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First-order thermodynamics of modified gravity

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Zusammenfassung

Einsteins Allgemeine Relativitätstheorie ist die erfolgreichste Gravitationstheorie, die je formuliert wurde, und sie wurde mit herausragender Präzision bereits auf einer Vielzahl von Skalen getestet. Das darauf basierende kosmologische Standardmodell erfordert jedoch eine unbekannte dunkle Energie, die als fein abgestimmte kosmologische Konstante modelliert wird, um die derzeitige beschleunigte Expansion des Universums zu erklären. Da die Gravitation auf großen kosmologischen Skalen experimentell nicht gleich gut erforscht ist wie in unserem Sonnensystem, bieten modifizierte Gravitationstheorien gute Alternativen. Auch jenseits der Kosmologie ist die wahre Natur der Schwerkraft weiterhin rätselhaft. Die Feldgleichungen von Gravitationstheorien lassen sich zum Beispiel als Zustandsgleichungen aus rein thermodynamischen Überlegungen ableiten. Dies führt dazu, dass die Allgemeine Relativitätstheorie mit einem Gleichgewichtszustand der Gravitation und die modifizierte Gravitation mit einem Nicht-Gleichgewichtszustand identifiziert werden kann. In dieser Doktorarbeit stellen wir einen neuen Ansatz für die Thermodynamik der modifizierten Gravitation vor, der eine konkrete Umsetzung dieser Idee ermöglicht. Durch die Anwendung einer thermodynamischen Beschreibung des Nicht-Gleichgewichts auf das effektive Fluid, welches die Skalar-Tensor-Gravitation beschreibt, entsteht auf natürliche Weise eine "Thermodynamik der Gravitationstheorien". Anwendungen dieses Modells auf die Kosmologie sowie Erweiterungen auf verschiedene Klassen modifizierter Gravitationstheorien und die Formulierung zweier komplementärer Beschreibungen basierend auf Temperatur und chemischem Potenzial zeichnen ein vereinheitlichendes Bild der Landschaft der Gravitationstheorien.

Abstract

Einstein's General Relativity is the most successful theory of gravity ever formulated and it has been tested to outstanding precision on a wide range of scales. However, the standard cosmological model based on it requires an unknown dark energy, modelled as a fine-tuned cosmological constant, to explain the universe's current accelerated expansion. Since gravity is not as well-tested on large cosmological scales as within our Solar System, modified gravity theories are a valid alternative. Even beyond cosmology, the true nature of gravity remains elusive. For example, the field equations of gravity theories can be derived as equations of state from purely thermodynamical considerations. This leads to identifying General Relativity with an equilibrium state of gravity and modified gravity with a non-equilibrium one. In this thesis, we present a novel approach to the thermodynamics of modified gravity which provides a concrete realisation of this idea. Applying a non-equilibrium thermodynamical description to the effective fluid describing scalar-tensor gravity, a "thermodynamics of gravitational theories" naturally emerges. Applications of this framework to cosmology, extensions to different classes of modified theories, and the formulation of two complementary descriptions based on temperature and chemical potential sketch a unifying picture of the landscape of gravity theories.

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Publication List

This thesis is based on the following publications:

[1] S. Giardino, V. Faraoni, and A. Giusti. "First-order thermodynamics of scalar-tensor cosmology". *JCAP* 04.04 (2022), p. 053. DOI: 10.1088/1475-7516/2022/04/053. arXiv: 2202.07393 [gr-qc].

[2] V. Faraoni, A. Giusti, S. Jose, and S. Giardino. "Peculiar thermal states in the first-order thermodynamics of gravity". *Phys. Rev. D* 106.2 (2022), p. 024049. DOI: 10.1103/PhysRevD.106.024049. arXiv: 2206.02046 [gr-qc].

[3] V. Faraoni, S. Giardino, A. Giusti, and R. Vanderwee. "Scalar field as a perfect fluid: thermodynamics of minimally coupled scalars and Einstein frame scalar-tensor gravity". *Eur. Phys. J. C* 83.1 (2023), p. 24. DOI: 10.1140/epjc/s10052-023-11186-7. arXiv: 2208.04051 [gr-qc].

[4] A. Giusti, S. Giardino, and V. Faraoni. "Past-directed scalar field gradients and scalar-tensor thermodynamics". *Gen. Rel. Grav.* 55.3 (2023), p. 47. DOI: 10.1007/s10714-023-03095-7. arXiv: 2210.15348 [gr-qc].

[5] S. Giardino, A. Giusti, and V. Faraoni. "Thermal stability of stealth and de Sitter spacetimes in scalar-tensor gravity". *Eur. Phys. J. C* 83.7 (2023), p. 621. DOI: 10.1140/epjc/s10052-023-11697-3. arXiv: 2302.08550 [gr-qc].

[6] S. Giardino and A. Giusti. "First-order thermodynamics of scalar-tensor gravity". *Ricerche Mat.* (2023). DOI: 10.1007/s11587-023-00801-0. arXiv: 2306.01580 [gr-qc].

[7] M. Miranda, S. Giardino, A. Giusti, and L. Heisenberg. "First-order thermodynamics of Horndeski cosmology". *Phys. Rev. D* 109.12 (2024), p. 124033. DOI: 10.1103/PhysRevD.109.124033. arXiv: 2401.10351 [gr-qc].

During the course of my PhD, I also worked on the following publication, which is not included in this thesis:

[8] L. Bohnenblust, S. Giardino, L. Heisenberg, and N. Nussbaumer. "To bounce or not to bounce in generalized Proca theory and beyond". (*In preparation.*)

Wer immer strebend sich bemüht,
Den können wir erlösen

Faust II
Johann Wolfgang von Goethe

Contents

Chapter 1

Introduction

1.1 Exploring the true nature of gravity

Out of the four fundamental forces of Nature, gravity has definitely something distinctive about it. It is by far the weakest, being tens of orders of magnitude weaker than the weak force. It becomes strong only at the Planck scale, which is, so far, experimentally inaccessible. This weakness translates to an unfathomably long range: gravity becomes single-handedly responsible for the evolution of our universe emerging from its early stages, and shapes the cosmic evolution. Gravity dictates the evolution of stars, galaxies, clusters and superclusters of galaxies, and the planetary systems just like our own that are born within them. Its all-encompassing reach includes both our everyday experience on Earth and the largest cosmological scales. Gravity cannot be escaped and it cannot be shielded. Even the weightlessness that can be experienced in space is not due to the absence of gravity, but instead to the manifestation of it.

Despite its importance, our understanding of gravity still presents a number of puzzles that make it less understood than the other forces at a fundamental level. More than 100 years after it was first formulated, Einstein's theory of General Relativity (GR) still provides the best description of the gravitational interaction. This elegant theory has been relentlessly tested in a wide range of settings and with remarkable precision, from the sub-mm scale in laboratory settings, to within our Solar System, and on increasingly large cosmological scales through its effects on the large-scale structure of the universe. In many recent scientific breakthroughs, such as the spectacular detections of gravitational waves, or the first ever image of a black hole, General Relativity always keeps center stage as one of the most successful theories in the history of science.

General Relativity is the best theory of gravity that we have at hand, yet it is not free from shortcomings. Its problems can be characterized as belonging either to the high-energy ultraviolet (UV) regime, or the low-energy infrared (IR) one. In the UV, by far the deepest and most fundamental issue with GR is the fact that it is not a renormalizable theory: while all other three fundamental interactions (electromagnetism, the weak force and the strong force) can be successfully unified in the Standard Model of Particle Physics, which can boast the most accurate predictions in the whole of physics (most famously, the prediction of the electron magnetic moment, accurate to one part in 10^{12} [9]), gravity cannot be made to fit into this framework. Unfortunately, for all its unifying power and its elegance, the Standard Model cannot account at all for some of the phenomena which are closest to our everyday experience, namely those of gravitation.

In the IR, the problems of GR show up in the realm of cosmology, which, with its unfathomably large scales, constitutes the setting at the lowest possible energy that is accessible to us. At the galactic scale, an unknown type of matter, called dark matter, must be introduced to make sense of the observed rotation curves of galaxies. The latter appear to be surrounded by halos of such mysterious matter, which interacts gravitationally but is completely inert with respect to all other forces. On cosmological scales, the even more puzzling dark energy needs to be postulated in order to account for the accelerated expansion of the universe, which became momentously evident around the turn of the millennium.

This separation of UV and IR scales, while necessary and useful for practical purposes, remains at the core of gravity's problems. Dark matter, dark energy and also cosmic inflation (a stage of quasi-exponential expansion postulated to occur after the Big Bang in order to solve some important problems) all do not have viable explanations within the framework of the Standard Model of Particle Physics based on Quantum Field Theory (QFT), and efforts to address these puzzles generally propose solutions that go beyond this model.

These missing pieces in our understanding of gravity based on General Relativity keep eluding the efforts of the physics community, despite many decades of research and the large number of proposed solutions for them. It is therefore natural to ask whether these puzzles might be solved, or at least these problems alleviated, by slightly modifying the laws of gravity so compellingly formulated by Einstein. Of course, such modifications need to still be consistent with the wealth of observational tests that GR has passed with flying colors since its inception. The many attempts at such modifications of gravity generally cluster around a few main ideas:

- adding new fields to the tensor gravitational degree of freedom that describes gravity, such as scalar, vector or additional tensor degrees of freedom. In section 1.2, we explore many of these theories and then focus on the first option, namely scalar-tensor theories, for the main body of this thesis;

- adding higher-order curvature invariant terms to the gravitational action. These terms generally arise as higher-order quantum corrections to GR and are therefore often used in a quantum gravity setting, in addition to a cosmological one. We explore them in 1.2.2;

- modifying the geometry of GR, for example through ascribing gravitational effects to torsion or non-metricity instead of curvature, and then extending these theories equivalent to GR by including higher-order terms as in the previous point. We briefly review these approaches in section 1.2.3;

- breaking some fundamental assumption of GR, such as the four dimensions of spacetime, Lorentz invariance, locality, or the masslessness of the graviton. The latter of these options is shortly reviewed in section 1.2.3.

The plethora of modified gravity theories available is sprawling. For this reason, it is desirable to find some unifying idea or formalism that makes it possible to embed several theories into a single framework. Not only does this answer to the theoretical physicists' deep need for the beauty and elegance of comprehensive descriptions, but it is also practically useful, since understanding GR as one special case in a broader framework can help shift our perspective and maybe view its problems from an angle that offers a glimpse of solution. While this is certainly a much too ambitious aim for this thesis, it is the big-picture idea that has guided the efforts presented here. A concept that will often come up in the following is that of a vast landscape of gravity theories, with General Relativity occupying a special role at its centre, and different modified gravity theories populating it in all directions. The main goal of this thesis is to construct a tool to chart this landscape, allowing different gravity theories to be seen through a single, unifying lens.

The main topic of the present work is a novel formalism called "first-order thermodynamics of modified gravity". It was first developed in [10, 11], extended in [1–5, 7, 12–18] and succintly reviewed in [6]. This proposal is based on the observation that the contribution of the scalar field to the field equations of scalar-tensor gravity can be described as an effective dissipative fluid, through a simple rewriting of the equations that does not entail extra assumptions [19, 20]. The novelty of this approach comes in when we apply a non-equilibrium thermodynamical description to this fluid. We choose the one developed by Eckart [21], entailing constitutive relations that are first-order in the dissipative variables and, despite problems related to causality that stem from this simplistic assumption, is still one of the most widely used models of dissipative thermodynamics. Giving a thermodynamical interpretation to the effective scalar fluid leads to the identification of its temperature, a sort of "temperature of scalar-tensor gravity", which is nothing but a temperature relative to GR, in addition to its bulk and shear viscosity coefficients. This temperature is the order parameter ruling the approach to equilibrium and it is positive definite for theories containing a scalar degree of freedom in addition to the two tensor ones of GR: Einstein's theory corresponds to the zero-temperature state in this "thermodynamics of gravitational theories", and all theories with a dynamical scalar degree of freedom have a positive, non-zero temperature. Dissipation corresponds to the relaxation of the effective fluid toward the GR state of equilibrium. This approach fits into the wider context of trying to gain physical intuition through an effective fluid description for theories involving complicated derivative self-interaction terms in their Lagrangians (see, for example, [22, 23]).

Our approach is inspired by and echoes the ideas of [24, 25], but follows a starkly different path. These previous works derived both Einstein's equations and the field equations of $f(R)$ gravity (a subclass of scalar-tensor theories [26]) as equations of state from purely thermodynamical considerations, leading to the identification of GR with an equilibrium state of gravity and modified gravity with a non-equilibrium one. These results made the interesting relationship between gravity and thermodynamics, originally explored in the context of black holes, even more intriguing. However, they left open the crucial questions of a precise description of the approach to equilibrium and the order parameter governing it, which our approach addresses.

This thesis is structured as follows: in the rest of the present chapter, we introduce the context of our results, further detailing the shortcomings of Einstein gravity (sections 1.1.1 and 1.1.2) in order to motivate the exploration of modified gravity theories. We provide some background knowledge on the most promising classes of modified theories, such as scalar-tensor theories in section 1.2.1 (mainly used for cosmological applications) and higher-order theories (also used in the context of quantum gravity) in section 1.2.2. Afterwards, we take a detour from modified gravity by exploring the intriguing relationship that exists between gravity and thermodynamics in section 1.3. In chapter 2, we first present the effective fluid formalism that is the basis of first-order thermodynamics and then explore the non-equilibrium description needed to implement our ideas. In chapter 3, we present the basics of the first-order thermodynamics of scalar-tensor theories, and define the crucial notion of effective temperature, additionally exploring how these basic results are affected by the choice of a past-directed scalar gradient. In chapter 4, we present our main results, namely the numerous extensions and applications of the thermodynamical formalism. Starting with the extension to Horndeski theories, we continue mapping the landscape by studying equilibrium states alternative to GR, corresponding to theories with non-dynamical scalar fields and stealth solutions. We also sketch an alternative and equivalent formulation of first-order thermodynamics, based on chemical potential instead of temperature, that can be applied to theories in the Einstein frame. Finally, we apply the formalism to the fruitful arena of cosmology, both for "old-school" scalar-tensor theories and Horndeski theories, and test the formalism on exact solutions of these classes, deriving some interesting physical implications. We close with some conclusions and an outlook on future research developments in 5.

We adopt the notation of Ref. [27]: the metric signature is $(-,+,+,+)$ and we use units in which the speed of light c and Newton's constant G are unity, unless differently stated.

1.1.1 UV regime: the quest for quantum gravity

In the modern perspective, General Relativity is a low-energy effective field theory, valid up to the Planck scale, which lacks a consistent UV completion and needs to be replaced at high energies by a full-fledged theory of quantum gravity. Within the formalism of Effective Field Theory (EFT), we are able to make predictions at the energies currently accessible to us, without making unwarranted assumptions about the theory's behaviour at high energies. Despite several paradigms that have enjoyed some success because of their apparent promise in the past decades, there is currently no universally accepted and fully consistent theory of quantum gravity. However, the effective field theory of gravity predicts its own breakdown at high energies or large curvature, such as around black holes and at the beginning of the universe, where quantum effects must come into play and cannot be ignored anymore. GR is understood, in the weak gravity regime, as the unique theory of a spin-2 particle, the graviton. This description is not valid for arbitrarily high energies or arbitrarily small length scales, as GR is non-renormalizable and the effective field theory description breaks down. This means that we cannot remove the infinities arising in the calculations of graviton scattering, since there is an infinite number of them, and the standard techniques that were successful for all other interactions are not helpful. The problem can also be seen as a problem of scales: GR spans the largest possible cosmological scales, and incurs into trouble at smaller distances.

Arguably the first physicist to grasp the problem with the quantization of gravity was Bronstein [28]. Already in 1936, he argued that gravity is in one important way fundamentally different from all other forces: it does not allow an arbitrarily high concentration of charge, meaning energy, in a tiny spacetime region, since this will unavoidably collapse to form a black hole. This is the first understanding of the existence of a minimum length that is taken into account in the so-called Generalized Uncertainty Principle, and seems to be a feature shared by various approaches to quantum gravity (see [29] for a review). This simple fact poses an unsurmountable and fundamental constraint to all our efforts of probing the UV regime of gravity with any type of experimental apparatus, and further distinguishes gravity from the other interactions. The ambiguity of gravity is also the fact that it is encoded into the curvature of spacetime, while standard QFT is formulated on a flat background: the gravitational field describes simultaneously the gravitational degrees of freedom and the background spacetime on which these degrees of freedom live.

Several approaches to quantum gravity have been explored over the decades, from the early one of geometrodynamics, to the more fashionable string theory, loop quantum gravity, the asymptotic safety program, causal dynamical triangulations etc. We will not go into the details of any of these, see [30] for a comprehensive review. In the currently available low-energy regime, GR can be generalized by introducing into its action some higher-order corrections that contain various combinations of curvature invariants. These lead, at the one-loop level, to a consistent and renormalizable theory. Such theories are a broad class of modified gravity theories with rich cosmological applications, and we discuss them in section 1.2.2. Some progress in the direction of a quantum description have been made in the context of black holes, using a semi-classical description based on QFT in curved spaces, finding the surprising result that black hole are thermal objects and emit Hawking radiation. We review black hole thermodynamics in section 1.3.

In recent years there have been increasing efforts in quantum gravity phenomenology, see for example the review [31]. Gravity at such small scales is often investigated with table-top experiments in the regime of quantum optics, but also astrophysics and cosmology with their increasingly sensitive and extensive surveys strengthen the hope for some phenomenological input, that might help make progress in quantum gravity in the near future. There is a remarkable diversity in the approaches to and opinions on quantum gravity by practitioners of the field (many of them can be sampled in [32]). The fact that each quantum gravity community speaks a completely different language makes it difficult to strive towards a common goal and recognise the similarities between different approaches, rather than being overwhelmed by the differences. Promising novel perspectives on this problem involve, for example, the efforts to find the overlap between different approaches and try to keep as agnostic as possible.

Cosmology encompasses both of the problematic regimes of GR: the Big Bang is by far the highest-energy regime we can imagine, but the low-energy realm of today's accelerating universe is all but free from problems. After sketching a rough picture of the quantum gravity conundrum, we elaborate extensively in the following on the cosmological IR problems, that are most commonly addressed with modified gravity.

1.1.2 IR regime: the standard model of cosmology and its discontents

The standard cosmological model, called ΛCDM, is based on the application of GR to the whole universe, together with the cosmological principle, which characterizes our universe as homogeneous and isotropic. Λ stands for the cosmological constant, a term in the Einstein equations that is the simplest candidate deemed responsible for the observed accelerated expansion of the universe, while CDM is the acronym for Cold Dark Matter, which needs to be cold to only interact gravitationally. These two components, making up a combined 75% of the energy content of the universe, remain unknown in standard cosmology, based only on GR and the cosmological principle. Moreover, we have no knowledge of what happened at the beginning of the universe, since a cosmic singularity is encountered and GR signals its own demise. Another unknown ingredient, the inflaton scalar field, is necessary for the phase of quasi-exponential expansion just after the beginning of the universe, cosmic inflation, which is supposed to solve several problems of the Big Bang model. Despite the great agreement with observations coming from a wide range of probes, ΛCDM presents several problems: apart from some recent tensions in the data, the most puzzling of them is by far the cosmological constant problem, that we explore in detail in the following.

For convenience, we recall here some basics of General Relativity and the Friedmann-Lemaître-Robertson-Walker (FLRW) background cosmology to set the notation needed for the next sections. The most general low-energy diffeomorphism-invariant action for the gravitational field, the Einstein-Hilbert action, includes the cosmological constant Λ (we make G explicit for the remainder of this section)

$$S_{EH} = \frac{1}{16\pi G} \int d^4\sqrt{-g}(R - 2\Lambda) + S_m, \tag{1.1}$$

where g is the metric determinant, R is the Ricci scalar and S_m is the matter action (we introduce G here for convenience). If $\Lambda = 0$, the Einstein equations are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.2}$$

The FLRW metric reads

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right], \tag{1.3}$$

where $a(t)$ is the scale factor, k is the curvature of the universe and $d\Omega^2 \equiv d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$. The Einstein equations for this metric are the Friedmann equations

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{1.5}$$

The stress-energy tensor of a perfect fluid reads

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{1.6}$$

where ρ is the energy density, P the pressure and u_μ its 4-velocity. The fluid respects the continuity equation

$$\dot{\rho} + 3H(P + \rho) = 0 \tag{1.7}$$

and its equation of state is $w \equiv \frac{P}{\rho}$. With a non-zero cosmological constant, the Einstein equations are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.8}$$

The Friedmann equations become

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{1.9}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{1.10}$$

It is practical to introduce the dimensionless density parameters Ω_i, expressing the energy densities of different components in the universe in units of the critical density

$$\rho_{\text{crit}} \equiv \frac{3H^2}{8\pi G} \tag{1.11}$$

The Ω-parameters for matter and radiation at the present epoch are in general given by

$$\Omega_{i0} \equiv \frac{\rho_{i0}}{\rho_{\text{crit}}} = \frac{8\pi G}{3H_0^2}\rho_{i0}. \tag{1.12}$$

In the same way, we also introduce a density parameter Ω_Λ for the cosmological constant, as well as an Ω_k for the curvature term. The matter energy density Ω_{m0} includes the contributions of baryonic matter, cold dark matter and massive neutrinos. The radiation density Ω_{r0} is negligible at the present epoch, but dominated in the early universe. The sum of the different density parameters is unity by construction $\Omega_{r0} + \Omega_{m0} + \Omega_\Lambda + \Omega_K = 1$, and therefore the Friedmann equation (1.9) can be also alternatively written as

$$H^2 = H_0^2 \left(\Omega_{r0} a^{-4} + \Omega_{m0} a^{-3} + \Omega_\Lambda + \Omega_K a^{-2} \right) \tag{1.13}$$

$$H^2 = H_0^2 \left(\Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 + \Omega_\Lambda + \Omega_K (1+z)^2 \right), \tag{1.14}$$

where the redshift z is defined by $a = 1/(1+z)$ and H_0 is the Hubble parameter at the present day.

Observational evidence for accelerated expansion

Type Ia Supernovae The first evidence of the universe's accelerated expansion dates back to the late 1990s and it was provided independently by two groups, the High-z Supernova Search Team [33] and the Supernova Cosmology project [34], using the observations of type Ia supernovae (SNIa). Supernova explosions are triggered when a white dwarf that has been accreting mass from a companion star in a binary system exceeds the Chandrasekhar limit of about 1.4 solar masses. Supernovae Type Ia are a specific class that does not exhibit hydrogen lines, but shows an absorption line of singly-ionized silicon in their spectrum. The explosion is an extremely violent event that causes a burst of radiation, and thus a strong peak in brightness, that is easily detectable, often making up a decent fraction of the luminosity of the star's host galaxy. SNIa are very suitable candidates to measure distances because their peak absolute luminosity is approximately constant across all supernovae, which means that they can be regarded as standard candles. However, in practice there are some complications in this picture: SNIa are best referred to as standardizeable candles, since their peak luminosity is not exactly the same, but strongly correlates with the shape of their light curves: brighter SNIa decline more slowly than dimmer ones. This causes scatter in the measurements which needs to be adjusted for, in order to measure distances. In an expanding universe, the luminosity distance as a function of redshift is

$$\mathrm{d_L}(z) = (1+z) \cdot \int_0^z \frac{\mathrm{d}z'}{H\left(z'\right)}. \tag{1.15}$$

The absolute magnitude M of an object in terms of its apparent magnitude m and the luminosity distance d_L is

$$m - M = 5 \log_{10} d_L + 25, \tag{1.16}$$

for distances expressed in Mpc. This means that the absolute magnitude corresponds to the apparent magnitude that the object would have if positioned at $d_L = 10\,\mathrm{pc}$. By independently measuring $d_L(z)$ and z, the history of the cosmic expansion can be reconstructed through $H(z)$. At the end of the 1990s, the availability of a high-quality sample of local (i.e., $z \ll 1$) SNIa allowed to understand the correlation between their absolute magnitude and light curve width, so that the absolute magnitude could be inferred by observing the apparent magnitude and the light curve. After these adjustments, all the supernovae have the same absolute magnitude M if they are standard candles, and all remaining differences in apparent magnitude m are attributed to their distance.

In the presence of an accelerated expansion, the luminosity distance becomes larger. Observations of SNIa measured the luminosity distance to be significantly larger than for a flat matter-dominated or a curvature-dominated Universe with the same Hubble constant. In other words, the most distant SNIa appear dimmer than one would expect in a matter-dominated universe, and this, after ruling out other causes, is best explained in a cosmological model describing a universe undergoing accelerated expansion, that has started recently. We remark that this finding *assumes* GR and a FLRW universe, therefore *a priori* not excluding the possibility that there might be something wrong with our understanding of gravity or with our assumptions of homogeneity and isotropy [35]. Naturally, since 1998, additional and more exhaustive supernova data have confirmed these findings. Some of the most recent and comprehensive SNIa surveys include the Pantheon+ sample [36] and other unified datasets [37].

Age of the universe Even before the evidence from supernovae, a discrepancy was found between the age of the universe that could be calculated without knowing about accelerated expansion and the age of the oldest observed stars. A simple calculation of the age of the universe t_0 for a FLRW universe containing no dark energy yields $8.2\,\mathrm{Gyr} < \mathrm{t}_0 < 10.2\,\mathrm{Gyr}$, for $h = 0.72 \pm 0.08$ (where h quantifies the uncertainty on the value of H_0) but the ages of some globular clusters in the Milky Way had been found to be, for example, $12.9 \pm 2.9\,\mathrm{Gyr}$ or $13.5 \pm 2\,\mathrm{Gyr}$ (see [38] and references therein). This discrepancy can of course be explained by an accelerating universe, but also in an open $k = 1$ universe. However, since the curvature of the universe has been constrained to be very close to $k = 0$ (for example through CMB measurements [39]), the presence of a cosmological constant (or dark energy with an equation of state $w_{DE} = -1$) is crucial in order to solve this problem.

Evidence for the cosmic acceleration that is complementary to SNIa results comes from the Cosmic Microwave Background (CMB) radiation and other probes. Not only do these other measurements independently confirm the results, but they also allow to break the degeneracy that is inevitably present between some cosmological parameters, especially the matter density Ω_{m0} and dark energy density $\Omega_{\Lambda 0}$, since the SNIa data alone allow for different combinations of these two parameters [40].

Often, such complementary measurements employ standard rulers instead of standard candles and focus on extracting the angular diameter distance $d_A(z)$, given by the ratio of the object's actual size to its angular size, from which $H(z)$ can be found. For a cautionary note about what dark energy measurements actually measure from these probes, see [35].

Cosmic Microwave Background The study of CMB radiation anisotropies does not provide very relevant information about dark energy on its own, but constitutes one of these complementary probes [41]. The early, radiation-dominated phase of the universe shortly after the Big Bang was characterized by the presence of a charged photon-baryon plasma, since photons were energetic enough to ionize hydrogen atoms. This plasma was opaque, as photon had short mean free paths and light could not travel freely. As the universe expanded and the temperature decreased, specifically around $z \sim 1090$, neutral atoms could form for the first time (the so-called time of recombination) and photons were able to decouple from the plasma and start their free streaming. This is the time when the CMB radiation originated, and its blackbody spectrum reached us as the first propagating light of the universe. On the one hand, the fluctuations we observe on the largest angular scales are the primordial ones, since the largest scales have only recently entered the horizon. On the other hand, on smaller scales we instead observe fluctuations which entered the horizon before decoupling happened. These fluctuations have been affected by the acoustic oscillations characteristic of the photon-baryon fluid (perturbations propagate as sound waves inside this plasma), which are the origin of the peaks in the CMB power spectrum. These acoustic oscillations are the damped oscillations of the pressure waves propagating in the photon-baryon plasma, which experiences the competing effects of radiation pressure and gravity. The oscillation of sound waves affect both photons and baryons: not only do they cause the peaks in the CMB spectrum, but they also imprint a characteristic scale onto the matter power spectrum, as detailed in the following. The first peak in the CMB power spectrum is related to the maximum amplitude of oscillations, reached by scales that entered the horizon at the time of last scattering, just before the decoupling started, at scale $R \sim H_{ls}^{-1}$. The location of the first and higher peak is then at an angular size corresponding to the size of the horizon at the time of last scattering.

The *physical* size of the horizon at this time is known, and so we can use it as a ruler, because the *angular* size will depend on the geometry of the universe. The agreement between the theoretical prediction for the position of the first peak in the CMB spectrum (calculated for a flat FLRW geometry) and its actual location gives very strong indications for the flat geometry of the universe. The CMB also strongly constraints the sum of matter and dark energy densities, in a complementary way to SNIa. If then one combines this with data about the matter distribution in the universe, the degeneracy can be broken and the dark energy parameter singled out. For example, through constraining the parameter σ_8 with galaxy cluster abundance, one obtains information about the matter density.

Moreover, information about dark energy in the CMB comes not only from the peaks, but also from the plateau observed at large angular scales. In this plateau, the so-called Integrated Sachs-Wolfe (ISW) effect is visible and constitutes a relevant probe of the expansion history, although it represents a small contribution to the total CMB signal and is not easy to single out. This effect comes from the variation of the gravitational potential that occurs during a phase of accelerated expansion, while in a matter-dominated phase these potentials would stay constant. The photons feel the effect since they gain energy when entering the potential well and lose it when exiting it. If the potentials vary during an accelerated expansion phase, photons will be blueshifted if the potential becomes more shallow and redshifted if it becomes deeper. Note that the anisotropies caused by this effect are called *secondary* since they intervene after the photons have left the surface of last scattering, while *primary* anisotropies are those imprinted on this same surface at the decoupling time. This effect can be singled out by correlating the CMB power spectrum with the matter power spectrum of the galaxy, since the ISW effect of course depends on the matter distribution exerting the gravitational potential. Recent observations of the ISW effect are compatible with the presence of a cosmological constant, but also provide an avenue to constrain dynamical dark energy models in the future (see, e.g. [42]).

Baryon Acoustic Oscillations The acoustic oscillations in the baryon-photon plasma additionally leave imprints on the matter power spectrum, called baryon acoustic oscillations (BAO). The specific length scale imprinted by the sound waves causes a clustering of matter at comoving length of $100h^{-1}$Mpc. Since galaxies trace the matter content of the universe, this scale is exhibited as a single localized enhancement of the galaxy correlation function and an oscillatory feature in the galaxy power spectrum, visible in the large scale structure. This length scale provides a standard ruler, completely independent from SNIa and complementary to them.

Because the BAO scale is known in absolute units (based on parameter values measured from the CMB), BAOs measure distances in absolute units, i.e., Mpc. Therefore, BAO and SNIa measurements at the same redshift carry different and complementary information [43]. BAOs provide an important cosmological probe, very sensitive to the cosmic expansion history and with systematics that are well under control.

Very recently, the Dark Energy Spectroscopic Instrument (DESI) collaboration published their results regarding BAO, which offer interesting up-to-date constraints on dark energy [44]. Unsurprisingly, the measurements collected so far by DESI are in good agreement with the simple flat ΛCDM scenario with dark energy being a cosmological constant, but given the precision of this new data, it is possible to test other dark energy models departing from this simple scenario and constrain them with unprecedented accuracy. In general, this is done through the introduction of a dark energy equation of state parameter, $w(a) \equiv \frac{P}{\rho}$, in addition to the dark energy density Ω_{DE}, therefore treating dark energy as an effective fluid (see section 2 for more details). In the early 2000s, the time variation of $w(a)$ was inaccessible to observations, and studies often kept w = const., which goes under the name of wCDM model. However, this clearly introduced some bias, making the analysis insensitive to possible variations of this parameter that the data might exhibit. Of course, measuring $w = -1$ when assuming a constant w does not provide the full picture if it indicates the cosmological constant scenario as correct. For this model, the tightest constraints are obtained from the combination of DESI BAO data with the SNIa dataset Pantheon+ [36] and CMB data from Planck [39], and read

$$\Omega_{m0} = 0.3095 \pm 0.0069 \qquad w = -0.997 \pm 0.025. \tag{1.17}$$

These data do not show any evidence for a constant equation of state parameter different from $w = -1$ when assuming a flat wCDM model.

Nevertheless, the physical dynamics of the dark energy effective fluid can be better taken into account by introducing the so-called Chevalier-Polarski-Linder (CPL) parametrization, namely $w(a) = w_0 + w_a(1 - a)$, which matches the background dynamics of the dark energy fluid quite accurately across a variety of dark energy models [45]. This model reduces to ΛCDM for $w_0 = -1$, $w(a) = 0$. BAO distance measurements are particularly important in constraining such extensions to the standard cosmological model, because they help break geometric degeneracies that limit the constraining power of the CMB [44]. For the CPL parametrization, DESI data favour solutions with $w_0 > -1$ and $w(a) < 0$. The combination of DESI and CMB data provides

$$w_0 = -0.45^{+0.34}_{-0.21}, \qquad w_a = -1.79^{+0.48}_{-1.00}, \tag{1.18}$$

indicating a $\sim 2.2\sigma$ difference with respect to ΛCDM. When combining also the data from SNIa, the preference is, in general, for $w_0 > -1$ and $w(a) < 0$. The level of tension with ΛCDM varies depending on which specific datasets are used, indicating that these results must be interpreted with caution. Specifically, the tension remains around $\sim 2.5\sigma$ level for combining DESI, CMB and Pantheon+ data, but increases to around 3.5σ and more for other SNIa datasets (that are still consistent with each other), such as Union3 [37]. At the time of writing, it is too early to say whether the presence of this slight tension means something significant for the scenarios beyond ΛCDM and indicates that dark energy might not be a cosmological constant after all. Inconsistencies between datasets and sources of systematic uncertainties might still have been unaccounted for, but, nonetheless, such Stage-IV surveys (also including the Rubin Observatory's Legacy Survey of Space and Time (LSST) [46], the ESA mission Euclid [47], the Roman Space Telescope from NASA [48]) hold the intriguing promise to get us closer to a deeper understanding of the source of cosmic acceleration.

The cosmological constant problem

The simplest candidate for dark energy is a cosmological constant (see [49, 50] for reviews), a form of energy that remains exactly the same across space and time, even as the universe expands and all other forms of energy get diluted. All local tests of GR are compatible with a zero cosmological constant, and it shows up only at large cosmological scales in the present cosmic epoch. The cosmological constant has a tiny energy density, that is negligible during the whole history of the universe until about our present epoch, when it starts dominating as all the other components have decayed. This dark energy candidate is not satisfactory, as we explore in the following. Nonetheless, even if we decided to accept the cosmological constant as the origin of the universe's accelerated expansion, it would still be important to keep an open mind to alternative explanations, understand their advantages, and to find convincing ways to possibly rule them out. The existence of dark energy and dark matter, constituting the most abundant components of the universe, has only been inferred through their gravitational interaction on large scales. Therefore, their very existence relies on a precise understanding of gravitational effects at those scales: our faulty grasp of these effects might well be the origin of these components.

The cosmological constant has $\rho_\Lambda = -P_\Lambda$ or $w_\Lambda = -1$ and its effects show up as a form of repulsive gravity, or anti-gravity. Following [49], this can be understood from the spatial part $\mathbf{g}$ of the geodesic acceleration (measuring the relative acceleration of two geodesics in spacetime). In GR, $\mathbf{g}$ satisfies the following equation

$$\nabla \cdot \mathbf{g} = -4\pi G(\rho + 3P), \tag{1.19}$$

showing that the source of geodesic acceleration is actually $(\rho + 3P)$ and not simply ρ. As long as $(\rho + 3P) > 0$, gravity remains attractive, while $(\rho + 3P) < 0$ can lead to repulsive gravitational effects. Since for the cosmological constant $(\rho_\Lambda + 3P_\Lambda) = -2\rho_\Lambda$, a $\Lambda > 0$ causes repulsive gravity. Take a simple universe made up only of a cosmological constant and pressureless, non-relativistic matter with $\rho_{\rm NR}$. According to (1.19), the geodesics will accelerate away from each other when $\rho_{\rm NR} < 2\rho_\Lambda$ due to the cosmological constant's repulsive effects. In an expanding universe, ρ_Λ stays constant while $\rho_{\rm NR}$ decreases. Therefore, ρ_Λ will eventually dominate over $\rho_{\rm NR}$ if the universe expands sufficiently.

Einstein's unstable universe When Einstein derived his theory of gravity in 1915, the fact that the universe is expanding was not known. Hence, Einstein thought, like the physics community at this time, that the universe must be in a steady state, and was interested in finding static solutions to his equations. This is the reason why he introduced the cosmological constant in 1917. The Friedmann equations (1.9) and (1.10) do admit a static solution, but it is unstable. We will briefly review this solution and its instability, following [51]. A static solution requires $\ddot{a} = 0$, which produces

$$\Lambda = 4\pi G\rho_m, \tag{1.20}$$

(where the subscript $_m$ indicates the matter quantities) thus such a solution is possible in a universe filled with pressureless matter if the cosmological constant is fine-tuned to match the matter density in this way. Additionally, a static solution also requires $\dot{a} = 0$ as well. With (1.20), the Friedmann equation (1.9) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda - \frac{k}{a^2}, \tag{1.21}$$

which tells us that the static universe needs to be positively curved ($k > 0$). This fine-tuned value of Λ required for a static universe of course renders the solution unstable, as we can easily check by considering a perturbed matter density

$$\rho_m = \rho_{m,\star}(1+\delta), \quad |\delta| \ll 1, \tag{1.22}$$

where $\rho_{m,\star} = \Lambda/4\pi G$ is the density of a static universe. The matter density evolves as a^{-3} in this universe, so that $\rho_m = \rho_{m,\star}a^{-3}$. Comparing the two previous equations, we can read off the density perturbation in terms of the scale factor as $\delta = a^{-3} - 1$. We can see that if δ is much smaller than unity, then a must also differ from the static value only by a small amount,

$$a(t) = 1 + \varepsilon(t), \quad \varepsilon \ll 1. \tag{1.23}$$

The Friedmann equation (1.10) can be written as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_{m,\star}a^{-3} + \frac{\Lambda}{3} \tag{1.24}$$

$$= \frac{4\pi G\rho_{m,\star}}{3}\left(1 - a^{-3}\right) \tag{1.25}$$

Using the perturbative expansions above, we find

$$\ddot{a} = \ddot{\varepsilon} = \frac{\Lambda}{3}\left(a - a^{-2}\right) \approx \frac{\Lambda}{3}(1 + \varepsilon - (1 - 2\varepsilon)) \approx \Lambda\varepsilon. \tag{1.26}$$

This equation has growing and decaying solutions,

$$\varepsilon(t) = c_1 e^{\sqrt{\Lambda}t} + c_2 e^{-\sqrt{\Lambda}t}. \tag{1.27}$$

The growing mode will grow exponentially and, depending on the sign of c_1 (influenced by the sign of the initial perturbation δ), will lead either to runaway expansion or collapse to a Big Crunch.

With Hubble's discovery of the expansion of the universe in 1929, there was no more need for a static universe and this solution faded away into oblivion. Einstein himself rejected the cosmological constant, and, according to Gamow's recollections, defined it his "biggest blunder". As [52] points out, Einstein's blunder was not the introduction of the cosmological constant *per se*, but rather the fact that he chose to fine-tune it to find a contrived and unstable static solution in order to follow the trend of his time, rather than embracing his theory's incredibly powerful prediction that the universe is indeed expanding. In the following decades, the cosmological constant came into and out of existence in the scientific discourse at various points in time, and was often accepted or rejected for reasons that are generally wrong or misguided, according to our current knowledge. Thereafter, the theoretical developments in high-energy physics were mostly invoked for providing an explanation of Λ.

The trouble with Λ In a purely classical GR setting, Λ is merely a constant of nature, with dimensions of $[\mathrm{L}]^{-2}$. In this setting, it is meaningless to enquire about the value of the constant, as it is simply determined through experiment. The fact that it shows up in the Einstein equations might just be a fact of nature without particularly deep implications. However, when we not only consider classical GR, but start taking into account the quantum nature of matter, it becomes clear that the cosmological constant poses a crucial puzzle for fundamental physics. The origin of the cosmological constant is nowadays identified with the energy of the vacuum, which is often quoted to be the worst prediction in all of physics, since the discrepancy between the observed value of Λ and the theoretical vacuum energy from standard QFT is of ~ 120 orders of magnitude. Before reviewing this argument, let us explore the connection between the cosmological constant and the vacuum energy.

The reason for this identification is the observation that a cosmological constant and the vacuum energy essentially act in the same way, once again because of the intrinsic freedom in the Einstein equations to transform a contribution to geometry into one to energy. Adding Λ to the left-hand side of the equations as in (1.8) turns out to be exactly equivalent to considering the zero-point energy of a matter field on the right-hand side, no matter what the field is. We can illustrate this with a simple scalar field ϕ, described by the action

$$S_\phi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{1.28}$$

where $V(\phi)$ is the scalar field potential and the energy-momentum tensor is

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\rho\phi\nabla_\rho\phi - g_{\mu\nu}V(\phi). \tag{1.29}$$

The vacuum configuration, with the lowest possible energy density, would not include the contributions from kinetic energy, so $T_{\mu\nu} = -V(\phi_0)g_{\mu\nu}$, where ϕ_0 minimizes the potential and is in general non-vanishing. The stress-energy tensor in vacuum therefore reads

$$T^{\mathrm{vac}}_{\mu\nu} = -\rho_{\mathrm{vac}}g_{\mu\nu} \tag{1.30}$$

with $\rho_{vac} = V(\phi_0)$, which is also the only Lorentz-invariant form for this tensor. Hence, the vacuum can be modeled as a perfect fluid with $p_{\mathrm{vac}} = -\rho_{\mathrm{vac}}$. The effect of introducing this vacuum stress-energy tensor on the right hand side of the Einstein equations is precisely the same as introducing a cosmological constant on the left-hand side, as can straightforwardly be checked by moving the $\Lambda g_{\mu\nu}$ term in (1.8) to the right-hand side and identifying

$$\rho_{\text{vac}} = \rho_\Lambda \equiv \frac{\Lambda}{8\pi G}. \tag{1.31}$$

This is the reason why the cosmological constant is identified with the vacuum energy, and this precise equivalence cannot be ignored whenever we consider anything other than pure classical GR. Of course, this result does not depend on the matter field being scalar, and furthermore one does not need an additional matter field at all in order to have vacuum energy contributions, since vacuum energy is the underlying background energy permeating everything. The vacuum energy of quantum fields is enormous: the first to notice this was Nernst [53], after whom Lenz [54] and Pauli (as recalled by [55]) noticed that if vacuum energy dominated over the dynamics of the universe, this "would not even reach to the moon", since the curvature it created would be tremendous (see [56] for a historical account). The universe is instead much larger because this vacuum energy is in a remarkable discrepancy with the cosmological constant.

As can be understood from (1.9), in order to achieve the current accelerated expansion we see from observations, we require that the cosmological constant Λ is of the order of the square of the Hubble parameter today, H_0. Since $H_0 = 100\,h\,\text{km}\,\text{s}^{-1}\text{Mpc}^{-1} = 2.1332\,\text{h} \times 10^{-42}\text{GeV}$, where h quantifies the uncertainty on the value of H_0 and $h \approx 0.7$. Hence, we require [38]

$$\Lambda \approx H_0^2 = \left(2.1332h \times 10^{-42}\text{GeV}\right)^2 . \tag{1.32}$$

If we interpret this as an energy density, it becomes

$$\rho_\Lambda \approx \frac{\Lambda M_{\text{Pl}}^2}{8\pi} \approx 10^{-47}\text{GeV}^4 \approx 10^{-123} M_{\text{Pl}}^4, \tag{1.33}$$

where $M_{\text{Pl}} \approx 10^{19}\text{GeV}$ is the Planck mass. If the energy density (1.33) comes from the vacuum energy $\langle\rho\rangle$, we know that the zero-point energy of any field of mass m, momentum k and frequency ω is given by $E = \omega/2 = \sqrt{k^2+m^2}/2$ (if $\hbar = c = 1$). A free quantum field can be thought of as a collection of an infinite number of harmonic oscillators in momentum space. Technically, the zero-point energy of this infinite collection is infinite, if we include all the modes with arbitrarily short wavelengths. But if we introduce an UV cutoff $k_{\text{max}}(\gg m)$ in the spirit of EFT, the vacuum energy density takes the form

$$\rho_{\text{vac}} = \int_0^{k_{\text{max}}} \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2}, \tag{1.34}$$

which can be approximated by

$$\rho_{\text{vac}} = \int_0^{k_{\text{max}}} \frac{4\pi k^2\ \mathrm{d}k}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2} \approx \frac{k_{\text{max}}^4}{16\pi^2}, \tag{1.35}$$

since the integral is dominated by the large-k mode. We have mentioned in section 1.1.1, that GR is believed to be valid up to the Planck scale M_{Pl}, so we can take $k_{\text{max}} \simeq M_{\text{Pl}}$, and thus the vacuum energy density becomes

$$\rho_{\text{vac}} \simeq 10^{74}\text{GeV}^4. \tag{1.36}$$

It is evident that this result is about 10^{121} times larger than the observed value (1.33), and this is the "worst prediction in physics". Furthermore, taking other energy scales in particle physics instead of the Planck scale does not ameliorate the problem. For example, the QCD scale is $k_{\max} \approx 0.1\text{GeV}$, so that $\rho_{\text{vac}} \approx 10^{-3}\text{GeV}^4$, which is still much larger than (1.33). Some authors argue that it might be fairer to express the vacuum energy in terms of a mass scale as $\rho_{\text{vac}} \approx M_{\text{vac}}^4$, in terms of which the value required to explain our observations is $M_{\text{vac}}^{(\text{obs})} \sim 10^{-3}\text{eV}$, translating to $M_{\text{vac}}^{(\text{obs})} \sim 10^{-30} M_{\text{vac}}^{(\text{theory})}$, but this does not alter the fact that there is an enormous discrepancy. If we suppose that the Λ in the Einstein equations is a "bare" cosmological constant of unknown value, then the tiny value (1.33) from observations has to be obtained by summing this bare contribution to the vacuum energy density, which is inevitably present, obtaining an effective cosmological constant. For this to vanish, then, the two contributions must cancel to an astonishing precision. Since the vacuum energy density is also expected to change wildly during the history of the universe, for example when cosmological phase transitions occur, this seems completely unrealistic.

The cosmological constant challenges the framework of EFT because scales do not decouple anymore: it is not possible to understand gravity in the IR without also understanding its UV behaviour, since the vacuum energy is now related to the cosmic expansion. The vacuum energy computed in QFT is UV-sensitive, just like the Higgs mass is highly UV-sensitive. This is why the cosmological constant problem is regarded as another hierarchy problem. The fine-tuned value for Λ is often regarded as technically unnatural, in the sense of 't Hooft [57] and unstable against quantum corrections. However, while naturalness arguments have been tremendously useful in predicting the mass of the Higgs boson and the existence of the W and Z bosons, they may be simply unsuitable to predict a reasonable value for the cosmological constant, given the discrepancy we find. One might argue that in purely classical, gravitational physics there is no problem with the cosmological constant having the value it has [52], and that it is simply misguided to identify the cosmological constant with the zero point energy since the prediction is so clearly wrong. After all, in deriving a prediction for Λ from QFT, we are trusting flat-space QFT to give us indications about a situation where the effects can only be detected on the largest scales, where the local flat space approximation obviously fails. Nonetheless, while this argument is reasonable, it is impossible to ignore that the cosmological constant acts just like a vacuum energy density and therefore that the calculation performed above is sensible.

