeq \mathrm{N}(\mathrm{Pro}(\mathcal{C})).
\end{aligned}$$

An argument for this without assuming $\mathcal{C}$ to admit finite limits can be found in [HTT, 5.3.5.6].

## 1.2 Σ-profinite homotopy theory

The goals of this section are to define the ∞-category of Σ-profinite anima, and to introduce homotopy-theoretic analogues of the group-theoretic (Σ-)profinite completion functors.

Throughout this section, let Σ denote a nonempty set of prime numbers.

**1.2.1.** In the following, we will mostly work with

(1) $\Sigma = \pi$ the set of all primes,

(2) $\Sigma = \{p\}$ the set containing a single prime $p$, and

(3) $\Sigma = p'$ the set containing all primes but $p$.

**1.2.2 Notation.**

(a) Given any set of primes Σ, we write $\Sigma' = \pi \smallsetminus \Sigma = \{p \text{ prime} \mid p \notin \Sigma\}$.

(b) By abuse of notation, we usually substitute $\Sigma = \{p\}$ with $p$ in formulas, i.e. we write $\mathbf{Ani}_p^{\wedge}$ instead of $\mathbf{Ani}_{\{p\}}^{\wedge}$, $(-)_p^{\wedge}$ instead of $(-)_{\{p\}}^{\wedge}$ etc. in the following.

### $\Sigma$-profinite group theory

To define the homotopy-theoretic concept of a $\Sigma$-(pro)finite anima, we first briefly discuss its group-theoretic counterpart.

**1.2.3 Definition** ($\Sigma$-finite groups)**.**

(a) A group $G$ is said to be $\Sigma$-*finite* if it is a finite group the cardinality of which lies in the multiplicative closure of $\Sigma$.

(b) We write $\mathbf{Grp}_\Sigma \subset \mathbf{Grp}$ for the full subcategory of the category of groups spanned by the $\Sigma$-finite groups.

Now, we pass to pro-objects:

**1.2.4 Definition** ($\Sigma$-profinite groups)**.** The category $\mathbf{Grp}_\Sigma^\wedge := \mathrm{Pro}(\mathbf{Grp}_\Sigma)$ is called the category of $\Sigma$-*profinite groups*.

**1.2.5 Lemma.** *Let* $1 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 1$ *be a short exact sequence of groups. The following are equivalent:*

(1) *$G$ is $\Sigma$-finite.*

(2) *$H$ and $K$ are $\Sigma$-finite.*

*Proof.* Immediately follows from $\#G = \#H \cdot \#K$. □

**1.2.6 Proposition** ($\Sigma$-profinite completion of groups)**.**

(1) *The inclusions* $\mathbf{Grp}_\Sigma \subset \mathbf{Grp}_\pi \subset \mathbf{Grp}$ *are stable under finite limits.*

(2) *The inclusions* $\mathbf{Grp}_\Sigma^\wedge \subset \mathbf{Grp}_\pi^\wedge \subset \mathrm{Pro}(\mathbf{Grp})$ *admit left adjoints*

$$(-)^\wedge = (-)^{\wedge\pi} : \ \mathrm{Pro}(\mathbf{Grp}) \to \mathbf{Grp}_\pi^\wedge \quad \textit{and} \quad (-)^{\wedge\Sigma} : \mathbf{Grp}_\pi^\wedge \to \mathbf{Grp}_\Sigma^\wedge,$$

*called* profinite completion *and* $\Sigma$-profinite completion *respectively. Under the identification* $\mathrm{Pro}(-) \simeq \mathrm{Fun}^{\mathrm{lex}}(-, \mathbf{Set})^{\mathrm{op}}$ *of Remark 1.1.15, they correspond to precomposition with the inclusions* $\mathbf{Grp}_\Sigma \subset \mathbf{Grp}_\pi \subset \mathbf{Grp}$.

*Proof.*

(1) The inclusions $\mathbf{Grp}_\Sigma \subset \mathbf{Grp}_\pi \subset \mathbf{Grp}$ are certainly stable under finite products. Given any cospan $G \to K \leftarrow H$ of groups, the fibre product $G \times_K H$ is a subgroup of $G \times H$. Therefore, by Lemma 1.2.5, they are also stable under fibre products. As fibre products and finite products generate all finite limits, the claim follows.

(2) In virtue of Theorem 1.1.13, this is a formal consequence of (1). □

**1.2.7 Remark.**

(a) Note that $\mathbf{Grp}_\pi$ simply denotes the category of finite groups.

(b) Write **PfGrps** for the category of "topological" profinite groups, i.e. compact Hausdorff and totally disconnected topological groups. Then the functor $\mathbf{Grp}_\pi^\wedge \to \mathbf{PfGrps}$ given as the extension by cofiltered limits of the functor

$$\mathbf{Grp}_\pi \to \mathbf{PfGrps}, \quad G \mapsto G^\delta,$$

where $G^\delta$ denotes $G$ equipped with the discrete topology, is an equivalence of categories by *Stone duality*, see [23, p.236].

(c) The composition $\mathbf{Grp} \subset \mathrm{Pro}(\mathbf{Grp}) \xrightarrow{(-)^\wedge} \mathbf{Grp}_\pi^\wedge \simeq \mathbf{PfGrps}$ recovers the usual profinite completion functor.

### $\Sigma$-profinite anima

The following is the homotopy-theoretic counterpart of $\Sigma$-finite groups:

**1.2.8 Definition** ($\Sigma$-finite anima)**.**

(a) An anima $K$ is said to be $\Sigma$-*finite* if the following conditions are satisfied:

(1) $K$ is truncated, i.e. there exists $N \in \mathbb{N}$ such that $\pi_n(K, k) = 0$ for all $k \in K$ and all $n \geq N$.

(2) The set $\pi_0(K)$ is finite.

(3) For each point $k \in K$ and each integer $n \geq 1$, the group $\pi_n(K, k)$ is a $\Sigma$-finite group.

(b) The full subcategory $\mathbf{Ani}_\Sigma \subset \mathbf{Ani}$ spanned by the $\Sigma$-finite anima is called the $\infty$-category of $\Sigma$-*finite anima*.

**1.2.9 Definition** ($\Sigma$-profinite anima)**.** The $\infty$-category $\mathbf{Ani}_\Sigma^\wedge := \mathrm{Pro}(\mathbf{Ani}_\Sigma)$ is called the $\infty$-category of $\Sigma$-*profinite anima*.

**1.2.10 Remark.** Let $\Sigma$ and $\Sigma'$ be two nonempty sets of primes.

(1) Note that $\Sigma' \subset \Sigma$ implies $\mathbf{Ani}_{\Sigma'} \subset \mathbf{Ani}_\Sigma$ and hence also $\mathbf{Ani}_{\Sigma'}^\wedge \subset \mathbf{Ani}_\Sigma^\wedge$.

(2) Since $\Sigma \subset \pi$ we in particular have an inclusion $\mathbf{Ani}_\Sigma^\wedge \subset \mathbf{Ani}_\pi^\wedge$.

**1.2.11 Lemma.** *Let*

$$\begin{array}{ccc} F & \longrightarrow & E \\ \big\downarrow & \lrcorner & \big\downarrow \\ * & \longrightarrow & B \end{array}$$

*be a fibre sequence of anima with $B$ connected. If two out of $F, E$ and $B$ are $\Sigma$-finite, so is the third.*

*Proof.* By splitting the induced long exact sequence on homotopy groups into short exact sequences, this follows from Lemma 1.2.5. A little extra care has to be taken in the case where $E$ and $B$ are assumed to be $\Sigma$-finite: Here, the induced long exact sequence on homotopy groups terminates in

$$\ldots \longrightarrow \pi_1(B) \longrightarrow \pi_0 F \twoheadrightarrow \pi_0 E \longrightarrow * = \pi_0 B.$$

and one has to show that $\pi_0 F$ is finite. This can be done by choosing different points of $F$: Indeed, since $\pi_0 E$ is assumed to be finite, the surjection $\pi_0 F \twoheadrightarrow \pi_0 E$ gives a finite decomposition

$\pi_0 F = \sqcup_{C\in\pi_0 E}(\pi_0 F)_C$, where $(\pi_0 F)_C$ denotes the subset of those connected components of $F$ that are mapped to $C \in \pi_0 E$. It is therefore enough to show that $(\pi_0 F)_C$ is finite for each $C \in \pi_0 E$. Given such a connected component $C$, choose a point $x \in F$ the component of which gets mapped to $C \in \pi_0 E$ (which is possible since $\pi_0 F \twoheadrightarrow \pi_0 E$ is surjective). The long exact sequence of homotopy groups induced at this point shows that

$$\begin{aligned}(\pi_0 F)_C &= (\pi_0 F \twoheadrightarrow \pi_0 E)^{-1}(C)\\ &= \operatorname{im}(\pi_1(B,b) \to \pi_0 F),\end{aligned}$$

where $b \in B$ denotes the image of $x \in F$. Thus $(\pi_0 F)_C$ is the image of a map with finite domain, hence finite. This shows that $\pi_0 F$ is finite as claimed. □

**1.2.12 Proposition** ($\Sigma$-profinite completion of anima)**.**

(1) *The inclusions* $\mathbf{Ani}_\Sigma \subset \mathbf{Ani}_\pi \subset \mathbf{Ani}$ *are stable under finite limits.*

(2) *The inclusions* $\mathbf{Ani}_\Sigma^\wedge \subset \mathbf{Ani}_\pi^\wedge \subset \operatorname{Pro}(\mathbf{Ani})$ *admit left adjoints*

$$(-)_\pi^\wedge\colon\ \operatorname{Pro}(\mathbf{Ani}) \to \mathbf{Ani}_\pi^\wedge \quad \textit{and} \quad (-)_\Sigma^\wedge\colon\ \mathbf{Ani}_\pi^\wedge \to \mathbf{Ani}_\Sigma^\wedge.$$

*Moreover, under the identifications* $\operatorname{Pro}(-) \simeq \operatorname{Fun}^{\mathrm{lex}}(-,\mathbf{Ani})^{\mathrm{op}}$ *of Theorem 1.1.10, they correspond to precomposition with the inclusions* $\mathbf{Ani}_\Sigma \subset \mathbf{Ani}_\pi \subset \mathbf{Ani}$.

For the proof, we need the following observation:

**1.2.13 Lemma** (Mayer-Vietoris sequence)**.** *Let*

$$\begin{array}{ccc} E' & \xrightarrow{\ p\ } & E \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ B' & \xrightarrow[\ g\ ]{} & B \end{array}$$

*be a pullback square of anima with $B$ connected. Then, for any choice of points $e \in E$ and $b' \in B'$, there is an induced fibre sequence*

$$\begin{array}{ccc} \Omega B & \longrightarrow & E' \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p\times q} \\ * & \xrightarrow[\ e\times b'\ ]{} & E\times B', \end{array}$$

*Proof of Lemma 1.2.13.* Write $b := g(b')$ and let $F := \operatorname{fib}_b(g)$ denote the fibre of $g$ at $b$. Note that, since $B$ is connected, the maps $E \overset{e}{\leftarrow} * \leftarrow F \to B'$ induce a map $F \to E'$ making the diagram

$$\begin{array}{ccccccc} F & \longrightarrow & E' & \xrightarrow{\ p\ } & E & \longrightarrow & * \\ \downarrow & (1) & \downarrow{\scriptstyle p\times q} & (2) & \downarrow{\scriptstyle \mathrm{id}_E\times f} & (3) & \downarrow{\scriptstyle b} \\ B' & \xrightarrow[\ e\times \mathrm{id}_{B'}\ ]{} & E\times B' & \xrightarrow{\ \mathrm{id}_E\times g\ } & E\times B & \xrightarrow[\ \mathrm{pr}_B\ ]{} & B \\ & & {\scriptstyle \mathrm{pr}_{B'}}\downarrow & (4) & \downarrow{\scriptstyle \mathrm{pr}_B} & & \\ & & B' & \xrightarrow[\ g\ ]{} & B & & \end{array}$$

commute. By assumption, the pasted diagrams $(1)+(2)+(3)$ and $(2)+(4)$ are cartesian. Squares (3) and (4) are also clearly cartesian. Because $(2)+(4)$ as well as (4) are cartesian, so is (2). Since $(1)+(2)+(3)$ as well as (2) and (3) are cartesian, it analogously follows that (1) is cartesian. Write $F^{\times}$ for the fibre of $p \times q$ at $e \times b'$. Then the above shows that we have a commutative diagram

$$\begin{array}{ccccc} F^{\times} & \longrightarrow & F & \longrightarrow & E' \\ \downarrow & (5) & \downarrow & (1) & \downarrow {\scriptstyle p\times q} \\ * & \xrightarrow[b']{} & B' & \xrightarrow[e\times \mathrm{id}_{B'}]{} & E \times B'. \end{array}$$

By definition, the pasted diagram $(5) + (1)$ is cartesian. Since, by the above, square (1) is also cartesian, it follows that (5) is cartesian too. The computation

$$\begin{aligned} * \times_{b',B'} F &\simeq * \times_{b',B'} (B' \times_{B,b} *) \\ &\simeq * \times_{b,B,b} * \\ &\simeq \Omega B, \end{aligned}$$

now lets us conclude. □

*Proof of Proposition 1.2.12.*

(1) It certainly suffices to show that $\mathbf{Ani}_{\Sigma} \subset \mathbf{Ani}$ is stable under finite limits, as we obtain the second claim in the special case $\Sigma = \pi$. Note that $\mathbf{Ani}_{\Sigma} \subset \mathbf{Ani}$ is clearly stable under finite products. It is therefore enough to show that it is also stable under fibre products. To this end, let $L \to K \leftarrow L'$ be any cospan of $\Sigma$-finite anima. If $K$ is not connected, say $K = \sqcup_{i=1}^{n} K_i$ with each $K_i$ a connected component of $K$, then

$$L \times_K L' = \sqcup_{i=1}^{n} (L_i \times_{K_i} L'_i),$$

where $L_i$ and $L'_i$ denote the unions of those connected components of $L$ and $L'$, respectively, mapping to $K_i$. Since $\Sigma$-finite anima are clearly stable under finite coproducts, we may assume $K$ to be connected. We may thus apply Lemma 1.2.13 to conclude that there is a fibre sequence

$$\begin{array}{ccc} \Omega K & \longrightarrow & L \times_K L' \\ \downarrow & \lrcorner & \downarrow \\ * & \xrightarrow[l\times l']{} & L \times L', \end{array}$$

for any choice of points $l \in L$ and $l' \in L'$. Since $L$ and $L'$ are $\Sigma$-finite, so is $L \times L'$. In particular, $\pi_0(L \times L') = \pi_0 L \times \pi_0 L'$ is finite. Choose a component $C \times C'$ of $L \times L'$ as well as a point $l \times l'$ lying in $C \times C'$ and observe that, since fibre products commute with products, square $(B)$ in

$$\begin{array}{ccccc} \Omega K & \longrightarrow & C \times_K C' & \longrightarrow & L \times_K L' \\ \downarrow & (A) & \downarrow & (B) & \downarrow \\ * & \xrightarrow[l\times l']{} & C \times C' & \longrightarrow & L \times L \end{array}$$

is cartesian. Since the pasted rectangle $(A) + (B)$ is also cartesian, it follows that $(A)$ is cartesian. Because $C \times C'$ is furthermore connected, Lemma 1.2.11 is applicable to $(A)$ and implies that $C \times_K C'$ is $\Sigma$-finite. Therefore, so is

$$L \times_K L' = \coprod_{C \times C' \in \pi_0 L \times \pi_0 L'} C \times_K C'$$

as a finite disjoint union of $\Sigma$-finite anima.

(2) In virtue of Theorem 1.1.13, this is a formal consequence of (1). □

**1.2.14 Definition** ($\Sigma$-profinite completion)**.** The functors constructed in Proposition 1.2.12

$$(-)^\wedge_\pi \colon \operatorname{Pro}(\mathbf{Ani}) \to \mathbf{Ani}^\wedge_\pi \quad \text{and} \quad (-)^\wedge_\Sigma \colon \mathbf{Ani}^\wedge_\pi \to \mathbf{Ani}^\wedge_\Sigma$$

are called *profinite completion* and $\Sigma$-*profinite completion*, respectively.

## 1.3 Profinite homotopy theory

In this section, we specialise to the set $\Sigma = \pi$ consisting of all primes. Since $\mathbf{Ani}^\wedge_\Sigma \subset \mathbf{Ani}^\wedge_\pi$ for all other choices of $\Sigma$, all concepts and results that hold at this level of generality automatically apply to $\Sigma$-profinite anima as well.

Our motivation for studying profinite anima is as follows: Since the étale fundamental group $\pi_1^{\text{ét}}(X, \bar{x})$ of a scheme $X$ is a profinite group, one cannot hope to adequately develop étale homotopy theory using ordinary anima, as their fundamental groups are ordinary groups. We will start by explaining how the homotopy groups of profinite anima are naturally *profinite groups*. We will then attach *cohomology groups* to any such profinite anima and close the section by studying their cohomological finiteness properties. We refer the reader to [SAG, §E] for a thorough treatment of profinite homotopy theory.

### Homotopy groups of profinite anima

As every profinite anima $K$ can be written as a formal cofiltered limit $K = \text{«lim»}_\alpha K_\alpha$ of $\pi$-finite anima $K_\alpha$, levelwise application of the ordinary homotopy group functors yield *profinite homotopy groups* $\hat{\pi}_n(K)$ of $K$ (see Definition 1.3.3). An important feature of profinite anima (that ceases to hold for the $\infty$-category $\operatorname{Pro}(\mathbf{Ani})$ of arbitrary proanima) is that one can check on homotopy groups whether a map of profinite anima is an equivalence (Theorem 1.3.9 & Corollary 1.3.10).

**1.3.1 Definition** (materialisation)**.** The *materialisation* functor $|-| \colon \mathbf{Ani}^\wedge_\pi \to \mathbf{Ani}$ is the extension by cofiltered limits of the inclusion $\mathbf{Ani}_\pi \subset \mathbf{Ani}$, i.e. $|\text{«lim»}_\alpha K_\alpha| = \lim_\alpha K_\alpha \in \mathbf{Ani}$.

**1.3.2 Definition** (points)**.** Let $K$ be a profinite anima.

(a) A *point* of $K$ is a map $k \colon * \to K$ of profinite anima. We sometimes write $k \in K$ in this case.

(b) The $\infty$-category of *pointed profinite anima* is given by $\mathbf{Ani}^\wedge_{\pi,*} = (\mathbf{Ani}^\wedge_\pi)_{*/} \simeq \operatorname{Pro}(\mathbf{Ani}_{\pi,*})$.

**1.3.3 Definition** (profinite homotopy groups)**.** Let $K$ be a profinite anima, $k$ a point of $K$ and $n \geq 1$.

(a) The *(profinite) set of connected components of* $K$ is the image of $K$ under the Pro-extension

$$\hat{\pi}_0 : \mathbf{Ani}_\pi^\wedge = \mathrm{Pro}(\mathbf{Ani}_\pi) \to \mathrm{Pro}(\mathbf{Set}_\pi) = \mathbf{Set}_\pi^\wedge$$

of the connected component functor $\pi_0 : \mathbf{Ani}_\pi \to \mathbf{Set}_\pi$.

(b) The *n-th (profinite) homotopy group of* $K$ *at* $k$ is the image of $(K, k)$ under the Pro-extension

$$\hat{\pi}_n : \mathbf{Ani}_{\pi,*}^\wedge \simeq \mathrm{Pro}(\mathbf{Ani}_{\pi,*}) \to \mathrm{Pro}(\mathbf{Grp}_\pi) = \mathbf{Grp}_\pi^\wedge$$

of the ordinary homotopy group functor $\pi_n : \mathbf{Ani}_{\pi,*} \to \mathbf{Grp}_\pi$. Note that $\hat{\pi}_n(K, k)$ is a profinite abelian group for $n \geq 2$.

