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The Random Greedy Hypergraph Matching Process

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Abstract

Consider the *random greedy hypergraph matching process* that, for a given hypergraph $\mathcal{H}$, iteratively chooses edges of a matching as follows. In every iteration, an edge is chosen uniformly at random among all edges of $\mathcal{H}$ that do not intersect a previously chosen edge. If no such edges exist, then the process terminates. We analyze the behavior of a variation and a special case of this process.

For the variation we consider, there is an initially given collection of forbidden edge sets that are not allowed to be subsets of the generated matching. Edges are then not chosen among all those that do not intersect with a previously chosen edge, but only among those that also do not form such a forbidden edge set with previously chosen edges. We call this variation the *conflict-free matching process*. Through an analysis of this process, we determine conditions on the hypergraph $\mathcal{H}$ and the collection of forbidden edge sets that still allow us to guarantee that, with high probability, the process generates a matching that covers all but a small fraction of the vertices of $\mathcal{H}$. This in turn allows us to obtain general theorems that describe settings where almost-perfect matchings with pseudorandom properties that also avoid given edge sets as subsets exist. Exploiting that a wide range of combinatorial problems can be phrased as hypergraph matching problems, these theorems can be applied in several different areas. As one application, we prove an approximate version of a generalization of a conjecture of Erdős about Steiner systems.

Perhaps one of the most obvious ways to construct a k-uniform hypergraph on n vertices containing no copies of a fixed k-uniform hypergraph $\mathcal{F}$, but preferably many edges, is the *$\mathcal{F}$-removal process*. Starting with a complete k-uniform hypergraph on n vertices, this iterative process proceeds as follows. In every iteration, all edges of a copy of $\mathcal{F}$ chosen uniformly at random among all remaining copies are removed and once no copies of $\mathcal{F}$ remain, the process terminates. While this process, which corresponds to the random greedy hypergraph matching process in an auxiliary hypergraph, is easy to formulate, proving that asymptotic runtime bounds with the correct order of magnitude hold with high probability is challenging. So far, ignoring the cases where $\mathcal{F}$ has at most one edge, such proofs were only available for one choice of $\mathcal{F}$, namely when $\mathcal{F}$ is a triangle. We extend this by proving such bounds whenever $\mathcal{F}$ comes from a large natural class of hypergraphs. As this class in particular includes all complete uniform hypergraphs, this confirms the major folklore conjecture in the area in a very strong form.

Zusammenfassung

Der *Random Greedy Hypergraph Matching Process* ist der Zufallsprozess, der bei gegebenem Hypergraphen $\mathcal{H}$ iterativ die Kanten eines Matchings wie folgt auswählt. In jeder Iteration wird eine Kante gleichverteilt zufällig unter all den Kanten von $\mathcal{H}$ ausgewählt, die keine der zuvor ausgewählten Kanten schneiden. Falls keine solchen Kanten existieren, so terminiert der Prozess. Wir analysieren das Verhalten einer Variante und eines Spezialfalls dieses Prozesses.

Bei der betrachteten Variante ist zu Beginn eine Familie verbotener Kantenmengen gegeben, die keine Teilmengen des konstruierten Matchings sein dürfen. Kanten werden dann nicht unter allen Kanten ausgewählt, die keine der zuvor ausgewählten Kanten schneiden, sondern nur unter denen, die zusätzlich keine verbotene Kantenmenge mit zuvor ausgewählten Kanten bilden. Wir bezeichnen diese Variante als *Conflict-Free Matching Process*. Aufbauend auf einer Analyse dieses Prozesses bestimmen wir Bedingungen für den Hypergraphen $\mathcal{H}$ und die Familie verbotener Kantenmengen, die es uns erlauben zu garantieren, dass der Prozess mit hoher Wahrscheinlichkeit ein Matching generiert, das bis auf einen kleinen Anteil alle Ecken von $\mathcal{H}$ überdeckt. Dies erlaubt es uns allgemeine Sätze zu formulieren, die Situationen beschreiben, in denen fast perfekte Matchings mit pseudozufälligen Eigenschaften existieren, die zusätzlich gegebene Kantenmengen als Teilmengen vermeiden. Da sich viele kombinatorische Probleme als Matchingprobleme in Hypergraphen beschreiben lassen, können diese Sätze in vielen verschiedenen Teilgebieten angewendet werden. Als eine Anwendung zeigen wir eine approximative Version einer Verallgemeinerung einer Vermutung von Erdős über Blockpläne.

Vielleicht eines der offensichtlichsten Verfahren für die Konstruktion k-uniformer Hypergraphen auf n Ecken, die keine Kopien eines festen k-uniformen Hypergraphen $\mathcal{F}$, aber möglichst viele Kanten enthalten, ist der als *$\mathcal{F}$-Removal Process* bezeichnete Zufallsprozess. Beginnend mit einem vollständigen k-uniformen Hypergraphen auf n Ecken verfährt dieser iterative Prozess wie folgt. In jeder Iteration werden alle Kanten einer gleichverteilt zufällig unter allen verbleibenden Kopien von $\mathcal{F}$ ausgewählten Kopie entfernt. Sobald keine Kopien von $\mathcal{F}$ verbleiben, terminiert der Prozess. Dieser Prozess, der dem Random Greedy Hypergraph Matching Process in einem Hilfshypergraphen entspricht, ist einfach zu formulieren, es ist aber eine Herausforderung zu zeigen, dass asymptotische Laufzeitschranken der richtigen Größenordnung mit hoher Wahrscheinlichkeit gelten. Abgesehen von den Fällen, bei denen $\mathcal{F}$ höchstens eine Kante hat, gab es solche Beweise bis jetzt nur für eine Wahl von $\mathcal{F}$, nämlich nur, falls $\mathcal{F}$ ein Dreieck ist. Als Erweiterung dessen zeigen wir solche Schranken für alle Fälle bei denen $\mathcal{F}$ Teil einer großen, natürlichen Klasse von Hypergraphen ist. Da diese Klasse insbesondere alle vollständigen uniformen Hypergraphen umfasst, bestätigt dies eine zentrale Vermutung in diesem Forschungsgebiet auf sehr allgemeine Weise.

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Contents

Chapter 1

Overview

1.1 Introduction

1.1.1 Hypergraph matchings

A *hypergraph* $\mathcal{H}$ consists of a finite set V together with a set E of subsets of V. The set V is the *vertex set* and the set E the *edge set* of $\mathcal{H}$. The elements of V are the *vertices* and the elements of E are the *edges* of $\mathcal{H}$. We use $V(\mathcal{H})$ to denote the vertex set and $E(\mathcal{H})$ to denote the edge set of $\mathcal{H}$. For $k \geq 1$, the hypergraph $\mathcal{H}$ is *k-uniform* if all edges of $\mathcal{H}$ are *k-sets*, that is have size k, and $\mathcal{H}$ is *uniform* if $\mathcal{H}$ is k-uniform for some k. A *k-graph* is a k-uniform hypergraph and a *graph* is a 2-graph. A *matching* in $\mathcal{H}$ is a set $\mathcal{M}$ of disjoint edges of $\mathcal{H}$. The matching $\mathcal{M}$ *covers* a vertex v if there exists an edge $e \in \mathcal{M}$ with $v \in e$ and $\mathcal{M}$ is a *perfect* matching in $\mathcal{H}$ if $\mathcal{M}$ covers every vertex of $\mathcal{H}$.

Suppose that given a hypergraph $\mathcal{H}$ and an integer m, the goal is to determine whether there exists a matching in $\mathcal{H}$ that has size at least m. If inputs are considered where $\mathcal{H}$ is a graph, then this is a computationally and structurally well understood problem. Indeed, there exist several polynomial-time algorithms for this problem as well as closely related variations, see for example [102], and several statements characterizing the structure of matchings in $\mathcal{H}$ are available, see for example [87]. However, the more general variant of this problem where the input hypergraphs are not necessarily 2-uniform appears to be more challenging. Restricting the inputs to certain 3-graphs yields one of Karp's 21 NP-complete problems [67] and there are several famous open conjectures concerning related questions, with important examples including the "Erdős Matching Conjecture" [29] and "Ryser's Conjecture" [100]. For recent progress and further open problems concerning matchings, see for example [38, 54, 116].

One reason why hypergraph matchings have been extensively studied, see for instance [70], is that many combinatorial problems can be phrased as hypergraph matching problems in the sense that many problems correspond to asking whether there exists a matching with certain properties in a specific hypergraph, see for example [64] for a survey with a focus on coloring problems. Another important class of such problems that play a central role for this thesis are *packing* and *decomposition* problems, see for example [51], where the general situation is as follows. Suppose that we are given two hypergraphs $\mathcal{H}$ and $\mathcal{F}$ and that the goal is to find a (large) $\mathcal{F}$*-packing* in $\mathcal{H}$, that is a collection $\mathscr{F}$ of edge-disjoint copies of $\mathcal{F}$ in $\mathcal{H}$. Here, we say that two hypergraphs $\mathcal{A}$ and $\mathcal{B}$ are *edge-disjoint* if their edge sets are disjoint, we say that a hypergraph $\mathcal{F}'$ is a *copy* of $\mathcal{F}$ if there exists a bijection $\beta\colon V(\mathcal{F}) \to V(\mathcal{F}')$ with $\beta(e) \in E(\mathcal{F}')$ for all $e \in E(\mathcal{F})$ and $\beta^{-1}(e') \in E(\mathcal{F})$ for all $e' \in E(\mathcal{F}')$ and we say that a hypergraph $\mathcal{F}'$ is a hypergraph *in* $\mathcal{H}$ if $\mathcal{F}'$ is a *subgraph* of $\mathcal{H}$, that is if $V(\mathcal{F}') \subseteq V(\mathcal{H})$ and $E(\mathcal{F}') \subseteq E(\mathcal{H})$ hold (we use the term subgraph here even if the hypergraphs we consider are not graphs). To see that such packing problems indeed can be phrased as hypergraph matching problems, consider the auxiliary hypergraph $\mathcal{H}^*$ with $V(\mathcal{H}^*) = E(\mathcal{H})$ whose edges are the edge sets of copies of $\mathcal{F}$ in $\mathcal{H}$. Then, collections of edge-disjoint copies of $\mathcal{F}$ in $\mathcal{H}$ precisely correspond to matchings in $\mathcal{H}^*$. A collection of edge-disjoint copies of $\mathcal{F}$ in $\mathcal{H}$ that corresponds to a perfect matching is called a *decomposition* of $\mathcal{H}$ as it partitions the edge set of $\mathcal{H}$ into edge sets of copies of $\mathcal{F}$.

The research in this thesis focuses on ways to obtain matchings or specifically packings, see Chapters 2 and 3. Chapter 2 closely corresponds to [46] and Chapter 3 closely corresponds to [61]. Furthermore, several parts and paragraphs in this overview (Chapter 1) are taken from [46] or [61].

1.1.2 Constructing hypergraph matchings

When the goal is to find a large matching, or more specifically a large packing, employing an iterative construction that randomly enlarges a collection of edges or copies often turns out to be a successful approach, see for example [2, 9, 66, 94, 96, 98, 105, 109, 113] or, for a more algorithmic overview, [82]. In fact, given suitable natural conditions, such constructions often yield matchings that are *almost-perfect* in the sense that when considering the hypergraph where a large matching is supposed to be constructed, the number of vertices that are not covered by the matching is only a small fraction of the number of all vertices. In the packing setting, collections of copies that correspond to almost-perfect matchings are called *approximate* decompositions. In some cases, such approximate constructions can be augmented to obtain non-approximate versions, that is perfect matchings in the case of matchings, using an approach known as *absorption*. Such techniques play a crucial role in several types of constructions. For examples and further discussion, we refer the reader to the surveys [76, 77, 97, 114, 115] mentioned in [70].

The goal of this thesis is to answer open questions regarding a variation and a special case of the perhaps most obvious way to implement such a random iterative construction that may possibly construct an almost-perfect matching, namely the *random greedy hypergraph matching process*. Given a hypergraph $\mathcal{H}$, this random process iteratively chooses edges of a matching as follows. In every iteration, an edge is chosen uniformly at random among all edges $e \in E(\mathcal{H})$ that do not intersect with a previously chosen edge. If no such edges exist, then the process terminates. By construction, when the process terminates, the set $\mathcal{M}$ of chosen edges is a matching in $\mathcal{H}$ which is maximal with respect to inclusion in the sense that there is no matching $\mathcal{M}'$ in $\mathcal{H}$ such that $\mathcal{M}$ is a proper subset of $\mathcal{M}'$. Many variants and special cases of this process have been investigated in the past, see for example [9, 14, 53, 98, 105, 113].

