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Geometric Algebra in General Relativity: A Tetrad-Based Formalism for Rotating Systems

With Applications to Rotational Dynamics and Black Hole Precession

Referees

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Dedicated to the all turtles all the way down.

“I believe that, so far as geometry is concerned, we need still another analysis which is distinctly geometrical or linear and which will express situation directly as algebra expresses magnitude directly”

— Leibniz, Gottfried, Letter to Christian Huygens. 8 September 1679, [Loemker and Leibniz, 1975]

“Theories of the known, which are described by different physical ideas, may be equivalent in all their predictions and are hence scientifically indistinguishable. However, they are not psychologically identical when trying to move from that base into the unknown. For different views suggest different kinds of modifications which might be made and hence are not equivalent in the hypotheses one generates from them in one’s attempt to understand what is not yet understood.”

— Richard Feynman, Nobel Lecture. December 11, 1965

Abstract

Abstract (English)

In this thesis, I develop a novel framework for General Relativity (GR) by combining tetrads with Geometric Algebra (GA), addressing some of the limitations present in traditional formalisms. GR is an inherently geometric theory, yet its conceptual clarity is often obscured by complicated notation and formalism. Tensor calculus, for instance, focuses on component-wise calculations rather than the abstract geometric structure of objects, while differential forms suffer from cumbersome notation and insufficient geometrical interpretation.

The motivation behind this novel approach stems from the success GA has shown in other areas of physics, combined with the underutilized use of tetrads in place of traditional coordinate frames. The reliance on coordinate frames unnecessarily complicates expressions and obscures physical insights. By leveraging tetrads within GA, I introduce a more intuitive and powerful approach to GR, offering clearer interpretations and computational advantages. These benefits are demonstrated through applications to FRW spacetimes, the Raychaudhuri equation, and precessing gyroscopes around black holes.

This new formalism captures the underlying geometry of physical objects in a more compact, intuitive, and computationally efficient manner. A key advantage lies in the geometric product, which naturally generalizes complex numbers to spaces of arbitrary dimension and signature. This greatly simplifies the treatment of Lorentz transformations, as exemplified in the case of gyroscopic precession. Here, the novel approach reduces the problem from solving a set of four coupled partial differential equations to a single, trivial differential equation in flat spacetime.

This thesis lays the groundwork for further exploration of GA in GR, offering new tools that could enhance both theoretical understanding and practical computations in the field.

Zusammenfassung (Deutsch)

In dieser Arbeit entwickele ich einen neuartigen formalismus für die Allgemeine Relativitätstheorie (GR), indem ich Tetraden mit Geometrischer Algebra (GA) kombiniere, um einige der in dem konventionellen Formalismus vorhandene Einschränkungen zu überwinden. Von Natur aus ist GR eine geometrische Theorie, deren konzeptionelle Klarheit jedoch oft durch komplizierte Notationen und Formalismen verdeckt wird. Das Tensor-Kalkül konzentriert sich beispielsweise auf komponentenweise Berechnungen, anstatt die abstrakte geometrische Struktur von Objekten zu berücksichtigen, während Differentialformen unter umständlicher Notation und einer unzureichenden geometrischen Interpretation leiden.

Die Motivation für diesen neuen Ansatz ergibt sich aus dem Erfolg, den GA in anderen Bereichen der Physik gezeigt hat, und der untergenutzten Verwendung von Tetraden anstelle traditioneller Koordinatensysteme. Die Abhängigkeit von Koordinatensystemen verkompliziert unnötig die mathematischen Ausdrücke und verschleiert physikalische Einsichten. Durch den Einsatz von Tetraden in Kombination mit GA präsentiere ich einen intuitiveren und leistungsstärkeren Ansatz für GR, der eine klarere physikalische Interpretation und rechnerische Vorteile bietet. Diese Vorteile werden anhand von Anwendungen auf FRW-Raumzeiten, die Raychaudhuri-Gleichung und präzedierende Kreisel in Orbits um Schwarze Löcher veranschaulicht.

Dieser neue Formalismus erfasst die zugrunde liegende Geometrie physikalischer Objekte in einer kompakteren, intuitiveren und rechnerisch effizientere Weise. Ein wesentlicher Vorteil liegt im geometrischen Produkt, das komplexe Zahlen auf natürliche Weise auf Räume beliebiger Dimension und Signatur verallgemeinert. Dies vereinfacht die Behandlung von Lorentz-Transformationen erheblich, wie am Beispiel der Präzession von Kreiseln gezeigt wird. Hier reduziert der neue Ansatz das Problem von der Lösungeines Systems aus vier gekoppelten partiellen Differentialgleichungen auf eine einzige, trivial zu lösende Differentialgleichung in einem flachen Raum.

Acknowledgments

The completion of a Ph.D. thesis is a long and arduous task that I could not have carried out without the help and support of many people. Some of these people have contributed directly by providing support during the darkest moments, suggesting new directions of research, or simply listening when I needed someone to talk to. In this regard, I would like to thank my family and closest people: Davide Legacci, Héctor Chuliá, Pablo Herrero, Lucía Quiles, Joaquín Pérez, Victor Boscá, and Sylvia Pscheidl. Without your regular support, this work would not have been completed.

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Contents

Introduction

Geometric Algebra (GA) is a powerful language capable of describing a wide range of physical phenomena [Doran and Lasenby, 2013, Doran, 1994, Lasenby et al., Lasenby and Doran, 2009, Hestenes, 2003b, 1986a]. Beyond its unifying capacity, GA can simplify descriptions and improve physical insights compared to conventional approaches. Traditional mathematical tools in General Relativity (GR), such as tensor calculus, often lead to cumbersome calculations and lack a direct geometric interpretation of physical quantities. GA offers an alternative with its compact, coordinate-free formalism, making it a promising tool for GR. Significant progress has already been achieved with GA in fields such as electromagnetism [Dressel et al., 2015], quantum mechanics [Doran et al., 1996], and quasicrystals [Hestenes and Holt, 2007], while its applications to GR remain relatively underexplored.

The initial attempts to apply GA to GR were made by Hestenes [2015, §21], who sought a direct translation of conventional GR formalism into GA, and by Lasenby et al. [1998], who developed a Gauge Theory of Gravity[1] within this framework. Hestenes' approach laid the groundwork by reformulating GR concepts using GA, while Lasenby's work took this further by describing gravity as a gauge field, which has implications for understanding GR through a new lens. Since then, however, limited progress has been made.

In this context, the aim of this work is to expand and deepen these initial efforts, using the tools of GA to improve our description of GR and extend the range of differential geometry tools available. In particular, this thesis focuses on practical applications and the physical interpretation of quantities within GA, while maintaining the GR framework. The formalism I present, which I term *tetrad-GA*, represents a natural integration of tetrads with

[1]Such theory has been criticized by Fernández and Rodrigues [2010, Appendix F] for containing a fundamental error in its formulation originating in a confusion between holonomic and non-holonomic indices, which makes it non compatible with GR.

GA, resulting in a framework particularly effective for describing GR. The specific contributions of this work are twofold: developing the tetrad-GA formalism and demonstrating its utility through applications to well-known physical systems, showcasing its advantages over conventional techniques.

Several reasons justify the use of tetrad-GA in GR. Among them, the following are particularly noteworthy:

- **Efficiency and simplification**: GA offers a compact mathematical formalism, where objects inherently reflect physical and geometric meanings, simplifying both notation and calculation. The unification of differential operators and the introduction of the geometric product result in shorter, more transparent calculations.
- **Coordinate-free expressions and covariant formulation**: Unlike tensor calculus, GA represents objects in contracted, "abstract" forms rather than component-wise expressions, e.g., representing a vector as $a = a^\mu g_\mu$ instead of merely its components a^μ. This feature facilitates a formulation of physical laws that connects physical objects directly, without reliance on specific coordinate systems.
- **Enhanced geometric interpretation**: The above points lead to a formalism that provides clearer geometric interpretations of equations than tensor calculus or differential forms, aiding in the understanding of physical phenomena.
- **Unified framework**: Working within a single mathematical framework enables the integration of diverse fields, such as electromagnetism and quantum mechanics, into a coherent structure.
- **Decoupling of frame and coordinate choices**: The use of tetrads allows meaningful interpretations of objects by working with orthonormal frames, which correspond to the frames of inertial observer. This approach separates the degrees of freedom associated with frame selection from those related to coordinate choice, revealing the intrinsic dependencies of objects and facilitating their analysis.

To explore this topic, this thesis is divided into six chapters. Chapter 1 presents the basics of GA and applies them to Minkowski spacetime and the description of electromagnetic phenomena. The goal here is not an exhaustive introduction to GA but rather to lay the groundwork and provide the essential tools necessary for understanding the subsequent chapters. Chapter 2 introduces the use of tetrads in GR and combines them with the GA of Minkowski spacetime established in the previous chapter. This chapter aims to rigorously present the mathematical foundation of the tetrad-GA formalism.

Chapter 3 serves as a bridge chapter, comparing GA with differential forms by highlighting their similarities and differences. Both frameworks employ the exterior algebra of Minkowski spacetime and share commonalities, yet GA introduces the geometric product, which allows for enhanced treatment of rotations and boosts, compact expressions, and improved geometric interpretation of objects. Differential forms, on the other hand, are particularly suitable for pre-metric geometry and benefit from extensive literature, which is a key challenge when dealing with advanced topics in GA.

In Chapter 4, I present the first practical application of our formalism by analyzing FRW spacetimes from various perspectives. Testing new tools on well-known systems allows us to evaluate their capabilities in comparison with traditional methods. This chapter begins with a metric-based analysis and progresses to exploring the relationship of FRW spacetimes with the Raychaudhuri equation, as well as their conformal properties and symmetries via the Lie derivative. The chapter concludes with an analysis of quintessence inflationary models.

In Chapter 5, I utilize GA's generalization of complex numbers to reformulate the Fermi-Walker transport equation in terms of rotors, yielding a powerful technique where the dynamics of a transported frame relate to the local frame via Lorentz transformations encoded in a rotor. The rotor-based analysis is simpler and more geometrically intuitive than component-based analysis. I illustrate these techniques with an example of a spaceship moving along an accelerated path near a wormhole's throat.

Finally, Chapter 6 investigates another application of the rotor-based techniques: the precession of gyroscopes around black holes. Beginning with the Schwarzschild case and extending to the Kerr-Newman black hole, I demonstrate how rotor-based calculations provide a straightforward method for determining the precession angle of gyroscopes. This approach involves solving a single first-order differential equation rather than a set of coupled partial differential equations, yielding equivalent results with clearer geometric insights.

In summary, this thesis aims to not only develop the mathematical tools of the tetrad-GA formalism but also to demonstrate its advantages and potential applications. Through specific examples, I show that this formalism is valuable for students and researchers alike, offering multiple benefits and a compelling alternative to conventional methods in GR and related fields. By highlighting its practical applications, I hope this work establishes tetrad-GA as a viable and valuable tool for advancing our understanding of gravitational phenomena and beyond.

Chapter 1

The Geometric Algebra of spacetime

In this chapter, I present an elementary construction of the Geometric Algebra (GA) of spacetime, along with several illustrative examples and common manipulations. For readers who are entirely new to the topic of GA, I recommend beginning with the series of pedagogical articles by Hestenes [2003b,a] and the more rigorous introduction by Macdonald [2010]. Conversely, readers who are already familiar with GA can explore more advanced discussions in [Macdonald, 2012, Doran and Lasenby, 2013, Hestenes and Sobczyk, 1987, Snygg, 2012, Kanatani, 2015], depending on their specific interests and preferred learning style.

Section 1.1 will present the constituting elements of the GA, which constitute the bases of this work. Then, in Section 1.2, I will use those tools to present the GA description of spacetime called Space-Time Algebra. I conclude the chapter with Section 1.3, in which I use electrodynamic theory as an example of a relativistic theory elegantly described by GA.

1.1 Introduction to GA

We denote the GA of a vector space $\mathcal{V}$ with dimension n and signature (p, q), where $p + q = n$, as $\mathcal{G}(p, q)$, and construct it as a direct sum of its exterior k-spaces for $k = 0, 1, \ldots, n$:

$$\mathcal{G}(p,q) = \bigoplus_{k=0}^{n} \bigwedge^{k}(\mathcal{V}). \tag{1.1.1}$$

This algebra is equipped with a bilinear form g between two vectors $u, v \in \mathcal{V}$, defined as

$$g(u, v) = u \cdot v, \tag{1.1.2}$$

which is known as the inner product of the space.

The dimension of $\mathcal{G}(p, q)$ is $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, and its elements are referred to as *multivectors*. A multivector M is a linear combination of k-vectors, which are elements of the k-th exterior power of $\mathcal{V}$, denoted as $\bigwedge^k(\mathcal{V})$:

$$M = \alpha + \beta^i e_i + \frac{1}{2!}\gamma^{ij} e_i \wedge e_j + \ldots + \frac{1}{n!}\delta^{ij\ldots n} e_i \wedge e_j \wedge \ldots \wedge e_n, \tag{1.1.3}$$

where $\{\alpha, \beta^i, \gamma^{ij}, \ldots\} \in \mathbb{R}$.

Each k-vector represents an oriented geometric object in the space. Therefore, $\mathcal{G}(p, q)$ can be understood as the space comprising all possible geometric elements of $\mathcal{V}$.

The number of k-vectors in a GA of dimension n is given by the binomial coefficient $\binom{n}{k}$, which can be visually represented using Pascal's triangle, as shown in Figure 1.1.

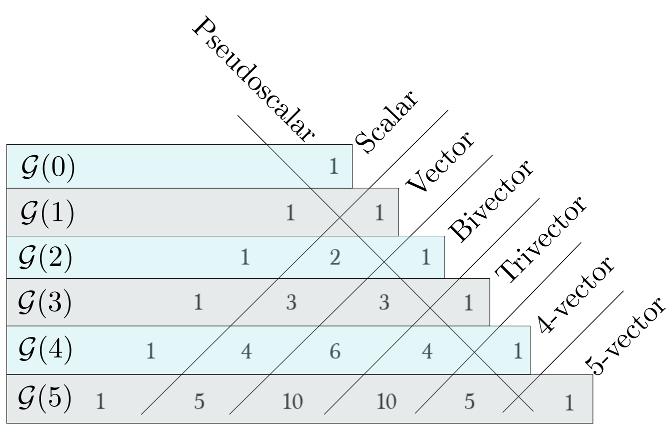


Figure 1.1: The number of k-vectors in a GA, $\mathcal{G}(p, q)$, of dimension $n = p + q$ is given by the binomial coefficient $\binom{n}{k}$. The element with the highest dimensionality in each space is called the pseudoscalar.

Because multivectors are composed of elements of different grades, certain equalities might initially appear confusing or incorrect. For instance, expressions like $e_i e_i = 1$ or $e_i e_j e_j e_k = e_i e_k$ may seem to equate elements of different grades. This confusion is resolved by recognizing that these equalities represent equivalence relations between multivectors, where null components are omitted.

The algebra $\mathcal{G}(p, q)$ is endowed with a bilinear operation, indicated by the absence of a symbol between vectors $u, v, w \in \mathcal{V}$, which satisfies the following properties:

- **Associativity**: $(uv)w = u(vw)$.
- **Left-distributivity**: $u(v + w) = uv + uw$.
- **Right-distributivity**: $(v + w)u = vu + wu$.
- **Relation to the inner product**: $u^2 = g(u, u)$.

This operation is known as the *geometric product* and can be expressed as the sum of the inner and outer products of two vectors $u, v \in \bigwedge^1(\mathcal{V})$:

$$uv := u \cdot v + u \wedge v. \tag{1.1.4}$$

Various aspects of this expression are noteworthy. First, $u \cdot v$ and $u \wedge v$ have different grades, 0 and 2, respectively. Second, the inner product and the outer product correspond to the symmetric and antisymmetric parts of the geometric product:

$$u \cdot v = \frac{1}{2}(uv + vu) \tag{1.1.5}$$

$$u \wedge v = \frac{1}{2}(uv - vu). \tag{1.1.6}$$

Third, the geometric product is neither commutative nor anticommutative. If the vectors are parallel ($u \propto v$), the geometric product reduces to the inner product and becomes commutative. Conversely, if the vectors are orthogonal ($u \cdot v = 0$), the geometric product reduces to the outer product and becomes antisymmetric. In general, the geometric product does not possess a defined symmetry.

It is important to note that the geometric product is initially defined by its action on vectors. Although generalizing this product to multivectors is non-trivial, it is possible; see [Macdonald, 2002] for details. However, in practice, this is often unnecessary because the associativity of the geometric product allows us to convert outer products into geometric products and then apply the associative property.

An essential characteristic of the geometric product, which is not shared by the inner or outer product, is its potential invertibility. For a non-null vector $a \in \bigwedge^1(\mathcal{V})$, where $a^2 = a \cdot a \neq 0$, we can define the inverse $a^{-1} \equiv a/a^2$ such that $aa^{-1} = 1$. The key properties enabling the existence of a^{-1} are the associativity of the geometric product and the possibility of adding elements of different grades in a multivector structure.

From a geometric standpoint, the inability to define an inverse using the inner or outer product arises because knowing solely the projection or area that two vectors span is insufficient to determine one, given the other. In contrast, Equation (1.1.4) encapsulates both element, thereby containing complete geometric information about their relative positions.

1.1.1 Conventions

To simplify expressions by removing unnecessary parentheses, I will adopt the common convention in GA of performing inner and outer products before geometric products. For instance, the following expressions are significantly simplified:

$$(a \cdot b)c = a \cdot bc, \quad (a \wedge b)I = a \wedge bI. \tag{1.1.7}$$

I will also use angle brackets to denote the projection into a specific grade, indicated by a subscript:

$$a \wedge b = \langle a \wedge b \rangle_2 = \langle ab \rangle_2, \quad a \cdot b = \langle ab \rangle_0 = \langle ab \rangle. \tag{1.1.8}$$

The scalar projection will be written without a subscript.

This notation sometimes offers the additional benefit of simplifying expressions by converting inner or outer products into geometric products, allowing us to utilize the associative property within the grade operation.

1.1.2 Complex numbers and rotor techniques

The use of the geometric product as the foundation of GA may initially appear to be a mere formal exercise. However, it significantly enhances the power of the formalism. Two notable examples are its application to complex numbers and rotations.

Let us begin with the GA of the Euclidean space $\mathbb{E}^2$, denoted by $\mathcal{G}(2)$. Consider an orthonormal basis $\{e_i\}$, from which we construct the k-vector basis $\{1, e_1, e_2, e_1 \wedge e_2\}$. We now calculate the square of the bivector $e_1 \wedge e_2$:

$$(e_1 \wedge e_2)^2 = e_1 e_2 e_1 e_2 = -(e_1 e_1)(e_2 e_2) = -1. \tag{1.1.9}$$

Here, we used the orthonormality of the basis to replace the outer product by a geometric product, reordered the expression, and then applied the associativity property to convert the geometric products into inner products.

Next, we can construct a general element of the even subalgebra of $\mathcal{G}(2)$, called $\mathcal{G}(2)^+$, as

$$z = \alpha + \beta e_1 e_2, \tag{1.1.10}$$

where $\alpha, \beta \in \mathbb{R}$. Given Equation (1.1.9) and the choice of notation, it is evident that z is isomorphic to the complex number $\alpha + \beta i$.

Furthermore, the bivector e_1e_2 can be used as a generator of rotations in the e_1e_2-plane. For example, right multiplication by e_1e_2 results in the expected $\pi/2$-rotations, analogous to multiplication by the complex unit:

$$e_1(e_1e_2) = e_2, \quad e_2(e_1e_2) = -e_1. \tag{1.1.11}$$

A rotation by an angle different that $\pi/2$ is achieved with an exponential function. We can define the exponential of a bivector using its series expansion, and construct an object called a *rotor* [Hestenes, 2003b]:

$$R(\theta) = \exp\left(-\frac{1}{2}\theta e_1 \wedge e_2\right) = \cos\frac{\theta}{2} - e_1 \wedge e_2 \sin\frac{\theta}{2}. \tag{1.1.12}$$

Because $|e_1 \wedge e_2| = 1$, the rotor R is normalized, meaning

$$RR^\dagger = 1. \tag{1.1.13}$$

Where the dagger symbol denotes the *reversion operation*, which reverses the order of all geometric products. Thus, $R^\dagger(\theta) = \exp\left(-\frac{1}{2}\theta e_2 \wedge e_1\right) = \exp\left(\frac{1}{2}\theta e_1 \wedge e_2\right)$.

The double-sided action of R and $R^\dagger$ on a multivector $M \in \mathcal{G}(\mathcal{V})$ generates a rotation of an angle θ in the $e_1 \wedge e_2$-plane:

$$M' = R(\theta)MR^\dagger(\theta). \tag{1.1.14}$$

Note that we have not imposed any restrictions on the dimension or signature of the space. Therefore, Equations (1.1.4, 1.1.14) are valid in spaces of any dimension and signature.

Moving to a higher dimension, we observe another remarkable isomorphism when examining the squares and permutation properties of the bivector basis generated by an orthonormal basis of $\mathbb{E}^3$:

$$e_1 \wedge e_2 \leftrightarrow i, \quad e_2 \wedge e_3 \leftrightarrow j, \quad e_3 \wedge e_1 \leftrightarrow -k. \tag{1.1.15}$$

Apart from a sign difference in k,[1] the (oriented) planes of $\mathbb{E}^3$ are isomorphic to the quaternions. This is a special case of a more general phenomenon in which the even subalgebra of a space provides a proper generalization of complex numbers.

The even subalgebras of GA provides a geometric interpretation that imaginary numbers inherently lack. Moreover, GA allows for the extension

[1]This sign discrepancy, originating from Hamilton's original left-handed basis, has caused considerable confusion.

of complex numbers and complex calculus theory to spaces of any dimension and signature [Hestenes and Sobczyk, 1987], granting access to powerful integral theorems that were previously achievable only through the complexification of spaces. This establishes a profound connection between complex numbers and geometry. We will further explore these topics in Sections 1.2.4 and 1.3 and chapter 6.

1.1.3 Pseudoscalar and duality

A k-vector with a grade equal to the dimension of the space, n, is called a pseudoscalar and we denote it by

$$e = \bigwedge_{k=1}^{n} e_k. \tag{1.1.16}$$

As the highest-grade element in the space, it is unique up to a sign and scaling, meaning that any n-volume V can be expressed as $V = \alpha e$, where $\alpha \in \mathbb{R}$.

Since the pseudoscalar is constructed from the coordinate basis, it is generally position-dependent, denoted as $e(x)$, and non-unitary, $e^2 \neq 1$. However, in orientable spaces, it is always possible to define a unit pseudoscalar I such that

$$I = \frac{e}{|e^2|}, \quad I^2 = \pm 1. \tag{1.1.17}$$

The pseudoscalar is of particular importance in GA because it determines the handedness of the space and enables the definition of the duality operation, which is performed in GA by right or left multiplication with the unit pseudoscalar $I = \bigwedge_{k=1}^{n} \hat{e}_k$, where $\hat{e}_k$ represents an orthonormal set of basis vectors.

The duality operation is equivalent to the Hodge dual of differential forms up to a sign, as explained in Section 3.2.1.

The commutation properties of the pseudoscalar depend on both the dimension of the other object and the dimension of the space: I commutes with odd k-vectors in spaces of odd dimension and anticommutes with them in spaces of even dimension. However, I always commutes with even k-vectors.

1.2 Space-Time Algebra

The GA of Minkowski spacetime was developed by Hestenes [2015], who coined the term Space-Time Algebra (STA). It has the metric $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$

and basis vectors $\{\gamma_\mu\}$.[2] These vectors satisfy the relationship:

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}. \tag{1.2.1}$$

The basis vectors are combined to construct the basis elements of $\mathcal{G}(1,3)$, as shown in Table 1.1.

Scalars	1					
4 Vectors	γ_0	γ_1	γ_2	γ_3		
6 Bivectors	γ_{10}	γ_{20}	γ_{30}	γ_{23}	γ_{31}	γ_{12}
4 Trivectors	γ_{123}	γ_{230}	γ_{310}	γ_{120}		
1 Pseudoscalar	γ_{0123}					

Table 1.1: Basis elements of $\mathcal{G}(1,3)$. The notation $\gamma_{\mu\nu} \equiv \gamma_\mu \wedge \gamma_\nu$ is used. Notice the dual relationship between scalars and pseudoscalars, vectors and trivectors, and between the first three bivectors and the last three bivectors.

The reciprocal basis, $\{\gamma^\mu\}$, is defined by the relationship:

$$\gamma_\mu \gamma^\nu = \delta_\mu^\nu. \tag{1.2.2}$$

Due to the orthonormality of the basis vectors, the reciprocal basis coincides with the coordinate basis, differing only in the signature:

$$\gamma^\mu = \eta^{\mu\nu}\gamma_\nu. \tag{1.2.3}$$

It is noteworthy to remark here that the reciprocal and coordinate bases span the same space. See Figure 2.2 for an example with non-orthonormal basis vectors.

In $\mathcal{G}(1,3)$, multivectors have $2^4 = 16$ components, and their general form is given by:

$$M = \alpha + v^\mu \gamma_\mu + \frac{1}{2} B^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + \frac{1}{6} T^\mu (I\gamma_\mu) + PI, \tag{1.2.4}$$

where $I = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3$ is the pseudoscalar. The sixteen elements correspond to one scalar, four vectors, six bivectors, four trivectors, and one pseudoscalar. The duality operation, explained in Section 1.1.3, is used to conveniently represent the trivector elements.

[2]This choice of notation for the basis vectors of the STA is non-standard. However, it is particularly suitable when considering the reformulation of Dirac theory developed by Hestenes [1975, 1997, 2003c], where the Dirac gamma matrices are isomorphic to the basis vectors of spacetime, and the pseudoscalar plays the role of γ_5.

1.2.1 Bivectors and rotations

Due to the mixed signature of the space, $\mathcal{G}(1,3)$ contains two types of basis bivectors: spatial bivectors, which have negative squares, $(\gamma_{ij})^2 = -1$, and spacetime bivectors, which have positive squares, $(\gamma_{0i})^2 = +1$. The notation $\gamma_{\mu\nu} \equiv \gamma_\mu \wedge \gamma_\nu$ is introduced here, and is valid only for orthogonal frames.

The bivectors of a space are the generators of its Lorentz group. In the STA case, with spatial bivectors generating spatial rotations and spacetime bivectors producing boosts.

To perform a Lorentz transformation on a multivector M, we consider the relevant bivector $\gamma_{\mu\nu}$ and apply the rotor techniques presented in Section 1.1.2. By sandwiching M between the corresponding rotor R we obtain the rotated multivector M', as illustrated in Figure 1.2:

$$M' = RMR^\dagger = e^{-\frac{\theta}{2}\gamma_{\mu\nu}} M e^{\frac{\theta}{2}\gamma_{\nu\mu}}, \tag{1.2.5}$$

where θ is the parameter of the transformation, and R is given by:

$$R = e^{-\frac{\theta}{2}\gamma_{\mu\nu}} = \cos\frac{\theta}{2} - \gamma_{\mu\nu}\sin\frac{\theta}{2}. \tag{1.2.6}$$

For example, a boost along the γ_1 axis with velocity $\vec{v} \in \mathbb{R}^3$ in the direction of γ_1 is calculated as follows:

$$\begin{aligned} \gamma_1' &= e^{-\gamma_{01}\alpha/2}\gamma_1 e^{\gamma_{01}\alpha/2} \\ &= e^{-\gamma_{01}\alpha}\gamma_1 = \cosh\alpha\,\gamma_1 + \sinh\alpha\,\gamma_0, \end{aligned} \tag{1.2.7}$$

where $\alpha = \tanh^{-1}(|\vec{v}|/c)$, and we have used the anticommutativity of γ_1 with γ_{01}.

The exponential treatment of boosts and rotations is analogous to quaternionic rotations in $\mathbb{E}^3$, providing simpler operational rules when compared to the 4×4 matrices from conventional tensor treatment, especially when dealing with their composition.

Note that the decomposition of the geometric product into scalar and bivector parts is the key element in the construction of rotors. This property distinguishes GA from other formalisms, such as differential forms, and is a major source of its powerful capabilities.

1.2.2 Time-Split

A remarkable feature of STA is the nested structure of subalgebras it generates[3], see Table 1.2. The even subalgebra of $\mathcal{G}(1,3)$, denoted as $\mathcal{G}(1,3)^+$,

[3] In most GA literature, the signature $(+,-,-,-)$ is used to define STA. This choice is motivated by the natural emergence of nested subalgebra structures under this signature

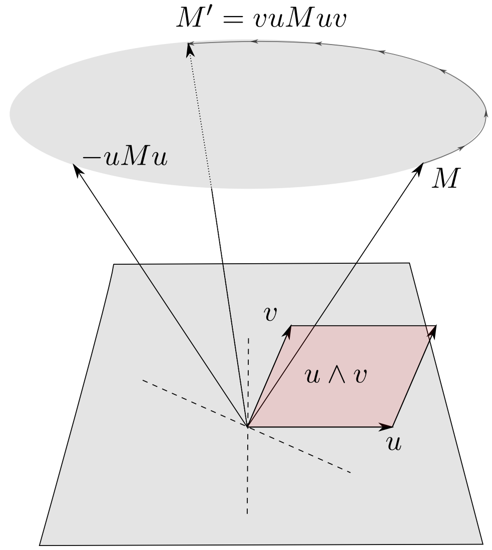


Figure 1.2: The rotation of the multivector M with the bivector $u \wedge v$ is performed by reflecting M with respect to the plane perpendicular to u and then reflecting the resulting vector with respect to the plane perpendicular to v (Equation (1.2.5)). This construction easily demonstrates that two reflections are equivalent to a rotation. See [Hestenes, 2003b] for details.

is isomorphic to $\mathcal{G}(3)$ by identifying the spacetime bivectors of $\mathcal{G}(1,3)$ with the basis vectors of $\mathcal{G}(3)$:

$$\boldsymbol{\sigma}_i \equiv \gamma_i \wedge \gamma_0. \tag{1.2.8}$$

This identification enables the so-called *time-split* of a vector p. Consider an observer moving with a 4-velocity given by the time-like basis vector γ_0. By right-multiplying the abstract vector p by the observer's 4-velocity, we obtain the projection of p into the observer's frame:

$$p\gamma_0 = \rho + p^i \gamma_i \wedge \gamma_0 = \rho + \vec{p}. \tag{1.2.9}$$

Where we used the arrow to denote the 3-dimensional components of p, $\vec{p} = p^i \boldsymbol{\sigma}_i$.

The identification of the observer's frame $\{\boldsymbol{\sigma}_i\}$ with the bivectors $\{\gamma_i \wedge \gamma_0\}$ explicitly shows a dependence on the 4-velocity of the observer, which is not present in conventional tensor calculus.

[Dressel et al., 2015]. However, Wu [2022] has shown that a similar structure can be obtained for the opposite signature by making a different isomorphism between the even subalgebra $\mathcal{G}(3,1)^+$ and $\mathcal{G}(3)$.

$Cl_{1,3}$	Space-Time Algebra
$Cl_{3,0}$	Relative 3-Space ($\mathbb{E}^3$)
$Cl_{0,2}$	Quaternions
$Cl_{0,1}$	Complex Numbers
$Cl_{0,0}$	Real Numbers

Table 1.2: Clifford Ladder. Each algebra is isomorphic to a subalgebra of the higher algebra.

1.2.3 Vector derivative operator

The calculus tools based on GA are known as Geometric Calculus (GC) and were developed by Hestenes and Sobczyk [1987]. In this section, I will review the main derivative operations, and in the next, I will present the primary integral theorems.

The fundamental derivative operator in $\mathcal{G}(1,3)$ is the *vector derivative*,[4] denoted by ∇. It is defined in terms of the reciprocal basis $\{\gamma^\mu\}$ and the directional derivatives $\partial_\mu = \partial/\partial x^\mu$ as

$$\nabla = \gamma^\mu \partial_\mu. \tag{1.2.10}$$

A crucial feature of ∇ is that it possesses the algebraic properties of a vector, allowing it to be treated as such in operations. This implies that the algebraic rules governing interactions between vectors and multivectors also apply to ∇, resulting in a wide range of differential properties.

For example, we can derive familiar differential operations by applying it with the inner and outer products to a vector field $v = v(x) = v^\mu(x)\gamma_\mu$:

- **Geometric derivative**: $\nabla v = \nabla \cdot v + \nabla \wedge v$.
- **Divergence**: $\nabla \cdot v = \partial_\mu v^\mu$.
- **Curl/Exterior derivative**:

$$\nabla \wedge v = \partial_\nu v^\mu \gamma^\nu \wedge \gamma_\mu$$

- **Laplace-Beltrami Operator**:

$$\nabla^2 v = \nabla(\nabla v) = \nabla \cdot \nabla v = \partial^\mu \partial_\mu v^\nu \gamma_\nu.$$

For more algebraic differential identities and their corresponding counterparts in vector calculus and differential forms, see Tables 3.1 to 3.3.

[4] This notation is chosen due to its correspondence with Dirac theory, where it is referred to as the Dirac operator.

Two particularly noteworthy aspects of these expressions are the constructions of the curl and the Laplace-Beltrami operator. In the case of the curl, the use of the outer product enables a consistent definition in spaces of any dimension. For the Laplace-Beltrami operator, the associativity of the geometric product allows ∇ to act on itself, forming a scalar operator capable of acting on any multivector field:

$$\begin{aligned}\nabla(\nabla M) &= (\nabla)^2 M = (\nabla\cdot\nabla)M \\ &= \partial^\mu\partial_\mu\alpha + (\partial^\mu\partial_\mu\beta^i)e_i + \frac{1}{2}(\partial^\mu\partial_\mu\gamma^{ij})e_i\wedge e_j + \dots\end{aligned} \tag{1.2.11}$$

Here, $\nabla\wedge\nabla = 0$ due to the commutativity of partial derivatives.

The ability to derive all differential operators from a single one is unique to Geometric Calculus. It greatly simplifies calculations, enhances the geometric interpretation of equations, and unifies algebraic and differential identities. For more details, see [Hestenes and Sobczyk, 1987, Ch. 2].

Finally, it is worth mentioning that ∇ has an inverse operator in the form of a first-order Green's function, analogous to the inverse of a vector. This property is fundamentally tied to the geometric product. Because it will not be necessary for the remainder of this work I will note expand further, but the interested reader can find the mathematical details in [Hestenes and Sobczyk, 1987, §7-3], and an elegant application to Dirac's and Maxwell's equations in [Gull et al., 1993].

1.2.4 Fundamental theorem of Geometric Calculus

Integration in GC offers two primary advantages over traditional vector calculus: directed integration and compact expressions. The first, directed integration, is a feature shared with differential forms, allowing significant simplification when dealing with geometric, oriented elements. The second advantage, unique to GC, is the use of the geometric product to combine multiple integral theorems into a single expression called *Fundamental theorem of Geometric Calculus.*

The fundamental theorem of geometric calculus is stated as follows by Macdonald [2012] and Hestenes [2015]: *Consider a closed p-dimensional surface*[5]*, Ω, bounded by the $(p-1)$-dimensional surface $\partial\Omega$. We can represent a differential oriented element of Ω by the p-vector $\mathrm{d}x_p$, and the differential oriented element of the boundary by $\mathrm{d}x_{p-1}$ (see Figure 1.3). If M is a*

[5] Non-necessarily simply-connected.

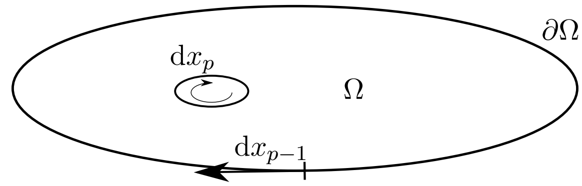


Figure 1.3: Depiction of a p-dimensional surface Ω with oriented differential element $\mathrm{d}x_p$, bounded by the $(p-1)$-dimensional surface $\partial\Omega$ with oriented differential element $\mathrm{d}x_{p-1}$.

differentiable multivector function on Ω, then we can write

$$\int_\Omega \mathrm{d}x_p \cdot \nabla M = \oint_{\partial\Omega} \mathrm{d}x_{p-1} M. \tag{1.2.12}$$

Notice that $\mathrm{d}x_p$ is a p-vector, which can be decomposed as the outer product of differential vectors $\mathrm{d}x^{(i)} = \mathrm{d}x^i \hat{e}_i$:

$$\mathrm{d}x_p = \mathrm{d}x^{(1)} \wedge \ldots \wedge \mathrm{d}x^{(p)} = \mathrm{d}x^1 \ldots \mathrm{d}x^p I^{(p)}, \tag{1.2.13}$$

where $I^{(p)} = \hat{e}_1 \wedge \ldots \wedge \hat{e}_p$ is the unitary pseudoscalar of the surface Ω, and $\mathrm{d}x^i$ are the scalar integration elements. The same decomposition applies for $\mathrm{d}x_{p-1}$.

In Equation (1.2.12), $\nabla = \sum_{i=1}^p \hat{e}^i \partial_i$ is the vector derivative in Ω. It is important to note that $\mathrm{d}x_p \wedge \nabla = 0$ because $\mathrm{d}x_p$ has the same dimension as Ω, and therefore $\mathrm{d}x_p \cdot \nabla = \mathrm{d}x_p \nabla$. The inner product with the vector ∇ lowers the grade of $\mathrm{d}x_p$ by one, making $\mathrm{d}x_p \cdot \nabla$ of the same grade as $\mathrm{d}x_{p-1}$, which is essential for Equation (1.2.12) to hold.

The fundamental theorem of geometric calculus applies to manifolds of any dimension, admits the presence of holes, and applies to non-scalar geometric objects. For more details, see [Macdonald, 2012, Part IV].

In Chapter 3, I will provide a detailed comparison between differential forms and GC. However, it is worth noting here the key differences between Equation (1.2.12) and the generalized Stokes' theorem of differential forms:

- In Equation (1.2.12), the product between $\mathrm{d}x_p \cdot \nabla$ and M is a geometric product, which decomposes in two parts corresponding to the generalized Stokes' theorem and its dual.
- M is not restricted to being a scalar-valued function, as in differential forms. It is a general map from multivectors to multivectors. This allows Equation (1.2.12) to incorporate Cauchy's integral theorem.

The simplicity of Equation (1.2.12) may obscure its power. To illustrate it, we will apply it to various manifolds to derive the fundamental theorem of calculus, the Divergence theorem, Curl theorem, Green's theorem, and Cauchy's integral theorems of complex analysis, as demonstrated by Macdonald [2012, Part IV].

Fundamental Theorem of Calculus If $p = 1$, Ω is a line and $\mathrm{d}\omega \to \mathrm{d}\vec{s}$ is a vector. M is a scalar function, $\partial\Omega$ represents the endpoints of the segment, and integration over them consists of multiplying by 1 or -1, depending on the orientation of the segment. Thus, Equation (1.2.12) reduces to the standard fundamental theorem of calculus:

$$\int_{x_1}^{x_2} \mathrm{d}\vec{s} \cdot M = M(x_2) - M(x_1). \tag{1.2.14}$$

Divergence Theorem Let M be a vector field v on a bounded p-dimensional manifold Ω in $\mathbb{R}^p$. Setting $\mathrm{d}\sigma = \vec{n}\mathrm{d}_{p-1}x$, Equation (1.2.12) becomes:

$$\int_\Omega \nabla \cdot v \, \mathrm{d}_p x = \oint_{\partial\Omega} v \cdot \mathrm{d}\sigma. \tag{1.2.15}$$

In a $\mathbb{E}^3$, this corresponds to Gauss' theorem. However, this only covers the inner product part of Equation (1.2.12). The outer product part, along with the dual operation, relates outer and cross products in $\mathbb{E}^3$, yielding:

$$\iiint_V \nabla \times v \, \mathrm{d}V = \oiint_S \mathrm{d}\sigma \times v. \tag{1.2.16}$$

Curl Theorem Let M be a $(p-1)$-vector valued field on an oriented, bounded p-dimensional manifold $\Omega \in \mathbb{R}^n$, where $p \leq n$. Then, Equation (1.2.12) is:

$$\int_\Omega \mathrm{d}_p x \cdot (\nabla_p \wedge M) = \oint_{\partial\Omega} \mathrm{d}_{p-1}x \cdot M. \tag{1.2.17}$$

In a 3-dimensional manifold, with a 2-dimensional surface Ω, differential area element $\mathrm{d}\sigma$, boundary $\partial\Omega$ as the enclosing curve, and M as a vector field v, this becomes the Curl theorem, commonly known as *Stokes' theorem*:

$$\iint_S (\nabla \times v) \cdot \mathrm{d}\sigma = \oint_C v \cdot \mathrm{d}s. \tag{1.2.18}$$

Green's Theorem Let R be a region in the xy-plane with boundary C oriented counterclockwise. Let M be a vector field $P(x,y)\hat{x}+Q(x,y)\hat{y}$ on R. Then:

$$\iint_R (\partial_x Q - \partial_y P)\,\mathrm{d}A = \oint_C (P\mathrm{d}x + Q\mathrm{d}y). \tag{1.2.19}$$

Cauchy's Theorem The concept of analytic functions in the complex plane can be generalized in GC to multivector fields M that satisfy the condition:

$$\nabla M = 0. \tag{1.2.20}$$

For such analytic functions, Equation (1.2.12) simplifies to:

$$\oint_{\partial\Omega} \mathrm{d}x_{p-1} M = 0, \tag{1.2.21}$$

which is called the *Generalized Cauchy's Theorem*.