**1.3.4 Lemma** [SAG, E.5.2.2]**.** *The functor* $\mathbf{Ani}_\pi^\wedge \to \mathbf{Set}$*, carrying a profinite anima* $K$ *to the set of connected components* $\pi_0(|K|)$ *of its materialisation, preserves cofiltered limits.*

**1.3.5 Corollary.** *Let* $n \geq 1$*. The functor* $\mathbf{Ani}_{\pi,*}^\wedge \to \mathbf{Grp}$*, carrying a pointed profinite anima* $(K, k)$ *to the n-th homotopy group* $\pi_n(|K|, |k|)$ *of its materialisation, preserves cofiltered limits.*

*Proof.* In virtue of Lemma 1.3.4, this follows from $\pi_n = \pi_0 \circ \Omega^n$, where $\Omega^n$ denotes the $n$-fold iterated loop anima functor $\Omega : \mathbf{Ani}_* \to \mathbf{Ani}_*, (K, k) \mapsto \Omega_k(K) := \mathrm{map}_*((\mathbb{S}^1, *), (K, k))$. □

**1.3.6 Corollary.** *Let* $K$ *be a profinite anima,* $k$ *a point of* $K$ *and* $n \geq 1$*. Then:*

(1) *The underlying set of* $\hat{\pi}_0 K$ *coincides with* $\pi_0|K|$*.*

(2) *The underlying group of* $\hat{\pi}_n(K, k)$ *coincides with* $\pi_n(|K|, |k|)$*.*

**1.3.7 Definition.** Let $n \geq 0$ and $K$ a profinite anima.

(a) $K$ is said to be *n-truncated* if for all points $k \in K$ and all $m > n$ the homotopy group $\hat{\pi}_m(K, k)$ vanishes.

(b) $K$ is said to be *truncated* if it is $n$-truncated for some $n \geq 0$.

(c) $K$ is said to be *n-connected* if it is connected and if for all points $k \in K$ and all $1 \leq i \leq n$ the homotopy group $\hat{\pi}_i(K, k)$ vanishes.

The following is a useful tool for reducing from profinite to $\pi$-finite statements:

**1.3.8 Lemma.** *Let* $K$ *be a profinite anima.*

(1) *The following are equivalent:*

  (a) $K$ *is n-truncated.*

  (b) $K$ *can be written as a formal cofiltered limit* $K = «\lim»_\alpha K_\alpha$ *with n-truncated* $\pi$*-finite anima* $K_\alpha$*.*

(2) *The following are equivalent:*

  (a) $K$ *is n-connected.*

  (b) $K$ *can be written as a formal cofiltered limit* $K = «\lim»_\alpha K_\alpha$ *with n-connected* $\pi$*-finite anima* $K_\alpha$*.*

*Proof.* The first statement follows immediately from the equivalences of conditions (2) and (4) in [SAG, E.4.6.2], the second one from [SAG, E.4.6.3]. In order to see this, note that Lurie defines the notions of $n$-truncatedness and $n$-connectedness by the conditions (b) above, see [SAG, §E.4.1.1]. □

A map of profinite groups $G \to H$ is an isomorphism if and only if the induced map on underlying groups is an isomorphism. One way to see this is through the topological incarnation of profinite groups mentioned in Remark 1.2.7: continuous bijections between compact Hausdorff spaces are already homeomorphisms. Crucially, a homotopy-theoretic analogue of this fact holds for profinite anima:

**1.3.9 Theorem** (Profinite Whitehead theorem; [SAG, E.3.1.6]**).**
*The materialisation functor* $|-|\colon \mathbf{Ani}_\pi^\wedge \to \mathbf{Ani}$ *is conservative. That is, the following are equivalent for a map* $f\colon K \to L$ *of profinite anima:*

(1) *The map* $f$ *is an equivalence of profinite anima.*

(2) *The map* $f$ *induces an equivalence of ordinary anima* $|K| \to |L|$.

Since it is not explicitly mentioned in [SAG, E.3.1.6], let us also record the following immediate corollary of the Profinite Whitehead theorem:

**1.3.10 Corollary.** *The following are equivalent for a map* $f\colon K \to L$ *of profinite anima:*

(1) *The map* $f$ *is an equivalence of profinite anima.*

(2) *The map* $f$ *induces isomorphisms on homotopy groups, more precisely:*

  (a) *The induced map* $f_*\colon \hat{\pi}_0 K \to \hat{\pi}_0 L$ *is an isomorphism of profinite sets.*

  (b) *For all* $k \in K$ *and* $n \geq 1$*, the induced map* $f_*\colon \hat{\pi}_n(K, k) \to \hat{\pi}_n(L, f(k))$ *is an isomorphism of profinite groups.*

*Proof.* The only non-obvious part is to prove that $f$ is already an equivalence if it induces isomorphisms on homotopy groups (and connected components). By Theorem 1.3.9, it suffices to show that $|f|\colon |K| \to |L|$ is an equivalence of ordinary anima in this case. But since, by Corollary 1.3.6, we have that $\pi_n(|K|, k)$ is the underlying group (resp. set, if $n = 0$) of $\hat{\pi}_n(K, k)$ and similarly for $L$, the claim follows from the ordinary Whitehead theorem [17, Thm. 4.5]. □

### Cohomology of profinite anima

Since filtered colimits of abelian groups are exact, one is able to define cohomology of profinite anima in terms of that of $\pi$-finite anima (Definition 1.3.12). See [SAG, §E.7.1] for more details.

**1.3.11 Notation.** For $K$ a Kan complex and $A$ an abelian group, we write $\mathrm{C}^*(K; A)$ for the chain complex of *$A$-valued (singular) cochains on $K$*. Since the construction $K \mapsto \mathrm{C}^*(K; A)$ carries equivalences to quasi-isomorphisms, it induces a functor $\mathrm{C}^*(-; A)\colon \mathbf{Ani}^{\mathrm{op}} \to \mathcal{D}(\mathbb{Z})$, where $\mathcal{D}(\mathbb{Z})$ denotes the derived $\infty$-category of $\mathbb{Z}$.

**1.3.12 Definition.** Let $A$ be an abelian group.

(a) We let $\mathrm{C}^*(-; A)\colon (\mathbf{Ani}_\pi^\wedge)^{\mathrm{op}} \to \mathcal{D}(\mathbb{Z})$ denote the extension of $\mathrm{C}^*(-; A)\colon \mathbf{Ani}_\pi^{\mathrm{op}} \to \mathcal{D}(\mathbb{Z})$ preserving filtered colimits.

(b) If $K$ is a profinite anima and $n \geq 0$ is an integer, we write $\mathrm{H}^n(K;A) = \mathrm{H}^n(\mathrm{C}^*(K;A))$ and refer to it as the *$n$-th cohomology group of $K$ with coefficients in $A$*.

**1.3.13 Remark.** If $K = «\lim»_\alpha K_\alpha$, then $\mathrm{C}^*(K;A) = \mathrm{colim}_\alpha \mathrm{C}^*(K_\alpha;A)$. In particular, $\mathrm{H}^n(K;A) = \mathrm{colim}_\alpha \mathrm{H}^n(K_\alpha;A)$.

As is the case for ordinary anima, cohomology is representable by Eilenberg-Maclane anima:

**1.3.14 Theorem** (representability of $\mathrm{H}^n(-;A)$, [SAG, E.7.1.6])**.** *Let $A$ be a finite abelian group and $n \geq 1$. The functor*

$$\mathbf{Ani}_\pi^\wedge \to \mathbf{Ab}, \quad K \mapsto \mathrm{H}^n(K;A)$$

*is represented by $\mathrm{K}(A,n) \in \mathbf{Ani}_\pi \subset \mathbf{Ani}_\pi^\wedge$ (in the homotopy category of $\mathbf{Ani}_\pi^\wedge$).*

*Proof.* Both functors

$$L \mapsto \mathrm{H}^n(L;A) \quad \text{and} \quad L \mapsto \pi_0 \operatorname{map}(L, \mathrm{K}(A,n))$$

coincide on $\mathbf{Ani}_\pi^{\mathrm{op}}$ and commute with filtered colimits. □

The following notion will be used extensively throughout the subsequent sections:

**1.3.15 Definition.** A map of (profinite) anima is said to be an *$\mathbb{F}_p$-equivalence* if it induces an isomorphism on $\mathbb{F}_p$-cohomology groups.

**1.3.16 Remark.** Because $K \mapsto \mathrm{C}^*(K;A)$ agrees with $K \mapsto \mathrm{C}^*(|K|;A)$ whenever $K$ is $\pi$-finite, Remark 1.3.13 shows that there is a natural comparison map $\mathrm{C}^*(K;A) \to \mathrm{C}^*(|K|;A)$ depending functorially on $K \in \mathbf{Ani}_\pi^\wedge$. In particular, there are natural maps $\mathrm{H}^*(K;A) \to \mathrm{H}^*(|K|;A)$ comparing "continuous" to ordinary cohomology. We refer the reader to [SAG, E.7.1.5] for more details.

### (Co-)homological finiteness properties of $\pi$-finite anima

In this section, we will collect some basic (co-)homological finiteness properties of $\pi$-finite and profinite anima that will be made use of later on. We also derive a Künneth theorem for profinite anima (Proposition 1.3.21).

**1.3.17 Lemma.** *For $K \in \mathbf{Ani}_\pi$ we have that $\mathrm{H}_n(K;\mathbb{Z})$ is finite for $n \geq 1$.*

*Proof.* We proceed in several steps:

(1) *$K$ simply connected.*

In the simply connected case, the claim is an immediate consequence of Serre's Hurewicz mod $\mathcal{C}$, Theorem 1.1.3, for $\mathcal{C}$ the class of finite abelian groups.

(2) *$K = \mathrm{B}G$ for a finite group $G$.*

When $K = \mathrm{B}G$ for a finite group $G$, we have $\mathrm{H}_*(\mathrm{B}G;\mathbb{Z}) = \mathrm{H}_*(G;\mathbb{Z})$, so the claim follows from the corresponding statement in group homology ($\mathrm{H}_n(G;\mathbb{Z})$ is finitely generated and annihilated by $\mathrm{ord}(G)$ provided that $n > 0$, hence finite). Furthermore, if we have a local system on $\mathrm{B}G$ with value a finite abelian group $A$, $\mathrm{H}_n(\mathrm{B}G;A) = \mathrm{H}_n(G;A)$ is finite provided that $n > 0$ (observe that this is already the case for the standard complex computing $\mathrm{H}_*(G;A)$).

(3) *Arbitrary $\pi$-finite $K$.*

We may assume $K$ to be connected. Consider the Serre spectral sequence

$$\mathrm{E}^2_{p,q} = \mathrm{H}_p(\mathrm{B}\pi_1 K; \mathrm{H}_q(\tau_{\geq 1}K; \mathbb{Z})) \Rightarrow \mathrm{H}_{p+q}(K; \mathbb{Z})$$

attached to the fibre sequence $\tau_{\geq 1}K \to K \to \mathrm{B}\pi_1 K$. Combining the first two steps, we see that $\mathrm{E}^2_{p,q}$ is finite provided that $p + q > 0$. Therefore $\mathrm{E}^\infty_{p,q}$ and thus also $\mathrm{H}_n(K; \mathbb{Z})$ are finite whenever $n = p + q > 0$ as claimed. □

We obtain the following immediate cohomological vanishing results:

**1.3.18 Corollary.** *Let $K$ be a connected $\pi$-finite anima. Then, for all but finitely many primes $\ell$, the cohomology group $\mathrm{H}^n(K; \mathbb{F}_\ell)$ vanishes for all $n > 0$.*

*Proof.* This immediately follows from Lemma 1.3.17 and the universal coefficient theorem. □

**1.3.19 Corollary.** *For $K$ a connected profinite anima, we have that*

$$\mathrm{H}^n(K; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & n = 0, \\ 0, & n > 0. \end{cases}$$

*Proof.* By writing $K$ as a cofiltered limit of connected $\pi$-finite anima, we may assume $K$ to be $\pi$-finite. The claim is now an easy consequence of the universal coefficient theorem and Lemma 1.3.17. □

**1.3.20 Proposition.** *Let $\varnothing \neq \Sigma$ be a set of primes, $\ell$ a prime number not in $\Sigma$, and $K$ a connected $\Sigma$-profinite anima. Then*

$$\mathrm{H}^n(K, \mathbb{F}_\ell) = \begin{cases} \mathbb{F}_\ell, & n = 0, \\ 0, & n > 0. \end{cases}$$

*Proof.* By writing $K = \text{«}\lim\text{»}_\alpha K_\alpha$ with connected and $\Sigma$-finite $K_\alpha$, we may assume $K$ to be $\Sigma$-finite. We proceed in several steps:

(1) *$K$ is simply connected and $\Sigma$-finite.*

If $K$ is simply connected, Serre's Hurewicz mod $\mathcal{C}$ 1.1.3 for $\mathcal{C}$ the class of $\Sigma$-finite abelian groups implies that also $\mathrm{H}_*(K; \mathbb{Z})$ are $\Sigma$-finite abelian groups. The claim is now an easy consequence of the universal coefficient theorem.

(2) *$K = \mathrm{B}G$ for $G$ a $\Sigma$-finite group.*

If we have any local system on $\mathrm{B}G$ with value a $\Sigma'$-finite abelian group $A$, $\mathrm{H}^n(\mathrm{B}G; A) = \mathrm{H}^n(G; A)$ vanishes, see e.g. [NSW, Proposition 1.6.2].

(3) *Arbitrary connected $\Sigma$-finite $K$.*

The Serre spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{B}\pi_1(K); \mathrm{H}^q(\tau_{\geq 1}K; \mathbb{F}_\ell)) \Rightarrow \mathrm{H}^{p+q}(K; \mathbb{F}_\ell).$$

attached to the fibre sequence $\tau_{\geq 1}K \to K \to \mathrm{B}\pi_1(K)$ degenerates at the $\mathrm{E}_2$-page by the preceeding steps. This shows the claim. □

Finally, we deduce a Künneth theorem for profinite anima:

**1.3.21 Proposition** (Künneth theorem)**.** *Let $K$ and $L$ be profinite anima. For every field $\kappa$, the canonical map*

$$\mathrm{C}^*(K;\kappa) \otimes_\kappa \mathrm{C}^*(L;\kappa) \to \mathrm{C}^*(K \times L;\kappa)$$

*is an equivalence. In particular, we have a canonical isomorphism*

$$\mathrm{H}^*(K;\kappa) \otimes_\kappa \mathrm{H}^*(L;\kappa) \to \mathrm{H}^*(K \times L;\kappa)$$

*Proof.* Using Proposition 1.1.14, we can write the diagram $K \to * \leftarrow L$ as a cofiltered limit of diagrams $K_\alpha \to * \leftarrow L_\alpha$ of $\pi$-finite anima. The canonical map of the statement can therefore be identified with the filtered colimit of the canonical maps

$$\mathrm{C}^*(K_\alpha;\kappa) \otimes_\kappa \mathrm{C}^*(L_\alpha;\kappa) \to \mathrm{C}^*(K_\alpha \times L_\alpha;\kappa)$$

The result now follows from the ordinary Künneth theorem [17, Thm. 3.15] (and the fact that, according to Lemma 1.3.17, it is applicable to $\pi$-finite anima). □

**1.3.22 Remark.** Alternatively, Proposition 1.3.21 can be deduced from [SAG, E.7.2.1]. We decided to give an alternative proof since the proof of [SAG, E.7.2.1] contains broken references as of right now.

### Profinite completion and materialisation

An essential property of profinite completion is that it remembers cohomology with finite coefficients. In order to give a precise statement, we first require the following formal observation:

**1.3.23 Lemma** [SAG, E.4.4.1, E.0.7.12]**.** *The restriction of the profinite completion functor*

$$(-)^\wedge_\pi \colon \mathbf{Ani} \subset \mathrm{Pro}(\mathbf{Ani}) \to \mathbf{Ani}^\wedge_\pi$$

*is left adjoint to the materialisation functor $|-| \colon \mathbf{Ani}^\wedge_\pi \to \mathbf{Ani}$.*

*Proof.* Write $|-| \colon \mathrm{Pro}(\mathbf{Ani}) \to \mathbf{Ani}$ for the extension by cofiltered limits of the identity functor on $\mathbf{Ani}$. Then this is clearly right adjoint to the inclusion $\mathbf{Ani} \subset \mathrm{Pro}(\mathbf{Ani})$. The profinite completion functor $(-)^\wedge_\pi \colon \mathrm{Pro}(\mathbf{Ani}) \to \mathbf{Ani}^\wedge_\pi$ is defined as the pro-left adjoint to the inclusion $\mathbf{Ani}_\pi \subset \mathbf{Ani}$. Since adjoints compose, we see that the restriction

$$(-)^\wedge_\pi \colon \mathbf{Ani} \subset \mathrm{Pro}(\mathbf{Ani}) \to \mathbf{Ani}^\wedge_\pi$$

is left adjoint to the restriction

$$|-| \colon \mathbf{Ani}^\wedge_\pi \subset \mathrm{Pro}(\mathbf{Ani}) \to \mathbf{Ani},$$

which coincides with the materialisation functor $|-| \colon \mathbf{Ani}^\wedge_\pi \to \mathbf{Ani}$ of Definition 1.3.1. □

The unit of the above adjunction allows us to compare the cohomology of an anima $K$ with the cohomology of its profinite completion $K^\wedge_\pi$.

**1.3.24 Proposition.** *Let $K$ be an anima and $A$ a $\Sigma$-finite abelian group. Then, the composition of the maps*

$$\mathrm{H}^*(K;A) \to \mathrm{H}^*(|K^\wedge_\Sigma|;A) \to \mathrm{H}^*(K^\wedge_\Sigma;A),$$

*where the first map comes from pulling back along the unit map $K \to |K^\wedge_\Sigma|$ and the second map is the one of Remark 1.3.16, is an isomorphism.*

*Proof.* This is an easy consequence of the representability of cohomology:

$$\begin{aligned} H^n(K^\wedge_\Sigma; A) &= \pi_0 \operatorname{map}(K^\wedge_\Sigma, K(A,n)) \\ &= \pi_0 \operatorname{map}(K, K(A,n)) \\ &= H^n(K, A), \end{aligned}$$

where the first and third equality hold by Theorem 1.3.14 (and the version of it for ordinary anima), and where the second equality holds even on the level of mapping anima since, by construction, $(-)^\wedge_\Sigma$ is left adjoint to $\mathbf{Ani}^\wedge_\Sigma \subset \operatorname{Pro}(\mathbf{Ani})$. □

## 1.4 $p$-profinite homotopy theory

In this section, we specialise to the set $\Sigma = \{p\}$ consisting of a single prime $p$. What makes $p$-profinite anima very pleasant is that their homotopy theory is entirely determined by their $\mathbb{F}_p$-cohomology (Theorem 1.4.1). As a corollary, *$p$-profinite completion* $K^\wedge_p$ (Definition 1.4.4) of a (profinite) anima $K$ is uniquely determined by the property that it exactly remembers all the information found in the $\mathbb{F}_p$-cohomology of $K$ (Proposition 1.4.6). More background on $p$-profinite homotopy theory can be found in [DAG, §3].

The following theorem can be found in [DAG, Lemma 3.3.15]:

**1.4.1 Theorem** (cohomological characterization of equivalences between $p$-profinite anima)**.** *The following are equivalent for a map $f : K \to L$ of $p$-profinite anima:*

(1) *The map $f$ is an equivalence of $p$-profinite anima.*

(2) *The map $f$ is an $\mathbb{F}_p$-equivalence (in the sense of Definition 1.3.15).*

As a consequence, we obtain a useful reformulation of Corollary 1.3.18:

**1.4.2 Corollary.** *Let $K$ be a connected $\pi$-finite anima. Then $K^\wedge_\ell$ is contractible for all but finitely many $\ell$.*

*Proof.* According to Proposition 1.4.6 and Theorem 1.4.1, this is an immediate consequence of Corollary 1.3.18. □

### $p$-profinite completion and materialisation

The profinite completion and materialisation adjunction carries over to the $p$-profinite setting:

**1.4.3 Lemma** ([DAG, Not. 3.3.14])**.** *The inclusion $\mathbf{Ani}^\wedge_p \subset \mathbf{Ani}^\wedge_\pi$ admits a left-adjoint*

$$(-)^\wedge_p : \mathbf{Ani}^\wedge_\pi \to \mathbf{Ani}^\wedge_p.$$

*Proof.* This is Proposition 1.2.12 in the case that $\Sigma = \{p\}$. □

**1.4.4 Definition.** The functor $(-)^\wedge_p : \mathbf{Ani}^\wedge_\pi \to \mathbf{Ani}^\wedge_p$ from Lemma 1.4.3 is called *$p$-profinite completion* functor.

**1.4.5 Notation.** We also simply write $(-)^\wedge_p : \mathbf{Ani} \to \mathbf{Ani}^\wedge_p$ for the composite

$$\mathbf{Ani} \xrightarrow{(-)^\wedge_\pi} \mathbf{Ani}^\wedge_\pi \xrightarrow{(-)^\wedge_p} \mathbf{Ani}^\wedge_p.$$

Since adjoints compose, this is a left adjoint to the materialisation functor $|-| : \mathbf{Ani}^\wedge_\pi \to \mathbf{Ani}$ restricted to $\mathbf{Ani}^\wedge_p \subset \mathbf{Ani}^\wedge_\pi$ (recall Lemma 1.3.23).