Historically, these processes were studied in great detail only after the introduction of a slightly different such random tool, the so called *Rödl nibble*, which had a significant impact on subsequent research in the area. The type of problem that led to the introduction of this technique dates back to one of the oldest theorems in combinatorics proved by Kirkman [74] concerning Steiner systems. For $m \geq 1$ and $s > t \geq 1$, a *partial* (m, s, t)-*Steiner system* is a collection $\mathcal{S}$ of s-sets $S \subseteq [m] := \{1, \dots, m\}$ such that for every t-set $T \subseteq [m]$, there exists at most one s-set $S \in \mathcal{S}$ with $T \subseteq S$. In this context, the elements of $[m]$ are often referred to as *points*. Note that, trivially, $|\mathcal{S}| \leq \binom{m}{t}/\binom{s}{t}$. The collection $\mathcal{S}$ is an (m, s, t)-*Steiner system* if every t-set of points is a subset of an s-set $S \in \mathcal{S}$. Occasionally, we omit the parameters m, s and t and refer to a partial (m, s, t)-Steiner system simply as a partial Steiner system and similarly for (m, s, t)-Steiner systems.

Every partial (m, s, t)-Steiner system $\mathcal{S}$ corresponds to a $K_s^{(t)}$-packing in $K_m^{(t)}$, where for $n \geq 0$ and $k \geq 1$, we use $K_n^{(k)}$ to denote a fixed *complete* k-graph on n vertices, that is a k-graph where every k-set of vertices is an edge. Indeed, the family $\mathscr{F}$ of complete t-graphs whose vertex sets are the s-sets in $\mathcal{S}$ is a $K_s^{(t)}$-packing in the complete t-graph with vertex set $[m]$ and thus also corresponds to a packing in $K_m^{(t)}$. Hence, asking for (partial) (m, s, t)-Steiner systems is again a hypergraph matching problem.

A *partial Steiner triple system* is simply a partial $(m, 3, 2)$-Steiner system and similarly, a *Steiner triple system* is an $(m, 3, 2)$-Steiner system. In 1847, Kirkman [74] proved that for this case, that is when $s = 3$ and $t = 2$, there exists an (m, s, t)-Steiner system if and only if m is congruent 1 or 3 modulo 6, which is an obvious necessary condition for the existence.

For general parameters $m > s > t \geq 1$, in 1963, Erdős and Hanani [31] conjectured that *approximate* (m, s, t)-Steiner systems, that is partial (m, s, t)-Steiner systems $\mathcal{S}$ with $|\mathcal{S}| \geq (1 - o(1))\binom{m}{t}/\binom{s}{t}$ where the $o(1)$ term is with respect to $m \to \infty$, exist. Introducing the aforementioned Rödl nibble method, this conjecture was proved in a breakthrough by Rödl [96] in 1985. The key idea here is to show that a partial Steiner system can be constructed by starting with an empty collection and iteratively adding small collections of admissible sets obtained based on random choices. With high probability, the random choices yield suitable collections such that the overall approach constructs a sufficiently large partial Steiner system. It is the iterative addition of small collections that can be thought of as "nibbles" that gives the method its name. The impact of Rödl's result and this proof method on combinatorics and beyond cannot be overstated. To mention just two outstanding examples, the result of Rödl was a key ingredient in the resolution of the "Existence conjecture" on combinatorial designs [69] (also in [26, 50, 71]) that generalize Steiner systems and the proof method was used to find the so-far largest gaps between primes [35].

In the years after Rödl's success with such a nibble approach, there has been a growing interest in the behavior and performance of the random greedy hypergraph matching process for obtaining large matchings. Notably, Spencer [105] as well as Rödl and Thoma [98] proved that such processes can also produce asymptotically optimal results with high probability.

1.1.3 Conflict-free hypergraph matchings

Concerning Steiner systems, further research was inspired by the following concept. The *girth* of a partial Steiner triple system $\mathcal{S}$ is the smallest integer $g \geq 4$ such that some g-set of points contains at least $g-2$ elements $S \in \mathcal{S}$ as subsets if such an integer exists and infinite otherwise. Equivalently, the girth of $\mathcal{S}$ is the smallest integer $g \geq 4$ such that some $(g-2)$-set $\mathcal{S}' \subseteq \mathcal{S}$ *spans* at most g points in the sense that the union of the elements of $\mathcal{S}'$ has size at most g if such an integer exists and infinite otherwise. Since by definition no two sets of a partial Steiner triple system can share two points, the girth of a partial Steiner triple system is at least 6. In 1973, Erdős [30] conjectured that for all $g \geq 4$ and all m that are congruent 1 or 3 modulo 6 and sufficiently large in terms of g, there exist not only Steiner triple systems spanning m points as proved by Kirkman [74], but even Steiner triple systems spanning m points with girth at least g. An approximate version of this statement where one asks for large partial Steiner triple systems with high girth was recently proved independently by Bohman and Warnke [18] as well as Glock, Kühn, Lo and Osthus [49] by analyzing a random greedy process. Subsequently, combining such an approach with absorption techniques, Kwan, Sah, Sawhney and Simkin [80] fully resolved the conjecture. Relying on the correspondence between Steiner systems and matchings in certain hypergraphs, the random greedy hypergraph matching process can be used to construct large Steiner systems and all three papers employ a variation of this process to obtain an approximate Steiner system. More specifically, a random greedy process is considered where the next matching edge is chosen uniformly at random not among all edges that do not intersect with previously selected edges, but among only those that also span a sufficient number of points with previously selected edges. The key observation here is that the restriction to viable matching edges dominates compared to the restriction to edges that avoid the creation of subfamilies that do not span a sufficient number of points, which causes the random greedy process to essentially behave as the ordinary random greedy hypergraph matching process. While such an approach for building Steiner systems as desired seems natural and has been suggested for instance in [75] (see also [28, 83]), the high-girth condition entails considerable technical difficulties in analyzing the process.

It is natural to ask for the existence of (m, s, t)-Steiner systems that satisfy an analogous high-girth condition for other values of s and t. Indeed, the existence of such Steiner systems was conjectured by Füredi and Ruszinkó [42], Glock, Kühn, Lo and Osthus [49] as well as Keevash and Long [72]. It were such questions that inspired the research presented in Chapter 2. Chapter 2 closely resembles [46] and specifically, we obtain the following statement.

Theorem 1.1.1. *For all $s > t \geq 2$ and $\ell \geq 0$, there exist $\varepsilon > 0$ and m_0 such that for all $m \geq m_0$, there exists a partial (m, s, t)-Steiner system $\mathcal{S}$ of size $(1 - m^{-\varepsilon})\binom{m}{t}/\binom{s}{t}$ such that any subset of $\mathcal{S}$ of size j, where $2 \leq j \leq \ell$, spans more than $(s-t)j + t$ points.*

We remark that the case where $s = t+1$ and $\ell = 3$ was already proved by Sidorenko [104] using an algebraic construction and that very recently, after the research Chapter 2 is based on was published, Delcourt and Postle [25] resolved the conjecture for general high-girth (m, s, t)-Steiner systems by again combining iterative techniques for obtaining approximate constructions with absorption. For our proof, instead of analyzing a modified packing process specifically for Steiner systems that avoids the forbidden configurations, we consider a general random greedy hypergraph matching process that avoids certain edge sets as subsets of the constructed matching. This enables us to obtain general conditions for such edge sets that still allow the process to generate an almost perfect matching with high probability and in turn leads us to more general hypergraph matching theorems, see Theorem 1.1.2 as well as the variations and extensions in Section 2.5. Theorem 1.1.1 is then a consequence of one of these theorems which are however general enough to have further applications beyond Steiner systems, see [6, 11, 47, 52, 63, 81] and [24] which builds on a result from [47]. For a more detailed discussion of directions for applications, see also Section 2.3.

Let us discuss our more general setting in detail. Overall, the shift from Steiner systems to general hypergraph matchings is similar to how shortly after Rödl's theorem [96], Frankl and Rödl [37] and Pippenger (see [94]) greatly generalized his result. Their fundamental observation was that Rödl's result is "just" the tip of the iceberg of a much more general phenomenon: every large regular hypergraph with small 2-degrees has a matching $\mathcal{M}$ which covers almost all vertices. Here, a hypergraph $\mathcal{H}$ is *d-regular* if all vertices $v \in V(\mathcal{H})$ are contained in exactly d edges, $\mathcal{H}$ is *regular* if $\mathcal{H}$ is d-regular for some integer d and the *2-degree* of distinct vertices $u, v \in V(\mathcal{H})$ is the number of edges $e \in E(\mathcal{H})$ with $u, v \in e$. Formally, in the following discussion, consider a d-regular k-graph $\mathcal{H}$, where k is fixed and asymptotics are with respect to $d \to \infty$. If the maximum 2-degree of $\mathcal{H}$ is small in the sense that for all distinct $u, v \in V(\mathcal{H})$, there are at most $o(d)$ edges $e \in E(\mathcal{H})$ with $u, v \in e$, then, the aforementioned results state that $\mathcal{H}$ has an *almost-perfect* matching, that is a matching which covers all but $o(|V(\mathcal{H})|)$ vertices. To see how this generalizes Rödl's theorem, consider parameters $m > s > t \geq 1$ and, following the correspondences between Steiner systems and packings as well as packings and matchings, construct a hypergraph $\mathcal{H}$ as follows: the vertices of $\mathcal{H}$ are all t-sets $T \subseteq [m]$, and for each s-set $S \subseteq [m]$, we create an edge of $\mathcal{H}$ which comprises all t-sets $T \subseteq S$. Then, matchings in $\mathcal{H}$ correspond exactly to partial (m, s, t)-Steiner systems of the same size, and an almost-perfect matching in $\mathcal{H}$ yields an approximate (m, s, t)-Steiner system. It is straightforward to check that $\mathcal{H}$ is k-uniform and d-regular with $k = \binom{s}{t}$ and $d = \binom{m-t}{s-t}$, and all 2-degrees are $o(d)$. Hence, the above result on matchings in hypergraphs indeed implies Rödl's theorem.

We now explain how one can capture the high-girth condition (and many other desired features) in the hypergraph matching setting. For simplicity in the discussion, we consider the case of Steiner triple systems. Hence, the vertices of $\mathcal{H}$ are the 2-sets $T \subseteq [m]$, and the edges of $\mathcal{H}$ correspond to the 3-sets $S \subseteq [m]$ in the sense that an edge e corresponding to a set S consists of the three 2-sets that are subsets of S. Now, suppose that we are given an ℓ-set C of 3-sets that spans at most $\ell + 2$ points. This means that if all 3-sets from C were contained in a partial Steiner triple system, then this system would have girth at most $\ell + 2$. Hence, for the system to have large girth, say at least $g > \ell + 2$, we have to make sure that not all 3-sets from C are contained in the system. Since 3-sets correspond to edges in $\mathcal{H}$, this gives us a set of edges of $\mathcal{H}$ which *conflict* in the sense that we want to find a matching which does not contain the set as a subset. We call such sets of conflicting edges *conflicts*. One can form a collection $\mathcal{C}$ consisting of all conflicts, that is, all those sets of edges that correspond to sets of 3-sets of points that span too few vertices. Then, the aim is to find a matching $\mathcal{M}$ in $\mathcal{H}$ such that no conflict $C \in \mathcal{C}$ is a subset of $\mathcal{M}$. It will be convenient to think of $\mathcal{C}$ as a hypergraph with vertex set $E(\mathcal{H})$ since elements of $\mathcal{C}$ are sets of edges. We do not assume $\mathcal{C}$ to be uniform. This reflects the fact that the girth condition can be violated by sets of edges of different size. We write $\mathcal{C}^{(j)}$ to denote the subgraph of $\mathcal{C}$ which consists of all those conflicts in $\mathcal{C}$ that have size j.