As shown by Hestenes [1986a], GC also encompasses the *residue theorem*: If M is analytic except at a pole x' with residue q in the 2-dimensional surface $\mathcal{A}$, then $\nabla M = 2\pi q\delta(x-x')$, and substituting into Equation (1.2.12) yields:

$$\oint \mathrm{d}xM = 2\pi q \int_{\mathcal{A}} e_1 \wedge e_2 |\mathrm{d}A(x)|\delta(x-x') = 2\pi e_1 \wedge e_2 q. \tag{1.2.22}$$

Note how $e_1 \wedge e_2$ plays the role of the imaginary unit i.

Details on the derivation of Cauchy's theorem for integral functions can be found in [Dressel et al., 2015, §4], [Doran and Lasenby, 2013, Ch. 4], and [Hestenes and Sobczyk, 1987, §7.3]. However, we would like to highlight three key points about this result:

- The theorem is derived without introducing complex numbers, as the complex plane is isomorphic to $\mathcal{G}(2)$.
- The result is applicable to spaces of arbitrary dimension.
- The theorem has a clear geometric interpretation.
- GC unifies the theory of poles and residues with that of Green's and delta functions.

The unifying power of the fundamental theorem of geometric calculus is unique to GC, providing a clear geometric interpretation of most integral theorems.

1.3 Electrodynamics

Geometric calculus offers a powerful framework for describing the laws of electromagnetism, as shown by Dressel et al. [2015]. In GA, the electromagnetic field is represented by the bivector field $F(x) \equiv F$, which can be expressed in components as follows:

$$\begin{aligned} F &= F^{\mu\nu}\gamma_{\mu\nu} \\ &= E^1\gamma_{10} + E^2\gamma_{20} + E^3\gamma_{30} + \left(B^1\gamma_{10} + B^2\gamma_{20} + B^3\gamma_{30}\right) I. \end{aligned} \tag{1.3.1}$$

The components $F^{\mu\nu}$ correspond to the entries of the Faraday electromagnetic tensor $\mathcal{F}^{\mu\nu}$, and I used the duality operation to express spatial bivectors as duals of spacetime bivectors. This formulation relates the spacetime components of the electromagnetic bivector to the electric field $F^{i0} \leftrightarrow E^i$ and the spatial components to the magnetic field $F^{ij} \leftrightarrow B^i$. By encoding the electromagnetic field in a bivector field, we can immediately identify its tracelessness and antisymmetric nature.

The time-split of F into the observer's frame[6] with 4-velocity γ_0 produces the $\vec{E}$ and $\vec{B}$ fields as measured in the local frame $\{\boldsymbol{\sigma}_i\}$:

$$F = E^1\boldsymbol{\sigma}_1 + E^2\boldsymbol{\sigma}_2 + E^3\boldsymbol{\sigma}_3 + \left(B^1\boldsymbol{\sigma}_1 + B^2\boldsymbol{\sigma}_2 + B^3\boldsymbol{\sigma}_3\right) I = \vec{E} + \vec{B}I. \tag{1.3.2}$$

In this 3-dimensional subspace, the electromagnetic bivector F decomposes into a vectorial electric field and the dual of the magnetic field. This vector-bivector representation of the electric and magnetic fields explains their different behavior under parity transformations and clarifies the classical distinction between axial and polar vectors.

Furthermore, noting that $I^2 = -1$ in Minkowski spacetime, we recognize that Equation (1.3.2) is precisely the Riemann-Silberstein vector, $\vec{E} + i\vec{B}$. This complex vector form of the electromagnetic field simplifies the treatment of certain electromagnetic systems and aids in the formulation and interpretation of quantum electrodynamics (QED) [Silberstein, 1907, Bialynicki-Birula and Bialynicka-Birula, 2013]. This underlying "complex" structure is not apparent when using the tensor components $\mathcal{F}^{\mu\nu}$ or the Faraday two-form $\mathbf{F}$.

We now proceed to derive Maxwell's equations in GA as the simplest differential equation for a bivector field:

$$\nabla F = \jmath, \tag{1.3.3}$$

where $\jmath$ is a multivector composed of grades 1 and 3, resulting from the geometric product between a "vector" and a bivector.

[6] Since F is a bivector, it does not need to be right-multiplied by γ_0 to decompose it into the observer's frame.

To show that Equation (1.3.3) is equivalent to the standard Maxwell equations, let's expand its left-hand side using the time-split of the vector derivative $\nabla = \gamma_0(\partial_0 + \vec{\nabla})$ and the time-split of F, as given in Equation (1.3.2):

$$\begin{aligned} \nabla F &= \gamma_0(\partial_0 + \vec{\nabla})(\vec{E} + \vec{B}I) \\ &= \gamma_0 \left[\vec{\nabla} \cdot \vec{E} + \partial_0 \vec{E} - \vec{\nabla} \times \vec{B}\right] \\ &+ \gamma_0 \left[\vec{\nabla} \cdot \vec{B} + \partial_0 \vec{B} + \vec{\nabla} \times \vec{E}\right] I. \end{aligned} \tag{1.3.4}$$

Where $\times$ means the usual cross-product which we obtained by using the algebraic property between vectors in $\mathbb{E}^3$: $a \times b = -I(a \wedge b)$. Therefore $\vec{\nabla} \times \vec{B}$ is the 3-dimensional curl of B.

For the right-hand side of Equation (1.3.3), we write $\jmath$ as the sum of a vector field $\jmath_e$ and a trivector field $\jmath_m I$:

$$\nabla F = \nabla \cdot F + \nabla \wedge F = \jmath_e + \jmath_m I. \tag{1.3.5}$$

The vector fields $\jmath_e$ and $\jmath_m$ can be decomposed into the observer's frame as:

$$\begin{aligned} \jmath_e &= c\rho_e\gamma_0 + j_e^i\gamma_i = (c\rho_e + \vec{J}_e)\gamma_0, \\ \jmath_m &= c\rho_m\gamma_0 + J_m^i\gamma_i = (c\rho_m + \vec{J}_m)\gamma_0. \end{aligned} \tag{1.3.6}$$

The time component of $\jmath_e$ and $\jmath_m$ represents the electric and magnetic charges, respectively, while the spatial components correspond to their currents. If we set $\jmath_m = 0$, thereby assuming no magnetic monopoles, and match terms by their grade between Equation (1.3.4) and Equation (1.3.6) we obtain the non-homogeneous Maxwell equations in the local observer's frame:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho_e, \\ \partial_0 \vec{E} - \vec{\nabla} \times \vec{B} &= \vec{J}_e, \\ \vec{\nabla} \cdot \vec{B} &= 0, \\ \partial_0 \vec{B} + \vec{\nabla} \times \vec{E} &= 0. \end{aligned} \tag{1.3.7}$$

The constants in SI units can be recovered by setting:

$$\begin{aligned} \vec{E} &\mapsto \sqrt{\epsilon_0}\, \vec{E}, \\ \vec{B} &\mapsto \vec{B}/\sqrt{\mu_0}, \\ \left(\rho_e, \vec{J}_e\right) &\mapsto \sqrt{\mu_0} \left(\rho_e, \vec{J}_e\right), \\ \left(\rho_m, \vec{J}_m\right) &\mapsto \sqrt{\mu_0}/c \left(\rho_m, \vec{J}_m\right). \end{aligned} \tag{1.3.8}$$

Equation (1.3.3) is not just an aesthetic exercise. The invertibility of ∇ allows us to directly obtain the fields from the sources without involving second-order derivatives, providing an expression where Huygens' principle of re-radiation is directly applicable and evident, and without ambiguities for the obliquity factor; see Gull et al. [1993] and Doran and Lasenby [2013, §7.5] for details.

The correct expressions for the interaction between charged particles and the field emerge naturally by contracting the bivector field F with the 4-current of a particle $\jmath = qu$, where q is the charge of the particle and u is its 4-velocity:

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}\tau} &= F \cdot \jmath \\ &= (\vec{E} \cdot \vec{\jmath})\gamma_0 + \left[q\vec{E} + \vec{\jmath} \times \vec{B}\right]\gamma_0. \end{aligned} \tag{1.3.9}$$

Here, $p = \xi\gamma_0 + p^i\gamma_i$ is the 4-momentum of the particle, and $\vec{\jmath} = q\vec{u}$ is its 3-velocity. The first term, $(\vec{E} \cdot \vec{\jmath})\gamma_0$, represents the power of the interaction, i.e., the rate at which energy is transferred from the electromagnetic field to the particle, and the second term is the Lorentz force

This compact and elegant formulation extends to other aspects of electromagnetism, such as potentials and the Poynting flux among others. A comprehensive and precise exposition of electromagnetism using GA can be found in [Dressel et al., 2015]. The first pages provide an impressive list of 33 characteristic advantages of using geometric calculus to describe electromagnetism.

Finally, it's worth mentioning that Gull et al. [1993] also showed that the GA formulation of Maxwell's equations can be mathematically solved as a particular case of Dirac's equation. This is the case because in the Dirac-Hestenes formulation of Dirac's theory, a spinor is described as an even element of $\mathcal{G}(1,3)$, composed by one scalar, six bivectors and one pseudoscalar, and the Dirac-Hestenes equation is mathematically similar to Equation (1.3.3)-illustrating the benefits of having a unified formalism for physics, where the same mathematical techniques can be applied to study different phenomena.

1.4 Conclusions

In this chapter, I introduced the basics of GA for a vector space. We explored its key features, such as the geometric product, the geometric interpretation of k-vectors, their generalization of complex numbers, and their simplified approach to handling rotations. For the sake of brevity and focus, I only covered those elements essential for the subsequent chapters of this work. However, GA encompasses many other fascinating and relevant aspects, and interested readers are encouraged to consult the referenced literature for a more comprehensive understanding.

I specifically focused on the GA of Minkowski spacetime, known as STA, where bivectors serve as generators for rotations and boosts, and the time-split operation plays a crucial role.

I then provided an overview of GC, including derivative and integral techniques that stem from GA. The vector derivative operator is central to these techniques, and many integral theorems traditionally seen in the literature are elegantly unified under the fundamental theorem of geometric calculus.

Lastly, I demonstrated the power of STA by reviewing the classical formulation of the laws of electromagnetism within the GA framework.

In the next chapter, I will extend these techniques to the tangent space of manifolds, developing a local STA that will enable us to apply the mathematical methods discussed here to a broader range of geometrical and physical problems.

Chapter 2

Space-time Algebra in curved space times

The extension of the techniques from Chapter 1 to curved manifolds is achieved by constructing a STA at every tangent space $T_p\mathcal{M}$. The basis vectors for $T_p\mathcal{M}$ are typically the generally non-orthonormal coordinate basis vectors. Although GA can handle these non-orthonormal bases effectively, introducing orthonormal frames (tetrads) to describe manifolds is particularly advantageous. It simplifies expressions and calculations, enhances the interpretation of quantities, and, as we will demonstrate, aligns seamlessly with GA techniques.

Therefore, I dedicate Section 2.1 to introduce tetrads, and integrate them in Section 2.2 with the GA formalism. Once the basic construction has been set up, I proceed by deriving the essential elements of differential geometry, such as the covariant vector operator, Section 2.3, the Riemann, Ricci and energy-moementum tensors, Section 2.4. Forming what I refer to as the *tetrad-GA* formalism. I conclude in Section 2.5 with a formulation of Einstein's equations in GA. The results in this chapter were originally presented in [Pérez and DeKieviet, 2024a].

2.1 The Lorentzian Spacetime

From this point forward, we will consider $\mathcal{M}$ to be a 4-dimensional, real-valued, differentiable manifold, with its *tangent space* at each point p denoted by $T_p\mathcal{M}$. The elements of $T_p\mathcal{M}$ are called *vectors*, and the union of all tangent spaces forms the *tangent vector bundle* $T\mathcal{M} = \bigcup_{p\in\mathcal{M}} T_p\mathcal{M}$. The sections of $T\mathcal{M}$ are referred to as *vector fields*.

We denote by $\bigwedge T\mathcal{M} = \bigoplus_{i=0}^{4} \bigwedge^i T\mathcal{M}$ the direct sum of exterior algebras over the tangent spaces, which we call the *bundle of multivector fields*.

It is useful to note that $\bigwedge^0 T_p\mathcal{M} = \mathbb{R}$, $\bigwedge^1 T_p\mathcal{M} = T_p\mathcal{M}$, and that the dimension of each exterior algebra of the tangent space is given by $\dim \bigwedge^i T_p\mathcal{M} = \binom{4}{i}$.

We equip $\mathcal{M}$ with a Lorentzian metric g, which is a bilinear map $g : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ with signature $(1, 3)$, and we refer to the pair $(\mathcal{M}, g)$ as a *Lorentzian manifold*.

We extend this Lorentzian manifold to a *Lorentzian spacetime* by adding the following structures to the pair $(\mathcal{M}, g)$: a linear connection D satisfying $Dg = 0$ (the metric compatibility condition), an oriented volume element $I \in \sec \bigwedge^4 T\mathcal{M}$, and an oriented time direction $\uparrow$. Altogether, these define the *Lorentzian spacetime*, denoted by the pentuple $(\mathcal{M}, g, D, I, \uparrow)$, within which we will work.

We will also assume that the torsion of the connection D vanishes, aligning our framework with GR rather than the Einstein-Cartan theory.

2.1.1 Coordinates and coordinate frames

As it is customary, we consider (U, φ) to be a coordinate chart of the maximal atlas $\mathcal{A}$ of $\mathcal{M}$, where U is an open set of $\mathcal{M}$, and φ is a differentiable mapping from U to an open set of $\mathbb{R}^4$. We denote the coordinate functions of (U, φ) by $x^\mu : U \to \mathbb{R}$, with $\mu = 0, 1, 2, 3$ (Figure 2.1).

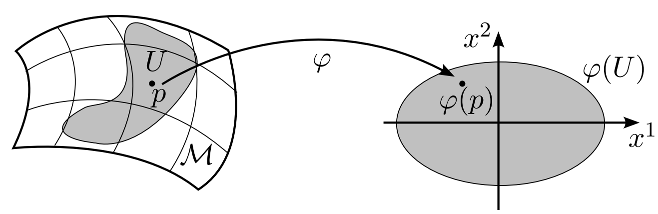


Figure 2.1: Two-dimensional depiction of a manifold $\mathcal{M}$ with an open subset U and a coordinate map $\{U, \varphi\}$. Each point $p \in U$ maps to a point $\varphi(p) \in \varphi(U) \subset \mathbb{R}^2$.

From the coordinate chart, we obtain a particularly important type of vector field, which forms a basis of the subbundle $TU \subset T\mathcal{M}$, called the *coordinate basis vector*. A set $\{g_\mu\} \in \sec TU$, with $\mu = 0, 1, 2, 3$, is called a coordinate basis vector of TU if there exists a coordinate chart (U, φ) and coordinate functions x^μ such that, for each differentiable function $f : \mathcal{M} \to$

$\mathbb{R}$,

$$g_\mu(f)\big|_p = g_\mu \cdot \nabla f\big|_p = \frac{\partial}{\partial x^\mu}\left(f \circ \varphi^{-1}\right)\bigg|_{x=\varphi(p)} = \frac{\partial f(x)}{\partial x^\mu}. \tag{2.1.1}$$

It is common practice to identify g_μ with ∂_μ, but to maintain a clear distinction between vectors and operators, we will avoid this identification.

From the coordinate basis vector, we can uniquely define a frame called the *reciprocal coordinate frame* (Figure 2.2), which we denote by $\{g^\mu\} \in \sec TU$. It is defined by the relationship:

$$g_\mu \cdot g^\nu = g(g_\mu, g^\nu) = \delta_\mu^\nu. \tag{2.1.2}$$

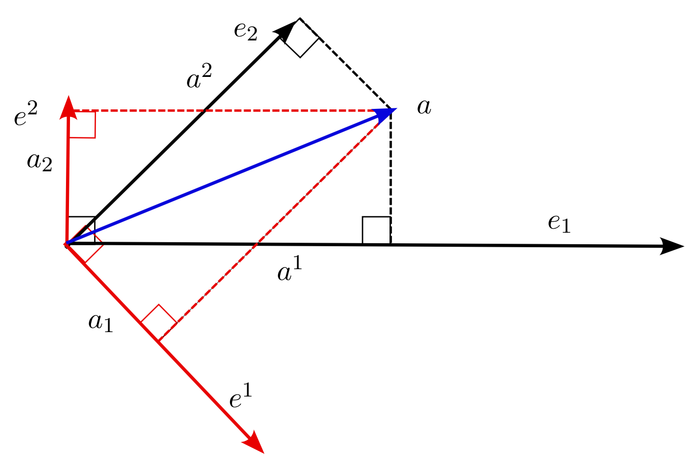


Figure 2.2: A non-orthonormal basis of $\mathbb{R}^2$, denoted by $\{e_i\}$, and its reciprocal basis, denoted by $\{e^i\}$, are depicted in black and red, respectively. Both bases are related by an orthonormality condition $e_i \cdot e^j = \delta_i^j$. A vector a is shown in blue, with components a^i in the $\{e_i\}$ basis, and components a_i in the reciprocal $\{e^i\}$ basis. Note that both bases span the same physical space.

At this point, one could introduce the dual space, dual basis, and their corresponding reciprocal frame, as done in other works such as [Rodrigues and de Oliveira, 2007]. However, since the presence of the metric induces a musical isomorphism relating the dual and tangent spaces, we can forgo this distinction and focus solely on the coordinate and reciprocal coordinate bases without losing generality or limiting the formalism. This decision follows the principle of parsimony in the construction of this framework.

The coordinate basis vectors $\{g_\mu\}$ are generally non-orthonormal, and their inner product gives the components of the metric g in the particular coordinate chart we are using:

$$g_\mu \cdot g_\nu|_x = g(g_\mu, g_\nu)|_x = g_{\mu\nu}(x). \tag{2.1.3}$$

Similar considerations apply to the reciprocal basis, producing the components of the inverse metric:

$$g^\mu \cdot g^\nu|_x = g(g^\mu, g^\nu)|_x = g^{\mu\nu}(x). \tag{2.1.4}$$

For simplicity, we will omit the x-dependence of the metric in our notation.

It is straightforward to show that the components of the metric and the inverse metric satisfy the condition:

$$g_{\mu\nu} g^{\mu\sigma} = \delta_\nu^\sigma. \tag{2.1.5}$$

2.1.2 Tetrads

From the coordinate basis vectors $\{g_\mu\}$, it is possible to perform an orthonormalization procedure[1,2] to obtain an orthonormal frame $\{\gamma_m\} \in \sec TU$, with $m = 0, 1, 2, 3$, called a tetrad. Physically, a tetrad represents the frame of reference of a local, inertial (free-falling) observer at point p, as shown in Figure 2.3.

The choice of tetrad is not uniquely determined by the coordinate frame, as there are infinitely many tetrads at each point p, related by local Lorentz transformations. In this work, I will restrict myself to right-handed tetrads with a positive time-orientation, all related by proper orthochronous Lorentz transformations. This choice ensures consistency in the orientation and time direction of the spacetime across different observers.

It is worth noting that in the gravitational literature, there exists a set of tetrads that do not satisfy these requirements. These are used in the Newman-Penrose formalism and are characterized by having a null "time" component (other null components are also possible). Such frames are very useful for describing gravitational radiation, but they will not be considered in this work see [Newman and Penrose, 1962, Hamilton, 2020] for details.

[1] The extension of the Gram–Schmidt procedure to pseudo-Riemannian spaces depends on the existence of a non-degenerate basis for the vector space. See Lee for comments on his book [Lee, 2019, p.30].

[2] Hestenes and Sobczyk [1987, §1-3] presents the Gram-Schmidt orthonormalization procedure in GA. However, due to the simplicity of the cases presented in this work, it is not necessary to resort to the general procedure to obtain an orthonormal basis.

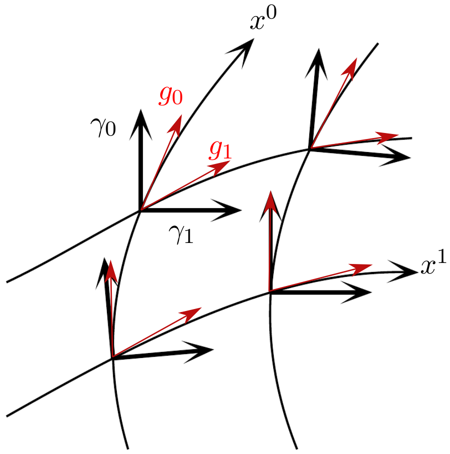


Figure 2.3: Manifold mapped with coordinates (x^1, x^2) and two bases for the tangent space at each point: in red, the coordinate tangent vectors $\{g_\mu\}$; in black, an orthonormal set of basis vectors $\{\gamma_m\}$, forming a tetrad.

Each choice of tetrad allows inertial observers to define four local coordinate maps $\mathbf{x}^m$, forming an *adapted coordinate system*, such that for a given differentiable function $f(x) : \mathcal{M} \to \mathbb{R}$,

$$\gamma_m \cdot \nabla f(x) = \frac{\partial}{\partial \mathbf{x}^m} f(x). \tag{2.1.6}$$

The components of the metric in the adapted coordinate system are given by

$$\gamma_m \cdot \gamma_n = \eta_{mn}, \tag{2.1.7}$$

where $\eta_{mn} = \mathrm{diag}(+1, -1, -1, -1)$ is the Minkowski metric. For tetrads representing the frame of inertial observers, Equation (2.1.7) implies that in a sufficiently small neighborhood, the perceived spacetime of an inertial observer is always flat.

Analogous to the reciprocal coordinate frame $\{g^\mu\}$, we can define the reciprocal tetrad frame $\{\gamma^m\} \in \sec TU$ by the relationship

$$\gamma_m \cdot \gamma^n = \delta_m^n. \tag{2.1.8}$$

Since the tetrad frame is orthonormal, the components of the inverse metric in the reciprocal tetrad frame are also given by $\eta^{mn} = \mathrm{diag}\,(+1, -1, -1, -1)$:

$$\gamma^m \cdot \gamma^n = \eta^{mn}. \tag{2.1.9}$$

In the preceding discussion, I have adopted the convention, common in tetrad literature, of using middle Latin indices, $\{m, n, l, \ldots\}$, to refer to tetrad components, ranging from 0 to 3. Greek indices $\{\mu, \nu, \lambda, \ldots\}$ will refer to coordinate indices, also ranging from 0 to 3, while other Latin indices, $\{i, j, k, \ldots\}$, will be used for spatial coordinate indices, running from 1 to 3. In cases where we refer to specific coordinate directions, such as $\{t, r, \theta, \phi\}$, a hat will denote the corresponding tetrad components $\{\hat{t}, \hat{r}, \hat{\theta}, \hat{\phi}\}$. See Appendix A for a more detailed explanation of our notation choices.

The tetrad and coordinate frames are related by a transformation called the vierbein, denoted by ${e^m}_\mu$:

$$g_\mu = {e^m}_\mu \gamma_m. \tag{2.1.10}$$

The components of the vierbein are determined by the metric $g_{\mu\nu}$ and the chosen tetrad. Often, when a "natural" choice of tetrad is available, the components of the vierbein can be read directly from the line element when written in an appropriate form:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{mn}{e^m}_\mu {e^n}_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{2.1.11}$$

In practical terms, the vierbein can be considered as the "positive square root" of the metric, as its elements correspond to the square roots of the metric components in diagonal metrics with the tetrad frame aligned with the coordinate frame.

The vierbein contains 16 degrees of freedom: 10 corresponding to the choice of coordinates, encapsulated in the metric, and 6 corresponding to the choice of tetrad. Therefore, in general, the vierbein does not possess symmetric or antisymmetric properties.

Using the reciprocal coordinate frame, we can define the inverse vierbein, ${e_m}^\mu$ in components. The vierbein and inverse vierbein satisfy the following relationships:

$$ {e^m}_\nu {e_m}^\mu = \delta^\mu_\nu, \quad {e^m}_\mu {e_n}^\mu = \delta^m_n. \tag{2.1.12}$$

We will use the inverse vierbein to relate the reciprocal coordinate and tetrad bases:

$$g^\mu = {e_m}^\mu \gamma^m, \tag{2.1.13}$$

as well as to invert the relationship in Equation (2.1.10):

$$\gamma_m = {e_m}^\mu g_\mu. \tag{2.1.14}$$

The choice of which transformations define the vierbein and which define the inverse vierbein, as in Equation (2.1.10) or Equation (2.1.14), is a matter of convention and may vary among different authors.

Compared to the traditional coordinate approach to GR, the use of the tetrad formalism offers the following advantages:

- **Separation of coordinate and frame-related degrees of freedom**: The tetrad formalism clearly distinguishes between coordinates, coordinate frames, and inertial frames. This separation helps isolate the degrees of freedom associated with coordinate choice from those related to frame choice, which is particularly useful in, for example, differentiating between physical and non-physical perturbations. For a comprehensive treatment of perturbations using the tetrad formalism, see [Hamilton, 2020, Ch. 26-30].
- **Local Lorentz invariance**: Tetrads provide a local description of spacetime geometry in terms of inertial frames at each point, making the local Lorentz invariance of spacetime explicit. This simplifies the analysis of physical quantities in different frames.
- **Treatment of spinors**: The tetrad formalism is essential for handling spinors in curved spacetimes, as spinors cannot be directly described in the coordinate basis.
- **Simplification of certain calculations**: In the tetrad frame, many tensors take on simpler forms, or even become trivial. This is because tensor components in the coordinate frame must compensate for the local variation of the coordinate basis vectors, leading to apparent, non-physical variations in tensor components.
- **Numerical relativity**: Tetrads are fundamental in some numerical approaches to GR, such as the ADM formalism, where they facilitate the separation of spatial and temporal components.
- **Extensions of GR**: Tetrads are crucial in various reformulations and extensions of GR, such as teleparallel gravity, the Einstein-Cartan theory, and some approaches to Quantum Gravity.

A more comprehensive treatment of the tetrad formalism in GR can be found in several texts, like [Misner et al., 1973, §6.4] or [Carroll et al., 2004, Appendix J].

2.2 Tetrads with GA

The tetrad formalism is further enhanced by promoting the local vector space $T_p\mathcal{M}$ to a GA, similar to the process described in Chapter 1 for Minkowski spacetime. This promotion of the tangent space $T_p\mathcal{M}$ to a GA is referred to as the *geometric tangent space at* p, denoted by $GT_p\mathcal{M} = \mathcal{G}(T_p\mathcal{M})$. We define the Clifford bundle as the collection of geometric algebras $GT\mathcal{M} = \bigcup_{p\in\mathcal{M}} GT_p\mathcal{M}$. Locally, $GT_p\mathcal{M}$ is isomorphic to the exterior algebra $\bigwedge T_p\mathcal{M} = \bigoplus_{k=0}^{4} \bigwedge^k T_p\mathcal{M}$, where $\bigwedge^k T_p\mathcal{M}$ represents the $\binom{4}{k}$-dimensional space of k-vectors (also known as k-blades). The sections of $GT\mathcal{M}$ are called multivector fields.

We can use the outer product of the tetrad field $\{\gamma_m\}$ to construct the basis of k-vector fields of $\bigwedge^k T\mathcal{M}$, which locally correspond to those presented in Table 1.1.

The extension of the geometric product between two vector fields $a, b \in \sec TM$, as defined in Equation (1.1.4), is achieved by interpreting the scalar product as the metric product, $a \cdot b = g(a, b)$:

$$ab = g(a, b) + a \wedge b, \tag{2.2.1}$$

where the position-dependence is implied.

The term $g(a, b)$ can be calculated in the coordinate basis frame $\{g_\mu\}$ as

$$g(a, b) = a \cdot b = a^\mu g_\mu \cdot b^\nu g_\nu = g_{\mu\nu} a^\mu b^\nu, \tag{2.2.2}$$

or in the tetrad frame as

$$a \cdot b = \eta_{mn} a^m b^n, \tag{2.2.3}$$

where $\eta_{mn} = \text{diag}(+1, -1, -1, -1)$, and a^m, b^n are the components of a and b in the tetrad frame $\{\gamma_m\}$.

2.2.1 Connection bivectors

When we parallel transport a tetrad frame $\{\gamma_m\}$ from point p to point q, the relationship between the local tetrad at q, $\{\gamma_m^{(q)}\}$, and the transported tetrad $\{\gamma_m'^{(q)}\}$ is necessarily a restricted Lorentz transformation. As described in Section 1.2, a Lorentz transformation in GA is performed by sandwiching vectors with the corresponding rotor $R(x)$:

$$\gamma_m'^{(q)} = R(x)\gamma_m^{(q)}\tilde{R}(x). \tag{2.2.4}$$

If the generator of the transformation is the bivector $\omega_\mu(x) = \omega_\mu$ and the transformation is infinitesimal with a parameter ϵ, we can express the rotor R as

$$R(x) = \exp\left(\frac{\epsilon}{2}\omega_\mu\right) \approx 1 + \frac{1}{2}\epsilon\omega_\mu, \tag{2.2.5}$$

and expand Equation (2.2.4) to first order as

$$\gamma_m'^{(q)} = \gamma_m^{(q)} + \frac{\epsilon}{2}\left[\omega_\mu, \gamma_m^{(q)}\right], \tag{2.2.6}$$

where the brackets denote the commutator between multivectors, $[A, B] = AB - BA$.[3]

We can use this result to express the action of the directional covariant derivative on a vector field $a \in \sec \bigwedge^1 T\mathcal{M}$. Typically, such a covariant directional derivative accounts for the variation of both the field components and the local basis. However, by decomposing the vector field in the tetrad frame, $a = a^m \gamma_m$, the infinitesimal variation of the tetrad from point to point is a restricted Lorentz transformation, which can be expressed as the commutator of the tetrad basis vectors with the generator of the transformation—the bivector field $\omega_\mu \in \sec \bigwedge^2 T\mathcal{M}$:

$$\begin{aligned} D_\mu a &= (D_\mu a^m)\gamma_m + a^m D_\mu \gamma_m \\ &= (\partial_\mu a^m)\gamma_m + a^m \frac{1}{2}\left[\omega_\mu, \gamma_m\right] \\ &= \partial_\mu a + \frac{1}{2}\left[\omega_\mu, a\right]. \end{aligned} \tag{2.2.7}$$

In the last line, it is understood that ∂_μ acts only on the components of a, not on the basis vectors γ_m. This definition of the directional covariant derivative applies similarly if a is a multivector field because the directional covariant derivative D_μ is a scalar differential operator, meaning it does not alter the grade of the object it acts upon.[4]

The field ω_μ, which we call the *bivector connection*, is the result of applying the map ω to the coordinate vector field g_μ. This map is linear and defined as:

$$\omega : \sec \bigwedge^1 T\mathcal{M} \to \sec \bigwedge^2 T\mathcal{M}, \quad \omega : g_\mu \mapsto \omega(g_\mu) = \omega_\mu. \tag{2.2.8}$$

The bivector ω_μ is determined by solving the equation $D_\mu \gamma_m = \omega_\mu \cdot \gamma_m$, and it depends entirely on the metric and the choice of tetrad field:

$$\omega_\mu = \frac{1}{2}\left(g^\lambda \wedge \nabla g_{\mu\lambda} + g_\alpha \wedge \partial_\mu g^\alpha\right), \tag{2.2.9}$$

where $\nabla = g^\mu \partial_\mu$ is the flat spacetime vector derivative operator, and the last term is computed as $\partial_\mu g^\alpha = \gamma^m \partial_\mu e_m{}^\alpha$. The derivation of Equation (2.2.9) is provided in Appendix B.

In the case of a diagonal metric, where the coordinate basis vectors are orthogonal, Equation (2.2.9) simplifies considerably if we choose the

[3]The commutator $[A, B]$ is not the Lie bracket, which is often denoted with the same symbol.

[4]It is easily shown that the commutator of any multivector with a bivector is a grade-preserving operation.

tetrad frame to be aligned with the coordinate frame.[5] In this case, ${e_m}^\mu = \mathrm{diag}(|g_{00}|^{-1/2}, |g_{11}|^{-1/2}, |g_{22}|^{-1/2}, |g_{33}|^{-1/2})$, making the second term in Equation (2.2.9) vanish, and the computation of the first term consists of a maximum of 12 derivatives—three for each coordinate direction of spacetime—to obtain all necessary connection coefficients

Physically, the map ω represents the Lorentz transformation experienced by an inertial frame when parallel transported along the integral lines of the μ-coordinate. This means that a free-falling observer moving along a worldline $c(\lambda)$ with tangent vector field $\mathcal{K} = \partial c(\lambda)/\partial\lambda$ will experience a rotation (or boost) of the frame attached to it, generated by the connection bivector $\omega(\mathcal{K})$.

Compared to the Christoffel symbols, Equation (2.2.9) is more straightforward to apply and, compared to the guess-and-check method used with differential forms [Misner et al., 1973, §14.6], they are systematic and clearer.

The decoupling of coordinate and frame-related degrees of freedom brought by the use of tetrads also has significant consequences in the number of connection coefficients at our disposal. In a torsion-free space, the Christoffel symbols have 40 degrees of freedom, whereas the bivector connections, mapping vectors to bivectors, have a maximum of only $4 \times 6 = 24$ coefficients, corresponding to the 3 proper Lorentz transformations and 3 rotations in each of the 4 possible directions of displacement. If the metric is diagonal and the tetrad is aligned with the coordinate axes, the maximum number of connection bivectors reduces to 12.

From the 24 degrees of freedom in the bivector connection, the remaining 16 needed to account for all 40 degrees of freedom in the Christoffel symbols are encoded in the vierbein ${e_m}^\mu$. These additional degrees of freedom relate to changes in the norm and relative position of the coordinate basis vectors, an issue not present in the tetrad formalism.

To connect with the tetrad literature, we can expand ω_μ in the tetrad basis and identify its components as the *spin connection coefficients*:

$$\omega_\mu = \frac{1}{2}\omega_{mn\mu}\gamma^m \wedge \gamma^n. \tag{2.2.10}$$

Since ω_μ is a bivector field, its components are automatically antisymmetric in their first two indices, $\omega_{mn\mu} = -\omega_{nm\mu}$, which is the expected symmetry for the generator of a Lorentz transformation.

[5] Some sources, such as [Hestenes and Sobczyk, 1987, p.235] and [Hestenes, 1986b], claim that this simplification occurs for any orthogonal coordinate frame. However, this is incorrect; the alignment of the tetrad with the coordinate frame is also necessary.

Finally, to relate the connection bivectors ω_μ with the Christoffel symbols $\Gamma_{\kappa\mu\nu}$ in the coordinate approach to GR, we note that the Christoffel symbols are not tensors and therefore are not directly related to the components $\omega_{mn\mu}$ through a simple change of basis using the vierbein $e_m{}^\mu$:

$$e^k{}_\kappa e^m{}_\mu \omega_{km\nu} = \omega_{\kappa\mu\nu} \neq \Gamma_{\kappa\mu\nu}. \tag{2.2.11}$$

The relationship between these quantities is more complex and involves derivatives of the vierbein:

$$\Gamma^\kappa{}_{\mu\nu} = e_l{}^\kappa \partial_\nu e^l{}_\mu + e^m{}_\mu e^n{}_\nu \omega^\kappa{}_{mn}. \tag{2.2.12}$$

Detailed calculations are provided in Appendix C.

2.3 Covariant vector derivative operator

Analogous to the vector derivative in flat spacetime, ∇, we define[6] the covariant derivative operator acting on multivector fields in $\sec G T\mathcal{M}$ as:

$$D = g^\mu D_\mu. \tag{2.3.1}$$

Being $\{g^\mu\}$ the reciprocal coordinate frame, and D_μ the covariant directional derivative defined in Equation (2.2.7). The covariant derivative operator, D, behaves algebraically as a vector, allowing us to apply the same techniques presented in Section 1.2 for the vector derivative operator to obtain a variety of first-order covariant differential operators. Consider the vector fields $a, b \in \sec \bigwedge^1 T\mathcal{M}$:

- **Covariant directional derivative in the $\boldsymbol{a}$-direction**: It is given abstractly by $a \cdot D$, and it is a scalar operator, meaning that it does not change the grade of the objects that it acts upon. Its action over a multivector $M \in \sec G T\mathcal{M}$ is given by:

$$a \cdot DM = a \cdot \nabla M + \frac{1}{2} \left[\omega(a), M\right]. \tag{2.3.2}$$

[6]An alternative route at this point is to avoid introducing the covariant derivative and instead apply the chain rule of the directional derivative ∇ to the components and basis of objects. This procedure is permissible due to the abstract representation of the objects we are using and naturally and consistently yields all necessary tensors of differential geometry. This approach is explored in more detail in Appendix D.

The reason I have chosen to use the covariant derivative in the remainder of this text is primarily to maintain consistency with the conventions of the broader general relativity literature.

Decomposing $a \cdot D$ and M in different frames yields various forms of the covariant directional derivative found in the literature. Consider the vector fields $a, b \in \sec \bigwedge^1 T\mathcal{M}$:

- **(Plain) Covariant directional derivative**: Obtained when $D = g^\mu D_\mu$, $a = a^\mu g_\mu$, and $b = b^\mu g_\mu$ are decomposed in the coordinate frame. The connection is given by the Christoffel symbols, and it is not possible to form any geometric structure over the connection. In this case, the components of the covariant directional derivative of a vector field b in the direction of the vector field a are:
$$a \cdot Db = a^\mu \left(\partial_\mu b^\alpha + \Gamma^\alpha{}_{\beta\mu} b^\beta \right) g_\alpha. \tag{2.3.3}$$
- **Fock-Ivanenko derivative**: Obtained when $D = g^\mu D_\mu$ and $a = a^\nu g_\nu$ are decomposed in the coordinate frame, and $b = b^m \gamma_m$ is decomposed in the tetrad frame. Then, ω_μ is defined by Equation (2.2.9), and the components of the covariant directional derivative of a vector field b in the direction of the vector field a are:
$$a \cdot Db = a^\mu \left(\partial_\mu b + \frac{1}{2}[\omega_\mu, b] \right). \tag{2.3.4}$$
- **Tetrad covariant derivative**: Obtained when $D = \gamma^m D_m$, $a = a^n \gamma_n$, and $b = b^l \gamma_l$ are decomposed in the tetrad frame. Then $\omega_\mu \to \omega_m = \frac{1}{2}\omega_{lnm}\gamma^l \wedge \gamma^n$, ω_{lnm} are called the Ricci rotation coefficients, and the components of the covariant directional derivative of a vector field b in the direction of the vector field a are:
$$a \cdot Db = a^m \left(\partial_m b + \frac{1}{2}[\omega_m, b] \right). \tag{2.3.5}$$

  Being $\partial_m = e_m{}^\mu \partial_\mu$ and $\omega_{lnm} = e_m{}^\mu \omega_{ln\mu}$.

- **Covariant divergence**: Obtained as the inner product between the covariant derivative operator and a vector field a. It is a scalar and independent of the frame in which it is calculated:
$$D \cdot a = D_\mu a^\mu = D_m a^m. \tag{2.3.6}$$
- **Covariant curl**: Obtained as the outer product between the covariant derivative operator and a vector field a:
$$D \wedge a = D_\mu a_\nu g^\mu \wedge g^\nu = \frac{1}{2}\left(D_\mu a_\nu - D_\nu a_\mu \right) g^\mu \wedge g^\nu. \tag{2.3.7}$$

An equivalent expression can be derived in the tetrad frame.

In practice, I have found that decomposing D into the coordinate frame $\{g_\mu\}$ and expressing the objects it acts on in the tetrad frame $\{\gamma_m\}$ is the most insightful combination. This approach clearly represents that the manifold is mapped by coordinates along which the geometric objects of $GT_p\mathcal{M}$ are displaced, while the transported objects are expressed in an orthonormal frame, eliminating non-physical variations due to changes in the coordinate frame.

The action of D on k-vector fields $M \in \sec \bigwedge^k T\mathcal{M}$ is straightforwardly obtained by decomposing M into its corresponding basis and applying the commutation rules of GA. For brevity, I will only explicitly provide the expressions for the covariant divergence and curl of M in the coordinate frame, where

$$M = \frac{1}{k!} M^{\lambda\mu\ldots\nu} \overbrace{g_\lambda \wedge g_\mu \wedge \ldots \wedge g_\nu}^{k \text{ elements}} \tag{2.3.8}$$

- The covariant divergence of a k-vector $M \in \sec \bigwedge^k T\mathcal{M}$, is denoted as $D \cdot M \in \sec \bigwedge^{k-1} T\mathcal{M}$, and it is expressed in the coordinate frame as:

$$D \cdot M = \frac{1}{(k-1)!} g^\mu \wedge \ldots \wedge g^\nu D^\lambda M_{\lambda\mu\ldots\nu}. \tag{2.3.9}$$

- The covariant curl of a k-vector $M \in \sec \bigwedge^k T\mathcal{M}$, is denoted as $D \wedge M \in \sec \bigwedge^{k+1} T\mathcal{M}$, and it is expressed in the coordinate frame as:

$$D \wedge M = \frac{1}{(k+1)!} g^\kappa \wedge g^\lambda \wedge g^\mu \wedge \ldots \wedge g^\nu D_{[\kappa} M_{\lambda\mu\ldots\nu]}. \tag{2.3.10}$$

The extension to multivector fields is straightforward by applying the distributive property of D over the individual k-vectors.