For completeness' sake, let us also record the following immediate corollary of Proposition 1.3.24:

**1.4.6 Proposition.** *Let $K$ be a (profinite) anima. Then the composition of the maps*

$$\mathrm{H}^*(K; \mathbb{F}_p) \to \mathrm{H}^*(|K^\wedge_p|; \mathbb{F}_p) \to \mathrm{H}^*(K^\wedge_p; \mathbb{F}_p),$$

*where the first map comes from pulling back along the unit map $K \to |K^\wedge_p|$ and where the second map is the one of Remark 1.3.16, is an isomorphism.*

*Proof.* This is Proposition 1.3.24 specialised to $\Sigma = \{p\}$ and $A = \mathbb{F}_p$. □

### Exactness properties of $p$-profinite completion

In this section, we record some elementary "exactness" properties of $p$-profinite completion-More precisely, we show that $p$-profinite completion preserves finite products and coproducts as well as connected components of profinite anima.

**1.4.7 Lemma.** *The $p$-profinite completion functor $(-)^\wedge_p : \mathbf{Ani} \to \mathbf{Ani}^\wedge_p$ preserves finite coproducts.*

*Proof.* Let $K_i, i = 1, \dots, n$ be a finite family of anima. Then $\amalg_i (K_i)^\wedge_p$ exists and is given by the levelwise coproduct. For any $p$-finite anima $L$, we have

$$\begin{aligned} \mathrm{map}(\amalg_i (K_i)^\wedge_p, L) &\simeq \Pi_i \,\mathrm{map}((K_i)^\wedge_p, L) \\ &\simeq \Pi_i \,\mathrm{map}(K_i, L) \\ &\simeq \mathrm{map}(\amalg_i K_i, L) \end{aligned}$$

and the result follows from the Yoneda lemma. □

**1.4.8 Corollary.** *Let $K$ be an anima. If $\pi_0 K$ is finite, the unit map $K \to |K^\wedge_p|$ induces a bijection $\pi_0 K = \pi_0 K^\wedge_p$. If $K$ is moreover 0-truncated, so is $|K^\wedge_p|$.*

*Proof.* By Lemma 1.4.7, it suffices to see that $\hat{\pi}_0 : \mathbf{Ani}^\wedge_\pi \to \mathbf{Set}$ preserves finite coproducts. But as $\hat{\pi}_0 = \pi_0 \circ |-|$, this follows from the fact that $|-|$ preserves finite coproducts [SAG, E.4.4.4]. If $K$ is moreover 0-truncated, we have $K \in \mathbf{Ani}_p$ so that $K^\wedge_p = K$. □

**1.4.9 Corollary.** *Let $K$ be a profinite anima.*

(1) *The unit map $K \to K^\wedge_p$ induces a bijection $\hat{\pi}_0 K = \hat{\pi}_0 K^\wedge_p$.*

(2) *If $K$ is 0-truncated, so is $K^\wedge_p$, i.e. $K = K^\wedge_p$ in this case.*

*Proof.* Since the canonical map $\hat{\pi}_0 K \to \hat{\pi}_0 K^\wedge_p$ is a continuous map between profinite sets, it suffices to show that it is a bijection on underlying sets. According to Lemma 1.3.4, $\pi_0(-)$ preserves cofiltered limits. By construction, so do $|-|$ as well as $(-)^\wedge_p$. Therefore, the functors $\pi_0(|-|)$ and $(-)^\wedge_p$ both preserve filtered limits, so we may assume $K$ to be $\pi$-finite. Now apply Corollary 1.4.8. Using Lemma 1.3.8, the second claim follows exactly the same way. □

**1.4.10 Lemma.** *The p-profinite completion functor* $(-)^\wedge_p \colon \mathbf{Ani}^\wedge_\pi \to \mathbf{Ani}^\wedge_p$ *preserves finite products.*

*Proof.* Since, according to Theorem 1.4.1, $p$-profinite anima are entirely determined by their $\mathbb{F}_p$-cohomology, it suffices to show that the canonical map $K \times L \to K^\wedge_p \times L^\wedge_p$ is an $\mathbb{F}_p$-equivalence. To this end, using Proposition 1.3.21 twice, we compute

$$\begin{aligned}
\mathrm{H}^*(K^\wedge_p \times L^\wedge_p; \mathbb{F}_p) &\cong \mathrm{H}^*(K^\wedge_p; \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{H}^*(L^\wedge_p; \mathbb{F}_p) \\
&\cong \mathrm{H}^*(K; \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{H}^*(L; \mathbb{F}_p) \\
&\cong \mathrm{H}^*(K \times L; \mathbb{F}_p).
\end{aligned}$$

Here we also used that $p$-profinite completion preserves $\mathbb{F}_p$-cohomology, see Proposition 1.4.6. □

Later, we will make use of the following straightforward observation:

**1.4.11 Corollary.** *The p-profinite completion functor* $(-)^\wedge_p$ *is symmetric monoidal as a functor*

$$(-)^\wedge_p \colon (\mathbf{Ani}^\wedge_\pi)^\times \to (\mathbf{Ani}^\wedge_p)^\times \subset (\mathbf{Ani}^\wedge_\pi)^\times$$

*Proof.* Lemma 1.4.10 shows that $(-)^\wedge_p \colon (\mathbf{Ani}^\wedge_\pi)^\times \to (\mathbf{Ani}^\wedge_p)^\times$ is symmetric monoidal. The result hence follows since, by Proposition 1.2.12 (1) for $\Sigma = \{p\}$, the inclusion $(\mathbf{Ani}^\wedge_p)^\times \subset (\mathbf{Ani}^\wedge_\pi)^\times$ is also symmetric monoidal. □

## 1.5 Homotopy fixed points

In this subsection, we finally introduce the notion of *homotopy fixed points*. For this, given a profinite group $G$, we first need to define an $\infty$-category $\mathbf{Ani}^\wedge_\pi(G)$ of *profinite anima with continuous G-action* (Definition 1.5.1). A central feature here is that the operation of taking a profinite anima $K$ with $G$-action to its homotopy quotient $K/\!\!/G$ does not lose any information, provided one remembers the induced map $K/\!\!/G \to \mathrm{B}G$ (Theorem 1.5.3). This result will be the basis for reformulating the Section Conjecture in terms of homotopy fixed points. We finish this section with discussing a version of Quick's homotopy fixed point spectral sequence (Proposition 1.5.6).

**1.5.1 Definition.** Let $G$ be a profinite group. The $\infty$-category of *profinite anima with (continuous) G-action* is given as $\mathbf{Ani}^\wedge_\pi(G) := \mathrm{RMod}_G(\mathbf{Ani}^\wedge_\pi)$. We usually write $\mathrm{map}_G$ instead of $\mathrm{map}_{\mathbf{Ani}^\wedge_\pi(G)}$.

**1.5.2 Remark.**

(1) The $\infty$-category $\mathbf{Ani}^\wedge_\pi$ is symmetric monoidal with respect to the product. As $G$ is a profinite group (i.e. a group object in profinite sets), it is also a group in $\mathbf{Ani}^\wedge_\pi$ and therefore an associative algebra in $(\mathbf{Ani}^\wedge_\pi)^\times$. The $\infty$-category $\mathrm{RMod}_G(\mathbf{Ani}^\wedge_\pi)$ of Definition 1.5.1 denotes the $\infty$-category of right modules over $G$, see [HA, §4.2] for more details.

(2) If $G$ is a finite group, the above definition unwinds to $\mathbf{Ani}^\wedge_\pi(G) \simeq \mathrm{Fun}(\mathrm{B}G, \mathbf{Ani}^\wedge_\pi)$.

We are now finally able to give a precise statement of Ingredient A of the introduction:

**1.5.3 Theorem** [SAG, E.6.5.1, E.6.4.4]**.** *Let $G$ be a profinite group. The construction*

$$G \curvearrowright K \quad \mapsto \quad K/\!\!/G \to */\!\!/G = \mathrm{B}G,$$

*where $K/\!\!/G$ denotes the* homotopy quotient *of $K$ under the action of $G$, determines an equivalence of $\infty$-categories $\mathbf{Ani}_\pi^\wedge(G) \to (\mathbf{Ani}_\pi^\wedge)_{/\,\mathrm{B}G}$. Furthermore, the inverse of this equivalence is given by taking fibres.*

We now come to the definition of homotopy fixed points.

**1.5.4 Definition.** Let $G$ be a *finite* group and $K \in \mathbf{Ani}_\pi^\wedge(G)$ a profinite anima with $G$-action. The *(profinite) homotopy fixed point anima* of $K$ is given by $K^{\mathrm{h}G} := \lim_{\mathrm{B}G} K \in \mathbf{Ani}_\pi^\wedge$.

**1.5.5 Remark.**

(1) Recall that by [SAG, E.6.0.8], given any map $q : K' \to K$ of $\pi$-finite anima, the pullback functor

$$q^* : (\mathbf{Ani}_\pi^\wedge)_{/K} \to (\mathbf{Ani}_\pi^\wedge)_{/K'}, L \mapsto L \times_K K'$$

admits a right adjoint $q_*$. Denote the unique map $\mathrm{B}G \to *$ by $t$. Then, under the equivalence $\mathbf{Ani}_\pi^\wedge(G) \simeq (\mathbf{Ani}_\pi^\wedge)_{/\,\mathrm{B}G}$ of Theorem 1.5.3, $(-)^{\mathrm{h}G}$ gets identified with $t_*$, i.e. $K^{\mathrm{h}G} \simeq t_* K/\!\!/G$. Indeed, unwinding the definitions we see:

$$\begin{aligned} \mathrm{map}(L, t_* K/\!\!/G) &\simeq \mathrm{map}_{\mathrm{B}G}(L \times \mathrm{B}G, K/\!\!/G) \\ &\simeq \mathrm{map}_G(L^{\mathrm{tr}}, K) \\ &\simeq \lim_{\mathrm{B}G} \mathrm{map}(L, K), \end{aligned}$$

where $L^{\mathrm{tr}}$ denotes $L$ equipped with the trivial $G$-action.

(2) Specialising to $L = *$, we in particular obtain

$$\begin{aligned} |K^{\mathrm{h}G}| &\simeq \mathrm{map}(*, K^{\mathrm{h}G}) \\ &\simeq \mathrm{map}_{\mathrm{B}G}(\mathrm{B}G, K/\!\!/G) \\ &\simeq \mathrm{map}_G(*^{\mathrm{tr}}, K). \end{aligned}$$

(3) One could use the last observation to at least define $|K^{\mathrm{h}G}|$ for any *profinite* group $G$ acting on $K$.

A central feature for us is the existence of a homotopy descent spectral sequence converging to the homotopy groups of homotopy fixed point anima:

**1.5.6 Proposition** (Quick, [36, Theorem 2.16])**.** *Let $G$ be a finite group and $K$ a connected profinite anima with $G$-action. Assume $K^{\mathrm{h}G} \neq \emptyset$ and let $x \in K^{\mathrm{h}G}$. Then there is a conditionally convergent homotopy descent spectral sequence*

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(G, \hat{\pi}_t(K)) \Rightarrow \hat{\pi}_{t-s}(K^{\mathrm{h}G}, x),$$

*where the action of $G$ on $\hat{\pi}_t(K)$ depends on the choice of $x$.*

**1.5.7.** In the following proof, we write $\mathcal{M}_\infty$ for the underlying $\infty$-category of a model category $\mathcal{M}$, i.e. $\mathcal{M}_\infty$ denotes the $\infty$-categorical localisation of the nerve of $\mathcal{M}$ with respect to the weak equivalences.

*Proof.* Denote Quick's model category of profinite spaces by $\hat{\mathcal{S}}$ (see [35] for more details). By [1, Corollary 7.4.9], there is an equivalence of $\infty$-categories $\mathbf{Ani}_\pi^\wedge \simeq \hat{\mathcal{S}}_\infty$. By [36, Corollary 2.11], there is a Quillen adjunction between Quick's category of profinite $G$-spaces $\hat{\mathcal{S}}(G)$ and the slice model category $\hat{\mathcal{S}}_{/\mathrm{B}G}$. Combining all of these results, we obtain a chain of equivalences of $\infty$-categories

$$\begin{aligned} \mathbf{Ani}_\pi^\wedge(G) &\simeq (\mathbf{Ani}_\pi^\wedge)_{/\,\mathrm{B}G} \qquad \text{by Theorem 1.5.3} \\ &\simeq (\hat{\mathcal{S}}_\infty)_{/\mathrm{B}G} \\ &\simeq (\hat{\mathcal{S}}_{/\mathrm{B}G})_\infty \\ &\simeq \hat{\mathcal{S}}(G)_\infty, \end{aligned}$$

where, in virtue of [9, Corollary 7.6.13], the second to last equivalence holds since B$G$ is fibrant in $\hat{\mathcal{S}}$. The result hence follows from Quick's profinite homotopy fixed point spectral sequence [36, Theorem 2.16]. ☐

**1.5.8 Remark.**

(1) The differentials in the above spectral sequence are of the form

$$d_r : \ \mathrm{E}_r^{s,t} \to \mathrm{E}_r^{s+r,t+r-1}.$$

Moreover, $\mathrm{E}_r^{s,t} = *$ if $t-s<0$ and the diagonal $\mathrm{E}_r^{s,s}$ is only receiving differentials.

(2) Also note that [36, Theorem 2.16] assumes that $K$ is pointed. This assumption is unnecessary: In [4], Bousfield explains how to obtain the spectral sequence of a *unpointed* cosimplicial space, as long as one chooses a point of its totalization ($\widehat{=}\ K^{\mathrm{h}G}$ above). So using the machinery of [4] instead of [BK, §X.6], one obtains the above unpointed variant.

(3) Quick puts unnecessary assumptions on $K$ (resp. $G$) in order to enforce strong convergence of the above spectral sequence (as he later noted himself in [37, Remark 4.3]). We refer the reader to [4, §4] for a detailed discussion of *convergence* of homotopy spectral sequences.

## 1.6 Nilpotent Σ-profinite anima

One often encounters simply-connectedness assumptions in homotopy theory. In a lot of cases, one can considerably relax these assumptions: First, a lot of the arguments assuming simply-connectedness also work for *simple* anima, i.e. anima with abelian fundamental group acting trivially on the higher homotopy groups. Second, using Postnikov towers, one can then often generalise to anima that are inductively build from such simple anima: So called *nilpotent* anima (we refer the reader to [27, §3, §4] for an overview of nilpotency in homotopy theory).

Unfortunately, except for some remarks here and there, [SAG, Appendix E] does not really develop the theory of nilpotent profinite anima. So in this section, we will provide a concise treatment of the theory of nilpotency in profinite homotopy theory.

**Nilpotent $\Sigma$-profinite groups**

Following the strategy of (§1.2), we start by discussing nilpotency in the world of profinite groups.

**1.6.1 Recollection** (nilpotent group action)**.** Let $\pi$ be a group acting on a (possibly nonabelian) group $G$.

(a) The action of $\pi$ on $G$ is called *nilpotent* if there exists a finite series of normal subgroups

$$1 = G_q \subset G_{q-1} \subset \ldots \subset G_0 = G$$

such that:

(1) $G_{j-1}/G_j$ is abelian and a central subgroup of $G/G_j$, and

(2) $G_j$ is a $\pi$-subgroup of $G$ and the induced action of $\pi$ on $G_{j-1}/G_j$ is trivial.

(b) $\pi$ is said to be *nilpotent* if the natural conjugation action of $\pi$ on itself is nilpotent.

**1.6.2 Definition** (nilpotent $\Sigma$-profinite groups)**.** Let $\emptyset \neq \Sigma$ denote a set of prime numbers.

(a) We write $\mathbf{Grp}_{\Sigma\text{-nil}} \subset \mathbf{Grp}_{\Sigma}$ for the full subcategory of nilpotent $\Sigma$-finite groups.

(b) We write $\mathbf{Grp}^{\wedge}_{\Sigma\text{-nil}} := \mathrm{Pro}(\mathbf{Grp}_{\Sigma\text{-nil}}) \subset \mathbf{Grp}^{\wedge}_{\Sigma}$ for the full subcategory of *(pro-)nilpotent* $\Sigma$-profinite groups.

**1.6.3 Lemma.** *Let $1 \longrightarrow N \longrightarrow G \xrightarrow{p} H \longrightarrow 1$ be a short exact sequence of $\pi$-groups with $N$ contained in the centre $\mathrm{C}(G)$ of $G$. If both $N$ and $H$ are nilpotent $\pi$-groups, so is $G$.*

*Proof.* Let $1 = H_s \subset H_{s-1} \subset \ldots \subset H_0 = H$ be a chain of subgroups witnessing the $\pi$-nilpotency of $H$. Similarly, let $1 = N_t \subset N_{t-1} \subset \ldots \subset N_0 = N$ be a chain witnessing the $\pi$-nilpotency of $N$. If we now write

$$G_q := \begin{cases} p^{-1}(H_q), & q \leq s \\ N_{q-s}, & s+t \geq q > s, \end{cases}$$

the chain $1 = G_{s+t} \subset G_{s+t-1} \subset \ldots \subset G_{s+1} \subset N = G_s \subset \ldots \subset G_0 = G$ witnesses the $\pi$-nilpotency of $G$. □

**1.6.4 Proposition.** *Any action of a finite $p$-group $\pi$ on another finite $p$-group $G$ is nilpotent.*

*Proof.* We prove this by induction on $\#G$. If $\#G = 1$ the claim is trivial. So assume the claim holds for all finite $p$-groups $H$ with $\#H < \#G$. Since $\mathrm{C}(G)$ is a characteristic subgroup of $G$, the short exact sequence $1 \longrightarrow \mathrm{C}(G) \longrightarrow G \longrightarrow G/\mathrm{C}(G) \longrightarrow 1$ is a short exact sequence of $\pi$-groups with all appearing groups being $p$-finite. Since, according to [16, Theorem 4.3.1], $1 \neq \mathrm{C}(G)$, we have that $\#\,\mathrm{C}(G), \#G/\mathrm{C}(G) < \#G$. Therefore, by our inductive assumption, both $\mathrm{C}(G)$ and $G/\mathrm{C}(G)$ are $\pi$-nilpotent. Hence so is $G$ by Lemma 1.6.3. □

**1.6.5 Lemma.** *Let $\pi$ and $\pi'$ be groups acting nilpotently on $G$ and $G'$ respectively. Then the diagonal action of $\pi \times \pi'$ on $G \times G'$ is nilpotent as well.*

*Proof.* Let $1 = G_q \subset G_{q-1} \subset \ldots \subset G_0 = G$, resp. $1 = G'_{q'} \subset G'_{q'-1} \subset \ldots \subset G'_0 = G'$, be chains of subgroups witnessing the $\pi$-nilpotency of $G$ and the $\pi'$-nilpotency of $G'$, respectively. By possibly switching $G$ and $G'$, we may assume that $q \geq q'$. Define $G'_i := 1$ for $q' < i \leq q$. Then clearly

$1 = G'_q \subset G'_{q-1} \subset \ldots \subset G'_0 = G'$ also witnesses the $\pi'$-nilpotency of $G'$. One now immediately verifies that

$$1 = G_q \times G'_q \subset G_{q-1} \times G'_{q-1} \subset \ldots \subset G_0 \times G'_0 = G \times G'$$

witnesses the $(\pi \times \pi')$-nilpotency of $G \times G'$. □

**1.6.6 Lemma** (nilpotent completion). *Let $\varnothing \neq \Sigma$ denote a set of prime numbers.*

(1) *The full subcategory $\mathbf{Grp}_{\Sigma\text{-nil}} \subset \mathbf{Grp}_{\Sigma}$ is stable under finite limits.*

(2) *The inclusion $\mathbf{Grp}^{\wedge}_{\Sigma\text{-nil}} \subset \mathbf{Grp}^{\wedge}_{\Sigma}$ admits a left adjoint*

$$(-)^{\mathrm{nil}} \colon \mathbf{Grp}^{\wedge}_{\Sigma} \to \mathbf{Grp}^{\wedge}_{\Sigma\text{-nil}}.$$

*Moreover, under the identifications* $\mathrm{Pro}(-) \simeq \mathrm{Fun}^{\mathrm{lex}}(-, \mathbf{Set})^{\mathrm{op}}$*, it corresponds to precomposition with the inclusion* $\mathbf{Grp}_{\Sigma\text{-nil}} \subset \mathbf{Grp}_{\Sigma}$.