Generally, given a hypergraph $\mathcal{C}$ with $V(\mathcal{C}) = E(\mathcal{H})$, we say that an edge set $E \subseteq E(\mathcal{H})$ is *$\mathcal{C}$-free* if no edge of $\mathcal{C}$ is a subset of E. One major goal of this work is to provide general conditions on $\mathcal{C}$ that allow us to guarantee that our random greedy algorithm generates a $\mathcal{C}$-free almost-perfect matching in $\mathcal{H}$ with high probability and that are satisfied in the Steiner system case. Note that a priori it is perhaps not even clear that this is sensible. For instance, the conflicts arising in the Steiner system application are inherently local in the sense that they forbid having too many 3-sets on small sets of points. However, when transferring the problem to the hypergraph matching setting, the information about points is lost in the sense that the point set of the Steiner system has no counterpart in the matching description. Moreover, in the case of high-girth Steiner triple systems, the proofs in [18, 49] extensively use the structural properties of the forbidden configurations of 3-sets. One of the key insights of the present work is that one can indeed formulate sensible general conditions on $\mathcal{C}$ which guarantee existence of an almost-perfect *conflict-free*, that is $\mathcal{C}$-free, matching. Not only are these conditions natural as evidenced by the fact that they are satisfied in the many applications referenced above and discussed in Section 2.5, but they are also necessary in the sense that Theorem 1.1.2 below would be false in general if one condition is omitted entirely. For more details, see Section 2.4.

We now state our main theorem. We remark that, although it captures the most important features, in Section 2.5 we state several variations which might be applicable in situations where the following is not. These variations include a version where $\mathcal{C}$ only needs to satisfy slightly weaker conditions and where dependencies of relevant parameters are stated more explicitly (see Theorem 2.5.1). For $i \geq 0$, we use

$$\Delta_i(\mathcal{H}) := \max_{U \subseteq V(\mathcal{H}) \colon |U|=i} |\{e \in E(\mathcal{H}) : U \subseteq e\}|$$

to denote the *maximum i-degree* of $\mathcal{H}$ and for $e \in E(\mathcal{H})$, we define

$$N^{(2)}_{\mathcal{C}}(e) := \{f \in E(\mathcal{H}) : \{e, f\} \in \mathcal{C}\}.$$

Theorem 1.1.2. *For all $k, \ell \geq 2$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, there exists d_0 such that the following holds for all $d \geq d_0$. Let $\mathcal{H}$ be a k-graph with $|V(\mathcal{H})| \leq \exp(d^{\varepsilon^3})$ such that every vertex is contained in $(1 \pm d^{-\varepsilon})d$ edges and $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon}$.*

Let $\mathcal{C}$ be a hypergraph with $V(\mathcal{C}) = E(\mathcal{H})$ such that every $C \in E(\mathcal{C})$ satisfies $2 \leq |C| \leq \ell$, and the following conditions hold.

(i) $\Delta_1(\mathcal{C}^{(j)}) \leq \ell d^{j-1}$ *for all $2 \leq j \leq \ell$;*

(ii) $\Delta_{j'}(\mathcal{C}^{(j)}) \leq d^{j-j'-\varepsilon}$ *for all $2 \leq j' < j \leq \ell$;*

(iii) $|\{f \in N_{\mathcal{C}}^{(2)}(e) : v \in f\}| \leq d^{1-\varepsilon}$ *for all* $e \in E(\mathcal{H})$ *and* $v \in V(\mathcal{H})$*;*

(iv) $|N_{\mathcal{C}}^{(2)}(e) \cap N_{\mathcal{C}}^{(2)}(f)| \leq d^{1-\varepsilon}$ *for all disjoint* $e, f \in \mathcal{H}$*.*

Then, there exists a $\mathcal{C}$*-free matching* $\mathcal{M}$ *in* $\mathcal{H}$ *which covers all but* $d^{-\varepsilon^3}|V(\mathcal{H})|$ *vertices of* $\mathcal{H}$*.*

Note that if there are no conflicts of size 2, the last two conditions are irrelevant and we are only left with simple degree conditions for $\mathcal{H}$ and $\mathcal{C}$. In addition, when applying Theorem 1.1.2 with some given conflict hypergraph $\mathcal{C}$ one may disregard conflicts that are not matchings and conflicts that contain at least one other conflict as a subset. In fact, it turns our that omitting such redundant conflicts is sometimes crucial to meet the degree conditions required in Theorem 1.1.2 (see our application to high-girth Steiner systems in Section 2.11).

Our strategy in proving Theorem 1.1.2 is to construct the matching $\mathcal{M}$ with a random greedy algorithm, which we call the *conflict-free matching process*. The process itself is as simple as it could be and is again a variation of the random greedy hypergraph matching process. Starting with an empty matching, the process iteratively adds an edge chosen uniformly at random among all edges that are *available* in the sense that adding them results in a matching which is still conflict-free. The process terminates when no such edges remain. The final matching is conflict-free by construction, so the crucial task is to show that it is as large as desired, which is to say that the process does not terminate too early (with high probability).

1.1.4 The hypergraph removal process

The special case of the random greedy hypergraph matchings process where packings are constructed by considering an appropriate auxiliary hypergraph has been of particular interest also from a different point of view. Instead of considering the collection of edge-disjoint copies of a hypergraph $\mathcal{F}$ in another hypergraph $\mathcal{H}$ that the process iteratively extends, we may also consider what remains of the hypergraph $\mathcal{H}$ after *removing* these copies in the sense that we consider the hypergraph obtained from $\mathcal{H}$ by removing all edges that are edges of a copy in the collection. From this point of view, the packing process starts with $\mathcal{H}$ and iteratively removes all edges of a copy of $\mathcal{F}$ chosen uniformly at random among all remaining copies of $\mathcal{F}$ until no copies are left. For this reason, this process may also be called the *$\mathcal{F}$-removal process starting at $\mathcal{H}$*. Usually, this process is considered for a k-graph $\mathcal{F}$ and a complete k-graph $\mathcal{H}$. In this case, we refer to the process as the *$\mathcal{F}$-removal process on n vertices* where n refers to the number of vertices of $\mathcal{H}$.

Note that, by definition of the process, there are no copies of $\mathcal{F}$ in the eventually generated hypergraph (provided that $\mathcal{F}$ has at least one edge). Such hypergraphs, that is hypergraphs without copies of a specific hypergraph $\mathcal{F}$, are often referred to as *$\mathcal{F}$-free* and we also adopt this terminology for our discussion of the hypergraph removal process. It will be clear from the context when we use the term in this sense and when we use it in the sense of a conflict-free hypergraph as defined above for conflict-free matchings. The fact that the $\mathcal{F}$-removal process generates $\mathcal{F}$-free graphs together with the random nature of the process makes it a possible candidate for a construction that might be useful when investigating problems such as the following. By Ramsey's Theorem [95], for all $s, t \geq 1$, there exists an integer $R(s,t)$ such that for all graphs $\mathcal{G}$ on at least $R(s,t)$ vertices, either there is a copy of K_s, where for $n \geq 0$ we set $K_n := K_n^{(2)}$, in $\mathcal{G}$ or there is a copy of K_t in the *complement* of $\mathcal{G}$, that is the graph with vertex set $V(\mathcal{G})$ where a 2-set of vertices is an edge if and only if it is not an edge of $\mathcal{G}$. Note that a set of vertices U of a graph $\mathcal{G}$ is the vertex set of a complete subgraph of the complement of $\mathcal{G}$ if and only if U is an *independent set* in $\mathcal{G}$, that is if no two vertices $u, v \in U$ form an edge in the sense that $\{u, v\} \in E(\mathcal{G})$. Suppose that we wish to obtain lower bounds for $R(s,t)$, more specifically, suppose that $s \geq 3$ is fixed and that we wish to find a function $r \colon \mathbb{N} \to \mathbb{N}$ such that $r(t) \leq R(s,t)$ for sufficiently large t. If for some strictly increasing function $\alpha \colon \mathbb{N} \to \mathbb{N}$ and all sufficiently large integers n we had a construction of a K_s-free graph $\mathcal{G}_n$ on n vertices such that the *independence number* of $\mathcal{G}_n$, that is the size of a largest independent set in $\mathcal{G}_n$, is at most $\alpha(n)$, then setting $r(t) := \max\{n \in \mathbb{N} : \alpha(n) \leq t\}$ would yield a function as desired.

The graphs generated by the K_s-removal process on n vertices are by construction K_s-free and it appears reasonable to expect that for large n, the independence number of the generated graphs behaves similar to the independence number of the *binomial random graph* on n vertices, that is the random graph where every 2-set of vertices is an edge independently with some probability p. This is a well-studied random graph construction, see for example [20, 40, 58, 107] and hence there is a natural guess for a corresponding function α and hence a lower bound r as above. Motivated by such considerations regarding lower bounds for $R(s,t)$, in 1990, as mentioned in [34] and [19], Bollobás and Erdős suggested studying the $\mathcal{F}$-removal process on n vertices as well as another random hypergraph process that generates $\mathcal{F}$-free hypergraphs, the *$\mathcal{F}$-free process on n vertices* which, given a k-graph $\mathcal{F}$, starts with a hypergraph on n vertices without edges and iteratively proceeds as follows. Among all k-sets of vertices that were not previously added and that do not form the edge set of a copy of $\mathcal{F}$ with previously added edges, a vertex set is chosen uniformly at random and added as an edge. The process terminates when no such vertex sets remain. Similarly to how the $\mathcal{F}$-removal process is a special case of the random greedy hypergraph matching process, the $\mathcal{F}$-free process is a special case of the *random greedy independent set process*. Indeed, for a k-graph $\mathcal{F}$, consider an auxiliary hypergraph $\mathcal{H}^*$ with vertex set $E(K_n^{(k)})$ where the edges are the edge sets of copies of $\mathcal{F}$ in $K_n^{(k)}$. Then, the $\mathcal{F}$-free process on n vertices corresponds to the random greedy process that constructs an independent set in $\mathcal{H}^*$ iteratively as follows. Starting with an empty set, a vertex chosen uniformly at random among all vertices that do not form an edge with previously added vertices is added until no such vertices remain.

Particularly concerning the $\mathcal{F}$-free process, a careful analysis indeed turned out to be a successful approach for obtaining lower bounds for $R(s,t)$, see [12, 15, 16, 34, 112]. However, with growing interest in the behavior of such random processes, both have been studied also independently of such questions related to Ramsey's theorem, see for example [8, 9, 14, 21, 32, 43, 53, 78, 90–93, 98, 99, 105, 110, 111, 113].

While random processes such as the $\mathcal{F}$-free and $\mathcal{F}$-removal process on n vertices are easy to formulate, in many cases a precise analysis is challenging. The central questions often concern structural properties that typically, that is, with high probability (with probability tending to 1 as $n \to \infty$), hold for the objects generated at termination. In particular, concerning the $\mathcal{F}$-free and $\mathcal{F}$-removal process, one may ask for asymptotic estimates for the number of edges or equivalently the number of iterations of the algorithm. For the $\mathcal{F}$-free process on n vertices, we use $F_n(\mathcal{F})$ to denote the (random) final number of edges present after termination and for the $\mathcal{F}$-removal process, we use $R_n(\mathcal{F})$.

For the special case of the K_3-free process, that is, where $\mathcal{F}$ is a *triangle*, Fiz Pontiveros, Griffiths and Morris [34] and independently Bohman and Keevash [16] famously proved that typically $F_n(K_3) = (\frac{1}{2\sqrt{2}} \pm o(1))(\log n)^{1/2} n^{3/2}$ (after Bohman determined the correct order of magnitude [12], answering a question of Spencer [106]) and through their analysis also obtained new lower bounds for $R(3,t)$. For the general case, again with an underlying analysis that for several cases yields the best known lower bounds for $R(s,t)$, a lower bound for $F_n(\mathcal{F})$ that holds with high probability is available whenever $\mathcal{F}$ comes from a large class of graphs or hypergraphs [8, 15]. At least for graphs, this lower bound is conjectured to be tight up to constant factors [15], however in general, the best upper bounds that are known to hold with high probability differ from this lower bound by logarithmic factors [78]. Estimates for $F_n(\mathcal{F})$ that are tight up to constant factors exist for a few specific choices of $\mathcal{F}$, see [16, 34, 91–93, 111, 112].