The covariant directional derivative in GA provides a compact and intuitive way of expressing the parallel transport equation of a vector a along a curve $c(\lambda)$ with tangent vector $t = \frac{\mathrm{d}c(\lambda)}{\mathrm{d}\lambda}$:

$$(t \cdot D)a = 0. \tag{2.3.11}$$

The geodesic equation is then given by:

$$(t \cdot D)t = 0. \tag{2.3.12}$$

Squaring the covariant derivative operator yields two second-order differential operators:

$$D^2 = D \cdot D + D \wedge D. \tag{2.3.13}$$

The first term, which we denote by $D \cdot D \equiv \Delta$, is a scalar operator that does not change the grade of the objects it acts on. It is equivalent to the Laplace-de Rham operator of differential forms, which we can see by expanding it in the coordinate frame to obtain the familiar form:

$$\Delta = D^{\mu} D_{\mu} - g^{\mu\nu} \Gamma^{\sigma}{}_{\mu\nu} D_{\sigma}. \tag{2.3.14}$$

The second term of Equation (2.3.13), $D \wedge D$, is a second-order diferential bivector operator, whose behavior is more interesting because it has no direct analog in differential forms or other formalisms. It is sometimes referred to as the *Ricci operator* [Rodrigues and Gomes de Souza, 2005], and expanding it in the coordinate frame, we can express it as the commutator of covariant directional derivatives:

$$D \wedge D = g^{\mu} \wedge g^{\nu} [D_{\mu}, D_{\nu}]. \tag{2.3.15}$$

Expansion of $D \wedge D$ into the tetrad frame would introduce a term $-f^{l}_{mn} D_l$, where f^{l}_{mn} are the structure coefficients, due to the anholonomy of the frame. Such term can be interpreted as a non-physical torsion originated in the non-closure of parallelograms formed by the adapted coordinates.

When acting on a vector field a, it is easy to show that $D \wedge D$ yields the Ricci tensor acting on a, justifying its name:

$$\begin{aligned} D \wedge Da &= g^{\mu} \wedge g^{\nu} (D_{\mu} D_{\nu} - D_{\nu} D_{\mu}) a^{m} \gamma_{m} \\ &= g^{\mu} R_{m\mu} a^{m} = R(a). \end{aligned} \tag{2.3.16}$$

In GA, the Ricci tensor is a map from vector fields to vector fields:

$$R : \sec \bigwedge^{1} T\mathcal{M} \to \sec \bigwedge^{1} T\mathcal{M}, \quad a \mapsto R(a), \tag{2.3.17}$$

where $R(a) = R_{\mu\nu} a^{\mu} g^{\nu}$ and $R_{\mu\nu}$ are the usual components of the Ricci tensor, as in the tensor formalism.

The geometric interpretation of the Ricci tensor emerges when considering its effect on a congruence of geodesics and its relation to the Raychaudhuri equations, as shown in Section 4.2. In brief, just as the geodesic deviation is governed by the Riemann tensor, the Ricci vector $R(a)$ quantifies the variation *due to curvature* of a small volume when displaced along the lines of the vector field a. The detailed proof can be found in [Loveridge, 2004], although I have not yet converted it into a simpler proof using GA.

2.4 Riemann map and Ricci scalar

2.4.1 The Riemann map

Applying the operator $D \wedge D$ to a multivector field M, we obtain the action of the Riemann tensor in GA as follows:

$$D \wedge DM = g^\mu \wedge g^\nu [D_\mu, D_\nu] M = g^\mu \wedge g^\nu [\mathbf{R}(g_\mu \wedge g_\nu), M]. \tag{2.4.1}$$

Here, we have defined the Riemann bivector map $\mathbf{R}(g_\mu \wedge g_\nu)$ in terms of the connection bivectors as:

$$\mathbf{R}(g_\mu \wedge g_\nu) = \mathbf{R}_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu]. \tag{2.4.2}$$

The Riemann bivector in GA is a map from bivector fields to bivector fields, defined as:

$$\mathbf{R} : \sec \bigwedge^2 T\mathcal{M} \to \sec \bigwedge^2 T\mathcal{M}, \quad B \mapsto \mathbf{R}(B), \tag{2.4.3}$$

with $\mathbf{R}(B) = B^{\mu\nu} \mathbf{R}_{\mu\nu mn} \gamma^m \wedge \gamma^n$.

Since bivectors represent areas and serve as the generators of rotations, the geometric meaning of the Riemann tensor becomes clear: it relates a differential area to the rotation experienced by a vector when parallel transported along its contour, as illustrated in Figure 2.4.

We can expand the Riemann bivector in either the coordinate or tetrad basis, obtaining the usual tensorial expression of the Riemann tensor, $\mathbf{R}_{\mu\nu\alpha\beta}$, or a mixed-index expression, $\mathbf{R}_{\mu\nu mn}$:

$$\mathbf{R}(g_\mu \wedge g_\nu) = \mathbf{R}_{\mu\nu} = \frac{1}{2} \mathbf{R}_{\mu\nu\alpha\beta} g^\alpha \wedge g^\beta = \frac{1}{2} \mathbf{R}_{\mu\nu mn} \gamma^m \wedge \gamma^n. \tag{2.4.4}$$

When expressed in the mixed-index form $\mathbf{R}_{\mu\nu mn}$, the Riemann tensor reveals some of its symmetries immediately:

1. The first two indices, related to the coordinate area, must be antisymmetric.
2. The second pair of indices, representing the basis expansion of the bivector, must also be antisymmetric.

The calculation of its degrees of freedom is also simplified. Since $\mathbf{R}$ maps bivectors to bivectors, in a 4-dimensional space it can have at most $6 \times 6 = 36$ degrees of freedom. This number is reduced by considering the protractionless property of the Riemann tensor:

$$\partial_a \wedge \mathbf{R}(a \wedge b) = 0. \tag{2.4.5}$$

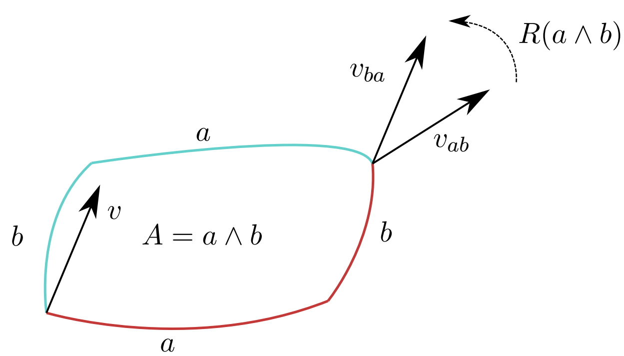


Figure 2.4: Representation of the effect of transporting a vector v to the same point through two different paths. When transported through the red path, $a \to b$, the resulting vector is v_{ab}. When transported through the blue path, $b \to a$, the resulting vector is v_{ba}. The vectors v_{ab} and v_{ba} are related by a rotation that depends on the area spanned between the paths, $A = a \wedge b$. This rotation is described by the Riemann tensor.

Equation (2.4.5) imposes a restriction on the function that maps vector fields b to trivector fields, resulting in a total of $4 \times 4 = 16$ equations, which reduce the number of degrees of freedom of the Riemann tensor from 36 to 20. The protractionless properties of the Riemann are explored in more detail in [Hestenes and Sobczyk, 1987, §3.9 and §5.1].

Equation (2.4.5) is equivalent to stating that $\mathbf{R}$ is symmetric under the pairwise interchange of indices and satisfies the algebraic Bianchi identity:

$$\mathbf{R}(a \wedge b) \cdot c + \mathbf{R}(c \wedge a) \cdot b + \mathbf{R}(b \wedge c) \cdot a = 0. \tag{2.4.6}$$

The differential Bianchi identity can be compactly written as:

$$\acute{D} \wedge \acute{\mathbf{R}}(a \wedge b) = g^{\mu} \left(D_{\mu}\mathbf{R}(a \wedge b) - \mathbf{R}(D_{\mu}(a \wedge b)) \right), \tag{2.4.7}$$

where I used the accent to denote the action of the covariant derivative operator on the Riemann tensor, but not on its arguments [Hestenes and Sobczyk, 1987, §5.1]. Although the presence of the accent can be confusing, it is necessary because the vector nature of the derivative operator prevents it from generally commuting with multivectors.

The components of the Ricci tensor are obtained from the Riemann tensor by contraction, which reduces the grade of the object it acts upon by 1. In our formalism, this is performed using the inner product:

$$R_{\mu} = g^{\mu} \cdot \mathbf{R}_{\mu\nu}. \tag{2.4.8}$$

The contraction and protraction properties of a tensor can be combined into the *traction* operation, defined as:

$$\partial_a \mathbf{R}(a \wedge b) = \partial_a \cdot \mathbf{R}(a \wedge b) + \partial_a \wedge \mathbf{R}(a \wedge b) = R(b) + 0. \tag{2.4.9}$$

As demonstrated with the Riemann tensor, many symmetries and algebraic properties of tensors can be derived from their traction operations.

2.4.2 The Ricci scalar

The Ricci scalar $R \in \sec \bigwedge^0 T\mathcal{M}$ is a scalar field that provides a measure of the curvature of the manifold $\mathcal{M}$ at a point p. It has a geometric interpretation analogous to the Gaussian curvature in two-dimensional surfaces but generalized to four-dimensional spacetime. The Ricci scalar can be obtained by contracting either the Ricci vector or the Riemann tensor as follows:

$$\mathcal{R} = g^{\mu} \cdot R_{\mu} = (g^{\nu} \wedge g^{\mu}) \cdot \mathbf{R}_{\mu\nu}. \tag{2.4.10}$$

This contraction can be performed in any desired frame, but it is often much simpler to compute in the tetrad frame, where the basis vectors are orthonormal.

2.5 Einstein's equations

Like the Ricci tensor, the Einstein tensor is a map from vector fields to vector fields:

$$G : \sec \bigwedge^1 T\mathcal{M} \to \sec \bigwedge^1 T\mathcal{M}, \quad a \mapsto G(a). \tag{2.5.1}$$

Here, $G(a)$ is defined as the combination:

$$G(a) = R(a) - \frac{1}{2} a\mathcal{R}, \tag{2.5.2}$$

where $R(a)$ represents the Ricci vector, and $\mathcal{R}$ is the Ricci scalar. The physical interpretation of the Einstein tensor is that it encodes the curvature of the various 3-dimensional subspaces of the manifold [Loveridge, 2004]. However, I have not found significant advantages in using its GA description for obtaining this interpretation.

The distribution of energy and momentum in spacetime is described by the energy-momentum tensor $T(a)$, which, similar to the Einstein tensor, is a map from vector fields to vector fields:

$$T : \sec \bigwedge^1 T\mathcal{M} \to \sec \bigwedge^1 T\mathcal{M}, \quad a \mapsto T(a). \tag{2.5.3}$$

The physical interpretation of $T(a) \in \bigwedge^1 T\mathcal{M}$ is that it represents the flux of 4-momentum passing through a hypersurface perpendicular to a [Doran and Lasenby, 2013, §7.2.3].

Using the Einstein tensor and the energy-momentum tensor, Einstein's Field Equations can be formulated in GA as:

$$G(a) = \kappa T(a) + a\Lambda, \tag{2.5.4}$$

where κ is the Einstein gravitational constant, and Λ is the cosmological constant. The trace-reversed form of these equations is given by:

$$R(a) = D \wedge Da = \kappa \left(T(a) - \frac{1}{2} a \mathrm{Tr}(T) \right) + \Lambda a, \tag{2.5.5}$$

with $\mathrm{Tr}(T) = \partial_a \cdot T(a)$ being the trace of the energy-momentum tensor.

2.6 Conclusions

In this chapter, I presented the tetrad-GA formalism of GR. I started by presenting the usual tetrad formalism and continue by generalizing the tangent space at point p, $T_p\mathcal{M}$ to a GA and form a bundle called $GT\mathcal{M}$. This construction properly generalizes the tools and objects presented in Chapter 1 to curved manifolds.

In deriving the most common objects of GR with GA, I found out that many of them emerge rather naturally when considering the different actions of the covariant vector D over vector and multivector fields. In the next chapters we will see that their explicit form gets also considerably simplified for the use of tetrads.

In this discussion, bivectors were particularly useful. Because of their role as generators of rotations, they allowed us to express the connection coefficients in as geometric objects, and interpret their action as restricted Lorentz transformations between observers.

The geometric interpretation of the Riemann tensor was also considerably enhanced by describing it as a map between bivector fields which relates areas to rotations. However, I found limited benefits in describing the Ricci and Einstein tensors, which I expected to reveal their geometric content more clearly in the tetrad-GA formalism.

The tools developed in this chapter lay the foundation for the subsequent chapters. However, since differential forms share many features with GA, and are widely used to describe GR, in the next chapter I will compare both formalisms as a bridge for the reader before delving into the applications of the tetrad-GA formalism.

Chapter 3

Differential Forms and Geometric Algebra

Differential forms are one of the most popular geometric formalism in physics, with applications spanning from electrodynamics to GR. Its framework extends vector calculus, providing a richer structure, simplified calculations and powerful integral theorems, particularly when dealing with fields on manifolds. They also allow for a formulation of physical laws relating abstract objects, instead of their components in a particular frame, and the properties of the exterior derivative provides a compact and natural way of dealing with gauge transformations.

Building on the discussion of GA in previous chapters, in this chapter I aim to bridge the gap between GA and differential forms. In Section 3.1 I will start with a brief historical note regarding the foundations and evolution of vector calculus, differential forms and GA, and in Section 3.2 I present a side-by-side comparison between differential forms and GA. On it, we will look at differential identities, integration theorems, and algebraic structures. Afterwards, in Section 3.3, I will use electrodynamics, already described in Section 1.3 in GA, as an example of application of both formalisms in flat spacetime. Finally, in Section 3.4 I will compare the tools introduced in Chapter 2 with those offered by differential forms in the context of GR, highlighting their respective strengths and weaknesses. The results in this chapter were originally presented in [Pérez and DeKieviet, 2024b].

3.1 Historical context

In 1870, William Kingdon Clifford [1878] laid the foundations of GA, building on the work of Hermann Grassmann . Clifford introduced the geometric product, correctly identified the geometrical nature of the algebra's elements, and incorporated quaternions and rotors. However, his early death in 1879, at the age of 33, kept his work from reaching a wider scientific community, which only gained wider recognition in the late 20th century, thanks to the revival by Hestenes [2002], Hestenes and Sobczyk [1987] and the Cambridge group led by Anthony and Joan Lasenby, Chris Doran, and others [Doran and Lasenby, 2013]. While GA and GC are rooted in Clifford's work, they have evolved by incorporating ideas from other mathematical frameworks, as illustrated in Figure 3.1.

Meanwhile, O. Heaviside [1893] and J. W. Gibbs and Wilson [1901] developed and popularized Vector Calculus, which was based on the quaternions developed by Hamilton [1844], and it became the most widespread mathematical formalism for expressing physical laws for over a century. The success of Vector Calculus was not only due to its suitable description of the three-dimensional space, the early death of Clifford combined with the popularity of Gibbs seems to have played a crucial role in its diffusion [Chappell et al., 2016].

At the same time, Élie Cartan [1899] developed the formalism known as differential forms, a geometrically sophisticated framework that proved to be especially effective for field theory on manifolds. Differential forms provided coordinate-independent descriptions, simplifying the formulation of physical laws such as Maxwell's equations and addressing some of the limitations inherent in vector and tensor calculus.

Because the connection between Vector Calculus and Differential forms is well known, see for example the didactic, though somewhat outdated, comparison made by Schleifer [1983], in this chapter I will focus on the comparison between GA and differential forms, a topic which has already been touched in literature [Hestenes, 1993], [Hestenes and Sobczyk, 1987, §6-4], [Francis and Kosowsky, 2003], although not always in a comprehensive or practical manner.

3.2 Differential Forms in Geometric Calculus

The correspondence between differential forms and geometric calculus can be confusing because differential forms are used in literature in two distinct

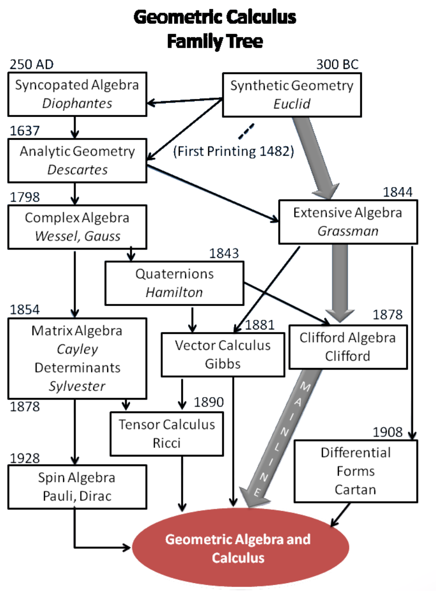


Figure 3.1: Diagram of the various mathematical frameworks that are part of Geometric Calculus [Rossi, 2012]

ways: As elements of the cotangent space and as differential elements for integration. Because the treatment in this chapter is not restricted to curved Riemanninan $(1+3)$-dimensional manifolds, I will use $\{e_i\}$ to refer to a general base of a given space with dimension n and $\hat{e}_i$ to denote a general orthonormal base of such space.

In the case, where a differential form is used to express a geometric element of the cotangent space, we can identify it with a k-vector expressed in the reciprocal frame. Then, a k-form represents a geometric object of dimension k:

- zero-form $\leftrightarrow$ Scalar.
- one-form $\leftrightarrow$ Vector $a = a^i e_i = a_i \hat{e}^i$
- two-form $\leftrightarrow$ Bivector $A = A^{ij} e_i \wedge e_j$.

The basis of the cotangent space in differential forms is $\{\mathrm{d}x^i\}$ and it acts over the basis of the tangent space $\{\partial_j\}$ as

$$\mathrm{d}x^i(\partial_j) = \delta^i_j. \tag{3.2.1}$$

This definition is equivalent to Equation (2.1.2), which we used to define the reciprocal base $\{e^i\}$ in terms of the base $\{e_j\}$. Therefore, we identify

$$\mathrm{d}x^i \leftrightarrow e^i \tag{3.2.2}$$

when $\{\mathrm{d}x^i\}$ is used as basis for forms.

Note that the presence of a metric is a necessary requirement to define a GA, and therefore in GA there is always a musical isomorphism between the tangent and cotangent elements. If this requirement is not fulfilled, we cannot define a GA and Equation (3.2.2) is not valid. In this sense, differential forms provide more general framework because they do not strictly require the presence of a metric. Nonetheless, in such case they are also limited in their power, since many of their applications and theorems rely on the Hodge star operator which is a metric-dependent operation.

The second use of differential forms is as directed integration elements, in this case we identify them with differential multivectors representing differential oriented geometric elements,

- zero-form $\leftrightarrow$ Scalar.
- one-form $\leftrightarrow$ Differential vector $\mathrm{d}x = \mathrm{d}x^i e_i$
- two-form $\leftrightarrow$ Differential of area $\mathrm{d}A = \mathrm{d}x^{(1)} \wedge \mathrm{d}x^{(2)} = \mathrm{d}x^1 \mathrm{d}x^2 e_1 \wedge e_2$.

Where $\mathrm{d}x^i$ are the scalar coordinate differentials, and $\mathrm{d}x^{(i)}$ are differential vectors in the i-direction.

E.g., In spherical coordinates, the differential of the $r\phi$-area is given by the outer product of the differential vectors $\mathrm{d}x^{(r)} = \mathrm{d}r e_r$ and $\mathrm{d}x^{(\phi)} = \mathrm{d}\phi e_\phi = \mathrm{d}\phi r \hat{e}_\phi$,

$$\mathrm{d}x^{(r)} \wedge \mathrm{d}x^{(\phi)} = \mathrm{d}r\, \mathrm{d}\phi\, e_r \wedge e_\phi = \mathrm{d}r\, \mathrm{d}\phi\, \hat{e}_r \wedge r\hat{e}_\phi = r\, \mathrm{d}r\, \mathrm{d}\phi\, \hat{e}_r \wedge \hat{e}_\phi. \tag{3.2.3}$$

Geometric interpretation Differential one-forms are typically depicted as surfaces that are pierced by vectors, as illustrated in Figure 3.2, while two-forms are often interpreted as “tubes” resulting from the intersections of two one-forms [Misner et al., 1973, §2.5 and Ch. 4]. Although this description is often regarded as intuitive in the literature, in my opinion, it is challenging to visualize and even more difficult to attribute a physical interpretation to it. This difficulty only increases with higher-dimensional forms, which become even harder to conceptualize.

Moreover, this description is vague and lacks the precision required for purely geometrical calculations, as the distance between surfaces is not determined, nor is there a clear way to account for how vectors pierce the surfaces. Thus, even though one can become accustomed to thinking about forms in this way with sufficient practice, this interpretation remains neither intuitive nor quantifiable.

In contrast, geometric calculus offers a direct and quantifiable geometric interpretation of objects and operations. Vectors are directed lines, bivectors represent directed planes, and trivectors correspond to oriented volumes, as shown in Figure 3.3. The outer product is a direct geometric combination of these elements, while the inner product represents the projection of one object onto another. Based on the experience in our group teaching GA, this approach is generally more intuitive and easier to grasp than differential forms. Students who have received minimal exposure to geometric algebra can often quickly explain the geometric meaning of various elements and operations, which is rarely the case with differential forms and their associated operations.

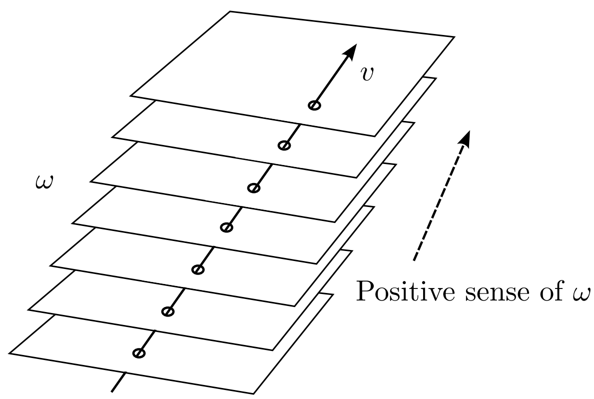


Figure 3.2: Depiction of a one-form ω pierced by the vector v. The resulting scalar, $\omega(v)$, is "equivalent" to the number of surfaces that v intersects.

Consider the inner product of a vector $a = \hat{e}_1 + \hat{e}_3$ with a bivector $B = \hat{e}_1 \wedge \hat{e}_2$, both expressed in the general orthonormal basis $\{\hat{e}_i\}$. This operation can be carried out algebraically as:

$$a \cdot B = (\hat{e}_1 + \hat{e}_3) \cdot \hat{e}_1 \wedge \hat{e}_2 = \hat{e}_2, \tag{3.2.4}$$

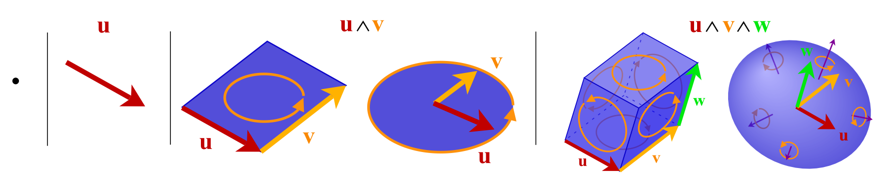


Figure 3.3: Geometrical elements of $\mathcal{G}(3)$. From top to bottom: scalars, vectors, bivectors, and trivectors (volumes). Note that k-vectors contain information only about the "area/volume" and its orientation, not the shape.

or geometrically, by projecting a onto the plane defined by B, and rotating it by $\pi/2$ in the direction defined by B, as shown in Figure 3.4. While the rotation might be surprising, recall that bivectors also serve as generators of rotations, which implies they contain orientation. Note that the resulting vector, $a \cdot B$, lies in the plane defined by B and is always orthogonal to a.

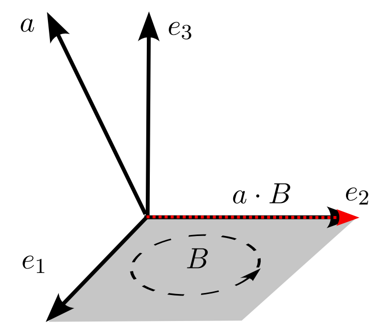


Figure 3.4: In red: the inner product of the vector $a = \hat{e}_1 + \hat{e}_3$ with the bivector $B = \hat{e}_1 \wedge \hat{e}_2$.

The equivalent operation in differential forms cannot be visualized, as the piercing analogy of one-forms and vectors is limited to scalar quantities.

3.2.1 Operations

In this section, we will explore several common operations in differential forms and their counterparts in GC.

Duality operation The duality operation in differential forms is performed using the Hodge star operator. In GC, the dual of an object is obtained by multiplying it by the pseudoscalar I. Depending on the grade of the object and the side on which I is multiplied, a minus sign may be involved according to the commutation rule for the pseudoscalar with a k-vector A_k in an n-dimensional space given by

$$IA_k = (-1)^{k(n-k)} A_k I. \tag{3.2.5}$$

For instance, in $\mathbb{E}^3$:

- The dual of a zero-form: $\star f = f \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \leftrightarrow If = f\hat{e}^1 \wedge \hat{e}^2 \wedge \hat{e}^3$.
- The dual of a one-form: $\star \mathrm{d}x^1 = \mathrm{d}x^2 \wedge \mathrm{d}x^3 \leftrightarrow I\hat{e}^1 = \hat{e}^1 I = \hat{e}^2 \wedge \hat{e}^3$.
- The dual of a two-form: $\star(\mathrm{d}x^1 \wedge \mathrm{d}x^2) = \mathrm{d}x^3 \leftrightarrow I\hat{e}^1 \wedge \hat{e}^2 = \hat{e}^1 \wedge \hat{e}^2 I = -\hat{e}^3$.

It is important to emphasize that both the Hodge dual and the duality in GC are metric-dependent operations. In GC, this is evident since the inner product is needed to contract with the pseudoscalar. Similarly, in differential forms, where the metric is an explicit part of the definition of the Hodge dual:

$$\alpha \wedge (\star\beta) = \langle \alpha, \beta \rangle (e_1 \wedge \cdots \wedge e_n), \tag{3.2.6}$$

where α and β are k-forms.

Outer product: The outer product in differential forms is equivalent to the outer product in GC. Both are superior to the cross-product in conventional vector calculus, as they are defined for spaces of any dimension. This allows the definition of the curl/exterior derivative in spaces with dimensions greater than one, and in GC, it is also the basis for its treatment of rotations.

For a pair of one-forms σ and ω, and their corresponding vectors a and b, the outer product is given by

$$\begin{aligned} \sigma \wedge \omega &= (\sigma_1\omega_2 - \sigma_2\omega_1)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ &\quad + (\sigma_2\omega_3 - \sigma_3\omega_2)\mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ &\quad + (\sigma_3\omega_1 - \sigma_1\omega_3)\mathrm{d}x^3 \wedge \mathrm{d}x^1 \\ &\quad \Updownarrow \\ a \wedge b &= (a_1b_2 - a_2b_1)\hat{e}^1 \wedge \hat{e}^2 \\ &\quad + (a_2b_3 - a_3b_2)\hat{e}^2 \wedge \hat{e}^3 \\ &\quad + (a_3b_1 - a_1b_3)\hat{e}^3 \wedge \hat{e}^1. \end{aligned} \tag{3.2.7}$$

In three dimensions, the relationship with the cross-product in vector calculus involves a duality operation:

$$a \times b \Leftrightarrow \star(a \wedge b) \Leftrightarrow -I(a \wedge b). \tag{3.2.8}$$

Inner product The inner product in differential forms is identical to the inner product in GC[1]:

$$\begin{gathered} \langle \sigma, \omega \rangle = \sigma_1\omega_1 + \sigma_2\omega_2 + \sigma_3\omega_3 \\ \Updownarrow \\ a \cdot b = a_1b_1 + a_2b_2 + a_3b_3. \end{gathered} \tag{3.2.9}$$

[1]In older texts on differential forms, the inner product was not incorporated directly, and the equivalent expression $\star(\sigma \wedge \star\omega)$ was used.

In differential forms, sometimes it is important to distinguish between the inner product and the interior product. The interior product, defined as the action of a one-form on a vector, is metric-independent, whereas the inner product is the usual metric-based product between two vectors. In a metric space, where there is an isomorphism between vectors and one-forms, both products are equivalent.

From the perspective of GC, the difference between the inner and interior products lies in the frame in which vectors are expressed. For example, given $a = a^i e_i = a_i e^i$:

$$a \cdot b = a^i b^j (e_i \cdot e_j) = a^i b^j g_{ij} = a_i b^j (e^i \cdot e_j) = a_i b^j \delta^i_j = a^i b_i. \tag{3.2.10}$$

Differential operations

In this section, I will demonstrate how the main differential operations in differential forms relate to their counterparts in GA. The fundamental differential operator in GA is the vector derivative, and various differential operations arise by performing different products or applying algebraic identities, as discussed in Section 1.2.

Exterior derivative: The exterior derivative of a k-form corresponds to the curl of a k-vector in GA. Consider a one-form $\omega = \omega_i \mathrm{d}x^i \in \bigwedge^1 \left(\mathbb{E}^3\right)$ and its corresponding vector field $a = a_i \hat{e}^i \in \bigwedge^1 \left(\mathbb{E}^3\right)$. Their exterior derivative and curl are given by:

$$\begin{aligned}
\mathbf{d}\omega &= (\partial_1 \omega_2 - \partial_2 \omega_1) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
&\quad + (\partial_2 \omega_3 - \partial_3 \omega_2) \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&\quad + (\partial_3 \omega_1 - \partial_1 \omega_3) \mathrm{d}x^3 \wedge \mathrm{d}x^1 \\
&\Updownarrow \\
\nabla \wedge a &= (\partial_1 a_2 - \partial_2 a_1) \hat{e}^1 \wedge \hat{e}^2 \\
&\quad + (\partial_2 a_3 - \partial_3 a_2) \hat{e}^2 \wedge \hat{e}^3 \\
&\quad + (\partial_3 a_1 - \partial_1 a_3) \hat{e}^3 \wedge \hat{e}^1.
\end{aligned} \tag{3.2.11}$$

The identity $\mathbf{d}^2 = 0$ in differential forms is equivalent to $\nabla \wedge \nabla = 0$ in GA. Both identities reflect the commutativity of coordinate directional derivatives, indicating the absence of torsion.

In three dimensions, the curl as defined in vector calculus is obtained using the dual operation over Equation (3.2.11),

$$\vec{\nabla} \times a \Leftrightarrow -I(\nabla \wedge a) \Leftrightarrow \star \mathbf{d}\omega. \tag{3.2.12}$$

Codifferential: The definition of divergence in differential forms is somewhat intricate, and has its roots in the original construction of the inner product in differential forms[1]. This operation, known as the *codifferential*, is denoted by δ. Its action on a k-form $\alpha \in \bigwedge^k(\mathcal{V})$, where $\mathcal{V}$ is a space of dimension n is defined as

$$\delta\alpha = (-1)^{n(k-1)+1} \star (\mathbf{d} \star \alpha). \tag{3.2.13}$$

The equivalent operation in GA is the inner product of the vector derivative, which we call divergence. For the a vector field $a = a^i e_i$ it reads

$$\nabla \cdot a = \sum_i \partial_i a^i, \tag{3.2.14}$$

and in the case of a k-vector field M, applying the commutation properties of the basis vectors we obtain:

$$\nabla \cdot M = \sum_{i,j} (-1)^{j+1} \partial_j M_{(i)}^{1...j...k} e_1 \wedge ... \wedge e_{j-1} \wedge e_{j+1} \wedge ... \wedge e_k. \tag{3.2.15}$$

Therefore, the codifferential equals the divergence in GA up to a sign which depends on the dimension of the space and the power, or grade, of the form.

Gradient: The gradient of a scalar function, or zero-form, f, corresponds to its exterior derivative and to its curl in GA:

$$\begin{gathered} \mathbf{d}f = \partial_1 f \mathrm{d}x^1 + \partial_2 f \mathrm{d}x^2 + \partial_3 f \mathrm{d}x^3 \\ \Updownarrow \\ \nabla f = \nabla \wedge f = (\partial_1 f)\, e^1 + (\partial_2 f)\, e^2 + (\partial_3 f)\, e^3. \end{gathered} \tag{3.2.16}$$

Laplacian: In differential forms, the Laplace-de Rham operator is expressed as a combination of the codifferential and exterior derivative. In GA, it is derived by acting twice with ∇ and applying the associativity property of the geometric product:

$$\begin{gathered} (\mathbf{d}\delta + \delta\mathbf{d}) f = \sum_{i=1}^{3} \partial_i^2 f \\ \Updownarrow \\ \nabla^2 f = (\nabla \cdot \nabla) f = \sum_{i=1}^{3} \partial_i^2 f. \end{gathered} \tag{3.2.17}$$

Where we used the property $\nabla \wedge \nabla = 0$.

It is worth noting the consistency in treating ∇ as a vector. The outer product of a vector with itself vanishes, while its inner product produces a scalar, i.e. the squared norm. Indeed, the Laplacian is a scalar operator in that it does not change the grade of the element it acts upon.

∇^2 correctly reproduces the Laplacian, d'Alembertian, or Laplace-de Rham operator, depending on the signature of the space in question.

3.2.2 Identities

This section lists a series of algebraic identities and their respective formulations in vector calculus, differential forms, and geometric calculus (GC). In cases where the cross-product appears in vector calculus, I will omit the dual operation in GC to produce algebraic relationships valid in any dimension.

Let f and g be scalar functions, σ and ω be differential one-forms, and a and b be vector fields. The symbol $\vec{\nabla}$ will represent the vector differential operator in vector calculus, while ∇ will denote the vector derivative operator in GC.

	Vector calculus
(1)	$\vec{\nabla}(fg) = (\vec{\nabla}f)g + f(\vec{\nabla}g)$
(2)	$\vec{\nabla} \cdot (fa) = (\vec{\nabla}f) \cdot a + f(\vec{\nabla} \cdot a)$
(3)	$\vec{\nabla} \cdot (a \times b) = b \cdot (\vec{\nabla} \times a) - a \cdot (\vec{\nabla} \times b)$
(4)	$\vec{\nabla} \times (fa) = \vec{\nabla}f \times a + f(\vec{\nabla} \times a)$
(5)	$a \cdot (b \times c) = c \cdot (a \times b) = b \cdot (c \times a)$
(6)	$a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$
(7)	$(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$

Table 3.1: Some algebraic identities in vector calculus. Corresponding identities in differential forms are shown in Table 3.2, and in GC in Table 3.3.

From these identities, we can observe that geometric calculus serves as a language that combines the best aspects of both vector calculus and differential forms. From vector calculus, it borrows geometric intuition and clarity. From differential forms, it incorporates the power of exterior algebra, computational efficiency, and directed integration. However, this fusion goes beyond a simple addition of features: the unification in GC enhances computational power and provides deeper insights into geometric structures.

	Differential forms
(1)	$\mathbf{d}fg = (\mathbf{d}f)g + f(\mathbf{d}g)$
(2)	$\star(\mathbf{d}\star(f\omega)) = \star(\mathbf{d}f \wedge \omega) + f\star(\mathbf{d}\star\omega)$
(3)	$\star(\mathbf{d}\star\star(\sigma\wedge\omega)) = \star(\omega\wedge\mathbf{d}\sigma) - \star(\sigma\wedge\mathbf{d}\omega)$
(4)	$\star(\mathbf{d}(f\omega)) = \star(\mathbf{d}f\wedge\omega) + f\star(\mathbf{d}\omega)$
(5)	$\star(\sigma_1\wedge\sigma_2\wedge\sigma_3) = \star(\sigma_2\wedge\sigma_3\wedge\sigma_1) = \star(\sigma_3\wedge\sigma_1\wedge\sigma_2)$
(6)	$\star(\sigma_1\wedge\star(\sigma_2\wedge\sigma_3)) = \sigma_2\star(\sigma_1\wedge\star\sigma_3) - \sigma_3\star(\sigma_1\wedge\star\sigma_2)$
(7)	$\star(\star(\sigma_1\wedge\sigma_2)\wedge\star(\sigma_3\wedge\sigma_4)) = \star(\sigma_1\wedge\sigma_3)\star(\sigma_2\wedge\sigma_4) - \star(\sigma_1\wedge\sigma_4)\star(\sigma_2\wedge\sigma_3)$

Table 3.2: Some algebraic identities in differential forms. Corresponding identities in vector calculus are shown in Table 3.1, and in GC in Table 3.3.

	Geometric calculus
(1)	$\nabla(fg) = (\nabla f)g + f(\nabla g)$
(2)	$\nabla\cdot(fa) = (\nabla f)\cdot a + f\nabla\cdot a$
(3)	$\nabla\cdot(a\wedge b) = (\nabla\cdot a)b - a(\nabla\cdot b)$
(4)	$\nabla\wedge(fa) = (\nabla f)\wedge a + f(\nabla\wedge a)$
(5)	$a\wedge b\wedge c = b\wedge c\wedge a = c\wedge a\wedge b$
(6)	Same as (3): $a\cdot(b\wedge c) = a\cdot bc - a\cdot cb$
(7)	$(a\wedge b)\cdot(c\wedge b) = (a\cdot(b\cdot(c\wedge b))) = (a\cdot d)(b\cdot c) - (a\cdot c)(b\cdot d)$

Table 3.3: Some algebraic identities in geometric calculus. Corresponding identities in vector calculus are shown in Table 3.1, and in differential forms in Table 3.2.

3.2.3 Theorems

In this section I will present some common theorems in differential forms and how do they relate to their corresponding versions in GC.

Poincaré Lemma: If a k-form ω satisfies $\mathbf{d}\omega = 0 \Rightarrow \exists$ a $(k-1)$-form α such that $\mathbf{d}\alpha = \omega$.

In GC: If a k-vector field E has null curl, $\nabla\wedge E = 0$, then $\exists$ a $(k-1)$-vector field, ϕ such that $\nabla\phi = \nabla\wedge\phi = E$.

Poincaré Lemma for the codifferential: If a k-form ω satisfies $\delta\omega = 0$, then $\exists$ a $(k+1)$-form β such that $\omega = \delta\beta$.

In GC: If a k-vector field B has null divergence, $\nabla\cdot B = 0$. Then $\exists$ a $(k+1)$-vector field A such that $\nabla\cdot A = B$.

Decomposition theorem: For any p-form $\omega \in \bigwedge^p(\mathcal{V})$, $\exists\ \alpha \in \bigwedge^{p-1}(\mathcal{V})$, $\beta \in \bigwedge^{p+1}(\mathcal{V})$ and $\gamma \in \bigwedge^p(\mathcal{V})$ with $(\mathbf{d}\delta + \delta\mathbf{d})\gamma = 0$, such that

$$\omega = \mathbf{d}\alpha + \delta\beta + \gamma \tag{3.2.18}$$

In GC [Roberts, 2022, Robson, 2023]: For any k-vector A_k, $\exists\ B_{k-1}$, C_{k+1}, D_k, with $\nabla^2 D_k = 0$ such that

$$A_k = \nabla \wedge B_{k-1} + \nabla \cdot C_{k+1} + D_k \tag{3.2.19}$$

Generalized Stokes Theorems: In the language of differential forms, the main integral theorems can be elegantly expressed using the generalized Stokes theorem and the Hodge star operator as follows:

$$\int_D \mathbf{d}\omega = \int_{\partial D} \omega \tag{3.2.20}$$

$$\int_D \mathbf{d}(\star\omega) = \int_{\partial D} \star\omega \tag{3.2.21}$$

In geometric calculus (GC), Gauss's and Stokes's theorems correspond to two parts of the fundamental theorem of calculus Equation (1.2.12) when applied to scalar-valued functions. We obtain Equation (3.2.20) by considering the outer product, $(\mathrm{d}x_p \cdot \nabla) \wedge M$, in the geometric product of Equation (1.2.12), and Equation (3.2.21) by considering the inner product, $(\mathrm{d}x_p \cdot \nabla) \cdot M$.

A key distinction between Equations (3.2.20, 3.2.21) and Equation (1.2.12) lies in their respective scopes of applicability. The generalized Stokes theorem in differential forms applies exclusively to scalar-valued differential forms, whereas the fundamental theorem of geometric calculus applies to multivector-valued fields, encompassing scalars, vectors, bivectors, and higher k-vectors. Thus, Equation (1.2.12) extends the applicability of the generalized Stokes theorem by allowing it to address a broader range of geometric entities, rendering it a more versatile tool in contexts where multivector fields naturally arise.

It is worth noting that both theorems hold in manifolds of any dimension and signature. However, while the applications of the generalized Stokes theorem to gravitational theories is a topic thoroughly investigated, the application of the fundamental theorem of geometric calculus to geometric problems in gravitational theories, to the best of my knowledge, has not been extensively explored, and it could open up intriguing possibilities. In particular, it could yield relevant insights in the study of gravitational waves, black hole spacetimes, and highly geometrical models beyond GR.