*Proof.*

(1) By Lemma 1.6.5, $\mathbf{Grp}_{\Sigma\text{-nil}} \subset \mathbf{Grp}_{\Sigma}$ is stable under finite products. Since subgroups of nilpotent groups are again nilpotent, $\mathbf{Grp}_{\Sigma\text{-nil}} \subset \mathbf{Grp}_{\Sigma}$ is also stable under fibre products (see the proof of Proposition 1.2.6). Since finite products and fibre products generate all finite limits, we conclude.

(2) In virtue of Theorem 1.1.13, this is a formal consequence of (1). □

The homotopy-theoretic analogue of the following group-theoretic characterization of nilpotency in the profinite world will be a crucial ingredient in our strategy for proving Theorem B of the introduction.

**1.6.7 Theorem** (characterization of pronilpotent groups; [38, Prop. 2.3.8]). *Let $G$ be a $\Sigma$-profinite group.*

(1) *$G$ is pronilpotent if and only if $G$ contains a unique $\ell$-Sylow subgroup $G_\ell \subset G$ for each prime $\ell \in \Sigma$.*

(2) *If $G$ is pronilpotent, then $G \cong \prod_{\ell \in \Sigma} G_\ell$.*

**1.6.8 Remark.** In [38, Prop. 2.3.8], Theorem 1.6.7 is only formulated in the case that $\Sigma = \pi$ is the set of all prime numbers. The $\Sigma$-profinite version is an immediate consequence of this version.

### Nilpotent ($\Sigma$-finite) anima

In this section, we collect some results concerning nilpotent ($\Sigma$-finite) anima. A key advantage of working with nilpotent instead of simple/simply connected anima is that they are stable under finite limits (Lemma 1.6.12). This allows us to define a nilpotent completion functor on profinite anima. We furthermore prove a homotopy-theoretic analogue of Theorem 1.6.7 (see Theorem 1.6.22 — Theorem C of the introduction).

**1.6.9 Recollection** ($\pi_1 \curvearrowright \pi_n$)**.** Let $K$ be an anima with basepoint $x$. Recall that there is a natural action of $\pi_1(K, x)$ on the homotopy groups $\pi_n(K, x), n \geq 1$. Indeed, $\pi_1(K, x)$ acts naturally on itself by conjugation. One way to see the action on higher homotopy groups is via the universal cover $\tau_{\geq 1} K$ of $K$. This sits in a fibre sequence

$$\begin{array}{ccc} \tau_{\geq 1} K & \longrightarrow & K \\ \big\downarrow & \lrcorner & \big\downarrow \\ * & \xrightarrow[x]{} & \mathrm{B}\pi_{\leq 1}(K), \end{array}$$

where $\pi_{\leq 1}(K)$ denotes the fundamental groupoid of $K$. One has $\pi_n(\tau_{\geq 1} K) = \pi_n(K, x)$ for $n \geq 2$ and $\pi_1(K, x) = \pi_1(\mathrm{B}\pi_{\leq 1}(K), x)$ naturally acts on the fibre $\tau_{\geq 1} K$ (by the classical version of Theorem 1.5.3). Since $\tau_{\geq 1} K$ is simply connected, this actions induces an action on $\pi_n(K, x)$ as desired.

**1.6.10 Definition** (nilpotent anima)**.** Let $K$ be an anima.

(a) $K$ is said to be *nilpotent* if it is connected and if for any basepoint $x \in K$, the action of $\pi_1(K, x)$ on $\pi_n(K, x)$ is nilpotent for all $n \geq 1$ (in the sense of Recollection 1.6.1).

(b) $K$ is said to be *componentwise nilpotent* if every connected component $K_\alpha$ of $K$ is nilpotent.

(c) The full subcategory of **Ani** determined by the *componentwise* nilpotent anima is denoted by $\mathbf{Ani}_{\mathrm{nil}}$.

There are plenty of nilpotent anima:

**1.6.11 Example.** Let $p$ be a prime number.

(1) Any simply connected anima is nilpotent.

(2) More generally, any simple anima (i.e. any connected anima $K$ such that the action of $\pi_1(K)$ on $\pi_n(K)$ is trivial for all $n \geq 1$) is nilpotent.

(3) Any $p$-finite anima is componentwise nilpotent, as follows from Proposition 1.6.4.

A huge technical advantage of allowing nilpotent anima in comparison of restricting oneself to simply connected anima are good categorical closure properties shared by the former but not the latter:

**1.6.12 Lemma.** *The subcategory* $\mathbf{Ani}_{\mathrm{nil}} \subset \mathbf{Ani}$ *is stable under finite limits.*

*Proof.* It suffices to show that $\mathbf{Ani}_{\mathrm{nil}} \subset \mathbf{Ani}$ is stable under finite products and fibre products. That it is stable under finite products follows immediately from its group-theoretic counterpart Lemma 1.6.5.

Stability under pullbacks is for example proven in [27, Proposition 4.4.3] for $\mathcal{C} = \mathbf{Ab}$ or also [10, Lemma 7.1]. □

### A characterization of nilpotent Σ-finite anima

The goal of this subsection is to show that nilpotent Σ-finite anima split as the product of their $p$-profinite completions, see Theorem 1.6.16. This is deduced from the classical arithmetic fracture square after comparing our $p$-profinite completion $(-)^\wedge_p$ to the Bousfield-Kan $p$-completion $\mathbb{F}_{p^\infty}(-)$.

**1.6.13 Notation.** We write $\mathbb{F}_{p^\infty}(-)$ for the *Bousfield-Kan mod $p$* completion functor, i.e. what Bousfield and Kan in [BK] denote by $R_\infty(-)$ for $R = \mathbb{F}_p$. Note that, in [BK], they use the notation $\mathbb{Z}_p$ to denote $\mathbb{F}_p$.

**1.6.14 Lemma.** *Let $K$ be a connected $\pi$-finite anima and $p$ a prime number.*

(1) *The Bousfield-Kan $p$-completion $\mathbb{F}_{p^\infty} K$ is $p$-finite.*

(2) *The canonical map $K_p^\wedge \to \mathbb{F}_{p^\infty} K$ induced by $K \to \mathbb{F}_{p^\infty} K$ is an equivalence.*

*Proof.*

(1) This is shown in [BK, VII.4.3 (i)].

(2) By (1), Proposition 1.4.6 and Theorem 1.4.1, it is enough to show that $K \to \mathbb{F}_{p^\infty} K$ is an $\mathbb{F}_p$-equivalence. This is an immediate consequence of the universal property of Bousfield-Kan $p$-completion for "$p$-good" anima [BK, VII.2.1] and the fact that $\pi$-finite anima are $p$-good [BK, VII.4.3 (iii)]. □

**1.6.15 Corollary.** *Let $K$ be a $\pi$-finite anima and $p$ a prime number. Then $K_p^\wedge$ is $p$-finite.*

*Proof.* We may assume $K$ to be connected. The result then follows by combining (1) and (2) of Lemma 1.6.14. □

**1.6.16 Theorem.** *The following are equivalent for a connected $\Sigma$-finite anima $K$:*

(1) *$K$ is nilpotent.*

(2) *The canonical map $K \to \prod_{\ell \in \Sigma} K_\ell^\wedge$ is an equivalence.*

*Proof.* Since $K$ is $\Sigma$-finite, so is $\prod_{\ell\in\Sigma} \mathbb{F}_{\ell^\infty} K = \prod_{\ell\in\Sigma} K_\ell^\wedge$ as is seen by combining Lemma 1.6.14 and Corollary 1.4.2. So, in virtue of Corollary 1.3.19, both $K$ and $\prod_{\ell\in\Sigma} K_\ell^\wedge$ have vanishing $\mathbb{Q}$-cohomology. Therefore (1) implies (2) by the classical arithmetic fracture square [BK, VI.8.1]. Combining Example 1.6.11 (3), Lemma 1.6.12 and Corollary 1.6.15, we see that also (2) implies (1). □

### Nilpotent $\Sigma$-profinite anima

By now, it should come as no surprise that we extend the notion of nilpotency into the world of profinite homotopy theory along cofiltered limits. We again fix a nonempty set of primes $\Sigma$.

**1.6.17 Definition** (nilpotent $\Sigma$-profinite anima)**.** Let $K$ a $\Sigma$-profinite anima.

(a) $K$ is said to be *(pro-)nilpotent* if it can be written as a formal cofiltered limit $K = «\lim»_\alpha K_\alpha$ with $K_\alpha$ componentwise nilpotent and $\Sigma$-finite.

(b) We write $\mathbf{Ani}_{\Sigma\text{-nil}}^\wedge := \mathrm{Pro}(\mathbf{Ani}_{\Sigma\text{-nil}}) \subset \mathbf{Ani}_\Sigma^\wedge$ for the full subcategory spanned by the nilpotent $\Sigma$-profinite anima.

**1.6.18 Lemma** (nilpotent completion)**.**

(1) *The full subcategory $\mathbf{Ani}_{\Sigma\text{-nil}} \subset \mathbf{Ani}_\Sigma$ is stable under finite limits.*

(2) *The inclusion* $\mathbf{Ani}^\wedge_{\Sigma\text{-nil}} \subset \mathbf{Ani}^\wedge_\Sigma$ *admits a left adjoint*

$$(-)^\wedge_{\mathrm{nil}} : \mathbf{Ani}^\wedge_\Sigma \to \mathbf{Ani}^\wedge_{\Sigma\text{-nil}}.$$

*Moreover, under the identifications* $\mathrm{Pro}(-) \simeq \mathrm{Fun}^{\mathrm{lex}}(-, \mathbf{Ani})^{\mathrm{op}}$ *of Theorem 1.1.10,* $(-)^\wedge_{\mathrm{nil}}$ *corresponds to precomposition with the inclusion* $\mathbf{Ani}_{\Sigma\text{-nil}} \subset \mathbf{Ani}_\Sigma$.

*Proof.*

(1) This is an immediate consequence of the fact that $\mathbf{Ani}_\Sigma \subset \mathbf{Ani}$ as well as $\mathbf{Ani}_{\mathrm{nil}} \subset \mathbf{Ani}$ are stable under finite limits, as was proven in Proposition 1.2.12 (1) and Lemma 1.6.12, respectively.

(2) In virtue of Theorem 1.1.13, this is a formal consequence of (1). □

**1.6.19 Notation.** We also write $(-)^\wedge_{\Sigma\text{-nil}}$ for the compositions

$$\mathbf{Ani}^\wedge_\pi \xrightarrow{(-)^\wedge_\Sigma} \mathbf{Ani}^\wedge_\Sigma \xrightarrow{(-)^\wedge_{\mathrm{nil}}} \mathbf{Ani}^\wedge_{\Sigma\text{-nil}} \quad \text{as well as} \quad \mathrm{Pro}(\mathbf{Ani}) \xrightarrow{(-)^\wedge_\pi} \mathbf{Ani}^\wedge_\pi \xrightarrow{(-)^\wedge_{\Sigma\text{-nil}}} \mathbf{Ani}^\wedge_{\Sigma\text{-nil}}.$$

and we call $K^\wedge_{\Sigma\text{-nil}}$ the $\Sigma$-*nilpotent completion* of the pro(-finite) anima $K$. Since adjoints compose, these $(-)^\wedge_{\Sigma\text{-nil}}$ are left adjoint to the inclusion $\mathbf{Ani}^\wedge_{\Sigma\text{-nil}} \subset \mathbf{Ani}^\wedge_\pi$ and $\mathbf{Ani}^\wedge_{\Sigma\text{-nil}} \subset \mathrm{Pro}(\mathbf{Ani})$, respectively.

**1.6.20 Remark.** Fix a nonempty set of primes $\Sigma$.

(1) Since the diagram

$$\begin{array}{ccccc} & & \mathbf{Ani}_\Sigma & & \\ & \overset{\subset}{\nearrow} & & \overset{\subset}{\searrow} & \\ \mathbf{Ani}_{\Sigma\text{-nil}} & & & & \mathbf{Ani}_\pi \\ & \underset{\subset}{\searrow} & & \underset{\subset}{\nearrow} & \\ & & \mathbf{Ani}_{\pi,\mathrm{nil}} & & \end{array}$$

of inclusions of full subcategories, that all are stable under finite limits, commutes, so does the corresponding diagram

$$\begin{array}{ccccc} & & \mathbf{Ani}^\wedge_\Sigma & & \\ & \overset{(-)^\wedge_\Sigma}{\nearrow} & & \overset{(-)^\wedge_{\mathrm{nil}}}{\searrow} & \\ \mathbf{Ani}^\wedge_\pi & & & & \mathbf{Ani}^\wedge_{\Sigma\text{-nil}} \\ & \underset{(-)^\wedge_{\mathrm{nil}}}{\searrow} & & \underset{(-)^\wedge_\Sigma}{\nearrow} & \\ & & \mathbf{Ani}^\wedge_{\pi,\mathrm{nil}} & & \end{array}$$

of left adjoints by uniqueness of adjoints. In other words, in order to form the $\Sigma$-nilpotent completion $K^\wedge_{\Sigma\text{-nil}}$ of a profinite anima $K$, it does not matter whether one first $\Sigma$-profinitely and then nilpotently completes $K$ or the other way around.

(2) Specialising to $\Sigma = \{p\}$, we furthermore observe the following:

(a) By Example 1.6.11 (3), the inclusion $\mathbf{Ani}_{p\text{-nil}} \subset \mathbf{Ani}_p$ is an equivalence. That is: All $p$-profinite anima $K$ are automatically nilpotent, the canonical map $K \to K^\wedge_{\mathrm{nil}}$ is an equivalence.

(b) In particular, we see that for all profinite anima $K$ the canonical map

$$K_p^\wedge \to (K_{\mathrm{nil}}^\wedge)_p^\wedge$$

is an equivalence of $p$-profinite anima.

(3) An immediate consequence of the last observation is that, for any profinite anima $K$, the canonical map

$$K \to K_{\mathrm{nil}}^\wedge$$

is an $\mathbb{F}_p$-equivalence for every prime $p$.

(4) More generally, $K \to K_{\Sigma\text{-nil}}^\wedge$ is an $\mathbb{F}_p$-equivalence for every prime $p \in \Sigma$.

A big class of examples of nilpotent profinite anima are given by simply connected ones:

**1.6.21 Lemma.** *Simply connected profinite anima are nilpotent.*

*Proof.* Let $K$ be any simply connected profinite anima. Then, according to Lemma 1.3.8, we can write $K = «\lim»_\alpha K_\alpha$ with each $K_\alpha$ being $\pi$-finite and simply connected. But since, by Example 1.6.11, simply connected anima are nilpotent, each $K_\alpha$ is nilpotent, hence so is $K$. □

Finally, we are able to prove a homotopy-theoretic analogue of Theorem 1.6.7 :

**1.6.22 Theorem** (characterization of nilpotent $\Sigma$-profinite anima, Theorem C)**.** *The following are equivalent for a connected $\Sigma$-profinite anima $K$:*

(1) *$K$ is nilpotent.*

(2) *The canonical map $K \to \prod_{\ell\in\Sigma} K_\ell^\wedge$ is an equivalence of $\Sigma$-profinite anima.*

*Proof.* Write $K = «\lim»_\alpha K_\alpha$ for connected $\Sigma$-finite anima $K_\alpha$.

We first prove that (1) implies (2). If $K$ is nilpotent, we may assume all the $K_\alpha$ to be nilpotent. Then $K_\alpha \simeq \prod_{\ell\in\Sigma}(K_\alpha)_\ell^\wedge$ by Theorem 1.6.16, hence

$$\begin{aligned} K &\simeq «\lim»_\alpha K_\alpha \\ &\simeq \lim_\alpha \prod_{\ell\in\Sigma}(K_\alpha)_\ell^\wedge \\ &\simeq \prod_{\ell\in\Sigma} \lim_\alpha (K_\alpha)_\ell^\wedge \\ &\simeq \prod_{\ell\in\Sigma} «\lim»_\alpha (K_\alpha)_\ell^\wedge \\ &\simeq \prod_{\ell\in\Sigma} K_\ell^\wedge, \end{aligned}$$

where the second to last equivalence holds in virtue of Corollary 1.6.15.

We now prove that also (2) implies (1). Using Corollary 1.3.18 and Corollary 1.6.15, we see that the canonical map

$$K \to \prod_{\ell\in\Sigma} K_\ell^\wedge$$

is given as «lim»$_\alpha(K_\alpha \to \prod_{\ell\in\Sigma}(K_\alpha)^\wedge_\ell)$. Since this is an equivalence, we may pass to a cofinal subset $\{\beta\} \subset \{\alpha\}$ such that each map

$$K_\beta \to \prod_{\ell\in\Sigma}(K_\beta)^\wedge_\ell$$

is an equivalence of connected $\Sigma$-finite anima. Theorem 1.6.16 now shows that each $K_\beta$ is nilpotent, hence so is $K =$ «lim»$_\beta K_\beta$. □

As a corollary, we obtain an alternative and very simple description of $(-)^\wedge_{\Sigma\text{-nil}}$:

**1.6.23 Corollary.** *Let $K$ be a connected profinite anima. Then the canonical map*

$$K^\wedge_{\Sigma\text{-nil}} \to \prod_{\ell\in\Sigma} K^\wedge_\ell$$

*is an equivalence.*

*Proof.* Since $K^\wedge_{\Sigma\text{-nil}}$ is a connected and nilpotent $\Sigma$-profinite anima, Theorem 1.6.22 implies that the canonical map

$$K^\wedge_{\Sigma\text{-nil}} \to \prod_{\ell\in\Sigma}(K^\wedge_{\Sigma\text{-nil}})^\wedge_\ell$$

is an equivalence. The result thus immediately follows from Remark 1.6.20 (4). □

### Exactness properties of $\Sigma$-nilpotent completion

In this section, we record some exactness properties of the $\Sigma$-nilpotent completion functor just as we did for $p$-profinite completion. The proofs are essentially the same.

**1.6.24 Lemma.** *The $\Sigma$-nilpotent completion functor $(-)^\wedge_{\Sigma\text{-nil}} : \mathbf{Ani} \subset \mathrm{Pro}(\mathbf{Ani}) \to \mathbf{Ani}^\wedge_{\Sigma\text{-nil}}$ preserves finite coproducts.*

*Proof.* Let $K_i, i = 1, \dots, n$ be a finite family of anima. Then $\amalg_i (K_i)^\wedge_{\Sigma\text{-nil}}$ exists and is given by the levelwise coproduct. For any componentwise nilpotent $\Sigma$-finite anima $L$, we have

$$\begin{aligned}
\mathrm{map}(\amalg_i (K_i)^\wedge_{\Sigma\text{-nil}}, L) &\simeq \Pi_i \,\mathrm{map}((K_i)^\wedge_{\Sigma\text{-nil}}, L) \\
&\simeq \Pi_i \,\mathrm{map}(K_i, L) \\
&\simeq \mathrm{map}(\amalg_i K_i, L)
\end{aligned}$$

and the result follows from the Yoneda lemma. □

**1.6.25 Corollary.** *Let $K$ be an anima. If $\pi_0 K$ is finite, the unit map $K \to |K^\wedge_{\Sigma\text{-nil}}|$ induces a bijection $\pi_0 K = \pi_0 K^\wedge_{\Sigma\text{-nil}}$. If $K$ is moreover 0-truncated, so is $|K^\wedge_{\Sigma\text{-nil}}|$.*

*Proof.* By Lemma 1.6.24, it suffices to see that $\hat{\pi}_0 : \mathbf{Ani}^\wedge_\pi \to \mathbf{Set}$ preserves finite coproducts. But as $\hat{\pi}_0 = \pi_0 \circ |-|$, this follows from the fact that $|-|$ preserves finite coproducts [SAG, E.4.4.4]. If $K$ is moreover 0-truncated, it is a finite set and hence $K \in \mathbf{Ani}_{\Sigma\text{-nil}}$ so that $K^\wedge_{\Sigma\text{-nil}} = K$. □

**1.6.26 Corollary.** *Let $K$ be a profinite anima.*

(1) *The unit map $K \to K^\wedge_{\Sigma\text{-nil}}$ induces a bijection $\hat{\pi}_0 K = \hat{\pi}_0 K^\wedge_{\Sigma\text{-nil}}$.*

(2) *If $K$ is 0-truncated, so is $K^\wedge_{\Sigma\text{-nil}}$, i.e. $K^\wedge_{\Sigma\text{-nil}} = K$ in this case.*

*Proof.* Since both $\hat{\pi}_0(-)$ and $(-)^\wedge_{\Sigma\text{-nil}}$ preserve cofiltered limits, we may assume $K$ to be $\pi$-finite. Now apply Corollary 1.6.25. □

Finally, we deduce that Σ-nilpotent completion is symmetric monoidal:

**1.6.27 Proposition.** *The functor $(\mathbf{Ani}^\wedge_\pi)^\times \to (\mathbf{Ani}^\wedge_\Sigma)^\times, K \mapsto K^\wedge_{\Sigma\text{-nil}}$ is symmetric monoidal.*

*Proof.* We need to show that, given two profinite anima $K$ and $L$, the canonical map

$$(K \times L)^\wedge_{\Sigma\text{-nil}} \to K^\wedge_{\Sigma\text{-nil}} \times L^\wedge_{\Sigma\text{-nil}}$$

is an equivalence. Since both sides are compatible with cofiltered limits, we may assume $K$ and $L$ to be $\pi$-finite. As $K$ and $L$ now also only have finitely many connected components, we may furthermore assume both to be connected. Using Corollary 1.6.23, we are reduced to showing that

$$\prod_{\ell\in\Sigma}(K \times L)^\wedge_\ell \to \prod_{\ell\in\Sigma} K^\wedge_\ell \times \prod_{\ell\in\Sigma} L^\wedge_\ell \simeq \prod_{\ell\in\Sigma}(K^\wedge_\ell \times L^\wedge_\ell)$$

is an equivalence in this case. This is an immediate consequence of Corollary 1.4.11 though. □

## 1.7 The Sullivan Conjecture

To conclude this section, we state Lurie's version of the Sullivan Conjecture as proven in his course [SC, Lecture 30: Theorem 4].