Attempted solutions The discrepancy between the vacuum energy and the expected value of Λ was present even before the observational discovery of dark energy at the end of the 1990s, but at the time, the consensus was that Λ was actually zero and we simply needed to find the right mechanism to explain why. The vanishing of a constant usually implies the existence of some symmetry, and it was thought for a long time that supersymmetry (SUSY) could fix the issue. SUSY is a symmetry relating fermions and bosons, postulating the existence of a superpartner for each elementary particle. This seems promising for the cosmological constant problem, because the contributions to vacuum energy from fermions and bosons would nicely cancel each other out. If supersymmetry was broken at an energy scale M_{SUSY}, we would expect a corresponding vacuum energy $\rho_\Lambda \sim M^4_{\mathrm{SUSY}}$. However, the particles predicted by supersymmetry have never been experimentally found, despite intense searches (most recently at the LHC [58]), so we must assume at the very least that supersymmetry is broken at low energies, if not an outright inadequate description of nature. To give a rough estimate, the fact that superpartners have not been found implies at least that M_{SUSY} is of order 10^3GeV or higher. Thus, we are left with a discrepancy

$$\frac{M_{\mathrm{SUSY}}}{M_{\mathrm{vac}}} \geq 10^{15},$$

meaning that supersymmetry still cannot solve the cosmological constant problem.

Even supergravity, the supersymmetric realization of GR, does not provide any hope. This theory is far more complicated and fine-tuning issues arise with renewed strength. In supergravity, the result that vacuum energy is always positive is not valid anymore, and there are no prospects at all to solve the cosmological constant problem.

A popular quantum gravity candidate, string theory, provides some ideas for addressing the problem, particularly in the context of anthropic arguments. The anthropic principle essentially asserts that in any formulation of the laws of nature, we must always take into account the fact that these laws are such that they have allowed for the evolution of observers. While this is a sensible consideration to make in order to account for selection effects in Nature, the concept is tautological and has no power to predict anything that is not already observed. Relying on this principle is a slippery slope that easily leads outside the realm of falsifiable science. In order to make sense of the idea that natural laws (or the value of fundamental constants) are a certain way because they need to allow for the development of life, we need a situation where many different values of physical parameters are possible, a sort of ensemble of universes, each with different probabilities of existing, and a criterion to estimate their likelihood. Specifically regarding the cosmological constant, the anthropic principle can provide at best a very weak constraint on Λ: its value cannot be so large that it would have prevented the formation of stars and galaxies and observers, and cannot be too small since we do see its effects. In other words, this gives a weak answer to the so-called *coincidence problem*, namely the problem of why the effects of Λ have become evident just now that we observe them, when ρ_Λ is of the same order of magnitude as the matter density ρ_{NR}. The answer is that if the epoch of $\rho_\Lambda \approx \rho_{\mathrm{NR}}$ had occurred very early in the evolution of the universe, the repulsive nature of a positive cosmological constant would have prevented the formation of galaxies, and if this epoch occurs too late in the future we would have obviously not have seen the effects of Λ. However, this provides absolutely no explanation for the value of the cosmological constant, despite the popularity of such arguments [50].

The resurgence of anthropic arguments in more recent times is due to the interplay of inflation and string theory. A hotly debated feature of many inflationary models is that they lead to the onset of eternal inflation: once the inflationary process begins, it continues for ever, ultimately generating a multiverse, a vast landscape of universes causally disconnected from our own, where the laws of physics and the fundamental constants are, in general, different from those of our universe. In this framework, "anything that can happen will happen, and it will happen an infinite number of times", as the pioneers of inflation assert [59]. Finding a probability measure that allows to compute predictions in this context, as a way of generating these different values for the constants, is a crucial and extremely challenging (if not downright impossible) task [59] [60]. While this consequence of inflation, if taken literally, leads to a problem with predictability, it also provides a mechanism to populate the 10^{500} possible vacua predicted by string theory, among which finding our universe is an arduous task [59]. However, an even more serious problem arises in string theory, namely the difficulty of obtaining a stable de Sitter space with $\Lambda > 0$ in accordance with observations, which becomes even more challenging in view of the recent swampland constraints (see [61] and references therein). Therefore, despite their popularity, anthropic arguments applied to string theory do not bring us any closer to understanding the true nature of the cosmological constant.

In summary, the fact that the vacuum energy is astonishingly larger than the observed cosmological constant might merely be a fact of nature, which renders irrelevant our aesthetic judgements about what value would be "natural". However, this cannot be easily dismissed as both QFT and gravity coexist in Nature and vacuum energy density acts precisely like a cosmological constant in the Einstein equations. We still lack a deeper understanding of the nature of the cosmological constant, and, given its fine-tuning, it is more than reasonable to look for less contrived alternatives.

1.1.3 Modified matter

The need for some form of dark energy to explain the accelerated expansion of the universe has been elaborated on in the previous section. Most of the solutions to the cosmological constant problem belong to two categories: either the right-hand, "matter" side or the left-hand, "geometric" side of Einstein's equations gets modified to try to account for the accelerated expansion without postulating dark energy. Since in the main topic of this thesis we will make extensive use of the freedom of moving terms on either side of the Einstein equations, specifically turning additional geometric terms into additional terms in the stress-energy tensor, it is relevant to consider both of these approaches, and in the following we review the two most representative classes of models belonging to the "modified matter" solutions, before focusing on modified gravity.

Quintessence

Quintessence [62–66] is the simplest dark energy model where the latter is not a constant, but is dynamically modeled as a scalar field minimally coupled to gravity evolving along a potential

$$S_Q = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}\nabla_a\phi\nabla^a\phi - V(\phi)\right]. \tag{1.37}$$

The equation of state is

$$w = \frac{P_\phi}{\rho_\phi} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}, \tag{1.38}$$

and the scalar field obeys the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \tag{1.39}$$

while the Friedmann equations (1.4) and (1.5) now read

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\left[\frac{\dot{\phi}^2}{2} + V\right] - \frac{k}{a^2}, \\ \dot{H} &= -H^2 - \frac{8\pi G}{3}\left[\dot{\phi}^2 - V\right], \end{aligned} \tag{1.40}$$

Given the variety of potentials that can be chosen, quintessence is generally studied making use of dynamical system analysis to explore the presence of fixed points and assess their stability.

A matter field with energy density ρ_M and pressure P_M is generally added to the action above in order to provide a solution to the coincidence problem of dark energy [67]. The so-called tracker field then corresponds to attractor solutions of the system and allows the dark energy evolution to closely mimic the behaviour of background matter for a wide range of initial conditions. This behaviour alleviates the fine-tuning issues that plague the cosmological constant scenario, since $\rho_M \gg \rho_\phi$ during the radiation- and matter-dominated eras, and dark energy starts dominating at late times. However, the tracking behaviour depends on the shape of the potential, which needs to be shallow enough to allow the field to evolve slowly along it, similarly to the slow-roll conditions in inflationary cosmology. This corresponds to $w_\phi < -1/3$ and $\dot{\phi} < V(\phi)$. However, the problem is not completely solved, since the tracker field has to change its behaviour abruptly to account for the recent onset of dark energy domination. A

s pointed out by Weinberg [68], this scenario incurs into another sort of fine-tuning to make the ρ_ϕ at which $\rho_\phi \approx \rho_M$ be close to the present critical density (1.11) at the present time.

Many quintessence models have been constructed inspired by particle physics scenarios, which is challenging because of the very low energy scale of dark energy $|m_\phi| \approx 10^{-33}$eV. Proposed models are numerous and include for example [69, 70]. Different potentials are often classified as either freezing or thawing: in the first case, the field slows down its evolution along the potential as it enters the cosmic acceleration, while in the second case the field is slowed down in its evolution by Hubble friction and begins to evolve only once the value of H decreases sufficiently (see [38] for details).

k-essence

Another very important class of modified matter models is k-essence [71, 72] involving a scalar field with non-canonical kinetic term

$$S_k = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} + P(\phi, X)\right], \tag{1.41}$$

where

$$X = -\frac{1}{2}(\nabla_\mu\phi\nabla^\mu\phi) \tag{1.42}$$

is the kinetic term responsible for the cosmic acceleration. The distinctive idea is that k-essence only tracks the background equation of state during the radiation-dominated era and not during the matter-dominated one. Instead, at the onset of matter domination, the energy density of k-essence quickly decays as the field approaches a new attractor solution, acting like a cosmological constant which overtakes the matter density and causes accelerated expansion at the present time. The stress-energy tensor of k-essence takes the form of a perfect fluid.

Specific examples under this broad class are, e.g., ghost condensates involving a slightly generalized version of a field with negative kinetic energy and an equation of state $w_\phi < -1$, characterizing the fluid as phantom. Such an equation of state is still marginally consistent with current observations [73]. However, despite the interesting cosmological consequences that would lead to a violent end for our universe, the so-called Big Rip [74] (see section 4.5.1 for an application of this model in the context of the main topic of this thesis), the phantom field is plagued by disastrous instabilities at high energies, since its energy is unbounded from below. The ghost condensate model involves a stabilised version of such a field [75], which still has quite exotic properties.

Other options for modified matter, distinct from k-essence, include for example the Chaplygin gas [76], which unifies dark matter and dark energy in a single perfect fluid with $P = -A/\rho$ (or $P = -A/\rho^{-\alpha}$), which is, however, challenged by CMB observations [77].

We have explored some options to modify the right-hand side of the Einstein equations in order to find an alternative to the cosmological constant. As will become clearer in section 2, the distinction between modified gravity and modified matter is, however, ambiguous, as one can recast the modifications to the Einstein tensor as giving rise to an *effective* stress-energy tensor. Modified gravity and modified matter cannot be distinguished by using gravity only [78] (see also [79]), but at the quantum level, of course, the field content of the two cases is different and they can be distinguished in principle.

1.2 Modified gravity

In this section we focus on modified gravity, which is the main topic of this thesis. We review scalar-tensor theories, that add a scalar degree of freedom to the tensor one of GR, and $f(R)$ theories containing higher-order curvature invariants, before mentioning a few other prominent theories that fall outside these two classes. Specifically, in the context of scalar-tensor theories, we focus on Brans-Dicke theories, the earliest example of modified gravity, gradually increasing in generality by analyzing first Galileons theories and then the broad class of Horndeski theories. Among the theories outside scalar-tensor and $f(R)$, we focus on massive gravity and theories belonging to the so-called "geometrical trinity". Modified gravity theories are reviewed, e.g., in [38, 80–83].

1.2.1 Scalar-tensor theories

The realm of scalar-tensor theories can be summarized as

- "Old-school" or traditional scalar tensor theories, characterized by first-order derivatives in the action and second-order derivatives in the equations of motion (for example, Brans-Dicke theories);
- Horndeski theories, characterized by second-order derivatives in the action and second-order in the equations;
- Beyond Horndeski theories, characterized by second-order derivatives in the action and higher-order derivatives in the equations.

The classification is motivated by Ostrogradski's theorem, which states that theories with second and higher time derivatives in the action generically introduce unstable degrees of freedom [84, 85]. However, there exist some loopholes to the theorem that make it possible to find theories beyond the Horndeski class, as detailed in the following.

Brans-Dicke

Brans-Dicke theory is the prototype of a modified gravity theory and was the first attempt to go beyond Einstein gravity. It was formulated in 1961 [86, 87] and is motivated by trying to incorporate Mach's principle into GR, which roughly states that the local inertial frame is affected by the distribution of matter on very large, even cosmological scales. Despite the inherent philosophical vagueness of the statement, it also non-negligibly influenced Einstein in the development of GR, although GR itself does not fully embody it. The ingenuity of Brans-Dicke theory and its main conceptual advance are contained in the fact that the gravitational constant is not a constant anymore, but rather a scalar field that varies in spacetime. The scalar field is of gravitational origin and not simply a matter field, which is manifested in the direct, non-minimal coupling between the scalar and the curvature, which conceptually differs from the minimally-coupled fields appearing for example in quintessence and k-essence models of section 1.1.3 (nonetheless, see [88] for some subtleties on this point). Brans-Dicke theory in the Jordan frame (see section 4.3 for a description in the conformally related Einstein frame) is described by the action

$$S_{\rm BD} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega}{\phi} \nabla^c\phi \nabla_c \phi - V(\phi) \right] + S^{(m)} \,, \tag{1.43}$$

where the Brans-Dicke scalar $\phi > 0$ is approximately the inverse of the effective gravitational coupling

$$G_{\rm eff}(\phi) \simeq \frac{1}{\phi}, \tag{1.44}$$

ω is the dimensionless constant Brans-Dicke coupling, $V(\phi)$ the scalar field potential and from here onward we use $G = 1$. In order to ensure a positive gravitational coupling, only $\phi > 0$ is considered. The essential term in this action is the direct coupling between the scalar and gravity in the first term, while matter remains minimally coupled to gravity. Theories with a gravitational coupling that vary across spacetime obviously find their natural application in cosmology, since the deviations from General Relativity that are small at present might have been more significant in the past, or even become so in the future. The natural time scale for the evolution of the scalar field is cosmological. However, there are tight experimental bounds on how much the gravitational constant can vary, at least within the Solar System. With lunar ranging, $|\dot{G}/G|$ (where the dot indicates a time derivative) is constrained to be at most $0.4 \cdot 10^{-11}$ to $1.0 \cdot 10^{-11}$ per year.

During the decades following its inception, despite its main applications in cosmology and thus in the IR regime, Brans-Dicke theory was also revived in connection to quantum gravity proposals: specifically, it was found that the low-energy limit of the bosonic string theory corresponds to a Brans-Dicke theory with $\omega = -1$. In four dimensions, the redefinition of the so-called "dilaton" scalar field as

$$\varphi = \mathrm{e}^{-2\phi}$$

yields the Brans-Dicke action (1.43) with $\omega = -1$ [89].

The limit to GR should be recovered (or at least so it goes in the standard lore [90]) as $\omega \to \infty$. This fact is also used to constrain Brans-Dicke theories within the Solar System, using the Parametrized Post Newtonian (PPN) formalism [91]. The best limits on Brans-Dicke gravity within the Solar System were provided by the Cassini probe and are $|\omega| > 40000$ at 2σ confidence [92]. On cosmological scales, such bounds are less stringent but more accurate (and arguably more relevant, given the cosmological applications of the theory): they give, for example, $\omega > 692$ at the 99% confidence level [93]. A more recent and very comprehensive analysis yields $\omega > 1540$ at 95% confidence level [94]. However, a not so well-known subtlety is that there are many solutions of Brans-Dicke gravity that do not reduce to GR with the same form of matter in the limit $\omega \to \infty$, for example the trivial vacuum solution with $T^{(m)} = 0$. This "anomalous" limit is influenced by the fact that in Brans-Dicke the gravitational coupling is technically determined by all matter in the universe, and therefore vacuum solutions with $T^{(m)} = 0$ are ill-defined [95]. For an analysis of this issue, see [96] and references therein. From the phenomenological point of view, such a large value of ω appears fine-tuned and therefore Brans-Dicke theory is historically valuable, but is not considered as a particularly attractive theory nowadays. Additionally, it only counts as one specific case of the much more general scalar-tensor classes described in the following sections.

For the main topic of this thesis, we will deal with a more general version of scalar-tensor theories that encompasses Brans-Dicke, but where ω is a function of the field, namely $\omega = \omega(\phi)$ [97, 98].

Galileons

As we will briefly review in section 1.2.3, the study of the non-linear interactions of the helicity-0 mode in the so-called decoupling limit of the Dvali-Gabadadze-Porrati (DGP) model, which contains the term $\mathcal{L}_3 = (\partial\pi)^2\Box\pi$ for the scalar π, spurred a flurry of activity to add more general non-linear interaction terms for the scalar field. The construction of all possible interaction terms for the scalar that, despite containing higher-order derivatives, yield second-order equations of motion due to a tuning of the coefficients (and therefore avoid the Ostrogradski instability) leads to the Galileon theories, called in this way because they are invariant under Galileian symmetry ($\pi \to \pi + c + x_\mu b^\mu$) in addition to shift symmetry [99].

The Lagrangian of Galileons reads

$$\mathcal{L}_{\mathrm{G}} = \sum_{i=1}^{5} c_i \mathcal{L}_i \tag{1.45}$$

where

$$\begin{aligned}
\mathcal{L}_1 &= \pi \\
\mathcal{L}_2 &= (\partial\pi)^2 \\
\mathcal{L}_3 &= (\partial\pi)^2 \Box\pi \\
\mathcal{L}_4 &= (\partial\pi)^2 \left[(\Box\pi)^2 - (\partial_\mu\partial_\nu\pi)^2\right] \\
\mathcal{L}_5 &= (\partial\pi)^2 \left[(\Box\pi)^3 - 3\Box\pi\,(\partial_\mu\partial_\nu\pi)^2 + 2\,(\partial_\mu\partial_\nu\pi)^3\right],
\end{aligned}$$

with arbitrary coefficients c_i (we denote the scalar Galileon field with π for historical reasons and to highlight the fact that it is in a flat background, while in the following we use ϕ for scalar fields in a curved background).

Galileons were developed in a flat background, and it was immediately noticed that the covariantization of Galileon interactions (i.e., promoting the flat Minkowski metric to a general curved metric $\eta_{\mu\nu} \to g_{\mu\nu}$ and partial derivatives to covariant derivatives $\partial_\mu \to \nabla_\mu$) performed in a naive way yields third-order equations of motion of π (incurring in the Ostrogradski instability mentioned above). Since these quartic and quintic interactions are non-linear in the connection, appropriate non-minimal couplings to curvature need to be introduced in $\mathcal{L}_4$ and $\mathcal{L}_5$ [100] to ensure that the equations of motion remain second-order. This breaks the Galileon invariance, but in curved spacetime its usefulness is limited. The Lagrangian of covariant Galileons reads

$$\mathcal{L}_{\mathrm{CG}} = \sum_{i=2}^{5} \mathcal{L}_i \tag{1.46}$$

with

$$\begin{aligned}
\mathcal{L}_2 &= c_2 X - \frac{c_1 M^3}{2}\phi \\
\mathcal{L}_3 &= 2\frac{c_3}{M^3} X \Box\phi, \\
\mathcal{L}_4 &= \left(\frac{M_{\mathrm{Pl}}^2}{2} + \frac{c_4}{M^6}X^2\right) R + 2\frac{c_4}{M^6} X \left[(\Box\phi)^2 - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\right], \\
\mathcal{L}_5 &= \frac{c_5}{M^9} X^2 G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{3}\frac{c_5}{M^9} X \left[(\Box\phi)^3 - 3(\Box\phi)\left(\nabla_\mu\nabla_\nu\phi\right)\left(\nabla^\mu\nabla^\nu\phi\right)\right. \\
&\quad \left. + 2\left(\nabla^\mu\nabla_\alpha\phi\right)\left(\nabla^\alpha\nabla_\beta\phi\right)\left(\nabla^\beta\nabla_\mu\phi\right)\right].
\end{aligned}$$

The mass scale $M^3 \equiv M_{\mathrm{Pl}} H_0^2$ ensures that the c_i coefficients remain dimensionless (and M_{Pl} is the Planck mass).

Covariant Galileons have been studied mostly for their cosmological applications, since they exhibit self-accelerating de Sitter solutions [101] and are so versatile that they can be employed for many different purposes, from inflation [102] to dark energy [103].

Observationally, cubic Galileons appear ruled out by the Integrated Sachs-Wolfe effect [104], while quartic and quintic Galileons are ruled out by gravitational wave observations in conjunction with their electromagnetic counterpart (see section 1.2.4 for details). Breaking the Galileon symmetry allowed for more freedom in the coefficients (although a generalised Galileon symmetry can be recovered on maximally symmetric backgrounds [105]), and led to the so-called Generalized Galileon action [106].

Horndeski theories

The exploration of the Generalized Galileons led to rediscovering the work of Horndeski [107] (see [108, 109] for reviews), who found the most general scalar-tensor theory respecting Lorentz invariance and diffeomorphism-invariance, while still keeping second-order equations of motion (see caveats below). Horndeski did this back in 1974, before leaving physics and embarking on a successful career as a painter. When the authors of [110] revisited Horndeski gravity in order to study it on FLRW backgrounds and exploring whether some subclasses of it could address the cosmological constant problem, it did not take very long before other authors [111] pointed out the equivalence between Generalized Galileons and the old Horndeski theory. In its modern form, it reads

$$S_{\mathrm{H}} = \int d^4x\sqrt{-g}\sum_{i=2}^{5}\mathcal{L}_i \tag{1.47}$$

with

$$\begin{aligned}
\mathcal{L}_2 &= G_2(\phi, X),\\
\mathcal{L}_3 &= -G_3(\phi, X)\Box\phi,\\
\mathcal{L}_4 &= G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right],\\
\mathcal{L}_5 &= G_5(\phi, X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi) - \frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right.\\
&\quad \left. +2\left(\nabla^\mu\nabla_\alpha\phi\right)\left(\nabla^\alpha\nabla_\beta\phi\right)\left(\nabla^\beta\nabla_\mu\phi\right)\right].
\end{aligned}$$

It reduces to the Covariant Galileons when $G_2 = -\frac{c_1}{2}M^3\phi + c_2X$, $G_3 = -\frac{2c_3}{M^3}X$, $G_4 = \frac{M_{\mathrm{Pl}}^2}{2} + \frac{c_4}{M^6}X^2$, $G_5 = -\frac{3c_5}{M^9}X^2$. As highlighted by the authors of [110], most of other scalar-tensor theories, especially those used for cosmological applications, are special cases of the Horndeski Lagrangian, including the models we discussed in the previous sections, such as k-essence, Brans-Dicke theories, Galileon models and many others. An additional intriguing subclass of models is dubbed "Kinetic Gravity Braiding" [22, 112] and includes up to cubic interaction terms. We will explore it because of its applications. Additionally, another interesting aspect of Horndeski and Galileons theories is that they can be interpreted as a proxy theory for massive gravity, since the decoupling limit of massive gravity contains Galileon interactions [113, 114].

Beyond Horndeski theories

Horndeski theories were thought to be the most general scalar-tensor theories with second-order equations of motion before some more recent theories were found. They still avoid the Ostrogradski instability despite having higher-order equations, by respecting a constraint equation, and further generalise the Horndeski class [115, 116]. The most general higher-order theories that do not incur into instabilities are the so-called Degenerate Higher-Order Scalar-Tensor (DHOST) theories [117, 118], which satisfy an additional degeneracy condition ensuring the propagation of only three degrees of freedom. They are interesting because of their invariance under disformal transformations $\tilde{g}_{\mu\nu} = C(\phi, X) g_{\mu\nu} + D(\phi, X)\partial_\mu\phi\partial_\nu\phi$, which is not a property of Horndeski and beyond Horndeski theories.

1.2.2 $f(R)$ theories

Another important class of modified gravity theories, $f(R)$ theories, provides an answer to the very natural question of whether higher-order curvature terms beyond the Ricci scalar are allowed into a gravitational Lagrangian. This class is remarkably versatile, as it has quite important applications in cosmology, but its exploration has been motivated mostly by the efforts to find a quantum theory of gravity and understand the UV regime of gravity (see [26, 80, 119] for reviews). The gravitational action reads

$$S_{f(R)} = \frac{1}{16\pi} \int d^4x \sqrt{-g} f(R), \tag{1.48}$$

where $f = f(R)$ is a generic function.

Naturally, the first extension of the Einstein-Hilbert term would be to consider additional quadratic terms in the curvature. The first attempt of this dates back to as early as 1918 with the work of Weyl [120], and subsequent interest in these theories was motivated by works in quantum gravity. It was shown that one-loop renormalisation required higher-order curvature terms in the Einstein-Hilbert action [121], and later proven that, unlike GR, higher-order (and, in particular, quadratic) theories are renormalisable [122]. Moreover, the study of these theories found a strong motivation in early-universe cosmology: Starobinsky inflation [123], despite being the oldest example of an inflationary model, devised *ante litteram*, not only achieves inflation without an additional scalar field, but also passes the tests posed by the most recent cosmological data with flying colors [73, 124]. Starobinsky inflation involves the specific choice $f(R) = R + \alpha R^2$. More recently, models within $f(R)$ have also been considered as candidates for explaining the late-time acceleration [125, 126], despite some challenges [127, 128].

There are three versions of $f(R)$ theories, depending on which variational principle is used to derive them: with the standard metric formalism, the metric and the connection are of course not independent variables. However, there exists also another formalism, called Palatini formalism, despite the fact that it was introduced by Einstein himself (see [80] for details). In this formulation, the metric and the connection are assumed to be independent variables and thus the variation is performed with respect to both of them. In GR (or any action linear in R), both formalisms give the same Einstein equations, since the field equations found for the connection would give the Levi-Civita connection of the metric [80], but this is not the case for a more general $f(R)$, and the resulting field equations are different. A more recent formulation is the metric-affine one, which makes use of the Palatini variation, but abandons the assumption that the matter action is independent of the connection [129]. This formulation is able to encompass both the metric and Palatini formalisms as specific cases.

Another particularly intriguing aspect of higher-order theories is that, through conformal transformations, they can always be recast into a scalar-tensor theory where the scalar is directly coupled to the curvature, similar to Brans-Dicke theories [26]. Therefore the distinction we are employing here between theories with higher-order curvature invariants and scalar-tensor theories is not as clear as it might have initially seemed.

Let us spend a few more words on this point. It will be relevant for the purposes of this thesis because we will work with a class of Brans-Dicke-like scalar-tensor theories, that includes metric $f(R)$ theories because of their equivalence. The formalism of first-order thermodynamics developed for scalar-tensor theories will then be valid for $f(R)$ theories as well.

The equivalence (in the metric formulation) can be seen as follows [26]: starting from the action

$$S_{eq} = \frac{1}{16\pi} \int d^4x\sqrt{-g} f(R) + S_M\left(g_{\mu\nu}, \psi\right), \tag{1.49}$$

where S_M is the matter action, we can introduce a new field χ and write the dynamically equivalent action

$$S_{eq} = \frac{1}{16\pi} \int d^4x\sqrt{-g}\left[f(\chi) + f'(\chi)(R-\chi)\right] + S_M\left(g_{\mu\nu}, \psi\right). \tag{1.50}$$

Variation with respect to χ leads to the equation

$$f''(\chi)(R-\chi) = 0. \tag{1.51}$$

Therefore, $\chi = R$ if $f''(\chi) \neq 0$, which reproduces the action above. Redefining the field χ by $\phi = f'(\chi)$ and setting

$$V(\phi) = \chi(\phi)\phi - f(\chi(\phi)), \tag{1.52}$$

the action takes the form

$$S_{eq} = \frac{1}{16\pi} \int d^4x\sqrt{-g}[\phi R - V(\phi)] + S_M\left(g_{\mu\nu}, \psi\right). \tag{1.53}$$

This is precisely the action of a Brans-Dicke theory with $\omega = 0$ in the Jordan frame. Of course, $\phi = f'(\chi)$ is not a conventional matter field, since it violates energy conditions, for example. The condition $f'' \neq 0$ ensures that the change of variable

$\phi = f'(R)$ is invertible. Following the same steps, Palatini $f(R)$ gravity is found to be equivalent to Brans-Dicke gravity with a different parameter, namely $\omega = -3/2$, and the theories are dynamically equivalent (see section 4.2.1 for application of these theories). $f(R)$ theories have field equations higher than second order. For example, quadratic terms in the curvature lead to fourth-order equations of motion, a $R\Box R$ term leads to sixth-order equations, and so on. The higher-order equations cause the graviton propagator to fall off more quickly in the UV, thus improving the renormalisability properties, but, as discussed before, lead to the appearance of ghost-like degrees of freedom because of the Ostrogradsky instability. However, this is not always the case: for example, in the $f(R) = R + R^2$, one assumption of Ostrogradsky's theorem is violated and there is a new degree of freedom that does not incur into a ghost instability [130]. Moreover, through a conformal transformation, these higher-order derivatives end up corresponding to a scalar field. More precisely, fourth-order gravity is equivalent to Einstein gravity with a minimally coupled scalar [131], sixth-order gravity to GR with two scalar fields, and so on [132, 133]. This is akin to the conformal transformation turning Jordan-frame scalar-tensor theory into GR plus a minimally coupled scalar in the Einstein frame (see section 4.3).

The strongest observational constraints on $f(R)$ theories come from cosmological probes. A particularly interesting cosmological model in this class is the Hu-Sawicki model [126], which can produce an accelerating universe and respect both the weaker Solar-System constraints and the tighter constraints coming from the growth of cosmic structure. These are around $|f_{R0}| \lesssim 10^{-6}$, where f_{R0} is the value of the cosmological field amplitude today. Additional late-time cosmological applications include [134].

1.2.3 Other theories

Geometrical trinity of gravity

Einstein formulated a geometric theory of gravity: his revolutionary insight is that the curved geometry of the spacetime is responsible for the gravitational effects. This geometrical nature of GR is connected to the universality of the gravitational force and is encapsulated in the equivalence principle, stating that all matter fields couple in the same way to gravity. However, ascribing geometrical concepts to gravity is not free from ambiguities: Einstein's approach of assigning gravity to the curvature of spacetime uniquely finds GR and both the torsion $T^{\alpha}_{\mu\nu} \equiv \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$ and the non-metricity $Q_{\alpha\mu\nu} \equiv \nabla_{\alpha} g_{\mu\nu}$ vanish, but this is not the only possible choice. Indeed, distinct yet equivalent representations of GR [83, 135] can be found in a flat spacetime by ascribing gravity entirely to torsion (Teleparallel Equivalent of GR) or in a flat and torsion-free spacetime by ascribing it entirely to non-metricity (Symmetric Teleparallel Equivalent of GR). One advantage of these formulations is the fact that they possess a well-defined variational principle without the need of the Gibbons-York-Hawking boundary term that GR needs [135]. Recently, modified gravity theories based on these equivalents to GR have received substantial attention in the literature. Just as one can generalize GR to $f(R)$ theories, the same can be done with $f(\mathbb{T})$ and $f(\mathbb{Q})$ theories, where

$$\mathbb{T} := \frac{1}{2}\left(\frac{1}{4}T_{\alpha\mu\nu} + \frac{1}{2}T_{\mu\alpha\nu} - g_{\alpha\mu}T_{\nu}\right)T^{\alpha\mu\nu} \tag{1.54}$$

is the torsion scalar ($T_{\nu} = T^{\alpha}_{\nu\alpha}$) and

$$\mathbb{Q} = \frac{1}{4}Q_{\alpha\mu\nu}Q^{\alpha\mu\nu} - \frac{1}{2}Q_{\alpha\mu\nu}Q^{\mu\alpha\nu} - \frac{1}{4}Q_{\alpha}Q^{\alpha} + \frac{1}{2}Q_{\alpha}\bar{Q}^{\alpha} \tag{1.55}$$

is the non-metricity scalar ($\tilde{Q}^{\nu} = Q_{\alpha}{}^{\alpha\nu}$ and $Q^{\mu} = Q^{\mu\alpha}{}_{\alpha}$). However, it is important to remark that, while GR and the other two formulations are equivalent, $f(R)$, $f(\mathbb{Q})$ and $f(\mathbb{T})$ are all distinct from each other. Despite some conceptual challenges such as the difficulty in determining the precise number of propagating degrees of freedom in these theories, they give rise to a rich phenomenology (see, for example, [136] for details) and are currently an active area of research.

Massive gravity

In this section, we give a short historical account of massive gravity, following [83] in order to highlight how its history is intertwined with that of Galileon theories and its goal is to explain the accelerated expansion of the universe without dark energy. We have mentioned in section 1.1 that, in addition to adding new fields and higher-order terms in the Lagrangian, a fair way of modifying gravity could be to modify one of its fundamental assumptions, for example by introducing more than four dimensions. One of these proposals is the Dvali-Gabadadze-Porrati (DGP) model [137], which allows for large-scale modifications of gravity, based on a three-brane embedded in a five-dimensional bulk. On small scales, gravity is recovered in four dimensions as the usual Einstein-Hilbert term is sourced by the brane curvature. On large cosmological scales, where constraints are much less stringent, gravity is weakened since the graviton acquires a soft mass. This “leaking” of gravity in the additional dimension is supposed to explain why gravity is so weak compared to the other fundamental forces. This models possesses a “self-accelerating” solution that is unfortunately plagued by ghost instabilities. A similar rationale is used in the case of massive gravity, where the graviton acquires a proper mass and the question of whether gravity is a force with infinite range or simply a very large but finite one is addressed. Since the range of the force is inversely related to the mass of its carrier particle, this is equivalent to asking whether the graviton is truly massless or if it possesses a tiny, but finite mass. This is not as counterintuitive as it might seem at first sight, since gravity is not as well-tested on the largest cosmological scales as it is on smaller scales, and the assumption that it behaves in the same way everywhere might be misplaced. Just as the classical description of gravity breaks down at high energies close to the Planck scale and needs to be replaced by an UV-complete theory of quantum gravity, the same might be true for the IR-limit: our lack of understanding of this regime might show up as the accelerating expansion of our universe that requires dark energy if we assume GR.

The Fierz-Pauli action was the first attempt to construct a unique mass term at the linear level and without ghost instabilities [138], but is unfortunately plagued by the van Dam-Veltman-Zakharov (vDVZ) discontinuity, namely the fact that the limit of vanishing graviton mass does not recover GR. This signals the need to go beyond the linear theory, but for more than forty years all non-linear extensions of the theory introduced ghosts. The breakthrough came only in 2010, where the correction [139] of a previous mistake allowed to finally perform the resummation of the non-linear interactions of massive gravity in the de Rham-Gabadadze-Tolley (dRGT) model, which yielded the first consistent example of a ghost-free non-linear covariant theory of massive gravity in four dimensions [140].

Both DGP and massive gravity provide ways to address the cosmological constant problem, since the mass of the graviton acts as a high-pass filter that weakens gravity in its IR limit, so that the vacuum energy has a weaker effect on the geometry than with GR, and therefore the value of the vacuum energy calculated from particle physics can be more easily reconciled with the observed acceleration. However, adding a mass to the graviton introduces more degrees of freedom and both theories have five propagating degrees of freedom (and no ghosts), the most important of which is the helicity-0 mode. This mode can mediate an extra fifth force and can be decoupled from the gravitational dynamics thanks to non-linear interactions. This is the essence of the Vainshtein mechanism that ensures the successful recovery of General Relativity in this decoupling limit [141–143]. Precisely these developments inspired the progress on Galileons, as discussed in 1.2.1.

1.2.4 Gravitational wave constraints on modified gravity

The relative freedom we seem to have in constructing modified theories of gravity should not make us believe, to quote Feyerabend, that "anything goes". Despite their appeal, the internal consistency and aesthetic value of a theory are nothing without agreement of the theory's predictions with observational data. Modified gravity theories at least make predictions that are easy to extract (which is not the case for quantum gravity theory candidates, for example), but many of them end up ruled out or severely constrained once new data are obtained, in a healthy process that is the essence of science. While we have mentioned some astrophysical and cosmological constraints on the modified theories of gravity surveyed in the previous sections, we aim to mention here the most powerful tool to constrain modified gravity that is becoming increasingly important in this day and age, namely gravitational waves. The first detection of gravitational waves [144] opened up a completely novel window on the universe, (complementary to that available with electromagnetic radiation) which has revolutionised gravitational physics. Combining the two in the promising field of multi-messenger astronomy holds the promise of gaining unprecedented knowledge about the true nature of gravity. With one multi-messenger detection only, that of two merging neutron stars detected both with gravitational waves and the electromagnetic counterpart, [145, 146] were able to constrain the speed of gravitational waves to $|c_T/c - 1| \lesssim 10^{-15}$, where c_T is the speed of gravitational waves. This single measurement ruled out or constrained the vast majority of modified gravity theories, especially scalar-tensor theories [147–149]. Other theories have been constrained, for example, by gravitational wave damping or oscillations, absence of detectable additional polarizations of the graviton (see [147] for details).

Restricting to the scalar-tensor theories we focus on, we have mentioned in section 1.2.1 that suitable non-minimal couplings to curvature need to be introduced in the Galileon/Horndeski Lagrangians, specifically in $\mathcal{L}_4$ and $\mathcal{L}_5$, to ensure that the equations of motion remain second-order in a curved background. Introducing the standard scalar, vector and tensor perturbations in a homogeneous FLRW background[1] (see [150] for a review) results in

$$q_T = \frac{1}{4}\left(2\left(G_4 - 2XG_{4,X}\right) - 2X\left(G_{5,X}\dot{\phi}H - G_{5,\phi}\right)\right) \tag{1.56}$$

$$c_T^2 = \frac{2G_4 - 2XG_{5,\phi} - 2XG_{5,X}\ddot{\phi}}{4q_T}, \tag{1.57}$$

where c_T is the coefficient in front of the kinetic term for tensor perturbations and c_T is the perturbation speed, namely the speed of gravitational waves. Thus, the speed of gravitational waves depends non-trivially on the derivative coupling functions, and this is the reason why constraining c_T translates into a constraint on the subclasses of Horndeski theories. Nonetheless, it is important to note that the LIGO/Virgo constraint on the speed of gravitational waves is restricted to frequencies of $10 - 100\,\mathrm{Hz}$, which is at the edge of the strong coupling scale of Horndeski theories, where the regime of validity of the effective field theory breaks down, and, potentially, new operators at this cutoff scale could affect the propagation speed [151].

If we neglect this concern, then the restrictions on Horndeski theories that ensure luminal propagation of gravitational waves (without any fine-tuning of the couplings) are

$$G_{4,X} = 0 \qquad \text{and} \qquad G_5 = 0 \tag{1.58}$$

(the validity of first-order thermodynamics interestingly requires the same constraint, see section 4.1). We will explore the class of Horndeski gravity that remains viable after the multi-messenger constraints in sections 4.1, 4.4.2 and 4.5.2.

Since Horndeski theories encompass other scalar-tensor theories we have discussed in section 1.2.1, also they are constrained by gravitational waves. The simplest Brans-Dicke models (as the coupling to gravity contained in $\mathcal{L}_4$ can be at most of the $f(\phi)R$ type) or Kinetic Gravity Braiding survive. Cubic Galileons are in contrast with ISW data, quartic and quintic Galileons ruled out, therefore all Galileons. In the remaining class, the only surviving theories exhibiting self-accelerating solutions must be shift symmetric or very nearly so, which include Kinetic Gravity Braiding and k-essence (see [148] for details). One specific model of beyond Horndeski survives. $f(R)$ theories receive only constraints that are about three orders of magnitude weaker than those coming from cosmological observations [152].

Theories including an additional tensor degree of freedom, such as massive gravity, are constrained as well. They do not predict an anomalous gravitational waves speed since the derivatives in the Lagrangian are not affected, but only the potential is. The mass of the graviton is theoretically predicted to be $m_g \simeq 10^{-32}\,\mathrm{eV}$, while constraints from gravitational waves yield $m_g < 7.7 \cdot 10^{-23}\,\mathrm{eV}$, much less stringent

[1]We will not cover standard cosmological perturbation theory in this thesis, since first-order thermodynamics is based on an analogy with an effective fluid at the background level.

than the cosmological constraint $m_g < 7.7 \cdot 10^{-33}$ eV [153]. Nonetheless, future gravitational wave experiments such as LISA are capable of tightening this constraint to a level closer to that of cosmology, but the real breakthrough promises to come from low-frequency observations, such as primordial gravitational waves, which could achieve $m_g < 10^{-29}$ eV.

1.3 Gravity and thermodynamics: a different perspective

In this section, we take a detour from the main path of this thesis, and explore the relationship between gravity and thermodynamics. This provides the necessary counterpart to the previous sections exploring modified gravity, since the main topic of this work, the formalism of first-order thermodynamics, unifies both topics. At first sight, it is very surprising that there is any relationship at all between gravity and thermodynamics. Thermodynamics deals with the properties that emerge from the collective behaviour of microscopic constituents, while gravity is one of the four fundamental forces of the universe, and therefore operates at a much more essential level. However, the exploration of this relationship will show that some of the problems that afflict our description of gravity, such as the lack of a consistent theory of quantum gravity, might be solved by a paradigm shift that sees gravity itself as emergent.

1.3.1 Induced gravity

The first exploration of the idea that gravity might be emergent was the pioneering work by Sakharov about induced gravity [154, 155]. This concept is starkly different with respect to approaches in quantum gravity and suggests a reason why quantizing gravity has not proven successful so far, and why it might be unnecessary or even misguided. The central idea involves taking a Lorentzian manifold as an arbitrary background and to consider one-loop Quantum Field Theory on it, so that matter fields are quantized, but refrain from introducing any gravitational dynamics explicitly. This construction gives rise to an effective action, which to one-loop order automatically contains the Einstein–Hilbert action with a cosmological constant, plus correction terms proportional to the curvature squared, in a sort of semi-classical limit. Such a result is quite striking: GR dynamics emerges from quantum field theory in a roughly similar way as hydrodynamics or continuum elasticity emerges from molecular physics. This is the first result that questions whether finding a theory of quantum gravity is actually reasonable. Even if we still want to find such a theory because we are motivated by the beauty of unified descriptions or to provide a solution to the problem of singularities, Sakharov's result shows that deriving Einsteinian gravity is almost trivial once a Lorentzian manifold with a Quantum Field Theory on it has been established, and therefore the real difficulty in bringing together GR and QFT must lie somewhere else. This surprising result highlights one of the ways in which gravity seems to be of an essentially different nature from the other fundamental forces.

The notion of gravity as emergent and not fundamental has been explored more recently from a variety of perspectives, mostly involving the study of the connection between gravity and thermodynamics, not included in Sakharov's pioneering work. The idea of emergent gravity was initially inspired by developments in the thermodynamics of black holes. Black holes are not only objects with puzzling interiors that harbor a singularity in their core, but also bodies that, regardless of their inner structure, can be described from the exterior by a handful of parameters. Similarly to the situation in thermodynamics, where, as long as the macroscopic properties such as temperature and entropy are the same and quantities are related through an equation of state characterizing the system, the microscopic structure is irrelevant, it does not matter how a black hole formed or what specific astrophysical conditions it is embedded in, since all black holes are described by three parameters. It is precisely this universality (or the fact that black holes "have no hair") that is the key to uncovering the thermodynamical properties of black holes [156, 157].

1.3.2 Black hole thermodynamics

During the 1970s, it was found that the so-called laws of black hole mechanics could be formulated in perfect analogy to the ordinary laws of thermodynamics [158, 159] (see also [160, 161] for reviews). Until then, it was believed that, since nothing can escape the black hole's gravitational pull, classical black holes are perfect absorbers but cannot emit anything. In this framework, the temperature of a black hole can only be absolute zero. However, some developments in the semi-classical treatment of black holes, based upon quantum field theory in curved space, led to the discovery that black holes actually emit Hawking radiation with a perfectly thermal spectrum and they are endowed with a temperature and an entropy. The latter is actually holographic, meaning that it scales with the area of the black hole's event horizon, at variance with ordinary entropy, which scales with volume.

The first result was Hawking's 1972 paper [162], showing that the area of an event horizon can never decrease, analogously to the entropy. This early result stood the test of time: it has been shown to hold in various settings [163] and seems to be even consistent with observations of the gravitational wave emission from black hole mergers [164]. Hawking's result led to the formulation of a "generalised" second law of thermodynamics by Bekenstein. He realised that, if any object is dropped into a black hole, and the black hole is not endowed with entropy, then the entropy carried by the object would have disappeared from the universe, since no information can be retrieved from a black hole. This clearly violates the second law of thermodynamics. He therefore suggested that black holes do possess an entropy, and he generalized the second law to account for it [165]. Specifically, it is the sum of the ordinary entropy of matter outside of a black hole plus the black hole entropy that never decreases. This black hole entropy, now called Bekenstein-Hawking entropy, is

$$S = \alpha A, \tag{1.59}$$

where A is the horizon area and $\alpha = \text{const.}$. (Note that the expression in physical units is $S = \alpha \frac{A}{\hbar G}$, containing $\hbar$ from quantum mechanics and G from gravity).

These findings culminated almost simultaneously in the discovery of the four laws of black hole mechanics [166]. For a stationary and asymptotically flat black hole in four dimensions, uniquely characterized by a mass M, an angular momentum J and a charge Q, the following laws hold, in analogy to the corresponding laws of thermodynamics:

0. The surface gravity κ is constant over the event horizon.

1. For two stationary black holes with slightly different M, J, and Q,

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \Phi_H \delta Q, \tag{1.60}$$

   where κ is the surface gravity (related to temperature, as shown in the following), Ω_H is the angular velocity and Φ_H is the electric potential at the horizon.

2. The area of the event horizon of a black hole never decreases,

$$\delta A \geq 0. \tag{1.61}$$

3. For a black hole, it is not possible to reduce κ to zero by a finite sequence of operations.

Initially, the similarity between these laws and those of thermodynamics was interpreted as a simple curiosity. Mostly, the resistance was due to the fact that black holes endowed with entropy must also have a non-vanishing temperature and therefore radiate, which seemed unthinkable at the time because of the common notion that nothing can escape a black hole. This prejudice changed with Hawking's discovery. While Zel'dovich had previously tentatively suggested that spinning black holes might radiate due to quantum effects, it was Hawking who employed the newly developed techniques of quantum field theory in curved space to show that all black holes radiate with a blackbody temperature [167]

$$T_H = \frac{\kappa}{2\pi}. \tag{1.62}$$

The Hawking effect described this emission as due to quantum particle creation effects. With this temperature and through the first law above, the constant in Bekenstein's entropy can be set to $\alpha = 1/4$. As [24] put it, the discovery of Hawking radiation turned the analogy between the laws of thermodynamics and those of black hole mechanics into an identity. Hawking's insight was based on a key feature of quantum field theory in curved spacetime, namely the fact that the vacuum depends on the choice of time. In particular, the Minkowski vacuum of a past observer watching the collapse of a star differs from the vacuum of a future observer looking at the resulting black hole. According to the Unruh effect [168], any observer with proper uniform acceleration a_U sees a thermal emission of particles with temperature

$$T_U = \frac{\hbar\, a_U}{2\pi}, \tag{1.63}$$

which is a sort of generalisation of Hawking's temperature. The temperature of Hawking radiation for common astrophysical black holes is of the order of a microkelvin and therefore way too low to be detected experimentally with current technologies. However, in recent years, exciting progress has been made in the context of analogue gravity experiments, pioneered by Unruh [169] and now carried out by several experimental groups, for example [170, 171] with interesting future prospects [172].

The realisation of the holographic nature of black hole entropy led to a flurry of research. The idea that physics in any spatial region can be fully described simply in terms of the degrees of freedom associated with the region's boundary was first explored in [173, 174] and is now interwoven with increasingly abstract developments in the AdS/CFT correspondence in string theory. There is yet no ultimate consensus in the literature as to the nature of the microscopic degrees of freedom which contribute to black hole entropy.

1.3.3 Thermodynamics of spacetime

The connection between gravity and thermodynamics discovered in the context of black holes can actually be extended to a more general setting. Jacobson [24] derived a thermodynamics of spacetime itself, showing that the Einstein equations can be derived as equations of state starting from purely thermodynamical considerations, exploiting the thermal nature of the vacuum and the thermodynamical properties of horizons that we have explored in the previous section. One of the most intriguing findings is that this construction holds beyond GR and can be recovered for modified theories of gravity, pointing at something deeper about the nature of gravity itself, regardless of the physical theory chosen to describe it. The catch is that, in order to generalize to modified gravity, one needs to go beyond standard equilibrium thermodynamics, allowing for dissipative, irreversible processes in the context of non-equilibrium thermodynamics.