For the $\mathcal{F}$-removal process, already getting close to the order of magnitude of $R_n(K_3)$ turned out to be challenging. After Spencer [105] as well as Rödl and Thoma [98] proved that $R_n(K_3) = o(n^2)$ typically holds, Grable [53] improved this to $R_n(K_3) \leq n^{11/6}$. Following these attempts to determine $R_n(K_3)$, Spencer conjectured that typically $R_n(K_3) = n^{3/2 \pm o(1)}$ holds and offered \$200 for a resolution [53, 113]. The breakthrough here happened when Bohman, Frieze and Lubetzky proved Spencer's conjecture [14]. Beyond the triangle, so far no results were known that give bounds that are somewhat close to the correct order of magnitude of $R_n(\mathcal{F})$ for any other non-trivial $\mathcal{F}$; in fact, obtaining asymptotic estimates for $R_n(K_4)$ was considered a central open problem in the area. (One reason for why this is a difficult problem may be that the technical complexity of the approach taken by Bohman, Frieze and Lubetzky to settle the triangle case seems to explode even for $\mathcal{F} = K_4$.) Following the same heuristic as for the triangle, Bennett and Bohman [9] state the following more general "folklore" conjecture predicting $R_n(K_\ell^{(k)})$.

Conjecture 1.1.3 ([9, Conjecture 1.2])**.** *Let $2 \leq k < \ell$. Then, for all $0 < \varepsilon < 1$, there exists $n_0 \geq 0$ such that for all $n \geq n_0$, with high probability,*

$$n^{k - \frac{\ell - k}{\binom{\ell}{k} - 1} - \varepsilon} \leq R_n(K_\ell^{(k)}) \leq n^{k - \frac{\ell - k}{\binom{\ell}{k} - 1} + \varepsilon}.$$

Our main result of Chapter 3 confirms Conjecture 1.1.3. Chapter 3 closely resembles [61] and in fact, we prove a significantly stronger result. For a k-graph $\mathcal{F}$, using $v(\mathcal{F})$ to denote the number of vertices of $\mathcal{F}$ and $e(\mathcal{F})$ to denote the number of edges of $\mathcal{F}$, the *k-density* of $\mathcal{F}$ is $\rho_{\mathcal{F}} := (e(\mathcal{F}) - 1)/(v(\mathcal{F}) - k)$ if $v(\mathcal{F}) \geq k+1$. As in [15], we say that $\mathcal{F}$ is *strictly k-balanced* if $\mathcal{F}$ has at least three edges and satisfies $\rho_{\mathcal{G}} < \rho_{\mathcal{F}}$ for all proper subgraphs $\mathcal{G}$ of $\mathcal{F}$ that have at least two edges. Here we say that a subgraph $\mathcal{G}$

of $\mathcal{F}$ if *proper* if $\mathcal{G} \neq \mathcal{F}$. Note that $K_\ell^{(k)}$ is strictly k-balanced for all $2 \leq k < \ell$. The following is a corollary of our main result (Theorem 1.1.5).

Theorem 1.1.4. *Let $k \geq 2$ and consider a strictly k-balanced k-graph $\mathcal{F}$ with k-density ρ. Then, for all $\varepsilon > 0$, there exists $n_0 \geq 0$ such that for all $n \geq n_0$, with probability at least* $1 - \exp(-(\log n)^{5/4})$*, we have*

$$n^{k-1/\rho-\varepsilon} \leq R_n(\mathcal{F}) \leq n^{k-1/\rho+\varepsilon}.$$

Observe that complete hypergraphs exhibit a very high degree of symmetry while most strictly k-balanced hypergraphs have locally and globally essentially no symmetries. This complicates the analysis and requires us to dedicate substantial parts of the proof to dealing with the extension from complete hypergraphs to general strictly k-balanced hypergraphs.

Furthermore, our analysis allows starting at any pseudorandom hypergraph, which may be a useful scenario for applications. For a k-graph $\mathcal{H}$, we use $R(\mathcal{H}, \mathcal{F})$ to denote the final number of edges of the $\mathcal{F}$-removal process starting at $\mathcal{H}$.

To formally describe the pseudorandomness we require for our theorem, we introduce the following definitions. A k-uniform *template* or *k-template* is a pair $(\mathcal{A}, I)$ where $\mathcal{A}$ is a k-graph and where $I \subseteq V(\mathcal{A})$. The *density* $\rho_{\mathcal{A},I}$ of $(\mathcal{A}, I)$ is $(e(\mathcal{A}) - e(\mathcal{A}[I]))/(v(\mathcal{A}) - |I|)$ if $V(\mathcal{A}) \neq I$ and 0 otherwise where we use $\mathcal{A}[I]$ to denote the subgraph of $\mathcal{A}$ induced by I, that is the subgraph with vertex set I and edge set $\{e \in E(\mathcal{A}) : e \subseteq I\}$. A template $(\mathcal{B}, J)$ is a *subtemplate* of $(\mathcal{A}, I)$ if $\mathcal{B} \subseteq \mathcal{A}$ and $J = I$. We write $(\mathcal{B}, J) \subseteq (\mathcal{A}, I)$ to mean that $(\mathcal{B}, J)$ is a subtemplate of $(\mathcal{A}, I)$. The template $(\mathcal{A}, I)$ is *strictly balanced* if $\rho_{\mathcal{B},I} < \rho_{\mathcal{A},I}$ holds for all $(\mathcal{B}, I) \subseteq (\mathcal{A}, I)$ with $V_\mathcal{B} \neq I$ and $\mathcal{B} \neq \mathcal{A}$. Note that for a k-graph $\mathcal{A}$ with $v(\mathcal{A}) \geq k+1$, the k-density of $\mathcal{A}$ is the density of the templates $(\mathcal{A}, e)$ with $e \in E(\mathcal{A})$ and that if $\mathcal{A}$ has at least three edges, then $\mathcal{A}$ is strictly k-balanced if and only if $(\mathcal{A}, e)$ is strictly balanced for all $e \in \mathcal{A}$. For $0 < \varepsilon, \delta < 1$ and $\rho \geq 1/k$, we say that a k-graph $\mathcal{H}$ on n vertices with $\vartheta n^k/k!$ edges is $(\varepsilon, \delta, \rho)$-*pseudorandom* if for all strictly balanced k-templates $(\mathcal{A}, I)$ with $v(\mathcal{A}) \leq 1/\varepsilon$ and all injections $\psi \colon I \to V(\mathcal{H})$, the number Φ of injections $\varphi \colon V(\mathcal{A}) \to V(\mathcal{H})$ with $\varphi|_I = \psi$ and $\varphi(e) \in E(\mathcal{H})$ for all $e \in E(\mathcal{A}) \setminus E(\mathcal{A}[I])$ satisfies the properties (P1)–(P4) below. Here, we set $\hat{\varphi} := n^{v(\mathcal{A})-v(\mathcal{A}[I])}\vartheta^{e(\mathcal{A})-e(\mathcal{A}[I])}$ and $\zeta := n^\delta/(n\vartheta^\rho)^{1/2}$.

(P1) If $\rho_{\mathcal{A},I} \leq \rho$, then $\Phi = (1 \pm \zeta)\hat{\varphi}$;

(P2) If $\hat{\varphi} \geq \zeta^{-\delta^{2/3}}$, then $\Phi = (1 \pm \zeta^\delta)\hat{\varphi}$;

(P3) If $1 \leq \hat{\varphi} \leq \zeta^{-\delta^{2/3}}$, then $\Phi = (1 \pm (\log n)^{3(v(\mathcal{A})-v(\mathcal{A}[I]))/2}\hat{\varphi}^{-\delta^{1/2}})\hat{\varphi}$.

(P4) If $\hat{\varphi} \leq 1$, then $\Phi \leq (\log n)^{3(v(\mathcal{A})-v(\mathcal{A}[I]))/2}$.

We remark that for all $k \geq 2$ and $0 < \varepsilon, \delta < 1$ and $\rho \geq 1/k$ where δ is sufficiently small in terms of $1/k$ and ε, the binomial random k-graph on n vertices where all vertex sets of size k are edges independently with probability $p \geq n^{-1/\rho+\delta^{1/2}}$ is $(\varepsilon, \delta, \rho)$-pseudorandom with high probability. Indeed, in this setting Chernoff's inequality (see for example Lemma 3.12.5) guarantees $\vartheta \geq n^{-1/\rho+3\delta}$ with high probability, and then sufficient lower tail bounds follow from Janson's inequality (see [57, Theorem 1]) and for the upper tails, one may apply [59, Corollary 4.1].

We are now ready to state our main theorem regarding the removal process.

Theorem 1.1.5. *Let $k \geq 2$ and consider a strictly k-balanced k-graph $\mathcal{F}$ with k-density ρ. Then, for all $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, there exists $n_0 \geq 0$ such that for all $n \geq n_0$, the following holds. If $\mathcal{H}$ is a $(\varepsilon^{20}, \delta, \rho)$-pseudorandom k-graph on n vertices with $e(\mathcal{H}) \geq n^{k-1/\rho+\varepsilon^5}$, then, with probability at least $1 - \exp(-(\log n)^{5/4})$, we have*

$$n^{k-1/\rho-\varepsilon} \leq R(\mathcal{H}, \mathcal{F}) \leq n^{k-1/\rho+\varepsilon}.$$

We prove the upper bound in Theorem 1.1.5 in a slightly more general setting in the sense that we only require a weaker notion of balancedness. We say that a k-graph $\mathcal{F}$ is *k-balanced* if $\mathcal{F}$ has at least one edge and satisfies $\rho_{\mathcal{G}} \leq \rho_{\mathcal{H}}$ for all subgraphs $\mathcal{G}$ of $\mathcal{H}$ on at least $k+1$ vertices.

Theorem 1.1.6. *Let $k \geq 2$ and consider a k-balanced k-graph $\mathcal{F}$ with k-density ρ. Then, for all $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, there exists $n_0 \geq 0$ such that for all $n \geq n_0$, the following holds. If $\mathcal{H}$ is a $(\varepsilon^{20}, \delta, \rho)$-pseudorandom k-graph on n vertices with $e(\mathcal{H}) \geq n^{k-1/\rho+\varepsilon^5}$, then, with probability at least $1 - \exp(-(\log n)^{5/4})$, we have*

$$R(\mathcal{H}, \mathcal{F}) \leq n^{k-1/\rho+\varepsilon}.$$

As part of our proof for Theorem 1.1.5, we obtain another theorem which describes the behavior of the $\mathcal{F}$-removal process starting at $\mathcal{H}$ for comparatively sparse $\mathcal{H}$ which complements Theorem 1.1.5. To formally describe the slightly different setup in the sparse setting, we introduce the following definitions. For $s, c \geq 0$, we say that a k-graph $\mathcal{H}$ with $\vartheta n^k/k!$ edges is *(s, c)-bounded* if for all strictly balanced templates $(\mathcal{A}, I)$ with $v(\mathcal{A}) \leq s$, all injections $\psi \colon I \to V(\mathcal{H})$ and $\hat{\varphi} := n^{v(\mathcal{A})-|I|}\vartheta^{e(\mathcal{A})-e(\mathcal{A}[I])}$, the number of injections $\varphi \colon V(\mathcal{A}) \to V(\mathcal{H})$ with $\varphi|_I = \psi$ and $\varphi(e) \in \mathcal{H}$ for all $e \in \mathcal{A}$ with $e \not\subseteq I$ is at most $c \cdot \max\{1, \hat{\varphi}\}$. We say that $\mathcal{H}$ is *$\mathcal{F}$-populated* if all edges of $\mathcal{H}$ are edges of at least two copies of $\mathcal{F}$ in $\mathcal{H}$.