**1.7.1 Theorem** (Sullivan Conjecture)**.** *Let $G$ be a finite $p$-group and $K$ a finite-dimensional $G$-CW complex. Then the composite map*

$$(K^G)^\wedge_p \to (K^{\mathrm{h}G})^\wedge_p \to (K^\wedge_p)^{\mathrm{h}G}$$

*is an equivalence of $p$-profinite anima.*

**1.7.2 Remark.**

(1) The map $(K^G)^\wedge_p \to (K^{\mathrm{h}G})^\wedge_p$ is obtained by applying $(-)^\wedge_p$ to the canonical map $K^G \to K^{\mathrm{h}G}$ comparing the (1-categorical) limit to the homotopy limit.

(2) The map $(K^{\mathrm{h}G})^\wedge_p \to (K^\wedge_p)^{\mathrm{h}G}$ is obtained as follows: The unit map $K \to |K^\wedge_p|$ is $G$-equivariant and therefore induces a map $K^{\mathrm{h}G} \to |K^\wedge_p|^{\mathrm{h}G} \simeq |(K^\wedge_p)^{\mathrm{h}G}|$. Since $(-)^\wedge_p$ is left adjoint to $|-|$, this determines a map $(K^{\mathrm{h}G})^\wedge_p \to (K^\wedge_p)^{\mathrm{h}G}$.

# 2 Higher-dimensional generalisations of the Section Conjecture

In this section, we discuss a generalisation of Grothendieck's Section Conjecture to schemes that are not of type $\mathrm{K}(\pi, 1)$. In this generality, one has to utilise the full (profinite) étale homotopy type $\Pi^{\text{ét}}_\infty(X)$, which we will briefly recall in (§2.1). We discuss the generalised Section Conjecture and its real variant in (§2.2). Finally, in (§2.3), we discuss the pro-$\ell$ variant of the generalised Section Conjecture.

## 2.1 The étale homotopy type

One major technical advantage of working with $\infty$-categories when developing étale homotopy theory is that the étale homotopy type $\Pi_\infty^{\text{ét}}(X)$ is uniquely determined in terms of a universal property. In order to make this precise, let us first discuss the $\infty$-categorical version of the étale topos of $X$.

### The étale $\infty$-topos of a scheme

First, we recall the definition of an étale sheaf in the language of $\infty$-categories:

**2.1.1 Recollection** (étale sheaves, [SAG, §A.3.3])**.** Let $X$ be a qcqs scheme and denote by $X^{\text{ét}}$ its (classical, 1-categorical) small étale site. Let $\mathcal{C}$ be any $\infty$-category.

(a) Recall that a functor $F\colon X^{\text{ét,op}} \to \mathcal{C}$ is an *étale sheaf on X with values in* $\mathcal{C}$ if the following two conditions are satisfied:

  (1) The functor $F$ preserves finite products.

  (2) Let $f\colon U_0 \twoheadrightarrow Y$ be an étale surjection and let $U_\bullet$ be a Čech nerve of $f$ (see [HTT, §6.1.2]), regarded as an augmented simplicial object of $X^{\text{ét}}$. Then the composite map

$$\mathbf{\Delta}_+ \xrightarrow{U_\bullet} X^{\text{ét,op}} \xrightarrow{F} \mathcal{C}$$

  is a limit diagram, i.e.

$$F(Y) \simeq \lim\Big(F(U_0) \rightrightarrows F(U_0 \times_Y U_0) \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots\Big).$$

(b) The $\infty$-*category of* $\mathcal{C}$*-valued étale sheaves on* $X$, denoted by $\mathrm{Sh}_{\text{ét}}(X, \mathcal{C}) = \mathrm{Sh}(X^{\text{ét}}, \mathcal{C})$, is the full subcategory of $\mathrm{Fun}(X^{\text{ét,op}}, \mathcal{C})$ spanned by the étale sheaves.

Plugging in $\mathcal{C} = \mathbf{Ani}$ in the above, we arrive at the étale $\infty$-topos of $X$:

**2.1.2 Definition** (étale $\infty$-topos)**.** Let $X$ be a qcqs scheme. The $\infty$-category $X_{\text{ét}} := \mathrm{Sh}_{\text{ét}}(X, \mathbf{Ani})$ of $\mathbf{Ani}$-valued sheaves for the étale topology on $X$ is called the *étale* $\infty$*-topos of* $X$.

We of course have an analogue of *global sections* and *constant sheaves*:

**2.1.3 Notation** (global sections)**.** The global sections functor

$$\Gamma_{X,*} := \Gamma_{\text{ét}}(X, -)\colon X_{\text{ét}} \to \mathbf{Ani}, \quad F \mapsto F(X)$$

admits a left-exact left adjoint $\Gamma_X^*\colon \mathbf{Ani} \to X_{\text{ét}}$ that carries an anima $K$ to the *constant sheaf on* $X$ *(with value* $K$*)*. The adjunction $\Gamma_X^* \dashv \Gamma_{X,*}$ determines a *geometric morphism* $X_{\text{ét}} \to \mathbf{Ani}$ of $\infty$-topoi (which is essentially unique since $\mathbf{Ani}$ is the terminal $\infty$-topos).

**2.1.4.** We refer the reader to [HTT, §6] and [SAG, §A.3] for an overview of the theory of $\infty$-topoi and sheaves.

With these notations in place, we are now able to give a concise definition of the étale homotopy type in terms of a universal property:

**2.1.5 Definition** (étale homotopy type)**.** The (profinite) *étale homotopy type* of a qcqs scheme $X$ is the profinite anima $\Pi_\infty^{\text{ét}}(X)$ prorepresenting the functor $\mathbf{Ani}_\pi \to \mathbf{Ani}$, $K \mapsto \Gamma_{\text{ét}}(X, \Gamma_X^* K)$.

**2.1.6 Remark.**

(a) Since the functor $\mathbf{Ani}_\pi \to \mathbf{Ani}, K \mapsto \Gamma_{\text{ét}}(X, \Gamma_X^* K)$ is left exact, there exists some $\Pi_\infty^{\text{ét}}(X)$ prorepresenting it by Theorem 1.1.10 (2).

(b) By unwinding the definition, we see that $\Pi_\infty^{\text{ét}}(X)$ is a profinite anima equipped with an identification
$$\operatorname{map}(\Pi_\infty^{\text{ét}}(X), K) \simeq \Gamma_{\text{ét}}(X, \Gamma_X^* K)$$
that is functorial in $K \in \mathbf{Ani}_\pi$.

To close this section, let us give some motivation for the above definition of the étale homotopy type.

**2.1.7 Remark.** Let $n \in \mathbb{N}$ and let $G$ be a finite (abelian if $n \geq 2$) group.

(a) By [HTT, §7.2.2.17], there is a natural identification
$$\mathrm{H}^n_{\text{ét}}(X; G) = \pi_0 \, \Gamma_*(X_{\text{ét}}, \Gamma^* \, \mathrm{K}(G, n)).$$
Combining this with the universal property of $\Pi_\infty^{\text{ét}}(X)$ as well as representability of cohomology (Theorem 1.3.14 and its non-abelian variant for $n = 1$), we see that
$$\begin{aligned} \mathrm{H}^n(\Pi_\infty^{\text{ét}}(X); G) &= \pi_0 \operatorname{map}(\Pi_\infty^{\text{ét}}(X), \mathrm{K}(G, n)) \\ &= \pi_0 \, \Gamma_*(X_{\text{ét}}, \Gamma^* \, \mathrm{K}(G, n)) \\ &= \mathrm{H}^n_{\text{ét}}(X; G). \end{aligned}$$
This in particular shows that $\Pi_\infty^{\text{ét}}(X)$ satisfies Desideratum 0.17:
One recovers the $\pi_1$-statement by observing that $\mathrm{H}^1_{\text{ét}}(X; G)$ classifies étale $G$-torsors on $X$.

(b) Given a group $G$ acting on an abelian group $A$, there is a twisted variant $\mathrm{K}(G \curvearrowright A, n)$ of Eilenberg–MacLane anima that satisfy:

(1) One has that $\pi_k \, K(G \curvearrowright A, n) = \begin{cases} G, & k = 1, \\ A, & k = n, \\ 0, & \text{otherwise.} \end{cases}$

(2) The action of $G = \pi_1 \, K(G \curvearrowright A, n)$ on $A = \pi_n \, \mathrm{K}(G \curvearrowright A, n)$ precisely recovers the action of $G$ on $A$ one started with.

These twisted Eilenberg–MacLane anima can be used to represent cohomology of local coeffiecient systems. One can leverage this to show that $\Pi_\infty^{\text{ét}}(X)$ similarly recovers the étale cohomology of $X$ with respect to local systems of finite abelian groups.

(c) Moreover, by strengthening Desideratum 0.17 to incorporate cohomology of local systems, one can actually realize these desiderata as a "homotopical representability criterion" that precisely boils down to the universal property of $\Pi_\infty^{\text{ét}}(X)$, as follows:

(1) Instead of asking for a comparison of cohomology, one should ask for a comparison of the form
$$\operatorname{map}(\Pi_\infty^{\text{ét}}(X), -) \simeq \Gamma_*(X_{\text{ét}}, \Gamma^* -)$$
for all $\pi$-finite twisted Eilenberg–MacLane anima.

(2) Using Postnikov towers, one can see that $\mathbf{Ani}_\pi \subset \mathbf{Ani}$ is the smallest subcategory of $\mathbf{Ani}$ containing all $\pi$-finite (twisted) Eilenberg–MacLane anima that is closed under finite disjoint unions as well as finite limits. Since both

$$\mathrm{map}(\Pi_\infty^{\text{ét}}(X), -) \quad \text{and} \quad \Gamma_*(X_{\text{ét}}, \Gamma^* -)$$

are compatible with these constructions, we arrive at the universal property of Definition 2.1.5.

### Definition via shape theory

In this section, we explain how our definition of the étale homotopy type $\Pi_\infty^{\text{ét}}(X)$ fits together with the definition via Lurie's shape theory, which is nowadays mostly used to define $\Pi_\infty^{\text{ét}}(X)$.

**2.1.8 Recollection** (shape of an $\infty$-topos)**.** Write $\mathbf{RTop}_\infty$ for the $\infty$-category of $\infty$-topoi and (right adjoints in) geometric morphisms. The *shape* is a left adjoint functor $\Pi_\infty\colon \mathbf{RTop}_\infty \to \mathrm{Pro}(\mathbf{Ani})$ admitting the following explicit description:

(a) Given an $\infty$-topos $\mathcal{X}$, the shape $\Pi_\infty(\mathcal{X}) \in \mathrm{Pro}(\mathbf{Ani})$ prorepresents the left exact functor

$$\mathbf{Ani} \to \mathbf{Ani}, \quad K \mapsto \Gamma_*(\mathcal{X}, \Gamma_{\mathcal{X}}^* K)$$

(see Theorem 1.1.13).

(b) Given a geometric morphism of $\infty$-topoi $f_*\colon \mathcal{Y} \to \mathcal{X}$ with unit $u\colon \mathrm{id}_{\mathcal{Y}} \to f_* \circ f^*$, the induced map $\Pi_\infty(\mathcal{Y}) \to \Pi_\infty(\mathcal{X})$ corresponds to the map

$$\Gamma_{\mathcal{X},*}\, u\, \Gamma_{\mathcal{X}}^*\colon \Gamma_{\mathcal{X},*}\Gamma_{\mathcal{X}}^* \to \Gamma_{\mathcal{X},*} f_* f^* \Gamma_{\mathcal{X}}^* \simeq \Gamma_{\mathcal{Y},*}\Gamma_{\mathcal{Y}}^*$$

in $\mathrm{Pro}(\mathbf{Ani})^{\mathrm{op}} \subset \mathrm{Fun}(\mathbf{Ani}, \mathbf{Ani})$.

**2.1.9 Definition** (étale shape)**.** Let $X$ be a qcqs scheme. The proanima $\Pi_\infty(X_{\text{ét}})$ is called the *étale shape of* $X$.

**2.1.10 Lemma.** *The profinite completion $\Pi_\infty(X_{\text{ét}})_\pi^\wedge$ of the étale shape $\Pi_\infty(X_{\text{ét}})$ of $X$ is equivalent to the étale homotopy type $\Pi_\infty^{\text{ét}}(X)$ in the sense of Definition 2.1.5.*

*Proof.* The explicit description of $(-)_\pi^\wedge\colon \mathrm{Pro}(\mathbf{Ani}) \to \mathbf{Ani}_\pi^\wedge$ given in Proposition 1.2.12 (2) combined with the description of $\Pi_\infty(X_{\text{ét}})$ of Recollection 2.1.8 (1) unravel to $\Pi_\infty(X_{\text{ét}})_\pi^\wedge$ satisfying the universal property of $\Pi_\infty^{\text{ét}}(X)$ given in Definition 2.1.5. □

In particular, we see that the assignment $X \mapsto \Pi_\infty^{\text{ét}}(X)$ is functorial in $X$:

**2.1.11 Corollary.** *The étale homotopy type defines a functor*

$$\Pi_\infty^{\text{ét}}(-)\colon \mathbf{Sch}^{\mathrm{qcqs}} \to \mathbf{Ani}_\pi^\wedge, \quad X \mapsto \Pi_\infty^{\text{ét}}(X).$$

*Proof.* Using Lemma 2.1.10, we can define $\Pi_\infty^{\text{ét}}(-)\colon \mathbf{Sch}^{\mathrm{qcqs}} \to \mathbf{Ani}_\pi^\wedge$ as the composition of the functors $\mathbf{Sch}^{\mathrm{qcqs}} \to \mathbf{RTop}_\infty, X \mapsto X_{\text{ét}}$, $\Pi_\infty\colon \mathbf{RTop}_\infty \to \mathrm{Pro}(\mathbf{Ani})$ and $(-)_\pi^\wedge\colon \mathrm{Pro}(\mathbf{Ani}) \to \mathbf{Ani}_\pi^\wedge$. □

**2.1.12 Warning.** Beware that many authors write $\Pi_\infty^{\text{ét}}(X)$ for what we denote by $\Pi_\infty(X_{\text{ét}})$ and instead write something like $\Pi_\infty^{\text{ét}}(X)_\pi^\wedge$ or $\widehat{\Pi}_\infty^{\text{ét}}(X)$ for what we denote by $\Pi_\infty^{\text{ét}}(X)$. We chose to write $\Pi_\infty^{\text{ét}}(X)$ for the profinitely completed invariant since this is the only one we ever use in this thesis.

**2.1.13 Remark** (comparison with $\Pi_{\text{AM}}^{\text{ét}}(X)$)**.**

(1) Hoyois proved in [20, Proposition 5.1] that the "protrunction" of $\Pi_\infty(X_{\text{ét}})$ (i.e. the pro-left adjoint to the inclusion $\mathbf{Ani}_{<\infty} \subset \mathbf{Ani}$ of *truncated* anima into all anima) recovers Artin and Mazur's classical construction of the étale homotopy type introduced in [AM].

(2) Consequently, $\Pi_\infty^{\text{ét}}(X)$ always recovers the profinitely completed étale homotopy type of Artin and Mazur. This of course also shows that our definition of $\Pi_\infty^{\text{ét}}(X)$ satisfies the Desideratum 0.17 — which should come to no surprise, given the preceeding discussions.

(3) If $X$ is geometrically unibranch, the protruncation of $\Pi_\infty(X_{\text{ét}})$ is profinite already, hence agrees with $\Pi_\infty^{\text{ét}}(X)$ in this case, as is shown in [AM, Theorem (11.1)]. This last statement is equivalent to saying that the equivalence

$$\operatorname{map}(\Pi_\infty^{\text{ét}}(X), K) \simeq \Gamma_*(X_{\text{ét}}; \Gamma^* K)$$

in fact holds for all *truncated* anima $K$, not the $\pi$-finite ones only.

### The fundamental fibre sequence in étale homotopy theory

**2.1.14.** Recall that if $X$ is a qcqs geometrically connected scheme over some field $k$, then the choice of algebraic closure $\bar{k}$ and geometric point $\bar{x} \to X_{\bar{k}}$ determines a short exact sequence

$$1 \longrightarrow \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \longrightarrow \pi_1^{\text{ét}}(X, \bar{x}) \longrightarrow G_k \longrightarrow 1$$

of profinite groups, the so-called *fundamental exact sequence* of étale fundamental groups.

**2.1.15.** Moreover, it is well-known that the projection $X_{\bar{k}} \to X$ induces isomorphisms

$$\pi_n^{\text{ét}}(X_{\bar{k}}) \to \pi_n^{\text{ét}}(X)$$

for all $n \geq 2$.

In joint work with Peter J. Haine and Sebastian Wolf, we proved the following homotopy-theoretical unification of the two preceeding phenomena:

**2.1.16 Theorem** (fundamental fibre sequence, [15, Corollary 0.5])**.** *Let $X$ be a qcqs scheme over a field $k$ with absolute Galois group $G_k$. Then the sequence*

$$\Pi_\infty^{\text{ét}}(X_{\bar{k}}) \to \Pi_\infty^{\text{ét}}(X) \to \mathrm{B}G_k,$$

*induced by the canonical maps $X_{\bar{k}} \to X \to \operatorname{Spec}(k)$, is a fibre sequence in $\mathbf{Ani}_\pi^\wedge$.*

## 2.2 The generalised Section Conjecture

As explained in the introduction, it is unreasonable to expect Grothendieck's original formulation of the Section Conjecture to be correct for schemes that are not étale $K(\pi, 1)$, as $\mathcal{S}(\pi_1^{\text{ét}}(X/k))$ does not contain any information whatsoever on the higher étale homotopy of $X/k$.

We start this section by formulating an appropriate variant of the Section Conjecture that remedies the above deficiency.

**Setup.** Throughout this section, let $X$ be some qcqs scheme over a field $k$ with algebraic closure $\bar{k}$ and absolute Galois group $G_k := \mathrm{Gal}(\bar{k}/k)$.

Motivated by the fundamental fibre sequence 2.1.16, we propose to replace the set of sections $\mathcal{S}(\pi_1^{\text{ét}}(X/k))$ with the following:

**2.2.1 Definition** (étale sections)**.** The *set of étale sections of $X$ over $k$*, denoted by $\mathcal{S}^{\text{ét}}(X/k)$, is given by the set of homotopy classes of sections of $\Pi_\infty^{\text{ét}}(X) \to \mathrm{B}G_k$, that is:

$$\mathcal{S}^{\text{ét}}(X/k) := \pi_0 \operatorname{map}_{\mathrm{B}G_k}(\mathrm{B}G_k, \Pi_\infty^{\text{ét}}(X)).$$

Using the functoriality of the étale homotopy type, any $k$-rational point $a \in X(k)$ induces a section $a_* : \mathrm{B}G_k \to \Pi_\infty^{\text{ét}}(X)$ of $\Pi_\infty^{\text{ét}}(X) \to \mathrm{B}G_k$. By taking the corresponding homotopy class, we obtain a replacement for the classical Kummer map:

**2.2.2 Definition** (Kummer map)**.** The map

$$\kappa_{X/k} : X(k) \to \mathcal{S}^{\text{ét}}(X/k), \quad a \mapsto [a_*],$$

is called the *Kummer map* of $X/k$.