3.3 Electrodynamics

In Section 1.3, we briefly reviewed the description of electromagnetism with GC. We recall that, in this framework, the electric field is described by space-time bivectors, while the magnetic field is described by space-space bivectors, accounting for their different behaviors under spatial parity transformations. These are the components of the electromagnetic bivector field F, whose geometric derivative produces Maxwell's equations, Equation (1.3.3). The time-split of F naturally takes the form of the Riemann-Silberstein vector, Equation (1.3.2), without the need for complex numbers. Altogether, this provides an intuitive, elegant, and geometric description of electromagnetic phenomena.

To compare this description with that of differential forms[2], we begin by considering the Faraday two-form $\mathbf{F} = \mathbf{F}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, which is evidently equivalent to the electromagnetic bivector field $F = F_{\mu\nu}g^\mu \wedge g^\nu$. The formulation of Maxwell's equations for $\mathbf{F}$ is not as straightforward as for F, but ultimately they reduce to a pair of equations which are directly relatable to the decomposition into trivector and vector part of Equation (1.3.3):

$$\begin{aligned} \mathbf{dF} = 0 &\leftrightarrow \nabla \wedge F = 0, \\ \star\mathbf{d} \star \mathbf{F} = \mathbf{J} &\leftrightarrow \nabla \cdot F = \jmath_e, \end{aligned} \tag{3.3.1}$$

where $\mathbf{J} = -J_\mu \star (\mathrm{d}x^\mu)$ is a 3-form representing the currents and charges, which is the dual of the vector $\jmath_e$.

As mentioned in Section 1.3, a key advantage of the unification provided by GC is the possibility of a direct inversion of ∇, enabling the calculation of F from the potentials without second-order derivatives.

While the introduction of the Lorentz force in GC is simple and straightforward, Equation (1.3.9), its formulation in differential forms is somewhat cumbersome. To properly define it, we need to introduce the interior product ι_X (not to be confused with the inner product), where X is a vector field.

[2]There has been a debate about the need to introduce twisted forms, also called impair forms, to properly describe EM, as twisted differential forms are necessary for integrations in non-orientable manifolds. If this would be the case, the GA description of electromagnetism would be incomplete, as defining a Clifford algebra on a non-orientable manifold presents challenges.

The debate is divided into two camps: one led by Itin, Obukhov, and Hehl, arguing for the necessity of twisted forms and the incompleteness of the GA description, and another led by Roldao da Rocha and Waldyr A. Rodrigues Jr., defending the non-necessity of twisted forms (since we live in an oriented universe) and the validity of GA. The debate is outlined in the following articles [da Rocha and Rodrigues Jr., 2010, Itin et al., 2010, da Rocha and Rodrigues Jr, 2010].

The interior product is a map that transforms a p-form ω into a $(p-1)$-form, $\iota_X\omega$, such that

$$\omega\left(X, X_1, \ldots, X_{p-1}\right) = \left(\iota_X\omega\right)\left(X_1, \ldots, X_{p-1}\right) \tag{3.3.2}$$

for any vector fields $X_1, \ldots, X_{p-1}$.

In tensor notation, it is written as

$$\left(\iota_X\omega\right)_{\beta\ldots\mu} = \omega_{\alpha\beta\ldots\mu}X^{\alpha}. \tag{3.3.3}$$

In a metric space, we can define a musical isomorphism, making the interior and inner product equivalent. Since we always work within a metric space when dealing with GAs, both operations are equivalent to the inner product $\cdot$ that we have been using so far. The complexity of defining the inner and interior product arises from working in tangent and cotangent spaces in a metric-free manner.

Once the interior product is defined, the Lorentz force can be expressed in the differential forms formalism in a manner similar to Equation (1.3.9):

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = q\iota_u\mathbf{F}, \tag{3.3.4}$$

where u is the 4-velocity of the particle.

To conclude this section, we'll examine the different geometric interpretations of the electromagnetic field in both formalisms. In GA, the electromagnetic bivector field follows the same geometric principles that we discussed previously: it is composed of two types of bivectors—spacetime for E and space-space for B, see Figure 3.5. Such interpretation allows to resolve geometrically electromagnetic phenomena, such as the interaction of a charged particle with the field.

In the differential forms formalism, the electromagnetic 2-form $\mathbf{F}$ is visualized as tubes formed by the intersection of two planes, see Figure 3.6. The "area" of these tubes is associated with the sense of rotation of the 2-form. This depiction arises from the sometimes-confusing description of 1-forms as surfaces. More importantly, and beyond the lack of intuition it offers, this approach fails to provide a geometrically quantifiable method for determining interactions with other phenomena.

In summary, description of electromagnetic phenomena by GC is clearly superior to that of differential forms in several respects including computational power and physical and geometric intuition.

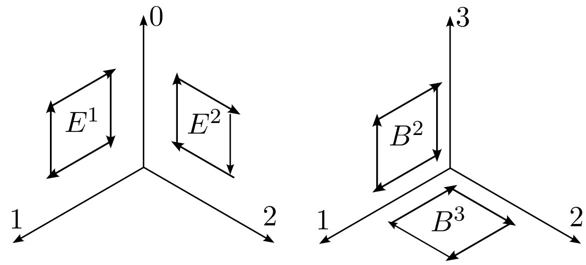


Figure 3.5: Depiction of the electric and magnetic field as interpreted by the Geometric Algebra formalisms. The electric field is a time-space bivector, while the magnetic field is a space-space bivector.

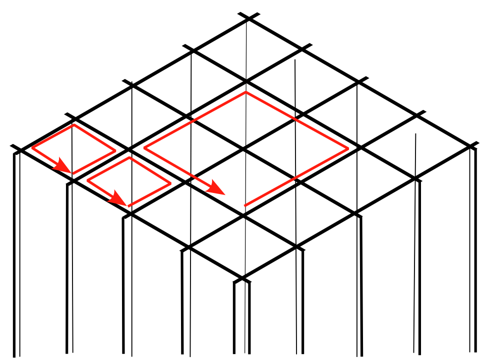


Figure 3.6: Depiction of the electromagnetic field as interpreted by the differential forms formalisms.

3.4 General Relativity and Cartan's formalism

Another domain where differential forms are widely used is GR. There, and in comparison to tensor formalism, it is undeniable that differential forms, together with Cartan's structure equations, provide an efficient way to describe the dynamics of curved manifoldas. But how does differential forms compare to the tetrad-GA formalism presented in Chapter 2? That is the aim of this section.

3.4.1 Frames

In both formalisms, the use of orthonormal frames is prevalent. These are denoted as $\{\theta_{\hat{a}}\}$ in differential forms and $\{\gamma_m\}$ in the tetrad-GA formalism. They relate to the coordinate frame, denoted as $\{\partial_\mu\}$ in differential forms and $\{g_\mu\}$ in tetrad-GA, via the vierbein $e^m{}_\mu$ as follows:

$$g_\mu = e^m{}_\mu \gamma_m \quad \leftrightarrow \quad \partial_\mu = e^{\hat{a}}{}_\mu \theta_{\hat{a}}, \tag{3.4.1}$$

with the inverse vierbein relating the reciprocal frames:

$$g^\mu = e_m{}^\mu \gamma^m \quad \leftrightarrow \quad \mathrm{d}x^\mu = e_{\hat{a}}{}^\mu \theta^{\hat{a}}, \tag{3.4.2}$$

and allowing for the inverse transformations, from the tetrad back to the coordinate frame. The coordinate one-forms are denoted as $\{\mathrm{d}x^\mu\}$ in the differential forms formalism, while in GA they are denoted by g^μ.

Non-holonomic frames and differential forms are sometimes used in the set up of the ADM formalism, so we will briefly explain its relation to our tetrad-GA formalism. The ADM formalism assumes that spacetime is foliated into a family of spacelike hypersurfaces Σ_t with induced metric $\mathbf{g}_{ij}$. So we can write the spacetime metric as

$$\mathbf{g}_{\mu\nu} = \pm \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{g}_{ij} \end{pmatrix}. \tag{3.4.3}$$

The spatial metric $\mathbf{g}_{ij}$ together with the conjugate momenta $\boldsymbol{\pi}^{ij}$, constitute the variables of the theory.

The spatial metric $\mathbf{g}_{ij}$ is generally not orthonormal, however one of the most common choices is

$$\mathbf{g}_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{3.4.4}$$

which corresponds to the same tetrad choice of inertial observers that we use in the tetrad-GA formalism. However, note that there is no restriction on choosing a different $\mathbf{g}_{ij}$ in the ADM formalism, or selecting a different local metric in the tetrad-GA formalism.

3.4.2 Connection elements

Given an arbitrary basis for vector fields $\{e_a\}$ and for one-forms $\{\theta^a\}$, the connection one-forms $\omega^a{}_b$ are defined in terms of the covariant directional derivative in the u-direction, D_u, as

$$D_u e_a = \omega^b{}_a(u) e_b, \tag{3.4.5}$$

where $\omega^b{}_a(u) \in \mathbb{R}$, and thus $\omega^b{}_a \in \bigwedge^1$, with no *a priori* symmetry properties.

For the tetrad-GA formalism we defined the connection bivectors $\omega(u)$ similarly in Equation (2.2.7). The relationship between both objects is simple: the components of $\omega(\gamma_l) = \frac{1}{2}\omega_{mnl}\gamma^m \wedge \gamma^n$ in the tetrad basis $\{\gamma^m\}$ are the same as the components of the connection one-forms in the orthonormal frame of the cotangent space $\{\theta^{\hat{a}}\}$, with $\omega^{\hat{a}}{}_{\hat{b}} = \omega^{\hat{a}}{}_{\hat{b}\hat{c}}\theta^{\hat{c}}$.

$$\omega_{\hat{a}\hat{b}\hat{c}} = \omega_{mnl}. \tag{3.4.6}$$

In terms of interpretation, the bivectorial nature of the connection bivector $\omega(u)$ provides a clearer geometric picture. As seen in Section 2.2.1, when expressed in the orthonormal frame of an inertial observer, $\omega(g_\mu)$ serves as the generator of the Lorentz transformation that such a frame experiences when parallel transported in the μ-direction. A similar geometric direct interpretation is missing for the connection one-forms.

For practical calculations, in torsion-free spaces, connection one-forms are typically obtained using *Cartan's first structure equation*:

$$\mathbf{d}\theta^b = -\omega^b{}_a \wedge \theta^a, \tag{3.4.7}$$

often through a *guess and check method*, as described in [Misner et al., 1973, §14.6].

In the tetrad-GA formalism, we obtain the connection bivectors using Equation (2.2.9). Which, in comparison to the *guess and check* method, has the advantage of being systematic, while providing a considerable reduction in calculation effort when compared with the Christoffel formula.

3.4.3 Curvature two-form

In differential forms, the curvature of the manifold is characterized by the curvature two-form $\Omega^a{}_b$, given by

$$\Omega^a{}_b = \mathbf{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{3.4.8}$$

It relates to the components of the Riemann tensor in the tetrad frame as

$$\Omega^a{}_b = \frac{1}{2}R^a{}_{bcd}\omega^c \wedge \omega^d. \tag{3.4.9}$$

This curvature two-form is completely equivalent to the Riemann tensor bivector discussed in Section 2.4.

The difference between these objects only lies in their interpretation. In the tetrad-GA formalism, the Riemann tensor is a linear map connectiong a coordinate area $A = g_\mu \wedge g_\nu$ to a bivector generator of the rotation experienced by a frame when transported around A. This interpretation is not feasible with differential forms, as only with the geometric product we can formulate a generalized version of complex numbers, which allows us to exponentiate bivectors and use them as generators of rotations. Thus, while the two objects are mathematically equivalent, only the tetrad-GA formalism provides a clear physical interpretation of it.

The description of the Ricci tensor R and energy-momentum tensor T illustrate a problem concerning both formalism, which is symmetric tensors. In GA we circumvent the problem by defining linear maps which satisfy the condition

$$R(a) \cdot b = R(b) \cdot a. \tag{3.4.10}$$

And we obtain the Ricci tensor by contracting the Riemann, Equation (2.4.8).

Differential forms has an equivalent expression using the inner product to define the Ricci one-form

$$\mathrm{Ric}_b = \iota_a \Omega^a{}_b, \tag{3.4.11}$$

while other texts simply restore to component tensor calculations when having to deal with symmetric tensors.

To conclude this section we can say that the treatment of differential geometry in differential forms and tetrad-GA is very similar. The main differences lying in their geometric interpretation and calculation methods, with the latter offering a clearer physical meaning and simpler calculation procedure. However, both formalisms can become a bit cumbersome when dealing with symmetric tensors.

3.5 Conclusion

In this chapter, we presented the fundamental elements of differential forms and their connections to their counterparts in GA and GC. The identification of differential forms with elements of GA depends on their application, as they can correspond either to the basis elements of the reciprocal space or to differential k-vectors.

While differential forms and GA share many similarities and both offer substantial improvements over traditional vector or tensor calculus, GA and GC provide several distinct advantages:

1. GA offers a more intuitive interpretation of quantities and operations.

2. The unification of the inner and outer products into the geometric product leads to more compact and cohesive expressions and integral theorems.
3. Integral theorems in GC are more general, containing more information than the generalized Stokes' theorem in differential forms.
4. An isomorphism between even subalgebras and complex numbers in any dimension extends the theory of analytic functions to spaces of arbitrary dimensions.
5. The unification of Cauchy's theorem from complex analysis with other integral theorems, along with its clear geometric interpretation, is unique to GC.
6. GC significantly simplifies the treatment of rotations and spinors compared to differential forms.

On the other hand, differential forms benefit from a well-established formalism with extensive literature, including numerous applications and textbooks. This maturity makes it accessible and widely used. In contrast, GA and GC suffer from a lack of comprehensive didactic materials, which can lead to a steeper learning curve for beginners.

As a concrete example, we compared the fundamental equations of electrodynamics in both formalisms. In differential forms, Maxwell's equations reduce to a pair of differential equations involving the Faraday two-form. In GC, however, Maxwell's equations achieve a higher level of unification, combining into a single differential equation that can be directly solved without requiring second-order derivatives (see Gull et al. [1993] and Doran and Lasenby [2013, §7.5] for details). An interesting connection, which is not present in differential forms, emerges when projecting the electromagnetic bivector in GC into the observer's frame: it takes the form of the Riemann-Silberstein vector, offering a neat geometric explanation and interpretation.

Furthermore, we found that relativistic electromagnetic fields are more intuitively understood through GC's interpretation of lines and planes, as opposed to the surfaces and tubes pierced by vectors in the differential forms formalism.

We also connected Cartan's formalism of GR with our tetrad-GA formalism, reaching conclusions similar to those in the electrodynamics case. While both formalisms can effectively describe GR, the tetrad-GA formalism better captures the symmetries of the involved objects and offers a clearer geometric understanding of physical processes.

However, both formalisms face challenges when dealing with symmetric tensors, particularly compared to the flexibility of tensor calculus. Tensor calculus, by working directly with coordinates without imposing constraints, is more adaptable but at the cost of cumbersome notation and a lack of clear geometric or physical interpretation.

This raises an important question: *What should we expect from a formalism?* From my perspective, more flexibility is not always the answer. Beyond accurately describing physical phenomena, a good formalism should provide a robust set of tools that minimize the need for external input. Ideally, a mathematical framework should handle the physical content automatically, deriving consequences with minimal ambiguity and computation. Additionally, a formalism should guide us towards meaningful and significant modifications of a theory, while restricting unhelpful ones.

In this light, I believe GC offers the best of all possible options. As noted in [Gu, 2018], "*Clifford algebra faithfully and exactly reflects intrinsic symmetry of spacetime and fields with no more or less content, and automatically classifies the parameters in field equations by grade, which is a definite guidance to set up dynamical equations and compatible constraints of fields.*" Conversely, tensor calculus provides maximum flexibility, though often at the expense of clarity and comprehension. Differential forms occupy a middle ground: their exterior algebra simplifies calculations, but their geometric interpretation of quantities is less transparent, which can present a barrier to newcomers and hinder a deep physical understanding of phenomena.

Chapter 4

The FLRW and Raychaudhuri Equations

As a first application of the tetrad-GA formalism, it might be a good idea to apply it to a well-studied system. This serves two purposes: to compare our formalism with traditional methods and to explore lesser-known aspects of the tetrad-GA formalism in a familiar context. With this scope in mind, I decided to investigate its description of Friedmann-Robertson-Walker (FRW) spacetimes and their relation with Raychaudhuri equations.

While the underlying theory remains unchanged, the GA approach offers a clear comparison with the standard Riemannian formulation, particularly highlighting the transparent geometric properties of FRW spacetimes as symmetric and solvable solutions in relativity.

In Section 4.1, I reiterate the standard approach to FRW spacetimes within GA, using comoving coordinates to derive Friedmann's equations and compare GA with tensor and differential form approaches. In Section 4.2 I employ the Raychaudhuri equations to derive Friedmann's second equation. Because the Raychaudhuri equations are naturally geometric, GA offers for each quantity a clear physical interpretation. Then, motivated by their important relationship to FRW-spacetimes, in Section 4.3 I explore conformal transformations in GA, with insights similar to those in Riemannian geometry regarding the Weyl tensor and spatial flatness.

Section 4.4 discusses spacetime symmetries and conservation laws, expressing Lie derivatives and Killing vectors in GA, noting that a general definition of Lie derivatives is problematic except for Killing fields. Lastly, Section 4.5 addresses scalar field dynamics in FRW backgrounds, showing how quintessence models can be easily expressed in GA and provide an alternative method to derive the slow-roll conditions for accelerated expansion.

The content of this chapter was originally published in [Pérez et al., 2024].

4.1 Friedmann's Equations with Coordinate Intuition

4.1.1 Choice of frame

We start with the FRW metric expressed in hyperspherical (or curvature-normalised) coordinates obtained under the assumptions of homogeneity and isotropy of the universe,

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right]. \tag{4.1.1}$$

Our hyperspherical coordinates are (t, r, θ, ϕ), and k is the usual constant quantifying the curvature of the universe, with the three cases $k = +1$ (open), $k = 0$ (flat) and $k = -1$ (closed).

Suggested by the line element, Equation (4.1.1), the usual choice of basis for the tangent space $T_p\mathcal{M}$ is the coordinate basis $\{g_\mu\}$. This choice of basis has the problem of not being orthonormal, as it can be seen by projecting them into each other, Equation (2.1.3).

To highlight the Minkowskian geometry of the tangent space $T_p\mathcal{M}$, simplify contractions and get a clear distinction in the dependence of objects on coordinates or on vector basis, which remains a problem frequently encountered in GR literature, we make a change of basis in $T_p\mathcal{M}$ to a tetrad, orthonormal, basis.

We recall from Chapter 2, that the transformation relating the coordinate and tetrad basis of $T_p\mathcal{M}$ is called the vierbein and is determined by Equation (2.1.10).

In the FRW case, the coordinate basis vectors are already orthogonal, so we simply need to normalize them to obtain the tetrad base. The matrix form of the vierbein for our FRW spacetime is

$$[e^m{}_\mu] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{a}{\sqrt{1-kr^2}} & 0 & 0 \\ 0 & 0 & ar & 0 \\ 0 & 0 & 0 & ar\sin\theta \end{pmatrix}. \tag{4.1.2}$$

From the conditions in Equation (2.1.12) we can obtain the inverse vierbein

$$[e_m{}^\mu] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{1-kr^2}}{a} & 0 & 0 \\ 0 & 0 & (ar)^{-1} & 0 \\ 0 & 0 & 0 & (ar\sin\theta)^{-1} \end{pmatrix}. \tag{4.1.3}$$

All this considered, the tetrad frame $\{\gamma^m\}$ can be obtained in terms of the coordinate base as

$$\left.\begin{aligned} g_\mu \cdot g_\nu &= g_{\mu\nu} \\ g^\mu g_\nu &= \delta^\mu_\nu \\ \gamma_m \cdot \gamma_n &= \eta_{mn} \\ \gamma^m \gamma_n &= \delta^m_n \\ g_\mu &= e^m{}_\mu \gamma_m \end{aligned}\right\} \quad \Rightarrow \quad \left.\begin{aligned} \gamma^t &= g^t \\ \gamma^r &= \tfrac{a}{\sqrt{1-kr^2}} g^r \\ \gamma^\theta &= a r g^\theta \\ \gamma^\phi &= a r \sin\theta g^\phi \end{aligned}\right\} \tag{4.1.4}$$

The tetrad basis vectors $\{\gamma_m\}$, and their reciprocals, $\{\gamma^m\}$, are related by the Minkowski metric $\eta_{mn} = \text{diag}(+1, -1, -1, -1)$, in our choice of signature, $\gamma_m = \eta_{mn}\gamma^n$.

4.1.2 Connection bivectors

The next element needed to obtain Friedman's equations are the connection bivectors, which we will obtain by direct application of Equation (2.2.9)

$$\begin{aligned} \omega_t &= \frac{1}{2} g^t \wedge g^\lambda \partial_\lambda g_{tt} = 0 \\ \omega_r &= \frac{1}{2} g^r \wedge g^\lambda \partial_\lambda g_{rr} = \frac{\dot{a}}{\sqrt{1-kr^2}} \gamma^t \wedge \gamma^r \\ \omega_\theta &= \frac{1}{2} g^\theta \wedge g^\lambda \partial_\lambda g_{\theta\theta} = \sqrt{1-kr^2}\gamma^r \wedge \gamma^\theta + \dot{a} r \gamma^t \wedge \gamma^\theta \\ \omega_\phi &= \frac{1}{2} g^\phi \wedge g^\lambda \partial_\lambda g_{\phi\phi} \\ &= r\sin(\theta)\dot{a}\gamma^t \wedge \gamma^\phi + \sin(\theta)\sqrt{1-kr^2}\gamma^r \wedge \gamma^\phi + \cos(\theta)\gamma^\theta \wedge \gamma^\phi. \end{aligned} \tag{4.1.5}$$

The overdot here represents a derivative with respect coordinate time $\dot{a} = \partial_t a$.

In an FRW universe in GA, we have 3 connection bivectors, with a total of 6 coefficients. A noticeable reduction in comparison with the 13 Christoffel symbols of tensor calculus.

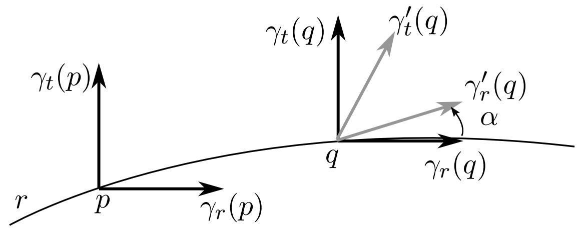


Figure 4.1: Picture to illustrate the transformation of a frame when parallel-displaced in the r direction. At point p we have an inertial frame $\{\gamma_\mu(p)\}$. The relationship between the local frame at q, $\{\gamma_\mu(q)\}$ and the parallel-displaced frame $\{\gamma_\mu'(p)\}$ in grey, is a Lorentz transformation with rapidity $\alpha = \frac{\dot{a}}{\sqrt{1-kr^2}}$, which is generated by the connection coefficient $\omega_r = \frac{\dot{a}}{\sqrt{1-kr^2}}\gamma^t \wedge \gamma^r$

As an example of the geometric interpretation of the connection bivectors let's consider the parallel transport of a tetrad frame along the r-coordinate. The infinitesimal transformation that it will experience when moving from a point p to the neighbor point q is given by the ω_r bivector. Since ω_r only has $\gamma^t \wedge \gamma^r$ components, the transformation will be solely a boost with rapidity $\frac{\dot{a}}{\sqrt{1-kr^2}}$, see Figure 4.1. Notice that $\omega_t = 0$, meaning that frames parallel-displaced in the t direction will experience no rotation or boost.

4.1.3 Riemann curvature, Ricci vector, Ricci scalar and Einstein tensor

From the connection bivectors we can directly use Equation (2.4.2) to obtain the components of the Riemann, $\mathbf{R}_{\mu\nu}$.

In the case of the FRW universe, the components of the Riemann take a particularly compact form in the tetrad basis. Which we obtain with $\mathbf{R}_{mn} = e_m{}^\mu e_n{}^\nu \mathbf{R}_{\mu\nu}$

$$\begin{aligned} \mathbf{R}_{\hat{t}m} &= -\frac{\ddot{a}}{a}\gamma^{\hat{t}} \wedge \gamma^m, \quad m = \hat{r}, \hat{\theta}, \hat{\phi} \\ \mathbf{R}_{mn} &= -\frac{\left(\dot{a}^2 + k\right)}{a^2}\gamma^m \wedge \gamma^n, \quad m, n = \hat{r}, \hat{\theta}, \hat{\phi} \end{aligned} \tag{4.1.6}$$

We now obtain the components of the Ricci map by contracting Equation (4.1.6), as shown in Equation (2.4.8). This task can be performed easily in the tetrad frame due to its orthonormality

$$\begin{aligned} R_{\hat{t}} &= \gamma^n \cdot \mathbf{R}_{n\hat{t}} = -3\frac{\ddot{a}}{a}\gamma^t \\ R_m &= \gamma^n \cdot \mathbf{R}_{nm} = \frac{\left(a\ddot{a} + 2\left(\dot{a}^2 + k\right)\right)}{a^2}\gamma^m, \quad m = \hat{r}, \hat{\theta}, \hat{\phi}. \end{aligned} \tag{4.1.7}$$

To obtain the Ricci scalar, we can perform either a second contraction of the Ricci vector or a direct bivector contraction with the Riemann, Equation (2.4.10). Just as before, contraction in the tetrad frame is easier

$$\mathcal{R} = (\gamma^n \wedge \gamma^m) \cdot \mathbf{R}_{mn} = -6\frac{\left(a\ddot{a} + \dot{a}^2 + k\right)}{a^2} \tag{4.1.8}$$

which recovers the correct result obtained from tensor calculus. Notice that, because the products in GA are non-commutative in general, one should be careful and perform the contractions in the right order, $(\gamma^n \wedge \gamma^m) \cdot \mathbf{R}_{mn} = -(\gamma^m \wedge \gamma^n) \cdot \mathbf{R}_{mn}$.

We are now in disposition to calculate the Einstein vector using Equation (2.5.2), for which we obtain

$$\begin{aligned} G_{\hat{t}} &= 3\frac{(k + \dot{a}^2)}{a^2}\gamma^t \\ G_m &= -\frac{\left(2a\ddot{a} + \dot{a}^2 + k\right)}{a^2}\gamma^m, \quad m = \hat{r}, \hat{\theta}, \hat{\phi} \end{aligned} \tag{4.1.9}$$

Remember from Section 2.5 that the Einstein tensor gives the different 3-dimensional subspaces of the manifold. In the FRW case, it means that the 3-space has curvature given by

$$G(g_t) \cdot g_t = G(e^{\hat{t}}{}_t\gamma_t) \cdot (e^{\hat{t}}{}_t\gamma_t) = G_{\hat{t}} \cdot \gamma_t = 3\frac{(k + \dot{a}^2)}{a^2}. \tag{4.1.10}$$

The fact that $G(g_t) \cdot g_t$ only depends on t, means that, for a given time slice, the curvature of the 3-space is constant in all directions.

4.1.4 Gravitational field equations

As defined by Doran and Lasenby [2013, §12.4.2], the expression for the energy-momentum tensor of a perfect-fluid in GA is

$$T(a) = (\rho + p)a \cdot uu - pa. \tag{4.1.11}$$

Where ρ is the energy density, p the pressure, and u is the 4-velocity of the fluid. We can make a particular choice of a frame where the fluid is at rest and therefore $u = \gamma^t$. From the time component of Einstein's equations, Equation (2.5.4), we get the first of Friedmann's equations

$$\gamma_t \left(3 \left(\frac{\dot{a}}{a} \right)^2 + 3 \frac{k}{a^2} \right) - \Lambda \gamma_t = 8\pi\rho\gamma^t \Rightarrow \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}. \quad (4.1.12)$$

And taking any spatial component of Einstein's equations

$$\left(2\frac{\ddot{a}}{a} + a^{-2} \left(k + \dot{a}^2 \right) \right) \gamma_i - \Lambda \gamma_i = -8\pi p \gamma_i, \quad (4.1.13)$$

we get the second of Friedmann's equation

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = -8\pi p + \Lambda \quad (4.1.14)$$

4.2 Friedmann's equations from Raychaudhuri congruences

An alternative and geometrically intuitive way of obtaining Friedmann's equations is from Raychaudhuri's equation, which describes the evolution of a congruence of geodesics by quantifying the evolution of their enclosed volume. We would like to point out that the Raychaudhuri-equation is simply a statement about the motion of a bundle of test particles through spacetime. How they will actually evolve and be accelerated relative to each other will be determined by the geometrical properties of the spacetime, which in turn are determined by the field equations.

In the FRW-case, the Raychaudhuri congruence has an intersection point as $a \to 0$ in the finite past, and given the current values of the densities and equations of state of the cosmological fluids, no further intersection point in the future. In an FRW-universe, shear and vorticity vanish due to the cosmological symmetries, as these terms effectively introduce anisotropies. Writing Raychaudhuri's equation as the evolution of the volume $\theta = \nabla_\mu u^\mu$ with proper time τ leads to:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{\theta^2}{3} - R_{\mu\nu} u^\mu u^\nu + \nabla_\mu (u^\nu \nabla_\nu u^\mu), \quad (4.2.1)$$

where in FRW-spacetimes proper time is equal to the coordinate or cosmic time t.

In Raychaudhuri's equation, the decomposition of the Riemann curvature into the Ricci- and Weyl-tensors has a particular intuitive interpretation, as only the Ricci curvature can change the enclosed volume. Weyl-curvature, which is absent in FRW spacetimes, would be responsible for a change in the shape of the enclosed volume, in violation of the cosmological symmetries. The Hubble expansion causes volumes to change proportionally to $a^3(\tau) = a^3(t)$, where $a(t)$ as a function depends on the densities and the equations of states of the cosmological fluids as encapsulated by the energy-momentum tensor, and by virtue of the field equation, by the Einstein-tensor, which reflects only Ricci-curvature.

4.2.1 Raychaudhuri congruences

The direct reformulation of Equation (4.2.1) in terms of GA is

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{\theta^2}{3} - R(g_t) \cdot g_t + D \cdot (u \cdot Du)\,, \tag{4.2.2}$$

with u being a vector field tangent to a geodesic congruence. The divergence of the vector field u is denoted by θ

$$\begin{aligned} \theta &= D \cdot u = D_m u^m = D_\mu u^\mu \\ &= \frac{1}{\sqrt{|g|}} \frac{\partial\left(\sqrt{|g|}u^\mu\right)}{\partial x^\mu} = u^\mu \frac{\partial \ln \sqrt{|g|}}{\partial x^\mu} = \frac{\mathrm{d} \ln \sqrt{|g|}}{\mathrm{d}\tau}, \end{aligned} \tag{4.2.3}$$

where $\sqrt{|g|}$ is the square root of the determinant of the metric.

In GA, the covolume $\sqrt{|g|}$ can be derived by considering the two relevant frames: the coordinate frame $\{g_\mu\}$, and the orthonormal tetrad frame $\{\gamma_m\}$ so that the unity pseudoscalar can be constructed with the tetrad vectors as

$$I = \gamma_t \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3, \quad |I| = 1. \tag{4.2.4}$$

Because there is only one pseudoscalar element in a space, the volume element of any coordinate basis vectors, denoted by e, must be a scalar multiplication of I, and its value will be $|e|$.

$$e = g_t \wedge g_1 \wedge g_2 \wedge g_3 = |e|I, \quad |e| = |g_t \wedge g_1 \wedge g_2 \wedge g_3|. \tag{4.2.5}$$

We can explicitly obtain $|e|$ using Equation (2.1.10), and obtaining $\sqrt{|g|} = |e|$ [Hamilton, 2020], which is consistent with the view of the vierbein as the "square root of the metric". Thus, we can write Equation (4.2.3) as

$$\theta = \frac{\mathrm{d} \ln \sqrt{|g|}}{\mathrm{d}\tau} = \frac{1}{|e|}\frac{\mathrm{d}|e|}{\mathrm{d}\tau} = \frac{|e|'}{|e|}. \tag{4.2.6}$$

paving the way to interpreting θ as the relative variation of the volume element of the coordinate basis with respect to the proper time. In this way, one arrives at the geometric interpretation of the left-hand side of Raychaudhuri's equation as

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{|e|}\frac{\mathrm{d}|e|}{\mathrm{d}\tau}\right) = \frac{1}{|e|}\frac{\mathrm{d}^2|e|}{\mathrm{d}\tau^2} - \frac{1}{|e|^2}\left(\frac{\mathrm{d}|e|}{\mathrm{d}\tau}\right)^2 = \frac{1}{|e|}\frac{\mathrm{d}^2|e|}{\mathrm{d}\tau^2} - \theta^2. \qquad (4.2.7)$$

The derivative of θ with respect to the proper time is composed of two terms, the first one is the relative acceleration in the change of the volume element, and the second one is the square of the relative variation of the volume element. With these geometric elements at hand, one can substitute Equations (4.2.6, 4.2.7) into Equation (4.2.2) and re-write Raychaudhuri's equation in a manner that fully reflects its geometric content and turns to be highly similar to the second Friedmann equation, Equation (4.1.14).

$$\frac{|e|''}{|e|} = \frac{2}{3}\left(\frac{|e|'}{|e|}\right)^2 - R(g_t)\cdot g_t + D\cdot(u\cdot Du) \qquad (4.2.8)$$

with $|e|' = \mathrm{d}|e|/\mathrm{d}\tau$ the derivative of the covolume with respect to the proper time.

4.2.2 Friedmann's second equation as a particular case

In an FRW-universe, the last term in Equation (4.2.8) vanishes due to homogeneity. In fact, the Euler-equation for the motion of the cosmological fluids is trivially fulfilled as there are no pressure gradients on spatial hypersurfaces. As a consequence, all fluid elements need to follow geodesics, which are defined through the autoparallelity condition $u\cdot Du = 0$, $u^\nu\nabla_\nu u^\mu = 0$ in tensor notation, producing

$$\frac{|e|''}{|e|} = \frac{2}{3}\left(\frac{|e|'}{|e|}\right)^2 - R(g_t)\cdot g_t. \qquad (4.2.9)$$

which requires the determination of $|e|$ and $R(g_t)\cdot g_t$. For obtaining $|e|$, one can consider for simplicity the vierbein of a FRW-spacetime in FRW-coordinates with a tetrad field aligned with the coordinate frame,

$$\left[e^m{}_\mu\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}. \qquad (4.2.10)$$

and the determinant $|e|$ can be derived immediately from Equation (4.2.5), giving

$$|e| = a^3 \tag{4.2.11}$$

such that volumes increase proportionally to a^3 in the Hubble-expansion, as intuitively expected. Furthermore, FRW-spacetimes possess the peculiarity that proper time and coordinate time are equal, $\tau = t$: The evolution of the volume element $|e|$ solely depends on time, and one can change the derivatives with respect to proper time time in Equation (4.2.9) by derivatives with coordinate time, $|e|' \rightarrow |\dot{e}|$, resulting in

$$3\frac{\ddot{a}}{a} = -R(g_t) \cdot g_t, \tag{4.2.12}$$

with the curvature term, $R(g_t) \cdot g_t$, given from Equations (2.5.2, 2.5.4) applied to g_t, and projecting into the observer's 4-velocity g_t,

$$R(g_t) \cdot g_t = \kappa T(g_t) \cdot g_t + \Lambda + \frac{\mathcal{R}}{2}, \tag{4.2.13}$$

where we used $g_t \cdot g_t = g_{tt} = +1$. For continuing, one requires explicit expressions for $T(g_t) \cdot g_t$ and $\mathcal{R}$. The former term, $T(g_t) \cdot g_t$, results directly from the expression of the energy-momentum tensor of a perfect fluid, Equation (4.1.11). For an FRW-universe in comoving coordinates, the fluid is at rest with respect to the observer's frame, which means $v = g_t$. Therefore, one immediately arrives at

$$T(g_t) \cdot g_t = \rho. \tag{4.2.14}$$

The latter term, $\mathcal{R}$, requires the calculation of the trace of Einstein's field equations, which in GA is computed as

$$g^\mu \cdot \left(R(g_\mu) - \frac{\mathcal{R}}{2} g_\mu = \kappa T(g_\mu) + \Lambda g_\mu \right) \quad \rightarrow \quad -\mathcal{R} = \kappa \operatorname{tr}(T) + 4\Lambda \tag{4.2.15}$$

relating, as expected, the Ricci-scalar with the trace of the energy-momentum tensor. This trace in particular reads as

$$\begin{aligned} \operatorname{tr}(T) &= g^\mu \cdot T(g_\mu) = g^\mu \left((\rho + p) g_t \cdot g_\mu g_t - p g_\mu \right) \\ &= (\rho + p) - p + (-p g^i g_i) = \rho - 3p. \end{aligned} \tag{4.2.16}$$

Combining both results implies for the Ricci scalar the well-known result

$$\mathcal{R} = -\kappa \operatorname{tr}(T) - 4\Lambda = -\kappa(\rho - 3p) - 4\Lambda. \tag{4.2.17}$$

The projection $R(g_t) \cdot g_t$ in Equation (4.2.13) can be obtained using Equations (4.2.14, 4.2.17) and Equation (4.2.13) as

$$R(g_t) \cdot g_t = \kappa\rho + \frac{1}{2}\left(-\kappa(\rho - 3p) - 4\Lambda\right) + \Lambda = \frac{\kappa}{2}(\rho + 3p) - \Lambda. \tag{4.2.18}$$

Combining all these results in Equation (4.2.9) produces exactly second's Friedmann's equation, with the substitution of $\dot{a}$ from Equation (4.1.12), as a particular case of Raychaudhuri's equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{4.2.19}$$

4.2.3 Varying the Friedmann action

A third method to obtain Friedmann's equations is by performing variations on the gravitational action, which is already symmetry reduced according to the cosmological principle. Interchanging variation and symmetry reduction is permissible under certain assumptions, the most notable being a compact symmetry group, but this condition is only necessary and not sufficient [Fels and Torre, 2002, Torre, 2011]. The action of a FRW-spacetime is given by the Ricci-scalar Equation (4.1.8) with a term called lapse function N.

$$S = \frac{1}{8\pi G}\int \mathrm{d}^{(4)}x \left[N\left(\Lambda - \frac{3k}{a^2}\right) + \frac{3\dot{a}^2}{Na^3}\right] \tag{4.2.20}$$

so that variation with respect to a and N as degrees of freedom recovers the Friedmann equations.

The purpose of the lapse function is to introduce the freedom to choose the time parameterization, and it is put in place as the g_{tt}-term of the metric. At first sight, Equation (4.2.20) looks the same as in conventional tensor formalism. However, notice the absence of $\sqrt{|g|}$. This is because $\mathrm{d}^{(4)}x$ is an oriented differential, which means that we can decompose it as

$$\mathrm{d}^{(4)}x = g_0 \wedge g_1 \wedge g_2 \wedge g_3 \, \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = |e|I \, \mathrm{d}^4x. \tag{4.2.21}$$

Where $g_0 \wedge g_1 \wedge g_2 \wedge g_3$ is the pseudoscalar construced from of the coordinates frame and $\mathrm{d}^\mu x$ are the coordinate differentials used to perform the integration. From Section 4.2.1 we recall that the scale factor is the volume element encased by our coordinate basis vectors $|e| = |g_0 \wedge g_1 \wedge g_2 \wedge g_3|$, according to Equation (4.2.5). In FRW-coordinates $|e| = a^3$ and in conformal coordinates $|e| = a^4$.

Because I is constant, it can be pulled out of the integral and we are left with the same scalar integral as in the usual treatment whose variation to recover Friedmann's equation we will note repeat here but can be found in [Düll, 2020].

4.3 Conformal flatness of FRW-spacetimes

FRW-spacetimes are conformally flat, meaning that their metric $g_{\mu\nu}$ result from the Minkowski metric $\eta_{\mu\nu}$ by scaling with a position dependent, strictly positive factor $\alpha^2(x)$,

$$g_{\mu\nu} = \alpha^2(x)\eta_{\mu\nu}. \tag{4.3.1}$$

This implies that, due to the symmetries of the problem, the tetrads can be derived as a scaling from a particular set of coordinates. This is not a general result, and usually an orthonormal frame needs to be defined locally as a transformation of the coordinate frame.

In particular, null-geodesics, characterised by a vanishing line element $\mathrm{d}s^2 = 0$, are invariant under conformal transformations, and conformally flat spacetimes allow a coordinate choice with manifestly Lorentzian light cones: The FRW-line element is commonly expressed in comoving coordinates and physical (or cosmic) time, as it measures the length of the world lines of comoving observers, as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right]. \tag{4.3.2}$$

Introducing conformal time $\mathrm{d}\eta$ through

$$\mathrm{d}\eta = \frac{\mathrm{d}t}{a(t)}, \tag{4.3.3}$$

the line element is reduced to that of Minkowski-spacetime. Here, in spherical coordinates,

$$\mathrm{d}s^2 = a^2(\eta)\left[\mathrm{d}\eta^2 - \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right] \tag{4.3.4}$$

with the scale factor $a(\eta)$ playing the role of the conformal factor $\alpha(x)$, which only depends on time in this case. This reduction can be done in curved FRW spacetimes as well; spatial flatness is not a necessity for conformal flatness.