Theorem 1.1.7. *Let $k \geq 2$ and suppose that $\mathcal{F}$ is a strictly k-balanced k-graph on m vertices with k-density ρ. For all $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$ and all $(4m, n^{\varepsilon^4})$-bounded and $\mathcal{F}$-populated k-graphs $\mathcal{H}$ on n vertices with $n^{k-1/\rho-\varepsilon^4} \leq e(\mathcal{H}) \leq n^{k-1/\rho+\varepsilon^4}$, with probability at least $1 - \exp(-n^{1/4})$, we have*

$$R(\mathcal{H}, \mathcal{F}) \geq n^{k-1/\rho-\varepsilon}.$$

Recall that by definition, $\mathcal{F}$ is strictly k-balanced if and only if $\mathcal{F}$ has at least three edges and satisfies $\rho_{\mathcal{G}} < \rho_{\mathcal{F}}$ for all proper subgraphs $\mathcal{G}$ of $\mathcal{F}$ that have at least two edges. Hence, Theorems 1.1.5 and 1.1.7 do not cover the case where $e(\mathcal{F}) = 2$, but it is possible to also obtain a similar statement for this case. If $\mathcal{F}$ is a matching of size 2, that is if $E(\mathcal{F})$ is a matching of size 2 and additionally has no *isolated* vertices, that is no vertices that are not contained in an edge of $\mathcal{F}$, then $\mathcal{F}$ has k-density $1/k$, so in this case the lower bounds in these theorems is always true (if we round down) and hence we ignore this case. For the case where $\mathcal{F}$ has exactly two edges and no isolated vertices but is not a matching, we obtain the following two theorems, where for $1 \leq k' \leq k$, we say that $\mathcal{H}$ is *k'-populated* if all sets $U \subseteq V(\mathcal{H})$ with $|U| = k'$ are subsets of at least two edges of $\mathcal{H}$.

Theorem 1.1.8. *Let $k \geq 2$ and consider a k-graph $\mathcal{F}$ with k-density ρ that is not a matching, has exactly two edges and no isolated vertices. Then, for all $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, there exists $n_0 \geq 0$ such that for all $n \geq n_0$, the following holds. If $\mathcal{H}$ is a $(\varepsilon^{20}, \delta, \rho)$-pseudorandom k-graph on n vertices with $e(\mathcal{H}) \geq n^{k-1/\rho+\varepsilon^5}$, then, with probability at least $1 - \exp(-(\log n)^{5/4})$, we have*

$$n^{k-1/\rho-\varepsilon} \leq R(\mathcal{H}, \mathcal{F}) \leq n^{k-1/\rho+\varepsilon}.$$

Theorem 1.1.9. *Let $k \geq 2$ and suppose that $\mathcal{F}$ is a k-graph with k-density ρ that is not a matching, has exactly two edges and no isolated vertices. Let $k' := |e \cap f|$ where e and f denote the edges of $\mathcal{F}$. For all $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$ and all $(4m, n^{\varepsilon^4})$-bounded and k'-populated k-graphs $\mathcal{H}$ on n vertices with $n^{k-1/\rho-\varepsilon^4} \leq e(\mathcal{H}) \leq n^{k-1/\rho+\varepsilon^4}$, with probability at least $1 - \exp(-n^{1/4})$, we have*

$$R(\mathcal{H}, \mathcal{F}) \geq n^{k-1/\rho-\varepsilon}.$$

1.1.5 The history of the $\mathcal{F}$-free and the $\mathcal{F}$-removal process

It is interesting to compare the history of the analysis of the $\mathcal{F}$-free and the $\mathcal{F}$-removal processes in detail. Modern research concerning the $\mathcal{F}$-free process began in 1992 when Ruciński and Wormald [99] answered a question of Erdős regarding the $\mathcal{F}$-free process where $\mathcal{F}$ is a *star*, that is a graph $\mathcal{G}$ where for some vertex $v \in V(\mathcal{G})$, the edges of $\mathcal{G}$ are the 2-sets $\{u, v\}$ with $u \in V(\mathcal{G}) \setminus \{v\}$. Concerning triangles, Spencer [106] conjectured in 1995 that with high probability, the K_3-free process terminates with $\Theta((\log n)^{1/2} n^{3/2})$ edges. This is the behavior one would expect when assuming that edges present in a hypergraph generated during the $\mathcal{F}$-free process are essentially distributed as if they were included independently with an appropriate probability. We discuss this heuristic in more detail in Section 3.18.

The K_3-free process as well as the variation of this process where not only triangles but all cycles of odd length are forbidden was investigated by Erdős, Suen and Winkler [32]. Here, a *cycle* is a graph $\mathcal{C}$ such that there exists an ordering $v_1, \dots, v_\ell$ of $V(\mathcal{C})$ such that the edges of $\mathcal{G}$ are the 2-sets $\{v_i, v_{i+1}\}$ where $1 \leq i \leq \ell$ with indices taken modulo ℓ and then the *length* of $\mathcal{C}$ is ℓ. Their result yields an upper and a lower bound for $F_n(K_3)$ that holds with high probability and these bounds are tight up to a $\log n$ factor. Bollobás and Riordan [21] obtained analogous bounds for $F_n(\mathcal{F})$ if $\mathcal{F}$ is a complete graph or cycle on four vertices. In 2001, Osthus and Taraz [90] generalized these results to all strictly 2-balanced graphs thus providing estimates for this large class of graphs that are tight up to logarithmic factors.

Guided by similar intuition as above, for the K_3-removal process, Spencer conjectured that with high probability, this process also terminates with $n^{3/2\pm o(1)}$ edges (see [53,113]). More generally, a special case of a conjecture of Alon, Kim and Spencer [2] about hypergraph matchings predicts $n^{k-1/\rho_{\mathcal{F}}\pm o(1)}$ as the expected value of $R_n(\mathcal{F})$ where $\rho_{\mathcal{F}}$ denotes the k-density of $\mathcal{F}$. Concerning estimates available around 2001 however, the situation for the $\mathcal{F}$-removal process was very different compared to the $\mathcal{F}$-free process. Only upper bounds for $R_n(\mathcal{F})$ that do not match the order of magnitude of $R_n(\mathcal{F})$

were known, namely $n^{11/6}$ for $R_n(K_3)$ due to Grable [53] and, as a consequence of a result about the random greedy hypergraph matching process due to Wormald [113], $n^{k-1/(9e(\mathcal{F})^2-9e(\mathcal{F})+3)+o(1)}$ for the general case (where no attempt was made to optimize the constant in the exponent). Intuitively, perhaps one reason that complicates the analysis of the $\mathcal{F}$-removal process compared to the $\mathcal{F}$-free process is the fact that to arrive at roughly $n^{3/2}$ edges, the K_3-free process needs to run for roughly $n^{3/2}$ iterations while the K_3-removal process requires $(1-o(1))n^2/6$ iterations.

It is worth mentioning that to obtain the general upper bound, Wormald [113] introduced a new approach to this area known as *differential equation method* that relies on closely following the evolution of carefully chosen key quantities throughout the process. This technique turned out to be a very valuable for later improvements and the core argumentations of our proofs in Chapters 2 and 3 resemble this method.

Using such an approach Bohman [12] was able to prove estimates for $F_n(K_3)$ that are tight up to constant factors, thereby confirming the aforementioned conjecture of Spencer. Shortly after this, Bohman and Keevash [15], again using similar techniques, obtained new lower bounds for $F_n(\mathcal{F})$ if $\mathcal{F}$ is a strictly 2-balanced graph and they conjecture that these bounds are tight up to constant factors. In the following years, these developments led to further progress for specific choices of $\mathcal{F}$ due to Picollelli [91–93] as well as Warnke [111, 112]. Eventually, by considering the random greedy independent set algorithm in hypergraphs, Bohman and Bennett [8] extended the lower bound to the hypergraph setting. Generalizing the results of Osthus and Taraz [90], upper bounds in the hypergraph setting were obtained by Kühn, Osthus and Taylor [78].

In contrast, concerning the $\mathcal{F}$-removal process, even with these new techniques available, there were no improvements until 2015. Only using a refined version of the differential equation method that exploits certain self-correcting behavior of key quantities to improve the precision of the analysis, Bohman, Frieze and Lubetzky [14] were able to confirm Spencer's conjecture for the K_3-removal process and show that with high probability, $R_n(K_3) = n^{3/2\pm o(1)}$. This refined version is known as *critical interval method*, for other examples, see [13, 16, 17, 34, 56, 108]. Using such an approach often requires an even more careful choice of key quantities to be able to rely on self-correcting behavior as some quantities may disturb the behavior of others. Indeed, for their analysis Bohman, Frieze and Lubetzky give explicit constructions of very specific substructures which they count. These substructures and their explicit descriptions are tailored towards the triangle case and it remained unclear how to generalize these structures that are already complicated for the triangle case.

Investigating again the random greedy hypergraph matching process, but without similarly sophisticated substructures, Bohman and Bennett [9] showed that with high probability, $R_n(\mathcal{F}) \le n^{k-1/(2e(\mathcal{F})-2)+o(1)}$. This upper bound improves on Wormald's previous result, and for hypergraph matchings takes the analysis to a natural barrier, but still has not the correct order of magnitude; without the appropriate substructures, it seems impossible to rely on self-correcting behavior to the same extent that was necessary to determine the order of magnitude of $R_n(K_3)$.

In a landmark result Fiz Pontiveros, Griffiths and Morris [34] and independently Bohman and Keevash [16] asymptotically determined the typically encountered final number of edges in the triangle-free process with the correct constant factor, that is, they showed that typically, the final number of edges is $(\frac{1}{2\sqrt{2}} \pm o(1))(\log n)^{1/2} n^{3/2}$. Furthermore, together with bounds for the independence number of the eventually generated graph, for large t, this yields an improved lower bound for the Ramsey numbers $R(3,t)$. These results also rely on the exploitation of self-correcting behavior by considering carefully chosen key quantities, which further highlights the power of this technique.

For our proof in Chapter 3, we also take such an approach. To overcome the seemingly exploding complexity of the necessary substructures, even when generalizing the approach of Bohman, Frieze and Lubetzky to the case where $\mathcal{F} = K_4$, instead of giving explicit constructions, we develop an implicit way of selecting the appropriate key quantities. This forces us to argue without explicit knowledge of the structures that we investigate which makes the nature of our proof significantly more abstract. One may argue that this implicit choice is the main step for the proof of Theorem 1.1.6 for complete hypergraphs. For general strictly k-balanced hypergraphs, we introduce a symmetrization approach as a further crucial ingredient for our proof.

1.2 Notation

In this section, we collect some basic notation that was previously introduced and explain general as well as standard notation that we use. More specific key concepts are defined in the section where they are first discussed.

If m and n are integers, we set $[n]_m := \{m, \dots, n\}$ if $m \leq n$, $[n]_m := \emptyset$ if $m > n$ and $[n] := [n]_1$. For a set A, we say that A is a *k-set* if $|A| = k$. We write $\binom{A}{k}$ for the set of k-sets that are subsets of A and $A^{\underline{k}}$ for the set of tuples $(a_1, \dots, a_k) \in A^k$ with $a_i \neq a_j$ for all $i \neq j$. We use $\mathbb{1}_A$ to denote the indicator function of A where a suitable choice for its domain will be obvious from the context. For example, for an event $\mathcal{E}$, we use $\mathbb{1}_{\mathcal{E}}$, to denote the indicator random variable of $\mathcal{E}$. For sets A, B, we write $\varphi\colon A \hookrightarrow B$ for an injective function φ from A to B and we write $\varphi\colon A \xrightarrow{\sim} B$ for a bijection from A to B.

For integers i, j, we set $i \wedge j := \min\{i, j\}$ and $i \vee j := \max\{i, j\}$. We write $\alpha \pm \varepsilon = \beta \pm \delta$ to mean that $[\alpha - \varepsilon, \alpha + \varepsilon] \subseteq [\beta - \delta, \beta + \delta]$. We extend this notation to similar expressions involving more uses of $\pm$ in the natural way. We occasionally only write α instead of $\lfloor \alpha \rfloor$ or $\lceil \alpha \rceil$ when the rounding is not important. Outside of proofs and numbered statements, we occasionally use standard Bachmann-Landau notation including $O(\cdot)$, $o(\cdot)$ and $\Omega(\cdot)$.