Using these higher-dimensional replacements for $\mathcal{S}(\pi_1^{\text{ét}}(X/k))$ and the Kummer map $\kappa_{X/k}$, we obtain the desired formulation of the generalised Section Conjecture for $X$:

**2.2.3 Conjecture** (generalised Section Conjecture)**.** *The Kummer map*

$$\kappa_{X/k} : X(k) \to \mathcal{S}^{\text{ét}}(X/k)$$

*is a bijection of sets.*

### Relation to the classical Section Conjecture

Conjecture 2.2.3 certainly fulfills the requirement of actually utilising all the étale homotopy-theoretic information contained in $X/k$. In order for it to adequately replace Grothendieck's Section Conjecture, we still have to make sure that it agrees with it for étale $\mathrm{K}(\pi, 1)$ schemes. To this end, we have to understand unpointed homotopy classes of sections of classifying anima of profinite groups.

**2.2.4 Proposition.** *Let $\pi' \xrightarrow{p'} G \xleftarrow{p} \pi$ be a cospan of profinite groups with $p$ surjective. Then the mapping $[\varphi] \mapsto [\hat{\pi}_1(\varphi)]$ defines a bijection of sets*

$$\pi_0 \operatorname{map}_{\mathrm{B}G}(\mathrm{B}\pi', \mathrm{B}\pi) \to \mathrm{Hom}_G(\pi', \pi)_\Delta,$$

*where $\Delta := p^{-1}(\mathrm{C}_G(\mathrm{im}(p')))$ denotes the preimage in $\pi$ of the centraliser in $G$ of the image of $p'$ and acts via conjugation.*

*Proof.* We use the equivalence $(\mathbf{Ani}_\pi^\wedge)_{/BG} \simeq (\hat{\mathcal{S}}_{/BG})_\infty$ with Quick's model category that we already saw in Proposition 1.5.6. Since $\pi \twoheadrightarrow G$ is surjective, $\mathrm{B}\pi \to \mathrm{B}G$ is a fibration in $\hat{\mathcal{S}}$ (see [35, Corollary 2.25]). Since every object is cofibrant, we thus have

$$\begin{aligned}\pi_0 \operatorname{map}_{\mathrm{B}G}(\mathrm{B}\pi', \mathrm{B}\pi) &= \pi_0 \, \mathbf{R}\mathrm{map}_{\hat{\mathcal{S}}_{/BG}}(\mathrm{B}\pi', \mathrm{B}\pi) \\ &= \pi_0 \operatorname{map}_{\hat{\mathcal{S}}_{/BG}}(\mathrm{B}\pi', \mathrm{B}\pi).\end{aligned}$$

Hence two maps $\varphi_0, \varphi_1 \colon \mathrm{B}\pi' \to \mathrm{B}\pi$ over $\mathrm{B}G$ are homotopic to each other if and only if there exists a homotopy $h\colon \mathrm{B}\pi' \times \Delta^1 \to \mathrm{B}\pi$ over $\mathrm{B}G$ such that the diagram

$$\begin{array}{ccc} \mathrm{B}\pi' \times \{0\} & & \\ \downarrow & \searrow^{\varphi_0} & \\ \mathrm{B}\pi' \times \Delta^1 & \xrightarrow{h} & \mathrm{B}\pi \\ \uparrow & \nearrow_{\varphi_1} & \\ \mathrm{B}\pi' \times \{1\} & & \end{array}$$

commutes. Such a homotopy corresponds to a continuous natural transformation $h\colon \varphi_0 \Rightarrow \varphi_1$ of maps of profinite 1-groupoids such that the whisker fulfills $p \circ h = \mathrm{id}_{p'}$ (observe that the category of profinite 1-groupoids embeds fully faithfully into $\hat{\mathcal{S}}$ via the "usual" nerve construction). Write $c_g\colon G \to G$ for the conjugation by $g$. The choice of such a continuous natural transformation boils down to the choice of an element $\gamma \in \pi$ with the properties that:

(1) For all $\sigma \in \pi'$ the diagram

$$\begin{array}{ccc} * & \xrightarrow{\varphi_0(\sigma)} & * \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ * & \xrightarrow[\varphi_1(\sigma)]{} & * \end{array}$$

commutes, i.e. $\varphi_0 = c_\gamma \circ \varphi_1$ (naturality of $h$).

(2) For all $\sigma \in \pi'$ the diagram

$$\begin{array}{ccc} * & \xrightarrow{p'(\sigma)} & * \\ {\scriptstyle p(\gamma)}\downarrow & & \downarrow{\scriptstyle p(\gamma)} \\ * & \xrightarrow[p'(\sigma)]{} & * \end{array}$$

commutes, i.e. $p(\gamma) \in \mathrm{C}_G(\mathrm{im}(p'))$ (since $p \circ h = \mathrm{id}_{p'}$). □

**2.2.5 Remark.** The notion of "profinite 1-groupoid" being used in the proof of Proposition 2.2.4 is to be understood as "groupoid object in the category of profinite sets".

The bridge between the generalised Section Conjecture 2.2.3 and Grothendieck's Section Conjecture 0.7 is completed by the following group-theoretic observation:

**2.2.6 Lemma.** *Let* $1 \longrightarrow \bar{\pi} \longrightarrow \pi \xrightarrow{p} G \longrightarrow 1$ *be a short exact sequence of profinite groups and write* $\Delta := p^{-1}(\mathrm{C}(G))$*. Then the canonical quotient map*

$$\mathcal{S}(\pi \twoheadrightarrow G) = \operatorname{Hom}_G(G, \pi)_{\bar{\pi}} \twoheadrightarrow \operatorname{Hom}_G(G, \pi)_{\Delta}, \quad [s]_{\bar{\pi}} \mapsto [s]_{\Delta}$$

*is a bijection.*

*Proof.* We obtain the quotient map since $\bar{\pi} \subset \Delta$. To prove bijectivity, let $s_0$ and $s_1$ be sections of $p$ such that there exists some $\sigma \in \Delta$ satisfying $s_0 = \sigma \cdot s_1 \cdot \sigma^{-1}$. We then have to show that there exists some $\bar{\sigma} \in \bar{\pi}$ with the same property. To this end, observe that $\bar{\sigma} := s_0(p(\sigma^{-1})) \cdot \sigma$ lies in $\bar{\pi}$ and satisfies

$$\begin{aligned}\bar{\sigma} \cdot s_1 \cdot \bar{\sigma}^{-1} &= s_0(p(\sigma^{-1})) \cdot (\sigma \cdot s_1 \cdot \sigma^{-1}) \cdot s_0(p(\sigma)) \\ &= s_0(p(\sigma^{-1})) \cdot s_0 \cdot s_0(p(\sigma)) \\ &= s_0 \circ c_{p(\sigma^{-1})} \\ &= s_0,\end{aligned}$$

where the last equality holds since $p(\sigma^{-1}) \in \mathrm{C}(G)$ by assumption. □

Combining Proposition 2.2.4 with Lemma 2.2.6, we conclude:

**2.2.7 Corollary.** *Let $X$ be a qcqs étale $\mathrm{K}(\pi, 1)$ scheme over $k$. Then the mapping $[s] \mapsto [\pi_1^{\text{ét}}(s)]$ defines a bijection of sets*

$$\mathcal{S}^{\text{ét}}(X/k) \longrightarrow \mathcal{S}(\pi_1^{\text{ét}}(X/k)).$$

*In particular, Conjecture 2.2.3 precisely recovers Grothendieck's Section Conjecture 0.7 in this case.*

### The generalised real Section Conjecture

When working over $k = \mathbb{R}$, injectivity of the (classical) Kummer map ceases to hold. Instead, two points $a, b \in X(\mathbb{R})$ induce the same section if and only if they lie in the same connected component of the underlying real analytification of $X$.

**2.2.8 Notation.** By abuse of notation, we write $X(\mathbb{R})$ for the *real analytification* of $X$. Similarly, we write $X(\mathbb{C})$ for the *complex analytification* of $X$.

The real Section Conjecture thus becomes:

**2.2.9 Conjecture** (real Section Conjecture)**.** *Let $X$ be a smooth projective curve of genus $g \geq 2$ over $\mathbb{R}$. Then the Kummer map $\kappa_{X/\mathbb{R}}$ induces a bijection*

$$\pi_0 X(\mathbb{R}) \to \mathcal{S}(\pi_1^{\text{ét}}(X/\mathbb{R})).$$

**2.2.10 Remark.** As mentioned in the introduction, there are several proofs of the above real Section Conjecture 2.2.9 by now.

Consequently, one would expect the correct generalised real Section Conjecture to be the following:

**2.2.11 Conjecture** (generalised real Section Conjecture)**.** *The Kummer map $\kappa_{X/\mathbb{R}}$ induces a bijection*

$$\pi_0 X(\mathbb{R}) \to \mathcal{S}^{\text{ét}}(X/\mathbb{R}).$$

## 2.3 Σ-nilpotent étale sections

In this section we turn our attention to a pro-$\ell$ version of Grothendieck's Section Conjecture 0.7. First, we recall the group-theoretic construction of the set $\mathcal{S}(\pi_1^{\text{ét}}(X/k)^{\wedge\Sigma\text{-nil}})$ of *geometrically Σ-nilpotent sections*. We then introduce a homotopy-theoretic analogue of this construction to obtain the set of *higher-dimensional Σ-nilpotent étale sections* $\mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(X/k)$ of $X/k$ and show that $\mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(X/k)$ recovers $\mathcal{S}(\pi_1^{\text{ét}}(X/k)^{\wedge\Sigma\text{-nil}})$ in the case of hyperbolic curves. Specialising to $\Sigma = \{\ell\}$, we obtain the desired pro-$\ell$ version.

**Group-theoretic geometrically $\Sigma$-nilpotent sections**

Before defining higher-dimensional $\Sigma$-nilpotent sections, we first recall the classical ones in this section. A thorough reference for the contents of this section is [43, §14].

**2.3.1 Construction.** Let $1 \longrightarrow \bar{\pi} \longrightarrow \pi \longrightarrow G \longrightarrow 1$ be a short exact sequence of profinite groups. Pushing out along $\bar{\pi} \twoheadrightarrow \bar{\pi}^{\wedge\Sigma\text{-nil}} = \prod_{\ell\in\Sigma} \bar{\pi}^{\wedge\ell}$, we obtain an induced short exact sequence

$$1 \longrightarrow \bar{\pi}^{\wedge\Sigma\text{-nil}} \longrightarrow \pi^{\wedge(\Sigma\text{-nil})} \longrightarrow G \longrightarrow 1,$$

where $\pi^{\wedge(\Sigma\text{-nil})} = \pi *_{\bar{\pi}} \bar{\pi}^{\wedge\Sigma\text{-nil}}$, together with a map

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \bar{\pi} & \longrightarrow & \pi & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & \ulcorner & \downarrow & & \downarrow = & & \\ 1 & \longrightarrow & \bar{\pi}^{\wedge\Sigma\text{-nil}} & \longrightarrow & \pi^{\wedge(\Sigma\text{-nil})} & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

of short exact sequences.

**2.3.2 Definition** ($\Sigma$-nilpotent $\pi_1^{\text{ét}}$-sections)**.** Let $X$ be a qcqs geometrically connected scheme over a field $k$ with absolute Galois group $G_k$. The set of $\Sigma$-*nilpotent* $\pi_1^{\text{ét}}$-*sections*, denoted by $\mathcal{S}(\pi_1^{\text{ét}}(X/k)^{\wedge\Sigma\text{-nil}})$, is the set of sections of the short exact sequence

$$1 \longrightarrow \pi_1^{\text{ét}}(X_{\bar{k}})^{\wedge\Sigma\text{-nil}} \longrightarrow \pi_1^{\text{ét}}(X)^{\wedge(\Sigma\text{-nil})} \longrightarrow G_k \longrightarrow 1.$$

The map of short exact sequences of 2.3.1 induces a map $\mathcal{S}(\pi_1^{\text{ét}}(X/k)) \to \mathcal{S}(\pi_1^{\text{ét}}(X/k)^{\wedge\Sigma\text{-nil}})$.

**2.3.3 Remark.**

(1) Usually, one only considers a "$\Sigma$-nilpotent Section Conjecture" in the case where $\Sigma = \{\ell\}$ consists of a single prime. In this case, one simply says *pro-$\ell$ section (resp. Section Conjecture)* instead of $\ell$-nilpotent section (resp. Section Conjecture).

(2) The reason for this is that the product decomposition $\bar{\pi}^{\wedge\Sigma\text{-nil}} = \prod_{\ell\in\Sigma} \bar{\pi}^{\wedge\ell}$ of Theorem 1.6.7 induces a decomposition

$$\mathcal{S}(\pi^{\wedge(\Sigma\text{-nil})} \twoheadrightarrow G) = \prod_{\ell\in\Sigma} \mathcal{S}(\pi^{\wedge(\ell)} \twoheadrightarrow G).$$

Therefore, two different points $a, b \in X(k)$ give rise to a plethora of group-theoretic sections by choosing one prime $p \in \Sigma$ and considering e.g. $(s_\ell)_{\ell\in\Sigma} \in \mathcal{S}(\pi_1^{\text{ét}}(X/k)^{\wedge\Sigma\text{-nil}})$ determined by

$$s_\ell = \begin{cases} \kappa_{X/k}(b)^{\wedge p}, & \ell = p, \\ \kappa_{X/k}(a)^{\wedge\ell}, & \text{otherwise.} \end{cases}$$

(3) Unfortunately, even the pro-$\ell$ Section Conjecture generally ceases to hold, as was first proven by Hoshi, see [19, Theorem A]. However, the counterexamples produced by Hoshi seem to be of a very specific nature. There is a slightly weaker version of the pro-$\ell$ Section Conjecture that throws out Hoshi's counterexamples by requiring every hyperbolic curve to admit a finite étale cover satisfying the pro-$\ell$ Section Conjecture, see [43, Question 221].

(4) Over $k = \mathbb{R}$, the pro-2 Section Conjecture in fact holds, as was first proven by Wickelgren in [46].

**$\Sigma$-nilpotent étale sections for higher-dimensional varieties**

We now turn our attention to formulating a higher-dimensional variant of the above. This boils down to finding a homotopy-theoretic analogue of Construction 2.3.1 and applying it to the fundamental fibre sequence

$$\Pi_\infty^{\text{ét}}(X_{\bar{k}}) \to \Pi_\infty^{\text{ét}}(X) \to \mathrm{B}G_k,$$

resulting in a definition of $\Sigma$-nilpotent étale sections $\mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(X/k)$, see Definition 2.3.6.

**Setup.** Let $k$ be a field and $X$ a qcqs scheme over $k$. Choose an algebraic closure $\bar{k}$ of $k$ and write $G_k := \mathrm{Gal}(\bar{k}/k)$ for the absolute Galois group of $k$ with respect to $\bar{k}$. Fix a nonempty set $\Sigma$ of prime numbers.

**2.3.4 Lemma.** *Let $G$ be a profinite group. The $\Sigma$-nilpotent completion functor $(-)^\wedge_{\Sigma\text{-nil}}$ induces a functor $(-)^\wedge_{\Sigma\text{-nil}} \colon \mathbf{Ani}^\wedge_\pi(G) \to \mathbf{Ani}^\wedge_\Sigma(G)$ such that the diagram*

$$\begin{array}{ccc} \mathbf{Ani}^\wedge_\pi(G) & \xrightarrow{(-)^\wedge_{\Sigma\text{-nil}}} & \mathbf{Ani}^\wedge_\Sigma(G) \\ \downarrow & & \downarrow \\ \mathbf{Ani}^\wedge_\pi & \xrightarrow[(-)^\wedge_{\Sigma\text{-nil}}]{} & \mathbf{Ani}^\wedge_\Sigma, \end{array}$$

*commutes. Here the vertical arrows $\mathbf{Ani}^\wedge_\pi(G) \to \mathbf{Ani}^\wedge_\pi$ and $\mathbf{Ani}^\wedge_\Sigma(G) \to \mathbf{Ani}^\wedge_\Sigma$ denote the functors forgetting the $G$-action.*

*Proof.* By Proposition 1.6.27, the functor $(-)^\wedge_{\Sigma\text{-nil}} \colon (\mathbf{Ani}^\wedge_\pi)^\times \to (\mathbf{Ani}^\wedge_\Sigma)^\times$ is symmetric monoidal. It therefore preserves group objects and induces a functor on associated module categories. As $G$, considered as an object of $\mathbf{Ani}^\wedge_\pi$, is 0-truncated, i.e. $G = \hat{\pi}_0\, G$, it follows that $G^\wedge_{\Sigma\text{-nil}} = G$ (see Corollary 1.6.26). This shows that $(-)^\wedge_{\Sigma\text{-nil}} \colon (\mathbf{Ani}^\wedge_\pi)^\times \to (\mathbf{Ani}^\wedge_\Sigma)^\times$ induces a functor

$$\mathbf{Ani}^\wedge_\pi(G) = \mathrm{RMod}_G(\mathbf{Ani}^\wedge_\pi) \to \mathrm{RMod}_G(\mathbf{Ani}^\wedge_\pi) = \mathbf{Ani}^\wedge_\pi(G), \quad K \mapsto K^\wedge_{\Sigma\text{-nil}}$$

satisfying the specified compatibility conditions as claimed. □

**2.3.5.** In particular, the natural action of $G_k$ on $\Pi_\infty^{\text{ét}}(X_{\bar{k}})$ induces an action of $G_k$ on $\Pi_\infty^{\text{ét}}(X_{\bar{k}})^\wedge_{\Sigma\text{-nil}}$. We write $\Pi_\infty^{\text{ét}}(X)^\wedge_{(\Sigma\text{-nil})} := \Pi_\infty^{\text{ét}}(X_{\bar{k}})^\wedge_{\Sigma\text{-nil}} /\!\!/ G_k$ so that, by Theorem 1.5.3, we have a fibre sequence

$$\Pi_\infty^{\text{ét}}(X_{\bar{k}})^\wedge_{\Sigma\text{-nil}} \longrightarrow \Pi_\infty^{\text{ét}}(X)^\wedge_{(\Sigma\text{-nil})} \longrightarrow \mathrm{B}G_k.$$

of profinite anima.

**2.3.6 Definition** ($\Sigma$-nilpotent étale sections)**.** The set of *$\Sigma$-nilpotent étale sections of $X$ over $k$* is given by

$$\mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(X/k) := \pi_0 \operatorname{map}_{\mathrm{B}G_k}(\mathrm{B}G_k, \Pi_\infty^{\text{ét}}(X)^\wedge_{(\Sigma\text{-nil})}).$$

**2.3.7 Notation.** To increase readability, we introduce the following abbreviations.

(a) We write $\mathcal{S}^{\text{ét}}_{\text{nil}}(X/k)$ instead of $\mathcal{S}^{\text{ét}}_{\pi\text{-nil}}(X/k)$ and refer to it as the set of *nilpotent étale sections of $X$ over $k$*.

(b) Given a prime $\ell$, we also write $\mathcal{S}^{\text{ét}}_{\ell}(X/k)$ instead of $\mathcal{S}^{\text{ét}}_{\ell\text{-nil}}(X/k)$ and refer to it as the set of *pro-$\ell$ étale sections of $X$ over $k$*.