4.3.1 Weyl curvature

Conformal transformations leave the Weyl curvature invariant, and conformal flatness implies a vanishing Weyl curvature. These two properties have particular relevance to FRW-spacetimes, as the symmetries of the cosmological principle, spatial homogeneity and isotropy, require the Weyl tensor to vanish. Because the dynamics with the scale factor $a(t)$ is a mere conformal transformation, the Weyl-tensor remains zero in time evolution. This is just another way of expressing the fact that the Hubble expansion maintains the FRW symmetries, commonly formulated in the way that the cosmological fluids remain at rest in the comoving frame. The condition for spatial flatness, i.e. that the densities of the fluids add up to $3H(t)^2/8\pi G$, is independent of the cosmological symmetries, so it is no surprise that spatial flatness and conformal flatness are two independent concepts.

In GA, the Weyl tensor is derived from algebraic arguments [Todoroff, Lasenby et al., 1998]. A possible starting point is considering the following property of the Ricci tensor:

$$\partial_a \cdot (R(a) \wedge b) = \partial_a \cdot R(a) b - b \cdot \partial_a R(a) = \mathcal{R} b - R(b), \tag{4.3.5}$$

with ∂_a being the derivative with respect to the vector a, not to be confused with $a \cdot D$, which is the directional derivative in the a direction. ∂_a is mathematically equivalent to ∇, but with the argument where it acts upon made explicit.

Because the Riemann tensor is a function of $a \wedge b$ such that $\partial_a \cdot \mathbf{R}(a \wedge b) = R(b)$, one term must be $R(a) \wedge b$, and to satisfy the anti-symmetry of the argument, it must have also a term $a \wedge R(b)$. Applying $\partial_a \cdot$ to this term one gets

$$\partial_a \cdot (R(a) \wedge b + a \wedge R(b)) = 2R(b) + b\mathcal{R}. \tag{4.3.6}$$

Furthermore, noting that

$$\partial_a (a \wedge b) = 4b - b = 3b, \tag{4.3.7}$$

one arrives at

$$\partial_a \cdot [\frac{1}{2}(R(a) \wedge b + a \wedge R(b) - \frac{1}{6}\mathcal{R} a \wedge b] = R(b) = \partial_a \cdot \mathbf{R}(a \wedge b). \tag{4.3.8}$$

Therefore, it is possible to rewrite the Riemann tensor as

$$\mathbf{R}(a \wedge b) = \frac{1}{2}(R(a) \wedge b + a \wedge R(b)) - \frac{1}{6}\mathcal{R} a \wedge b + C(a \wedge b), \tag{4.3.9}$$

where $C(a \wedge b)$ is an arbitrary traceless, $\partial_a \cdot C(a \wedge b) = 0$, function called the Weyl tensor. Then we can proceed by isolating this particular tensor,

$$C(a \wedge b) \equiv \mathbf{R}(a \wedge b) - \frac{1}{2}\left(R(a) \wedge b + a \wedge R(b) - \frac{1}{6} a \wedge b R \right). \tag{4.3.10}$$

and, as it is shown in [Hestenes and Sobczyk, 1987, §5-5], its traceless and protractionless properties can be expressed as

$$\partial_a C(a \wedge b) = \partial_a \cdot C(a \wedge b) + \partial_a \wedge C(a \wedge b) = 0. \tag{4.3.11}$$

Its application to the coordinate bivector $g_\mu \wedge g_\nu$ and its decomposition into a coordinate base provides the conventional components of the Weyl tensor in tensorial formalism

$$C(g_\mu \wedge g_\nu) = C_{\mu\nu} = \frac{1}{2} C_{\mu\nu\alpha\beta}\, g^\alpha \wedge g^\beta. \tag{4.3.12}$$

Unfortunately, there does not seem to be a computational advantage over conventional Riemannian geometry to prove that the Weyl-tensor is zero for FRW-spacetimes, nor to show its invariance under conformal transformations.

4.3.2 Conformal transformations in GA and conformal flatness

There is a notational advantage in GA with respect to conformal transformations, though: The relative change of the coordinate basis vectors $\{g_\mu\}$ due to curvature can be easily written as

$$g_{\mu\nu} = g_\mu \cdot g_\nu, \tag{4.3.13}$$

where each of the basis vectors is obtained by applying the directional derivative on the coordinate function x,

$$g_\mu = \frac{\partial x}{\partial x^\mu} = \partial_\mu x, \tag{4.3.14}$$

such that we can obtain their reciprocal frame by applying the vector derivative $\nabla = g^\mu \partial_\mu$ to each of the inverse scalar mappings $x^\mu = x^\mu(x)$

$$g^\mu = \nabla x^\mu. \tag{4.3.15}$$

Therefore, the line element is the product of two differential vectors,

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}x^{(1)} \cdot \mathrm{d}x^{(2)}, \quad \text{with} \quad \mathrm{d}x^{(i)} = (\mathrm{d}x^\mu)^{(i)} g_\mu^{(i)} \tag{4.3.16}$$

where $\mathrm{d}x^{(i)}$ is a differential vector with scalar components $\mathrm{d}x^\mu$. Conformal transformations of the metric with a conformal factor $\alpha^2(x)$ are now equivalent to transformation of the coordinate differential $\mathrm{d}x$ as a GA-vector with a single power of $\alpha(x)$:

$$x \mapsto x' \quad \text{implies} \quad g'_\mu = \partial_\mu x' = \alpha(x) g_\mu \tag{4.3.17}$$

for any $\alpha(x) \in \mathbb{R}$. Then, we can write the new metric as a scalar multiplication of the old one,

$$g'_{\mu\nu} = g'_\mu \cdot g'_\nu = \alpha(x)^2 g_{\mu\nu}. \tag{4.3.18}$$

We can also straightforwardly obtain the change in the volume element after a conformal transformation. As explained in Section 4.2.1, the coordinate volume element $\sqrt{|g|}$ is equivalent to the determinant of the vierbein $|e|$. After a conformal transformation, the new volume element e' is given by the outer product of the new coordinate basis vectors, and we can write it in terms of the old coordiante basis volume element e as

$$e' = g'_0 \wedge g'_1 \wedge g'_2 \wedge g'_3 = \alpha(x)^4 g_0 \wedge g_1 \wedge g_2 \wedge g_3 = \alpha(x)^4 e. \tag{4.3.19}$$

Where we can see that a conformal transformation changes the volume element as $\sqrt{|g'|} \to \alpha(x)^4 \sqrt{|g|}$.

In the FRW universe, the volume element e in FRW-coordinates was obtained in Section 4.2.2, with determinant $|e(t)| = a^3(t)$. Under the conformal transformation Equation (4.3.3), the new volume element will be

$$|e'(\eta)| = a(\eta)^4. \tag{4.3.20}$$

In both cases, we can see that the volume element only depends on the time parameter, reflecting the isotropy and homogeneity assumptions.

4.4 Spacetime symmetries and conservation laws

FRW spacetimes are highly symmetric due to the shift- and rotation invariance required by the cosmological principle. There are different possible interpretations of how these symmetries are maintained in the course of the time evolution, or equivalently, how the Hubble expansion is the only dynamical evolution of FRW-spacetime that is compatible with the cosmological symmetries: From a geometric point of view one could argue that the scale factor introduces a conformal scaling of the metric leaving the Weyl-curvature invariant and in fact zero, making sure that the FRW-spacetime pertains only Ricci-curvature in agreement with the cosmological principle. From the point of view of fluid mechanics, diluting the cosmological fluids with the scale factor would lead to the only admissible continuity equation that would not change the spatial uniformity of the fluids, ensured by a trivially fulfilled Euler-equation.

4.4.1 Energy-momentum conservation, continuity and Euler equations

Conservation of energy and momentum can be expressed in terms of a continuity equation, which in GA takes on the form

$$D \cdot T(a) = 0. \tag{4.4.1}$$

Because $T(a)$ is a symmetric tensor, $T(a)$ and its adjoint $\overline{T}(a)$, are identical[1]. By using the identity, $T(a) \cdot b = \overline{T}(b) \cdot a = T(b) \cdot a$ one can rewrite Equation (4.4.1) as

$$D \cdot T(a) = a \cdot T(D) = 0 \Rightarrow T(D) = 0. \tag{4.4.2}$$

Specifying for the energy-momentum tensor of a perfect fluid, Equation (4.1.11), and making D act left and right, we obtain

$$(u \cdot \acute{D})(\acute{\rho} + \acute{p})u + (\rho + p)u(\acute{D} \cdot \acute{u}) + (\rho + p)(u \cdot \acute{D})\acute{u} - Dp = 0 \tag{4.4.3}$$

Where we used the tilde to denote over which terms D acts. Notice that all terms are vectors, including last term which is the gradient of p.

We can reduce this expression by making the following considerations: third term vanishes due to geodesic equation $(u \cdot D)u = 0$, expanding the covariant divergence in second term, with Equation (4.2.6), choosing in FRW-coordinates with $|e| = a^3$, being $u = \gamma_t$ a 4-velocity and considering that the gradient of p reduces to its time derivative due to isotropy. Then, we obtain the continuity equation for FRW

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{4.4.4}$$

4.4.2 Lie derivatives

The point of whether spacetime exhibits certain symmetries is independent of the coordinate choice, which might or might not be adapted to the symmetries at hand. In either case, due to the diffeomorphism invariance, general relativity is perfectly capable to determine the geometric and dynamical properties of spacetime. It is possible, though, to find directions in which a geometric object like the metric is shift-invariant, and this invariance corresponds exactly to a vanishing Lie derivative.

[1] See [Hestenes and Sobczyk, 1987, §2.1] for details about the adjoint map defined in GA.

Because the Lie derivative is a pre-metric construction and Clifford bundles depend on the metric to be defined, defining Lie derivatives on a Clifford bundle in general is a subtle issue. The reason is that varying the metric modifies the inherent structure of the Clifford bundle, which depends fundamentally on the metric. Another important point to consider is that when working with a pre-metric structure, one must differentiate between tangent and cotangent spaces, as the musical isomorphism also relies on the metric. Consequently, different Lie derivatives must be defined for multivector and co-multivector fields(or forms). For more details, see [Rodrigues and de Oliveira, 2007, §4.1 and 9.5], [Leão et al., 2017], and [Heisler, §4.6].

All problems that arise from defining the Lie derivative for general multivector fields vanish if we restrict ourselves to derivatives along Killing vector fields, in the direction of which the metric remains invariant. In this setting, we define the Lie derivative of a vector field a, in the direction of the Killing vector ξ, as

$$\mathcal{L}_\xi a = \xi \cdot \nabla a - a \cdot \nabla \xi, \tag{4.4.5}$$

where $\xi \cdot \nabla$, understood as an interior (not inner) product between the vector field ξ and the vector operator $\nabla = \gamma^\mu \partial_\mu$, is equivalent to the directional derivative in the ξ-direction.

If the spacetime is torsion-free and metric-compatible, one can choose the Levi-Civita connection and write the Lie derivative in terms of the covariant derivative D

$$\mathcal{L}_\xi a = \xi \cdot Da - a \cdot D\xi. \tag{4.4.6}$$

This definition can be generalised to obtain the Lie derivative of multivector fields, which we will need to find the Lie derivative of the metric [Wilson, 2022],

$$\mathcal{L}_\xi A = \xi \cdot DA - \acute{\xi} \wedge (\acute{D} \cdot A). \tag{4.4.7}$$

Because, in general, the Lie derivative does not commute with the metric, it does not follow the Leibnitz rule either with the inner or the geometric product. However, it does so with the outer product because it is a metric-independent operation.

$$\mathcal{L}_\xi (A \wedge B) = (\mathcal{L}_\xi A) \wedge B + A \wedge (\mathcal{L}_\xi B). \tag{4.4.8}$$

We can use the Lie derivative of a multivector to define its action on a tensor field $T(a, b)$ as

$$\mathcal{L}_\xi T(a, b) = \dot{\mathcal{L}}_\xi \dot{T}(a, b) = \mathcal{L}_\xi [T(a, b)] - T(\mathcal{L}_\xi a, b) - T(a, \mathcal{L}_\xi b) \tag{4.4.9}$$

where $\mathcal{L}_v[T(a,b)]$ means evaluating the derivative as a tensor function – that is, deriving also the arguments.

An interesting application of the Lie derivative is that of the metric tensor $g(a,b)$ along with the killing vector ξ

$$\begin{aligned}\mathcal{L}_\xi g(a,b) &= \xi \cdot D(a \cdot b) - g(\mathcal{L}_\xi a, b) - g(a, \mathcal{L}_\xi b) \\ &= (\xi \cdot Da) \cdot b + a \cdot (\xi \cdot Db) \\ &- b \cdot (\xi \cdot Da - a \cdot D\xi) - a \cdot (\xi \cdot Db - b \cdot D\xi) \\ &= a \cdot (b \cdot D\xi) + b \cdot (a \cdot D\xi).\end{aligned} \tag{4.4.10}$$

Spacetime symmetries would now correspond to directions along which the metric does not change. Observing Equation (4.4.10), we see that $\mathcal{L}_\xi g(a,b) = 0$ if and only if ξ satisfies

$$a \cdot (b \cdot D\xi) + b \cdot (a \cdot D\xi) = 0, \tag{4.4.11}$$

which is known as Killing equation. The usual form of it can be obtained replacing $(a,b) \to (g_\mu, g_\nu)$

$$g_\mu \cdot (g_\nu \cdot D\xi) + g_\nu \cdot (g_\mu \cdot D\xi) = 0 \;\Rightarrow\; (\partial_\nu \xi)_\mu + (\partial_\mu \xi)_\nu = 0, \tag{4.4.12}$$

As such, the Killing equation is an eigenvalue equation. This allows us to isolate the symmetries of spacetime in terms of the Killing vectors. Then, we can use that knowledge to choose a coordinate system where one of the killing vectors is a coordinate basis. In that coordinate system, the metric would not depend on that particular coordinate.

4.4.3 Killing vectors

From the GA framework, we can obtain some of the properties of Killing vectors fields immediately. First, Equation (4.4.11) means that $a \cdot D\xi$ is antisymmetric. Therefore, $D\xi$ is completely determined by its protraction, resulting in the bivector Ω,

$$D\xi = D \wedge \xi = 2\Omega. \tag{4.4.13}$$

By recalling that bivectors are the generators of the Lorentz group, it follows that the the bivectors Ω are the generators of the symmetries associated with ξ.

Equation (4.4.13) automatically means that Killing vectors are divergence-free,

$$D \cdot \xi = 0. \tag{4.4.14}$$

This reflects that Killing vector fields are the associated with of a one-parameter family of curves, the trajectories of the isometry, along which the geometry is invariant. If the Killing vector field would not be divergence-free, the trajectories of the isometry may converge or expand, and thus would not keep lengths invariant.

This can be seen by exploring the Lie Derivative of the pseudoscalar I along a vector field v. By applying Equation (4.4.7) we obtain

$$\mathcal{L}_v I = -ID \cdot v. \tag{4.4.15}$$

Meaning that the variation in the pseudoscalar I along the direction of v is governed by the divergence of the vector field v.

From Equation (4.4.13) we see that Killing's equation is equivalent to the requirement of ξ to satisfy

$$a \cdot D\xi = a \cdot \Omega. \tag{4.4.16}$$

This can easily be checked by expanding in coordinates,

$$\begin{aligned} \Omega &= \frac{1}{2}\left(D_\mu \xi_\nu - D_\nu \xi_\mu\right) g^\mu \wedge g^\nu \\ g_\alpha \cdot \Omega &= \frac{1}{2}\left(D_\alpha \xi_\nu - D_\nu \xi_\alpha\right) g^\nu = D_\alpha \xi_\nu g^\nu = D_\alpha \xi \end{aligned} \tag{4.4.17}$$

in the last step, we used Killing's equation Equation (4.4.11).

As a note, Sobczyk [1986] used Equation (4.4.16) to define the Killing vector fields in GA, which is a different route to the one taken here of deriving them from the requirement of vanishing Lie derivative of the metric.

Killing vectors of FRW-spacetimes

Solving Killing's equation for a given spacetime is in general a difficult task. However, we can use the known symmetries of the spacetime to try to guess its Killing vectors. For FRW this is an easy task. Due to the cosmological assumptions of isotropy and homogeneity, we know that the 3-dimensional time slices are maximally symmetric. Therefore, we can use the Killing vectors of the Euclidean space $\mathbb{E}^3$, which are easily obtained in cartesian coordinates, and transform them to spherical coordinates [Deruelle et al., 2018].

The FRW universe exhibits 6 Killing vectors, corresponding to 3 translations and 3 rotations in each time slice. Any linear combination of Killing vectors is a Killing vector, so we can express the general Killing vectors of FRW as

$$\xi = \sum_{i=1}^{3} \alpha_i \xi^{(i)} + \sum_{j=1}^{3} \beta_j \xi^{(j)} \tag{4.4.18}$$

where $\xi^{(i)}$ are the Killing vectors related to translations and $\xi^{(j)}$ are related to rotations.

For the flat FRW universe, the components of the general Killing vector ξ are

$$\begin{aligned}
\xi^t &= 0\\
\xi^r &= \alpha_3\cos(\theta) + \sin(\theta)\,(\alpha_2\sin(\phi)\\
&\quad + \cos(\phi)(\alpha_1 + 2\beta_3 r\cos(\theta)))\\
\xi^\theta &= \frac{1}{r}\,(\cos(\theta)(\alpha_1\cos(\phi) + \alpha_2\sin(\phi)) - \alpha_3\sin(\theta)\\
&\quad + \beta_2 r\sin(\phi) + \beta_3 r\cos(2\theta)\cos(\phi))\\
\xi^\phi &= \frac{1}{r\sin\theta}\,(-\sin(\phi)(\alpha_1 + \beta_3 r\cos(\theta)) + \cos(\phi)(\alpha_2 + \beta_2 r\cos(\theta))\\
&\quad + \beta_1 r\sin(\theta))\,.
\end{aligned} \tag{4.4.19}$$

Notice that there is no time-like Killing vector, reflecting the fact that the FRW universe is curved in the time direction.

As an example, we show the case for $\alpha_1 = 1, \alpha_{i\neq 1} = \beta_i = 0$, which corresponds to a constant translation in the cartesian x-direction,

$$\xi^{(1)} = \sin(\theta)\cos(\phi) g_r + \frac{\cos(\theta)\cos(\phi)}{r} g_\theta - \frac{\csc(\theta)\sin(\phi)}{r} g_\phi. \tag{4.4.20}$$

And we obtain the corresponding bivector by rising the indices of $\xi^{(1)}$ and using Equation (4.4.13)

$$\begin{aligned}
\Omega^{(1)} &= a\dot{a}\sin(\theta)\cos(\phi) g^t \wedge g^r\\
&\quad + a\dot{a}r\cos(\theta)\cos(\phi) g^t \wedge g^\theta\\
&\quad - a\dot{a}r\sin(\theta)\sin(\phi) g^t \wedge g^\phi.
\end{aligned} \tag{4.4.21}$$

Conserved quantities

By Noether's theorem, we know that any continuous symmetry is associated with a conserved quantity. For an isometry of the metric, the conserved quantity associated with it is the projection of the Killing vector along a geodesic with tangent vector v,

$$v \cdot D(v \cdot \xi) = (v \cdot Dv) \cdot \xi + v \cdot (v \cdot D\xi) = v \cdot (v \cdot \Omega) = (v \wedge v) \cdot \Omega = 0. \quad (4.4.22)$$

where $(v \cdot Dv) \cdot \xi$ vanishes due to auto-parallelity expressed by the geodesic equation, $v \cdot Dv = 0$. The last term vanishes too because $v \wedge v = 0$.

The prior example provides an illustrative case. If we express $\xi^{(1)}$ in cartesian coordinates (t, x, y, z), $\xi^{(1)} = g_x$. For a geodesic with tangent vector v

$$v \cdot D(v \cdot \xi) = v \cdot D(v_x) = 0. \quad (4.4.23)$$

Meaning that the v_x component is conserved along the geodesic. If we take into account that free-falling bodies follow geodesics, Equation (4.4.23) means that the momentum in the x-direction is conserved.

Repeating the argument for the general Killing vector ξ in Equation (4.4.19) we find that linear and angular momentum are conserved for each time slice of the FRW universe.

Another conserved quantity along Killing vector fields was obtained in Equation (4.4.15), where we saw that by taking into account that Killing vector fields are divergence-less we obtain

$$\mathcal{L}_\xi I = -ID \cdot \xi = 0. \quad (4.4.24)$$

Meaning that, along Killing vector fields, the spacetime geometry is invariant and therefore the pseudoscalar is conserved.

Tensor symmetries

Another important consequence of the existence of Killing vector fields is the restriction that they impose on the degrees of freedom of the Riemann tensor. Consider the following equation relating covariant derivatives and the Riemann tensor,

$$\mathbf{R}(a \wedge b) \cdot c = b \cdot D(a \cdot Dc) - a \cdot D(b \cdot Dc) - (\mathcal{L}_a b) \cdot Dc = c_{;ab} - c_{;ba}. \quad (4.4.25)$$

Where we have introduced the common notation, $a \cdot Dc = c_{;a}$.

If choose c to be a Killing vector ξ, we obtain

$$\mathbf{R}(a \wedge b) \cdot \xi = b \cdot D(a \cdot D\xi) - a \cdot D(b \cdot D\xi) = b \cdot D(a \cdot \Omega) - a \cdot D(b \cdot \Omega). \quad (4.4.26)$$

Dotting with a general vector c and using $R(A) \cdot B = R(B) \cdot A$ we arrive at

$$\mathbf{R}(a \wedge b) \cdot (\xi \wedge c) = (a \wedge b) \cdot \mathbf{R}(\xi \wedge c) = a \cdot (\Omega_{;b}) \cdot c - b \cdot (\Omega_{;a}) \cdot c. \quad (4.4.27)$$

By taking derivatives with respect to a, followed by derivatives with respect to b and then using

$$D \wedge D \wedge \xi = 0 \;\leftrightarrow\; \Omega_{;a} + \partial_b \wedge (\Omega_{;b} \cdot a) = 0, \tag{4.4.28}$$

we find

$$\mathbf{R}(c \wedge \xi) = -\partial_b \wedge (\Omega_{;b} \cdot c) = -(\partial_b \wedge \Omega_{;b}) \cdot c + \Omega_{;c} = c \cdot D\Omega. \tag{4.4.29}$$

From this relation, it is straightforward to recover the known equation relating the second directional covariant derivative of Killing vectors and the Riemann tensor

$$\mathbf{R}(\xi \wedge c) \cdot a = \xi_{;ac} = c \cdot \acute{D}(a \cdot D\acute{\xi}). \tag{4.4.30}$$

Another useful equation relating the Ricci tensor and the Laplacian of Killing vector fields can be obtained by contracting Equation (4.4.29)

$$R(\xi) = D \cdot \Omega = D\Omega = D^2\xi. \tag{4.4.31}$$

The last proof that we present here is that the Lie derivative of a Killing vector is also a Killing vector,

$$\xi_3 = \mathcal{L}_{\xi_1}\xi_2 = \xi_1 \cdot D\xi_2 - \xi_2 \cdot D\xi_1 = \xi_1 \cdot \Omega_2 - \xi_2 \cdot \Omega_1. \tag{4.4.32}$$

Calculating now the directional derivative of ξ_3,

$$\begin{aligned} a \cdot D\xi_3 &= (a \cdot \Omega_1) \cdot \Omega_2 - (a \cdot \Omega_2) + \xi_1 \cdot \Omega_{1;a} - \xi_2 \cdot \Omega_{2;a} \\ &= a \cdot (\Omega_1 \times \Omega_2) + \xi_1 \cdot R(a \wedge \xi_2) - \xi_2 \cdot R(a \wedge \xi_1) \\ &= a \cdot (\Omega_1 \times \Omega_2) + a \cdot 2R(\xi_1 \wedge \xi_2) = a \cdot \Omega_3, \end{aligned} \tag{4.4.33}$$

one sees that ξ_3 is a Killing vector with bivector $\Omega_3 = (\Omega_1 \times \Omega_2) + 2\mathbf{R}(\xi_1 \wedge \xi_2)$.

4.5 Quintessence Lagrange density

The formalism of GA is foremost a tool for geometric objects with internal degrees of freedom and has limited additional power over conventional methods when dealing with scalar quantities. Nevertheless, because of the importance of scalar fields in cosmology, in particular at early times driving cosmic inflation and at late times in the context of quintessence dark energy, we revisit their fundamental constructions [Wetterich, 1988, Peebles and Ratra, 1988, Ratra and Peebles, 1988, Peebles and Ratra, 2003] from the point of view of GA.

4.5.1 Quintessence equation of motion

The field equation of a scalar field ϕ is given by

$$\Box\phi = D^2\phi = D(D\phi) = D\cdot v = \frac{1}{|e|}\frac{\partial\left(|e|(D\phi)^\mu\right)}{\partial x^\mu}. \tag{4.5.1}$$

Where one can write $D^2\phi = D\cdot v$ because $D\wedge D\phi = 0$, which is equivalent to chosing the torsion-free condition $D\wedge g^\mu = 0$, and requiring integrability condition for scalar fields [Hestenes, 2020]:

$$D\wedge D\phi = D\wedge g^\mu\partial_\mu\phi = g^\nu\wedge g^\mu\partial_\nu\partial_\mu\phi \overset{!}{=} 0 \;\rightarrow\; \partial_\nu\partial_\mu\phi = \partial_\mu\partial_\nu\phi \tag{4.5.2}$$

For the particular case of FRW spacetimes in FRW-coordinates, one has $|e| = a^3$ and the homogeneity assumption only allows time derivatives, $\partial_\mu \rightarrow \partial_t$, such that one can write Equation (4.5.1) as

$$D^2\phi = \frac{1}{a^3}\partial_t\left(a^3\dot\phi\right) = \frac{1}{a^3}\left(3\dot a a^2\dot\phi + a^3\ddot\phi\right) = 3\frac{\dot a}{a}\dot\phi + \ddot\phi, \tag{4.5.3}$$

where the dot represents derivatives with respect to cosmic time, ∂_t.

As shown by Lasenby et al. [1993], the Euler-Lagrange equations for multivector fields ψ are

$$\partial_\psi\mathcal{L} - (\partial_{D\psi}\acute{\mathcal{L}})\acute{D} = 0, \tag{4.5.4}$$

which, in the case of scalar fields ϕ, reduces to the familiar result

$$\partial_\phi\mathcal{L} - D\left(\partial_{D\phi}\mathcal{L}\right) = 0. \tag{4.5.5}$$

Then, for a standard Lagrangian defining the dynamics of the scalar field ϕ

$$\mathcal{L} = \mathcal{L}(\phi, D\phi) = \frac{1}{2}\left(D\phi\right)^2 - V(\phi) \tag{4.5.6}$$

with $V(\phi)$ being a potential. Evaluation of the Euler-Lagrange Equation (4.5.5) yields

$$\frac{\partial\mathcal{L}}{\partial\phi} = -\frac{V(\phi)}{\partial\phi} \quad\text{and}\quad \frac{\partial\mathcal{L}}{\partial(D\phi)} = D\phi. \tag{4.5.7}$$

Which immediately suggests that the equation of motion for ϕ is,

$$\ddot\phi + 3\frac{\dot a}{a}\dot\phi = -\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi}. \tag{4.5.8}$$

4.5.2 Slow-roll conditions

Accelerated expansion takes place if the equation of state of the dominating cosmological fluid is sufficiently negative, $w = p/\rho < -1/3$, and for solving the flatness problem one needs this accelerated expansion to be maintained for a sufficiently long time. Both conditions can be formulated in terms of derivatives of the quintessence potential $V(\phi)$, which becomes quite apparent in the Raychaudhuri equation, which relates the acceleration in the change of the volume element $|e|$ to the Ricci-curvature

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 \ln|e|}{\mathrm{d}\tau^2} = -\frac{\theta^2}{3} - R(g_t)\cdot g_t. \tag{4.5.9}$$

As the Ricci-curvature is determined by the energy-momentum tensor, one can use Equation (4.2.9) to obtain constraints on ϕ: Specifically, rewriting the energy-momentum tensor of the field $\phi = \phi(t)$ in the language of GA yields

$$T^\phi(a) = D\phi(a\cdot D)\phi - a\mathcal{L}. \tag{4.5.10}$$

From this expression, one can recover the usual tensorial components as

$$T^\phi_{\mu\nu} = g_\mu \cdot T^\phi(g_\nu) = (g_\mu\cdot D)\phi(g_\nu\cdot D)\phi - g_\mu\cdot g_\nu\mathcal{L}. \tag{4.5.11}$$

By component-wise comparison of Equation (4.5.10) with the energy-momentum tensor of a perfect fluid, Equation (4.1.11), it is possible to identify the energy density and pressure associated with the scalar field ϕ,

$$\rho = g_t\cdot T^\phi(g_t) = (\partial_0\phi)^2 - \frac{1}{2}(D\phi)^2 + V(\phi) = \frac{1}{2}\dot\phi^2 + V(\phi) \tag{4.5.12a}$$

$$p = \frac{1}{3}\mathrm{tr}(T^\phi) = \frac{1}{3}g^i\cdot T^\phi(g_j) = \frac{1}{2}\dot\phi^2 - V(\phi). \tag{4.5.12b}$$

Where, in the particular case of a spatially homogeneous scalar field ϕ, it can only depend on time. Combining Equation (4.2.13) with Equation (4.5.12a) and Equation (4.2.17) yields an expression for the Ricci-curvature that directly depends on the scalar field ϕ and its derivative $\dot\phi$,

$$\mathcal{R} = \kappa(\dot\phi^2 - 4V(\phi)) - 4\Lambda, \tag{4.5.13}$$

and similarly, one can obtain the Ricci vector over g_t as

$$R(g_t)\cdot g_t = \kappa T(g_t)\cdot g_t + \frac{\mathcal{R}}{2} + \Lambda = \kappa(\dot\phi^2 - V(\phi)) - \Lambda. \tag{4.5.14}$$

Therefore, the evolution of the volume element as given by Equation (4.5.9) results in

$$\frac{\mathrm{d}^2 \ln |e|}{\mathrm{d}\tau^2} = -\frac{\theta^2}{3} - \kappa(\dot{\phi}^2 - V(\phi)) + \Lambda, \tag{4.5.15}$$

with accelerated expansion taking place under the condition that $\Lambda > 0$ or that the kinetic energy $\dot{\phi}^2$ of the quintessence field is smaller than its potential energy $V(\phi)$,

$$\dot{\phi}^2 \ll V(\phi), \tag{4.5.16}$$

for obtaining the right sign in equation Equation (4.5.15).

4.6 Conclusions

In this chapter we presented the description of Friedmann-Robertson-Walker spacetimes in the language of geometric algebra. GA simplifies many calculations and allows for a clear interpretation of the physical situation. We did not aim for finding unknown physical properties of FRW-spacetimes, which would be a surprise given that they are well-known and well-investigated. Rather, our intention was to apply the formalism of GA to a simple, physically well-defined system in order to demonstrate the power of the formalism and its geometric concepts.

This is particularly true for the Raychaudhuri-equation for the evolution of congruences, i.e. of spacetime volumes bounded by geodesics. Within GA the velocity divergence $\theta = \nabla_\mu u^\mu$ turns out to be the relative expansion of the spacetime volume element, as influenced by the presence of Ricci-curvature, while vorticity and shear are absent in FRW-spacetimes. It is possible to recast the Raychaudhuri-equation into a form highly similar to the second Friedmann's equation, with the interpretation of volume evolution as a consequence of the Hubble-expansion.

The cosmological principle imposes a very high level of symmetry onto FRW-spacetimes and renders them conformally flat. Unfortunately, there does not seem to be a computational advantage to using GA in terms of conformal invariance of the Weyl-tensor. However, the discussion showed that the square root of the determinat of the metric, $\sqrt{|g|}$, is simply the value of the coordinate volume element within GA. Given the highly symmetric nature of FRW-spacetimes, the role of Lie-derivatives and Killing-vectors should be clarified: One can not define a general Lie derivative due to the necessity of having to fix the metric prior in order to define a Clifford space. In Clifford spaces, the Lie derivative can be properly defined only for Killing fields. Effects of repulsive gravity caused by scalar, self-interacting fields in a state of slow roll can easily be described in GA. In particular, in combination with Raychaudhuri's equations, the slow-roll conditions are recovered.

In summary, GA proved to be an excellent coordinate-free formalism to deal with the geometry and dynamics of FRW-spacetimes. It is concise and helps could help both students and researchers to develop better intuition for the topic. In the next chapter we will explore another well-known system, the Morris-Thorne wormhole and use it to explore the Fermi transport equations in GA.

Chapter 5

Fermi-Walker transport and frame rotations

Among the various advantages of the tetrad-GA formalism discussed in Chapter 1, one of the most distinct and powerful features is its handling of rotations and Lorentz transformations using bivectors and rotors. In this chapter, we will further explore these techniques, starting with an examination of the Fermi-Walker equations within the tetrad-GA framework in Section 5.1. We will then apply these concepts to determine the acceleration required to keep a spaceship in orbit around the throat of a wormhole, as detailed in Sections 5.2 and 5.3.

We will demonstrate that the bivector formulation of the tetrad-GA formalism is particularly effective in describing non-geodesic motion by introducing a general bivector term into the geodesic equation, which naturally accounts for external rotations or boosts experienced by the transported frame.

This chapter will establish the foundation and set the stage for calculating the frame precession of an observer as it orbits a Schwarzschild and a Kerr black hole in the following chapter.

5.1 Rotor techniques and Fermi-Walker transport

The method for describing the transport of frames along curves using rotors was originally developed by Hestenes and Sobczyk [1987, §6-3], and we will follow their derivation. We begin by considering a time-like curve $c = x(\tau)$ with tangent vector $u = \mathrm{d}x/\mathrm{d}\tau$, where $u^2 > 0$. By the chain rule, the directional covariant derivative along the curve is equivalent to the total

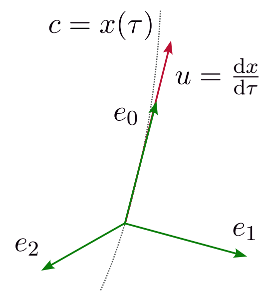


Figure 5.1: Orthonormal frame $\{e_k\}$ along the curve c, aligned such that $e_0 \propto u = \dot{c} = \mathrm{d}x/\mathrm{d}\tau$.

derivative with respect to the affine parameter τ,

$$u \cdot D = \frac{\mathrm{d}x}{\mathrm{d}\tau} g_0 \cdot g^\mu D_\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}. \tag{5.1.1}$$

Now, suppose we have an orthonormal frame $\{e_k\}$ with one of its axes aligned with the tangent vector of the curve,

$$e_0 = u = \frac{\mathrm{d}x}{\mathrm{d}\tau}. \tag{5.1.2}$$

See Figure 5.1. We aim to transport this frame along the curve such that the vector e_0 remains tangent to the curve.

We write this condition as the directional covariant derivative of the frame $\{e_k\}$ along $c(\tau)$. To maintain the orthonormality of the frame, the variation of the frame must correspond to a proper, orthochronous Lorentz transformation. Following the approach used to derive connection bivectors, we can express this variation as the inner product of the frame with a general bivector $\Omega = \Omega(\tau)$, which we seek to determine:

$$(u \cdot D)e_k = \frac{\mathrm{d}}{\mathrm{d}\tau} e_k = \dot{e}_k = \Omega \cdot e_k. \tag{5.1.3}$$

Solving for Ω is straightforward by right-multiplying by e^k and using the property $e_k e^k \cdot M_r = r M_r$ (where M_r is a blade of grade r),

$$\Omega = \frac{1}{2}\dot{e}_k e^k = \frac{1}{2}\dot{e}_k \wedge e^k. \tag{5.1.4}$$

Since our curve is time-like, u represents the 4-velocity of the particle describing the curve, and thus $\dot{e}_0 = \dot{u}$. This allows us to expand Ω to include both time and space components:

$$\Omega = \dot{u} \wedge u + B, \tag{5.1.5}$$

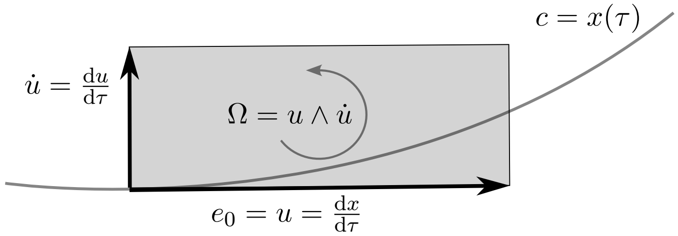


Figure 5.2: A time-like curve $c(\tau)$ with its corresponding tangent vector $u = \frac{dc}{d\tau}$. The angular acceleration required to keep e_0 tangent to $c = x(\tau)$ is given by the bivector $\Omega = u \wedge \dot{u}$.

where $B = \dot{e}_i \wedge e^i$ is a general spatial bivector satisfying $B \cdot u = 0$, representing the spatial rotation of the frame.

A key physical element of this approach is the normalization of the 4-velocity. With this normalization, the 4-acceleration is always orthogonal to the 4-velocity, enabling us to represent it as a rotor that changes the direction of the tangent vector to the worldline of the particle; see Figure 5.2. Techniques for handling non-constant 4-velocities remain to be developed.

In the case where $B = 0$, meaning transport without any additional spatial rotation, the bivector $\Omega = \dot{u} \wedge u$ acts as the generator of the transformation along the curve and represents the angular velocity necessary to keep e_0 tangent to the curve. When this condition is met, we say that $\{e_k\}$ is *Fermi-Walker* transported along the curve, and Equation (5.1.3) becomes the Fermi-Walker transport equation in GA.

If we describe our spacetime using local orthonormal frames (tetrads) $\{\gamma_k\}$, we can relate the transported frame $\{e_k\}$ to the local tetrad at each point by another Lorentz transformation, given by

$$e_k = R(\tau)\gamma_k R(\tau)^\dagger. \tag{5.1.6}$$

It is important to note that this is a local equation, valid at each point of the trajectory, but not globally.

Therefore, the dynamics of $\{e_k\}$ are entirely contained in the rotor $R(\tau)$, allowing us to focus on the dynamics of $R(\tau)$ rather than analyzing the evolution of each axis individually. This significantly simplifies calculations.

To begin, we express the variation of $\{e_k\}$ along the curve $x(\tau)$ as a covariant directional derivative and as a rotation with the bivector Ω:

$$u \cdot De_k = \partial_u e_k + \frac{1}{2}\left[\omega(u), e_k\right] = \Omega \cdot e_k, \tag{5.1.7}$$

where $\partial_u = u \cdot \nabla = u^\mu \partial_\mu$ is the partial derivative in the u-direction.

We obtain an expression for $\partial_u e_k$ by differentiating Equation (5.1.6) with respect to the proper time τ and substituting the result into Equation (5.1.7), yielding a differential equation for the rotor $R(\tau)$:

$$\dot{R}(\tau) = \frac{1}{2}\left(\Omega - \omega(u)\right) R(\tau). \tag{5.1.8}$$

This is the central equation we need to solve in the following sections. Fortunately, because $\partial_u \gamma_m = 0$ by definition, the techniques for solving this equation locally are identical to those used on a flat manifold, which further simplifies the calculations. Furthermore, as we will see in the following sections, it is often the case that $\omega(u)$ and Ω are independent of τ, making the solution to Equation (5.1.8) straightforward.

The advantages of this approach become evident when compared to the conventional formalism of tensor or differential forms, where considerable index manipulation is required to describe frame precession. These complexities arise not only from component-based descriptions but also from the use of cross products and the representation of rotations and angular velocities as vectors, rather than bivectors.

To circumvent these difficulties, conventional methods project into three-dimensional subspaces, introduce pseudovector notation, and use the cross product. These approaches hinder a fully covariant treatment and obscure the geometric and physical significance of the equations.

A particularly illuminating example is the calculation of the Lense-Thirring effect. In [Straumann, 2013, p. 57], for example, the one-form representing the spin rotation in a stationary field is given by

$$\mathbf{\Omega} = \frac{1}{2}\frac{\star\left(\mathbf{K} \wedge \mathrm{d}\mathbf{K}\right)}{\langle K, K\rangle}, \tag{5.1.9}$$

where $K = \partial/\partial t$ (or g_t in our notation) is a Killing vector field, and $\mathbf{K} = g_{00}\,\mathrm{d}t + g_{0i}\,\mathrm{d}x^i$ is the corresponding differential form. The derivation of Equation (5.1.9) is lengthy and involves several pages of differential form manipulations.

For instance, consider the long-distance approximation of a stationary field:

$$g_{tt} \approx 1, \quad g_{ij} \approx -\delta_{ij}, \quad \frac{g_{ti}}{g_{tt}} \ll -1. \tag{5.1.10}$$

In the tensor or differential form formalism with adapted coordinates,[1] Equation (5.1.9) reduces to

$$\mathbf{\Omega} \approx \frac{1}{2}\vec{\nabla} \times \vec{g}, \tag{5.1.11}$$

[1] These are local coordinates induced by the local tetrad.

the curl of the three-dimensional gravitational field in vector notation.