Jacobson's construction requires the introduction of a notion of local horizon and of the thermodynamical system at hand. Taking a point p in a generic spacetime manifold $(\mathcal{M}, g_{ab})$, positioned on one side of the past boundary of a spacelike 2-surface element $\mathcal{P}$ (see [24, 175] for details). Near p, the local horizon is constituted by the congruence of null geodesics orthogonal to $\mathcal{P}$. Invoking the equivalence principle, a local inertial frame (with zero expansion and shear, see below for details) can be introduced near p. Of course, this is valid as long as the radius of the region considered is smaller than the radius of curvature at p. In this region, the metric will be approximately Minkowski, with the order of approximation determined by the local curvature as $g_{ab} = \eta_{ab} + \mathcal{O}(\epsilon^2)$. One can then define coordinates in this patch such that p is at $x^a = 0$ and connect the local inertial frame to a local Rindler frame of uniformly accelerated observers through the coordinate transformation

$$x = \chi \cosh \eta\kappa \qquad t = \chi \sinh \eta\kappa, \tag{1.64}$$

where κ is an arbitrary rescaling factor for the proper time. The spacetime in the local inertial frame will be described by the Rindler metric

$$ds^2 = -\kappa^2 \chi^2 d\eta^2 + d\chi^2 + dy^2 + dz^2, \tag{1.65}$$

where the acceleration is $a = \frac{1}{\chi}$. The past horizon of $\mathcal{P}$ is called the local Rindler horizon of $\mathcal{P}$ and is used to define a thermodynamical system, made up of the spacetime region beyond the horizon itself. One exploits the fact that at any point in spacetime there are local Rindler horizons in all null directions. Additionally, in that locally flat spacetime patch, the usual Poincaré symmetries hold, and therefore there is a Killing vector field χ^a future-pointing on the causal horizon that generates boosts vanishing at $\mathcal{P}$ and orthogonal to it. Any observer accelerating uniformly in this frame will follow an orbit associated to a group of Lorentz boost isometries generated by a Killing field χ^a. The requirement that this system is in local equilibrium at p will be crucial for the whole construction. The local flatness has to be complemented by considerations on the vacuum: it is assumed that the ground state of the fields of this spacetime can be locally approximated by the Minkowski vacuum, since this is known to be a thermal state with temperature T_U (1.63), which in this case becomes

$$T = \frac{\hbar\kappa}{2\pi} \tag{1.66}$$

(we call it simply T for the rest of this chapter). This is the Unruh temperature for the Rindler observer, κ being here the acceleration of the Killing trajectory for which χ^a has unit norm. Due to the Unruh effect, the vacuum fluctuations, seen by a uniformly accelerated observer, have thermal character [168]. The temperature of the system will then be the Unruh temperature associated to the observer, and the temperature will be an observer-dependent notion. This temperature is well-defined on the horizon and is the same within the entire Rindler wedge described by $x > |t|$.

The thermodynamical system thus defined must crucially be endowed with an entropy: this is conjectured to originate from the entanglement entropy, related to the information stored in the correlations between vacuum fluctuations on the inside and outside of the horizon. Causal horizons have entropy because they hide information [158]. This entanglement entropy scales with the area of the local boundary and diverges in the UV because the number of short-wavelength degrees of freedom diverges close to the horizon. This entropy becomes finite with the introduction of a cut-off length, of the order of Planck length. As in the case of black holes, we take the proportionality between entropy and area (1.59) to hold for Rindler horizons as well. Defining a temperature and an entropy in this way makes it possible to consider the local Rindler wedge and its Killing horizon as an analogue to a canonical ensemble, where the horizon plays a role akin to a diathermic wall. All the thermal information of the Rindler wedge is encoded in the horizon boundary, and therefore the notion of thermal equilibrium is related to the stationarity of this horizon, depending on the absence of expansion and shear around p.

The equilibrium condition is a crucial requirement in [24], and it is not a neutral choice: it selects which theory of gravity is put in correspondence with the thermodynamics, as detailed in the following.

Equilibrium thermodynamics and General Relativity

The assumed framework in Jacobson's approach is the local inertial frame for the Rindler observers described above. The two fundamental assumptions that the thermodynamics of spacetime relies on are the Clausius relation and the proportionality between area and entropy. As long as the departure from equilibrium is small, classical equilibrium thermodynamics is valid and the Clausius relation between the entropy S, the temperature T and the heat Q holds

$$dS = \frac{\delta Q}{T}. \tag{1.67}$$

In Jacobson's construction, this is required to hold for *all* local Rindler horizons, essentially amounting to a local equilibrium condition. The first assumption by Jacobson is that (1.67) relates the horizon entropy with the boost energy across the horizon itself, and T is the Unruh temperature. The second assumption is the proportionality between entropy and area, encoded in

$$dS = \alpha \delta A, \tag{1.68}$$

which implicitly takes into account a constant UV cut-off because α is chosen constant. As [175] elucidated, this choice also corresponds to a specific formulation of the Equivalence Principle and thus to the specific gravity theory at hand. In GR, the cut-off, identified with the quantum gravity scale, is the Planck length $l_{\rm Pl} = \sqrt{\frac{G\hbar}{c^3}}$, in physical units. Due to the fact that Newton's constant appears, this assumption is laden with the choice of a theory of gravity with constant gravitational coupling. In modified gravity, for example in Brans-Dicke theory, we have discussed that G is not constant anymore, but becomes a spacetime field, and therefore the UV cut-off is in general coordinate-dependent. Assuming $\alpha =$ const. corresponds to subscribing to the strong version of the Equivalence Principle (SEP) [91], which is respected only by GR (and Nordström gravity, an attempt to a scalar theory of gravity, briefly explored in 4.2.1). A theory in which G is a spacetime field is compatible merely with the so-called Einstein Equivalence Principle, corresponding to a weaker formulation [91].

It is thus assumed that the heat flow across the horizon is boost energy carried by matter, given by

$$\delta Q = \int_{\mathcal{H}} T_{ab} \chi^a d\Sigma^b, \tag{1.69}$$

where T_{ab} is the matter stress-energy tensor (whose quantum fluctuations are considered negligible) and the integral is evaluated at the horizon. The volume element is $d\Sigma^b = k^b d\lambda dA$, k^b is a tangent vector to the horizon generators for an affine parameter λ, $\chi^a = -\kappa \lambda k^a$, and dA is the area element on a cross-section of the horizon. The heat flux can thus be written as

$$\delta Q = -\kappa \int_{\mathcal{H}} \lambda T_{ab} k^a k^b d\lambda d\mathcal{A}. \tag{1.70}$$

With the assumption of the proportionality between area and entropy, the relationship between the two variations of the area is

$$\delta A = \int_{\mathcal{H}} \theta d\lambda dA, \tag{1.71}$$

and θ is the expansion of the horizons generators. Requiring the Clausius relation to hold essentially requires that the energy flux is associated to a focusing of the horizon generators. The evolution of the null geodesic congruence that generates the horizon is given by the Raychaudhuri equation

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^2 - R_{ab}k^a k^b, \tag{1.72}$$

where $\sigma^2 = \sigma^{ab}\sigma_{ab}$ is the shear scalar, which vanishes together with the expansion at $\mathcal{P}$, since we made this choice at the beginning in order to ensure local equilibrium. For small λ, we can integrate this to obtain

$$\theta = -\lambda R_{ab}k^a k^b, \tag{1.73}$$

which we can substitute into (1.71) to yield

$$\delta A = -\int_{\mathcal{H}} \lambda R_{ab}k^a k^b d\lambda d\mathcal{A}. \tag{1.74}$$

Substituting the Unruh temperature (1.66) into the Clausius relation (1.67), we obtain

$$\delta Q = TdS = (\hbar\kappa/2\pi)\alpha\delta A, \tag{1.75}$$

which, taking into account (1.74) and (1.70), can only hold if

$$T_{ab}k^a k^b = (\hbar\alpha/2\pi)R_{ab}k^a k^b \tag{1.76}$$

for all null k^a, implying that

$$(2\pi/\hbar\alpha)T_{ab} = R_{ab} + f g_{ab} \tag{1.77}$$

for some generic function f. Assuming the local energy conservation $\nabla^b T_{ab} = 0$, applying the divergence on both sides of (1.77), and using the contracted Bianchi identity $\nabla^b R_{ab} = \frac{1}{2}\nabla_a R$, one obtains

$$f = -\frac{R}{2} - \Lambda, \tag{1.78}$$

for an arbitrary integration constant Λ. The Einstein equation then holds automatically:

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = \frac{2\pi}{\hbar\alpha}T_{ab}. \tag{1.79}$$

The constant α, left undetermined in the previous steps, can now be fixed as

$$\alpha = \frac{1}{4\hbar}, \tag{1.80}$$

so that $\sqrt{\alpha} = \dfrac{1}{2l_{\mathrm{Pl}}}$. Therefore, assuming entropy and energy conservation for this system, the local equilibrium condition (the Clausius relation (1.67), which can be recast in the form of an equation of state by using the first law of thermodynamics) is equivalent to the Einstein equations.

The reasoning above can be applied to any spacetime point because of the equivalence principle, and it is in this profound sense that Jacobson constructed a thermodynamics of the whole spacetime, highlighting the intrinsic thermodynamical properties interwoven in the fabric of spacetime itself and in the laws of gravity. This analogy is extremely suggestive, and reinforces the idea mentioned in the previous sections that, looking at the laws of gravity from this thermodynamic perspective, it is reasonable to wonder whether it makes sense at all to try to quantize it. Jacobson uses the expression that this would amount to "quantizing the wave equation for sound". The question that Jacobson asks, considering the correspondence between the laws of black hole mechanics and those of thermodynamics is "how did classical GR know that the horizon area would turn out to be a form of entropy, and that surface gravity is a temperature?" [24]. The answer he provides turns the logic around: it is because the Einstein equations are equations of state, connecting these two realms, that these correspondences can hold. Gravitational phenomena on a macroscopic scale arise from the thermodynamical behaviour of the spacetime vacuum at microscopic scales.

Non-equilibrium thermodynamics and modified gravity

The reasoning above is even more general than has been presented so far, and intriguingly holds beyond General Relativity for modified gravity theories. It is straightforward to see that allowing for the presence of shear at p disrupts the whole construction above, as the equilibrium condition strictly does not hold anymore. Choosing a local inertial frame without shear is, however, arbitrary. Hence, the question of what happens in cases with shear naturally arises. Additionally, the assumption of $\alpha = \text{const.}$ also sets a considerable restriction. It is also fair to wonder whether the case of GR is special, whether it is merely an accident, or if maybe the answer was implicitly assumed in the question. In the following, based on [25], we show that this is not the case.

An extension to non-equilibrium thermodynamics allows for a generic description that encompasses not only less fine-tuned situations, but also other theories of gravity. In [25], the derivation above was generalized to a non-equilibrium setting, where

$$dS > \delta Q/T, \tag{1.81}$$

and specifically

$$dS = \delta Q/T + d_i S, \tag{1.82}$$

where the internal entropy production $d_i S$ is caused by the fact that the system is out of equilibrium. This internal energy production can further be associated to the non-vanishing shear terms and be interpreted as being due to internal viscosity [175] (see also 2.3 for a more general discussion of non-equilibrium thermodynamics). We now set out to find the equation of state if the entropy density is α times a function of the Ricci scalar $f(R) = 1+\mathcal{O}(R)$, instead of simply a constant α as in the previous case. The change in entropy, analogously to (1.71) is

$$\delta S = \alpha \int (\theta f + \dot{f}) d\lambda dA, \tag{1.83}$$

where f is distinct from that of the previous section and $\dot{f} \equiv \mathrm{d}f/\mathrm{d}\lambda$. Now when the expansion vanishes, this integral is non-zero at p, but it cannot match the $\delta Q/T$ integrand anymore as it was possible for (1.70), because that is of order λ. This means that $\theta(p)$ cannot vanish because it needs to cancel with the derivative of f, namely

$$(\theta f + \dot{f})(p) = 0. \tag{1.84}$$

This translates to the fact that the horizon area is now changing at p because the equilibrium condition is not valid anymore, together with the assumption of vanishing expansion. In order to obtain the $\mathcal{O}(\lambda)$ term in the integrand of the following δS,

$$\delta S = \alpha \int (\theta f + \dot{f}) d\lambda d^2 A \tag{1.85}$$

we derive with respect to λ and use (1.84)

$$\frac{d}{d\lambda}(\theta f + \dot{f})\bigg|_{\lambda=0} = \dot{\theta} f - f^{-1}\dot{f}^2 + \ddot{f}. \tag{1.86}$$

Upon use of the Raychaudhuri equation (1.72) and the geodesic equation $k^a \nabla_a k^b = 0$, this becomes

$$-k^a k^b \left(f R_{ab} - \nabla_a \nabla_b f + f^{-1} \nabla_a f \nabla_b f \right) - \frac{1}{2} f \theta^2. \tag{1.87}$$

The argument used to prove that there must be some internal energy production $d_i S$ is a *reductio ad absurdum*, showing that $d_i S = 0$ would lead to inconsistency with the conservation of energy. Namely, without the $d_i S$ term, (1.87) must equal the λ coefficient in the heat flux integrand of

$$\frac{\delta Q}{T} = (2\pi/\hbar) \int T_{ab} k^a k^b (-\lambda) d\lambda dA \tag{1.88}$$

for all null vectors k^a (where T is again the Unruh temperature in the case where the acceleration $a = 1$). Thus, instead of (1.77) for the case of equilibrium and GR, we would now have

$$f R_{ab} - \nabla_a \nabla_b f + \frac{3}{2} f^{-1} \nabla_a f \nabla_b f + \Psi g_{ab} = (2\pi/\hbar\alpha) T_{ab}, \tag{1.89}$$

where (1.84) was used to rewrite θ^2 and Ψ is an unknown function. However, this leads to a contradiction. If T_{ab} is divergence free (which means that energy is conserved, or equivalently, that the first law of thermodynamics holds) and we use the contracted Bianchi identity as in the equilibrium case, in addition to exploiting the commutator of covariant derivatives $[\nabla_a, \nabla_b] v^c = R_{abd}{}^c v^d$, and defining $\mathcal{L}$ through $f = d\mathcal{L}/dR$, we obtain

$$\nabla^a (f R_{ab} - \nabla_a \nabla_b f) = \nabla_b \left(\frac{1}{2}\mathcal{L} - \Box f \right). \tag{1.90}$$

This implies

$$\Psi = \Box f - \frac{1}{2}\mathcal{L} - \Theta, \tag{1.91}$$

where the gradient of Θ must equal the divergence of the residual term in (1.89), namely

$$\nabla_b \Theta = \nabla^a \left(\frac{3}{2f} \nabla_a f \nabla_b f \right). \tag{1.92}$$

However, this incurs into a contradiction, since the right-hand side of the above equation is in general not the gradient of a scalar. The proposition of [25] is that such contradiction is resolved by the introduction of the entropy production term $d_i S$. Specifically, the troublesome term would vanish upon setting

$$d_i S = \int \sigma d\lambda dA \tag{1.93}$$

where σ is the entropy production density

$$\sigma = -\frac{3}{2}\alpha f^{-1} \dot{f}^2 \lambda = -\frac{3}{2}\alpha f \theta^2 \lambda. \tag{1.94}$$

This has the same form of the entropy production term for a fluid with temperature T, if it is due to the presence of bulk viscosity with coefficient $\zeta = (3/2)\alpha f T$. If $T = \hbar/2\pi$, then

$$\zeta = 3\hbar\alpha f/4\pi. \tag{1.95}$$

With the above choice of $d_i S$, (1.82) at $\mathcal{O}(\lambda)$ implies the equation of state

$$f R_{ab} - \nabla_a \nabla_b f + \left(\Box f - \frac{1}{2}\mathcal{L} \right) g_{ab} = (2\pi/\hbar\alpha) T_{ab}, \tag{1.96}$$

which is the equation of motion arising from the Lagrangian $(\hbar\alpha/4\pi)\mathcal{L}(R)$ and again $G = (4\hbar\alpha)^{-1}$. Therefore, the field equations of a generic $f(R)$ theory of gravity (as denoted in section 1.2.2) are recovered as well as equations of state in a non-equilibrium setting.

In [175], it was shown that the dissipative terms present in such a non-equilibrium description are related to non-local heat fluxes associated to the internal degrees of freedom of purely gravitational nature. This spacetime viscosity is interpreted as being due to gravitational fluxes, intended as local curvature perturbations, related to the Weyl conformal contributions and not appearing in the field equations. $d_i S$ is positive for irreversible processes, and the generalized form of the Clausius relation (1.82) is written as

$$dS = \frac{\delta Q}{T} + \delta N, \tag{1.97}$$

where the authors make a distinction between compensated heat δQ and uncompensated heat δN, arising from the contribution of irreversible processes. These processes cannot be anything but the effect of the internal and purely gravitational degrees of freedom of the theory. The authors of [175] physically interpret the entropy production as caused by the viscous stresses and the shear contribution. Within this framework, the uncompensated heat term, quantifying the energy of the system which gets dissipated in viscous processes is found to coincide with the expression for tidal heating of a classical black hole. This internal entropy production can therefore be identified with the work done on the horizon by the tidal field described by the electric part of the Weyl curvature tensor. In the modified gravity case, the authors find a straightforward generalisation of this tidal heating, which is the term related to the shear. One can also introduce a bulk viscosity in the GR case, as in [25], which considered the dissipative entropy contribution to be due to bulk viscosity.

In the modified gravity context, one can exploit the equivalence between $f(R)$ and scalar-tensor theories described in section 1.2.2 to give a neat interpretation of the viscous terms as those responsible for the internal energy loss due to gravitational energy fluxes through the horizon of both scalar and tensorial origin [175]. The expression for the viscous contribution looks very similar to that of the energy loss rate due to the emission of gravitational waves in scalar-tensor gravity. The reversible and irreversible contributions appear neatly separated: the reversible ones are of local nature and are always related to the Ricci curvature (and therefore the sources of the gravitational field), while the non-equilibrium terms are intrinsically non-local and related to the curvature components of the Weyl tensor, independent from the source distribution. Such dissipative effects should be interpreted as consequences of the presence of some underlying spacetime fluctuations at the UV cutoff, since the bulk and shear viscosity coefficients are always related to the UV cutoff scale through the energy density. The equivalence between $f(R)$ and Brans-Dicke theories was further explored in [176], arguing that only shear viscosity should be attributed to the irreversible part of the thermodynamics, while bulk viscosity turns out to belong to the reversible part.

1.3.4 Emergent gravity

An approach that has elucidated very general properties of the relationship between gravity and thermodynamics, after Jacobson's developments, is that of Padmanabhan, who laid down a programme for a deeper understanding of gravity through thermodynamics (see [177] for a review of these efforts). These works pinned down the fundamental threads along which gravity and thermodynamics are related and generalized the description as much as possible to cover a diverse set of situations.

Firstly, one of the main results of this line of research is that the gravitational field equations in a wide variety of theories (GR with several types of horizons, Lovelock gravity with different types of horizons, theories formulated in a different number of dimensions, various theories in a cosmological background, Hořava-Lifshitz gravity) can be recast as the thermodynamics identity $TdS = dE + PdV$, when evaluated on a horizon [177, 178]. Secondly, the actions of wide classes of theories can be divided into a bulk and a surface term. The latter is generally ignored and the field equations are obtained purely from the bulk term. It turns out that when the surface term is evaluated at the horizon of any solution of the theory (computed ignoring this same surface term), this yields the entropy of the horizon. This result seems to hold also beyond Einstein theory, and even in situations where the entropy is not proportional to the horizon area. This particular property points to the holographic relationship between the bulk and surface terms and the holographic character of gravity itself [179]. Thirdly, The Euclidean action in any static spacetime has an interpretation as the free energy, connecting the minimization of the action with the minimization of the free energy, again in a wide class of theories of gravity.

Inspired by Boltzmann's insight that any material that can be heated and is endowed with a temperature must have a microstructure (even if this cannot directly be observed) since the energy in form of heat must be stored in these microscopic degrees of freedom, this emergent gravity approach by Padmanabhan also suggested that spacetime must contain a microstructure, the quantum degrees of freedom from which gravity emerges, since it too can be "heated" and possesses a temperature. Based on those three main results, that cannot be explained in the conventional approach to gravity, where the thermodynamical aspects of gravity are simply interpreted as by-products of a semi-classical description involving QFT in a curved spacetime, Padmanabhan developed a new paradigm where these hints are taken to mean something more radical, although the final answer of what the microscopic constituents of spacetime giving rise to these thermodynamical properties still remains unanswered.

Jacobson's and Padmanabhan's approach are similar, but also exhibit some key differences (see, for example, [180]), especially in how matter fluxes across the horizon are defined and treated. Padmanabhan's analysis emphasizes the change in quasi-local energy associated with the horizon's surface. In contrast, Jacobson's derivation is based on the null generators and the Clausius relation, which involves additional terms and assumptions such as the vanishing expansion of the null congruence. Overall, the differences in the treatment of matter fluxes and horizon deformations result in distinct interpretations and implications for the thermodynamic structure of the gravitational field equations. Additionally, a subtle difference is that Jacobson performs a derivation of the field equations from thermodynamical arguments, while Padmanabhan provides a thermodynamical interpretation of them [181]. Padmanabhan takes the stance that the assumption of proportionality between area and entropy, made by Jacobson, might bias the results towards Einstein gravity and prevent a generalisation to broader classes of theories. He thus considers the Wald entropy of the horizon, which is not always proportional to the area and arises as the conserved Noether charge from the diffeomorphism invariance of the theory [182].

After these developments, another intriguing line of research was initiated by Verlinde [183], building upon the holographic properties that gravity seems to possess. In this view, Newton's law of gravitation arises naturally if space emerges from a holographic scenario, and gravity can be understood as an entropic force, caused by the change of information stored on a holographic screen, when material bodies are moving with respect to it. Einstein's equations are then found when these arguments are generalized to a relativistic setting. Despite more work in this direction, breakthroughs have arguably not been achieved yet.

Concluding this section, we remark that, despite the several lines of research investigating the relationship between gravity and thermodynamics and the progress made in our understanding of it, many questions still remain open. In the following chapters, our goal will be to lay down the foundations for a novel idea that provides a fresh perspective on this issue, especially in the realm of modified gravity. While we do not attempt to provide any definitive answer to these deep questions about the true nature of gravity, we will show how this new perspective can yield a unifying description of modified gravity theories based on thermodynamical considerations.

Chapter 2

Effective fluid formalism and non-equilibrium thermodynamics

2.1 Motivations and applications

Now that we have motivated the study of extensions and generalisations of Einstein's General Relativity, it is fruitful to make some considerations about the form of the metric field equations for such theories. In the previous chapter, we have asserted that modifying gravity translates into a modification of the left-hand side of the Einstein field equations, through the addition of more curvature terms or additional degrees of freedom there. At least in principle, this approach is distinct from altering the right-hand side of the equations, which would amount to introducing some additional form of matter or energy through the stress-energy tensor.

However, altering the right-hand side or the left-hand side are not fundamentally different approaches, because one can always reformulate one into the language of the other. This is the essence of GR, and is precisely what we exploit in the main topic of this thesis: bringing modified gravity into the form of modified matter. This is valid especially in modified theories including additional degrees of freedom to GR, such as scalar-tensor or vector-tensor theories (although theories with higher order curvature invariants in the action, i.e., $f(R)$ theories, can always be recast as scalar-tensor theories, hence these considerations are valid for them as well). These additional fields are responsible for the gravitational effects and therefore should be on the left-hand side of the Einstein equations. Generally, the distinction is that fields non-minimally coupled to the curvature are gravitational, while minimally coupled fields are matter fields. However, as pointed out in [88], since this distinction falls apart when switching from the Jordan to the Einstein frame through a conformal transformation and a field redefinition (we deal with this in more detail in section 4.3), it is not as fundamental as one might think and harbors some essential ambiguity. Additionally, [88] points out that the distinction between minimally and non-minimally coupled fields as matter or gravitational fields incurs into problems when, quantizing a minimally coupled field, the first loop corrections turn out to involve non-minimally coupling terms, if the theory is renormalizable.

We will therefore exploit the freedom of bringing "gravitational" terms onto the right-hand side of the field equations and considering them as matter-energy terms.

Specifically, the metric field equations of scalar-tensor gravity can be recast as simpler *effective* Einstein equations, by keeping only the geometric terms present in the Einstein tensor G_{ab} on the left-hand side, and moving all the other terms to the right-hand side, including them into an *effective* stress-energy tensor. This approach has the advantage of translating complicated problems in modified gravity to simpler problems in GR with a specific matter content and has proven to be quite fruitful.

To the best of our knowledge, the first instance of this approach is [184], where the energy-momentum tensor of a scalar field was interpreted as that of a perfect fluid and its equation of state was studied. A few years later, the fact that the stress-energy tensor of a real scalar coupled to Einstein gravity with a coupling of the form $\xi\phi^2 R$ has the form of an imperfect fluid was established by the same author [185], and the fluid's hydrodynamical quantities were explicitly computed. This work clearly spells out that the only necessary assumption for the validity of the fluid interpretation is the "timelike nature of the spacetime gradient of the scalar field". The paper remarks the lack of a satisfactory rigorous argument for why the gradient should always be timelike, and connects it to the weak energy condition being satisfied, although dealing with an effective fluid, the validity of such energy conditions is not expected in general. Following this, [20] generalised the result to "old-school" scalar-tensor theories including Brans-Dicke, which, at the time, were thought to be the most general scalar-tensor theories.

The results mentioned so far are valid in general and do not make any reference to the presence of particular symmetries of the Lagrangian, but the effective fluid formalism has a rich literature especially in the context of FLRW spacetime. This is explained not only with the fact that cosmology is the natural arena for scalar-tensor theories, but also with the relevance of scalar-tensor theories as alternatives to dark energy, as explained in the previous chapter. However, it is crucial to note that the effective fluid formalism we are interested in for the purposes of this thesis is technically different from the formalism of the same name, which is commonly used in the context of dark energy (see, for example, [186] and [187] for reviews). This other effective fluid formalism allows for the comparison between modified gravity theories as dark energy with cosmological data, and has been invaluable for constraining the plethora of different models. Of course, the two approaches start from the same premise: treating modified gravity models as GR with an effective fluid that includes the terms with additional degrees of freedom. The formalism we will use in the following and this formalism employed for confrontation with observational data both involve writing the effective Einstein equations and separating the contributions to the stress-energy that are due to the additional degrees of freedom. Interpreting the additional terms for gravity as an effective dark energy fluid, one studies its perturbations and its time evolution, to see if it matches observational data. In this context, the fluid is an ideal dark energy fluid, described by an equation of state, a sound speed and other properties that can be simplified under some approximations. This allows for the inclusion of many modified gravity theories into the Einstein-Boltzmann codes that compute the cosmological perturbations. Modified gravity can thus be tested against observational data, most notably the surveys of the universe's large scale structure, which represent crucial tests for constraining these theories and will be increasingly important in the future, such as [47]. However, it is not possible to discriminate between the effects of modified gravity and GR with an additional exotic dark energy fluid, because of this inherent ambiguity in moving the modifications of gravity on the right-hand or left-hand side of the Einstein equations. Other interesting fluid scenarios for (unified) dark energy and dark matter include [188–190]. The main difference with our approach is that we do not take into account perturbations of the fluid and only stick to the background level, because we use the fluid for a thermodynamical analogy, as we will see in the following, and are not interested in phenomenology at this stage. Our goal is not to provide an explanation of the evolution of the universe, but rather a unifying perspective on modified gravity theories exploiting the thermodynamical description of the fluid, at a purely theoretical level. Other works with a similar approach include [191, 192].

The goal of our work is more similar to that of [22], which gives the best supporting argument for the advantages presented by the effective fluid approach and motivates our study. These authors investigated the fluid "behind" scalar-tensor theories exhibiting kinetic gravity braiding, or KGB (characterized by the fact that the derivatives of the scalar and of the metric are mixed in an essential way which cannot be eliminated by performing a field redefinition, thus necessarily modifying gravity). This approach greatly simplifies the study of such theories and obscure combinations of terms in the Lagrangian are given a physically meaningful interpretation, clarifying the complex dynamics of the system. The authors of [22] follow the work of Schutz [193] to interpret the derivative of the scalar field with respect to proper time as the chemical potential and base their description on this notion. In section 4.3 we show that this is an alternative but equivalent approach to ours, which is instead based on the notion of temperature. The choice in [22] is not unique, as the conserved charges arising from shift-symmetry could correspond to entropy instead of particles, thus exchanging number density with entropy density and chemical potential with temperature, through the first law of thermodynamics. Their choice is motivated by the fact that entropy is not conserved if we make this exchange. We find the same result but explain in the following why this lack of conservation is not an issue.

In general, the fluid arising from KGB is out of equilibrium and imperfect, like the one we will describe in the following sections, but does not exhibit dissipation, at variance with our description. The energy flow is the diffusion along gradients of the chemical potential, and diffusion occurs without dissipation. The fluid has zero temperature and shear viscosity, and is dubbed "imperfect superfluid". Theories exhibiting KGB and their fluid interpretation have particularly interesting applications to dark energy [112]. For more generic scalar-tensor theories, approaches like that of [194] have been developed, for example for Horndeski and DHOST theories.

2.2 Effective fluid formalism for scalar-tensor theories

While the fact that modified gravity can be described by an effective fluid was realized in the previous works analysed above, in [19], and subsequently in [10, 11], the effective fluid picture was extended and completed to set the stage for the development of first-order thermodynamics. The novelty with respect to previous works is that for the first time the effective fluid was considered endowed with thermodyamical properties.

In [19], the authors found expressions for the thermodynamical quantities derived from the stress-energy tensor, but stopped short of interpreting them as a tool to give a concrete realisation of the "thermodynamics of gravitational theories", which was hinted at in broad strokes by Jacobson. In [10, 11], this aim was stated clearly and the formalism that we call "first-order thermodynamics of scalar-tensor theories" was initially constructed.

Let us consider a generic scalar-tensor theory, made up of the tensor degree of freedom of GR and an additional scalar. For the purposes of the following paragraph, it suffices to know that a scalar degree of freedom is present in the theory, as the definitions given in the following only rely on the presence of a scalar field gradient, are purely kinematic and do not depend on the specific action at hand. Hence, they remain the same also in extensions of the formalism, such as that to Horndeski theories, studied in section 4.1.

2.2.1 Kinematic quantities

As recognised already in [185], the only fundamental assumption on which the effective fluid formalism rests is the presence of a scalar field gradient $\nabla_c\phi$ that is timelike (i.e., $\nabla^c\phi\nabla_c\phi < 0$). Additionally, it is crucial that the gradient is also future-oriented (i.e., $g_{ab}\, u^a\, (\partial_t)^b > 0$ for the time coordinate t). In other words, the vector lies in the future half of the light cone (see section 3.2 for a generalisation to past-oriented gradients). Both of these requirements are essential to define

$$u^a = \frac{\nabla^a\phi}{\sqrt{-\nabla^e\phi\nabla_e\phi}} \tag{2.1}$$

as the 4-velocity of the effective fluid, with normalisation $u^a u_a = -1$ [10, 11]. This choice highlights the presence of a built-in arrow of time, with the scalar field playing the role of an internal clock for the fluid, which will be relevant when describing dissipative processes. Choosing a velocity parallel to the gradient of ϕ makes the fluid irrotational by definition. The existence of a preferred velocity at each point additionally implies the existence of a preferred rest frame at each point.

This definition allows for the standard $3+1$ splitting of spacetime into the time direction u^c and the 3-dimensional space of comoving observers of this effective fluid (see [195] for details). This 3-space is endowed with the induced metric

$$h_{ab} \equiv g_{ab} + u_a u_b \tag{2.2}$$

and ${h_a}^b$ is the projection operator on this 3-space orthogonal to u^a, so that

$$h_{ab}u^a = h_{ab}u^b = 0, \tag{2.3}$$

$$h^a{}_b h^b{}_c = h^a{}_c, \quad h^a{}_a = 3. \tag{2.4}$$

The 4-acceleration of the fluid is $\dot{u}^a \equiv u^b\nabla_b u^a$ and is orthogonal to the four-velocity, i.e., $\dot{u}^c u_c = 0$. Projecting the velocity gradient onto the 3-space of the comoving observers, we obtain

$$V_{ab} \equiv h_a{}^c h_b{}^d \nabla_d u_c, \tag{2.5}$$

which can be decomposed as

$$V_{ab} = \theta_{ab} + \omega_{ab} = \sigma_{ab} + \frac{\theta}{3} h_{ab} + \omega_{ab}, \tag{2.6}$$

where the symmetric part $V_{(ab)} \equiv \theta_{ab}$ has trace $\theta \equiv \theta^c{}_c = \nabla^c u_c$, which is the expansion scalar, and the antisymmetric part $V_{[ab]} \equiv \omega_{ab}$ is the vorticity tensor, while

$$\sigma_{ab} \equiv \theta_{ab} - \frac{\theta}{3} h_{ab} \tag{2.7}$$

is the trace-free shear tensor. $V_{ab}, \theta_{ab}, \sigma_{ab}$, and ω_{ab} are purely spatial, meaning that

$$\theta_{ab} u^a = \theta_{ab} u^b = \omega_{ab} u^a = \omega_{ab} u^b = \sigma_{ab} u^a = \sigma_{ab} u^b = 0, \tag{2.8}$$

and $\sigma^a{}_a = \omega^a{}_a = 0$. One can also define the squared shear scalar σ and vorticity scalar ω as

$$\sigma^2 \equiv \frac{1}{2} \sigma_{ab} \sigma^{ab} \geq 0$$
$$\omega^2 \equiv \frac{1}{2} \omega_{ab} \omega^{ab} \geq 0.$$

This $3+1$ decomposition is of course valid for any generic u^a, but, with the choice of (2.1) as the gradient of a scalar field, we restrict to an irrotational fluid, where $\omega_{ab} = \omega^2 = 0$.

The gradient of the velocity can therefore be written as

$$\nabla_b u_a = \sigma_{ab} + \frac{\theta}{3} h_{ab} + \omega_{ab} - \dot{u}_a u_b = V_{ab} - \dot{u}_a u_b. \tag{2.9}$$

The projection of this gradient onto u^c recovers $\dot{u}_a$, while its projection onto the 3-space of comoving observers yields V_{ab}.

So far, these definitions are purely kinematic and completely generic. Specifying to the u^a relevant throughout this thesis and valid for any scalar-tensor theory, namely (2.1), the definitions of the induced metric and the velocity gradient become

$$h_{ab} = g_{ab} - \frac{\nabla_a \phi \nabla_b \phi}{\nabla^e \phi \nabla_e \phi} \tag{2.10}$$

and

$$\nabla_b u_a = \frac{1}{\sqrt{-\nabla^e \phi \nabla_e \phi}} \left(\nabla_a \nabla_b \phi = \frac{\nabla_a \phi \nabla^c \phi \nabla_b \nabla_c \phi}{\nabla^e \phi \nabla_e \phi} \right). \tag{2.11}$$

The 4-acceleration then becomes

$$\dot{u}_a = u^c \nabla_c u_a = \left(-\nabla^e \phi \nabla_e \phi\right)^{-2} \nabla^b \phi \left[\left(-\nabla^e \phi \nabla_e \phi\right) \nabla_a \nabla_b \phi + \nabla^c \phi \nabla_b \nabla_c \phi \nabla_a \phi \right]. \tag{2.12}$$

The condition for the worldlines of the ϕ-fluid to be geodesics reads

$$\nabla^e \phi \nabla_{[e} \phi \nabla_{a]} \nabla_b \phi \nabla^b \phi = 0, \tag{2.13}$$

and therefore, for a geodesic flow, the following equation holds

$$\nabla^b\nabla^c\phi\nabla_b\nabla_c\phi = -\frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}, \tag{2.14}$$

so that V_{ab} reduces to

$$\begin{aligned} V_{ab} =& \frac{\nabla_a\nabla_b\phi}{\left(-\nabla^e\phi\nabla_e\phi\right)^{1/2}} + \frac{\left(\nabla_a\phi\nabla_b\nabla_c\phi + \nabla_b\phi\nabla_a\nabla_c\phi\right)\nabla^c\phi}{\left(-\nabla^e\phi\nabla_e\phi\right)^{3/2}} \\ &+ \frac{\nabla_d\nabla_c\phi\nabla^c\phi\nabla^d\phi}{\left(-\nabla^e\phi\nabla_e\phi\right)^{5/2}}\nabla_a\phi\nabla_b\phi. \end{aligned} \tag{2.15}$$

For an irrotational fluid,

$$V_{ab} = \theta_{ab}, \quad \nabla_b u_a = \theta_{ab} - \dot{u}_a u_b. \tag{2.16}$$

Due to the fact that $u_a\dot{u}^a = 0$, the expansion scalar becomes

$$\theta = \nabla_a u^a = \frac{\Box\phi}{\left(-\nabla^e\phi\nabla_e\phi\right)^{1/2}} + \frac{\nabla_a\nabla_b\phi\nabla^a\phi\nabla^b\phi}{\left(-\nabla^e\phi\nabla_e\phi\right)^{3/2}}, \tag{2.17}$$

while the shear tensor reads

$$\begin{aligned} \sigma_{ab} =& \left(-\nabla^e\phi\nabla_e\phi\right)^{-3/2}\left[-\left(\nabla^e\phi\nabla_e\phi\right)\nabla_a\nabla_b\phi\right. \\ &-\frac{1}{3}\left(\nabla_a\phi\nabla_b\phi - g_{ab}\nabla^c\phi\nabla_c\phi\right)\Box\phi - \frac{1}{3}\left(g_{ab} + \frac{2\nabla_a\phi\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right)\nabla_c\nabla_d\phi\nabla^d\phi\nabla^c\phi \\ &\left.+\left(\nabla_a\phi\nabla_c\nabla_b\phi + \nabla_b\phi\nabla_c\nabla_a\phi\right)\nabla^c\phi\right], \end{aligned} \tag{2.18}$$

and the shear scalar reads

$$\begin{aligned} \sigma =& \left(-\nabla^e\phi\nabla_e\phi\right)^{-3/2}\left\{\frac{1}{2}\left(\nabla^e\phi\nabla_e\phi\right)^2\left[\nabla^a\nabla^b\phi\nabla_a\nabla_b\phi - \frac{1}{3}(\Box\phi)^2\right]\right. \\ &\left.+\frac{1}{3}\left(\nabla_a\nabla_b\phi\nabla^a\phi\nabla^b\phi\right)^2 - \left(\nabla^e\phi\nabla_e\phi\right)\left(\nabla_a\nabla_b\phi\nabla^b\nabla_c\phi - \frac{1}{3}\Box\phi\nabla_a\nabla_c\phi\right)\nabla^a\phi\nabla^c\phi\right\}^{1/2}. \end{aligned} \tag{2.19}$$

We can now specialize to the action for scalar-tensor theories that will be mainly employed throughout this thesis. This is the context in which first-order thermodynamics was first developed in [11] and serves as the prototype for all other extensions of the formalism. The initial idea was to start from a scalar-tensor action sufficiently generic to include metric $f(\mathcal{R})$ gravity, which was the case studied in [25] and made it possible to connect modified gravity to a non-equilibrium thermodynamical description, as reviewed in section 1.3. A slightly generalised version of the Brans-Dicke action serves this purpose: the Brans-Dicke parameter ω present in (1.43) is promoted to a function of the scalar field: $\omega(\phi)$. The action then reads (in the Jordan frame)

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla^c\phi\nabla_c\phi - V(\phi)\right] + S^{(\mathrm{m})}, \tag{2.20}$$

where the Brans-Dicke scalar $\phi > 0$ is approximately the inverse of the effective gravitational coupling $G_{\text{eff}}(\phi) \simeq \frac{1}{\phi}$, $\omega(\phi)$ is the Brans-Dicke coupling and $S^{(\text{m})}$ is the matter action. This action is sufficiently generic to encompass Brans-Dicke theories described by (1.43) and $f(R)$ gravity. It is not as general as Horndeski, but it is much simpler and ensures that the calculations remain straightforward and a physical interpretation of the meaning behind the formalism can be attempted.

The field equations are [86, 97, 98]

$$\begin{aligned} G_{ab} \equiv\; & R_{ab} - \frac{1}{2}\, g_{ab} R = \frac{8\pi}{\phi}\, T_{ab}^{(\text{m})} + \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}\, g_{ab}\nabla_c\phi\nabla^c\phi\right) \\ & + \frac{1}{\phi}\left(\nabla_a\nabla_b\phi - g_{ab}\Box\phi\right) - \frac{V}{2\phi}\, g_{ab} \end{aligned} \tag{2.21}$$

$$\Box\phi = \frac{1}{2\omega+3}\left(8\pi T^{(\text{m})} + \phi\, V_{,\phi} - 2V - \omega_{,\phi}\, \nabla^c\phi\nabla_c\phi\right)\,, \tag{2.22}$$

where R_{ab} is the Ricci tensor, $T^{(\text{m})} \equiv g^{ab} T_{ab}^{(\text{m})}$ is the trace of the matter stress-energy tensor $T_{ab}^{(\text{m})}$, and $\omega_{,\phi} \equiv d\omega/d\phi$, $V_{,\phi} \equiv dV/d\phi$.

The stress-energy tensor of the effective fluid represented by the scalar contributions can be read off the right-hand side of (2.21):

$$8\pi T_{ab}^{(\phi)} = \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}\, g_{ab}\nabla^c\phi\nabla_c\phi\right) + \frac{1}{\phi}\left(\nabla_a\nabla_b\phi - g_{ab}\Box\phi\right) - \frac{V}{2\phi}\, g_{ab}. \tag{2.23}$$

2.2.2 Effective stress-energy tensor

$T_{ab}^{(\phi)}$ has the form of the stress-energy tensor of an imperfect fluid [19, 20]

$$T_{ab} = \rho\, u_a u_b + q_a u_b + q_b u_a + \Pi_{ab}\,, \tag{2.24}$$

where the comoving effective energy density, heat flux density, stress tensor, isotropic pressure, and anisotropic stresses (the trace-free part π_{ab} of the stress tensor Π_{ab}) are, respectively,

$$\rho = T_{ab} u^a u^b \tag{2.25}$$

$$q_a = -T_{cd}\, u^c h_a{}^d \tag{2.26}$$

$$\Pi_{ab} = P h_{ab} + \pi_{ab} = T_{cd}\, h_a{}^c\, h_b{}^d \tag{2.27}$$

$$P = \frac{1}{3}\, g^{ab}\Pi_{ab} = \frac{1}{3}\, h^{ab} T_{ab} \tag{2.28}$$

$$\pi_{ab} = \Pi_{ab} - P h_{ab}. \tag{2.29}$$

From the physical point of view, in an imperfect fluid the mean free paths and mean free times of the fluid molecules are so short that isotropy is maintained to a good approximation, at any point in the fluid. This is equivalent to say that the mean time between collisions is much less than any characteristic macroscopic time.

Equivalently, the mean free path is much less than any characteristic macroscopic length. If the fluid quantities such as pressure and density vary to an appreciable degree over distances of the order of a mean free path, or over times of the order of a mean free time, equilibrium cannot be easily maintained in the fluid, and dissipative phenomena occur [90]. Properly dealing with dissipative effects occurring in relativistic fluids is far from straightforward and delicate considerations have to be made with respect to the non-relativistic case (this is covered in section 2.3).

The heat flux density is purely spatial,

$$q_c u^c = 0 \tag{2.30}$$

and

$$\Pi_{ab}u^b = \pi_{ab}u^b = \Pi_{ab}u^a = \pi_{ab}u^a = 0, \quad \pi^a{}_a = 0. \tag{2.31}$$

Starting from the action (2.20) we can compute the effective thermodynamical quantities (2.25)-(2.29) as in [19]

$$8\pi\rho^{(\phi)} = -\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi + \frac{V}{2\phi} + \frac{1}{\phi}\left(\Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right), \tag{2.32}$$

$$\begin{aligned} 8\pi q_a^{(\phi)} &= \frac{\nabla^c\phi\nabla^d\phi}{\phi\left(-\nabla^e\phi\nabla_e\phi\right)^{3/2}}\left(\nabla_d\phi\nabla_c\nabla_a\phi - \nabla_a\phi\nabla_c\nabla_d\phi\right) \\ &= -\frac{\nabla^c\phi\nabla_a\nabla_c\phi}{\phi\left(-\nabla^e\phi\nabla_e\phi\right)^{1/2}} - \frac{\nabla^c\phi\nabla^d\phi\nabla_c\nabla_d\phi}{\phi\left(-\nabla^e\phi\nabla_e\phi\right)^{3/2}}\nabla_a\phi, \end{aligned} \tag{2.33}$$

$$\begin{aligned} 8\pi\Pi_{ab}^{(\phi)} &= \left(-\nabla^e\phi\nabla_e\phi\right)^{-1}\left[\left(-\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi - \frac{\Box\phi}{\phi} - \frac{V}{2\phi}\right)\right. \\ &\left(\nabla_a\phi\nabla_b\phi - g_{ab}\nabla^e\phi\nabla_e\phi\right) - \frac{\nabla^d\phi}{\phi}\left(\nabla_d\phi\nabla_a\nabla_b\phi - \nabla_b\phi\nabla_a\nabla_d\phi\right. \\ &\left.\left. -\nabla_a\phi\nabla_d\nabla_b\phi + \frac{\nabla_a\phi\nabla_b\phi\nabla^c\phi\nabla_c\nabla_d\phi}{\nabla^e\phi\nabla_e\phi}\right)\right] \\ &= \left(-\frac{\omega}{2\phi^2}\nabla^c\phi\nabla_c\phi - \frac{\Box\phi}{\phi} - \frac{V}{2\phi}\right)h_{ab} + \frac{1}{\phi}h_a{}^c h_b{}^d\nabla_c\nabla_d\phi, \end{aligned} \tag{2.34}$$

$$8\pi P^{(\phi)} = -\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi - \frac{V}{2\phi} - \frac{1}{3\phi}\left(2\Box\phi + \frac{\nabla^a\phi\nabla^b\phi\nabla_b\nabla_a\phi}{\nabla^e\phi\nabla_e\phi}\right), \tag{2.35}$$

$$\begin{aligned} 8\pi\pi_{ab}^{(\phi)} &= \frac{1}{\phi\nabla^e\phi\nabla_e\phi}\left[\frac{1}{3}\left(\nabla_a\phi\nabla_b\phi - g_{ab}\nabla^c\phi\nabla_c\phi\right)\left(\Box\phi - \frac{\nabla^c\phi\nabla^d\phi\nabla_d\nabla_c\phi}{\nabla^e\phi\nabla_e\phi}\right)\right. \\ &\left. +\nabla^d\phi\left(\nabla_d\phi\nabla_a\nabla_b\phi - \nabla_b\phi\nabla_a\nabla_d\phi - \nabla_a\phi\nabla_d\nabla_b\phi + \frac{\nabla_a\phi\nabla_b\phi\nabla^c\phi\nabla_c\nabla_d\phi}{\nabla^e\phi\nabla_e\phi}\right)\right], \end{aligned} \tag{2.36}$$

and the trace of the effective stress-energy tensor reads

$$8\pi T^{(\phi)} \equiv 8\pi g^{ab}T_{ab}^{(\phi)} = -\frac{\omega}{\phi^2}\nabla^c\phi\nabla_c\phi - \frac{3\Box\phi}{\phi} - \frac{2V}{\phi}. \tag{2.37}$$

In general, the effective fluid stress-energy tensor $T_{ab}^{(\phi)}$ cannot be expected to satisfy any energy condition, since, in addition to positive-definite squares of first derivatives, it also contains second derivatives of ϕ, whose sign cannot be established *a* *priori*. Moreover, it is important to emphasize that the fluid considered here is not a real fluid, but merely an effective one, which makes it unlikely that it fulfils these conditions in general. In any case, the weak energy condition, ($T_{ab}t^a t^b \geq 0$ for all timelike vectors t^a), reads [11]

$$T_{ab}^{(\phi)} u^a u^b = -\frac{\omega}{2\phi}\nabla^e\phi\nabla_e\phi + \frac{V}{2} + \Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi} \geq 0. \tag{2.38}$$

The strong energy condition, $((T_{ab} - T g_{ab}/2)\, t^a t^b \geq 0$ for all timelike vectors t^a) yields

$$\left(T_{ab}^{(\phi)} - \frac{1}{2}T^{(\phi)} g_{ab}\right) u^a u^b = \frac{1}{2}\left(\rho^{(\phi)} + 3P^{(\phi)}\right) \tag{2.39}$$

$$= -\frac{\omega}{\phi^2}\nabla^e\phi\nabla_e\phi - \frac{V}{2\phi} + \frac{1}{\phi}\left[-\frac{1}{2}\Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right] \geq 0. \tag{2.40}$$

2.2.3 Looking for a "thermodynamics of gravitational theories"

The novelty of first-order thermodynamics, which is the focus of this thesis, comes in when we apply a non-equilibrium thermodynamical description to this effective fluid. As reviewed in section 1.3, the seminal papers [24, 25] sketch a suggestive picture in which GR is related to equilibrium, while a non-equilibrium description is needed whenever we are dealing with a modified gravity theory. What emerges from this works is the idea that it might be possible to construct a "thermodynamics of gravitational theories", a meta-theory that encompasses both GR and modified gravity as special cases in a broader framework.