A *hypergraph* $\mathcal{H}$ consists of a finite set V, its *vertex set*, together with a set E, its *edge set*, of subsets of V. The elements of V are the *vertices* of $\mathcal{H}$ and the elements of E are the *edges* of $\mathcal{H}$. We use $V(\mathcal{H})$ or $V_{\mathcal{H}}$ to denote the vertex set and $E(\mathcal{H})$ to denote the edge set of $\mathcal{H}$. We set $v(\mathcal{H}) := |V_{\mathcal{H}}|$ and $e(\mathcal{H}) := |\mathcal{H}|$. We say that $\mathcal{H}$ is a hypergraph *on n* vertices if $|V_{\mathcal{H}}| = n$. We often simply write $\mathcal{H}$ instead of $E(\mathcal{H})$, so in particular, we often write $|\mathcal{H}|$ for the size of $E(\mathcal{H})$. For $k \geq 1$, the hypergraph $\mathcal{H}$ is *k-uniform* if all its edges have size k and we refer to k as the *uniformity* of $\mathcal{H}$. A *k-graph* is a k-uniform hypergraph and a *graph* is a 2-graph. A k-graph $\mathcal{H}$ is *complete* if $\mathcal{H} = \binom{\mathcal{H}}{k}$ and *empty* if $\mathcal{H} = \emptyset$. A complete uniform hypergraph is also called a *clique* and an *n-clique* is a clique on n vertices. For $n \geq 0$ and $k \geq 1$, we use $K_n^{(k)}$ to denote a fixed k-uniform n-clique.

An edge e of a hypergraph $\mathcal{H}$ is *incident* to a vertex $v \in V_{\mathcal{H}}$ if $v \in e$. A *matching* in a hypergraph $\mathcal{H}$ is a set $\mathcal{M} \subseteq \mathcal{H}$ of disjoint edges, $\mathcal{M}$ *covers* a vertex $v \in V_{\mathcal{H}}$ if there exists an edge $e \in \mathcal{M}$ which is incident to v and $\mathcal{M}$ is *perfect*, if $\mathcal{M}$ covers every vertex $v \in V_{\mathcal{H}}$. An *independent set* in $\mathcal{H}$ is a set $U \subseteq V_{\mathcal{H}}$ such that no edge $e \in \mathcal{H}$ is a subset of U. The *independence number* of $\mathcal{H}$ is the size of a largest independent set in $\mathcal{H}$. For $v \in V_{\mathcal{H}}$, we use $N^{(2)}_{\mathcal{H}}(v)$ to denote the set $\{u \in V_{\mathcal{H}} : \{u, v\} \in \mathcal{H}\}$. For a j-set $U = \{u_1, \ldots, u_j\} \subseteq V_{\mathcal{H}}$, we write $d_{\mathcal{H}}(U)$ or $d_{\mathcal{H}}(u_1 \ldots u_j)$ for the *j-degree*, sometimes simply called *degree*, $|\{e \in \mathcal{H} \colon U \subseteq e\}|$ of U in $\mathcal{H}$. The degree $d(v)$ of $v \in V_{\mathcal{H}}$ is also called the *vertex degree* of v in $\mathcal{H}$. We define $\delta_j(\mathcal{H})$ to be the minimum j-degree $\min\bigl\{d_{\mathcal{H}}(U) \colon U \in \binom{V_{\mathcal{H}}}{j}\bigr\}$ of $\mathcal{H}$ and we define $\Delta_j(\mathcal{H})$ to be the maximum j-degree $\max\bigl\{d_{\mathcal{H}}(U) \colon U \in \binom{V_{\mathcal{H}}}{j}\bigr\}$ of $\mathcal{H}$. We set $\delta(\mathcal{H}) := \delta_1(\mathcal{H})$ and $\Delta(\mathcal{H}) := \Delta_1(\mathcal{H})$. The hypergraph $\mathcal{H}$ is *d-regular* if $\delta(\mathcal{H}) = \Delta(\mathcal{H}) = d$ and *regular* if it is d-regular for some d.

For an integer $j \geq 2$, we write $\mathcal{H}^{(j)}$ for the subgraph of $\mathcal{H}$ with vertex set $V_{\mathcal{H}}$ and edge set $\{e \in \mathcal{H} : |e| = j\}$. We use $\mathcal{H}_v$ to denote the *link* of v in $\mathcal{H}$, that is, the hypergraph with vertex set $V_{\mathcal{H}} \setminus \{v\}$ and edge set $\{e \setminus \{v\} : e \in \mathcal{H}, v \in e\}$ and for an integer $j \geq 1$ and $v \in V_{\mathcal{H}}$, we write $\mathcal{H}^{(j)}_v$ as a shorthand for $(\mathcal{H}_v)^{(j)} = (\mathcal{H}^{(j+1)})_v$. For $U \subseteq V_{\mathcal{H}}$, we write $\mathcal{H}[U]$ for the subgraph of $\mathcal{H}$ induced by U, that is, the subgraph with vertex set U and edge set $\{e \in \mathcal{H} : e \subseteq U\}$ and we use $\mathcal{H} - U$ to denote the k-graph $\mathcal{H}[V_{\mathcal{H}} \setminus U]$. The *complement* of a k-graph $\mathcal{H}$ is the k-graph with vertex set $V_{\mathcal{H}}$ and edge set $\binom{V_{\mathcal{H}}}{k} \setminus \mathcal{H}$. A hypergraph $\mathcal{H}_1$ is a *subgraph* of a hypergraph $\mathcal{H}_2$ if $V_{\mathcal{H}_1} \subseteq V_{\mathcal{H}_2}$ and $\mathcal{H}_1 \subseteq \mathcal{H}_2$ and $\mathcal{H}_1$ is a *proper* subgraph of $\mathcal{H}_2$ if at least one of these inclusions is proper. We write $\mathcal{H}_1 \subseteq \mathcal{H}_2$ to mean that $\mathcal{H}_1$ is a subgraph of $\mathcal{H}_2$ and we write $\mathcal{H}_1 \subsetneq \mathcal{H}_2$ to mean that $\mathcal{H}_1$ is a proper subgraph of $\mathcal{H}_2$. Note that if every vertex of $\mathcal{H}_1$ is contained in an edge of $\mathcal{H}_1$, then $E(\mathcal{H}_1) \subseteq E(\mathcal{H}_2)$ holds if and only if $\mathcal{H}_1$ is a subgraph of $\mathcal{H}_2$. If this is not the case when we write $\mathcal{H}_1 \subseteq \mathcal{H}_2$, it will be clear from the context whether we mean an inclusion of edge sets or that $\mathcal{H}_1$ is a subgraph of $\mathcal{H}_2$. The hypergraph $\mathcal{H}_1$ is a *spanning* subgraph of $\mathcal{H}_2$ if $\mathcal{H}_1$ is a subgraph of $\mathcal{H}_2$ with $V_{\mathcal{H}_1} = V_{\mathcal{H}_2}$. We write $\mathcal{H}_1 + \mathcal{H}_2$ for the *union* of $\mathcal{H}_1$ and $\mathcal{H}_2$, that is the hypergraph with vertex set $V_{\mathcal{H}_1} \cup V_{\mathcal{H}_2}$ and edge set $\mathcal{H}_1 \cup \mathcal{H}_2$. The hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ are *disjoint* if $V_{\mathcal{H}_1} \cap V_{\mathcal{H}_2} = \emptyset$ and *edge-disjoint* if $\mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$. We say that $\mathcal{H}_1$ is a *copy* of $\mathcal{H}_2$ if there exists a bijection $\beta \colon V_{\mathcal{H}_1} \xrightarrow{\sim} V_{\mathcal{H}_2}$ with $\beta(e_1) \in \mathcal{H}_2$ for all $e_1 \in \mathcal{H}_1$ and $\beta^{-1}(e_2) \in \mathcal{H}_1$ for all $e_2 \in \mathcal{H}_2$ and $\mathcal{H}_1$ is a copy of $\mathcal{H}_2$ *in* a hypergraph $\mathcal{H}$ if additionally $\mathcal{H}_1$ is a subgraph of $\mathcal{H}$. An *$\mathcal{H}_1$-packing* in $\mathcal{H}_2$ is a collection $\mathscr{H}$ of edge-disjoint copies of $\mathcal{H}_1$ in $\mathcal{H}_2$ and $\mathscr{H}$ is an *$\mathcal{H}_1$-decomposition* of $\mathcal{H}_2$ if for every edge $e \in \mathcal{H}_2$, there exists a hypergraph $\mathcal{H} \in \mathscr{H}$ with $e \in \mathcal{H}$.

For a statement φ that is true or false (usually depending on random choices), we use $\{\varphi\}$ to denote the event that φ is true. When describing our probabilistic constructions for a given approximately d-regular hypergraph, we occasionally use the phrase "with high probability" to refer to an event that happens with probability asymptotically tending to 1 where the asymptotics will be clear from the context. However in all formal statements and proofs, we provide explicit bounds for all relevant probabilities.

Chapter 2

Conflict-free Hypergraph Matchings

2.1 Introduction

The intuition for our conflict-free matching process is that, through the fact that we keep adding edges to our matching in a random fashion, the current matching after each iteration essentially behaves like a uniformly random subset of edges. Concretely, we show that certain random variables associated with this process follow with high probability a deterministic trajectory given by the above intuition which allows us to deduce that the process only stops when almost all vertices of $\mathcal{H}$ are covered. On this high level, our proof of Theorem 1.1.2 is similar to the proofs in [18, 49].

We remark that the results of Frankl and Rödl [37] and Pippenger (see [94]) mentioned in Chapter 1 are proved using the Rödl nibble, also known as the semi-random method, which only consists of a constant number of rounds and in each round a linear proportion of the matching is constructed. In contrast, a random greedy algorithm only adds one edge at a time. Both methods can be applied in similar situations, and each has its advantages. In fact, Delcourt and Postle [23] independently obtained results similar to those in this chapter by pursuing a nibble argument instead of analyzing a random process. In particular, they were also motivated by and proved the existence of approximate high-girth Steiner systems and observed that this is just a special case of a general hypergraph matching theorem. The precise statements of the obtained matching theorems differ. For instance, the result of Delcourt and Postle does not require an upper bound on the number of vertices and they also obtain guarantees providing control over the uncovered vertices in a bipartite setting. Moreover they observed that the classical theorem of Ajtai, Komlós, Pintz, Spencer and Szemerédi [1] for finding an independent set in girth five hypergraphs can also be deduced from the general matching theorem (in our case we need polylogarithmic degree). Our result has the advantage that it allows tracking test functions (which we believe is crucial for potential applications based on the absorption method) and our analysis enables a more precise counting of matchings, see Theorem 2.5.5.

The assumptions on $\mathcal{H}$ we make are (qualitatively) the same as usual: we want $\mathcal{H}$ to be almost-regular and to have small 2-degrees. This would, as discussed before, imply the existence of an almost-perfect matching in $\mathcal{H}$. As far as the conditions on $\mathcal{C}$ in Theorem 1.1.2 are concerned, observe first that the order of magnitude which bounds the degrees of $\mathcal{C}$ is natural, as the following probabilistic deletion argument shows. Indeed, it means that in total, there are $O(nd^j)$ conflicts of size j, where n is the number of vertices of $\mathcal{H}$. If we select every edge of $\mathcal{H}$ with probability $p = \delta/d$, the expected number of chosen edges is roughly $pnd/k = \delta n/k$. Call an edge *bad* if it overlaps with another chosen edge or participates in a conflict where all edges have been chosen. The expected number of overlapping pairs of chosen edges is $O(p^2nd^2) = O(\delta^2 n)$, and the expected number of completely chosen conflicts of size j is $O(p^j nd^j) = O(\delta^j n)$. Hence, the expected total number of bad edges is only $O(\delta^2 n)$. Consequently, choosing δ small enough, there exists an outcome of this random experiment such that after removing all bad edges, $\delta n/k - O(\delta^2 n)$ edges still remain. In other words, we have found a $\mathcal{C}$-free matching covering a constant proportion of all vertices. One can think of the above procedure as one "bite" of the Rödl nibble.