The same calculation becomes significantly simpler using the GA techniques. First we note that in adapted coordinates, the orbiting body only moves in the g_t-direction, so the only relevant connection coefficient is $\omega(g_t) = \omega_t$, describing frame rotation when displaced along the t-direction.

To obtain ω_t, we observe that because the metric is static, the second term of Equation (2.2.9) vanishes:

$$\omega_t = \frac{1}{2}\left(g^\nu \wedge \nabla(g_{t\nu}) + g^\sigma \wedge \cancelto{0}{\partial_t g_\sigma}\right). \tag{5.1.12}$$

The remaining term simplifies to a curl that matches Equation (5.1.11):

$$\begin{aligned}\omega_t &= \frac{1}{2} g^\nu \wedge \nabla(g_{t\nu})\\ &= \frac{1}{2}\left(g^t \wedge \cancelto{0}{\nabla(g_{tt})} + g^i \wedge \nabla(g_{ti})\right) \approx \frac{1}{2}\nabla \wedge (g^i g_{ti})\\ &= \frac{1}{2}\left(g^t \cancelto{0}{\partial_t \wedge (g^i g_{ti})} + \vec{\nabla} \wedge \vec{g}\right) = \frac{1}{2}\vec{\nabla} \wedge \vec{g}.\end{aligned} \tag{5.1.13}$$

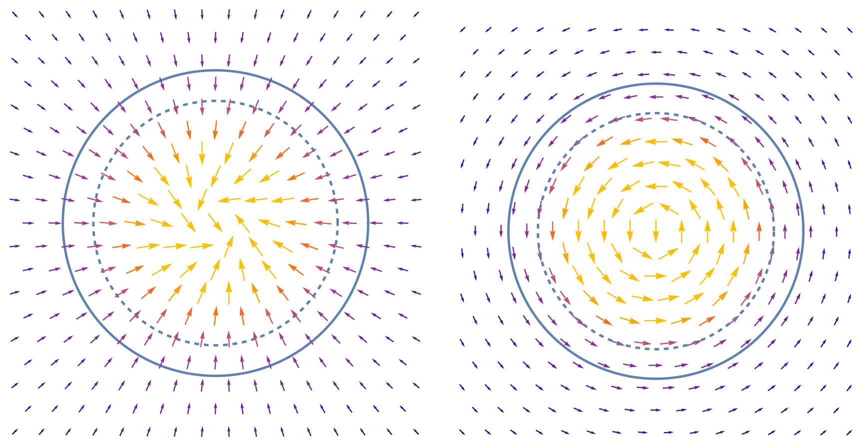
In the third step, we pushed g^j inside the derivative because the Lens-Thirring metric is asymptotically flat in the long-distance regime and therefore the coordinate basis vectors are approximately constant. Also, we defined $\vec{g} := g^i g_{ti}$ with $i = 1, 2, 3$ and used $\vec{\nabla} = g^i \partial_i$.

We can obtain the rotor responsible for the frame rotation with respect to the local tetrad by integrating Equation (5.1.8). Since nothing depends on t, the integration is straightforward:

$$R(\tau) = \exp\left(-\frac{1}{2}\omega_t \tau\right). \tag{5.1.14}$$

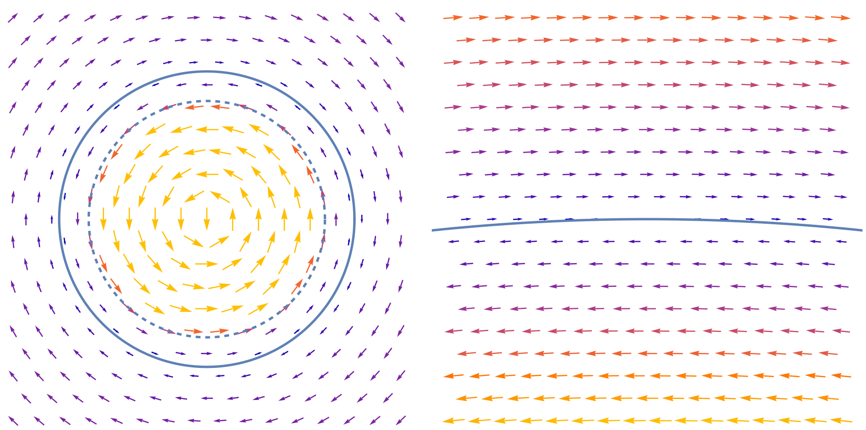
Equation (5.1.14) allows us to determine the orientation of the orbiting frame at any point along its trajectory.

To visualize the field's curl as experienced by a local orbiting frame see the following figures: Figure 5.3a illustrates the far field approximation of the Lens-Thirring while Figure 5.3b shows only the angular component.Then Figure 5.4a illustrates the angular part in adapted coordinates and arbitrary units obtained by substracting the angular velocity of the orbiting particle to the angular component of the field. Evidently, the field obtained by such operation is only valid in the neighborhood of the orbit, which is showed in Figure 5.4b, and from where we can see that the orbiting body experiences



(a) Full field. (b) Angular part of the field.

Figure 5.3: Graphical depiction of the distant gravitational field generated by a slowly rotating massive source. The solid line, represents the orbit and the dashed line the boundary of the source of the field.



(a) Angular part of the field with the angular velocity of at the orbit substracted. (b) Local field as experienced by the orbiting particle

Figure 5.4: Graphical depiction of the angular part of the distant field generated by a slowly rotating massive source in adapted coordinates as experienced by an orbiting particle. The solid line, represents the orbit and the dashed line the boundary of the source of the field.

a local field with a non-zero curl due to the radial gradient of the field's angular component.

As an example of the application of these techniques, in the following sections, I will examine the acceleration necessary to maintain a spaceship's orbit around a wormhole.

5.2 Traversable wormhole

In the following two sections, I draw inspiration from the influential paper *Wormholes in Spacetime and Their Use for Interstellar Travel: A Tool for Teaching General Relativity* [Morris and Thorne, 1988]. Michael Morris and Kip Thorne's 1987 article is a pedagogical gem that employs wormhole metrics to teach basic GR concepts interactively and intuitively. The simplicity of their models facilitates physical intuition in GR, and their rigorous yet intuitive presentation mirrors the style of their classic book *Gravitation* [Misner et al., 1973]. Their use of tetrads, while not standard in initial GR expositions, effectively illustrates physical effects and simplifies the treatment of spacetime by decoupling coordinate from frame degrees of freedom. Building on these pedagogical strengths, I apply the tetrad-GA formalism to their wormhole models, aiming to recreate a compelling and educational resource for teaching and familiarizing students and researchers with this formalism.

Utilizing the tools introduced in Chapter 2, I derive the connection bivectors, Riemann tensor, Ricci tensor, Ricci scalar, and Einstein tensor for a general wormholes solution and arrive at the same set of scalar equations governing the wormhole characteristics. Then, I specify our calculations for a particularly simple and analytical case which I use to illustrate the Fermi-Walker transport techniques that I presented before. This section benefited greatly from the contributions of Cheyenne Leize, as part of her bachelor thesis [Leize, 2022]. Her exceptional work, especially given her initial lack of experience in GA and GR, highlights both her capabilities and the pedagogical effectiveness of our approach.

As it was originally explained in [Morris and Thorne, 1988], we require the following characteristics to the metric for it to describe the geometry a traversable wormhole:

- To simplify calculations, we require the metric should be spherically symmetric and static.
- The solution must have a throat connecting two flat regions of spacetime.

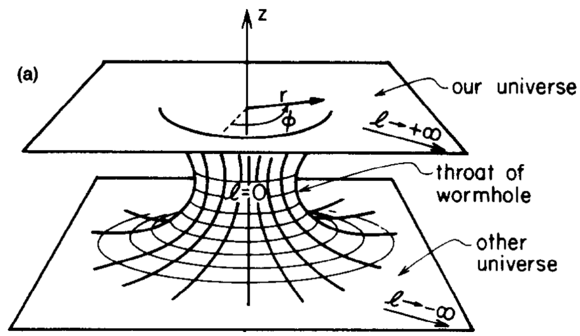


Figure 5.5: Diagram of a toy model wormhole with metric corresponding to Equation (5.3.3) in the text. Notice that the system is spherically symmetric, even though the depiction is cylindrical. Image retrieved from [Morris and Thorne, 1988].

- There should be no horizon, since the presence of one would impede two-way traveling.
- The experienced tidal gravitational forces must be bearable for humans.
- A trip through the wormhole should be of the order of a year both for the traveler and outside observers.
- The energy-momentum tensor generating the desired geometry should have a positive energy-density, as measured by any observer.
- The solution must be perturbatively stable.
- The creation of the wormhole must be relatively feasible. That is, it must requires less energy and mass than that in the universe, and should take less time than the age of the universe.

By requiring the solution to be spherically symmetric, we can write the metric as

$$\mathrm{d}s^2 = e^{2\Phi}\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1-b/r} - r^2(\mathrm{d}\theta^2 + \sin\theta\mathrm{d}\phi^2). \tag{5.2.1}$$

Where $\Phi = \Phi(r)$, called the redshift function, and $b = b(r)$, called the shape function, are two functions of r, whose properties need to be determined. The coordinates here, $\{t, r, \theta, \phi\}$, correspond to the usual spherical coordinates, except for the radial coordinate r, which in our case is non-monotonic ranging from $-\infty$ to a minimum value r_0 and then increases again to $+\infty$. r is the radial coordinate in a diagram like the one presented in Figure 5.5.

Because the chosen coordinates naturally produce an orthogonal coordinate frame $\{g_\mu\}$, the natural choice of tetrad frame is the one resulting from a simple normalisation procedure of the coordinate frames. Resulting in a frame with local Minkowski metric $\eta_{mn} = \mathrm{diag}(+1, -1, -1, -1)$ and axes aligned with the coordinate frame. Because the metric is diagonal, we can immediately obtain the vierbein relating both frames (and its inverse) as the square-root (and its inverse) of the metric elements

$$[e^m{}_\mu] = \begin{pmatrix} e^\Phi & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-b/r}} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin(\theta) \end{pmatrix}, \tag{5.2.2}$$

$$[e_m{}^\mu] = \begin{pmatrix} e^{-\Phi} & 0 & 0 & 0 \\ 0 & \sqrt{1-b/r} & 0 & 0 \\ 0 & 0 & r^{-1} & 0 \\ 0 & 0 & 0 & r^{-1}\sin(\theta)^{-1} \end{pmatrix}. \tag{5.2.3}$$

With Equation (2.2.9) we obtain the connection coefficients,

$$\begin{aligned} \omega_t &= -e^\Phi\sqrt{1-\frac{b}{r}}\Phi'\gamma_t\wedge\gamma_r \\ \omega_r &= 0 \\ \omega_\theta &= \sqrt{1-\frac{b}{r}}\gamma_\theta\wedge\gamma_\phi \\ \omega_\phi &= \sqrt{1-\frac{b}{r}}\sin(\theta)\gamma_r\wedge\gamma_\phi + \cos(\theta)\gamma_\theta\wedge\gamma_\phi \end{aligned} \tag{5.2.4}$$

Where the primed quantities represent derivatives with respect to r, $\Phi' = \partial_r\Phi$ and $b' = \partial_r b$.

Observing the connection bivectors, we notice that spacetime is curved in the time direction, as evidenced by the non-null bivector ω_t, which is a function of the redshift function Φ. However, it is "flat" along the radial coordinate, contrary to what we might naively expect from observing Figure 5.5. This implies that an observer traveling along the radial direction, while also displacing in the t dimension, would experience a boost in the radial direction governed by the Φ function.

The displacement in the angular directions is governed by the bivectors $\{\omega_\theta, \omega_\phi\}$ and it has components both from the spherical coordinates and the shape function b.

From the connection bivectors we obtain the Riemann tensor for the spacetime with Equation (2.4.2), which we will express in the tetrad frame for simplicity.

$$\begin{aligned}
\mathbf{R}_{\hat{t},\hat{r}} &= \frac{\Phi'\left(-rb' + 2r(r-b)\Phi' + b\right) + 2r(r-b)\Phi''}{2r^2}\gamma_t \wedge \gamma_r \\
\mathbf{R}_{\hat{t},\hat{m}} &= \frac{(r-b)\Phi'}{r^2}\gamma_t \wedge \gamma_m, \quad m = \hat{\theta}, \hat{\phi} \\
\mathbf{R}_{\hat{r},\hat{m}} &= \frac{b - rb'}{2r^3}\gamma_r \wedge \gamma_m, \quad m = \hat{\theta}, \hat{\phi} \\
\mathbf{R}_{\hat{\theta},\hat{\phi}} &= -\frac{b}{r^3}\gamma_\theta \wedge \gamma_\phi.
\end{aligned} \tag{5.2.5}$$

To obtain the Ricci vector we contract the components of the Riemann tensor in the tetrad frame, Equation (2.4.8),

$$\begin{aligned}
R_{\hat{t}} &= \frac{\Phi'\left(-rb' + 2r(r-b)\Phi' - 3b + 4r\right) + 2r(r-b)\Phi''}{2r^2}\gamma_t \\
R_{\hat{r}} &= \frac{1}{2r^3}\left(-rb'\left(2 + r\Phi'\right) + b\left(r\left(-2r\Phi'' - 2r\Phi'^2 + \Phi'\right) + 2\right)\right. \\
&\quad \left. + 2r^3\left(\Phi'' + \Phi'^2\right)\right)\gamma_r \\
R_m &= -\frac{rb' + 2r(b-r)\Phi' + b}{2r^3}\gamma_m, \quad m = \hat{\theta}, \hat{\phi}.
\end{aligned} \tag{5.2.6}$$

A second contraction provides us withe to the Ricci scalar

$$\mathcal{R} = \frac{-b'\left(r\Phi' + 2\right) + (4r - 3b)\Phi' + 2r(r-b)\left(\Phi'' + \Phi'^2\right)}{r^2}. \tag{5.2.7}$$

Finally the calculation of Einstein's tensor produces

$$\begin{aligned}
G_{\hat{t}} &= \frac{b'}{r^2}\gamma_t \\
G_{\hat{r}} &= \frac{2r(b-r)\Phi' + b}{r^3}\gamma_r \\
G_m &= \frac{\left(1 + r\Phi'\right)\left(rb' + 2r(b-r)\Phi' - b\right) + 2r^2(b-r)\Phi''}{2r^3}\gamma_m, \quad m = \hat{\theta}, \hat{\phi}
\end{aligned} \tag{5.2.8}$$

The resulting components of the Riemann, Ricci and Einstein tensor might look complicated, but nonetheless they are considerable simpler than in the coordinate frame.

5.2.1 Energy-momentum vector

By virtue of Birkhoff's theorem, we know that any exterior vacuum solution of Einstein's equations must be given by the Schwarzschild metric. This solution corresponds to what is called a Schwarzschild wormhole, which is non-traversable [Morris and Thorne, 1988, §1-B]. Therefore, it follows that a traversable wormhole must be created by a non-zero energy-momentum tensor.

Because Einstein's equations relate the geometry to the energy-momentum tensor, we can use the Einstein tensor to impose constraints on the energy-momentum tensor of the matter required to create a wormhole. Since all quantities are expressed in the tetrad frame, and they have simple interpretations as the energy density $\hat{\rho}(r)$, radial tension $\hat{\tau}(r)$, and lateral pressure $\hat{p}(r)$ that a local observer would experience, and therefore we hat following the tetrad convention. Thus, we can write the energy-momentum tensor of a spherically symmetric system as

$$T(a) = \hat{\rho}(r)\gamma_t + \hat{\tau}(r)\gamma_r + \hat{p}(r)(\gamma_\theta + \gamma_\phi). \tag{5.2.9}$$

By using Einstein's equations, Equation (2.5.4), we immediately obtain the following relationships:

$$8\pi G\hat{\rho}(r) = \frac{b'}{r^2}, \tag{5.2.10a}$$

$$8\pi G\hat{\tau}(r) = \frac{2r(b-r)\Phi' + b}{r^3}, \tag{5.2.10b}$$

$$8\pi G\hat{p}(r) = \frac{(1 + r\Phi')\,(rb' + 2r(b-r)\Phi' - b) + 2r^2(b-r)\Phi''}{2r^3}. \tag{5.2.10c}$$

First order differential equations for b and Φ are obtained directly:

$$b' = 8\pi G\hat{\rho}(r)r^2, \tag{5.2.11a}$$

$$\Phi' = \frac{8\pi G\hat{\tau}(r)r^3 - b}{2r(b-r)}, \tag{5.2.11b}$$

while a first-order differential equation for $\tau'(r)$ requires additional work. First, we derive Equation (5.2.10b). To eliminate Φ'', we perform a second derivative of Equation (5.2.11b), and substitute b' from Equation (5.2.11a) and b from Equation (5.2.10b). After a considerable amount of algebra, the result simplifies to

$$\hat{\tau}' = (\hat{\rho} - \hat{\tau}) - \frac{2}{r}(\hat{p} + \hat{\tau}). \tag{5.2.12}$$

Given that we have three equations for five functions, we need to decide which to fix and which to derive. Following the original paper, we fix the geometry of the system by choosing the redshift Φ and shape function b in advance, and derive from them the characteristics of the matter $\{\hat{\rho}, \hat{\tau}, \hat{p}\}$. With this approach in mind, we rearrange Equations (5.2.11, 5.2.12) as

$$\hat{\rho} = \frac{b'}{8\pi G r^2}, \tag{5.2.13a}$$

$$\hat{\hat{\tau}} = \frac{b/r - 2(r-b)\Phi'}{8\pi G r^2}, \tag{5.2.13b}$$

$$\hat{p} = \frac{r}{2}\left((\hat{\rho}^2 - \hat{\tau})\Phi' - \hat{\tau}'\right) - \hat{\tau}. \tag{5.2.13c}$$

From here, what remains is to choose our $\Phi(r)$ and $b(r)$ functions and study their effect on the components of the energy-momentum tensor and on the travelers. Because such analysis only involves scalar quantities, our formalism provides no further advantage in either the computational or geometrical aspects. Therefore, I refer the reader to [Morris and Thorne, 1988, §4 and onward] for the analysis and various reasonable choices for $\Phi(r)$ and $b(r)$.

5.3 Wormhole - Toy model

To illustrate the methods used to describe Fermi-Walker transport presented in Section 5.1, let's consider the following choice of the redshift and shape functions:

$$\Phi(r) = 0, \quad b(r) = \frac{b_0^2}{r}, \tag{5.3.1}$$

where b_0 is a constant, and perform a radial coordinate change $r \to l$ with

$$l^2 = r^2 - b_0^2, \tag{5.3.2}$$

such that $-\infty < l < \infty$, with positive values corresponding to the upper universe, and negative values to the lower universe.

Under these assumptions, Equation (5.2.1) reads

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}l^2 - (b_0^2 + l^2)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{5.3.3}$$

This metric corresponds to a simplified model of a wormhole presented in Box 2 of the original paper and it is known as the Morris-Thorne wormhole.

Equation (5.3.3) describes a static and spherically symmetric system. Here, t is the coordinate time, θ and ϕ are the usual spherical coordinates, and l serves as the radial coordinate. The spacetime is asymptotically flat as $l \to \pm\infty$ and has a throat with radius b_0 (in the original radial coordinate r) at $l = 0$; see Figure 5.5.

We obtain the vierbein and the inverse vierbein transformations, which correspond to a change from coordinate to the stationary tetrad frame aligned with the coordinate vectors, directly from Equations (5.2.2, 5.2.3):

$$[e^m{}_\mu] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{b_0^2 + l^2} & 0 \\ 0 & 0 & 0 & \sqrt{b_0^2 + l^2}\sin(\theta) \end{pmatrix}, \tag{5.3.4}$$

$$[e_m{}^\mu] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{b_0^2+l^2}} & 0 \\ 0 & 0 & 0 & \frac{\csc(\theta)}{\sqrt{b_0^2+l^2}} \end{pmatrix}. \tag{5.3.5}$$

For this choice of functions, the expression for the connection bivectors in Equation (5.2.4) simplifies to

$$\begin{aligned} \omega_t &= 0, \\ \omega_l &= 0, \\ \omega_\theta &= \frac{l}{\sqrt{b_0^2 + l^2}}\, \gamma_l \wedge \gamma_\theta, \\ \omega_\phi &= \frac{l\sin(\theta)}{\sqrt{b_0^2 + l^2}}\, \gamma_l \wedge \gamma_\phi + \cos(\theta)\, \gamma_\theta \wedge \gamma_\phi. \end{aligned} \tag{5.3.6}$$

Observing the new form of the connection bivectors, we see that an observer parallel-displaced in the t or l directions would experience no rotation or boost. However, an inertial frame parallel-transported in the θ direction would rotate in the $l-\theta$ plane at a rate of $l/\sqrt{b_0^2 + l^2}$ with respect to the local tetrads. Similarly, in the ϕ direction, parallel-transported frames would rotate in the $l - \phi$ and $\theta - \phi$ planes.

This presents an interesting view of spacetime with metric Equation (5.3.3). As in the general case, the radial coordinate is “flat”, but now the time coordinate is also flat. This implies that observers traveling in the radial direction would not “fall” through the wormhole, as there is no force pulling them in; they would move as in a perfectly Minkowskian spacetime. This result is counterintuitive when considering the embedded depiction of the metric in Figure 5.5.

The components of the Riemann tensor $\mathbf{R}_{\mu\nu}$ are also considerably simplified, and we can express them in the tetrad frame as

$$\mathbf{R}_{mn} = \frac{b_0^2}{\left(b_0^2 + l^2\right)^2}\,\gamma_m \wedge \gamma_n, \quad m, n = l, \theta, \phi, \quad m \neq n. \tag{5.3.7}$$

To obtain the Ricci vector, we simplify Equation (5.2.6) or contract Equation (5.3.7), producing only one non-zero term:

$$R_{\hat{l}} = \frac{2b_0^2}{\left(b_0^2 + l^2\right)^2}\,\gamma_l. \tag{5.3.8}$$

Further contraction yields the Ricci scalar $\mathcal{R} = \gamma^m \cdot R_m$,

$$\mathcal{R} = \frac{2b_0^2}{\left(b_0^2 + l^2\right)^2}. \tag{5.3.9}$$

And finally, the Einstein vector reduces to

$$\begin{aligned} G_m &= -\frac{b_0^2}{\left(b_0^2 + l^2\right)^2}\,\gamma_m, \quad m = \hat{t}, \hat{\theta}, \hat{\phi}, \\ G_{\hat{l}} &= \frac{b_0^2}{\left(b_0^2 + l^2\right)^2}\,\gamma_l. \end{aligned} \tag{5.3.10}$$

5.3.1 Energy-momentum tensor

Because the redshift and shape functions are so simple, we can immediately obtain expressions for the components of the energy-momentum as measured by local observers:

$$\begin{aligned} \hat{\rho}(l) &= -\frac{b_0^2}{8\pi G\left(b_0^2 + l^2\right)^2} \\ \hat{\tau}(l) &= \frac{b_0^2}{8\pi G\left(b_0^2 + l^2\right)^2} \\ \hat{p}_\theta(l) = \hat{p}_\phi(l) &= -\frac{b_0^2}{8\pi G\left(b_0^2 + l^2\right)^2} \end{aligned} \tag{5.3.11}$$

One immediate element that should get our attention is that to create this particular geometry we would need matter with negative energy-density, therefore making it an non-viable possibility to create a wormhole. And as a second observation we notice that the radial tension is positive, meaning that an observer would feel a stretch in the radial direction. Such radial tension vanishes as $l \to \pm\infty$, but it would mean that even though there is no “in-falling” for radial observers, their experience would not be completely equivalent to a flat spacetime. Third, the lateral pressure is also positive, representing the curvature in the angular dimensions, meaning that compressive forces are experienced by the observer. Those forces tend also to zero the further we are from the center, in the asymptotically flat region.

5.3.2 Free-falling observers

Because of the flatness of the time and radial dimensions, free-falling observers maintain their original velocity. To investigate the experience of such observers, we perform a Lorentz transformation from stationary to free-falling frames. This transformation is generated by the bivector $\gamma_l \wedge \gamma_t$ and has $\xi = \tanh^{-1}|v|$ as the rapidity parameter. Therefore, we can immediately construct the rotor $R(\xi) = -\exp\left(\frac{\xi}{2}\gamma_l \wedge \gamma_t\right)$ and obtain the components of the free-falling frame $\{\gamma'_m\}$ in terms of the resting frame $\{\gamma_m\}$ using Equation (1.2.5).

$$\begin{aligned}\gamma'_m &= e^{-\frac{\xi}{2}\gamma_l\wedge\gamma_t}\gamma_m e^{\frac{\xi}{2}\gamma_l\wedge\gamma_t}\\ &= e^{-\xi\gamma_l\wedge\gamma_t}\gamma_m \\ &= \left(\cosh\xi - \gamma_l\wedge\gamma_t\sinh\xi\right)\gamma_m.\end{aligned} \tag{5.3.12}$$

Taking into account the permutation properties of $\{\gamma_m\}$, this results in

$$\begin{aligned}\gamma'_t &= \cosh\xi\gamma_t - \sinh\xi\gamma_l = \boldsymbol{\gamma}\gamma_t - |v|\boldsymbol{\gamma}\gamma_l,\\ \gamma'_l &= \cosh\xi\gamma_l - \sinh\xi\gamma_t = \boldsymbol{\gamma}\gamma_l - |v|\boldsymbol{\gamma}\gamma_t,\\ \gamma'_\theta &= \gamma_\theta,\\ \gamma'_\phi &= \gamma_\phi,\end{aligned} \tag{5.3.13}$$

where $\boldsymbol{\gamma} = \cosh\xi$, denotes the Lorentz factor of the transformation.

To obtain the forces that the free-falling observer experiences, we use the geodesic deviation equation, which in GA reads

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} = v\cdot D\left(v\cdot D\eta\right) = \mathbf{R}'\left(v\wedge\eta\right)\cdot v, \tag{5.3.14}$$

where η is the separation vector between two geodesics, and $\mathbf{R}'$ is the Riemann tensor experienced by the free-falling observer. We could obtain its components by inverting Equation (5.3.13) and substituting into Equation (5.3.7). However, we can also exploit the fact that Equation (1.2.5) applies to any multivector to express the transformation as

$$\mathbf{R}'_{mn} = e^{-\frac{\xi}{2}\gamma_l \wedge \gamma_t} \mathbf{R}_{mn} e^{\frac{\xi}{2}\gamma_l \wedge \gamma_t}. \tag{5.3.15}$$

This yields

$$\begin{aligned} \mathbf{R}'_{l\theta} &= \frac{b_0^2}{\left(b_0^2 + l^2\right)^2} \left(\boldsymbol{\gamma}\gamma'_l + |v|\boldsymbol{\gamma}\gamma'_t\right) \wedge \gamma'_\theta, \\ \mathbf{R}'_{l\phi} &= \frac{b_0^2}{\left(b_0^2 + l^2\right)^2} \left(\boldsymbol{\gamma}\gamma'_l + |v|\boldsymbol{\gamma}\gamma'_t\right) \wedge \gamma'_\phi, \\ \mathbf{R}'_{\theta\phi} &= \mathbf{R}_{\theta\phi}. \end{aligned} \tag{5.3.16}$$

Specifying Equation (5.3.14) for $v = \gamma'_l$, corresponding to radial in-fall, we obtain

$$\begin{aligned} \frac{d^2\eta}{d\tau^2} &= \mathbf{R}'\left(\gamma'_l \wedge \eta\right) \cdot \gamma'_l \\ &= \mathbf{R}'_{lm} \cdot \gamma'_l \\ &= \frac{|v|^2}{1 - |v|^2} \frac{b_0^2}{\left(b_0^2 + l^2\right)^2} \gamma'_m, \quad m = \theta, \phi. \end{aligned} \tag{5.3.17}$$

This means that, in the free-falling observer's local frame, the only nonzero tide-producing components of the Riemann tensor are $\mathbf{R}'_{l\theta}$ and $\mathbf{R}'_{l\phi}$. Thus, the observer would experience compressive forces only in the spatial directions perpendicular to the direction of travel. These forces vanish when the observer is far enough from the wormhole and reach their maximum when traveling through the throat at $l = b_0$. The forces also tend to zero as the velocity decreases, indicating that one could always travel through the wormhole regardless of the throat size b_0, provided the journey is slow enough.

5.3.3 Accelerated observers

The trajectory of non-inertial observers (meaning those not in free-fall) is not determined by the parallel transport equation; instead, it is governed by the *Fermi-Walker transport*, Equation (5.1.3). In this section, I will use the Morris-Thorne metric to illustrate the techniques presented in Section 5.1 and calculate the acceleration needed to maintain a spaceship circling the throat of a wormhole.

The reason I chose the Morris-Thorne metric, Equation (5.3.3), as an example is that it does not have any stable circular orbit. The only possible circular orbit is unstable and located at $l = 0$, right at the throat [Müller, 2008]. Therefore, if an observer wants to circle around the wormhole at $l \neq 0$ with a constant angular velocity, to obtain information about it without crossing the throat, they would need to constantly accelerate to correct their direction.

Because the 4-velocity is normalized, the acceleration $a = \dot{u}$ is orthogonal to it, and we can construct the 4-acceleration bivector with Equation (5.1.5) as

$$\Omega = u \wedge a. \tag{5.3.18}$$

where we consider, for simplicity, that no extra spatial rotation takes place and thus we set $B = 0$.

The desired trajectory is a circular motion determined by the constraints $\dot{r} = \dot{l} = 0$, and we choose $\theta = \frac{\pi}{2}$ for simplicity. If we parametrize the curve with the coordinate time, we can describe it using the function $\phi(t)$. Therefore, the 4-velocity vector tangent to the trajectory is

$$u = u^t g_t + u^\phi g_\phi = u^t(g_t + \Omega_o g_\phi), \tag{5.3.19}$$

where we have defined the angular velocity of the orbit as $\Omega_o = u^\phi / u^t = \mathrm{d}\phi/\mathrm{d}t$, which is a free constant. With the help of the normalization condition of the 4-velocity, we obtain u^t as a function of l and Ω_o:

$$u^t = \frac{1}{\sqrt{1 - \Omega_o^2\left(b_0^2 + l^2\right)}}. \tag{5.3.20}$$

To obtain the 4-acceleration a, we calculate the directional covariant derivative of u in the u-direction,

$$\begin{aligned}
a = u \cdot Du &= \left(u^t D_t + u^\phi D_\phi\right)\left(u^t g_t + u^\phi g_\phi\right) \\
&= \left(u^t D_t + u^\phi D_\phi\right) u^t \left(g_t + \Omega_o g_\phi\right) \\
&= \left(u^t\right)^2 \left(D_t + \Omega_o D_\phi\right)\left(\gamma_t + \Omega_o e^\phi{}_\phi \gamma_\phi\right) \\
&= \frac{\left(u^t\right)^2}{2}\left(\left[\cancelto{0}{\omega_t}, \gamma_t\right] + \Omega_o \cancelto{0}{\left[\omega_\phi, \gamma_t\right]} + \Omega_o e^\phi{}_\phi \left(\left[\cancelto{0}{\omega_t}, \gamma_\phi\right] + \Omega_o\left[\omega_\phi, \gamma_\phi\right]\right)\right) \\
&= -\left(u^t\right)^2 \Omega_o^2 e^\phi{}_\phi \gamma_l \\
&= -\frac{\Omega_o^2 l}{1 - \Omega_o^2\left(b_0^2 + l^2\right)}\gamma_l,
\end{aligned} \tag{5.3.21}$$

where we have used the fact that u^t, Ω_o, and ${e^\phi}_\phi$ are independent of t and ϕ to pull them out of the derivatives, and that $\partial_m \gamma_n = 0$.

As we would expect, a is negative and oriented in the radial direction and it vanishes for $l \to 0$, corresponding to the middle of the throat. This is expected because, in that case, the observer would be in an orbit and no acceleration would be needed to maintain the movement.

We can now construct the 4-acceleration bivector Ω using the outer product between Equation (5.3.19) and Equation (5.3.21):

$$\Omega(l, \Omega_o) = u \wedge a = \frac{\Omega_o^2 l \gamma_l}{\left(1 - \Omega_o^2 \left(b_0^2 + l^2\right)\right)^{3/2}} \wedge \left(\gamma_t + \Omega_o \sqrt{b_0^2 + l^2}\gamma_\phi\right), \quad (5.3.22)$$

where we can see that the 4-acceleration bivector Ω depends solely on the free parameters: radial distance l and angular velocity Ω_o.

The second term in Equation (5.1.8), corresponding to the non-accelerated transport of the frame, $\omega(u)$, is easily obtained because the connection bivectors are a linear map:

$$\begin{aligned} \omega(u) &= u^t \omega_t + u^\phi \omega_\phi = u^t \Omega_o \omega_\phi \\ &= \frac{\Omega_o}{\sqrt{\left(b_0^2 + l^2\right)\left(1 - \Omega_o^2\left(b_0^2 + l^2\right)\right)}} \gamma_l \wedge \gamma_\phi . \end{aligned} \quad (5.3.23)$$

And because neither Ω nor $\omega(u)$ depend on t, the integration of Equation (5.1.8) is straightforward, allowing us to obtain an expression for the rotor $R(t)$ as a simple exponential:

$$R(t; l, \Omega_o) = \exp\left(-\frac{t}{2}\left(\Omega(l, \Omega_o) - \omega(u)\right)\right), \quad (5.3.24)$$

which relates the local tetrad $\{\gamma_k\}$ frame with the traveling frame $\{e_k\}$ via Equation (5.1.6), and has a contribution due to the accelerated motion and another due to the local rotation of tetrads.

Comparing the usual 4-acceleration approach with the GA approach, the bivector Ω provides more geometrical information, as it directly gives the planes of rotation of u, and, by observing their coefficients, the angular velocity on each plane independently.

Note that because $R(t)$ is a normalized bivector, it only modifies the direction of u, which is convenient in the relativistic treatment of problems where the 4-velocity is normalized. However, in cases involving a system with tangential acceleration, the approach needs to be modified by dropping the normalization condition on R, which would lead to both rotation and scaling of the velocity.

5.4 Conclusions

In this chapter, I presented the formulation of the Fermi-Walker transport equation within the tetrad-GA formalism and explored its application to a wormhole model. The description of non-geodesic motion with this formalism is particularly insightful because, due to the normalization of the 4-velocity, the 4-acceleration can be expressed as a bivector. This allows us to gather all the dynamic properties of the system in a rotor relating the tetrad and transported frames.

As an application of these techniques, I investigated the 4-acceleration required to maintain a spaceship in orbit around a wormhole described by the Morris-Thorne metric. While the calculations were not significantly simplified relative to the conventional tensor approach, the bivector description of the 4-acceleration provides geometrical insights into the planes of rotation and their angular velocities.

In the next chapter, I will apply the same techniques to calculate the angular precession of a gyroscope around various black hole models. In this case, the calculations are significantly simplified, and we gain substantial geometrical insights.

Chapter 6

Gyroscopic precession

In this chapter, I apply the Fermi-Walker techniques developed in Section 5.1 to obtain the precession angle of a gyroscope orbiting various black hole models.

In Section 6.1, I begin with the simple and well-known Schwarzschild black hole and present the procedure to obtain the precession angle using both the conventional tensor formalism and the tetrad-GA formalism. As we will see, the ability to encode the dynamics into a rotor greatly simplifies the calculations, reducing the set of four coupled differential equations of the tensor formalism to a single, first-order differential equation.

In Section 6.2, I apply the same techniques to the Kerr-Newman black hole, where we can easily obtain an expression for the precession angle. The result is remarkably simple given the complexity of the system and provides a clear example of how the tetrad-GA formalism simplifies the calculations and offers numerous geometric insights.

6.1 The Schwarzschild black hole

The metric describing a Scharwzschild black hole in Schwarzschild coordinates, (t, r, θ, ϕ) is

$$\mathrm{d}s^2 = \left(1 - \frac{r_s}{r}\right) \mathrm{d}t^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\theta^2 - r^2 \sin^2(\theta) \mathrm{d}\phi^2 \qquad (6.1.1)$$

where $r_s = 2GM$ is the Schwarschild radius corresponding to the position of the horizon in Schwarzschild coordinates.

Choosing a tetrad frame aligned with the coordinate frame, we read the elements of vierbein directly from the metric, Equation (2.1.11),

$$e^0{}_\mu \mathrm{d}x^\mu = \left(1 - \frac{r_s}{r}\right)^{1/2} \mathrm{d}t \tag{6.1.2a}$$

$$e^1{}_\mu \mathrm{d}x^\mu = \left(1 - \frac{r_s}{r}\right)^{-1/2} \mathrm{d}r \tag{6.1.2b}$$

$$e^2{}_\mu \mathrm{d}x^\mu = r\mathrm{d}\theta \tag{6.1.2c}$$

$$e^3{}_\mu \mathrm{d}x^\mu = r\sin\theta \mathrm{d}\phi. \tag{6.1.2d}$$

And we can write the vierbein in matrix form as

$$\left[e^m{}_\mu\right] = \begin{pmatrix} \left(1 - \frac{r_s}{r}\right)^{1/2} & 0 & 0 & 0 \\ 0 & \left(1 - \frac{r_s}{r}\right)^{-1/2} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}. \tag{6.1.3}$$

The inverse vierbein follows immediately, and in matrix form reads,

$$[e_m{}^\mu] = \begin{pmatrix} \left(1 - \frac{r_s}{r}\right)^{-1/2} & 0 & 0 & 0 \\ 0 & \left(1 - \frac{r_s}{r}\right)^{1/2} & 0 & 0 \\ 0 & 0 & r^{-1} & 0 \\ 0 & 0 & 0 & (r\sin\theta)^{-1} \end{pmatrix}. \tag{6.1.4}$$

Such a tetrad choice naturally emerges from Schwarzschild coordinates. Nonetheless, other interesting orthonormal frame choices exist. One notable option is to select orthonormal frames that are free-falling at each point with a velocity corresponding to an observer who would begin traveling from infinity with zero initial velocity. These observers are known as "Gullstrand-Painlevé fishes" because they are the natural choice of tetrad for the Gullstrand-Painlevé metric. The term fishes emerges from the analogy of the Gullstrand-Painlevé coordinates with a river, which flows into the black hole with velocity dependent on the radius and reaches the speed of light at the black hole horizon. At that point, free-falling observers, which "swim" through the space-time river, cannot resist the inflow of spacetime anymore and are no longer able to escape the gravitational pull.

Remarkably, and contrary to what one might expect, the relationship between static tetrads (as chosen here) and the Gullstrand-Painlevé tetrads is not a Lorentz transformation but a Galilean one, which leaves the time component invariant. Thus, the Gullstrand-Painlevé metric can be understood as a Galilean river of space flowing into the black hole, with observers moving in the river according to the rules of special relativity. See [Hamilton and Lisle, 2008] for more details.

Because the metric is diagonal, the application of Equation (2.2.9) is particularly simple and we obtain the connection bivectors:

$$\omega_t = -\frac{r_s}{2r^2}\gamma_t \wedge \gamma_r \tag{6.1.5a}$$

$$\omega_r = 0 \tag{6.1.5b}$$

$$\omega_\theta = \sqrt{\frac{1}{1-\frac{r_s}{r}}}\gamma_r \wedge \gamma_\theta \tag{6.1.5c}$$

$$\omega_\phi = \cos(\theta)\gamma_\theta \wedge \gamma_\phi + \sin(\theta)\sqrt{\frac{r-r_s}{r}}\gamma_r \wedge \gamma_\phi \tag{6.1.5d}$$

The calculation of the Riemann, Ricci and Einstein tensor follow the same procedure than in the previous cases. Because they are not relevant to the current discussions I will leave their result to Appendix E.

6.1.1 The parallel transport of the angular momentum

The conventional approach to obtain the precession of a gyroscope around a Schwarzschild black hole consists of representing the angular velocity of the gyroscope by a 3D vector $\vec{s}$, which we then promote to a 4-vector to parallel transport along the orbit. The result is a set of four coupled differential equations, one for each component of the angular momentum, which we must solve.

The same result in the tetrad-GA approach can be obtained in two different ways: one by following the parallel-transport steps and the other by using rotors. To compare both methods side by side, I present the former here and, in Section 6.1.2, the latter.

The resolution of this problem using the conventional tensor formalism, without tetrads, can be found in [Hobson et al., 2006, §10.5].

We start by considering a test particle orbiting a Schwarzschild black hole. The test particle has negligible mass and angular momentum compared to the mass of the black hole. Therefore, it can be treated as a small gyroscope whose movement does not affect the metric under consideration.

For the purpose, of parallel transporting the angular momentum 4-vector s along the trajectory we use Equation (2.3.11). For simplicity, we will choose a circular orbit in the equatorial plane, $\theta = \pi/2$. The path of this orbit is a spiral with tangent vector $u = u^t g_t + u^\phi g_\phi$, with components

$$u^t = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{3r_s}{2r}\right)^{-1/2}, \quad u^\phi = u^t \Omega_o = u^t \left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) = u^t \sqrt{\frac{r_s}{2r^3}}, \quad (6.1.6)$$

where $\Omega_o = r_s/2r^3$ is the orbital angular velocity of the gyroscope [Hobson et al., 2006, §9.8 and §10.5].