Despite the influence of [24, 25] on the subsequent literature and the elegance of their insights, two important questions raised by their approach remained unanswered:

1. What is the dissipative process leading from non-equilibrium to equilibrium?

2. What is the order parameter (presumably, the temperature) that measures the closeness to equilibrium?

The follow-up work [175] asserted that the thermodynamical approach of [24, 25] seemed to suggest that different gravitational theories might be interpreted as "different regimes of some more general effective description of gravity", which might lead to "a unified framework which associates different gravitational theories with different hydrodynamical regimes".

It is in order to answer the questions above that the approach we dub "first-order thermodynamics" was developed, initially in [10, 11], and subsequently consolidated and extended in the works presented throughout this thesis. The formalism is inspired by Jacobson's approach in that it finds a concrete realization of the idea of the thermodynamics of gravitational theories where GR is an equilibrium state and modified gravity a non-equilibrium state, but it follows a starkly different path. Jacobson's works required several assumptions, for example the fact that the Clausius relation must hold at each spacetime point as seen from the uniformly accelerated Rindler observers, and also the proportionality between area and entropy, as an extension of black hole thermodynamics. Our approach is arguably much simpler, since all that is needed is the well-established effective fluid formalism spelled out in the previous sections, which only requires a timelike and future-oriented velocity. Additionally, a formulation of non-equilibrium thermodynamics must be chosen, since this is not as standard as equilibrium thermodynamics. This choice is the topic of the following section.

2.3 Non-equilibrium thermodynamics

We are interested in the description of imperfect fluids which exhibit dissipation and therefore need to go beyond standard, equilibrium thermodynamics to a non-equilibrium description (see [196] for a review). Additionally, we are dealing with fluids in a relativistic context (reviewed, for example, in [197]), relying on the standard $3+1$ covariant formalism illustrated in the previous sections. Technically speaking, most systems in nature are subject to physical processes that do not occur in perfect equilibrium, as there are energy or matter exchanges, chemical reactions and so on. However, often these systems and processes can be considered to be at least in local equilibrium, and the deviations from equilibrium are negligible. The most important difference between equilibrium and non-equilibrium thermodynamics is that the former neglects the time evolution of physical processes: while initial and final states are considered, or isolated snapshots of the system's evolution can be obtained at specific times, the actual change of physical quantities and the system's properties with time is ignored. The central feature of non-equilibrium thermodynamics is that it incorporates the relaxation or dissipation times which are always vanishing in equilibrium thermodynamics. Such a non-equilibrium description is also needed to describe any irreversible (i.e., non-isentropic) process.

Dissipative thermodynamics has interesting and varied applications in cosmology, because most processes in the evolution of the universe are indeed dissipative, such as reheating at the end of inflation, primordial nucleosynthesis, the gravitational collapse of overdensities in cosmological structure formation, to name but a few. While in most cosmological situations perfect fluids are adequate in capturing the dynamics and serve as a basis for accurate models, as the anisotropic dissipative terms are negligible in a FLRW universe on scales that respect homogeneity and isotropy, in specific situations, the use of relativistic fluids and dissipative, non-equilibrium thermodynamics is necessary. For example, the role of bulk viscosity in cosmology is quite important (since it can arise also in homogeneous and isotropic situations, while shear viscosity cannot), and arises especially when dealing with mixtures of fluids, such as a radiation and a dust fluid, or a mixture of particle species in the early universe. In the context of the late-time universe, viscosity is often employed in a unifying approach to dark matter and dark energy, describing both components as a single viscous fluid.

2.3.1 Different choices of frame

The first instance of non-equilibrium thermodynamics describing relativistic fluids was devised by Eckart in 1940 [21], and then in a slightly different form by Landau and Lifshitz in the 1950s [198]. In this paragraph, we present the relationship between the two, and then from the next paragraph onward, we only deal with Eckart's formulation. The two approaches are completely equivalent, but the difference between them is in the choice of the fluid's 4-velocity: in Eckart's approach, u^α is the velocity of particle flow in the fluid, so that the particle flux vanishes in the frame of a comoving observer. In Landau and Lifshitz's approach, u^α is instead the velocity of energy transport, so that the stress-energy components T^{i0} vanish in a comoving frame. This ambiguity does not occur for a perfect fluid, since its velocity u^α is uniquely defined as the 4-velocity relative to which there is no particle current, defined as

$$n^\alpha = n u^\alpha, \tag{2.41}$$

where n is the number density. If the fluid is out of equilibrium as a result of dissipative effects, then there is no unique average 4–velocity and the freedom to choose the frame can be exploited.

Whether the Eckart of the Landau frame is more convenient depends on the specifics of the problem at hand. For the purposes of this thesis, we will work in the Eckart frame, in order to avoid problems related to the identification of suitable "particles" that characterize the particle flux. Since we rely on non-equilibrium thermodynamics to describe an effective (and not a real) fluid, this seems to us the most straightforward approach. Nonetheless, through a suitable change of frame, it is always possible to switch from one description to the other, bearing in mind that different choices of frames translate to different thermodynamical properties of the fluid. The choice of frame also influences the perfect or imperfect properties of the fluid: an observer moving with a given velocity relative to a perfect fluid will see an effective non-zero momentum density and anisotropic stress tensor. For example, the observed dipole anisotropy in the CMB radiation is interpreted as arising from the peculiar velocity of our galaxy relative to the CMB rest frame.

In order to illustrate the consequences of choosing the Eckart or Landau-Lifshitz frame, let us consider, following [17], a perfect fluid with four-velocity u^*_μ described in its comoving frame by the perfect fluid stress-energy tensor

$$T_{\mu\nu} = \rho^* u^*_\mu u^*_\nu + P^* h^*_{\mu\nu}. \tag{2.42}$$

In a second frame, moving with 4-velocity u^μ related to u^μ_* by

$$u^*_\mu = \gamma \left(u_\mu + v_\mu \right), \tag{2.43}$$

where

$$\begin{gathered} \gamma = \frac{1}{\sqrt{1-v^2}} = -u^*_\mu u^\mu, \\ v^2 \equiv v^\alpha v_\alpha > 0, \quad v^\alpha u_\alpha = 0, \quad 0 \leq v^2 < 1, \end{gathered} \tag{2.44}$$

this perfect fluid is now "tilted" and appears dissipative, so that its stress-energy tensor can be decomposed as (2.24) (with $h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$) where the energy density is

$$\rho = \rho^* + \gamma^2 v^2 \left(\rho^* + P^*\right) = \gamma^2 \left(\rho^* + v^2 P^*\right), \tag{2.45}$$

the pressure is

$$P = P^* + \frac{\gamma^2 v^2}{3} \left(\rho^* + P^*\right), \tag{2.46}$$

the energy flux is

$$q^\mu = \left(1 + \gamma^2 v^2\right) \left(\rho^* + P^*\right), v^\mu = \gamma^2 \left(\rho^* + P^*\right) v^\mu \tag{2.47}$$

and the anisotropic stress tensor is

$$\pi^{\mu\nu} = \gamma^2 \left(\rho^* + P^*\right) \left(v^\mu v^\nu - \frac{v^2}{3} h^{\mu\nu}\right). \tag{2.48}$$

Clearly, the purely spatial vector q^μ arises only due to the relative motion between the two frames, namely due to the purely spatial vector v^μ. Therefore, the interpretation of q^μ as heat flow is ambiguous, and care should be exercised when using this interpretation. In the following, we will always deal with the Eckart frame.

2.3.2 Eckart's first-order theory

Coming back to the central topic of this thesis, we want to apply a non-equilibrium thermodynamics description to the effective fluid arising from scalar-tensor gravity. We choose the simple formulation by Eckart and lay out in this section the basic assumptions that it rests on [196].

Several conservation laws are valid for a perfect fluid. We review them here in order to understand how they change in a fluid with dissipation and so that we can use them to derive the constitutive equations of Eckart's thermodynamics. In a fluid where no creation or annihilation processes occur, particle number is conserved, as expressed by the continuity equation

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{v}) = 0, \tag{2.49}$$

or, in covariant form for the particle current,

$$\nabla_\alpha n^\alpha. \tag{2.50}$$

Of course, also the stress-energy tensor obeys a conservation law

$$\nabla_\beta T^{\alpha\beta} = 0. \tag{2.51}$$

The component parallel to u^α is the energy conservation equation, which, for example, for a perfect fluid in FLRW spacetimes becomes (1.7), while the projection orthogonal to u^α provides the momentum conservation equation. The energy conservation equations generalises to a relativistic context the mass conservation equation in the Newtonian treatment of fluids, and the momentum conservation equation generalises the Euler equation.

In non-equilibrium thermodynamics, the conservation laws for the particle number and stress-energy tensor also hold, and the particle number conservation (2.50) does not change, but additional terms come into play into the energy and momentum conservation equations due to the introduction of dissipative quantities into the imperfect form of the stress-energy tensor (2.24). In non-equilibrium and irreversible thermodynamics, the entropy is no longer conserved but grows, according to the second law of thermodynamics. The rate of entropy production is provided by the divergence of the entropy 4-current, and the covariant form of the second law reads

$$\nabla_\alpha S^\alpha \geq 0, \tag{2.52}$$

where

$$S^\alpha = Snu^\alpha + \frac{R^\alpha}{\mathcal{T}} \tag{2.53}$$

and the second term is purely dissipative (for an isolated system in a non-dissipative fluid, the second law would simply be $\nabla_\alpha S^\alpha = 0$. (We denote the temperature as $\mathcal{T}$ as it is the same that we use in our formalism).

The dissipative term R^α in S^α is assumed to be an algebraic function (that does not include derivatives) of n^α and $T^{\alpha\beta}$, $R^\alpha = R^\alpha\left(n^\beta, T^{\mu\nu}\right)$ that vanishes in equilibrium. The standard Eckart theory is defined by the simplest possible choice for R^α, namely that it is linear in the dissipative quantities.

This term is generally taken to be $\dfrac{q^\alpha}{\mathcal{T}}$ as this implements the simplest vector constructed from the dissipative quantities, so that

$$S^\alpha = Snu^\alpha + \frac{q^\alpha}{\mathcal{T}}. \tag{2.54}$$

Using the particle conservation equation and the energy conservation equation, in addition to the first law $\mathcal{T}dS = dU + pdV$ with U the internal energy (see [196] for more details), the divergence of (2.54) yields

$$\mathcal{T}\nabla_\alpha S^\alpha = -\left[\theta P_{\text{visc}} + \left(h_{\alpha\beta}\nabla^\beta \ln\mathcal{T} + \dot{u}_\alpha\right) q^\alpha + \sigma_{\alpha\beta}\pi^{\alpha\beta}\right], \tag{2.55}$$

where P_{visc} is the bulk viscous pressure, contained in $P = P_{\text{visc}} + P_{\text{non-visc}}$ together with the usual, non-viscous pressure (see section 4.4 for details).

Equation (2.55) shows that the simplest way to satisfy (2.54) is to assume that the following linear relationships hold between the thermodynamic "fluxes" $P_{\text{visc}}, q_\alpha, \pi_{\alpha\beta}$ and the corresponding thermodynamic "forces" $\theta, \dot{u}_\alpha + h_{\alpha\beta}\nabla^\beta \ln\mathcal{T}, \sigma_{\alpha\beta}$:

$$P_{\text{visc}} = -\theta\zeta \tag{2.56}$$

$$q_\alpha = -\mathcal{K}\left(h_{\alpha\beta}\nabla^\beta\mathcal{T} + \mathcal{T}\dot{u}_\alpha\right) \tag{2.57}$$

$$\pi_{\alpha\beta} = -2\eta\sigma_{\alpha\beta} \tag{2.58}$$

These three equations are the *constitutive equations* (which, in general, characterize the response of a given fluid to external forces and stresses) of Eckart's thermodynamics and they encapsulate its essence. This formulation of non-equilibrium thermodynamics is also called *first-order* thermodynamics, as it makes the simplest possible assumptions to satisfy the covariant second law of thermodynamics (2.54), namely that the entropy contains first-order (linear) terms in the dissipative variables $P_{\text{visc}}, q_\alpha, \pi_{\alpha\beta}$. The constitutive equations are relativistic generalisations of the corresponding laws in Newtonian, non-relativistic thermodynamics

$$\begin{aligned} P_{\text{visc}} &= -3\zeta\vec{\nabla}\cdot\vec{v} \\ \vec{q} &= -\mathcal{K}\vec{\nabla}\mathcal{T} \\ \pi_{ij} &= -2\eta\sigma_{ij} \end{aligned} \tag{2.59}$$

which are, respectively, Stokes' law, the Fourier's law of heat conduction and Newton's law of viscosity, and where $\vec{v}$ is the fluid velocity and $\vec{q}$ is the vector representing the heat flow. As first discovered by Eckart [21], there is an acceleration term $T\dot{u}_\alpha$ in (2.57) arising from the inertia of heat energy. Physically, this means that a heat flux will arise from accelerated matter even in the absence of a temperature gradient. Comparison with the Newtonian laws allow us to identify the thermodynamic coefficients: ζ is the bulk viscosity, $\mathcal{K}$ is the thermal conductivity and η is the shear viscosity. Given the linear constitutive equations above, the entropy production rate (2.55) can be recast as

$$\nabla_\alpha S^\alpha = \frac{P_{\text{visc}}^2}{\zeta\mathcal{T}} + \frac{q_\alpha q^\alpha}{\mathcal{K}\mathcal{T}^2} + \frac{\pi_{\alpha\beta}\pi^{\alpha\beta}}{2\eta\mathcal{T}}. \tag{2.60}$$

The second law is therefore satisfied as long as

$$\zeta \geq 0, \quad \mathcal{K} \geq 0, \quad \eta \geq 0. \tag{2.61}$$

Eckart's first-order thermodynamics is the most widely used formulation of irreversible thermodynamics in relativity, but it is not free from problems, as the simplicity of the assumptions above comes at a cost. The theory cannot account for relaxation times and therefore as soon as one of the thermodynamical forces is turned off, the corresponding flux immediately vanishes. This means that there is non-causal propagation in this formulation, and this motivated the search for the so-called *second-order* causal irreversible thermodynamics. This more complex, fully causal relativistic formulation is based on kinetic theory, which relates the microscopic properties of atoms and molecules to the macroscopic properties of matter, and provides a substantial improvement on the first-order formulation.

However, for the purposes of this thesis, we shall only restrict to Eckart's first-order formulation. This is mainly due to the fact that we are interested in applying a non-equilibrium thermodynamics to an *effective* and not a real fluid. Therefore, despite the shortcomings of Eckart's theory, it is more than adequate for the analogy that we are constructing. In the following paragraph, we sketch the second-order formulation that we may consider as an outlook of this thesis.

2.3.3 Second-order theories

The problem with postulating the simple form (2.54) for R^α is that kinetic theory indicates that in fact R^α is second-order in the dissipative fluxes. Instead Eckart's assumption truncates at first order, removing the terms that are necessary to ensure causality and stability. A formulation of *second-order*, causal and stable thermodynamics was first developed by Israel and Stewart in the 1970s [199], and then improved upon with more sophisticated formulations by Müller and Ruggeri [200], among others.

The idea is to keep R^α still algebraic, but use an Ansatz that is at most second-order in the dissipative fluxes, namely

$$\begin{aligned} S^\mu = Sn u^\mu + \frac{q^\mu}{\mathcal{T}} - \left(\beta_0 P_{\text{visc}}^2 + \beta_1 q_\nu q^\nu + \beta_2 \pi_{\nu\kappa}\pi^{\nu\kappa}\right)\frac{u^\mu}{2\mathcal{T}} \\ + \frac{\alpha_0 P_{\text{visc}} q^\mu}{\mathcal{T}} + \frac{\alpha_1 \pi^{\mu\nu} q_\nu}{\mathcal{T}}, \end{aligned} \tag{2.62}$$

where $\beta_i \geq 0$ are thermodynamic coefficients for scalar, vector and tensor dissipative contributions to the entropy density, and α_i are viscous or heat coupling coefficients. If we assume no viscous or heat coupling, therefore setting

$$\alpha_0 = 0 = \alpha_1, \tag{2.63}$$

the simplest way to satisfy the second law of thermodynamics is to impose again linear relationships between the thermodynamical fluxes and forces, but this time using the extended form (2.62). This yields the following constitutive equations, sometimes dubbed "extended" Israel-Stewart constitutive equations:

$$\tau_0 \dot{P}_{\text{visc}} + P_{\text{visc}} = -\theta\zeta - \left[\frac{1}{2}\zeta\mathcal{T}\nabla_\alpha\left(\frac{\tau_0}{\zeta\mathcal{T}}u^\alpha\right)P_{\text{visc}}\right] \tag{2.64}$$

$$\tau_1 h_\alpha{}^\beta \dot{q}_\beta + q_\alpha = -\mathcal{K}\left(h_{\alpha\beta}\nabla^\beta\mathcal{T} + \mathcal{T}\dot{u}_\alpha\right) - \left[\frac{1}{2}\mathcal{K}\mathcal{T}^2\nabla_\beta\left(\frac{\tau_1}{\mathcal{K}\mathcal{T}^2}u^\beta\right)q_\alpha\right] \tag{2.65}$$

$$\tau_2 h_\alpha{}^\mu h_\beta{}^\nu \dot{\pi}_{\mu\nu} + \pi_{\alpha\beta} = -2\eta\sigma_{\alpha\beta} - \left[\eta\mathcal{T}\nabla_\nu\left(\frac{\tau_2}{2\eta\mathcal{T}}u^\nu\right)\pi_{\alpha\beta}\right], \tag{2.66}$$

which introduce the relaxation times τ_i, given by

$$\tau_0 = \zeta\beta_0, \quad \tau_1 = \mathcal{K}\mathcal{T}\beta_1, \quad \tau_2 = 2\eta\beta_2. \tag{2.67}$$

In general, such relaxation times are phenomenological and have to be determined experimentally. The resulting equations are quite involved, but in practice some simplifications are often made, namely the terms in square brackets on the right of (2.64), (2.65) and (2.66) are omitted. This choice amounts to the assumption that these terms are negligible if compared with the other terms in the equations. These equations are then dubbed "truncated" and are sufficiently accurate to be used in many contexts, from relativistic quantum gases to the treatment of bulk viscosity in cosmology [201].

The truncated equations read

$$\begin{aligned} \tau_0 \dot{P}_{\text{visc}} + P_{\text{visc}} &= -\theta\zeta \\ \tau_1 h_\alpha{}^\beta \dot{q}_\beta + q_\alpha &= -\mathcal{K}\left(h_{\alpha\beta}\nabla^\beta\mathcal{T} + \mathcal{T}\dot{u}_\alpha\right) \\ \tau_2 h_\alpha{}^\mu h_\beta{}^\nu \dot{\pi}_{\mu\nu} + \pi_{\alpha\beta} &= -2\eta\sigma_{\alpha\beta}. \end{aligned} \tag{2.68}$$

The essential difference between the first-order constitutive equations by Eckart and those of the extended Israel-Stewart formulation is that the latter are differential equations describing the time evolution of the dissipative quantities, while the former are simply algebraic relations without dynamics. The presence of evolution terms in the extended Israel-Stewart formulation is precisely what ensures causality, through the relaxation time coefficients τ_i. Of course, the cost of the greater power of these equations is the introduction of new thermodynamic coefficients, which may be evaluated or estimated through kinetic theory. The relaxation times τ_i, for example, are usually estimated as mean collision times, of the form

$$\tau \approx \frac{1}{n\tilde{\sigma}v}, \tag{2.69}$$

where $\tilde{\sigma}$ is a collision cross section and v the mean particle speed. The derivation of the second-order constitutive equations is based on the assumption that the fluid is close to equilibrium, and thus the dissipative fluxes are small:

$$|P_{\text{visc}}| \ll P_{\text{non-visc}}, \quad \left(\pi_{\alpha\beta}\pi^{\alpha\beta}\right)^{1/2} \ll P_{\text{non-visc}}, \quad (q_\alpha q^\alpha)^{1/2} \ll \rho. \tag{2.70}$$

A complete description of dissipative processes in a fully causal fashion requires a very sophisticated formalism involving 14 coefficients based on kinetic theory [199]. Fortunately, nine of these modes are strongly damped in the long-wave limit (compared to the average mean free path) and three modes decay, making it possible for the equations above to be adequate in most practical scenarios.

Chapter 3

First-order thermodynamics of scalar-tensor gravity

This chapter is mostly based on [10, 11] and presents the basic formalism of first-order thermodynamics in its original developments, before we consider its extensions and generalizations in the following chapter.

3.1 Basics of the formalism

The main goal is to apply Eckart's first-order thermodynamics to the effective fluid (2.23) arising from the scalar-tensor action (2.20), explored in section 2.2.2. Rewriting the constitutive equations (2.56)-(2.58) for our effective fluid $T^{(\phi)}_{\mu\nu}$ for convenience, we have

$$P^{(\phi)}_{\text{vis}} = -\zeta\,\theta \tag{3.1}$$

$$q^{(\phi)}_a = -\mathcal{K}\left(h_{ab}\nabla^b\mathcal{T} + \mathcal{T}\dot{u}_a\right) \tag{3.2}$$

$$\pi^{(\phi)}_{ab} = -2\eta\,\sigma_{ab}\,. \tag{3.3}$$

Comparing the definition of the 4-acceleration, $\dot{u}_a \equiv u^b\nabla_b u_a$ and the heat flux density $q^{(\phi)}_a$ (2.26) found for our effective fluid leads to the identification [10, 11, 19]

$$q^{(\phi)}_a = -\frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi\phi}\,\dot{u}_a \tag{3.4}$$

Further comparison with Eckart's generalized Fourier law (3.2) yields an expression for the product of thermal conductivity and temperature, namely

$$\mathcal{K}\mathcal{T} = \frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi\phi}. \tag{3.5}$$

Additionally, one finds

$$h_{ab}\nabla^b\mathcal{T} = 0, \tag{3.6}$$

highlighting the fact that a heat flux arises from accelerated matter even in absence of a temperature gradient, as first identified by Eckart [21]. Note that $\mathcal{K}\mathcal{T}$ is positive-definite, as a temperature should be, which could not be expected *a priori* from this formal identification of quantities and the proportionality between q^a and $\dot{u}^a$.

3.1.1 Temperature of scalar-tensor gravity

This temperature is clearly different from the Unruh temperature seen by an accelerated observer hovering above the Rindler causal horizon that was used in Jacobson's thermodynamics of spacetime. On the one hand, that temperature is physical and observable in principle, just like the Hawking temperature of a black hole is observable, despite being so low that it is practically dwarfed by the CMB temperature, for astrophysical black holes. On the other hand, (3.5) represents the temperature of an effective fluid, and, if we considered a black hole solution of scalar-tensor theories described by the action (2.20), it would be distinct from the physical Hawking temperature associated to its horizon through black hole thermodynamics.

The temperature of the effective ϕ-fluid in scalar-tensor gravity is one of the two cornerstones of the whole first-order thermodynamical approach (the other being the effective heat equation, see section 3.1.3). This provides a rather straightforward answer to the second question left open by Jacobson's thermodynamics of spacetime approach (see section 2.2.3), namely what is the order parameter measuring the closeness to equilibrium. In general, this parameter is the temperature, and providing an explicit expression for this quantity is a promising first step. In order to understand the role of this temperature as an order parameter measuring closeness to equilibrium, however, we need to clarify what this equilibrium state is.

It is straightforward to see that $\mathcal{KT}$ vanishes when $\phi = \text{const.}$, namely in the limit that recovers GR, where there is no ϕ-fluid. GR can therefore be identified with the $\mathcal{KT} = 0$, equilibrium state of this thermodynamics of gravitational theories. This provides a concrete realisation of Jacobson's powerful first idea that GR corresponds to the equilibrium state of gravity. Let us remark that this thermodynamical analogy provides a meta-description of gravitational theories. We are not dealing with a real fluid, but with an effective fluid, whose temperature does not arise from the microscopic motion of fluid particles, but is simply a notion of the "distance" from equilibrium. Indeed, as we show in the following, we have found that whenever we consider modified gravity theories with a (scalar) degree of freedom *in addition* to the two tensor degrees of freedom of GR, the theory exhibits $\mathcal{KT} > 0$. The temperature is just a parameter relative to the GR equilibrium state, which provides a realization of Jacobson's second idea, namely that modified gravity corresponds to a non-equilibrium state in the thermodynamics of gravitational theories. To provide an intuitive picture, let us consider again the landscape of gravity theories, populated by GR and its generalisations and extensions (so far we restrict to scalar-tensor theories only). The GR state at $\mathcal{KT} = 0$ is the lowest energy state at the center of this landscape and is an attractor to other theories in non-equilibrium situations at $\mathcal{KT} > 0$. We describe the dissipation process leading from non-equilibrium to equilibrium in section 3.1.3. The additional scalar degree of freedom on top of the two tensor degrees of freedom of GR makes the temperature positive definite, since without it, it would be $\mathcal{KT} = 0$. This seems to be quite a generic property, regardless of which specific scalar-tensor theory is considered, as we describe in section 4.2.

So far we have not addressed the fact that, in general, the thermal conductivity $\mathcal{K}$ and the temperature $\mathcal{T}$ always appear coupled in the above equations. It is technically $\mathcal{KT}$ that we mean when we say the "temperature of scalar-tensor gravity", and interpret the thermal conductivity is simply a coefficient in front of it (which still depends on the energy density ρ and particle density n of the fluid). In the following, we try to separate the thermal conductivity from the temperature to provide a straightforward physical interpretation. Unfortunately, this can only be achieved under severely simplifying assumptions, which depend on the specific choice of the solution of the system composed by (3.5) and (3.6). Probably the simplest physical interpretation of the quantities $\mathcal{K}$ and $\mathcal{T}$ arises if we isolate the temperature from (3.5) and insert it into $h_{ab}\nabla^b\mathcal{T}=0$, finding the simple solution

$$\mathcal{K}=C\sqrt{-\nabla^c\phi\nabla_c\phi}, \tag{3.7}$$

with C a positive constant that can be set to $C=1/8\pi$, yielding

$$\mathcal{T}=1/\phi=G_{\text{eff}} \tag{3.8}$$

and

$$\mathcal{K}=C\sqrt{-\nabla^c\phi\nabla_c\phi}. \tag{3.9}$$

We recall from section 1.2.1 that the main motivation for Brans-Dicke theory was the implementation of Mach's principle, and that the main property of the theory is a gravitational coupling $G_{\text{eff}}(\phi)\simeq 1/\phi$ dependent on spacetime coordinates, which leads to a generalisation of the concept of Newton's constant. In this simple setup, since the action (2.20) we started from is a generalization of the Brans-Dicke action (1.43), the effective temperature of the ϕ-fluid (3.8) precisely corresponds to this generalisation of Newton's constant, thus measuring the effective strength of the gravitational interaction. The thermal conductivity instead keeps track of the variability of ϕ since it contains its gradient. The analysis above sheds light on the GR limit of our description, which recovers the perfect insulator limit of the effective fluid: $\mathcal{T}$ reduces to the Newton constant G_{N}, while $\mathcal{K}$ vanishes. Considering a more general solution involving $\mathcal{K}\neq 0$ is much more troublesome. The GR limit in this case would correspond to $\mathcal{T}\to 0$, or the minimum possible temperature of the fluid. We also rely on this physical interpretation in the understanding of the dissipative process in section 3.1.3.

Even without separating the temperature from the thermal conductivity, however, we can understand that GR is recovered in the limit where there is no ϕ-fluid, so that $\phi=$ const., $\nabla_a\phi=0$, $\mathcal{K}\mathcal{T}=0$ and the minimum possible temperature of the effective fluid is reached. In this sense, we can characterize GR as an *equilibrium state* in the thermodynamics of gravitational theories, the state with the lowest possible temperature, a ground state (since we are dealing with an effective temperature, here we do not dwell on the details of whether reaching zero temperature is practically feasible). Having vanishing $\mathcal{K}\mathcal{T}$ of course means vanishing heat flux q_a, since there are no dissipative quantities. In pure GR there is of course no effective fluid at all, but if we add a minimally coupled scalar field, the fluid arising from it has the form of a perfect fluid (see section 4.3).

A further aspect of the thermodynamical description that we are sketching here is that it is flexible enough that it can be applied both to entire classes of theories and specific solutions within them. The thermodynamical quantities of the effective fluid can of course be computed starting from a different scalar-tensor action than (2.20), therefore extending the formalism to a different theory or class thereof (see section 4.1), but the quantities like $\mathcal{KT}$ and η can also be calculated for exact solutions of the theory at hand (see, e.g. sections 3.1.3 and 4.5).

In summary, the effective temperature is nothing but a temperature relative to GR, the zero-temperature state at equilibrium.

3.1.2 Viscosity of scalar-tensor gravity

The structure of the imperfect fluid (2.24) and of the field equation (2.22) makes the explicit derivation of the bulk viscosity from the thermodynamic analogy (feasible but) nontrivial. For the sake of simplicity we shall set the bulk viscosity to zero as in the original proposal [10, 11].(For a more precise analysis on this matter we refer the reader to [23].) Nonetheless, one can still easily infer the shear viscosity coefficient in a similar way as done for the temperature, from the comparison between the anisotropic stress tensor (2.29) and the shear tensor (2.18), obtaining [10, 11]

$$\eta = -\frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{16\pi\phi} \tag{3.10}$$

or $\eta = -\dfrac{\mathcal{KT}}{2}$. The simplicity of this result should not detract from the fact that it is rather surprising that such a proportionality between the anisotropic stress tensor and the shear tensor is realized in the first place, allowing to find a simple expression for the shear viscosity coefficient.

Despite the fact that the coefficients in the constitutive equations must be positive to satisfy the second law of thermodynamics (2.61), the shear viscosity coefficient (3.10) in this case can be negative. In order to understand this, we need to keep in mind that the ϕ-fluid is clearly not isolated, given the explicit coupling to gravity in the action (2.20) which involves the mixing of scalar and tensor degrees of freedom. Therefore, there is an exchange of energy which necessarily affects the entropy balance. Negative viscosities also occur in various phenomena in fluid mechanics, including jet streams, ocean currents and liquid crystals, to name but a few.

The effective shear viscosity vanishes at $\mathcal{KT} = 0$, in the GR limit corresponding to equilibrium, at $\phi = $ const., where the effective ϕ-fluid disappears. In the context of thermodynamics of spacetime in section 1.3, [176] stressed the role of shear viscosity as the source of dissipation associated to the purely gravitational irreversible contributions in a non-equilibrium setting, correcting the original reference [25] which focused on bulk viscosity. The results presented so far echo this approach, since the choice was made in the form of the effective stress-energy tensor (2.24) to neglect bulk viscosity ($P^{(\phi)}$ includes both viscous and non-viscous contributions, but if the bulk viscosity coefficient ζ vanishes, so does the viscous pressure P_{visc}). We make a different choice in the applications to cosmology in section 4.4, since shear viscosity must vanish to respect spatial isotropy, but isotropic bulk viscosity is possible.

It is important to note that the decomposition in (2.24) applies to any symmetric second-order tensor, although of course the dissipative quantities would vanish if the effective stress-energy tensor for the theory at hand takes the form of a perfect fluid (see section 4.3 for more details). The special feature of scalar-tensor gravity in first-order thermodynamics is not that the decomposition above can be performed, but rather that the constitutive relations of Eckart's thermodynamics hold [18].

Recalling the dissipative contributions to the entropy in Eckart's non-equilibrium thermodynamics (2.60), we are now in the position to find an expression for the entropy density in the context of scalar-tensor gravity. Since

$$s = \frac{dS}{dV} = \frac{\rho + P}{\mathcal{T}}, \tag{3.11}$$

we can use (2.25), (2.28) and (3.5) to obtain

$$s = \frac{\mathcal{K}}{\sqrt{-\nabla^e\phi\nabla_e\phi}}\left[-\frac{\omega}{\phi}\nabla^e\phi\nabla_e\phi + \frac{\Box\phi}{3} - \frac{4}{3}\frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right]. \tag{3.12}$$

From (2.60), the dissipative contributions to the entropy are

$$\nabla_c s^c = \frac{P^2_{\text{visc}}}{\zeta\mathcal{T}} + \frac{q_c q^c}{\mathcal{K}\mathcal{T}^2} + \frac{\pi_{ab}\pi^{ab}}{2\eta\mathcal{T}}, \tag{3.13}$$

and using the fact that $q^a = -\mathcal{K}\mathcal{T}\dot{u}^a$, the term accounting for entropy production in (2.53) is

$$R^a = -\mathcal{K}\dot{u}^a. \tag{3.14}$$

We can then use (3.4) and (2.29) to compute $\nabla_c s^c$, obtaining

$$\begin{aligned}\frac{q_c q^c}{\mathcal{K}\mathcal{T}^2} = \mathcal{K}\dot{u}_c\dot{u}^c = \frac{\mathcal{K}}{(-\nabla^e\phi\nabla_e\phi)^3}&\left[-\nabla^e\phi\nabla_e\phi\nabla_b\phi\nabla^d\phi\nabla^b\nabla^a\phi\nabla_d\nabla_a\phi\right.\\ &\left.+\left(\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi\right)^2\right]\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\pi_{ab}\pi^{ab} = 8\eta^2\sigma^2 = 2\mathcal{K}^2\mathcal{T}^2\sigma^2 = \frac{(-\nabla^e\phi\nabla_e\phi)^{-2}}{32\pi^2\phi^2}&\left\{\frac{1}{2}(-\nabla^e\phi\nabla_e\phi)^2\right.\\ \left[\nabla^a\nabla^b\phi\nabla_a\nabla_b\phi - \frac{(\Box\phi)^2}{3}\right] &+ \frac{1}{3}\left(\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi\right)^2\\ -(\nabla^e\phi\nabla_e\phi)\left(\nabla_a\nabla_b\phi\nabla^b\nabla_c\phi - \frac{\Box\phi}{3}\nabla_a\nabla_c\phi\right)&\left.\nabla^a\phi\nabla^c\phi\right\},\end{aligned} \tag{3.16}$$

and finally

$$\nabla_c s^c = \mathcal{K}\left(\dot{u}^a\dot{u}_a + \frac{\mathcal{K}\mathcal{T}\sigma^2}{\eta}\right) = \mathcal{K}\left(\dot{u}^a\dot{u}_a - \sigma_{ab}\sigma^{ab}\right). \tag{3.17}$$

Since the second term in the above equation is negative, one cannot conclude that the entropy never decreases, in accordance to the fact that the effective fluid is not isolated, as mentioned above. A special situation would occur if ϕ-fluid's flow is geodesic, namely $\dot{u}^a = 0$, which always corresponds to $q^a = 0$ and decreasing entropy density, consistent with the fact that the entropy production vector (3.14) vanishes and shear viscosity contributes to decreasing s in the equation above because of the negative η from (3.10).

3.1.3 Approach to equilibrium

It is natural to ask how equilibrium might be approached starting from a non-equilibrium state, as the understanding of this dissipative process is crucial to establish the picture we are trying to construct. This provides an answer to the second question left open by Jacobson's approach (section 2.2.3).

An effective heat equation for the ϕ-fluid can be found by differentiating (3.5). Although this might seem redundant, the resulting equation provides the evolution of $\mathcal{KT}$ with time and allows us to understand the circumstances where the dissipation to equilibrium takes place. Computing

$$\frac{d(\mathcal{KT})}{d\tau} \equiv u^c \nabla_c (\mathcal{KT}), \tag{3.18}$$

one obtains [10, 11]

$$\begin{aligned} \frac{d(\mathcal{KT})}{d\tau} &= -\frac{\sqrt{-\nabla^e\phi\nabla_e\phi}}{8\pi\phi}\frac{1}{\phi}\frac{\nabla^c\phi\nabla_c\phi}{\sqrt{-\nabla^e\phi\nabla_e\phi}} - \frac{u^c}{8\pi\phi}\frac{\nabla^e\phi\nabla_c\nabla_e\phi}{\sqrt{-\nabla^e\phi\nabla_e\phi}} \\ &= \frac{\mathcal{KT}}{\phi}\sqrt{-\nabla^e\phi\nabla_e\phi} - \mathcal{KT}\left(\theta - \frac{\Box\phi}{\sqrt{-\nabla^e\phi\nabla_e\phi}}\right). \end{aligned} \tag{3.19}$$

Using again the definition (3.5), one has

$$\frac{d(\mathcal{KT})}{d\tau} = 8\pi(\mathcal{KT})^2 - \theta(\mathcal{KT}) + \frac{\Box\phi}{8\pi\phi}. \tag{3.20}$$

A general interpretation of this equation is challenging since $\Box\phi$ does not have definite sign and the dependence of θ on ϕ and its derivatives is not straightforward, but one can gain some physical intuition by considering the vacuum case, with $\omega =$ const. and $V(\phi) = 0$, so that $\Box\phi = 0$.

On the one hand, if $\theta < 0$, $\mathcal{KT}$ grows out of control in a finite time and diverges away from equilibrium. This behaviour is relevant around spacetime singularities, where $\theta < 0$ because the worldlines of the field ϕ converge: in our formalism, this means that the deviations of scalar-tensor gravity from GR will be extreme. In this spirit, we interpret this result as the fact that singularities are "hot" in the context of first-order thermodynamics, in the sense that $\mathcal{KT}$ diverges there.

On the other hand, if $\theta > 0$, depending on which term dominates in (3.20), $\mathcal{KT}$ could either asymptotically tend to zero and approach the equilibrium state, or not. Thus, the approach to equilibrium is not granted: we show in chapter 4 that it occurs in most scenarios, but we cover two special situations where it does not occur, one in the following paragraph and one in the context of cosmology in section 4.4.

Behaviour at singularities An example of the "hot" behaviour of singularities in our formalism is illustrated in [11], which studies a Brans-Dicke solution with a central naked singularity, conformal to a GR solution. Of course, this solution is not physically realistic, but it is one of the few dynamical solutions of the Brans-Dicke field equations (2.21)-(2.22) with timelike $\nabla_a\phi$ that are not expanding FLRW universes (for the analysis of such cosmological solutions, see section 4.4). Hence, this solution serves as a useful practical example to test the physical intuition about first-order thermodynamics that we gained in the previous section. This solution generalizes the one in [202] by including the cosmological constant Λ. The scalar field potential is

$$V(\phi) = \frac{m^2\phi^2}{2}, \tag{3.21}$$

with $m^2 = 2\Lambda/\kappa > 0$ and $\kappa = 8\pi G$. The line element reads

$$ds^2 = -\kappa r^2 d\tau^2 + \left(1 - \frac{\tau}{\tau_*}\right)^2 \left(\frac{2dr^2}{1-\frac{2\Lambda r^2}{3}} + r^2 d\Omega^2_{(2)}\right). \tag{3.22}$$

The Brans-Dicke scalar has a non-trivial time profile and reads

$$\phi(\tau) = \frac{\phi_*}{\left(1 - \frac{\tau}{\tau_*}\right)^2}, \tag{3.23}$$

where ϕ_* arises from an initial condition at τ_*. The Ricci scalar is (we denote it differently in this section in order to distinguish it from the areal radius)

$$\mathcal{R} = \frac{\omega}{\phi^2}\nabla^c\phi\nabla_c\phi + \frac{3\Box\phi}{\phi} + \frac{2V}{\phi} = \frac{1}{\kappa\left(1-\frac{\tau}{\tau_*}\right)^2}\left(2\Lambda\phi_* - \frac{4\omega}{\tau_*^2 r^2}\right), \tag{3.24}$$

which diverges as $\tau \to \tau_*^-$ for any value of ω. τ_* corresponds to a Big Crunch singularity, where also the scalar (3.23) diverges. On the one hand, when $\omega \neq 0$, (3.24) diverges also when $r \to 0^+$. The areal radius $R(\tau, r) = \left(1 - \frac{\tau}{\tau_*}\right) r$ tends to zero as $r \to 0$ and there is a central singularity in this case. The constant time slices are finite with $0 \leq r \leq r_*$, where

$$r_* = \sqrt{\frac{3}{2\Lambda}}\sqrt{1 - \frac{2}{\kappa\tau_*^2}}. \tag{3.25}$$

Hence, there is a naked central singularity embedded in a finite inhomogeneous universe created by Λ and ϕ, ending at a finite future time τ_*. On the other hand, when $\omega = 0$, the curvature invariant

$$\mathcal{R}_{ab}\mathcal{R}^{ab} = \frac{1}{\phi^2}\left(\nabla_a\nabla_b\phi\nabla^a\nabla^b\phi + \frac{\Lambda^2}{\kappa^2}\right) \tag{3.26}$$

$$= \frac{1}{\tau_*^4\kappa r^4\left(1-\frac{\tau}{\tau_*}\right)^4}\left(\frac{9}{\kappa\tau_*^2} - 4 + \frac{8\Lambda r^2}{3}\right) + \frac{\Lambda^2}{\kappa^2\phi_*^2}\left(1 - \frac{\tau}{\tau_*}\right)^4 \tag{3.27}$$

diverges as $r \to 0^+$ (or as the areal radius $R \to 0^+$), so that the naked central singularity persists for $\omega = 0$ as well.

This Brans-Dicke solution is also a solution of quadratic $f(\mathcal{R})$ gravity with $f(\mathcal{R}) = \frac{\kappa\mathcal{R}^2}{4\Lambda}$. This theory is not endowed with a Newtonian limit, but it approximates Starobinsky inflation with $f(\mathcal{R}) = \mathcal{R} + \alpha\mathcal{R}^2$ The gradient $\nabla_c\phi$ of this solution is timelike and we can easily calculate (3.5), which becomes

$$\mathcal{KT} = \frac{2}{(8\pi)^{3/2} r\,(\tau_* - \tau)}. \tag{3.28}$$

It is straightforward to see that $\mathcal{KT}$ diverges as $r \to 0$ at the central singularity, but also at the Big Crunch singularity $\tau \to \tau_*^-$. The corresponding GR solution with $\Lambda > 0$ and $\phi = \text{const.}$ is de Sitter spacetime, where no singularities are present at all. This example confirms the observation made in section (3.1.1) that, at singularities, not only $\mathcal{KT}$ diverges, but also that the deviation of scalar-tensor gravity from GR becomes extreme, as confirmed by the completely different behaviour of the solution studied above from de Sitter spacetime.

3.2 Past-directed scalar gradients

The first-order thermodynamics of scalar-tensor gravity relies on a timelike and future-oriented scalar gradient in the fluid's 4-velocity (2.1) and (3.5) shows that the sign of the temperature depends on the time orientation of the scalar gradient. The temperature is positive for future-oriented gradients. If the gradient were past-oriented instead of future-oriented, the temperature would be negative-definite, with drastic consequences.

In this section, based on [4], we aim to show what happens to the formalism if the scalar field gradient is past-oriented, since past-directed scalar field gradients do naturally arise in analytical solutions of scalar-tensor gravity. Time reversal $t \to -t$ of the whole framework presented so far achieves the same result of a negative temperature, turning future-directed scalar field gradients into past-directed ones.

The main result of [4] is that, with past-oriented gradients, the kinematic fluid quantities remain unchanged, but certain thermodynamical variables such as heat fluxes change sign, leading to a negative temperature and a positive shear viscosity, at variance with previous works. A negative temperature is problematic in first-order thermodynamics, where additional degrees of freedom to those of GR give modified theories a positive-definite temperature. We cannot provide an assessment of the physical viability of solutions in scalar-tensor gravity through the sign of the temperature within our formalism, but in the following we show the need to restrict the applications of the formalism to situations with future-directed scalar field velocity only, to ensure a meaningful interpretation of it.

In order to show how the formalism changes with past-oriented gradients, let us define the timelike vector field $\tilde{u}^a$ as

$$\tilde{u}^a := \frac{\nabla^a\phi}{\sqrt{2X}}\,, \qquad X := -\frac{1}{2}\,\nabla_a\phi\nabla^a\phi > 0\,. \tag{3.29}$$

(only in this section will $\tilde{u}^a$ denote a velocity with past-directed gradient). Assuming that the spacetime manifold $(\mathcal{M}, g_{ab})$ admits a chart (t, x) with time coordinate t, then $g_{ab}\,\tilde{u}^a\,(\partial_t)^b > 0$ implies that $\tilde{u}^a$ is past-directed and it cannot be identified with the 4-velocity of an effective fluid, which is defined as timelike *future-directed* vector field. Now, let the scalar field ϕ be such that $\nabla^a\phi$ is past-directed: we can then define a future-directed vector field as

$$v^a := -\tilde{u}^a = -\frac{\nabla^a\phi}{\sqrt{2X}}\,. \tag{3.30}$$

The corresponding projection operator onto the 3-space orthogonal to v^a is $\mathfrak{h}^a{}_b$, where

$$\mathfrak{h}_{ab} := g_{ab} + v_a v_b = g_{ab} + \tilde{u}_a\tilde{u}_b = g_{ab} + \frac{\nabla_a\phi\nabla_b\phi}{2X} = h_{ab}\,, \tag{3.31}$$

and $h^a{}_b$ is the projection operator onto the 3-space orthogonal to $\tilde{u}^a$. Thus, h_{ab} remains unaffected by the change of sign in the definition of the 4-velocity when the timelike gradient $\nabla^a\phi$ is past-directed instead of being future-directed.

Let us examine now how the kinematic quantities associated with the effective scalar-tensor dissipative fluid [19, 20, 194] change when the definition of 4-velocity is modified to account for a past-directed gradient $\nabla^a\phi$. In particular, we make explicit the relations between the kinematic quantities associated with v^a (denoting them with $^{(v)}$) and those corresponding to $\tilde{u}^a = -v^a$ (denoting them with $^{(u)}$). For the 4-velocity gradient, we have

$$\nabla_a v_b = -\nabla_a\tilde{u}_b = -\frac{1}{\sqrt{2X}}\left(\nabla_a\nabla_b\phi - \frac{\nabla_a X\nabla_b\phi}{2X}\right)\,, \tag{3.32}$$

which implies

$$\theta_{(v)} = \nabla_a v^a = -\nabla_a\tilde{u}^a = -\theta_{(u)} \tag{3.33}$$

for the expansion scalar of the effective fluid,

$$a^a_{(v)} := v^c\nabla_c v^a = \tilde{u}^c\nabla_c\tilde{u}^a = a^a_{(u)} \tag{3.34}$$

for its 4-acceleration, while the projection of the velocity gradient onto the 3-space of the comoving observers reads

$$V^{(v)}_{ab} := h_a{}^c h_b{}^d\nabla_d v_c = -h_a{}^c h_b{}^d\nabla_d\tilde{u}_c = -V^{(u)}_{ab}\,, \tag{3.35}$$

and the new shear tensor is

$$\sigma^{(v)}_{ab} := V^{(v)}_{(ab)} - \frac{\theta^{(v)}}{3}h_{ab} = -\left(V^{(u)}_{ab} - \frac{\theta^{(u)}}{3}h_{ab}\right) = -\sigma^{(u)}_{ab}\,. \tag{3.36}$$

These kinematic quantities do not depend on the field equations and are the same in all scalar-tensor gravity theories.