As observed by Frankl and Rödl, and Pippenger, the small 2-degree assumption for $\mathcal{H}$ is enough to ensure that one can repeatedly take such small "bites" until almost all vertices are covered. When dealing in addition with a conflict system in the form of a hypergraph with vertex set $\mathcal{H}$, the obstructions coming from the exclusion of conflicts additionally influence the behavior of this procedure. For instance, roughly speaking, if an edge of $\mathcal{H}$ participates in many conflicts, it is much more likely to become unavailable at some point than an edge which participates in few conflicts. Hence, in order to ensure that the uncovered part of $\mathcal{H}$ remains almost-regular throughout, we also wish to know quite precisely in how many conflicts which may still cause unavailability a given edge participates in each step. Controlling the regularity of the conflict hypergraph is complicated by the fact that conflicts consist of several edges, some of which might already be included in the matching, while others are not, or might even be unavailable due to an overlap with an edge in the matching or another conflict.

One important point in Theorem 1.1.2 in terms of its applicability (see Section 2.3.2) is that we only require upper bounds for the conflict hypergraph $\mathcal{C}$. While this seems natural as having fewer conflicts should only be advantageous when finding a conflict-free matching, one has to be careful since, as described above, the degrees of the conflict hypergraph significantly influence the evolution of the degrees of $\mathcal{H}$. In our proof, we actually show that given a conflict system with upper bounds on the degrees, one can artificially add conflicts to "regularize" the conflict system, and then we analyze the random process with respect to this enlarged conflict system. Our approach for obtaining an approximately regular conflict system (see Lemmas 2.10.3, 2.10.4 and 2.10.6) shares similarities with arguments in the independent work of Kwan, Sah, Sawhney and Simkin [80] (see Lemma 5.2 and Section 10.1.2), which also was inspired by earlier works that employ a "regularity boosting approach" [2, 50]. The idea of regularization has turned out to be crucial in other places, see for example [50].

An additional point is that during the process, we also allow to track certain "test functions." This is not necessary to prove our main result, but we provide this additional feature to facilitate future applications. Roughly speaking, the idea is that we also want to be able to claim that the obtained conflict-free matching $\mathcal{M}$ behaves as one would expect by considering probabilistic heuristics. In the usual setting, without a conflict system, such a tool was provided in [27], thereby extending [4], and has already found a number of applications, see for example [65, 73, 101]. Inevitably, this additional feature adds in technicality and length to our proof, but it is essential for some future applications, for example, the application mentioned in Section 2.3.6 relies heavily on these test functions. For instance, one can utilize our test functions to see that not only can one find a high-girth partial (m, s, t)-Steiner system with $o(m^t)$ uncovered t-sets (as stated in Theorem 1.1.1), but even one where every $(t-1)$-set is contained in $o(m)$ uncovered t-sets.

Our analysis of the conflict-free matching process, which constitutes the core part of the chapter, relies on identifying sequences of key random variables that govern the evolution of the process and subsequently obtaining precise deterministic estimates for these key random variables that hold with high probability. To achieve this, we rely on supermartingale concentration techniques, for example a version of Freedman's inequality for supermartingales, see Lemma 2.9.4, in an approach that resembles the differential equation method introduced by Wormald [113].

One obvious choice for a sequence of such key random variable that we wish to track is, for each step, the number of edges that are still available for adding them to the conflict-free matching constructed so far as the evolution of this number of edges determines when the process terminates and hence the size of the eventually generated matching. More specifically, our goal is to show that with high probability, these numbers of edges are closely concentrated around deterministic values that describe a trajectory for the sequence of random variables.

As we aim to employ supermartingale concentration inequalities, we require insights concerning the expected one-step changes of the random process given by the number of available edges. In fact it is such a reliance on expected one-step changes that gives the differential method its name. Using quantities to express the expected one-step changes of others leads to dependencies that can be interpreted as a system of differential equations whose deterministic solutions constitute the trajectories that the random quantities are typically concentrated around. We do not explicitly formulate and solve such a system of differential equations and obtain our trajectories based on heuristic considerations. Nevertheless, for our formal argument based on supermartingale concentration where we aim to show that the random variables are close to the trajectories with high probability, we need to rely on a further analysis of the expected one-step changes.

In our setting, an edge e can become unavailable and thus contribute to the change of the number of available edges for two reasons. Firstly, because an edge intersecting e is selected for the matching and secondly, because an edge is selected causing e to be the only edge in a conflict C that is not already selected for the matching. Hence, motivated by the goal to express the probability that a fixed available edge e is no longer available in the next step of the conflict-free matching process, we are interested in the number of available edges intersecting e and the number of conflicts C that contain e, another available edge f and $|C| - 2$ edges that are already selected for the matching. This introduces a collection of further sequences of key quantities that we wish to control in the sense that we show that they similarly follow deterministic trajectories, which highlights one of the central challenges of such an approach: we need a collection of key quantities that is *closed* in the sense that given a sequence of key random variables from our collection, we can express the expected one-step changes using other key quantities from the collection, at least up to acceptable estimation errors in the form of error terms that we choose and that need to be carefully calibrated across the collection.

Specifically, in our setting, our choice of such a closed collection is based on the following idea. If in every step, for every uncovered vertex, we have estimates for the number of available edges containing v, then, since $\mathcal{H}$ has small 2-degrees, we can essentially express the number of available edges that intersect a fixed available edge. Furthermore, immediate effects of conflicts on availability can be controlled based on estimates for the numbers of conflicts C containing a fixed available edge, another available edge and $|C|-2$ edges that are already selected for the matching as above. Finally, to express one-step changes of such random variables, in our collection we more generally for all s consider all numbers of conflicts C containing a fixed available edge, s other available edges and $|C|-s-1$ edges that are already selected for the matching.

2.2 Organization of the chapter

First, in Section 2.3, we collect several possible directions and examples for applications of the main results in this chapter that highlight the flexibility and generality and that further illustrate that our conditions are natural. We then briefly discuss the optimality of our conditions in Section 2.4 and present variations and extensions of Theorem 1.1.2 in Section 2.5. Then we formally describe our conflict-free matching process in detail in Section 2.6 and discuss key random variables in Section 2.7. In our supermartingale argument, we require some contributions to expected one-step changes that arise from more delicate local interactions to be negligible. We dedicate Section 2.8 to bounding the effect of these contributions which subsequently allows us to deduce that these contributions are indeed negligible compared to the carefully chosen estimation errors we allow in our argument. The supermartingale argument itself which constitutes the main part of this chapter can be found in Section 2.9. Finally, we prove the main theorems stated in Section 2.10, discuss our application to Steiner systems in detail in Section 2.11 and conclude the chapter with some further remarks concerning applications and open questions, see Section 2.12.

2.3 Applications

We now discuss some applications of Theorem 1.1.2. In addition to discussing further details of various new applications, we also point out that some results which are already known in the literature, proved ad-hoc and with no obvious connection to, say, Steiner triple systems of large girth, are implied by Theorem 1.1.2. Together with the various new results [6, 11, 24, 47, 52, 63, 81] mentioned in Chapter 1, this underpins the fact that Theorem 1.1.2 reveals a very general phenomenon.

2.3.1 Steiner systems and Latin squares

As already discussed, Theorem 1.1.2 implies the existence of high-girth approximate Steiner systems as stated in Theorem 1.1.1, which generalizes the results from [18, 49]. We provide the details of this deduction, together with some additional extensions such as growing girth, in Section 2.11.

We remark that similar results can be obtained in the "partite" setting. For instance, while Steiner triple systems are equivalent to triangle decompositions of complete graphs, Latin squares, for an introduction to the topic, see for example [68], are equivalent to triangle decompositions of complete balanced tripartite graphs.

In particular, since every partial Latin square is also a partial Steiner triple system, the definition of girth also applies for Latin squares. More concretely, the girth of a partial Latin square $\mathcal{L}$ is the smallest $g \geq 4$ such that there exists a set of rows, columns and symbols that has size g, such that there are at least $g-2$ cells whose row, column and symbol is contained in the given set (if no such g exists, the girth is infinite). For instance, it is easy to see that $\mathcal{L}$ has girth greater than 6 if and only if it contains no *intercalate* (a 2×2 sub-Latin square). The same argument used to prove Theorem 1.1.1 gives the existence of partial $m \times m$ Latin squares which are almost complete (all but $o(m^2)$ cells are filled) and have arbitrarily large girth. This yields an approximate solution to a question of Linial who conjectured that $m \times m$ Latin squares of arbitrarily large girth exist for all sufficiently large m. Linial's conjecture was very recently confirmed in full by Kwan, Sah, Sawhney and Simkin in [79], where they adopted the methods they used for Steiner triple systems in [80]. Finally, we remark that analogous (approximate) results hold for "high-dimensional permutations", see [84–86], which are a generalization of Latin squares (and correspond to Steiner systems with arbitrary parameters).

2.3.2 Erdős meets Nash-Williams

A famous conjecture of Nash-Williams [89] says that every graph $\mathcal{G}$ with minimum degree at least $3|V_{\mathcal{G}}|/4$ has a triangle decomposition, subject to the necessary conditions that $|\mathcal{G}|$ is divisible by 3 and all the vertex degrees are even. In [51], a combination of the conjectures of Erdős and Nash-Williams was proposed: that every sufficiently large graph as above in fact has a triangle decomposition with arbitrarily high girth. In this context, it was asked whether minimum degree $9|V_{\mathcal{G}}|/10$, say, is at least enough to guarantee an approximate triangle decomposition with arbitrarily high girth. We can answer this question, see Theorem 2.3.2. This is a consequence of the aforementioned feature that we only require upper bounds on the degrees of the conflict hypergraph. Recall that in the hypergraph matching setting, the vertices of $\mathcal{H}$ are the edges of $\mathcal{G}$ and the edges of $\mathcal{H}$ correspond to the triangles of $\mathcal{G}$. In general, $\mathcal{H}$ will not be regular. However, if $\mathcal{G}$ contains a suitable collection of triangles such that every edge is contained in roughly the same number of triangles from this collection, then we can simply define $\mathcal{H}$ by only keeping the triangles from the collection. The conflict hypergraph might become irregular through this sparsification, but since we only require upper bounds, this does not cause any problem.

Theorem 2.3.1. *For all $c_0 > 0$, $s > t \geq 2$ and $\ell \geq 2$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, there exists m_0 such that the following holds for all $m \geq m_0$ and $c \geq c_0$. Let $\mathcal{G}$ be a t-graph on m vertices and let $\mathcal{K}$ be a collection of s-sets which induce cliques in $\mathcal{G}$ such that every edge is a subset of $(1 \pm m^{-\varepsilon})cm^{s-t}$ elements of $\mathcal{K}$.*

Then, there exists a partial (m, s, t)-Steiner system $\mathcal{S} \subseteq \mathcal{K}$ of size $(1 - m^{-\varepsilon^3})|\mathcal{G}|/\binom{s}{t}$ such that any subset of $\mathcal{S}$ of size j, where $2 \leq j \leq \ell$, spans more than $(s - t)j + t$ points.

In Section 2.11, we outline how to obtain Theorem 2.3.1 from our main theorem. Note that by specifying $\mathcal{G}$ to be the complete t-graph and $\mathcal{K}$ the collection of all s-sets, we recover Theorem 1.1.1. The above could also be used when $\mathcal{G}$ is a sufficiently dense binomial random t-graph.

Concerning the setting where the minimum degree is sufficiently large, we may furthermore argue as follows. An *approximate fractional decomposition* of a t-graph $\mathcal{G}$ into s-cliques is a function $w\colon \mathscr{K}_s \to [0, \infty)$ such that for all $e \in \mathcal{G}$, the sum $\sum_{\mathcal{K} \in \mathscr{K}_s\colon e \in \mathcal{K}} w(\mathcal{K})$, where $\mathscr{K}_s$ denotes the set of s-cliques in $\mathcal{G}$, is sufficiently close to 1 and w is a *fractional decomposition* of $\mathcal{G}$ into s-cliques if for all $e \in \mathcal{G}$, the above sum is exactly 1. If there exists an approximate fractional decomposition w of $\mathcal{G}$ into s-cliques such that the largest weight $w_{\max}$ assigned by w is sufficiently small, then, to find a collection of s-cliques that is regular enough, one may consider the random collection $\mathscr{K}_s'$ where every s-clique $\mathcal{K} \in \mathscr{K}_s$ is included independently at random with probability $w(\mathcal{K})/w_{\max}$. Such an (even non-approximate) fractional decomposition exists in particular if $\mathcal{G}$ has very large $(t-1)$-degree, say $\delta_{t-1}(\mathcal{G}) \geq (1 - (4s)^{-2t})m$ (see for example Theorem 1.5 and its proof in [7] and [50, Lemma 6.3]).