**2.3.8.** Let $\varnothing \neq \Sigma' \subset \Sigma$ denote sets of primes. The canonical and $G_k$-equivariant maps

$$\Pi^{\text{ét}}_{\infty}(X_{\bar{k}}) \xrightarrow{c} \Pi^{\text{ét}}_{\infty}(X_{\bar{k}})^{\wedge}_{\Sigma\text{-nil}} \xrightarrow{\mathrm{pr}^{\Sigma}_{\Sigma'}} \Pi^{\text{ét}}_{\infty}(X_{\bar{k}})^{\wedge}_{\Sigma'\text{-nil}}$$

induce maps of fibre sequences

$$\begin{array}{ccccc}
\Pi^{\text{ét}}_{\infty}(X_{\bar{k}}) & \longrightarrow & \Pi^{\text{ét}}_{\infty}(X_{\bar{k}}) & \longrightarrow & \mathrm{B}G_k \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c/\!\!/G_k} & & \downarrow{\scriptstyle =} \\
\Pi^{\text{ét}}_{\infty}(X_{\bar{k}})^{\wedge}_{\Sigma\text{-nil}} & \longrightarrow & \Pi^{\text{ét}}_{\infty}(X)^{\wedge}_{(\Sigma\text{-nil})} & \longrightarrow & \mathrm{B}G_k \\
\downarrow{\scriptstyle \mathrm{pr}^{\Sigma}_{\Sigma'}} & & \downarrow{\scriptstyle \mathrm{pr}^{\Sigma}_{\Sigma'}/\!\!/G_k} & & \downarrow{\scriptstyle =} \\
\Pi^{\text{ét}}_{\infty}(X_{\bar{k}})^{\wedge}_{\Sigma'\text{-nil}} & \longrightarrow & \Pi^{\text{ét}}_{\infty}(X)^{\wedge}_{(\Sigma'\text{-nil})} & \longrightarrow & \mathrm{B}G_k
\end{array}$$

of profinite anima. In particular, postcomposition with $c/\!\!/G_k$ and $\mathrm{pr}^{\Sigma}_{\Sigma'}/\!\!/G_k$ define maps

$$\mathcal{S}^{\text{ét}}(X/k) \to \mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(X/k) \quad \text{and} \quad \mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(X/k) \to \mathcal{S}^{\text{ét}}_{\Sigma'\text{-nil}}(X/k).$$

Fix a prime $\ell$. We are now able to formulate the appropriate analogue of the pro-$\ell$ Section Conjecture:

**2.3.9 Conjecture** (generalised pro-$\ell$ Section Conjecture)**.** *The canonical map*

$$X(k) \to \mathcal{S}^{\text{ét}}_{\ell}(X/k)$$

*is a bijection of sets.*

Again, one obtains a real variant of the above conjecture by looking at $\pi_0 X(\mathbb{R})$ instead of $X(\mathbb{R})$.

As a corollary of Theorem 1.6.22, we obtain a product decomposition of the set of $\Sigma$-nilpotent étale sections.

**2.3.10 Lemma.** *The canonical map*

$$\mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(X/k) \to \prod_{\ell\in\Sigma} \mathcal{S}^{\text{ét}}_{\ell}(X/k)$$

*is a bijection of sets.*

*Proof.* The map in question is obtained as the product over $\ell \in \Sigma$ of the maps induced by taking $\{\ell\} \subset \Sigma$ in 2.3.8. Using the equivalence of Theorem 1.5.3, we compute: Taking homotopy fixed points of the $G_k$-equivariant equivalence $\Pi^{\text{ét}}_{\infty}(X_{\bar{k}})^{\wedge}_{\Sigma\text{-nil}} \simeq \prod_{\ell\in\Sigma} \Pi^{\text{ét}}_{\infty}(X_{\bar{k}})^{\wedge}_{\ell}$ yields an equivalence

$$(\Pi^{\text{ét}}_{\infty}(X_{\bar{k}})^{\wedge}_{\Sigma\text{-nil}})^{\mathrm{h}G_k} \simeq \prod_{\ell\in\Sigma} (\Pi^{\text{ét}}_{\infty}(X_{\bar{k}})^{\wedge}_{\ell})^{\mathrm{h}G_k}.$$

The result now follows by applying $\pi_0$. □

**2.3.11 Remark.**

(a) In view of the product decomposition of $\mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(X/k) = \prod_{\ell\in\Sigma} \mathcal{S}^{\text{ét}}_{\ell}(X/k)$ from above, we expect the "$\Sigma$-nilpotent Section Conjecture" to usually fail for similar reasons as in Remark 2.3.3. This is why we only stated it in the case where $\Sigma = \{\ell\}$ consists of a single prime.

(b) Somewhat surprisingly, the $(2 \in \Sigma)$-nilpotent Section Conjecture often holds over $k = \mathbb{R}$: We will later see in Corollary 3.2.2 that, over $\mathbb{R}$, it is equivalent to the pro-2 Section Conjecture.

**Comparison with the classical $\Sigma$-nilpotent sections**

In this section, we show that our set of $\Sigma$-nilpotent étale sections $\mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(X/k)$ agrees with the classically considered $\mathcal{S}(\pi_1^{\text{ét}}(X/k)^{\wedge\Sigma\text{-nil}})$ in the case of hyperbolic curves. With all the preliminary results proven so far, this quickly reduces to the following result of Schmidt:

**2.3.12 Theorem** (Schmidt, [39, Prop. 15])**.** *Let $k$ be a field and $C$ a connected, smooth curve over $k$ that is either incomplete or of strictly positive genus. Then:*

(1) *$C$ is an étale $\mathrm{K}(\pi,1)$, i.e. $\Pi^{\text{ét}}_{\infty}(C) = \mathrm{B}\pi_1^{\text{ét}}(C)$.*

(2) *If $k$ is separably closed and $\ell$ is any prime number, then also $\Pi^{\text{ét}}_{\infty}(C)^{\wedge}_{\ell} = \mathrm{B}\pi_1^{\text{ét}}(C)^{\wedge\ell}$.*

**2.3.13 Corollary.** *Let $\Sigma$ denote a nonempty set of prime numbers. Then, under the additional assumptions of Theorem 2.3.12, one has*

$$\Pi^{\text{ét}}_{\infty}(C)^{\wedge}_{\Sigma\text{-nil}} = \mathrm{B}\pi_1^{\text{ét}}(C)^{\Sigma\text{-nil}}.$$

*In particular, we have that $\mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(C/k) = \mathcal{S}(\pi_1^{\text{ét}}(C/k)^{\wedge\Sigma\text{-nil}})$ in this case.*

*Proof.* The first part of the statement follows immediately from Theorem 2.3.12 and the product decompositions

$$\Pi^{\text{ét}}_{\infty}(C)^{\wedge}_{\Sigma\text{-nil}} = \prod_{\ell\in\Sigma} \Pi^{\text{ét}}_{\infty}(C)^{\wedge}_{\ell} \quad \text{and} \quad \pi_1^{\text{ét}}(C)^{\Sigma\text{-nil}} = \prod_{\ell\in\Sigma} \pi_1^{\text{ét}}(C)^{\wedge\ell}.$$

of Theorem 1.6.22 and Theorem 1.6.7, respectively. The second part follows from combining the first part with Lemma 2.2.6. □

**2.3.14 Remark.** Let $\pi$ be a profinite group and $\ell$ a prime.

(1) It is generally *not* true that

$$\mathrm{H}^*(\pi, \mathbb{F}_\ell) \cong \mathrm{H}^*(\pi^{\wedge\ell}; \mathbb{F}_\ell)$$

and therefore it is also generally *not* true that

$$(\mathrm{B}\pi)^{\wedge}_{\ell} \simeq \mathrm{B}\pi^{\wedge\ell}.$$

(2) In particular, it is a special property of algebraic curves $C$ (satisfying the conditions of Theorem 2.3.12) that it in fact holds that

$$(\mathrm{B}\pi_1^{\text{ét}}(C_{\bar{k}}))^{\wedge}_{\ell} = \mathrm{B}\pi_1^{\text{ét}}(C_{\bar{k}})^{\wedge\ell}.$$

We currently do not know if a similar property holds for a wider range of étale $\mathrm{K}(\pi,1)$-schemes.

# 3 Proof of the generalised (pro-2) real Section Conjecture

## 3.1 A vanishing theorem for nonabelian cohomology

Our proof of the various versions of the real Section Conjecture heavily relies on the following group-theoretic observation:

**3.1.1 Theorem** (Theorem E)**.** *Let $\Gamma$ be a profinite group acting continuously on another profinite group $N$. If the supernatural orders of $\#\Gamma$ and $\#N$ are coprime, then* $\mathrm{H}^1(\Gamma, N) = *$.

Our proof of Theorem 3.1.1 relies on a characterization of $\mathrm{H}^1(\Gamma, N)$ in terms of *complements*, which we will recall first.

### Nonabelian cohomology, sections and complements

First of all, recall that there is a characterization of $\mathrm{H}^1(\Gamma, N)$ in terms of group-theoretic sections.

**3.1.2 Recollection** (nonabelian $\mathrm{H}^1$ and sections, see [43, §1, Prop. 8])**.** Let $\Gamma$ be a profinite group acting continuously on another profinite group $N$. Then we obtain a canonically split short exact sequence

$$1 \longrightarrow N \longrightarrow N \rtimes \Gamma \longrightarrow \Gamma \longrightarrow 1$$

of profinite groups. Recall that $\mathrm{H}^1(\Gamma, N)$ classifies sections of $N \rtimes \Gamma \twoheadrightarrow \Gamma$ up to conjugation by $N$, i.e. there is a canonical identification

$$\begin{aligned} \mathrm{H}^1(\Gamma, N) &= \mathrm{Hom}_\Gamma(\Gamma, N \rtimes \Gamma)_N \\ &= \mathcal{S}(N \rtimes \Gamma \twoheadrightarrow \Gamma). \end{aligned}$$

**3.1.3 Definition** (complement of a subgroup)**.** Let $H \leq G$ be a closed subgroup of the profinite group $G$.

(a) A *complement of $H$ in $G$* is a closed subgroup $K \leq G$ such that:

   (1) $H \cdot K = G$

   (2) $H \cap K = 1$

(b) We denote by $\mathrm{Comp}_G(H)$ the set of all complements of $H$ in $G$.

We are interested in complements since they encode sections:

**3.1.4 Lemma.** *Let $1 \longrightarrow N \longrightarrow G \xrightarrow{p} H \longrightarrow 1$ be a short exact sequence of profinite groups. Then there is a bijection*

$$\mathrm{Hom}_H(H, G) \xleftrightarrow{1:1} \mathrm{Comp}_G(N)$$

*carrying a section $s$ to the complement $K_s := \mathrm{im}(s)$ and a complement $K$ to the section $s_K$ given by*

$$s_K : H \xrightarrow[\sim]{(p|_K)^{-1}} K \subset G$$

*Proof.* We first show that both maps are well-defined, and then that they are mutually inverse to each other.

(1) Let $s \in \mathrm{Hom}_H(H, G)$ be a section of $p$. Then $K_s = \mathrm{im}(s)$ certainly is a closed subgroup of $G$. Furthermore, $N \cap K_s = 1$ as $N = \ker(p)$. Finally, we also have $N \cdot K_s = G$: To see this, let $g$ be an element of $G$. Then $g = (g \cdot s(p(g))^{-1}) \cdot s(p(g)) \in N \cdot K_s$ as required. This shows that $K_s$ indeed is a complement of $N$ in $G$.

(2) Let $K \in \mathrm{Comp}_G(N)$ be a complement of $N$ in $G$.

Then the composition $K \subset G \xrightarrow{p} H$ is an isomorphism: Note that

$$\begin{aligned}\ker(K \subset G \xrightarrow{p} H) &= \ker(p) \cap K \\ &= N \cap K \\ &= 1,\end{aligned}$$

i.e. $K \subset G \xrightarrow{p} H$ is injective. Furthermore,

$$\begin{aligned}H &= p(G) \\ &= p(N \cdot K) \\ &= p(K),\end{aligned}$$

so $K \subset G \xrightarrow{p} H$ is seen to also be surjective. This shows that $s_K : H \xrightarrow[\sim]{(p|_K)^{-1}} K \subset G$ is well-defined and a section of $p$.

(3) Finally, let us prove that both constructions are inverse to each other:

(a) Let $s$ be a section of $p$. Then $s$ factorises as

$$H \xrightarrow[\sim]{s = (p|_{K_s})^{-1}} K_s = \mathrm{im}(s) \subset G.$$

This shows that $s_{K_s} = s$.

(b) Let $K$ be a complement of $N$ in $G$. Then

$$\begin{aligned}K_{s_K} &= \mathrm{im}(\ s_K : H \xrightarrow[\sim]{(p|_K)^{-1}} K \subset G\ ) \\ &= K\end{aligned}$$

as claimed. □

To use complements to determine $\mathrm{H}^1$, we still have to control the effect of conjugation by elements of $N$ to the above bijection.

**3.1.5 Lemma.** *Let $1 \longrightarrow N \longrightarrow G \xrightarrow{p} H \longrightarrow 1$ be a short exact sequence of profinite groups. The bijection of Lemma 3.1.4 induces a bijection*

$$\mathcal{S}(G \twoheadrightarrow H) = \mathrm{Hom}_H(H, G)_N \xleftrightarrow{1:1} \mathrm{Comp}_G(N)/_{conj.}.$$

*Proof.* We proceed in two steps:

(1) Let $s \in \mathrm{Hom}_H(H, G)$ be a section and $\sigma \in N$ an element. Then

$$\begin{aligned} K_{\sigma\cdot s\cdot\sigma^{-1}} &= \mathrm{im}(\sigma \cdot s \cdot \sigma^{-1}) \\ &= \sigma \cdot \mathrm{im}(s) \cdot \sigma^{-1} \\ &= \sigma \cdot K_s \cdot \sigma^{-1}, \end{aligned}$$

i.e. $K_{\sigma\cdot s\cdot\sigma^{-1}}$ is conjugate to $K_s$.

(2) Let $K \in \mathrm{Comp}_G(N)$ be a complement and $g \in G$ any element. Then, since $G = N \cdot K$, there exist elements $\sigma \in N$ and $k \in K$ such that $g = \sigma \cdot k$. Thus

$$\begin{aligned} g \cdot K \cdot g^{-1} &= \sigma \cdot (k \cdot K \cdot k^{-1}) \cdot \sigma^{-1} \\ &= \sigma \cdot K \cdot \sigma^{-1}, \qquad \text{since } k \in K, \end{aligned}$$

which readily implies that $s_{g\cdot K\cdot g^{-1}} = \sigma \cdot s_K \cdot \sigma^{-1}$ is conjugate to $s_K$ via $N$. □

Finally, we are able to deduce Theorem 3.1.1 via the following profinite version of a classical result of Schur-Zassenhaus.

**3.1.6 Theorem** (Profinite Schur-Zassenhaus, [38, Thm 2.3.15])**.** *Let $N$ be a closed normal subgroup of a profinite group $G$ satisfying $(\#N, \#G/N) = 1$. Then:*

(1) *There exists a complement $K$ of $N$ in $G$.*

(2) *Any two complements of $N$ in $G$ are conjugate.*

*Here, we write $\#G$ for the* supernatural order *of a profinite group $G$.*

*Proof of Theorem 3.1.1.* Since $(\#\Gamma, \#N) = 1$, the above theorem is applicable to the closed subgroup $N \leq N \rtimes \Gamma$. Therefore

$$\begin{aligned} \mathrm{H}^1(\Gamma, N) &= \mathrm{Comp}_{N\rtimes\Gamma}(N)/_{\mathrm{conj.}} \\ &= 1, \end{aligned}$$

where the first equality follows from combining Recollection 3.1.2 and Lemma 3.1.5. □

**3.1.7 Remark.**

(a) The proof of the second part of the above theorem is elementary, provided that either $N$ or $G/N$ is solvable, see e.g. [21, Thm. §I.18.2]. All known proofs of the above theorem reduce to the solvable case by the celebrated Feit–Thompson theorem [12], applicable since either $N$ or $G/N$ are of odd order.

(b) Note that there also is a natural definition of $\mathrm{H}^2(\Gamma, N)$ via extensions of $\Gamma$ by $N$, see e.g. [43, Def. 9]. Since, by Theorem 3.1.6 (1) combined with Lemma 3.1.4, any short exact sequence $1 \longrightarrow N \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$ necessarily splits, one actually also has that $\mathrm{H}^2(\Gamma, N)$ vanishes.

## 3.2 Proof of Theorem D

We are interested in Theorem 3.1.1 because it has the following useful homotopy-theoretic application:

**3.2.1 Proposition.** *Let $p$ be a prime number and let $G$ be a finite $p$-group acting on a connected $p'$-profinite anima $K$. Assume that $K^{\mathrm{h}G} \neq \varnothing$ and let $x \in K^{\mathrm{h}G}$. Then $\hat{\pi}_n(K^{\mathrm{h}G}, x) = \hat{\pi}_n(K)^G$. In particular, $K^{\mathrm{h}G}$ is connected.*

*Proof.* The homotopy fixed point spectral sequence 1.5.6

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(G, \hat{\pi}_t(K)) \Rightarrow \hat{\pi}_{t-s}(K^{\mathrm{h}G}, x)$$

attached to $G \curvearrowright K$ collapses on the $\mathrm{E}_2$-page:

Since, by construction, $\mathrm{E}_2^{s,t} = *$ whenever $t - s < 0$, it suffices to discuss the cases $t \geq s \geq 0$. Note that $\mathrm{E}_2^{s,t} = \mathrm{H}^s(G, \hat{\pi}_t(K))$ vanishes whenever $t \geq 2$, as we are looking at group cohomology of a finite $p$-group with values in an abelian group prime to $p$. As $K$ is assumed to be connected, there is furthermore nothing to show whenever $t = 0$. When $t = 1$, the only potentially nontrivial term is $\mathrm{E}_2^{1,1} = \mathrm{H}^1(G, \hat{\pi}_1(K))$, which vanishes by Theorem 3.1.1. □

As a corollary, we obtain that the pro-2 Section Conjecture is equivalent to the full nilpotent Section Conjecture over the real numbers. More precisely:

**3.2.2 Corollary.** *Let $\varnothing \neq \Sigma$ denote a set of prime numbers and $X/\mathbb{R}$ any geometrically connected qcqs scheme. Assume that $X(\mathbb{R}) \neq \varnothing$. Then*

$$\mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(X/\mathbb{R}) = \begin{cases} \mathcal{S}^{\text{ét}}_2(X/\mathbb{R}), & \text{if } 2 \in \Sigma \\ \{*\}, & \text{otherwise.} \end{cases}$$

*Proof.* Using Lemma 2.3.10, it suffices to see that $\mathcal{S}^{\text{ét}}_\ell(X/\mathbb{R}) = \{*\}$ for all primes $\ell \neq 2$. Since $X(\mathbb{R}) \neq \varnothing$, we also have that $\Pi^{\text{ét}}_\infty(X_{\mathbb{C}})^{\mathrm{h}\mathbb{Z}/2} \simeq \mathrm{map}_{\mathrm{B}\mathbb{Z}/2}(\mathrm{B}\mathbb{Z}/2, \Pi^{\text{ét}}_\infty(X)) \neq \varnothing$. This shows that in particular $(\Pi^{\text{ét}}_\infty(X_{\mathbb{C}})^\wedge_\ell)^{\mathrm{h}\mathbb{Z}/2} \neq \varnothing$, so that Proposition 3.2.1 is applicable. Therefore, for any $\ell \neq 2$,

$$\begin{aligned} \mathcal{S}^{\text{ét}}_\ell(X/\mathbb{R}) &= \pi_0(\Pi^{\text{ét}}_\infty(X_{\mathbb{C}})^\wedge_\ell)^{\mathrm{h}\mathbb{Z}/2} \\ &= \{*\} \end{aligned}$$

as claimed. □

This also lets us settle the "prime-to-2 nilpotent real Section Conjecture":

**3.2.3 Corollary** (prime-to-2 nilpotent real Section Conjecture)**.** *Let $2 \notin \Sigma$ be a nonempty set of prime numbers. Then the following are equivalent for a qcqs scheme $X/\mathbb{R}$ such that $X(\mathbb{R}) \neq \varnothing$:*

(1) *$X$ satisfies the $\Sigma$-nilpotent real Section Conjecture, i.e. the map*

$$\pi_0 X(\mathbb{R}) \to \mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(X/\mathbb{R})$$

*is a bijection.*

(2) *The topological space $X(\mathbb{R})$ is connected.*

*Proof.* By Corollary 3.2.2, we have that $\mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(X/\mathbb{R}) = \{*\}$ since we assumed $2 \notin \Sigma$. □

The following result is the key homotopical input that lets us derive the full Section Conjecture from the pro-2 Section Conjecture:

**3.2.4 Theorem** (Theorem D)**.** *Let $G$ be a finite $p$-group and $K$ a connected nilpotent profinite anima with $G$-action. Assume that $K^{\mathrm{h}G} \neq \varnothing$. Then the canonical map*

$$(K^{\mathrm{h}G})^{\wedge}_{p} \to (K^{\wedge}_{p})^{\mathrm{h}G}$$

*is an equivalence of $p$-profinite anima.*

*Proof.* As $K$ is nilpotent, the canonical map $K \to \prod_{\ell} K^{\wedge}_{\ell}$ is an equivalence of profinite anima with $G$-action. Hence

$$\begin{aligned} K^{\mathrm{h}G} &\simeq \Big(\prod_{\ell} K^{\wedge}_{\ell}\Big)^{\mathrm{h}G} \\ &\simeq \prod_{\ell} (K^{\wedge}_{\ell})^{\mathrm{h}G}. \end{aligned}$$

Given any prime $\ell$, $(K^{\wedge}_{\ell})^{\mathrm{h}G}$ remains $\ell$-profinite as $\mathbf{Ani}^{\wedge}_{\ell} \subset \mathbf{Ani}^{\wedge}_{\pi}$ is stable under limits. Therefore, since by Lemma 1.4.10 $p$-profinite completion preserves finite products, we have that

$$(K^{\mathrm{h}G})^{\wedge}_{p} \simeq (K^{\wedge}_{p})^{\mathrm{h}G} \times \Big(\prod_{\ell \neq p} (K^{\wedge}_{\ell})^{\mathrm{h}G}\Big)^{\wedge}_{p}.$$

By Proposition 3.2.1, $\prod_{\ell \neq p} (K^{\wedge}_{\ell})^{\mathrm{h}G}$ is a product of connected profinite anima, hence itself connected. Since it is furthermore $p'$-profinite, Proposition 1.3.20 implies that $(\prod_{\ell \neq p} (K^{\wedge}_{\ell})^{\mathrm{h}G})^{\wedge}_{p}$ is contractible. Thus $(K^{\mathrm{h}G})^{\wedge}_{p} \simeq (K^{\wedge}_{p})^{\mathrm{h}G}$ as claimed. □

## 3.3 Proofs of Theorems A and B

We can now finally prove the (pro-2) real Section Conjecture. We identify $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2$.