The effective energy-momentum tensor for scalar-tensor gravity is (2.23) and it has been recognised to have the form of an imperfect fluid stress-energy tensor. In the case of past-directed gradients of ϕ, we can write it as

$$T^{(v)}_{ab} = \rho^{(v)}\,v_a v_b + q^{(v)}_a v_b + q^{(v)}_b v_a + \Pi^{(v)}_{ab}\,, \tag{3.37}$$

where the effective energy density, heat flux density, stress tensor, isotropic pressure, and anisotropic stress tensor (the trace-free part π_{ab} of the stress tensor Π_{ab}) in the comoving frame of the effective fluid are, respectively,

$$\rho^{(v)} = T_{ab}v^a v^b \,, \tag{3.38}$$

$$q_a^{(v)} = -T_{cd}\, v^c {h_a}^d \,, \tag{3.39}$$

$$\Pi_{ab}^{(v)} = P^{(v)} h_{ab} + \pi_{ab}^{(v)} = T_{cd}\, {h_a}^c\, {h_b}^d \,, \tag{3.40}$$

$$P^{(v)} = \frac{1}{3}\, g^{ab}\Pi_{ab}^{(v)} = \frac{1}{3}\, h^{ab} T_{ab} \,, \tag{3.41}$$

$$\pi_{ab}^{(v)} = \Pi_{ab}^{(v)} - P^{(v)} h_{ab} \,. \tag{3.42}$$

It is straightforward to see that some of these quantities are not altered with respect to those arising from future-directed scalar field gradients:

$$\rho^{(v)} = \rho^{(u)} \,, \qquad \Pi_{ab}^{(v)} = \Pi_{ab}^{(u)} \,, \qquad P^{(v)} = P^{(u)} \,, \qquad \pi_{ab}^{(v)} = \pi_{ab}^{(u)} \,. \tag{3.43}$$

However, the heat flux density changes sign when the 4-velocity changes orientation:

$$q_a^{(v)} = -T_{cd}\, v^c {h_a}^d = T_{cd}\, \tilde{u}^c {h_a}^d = -q_a^{(u)} \,, \tag{3.44}$$

which has important consequences for the definition of a meaningful temperature, as we detail in the following. Since, as we have seen previously,

$$q_a^{(u)} = -\frac{\sqrt{2X}}{8\pi\phi}\, a_a^{(u)} \,, \qquad \pi_{ab}^{(u)} = \frac{\sqrt{2X}}{8\pi\phi}\, \sigma_{ab}^{(u)} \,, \tag{3.45}$$

given (3.43) and (3.44), we have

$$q_a^{(v)} = -q_a^{(u)} = \frac{\sqrt{2X}}{8\pi\phi}\, a_a^{(u)} = \frac{\sqrt{2X}}{8\pi\phi}\, a_a^{(v)} \tag{3.46}$$

and

$$\pi_{ab}^{(v)} = \pi_{ab}^{(u)} = \frac{\sqrt{2X}}{8\pi\phi}\, \sigma_{ab}^{(u)} = -\frac{\sqrt{2X}}{8\pi\phi}\, \sigma_{ab}^{(v)} \,. \tag{3.47}$$

This means that, for a scalar field with timelike past-directed gradient, one finds the "temperature of scalar-tensor gravity"

$$(\mathcal{K}\mathcal{T})^{(v)} = -(\mathcal{K}\mathcal{T})^{(u)} = -\frac{\sqrt{2X}}{8\pi\phi} < 0 \tag{3.48}$$

The shear viscosity coefficient η reads

$$\eta^{(v)} = -\eta^{(u)} = \frac{\sqrt{2X}}{16\pi\phi} > 0 \,. \tag{3.49}$$

The effective heat equation describing the approach to (or the departure from) thermal equilibrium reads

$$\frac{d}{d\tau}(\mathcal{K}\mathcal{T})^{(v)} = 8\pi(\mathcal{K}\mathcal{T})^2_{(v)} - \theta_{(v)}(\mathcal{K}\mathcal{T})_{(v)} + \frac{\Box\phi}{8\pi\phi} \tag{3.50}$$

and is, therefore, not affected by the replacement $\tilde{u}^a \mapsto -\tilde{u}^a = v^a$.

Thus, for past-directed gradients, we find a negative temperature and positive shear viscosity, opposite to the result for future-directed gradients. This is precisely the reason why making sure the velocity of ϕ is future-directed is crucial: the thermodynamical analogy built in [10–12] itself relies on a meaningful notion of temperature. The fact that such a temperature naturally arose to be positive-definite in the case of future-directed velocity is one of the promising features of the formalism. Moreover, modified gravity theories with degrees of freedom additional to those of GR always have a positive temperature with respect to GR, which is quite intuitive (see section 4.2.1 for a pathological instance of negative temperature studied within the standard formalism with future-directed gradients). Therefore, we conclude that the formalism remains valid, provided that we restrict to future-directed timelike gradients.

Chapter 4

Mapping the landscape of gravity theories: extensions and applications of first-order thermodynamics

In the previous chapter, we have laid out the basics of first-order thermodynamics, exploring how it provides a well-defined notion of temperature that characterizes scalar-tensor gravity with $\mathcal{KT} > 0$ and its deviation from GR, the equilibrium state at $\mathcal{KT} = 0$. The effective heat equation (3.20) described the approach to the equilibrium state or the departure from it. In this chapter, we use these two cornerstones of the formalism as tools to draw a map of the landscape of gravity theories. Recalling that first-order thermodynamics is flexible enough that we can study both entire classes of theories and specific solutions within them, in the following we probe how far the formalism can be extended beyond the generalised Brans-Dicke class described by (2.20) explored so far.

In section 4.1, based on [12], we review the extension to Horndeski theories. In section 4.2, we explore the equilibrium states alternative to GR found through the fixed points of the effective heat equation. Section 4.2.1 deals with zero-temperature states and is based on [2], while section 4.2.2 deals with constant-temperature states and is based on [5]. Section 4.3 shows that an alternative formulation of first-order thermodynamics based on chemical potential instead of temperature can be found, and is based on [3]. The last part of this chapter focuses on cosmology: in section 4.4, based on [1], we study the cosmological applications to "old-school" scalar-tensor gravity, and in 4.5.1 explore some exact solutions. The formalism is finally extended to Horndeski cosmology in section 4.4.2 and some exact solutions are studied in 4.5.2, based on [7].

4.1 First-order thermodynamics of Horndeski theories

An application of the thermodynamical formalism that yielded compelling results was that to Horndeski gravity in [12], the most general class of scalar-tensor theories exhibiting second-order equations of motion and thus avoiding Ostrogradsky instabilities. Theories within the Horndeski class are employed in numerous models of dynamical dark energy and as late-time modifications of GR, but have been severely constrained by the multi-messenger gravitational wave event GW170817 [147] that showed that gravitational waves propagate at the speed of light up to remarkable precision (see 1.2.4). In the following, we dub the subclass of Horndeski theories respecting these constraints "viable Horndeski".

Not only did the extension to Horndeski theories substantially widen the realm of applicability of first-order thermodynamics, with respect to the generalized Brans-Dicke theories where it was originally formulated, but it was also found that the thermodynamical formalism does not work for the most general Horndeski theories, as some terms in their field equations explicitly break the thermodynamical analogy. Strikingly, these terms are precisely those that violate the equality between the propagation speeds of gravitational and electromagnetic waves. Therefore, the crucial finding is that first-order thermodynamics indicates the direction of the physical constraints on Horndeski gravity, which paves the way for intriguing further developments. The analogy is spoiled for those operators which contain derivative non-minimal couplings and nonlinear contributions in the connection. This relates to the well-known but hard to tackle problem of separating matter from gravity degrees of freedom in terms of a local description, as we detail in the following.

For more details on the thermodynamics of Horndeski theories in the context of a FLRW cosmological background, see sections 4.4.2 and 4.5.2.

Viable Horndeski The full Horndeski action is given by (1.47). The viable subclass of Horndeski theories that restricts to a luminal propagation of gravitational waves is given by $G_{4X} = 0$ and $G_5 = 0$. The field equations in this case read [7, 111]

$$\begin{aligned} G_4\, G_{ab} - \nabla_a \nabla_b G_4 + \left[\Box G_4 - \frac{G_2}{2} - \frac{1}{2} \nabla_c \phi \nabla^c G_3 \right] g_{ab} \\ + \frac{1}{2} \left[G_{3X}\, \Box\phi - G_{2X} \right] \nabla_a \phi \nabla_b \phi + \nabla_{(a} \phi \nabla_{b)} G_3 = T^{(\mathrm{m})}_{ab} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} G_{4\phi} R + G_{2\phi} + G_{2X} \Box\phi + \nabla_c \phi \nabla^c G_{2X} - G_{3X} (\Box\phi)^2 - \nabla_c \phi \nabla^c G_{3X} \Box\phi \\ - G_{3X} \nabla^c \phi \Box \nabla_c \phi + G_{3X} R_{ab} \nabla^a \phi \nabla^b \phi - \Box G_3 - G_{3\phi} \Box\phi = 0, \end{aligned} \tag{4.2}$$

where round brackets indicate symmetrization. We can recast the field equations (4.1) as effective Einstein equations,

$$G_{ab} = T^{(\mathrm{eff})}_{ab} \,, \tag{4.3}$$

where

$$T^{(\mathrm{eff})}_{ab} = \frac{T^{(\mathrm{m})}_{ab}}{G_4} + T^{(\phi)}_{ab} \,, \tag{4.4}$$

$$T^{(\phi)}_{ab} = T^{(2)}_{ab} + T^{(3)}_{ab} + T^{(4)}_{ab} \,, \tag{4.5}$$

and the individual contributions are

$$T^{(2)}_{ab} = \frac{1}{2G_4}\left(G_{2X}\nabla_a\phi\nabla_b\phi + G_2\, g_{ab}\right)\,, \tag{4.6}$$

$$\begin{aligned} T^{(3)}_{ab} =& \frac{1}{2G_4}\left(G_{3X}\nabla_c X\nabla^c\phi - 2XG_{3\phi}\right) g_{ab} \\ & - \frac{1}{2G_4}\left(2G_{3\phi} + G_{3X}\Box\phi\right)\nabla_a\phi\nabla_b\phi - \frac{G_{3X}}{G_4}\nabla_{(a}X\nabla_{b)}\phi\,, \end{aligned} \tag{4.7}$$

$$T^{(4)}_{ab} = \frac{G_{4\phi}}{G_4}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi) + \frac{G_{4\phi\phi}}{G_4}(\nabla_a\phi\nabla_b\phi + 2X\, g_{ab})\,. \tag{4.8}$$

The equation of motion for the scalar field can be written as

$$\mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 = 0\,, \tag{4.9}$$

where

$$\mathcal{S}_2 = \left(G_{2X}g^{ab} - G_{2XX}\nabla^a\phi\nabla^b\phi\right)\nabla_a\nabla_b\phi + G_{2\phi} - 2XG_{2\phi X}\,, \tag{4.10}$$

$$\begin{aligned} \mathcal{S}_3 =& G_{3X}R_{ab}\nabla^a\phi\nabla^b\phi - 2\left(G_{3X}g^{ab}g^{cd} - G_{3XX}\nabla^a\phi\nabla^b\phi\, g^{cd}\right)\nabla_{[a|}\nabla_b\phi\nabla_{|c]}\nabla_d\phi \\ & - 2\left[\left(G_{3\phi} - XG_{3\phi X}\right)g^{ab} - G_{3\phi X}\nabla^a\phi\nabla^b\phi\right]\nabla_a\nabla_b\phi\, + 2XG_{3XX}, \end{aligned} \tag{4.11}$$

$$\mathcal{S}_4 = G_{4\phi}R\,. \tag{4.12}$$

Building the thermodynamical analogy through Eckart's constitutive equations (3.1), (3.2) and (3.3) starting from the Horndeski Lagrangian (and making the choice of neglecting bulk viscosity), one obtains the following effective energy density and pressure [7, 12]:

$$\rho^{(\phi)} = \frac{1}{2G_4}\left(2XG_{2X} - G_2 - 2XG_{3\phi}\right) + \frac{\sqrt{2X}}{G_4}\left(G_{4\phi} - XG_{3X}\right)\theta\,, \tag{4.13}$$

$$\begin{aligned} P^{(\phi)} =& \frac{1}{2G_4}\left(G_2 - 2XG_{3\phi} + 4XG_{4\phi\phi}\right) - \frac{\left(G_{4\phi} - XG_{3X}\right)}{G_4\sqrt{2X}}\dot{X} - \frac{2G_{4\phi}}{3G_4}\sqrt{2X}\,\theta \\ =& \frac{1}{2G_4}\left(G_2 - 2XG_{3\phi} + 4XG_{4\phi\phi}\right) - \frac{\left(G_{4\phi} - XG_{3X}\right)}{G_4}\Box\phi + \frac{\left(G_{4\phi} - 3XG_{3X}\right)}{3G_4}\sqrt{2X}\,\theta\,. \end{aligned} \tag{4.14}$$

The effective heat flux reads

$$q_a^{(\text{eff})} = \frac{G_{4\phi} - XG_{3X}}{G_4\sqrt{2X}}\left(\nabla_a X + \frac{\nabla\phi\cdot\nabla X}{2X}\nabla_a\phi\right), \tag{4.15}$$

which is

$$q_a^{(\text{eff})} = -\frac{\sqrt{2X}\left(G_{4\phi} - XG_{3X}\right)}{G_4}\dot{u}_a, \tag{4.16}$$

so that (3.5) becomes

$$\mathcal{K}\mathcal{T} = \frac{\sqrt{2X}(G_{4\phi} - XG_{3X})}{G_4}. \tag{4.17}$$

The anisotropic stress tensor reads

$$\pi_{ab}^{(\text{eff})} = \frac{G_{4\phi}}{G_4}\left[\nabla_a\nabla_b\phi - \frac{\nabla_{(a}X\nabla_{b)}\phi}{X}\right. \tag{4.18}$$

$$\left. - \frac{\nabla X\cdot\nabla\phi}{4X^2}\nabla_a\phi\nabla_b\phi - \frac{h_{ab}}{3}\left(\Box\phi - \frac{\nabla X\cdot\nabla\phi}{2X}\right)\right], \tag{4.19}$$

which is equivalent to

$$\pi_{ab}^{(\text{eff})} = \frac{G_{4\phi}\sqrt{2X}}{G_4}\sigma_{ab}, \tag{4.20}$$

so that the shear viscosity coefficient is

$$\eta = -\frac{\sqrt{X}\,G_{4\phi}}{\sqrt{2}\,G_4}. \tag{4.21}$$

In the shear viscosity, $G_4 > 0$ ensures a positive gravitational coupling of gravity to matter. There is always the freedom to redefine the scalar field ϕ through $\psi = G_4(\phi)$ (which is invertible whenever $G_{4\phi} \neq 0$), hence the shear viscosity would be positive whenever $G_{4\phi} < 0$ and negative otherwise, for example in Brans-Dicke theory where $G_4(\phi) = \phi$, which recovers the negative viscosity found in section 3.1.2. The expressions for $\mathcal{K}$ and η of course reduce to (3.5) and (3.10), respectively, for $G_4 = 8\pi\phi$ and $G_3 = 0$.

A general interpretation of $\mathcal{K}$ and $\mathcal{T}$ emerges from the results above. Choosing

$$\mathcal{K} \equiv \sqrt{2X}\,(G_{4\phi} - XG_{3X}) \tag{4.22}$$

and

$$\mathcal{T} \equiv \frac{1}{G_4}, \tag{4.23}$$

then $\mathcal{T}$ automatically satisfies $h_{ab}\nabla^b\mathcal{T} = 0$. Indeed, $\mathcal{T} = \mathcal{T}(\phi)$ since $G_4 = G_4(\phi)$, thus $\nabla_a\mathcal{T} \propto \nabla_a\phi$. Furthermore, it must be $G_4 > 0$ to guarantee a positive coupling strength of gravity to matter, as is clear from (4.23), and the temperature of gravity $\mathcal{T}$ is non-negative, which was not granted. GR corresponds to $\phi =$ const. and, therefore, to a unit value of the temperature (if coupling with matter is considered) and vanishing thermal conductivity.

The approach to equilibrium is described by the effective heat equation, which reads

$$\frac{d(\mathcal{K}\mathcal{T})}{d\tau} = \left(\frac{\Box\phi}{\sqrt{2X}} - \theta\right)\left[\mathcal{K}\mathcal{T} - \frac{(2X)^{3/2}}{G_4}(G_{3X} + XG_{3XX})\right] \tag{4.24}$$

$$- \frac{2X}{G_4^2}\left[G_4G_{4\phi\phi} - XG_4G_{3X\phi} - G_{4\phi}\left(G_{4\phi} - XG_{3X}\right)\right]. \tag{4.25}$$

The physical interpretation is complex because of the many free functions, but reduces to the simple case studied in section 3.1.3 for $G_4 = \phi$ and $G_3 = 0$, when Horndeski recovers Brans-Dicke theories.

Beyond viable Horndeski It is still highly non-trivial that the constitutive relations of Eckart's thermodynamics hold also in this subclass of Horndeski theories, since this is all based on the proportionality between some kinematic quantities and the dissipative terms. However, the extension to viable Horndeski reached the limits of applicability of the formalism: [12] shows that, whenever we try to apply the thermodynamical formalism to theories beyond the viable class considered above, whose effective stress-energy tensor contains the term

$$T^{(\phi)}_{ab} \supset \zeta(\phi, X)\, R_{acbd} \nabla^c\phi\nabla^d\phi\,, \tag{4.26}$$

where $\zeta(\phi, X)$ is a generic function, the Riemann tensor R_{acbd} ends up breaking the proportionality between the traceless shear tensor σ_{ab} and the anisotropic stress tensor $\pi^{(\phi)}_{ab}$, so that Eckart's constitutive equations no longer hold.

For the stress-energy tensor above, the stress tensor reads

$$\begin{aligned}\Pi_{ab} = T_{cd}{h_a}^c{h_b}^d &\supset \zeta(\phi, X){h_a}^c{h_b}^d R_{cedf}\nabla^e\phi\nabla^f\phi \\ &= \zeta(\phi, X)R_{aebf}\nabla^e\phi\nabla^f\phi,\end{aligned} \tag{4.27}$$

the isotropic pressure

$$\begin{aligned}P = \frac{1}{3}g^{ab}\Pi_{ab} &\supset \frac{\zeta(\phi, X)}{3}g^{ab}R_{aebf}\nabla^e\phi\nabla^f\phi \\ &= \frac{\zeta(\phi, X)}{3}R_{ef}\nabla^e\phi\nabla^f\phi,\end{aligned} \tag{4.28}$$

and the anisotropic stress tensor

$$\begin{aligned}\pi_{ab} =& \Pi_{ab} - Ph_{ab} \supset \zeta(\phi, X)R_{aebf}\nabla^e\phi\nabla^f\phi \\ &- \frac{\zeta(\phi, X)}{3}h_{ab}R_{ef}\nabla^e\phi\nabla^f\phi.\end{aligned} \tag{4.29}$$

The terms containing the Ricci tensor could, in principle, cancel out with similar terms originating in the field equations and showing up in the energy-momentum tensor, but the contributions proportional to the Riemann tensor cannot be traced away without involving unreasonable fine-tuning. This general feature of the first-order thermodynamics of Horndeski theories becomes evident already as soon as G_4 is allowed to be a function of X, such as in the simple case

$$\mathcal{L} = G_4(X)R = XR. \tag{4.30}$$

The heat flux reads

$$\begin{aligned}q^{(\text{eff})}_a = &-\frac{1}{X}\left[\Box\phi\nabla_c\nabla_d\phi - \nabla_c\nabla_e\phi\nabla^e\nabla_d\phi\right. \\ &\left.-R_{cedf}\nabla^e\phi\nabla^f\phi\right]u^c{h_a}^d,\end{aligned} \tag{4.31}$$

which cannot be reduced to the Eckart constitutive relation (3.2), while the anisotropic stress tensor

$$
\begin{aligned}
\pi^{(\text{eff})}_{ab} \equiv{}& \Pi^{(\text{eff})}_{ab} - P^{(\text{eff})} h_{ab} \qquad (4.32)\\
={}& \frac{1}{X}\left\{\Box\phi\left[\nabla_a\nabla_b\phi - \frac{\nabla_{(a}\phi\nabla_{b)}X}{X} - \frac{\nabla^c\phi\nabla_c X}{4X^2}\nabla_a\phi\nabla_b\phi + \frac{\nabla^c\phi\nabla_c X}{6X}h_{ab}\right]\right.\\
&- \nabla_a\nabla_e\phi\nabla^e\nabla_b\phi + \frac{\nabla_{(a}\phi\nabla_{b)}\nabla_e\phi\nabla^e X}{X} - \nabla_a\phi\nabla_b\phi\frac{\nabla^e X\nabla_e X}{4X^2} + \frac{\nabla^e X\nabla_e X}{6X}h_{ab}\\
&\left.-\frac{(\Box\phi)^2}{6}h_{ab} - R_{cdef}h_a{}^c h_b{}^d\nabla^e\phi\nabla^f\phi + \frac{1}{3}\left(\nabla^a\nabla^e\phi\nabla_a\nabla_e\phi\right)h_{ab}\right\}.
\end{aligned}
$$

is not proportional to the shear tensor at all. This means that the proportionality between dissipative and kinematic quantities encoded into Eckart's constitutive relations is broken, and no thermodynamical analogy can be found.

These results spurred some further developments, such as the considerations in [18] that pave the way for extending the study of first-order thermodynamics of Horndeski gravity to anisotropic Bianchi universes, going beyond FLRW. The imperfect fluid analogy developed for this class of theories has also been exploited with the goal of attempting to classify Horndeski theories based on the nature of the effective fluid, specifically on its requirement to be a Newtonian fluid [23].

Constraints from Newtonian fluid nature A further aspect of the effective fluid, related to the deeper meaning behind the formal rewriting of $T^{(\phi)}_{ab}$ as an imperfect fluid, is also visible in the application to Horndeski theories and was explored in [23]. In general, effective fluids can be classified according to their constitutive relations. As compellingly shown in [22], exploring the properties of the imperfect fluid behind modified theories of gravity allows one to obtain an intuitive picture of their physical meaning, often obfuscated by cumbersome expressions. The effective fluid approach described so far is not only useful for constructing a "thermodynamics of gravitational theories", but also provides a promising way to classify different subclasses of the very general Horndeski theories based on the nature of this fluid. Specifically, in [23], the requirement that the effective fluid be Newtonian (i.e., with the viscous stresses depending only on the first derivatives of the fluid's 4-velocity) was explored. This requirement is relevant since this is the same requirement of Eckart's non-equilibrium thermodynamics. It was found that the requirement of a Newtonian fluid is quite stringent and selects two specific subclasses of viable Horndeski: one is characterized by $G_3 = G_{4\phi}\ln(X/X_*)$, where X_* is a constant, and the other is identified with $G_3 = 0$. This way, all the non-linear contributions in the dissipative quantities disappear from the constitutive equations. These subclasses are disconnected with respect to conformal transformations of the metric tensor, and the second one exists only for a dynamical scalar field. More general theories correspond to effective fluids that are non-Newtonian, and therefore exotic and less easily interpretable from the physical point of view (see 4.4.2 for a study of these subclasses).

4.2 Analysis of equilibrium states other than GR

GR is the fundamental equilibrium state that all other states refer to, but it is not unique. In this section, we use the effective heat equation (3.20) to find equilibrium states alternative to GR. We can gain more insight into this equation describing the approach to or departure from equilibrium by studying its fixed points, i.e., those with

$$\frac{d(\mathcal{KT})}{d\tau} = 0. \tag{4.33}$$

Of course, they correspond to situations where either $\mathcal{KT} = 0$ or $\mathcal{KT} =$ const. and constitute equilibrium states in the thermodynamics of gravitational theories. Our goal is to analyse these other possible equilibrium states that might challenge the uniqueness of the GR equilibrium state and its special role in this landscape of gravity theories. In turn, this will provide additional tests of the formalism and its physical interpretation illustrated in chapter 3.

4.2.1 States with $\mathcal{KT} = 0$: non-dynamical scalars

Let us start with the study of other situations where $\mathcal{KT} = 0$, in addition to the zero-temperature GR state: such states would correspond to theories of gravity that are special in some physical sense and they were studied in [2]. In particular, we want to probe the possibility of states of equilibrium other than GR, corresponding to $\mathcal{KT} =$ constant. Here, we apply the formalism to entire classes of gravitational theories rather than to specific solutions, with the aim of better understanding the regime of validity of the formalism. We test it on theories that, while not always physically viable, allow us to clarify the possible existence of other equilibrium states.

Intuitively, we expect that theories of gravity containing non-dynamical fields in addition to the two spin-2 massless modes of GR will have either zero $\mathcal{KT}$ or that the latter will be completely arbitrary, if these extra non-dynamical fields are. Indeed, we show in the following that this intuition is accurate, by analysing several instances of theories including non-dynamical scalars: Brans-Dicke theory with $\omega = -3/2$, Palatini $f(R)$ gravity, and cuscuton gravity. While not always physically viable, these theories help us to test the boundaries of the new thermodynamical formalism and to better grasp the meaning of the zerotemperature equilibrium states. All these theories with a non-dynamical scalar field ϕ are all contained in the subclass of viable Horndeski theories, hence the first-order thermodynamics presented so far can be applied without changes. It is also natural to wonder what a theory with *less* degrees of freedom than GR would look like from the point of view of the thermodynamics of modified gravity. In particular, if one can define a concept of temperature as done in scalar-tensor and Horndeski gravity, this temperature should be negative, corresponding to the excitation of less degrees of freedom than GR. To test this intuition, we study Nordström's theory of gravity [203], in which the metric is forced to be conformally flat and only a scalar field degree of freedom (but not the two spin two modes of GR) is excited. This theory was considered as a serious candidate for the description of gravity only for a very brief period of time and is of course completely ruled out. However, it is still useful as a toy model when studying fundamental questions such as the validity of different equivalence principles (since only GR and Nordström's gravity respect the strong equivalence principle) [204, 205] or, in our case, the thermodynamics of gravity in a landscape of theories.

Brans-Dicke gravity with non-dynamical scalar

It is well-known that Brans-Dicke gravity with $\omega = -3/2$ and Palatini $f(R)$ gravity become non-dynamical. The (Jordan frame) field equations of Brans-Dicke gravity are [86, 98] (with a constant ω, at variance with the generalised Brans-Dicke (2.20) we consider in most of this thesis, which has $\omega(\phi)$)

$$G_{ab} = \frac{8\pi}{\phi} T_{ab}^{(\mathrm{m})} + \frac{\omega}{\phi^2} \left(\nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \right) + \frac{1}{\phi} \left(\nabla_a \nabla_b \phi - g_{ab} \Box \phi \right) - \frac{V}{2\phi} g_{ab} \tag{4.34}$$

$$(2\omega + 3) \Box \phi = 8\pi T^{(\mathrm{m})} + \phi V_{,\phi} - 2V, \tag{4.35}$$

with $V' \equiv \frac{dV}{d\phi}$ and, for the rest, the same notation as in section 2.2. Setting $\omega = -3/2$, the field equation (4.35) for ϕ is not a wave equation anymore, but reduces to the algebraic identity

$$8\pi T^{(\mathrm{m})} = 2V - \phi V', \tag{4.36}$$

making it clear that the scalar ϕ is not dynamical in this theory. The other field equation (4.34) becomes

$$R_{ab} - \frac{R}{2} g_{ab} = \frac{8\pi}{\phi} T_{ab}^{(\mathrm{m})} - \frac{3}{2\phi^2} \left(\nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \right) + \frac{1}{\phi} \left(\nabla_a \nabla_b \phi - g_{ab} \Box \phi \right) - \frac{V}{2\phi} g_{ab} \tag{4.37}$$

which, upon contraction, gives

$$R = -\frac{8\pi T^{(\mathrm{m})}}{\phi} - \frac{3}{2\phi^2} \nabla^c \phi \nabla_c \phi + \frac{3}{\phi} \Box \phi + \frac{2V}{\phi}. \tag{4.38}$$

Substituting (4.36) gives

$$R = V' - \frac{3}{2\phi^2} \nabla^c \phi \nabla_c \phi + \frac{3\Box\phi}{\phi}. \tag{4.39}$$

If we differentiate (4.36), we obtain

$$\left(V' - \phi V'' \right) \nabla_c \phi = 8\pi \nabla_c T^{(\mathrm{m})}. \tag{4.40}$$

In the absence of a potential, or when the latter is a pure mass term $V = m^2\phi^2/2$, then

$$T^{(\mathrm{m})} = 0 \tag{4.41}$$

i.e., we can only have vacuum or conformally invariant matter.

If the gradient $\nabla^c\phi$ is timelike and $V(\phi) \neq m^2\phi^2/2$ (with $m^2 \geq 0$), then one can rewrite $\nabla^c\phi$ in terms of $\nabla^c T^{(\mathrm{m})}$ by taking advantage of (4.40). Therefore, the effective temperature of $\omega = -3/2$ Brans-Dicke theory with non-dynamical scalar is given by

$$\mathcal{K}\mathcal{T} = \frac{\sqrt{|\nabla^c\phi\nabla_c\phi|}}{8\pi\phi} = \frac{\sqrt{|\nabla^c T^{(\mathrm{m})}\nabla_c T^{(\mathrm{m})}|}}{\phi\,|V' - \phi V''|}. \tag{4.42}$$

If, instead, $V(\phi) = m^2\phi^2/2$, then $T^{(\mathrm{m})} = 0$ and there is no relation between $\nabla_c\phi$ and $\nabla_c T^{(\mathrm{m})}$. For general forms of matter, in both cases the temperature is almost completely arbitrary. This is not too surprising because the scalar field itself is non-dynamical and, essentially, is also arbitrary. The temperature $\mathcal{K}\mathcal{T}$ relative to GR is found with this non-dynamical scalar field and is ill-defined as a consequence of its arbitrariness. The situation changes in vacuum, possibly in the presence a cosmological constant. In this case,

$$T^{(\mathrm{m})}_{ab} = -\Lambda g_{ab}, \tag{4.43}$$

hence one has that $T^{(\mathrm{m})} = -4\Lambda$ is constant, which implies $\nabla_c T^{(\mathrm{m})} = 0$ and $\mathcal{K}\mathcal{T} = 0$.

Palatini $f(R)$ gravity

It is well known that Palatini $f(R)$ gravity is equivalent to $\omega = -3/2$ Brans-Dicke theory with a complicated potential [26] and that, in vacuum, it reduces to GR with (possibly) a cosmological constant. Therefore, vacuum Palatini $f(R)$ gravity has effective "temperature of gravity" given by $\mathcal{K}\mathcal{T} = 0$. In any case, the scalar field is non-dynamical and, in the presence of matter, the theory runs into all sorts of problems, including unacceptably strong couplings to the Standard Model, impossibility to build polytropic stars, ill-posed Cauchy problem and so on [26].

Cuscuton gravity

Cuscuton gravity [206–210] is interesting from various points of view: it is a special case of Hořava-Lifschitz theory, a model of a Lorentz-violating theory, it can implement the idea of limiting curvature without cosmological instabilities [207, 211] and cosmological singularities [212, 213] (this is not true in more general Horndeski theories [214–216]), and has been obtained as the ultraviolet limit of an anti-Dirac-Born-Infeld theory [217]. Other phenomenological properties are studied in [218–220]. The cuscuton is realized by a scalar field that does not propagate new degrees of freedom with respect to GR (at least in the unitary gauge [221], but this property is believed to hold in any gauge [222]). This scalar (cuscuton field) satisfies a first-order equation of motion, i.e., a constraint and the perturbed scalar action does not contain a kinetic term for this field, at all orders [221]. Denoting the cuscuton field with ϕ, its potential with $V(\phi)$, and using $f_{,\phi} \equiv \frac{\partial f}{\partial\phi}$, $f_{,X} \equiv \frac{\partial f}{\partial X}$, for any $f = f(\phi, X)$, the cuscuton Lagrangian density is

$$\mathcal{P}(\phi, X) = \pm\mu^2\sqrt{2X} - V(\phi), \tag{4.44}$$

where μ is a mass scale. The total action is

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi} + \mathcal{P}\right) + S_{\text{matter}} \tag{4.45}$$

The cuscuton satisfies the equation of motion

$$g^{ab}\nabla_a\left(\mathcal{P}_{,X}\nabla_b\phi\right) + \mathcal{P}_{,\phi} = 0 \tag{4.46}$$

or

$$\pm\mu^2\nabla^b\left(\frac{\nabla_b\phi}{\sqrt{2X}}\right) = V_{,\phi} \tag{4.47}$$

which reduces to a first-order constraint [213]. The field equations for g_{ab} can be written in the form of effective Einstein equations with the effective stress-energy tensor

$$T_{ab}^{(\phi)} = \mathcal{P}g_{ab} + \mathcal{P}_{,X}\nabla_a\phi\nabla_b\phi = \left[\pm\mu^2\sqrt{2X} - V\right]g_{ab} \pm \mu^2\frac{\nabla_a\phi\nabla_b\phi}{\sqrt{2X}} \tag{4.48}$$

on the right-hand side as the effective source. $T_{ab}^{(\phi)}$ has the form a perfect fluid stress-energy tensor (1.6), with energy density, pressure, and 4-velocity, respectively

$$\rho^{(\phi)}(\phi, X) = 2X\mathcal{P}_{,X} - P^{(\phi)} = V(\phi) \tag{4.49}$$

$$P^{(\phi)}(\phi, X) = \mathcal{P}(\phi, X) = \pm\mu^2\sqrt{2X} - V(\phi), \tag{4.50}$$

$$u^a = \pm\frac{\nabla^a\phi}{\sqrt{2X}}. \tag{4.51}$$

The fact that the Lagrangian $\mathcal{P}(\phi, X)$ coincides with the pressure is a trademark of a perfect fluid, for which a Lagrangian description is known [193]. The $\pm$ sign in (4.51) ensures that u^c can be chosen so that it is future pointing. The speed of sound in the cuscuton fluid, given by

$$c_s^2 = \frac{P_{,X}^{(\phi)}}{\rho_{,X}^{(\phi)}} = \frac{\mathcal{P}_{,X}}{\mathcal{P}_{,X} + 2X\mathcal{P}_{,XX}} \tag{4.52}$$

diverges because the denominator vanishes, highlighting the typical rigidity of the incompressible cuscuton fluid [206]. In the unitary gauge, where $\phi = \phi(t)$, it is obvious that $\nabla^c\phi$ is timelike. Since there is no dissipation, the cuscuton field corresponds to a state of equilibrium: one can argue that no dissipation occurs in this fluid because it is already in a state of equilibrium. This is not really surprising, since no propagating degree of freedom is excited in addition to the two massless spin two modes of GR.

From a more general point of view, the cuscuton is a special case of the viable class of Horndeski gravity (see section 4.1), corresponding to the choice of functions

$$G_4(\phi, X) = \frac{1}{16\pi} \tag{4.53}$$

$$G_2(\phi, X) = \mathcal{P}(\phi, X) = \pm\mu^2\sqrt{2X} - V(\phi) \tag{4.54}$$

$$G_3(\phi, X) = G_5(\phi, X) = 0. \tag{4.55}$$

Since for viable Horndeski, $\mathcal{KT}$ is (4.17), taking the limit in which $G_3 \to 0, G_4 \to$ const., and G_2 as above, one obtains the cuscuton theory without dissipation. In this limit, (4.17) yields $\mathcal{KT} \to 0$.

Extended cuscuton theories

By contrast, consider extended cuscuton theories [223–225], which include a Galileon generalization of the cuscuton (also called cuscuta-Galileon [223]): these generally contain a dynamical scalar field. For example, the specific theory [213] identified by

$$G_4(\phi, X) = \frac{1}{16\pi}, \tag{4.56}$$
$$G_2(\phi, X) = \pm\mu^2\sqrt{2X} - V(\phi), \tag{4.57}$$
$$G_3(\phi, X) = -a_3 \ln\left(\frac{X}{\Lambda^4}\right), \tag{4.58}$$
$$G_5(\phi, X) = 0 \tag{4.59}$$

is endowed with a nonzero effective temperature from (4.17), namely

$$\mathcal{KT} = 16\pi a_3\sqrt{-\nabla^c\phi\nabla_c\phi}. \tag{4.60}$$

This model has three dynamical degrees of freedom, unlike the original cuscuton theory, meaning that the scalar degree of freedom is excited and propagates [224]. Taking the limit $a_3 \to 0$ recovers the usual cuscuton, hence sending $\mathcal{KT}$ to zero.

Nordström gravity

In Nordström's scalar theory of gravity [203], the spacetime metric $\tilde{g}_{ab}$ is conformally flat,

$$\tilde{g}_{ab} = \Omega^2 g_{ab} \tag{4.61}$$

where g_{ab} is here the Minkowski metric and the conformal factor Ω satisfies

$$\Box\Omega = 0. \tag{4.62}$$

Under a generic conformal map (4.61), geometric quantities transform according to the well-known rules [27] (all quantities with tilde derive from $\tilde{g}_{\mu\nu}$)

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \frac{1}{\Omega}\left(\delta^a_b\nabla_c\Omega + \delta^a_c\nabla_b\Omega - g_{bc}\nabla^a\Omega\right), \tag{4.63}$$
$$\begin{aligned}\tilde{R}_{ab} = {}& R_{ab} - 2\nabla_a\nabla_b\ln\Omega - g_{ab}g^{ef}\nabla_e\nabla_f\ln\Omega \\ &+ 2\nabla_a\ln\Omega\nabla_b\ln\Omega - 2g_{ab}g^{ef}\nabla_e\ln\Omega\nabla_f\ln\Omega,\end{aligned} \tag{4.64}$$
$$\tilde{R} = \frac{1}{\Omega^2}\left(R - \frac{6\Box\Omega}{\Omega}\right). \tag{4.65}$$

In our case $\Box\Omega = 0$ and g_{ab} is the Minkowski metric, thus $R_{ab} = 0$ and $R = 0$. This implies that $\tilde{R} = 0$ and the Einstein tensor transforms as

$$\tilde{G}_{ab} = \tilde{R}_{ab} - \frac{\tilde{R}}{2}\tilde{g}_{ab} \tag{4.66}$$
$$= -\frac{2\nabla_a\nabla_b\Omega}{\Omega} + \frac{4\nabla_a\Omega\nabla_b\Omega}{\Omega^2} - g_{ab}\frac{\nabla^c\Omega\nabla_c\Omega}{\Omega^2}. \tag{4.67}$$

Inverting (4.63) one has that

$$\Gamma^a_{bc} = \tilde{\Gamma}^a_{bc} - \frac{1}{\Omega}\left(\delta^a_b \tilde{\nabla}_c \Omega + \delta^a_c \tilde{\nabla}_b \Omega - \tilde{g}_{bc} \tilde{\nabla}^a \Omega\right), \tag{4.68}$$

where we recall that $\nabla_a \Omega = \partial_a \Omega = \tilde{\nabla}_a \Omega$ since $\Omega = \Omega(x)$ is a scalar function. Therefore, it is easy to see that

$$\nabla_a \nabla_b \Omega = \tilde{\nabla}_a \tilde{\nabla}_b \Omega + \frac{1}{\Omega}\left(\delta^c_a \tilde{\nabla}_b \Omega + \delta^c_b \tilde{\nabla}_a \Omega - \tilde{g}_{ab} \tilde{\nabla}^c \Omega\right) \tilde{\nabla}_c \Omega. \tag{4.69}$$

Then, taking advantage of the fact that $g^{ab} = \Omega^2 \tilde{g}^{ab}$ (derived from (4.61)), we see that

$$\Box \Omega = \Omega^2 \tilde{\Box} \Omega - 2\Omega \tilde{g}^{ef} \tilde{\nabla}_e \Omega \tilde{\nabla}_f \Omega, \tag{4.70}$$

which reduces to

$$\tilde{\Box} \Omega = \frac{2}{\Omega} \tilde{g}^{ef} \tilde{\nabla}_e \Omega \tilde{\nabla}_f \Omega \tag{4.71}$$

by using (4.62). Additionally, one can use (4.69) to rewrite (4.66) as

$$\tilde{G}_{ab} = -\frac{2\tilde{\nabla}_a \tilde{\nabla}_b \Omega}{\Omega} - \tilde{g}_{ab} \frac{\tilde{\nabla}^c \Omega \tilde{\nabla}_c \Omega}{\Omega^2}. \tag{4.72}$$

We can now use this Einstein tensor for the conformally flat solutions of Nordström theory to write the vacuum $\tilde{G}_{ab}$ in the form of effective Einstein equations

$$\tilde{G}_{ab} = 8\pi \tilde{T}^{(\Omega)}_{ab}, \tag{4.73}$$

where

$$8\pi \tilde{T}^{(\Omega)}_{ab} = -\frac{2\tilde{\nabla}_a \tilde{\nabla}_b \Omega}{\Omega} + \tilde{g}_{ab} \frac{\tilde{\nabla}^c \Omega \tilde{\nabla}_c \Omega}{\Omega^2}. \tag{4.74}$$

This tensor is traceless, $\tilde{T}^{(\Omega)} = 0$, as a result of (4.71). Assuming the gradient $\nabla^c \Omega$ to be timelike and following the usual procedure to associate an effective fluid with a scalar field as described in section 2.2.2, we introduce the effective fluid 4-velocity

$$\tilde{u}_a \equiv \pm \frac{\tilde{\nabla}_a \Omega}{\sqrt{-\tilde{g}^{cd} \tilde{\nabla}_c \Omega \tilde{\nabla}_d \Omega}}, \tag{4.75}$$

where the sign of the right-hand side is chosen so that $\nabla^c \Omega$ is future oriented. The Nordström metric undergoes the $3+1$ splitting

$$\tilde{g}_{ab} = \tilde{h}_{ab} - \tilde{u}_a \tilde{u}_b, \tag{4.76}$$

with h_{ab} as in section 2.2.1. The effective stress-energy tensor $\tilde{T}^{(\Omega)}_{ab}$ has the structure (2.24) of an imperfect fluid. The heat flux reads

$$\tilde{q}^{(\Omega)}_a = -\tilde{T}^{(\Omega)}_{cd} \tilde{u}^c \tilde{h}_a{}^d = \frac{\sqrt{2\tilde{X}}}{4\pi\Omega} \dot{\tilde{u}}_a, \tag{4.77}$$

where

$$\tilde{X} \equiv -\frac{1}{2} \tilde{g}^{ef} \tilde{\nabla}_e \Omega \tilde{\nabla}_f \Omega. \tag{4.78}$$

Using Eckart's constitutive relation (3.2), we find

$$\mathcal{K}\mathcal{T} = -\frac{\sqrt{2\tilde{X}}}{4\pi\Omega} = -\frac{\sqrt{-\tilde{g}^{ef}\tilde{\nabla}_e\Omega\tilde{\nabla}_f\Omega}}{4\pi\Omega}, \tag{4.79}$$

which is negative. A negative temperature (in a context different from that of section 3.2) is explained here since $\mathcal{K}\mathcal{T} > 0$ for scalar-tensor theories with dynamical scalars, and in a theory with *less* degrees of freedom than GR a negative temperature makes sense. However, this is a pathological case not only with respect to physical implications of the theory, but also in the formalism, which is based on a positive-definite temperature.

In order to draw an explicit parallel with scalar-tensor gravity it is useful to remember that, if g_{ab} is an electrovacuum solution of the Einstein equations, the conformally transformed metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$ is a solution of $\omega = -3/2$ Brans-Dicke theory with the following Brans-Dicke field [226]

$$\phi = \frac{1}{\Omega^2}. \tag{4.80}$$

Here however, contrary to $\omega = -3/2$ Brans-Dicke gravity considered above, the scalar field ϕ is not arbitrary but must satisfy (4.62), equivalent to

$$\Box\phi = \frac{3}{2\phi}\nabla^c\phi\nabla_c\phi. \tag{4.81}$$

The temperature (4.79) found for Nordström gravity then becomes

$$\mathcal{K}\mathcal{T} = -\frac{\sqrt{-\tilde{g}^{ab}\tilde{\nabla}_a\phi\tilde{\nabla}_b\phi}}{8\pi\phi}, \tag{4.82}$$

which is exactly the result found in scalar-tensor gravity in section 3.1.1, but with opposite sign. Solutions with constant ϕ, or constant Ω, correspond to the Minkowski metric and the absence of (scalar) gravity.

Next, one can consider the stability of Nordström gravity seen as a (peculiar) thermal state of gravity. The Nordström field equation (4.71) can be rewritten in the form of an effective Klein-Gordon equation

$$\tilde{\Box}\Omega - m^2_{\text{eff}}\Omega = 0, \tag{4.83}$$

where

$$m^2_{\text{eff}} \equiv \frac{2}{\Omega^2}\tilde{g}^{cd}\tilde{\nabla}_c\Omega\tilde{\nabla}_d\Omega \tag{4.84}$$

can be taken to represent an effective mass that must be non-negative for stability. Since $\tilde{\nabla}^c\Omega\tilde{\nabla}_c\Omega < 0$, we have an effective thermal instability of Nordström gravity. This stability criterion derived purely from first-order thermodynamics will be crucial in assessing the stability of the constant-temperature states and will be explored in more detail in section 4.2.2, based on [5].