Specifying Equation (2.3.11) to our case and expanding in components, we obtain

$$\begin{aligned}(u \cdot D)s &= (u^t D_t + u^\phi D_\phi)(s^m \gamma_m) \\ &= (u^t \partial_t + u^\phi \partial_\phi)s + \frac{u^t}{2}[\omega_t, s] + \frac{u^\phi}{2}[\omega_\phi, s] = 0.\end{aligned} \quad (6.1.7)$$

Since our path is a geodesic curve, u is a 4-velocity, and we can also express the left-hand side as a total derivative with respect to proper time, $u \cdot D = \mathrm{d}/\mathrm{d}\tau$. By separating the components of the angular momentum 4-vector in Equation (6.1.7), we obtain the system of four coupled differential equations:

$$\frac{ds^t}{d\tau} + u^t s^r \frac{r_s}{2r^2} = 0, \quad (6.1.8a)$$

$$\frac{ds^r}{d\tau} + u^t s^t \frac{r_s}{2r^2} - u^\phi s^\theta \sqrt{1 - \frac{r_s}{r}} = 0, \quad (6.1.8b)$$

$$\frac{ds^\theta}{d\tau} = 0, \quad (6.1.8c)$$

$$\frac{ds^\phi}{d\tau} + u^\phi s^r \sqrt{1 - \frac{r_s}{r}} = 0. \quad (6.1.8d)$$

This system of differential equations is similar to the one obtained by tensor calculus and should be solved using standard techniques. If we assume initial conditions with the angular momentum oriented in the r-direction, $s(0) = s^r(0)\gamma_r$, the solution to the system of equations is:

$$s^t(t) = \frac{-r_s}{2\sqrt{\frac{r_s}{2}(r - \frac{3r_s}{2})}} s^r(0) \sin \Omega t, \quad (6.1.9a)$$

$$s^r(t) = s^r(0) \cos \Omega t, \quad (6.1.9b)$$

$$s^\theta(t) = s^\theta(0), \quad (6.1.9c)$$

$$s^\phi(t) = -\sqrt{1 - \frac{r_s}{r}} \frac{\Omega_o}{\Omega} s^r(0) \sin \Omega t, \quad (6.1.9d)$$

where $\Omega = \sqrt{\frac{r_s}{2r^4}\left(r - \frac{3r_s}{2}\right)}$ is the precession angular velocity of the angular momentum with respect to the coordinate frame.

To obtain the precession angle α after one orbit, we first note that one orbit takes a coordinate time

$$t = \frac{2\pi}{\Omega_o} = 2\pi\sqrt{\frac{2r^3}{r_s}}. \tag{6.1.10}$$

During this time interval, the angular momentum will have precessed by an angle $2\pi + \alpha$. Equating time expressions, we obtain α as a function of the orbital radius:

$$\frac{2\pi}{\Omega_o} = \frac{2\pi + \alpha}{\Omega} \Rightarrow \alpha = 2\pi\left(\frac{\Omega}{\Omega_o} - 1\right) = 2\pi\left(\sqrt{\frac{1}{r}\left(r - \frac{3r_s}{2}\right)} - 1\right). \tag{6.1.11}$$

Notably, α acquires an imaginary component if $r < 3r_s/2$, as $3GM$ is the radius below which no stable circular orbits exist.

6.1.2 Schwarzschild precession in GA

An alternative procedure to obtain the precession angle α, which does not involve solving Equation (6.1.8), is to exploit the techniques from Section 5.1. The key element is identifying that the precession of the angular momentum vector is equivalent to the precession of the gyroscopic frame. Thus, we can relate the gyroscopic frame at each point of the orbit $\{e_k(\tau)\}$ with the local tetrad $\{\gamma_m\}$ by a Lorentz transformation. This Lorentz transformation is determined by a rotor satisfying Equation (5.1.8). By changing variables from proper time to coordinate time and considering that the gyroscope is in free-fall $\dot{u} = 0$, the resulting rotor equation reads

$$u^t \frac{\mathrm{d}}{\mathrm{d}t} R(t) = -\frac{1}{2}\omega(u)R(t). \tag{6.1.12}$$

With $\omega(u)$ being the connection bivector along the trajectory, which we can expand by linearity as

$$\omega(u) = u^t\omega_t + u^\phi\omega_\phi = u^t\left(\omega_t + \Omega_o\omega_\phi\right), \tag{6.1.13}$$

where $\omega_t = \omega(g_t)$ and $\omega_\phi = \omega(g_\phi)$ are given by Equation (6.1.5).

Because neither ω_t, ω_ϕ, u^t, nor u^ϕ depend on t, the integral in Equation (6.1.12) is straightforward, and the rotor governing the frame's precession along the orbit is

$$R(t) = \exp\left(-\frac{1}{2}\frac{\omega(u)}{u^t}t\right). \tag{6.1.14}$$

To obtain the total rotation after one orbit, we start by calculating the magnitude[1] of $\omega(u)$ as

$$\begin{aligned} |\omega(u)| =& \sqrt{(u^t)^2 \omega_t \omega_t^\dagger + (u^\phi)^2 \omega_\phi \omega_\phi^\dagger}, \\ =& u^t \sqrt{(\Omega_o)^2 \, |\omega_\phi|^2 - |\omega_t|^2}, \\ =& u^t \sqrt{\frac{r_s}{2r^4}\left(r - \frac{3r_s}{2}\right)}. \end{aligned} \tag{6.1.15}$$

Thus, we can identify $|\omega(u)|/u^t$ as the angular velocity of the parallel transported angular momentum, which in Equation (6.1.9) we called Ω.

From this, the precession angle follows immediately by the same considerations as before:

$$\alpha_S = 2\pi \left(\frac{\omega(u)/u^t}{\Omega_o} - 1 \right) = 2\pi \left(\sqrt{\frac{1}{r}\left(r - \frac{3r_s}{2}\right)} - 1 \right). \tag{6.1.16}$$

With this approach, we replicated the previous result while only needing to solve a single, first-order differential equation in flat spacetime, Equation (6.1.12), instead of a system of four coupled differential equations, Equation (6.1.8). As in the previous example in Section 5.3, this procedure also directly provides geometric information about the involved planes of rotation and their corresponding angular velocities, which would be difficult to recover directly from Equation (6.1.9).

In the following section, we will apply this same technique to the general case of a Kerr-Newman black hole.

6.2 The Kerr-Newman black hole

To further investigate the capabilities of this method, I used it to explore the precession around the most general asymptotically flat and stationary solution of Einstein's equations: the Kerr-Newman metric, which describes the dynamics of a charged, rotating black hole. We will express the metric of this system using the Boyer-Lindquist coordinates. The reasons being that they adapt properly to the geometry of the system, making expressions relatively simple, and that they easily reduce to Schwarzschild coordinates

[1] Because in spacetime the square of bivectors can be positive or negative, the magnitude of a bivector B is defined as $|B| = \sqrt{BB^\dagger}$, with the dagger representing the reversion operation defined in Section 1.1.2.

when $a \to 0$ and $Q \to 0$, allowing for simple checks of our results. The Kerr-Newman metric in Boyer-Lindquist coordinates reads

$$\begin{aligned} \mathrm{d}s^2 = &\frac{R^2\Delta}{\rho^2}\left(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi\right)^2 - \frac{\rho^2}{R^2\Delta}\mathrm{d}r^2 \\ &- \rho^2\mathrm{d}\theta^2 - \frac{R^4\sin^2\theta}{\rho^2}\left(\mathrm{d}\phi - \frac{a}{R^2}\mathrm{d}t\right)^2. \end{aligned} \tag{6.2.1}$$

Where $a = J/M$ is a constant parameter representing specific angular momentum of the system, and we have employed the following definitions

$$R^2 = r^2 + a^2, \quad \rho^2 = r^2 + a^2\cos\theta, \quad \Delta = 1 - \frac{2Mr}{R^2} + \frac{Q^2}{R^2}. \tag{6.2.2}$$

We also employed the convention $c = G = 1$, and we should remark that the coordinates $\{r, \theta, \phi\}$ correspond to oblate-spheroidal, and not spherical, coordinates.

From the condition $\Delta = 0$, we obtain the position of the horizons

$$r_\pm \equiv M \pm \sqrt{M^2 - a^2 - Q^2}. \tag{6.2.3}$$

With r_+ and r_- representing the radius of the outer and inner horizons. The horizons are only real if $M^2 > a^2 + Q^2$. If $M^2 < a^2 + Q^2$, Δ has no zeros meaning that there are no horizons. In this case, the spacetime corresponds to a naked ring singularity located at

$$r^2 + a^2\cos^2\theta = 0. \tag{6.2.4}$$

More details about the Kerr-Newman metric in Boyer-Lindquist coordinates can be found in [Misner et al., 1973, §33.2] and [Straumann, 2013, §7.3].

With the metric written in the form of Equation (6.2.1), and by choosing a Minkowskian metric to conform our tetrad frame, $\eta_{mn} = \mathrm{diag}(+1, -1, -1, -1)$, we can immediately obtain the elements of the vierbein by using the equation Equation (2.1.11):

$$\begin{aligned} e^0{}_\mu dx^\mu &= \frac{R\sqrt{\Delta}}{\rho}\mathrm{d}t - \frac{a\sin^2\theta R\sqrt{\Delta}}{\rho}\mathrm{d}\phi, \\ e^1{}_\mu dx^\mu &= \frac{\rho}{R\sqrt{\Delta}}\mathrm{d}r, \\ e^2{}_\mu dx^\mu &= \rho\mathrm{d}\theta, \\ e^3{}_\mu dx^\mu &= \frac{\sin\theta}{\rho}\left(-a\mathrm{d}t + R^2\mathrm{d}\phi\right). \end{aligned} \tag{6.2.5}$$

Which we can express in matrix form as

$$\left[e^m{}_\mu\right] = \begin{pmatrix} \frac{R\sqrt{\Delta}}{\rho} & 0 & 0 & -\frac{a\sin^2\theta R\sqrt{\Delta}}{\rho} \\ 0 & \frac{\rho}{R\sqrt{\Delta}} & 0 & 0 \\ 0 & 0 & \rho & 0 \\ -\frac{a\sin\theta}{\rho} & 0 & 0 & \frac{R^2\sin\theta}{\rho} \end{pmatrix}. \tag{6.2.6}$$

With the inverse vierbein being determined by Equation (5.2.3),

$$[e_m{}^\mu] = \frac{1}{\rho}\begin{pmatrix} \frac{R}{\sqrt{\Delta}} & 0 & 0 & \frac{a}{R\sqrt{\Delta}} \\ 0 & R\sqrt{\Delta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a\sin\theta & 0 & 0 & \frac{1}{\sin\theta} \end{pmatrix}. \tag{6.2.7}$$

Similarly to the previous case, where we discussed two relevant tetrad choices for Schwarzschild geometry, in the Kerr-Newman spacetime there are also two primary tetrad options. The one we selected here, which naturally emerges from Equation (6.2.1), is commonly referred to in the literature as a *static tetrad*. This choice corresponds to the frames of *static observers* as perceived by distant observers. These observers remain stationary relative to the Boyer-Lindquist coordinates, implying that they must constantly apply force to counteract the rotational frame-dragging effect induced by the Kerr-Newman black hole.

An alternative and physically significant tetrad choice was introduced by Bardeen et al. [1972], representing observers who co-rotate with the black hole at the precise angular velocity of the surrounding spacetime. These frames are known as *locally non-rotating frames* and are particularly well-suited for examining energy-extraction processes, accretion disk dynamics, and similar phenomena.

In this study, we choose the static tetrad to facilitate direct comparison with the Schwarzschild case.

From the vierbein, the connection coefficients can be directly calculated using Equation (2.2.9). While the calculations for previous examples were manageable by hand, this is not feasible with the Kerr-Newman metric due to its complexity. Consequently, I developed a code based on the Mathematica package `BasicClifford` [Aragon-Camarasa et al., 2018] to facilitate GA calculations in curved spacetimes. The code is publicly accessible in [Pérez].

Because the complete expressions for the connection bivectors are algebraically convoluted and we will be considering a gyroscope orbiting in the equatorial plane, I will only give the result of the connection bivectors for the case of $\theta = \pi/2$:

$$\omega_t\big|_{\theta=\pi/2} = -\frac{Q^2 - Mr}{r^3}\gamma_t \wedge \gamma_r \tag{6.2.8a}$$
$$\omega_r\big|_{\theta=\pi/2} = \frac{a}{rR\sqrt{\Delta}}\gamma_t \wedge \gamma_\phi \tag{6.2.8b}$$
$$\omega_\theta\big|_{\theta=\pi/2} = \frac{R\sqrt{\Delta}}{r}\gamma_r \wedge \gamma_\theta \tag{6.2.8c}$$
$$\omega_\phi\big|_{\theta=\pi/2} = \frac{a\left(r(r+M) - Q^2\right)}{r^3}\gamma_t \wedge \gamma_r + \frac{R\sqrt{\Delta}}{r}\gamma_r \wedge \gamma_\phi \tag{6.2.8d}$$

The Riemann, Ricci tensor, Ricci scalar and Einstein tensors, also present a particularly simple form when expressed in the static tetrad, however they play no role in the following derivation and thus I leave their expression to Appendix E.

6.2.1 Orbiting angular velocity

Our goal in the following sections is to obtain precession angular velocity of the transported frame with respect to the static tetrads. Fortunately, the derivation in Section 6.1.2 is independent of the metric, and the necessary expression is given by

$$\frac{|\omega(u)|}{u^t} = \sqrt{(\Omega_o)^2\,|\omega_\phi|^2 - |\omega_t|^2}. \tag{6.2.9}$$

Once we have the connection bivectors, the next element needed is the coordinate orbital angular velocity of the gyroscope, defined as $\Omega_o = \mathrm{d}\phi/\mathrm{d}t$. To obtain it, we will analyze the potential experienced by the orbiting particle and identify the locations of its minima. This process will yield the energy and angular momentum of a particle orbiting at a particular radius, from which we will obtain an expression for the orbital angular velocity for both co- and counter-rotating cases.

The procedure is significantly simplified by a series of transformations presented in [Ulbricht and Meinel, 2015]: First, we cast the metric into a dimensionless form by dividing by M, and then, we reduce it to the equatorial plane by setting $\theta = \pi/2$. The resulting metric is:

$$\mathrm{d}s' = g_{11}\mathrm{d}x^2 + g_{33}\mathrm{d}\phi^2 + 2g_{34}\mathrm{d}\phi\mathrm{d}t' + g_{44}\mathrm{d}t'^2, \tag{6.2.10}$$

where $x = r/M$, $t' = t/M$, $\mathrm{d}s' = \mathrm{d}s/M$, and the metric coefficients are given by

$$\begin{aligned} g_{11} &= \frac{x^2}{x^2 - 2x + a'^2 + q^2}, \\ g_{33} &= x^2 + a'^2 + \frac{\left(2x - q^2\right) a'^2}{x^2}, \\ g_{34} &= -\left(2x - q^2\right) \frac{a'}{x^2}, \\ g_{44} &= -\frac{x^2 - 2x + q^2}{x^2}. \end{aligned} \tag{6.2.11}$$

Here, $q = Q/M$ and $a' = a/M$. The values of q and a' are constrained to the parameter space $\{(a', q) \,|\, a'^2 + q^2 \leq 1 \text{ and } a, q \geq 0\}$ to keep Δ real and prevent naked singularities.

Since the metric is independent of t' and ϕ, we can define the conserved quantities E and L, associated with the specific energy and angular momentum of the orbiting particle, as

$$E = -g_{44}\dot{t}' - g_{34}\dot{\phi}, \quad L = g_{33}\dot{\phi} + g_{34}\dot{t}', \tag{6.2.12}$$

where dotting denotes differentiation with respect to proper time.

For circular orbits, we analyze the potential $V(x)$ and its derivative $V'(x)$ to locate values of the reduced radius x that minimize $V(x)$. Substituting those results into Equation (6.2.12), we obtain algebraically complex expressions for E and L. These, however, can be compactly expressed in terms of $\xi = x - q^2$ to obtain polynomials of $\sqrt{\xi}$:

$$\begin{aligned} E^{\pm}(\xi, a, q) &= \frac{q^4 + q^2(2\xi - 1) + \xi^2 - 2\xi \pm a\sqrt{\xi}}{(\xi + q^2)\sqrt{q^4 + q^2(2\xi - 1) + \xi^2 - 3\xi \pm 2a\sqrt{\xi}}}, \\ L^{\pm}(\xi, a, q) &= \pm\frac{a^2\sqrt{\xi} + \sqrt{\xi}\left(\xi + q^2\right)^2 \mp a\left(2\xi + q^2\right)}{(\xi + q^2)\sqrt{q^4 + q^2(2\xi - 1) + \xi^2 - 3\xi \pm 2a\sqrt{\xi}}}. \end{aligned} \tag{6.2.13}$$

Here, the upper sign corresponds to co-rotating orbits, while the lower sign corresponds to counter-rotating orbits.

We can now obtain the coordinate orbital angular velocity as a function of E and L by solving Equation (6.2.12) for $\dot{\phi}$ and $\dot{t}'$, and applying the chain rule:

$$\Omega_o = \frac{\dot{\phi}}{\dot{t}'} = \frac{\mathrm{d}\phi}{\mathrm{d}t'} = -\frac{g_{44}L + g_{34}E}{g_{33}E + g_{34}L}. \tag{6.2.14}$$

Finally, by substituting the values for E and L from Equation (6.2.13) we obtain a surprisingly simple expression for the coordinate orbital angular velocity as a function of the orbital radius and the parameters of the space-time

$$\Omega_o^{\pm}(\xi, a', q) = \pm \frac{\sqrt{\xi}}{\left(q^2+\xi\right)^2 \pm a'\sqrt{\xi}}. \tag{6.2.15}$$

6.2.2 Kerr-Newman precession

At this point, it is convenient to define the reduced version of the connection coefficients from Equation (6.2.8):

$$\begin{aligned} \omega_t' &= M\omega_t = \frac{\left(q^2-x\right)}{x^3}\gamma_t \wedge \gamma_r, \\ \omega_\phi &= \frac{a'\left(x(x+1)-q^2\right)}{x^3}\gamma_t \wedge \gamma_r + \frac{(x^2+a'^2)\sqrt{\Delta}}{x}\gamma_r \wedge \gamma_\phi. \end{aligned} \tag{6.2.16}$$

The square magnitudes of these are

$$\begin{aligned} |\omega_t'|^2 &= \tilde{\omega}_t'\tilde{\omega}_t'^{\dagger} = -\frac{\left(q^2-x\right)^2}{x^6}, \\ |\omega_\phi|^2 &= \omega_\phi \omega_\phi^{\dagger} = \frac{x(x-2)+q^2}{x^2} - \frac{a'^2}{x^6}\left(q^2-x^2\right)\left(q^2-x^2\left(2x+1\right)\right). \end{aligned} \tag{6.2.17}$$

Now, we can calculate the magnitude of the precession angular velocity from Equation (6.2.9) in terms of the reduced quantities $\{x, q, a'\}$:

$$\begin{aligned} \frac{|\omega'^{\pm}(u)|}{u^t} &= \sqrt{\left(\Omega_o^{\pm}\right)^2|\omega_\phi|^2 + |\omega_t'|^2} \\ &= \frac{\left(x-q^2\right)}{x^3}\times \\ &\sqrt{\pm\frac{x^4\left(q^2+(x-2)x\right)-a^2q(q-x)\left(q^2-x(2x+1)\right)}{\left(x-q^2\right)\left(x^2 \pm a\sqrt{x-q^2}\right)^2} - 1}. \end{aligned} \tag{6.2.18}$$

Once we have the orbital and precession angular velocities of the gyroscope, we can obtain the precession angle around a Kerr-Newman black hole as:

$$\alpha_{KN} = 2\pi\left(\frac{|\omega'^{\pm}(u)|/u^t}{\Omega_o^{\pm}} - 1\right). \tag{6.2.19}$$

And since all quantities are in their reduced, dimensionless form, we can plot the precession angle α_{KN} as a function of the reduced radius for various values of a' and q: Figures 6.1 to 6.3.

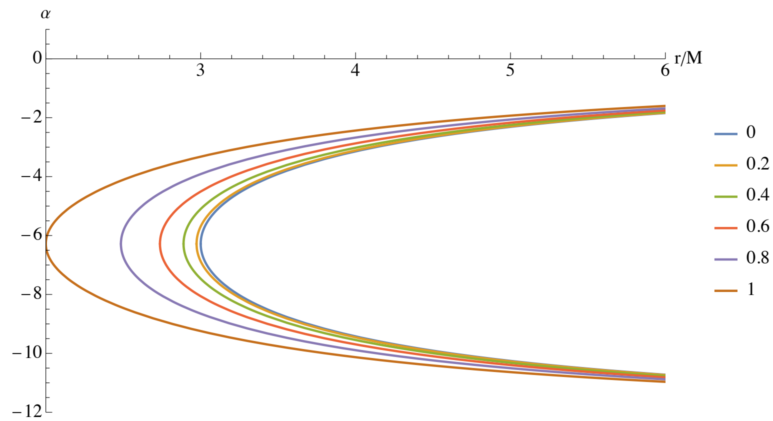


Figure 6.1: Frame precession around a Kerr-Newman black hole with $a/M = 0$ (Reissner-Nördstrom solution), for various values of $q = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. The upper branch corresponds to particles rotating counter-clock, and the lower branch to particles rotating clock-wise in the equatorial plane. The r/M axis starts at 2.

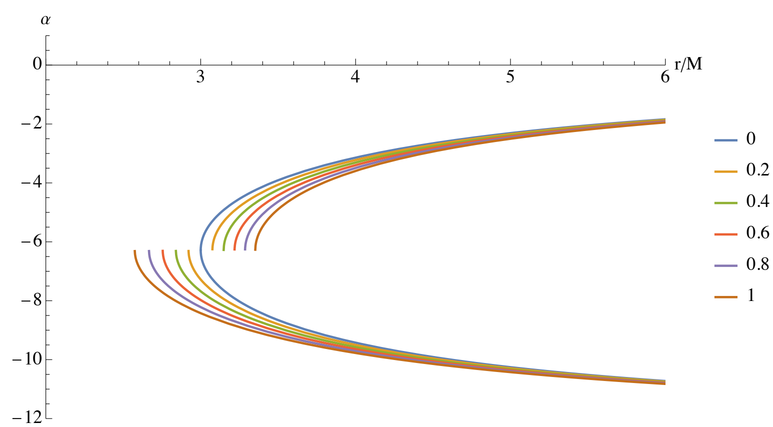


Figure 6.2: Frame precession around a Kerr-Newman black hole with $Q = 0$ (Kerr solution), and various values of $a/M = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. The upper branch corresponds to particles co-rotating with the black hole, and the lower branch to particles counter-rotating in the equatorial plane. The r/M axis starts at 2.

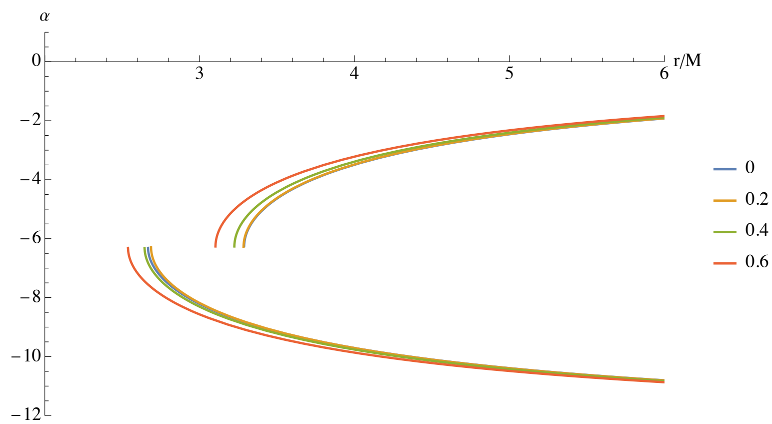


Figure 6.3: Frame precession around a Kerr-Newman black hole with $a = 0.8$, and various values of $q = \{0, 0.2, 0.4, 0.6\}$. The values of q are constrained to avoid a naked singularity, so $q \in \{0, \sqrt{1-a^2}\}$. The upper branch corresponds to particles co-rotating with the black hole, and the lower branch to particles counter-rotating in the equatorial plane. The r/M axis starts at 2.

As expected, increasing the angular momentum of the black hole, a, increases the asymmetry between co- and counter-rotating orbits, resulting in different precession angles at the same radius. For a given a and r, the counter-rotating orbit will accumulate a greater precession angle than the co-rotating one. This phenomena can be understood by noting that the counter-rotating orbit travels through "more spacetime" before returning to its original position.

The electric charge parameter q has a distinctly different effect, creating an isotropic distortion that equally affects the precession angle of both the co-rotating and counter-rotating orbits. As the electric charge increases, both orbits shift closer to the origin. However, they never reach the outer event horizon r_+.

This leads to another observation: the orbits cease to exist well before reaching the outer event horizon r_+. This feature is analogous to the Schwarzschild case, where orbits exist only for $r > 3r_s/2$, and are stable only if $r > 3r_s$. In the Kerr-Newman case, Liu et al. [2017] showed that the location of the last stable orbit varies from $r = M$ to $r = 9M$, depending on a, q, and the orbit's direction.

Two important limits are worth noting: First, both the counter- and co-rotating solutions converge to the Schwarzschild solution when both charge and angular momentum vanish:

$$\alpha_{KN}\Big|_{\substack{q=0\\a'=0}} = 2\pi\left(\sqrt{1-\frac{3r_s}{2r}}-1\right). \tag{6.2.20}$$

Second, at distances sufficiently far from the black hole, the co-rotating solution tends to zero, while the counter-rotating solution approaches 4π, indicating that, far from the black hole, there is no frame precession, reflecting the fact that the Kerr-Newman spacetime is asymptotically flat.

As in the previous case, the rotor method presented here not only provides the frame precession but also directly gives the planes of rotation involved in the motion, $\gamma^t \wedge \gamma_r$ and $\gamma^r \wedge \gamma_\phi$, as well as the angular velocities within these planes, $|\omega_t|$ and ω_ϕ.

6.3 Conclusions

In this chapter, we obtained the precession angle of a gyroscope orbiting both Schwarzschild and Kerr-Newman black holes along their equatorial planes. As a novel approach, we performed these calculations using the tetrad-GA formalism, where, through a wise choice of tetrads and the application of rotor techniques, we were able to considerably simplify the computations.

To connect with the conventional approach, we examined the precession around a Schwarzschild black hole using two methods. First, we solved the parallel-transport equation within our formalism, obtaining a set of four coupled differential equations analogous to those derived using standard tensor calculus. Second, in a more geometric approach, we related the gyroscope frame to the local tetrad frame via rotor transformation, and then solved the governing rotor's single differential equation.

Because the tetrad is a locally defined orthonormal frame, the integration of the relevant differential equation could be carried out as if it were in flat space, considering that the solution applies only locally. Additionally, since the rotor depends solely on the coordinate time, and the system is t-independent, the integration was straightforward.

We then applied the same technique to calculate the gyroscopic precession around a Kerr-Newman black hole. Due to similar symmetries as in the Schwarzschild case, the rotor differential equation remained equally solvable. The primary challenge here arose from the inherently lengthy algebraic expressions for the connection bivectors and the complexity involved in determining the orbital angular velocity.

Overall, this new approach was shown to significantly simplify the calculation of orbital precession around massive objects, while also providing more geometric insight into the rotational planes of the transported frame. In summary, the tetrad-GA formalism offers powerful tools for addressing systems involving rotations or Lorentz transformations. This is because it exploits the system symmetries, decouples coordinate degrees of freedom from those of the frames, and directly reveals geometric behavior, thereby enabling a powerful relativistic treatment of rotations.

Chapter 7

Conclusions and outlook

In this dissertation, I presented the basics of the tetrad-GA formalism and applied it to GR. Compared to the usual approach that employs coordinate frames, using tetrads removes undesired degrees of freedom from the local decomposition of objects. These orthonormal frames represent local observers, meaning that the decomposition of quantities corresponds to the results of local measurements. The definition of tetrad frames from the coordinate frame is not unique, with different tetrads being related by Lorentz transformations—a feature that we thoroughly exploited throughout this work.

The use of GA, on the other hand, provides a powerful set of mathematical tools for describing geometrical objects. Among the multiple benefits of GA, its description of rotations and Lorentz transformations through rotors, rather than 4×4-matrices, stands out. The efficiency of GA directly results from the geometric product, which immediately extends tools that were previously limited to the complex plane to spaces of arbitrary dimension and signature.

Besides this advantage, GA's compact notation tends to simplify calculations and offers greater geometrical insights than either tensor calculus or differential forms. However, this concise description can sometimes be challenging to understand and relate to other formalisms, especially given the limited literature on the field.

This work began by defining GA, presenting some of its properties and providing some introductory examples in Chapter 1. We then extended its application to curved manifolds in Chapter 2, demonstrating that it aligns particularly well with the tetrad description of GR. In this new framework, the various differential operations in GR are condensed into a single operator that possesses the algebraic properties of a vector. This approach allowed us to derive the connection elements as bivectors, which generate the Lorentz transformations of frames moving through the manifold, and naturally obtain other objects like the Ricci vector or the Riemann tensor, which got reinterpreted as a map from bivectors to bivectors–providing a direct connection between areas and rotations and revealing some of its symmetries in a particularly clear way.

To clarify its similarities and differences with another popular formalism, differential forms, and to provide a bridge for new readers, I compared both formalisms side-by-side in Chapter 3. I found that although GA and differential forms share many tools, the geometric product provides GA with a clear computational advantage. Furthermore, GA also tends to offer a more intuitive geometric interpretation of objects.

Moving to applications of the tetrad-GA formalism, in Chapter 4 I explored its use in the well-known FRW spacetimes. We derived Friedmann's equations in two ways: from the metric and from Raychaudhuri's equation. The latter took on a particularly suggestive form in the tetrad-GA formalism, directly relating the relative acceleration of the volume element to the Ricci tensor in a form highly similar to the second Friedmann equation. We also examined the conformal properties of the spacetime, finding no meaningful advantage over tensor calculus or differential forms. Subsequently, we analyzed the spacetime symmetries via the Lie derivative and Killing's equation. Here, in agreement with other literature, we found that defining the Lie derivative for general vector fields within a Clifford bundle is problematic due to its dependence on the metric. However, the Lie derivative can be defined properly for Killing vector fields, along which the metric remains invariant. Finally, we discussed scalar inflationary models, finding no major differences from the conventional treatment, as GA is most advantageous in describing higher-dimensional fields. Nonetheless, the covariant description proved to be both elegant and concise.

In Chapter 5, I examined the formulation of Fermi-Walker transport within the tetrad-GA formalism. Since in GR, 4-acceleration is always orthogonal to the 4-velocity, its effects can be encoded in a rotor that modifies the direction of the 4-velocity. And given that orthonormal frames are related by Lorentz transformations, we can use the rotor constructed from the 4-acceleration to encapsulate the dynamics of non-geodesic motion, simplifying the analysis of certain problems.

To assess the efficacy of this approach, I applied it to describe the 4-acceleration needed for a spaceship to orbit a Morris-Thorne wormhole. In comparison to the conventional description, the tetrad-GA treatment proved to be powerful, concise and particularly insightful by separating contributions to the rotation of the frame originating from the accelerating motion from those coming from spacetime curvature.

Finally, in Chapter 6, I used similar techniques to describe the precession angle of a gyroscope around a black hole. I began with the Schwarzschild solution and then explored the more general Kerr-Newman case. In both cases, the problem reduced to a single differential equation, which could be solved immediately and analytically due to the system's symmetries. Compared to the tensor formalism approach, the tetrad-GA method was shown to be simpler, more manageable, and more insightful geometrically.

From the perspective provided by this work, I can confidently assert that the tetrad-GA formalism is significantly easier to handle than tensor calculus, it provides a clearer geometric interpretation than differential forms, it effectively decouples the degrees of freedom corresponding to coordinate choice from those corresponding to frame choice, and provides a robust set of tools for differential geometry.

Nonetheless, it is essential to highlight some of GA's limitations. Primarily, GA has a steep learning curve due to the limited literature and worked examples available. This is especially relevant for advanced topics in physics. Those interested in applications not present in the GA literature could benefit from exploring the more abstract and mathematical field of Clifford Algebras.

Other challenges to GA include the high level of abstraction within the formalism, which can make it difficult to directly relate to practical applications. Additionally, there is the scarcity of computational programs specifically designed for handling multivectors and geometric products, and the rapidly growth of the dimensionality of the algebra with the dimension of the space.

Potential applications of the tools presented here are as vast as physics itself. However, I believe that fields of particular interest include theoretical work with substantial geometric content, as well as fields where tensor calculus and differential forms already play a fundamental role.

In gravitation, specifically, the extension of complex numbers that GA provides to spaces of arbitrary dimension is a tool whose full potential remains largely untapped. This is particularly promising for applications involving integral theorems, such as those on the boundaries of black holes. Another interesting avenue, given GA's elegant description of electromagnetism, would be extending its description to curved spacetimes with the tetrad-GA formalism.

Beyond GR, there are many theories that could benefit from these tools, including modifications of the kind similar to Scalar–Tensor–Vector models or others like Loop Quantum Gravity.

Finally, I would like to stress that while the physical content of a theory remains unchanged when written in GA, the endeavor can still offer an improved description and enhanced geometric insights, even in already well-known fields. Whether these insights could deepen the understanding of a field and inspire further theoretical expansions is an intriguing possibility that should not be dismissed.

Appendix A

Notation

Using the appropriate notation can be as important as using the proper formalism. The notation that we use in this thesis has been carefully selected and crafted to be as meaningful, simple and intuitive as possible. We also tried to use a common notation with other fields where GA has been applied to make the connection between fields as seamless and intuitive as possible. In this section, we will argue our choice of notation and briefly connect with other fields.

- Our choice of notation for the inner, outer and geometric product follows the usual convention in GA [Doran and Lasenby, 2013].
- For the coordinate frame we chose $\{g_\mu\}$ because their inner product produces the components of the metric, $g_\mu \cdot g_\nu = g_{\mu\nu}$.
- For the tetrad frame we chose $\{\gamma_m\}$ because the most common tetrad frame is Minkowskian and because it establishes a nice correspondence between Greek and Latin letters to change between coordinate and tetrad quantities. The choice of $\{\gamma_m\}$ as the basis frame for the Minkowski space-time might seem arbitrary until one realizes that the basis vectors of flat space-time can be identified with the Dirac matrices and that allows for a neat interpretation of Dirac theory without complex numbers[Hestenes, 1975, Doran et al., 1996]. In the GR context, this choice of notation creates a nice correspondence with the treatment of spinors in curved backgrounds, which necessitates tetrads to be properly included.

  This notation is also in line with other work treating electromagnetism with GA[Dressel et al., 2015] and facilitates the treatment of electromagnetism in curved space-times
- The indices follow the opposite correspondence than the gammas,

Latin middle indices $\{m, n, l, ...\}$ refer to tetrad indices while their Greek counterparts, $\{\mu, \nu, \lambda, ...\}$ refer to coordinate indices. This is in line with the conventional treatment of GR and some of the literature on tetrads.

A problem in notation arises when an index takes a particular value. For these situations, we hat the tetrad indices, $a^{\hat{r}}$, and leave un-hatted the coordinate ones, a^r.

- The choice of $\omega(g_\mu) = \omega_\mu$ for the connection bivectors follow two purposes. One, to distinguish them from the Christoffel symbols, denoted by Γ, and two, to reflect the fact that they are the generators of rotations and their value corresponds to the angular velocity of rotation of a frame displaced in the g_μ direction.
- The choice of $D = g^\mu D_\mu$ for the covariant vector derivative might seem strange, but it follows when considering that ∇ is used ubiquitously in the GA literature as the vector derivative in flat space-time. The reason for this choice being that ∇ can be identified with the Dirac operator and it neatly matches our current notations for gradient, divergence and curl: $\vec{\nabla}\phi, \nabla \cdot a, \nabla \wedge a$.
- The use of three different fonts to describe the Riemann, Ricci vector and Ricci scalar, is necessary because, in the treatment that I present, confusion can arise when referring to them in their abstract form where indices are not present.

Appendix B

Derivation of connection coefficients

As far as I am aware of, the first appearance of the following derivation was by Snygg [2012]. The following is a slight modification of his steps.

To obtain an expression to calculate directly the connection coefficients we'll start with a tetrad basis frame expressed in the basis of our coordinate frame

$$\gamma_m = e_m{}^{\mu} g_{\mu}. \tag{B.0.1}$$

We perform a covariant directional derivative in the g_α direction,

$$D_\alpha \gamma_m = e_m{}^{\mu} D_\alpha g_\mu + \left(\partial_\alpha e_m{}^{\mu}\right) g_\mu. \tag{B.0.2}$$

Where we wrote ∂_α when D_α acts over a scalar.

We can write on the left-hand side as $1/2[\omega_\alpha, \gamma_m] = \omega_\alpha \cdot \gamma_m$, by definition of ω_α, Equation (2.2.7). And, on the right-hand side, $D_\alpha g_\mu = \Gamma^{\beta}_{\mu\alpha} g_\beta$, from the Christoffel symbols definition.

Now we need to left multiply by γ^m to isolate ω_α

$$\gamma^m \omega_\alpha \cdot \gamma_m = \gamma^m e_m{}^{\mu} \Gamma^{\beta}_{\mu\alpha} g_\beta + \gamma^m \left(\partial_\alpha e_m{}^{\mu}\right) g_\mu. \tag{B.0.3}$$

We use the GA identity $\gamma^m \left(\gamma_m \cdot A_r\right) = r A_r$, being A_r a multivector of grade r, to simplify the left-hand side, and, on the right-hand side, we expand the Christoffel symbols into the derivatives of the metric.

$$-2\omega_\alpha = e_m{}^{\mu} \gamma^m \frac{g^{\beta\lambda}}{2} \left(\partial_\mu g_{\alpha\lambda} + \partial_\alpha g_{\mu\lambda} - \partial_\lambda g_{\alpha\mu}\right) g_\beta + \partial_\alpha \left(e_m{}^{\mu} \gamma^m\right) g_\mu \tag{B.0.4}$$

Because γ^m is constant we pulled it inside of the derivative in the last term, and now we can identify the terms $e_m{}^{\mu} \gamma^m = g^\mu$. And, considering that

ω_α is a bivector, we discard terms of grade other than 2 on the right-hand side, and write the geometric products as outer products to get

$$-2\omega_\alpha = \frac{1}{2} g^\mu \wedge g^\lambda \left(\partial_\mu g_{\alpha\lambda} + \partial_\alpha g_{\mu\lambda} - \partial_\lambda g_{\alpha\mu}\right) + \left(\partial_\alpha g^\mu\right) \wedge g_\mu. \tag{B.0.5}$$

The only remaining step is to realize that the $\partial_\alpha g_{\mu\lambda}$ cancels with $g^\lambda \wedge g^\mu$ due to symmetry. And the remaining term $(\partial_\mu g_{\alpha\lambda} - \partial_\lambda g_{\alpha\mu})$ can be written as the vector operator acting on the metric. All in all we obtain:

$$\omega_\alpha = \frac{1}{2}\left(g^\lambda \wedge \nabla g_{\alpha\lambda} + g_\mu \wedge \partial_\alpha g^\mu\right) \tag{B.0.6}$$

This is our final expression to obtain the connection coefficients from the metric and the vierbein, which is hidden in the last term as $\partial_\alpha g^\mu = \gamma^m \partial_\alpha {e_m}^\mu$, and where $\nabla = g^\nu \partial_\nu$ is the flat spacetime vector derivative operator.

In the case of having a diagonal metric, where the coordinate basis vectors are orthogonal, Equation (B.0.6) simplifies considerably if we choose the tetrad frame to be aligned with the coordinate frame.[1]

In this case

$${e_m}^\mu = \mathrm{diag}(|g_{00}|^{-1/2}, |g_{11}|^{-1/2}, |g_{22}|^{-1/2}, |g_{33}|^{-1/2}), \tag{B.0.7}$$

making the second term in Equation (B.0.6) vanish

$$g_\mu = |g_\mu|\gamma_m \Rightarrow g_\mu \wedge \partial_\alpha g^\mu = \sum_\mu |g_\mu| \left(\partial_\alpha |g_\mu|^{-1}\right) \eta^{mm} \gamma_m \wedge \gamma_m = 0. \tag{B.0.8}$$

In this case the first term also gets simplified, because $g^\mu \wedge g^\mu = 0$, and its computation reduces to a maximum of 12 derivatives—three for each coordinate direction of spacetime.

In an abuse of notation, we can write the vierbein as ${e^m}_\mu = \mathrm{diag}((g_{\mu\mu})^{1/2})$ and its inverse as ${e_m}^\mu = \mathrm{diag}((g^{\mu\mu})^{1/2}) = \mathrm{diag}((g_{\mu\mu})^{-1/2})$. Then, Equation (B.0.6) reduces to

$$\omega_\alpha = \frac{1}{2} g^\alpha \wedge \nabla g_{\alpha\alpha} = \frac{1}{2} g^\alpha \wedge g^\mu \partial_\mu g_{\alpha\alpha}, \tag{B.0.9}$$

where α is not a summed index.