In order to follow through with our strategy, we need to make sure that the Sullivan Conjecture is applicable to $X(\mathbb{C})$, i.e. we need to make sure that $X(\mathbb{C})$ is a finite-dimensional $\mathbb{Z}/2$-CW complex. For the moment, we will turn this property into an auxiliary adjective:

**3.3.1 Definition** (equivariantly triangulable)**.** A scheme $X$ over $\mathbb{R}$ is *equivariantly triangulable* if there exists a finite-dimensional $\mathbb{Z}/2$-CW complex $K$ and a $\mathbb{Z}/2$-equivariant homeomorphism $K \to X(\mathbb{C})$, where $\mathbb{Z}/2$ acts on $X(\mathbb{C})$ via complex conjugation.

**3.3.2 Remark.** In Proposition 3.3.11, we will see that the following classes of varieties over $\mathbb{R}$ are equivariantly triangulable:

(1) $X/\mathbb{R}$ smooth.

(2) $X/\mathbb{R}$ affine and of finite type.

(3) $X/\mathbb{R}$ projective.

As explained in the introduction, we need to relate the homotopy type of $X(\mathbb{C})$ with the étale homotopy type $\Pi^{\text{ét}}_{\infty}(X_{\mathbb{C}})$. This is achieved by the following theorem of Artin and Mazur (see [AM, Theorem (12.9)]):

**3.3.3 Theorem** (generalised Riemann existence)**.** *Let $X$ be a scheme of finite type over $\mathbb{C}$. Then there is a canonical equivalence $X(\mathbb{C})^\wedge_\pi \simeq \Pi^{\text{ét}}_\infty(X)$.*

**3.3.4 Remark.** Artin and Mazur's version of the generalised Riemann existence theorem assumed $X$ to be pointed and connected. Using the modern shape-theoretic definition of the étale homotopy type, these assumptions can be dropped (as we did in the formulation of Theorem 3.3.3), see e.g. [2, Theorem 11.5.3], [6, Theorem 4.12] and [8, Theorem 4.3.10].

The following constitutes the homotopical incarnation of the real Section Conjecture:

**3.3.5 Theorem** (homotopical real Section Conjecture, equivariantly triangulable version)**.** *Let $X$ be any geometrically connected and equivariantly triangulable qcqs scheme of finite type over $\mathbb{R}$ and $2 \in \Sigma$ a set of primes. There are natural equivalences*

$$\begin{aligned} X(\mathbb{R})^\wedge_2 &\simeq (\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_2)^{\mathrm{h}\mathbb{Z}/2} \\ &\simeq ((\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_{\Sigma\text{-nil}})^{\mathrm{h}\mathbb{Z}/2})^\wedge_2 \end{aligned}$$

*of 2-profinite anima.*

*Proof.* We will apply the Sullivan Conjecture in the form of Theorem 1.7.1. First of all, it holds that

$$X(\mathbb{R})^\wedge_2 = (X(\mathbb{C})^{\mathbb{Z}/2})^\wedge_2.$$

Since $X(\mathbb{C})$ is a finite-dimensional $\mathbb{Z}/2$-CW-complex, Theorem 1.7.1 is applicable and shows that furthermore

$$(X(\mathbb{C})^{\mathbb{Z}/2})^\wedge_2 = (X(\mathbb{C})^\wedge_2)^{\mathrm{h}\mathbb{Z}/2}\ .$$

By the generalised Riemann existence theorem (3.3.3),

$$\begin{aligned} X(\mathbb{C})^\wedge_2 &= (X(\mathbb{C})^\wedge_\pi)^\wedge_2 \\ &= \Pi^{\text{ét}}_\infty(X(\mathbb{C}))^\wedge_2 \end{aligned}$$

and thus

$$(X(\mathbb{C})^\wedge_2)^{\mathrm{h}\mathbb{Z}/2} = (\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_2)^{\mathrm{h}\mathbb{Z}/2},$$

which shows that $X(\mathbb{R})^\wedge_2 \simeq (\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_2)^{\mathrm{h}\mathbb{Z}/2}$ as claimed. Regarding the second equivalence, note that we also have a $\mathbb{Z}/2$-equivariant identification

$$\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_2 = (\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_{\Sigma\text{-nil}})^\wedge_2.$$

If $(\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_{\Sigma\text{-nil}})^{\mathrm{h}\mathbb{Z}/2} \neq \varnothing$, the second claim now follows as we then have that

$$((\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_{\Sigma\text{-nil}})^\wedge_2)^{\mathrm{h}\mathbb{Z}/2} = ((\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_{\Sigma\text{-nil}})^{\mathrm{h}\mathbb{Z}/2})^\wedge_2$$

by Theorem 3.2.4. However, if $\left(\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_{\Sigma\text{-nil}}\right)^{\mathrm{h}\mathbb{Z}/2}$ is empty, so is $X(\mathbb{R})$: any real point $a \in X(R)$ induces a map $\mathrm{B}\mathbb{Z}/2 \xrightarrow{a_*} \Pi^{\text{ét}}_\infty(X) \to \Pi^{\text{ét}}_\infty(X)^\wedge_{(\Sigma\text{-nil})}$ over $\mathrm{B}\mathbb{Z}/2$ and thus a point of

$$(\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_{\Sigma\text{-nil}})^{\mathrm{h}\mathbb{Z}/2} = \mathrm{map}_{\mathrm{B}\mathbb{Z}/2}(\mathrm{B}\mathbb{Z}/2, \Pi^{\text{ét}}_\infty(X)^\wedge_{(\Sigma\text{-nil})}).$$

This finishes the proof. □

Putting everything together, the pro-2 real Section Conjecture follows:

**3.3.6 Corollary** (($2 \in \Sigma$-) Theorem A, equivariantly triangulable version)**.** *Let $X$ be any equivariantly triangulable qcqs scheme of finite type over $\mathbb{R}$ and $2 \in \Sigma$ a set of primes. Then the ($\Sigma$-)nilpotent Section Conjecture holds for $X$, i.e. the natural map*

$$\pi_0 X(\mathbb{R}) \to \mathcal{S}^{\text{ét}}_{\text{nil}}(X/\mathbb{R}) = \mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(X/\mathbb{R}) = \mathcal{S}^{\text{ét}}_{2}(X/\mathbb{R})$$

*is a bijection of finite sets.*

*Proof.* By (2.3.5), $\Pi^{\text{ét}}_\infty(X)^\wedge_{(\Sigma\text{-nil})}$ sits in a fibre sequence $\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_{\Sigma\text{-nil}} \to \Pi^{\text{ét}}_\infty(X)^\wedge_{(\Sigma\text{-nil})} \to \mathrm{B}\mathbb{Z}/2$ of profinite anima. Therefore, under the equivalence of Theorem 1.5.3, the map

$$\Pi^{\text{ét}}_\infty(X)^\wedge_{(\Sigma\text{-nil})} \to \mathrm{B}\mathbb{Z}/2$$

corresponds to $\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_{\Sigma\text{-nil}}$ with the $\mathbb{Z}/2$-action also constructed in (2.3.5). Combining this with Remark 1.5.5, we obtain

$$|(\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_{\Sigma\text{-nil}})^{\mathrm{h}\mathbb{Z}/2}| = \mathrm{map}_{\mathrm{B}\mathbb{Z}/2}(\mathrm{B}\mathbb{Z}/2, \Pi^{\text{ét}}_\infty(X)^\wedge_{(\Sigma\text{-nil})}).$$

Taking $\pi_0(-)$, we conclude

$$\begin{aligned}\pi_0(\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_{\Sigma\text{-nil}})^{\mathrm{h}\mathbb{Z}/2} &= \pi_0\, \mathrm{map}_{\mathrm{B}\mathbb{Z}/2}(\mathrm{B}\mathbb{Z}/2, \Pi^{\text{ét}}_\infty(X)^\wedge_{(\Sigma\text{-nil})}) \\ &= \mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(X/\mathbb{R}),\end{aligned}$$

and, in virtue of Corollary 1.4.9, the claim follows from Theorem 3.3.5. □

In order to be able to state the most general form of the real Section Conjecture that we can prove, let us introduce the following notation:

**3.3.7 Definition** (geometrically étale nilpotent)**.** A qcqs scheme $X/k$ is said to be *geometrically étale nilpotent* if the étale homotopy type $\Pi^{\text{ét}}_\infty(X_{\bar{k}})$ is nilpotent (in the sense of Definition 1.6.17).

**3.3.8 Remark.** By Lemma 1.6.21, any geometrically étale simply connected scheme is geometrically étale nilpotent.

Finally, we deduce the real Section Conjecture in the nilpotent (and therefore simply connected) case:

**3.3.9 Corollary** (Theorem B, equivariantly triangulable version)**.** *Let $X$ be any equivariantly triangulable qcqs and geometrically étale nilpotent scheme of finite type over $\mathbb{R}$. Then the Section Conjecture holds for $X$, i.e. the canonical map*

$$\pi_0 X(\mathbb{R}) \to \mathcal{S}^{\text{ét}}(X/\mathbb{R})$$

*is a bijection.*

*Proof.* Since $\Pi^{\text{ét}}_\infty(X_\mathbb{C})$ is nilpotent by assumption, the canonical map

$$\mathcal{S}^{\text{ét}}(X/\mathbb{R}) \to \mathcal{S}^{\text{ét}}_{\text{nil}}(X/\mathbb{R})$$

is a bijection and the claim thus immediate from Corollary 3.3.6. □

**3.3.10 Remark.** Unfortunately, Lurie's version of the Sullivan Conjecture 1.7.1 has not been published outside of his set of lecture notes. Based on the comparison of Sullivans $p$-adic completion and the Bousfield-Kan $p$-completion $\mathbb{F}_{p\infty}(-)$, we are still able to derive Corollary 3.3.6 as well as Corollary 3.3.9, using the more classical versions of the Sullivan Conjecture as follows.

(1) Carlsson's version of the Sullivan Conjecture, [7, Theorem B], states that the canonical map

$$\mathbb{F}_{p\infty}(K^G) \to \mathbb{F}_{p\infty}(K^{\mathrm{h}G}) \to (\mathbb{F}_{p\infty} K)^{\mathrm{h}G}$$

is an equivalence of ordinary anima for every finite $p$-group $G$ and finite-dimensional $G$-CW complex $K$. Starting the first step of the proof of Theorem 3.3.5 with this version of the Sullivan Conjecture, we obtain an equivalence

$$\mathbb{F}_{2\infty}X(\mathbb{R}) \simeq (\mathbb{F}_{2\infty}X(\mathbb{C}))^{\mathrm{h}\mathbb{Z}/2}.$$

We can thus continue as in Theorem 3.3.5, provided we have an equivalence

$$(\dagger) \qquad \mathbb{F}_{2\infty}\, X(\mathbb{C}) \simeq |X(\mathbb{C})^{\wedge}_2|,$$

in which case, because $(-)^{\mathrm{h}\mathbb{Z}/2}$ commutes with $|-|$ (both are limits), the $\mathbb{F}_{2\infty}$-version of Theorem 3.3.5 will yield equivalences

$$\begin{aligned}\mathbb{F}_{2\infty}(X(\mathbb{R})) &\simeq |((\Pi^{\text{ét}}_{\infty}(X_{\mathbb{C}})^{\wedge}_2)^{\mathrm{h}\mathbb{Z}/2})| \\ &\simeq |((\Pi^{\text{ét}}_{\infty}(X_{\mathbb{C}})^{\wedge}_{\Sigma\text{-nil}})^{\mathrm{h}\mathbb{Z}/2})^{\wedge}_2|\end{aligned}$$

of ordinary anima. This is still strong enough to deduce Theorems A and B, the proofs of which evidently only depend on |Theorem 3.3.5| to begin with.

(2) The functor $\mathbf{Ani} \to \mathbf{Ani}, K \mapsto |K^{\wedge}_p|$ recovers Sullivan's $p$-profinite completion functor. Bousfield-Kan $p$-completion and Sullivan's $p$-profinite completion are known to agree for spaces with degreewise finitely generated $\mathbb{F}_p$-homology (see [13, Thm 6.10]). Therefore, since

$$\mathrm{H}^*(X(\mathbb{C}); \mathbb{F}_p) \cong \mathrm{H}^*_{\text{ét}}(X_{\mathbb{C}}; \mathbb{F}_p)$$

the comparison (†): $\mathbb{F}_{2\infty}\, X(\mathbb{C}) \simeq |X(\mathbb{C})^{\wedge}_2|$ indeed holds in virtue of [SGA 4, §XIX, Thm 5.1].

We however chose to stick with using Lurie's version of the Sullivan Conjecture 1.7.1, since this better reflects the way we actually arrived at the proofs of Theorems A and B.

### Examples of equivariantly triangulable schemes

In order to bring the above theorems to life, we of course need to address the question for which schemes $X/\mathbb{R}$ the complex points $X(\mathbb{C})$ actually admit the structure of a finite-dimensional $\mathbb{Z}/2$-CW complex, i.e. which $X/\mathbb{R}$ are equivariantly triangulable.

The most general results in this direction that we could find in the literature are summarized in the following proposition:

**3.3.11 Proposition.** *Let $X$ be a scheme over $\mathbb{R}$. Then $X$ is equivariantly triangulable, provided one of the following conditions hold:*

(1) *$X$ is smooth over $\mathbb{R}$.*

(2) *$X(\mathbb{C})$ can be obtained as the zero-locus of a finite family of polynomials in $\mathbb{R}[T_1, \dots, T_n]$ for some $n \in \mathbb{N}$. This includes the cases:*

   (a) *$X$ is affine and of finite type over $\mathbb{R}$.*

   (b) *$X$ is projective over $\mathbb{R}$.*

*Proof.* We prove the statement case by case.

(1) If $X$ is smooth, $X(\mathbb{C})$ admits the structure of a smooth $\mathbb{Z}/2$-manifold, hence admits a finite-dimensional equivariant $\mathbb{Z}/2$-triangulation by the main result of [22]. Note that, by the conventions regarding equivariant simplicial complexes made in [22, §1], all simplicial complexes considered by Illman are finite-dimensional.

(2) For $X(\mathbb{C}) \subset \mathbb{R}^n$ cut out by polynomial equations, the result follows from [33, Theorem 1.3].

   (a) If $X$ is affine and of finite type over $\mathbb{R}$, then $X(\mathbb{C}) \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ for suitable $n \in \mathbb{N}$.

   (b) If $X$ is projective, then $X(\mathbb{C}) \subset \mathbf{P}^n(\mathbb{C})$ for suitable $n \in \mathbb{N}$. The claim now follows from the embedding

$$\mathbf{P}^n(\mathbb{C}) \hookrightarrow \mathbb{R}^{2n^2}, \quad [z_0 : \dots : z_n] \mapsto \left(\frac{z_i \cdot \bar{z}_j}{\sum_k |z_k|^2}\right)_{1 \le i,j \le n}$$

   of complex projective $n$-space into real Euclidean $2n^2$-space as a compact, algebraic subset, originally due to Mannoury [26] (see also [3, Prop. 3.4.6]).

□

**3.3.12 Remark.** In his PhD thesis [18], Hofmann shows that the complex points of any separated scheme $X$ of finite type over $\mathbb{R}$ admit the structure of a finite-dimensional CW complex. It seems extremely likely that one can actually refine this into the structure of a $\mathbb{Z}/2$-CW complex, but we have not been able to find a reference for this fact.

We therefore obtain the following concrete incarnations of Theorem 3.3.5, Corollary 3.3.6 and Corollary 3.3.9:

**3.3.13 Theorem** (homotopical pro-2 Section Conjecture)**.** *Let $X$ be any qcqs scheme that is either smooth, or affine and of finite type, or projective over $\mathbb{R}$ and $2 \in \Sigma$ a set of primes. There are natural equivalences*

$$\begin{aligned} X(\mathbb{R})^\wedge_2 &\simeq (\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_2)^{\mathrm{h}\mathbb{Z}/2} \\ &\simeq ((\Pi^{\text{ét}}_\infty(X_\mathbb{C})^\wedge_{\Sigma\text{-nil}})^{\mathrm{h}\mathbb{Z}/2})^\wedge_2 \end{aligned}$$

*of 2-profinite anima.*

**3.3.14 Corollary** (($2 \in \Sigma$-) Theorem A)**.** *Let $X$ be any qcqs scheme that is either smooth, or affine and of finite type, or projective over $\mathbb{R}$ and $2 \in \Sigma$ a set of primes. Then the ($\Sigma$-)nilpotent Section Conjecture holds for $X$, i.e. the natural map*

$$\pi_0 X(\mathbb{R}) \to \mathcal{S}^{\text{ét}}_{\text{nil}}(X/\mathbb{R}) = \mathcal{S}^{\text{ét}}_{\Sigma\text{-nil}}(X/\mathbb{R}) = \mathcal{S}^{\text{ét}}_2(X/\mathbb{R})$$

*is a bijection of finite sets.*

**3.3.15 Corollary** (Theorem B)**.** *Let $X$ be any qcqs scheme that is either smooth, or affine and of finite type, or projective over $\mathbb{R}$. Assume further that $X$ is geometrically étale simply connected. Then the Section Conjecture holds for $X$, i.e. the canonical map*

$$\pi_0 X(\mathbb{R}) \to \mathcal{S}^{\text{ét}}(X/\mathbb{R})$$

*is a bijection.*

*Proofs of Theorem 3.3.13, Corollary 3.3.14 and Corollary 3.3.15.*
Combine Proposition 3.3.11 with Theorem 3.3.5, Corollary 3.3.6 and Corollary 3.3.9. □

Finally, let us note that the above Theorem A also provides yet another proof of the classical real Section Conjecture for hyperbolic curves over $\mathbb{R}$:

**3.3.16 Corollary.** *Let $X/\mathbb{R}$ be a hyperbolic curve. Then $X$ satisfies the Section Conjecture, i.e. the Kummer map*

$$\kappa_{X/\mathbb{R}} \colon \ \pi_0 X(\mathbb{R}) \to \mathcal{S}(\pi_1^{\text{ét}}(X/\mathbb{R})) = \mathcal{S}^{\text{ét}}(X/\mathbb{R})$$

*is a bijection.*

*Proof.* By [43, Lemma 228] it suffices to show that the existence of a section $s \in \mathcal{S}(\pi_1^{\text{ét}}(X/\mathbb{R}))$ implies that $X(\mathbb{R}) \neq \emptyset$. But since, by Theorem 2.3.12 and Corollary 2.2.7, $\mathcal{S}(\pi_1^{\text{ét}}(X/\mathbb{R})) = \mathcal{S}^{\text{ét}}(X/\mathbb{R})$, any such $s$ induces a section $s_2^\wedge \in \mathcal{S}_2^{\text{ét}}(X/\mathbb{R})$ via the map of 2.3.8 (for $\Sigma = \{2\}$). Thus $X(\mathbb{R}) \neq \emptyset$ by Corollary 3.3.14. □

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