To complete the first-order thermodynamical description of Nordström gravity, we note that the kinematic quantities such as acceleration, expansion, and shear derived from the 4-velocity and its gradient coincide with those already derived in scalar-tensor gravity (see section 2.2), with the provision that the Brans-Dicke-like field ϕ must be replaced with Ω and that (4.71) be substituted into the equations of section 2.2 (see [2] for all the dissipative quantities). In the following, we only report the ones necessary for the two remaining constitutive relations. The 4-acceleration reads

$$\dot{\tilde{u}}_a \equiv \tilde{u}^c\tilde{\nabla}_c\tilde{u}_a = \frac{\tilde{\nabla}^b\Omega}{(2\tilde{X})^2}\left[2\tilde{X}\tilde{\nabla}_a\tilde{\nabla}_b\Omega + \tilde{\nabla}^d\Omega\tilde{\nabla}_b\tilde{\nabla}_d\Omega\tilde{\nabla}_a\Omega\right], \tag{4.85}$$

and the expansion is

$$\tilde{\theta} = \tilde{\nabla}_c\tilde{u}^c = -\frac{2\sqrt{2\tilde{X}}}{\Omega} + \frac{\tilde{\nabla}^a\Omega\tilde{\nabla}^b\Omega\tilde{\nabla}_a\tilde{\nabla}_b\Omega}{(2\tilde{X})^{3/2}}. \tag{4.86}$$

The shear tensor reads

$$\begin{aligned}\tilde{\sigma}_{ab} = {} & \frac{1}{(2\tilde{X})^{3/2}}\Bigg[2\tilde{X}\tilde{\nabla}_a\tilde{\nabla}_b\Omega + \frac{4\tilde{X}}{3\Omega}\left(\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega + 2\tilde{X}\tilde{g}_{ab}\right) \\ & -\frac{1}{3}\left(\tilde{g}_{ab} - \frac{\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega}{\tilde{X}}\right)\tilde{\nabla}^c\Omega\tilde{\nabla}^d\Omega\tilde{\nabla}_c\tilde{\nabla}_d\Omega + 2\tilde{\nabla}^c\Omega\tilde{\nabla}_{(a}\Omega\tilde{\nabla}_{b)}\tilde{\nabla}_c\Omega\Bigg] \\ = {} & \frac{1}{\sqrt{2\tilde{X}}}\Bigg[\tilde{\nabla}_a\tilde{\nabla}_b\Omega + \frac{\tilde{\nabla}^e\Omega\tilde{\nabla}_{(a}\Omega\tilde{\nabla}_{b)}\tilde{\nabla}_e\Omega}{\tilde{X}} + \frac{\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega\tilde{\nabla}^c\Omega\tilde{\nabla}^d\Omega\tilde{\nabla}_c\tilde{\nabla}_d\Omega}{4\tilde{X}^2} \\ & + h_{ab}\left(\frac{4\tilde{X}}{3\Omega} - \frac{\tilde{\nabla}^c\Omega\tilde{\nabla}^d\Omega\tilde{\nabla}_c\tilde{\nabla}_d\Omega}{6\tilde{X}}\right)\Bigg]\end{aligned} \tag{4.87}$$

and the spatial stress tensor reads

$$\begin{aligned}8\pi\tilde{\Pi}^{(\Omega)}_{ab} &= 8\pi\tilde{T}^{(\Omega)}_{cd}\tilde{h}^c_a\tilde{h}^d_b \\ &= -\frac{2}{\Omega}\Bigg[\tilde{\nabla}_a\tilde{\nabla}_b\Omega + \frac{\tilde{\nabla}^e\Omega\tilde{\nabla}_{(a}\Omega\tilde{\nabla}_{b)}\tilde{\nabla}_e\Omega}{\tilde{X}} \\ &\quad + \frac{\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega\tilde{\nabla}^c\Omega\tilde{\nabla}^d\Omega\tilde{\nabla}_c\tilde{\nabla}_d\Omega}{4\tilde{X}^2} + \frac{\tilde{X}}{\Omega}h_{ab}\Bigg],\end{aligned} \tag{4.88}$$

while the isotropic pressure is

$$8\pi\tilde{P}^{(\Omega)} = \frac{8\pi}{3}\tilde{h}^{ab}\tilde{\Pi}^{(\Omega)}_{ab} = \frac{2\tilde{X}}{3\Omega^2} - \frac{\tilde{\nabla}^a\Omega\tilde{\nabla}^b\Omega\tilde{\nabla}_a\tilde{\nabla}_b\Omega}{3\tilde{X}\Omega}, \tag{4.89}$$

and the anisotropic stress tensor is

$$\begin{aligned}8\pi\tilde{\pi}^{(\Omega)}_{ab} &= 8\pi\left(\tilde{\Pi}^{(\Omega)}_{ab} - \tilde{P}^{(\Omega)}\tilde{h}_{ab}\right) \\ &= -\frac{2}{\Omega}\Bigg[\tilde{\nabla}_a\tilde{\nabla}_b\Omega + \frac{\tilde{\nabla}^e\Omega\tilde{\nabla}_{(a}\Omega\tilde{\nabla}_{b)}\tilde{\nabla}_e\Omega}{\tilde{X}} + \frac{\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega\tilde{\nabla}^c\Omega\tilde{\nabla}^d\Omega\tilde{\nabla}_c\tilde{\nabla}_d\Omega}{4\tilde{X}^2} \\ &\quad + h_{ab}\left(\frac{4\tilde{X}}{3\Omega} - \frac{\tilde{\nabla}^c\Omega\tilde{\nabla}^d\Omega\tilde{\nabla}_c\tilde{\nabla}_d\Omega}{6\tilde{X}}\right)\Bigg].\end{aligned} \tag{4.90}$$

The shear tensor is therefore proportional to the contribution of the anisotropic stresses. Indeed, we see that

$$\tilde{\pi}_{ab}^{(\Omega)} = -\frac{\sqrt{2\tilde{X}}}{4\pi\Omega}\tilde{\sigma}_{ab}, \tag{4.91}$$

and thus, using Eckart's constitutive relation (3.3), the shear viscosity is

$$\eta = \frac{\sqrt{2\tilde{X}}}{8\pi\Omega} = -\frac{\mathcal{K}\mathcal{T}}{2}, \tag{4.92}$$

analogously to scalar-tensor gravity. One can also compute the bulk viscosity coefficient, and distinguish the two contributions o the total isotropic pressure (non-viscous and viscous, respectively):

$$\tilde{P}^{(\Omega)} = \tilde{P}^{(\Omega)}_{non-visc} + \tilde{P}^{(\Omega)}_{visc} = -\frac{\tilde{X}}{4\pi\Omega^2} - \frac{\sqrt{2\tilde{X}}}{12\pi\Omega}\tilde{\theta}. \tag{4.93}$$

Recalling Eckart's constitutive relation (3.1), we find the bulk viscosity coefficient for Nordström gravity:

$$\zeta = \frac{\sqrt{2\tilde{X}}}{12\pi\Omega} = \frac{2}{3}\eta = -\frac{\mathcal{K}\mathcal{T}}{3}. \tag{4.94}$$

The same procedure to find the bulk viscosity coefficient is employed in a cosmological setting in section 4.4.

The theories analyzed above and corresponding to additional equilibrium states are all quite peculiar and, in some cases, even pathological. This confirms the expected conclusion that, in the thermodynamical formalism, GR does retain a special status as an equilibrium state, just as it does in the landscape of gravity theories.

Other theories of gravity could be examined from the point of view of the effective thermodynamics, provided that their field equations can be written as effective Einstein equations with effective dissipative fluids and that the Eckart constitutive relations deliver a meaningful effective temperature. It is easy to include in the list Rastall theory, which has seen a recent resurgence of interest: it is shown in [227] that this theory is just GR with a cosmological constant, so we have trivially $\mathcal{K}\mathcal{T} = 0$. Similarly, Eddington-inspired Born-Infeld gravity is very similar to Palatini $f(R)$ gravity and in vacuo it is equivalent to GR plus a cosmological constant [228], so one expects a similar conclusion. Likewise, unimodular gravity [229] is equivalent to GR with Λ [230], yielding $\mathcal{K}\mathcal{T} = 0$.

4.2.2 States with $\mathcal{K}\mathcal{T} = \text{const.}$: stealth and degenerate de Sitter solutions

In this section, we continue the exploration of equilibrium states different than GR, extending the study of scalar-tensor thermodynamics to uncharted territory. We focus on the second class of fixed points of the effective heat equation (3.20), namely states with $\mathcal{K}\mathcal{T} = \text{const}$, explored in [5].

A peculiar solution of Brans-Dicke gravity exhibiting $\mathcal{KT}$ = const. was first noticed in [11]. This is interesting because it provides an example of situations where the $\mathcal{KT} = 0$ equilibrium state is not approached, at variance with situations presented in the previous section, and it encourages the exploration of other equilibrium states that might compete with GR. The solution studied in [11] is a stealth solution, namely a solution with the same geometry of GR solutions but with a nontrivial scalar field profile that does not contribute to the effective stress-energy tensor. Current motivation to study stealth solutions comes from the possibility of detecting black hole hair in stealth black holes through gravitational wave observations [231]. Indeed, "old-school" scalar-tensor and Horndeski theories allow for stealth solutions that violate some assumptions of the no-hair theorems and for which the scalar field does not gravitate. This would in principle make it possible to observationally distinguish GR from scalar-tensor theories. Such solutions include stealth Schwarzschild (-de Sitter) black holes with a scalar field linearly dependent on time in the context of Horndeski and beyond-Horndeski gravity [232–237]. Other stealth solutions include those studied in [231, 238–250]. Often these are degenerate cases of de Sitter spaces with non-constant scalar fields, which are not as well-known as stealth solutions of the field equations. De Sitter spaces with constant scalar fields are fixed points of the dynamical system of scalar-tensor cosmology [251] and are also common in GR cosmology sourced by scalar fields. On the contrary, de Sitter spaces with a non-constant scalar field are a signature of modified gravity.

For the purposes of first-order themrodynamics, it is interesting to study whether stealth solutions always constitute states at $\mathcal{KT}$ = const, and in [5] we undertook a more extensive analysis of stealth solutions and their stability. Assessing the stability of such states is crucial: it is reasonable to expect that, due to the special status of the GR equilibrium state in the landscape of gravity theories, these other equilibria would be unstable, thus less relevant than GR. Of course, in different systems, distinct types of stability can arise (thermal, dynamical, *etc.*) which are not necessarily expected to coincide: in the following, we make use of different types of stability criteria, but mostly rely on the thermal stability criterion sketched in section 4.2.1, which is derived purely from first-order thermodynamics. If the equilibrium states we study are not stable, it means they cannot compete with the zero-temperature state constituted by GR, further strengthening its special role. Of course, it is not practically feasible to study all possible equilibrium states and, for the time being, first-order thermodynamics is not in the position to turn the statement that GR is the only possible equilibrium state into a formal theorem. Therefore, the nature of this statement is an inductive result based on the most relevant theories that we studied.

Thermal stability criterion As briefly illustrated in 4.2.1 in the specific case of Nordström gravity, the effective heat equation (3.20) can be recast in a Klein-Gordon-like form as

$$\Box\phi - m_{\text{eff}}^2\phi = 0, \tag{4.95}$$

where

$$m_{\text{eff}}^2 \equiv 8\pi \left[\frac{d\left(\mathcal{KT}\right)}{d\tau} - 8\pi\left(\mathcal{KT}\right)^2 + \theta\,\mathcal{KT}\right], \tag{4.96}$$

where we considered the standard form of (3.20) derived from the generalized Brans-Dicke action (2.20) that we mainly use throughout this thesis. The effective mass m_{eff} is clearly not a physical mass, but simply an effective quantity derived in the context of the thermodynamical analogy explained above. The sign of m_{eff}^2 can be used to construct a stability criterion, based on the need to avoid tachyonic instabilities. Indeed, we have *instability* if the square of the effective mass (that we call "thermal mass") is $m_{\text{eff}}^2 < 0$ and *stability* if

$$m_{\text{eff}}^2 \geq 0\,. \tag{4.97}$$

Since $\mathcal{KT}$ is a scalar, this stability criterion is covariant and gauge-invariant. Of course, this notion of stability is only meaningful in the context of the thermodynamical analogy at hand to assess theories or solutions with a given $\mathcal{KT}$, whose time evolution is described by (3.20). It is distinct from and unrelated to an assessment of the perturbative stability of the solution. This effective mass of scalar-tensor gravity differs from those explored in [252, 253].

It may seem odd that thermal stability reduces to avoiding tachyonic instabilities determined by an effective mass that depends on $\mathcal{KT}$ and its derivative. However, this is exactly the same philosophy used to deduce the Dolgov-Kawasaki stability criterion for metric $f(R)$ gravity [254, 255]. Since the effective mass m_{eff} is built only out of $\mathcal{KT}$ and its time derivative and is deduced from the effective thermal description of scalar-tensor gravity, this stability criterion is definitely thermal (in the sense of said effective thermodynamics). It is made possible by the fact that an equation describing the approach to thermal equilibrium (or departure from it) exists in the theory. Near states of thermal equilibrium, it has essentially the same physical content as the effective heat equation.

The thermal stability criterion (4.97) is not particularly useful in the general thermodynamics of scalar-tensor gravity because one does not *a priori* know the quantities appearing in (4.96). However, if one wants to assess the stability *of specific solutions* (or classes of solutions) of the field equations, (4.97) is indeed suitable. This is the goal of the rest of this section. We note that the stability of certain stealth geometries has been previously studied with the Bardeen-Ellis-Bruni-Hwang [256–260] approach for cosmological perturbations in modified gravity [261–266], covered in appendix A.

The significance of the new thermal stability criterion derived here lies in the fact that it can reject certain solutions of scalar-tensor gravity that, although mathematically possible, cannot occur in nature because they are unstable. Our stability criterion expresses thermal stability because it is derived from the equation describing the approach to thermal equilibrium or the departures from it in scalar-tensor gravity. Near thermal equilibrium states, this criterion expresses the physical content of the equation describing the approach to equilibrium in a way that makes it easier and practical to assess the stability of analytical solutions of the scalar-tensor field equations.

Stealth solutions

The stealth solutions we are interested in here are special cases where Minkowski space results not from the absence of matter, but from a tuned balance between matter and the Brans-Dicke scalar or, in vacuum, between different terms in the scalar contribution to the stress-energy tensor. Stealth solutions like those studied in [238, 243–245] are interesting since they show that Minkowski space is not necessarily devoid of matter, and the effect of gravitational coupling persists in the energy-momentum tensor even when this coupling is switched off.

Stealth solutions encountered in the literature in the context of the scalar-tensor theories described by (2.20) are usually of two kinds:

1. $g_{ab} = \eta_{ab}$ and $\phi = \phi_0 \, e^{\alpha t}$;
2. $g_{ab} = \eta_{ab}$ and $\phi = \phi_0 \, |t|^\beta$,

where η_{ab} is the Minkowski metric in Cartesian coordinates, ϕ_0, α, β are constants, and $\phi_0 > 0$ so that gravity is always attractive. Differentiation yields

$$\dot{\phi} = \phi \times \begin{cases} \alpha \, , \\ \dfrac{\beta}{t} \, , \quad \text{if } t \neq 0 \, , \end{cases} \tag{4.98}$$

thus the requirement of future-directed scalar gradient translates into the conditions

$$\phi > 0 \quad \text{and} \quad g_{ab} \nabla^a \phi \, (\partial_t)^b < 0 \tag{4.99}$$

or, for the specific scenarios above,

$$\begin{aligned} 0 > g_{ab} \nabla^a \phi \, (\partial_t)^b &= g_{ab} \left(g^{a0} \dot{\phi} \right) \, \delta^b{}_0 = g_{00} \, g^{00} \dot{\phi} = \dot{\phi} \\ &= \phi \times \begin{cases} \alpha \, , \\ \dfrac{\beta}{t} \quad \text{if } t \neq 0 \, . \end{cases} \end{aligned} \tag{4.100}$$

Thus, enforcing the future orientation of the scalar field gradient, we shall restrict to cases that satisfy the conditions

1. $\alpha < 0$;
2. $\beta < 0$ if $t > 0$ or $\beta > 0$ if $t < 0$.

In the first case

$$\mathcal{KT} = \frac{\sqrt{-\nabla^c \phi \nabla_c \phi}}{8\pi \phi} = \frac{|\alpha|}{8\pi} = \text{const.} > 0 \, , \tag{4.101}$$

which means that this solution never approaches the GR equilibrium state. If we now consider its stability from the point of view of first-order thermodynamics, we see that the effective mass is constant and given by

$$\begin{aligned} m^2_{\text{eff}} &= \frac{\Box \phi}{\phi} = \frac{\partial^\mu \partial_\mu \phi}{\phi} \\ &= \frac{\partial^\mu \left(\alpha \, \delta^0{}_\mu \phi \right)}{\phi} = -\alpha^2 < 0 \, , \end{aligned} \tag{4.102}$$

which makes this stealth solution *unstable*. This is the type of stealth solution mentioned in [11], analysed in [15] with the gauge-invariant criterion for cosmological perturbations and shown to be a metastable state.

In the second case, $\beta = 1$ and $\beta = 2$ are the most relevant situations encountered in the literature. Therefore, according to our conventions, in order to have $G_{\text{eff}} = \phi^{-1} > 0$ and $u^a = \nabla^a\phi/\sqrt{-\nabla^c\phi\nabla_c\phi}$ future-oriented, it must be $\phi_0 > 0$ in conjunction with $t < 0$ if $\beta > 0$. Then, if $\beta > 0$ the effective gravitational coupling behaves as

$$G_{\text{eff}} = \frac{1}{\phi} = \frac{1}{\phi_0\,|t|^\beta} \to +\infty \quad \text{as } t \to 0^- \,, \tag{4.103}$$

the effective temperature of gravity (3.5) is

$$\mathcal{K}\mathcal{T} = \frac{\beta}{8\pi|t|} \to +\infty \quad \text{as } t \to 0^- \,, \tag{4.104}$$

and the effective mass reads

$$m_{\text{eff}}^2 = \frac{\Box\phi}{\phi} = -\frac{\beta(\beta-1)}{t^2} \,. \tag{4.105}$$

If $\beta = 1$, we get $m_{\text{eff}}^2 = 0$. Therefore this constant "mass" solution is *marginally stable*. As $t \to 0^-$, we approach a singularity of the theory where $G_{\text{eff}} \to +\infty$, $\mathcal{K}\mathcal{T} \to +\infty$, gravity becomes infinitely strong and deviates from GR drastically. Indeed, nothing could be further from a GR situation than infinitely strong gravity with Minkowski spacetime! This solution matches the idea that singularities are "hot" in the sense of the thermodynamics of scalar-tensor gravity [10, 11]. This situation is stable according to the thermal stability criterion (4.97). Hence, barring instabilities of a different kind, one expects this behaviour to occur in nature if singularities are present. The implication is that the GR equilibrium state is not always approached and gravity indeed departs from GR near singularities. Of course, the final theory of gravity should remove singularities, but it is clear that scalar-tensor gravity is not this final theory since it does contain spacetime singularities and singularities of G_{eff}.

The situation where $\beta = 2$, exemplified in section 4.2.2, entails $m_{\text{eff}}^2 = -2/t^2 < 0$, meaning *instability* from the thermal point of view, while $\mathcal{K}\mathcal{T} = 1/4\pi|t|$ and $G_{\text{eff}} = 1/\phi_0 t^2$ both diverge as $t \to 0^-$, thus departing from GR at this singularity of G_{eff}. In our formalism the $t > 0$ branch of the solution is not meaningful.

Most exact solutions of Brans-Dicke theories in cosmology exhibit the power-law behaviour $\phi = \phi_0\, t^\beta$ [267], such as those found by O'Hanlon and Tupper [268] and Nariai [269, 270]. These were studied from the point of view of first-order thermodynamics in [1], and in the following we consider two degenerate cases of such solutions that reduce to a Minkowski background with a non-trivial scalar field profile.

Other types of stealth solutions with Minkowski metric and non-trivial scalar include those found for a nonminimally coupled ϕ [243], where the field is inhomogeneous, wave-like, and does not gravitate. Their stability was studied in [271] using the Bardeen-Ellis-Bruni-Hwang gauge-invariant formalism for cosmological perturbations [256–260], showing mixed stability results depending on the specific choice of parameters. These solutions either do not correspond to future-oriented four-velocity u^c, or are very cumbersome to discuss because $\nabla^a\phi$ is timelike only in very restricted spacetime regions and for special combinations of their parameters. Therefore, they are not examined here.

Metastable Brans-Dicke stealth solution The stealth solution studied in [11] is a static solution in vacuum Brans-Dicke theory described by the action (2.20), with $\omega = -1$ and $V(\phi) = V_0\,\phi$, where $V_0 > 0$. The line element and scalar field in spherical coordinates (t, r, θ, φ) read

$$ds^2 = -dt^2 + A^{-\sqrt{2}}(r)\,dr^2 + A^{1-\sqrt{2}}(r)\,r^2 d\Omega^2_{(2)}\,, \tag{4.106}$$

and

$$\phi(t, r) = \phi_0\,\mathrm{e}^{2a_0 t} A^{1/\sqrt{2}}(r)\,, \tag{4.107}$$

where $A(r) = 1 - 2m/r$, while m, a_0, ϕ_0 are constants. Taking the limit $m \to 0$ produces the Minkowski metric with $\phi(t) = \phi_0\,\mathrm{e}^{2a_0 t}$. The scalar profile is non-trivial but ϕ does not gravitate, so this represents a stealth solution, which interestingly has a constant $\mathcal{KT}$, namely

$$\mathcal{KT} = \frac{|a_0|}{4\pi}\,. \tag{4.108}$$

This solution can be interpreted as a metastable state of the theory which always remains away from the GR equilibrium state. It was shown in [15] that it is unstable with respect to tensor perturbations (on a timescale $\Delta t = (|a_0|)^{-1}$ using the Bardeen-Ellis-Bruni-Hwang gauge-invariant formalism developed for cosmological perturbations in [256, 257, 259, 260] and adapted to modified gravity in [261–266]. This formalism can be applied because the stealth Minkowski spacetime found taking the $m \to 0$ limit is a trivial FLRW spacetime. The only exception to this instability occurs for $a_0 = 0$, corresponding to constant scalar field $\phi = \phi_0$, which recovers the GR case and is in agreement with GR being the state of equilibrium for scalar-tensor gravity.

O'Hanlon & Tupper solution with $\omega = 0$ The O'Hanlon & Tupper spatially flat FLRW solution of Brans-Dicke cosmology is obtained from the usual action (2.20) for $\omega = \text{const.} > -3/2$ and $\omega \neq -4/3$ and $V = 0$ [268]. The scale factor and scalar field read

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{q_\pm}\,, \tag{4.109}$$

$$\phi(t) = \phi_0\left(\frac{t}{t_0}\right)^{s_\pm}\,, \tag{4.110}$$

with

$$q_\pm = \frac{\omega}{3(\omega+1) \mp \sqrt{3(2\omega+3)}}\,, \tag{4.111}$$

$$s_\pm = \frac{1 \mp \sqrt{3(2\omega+3)}}{3\omega+4}\,, \tag{4.112}$$

and $3q_{\pm}+s_{\pm}=1$. This solution has a "hot" singularity at $t\to 0^+$, where Brans-Dicke theory departs from the GR behaviour. Although the value $\omega=0$ was not contemplated in [268], it is straightforward to check that it corresponds to a Minkowski space solution of the equations of vacuum Brans-Dicke cosmology with $V=0$, $q=0$, $a(t)=1$, and linear scalar field $\phi(t)=\phi_0\,t$ (choosing $t_0=1$ for convenience). This is a *bona fide* stealth solution, which could have been introduced in [268] long before solutions with this name were noticed and appreciated [231, 238–250]. In order for the 4-velocity to be future-oriented and for G_{eff} to be positive, it must be $\phi_0<0$ and $t<0$. This situation is akin to case 2. with $\beta=1$ considered above, hence the $\omega=0$ O'Hanlon & Tupper solution turns out to be *marginally stable* according to the thermal stability criterion.[1] This universe has

$$\mathcal{K}\mathcal{T}=\frac{1}{8\pi|t|}\to+\infty \tag{4.113}$$

as $t\to 0^-$, deviating from GR. This solution will also be studied from the point of view of first-order thermodynamics in FLRW cosmology in section 4.5.1.

Nariai solution with $\omega=-1/2$ The Nariai solution [269, 270] is a particular power-law solution for a $K=0$ FLRW universe with perfect fluid matter that has $P=(\gamma-1)\,\rho$ (with $\gamma=$ const.), $V(\phi)=0$ and $\omega\neq-4\left[3\gamma\left(2-\gamma\right)\right]^{-1}<0$. Here we are interested in a cosmological constant fluid with $\gamma=0$, $P^{(\text{m})}=-\rho^{(\text{m})}$, and

$$a(t)=a_0\left(1+\delta t\right)^{\omega+1/2}\,, \tag{4.114}$$

$$\phi(t)=\phi_0\left(1+\delta t\right)^2\,, \tag{4.115}$$

$$\delta=\left[\frac{32\pi\rho_0}{\phi_0}\frac{1}{\left(6\omega+5\right)\left(2\omega+3\right)}\right]^{1/2}\,. \tag{4.116}$$

This solution is an attractor in phase space and was used in the extended inflationary scenario [272, 273]. For $\omega=-1/2$, $\delta=\sqrt{8\pi\rho_0/\phi_0}$, the scale factor is constant and $H=0$, making this a Minkowski stealth solution with non-trivial (polynomial) scalar field profile. It is a straightforward generalisation of the type 2. stealth solutions described above.[2] It must be $\phi_0>0, (1+\delta t)<0$ and

$$\mathcal{K}\mathcal{T}=\frac{\delta}{4\pi|1+\delta t|}\to+\infty \tag{4.117}$$

as $(1+\delta t)\to 0^-$. In the far past $t\to-\infty$, $\mathcal{K}\mathcal{T}\to 0$ and GR is approached, but the instability prevents this state from being an equilibrium alternative to GR. In fact,

[1] In the analysis at the beginning of section 4.2.2, we conventionally denoted $\phi(t)=\phi_0\,|t|^\beta$ with $\phi_0>0$. In this section we instead employ the usual notation that can be found in the literature, i.e., $\phi(t)=\phi_0\,t^\beta$, where ϕ_0 and t can both be either positive or negative, provided that ϕ remains positive.

[2] Here again we implicitly adapted our notation to the one which is typically employed in the literature. See footnote 1.

the thermal stability criterion yields

$$m_{\text{eff}}^2 = \frac{\Box\phi}{\phi} = -\frac{2\delta^2}{(1+\delta t)^2} < 0 \tag{4.118}$$

and this solution is thermally *unstable*. This solution will also be studied from the point of view of first-order thermodynamics in FLRW cosmology in section 4.5.1.

Degenerate de Sitter solutions

Other common solutions of scalar-tensor gravity are de Sitter ones with line element

$$ds^2 = -dt^2 + a_0^2 \, \mathrm{e}^{2H_0 t} \left(dx^2 + dy^2 + dz^2\right) \tag{4.119}$$

in comoving coordinates, with scale factor $a(t) = a_0 \, \mathrm{e}^{H_0 t}$, where a_0, H_0 are constants. In GR with a minimally coupled scalar field as the only matter source, the only possible de Sitter spaces are obtained for a constant scalar field, $(H, \phi) = (H_0, \phi_0)$, with both H_0 and ϕ_0 constant. In spatially flat FLRW cosmology, the independent dynamical variables are[3] (H, ϕ) and the phase space is a 2-dimensional subset of the 3-dimensional space $\left(H, \phi, \dot{\phi}\right)$ identified by the Hamiltonian constraint. This 2-dimensional subset is analogous to an energy surface in point particle mechanics [274, 275]. The points (H_0, ϕ_0) are then all the equilibrium points of the dynamical system.

For spatially flat FLRW universes in scalar-tensor cosmology, the independent variables are still H and ϕ and there can be fixed points (H_0, ϕ_0) of this dynamical system. The structure of the phase space and the fixed points for specific scalar-tensor theories are discussed extensively in [274] and [251], respectively. Gauge-invariant criteria for the stability of these de Sitter fixed points (and of their degenerate Minkowski cases) are given in [276–280]. In addition to de Sitter fixed points, in scalar-tensor cosmology there can be de Sitter spaces with non-constant scalar field, usually exponential or power-law in time. Since these are only admissible in modified gravity and not in GR, they are interesting for first-order thermodynamics. Degenerate cases of such de Sitter solutions can reproduce Minkowski space with a non-trivial scalar field and are therefore another kind of stealth solutions similar to those of the previous section.

This type of solution, found starting from the usual action (2.20) has

$$H = H_0 = \text{const.}\,, \tag{4.120}$$

$$\phi(t) = \phi_0 \, \mathrm{e}^{\alpha t}\,, \tag{4.121}$$

with ϕ_0 a positive constant. The constants H_0 and α are related to the parameters of the specific scalar-tensor theory. These solutions have been known for a long time,

[3] In the field equations for spatially flat FLRW universes, the scale factor only appears in the combination $H \equiv \dot{a}/a$.

but here we consider them from the novel point of view of scalar-tensor thermodynamics. In order to get a future-directed 4-velocity of the effective ϕ-fluid and an attractive gravitational interaction we need to require, again, that

$$\phi > 0 \quad \text{and} \quad g_{ab}\nabla^a\phi\,(\partial_t)^b < 0\,, \tag{4.122}$$

which implies $\phi_0 > 0$ and $\alpha < 0$. We have (as in (4.101))

$$\mathcal{K}\mathcal{T} = \frac{|\alpha|}{8\pi} = \text{const.} \tag{4.123}$$

and this solution remains away from the zero-temperature GR state of equilibrium at all times. Is it thermally stable? We find

$$\begin{aligned} m^2_{\text{eff}} &= \frac{\Box\phi}{\phi} = \frac{-\left(\ddot{\phi} + 3H_0\dot{\phi}\right)}{\phi} = -\alpha\left(\alpha + 3H_0\right) \\ &= |\alpha|\left(3H_0 - |\alpha|\right)\,; \end{aligned} \tag{4.124}$$

therefore, we have *stability* for $3H_0 \geq |\alpha|$ and *instability* for $|\alpha| > 3H_0$.

In particular, it is clear that exponentially contracting FLRW universes ($H_0 < 0$) are always unstable. This conclusion, obtained with simple considerations in scalar-tensor thermodynamics, matches the result found in the previous literature on scalar-tensor cosmology [276] through a dynamical systems analysis which, however, requires the complete specification of the theory.

Kolitch solutions of vacuum Brans-Dicke cosmology with cosmological constant
Kolitch [281] found solutions of vacuum Brans-Dicke cosmology with positive cosmological constant Λ, equivalent to the linear potential $V(\phi) = 2\Lambda\phi$. These solutions were previously noted in [282, 283] and read

$$a(t) = a_0 \exp\left[\pm\left(\omega+1\right)\sqrt{\frac{2\Lambda}{(2\omega+3)(3\omega+4)}}\,t\right] \tag{4.125}$$

$$\phi(t) = \phi_0 \exp\left[\pm\sqrt{\frac{2\Lambda}{(2\omega+3)(3\omega+4)}}\,t\right]\,. \tag{4.126}$$

For $\omega = -1$, they reduce to the stealth solution with

$$H = 0\,, \qquad a(t) = 1\,, \qquad \phi(t) = \phi_0\, \mathrm{e}^{\pm\sqrt{2\Lambda}\,t}, \tag{4.127}$$

where, again, we must choose the lower sign to have a future-oriented four-velocity. This solution deviates from GR at all times since $\mathcal{K}\mathcal{T} = \text{const.} > 0$, but it corresponds to $m^2_{\text{eff}} = -\alpha^2 < 0$ and is *unstable*. Its stability has also been studied with respect to both homogeneous and inhomogenous metric perturbations in [271], where the solution with the upper sign is found to be stable and the one with the lower sign unstable. However, the solution with the upper sign cannot be analysed in the framework of scalar-tensor thermodynamics since it entails a past-oriented $\nabla^a\phi$.

Let us consider now the de Sitter spaces (4.125), (4.126) for $\omega \neq -1$: taking the lower sign we have

$$H_0 = -(\omega+1)\sqrt{\frac{2\Lambda}{(2\omega+3)(3\omega+4)}} \equiv -(\omega+1)\,\mathrm{C} \tag{4.128}$$

and

$$\alpha = -\sqrt{\frac{2\Lambda}{(2\omega+3)(3\omega+4)}} \equiv -\mathrm{C}\,, \tag{4.129}$$

where C is a positive real constant if $\omega < -3/2$ and $\omega > -4/3$. Therefore, the effective mass reads

$$m^2_{\text{eff}} \;=\; |\alpha|\,(3H_0 - |\alpha|) = -\mathrm{C}^2\,(3\omega+4) \tag{4.130}$$

Then, if $\omega < -3/2$ we have an expanding de Sitter universe which is thermodynamically *stable*, although the scalar field for such values of the coupling is phantom and therefore suffers from different types of instabilities [95]. Other configurations are otherwise *unstable*.

O'Hanlon & Tupper solution in the $\omega \to -4/3$ limit It is often mentioned in the literature that the O'Hanlon & Tupper solution (4.109)-(4.112) approaches de Sitter space in the limit $\omega \to -4/3$, recovering

$$a(t) \;=\; a_0 \exp\left(H_0\, t\right), \tag{4.131}$$

$$\phi(t) \;=\; \phi_0 \exp\left(-3H_0\, t\right), \tag{4.132}$$

with H_0 a positive constant. Technically, this statement is not accurate since the above result is recovered by simultaneously choosing the values q_+ and s_- of the exponents, which correspond to two distinct solutions. However, the solution above is the only de Sitter one for flat FLRW and vacuum [95]. Given that $\alpha < 0$, the velocity of the scalar field fluid is future-oriented and $3H_0 - |\alpha| = 0$, so this solution is *marginally stable* according to the thermal criterion.

This solution describes expanding universes for which the effective fluid four-velocity is only future-oriented. These expanding universes are unstable with respect to tensor modes, as can be concluded using the Bardeen-Ellis-Bruni gauge-invariant formalism for cosmological perturbations [256–260] in Hwang's version adapted to modified gravity [261–266]. The relevant equations are summarized in appendix A. We only need (A.13) for the gauge-invariant variable H_T associated with the tensor modes which, in the background (4.131) and (4.132), becomes

$$\ddot{H}_T + \left(3H + \frac{\dot{\phi}}{\phi}\right)\dot{H}_T + \frac{k^2}{a^2(t)}\,H_T = 0\,, \tag{4.133}$$

where k is the mode's momentum and the coefficients are given by the unperturbed $a(t)$ and $\phi(t)$, which yields $3H+\dot{\phi}/\phi=0$ to zero order. With $H_0>0$, the asymptotic equation at late times $t\to+\infty$ reduces to

$$\ddot{H}_T+\frac{k^2}{a^2}\,H_T\simeq\ddot{H}_T=0\,, \tag{4.134}$$

with linear solution $H(t)=\alpha\,t+\text{const.}$ The tensor perturbation diverges and this universe is *unstable*.

Constant curvature spaces in $f(R)$ gravity

We have mentioned in section 1.2.2 that metric $f(R)$ gravity is equivalent to a Brans-Dicke theory with $\phi=f'(R)$ (a prime denotes differentiation with respect to R), $\omega=0$, and a specific potential. Assuming that $\nabla^c R$ is timelike and future-oriented, the effective dissipative fluid associated with $f(R)$ gravity has [10, 19]

$$\mathcal{KT}=\frac{f''(R)\sqrt{-\nabla^c R\nabla_c R}}{8\pi f'(R)}\,, \tag{4.135}$$

where it is required that $f'(R)>0$ in order for the effective gravitational coupling $G_{\text{eff}}=1/\phi$ to be positive and for the graviton to carry positive kinetic energy, while $f''(R)\geq0$ is required for local stability [255] (here $\nabla^c\phi$ is timelike and future-oriented if $\nabla^c R$ is). The fact that the effective Brans-Dicke scalar field ϕ in $f(R)$ gravity is tied so intimately with the Ricci scalar makes all constant curvature spaces in these theories zero-temperature states indistinguishable from GR, because this means that $\phi=f'(R)=\text{const.}$ and $\nabla_c\phi$ vanishes identically, together with $\mathcal{KT}$. Furthermore, these states are *(marginally) stable* in our thermal sense because $\Box\phi=0$ and the effective mass $m^2_{\text{eff}}=\Box\phi/\phi$ also vanishes identically. The condition $m^2_{\text{eff}}\geq0$ for the thermal stability of $f(R)$ gravity does not coincide with the stability condition of de Sitter space with respect to first order local perturbations, obtained in a gauge-invariant way ([280] and references therein),

$$\left(f_0'\right)^2-2f_0f_0''\geq0\,, \tag{4.136}$$

where a zero subscript denotes a quantity evaluated on the de Sitter background. Therefore, the thermal stability condition $m^2_{\text{eff}}\geq0$ does not necessarily coincide with other stability notions, as could be expected. Indeed, also in Newtonian systems and in GR one has different notions of stability (thermal, dynamical, *etc.*) and the thermodynamics of modified gravity evidently cannot account for all possible notions of stability.

The results obtained for the solutions of scalar-tensor gravity analyzed here are summarized in Table 4.1. Overall, the two general principles of first-order thermodynamics of scalar-tensor gravity are confirmed: *i*) gravity deviates wildly from GR near spacetime singularities and near singularities of the gravitational coupling; *ii*) the convergence of gravity to GR at late times is marked by $\mathcal{KT}\to0$. No states of equilibrium $\mathcal{KT}=\text{const.}$ other than GR (corresponding to $\mathcal{KT}=0$) have been

Solution	Type	Thermal Stability
OHT $\omega = 0$	Minkowski stealth	marginally stable (departs from GR as $t \to 0^-$)
Nariai $\omega = -1/2$	Minkowski stealth	unstable
Kolitch $\omega = -1$	Minkowski stealth	unstable
Kolitch $\omega < -3/2$	de Sitter	stable (but ϕ phantom)
OHT $\omega \to -4/3$	de Sitter	marginally stable
$f(R)$ gravity	(Anti-)de Sitter, Minkowski	marginally stable

Table 4.1: Summary of the analytical solutions studied and their thermal stability.

found here, except for solutions that are unstable according to various criteria and are, therefore, physically irrelevant. This result reinforces the special role of general relativity as an equilibrium state in the landscape of gravity theories, seen through the lens of first-order thermodynamics.

4.3 Alternative formulation based on chemical potential

The entire thermodynamical formalism so far has been developed starting from the Jordan frame action (2.20), where there is an explicit coupling between the Ricci scalar and the ϕ field. However, scalar-tensor theories can also be studied in the (conformally related) Einstein frame, where the scalar couples minimally to gravity but nonminimally to matter. The thermodynamical formalism based on the notion of temperature that we have detailed in the previous sections could not be applied to theories in the Einstein frame, since the minimally coupled scalar gives rise to an effective fluid that is perfect. Since all imperfect fluid quantities vanish, the analogy built in the previous sections becomes trivial: $\mathcal{KT}$ is always zero and no approach to equilibrium (or departure from it) can be analysed.

However, in [3], we realised that an alternative but equivalent picture of first-order thermodynamics, based instead on the notion of chemical potential, can address this problem. Although $\mathcal{KT}$ vanishes for the fluid describing the minimally coupled scalar, the chemical potential, defined as $\tilde{\mu} = \sqrt{2\tilde{X}}$ does not, and the dissipation to equilibrium can be characterised as $\tilde{\phi} \to$ const. and $\tilde{\mu} \to 0$.

This approach is reminiscent of the influential one in [22], that exploited the analogy between an imperfect fluid and a special class of scalar-tensor theories to help gain physical insight into such theories and their interesting cosmological applications. As explicitly mentioned by these authors, the identification of shift charges with particles, on which the identification of the scalar field gradient with the chemical potential relies, is not unique. An equivalent choice would have been to assume that these charges correspond to entropy, and then from the first law of thermodynamics, one could construct a description in which $n \to s$ and $\mu \to \mathcal{T}$, which is precisely what we do in first-order thermodynamics. In this section, we show that, also starting from our description, we can find this equivalence.

Before that, we revisit the analogy between a minimally coupled scalar field and a perfect fluid, which is well-known but still leaves room for interesting developments. Specifically, a thermodynamical description for this fluid was presented [192], introducing the notions of temperature and chemical potential for the fluid. However, these results pose problems that we aim to solve. Addressing these issues also makes it possible to understand our thermodynamical analogy in a broader picture, by connecting it with the more general thermodynamical description of imperfect fluids in the context of scalar-tensor gravity. This link additionally allows one to shed light on the Einstein frame formulation of first-order thermodynamics, which has so far remained elusive.

The action of gravity with a minimally coupled scalar field can be written generically as

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi} + \mathcal{L}(\phi, X)\right] + S^{(\mathrm{m})}. \tag{4.137}$$

We denote with $\mathcal{L}(\phi, X)$ the scalar field Lagrangian density, and $S^{(\mathrm{m})}$ describes matter other than the scalar field. We assume that the scalar field has timelike gradient, $X > 0$. One can then establish an analogy between the scalar and a perfect fluid by taking the normalized gradient of ϕ as the fluid's four-velocity, as we have previously stated. Several works have derived the fluid-mechanical quantities corresponding to the minimally coupled scalar field [22, 112, 188, 189, 192, 284–287] and its thermodynamics has been studied in [192], which found the following thermodynamical quantities associated to this perfect fluid in terms of $\mathcal{L}$ and X:

$$\rho = 2X\mathcal{L}_X - \mathcal{L} \tag{4.138}$$
$$P = \mathcal{L} \tag{4.139}$$
$$n = \sqrt{2X}\,\mathcal{L}_X \tag{4.140}$$
$$\frac{s}{n} = \phi \tag{4.141}$$
$$\mathcal{T} = \frac{-\mathcal{L}_\phi}{\sqrt{2X}\,\mathcal{L}_X} \tag{4.142}$$
$$\mu = \frac{2X\mathcal{L}_X + \phi\mathcal{L}_\phi}{\sqrt{2X}\,\mathcal{L}_X} = \sqrt{2X} - \phi\mathcal{T}, \tag{4.143}$$

where $\mathcal{L}_\phi \equiv \partial\mathcal{L}/\partial\phi$ and $\mathcal{L}_X \equiv \partial\mathcal{L}/\partial X$. There are some mistakes in these expressions, that we correct in the following, with the aid of the formalism developed in the previous sections. Assuming $\mathcal{L}_X > 0$, i.e., that the field ϕ is not phantom, the energy density ρ and the particle number density n are non-negative. Using (4.138), and (4.139), the stress-energy tensor of the scalar field reads

$$T^{(\phi)}_{ab} = \mathcal{L}_X\nabla_a\phi\nabla_b\phi + \mathcal{L}g_{ab}\,, \tag{4.144}$$

which is conserved ($\nabla^b T^{(\phi)}_{ab} = 0$), is rewritten in the perfect fluid form (1.6). The equation of motion for ϕ

$$\nabla_a \Big(\mathcal{L}_X \nabla^a \phi \Big) = -\mathcal{L}_\phi \tag{4.145}$$

is written as

$$\nabla_a \Big(n u^a \Big) \equiv \nabla_a N^a = -\mathcal{L}_\phi \,, \tag{4.146}$$

which reduces to the familiar Klein-Gordon equation $\Box\phi = V_\phi$ if $\mathcal{L}(\phi, X) = X - V(\phi)$, where $V(\phi)$ is the scalar field potential.

If $\mathcal{L} = \mathcal{L}(X)$, the scalar field theory is invariant under the shift symmetry $\phi \to \phi + C$ (where C is a constant) and there is a conserved Noether current

$$N^a = \mathcal{L}_X \nabla^a \phi = n u^a \tag{4.147}$$

satisfying

$$\nabla_a N^a = 0 \,. \tag{4.148}$$

N^a is the analogue of the particle number current density. The particle number density in the comoving frame is the corresponding Noether charge

$$n = -N^0 = -u^c N_c = -\frac{\nabla^a \phi}{\sqrt{2X}} \, \mathcal{L}_X \nabla_a \phi = \sqrt{2X} \, \mathcal{L}_X \,, \tag{4.149}$$

consistently with (4.140). If $\mathcal{L}_\phi \neq 0$ (for example, if there is a potential $V(\phi)$), the analogue

$$N^a = n u^a \tag{4.150}$$

of the particle current density is not conserved, $\nabla_a N^a = -\mathcal{L}_\phi \neq 0$. Being derived from a scalar field, the ϕ-fluid is, of course, irrotational. As mentioned in section 2.3.1, in a dissipative fluid, the directions of the particle flow and of the energy flow are different. As a consequence, N^a coincides with $n u^a$ in the comoving (or Eckart) frame [21] which is adapted to follow the total flux of particles, while $N^a = n u^a + v^a_{(L)}$ in the Landau (or energy frame) [198], where $v^a_{(L)}$ is the diffusive current density of particles caused by gradients of the chemical potential μ. The Eckart and the Landau frames coincide for a perfect fluid, which is the case we are interested in here.

The introduction of temperature $\mathcal{T}$ and chemical potential μ in the correspondence between minimally coupled scalar field and perfect fluid is quite recent (appearing only in [192] to the best of our knowledge) and has not been tested as well as the rest of the analogy. Indeed, the derivation of $\mathcal{T}$ and, as a consequence, of μ in [192] exhibits an inconsistency (that does not affect the other fluid quantities), that we correct here. Specifically, the $\mathcal{T}$ and μ given (4.142) and (4.143) suffer from three problems.

1. The first issue (already noted in [192]) is that both $\mathcal{T}$ and μ can be negative. This fact is surprising because, contrary to the nonminimally coupled scalars of scalar-tensor gravity, the effective ϕ-fluid is otherwise well-behaved and satisfies the weak and null energy conditions, hence one expects $\mathcal{T}$ and μ to be non-negative like ρ and n.

2. The most serious problem is that, according to (4.142), there is a temperature gradient. Moreover, in general the effective ϕ-fluid is non-geodesic, with non-zero acceleration

$$\dot{u}_a \equiv u^c \nabla_c u_a = -\frac{1}{2X}\left(\nabla_a X + \frac{\nabla_c X \nabla^c \phi}{2X}\nabla_a \phi\right). \tag{4.151}$$

   Then, there must necessarily be a heat current with density [21]

$$q_a = -\mathcal{K}\left(h_{ab}\nabla^b \mathcal{T} + \mathcal{T}\dot{u}_a\right). \tag{4.152}$$

   This generalized Fourier law is one of the three constitutive equations of Eckart's first-order thermodynamics [21] and a minimal assumption. The first term in the right-hand side of (4.152) is nothing but the usual Fourier law, while the second one is a purely relativistic "inertial" contribution discovered by Eckart [21]. The heat conduction described by q_a makes a fluid dissipative and endows its stress-energy tensor with the dissipative terms appearing in (2.24). Then, it is not possible that the fluid equivalent of a minimally coupled ϕ be a perfect fluid described by (1.6). A heat current would necessarily show up in the comoving (or Eckart) frame based on the usual four-velocity. The *only* way for q_a to vanish identically is if $\mathcal{T} = 0$.

3. From the point of view of first-order thermodynamics, developed in the previous sections, the fact that a minimally coupled scalar field fluid is endowed with a non-zero temperature appears unexplainable, since the nonminimal coupling with R is responsible for a nonvanishing fluid temperature, therefore the fluid equivalent to a minimally coupled ϕ and with Lagrangian depending only on ϕ and X should always have zero temperature.

In the rest of this section, we address three points. Firstly, we correct the temperature (4.142), establishing the fact that the fluid equivalent to a minimally coupled ϕ has always zero temperature. As a consequence, only the first term in the chemical potential $\mu = \sqrt{2X} - \phi\mathcal{T}$ remains, which makes this quantity positive-definite. The second and third issue listed above are also solved because the heat flux density q_a then vanishes identically and the fluid becomes non-dissipative.

Secondly, we are also able to comment on the thermodynamics of phantom scalar fields with $\mathcal{L}_X < 0$, which have been the subject of an extensive literature, in conjunction with studies considering the possibility of a very negative equation of state ($w \equiv P/\rho < -1$) for the dark energy driving the present acceleration of the cosmic expansion, *e.g.* [288–290]. Although the claims of a phantom equation of state are disputed, the possibility of $w < -1$ is not excluded by present cosmological observations [39]. Phantoms are unstable from the classical, and even more from the quantum, point of view but they are still accepted in the cosmological literature as the expression of a truncated action that will be cured if all terms are included. The literature on phantom field thermodynamics, now mostly a decade old, has not been conclusive and we contribute to a clearer picture.

Thirdly, we can extend the fluid-ϕ correspondence to include scalar fields coupled nonminimally *to matter* (but not to R). The effect of these couplings is analogous to that of a scalar field potential which constitutes a source of fluid "particles" making the "particle number density" n a non-conserved quantity, but has no drastic effect on the rest of the analogy. This extension allows one to discuss the Einstein frame version of scalar-tensor gravity in which the gravitational Brans-Dicke-like field $\tilde{\phi}$ couples explicitly to matter but not to R (contrary to the Jordan frame formulation of the same theory in which the scalar $\phi \neq \tilde{\phi}$ couples to R but not to matter). This development makes it possible to find a missing piece in the first-order thermodynamics of scalar-tensor gravity which, being based on the notion of temperature, was thus far unable to deal with the Einstein frame description.

Perfect fluid analogy and approach to the diffusive equilibrium The first law of thermodynamics can be written in the form chosen by [291] in Box 22.1, namely

$$d\left(\frac{\rho}{n}\right) + P\, d\left(\frac{1}{n}\right) = \mathcal{T}\, d\left(\frac{s}{n}\right) , \tag{4.153}$$

The symbol s in [291] corresponds to s/n in our notation.