Equation (B.0.9) contains a maximum of 12 derivatives and is, by far, the fastest way of computing the connection coefficients that I am are aware of.

[1] Some sources, such as [Hestenes and Sobczyk, 1987, p.235] and [Hestenes, 1986b], claim that this simplification occurs for any orthogonal coordinate frame. However, this is incorrect; the alignment of the tetrad with the coordinate frame is also necessary.

Appendix C

Relation between Ricci rotation coefficients, Christoffel symbols and spin rotation coefficients.

Because the Christoffel symbols $\Gamma_{\kappa\mu\nu}$ are not tensors, it is not possible to change their indices using the vierbein ${e_m}^{\mu}$ to relate them with the connection coefficients $\omega_{km\nu}$,

$$\Gamma_{\kappa\mu\nu} \neq {e^k}_{\kappa}{e^m}_{\mu}\omega_{km\nu}. \tag{C.0.1}$$

To obtain their relationship, we start with the definition of the Christoffel symbols and expand in the tetrad frame

$$\begin{aligned}
{\Gamma^{\kappa}}_{\mu\nu} g_{\kappa} &= \partial_\nu g_\mu = \partial_\nu {e^m}_{\mu}\gamma_m \\
&= (\partial_\nu {e^m}_{\mu})\gamma_m + {e^m}_{\mu}\partial_\nu \gamma_m \\
&= (\partial_\nu {e^l}_{\mu})\gamma_l + {e^m}_{\mu}{\omega_m}^{n}{}_{\nu}\gamma_n \\
&= (\partial_\nu {e^l}_{\mu})\gamma_l + {e^m}_{\mu}{\omega_m}^{n}{}_{\nu}{e_n}^{\kappa} g_\kappa \\
&= \left[{e_l}^{\kappa}\partial_\nu {e^l}_{\mu} + {e^m}_{\mu}{e_n}^{\kappa}{\omega_m}^{n}{}_{\nu}\right] g_\kappa.
\end{aligned} \tag{C.0.2}$$

Thus, we obtain the relationship between the Christoffel symbols and the rotation coefficients as:

$${\Gamma^{\kappa}}_{\mu\nu} = {e_l}^{\kappa}\partial_\nu {e^l}_{\mu} + {e^m}_{\mu}{e_n}^{\kappa}{\omega_m}^{n}{}_{\nu}. \tag{C.0.3}$$

We can invert Equation (C.0.3) with the inverse vierbein to obtain the connection coefficients in terms of the Christoffel symbols:

$$\omega_m{}^n{}_\nu = e^n{}_\kappa e_m{}^\mu \Gamma^\kappa{}_{\mu\nu} - e^n{}_\kappa e_m{}^\mu e_l{}^\kappa \partial_\nu e^l{}_\mu. \tag{C.0.4}$$

The relationship between the spin rotation coefficients $\omega_{kn\mu}$ and the Ricci connection coefficients ω_{knm}, corresponding to the covariant directional derivative in the tetrad frame

$$D_m \gamma_n = \omega_m \cdot \gamma_n = (e_m{}^\mu \omega_\mu) \cdot \gamma_m, \tag{C.0.5}$$

and it is a direct transformation with the vierbein:

$$\omega_{kn\mu} = e_\mu{}^m \omega_{knm}. \tag{C.0.6}$$

Appendix D

On the necessity of covariant derivatives

When working with non-flat manifolds, or non-cartesian coordinates in flat manifolds, it is customary to introduce the covariant derivative operator D to account for the local variation in basis vectors, which the conventional directional derivative does not address. However, as I will show in this appendix, this step can be bypassed by using contracted, abstract objects rather than their component forms. Although this approach is rarely discussed in existing literature, it warrants consideration for its simplicity and coherence.

In Section 2.3, I defined the covariant derivative D and applied the rule that partial derivatives ∂_μ do not act on basis vectors. Instead, one could avoid introducing a distinct covariant derivative operator and use ∇, as defined in Section 1.2, applying partial derivatives to both components and basis vectors directly.

This raises the question: why introduce the covariant derivative at all? The answer lies in the component-based formalisms of tensor calculus, where the covariant derivative becomes necessary because basis vectors are omitted from expressions. This omission precludes deriving connection coefficients from basis vector variations, thus requiring the covariant derivative operator. This approach, still prevalent even in modern texts, underscores the traditional view that treats objects and components interchangeably. See for example [Carroll et al., 2004, p. 484]: "*In accord with our usual practice of blurring the distinction between objects and their components...*"

Although the concept of a connection is standard in mathematics, it remains a source of confusion in many physics texts. For clarity, this section demonstrates how various covariant operators and tensors naturally emerge from different applications of ∇, offering a unified perspective that applies to both flat and curved spacetimes. This method enhances consistency, provides correct ordering in expressions involving second-order derivatives, and inherently satisfies covariance requirements without additional checks. In the following, ∇ is defined as in Equation (1.2.10).

D.1 First order derivative operators

We begin by examining various forms of the directional derivative, a scalar operator, and how multiple covariant derivatives arise from selecting different decomposition bases for the vector derivative and the object upon which it acts:

- **Covariant derivative** The standard covariant derivative, $\tilde{\nabla}_\mu$, is obtained by projecting the vector derivative ∇ and decomposing the vector a in the coordinate basis $\{g_\mu\}$:

$$g_\mu \cdot \nabla a = \partial_\mu (a^\nu g_\nu) = \partial_\mu (a^\nu) g_\nu + a^\nu \partial_\mu g_\nu = (\partial_\mu a^\nu + a^\lambda \Gamma^\nu_{\lambda\mu}) g_\nu. \quad \text{(D.1.1)}$$

  The connection coefficients $\Gamma^\nu_{\lambda\mu}$ are called the *Christoffel symbols.*
- **Fock-Ivanenko covariant derivative** The Fock-Ivanenko covariant derivative is introduced to define a covariant derivative for spinors [Capozziello et al., 2022, Aldrovandi and Pereira, 2012].
  The Fock-Ivanenko derivative is obtained by projecting the vector derivative ∇ into the coordinate frame and expanding the vector a in the tetrad basis $\{\gamma_m\}$:

$$\begin{aligned} g_\mu \cdot \nabla a &= \partial_\mu (a^m \gamma_m) = (\partial_\mu a^m) \gamma_m + a^m \partial_\mu \gamma_m \\ &= (\partial_\mu a^m + a^n {\omega^m}_{n\mu}) \gamma_m. \end{aligned} \quad \text{(D.1.2)}$$

  These connection coefficients are called the *spin connection coefficients* ${\omega^m}_{n\mu}$.
- **Tetrad covariant derivative** The tetrad covariant derivative is less common and corresponds to projecting the vector derivative in the γ_l direction and expanding a in the tetrad basis $\{\gamma_m\}$:

$$\begin{aligned} \gamma_l \cdot \nabla a &= \partial_l (a^m \gamma_m) = \partial_l (a^m) \gamma_m + a^m \partial_l \gamma_m \\ &= (\partial_l a^m + a^n \omega^m_{nl}) \gamma_m. \end{aligned} \quad \text{(D.1.3)}$$

  These connection coefficients are called the *Ricci rotation coefficients* ${\omega^m}_{nl}$, and are given by $\partial_l \gamma_m = {\omega_{nl}}^m \gamma_m$. They are related to the spin connection coefficients by the vierbein:

$$\omega^m{}_{nl} = e^\lambda{}_l \omega^m{}_{n\lambda}. \tag{D.1.4}$$

In some texts, such as [Capozziello et al., 2022], these three objects are treated as distinct operators, whose equivalence is asserted only by the *tetrad principle*. This equivalence principle is unnecessary and ambiguous and has been criticized in [Rodrigues and Gomes de Souza, 2005]. In his text, W.A. Rodrigues rigorously presents several misconceptions and ambiguities regarding the definition and treatment of tetrads in the literature.

We proceed with differential operators resulting from different operations involving ∇:

- **Exterior derivative** The exterior derivative of differential forms, $\mathbf{d}$, is equivalent to the curl of the vector derivative $\nabla\wedge$ in flat space. When acting on a scalar field ϕ, it produces the gradient:

$$\mathbf{d}\phi = \partial_\mu \phi g^\mu = \nabla \wedge \phi = \nabla \phi = g^\mu \partial_\mu \phi, \tag{D.1.5}$$

  and when acting on a vector field, in the reciprocal coordinate basis $a = a_\mu g^\mu$, equivalent to a 1-form, it produces the curl of a:

$$\mathbf{d}a = \partial_\nu a_\mu g^\nu \wedge g^\mu = \nabla \wedge a. \tag{D.1.6}$$

  The condition $\mathbf{d}^2 = 0$ is equivalent to $\nabla\wedge\nabla = 0$ for torsion-free spaces.

- **Exterior covariant derivative** The exterior covariant derivative $\mathbf{D}$ is also equivalent to $\nabla\wedge$ in non-flat spaces, where we also derive the coordinate basis $\{g_\mu\}$:

$$\begin{aligned}\nabla \wedge a &= (\partial_\nu a_\mu) g^\nu \wedge g^\mu + a_\mu g^\nu \wedge (\partial_\nu g^\mu)\\ &= \partial_\nu a_\mu g^\nu \wedge g^\mu + a_\mu \Gamma^\mu_{\nu\lambda} g^\nu \wedge g^\lambda\\ &= (\partial_\nu a_\mu + a_\lambda \Gamma^\lambda_{\nu\mu}) g^\nu \wedge g^\mu.\end{aligned} \tag{D.1.7}$$

- **Covariant Divergence** The operator ∇ produces the correct divergence in both flat and non-flat spaces when applied with the inner product. If $a = a^\nu g_\nu$ is a vector field in a curved manifold, its covariant divergence is:

$$\begin{aligned}\nabla \cdot a &= \nabla \cdot a^\nu g_\nu\\ &= g^\mu \partial_\mu \cdot (a^\nu g_\nu) = g^\mu \cdot \partial_\mu (a^\nu g_\nu)\\ &= g^\mu \cdot (\partial_\mu a^\nu + a^\lambda \Gamma^\nu_{\lambda\mu}) g_\nu\\ &= \delta^\mu_\nu (\partial_\mu a^\nu + a^\lambda \Gamma^\nu_{\lambda\mu})\\ &= \partial_\mu a^\mu + a^\lambda \Gamma^\mu_{\lambda\mu}.\end{aligned} \tag{D.1.8}$$

- **Covariant Curl** The operator ∇ produces the correct curl in both flat and non-flat spaces when applied with the outer product. This operation is equivalent to the covariant exterior derivative. If $a = a^\nu g_\nu$ is a vector field in a curved manifold, its covariant curl is:

$$\begin{aligned} \nabla \wedge a &= \nabla \wedge a^\nu g_\nu \\ &= g^\mu \partial_\mu \wedge (a^\nu g_\nu) = g^\mu \wedge \partial_\mu (a^\nu g_\nu) \\ &= (\partial_\mu a^\nu + a^\lambda \Gamma^\nu_{\lambda\mu}) g^\mu \wedge g_\nu . \end{aligned} \tag{D.1.9}$$

From this perspective we conclude that the only relevant, first-order, physically meaningful derivative operations are:

- **Directional derivative** in the direction of the vector field b, $b\cdot\nabla = \partial_b$.
- **Geometric derivative** resulting from applying the geometric product of ∇, which decomposes into divergence, $\nabla\cdot$, and curl, $\nabla\wedge$.

The gradient is obtained if ∇ acts over is a scalar field ϕ, in which case its divergence is zero and the application of the geometric derivative reduces to the curl, $\nabla\phi = \nabla \wedge \phi$.

D.2 Derivative of linear functions

To describe the derivative of multilinear maps, or tensors, we introduce a notational tool called the *overdot convention*[1]. Since the vector derivative operator is algebraically a vector, it does not generally commute with geometric objects. Therefore, we need a method to denote the derivative of an object that is not immediately adjacent to it. This is achieved by using the overdot notation:

$$\nabla(AB) = (\nabla A)B + \dot{\nabla} A \dot{B}. \tag{D.2.1}$$

This indicates that one must permute the basis vectors of ∇ across the basis vectors of A to obtain the desired result.

$$\dot{\nabla} A \dot{B} = \gamma^\lambda A^{ab\ldots} \gamma_a \wedge \gamma_b \ldots \partial_\lambda B^{cd\ldots} \gamma_c \wedge \gamma_d \ldots . \tag{D.2.2}$$

With this tool, we can express the derivative of a linear function $T(A)$ as:

$$\dot{\nabla} \dot{T}(A) = \nabla T(A) - e^k T(\partial_k A). \tag{D.2.3}$$

Here, the term $\nabla T(A)$ represents the derivative applied to the geometric object $T(A)$ (without differentiating the argument). Thus, $\dot{\nabla}\dot{T}(A)$ only

[1] In the main text, an accent is used to avoid confusion with time derivatives. However, the overdot notation is more common in GA texts.

differentiates the positional dependence within the function and not within the argument [Doran and Lasenby, 2013, §6.1.3].

As an example, consider the directional derivative of T, where T is a linear map from vector fields to vector fields, such that $T(a) = T^{\mu}{}_{\alpha}\, a^{\alpha} g_{\mu}$:

$$\begin{aligned}\dot{\partial}_{\mu}\dot{T}(g_{\alpha}) &= \partial_{\mu}(T^{\lambda}_{\alpha} g_{\lambda}) - T(\partial_{\mu} g_{\alpha}) \\ &= (\partial_{\mu} T^{\lambda}_{\alpha}) g_{\lambda} + T^{\lambda}_{\alpha} \partial_{\mu} g_{\lambda} - T(\partial_{\mu} g_{\alpha}) \\ &= (\partial_{\mu} T^{\lambda}_{\alpha}) g_{\lambda} + T^{\lambda}_{\alpha} \Gamma^{\nu}_{\lambda\mu} g_{\nu} - T(\Gamma^{\lambda}_{\alpha\mu} g_{\lambda}) \\ &= (\partial_{\mu} T^{\nu}_{\alpha} + T^{\lambda}_{\alpha} \Gamma^{\nu}_{\lambda\mu} - T^{\nu}_{\lambda} \Gamma^{\lambda}_{\alpha\mu}) g_{\nu}.\end{aligned} \tag{D.2.4}$$

As demonstrated, we recover the usual rule of contracting with the connections: using a "+" sign for each contravariant index and a "−" sign for each covariant index. Although this procedure may seem intricate, it is equivalent to the tensor formalism. The complexities arise from the fact that the general procedure for representing tensors in GA is not straightforward, and its theory of linear functions, while powerful, is not trivial.

D.2.1 Second order derivatives

In this section, I will demonstrate how various tensors, such as the torsion, Riemann, and Ricci tensors, naturally emerge from the commutation properties of second-order derivatives.

Consider M to be a multivector field in $GT_p\mathcal{M}$. Due to the associativity of the geometric product, we can express its second derivative as:

$$\nabla(\nabla M) = \nabla^2 M = (\nabla \cdot \nabla + \nabla \wedge \nabla) M. \tag{D.2.5}$$

In this form, we can independently analyze the actions of the scalar and bivector operators. To proceed in the most general manner, let us consider a general basis of $T_p\mathcal{M}$, $\{e_a\}$, that satisfies the symmetric relationship:

$$e_a \cdot e_b = \hat{g}_{ab}. \tag{D.2.6}$$

To distinguish these indices from the coordinate indices (represented with Greek symbols) and the tetrad indices (represented with middle Latin indices), we use early Latin indices for the general coordinates $\{a, b, c, \dots\}$, running from 0 to 3.

In this basis, the directional derivatives in the e_a direction are denoted as $\partial_a = e_a \cdot \nabla$. The basis has the connection coefficients Υ such that:

$$\partial_a e_b = \Upsilon^{c}{}_{ba} e_c. \tag{D.2.7}$$

We assume no symmetry in ${\Upsilon^c}_{ba}$, aside from the condition relating to the derivative of the reciprocal basis:

$$\partial_a e^b = -{\Upsilon^b}_{ca} e^c, \tag{D.2.8}$$

which arises from the usual procedure of considering the derivatives of the reciprocal basis e^a, constructed such that $e^a \cdot e_b = \delta^a_b$.

D.2.2 The scalar operator

Starting with the scalar term of Equation (D.2.5), $\nabla \cdot \nabla$, we can observe that it corresponds to the Laplace-de Rham operator:

$$\begin{aligned}
\nabla \cdot \nabla &= (e^a \partial_a) \cdot (e^b \partial_b) \\
&= e^a \cdot \Big[(\partial_a e^b) \partial_b + e^b \partial_a \partial_b \Big] \\
&= e^a \cdot \Big[(-\Upsilon^b_{ca} e^c) \partial_b + e^b \partial_a \partial_b \Big] \\
&= \Big(\partial^a \partial_a - \hat{g}^{cb} \Upsilon^a_{cb} \partial_a \Big) \\
&= \Delta.
\end{aligned} \tag{D.2.9}$$

In the coordinate basis $e_a \to g_\mu$ and $\Upsilon^a_{cb} \to \Gamma^\sigma_{\mu\nu}$, we have:

$$\Delta = \partial^\mu \partial_\mu - g^{\mu\nu} \Gamma^\sigma_{\mu\nu} \partial_\sigma. \tag{D.2.10}$$

It is noteworthy that due to the associativity of the geometric product we obtain the Laplace-de Rham operator, which acts on multivector fields, rather than the Laplace-Beltrami operator, which is defined solely on twice-differentiable real-valued functions.

D.2.3 The Ricci operator

The bivector term of our second-order derivative operator, $\nabla \wedge \nabla$, represents the curl of the vector derivative, and as we will see it is related to the commutation of directional derivatives. When applied to a scalar field, we obtain the torsion tensor, while the Ricci tensor appears when it is applied to a vector, hence its name its name. And the Riemann tensor only arises when applied to a multivector.

To analyze this, we expand ∇ in the general basis $\{e_a\}$:

$$\begin{aligned}\nabla \wedge \nabla &= (e^a\partial_a) \wedge (e^b\partial_b) \\ &= e^a \wedge \left[(\partial_a e^b)\partial_b + e^b\partial_a\partial_b\right] \\ &= e^a \wedge \left[(-\Upsilon^b_{ca}e^c)\partial_b + e^b\partial_a\partial_b\right] \\ &= e^a \wedge e^b\left(\partial_a\partial_b - \Upsilon^c_{ba}\partial_c\right).\end{aligned} \tag{D.2.11}$$

Because $e^a \wedge e^b$ is antisymmetric, we can rewrite Equation (D.2.11) as:

$$\nabla \wedge \nabla = e^a \wedge e^b\left([\partial_a, \partial_b] + (\Upsilon^c_{ab} - \Upsilon^c_{ba})\,\partial_c\right), \tag{D.2.12}$$

where $[\partial_a, \partial_b] = \partial_a\partial_b - \partial_b\partial_a$ is the commutator of directional derivatives, and we proceed to analyze each term separately.

The commutator of directional derivatives, $[\partial_a, \partial_b]$, can be expanded in terms of the coordinate frame and the vierbein, $g_\mu = e^a{}_\mu e_a$, along with similar relations, to obtain the commutator of coordinate derivatives and the *coefficients of anholonomy*:

$$\begin{aligned}[\partial_a, \partial_b] &= [e_a{}^\mu\partial_\mu, e_b{}^\nu\partial_\nu] \\ &= e_a{}^\mu(\partial_\mu e_b{}^\nu)\partial_\nu + e_a{}^\mu e_b{}^\nu\partial_\mu\partial_\nu \\ &- e_b{}^\nu(\partial_\nu e_a{}^\mu)\partial_\mu - e_b{}^\nu e_a{}^\mu\partial_\nu\partial_\mu \\ &= e_a{}^\mu(\partial_\mu e_b{}^\nu)e^c{}_\nu\partial_c - e_b{}^\nu(\partial_\nu e_a{}^\mu)e^c{}_\mu\partial_c \\ &+ e_a{}^\mu e_b{}^\nu(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu) \\ &= e_a{}^\mu e_b{}^\nu(\partial_\nu e^c{}_\mu - \partial_\mu e^c{}_\nu)\partial_c + e_a{}^\mu e_b{}^\nu[\partial_\mu, \partial_\nu] \\ &= f^c_{ab}\partial_c + e_a{}^\mu e_b{}^\nu[\partial_\mu, \partial_\nu].\end{aligned} \tag{D.2.13}$$

The coefficients of anholonomy, or *structure coefficients*, f^l_{ab}, represent the non-closure of a circuit using steps the size of the basis $\{e_a\}$, while $[\partial_\mu, \partial_\nu]$ denotes the commutator of coordinate directional derivatives.

In GR we set torsion to zero, consequently the Christoffel symbols of second kind are symmetric in their lower indices and the commutator of directional derivatives, when acting over scalar fields, is zero. This is carelessly extended to the rule that coordinate directional derivatives commute.

The approach presented here is more consistent: When setting torsion to zero, directional derivatives only commute when acting over scalar fields because those contain no basis vectors,

$$[\partial_\mu, \partial_\nu]\phi = 0. \tag{D.2.14}$$

But when acting over vector or multivector fields there is a dependence on the path taken and coordinate derivatives do not commute. In fact, by following the chain rule and expanding such operation we obtain the Ricci and Riemann tensors.

$$\begin{aligned}
[\partial_\mu, \partial_\nu]a &= (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)a^m\gamma_m \\
&= \cancelto{0}{(\partial_\mu\partial_\nu a^m - \partial_\nu\partial_\mu a^m)} \quad \gamma_m \\
&+ \cancel{(\partial_\nu a^m)\omega^l_{m\mu}\gamma_l} - \cancel{(\partial_\mu a^m)\omega^l_{m\nu}\gamma_l} \\
&+ \cancel{(\partial_\mu a^m)\omega^l_{m\nu}\gamma_l} - \cancel{(\partial_\nu a^m)\omega^l_{m\mu}\gamma_l} \\
&+ a^m\gamma_n(\partial_\mu\omega^n_{m\nu}) - a^m\gamma_n(\partial_\nu\omega^n_{m\mu}) \\
&+ \omega^n_{m\nu}\omega^l_{n\mu}a^m\gamma_l - \omega^n_{m\mu}\omega^l_{n\nu}a^m\gamma_l \\
&= a^m\gamma_n(\partial_\mu\omega^n_{m\nu}) - a^m\gamma_n(\partial_\nu\omega^n_{m\mu}) \\
&+ \omega^l_{m\nu}\omega^n_{l\mu}a^m\gamma_n - \omega^l_{m\mu}\omega^n_{l\nu}a^m\gamma_n \\
&= \left(\partial_\mu\omega^n_{m\nu} - \partial_\nu\omega^n_{m\mu} + \omega^l_{m\nu}\omega^n_{l\mu} - \omega^l_{m\mu}\omega^n_{l\nu}\right) a^m\gamma_n \\
&= R^n{}_{m\mu\nu}a^m\gamma_n = [R_{\mu\nu}, a] = \mathbf{R}_{\mu\nu}\cdot a.
\end{aligned} \tag{D.2.15}$$

Here, $R^n{}_{m\mu\nu}$ are the components of the Riemann tensor in mixed coordinates, and $\mathbf{R}_{\mu\nu}$ denotes the Riemann tensor as defined in Section 2.4. The final step holds only for a being a vector field. Following the same steps for a multivector field M, we obtain:

$$[\partial_\mu, \partial_\nu]M = [\mathbf{R}(g_\mu \wedge g_\nu), M]. \tag{D.2.16}$$

Meaning that the commutator of coordinate directional derivatives is equivalent to the commutator of the Riemann map, acting over the coordinate bivector $g_\mu \wedge g_\nu$, with the multivector M.

Notice, however, that in Equation (D.2.12), $[\partial_\mu, \partial_\nu]$ appears contracted with $g^\mu \wedge g^\nu$. This contraction further simplifies Equation (D.2.15) by the first Bianchi identity, Equation (2.4.6), and the definition of the Ricci vector, Equation (2.4.8), as follows:

$$\begin{aligned}
g^\mu \wedge g^\nu R^\alpha{}_{\beta\mu\nu}a^\beta g_\alpha &= R^\alpha{}_{\beta\mu\nu}a^\beta\,(g^\mu \wedge g^\nu \cdot g_\alpha + g^\mu \wedge g^\nu \wedge g_\alpha) \\
&= R^\alpha{}_{\beta\mu\alpha}a^\beta g^\mu + \cancelto{0}{R_{\alpha\beta\mu\nu}a^\beta g^\mu \wedge g^\nu \wedge g^\alpha} \\
&= R_{\beta\mu}a^\beta g^\mu = R(a).
\end{aligned} \tag{D.2.17}$$

Thus, we obtain the action of the Ricci tensor on the vector field a.

By decomposing the commutator of directional derivatives into the commutator of coordinate directional derivatives plus the structure coefficients, we can rewrite Equation (D.2.11) as:

$$\nabla \wedge \nabla = e^a \wedge e^b \left({e_a}^\mu {e_b}^\nu [\partial_\mu, \partial_\nu] + (f^c_{ab} + \Upsilon^c_{ab} - \Upsilon^c_{ba})\, \partial_c \right). \tag{D.2.18}$$

Examining the second term on the right-hand side, we can directly identify it with the torsion tensor:

$$(f^c_{ab} + \Upsilon^c_{ab} - \Upsilon^c_{ba})\, \partial_c = (f^c_{ab} + \Upsilon^c_{ab} - \Upsilon^c_{ba})\, e_c \cdot \nabla = S_{ab} \cdot \nabla. \tag{D.2.19}$$

In GA, the torsion tensor is a map from bivectors to vectors, such that it takes an bivector area $a \wedge b$ and returns the corresponding non-closure vector $S(a \wedge b) = S_{ab} = {S^c}_{ab} e_c$:

$$S : \Lambda^2 \in GT_p\mathcal{M} \to \Lambda^1 \in GT_p\mathcal{M}. \tag{D.2.20}$$

This construction of torsion gives immediately its antisymmetric character in its bottom two indices.

We can express the torsion tensor in components as:

$$S = g^\mu \wedge g^\nu \wedge g_\alpha {S^\alpha}_{\mu\nu}. \tag{D.2.21}$$

The torsion tensor acts as a first-order differential operator. Therefore, the variation due to torsion of a vector field a, as a result of transporting it around a coordinate area $g_\mu \wedge g_\nu$, is given by:

$$S_{\mu\nu} \cdot \nabla a = {S^\alpha}_{\mu\nu} \partial_\alpha a = {S^\alpha}_{\mu\nu} (\partial_\alpha a^\beta + a^\lambda \Gamma^\beta_{\lambda\alpha}) g_\beta. \tag{D.2.22}$$

The torsion tensor quantifies the non-closure of parallelograms, as illustrated in Figure D.1. From Equation (D.2.19), we see that this dislocation, which is analogous to the Burgers vector in solid-state physics, can have two distinct origins.

First, torsion may arise from the choice of an anholonomic basis, in which case $f^c_{ab} \neq 0$. This type of torsion is not physical, as evident from its appearance in flat spaces when constructing an orthonormal tetrad in cylindrical coordinates and transporting a vector around a loop taking normalized steps along the coordinate grid.

Second, torsion, like in solid-state physics, can originate from "defects" in spacetime, which impede loops from "closing" everywhere. This type of torsion is physically meaningful. In the Einstein-Cartan theory of gravitation, spin generates this torsion in the same way that mass generates

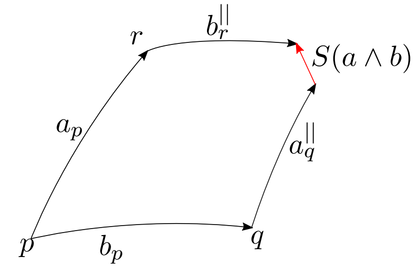


Figure D.1: Representation of torsion. Given two vector fields a and b that coexist at point p, we can parallel-transport one along the other to obtain $a_q^{||}$ and $b_r^{||}$. The failure of the transported vectors to close is quantified by the torsion vector $S(a_p \wedge b_p)$.

curvature. In such cases, it is impossible to globally close parallelograms by changing the choice of frame, just as curvature effects cannot be globally removed by changing coordinates.

In GR, we axiomatically set torsion to zero. Gravitational theories that include both curvature and torsion are called Einstein-Cartan theories. As they are currently developed, such theories predict torsion to be non-propagating outside of sources, making it either non-measurable or irrelevant for current gravitational calculations.

Finally, we can elegantly express the action of the second derivative of a vector field as:

$$\nabla^2 a = \Delta a + R(a) + S_{bc} \cdot \nabla a, \tag{D.2.23}$$

where $S_{bc} = S(b \wedge c)$ is the torsion tensor applied to the corresponding bivector, as defined in Equation (D.2.19).

D.2.4 Einstein Equations

Under these considerations, we can recast the Einstein tensor to highlight its second-order nature. To do so, we introduce the following notation for the derivative with respect to a vector:

$$\partial_{(a)} = g^\mu \frac{\partial}{\partial a^\mu}. \tag{D.2.24}$$

This operator is analogous to ∇, with the difference that $\nabla = \partial^\mu \frac{\partial}{\partial x^\mu}$ performs derivatives with respect to coordinates, while $\partial_{(a)}$ operates with respect to the components of a vector.

Using this, we can remove the a dependence from the Ricci vector in a contraction operation:

$$\partial_{(a)} \cdot R(a) = \partial_{(a)} \cdot (\nabla \wedge \nabla a) = \mathcal{R}, \tag{D.2.25}$$

to obtain the Ricci scalar.

Thus, we can express the Einstein tensor as:

$$G(a) = R(a) - \frac{1}{2}\, a\, \mathcal{R} = \nabla \wedge \nabla a - \frac{1}{2}\, a\, \partial_{(b)} \cdot (\nabla \wedge \nabla b). \tag{D.2.26}$$

This form of the Einstein tensor makes its second-order nature explicit and shows its relationship to the circulation of vector fields.

Einstein's equations then take the form:

$$\nabla \wedge \nabla a - \frac{1}{2}\, a\, \partial_{(a)} \cdot (\nabla \wedge \nabla a) = T(a), \tag{D.2.27}$$

where $T(a)$ denotes the energy-momentum tensor of the system as defined in Section 2.5.

D.3 Conclusions

Based on these calculations, we conclude that when physical laws are expressed in their abstract, contracted form, they do not require alteration for application to curved manifolds. The systematic application of the chain rule to directional derivatives acting on the components and corresponding basis vectors of objects naturally produces all necessary connection terms. Therefore, the *minimal coupling principle* can be viewed as a non-fundamental rule that does not need to be imposed, but rather arises from using an inadequate formalism.

An illustrative example of this is the expansion of Maxwell's equations, Equation (1.3.3), into components using these rules. Following this approach, it yields all the correct terms for its generalization to curved manifolds. Moreover, all connection terms emerge without ambiguity in their order, unlike with the minimal coupling procedure.

Even if this presentation may seem trivial, it is noteworthy how smoothly the results are obtained. I believe we should adhere to the principle of parsimony when selecting formalisms, introducing new operators and rules *only when strictly necessary*. In this regard, the approach developed here proves to be remarkably simple and consistent.

Appendix E

Schwarzschild and Kerr-Newman tensors

In this appendix, I present the Riemann tensor for the Schwarzschild and Kerr-Newman black holes as calculated using the tetrad-geometric algebra (GA) formalism presented in Chapter 2.

Since we are dealing with multiple bases, and the mixture of indices and abstract notation can be confusing, it is helpful to clarify the components of the Riemann tensor explicitly.

The Riemann tensor is a multilinear function between bivectors, meaning it acts on a bivector and returns another bivector. We can express its action over the bivectors $a \wedge b \in \bigwedge^2 GT_p\mathcal{M}$ as:

$$\mathbf{R}(a \wedge b) = a^{\mu} b^{\nu} \left(\mathbf{R}(g_\mu \wedge g_\nu)\right)^{\alpha\beta} e_\alpha \wedge e_\beta. \tag{E.0.1}$$

Where we decomposed $a \wedge b$ in the coordinate base and expressed $\mathbf{R}(g_\mu \wedge g_\nu)$ in the general base $\{e_\alpha\}$.

We usually denote the bivector resulting of making the Riemann act over the coordinate bivectors as $\mathbf{R}(g_\mu \wedge g_\nu) = \mathbf{R}_{\mu\nu}$, and its expansion in the coordinate base,

$$\mathbf{R}_{\mu\nu} = \left(\mathbf{R}_{\mu\nu}\right)_{\alpha\beta} g^{\alpha} \wedge g^{\beta}, \tag{E.0.2}$$

gives back the components $\left(\mathbf{R}_{\mu\nu}\right)_{\alpha\beta}$ corresponding to the expression of the Riemann tensor in conventional tensor calculus $R_{\mu\nu\alpha\beta}$.

In this context, confusion can arise when discussing a change of basis, as it is possible to change either the bivector on which the Riemann tensor acts or the basis in which it is expanded.

Another common choice is, $\mathbf{R}(\gamma_m \wedge \gamma_n) = \mathbf{R}_{mn}$, representing the bivectors resulting of making the Riemann map act over the tetrad bivectors. The decomposition of this bivectors in the tetrad frame is what is refer to in literature as the Riemann tensor in the tetrad frame R_{mnab}.

E.1 Schwarzschild black hole

By working in Schwarzschild coordinates and choosing a tetrad with a minkowski metric whose axes are aligned with the coordinate axes we obtain the following expressions

The Riemann tensor over the coordinate bivectors, $\mathbf{R}_{\mu\nu} = \mathbf{R}(g_\mu \wedge g_\nu)$, as expressed in the tetrad and the coordinate frame is

$$\mathbf{R}_{tr} = -\frac{2GM}{r^3}\gamma_t \wedge \gamma_r = -\frac{2GM}{r^3} g_t \wedge g_r \tag{E.1.1}$$

$$\mathbf{R}_{t\theta} = \frac{GM}{r^2}\sqrt{1-\frac{2GM}{r}}\gamma_t \wedge \gamma_\theta = \frac{GM}{r^3} g_t \wedge g_\theta \tag{E.1.2}$$

$$\mathbf{R}_{t\phi} = \frac{GM\sin(\theta)}{r^2}\sqrt{1-\frac{2GM}{r}}\gamma_t \wedge \gamma_\phi = \frac{GM}{r^3} g_t \wedge g_\phi \tag{E.1.3}$$

$$\mathbf{R}_{r\theta} = \frac{GM}{r^2}\frac{1}{\sqrt{1-\frac{2GM}{r}}}\gamma_r \wedge \gamma_\theta = \frac{GM}{r^3} g_r \wedge g_\theta \tag{E.1.4}$$

$$\mathbf{R}_{r\phi} = \frac{GM\sin(\theta)}{r^2}\frac{1}{\sqrt{1-\frac{2GM}{r}}}\gamma_r \wedge \gamma_\phi = \frac{GM}{r^3} g_r \wedge g_\phi \tag{E.1.5}$$

$$\mathbf{R}_{\theta\phi} = -\frac{2GM\sin(\theta)}{r}\gamma_\theta \wedge \gamma_\phi = -\frac{2GM}{r^3} g_\theta \wedge g_\phi \tag{E.1.6}$$

The Riemann tensor over the tetrad bivectors, $\mathbf{R}_{mn} = \mathbf{R}(\gamma_m \wedge \gamma_n)$ as expressed in the tetrad frame are

$$\mathbf{R}_{\hat{t}\hat{r}} = -\frac{2GM}{r^3}\gamma_t \wedge \gamma_r \tag{E.1.7}$$

$$\mathbf{R}_{\hat{t}m} = \frac{GM}{r^3}\gamma_t \wedge \gamma_m, \quad m = \theta, \phi \tag{E.1.8}$$

$$\mathbf{R}_{\hat{r}m} = \frac{GM}{r^3}\gamma_r \wedge \gamma_m, \quad m = \theta, \phi \tag{E.1.9}$$

$$\mathbf{R}_{\hat{\theta}\hat{\phi}} = -\frac{2GM}{r^3}\gamma_\theta \wedge \gamma_\phi \tag{E.1.10}$$

Note that the components of the Riemann acting over the coordinate bivectors expressed in the coordinate frame has the same components than the Riemann acting over the tetrad components expressed in the tetrad basis. This is a consequence of the high degree of symmetry of the system and it is not a general result.

Because the Schwarzschild metric is a vacuum solution of Einstein's equations, the Ricci vector, Ricci scalar and Einstein tensor are identically zero.

E.2 Kerr-Newman

For simplicity I will only present the components of the Riemann tensor for the equatorial plane $\theta = \pi/2$ as calculated from the Boyer-Lindquist coordinates, Equation (6.2.1). As explained in Section 6.2, we choose tetrads which are static with respect to the background stars as observed by distant observers. This is known in literature as static tetrads.

The Riemann over the coordinate bivectors, $\mathbf{R}_{\mu\nu} = \mathbf{R}(g_\mu \wedge g_\nu)$, as expressed in the tetrad frame and coordinate frame is

$$\begin{aligned}\mathbf{R}_{tr} &= \frac{3Q^2 - 2Mr}{r^4}\gamma_t \wedge \gamma_r + \frac{a\left(Mr - Q^2\right)}{r^4 R\sqrt{\Delta}}\gamma_r \wedge \gamma_\phi \\ &= \frac{a^2\left(4Q^2 - 3Mr\right) + r^2\left(3Q^2 - 2Mr\right)}{r^6} g_t \wedge g_r \\ &+ \frac{a\left(3Mr - 4Q^2\right)}{r^6} g_r \wedge g_\phi\end{aligned} \tag{E.2.1}$$

$$\begin{aligned}\mathbf{R}_{t\theta} &= \frac{\left(Mr - Q^2\right)R\sqrt{\Delta}}{r^4}\gamma_t \wedge \gamma_\theta + \frac{a\left(Q^2 - 2Mr\right)}{r^4}\gamma_\theta \wedge \gamma_\phi \\ &= \frac{a^2\left(3Mr - 2Q^2\right) + r^2\left(Mr - Q^2\right)}{r^6} g_t \wedge g_\theta \\ &+ \frac{a\left(2Q^2 - 3Mr\right)}{r^6} g_\theta \wedge g_\phi\end{aligned} \tag{E.2.2}$$

$$\mathbf{R}_{t\phi} = \frac{\left(Mr - Q^2\right)R\sqrt{\Delta}}{r^4}\gamma_t \wedge \gamma_\phi = \frac{Mr - Q^2}{r^4} g_t \wedge g_\phi \tag{E.2.3}$$

$$\mathbf{R}_{r\theta} = \frac{Mr - Q^2}{r^2 R\sqrt{\Delta}}\gamma_r \wedge \gamma_\theta = \frac{Mr - Q^2}{r^4} g_r \wedge g_\theta \tag{E.2.4}$$

$$\begin{aligned}\mathbf{R}_{r\phi} &= \frac{a\left(3Q^2 - 2Mr\right)}{r^4}\gamma_t \wedge \gamma_r + \frac{\left(Mr - Q^2\right)R}{r^4\sqrt{\Delta}}\gamma_r \wedge \gamma_\phi \\ &= \frac{a\left(a^2 + r^2\right)\left(4Q^2 - 3Mr\right)}{r^6} g_t \wedge g_r \\ &+ \frac{a^2\left(3Mr - 4Q^2\right) + r^2\left(Mr - Q^2\right)}{r^6} g_r \wedge g_\phi\end{aligned} \tag{E.2.5}$$

$$
\begin{aligned}
\mathbf{R}_{\theta\phi} &= \frac{a\left(Mr - Q^2\right) R\sqrt{\Delta}}{r^4}\gamma_t \wedge \gamma_\theta + \frac{R^2\left(Q^2 - 2Mr\right)}{r^4}\gamma_\theta \wedge \gamma_\phi \\
&= \frac{a\left(a^2 + r^2\right)\left(3Mr - 2Q^2\right)}{r^6} g_t \wedge g_\theta \\
&+ \frac{a^2\left(2Q^2 - 3Mr\right) + r^2\left(Q^2 - 2Mr\right)}{r^6} g_\theta \wedge g_\phi
\end{aligned}
\tag{E.2.6}
$$

The Riemann acting over the tetrad bivectors, $\mathbf{R}_{mn} = \mathbf{R}(\gamma_m \wedge \gamma_n)$ as expressed in the tetrad frame is

$$
\mathbf{R}_{\hat{t}\hat{r}} = \frac{3Q^2 - 2Mr}{r^4}\gamma_t \wedge \gamma_r \tag{E.2.7}
$$

$$
\mathbf{R}_{\hat{t}m} = \frac{Mr - Q^2}{r^4}\gamma_t \wedge \gamma_m, \quad m = \theta, \phi \tag{E.2.8}
$$

$$
\mathbf{R}_{\hat{r}m} = \frac{Mr - Q^2}{r^4}\gamma_r \wedge \gamma_m, \quad m = \theta, \phi \tag{E.2.9}
$$

$$
\mathbf{R}_{\hat{\theta}\hat{\phi}} = \frac{Q^2 - 2Mr}{r^4}\gamma_\theta \wedge \gamma_\phi \tag{E.2.10}
$$

The expressions in the tetrad base are more compact and observing the $\theta - \phi$ symmetry in the $\mathbf{R}_{\hat{t}m}$ and $\mathbf{R}_{\hat{r}m}$ we can see that they also reflect better the spherical symmetry of the system.

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