Now, taking s and n as independent variables as done in [291], one finds

$$\mathcal{T}(s,n) = \frac{1}{n}\frac{\partial\rho}{\partial(s/n)}\bigg|_n = \frac{\partial\rho}{\partial s}\bigg|_n \quad . \tag{4.154}$$

Since (4.144) maps into a perfect fluid, the absence of any dissipative effects, hence vanishing heat fluxes, requires $\mathcal{T} = 0$, in accordance with first-order thermodynamics. Therefore, assuming $\phi = \phi(s,n)$ and $X = X(s,n)$, one has that

$$0 = \frac{\partial\rho}{\partial s}\bigg|_n = -\mathcal{L}_\phi\, \frac{\partial\phi}{\partial s}\bigg|_n \quad , \tag{4.155}$$

where we have taken advantage of (4.138) and (4.140). The condition in (4.155) is then satisfied if $\mathcal{L}_\phi = 0$ or $\frac{\partial\phi}{\partial s}\Big|_n = 0$. Since, in general, $\mathcal{L}$ will contain a potential term, consistency with first-order thermodynamics translates into the condition

$$\frac{\partial\phi}{\partial s}\bigg|_n = 0\,, \tag{4.156}$$

so that the temperature of gravity vanishes for a scalar field non-minimally coupled to Einstein gravity.

On a similar note, it is easy to see that combining

$$P\,(n,s) = n\,\frac{\partial\rho}{\partial n}\bigg|_{s/n} - \rho\,, \tag{4.157}$$

with (4.138) and (4.139) recovers the perfect fluid identification $P = \mathcal{L}$ if and only if

$$\frac{\partial\phi}{\partial n}\bigg|_{s/n} = 0\,, \tag{4.158}$$

when $\mathcal{L}_\phi \neq 0$. It is then easy to identify the chemical potential of the system, which reads [291]

$$\mu = \frac{P+\rho}{n} = \sqrt{2X}\,, \tag{4.159}$$

where we have again taken advantage of (4.138)-(4.140). The condition in (4.141) is incompatible with both the thermodynamic analogy presented here and the requirement of conservation of the entropy per particle along perfect fluid lines. This condition is, however, marginal in our discussion since it is not used.

One can *a posteriori* derive an equation describing the approach to diffusive equilibrium along the fluid lines. For relativistic fluids, the chemical potential μ and the (purely spatial) diffusive flux density of particles $q_a^{(p)}$ will obey a generalization of Fick's law analogous to Eckart's generalization (4.152) of Fourier's law (see [292])

$$q_a^{(p)} = -\mathcal{D}\left(h_{ab}\nabla^b\mu + \mu\,\dot{u}_a\right) , \tag{4.160}$$

where $\mathcal{D}$ is a diffusion coefficient analogous to the thermal conductivity $\mathcal{K}$. Diffusive equilibrium is reached when the chemical potential μ vanishes identically (in the presence of acceleration $\dot{u}^a$, a constant μ would still generate particle diffusion due to the second term in the right-hand side of (4.160). This equation is reminiscent of a relativistic version of the drift-diffusion equation [293]. Let us compute the derivative $d\mu/d\tau$, where τ is the proper time along the flow lines of the effective ϕ-fluid:

$$\frac{d\mu}{d\tau} \equiv u^c\nabla_c\mu = \frac{\nabla^c\phi}{\sqrt{2X}}\,\nabla_c\left(\sqrt{2X}\right) = \frac{\nabla^c\phi\,\nabla_c X}{2X}\,. \tag{4.161}$$

Now use the expression of the expansion scalar of the ϕ-fluid [12, 19]

$$\theta \equiv \nabla_a u^a = \frac{1}{\sqrt{2X}}\left(\Box\phi - \frac{\nabla_c X\nabla^c\phi}{2X}\right) \tag{4.162}$$

to eliminate the term containing second derivatives of ϕ in (4.161), obtaining

$$\frac{d\mu}{d\tau} = -\mu\,\theta + \Box\phi\,. \tag{4.163}$$

This equation is not so simple because of the d'Alembertian of ϕ in the right-hand side. However, to gain some insight, we can consider the situation in which $\mathcal{L}$ does not depend on ϕ, for example a free scalar field with $\mathcal{L} = X$, in which case $\Box\phi = 0$ and (4.163) reduces to[4]

$$\dot{\mu} = -\mu\,\theta\,. \tag{4.164}$$

One can introduce a representative length ℓ by [195]

$$\frac{\dot{\ell}}{\ell} \equiv \frac{\theta}{3} \tag{4.165}$$

and then

$$\frac{\dot{\mu}}{\mu} = -\frac{3\dot{\ell}}{\ell} \tag{4.166}$$

so that $\mu = \text{const.}/\ell^3$. The simplified evolution equation of μ then describes the fact that when the flow expands and dilutes, μ decreases and the state of equilibrium

[4] If $\mathcal{L}_X = 1$, then $n = \mu$ satisfies the same equation, which is reported in [294].

$\mu = 0$ is approached, while when the flow gets concentrated, the chemical potential increases and there is departure from the equilibrium state. In particular, the ϕ-flow is diluted in an expanding universe, which will approach the diffusive equilibrium state $\mu = 0$ as $\ell \to +\infty$. Near spacetime singularities, instead, this flow is focused, the flow lines become closer and closer, and there are extreme departures from the equilibrium state $\mu = 0$. In principle, this understanding of the approach to equilibrium in the thermodynamical picture based on μ is equivalent to that obtained in the context of scalar-tensor thermodynamics. However, in the comoving frame one does not see particle diffusion, as explained in the next paragraph. When the second term $\Box\phi$ is included in the right-hand side of (4.163), the situation becomes more complex since this term could in principle be positive or negative, hence it can favour the approach to equilibrium or oppose it depending on its sign.

The effective stress-energy tensor of the effective ϕ-fluid has the perfect fluid form (1.6), yet the chemical potential $\mu = \sqrt{2X}$ depends on the spacetime position so its variation must give rise to a diffusive ϕ- (or "particle") current (the acceleration also contributes to this diffusive flow according to (4.160)). Then it is natural to ask why we do not see a vector $q^a_{(\mathrm{p})}$ describing this diffusion in the effective fluid stress-energy tensor. As mentioned before, in dissipative fluids, the Eckart (or comoving) frame is based on the particle 4-velocity and is adapted to follow the total flow of particles, the diffusive particle current vanishes. The Landau or energy frame, instead, is the frame of observers with four-velocity $u^a_{(\mathrm{L})} \neq u^a$ moving with the energy flow. In this frame, Landau observers see a diffusive particle flow described by a current $q^a_{(\mathrm{p})}$ but not an energy flow, since the heat current density $q^a_{(\mathrm{L})}$ vanishes identically. For a perfect fluid, the Eckart (comoving) and the Landau (energy) frames coincide and both the heat and the particle diffusion currents are zero.

We have shown that $\mathcal{T} = 0$ but $\mu = \sqrt{2X} \neq 0$ in the comoving frame of the effective fluid associated with a minimally coupled scalar field. Here we check explicitly that this fact does not contradict the vanishing of the diffusion current because the two terms in the right hand side of (4.160) cancel each other out. We have

$$\nabla_a \mu = \nabla_a \left(\sqrt{2X}\right) = \frac{\nabla_a X}{\sqrt{2X}} \tag{4.167}$$

and the spatial gradient of μ is

$$h_{ab}\nabla^b\mu = \left(g_{ab} + \frac{\nabla_a\phi\nabla_b\phi}{2X}\right)\frac{\nabla^b X}{\sqrt{2X}} = \frac{\nabla_a X}{\sqrt{2X}} + \frac{\nabla_b\phi\nabla^b X}{(2X)^{3/2}}\nabla_a\phi \,. \tag{4.168}$$

Adding to this quantity the acceleration term $\mu\,\dot{u}_a$ and using the expression (4.151) of the acceleration yields

$$h_{ac}\nabla^c\mu + \mu\,\dot{u}_a = 0\,, \tag{4.169}$$

which ensures that there is no diffusive "particle" current in this frame in spite of the non-uniform chemical potential.

Einstein frame formulation of scalar-tensor gravity Let us consider now the situation in which the scalar field couples nonminimally to other forms of matter, which are described by the Lagrangian density $\mathcal{L}^{(\mathrm{m})}$ through a coupling function $f(\phi)$ (this coupling is non-trivial if $f(\phi) \neq \text{const.}$). For simplicity, we restrict to the scalar field Lagrangian $\mathcal{L} = X - V(\phi)$. The total Lagrangian density is then

$$\mathcal{L} = X - V(\phi) + f(\phi)\mathcal{L}^{(\mathrm{m})} \,. \tag{4.170}$$

The equation of motion of ϕ becomes

$$\Box\phi = V_\phi - f_\phi \mathcal{L}^{(\mathrm{m})} \,. \tag{4.171}$$

The extra term acts as a source of ϕ, hence as a source of "particles" in the effective ϕ-fluid. As a consequence, the stress-energy tensors of ϕ and of the other matter are not conserved ($\nabla^b T^{(\phi)}_{ab} \neq 0$, $\nabla^b T^{(\mathrm{m})}_{ab} \neq 0$) but their sum is, $\nabla^b \left(T^{(\phi)}_{ab} + T^{(\mathrm{m})}_{ab} \right) = 0$. The coupling term on the right-hand side of (4.171) acts in the same way as the potential $V(\phi)$, preventing the conservation of the "particle" current density $N^a = nu^a = \nabla^a\phi$ according to (4.148). Indeed, this extra term breaks the shift invariance $\phi \to \phi + C$ of the scalar field Lagrangian $\mathcal{L} = \mathcal{L}(X)$ in the absence of a potential $V(\phi)$ and prevents N^a from being a conserved Noether current even when $V(\phi) \equiv 0$.

Since in the Einstein frame the scalar couples minimally to gravity but non-minimally to matter, these considerations open up the possibility of discussing the Einstein frame formulation of the thermodynamics of scalar-tensor gravity, which has so far been developed in the Jordan frame, in the previous chapters and in [1, 10, 11, 15].

This alternative and complementary picture based on chemical potential can handle the Einstein frame. We switch from the Jordan to the Einstein frame by performing the well-known conformal transformation of the metric [295]

$$g_{ab} \to \tilde{g}_{ab} \equiv \phi\, g_{ab} \tag{4.172}$$

and the scalar field redefinition $\phi \to \tilde{\phi}$ with

$$d\tilde{\phi} = \sqrt{\frac{|2\omega + 3|}{16\pi}}\, \frac{d\phi}{\phi} \,. \tag{4.173}$$

The action then reads

$$S_{\mathrm{EF}} = \int d^4x \sqrt{-g} \left[\frac{\tilde{R}}{16\pi} - \frac{1}{2}\, \tilde{g}^{ab}\, \nabla_a\tilde{\phi}\nabla_b\tilde{\phi} - U(\tilde{\phi}) + \frac{\mathcal{L}^{(\mathrm{m})}}{\phi^2(\tilde{\phi})} \right] , \tag{4.174}$$

with

$$U(\tilde{\phi}) = \frac{V(\phi)}{16\pi\phi^2} \Big|_{\phi=\phi(\tilde{\phi})} \,. \tag{4.175}$$

It might be advantageous in some situations to deal with theories in the Einstein frame, although it introduces the unpleasant feature of modifying the equation of massive particles by the addition of an extra force proportional to $\nabla^\mu\Omega$. Photon trajectories, on the other hand, are not modified as null rays are unchanged under conformal rescaling. Brans was already aware of this in the 1960s [87].

All Einstein frame variables $\left(\tilde{g}_{ab}, \tilde{\phi}\right)$ are denoted by a tilde. The Einstein frame field equations read

$$\tilde{R}_{ab} - \frac{1}{2}\,\tilde{g}_{ab}\tilde{R} = 8\pi\left(\mathrm{e}^{-\sqrt{\frac{64\pi}{|2\omega+3|}}\,\tilde{\phi}}\,T^{(\mathrm{m})}_{ab} + \nabla_a\tilde{\phi}\nabla_b\tilde{\phi}\right. \\ \left. -\frac{1}{2}\,\tilde{g}_{ab}\,\tilde{g}^{cd}\nabla_c\tilde{\phi}\nabla_d\tilde{\phi} - U(\tilde{\phi})\,\tilde{g}_{ab}\right) \tag{4.176}$$

$$\tilde{g}^{ab}\nabla_a\nabla_b\tilde{\phi} - \frac{dU}{d\tilde{\phi}} + 8\sqrt{\frac{\pi}{|2\omega+3|}}\,\mathrm{e}^{-\sqrt{\frac{64\pi}{|2\omega+3|}}\,\tilde{\phi}}\,\mathcal{L}^{(\mathrm{m})} = 0\,. \tag{4.177}$$

The scalar contribution to the stress-energy tensor arising from this action is of course that of a perfect fluid (1.6). However, this presents a puzzle for the first-order thermodynamics of scalar-tensor theories. The thermodynamical formalism based on the temperature description is not suitable for a perfect fluid, since all imperfect fluid quantities vanish and the theory becomes trivial. This means that the approach to equilibrium cannot be studied. The formalism based on temperature only works for gravitational theories in representations where an effective imperfect fluid description can be found, which is possible only if the scalar is directly coupled to R in the action. These considerations relate to the well-known but hard-to-tackle problem of the ambiguity that arises in distinguishing between "gravitational" and "matter" degrees of freedom whenever we switch representation through a conformal transformation [88].

However, the notion of chemical potential comes to the rescue. Although the temperature $\mathcal{T}$ of the Einstein frame scalar field effective fluid is zero, according to the previous sections, the chemical potential $\tilde{\mu} = \sqrt{2\tilde{X}}$ is not. Now the scalar field $\tilde{\phi}$ has gravitational nature and is always present in spacetime, that is, one cannot decide to set it to zero or replace it with other forms of matter. The state of diffusive equilibrium corresponds to $\tilde{\mu} = 0$ and $\tilde{\phi} = \text{const.}$, but this condition automatically recovers GR (possibly, with a cosmological constant if a potential for the scalar field is present), as a limiting case given that a timelike gradient for the scalar field represents our starting assumption for this analogy. This result goes hand-in-hand with that of first-order thermodynamics of scalar-tensor gravity formulated in the Jordan frame, where GR is the zero-temperature state of equilibrium [1, 10–12]. In the Einstein frame, instead, $\tilde{\mathcal{K}}\tilde{\mathcal{T}}$ is identically zero but GR is the state of equilibrium of scalar-tensor gravity corresponding to vanishing chemical potential $\tilde{\mu} = \sqrt{2\tilde{X}}$. Therefore, the dissipative process leading to equilibrium can be characterized by $\mu \to 0$ instead of $\mathcal{K}\mathcal{T} \to 0$. This thermodynamical picture based on the chemical potential makes it possible to gain an understanding of the approach to equilibrium even for theories described by perfect fluids.

4.4 Cosmological applications

Cosmology is naturally the most fruitful arena for the study of extended theories of gravity, as explored in the Introduction. This encourages the application of first-order thermodynamics to a cosmological setting. The main result that this application produces is a notion that the expansion of the universe generally coincides with a dissipation towards equilibrium in our formalism: namely, scalar-tensor gravity at $\mathcal{KT} > 0$ tends to the GR equilibrium $\mathcal{KT} = 0$ as the expansion progresses. Interestingly, the idea of scalar-tensor theories relaxing towards GR in a cosmological setting has been explored in [296, 297] (albeit with a very different scope from that of our work). The authors found that, during the matter-dominated era, the expansion of the universe drives the scalar field toward a state where scalar-tensor gravity becomes effectively indistinguishable from GR: the expected present deviations from GR would therefore be small, but not unmeasurably so, which has since been corroborated further.

4.4.1 Generalised Brans-Dicke cosmology

In a FLRW universe, because of spatial homogeneity and isotropy, we have $\phi = \phi(t)$, $q_a^{(\phi)} = 0$ and $\pi_{ab}^{(\phi)} = 0$. This implies that also the shear viscosity vanishes; however, it still makes sense to consider bulk viscosity, which is isotropic. Thus, the only two non-vanishing contributions to (2.24) are

$$8\pi\rho^{(\phi)} = 8\pi T_{ab}^{(\phi)} u^a u^b = -\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi + \frac{V}{2\phi} + \frac{1}{\phi}\left(\Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right) \tag{4.178}$$

$$8\pi P^{(\phi)} = \frac{8\pi}{3}h^{ab}T_{ab}^{(\phi)} = -\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi - \frac{V}{2\phi} - \frac{1}{3\phi}\left(2\Box\phi + \frac{\nabla^a\phi\nabla^b\phi\nabla_b\nabla_a\phi}{\nabla^e\phi\nabla_e\phi}\right). \tag{4.179}$$

Since $\phi = \phi(t)$, then

$$\nabla_a\phi = \delta_a^0\,\dot\phi\,, \qquad \nabla^a\phi = g^{0a}\,\dot\phi\,, \qquad 2X = -\nabla^e\phi\nabla_e\phi = \dot\phi^2\,. \tag{4.180}$$

Furthermore, one has that

$$\Box\phi = -\left(\ddot\phi + 3H\dot\phi\right)\,, \tag{4.181}$$

$$\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi = \left(\nabla^0\phi\right)^2\left(\partial_a\partial_b\phi - \Gamma^0_{00}\partial_0\phi\right) = \ddot\phi\,\dot\phi^2 \tag{4.182}$$

where, for the latter, we have used the fact that $\Gamma^0_{00} = 0$. From (4.178) and (4.179) we can therefore infer that

$$8\pi\rho^{(\phi)} = \frac{\omega\,\dot\phi^2}{2\phi^2} + \frac{V}{2\phi} - 3H\,\frac{\dot\phi}{\phi}\,, \tag{4.183}$$

$$8\pi P^{(\phi)} = \frac{\omega\,\dot\phi^2}{2\phi^2} - \frac{V}{2\phi} + \frac{\ddot\phi}{\phi} + 2H\,\frac{\dot\phi}{\phi}\,. \tag{4.184}$$

Moving on to the kinematic properties of the ϕ-fluid, we assume a scalar field ϕ strictly monotonic in t, the vector field

$$v^a \equiv \frac{\nabla^a\phi}{\sqrt{2X}} = g^{a0}\,\mathrm{Sign}(\dot\phi) = (-\mathrm{Sign}(\dot\phi), \mathbf{0}) \tag{4.185}$$

is timelike, though it is not necessarily future-directed. Therefore, we define the 4-velocity of the comoving observer as

$$u^a = -\mathrm{Sign}(\dot{\phi})\, v^a \,, \tag{4.186}$$

so that u^a is a timelike, future-directed vector field with $u^a u_a = -1$. The $3+1$ splitting of spacetime is obtained by identifying the Riemannian metric of the 3-space orthogonal to the 4-velocity of the comoving observers with

$$h_{ab} \equiv g_{ab} + u_a u_b = g_{ab} + v_a v_b, \tag{4.187}$$

as usual. The trace-free shear tensor σ_{ab} vanishes because of spatial homogeneity and isotropy, and the expansion scalar reduces to $\theta = 3H$.

Going back to the effective pressure (4.184) of the ϕ-fluid, one can now use the equation of motion (2.22) of the Brans-Dicke-like scalar field to eliminate $\ddot{\phi}$. We use the Hubble function $H \equiv \dot{a}/a$ and denote differentiation with respect to the comoving time t with an overdot. Substituting

$$\frac{\ddot{\phi}}{\phi} = -\frac{3H\dot{\phi}}{\phi} - \frac{8\pi T^{(\mathrm{m})}}{(2\omega+3)\phi^2} + \frac{2V - \phi V_{,\phi}}{(2\omega+3)\phi} - \frac{\dot{\phi}^2\, \omega_{,\phi}}{(2\omega+3)\phi} \tag{4.188}$$

into (4.184) yields

$$\begin{aligned} P^{(\phi)} &= P_{\mathrm{non-visc}} + P_{\mathrm{visc}} \qquad\qquad (4.189)\\ &= \frac{1}{8\pi}\left[\frac{(2\omega+3)\omega - 2\phi\,\omega_{,\phi}}{2(2\omega+3)}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{4V - 2\phi V_{,\phi} - (2\omega+3)V}{2(2\omega+3)\phi} - \frac{8\pi T^{(\mathrm{m})}}{(2\omega+3)\phi^2}\right]\\ &\quad - \frac{H\dot{\phi}}{8\pi\phi}\,. \end{aligned}$$

According to Eckart's constitutive relation (3.1), the viscous pressure is

$$P_{\mathrm{visc}} = -3\zeta H, \tag{4.190}$$

which leads to the straightforward identification of the bulk viscosity coefficient

$$\zeta = \frac{\dot{\phi}}{24\pi\phi}\,. \tag{4.191}$$

One wonders whether the splitting of the pressure into a non-viscous part $P_{\mathrm{non-visc}}$ and a viscous part $-3\zeta H$ could be performed differently, leading to a different result for P_{visc}, in which we are interested here. We argue that the identification performed is the only natural one. Let us consider, for the sake of simplicity, vacuum scalar-tensor gravity in a spatially flat FLRW universe. The variables appearing in the field equations equations are the scale factor $a(t)$ and the scalar field $\phi(t)$. While the acceleration equation for $a(t)$ and the Klein-Gordon-like equation for $\phi(t)$ are of second order, when $k = 0$ the scale factor only appears in the combination $H(t) \equiv \dot{a}/a$. The dynamical variables are, therefore, $H(t)$ and $\phi(t)$ and the phase space reduces to the three-dimensional space $\left(H, \phi, \dot{\phi}\right)$ [274]. Furthermore, the orbits of the solutions are forced to lie on a two-dimensional submanifold of this space identified by the first order Hamiltonian constraint, which is analogous to an energy constraint in point particle dynamics [274]. The right-hand side of (4.189) contains only the phase space variables H, ϕ, and $\dot{\phi}$ (in addition to the functions $V(\phi), \omega(\phi)$), while only the last term contains $H = \theta/3$. It is natural to identify the viscous pressure with this term only, and any attempt to split $P^{(\phi)}$ differently into viscous and non-viscous parts would be contrived.

In a spatially curved ($k \neq 0$) FLRW universe, one cannot eliminate $a(t)$ in terms of $H(t)$ and the phase space variables are $\left(a, \dot{a}, \phi, \dot{\phi}\right)$; again, the Hamiltonian constraint forces the orbits of the solutions to lie in a 3-dimensional subspace but the previous argument still applies because, again, in the right-hand side of (4.189) only the last term depends on the scale factor $a(t)$ and the splitting performed is the only natural one (doing otherwise would require to add and subtract terms containing the scale factor or its derivatives, which would be completely arbitrary and unmotivated).

An immediate consequence of (4.191) is that GR, obtained for $\phi =$ const., corresponds to zero viscosity ζ and can still be regarded as a state of equilibrium. Increasing ϕ corresponds to decreasing strength of gravity $G_{\text{eff}} = 1/\phi$ and to increasing bulk viscosity coefficient, moving away from the GR equilibrium state. *Vice-versa*, decreasing ϕ (with increasing gravitational coupling) leads to the GR equilibrium state and to the decrease of bulk viscosity dissipation.

Since the heat flux density $q_a^{(\phi)}$ vanishes identically in FLRW universes by virtue of spatial isotropy, (3.5) and the concept of effective temperature of scalar-tensor gravity lose meaning. However, (3.5) is deduced in the general theory without reference to particular geometries and one may want to regard this temperature as a general concept holding even in FLRW spacetimes. The possibility of considering the heat flux as a timelike vector aligned with the four-velocity of comoving observers would preserve the spatial homogeneity and isotropy of FLRW spaces. In this case, Eckart's (3.4) would hold only for a timelike four-acceleration of the fluid, which is the case for FLRW spacetimes sourced by a perfect fluid. However, dealing with a timelike heat current density would require an extension of the formalism presented in the previous chapters. If one assumes (3.5) to hold, then it reduces to

$$\mathcal{KT} = \frac{|\dot{\phi}|}{8\pi\phi} \tag{4.192}$$

in FLRW universes, and then the bulk viscosity coefficient $\zeta = \mathcal{KT}/3$ is linear in the temperature and vanishes in the GR equilibrium state together with it.

4.4.2 Horndeski cosmology

One of the most interesting applications of first-order thermodynamics is that to Horndeski theories, where [12] found that the formalism breaks down beyond the subclass that respects constraints from gravitational wave observations. Motivated by the results of the previous section on Brans-Dicke cosmology, the goal of the present section, based on [7], is to extend the first-order thermodynamics of Horndeski theories to the fruitful setting of FLRW spacetime, in order to put this version of the formalism to the test and gain physical intuition in this context. In flat FLRW cosmology, due to the symmetries of the background, the heat flux (4.15) and the anisotropic stress tensor (4.18) vanish identically. However, the viscous contribution remains and is visible through the isotropic pressure giving rise to a non-vanishing bulk viscosity. In [7], we computed the effective bulk viscosity for viable Horndeski in FLRW, while the temperature and thermal conductivity are naturally inherited from the general (background-independent) approach from [12], reviewed in section 4.1. Additionally, we explored exact solutions of particularly interesting subclasses of viable Horndeski, also finding a new solution thereof. These results are included in section 4.5.2.

In section 4.1 based on [12], the role of bulk viscosity was neglected in the stress-energy tensor, because this simple choice is the one that was made when developing the original formalism detailed in section 2.2. However, the role of bulk viscosity becomes important in a FLRW background, since anisotropic contributions vanish and only the isotropic pressure matters. In the following, we first set the stage for finding an expression of the total pressure, which includes viscous and non-viscous contributions. We then restrict to a FLRW background and find explicit expression for this pressure, allowing us to determine the bulk viscosity coefficient from Eckart's constitutive relation (3.1), similarly to the procedure in section 4.4.1, but with inherent complications introduced due to the more complicated Horndeski equations of motion.

In order to understand the dissipative properties of the effective scalar fluid we are dealing with, we need to write the derivatives of the scalar field in terms of 4-velocity gradients. The derivatives acting on the scalar field can be written as

$$\nabla_a \phi = \epsilon\sqrt{2X}\, u_a\,, \qquad \nabla_a X = -\dot{X}\, u_a - 2X\, \dot{u}_a\,, \tag{4.193}$$

$$\nabla_a \nabla_b \phi = \epsilon\sqrt{2X}\left(\nabla_a u_b - \dot{u}_a u_b\right) - \epsilon\frac{\dot{X}}{\sqrt{2X}} u_a u_b, \quad \Box\phi = \epsilon\left(\sqrt{2X}\theta + \frac{\dot{X}}{\sqrt{2X}}\right),$$

where $\theta \equiv \nabla_a u^a$ is the expansion scalar, $\dot{u}^a \equiv u^b \nabla_b u^a$ is the 4-acceleration of the fluid and $\epsilon = \pm 1$ is used to ensure a future-oriented velocity (details in the following). Eckart's constitutive relations are linear in the velocity gradient $\nabla_b u_a$, and this is also the requirement for the fluid behind viable Horndeski theories to be Newtonian, as found in [17] and reviewed in section 4.1. The only problem in writing the derivatives of the scalar field present in the dissipative quantities derived from the Horndeski effective fluid (4.5) in terms of $\nabla_b u_a$ arises when considering the pressure. Indeed, inside (4.14) (provided in section 4.1, together with all other dissipative quantities that will be useful in the following) there is a contribution that includes $\Box\phi$. This is equivalent to having a term containing $\dot{X}$, because of the linearity of (4.193). Therefore, we need to into account the equation of motion of the scalar field (4.9). Using the metric field equation (4.3) to rewrite the curvature contributions inside (4.9) in terms of the total effective stress-energy tensor (4.4) through $R = -T^{(\mathrm{eff})}$ and $R_{ab} = \left(T^{(\mathrm{eff})}_{ab} - \frac{1}{2}T^{(\mathrm{eff})} g_{ab}\right)$, it is possible to algebraically solve the scalar field equation of motion and obtain $\Box\phi$. This yields

$$\Box\phi = \frac{A + B\,\theta + C\theta^2 + D\,\sigma_{ab}\sigma^{ab} + E\,\dot{u}^c\dot{u}_c}{J + K\,\theta}, \tag{4.194}$$

where

$$\begin{aligned} A = {} & T^{(\mathrm{m})}(G_{4\phi} - XG_{3X}) - 2\left(T^{(\mathrm{m})}_{ab}u^a u^b\right) XG_{3X} + G_2(2G_{4\phi} - XG_{3X}) - G_4 G_{2\phi} \\ & X\left[(G_{4\phi} - XG_{3X})(6G_{4\phi\phi} + G_{2X} - 4G_{3\phi}) + 2G_{4\phi}G_{3\phi} + 2G_4(G_{2\phi X} - G_{3\phi\phi})\right], \end{aligned} \tag{4.195}$$

$$B = \epsilon(2X)^{3/2}\left[2XG_{3X}^2 - 2G_{4\phi}G_{3X} + G_4(G_{2XX} - 2G_{3\phi X})\right], \tag{4.196}$$

$$C = -\frac{4}{3}XG_4(2G_{3X} + 3XG_{3XX}), \tag{4.197}$$

$$D = -2XG_4G_{3X}, \tag{4.198}$$

$$E = 4XG_4(G_{3X} + XG_{3XX}), \tag{4.199}$$

$$J = 3(G_{4\phi} - XG_{3X})^2 + G_4\left[G_{2X} + 2XG_{2XX} - 2(G_{3\phi} + XG_{3\phi X})\right], \tag{4.200}$$

$$K = -2\epsilon\sqrt{2X}G_4(G_{3X} + XG_{3XX}). \tag{4.201}$$

If we treat (4.194) as an algebraic equation for $\Box\phi$, intending to rewrite it in terms of the kinematic quantities, we must note that this holds as long as the denominator $J + K\theta$ is not vanishing. Indeed, $J + K\theta = 0$ entails $K = 0$ and $J = 0$, separately [23]. On the one hand, given Eq. (4.201), $K = 0$ implies $G_{3X} + XG_{3XX} = 0$, which has

$$G_3(\phi, X) = F(\phi)\ln(X/X_*) + V(\phi) \tag{4.202}$$

as a general solution, with X_* constant. On the other hand, $J = 0$ provides the functional form of G_2,

$$\begin{aligned} G_2(\phi, X) = {} & \mu(\phi)\sqrt{2X} + \nu(\phi) - 4X\left(F_\phi(\phi) + \frac{3\left[F(\phi) - G_{4\phi}(\phi)\right]^2}{4G_4(\phi)} + \frac{1}{2}V_\phi(\phi)\right) \\ & + 2F_\phi(\phi)X\ln(X/X_*), \end{aligned} \tag{4.203}$$

where $F(\phi)$, $\mu(\phi)$ and $\nu(\phi)$ are generic functions. It is straightforward to verify that the potential $V(\phi)$ does not play any role since it can be eliminated by performing an integration by parts, namely $-V(\phi)\,\Box\phi \simeq 2X\,V_\phi(\phi)$ up to a total derivative. Therefore, $G_3 = V(\phi)$ is equivalent to considering $\tilde{G}_3 = 0$ and $\tilde{G}_2 = G_2 + 2XV_\phi$. Redefining $F \to G_{4\phi} + \frac{1}{2}F$, (4.203) (and (4.209) in the following) turn into the well-known form used in [210, 298, 299]. It is worth stressing that we need (4.194) only in the case of $G_{4\phi} \neq X\, G_{3X}$. Indeed, when $G_{4\phi} = X\, G_{3X}$ (or, equivalently, $F = G_{4\phi}$), $\Box\phi$ disappears from (4.14) thus making (4.194) no longer necessary for carrying out the thermodynamic analogy, and the fluid behaves as a Newtonian fluid [23]. An example of such a scenario is given by k-essence, for which one has $G_{4\phi} = G_{3X} = 0$.

The viable Horndeski effective fluid is characterized by the usual linear constitutive relations for the heat flux density (3.2), and the anisotropic stress (3.3), that lead to the identification of temperature (4.17) and shear viscosity (4.21). After substituting (4.194) into (4.14) to obtain an expression for the pressure, we can make the dependence on the 4-velocity gradients apparent

$$P^{(\phi)} = P_0 + \xi \left(\frac{A + B\theta + C\theta^2 + D\sigma^2 + E\dot{u}^2}{J' + K'\theta} \right) - \left(\xi - \frac{4}{3}\eta \right) \theta. \tag{4.204}$$

where $J' = \epsilon\sqrt{2X}J$, $K' = \epsilon\sqrt{2X}K$, $P_0 = \left(G_2 - 2XG_{3\phi} + 4XG_{4\phi\phi}\right)/2G_4$, $\xi = -\epsilon\sqrt{2X}\left(G_{4\phi} - XG_{3X}\right)/G_4$.

The non-Newtonian behaviour of the fluid can arises precisely from this pressure (4.204). The requirement of a Newtonian fluid is quite stringent and selects two specific subclasses of viable Horndeski: one is characterized by $G_3 = G_{4\phi} \ln(X/X_*)$ (associated to $\xi = 0$), and the other is identified with $G_3 = 0$ [23]. This way, all the non-linear contributions in the dissipative quantities due to the presence of $\Box\phi$ in (4.204) disappear. These classes are disconnected with respect to conformal transformations of the metric tensor, and the second one exists only for a dynamical scalar field. More general theories correspond to effective fluids that are non-Newtonian, and therefore exotic and less easily interpretable from the physical point of view.

However, here we are interested in applying Eckart's thermodynamics in the context of cosmology, i.e., with a particular fixed background. From now on, we restrict to a FLRW spacetime. This restriction also has an effect on the Horndeski subclasses satisfying linear constitutive equations like those of Eckart's thermodynamics. Specifically, for some particular geometries, such as FLRW, it is possible to satisfy these constitutive relations in a larger subclass of viable Horndeski, which contains the classes mentioned earlier as sub-cases. This is one of the reasons why studying the effective fluid description of Horndeski theories is particularly interesting in a cosmological setting.

We are now in the position to perform the same feat as section 4.4.1 with the more general class of viable Horndeski theories. Taking the usual FLRW line element with $k = 0$, the 4-velocity of the effective fluid in a FLRW setting can be written as

$$u^a \equiv \epsilon \frac{\nabla^a \phi}{\sqrt{2X}} = \left(-\epsilon\, \mathrm{Sign}(\dot{\phi}), 0, 0, 0 \right), \tag{4.205}$$

where we assume that ϕ is strictly monotonic in t and $\dot{\phi} = -\sqrt{2X}$, since $X = \frac{1}{2}\dot{\phi}^2$ and the velocity is future-oriented if $\epsilon = -\mathrm{Sign}(\dot{\phi})$. As mentioned above, once we restrict to a fixed background, the constraint that an effective fluid is linear in $\nabla_b u_a$ becomes less stringent than in case without specifying to a particular geometry, as the symmetries of the FLRW metric allow us to find a larger subclass of viable Horndeski theories. We recall that $\theta = 3H$, and the shear tensor and the 4-acceleration vanish ($\sigma^2 = 0$, $\dot{u}^2 = 0$). The Friedmann constraint reads

$$H^2 = \frac{1}{3}\left(\frac{\rho^{(\mathrm{m})}}{G_4} + \rho^{(\phi)}\right) = \frac{1}{3}\left(\frac{\rho^{(\mathrm{m})}}{G_4} + \rho_0 - 3H\xi\right) . \tag{4.206}$$

Therefore, since $\rho^{(\phi)}$ is always linear in H, i.e., linear in the expansion scalar, we can rewrite the $\theta^2 = 9H^2$ term in (4.194) and (4.204) as a linear expression in terms of $\theta = 3H$. Then, the general expression for the energy density and pressure take the form

$$\rho^{(\phi)} = \rho_0 - \xi\theta \tag{4.207}$$

$$P^{(\phi)} = P_0 + \xi\left(\frac{A' + B'\theta}{J' + K'\theta}\right) - \left(\xi - \frac{4}{3}\eta\right)\theta\,. \tag{4.208}$$

At this point, Eckart's constitutive relation (3.1) can be realised by imposing $K' = 0$, which corresponds to assuming

$$G_3(\phi, X) = F(\phi)\ln(X/X_*)\,. \tag{4.209}$$

This functional form is a solution of the partial differential equation $G_{3X} + XG_{3XX} = 0$ (see (4.201)), which eliminates the non-linear contribution due to the denominator in (4.208). As mentioned above, an additional function of the scalar field, $V(\phi)$, in G_3 is neglected since it can be reabsorbed through a redefinition of G_2. Therefore, in order to deal with a linear effective fluid and apply the Eckart's thermodynamics, in the following we assume the above functional form of G_3. This particular choice is not just attractive for its simplicity, but also includes interesting applications like shift-symmetric theories exhibiting hairy black holes [300, 301], vanishing braiding theories [302], and the case of $XG_{3X} \propto G_{4\phi}$ which appears favoured by observational data [303].

Recalling that, in a homogeneous and isotropic background, the matter stress-energy tensor is $T^{(\mathrm{m})}_{ab} = \rho^{(\mathrm{m})}u_a u_b + P^{(\mathrm{m})}h_{ab}$, the pressure in the effective ϕ-fluid (4.208) is comprised of three contributions,

$$P^{(\phi)} = P_{\mathrm{int}} + P_{\mathrm{non-visc}} + P_{\mathrm{visc}}. \tag{4.210}$$

We denote with P_{int} the "interaction" pressure, namely the contribution to the pressure arising from the interaction between the scalar field and the components of the matter stress-energy tensor $T^{(\mathrm{m})}_{ab}$, while P_{visc} and $P_{\mathrm{non-visc}}$ denote the viscous and non-viscous pressure contributions (note that for Horndeski theories we single out the "interaction" contribution, in order to better classify the numerous terms in the expression, while there was no need to do so in section 4.4.1). The viscous pressure, similarly to the case in [1], is taken as proportional to H because of Eckart's constitutive relation (3.1).[5] Taking into account (4.209), the explicit expressions for the different contributions to the pressure are

$$P_{\mathrm{int}} = \frac{(G_{4\phi} - F)}{G_4\Delta}\left[G_{4\phi}\left(\rho^{(\mathrm{m})} - 3P^{(\mathrm{m})}\right) - 3F\left(\rho^{(\mathrm{m})} - P^{(\mathrm{m})}\right)\right] , \tag{4.211}$$

[5] For a detailed discussion on the splitting of viscous and non-viscous terms within this thermodynamic analogy for maximally symmetric spaces, see [1].

$$
\begin{aligned}
P_{\text{non-visc}} = \frac{1}{G_4\Delta}\{ & 2XG_4\left(G_{2X}+2XG_{2XX}\right)\left(2G_{4\phi\phi}-F_\phi\ln(X/X_*)\right) \\
& + G_2\left[G_4\left(G_{2X}+2XG_{2XX}\right)-2G_4(1+\ln(X/X_*))F_\phi+4FG_{4\phi}-3F^2-G_{4\phi}^2\right] \\
& + 2XG_{2X}\left(3F^2-4FG_{4\phi}+G_{4\phi}^2\right)-2G_4\left(G_{2\phi}-2XG_{2\phi X}\right)\left(F-G_{4\phi}\right) \\
& - 2X\left[\ln(X/X_*)\left(2G_4F_{\phi\phi}\left(F-G_{4\phi}\right)+F_\phi\left(-2G_4\left(F_\phi-2G_{4\phi\phi}\right)-4FG_{4\phi}\right.\right.\right. \\
& \left.\left.\left.+3F^2+G_{4\phi}^2\right)\right)+4G_4F_\phi G_{4\phi\phi}-2G_4\ln^2(X/X_*)F_\phi^2\right]\}\,,
\end{aligned}
\tag{4.212}
$$

$$
\begin{aligned}
P_{\text{visc}} = & -\frac{\epsilon\sqrt{2X}H}{G_4\Delta}\left\{G_{4\phi}\left[G_4\left(G_{2X}-4XG_{2XX}+2(5-\ln(X/X_*))F_\phi\right)+21F^2\right]\right. \\
& \left.-3FG_4\left[2(1-\ln(X/X_*))F_\phi+G_{2X}\right]-3\left(5FG_{4\phi}^2+3F^3-G_{4\phi}^3\right)\right\}\,,
\end{aligned}
\tag{4.213}
$$

where

$$
\Delta = G_4\left[G_{2X}+2XG_{2XX}-2(1+\ln(X/X_*))F_\phi\right]+3\left(F-G_{4\phi}\right)^2\,. \tag{4.214}
$$

From the viscous component P_{visc} of the pressure, we can extract the bulk viscosity coefficient ζ as defined in (3.1), which is proportional to $\dot{\phi}$ (as in section 4.4.1) and reads

$$
\begin{aligned}
\zeta = & \frac{\sqrt{2X}}{3G_4\Delta}\left\{G_{4\phi}\left[G_4\left(G_{2X}-4XG_{2XX}+2(5-\ln(X/X_*))F_\phi\right)+21F^2\right]\right. \\
& \left.-3FG_4\left[2(1-\ln(X/X_*))F_\phi+G_{2X}\right]-3\left(5FG_{4\phi}^2+3F^3-G_{4\phi}^3\right)\right\}\,.
\end{aligned}
\tag{4.215}
$$

Focusing on dynamical scalar fields, we can have either a vanishing bulk viscosity, corresponding to

$$
\begin{aligned}
G_2(\phi,X) = & \mu(\phi)\left(\frac{5G_{4\phi}(\phi)-3F(\phi)}{4G_{4\phi}(\phi)}\right)^{-1}X^{\frac{5G_{4\phi}(\phi)-3F(\phi)}{4G_{4\phi}(\phi)}}+\nu(\phi) \\
& -4X\left(F_\phi(\phi)+\frac{3\left[F(\phi)-G_{4\phi}(\phi)\right]^2}{4G_4(\phi)}\right)+2F_\phi(\phi)X\ln(X/X_*)\,,
\end{aligned}
\tag{4.216}
$$

or a vanishing interaction term, associated with $F(\phi)=G_{4\phi}(\phi)$. If one imposes both vanishing P_{int} and P_{visc}, the scalar field becomes non-dynamical.

Following the argument in [1], we can still find the $\mathcal{KT}$ of Horndeski gravity in FLRW, despite the fact that the heat flux density q_a vanishes identically due to homogeneity. Indeed, the general expression (4.17) is found in [12] for Horndeski theories without specifying to particular geometries. Using the notation $XG_{3X}=F$, coming from substituting (4.209) into (4.17), we find

$$
\mathcal{KT} = \epsilon\sqrt{2X}\frac{(G_{4\phi}-F)}{G_4}\,. \tag{4.217}
$$

The above quantity is closely related to the braiding parameter used in Horndeski parametrizations, that quantifies kinetic mixing between the metric and scalar perturbations and receives contributions from G_3 and G_4 [304, 305]. We notice that the relationship $\zeta = \mathcal{K}\mathcal{T}/3$, valid for "traditional" scalar-tensor theories [1] is not valid for Horndeski theories.

An intriguing and novel feature that went unnoticed in [12] and that emerges from (4.217) is that $\mathcal{K}\mathcal{T} = 0$ both for $\dot{\phi} = 0$ (which is the usual GR equilibrium) and $F = G_{4\phi}$. It is interesting because it means there are equilibrium states at $\mathcal{K}\mathcal{T} = 0$ in the theory that are different than GR. In general, such alternative equilibrium states are found to be unstable [5] and are therefore unable to compete with the special role of GR in the landscape of gravity theories seen through the lens of the first-order thermodynamics. The stability of such states is assessed (generally after reducing to an exact solution of the theory) through the effective heat equation that provides the precise description of the dissipative process leading from non-equilibrium to equilibrium. For Horndeski theories this equation reads [12]

$$\frac{\mathrm{d}(\mathcal{K}\mathcal{T})}{d\tau} = \left(\epsilon\frac{\Box\phi}{\sqrt{2X}} - \theta\right)\mathcal{K}\mathcal{T} + \nabla^c\phi\nabla_c\left(\frac{G_{4\phi} - XG_{3X}}{G_4}\right). \tag{4.218}$$

4.5 Study of cosmological exact solutions

As mentioned in the previous sections, the formalism of first-order thermodynamics can be applied both to entire classes of theories and specific solutions within them. While in the previous section 4.4 we studied two classes of theories in a FLRW settings, in the present section we focus on exact solutions of these theories (some well-known and some new) which is a useful test of the formalism. The solutions of generalised Brans-Dicke theory from the point of view of first-order thermodynamics were explored in [1] and those of viable Horndeski theories in [7].

The results found for the exact solutions of generalized Brans-Dicke theory support the previous ideas, formulated in [10, 11], that gravity approaches the GR equilibrium state of zero temperature with the universe's expansion, while it departs from it near spacetime singularities. A peculiar situation arises with Big Rip singularities, in which the universe expands explosively in a pole-like singularity: here (extreme) expansion occurs simultaneously with a spacetime singularity. By analyzing an exact solution in a conformally coupled scalar field model [306], it is found that gravity still departs from the GR equilibrium state at the Big Rip.

4.5.1 Brans-Dicke theory

The well-known exact FLRW solutions of scalar-tensor gravity are collected in [95], and we focus particular attention to the simpler $k = 0$ case. The analytical solutions chosen, which are specifically solutions of Jordan frame Brans-Dicke theory, exhibit interesting features for the purposes of cosmology, once particular forms of the cosmic matter are chosen. The latter is described by a perfect fluid with stress-energy tensor and equation of state

$$T^{(\mathrm{m})}_{ab} = \left(P^{(\mathrm{m})} + \rho^{(\mathrm{m})}\right) u_a u_b + P^{(\mathrm{m})}\, g_{ab}\,, \tag{4.219}$$

$$P^{(\mathrm{m})} = (\gamma - 1)\rho^{(\mathrm{m})}, \qquad \gamma = \mathrm{const.} \tag{4.220}$$

Most of the solutions in the following have a power-law behaviour: such solutions play a role analogous to that of the inflationary de Sitter attractor in GR.

O'Hanlon and Tupper solution

This solution [268] corresponds to vacuum, $V(\phi) = 0$, and $\omega > -3/2$, $\omega \neq 0, -4/3$ (the case of $\omega = 0$ corresponds to a stealth solution and was studied in 4.2.2):

$$a(t) = a_0 \left(\frac{t}{t_0} \right)^{q_\pm} , \tag{4.221}$$

$$\phi(t) = \phi_0 \left(\frac{t}{t_0} \right)^{s_\pm} , \tag{4.222}$$

where

$$q_\pm = \frac{\omega}{3(\omega+1) \mp \sqrt{3(2\omega+3)}} , \tag{4.223}$$

$$s_\pm = \frac{1 \mp \sqrt{3(2\omega+3)}}{3\omega+4} , \tag{4.224}$$

satisfying $3q+s=1$. The two sets of exponents with upper or lower sign correspond to the so-called fast and slow solutions, respectively, a nomenclature tied to the behaviour of the Brans-Dicke scalar at early times [95]. This solution is endowed with a Big Bang singularity for $t \to 0$ and its limit for $\omega \to +\infty$, namely $a(t) \propto t^{1/3}$ and $\phi =$ const., does not reproduce the corresponding GR solution, which is Minkowski space (this is a well-known anomaly of the $\omega \to \infty$ limit of Brans-Dicke theory [95], which can be approached also with the effective fluid formulation of this theory [96]).

The behaviours of the scale factor and the scalar field yield

$$\frac{\dot{\phi}}{\phi} = \frac{s_\pm}{t} , \tag{4.225}$$

$$\frac{\dot{a}}{a} = \frac{q_\pm}{t} , \tag{4.226}$$

hence the viscous pressure (4.190)

$$P_{\text{visc}} = -\frac{q_\pm \, s_\pm}{8\pi t^2} \tag{4.227}$$

and the bulk viscosity coefficient

$$\zeta = \frac{s_\pm}{24\pi \, t} \tag{4.228}$$

vanish at late times, recovering the GR limit. Since $T^{(\mathrm{m})} = 0$, the total pressure is

$$P^{(\phi)} = \frac{s_\pm}{8\pi t^2} \left(\frac{\omega \, s_\pm}{2} - q_\pm \right) \tag{4.229}$$

and the ratio $P_{\text{visc}}/P_{\text{non-visc}}$ is time-independent,

$$\frac{P_{\text{visc}}}{P_{\text{non-visc}}} = -\frac{2q_{\pm}}{\omega\, s_{\pm}} \,. \tag{4.230}$$

The product of the effective temperature and the thermal conductivity

$$\mathcal{KT} = \frac{|s_{\pm}|}{8\pi\, t} \tag{4.231}$$

vanishes for $t \to +\infty$ similarly to the bulk viscosity coefficient, recovering the GR limit. The limit $\mathcal{KT} \to +\infty$ is obtained at early times $t \to 0^+$, in accordance with the existence of a Big Bang for this solution and the hypothesis that gravity is "hot" near spacetime singularities [10, 11], as explained in section 3.1.1.

Brans-Dicke dust solution

This solution [86] corresponds to a pressureless dust fluid ($\gamma = 1$) and a matter-dominated universe characterised by $V(\phi) = 0$ and $\omega \neq -4/3$. The scale factor, scalar field, and matter energy density behave as

$$a(t) = a_0\, t^q \,, \tag{4.232}$$

$$\phi(t) = \phi_0\, t^s \,, \tag{4.233}$$

$$\rho^{(\text{m})} = \rho_0\, t^r \,, \tag{4.234}$$

where $\rho_0 = C/a_0^3$, C is an integration constant related to initial conditions, and

$$q = \frac{2(\omega+1)}{3\omega+4} \,, \tag{4.235}$$

$$s = \frac{2}{3\omega+4} \,, \tag{4.236}$$

$$r = -3q \,, \tag{4.237}$$

satisfying $3q + s = 2$.

In order to find the expressions needed for writing (4.189), we use the fact that a pressureless fluid has $T^{(\text{m})} = -\rho^{(\text{m})}$. The scale factor and scalar field of this solution yield

$$\frac{\dot{\phi}}{\phi} = \frac{s}{t} = \frac{2}{(3\omega+4)t} \,, \tag{4.238}$$

$$H = \frac{q}{t} = \frac{2(\omega+1)}{(3\omega+4)t} \,, \tag{4.239}$$

while the viscous pressure is

$$P_{\text{visc}} = -\frac{H\dot{\phi}}{8\pi\phi} = -\frac{\omega+1}{2\pi(3\omega+4)^2 t^2} \,. \tag{4.240}$$

The bulk viscosity coefficient is thus

$$\zeta = \frac{1}{12\pi(3\omega+4)t} \tag{4.241}$$

and it vanishes at late times, meaning that this cosmology approaches the GR equilibrium state.

The full expression of the effective ϕ-fluid pressure reads

$$P^{(\phi)} = \frac{1}{8\pi}\left[\frac{\omega}{2}\left(\frac{2}{(3\omega+4)t}\right)^2 + \frac{8\pi\rho_0\, t^r}{(2\omega+3)\,\phi_0^2\, t^{2s}}\right] - \frac{\omega+1}{2\pi(3\omega+4)^2t^2}\,. \tag{4.242}$$

The ratio $P_{\text{visc}}/P_{\text{non-visc}}$ goes to zero as $t \to +\infty$ if $s < 0$, to $-1/2$ if $s = 0$, and to -1 if $s > 0$. An alternative way to see this limit uses the relationship $-r + s = 2$ between the exponents, which yields

$$\frac{P_{\text{visc}}}{P_{\text{non-visc}}} \propto -\frac{1}{1+t^{r-2s+2}} = -\frac{1}{1+t^{-s}}\,. \tag{4.243}$$

The ratio tends to -1 as $t \to +\infty$ if $s > 0$; this choice of sign for s is supported by the observational constraints on the Brans-Dicke coupling ω, which provide a lower bound $\omega \gtrsim 10^3$ (for recent results, see for example [94] and section 1.2.1).

As for the temperature of scalar-tensor gravity, if we assume 3.5) to hold even in FLRW spacetimes (see the argument in 4.4), we have

$$\mathcal{K}\mathcal{T} = \frac{1}{4\pi\,|3\omega+4|\,t}\,. \tag{4.244}$$

Since $\omega \neq -4/3$, $\mathcal{K}\mathcal{T} \to \infty$ at the Big Bang $t \to 0^+$, which agrees again with the hypothesis that gravity is "hot" near spacetime singularities. The GR equilibrium state $\mathcal{K}\mathcal{T} \to 0$ is approached at late times $t \to +\infty$.

Nariai solution

The power-law Nariai solution [269, 270] describes a $K = 0$ FLRW universe filled by a perfect fluid, $V(\phi) = 0$, and $\omega \neq -4[3\gamma(2-\gamma)]^{-1} < 0$, given by

$$a(t) = a_0(1+\delta t)^q\,, \tag{4.245}$$

$$\phi(t) = \phi_0(1+\delta t)^s\,, \tag{4.246}$$

$$\rho^{(\mathrm{m})}(t) = \rho_0(1+\delta t)^r\,, \tag{4.247}$$

where

$$q = \frac{2[\omega(2-\gamma)+1]}{3\omega\gamma(2-\gamma)+4}\,, \tag{4.248}$$

$$s = \frac{2(4-3\gamma)}{3\omega\gamma(2-\gamma)+4}\,, \tag{4.249}$$

$$r = -3\gamma q\,. \tag{4.250}$$

Using $\alpha \equiv \dfrac{2(4-3\gamma)}{(2\omega+3)(2-\gamma)+3\gamma-4}$ and $A \equiv \dfrac{2\omega+3}{12}$, we write

$$\delta = \left(\frac{\alpha+3\gamma}{2}\right)\frac{8\pi\rho_0}{3\phi_0\left[(1+\alpha/2)^2 - A\alpha^2\right]} . \tag{4.251}$$

The Nariai solution contains the Brans-Dicke dust solution already discussed as a special case. Other special cases of interest include a radiative fluid and the cosmological constant.

Radiative fluid This solution corresponds to $\gamma = 4/3$, $P^{(\mathrm{m})} = \rho^{(\mathrm{m})}/3$, $\alpha = 0$, and

$$a(t) = a_0\sqrt{1+\delta t}\,, \tag{4.252}$$

$$\phi(t) = \phi_0 = \text{const.}\,, \tag{4.253}$$

$$\rho^{(\mathrm{m})}(t) = \frac{\rho_0}{(1+\delta t)^2}\,, \tag{4.254}$$

with $\delta = \left(\dfrac{32\pi\rho_0}{3\phi_0}\right)^{1/2}$. The constant scalar field translates into $P_{\text{visc}} = 0$ and $\mathcal{KT} = 0$ at all times, which reproduces the GR equilibrium state. Moreover, $P_{\text{non}-\text{visc}}$ also vanishes, since the first three terms in (4.189) are zero.

Cosmological constant In this case $\gamma = 0$, $P^{(\mathrm{m})} = -\rho^{(\mathrm{m})}$, $\alpha = \dfrac{4}{2\omega+1}$, and

$$a(t) = a_0(1+\delta t)^{\omega+1/2}\,, \tag{4.255}$$

$$\phi(t) = \phi_0(1+\delta t)^2\,, \tag{4.256}$$

$$\delta = \left[\frac{32\pi\rho_0}{\phi_0}\,\frac{1}{(6\omega+5)(2\omega+3)}\right]^{1/2}\,, \tag{4.257}$$

while $\rho^{(\mathrm{m})}(t) = \rho_0$ due to (4.247). This is not the only solution describing a universe driven by a cosmological constant in scalar-tensor cosmology but it is an attractor in phase space, which makes it relevant for the extended inflationary scenario [272]. For this solution,

$$\frac{\dot{\phi}}{\phi} = \frac{2\delta}{1+\delta t}\,, \tag{4.258}$$

$$H = \frac{\delta(\omega+1/2)}{1+\delta t}\,, \tag{4.259}$$

while the trace of the matter stress-energy tensor is $T^{(\mathrm{m})} = -4\rho^{(\mathrm{m})}$. The viscous pressure reads

$$P_{\text{visc}} = -\frac{\delta^2(\omega+1/2)}{4\pi(1+\delta t)^2}\,, \tag{4.260}$$

yielding the bulk viscosity coefficient

$$\zeta = \frac{\delta}{12\pi(1+\delta t)} \tag{4.261}$$

which tends to the GR equilibrium state at late times. The total pressure is

$$P^{(\phi)} = \frac{1}{8\pi}\left[\frac{\omega}{2}\left(\frac{2\delta}{1+\delta t}\right)^2 + \frac{32\pi\rho_0}{(2\omega+3)\,\phi_0^2\,(1+\delta t)^4}\right] - \frac{\delta^2(\omega+1/2)}{4\pi(1+\delta t)^2}\,, \tag{4.262}$$

while the ratio between viscous and non-viscous pressures is

$$\frac{P_{\text{visc}}}{P_{\text{non-visc}}} = -\frac{\delta^2(2\omega+1)\,(1+\delta\,t)^2}{\frac{32\pi\rho_0}{(2\omega+3)\phi_0^2} + 2\omega\,\delta^2\,(1+\delta\,t)^2} \to -\frac{(2\omega+1)}{2\omega} \quad \text{as } t \to +\infty\,. \tag{4.263}$$

The product of effective temperature and thermal conductivity

$$\mathcal{K}\mathcal{T} = \frac{|\delta|}{4\pi(1+\delta t)} \tag{4.264}$$

vanishes as $t \to +\infty$, recovering the GR equilibrium state.

Big Rip with conformally coupled scalar field

At the end of the 1990s and in the early 2000s, inspired by the first observational constraints from cosmological probes on the dark energy equation of state (which approached the boundary $w \equiv P/\rho = -1$), the possibility of a phantom equation of state parameter $w < -1$ was first explored. Such values of w cannot be explained by Einstein gravity coupled minimally with a scalar field of positive energy density. A regime with $w < -1$ is associated with $\dot{H} > 0$ ("superacceleration", as dubbed at the time) [307], while a dark energy fluid that could exhibit superacceleration is named phantom energy or superquintessence. The phantom energy density would grow in time instead of redshifting and would quickly come to dominate all other forms of energy, leading to a scale factor diverging in a finite amount of time, reaching a peculiar end of the universe (Big Rip) in which gravitationally bound structures would be ripped apart [74, 308]. The Big Rip is not unavoidable in models with a time-dependent equation of state and could occur or not, depending on the specific model adopted.

The current observational bounds on the dark energy equation of state are more precise thanks to surveys such as Planck and $w < -1$ is no longer favoured. However, the value of w still hovers around -1 and a Big Rip is not completely ruled out, although the closer w is to -1, the further in the future the Big Rip would be. For example, combining Planck data with data coming from supernovae, Baryon Acoustic Oscillations and other datasets, one has $w = -1.028 \pm 0.031$ [39]. Models that could exhibit superacceleration have been studied in various contexts, including Brans-Dicke-like fields in scalar-tensor gravity. This makes such models interesting as applications of our thermodynamical formalism to analytical solutions of scalar-tensor cosmology. In the following, we consider one of the simplest superquintessence models consisting of a single scalar field ϕ nonminimally coupled to the Ricci curvature, with action

$$S = \int d^4x\sqrt{-g}\left[\left(\frac{1}{8\pi} - \xi\phi^2\right)\frac{R}{2} - \frac{1}{2}\nabla^c\phi\nabla_c\phi - V(\phi)\right] + S^{(\mathrm{m})}\,, \tag{4.265}$$

where ξ is a dimensionless coupling constant. We can rewrite this action in the general scalar-tensor form (2.20) with scalar ψ

$$S_{\mathrm{ST}} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\psi R - \frac{\omega(\psi)}{\psi}\nabla^c\psi\nabla_c\psi - V(\psi)\right] + S^{(\mathrm{m})}\,, \tag{4.266}$$

where the scalar fields ϕ and ψ are related by

$$\psi = 1 - 8\pi\xi\phi^2\,. \tag{4.267}$$

Since we now start from the different action (4.265), we consider an expression for the effective pressure which is different from (4.189) used in the previous sections. The action (4.265) can be explicitly recast as a scalar-tensor action with a variable Brans-Dicke parameter. From [306], the expression for the pressure of the effective fluid equivalent to the nonminimally coupled scalar (in a flat FLRW universe) reads

$$P^{(\phi)} = \frac{\dot{\phi}^2}{2} - V(\phi) - \xi[4H\phi\dot{\phi} + 2\dot{\phi}^2 + 2\phi\ddot{\phi} + (2\dot{H} + 3H^2)\phi^2]\,. \tag{4.268}$$

As before, we use the equation of motion

$$\ddot{\phi} = -3H\dot{\phi} - \xi R\phi - V_{,\phi} \tag{4.269}$$

to substitute for $\ddot{\phi}$, obtaining

$$P^{(\phi)} = \frac{\dot{\phi}^2}{2} - V(\phi) - \xi\left[-2H\phi\dot{\phi} + 2\dot{\phi}^2 - 2\xi R\phi^2 - 2\phi V_{,\phi} + (2\dot{H} + 3H^2)\phi^2\right]\,. \tag{4.270}$$

Further use of the expression $R = 6\left(\dot{H} + 2H^2\right)$ of the Ricci scalar yields

$$P^{(\phi)} = \left(\frac{1}{2} - 2\xi\right)\dot{\phi}^2 + 2\xi\phi V_{,\phi} - V + \xi\phi^2\left[2\left(6\xi - 1\right)\dot{H} + 3\left(8\xi - 1\right)H^2\right] + 2\xi H\phi\dot{\phi}\,. \tag{4.271}$$

Following [306], we consider a simple toy model with conformal coupling $\xi = 1/6$, potential $V(\phi) = \dfrac{m^2\phi^2}{2} + \lambda\phi^4$, and $\lambda < 0$.[6] Since $H(t)$ and $\phi(t)$ grow very quickly in the superacceleration regime, we consider solutions for the scale factor and scalar field that assume large values of these quantities. Such solutions have the pole-like form

$$a(t) = \frac{a_*}{|t - t_0|^{\alpha_\pm}} \tag{4.272}$$

[6] A negative potential yields a negative energy density for a minimally coupled scalar field but, since here $\xi \neq 0$, the positivity of $\rho^{(\phi)}$ is not spoiled.

and

$$\phi(t) = \frac{\phi_*}{|t-t_0|^{\beta_\pm}}\,, \tag{4.273}$$

where we restrict to $t < t_0$ and where $\alpha_\pm, \beta_\pm > 0$, while t_0, a_*, and ϕ_* are constants. If $\mu = 4\pi m^2/3$, then

$$\alpha_\pm = \frac{\pm\sqrt{-\lambda(2\mu+\lambda)} - \mu - \lambda}{\mu + 4\lambda}\,, \tag{4.274}$$

$$\beta_\pm = 1\,, \tag{4.275}$$

$$\phi_*^\pm = \pm\frac{(1+\alpha_\pm)}{\sqrt{-2\lambda}}\,, \tag{4.276}$$

leading to

$$H = \frac{\alpha_\pm}{t_0 - t}\,, \tag{4.277}$$

$$\dot{\phi} = \frac{\phi_*}{(t_0-t)^2}\,, \tag{4.278}$$

which we substitute in (4.271). The only term containing $\theta = 3H$ is the third one on the right-hand side of (4.271), giving the viscous pressure

$$P_{\text{visc}} = 2\xi H\phi\dot{\phi} = \frac{\alpha_\pm\,\phi_*^2}{3\,(t_0-t)^4} \tag{4.279}$$

and the bulk viscosity coefficient

$$\zeta = -\frac{2\xi\phi\dot{\phi}}{3} = -\frac{\phi_*^2}{9\,(t_0-t)^3}\,. \tag{4.280}$$

An expanding universe ends its existence in the Big Rip as $t \to t_0^-$, where ζ diverges. This behaviour is interesting because, while gravity is "hot" near spacetime singularities, it is "cooled" by expansion and it is not *a priori* clear what to expect at a Big Rip singularity in which the universe expands explosively.

Substituting the scale factor and scalar field in the total pressure (4.271) yields

$$P^{(\phi)} = -\frac{m^2\phi_*^2}{6\,(t_0-t)^2} + \frac{\phi_*^2}{3\,(t_0-t)^4}\left[\frac{(\alpha_\pm+1)^2}{2} + \lambda\phi_*^2\right]. \tag{4.281}$$

The ratio $P_{\text{visc}}/P_{\text{non-visc}}$

$$\frac{P_{\text{visc}}}{P_{\text{non-visc}}} = \frac{\alpha_\pm\phi_*^2}{-\alpha_\pm\phi_*^2 - \frac{m^2\phi_*^2}{2}\,(t_0-t)^2 + \phi_*^2\left[\frac{(\alpha_\pm+1)^2}{2} + \lambda\phi_*^2\right]} \tag{4.282}$$

has the $t \to t_0$ limit

$$\to \frac{2\alpha_\pm}{\alpha_\pm^2 + 1 + 2\lambda\phi_*^2} \qquad \text{as } t \to t_0^-\,. \tag{4.283}$$

Considering now the product of thermal conductivity and effective temperature, the 4-velocity and the 4-acceleration of the effective fluid in section 2.2.1 have the same form for ϕ and ψ in the actions (4.265) and (2.20). Therefore, only the factor in front of $\dot{u}_a$ in (3.4) is different if we start from the action (4.265), and the expression for the heat flux density now reads

$$q_a^{(\phi)} = -\frac{2\,|\xi\phi|\,\sqrt{-\nabla^e\phi\nabla_e\phi}}{1-8\pi\xi\phi^2}\,\dot{u}_a\,, \tag{4.284}$$

so that

$$\mathcal{KT} = \frac{2\,|\xi\phi|\,\sqrt{-\nabla^e\phi\nabla_e\phi}}{1-8\pi\xi\phi^2}\,. \tag{4.285}$$

Substituting the solution $\phi(t)$ yields

$$\mathcal{KT} = \frac{\phi_*^2}{3\,(t_0-t)^3}\left[1-\frac{4\pi\phi_*^2}{3\,(t_0-t)^2}\right]^{-1} = \frac{\phi_*^2}{(t_0-t)\left[3\,(t_0-t)^2-4\pi\phi_*^2\,(t_0-t)\right]}\,. \tag{4.286}$$

This expression diverges as $t \to t_0^-$, recovering the expected result for the approach to a singularity. $\mathcal{KT}$ diverges also at an earlier time when $\phi(t)$ approaches the critical value $\phi_c \equiv (8\pi\xi)^{-1/2}$, which is always present for $\xi > 0$ (in our case, $\phi_c = (4\pi/3)^{-1/2}$). At this critical value, the effective gravitational coupling

$$G_{\text{eff}} = \frac{G}{1-8\pi\xi\phi^2} \tag{4.287}$$

diverges and its sign changes for $|\phi| > \phi_c$, together with the sign of $\mathcal{KT}$. Indeed, for $\mathcal{KT}$ to make sense for nonminimally coupled scalar fields, it must be $|\phi| < \phi_c$, but this limitation coincides with the familiar one requiring that G_{eff} be positive [95].

4.5.2 Viable Horndeski

In [7], we tested the general results for the thermodynamics of viable Horndeski cosmology on exact solutions of interesting subclasses of the general theory that are also favoured by cosmological observations. The considered examples differ significantly from the results obtained for "traditional" scalar-tensor cosmologies in [1], since they display a non-vanishing effective temperature at all times in the cosmic evolution and asymptotically approach a constant effective temperature at late times. These results have been obtained, in particular, for classes of shift-symmetric and asymptotically shift-symmetric theories (the latter being shift-symmetric as the non-minimal coupling function G_4 approaches unity), both characterised by a non-vanishing braiding parameter.

In addition to showing the existence of subclasses of viable Horndeski gravity that never relax to the GR equilibrium state, our analysis further confirms previous findings according to which curvature singularities are "hot" [11], exhibiting a diverging temperature. This suggests that the deviations of these models from GR become extreme at spacetime singularities. An additional intriguing consequence of finding the effective temperature associated to these viable Horndeski subclasses is that imposing its positivity recovers the weak energy condition for the ϕ-fluid, which is characteristic of a real fluid and was not expected to hold for an effective fluid. In the following, we focus on background cosmologies in cubic shift-symmetric Horndeski theories with a vanishing scalar current. Galileons possess shift symmetry (in addition to Galileian symmetry), and this class of theories has some of the most interesting and well-explored cosmological consequences. We start in the following from a shift-symmetric solution and follow the strategy in [301] to find a cosmological solution with the desired expansion behaviour.

Shift-symmetric solutions

Shift symmetry refers to the invariance under $\phi \to \phi + \phi_0$, where ϕ_0 is a constant. The shift-symmetric subclass of the Horndeski theory corresponds to the choice $G_i = G_i(X)$, i.e., the Lagrangian does not explicitly depend on ϕ. In this case, the theory is characterized by the presence of a Noether conserved current, J^a, and the scalar field equation of motion becomes $\nabla_a J^a = 0$. The shift-symmetric viable Horndeski scalar current in spatially flat FLRW reads

$$J^a = \delta^a_0 \dot{\phi} \left(G_{2X} + 3HG_{3X}\dot{\phi} \right). \tag{4.288}$$

Combining the shift-symmetric restriction $G_i = G_i(X)$ with $G_4 =$ const. required for the viable class, all nonminimal couplings disappear. The following action for the gravitational sector describes the shift-symmetric subclass of the linear model selected above

$$S_g = \frac{1}{2} \int d^4x \sqrt{-g} \left[R + G_2(X) - \lambda \ln{(X/X_*)} \Box\phi \right], \tag{4.289}$$

so that $G_2 = G_2(X)$, $G_3 = \lambda \ln{(X/X_*)}$, and $G_4 = 1$. The study of this choice of couplings is also phenomenologically motivated, since it provides a good fit to cosmological data from standard probes [303, 309]. The associated scalar current reduces to

$$J^a = \delta^a_0 \left(\dot{\phi}\, G_{2X} + 6\lambda H \right). \tag{4.290}$$

We now restrict to solutions with vanishing scalar current, in order to provide some simple examples, similarly to [301]. Excluding the trivial case of $\dot{\phi} = 0$ which is equivalent to GR, the vanishing scalar current translates to

$$\dot{\phi}\, G_{2X} + 6\lambda H = 0. \tag{4.291}$$

Using the above equation, and assuming that the standard matter content is described by the usual barotropic equation of state $P^{(m)} = w\rho^{(m)}$, we obtain the following expressions for the scalar field energy density and pressure

$$\rho^{(\phi)} = -\frac{1}{2} G_2 \tag{4.292}$$

$$P^{(\phi)} = \frac{1}{2} G_2 - \frac{X G_{2X}^2 + 3\lambda^2 G_2}{G_{2X} + 2X G_{2XX} + 3\lambda^2} - \frac{3\lambda^2 (w-1)\rho^{(\mathrm{m})}}{G_{2X} + 2X G_{2XX} + 3\lambda^2}. \tag{4.293}$$

The effective temperature of shift-symmetric viable Horndeski then reads

$$\mathcal{K}\mathcal{T} = -\lambda\epsilon\sqrt{2X}\,. \tag{4.294}$$

To ensure a positive-definite $\mathcal{K}\mathcal{T}$, $\lambda\epsilon$ must be negative, i.e., $\lambda\,\text{Sign}(\dot{\phi}) > 0$. Since X must be strictly positive, (4.294) does not tend to the GR equilibrium state at $\mathcal{K}\mathcal{T} = 0$. This shows that the approach to equilibrium is not always granted, as it was also found in [1] and section 4.5.1 for the Big Rip solution to Brans-Dicke theory.

In the following, we employ the method devised by [301] to find exact cosmological background solutions within the shift-symmetric Horndeski class that exhibit a desirable expansion behaviour, such as power-law or de Sitter expansion. The method consists in tailoring a cubic shift-symmetric Horndeski theory (with vanishing scalar current) to a flat FLRW spacetime with a given dynamics. We assume $P^{(\text{m})} = w\rho^{(\text{m})}$ and a strictly monotonic H (such that $t = t(H)$). Then the Friedmann constraint (4.206) and the scalar equations of motion (4.291) respectively read

$$G_2\,[H] = 2\rho^{(\text{m})}\,[H] - 6H^2\,, \tag{4.295}$$

$$G_{2X}\,[H] = -\frac{6\lambda H}{\dot{\phi}\,[H]} = \frac{6\lambda\epsilon H}{\sqrt{2X[H]}}\,, \tag{4.296}$$

where $[H]$ indicates the functional dependence on H and $\epsilon = -\text{Sign}(\dot{\phi})$. Combining the differentiation of G_2 and that of (4.295), both with respect to H, yields

$$2\frac{d\rho^{(\text{m})}}{dH} - 12H = G_{2X}\frac{dX}{dH}. \tag{4.297}$$

Then, taking advantage of (4.296), the latter can be rewritten as

$$2\frac{d\rho^{(\text{m})}}{dH} - 12H = \left(\frac{6\lambda H}{\epsilon\sqrt{2X}}\right)\frac{dX}{dH}, \tag{4.298}$$

and be formally integrated in H on both sides as

$$\int \left(\frac{2}{H}\frac{d\rho^{(\text{m})}}{dH} - 12\right) dH = 6\lambda\epsilon \int \frac{dX}{\sqrt{2X}}\,, \tag{4.299}$$

or, equivalently,

$$\int \left(\frac{2}{H}\frac{d\rho^{(\text{m})}}{dH}\right) dH - 12H = 6\lambda\epsilon\,\sqrt{2X}\,. \tag{4.300}$$

At this point, one chooses a specific cosmological evolution given by $H(t)$ and then solves the continuity equation for the matter energy density. In turn, this is used to analytically obtain the function G_{2X} by inverting the relation $X[H]$ (when possible) and integrating the vanishing scalar current condition.

Power-law expansion Choosing a power-law expansion

$$a(t) = a_* \left(\frac{t}{t_*}\right)^n , \quad n > 0 \,, \;\; t \geq 0 \,, \tag{4.301}$$

where all the quantities with an asterisk are constant, we have

$$H(t) = \frac{n}{t} \qquad \longleftrightarrow \qquad t(H) = \frac{n}{H} \tag{4.302}$$

$$\rho^{(\mathrm{m})}(t) = \rho_* \left(\frac{t_*}{t}\right)^{3n(w+1)} \qquad \longleftrightarrow \qquad \rho^{(\mathrm{m})}(H) = \rho_* \left(\frac{H}{H_*}\right)^{3n(w+1)} . \tag{4.303}$$

Using (4.302) and (4.303) into (4.300), and integrating, we obtain

$$6\lambda\epsilon\sqrt{2X} = 6\lambda\epsilon\, c_X - 12H + \frac{6n\rho_*(w+1)}{H_*[3n(w+1)-1]} \left(\frac{H}{H_*}\right)^{3n(w+1)-1} , \tag{4.304}$$

where c_X is an integration constant, associated with the non-vanishing asymptotic value of $\sqrt{2X}$ reached when $H = 0$ (in the limit $t \to \infty$).

The simplest case admitting an analytical solution corresponds to $n = 2/3(w+1)$, with $w \neq -1$. Then, it turns out that

$$6\lambda\epsilon\sqrt{2X} = 12H \left(\frac{\rho_*}{3H_*^2} - 1\right) + 6\lambda\epsilon\, c_X \Rightarrow H = \frac{\lambda\epsilon}{2} \left(\sqrt{2X} - c_X\right) \left(\frac{\rho_*}{3H_*^2} - 1\right)^{-1} . \tag{4.305}$$

Thus the system is analytically solvable, yielding

$$G_2(X) = \frac{3\lambda^2}{2} \left(\sqrt{2X} - c_X\right)^2 \left(\frac{\rho_*}{3H_*^2} - 1\right)^{-1} , \tag{4.306}$$

with the scalar field

$$\phi(t) = \phi_* - \epsilon\, c_X(t - t_*) - \frac{2n}{3\lambda} \left(\frac{\rho_*}{3H_*^2} - 1\right) \ln(t/t_*). \tag{4.307}$$

Therefore, the effective scalar fluid temperature for this exact solution reads

$$\mathcal{K}\mathcal{T} = \frac{2n}{t} \left(1 - \frac{\rho_*}{3H_*^2}\right) - \lambda\epsilon\, c_X \,. \tag{4.308}$$

As expected for cosmological solutions with an initial singularity [1], $\mathcal{K}\mathcal{T} \to +\infty$ for $t \to 0$, indicating an extreme deviation of Horndeski theory from the GR equilibrium state as the singularity is approached.

It is interesting to notice that the (4.294) implies the weak energy condition for the effective fluid, which was not the case for Brans-Dicke theories in 4.4.1. Given (4.294), $\mathcal{K}\mathcal{T} > 0$ translates to $\lambda\epsilon < 0$, but in order to have a positive definite $\mathcal{K}\mathcal{T}$ for any $t > 0$,also

$$\left(1 - \frac{\rho_*}{3H_*^2}\right)(w+1) > 0, \tag{4.309}$$

associated to the requirement of a non-vanishing kinetic term, must be satisfied. Then, assuming $w > -1$ (corresponding to $n > 0$),

$$\left(1 - \frac{\rho_*}{3H_*^2}\right) > 0. \tag{4.310}$$

The term $\sqrt{2X}$ diverges at $t = 0$ and approaches c_X for $t \to \infty$, which is consistent with (4.302) and (4.305), where $H = \dfrac{n}{t}$ must be positive. As a consequence, G_2 is actually negative definite, and this implies $\rho^{(\phi)} > 0$ from (4.292). Since

$$\rho^{(\phi)} = \frac{3n^2}{t^2}\left(1 - \frac{\rho_*}{3H_*^2}\right), \tag{4.311}$$

and $P^{(\phi)} = w\rho^{(\phi)}$, also

$$\rho^{(\phi)} + P^{(\phi)} = \rho^{(\phi)}(1 + w) > 0. \tag{4.312}$$

is valid, satisfying the weak energy condition, which is in general not expected to hold for an effective fluid, as mentioned in section 2.2.2.

The bulk viscosity coefficient ζ (from (4.215)) yields

$$\zeta = \lambda\epsilon\left(\sqrt{2X} - c_X\frac{3H_*^2}{\rho_*}\right) = -\frac{2n}{t}\left(1 - \frac{\rho_*}{3H_*^2}\right) + \lambda\epsilon c_X\left(1 - \frac{3H_*^2}{\rho_*}\right). \tag{4.313}$$

The above equation shows that the effective fluid starts off with a negative (and diverging to $-\infty$) bulk viscosity approaching the initial singularity, then ζ vanishes as the gradient of the scalar field approaches $\sqrt{2X} = c_X\dfrac{3H_*^2}{\rho_*}$, or, equivalently, as the cosmological time approaches $t = -\dfrac{1}{\lambda\epsilon}\dfrac{2n\rho_*}{3c_X H_*^2}$, and finally it becomes positive as t increases from that point.

de Sitter expansion Let us now employ the same method to a spatially flat de-Sitter spacetime,

$$a(t) = a_* \exp(H_* t)\,. \tag{4.314}$$

The continuity equation gives

$$\rho^{(\mathrm{m})}(t) = \rho_* \exp\left[-3H_*(w+1)t\right] \tag{4.315}$$

and, from Eq. (4.296), we obtain

$$G_2(X) = 6\lambda H_*\left(\epsilon\sqrt{2X} + c_X\right). \tag{4.316}$$

Then, from the temporal component of the field equations (4.295), we obtain

$$6\lambda H_*(\epsilon\sqrt{2X} + c_X) = 2\rho^{(\mathrm{m})} - 6H_*^2, \tag{4.317}$$

which is equivalent to

$$\dot{\phi} = c_X + \frac{H_*}{\lambda}\left(1 - \frac{\rho_*}{3H_*^2}\exp\left[-3H_*(w+1)t\right]\right). \tag{4.318}$$

Integrating the equation above, the scalar field reads

$$\phi(t) = \phi_* + t\left(c_X + \frac{H_*}{\lambda}\right) + \frac{1}{3(w+1)\lambda}\frac{\rho_*}{3H_*^2}\exp\left[-3H_*(w+1)t\right]. \tag{4.319}$$

The effective scalar fluid temperature is

$$\mathcal{KT} = (\lambda\epsilon\, c_X + H_*) - \frac{\rho_*}{3H_*}\exp\left[-3H_*(w+1)t\right]. \tag{4.320}$$

Also in this case, the condition

$$\left(1 - \frac{\rho_*}{3H_*^2}\right) > 0 \tag{4.321}$$

ensures the positivity of $\rho^{(\phi)}$ and of $\mathcal{KT}$, under the assumption of $\epsilon\lambda < 0$ and

$$c_X < -\frac{H_*}{\epsilon\lambda}\left(1 - \frac{\rho_*}{3H_*^2}\right). \tag{4.322}$$

The constant c_X can be chosen so that the condition $\rho^{(\phi)} + P^{(\phi)} > 0$ is satisfied as well. Therefore, the effective fluid can be easily tuned to satisfy the weak energy condition, connected to the positivity of (4.294).

Chapter 5

Conclusions and outlook

5.1 Summary and results

In this thesis, we presented the first-order thermodynamics of modified gravity, a novel and promising formalism that provides a unifying perspective on the landscape of gravity theories. We set the stage by discussing the flaws of Einstein's General Relativity, which, despite being the most accomplished theory of gravity ever formulated, is not UV-complete and necessitates the introduction of the unknown dark matter and dark energy in order to describe the universe that we see around us in accordance with observations. We briefly reviewed how the first problem entails the lack of a consistent quantum theory of gravity. Afterwards, we described in detail the cosmological constant problem, reviewing the observational evidence for the cosmic accelerated expansion, to the fine-tuning issues posed by the simplest dark energy candidate Λ, to possible mechanisms proposed to ameliorate this problem.

We then moved on to alternatives, which either fall in the category of modified matter (i.e., entailing modifications of the right-hand side of the Einstein equations) or modified gravity (i.e., entailing modifications of the left-hand side of the equations). From then on, we focused on modified gravity and sketched a picture of the landscape of gravity theories, including scalar-tensor theories and theories with higher-order curvature terms. We briefly touched upon the cosmological consequences of these theories and their observational constraints from cosmological probes, but especially those imposed by gravitational waves, which are an increasingly powerful tool to put modified gravity to the test.

Subsequently, we took a detour from cosmology, by focusing on the intriguing but poorly understood relationship between gravity and thermodynamics. We reviewed the ideas of black hole thermodynamics, where this connection was first discovered, and then presented several approaches that make use of this connection to explain why gravity is so different from the other fundamental forces and why it cannot be so easily quantized. Out of these approaches, we delved into that of Jacobson, who formulated a thermodynamics of spacetime, finding that both Einstein's equations and the field equations of metric $f(R)$ theories can be recovered as equations of state from purely thermodynamical assumptions. In the first case, the standard equilibrium thermodynamics is sufficient, while in the second case, an extension to non-equilibrium, dissipative thermodynamics is necessary. These suggestive findings opened up the possibility that one might find a "thermodynamics of gravitational theories", a meta-description of theories that unifies the plethora of modified gravity models under a single idea. Additionally, Jacobson's results left the questions about the dissipative process leading from non-equilibrium to equilibrium unanswered.

This is precisely what this thesis set out to explain. After reviewing the well-known effective fluid formalism, that exploits the freedom to translate a modification to gravity (on the left-hand side of Einstein's equations) into a modification of matter (on the right-hand side), we used this framework to recover the imperfect dissipative fluid that originates from scalar-tensor gravity when all the terms containing the scalar are collected into an effective stress-energy tensor. We then presented the generalities of non-equilibrium, irreversible thermodynamics and specifically the simple formulation by Eckart, that is dubbed first-order as it only entails first-order terms in the dissipative variables to satisfy the second law of thermodynamics through its constitutive equations.

With all the necessary notions in place, we introduced the formalism of first-order thermodynamics of scalar-tensor gravity, which was originally found in a slightly generalised version of Brans-Dicke gravity. Applying Eckart's non-equilibrium thermodynamics to the effective fluid describing scalar-tensor gravity, one can find a positive-definite effective temperature, which is nothing but a temperature relative to GR. Einstein's theory indeed corresponds to the equilibrium state at zero temperature, while any theory containing an additional scalar degree of freedom has a finite positive temperature. The dissipative process leading to equilibrium can be precisely characterized through an effective heat equation, describing the time evolution of this temperature. The only assumption this whole construction relies on is a timelike and future-oriented 4-velocity for the fluid defined through the scalar gradient. We showed that, if the gradient is instead past-oriented, the thermodynamical description still holds but it is pathological.

The goal of this thesis was to extend and apply the first-order thermodynamical formalism to as many situations as possible, in order to test the physical intuition behind it and understand its limits of applicability. The formalism is versatile enough that it can be employed both for entire classes of theories and for specific solutions within them. After reviewing the interesting extension to Horndeski theories, we explored the fixed points of the effective heat equation constituting alternative equilibrium states distinct from GR. States with zero temperature were found to correspond to theories with non-dynamical scalars, such as specific cases of Brans-Dicke theories and cuscuton theories, while those with constat temperature instead correspond to stealth solutions and degenerate de Sitter solutions of scalar-tensor gravity. We performed many case studies and concluded that, in the thermodynamics of gravitational theories built so far, the approach to the GR state of equilibrium is often recovered, but is not granted. The other equilibrium states distinct from GR, however, are unstable according to various criteria and thus cannot compete with the central role occupied by Einstein's theory in the landscape of gravity theories, seen through the lens of first-order thermodynamics. In the following, it was shown that the formalism based on the notion of temperature presented so far can also be recast into an alternative formulation, based on the notion of chemical potential. In this picture, we can deal with scalar-tensor theories in the Einstein frame, whose description proved elusive in the original framework. Even if these theories are described by a perfect fluid with no dissipation, the approach to GR equilibrium can be interpreted as the process of the chemical potential approaching zero, instead of the temperature. The last and most comprehensive application of the formalism was to cosmology, both for generalised Brans-Dicke and Horndeski theories. In this context, we found that the the expansion of the universe brings the effective temperature of scalar-tensor theories closer to the GR equilibrium, echoing some previous results that scalar-tensor theories relaxes to GR at late cosmological times. Additionally, spacetime singularities exhibit the peculiar behaviour characterized by a diverging temperature, hence scalar-tensor theories and GR diverge from each other in extreme ways. Finally, we studied some exact solutions of both Brans-Dicke and Horndeski theories, confirming these findings.

The topic of this thesis, the formalism of first-order thermodynamics, unifies two realms that are quite distinct: scalar-tensor gravity and thermodynamics. We take inspiration from Jacobson's thermodynamics of spacetime, but reject the assumptions of that construction to make a more minimalistic choice: the only assumption we need is a timelike and future-oriented scalar gradient in the fluid's velocity, upon which the whole formalism rests. The big picture emerging from the application of this formalism is a map of the landscape of gravity theories, constructed through the notion of temperature: GR occupies a central role and modified gravity theories with an additional, dynamical scalar degree of freedom are scattered around the landscape and tend to the GR equilibrium in most situations. Our work is motivated by the fact that, given the plethora of modified gravity theories in the literature, it is useful to find ideas under which they can be seen as special cases in a broader framework. Understanding the role of GR in this broader context could also offer some promising ideas to address its shortcomings, although this is definitely a far-fetched and too ambitious goal we have not dealt with in this thesis.

The applications of first-order thermodynamics detailed in the previous sections can be visualized in a map.

5.2 Outlook

Given the diverse applications of first-order thermodynamics, it is important to also understand its limits of applicability. Some of these are reported below, as trying to overcome them provides opportunities for future research.

5.2.1 Formalism breakdown beyond viable Horndeski

One of the most intriguing findings of first-order thermodynamics was that the formalism breaks down beyond viable Horndeski, namely whenever $G_4 = G_4(X)$, as detailed in section 4.1. The reason for this is not entirely clear: some terms proportional to the Riemann tensor break the proportionality required for Eckart's constitutive relations to hold, but a deeper understanding of the causes of this fact are missing. The understanding of this is crucial to establish the limits of applicability of the formalism. One might wonder what exactly is in common between

Figure 5.1: Map showing the applications of first-order thermodynamics covered in this thesis.

all theories where the first-order thermodynamical analogy holds. The fact that the effective fluid can be recast in the form of a dissipative fluid is necessary, but not sufficient. It is natural to ask whether the validity of first-order thermodynamics, which relies on an effective imperfect fluid description, might be related to the presence of an essential kinetic gravity braiding [22], since kinetic braiding translates to the impossibility of bringing the energy-momentum tensor of the theory to the perfect fluid form. However, one quickly realizes that there is not relationship between this and first-order thermodynamics, since, for example, both viable and non-viable Horndeski have kinetic braiding, while the analogy only holds for viable Horndeski. However, a preliminary analysis suggests that it is related instead to the presence of a residual Weyl curvature that remains after a covariant debraiding procedure (see [305] for details). The presence of residual Weyl curvature (or lack thereof) creates a more subtle braiding structure, and allows for a classification of scalar-tensor theories in "stirred" and "shaken" (the terminology comes from whether the Weyl tensor can be rotated away or not). The thermodynamical formalism seems to hold only in "shaken" theories, but more robust calculations are needed to understand the meaning and implications of this relation.

5.2.2 Vector-tensor theories

The range of applicability of first-order thermodynamics would be substantially widened by extending it to vector-tensor theories, and it is natural to wonder whether an additional vector field still leads to a positive, non-zero temperature. The preliminary study of the simplest case of vector-tensor theories, Einstein-aether theory [310], seems to suggest that the analogy cannot be so easily generalized. The proportionalities required for the constitutive equations to hold are not easily recovered, even in the case where the fluid velocity is restricted to be the gradient of a scalar. The situation seems to change with Generalized Proca theories [311], containing derivative self-interactions for the vector field and still ensuring second-order equations of motion. However, the analogy seems to hold only when restricting to the subclass of Generalized Proca that is equivalent to viable Horndeski, recovered if the vector field is the gradient of the scalar, namely $A_\mu = \partial_\mu \phi$ and$G_{4X} = G_5 = 0$, which makes the extension trivial. This might signal the insurmountable limits to the range of applicability of the formalism and deserves to be investigated further to establish those limits.

5.2.3 Black-hole stealth solutions

The formalism as it stands is limited by the reliance on a timelike gradient for the scalar field, which ensures the effective fluid we consider has a meaningful velocity. However, an extension to null and spacelike gradients would allow to study not only stealth solutions with timelike gradients as those studied in 4.2.2, but also the more interesting black hole stealth solutions in scalar-tensor theories, that generally have a radial dependence of the scalar field, such as those in [232]. Preliminary results about the extension to null gradients suggest that we might be able to define an (ultra)-local notion of temperature that plays the same role of the effective temperature in the standard formalism. The extension to null gradients is as a stepping stone towards the treatment of spacelike gradients, which is the most interesting development. This would allow to deal with black hole stealth solutions which could not be considered earlier because of the scalar field's radial dependence. It is compelling to understand if such solutions still represent fixed points of the dissipation equation, and if they are also unstable according to various criteria, therefore confirming the special role of the GR equilibrium in the thermodynamical formalism. Additionally, since some of the modified gravity theories we studied with our formalism, such as Horndeski, violate some assumptions of the no-hair theorem, it would be fruitful to understand what the allowed hair correspond to in our thermodynamical picture. A very exciting development involves the process of spontaneous scalarization (a screening mechanism that makes fields appear at strong curvatures, while they remain undetectable at small spacetime curvatures), which seems conceptually related to the dissipative process we describe. From preliminary analysis, it appears plausible that the onset of scalarization could correspond to a phase transition in the thermodynamical formalism, where the effective properties of the fluid change abruptly. Since this happens as gravity becomes strong close to singularities, where we found that the effective temperature diverges, it would be of utmost interest to understand the evolution of this order parameter and test the consistency with our previous results.

5.2.4 Second-order thermodynamics

Ultimately, as described in section (2.3.3), Eckart's first-order thermodynamics suffers from shortcomings related to its excessively simple assumptions. A fully causal and stable thermodynamics is provided by theories containing second-order terms in the dissipative variables, which are, however, much more cumbersome to deal with, especially in the setting of an effective fluid. These considerations notwithstanding, it is interesting to attempt to apply such a thermodynamical description to the effective fluid of scalar-tensor gravity. There are two main approaches that seem promising. First, one could start with the truncated version of the Israel-Stewart formalism, for which the calculations in a similar fashion to first-order thermodynamics still appear manageable, and apply this more comprehensive description to a simple scalar-tensor theory class such as the original Brans-Dicke-like class where the formalism was originally developed. Second, one could take the opposite perspective altogether, and ask which modified gravity theory can be put in correspondence with the constitutive equations of second-order thermodynamics. Presumably, this should be some beyond Horndeski theory. Both of these points will be investigated in future research.

Appendix A

Gauge-invariant perturbations for scalar-tensor cosmology

Consider the modified gravity described by the action

$$S = \int d^4x \sqrt{-g} \left[\frac{f(\phi, R)}{2} - \frac{\bar{\omega}(\phi)}{2} \nabla^c \phi \nabla_c \phi - V(\phi) \right] \tag{A.1}$$

and a spatially flat unperturbed FLRW universe. The background field equations are

$$H^2 = \frac{1}{3F} \left(\frac{\bar{\omega}}{2} \dot{\phi}^2 + \frac{RF}{2} - \frac{f}{2} + V - 3H\dot{F} \right) , \tag{A.2}$$

$$\dot{H} = -\frac{1}{2F} \left(\bar{\omega} \dot{\phi}^2 + \ddot{F} - H\dot{F} \right) , \tag{A.3}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{2\bar{\omega}} \left(\frac{d\bar{\omega}}{d\phi} \dot{\phi}^2 - \frac{\partial f}{\partial \phi} + 2 \frac{dV}{d\phi} \right) = 0 , \tag{A.4}$$

where an overdot denotes differentiation with respect to the comoving time t, $H \equiv \dot{a}/a$ is the Hubble function, and $F \equiv \partial f / \partial R$. Quantities denoted with A, B, H_L, and H_T define the metric perturbations in the Bardeen-Ellis-Bruni-Hwang formalism [256–260] according to

$$g_{00} = -a^2 (1 + 2AY) , \tag{A.5}$$

$$g_{0i} = -a^2 B Y_i , \tag{A.6}$$

$$g_{ij} = a^2 \left[h_{ij} (1 + 2H_L) + 2H_T Y_{ij} \right] , \tag{A.7}$$

where h_{ij} is the 3-metric of the unperturbed FLRW space seen by the comoving observer, the scalar harmonics Y satisfy the eigenvalue problem $\bar{\nabla}_i \bar{\nabla}^i Y = -k^2 Y$ with eigenvalue k, and $\bar{\nabla}_i$ is the covariant derivative operator of h_{ij}. The vector and tensor harmonics Y_i and Y_{ij} satisfy

$$Y_i = -\frac{1}{k}\,\bar{\nabla}_i Y \ , \tag{A.8}$$

$$Y_{ij} = \frac{1}{k^2}\,\bar{\nabla}_i\bar{\nabla}_j Y + \frac{1}{3} Y h_{ij} \ . \tag{A.9}$$

$$\Phi_H = H_L + \frac{H_T}{3} + \frac{\dot{a}}{k}\left(B - \frac{a}{k}\,\dot{H}_T\right) \ , \tag{A.10}$$

$$\Phi_A = A + \frac{\dot{a}}{k}\left(B - \frac{a}{k}\,\dot{H}_T\right) + \frac{a}{k}\left[\dot{B} - \frac{1}{k}\left(a\dot{H}_T\right)^{\cdot}\right] \ , \tag{A.11}$$

are the Bardeen gauge-invariant potentials [256],

$$\Delta\phi = \delta\phi + \frac{a}{k}\,\dot{\phi}\left(B - \frac{a}{k}\,\dot{H}_T\right) \tag{A.12}$$

is the Ellis-Bruni variable [257, 258], and similar relations define the other gauge-invariant variables Δf, ΔF, and ΔR. We refer the reader to Refs. [261–266] for the complete set of equations for the gauge-invariant perturbations. In section 4.2.2, we only need the equation for the tensor modes

$$\ddot{H}_T + \left(3H + \frac{\dot{F}}{F}\right)\dot{H}_T + \frac{k^2}{a^2}\,H_T = 0\,. \tag{A.13}$$

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