Additional arguments concerning fractional decompositions allow us to gain further insights. For $s > t \geq 2$, let $\delta_{t,s}$ denote the *fractional decomposition threshold* for the t-uniform s-clique, that is the infimum of all δ such that for all $\varepsilon > 0$, there exists m_0 such that all t-graphs $\mathcal{G}$ on $m \geq m_0$ vertices with $\delta_{t-1}(\mathcal{G}) \geq (\delta + \varepsilon)m$ admit a fractional decomposition into s-cliques. For all $\varepsilon > 0$ and all sufficiently large t-graphs $\mathcal{G}$ with $\delta_{t-1}(\mathcal{G}) \geq (\delta_{t,s} + \varepsilon)m$, there exists an approximate fractional decomposition into s-cliques where the largest assigned weight is sufficiently small (see Lemma 2.11.4). In Section 2.11, we outline how the existence of such an approximate fractional decomposition may be used to obtain the following statement.

Theorem 2.3.2. *For all $s > t \geq 2$, $\varepsilon > 0$ and $\ell \geq 2$, there exists m_0 such that for all t-graphs $\mathcal{G}$ on $m \geq m_0$ vertices with $\delta_{t-1}(\mathcal{G}) \geq (\delta_{t,s} + \varepsilon)m$, the following holds. There exists a partial (m, s, t)-Steiner system $\mathcal{S}$ with size $(1 - \varepsilon)|\mathcal{G}|/\binom{s}{t}$ whose elements induce cliques in $\mathcal{H}$ such that every subset of $\mathcal{S}$ of size j, where $2 \leq j \leq \ell$, spans more than $(s - t)j + t$ points.*

2.3.3 Excluding grids

Another application is a Turán-type question that was already studied by Füredi and Ruszinkó [42]. A hypergraph $\mathcal{H}$ is called *linear* if $|e \cap f| \leq 1$ for all distinct $e, f \in \mathcal{H}$. An *s-grid* is an s-graph on s^2 vertices with $2s$ edges $e_1, \ldots, e_s, f_1, \ldots, f_s$ such that $\{e_1, \ldots, e_s\}$,

$\{f_1, \ldots, f_s\}$ are matchings and $|e_i \cap f_j| = 1$ for all $i, j \in [s]$. An s-graph is *grid-free* if it does not contain an s-grid as a subgraph.

Theorem 2.3.3 ([42, Theorem 1.2])**.** *For all $s \geq 4$, there exists $\varepsilon > 0$ such that there are linear grid-free s-graphs $\mathcal{H}$ on m vertices with $(1 - m^{-\varepsilon})\binom{m}{2}/\binom{s}{2}$ edges.*

Our results yield the following generalization that allows multiple forbidden subgraphs. Here, we call a hypergraph $\mathcal{H}$ *t-linear* if $|e \cap f| \leq t - 1$ for all distinct $e, f \in \mathcal{H}$ and for a given collection $\mathscr{F}$ of hypergraphs, we say that $\mathcal{H}$ is *$\mathscr{F}$-free* if no subgraph of $\mathcal{H}$ is a copy of an element of $\mathscr{F}$.

Theorem 2.3.4. *Let $s \geq 2$ and $t \in [s-1]$. Suppose $\mathscr{F}$ is a finite collection of t-linear s-graphs $\mathcal{F}$ with $|\mathcal{F}| \geq 2$ and $|V_{\mathcal{F}}| \leq (s-t)|\mathcal{F}| + t$. Then, there exist $\varepsilon > 0$ and m_0 such that for all $m \geq m_0$, there is an $\mathscr{F}$-free t-linear s-graph $\mathcal{H}$ on m vertices with $(1 - m^{-\varepsilon})\binom{m}{t}/\binom{s}{t}$ edges.*

Note that in particular, this applies to s-grids with $s \geq 4$ and (2-)linear hypergraphs, so it implies Theorem 2.3.3. Note that for a 3-grid $\mathcal{F}$, we have $|V_{\mathcal{F}}| = 9 > 8 = |\mathcal{F}| + 2$, so similarly to Theorem 2.3.3, this theorem cannot be applied for grid-free 3-graphs. However, Füredi and Ruszinkó conjectured that similar asymptotics also hold for grid-free 3-graphs. For significant progress on this, see [44].

To see that Theorem 2.3.4 is true, note that a hypergraph with vertex set $[m]$ is a t-linear s-graph if and only if its edge set is a partial (m, s, t)-Steiner system $\mathcal{S}$ and that the high girth condition in Theorem 1.1.1 means that any subgraph $\mathcal{F}$ of the s-graph with vertex set $[m]$ and edge set $\mathcal{S}$ with $2 \leq |\mathcal{F}| \leq \ell$ has more than $(s-t)|\mathcal{F}| + t$ vertices. Hence, for a given finite collection $\mathscr{F}$ as in Theorem 2.3.4, Theorem 1.1.1 implies the existence of suitable $\mathscr{F}$-free t-linear s-graphs, provided that ℓ is sufficiently large. In fact, Theorems 1.1.1 and 2.3.4 are equivalent, so Theorem 2.3.4 is essentially just a rephrasing of Theorem 1.1.1 using the terminology of t-linear s-graphs instead of partial Steiner systems.

2.3.4 Well-separated packings

Another straightforward application is the existence of asymptotically optimal $\mathcal{F}$-packings which are "well-separated." The following theorem was proved by Frankl and Füredi [36] and has turned out to be useful for many applications, see for example [3, 41, 103].

Theorem 2.3.5. *Let $\mathcal{F}$ be a t-graph with $|\mathcal{F}| \geq 2$. There exist $\varepsilon > 0$ and m_0 such that for all $m \geq m_0$, there exists a collection $\mathscr{F}$ of copies of $\mathcal{F}$ whose vertex sets are subsets of an m-set with $|\mathscr{F}| \geq (1 - m^{-\varepsilon})\binom{m}{t}/|\mathcal{F}|$ such that the following holds. For all distinct $\mathcal{F}_1, \mathcal{F}_2 \in \mathscr{F}$, we have $|V_{\mathcal{F}_1} \cap V_{\mathcal{F}_2}| \leq t$, and if $|V_{\mathcal{F}_1} \cap V_{\mathcal{F}_2}| = t$, then this t-set is neither an edge of $\mathcal{F}_1$ nor of $\mathcal{F}_2$.*

One can deduce this theorem from our Theorem 1.1.2. Since the result is already known, we omit the details.

2.3.5 Counting

We remark that, by analyzing our proof, one can also obtain a lower bound on the number of conflict-free almost-perfect matchings. This is a consequence of our tight control over the number of choices which the algorithm has in each step (with high probability). As discussed in Section 2.1, the degrees of the conflict system $\mathcal{C}$ significantly influence the trajectories of the process, hence the number of choices in each step and thus the total number of choices depends on the given conflict system $\mathcal{C}$. For a precise counting statement, see Theorem 2.5.5.

2.3.6 Ramsey theory

Erdős and Gyárfás [33] investigated the following Ramsey problem: given $s, t \in \mathbb{N}$, what is the minimum number of colors $f(n, s, t)$ such that the edges of K_n can be coloured with $f(n, s, t)$ colors in such a way that the edge set of every s-clique receives at least t distinct colors? They in particular asked for the value of $f(n, 4, 5)$. Very recently, Bennett, Cushman, Dudek and Prałat [10] solved the Erdős-Gyárfás question by determining the asymptotic value of $f(n, 4, 5)$. To this end, they considered a modification of the triangle removal process, where each chosen triangle receives exactly two colors, where 2-chromatic 4-cycles are forbidden, and where, among further conditions, each vertex has a list of forbidden colors. Joos and Mubayi [63] found a way to give a very short alternative proof for the Erdős-Gyárfás question using the results in this chapter. More specifically, they show that the result of the random process in [10] can be equivalently phrased as a conflict-free hypergraph matching. The important message thereby is that even intricate random processes can be captured by the main results of this chapter and it is likely that this may be beneficial for other applications as well. In [63] one can also find a short proof for a similar edge coloring problem in the complete bipartite graph $K_{n,n}$ avoiding 2-chromatic 4-cycles, thereby solving the main open problem posed in [5].

2.3.7 Degenerate Turán densities

Theorem 1.1.2 turns out to be useful for solving the following problem of Brown, Erdős and Sós [22]. Let $f^{(r)}(n, s, k)$ be the maximum number of edges of an r-graph on n vertices not containing a subgraph on s vertices with k edges. Brown, Erdős and Sós conjectured that $n^{-2} f^{(3)}(n, k+2, k)$ converges for all $k \geq 2$ and confirmed it for $k = 2$. In [45], the conjecture is verified for $k = 3$ and in [47] for $k = 4$. Building on a result from [47], Delcourt and Postle [24] confirmed the conjecture for all k.

While for $k \in \{2, 3, 4\}$, the aforementioned work where the existence of the limit was proved also yields its value, this is not the case for [24]. Very recently, Glock, Kim, Lichev, Pikhurko and Sun [48] determined the limit for $k \in \{5, 6, 7\}$.

2.4 Optimality of the conditions

In the previous section, we provide several examples showing that the conditions on $\mathcal{C}$ are general enough to have many interesting applications. Here, we additionally demonstrate that our conditions are necessary in the sense that Theorem 1.1.2 would be false if any of the four conditions listed for $\mathcal{C}$ is omitted.

For the first condition, we consider a random construction of $\mathcal{C}$ for some fixed uniformity $j \geq 3$. The structure of $\mathcal{H}$ can be quite arbitrary, we only use that it has roughly nd/k edges, which are the vertices of $\mathcal{C}$. Let $\mathcal{C}$ be the binomial random j-graph where every j-set of vertices is an edge independently with probability $p = (K \log d)/n^{j-1}$. Then, assuming K is a large enough constant (depending only on j and k), the following holds with high probability.

(i) $\Delta(\mathcal{C}) \leq 2Kd^{j-1} \log d$;
(ii) $\Delta_{j'}(\mathcal{C}) \leq d^{j-j'-\varepsilon}$ for all $2 \leq j' < j$;
(iii) there is no independent set in $\mathcal{C}$ of size larger than $n/2k$.

Here, the first two properties follow from standard concentration inequalities and the third from a simple first moment argument.

In particular, the condition for the j'-degrees in $\mathcal{C}$ with $2 \leq j' \leq j$ is satisfied, but the largest $\mathcal{C}$-free subset of $\mathcal{H}$ has size at most $n/2k$. This shows that the maximum degree condition for $\mathcal{C}$ cannot be omitted. In fact, a stronger version of Theorem 1.1.2, namely Theorem 2.5.1, allows for maximum degree $\Delta(\mathcal{C}^{(j)}) \leq \alpha d^{j-1} \log d$ for some small enough constant α (provided μ and ℓ in Theorem 2.5.1 are both constant), and (i) shows that this is tight up to the constant factor.

Let us now check that the condition for the j'-degrees in $\mathcal{C}$ is also necessary. Again mostly ignoring the structure of $\mathcal{H}$, we fix some $j \geq 3$ and simply choose $\mathcal{C}$ as a spanning j-graph that is the disjoint union of cliques on $2jd$ vertices each. Then the maximum degree condition is satisfied, but a $\mathcal{C}$-free edge set can contain at most $j-1$ vertices from each clique and hence will have size at most $n/2k$.

The third condition is also necessary. Indeed, consider the complete bipartite graph $\mathcal{H} = K_{n,n}$ with parts A, B, that is the graph with vertex set $A \cup B$ where A and B are disjoint n-sets and edge set $\{\{a, b\} : a \in A, b \in B\}$. Take an arbitrary partition of A into 2-sets and for each such 2-set $\{u, v\}$, add a conflict between all pairs of distinct edges where one edge is incident to u and the other to v. This construction satisfies all but the third condition, but any conflict-free matching can only cover half of the vertices of A.

To see that the fourth condition is necessary, consider as $\mathcal{H}$ a graph that is the disjoint union of $2\sqrt{n}$ cliques on $\sqrt{n}/2$ vertices each. Let $\mathcal{C}$ be the disjoint union of $\binom{\sqrt{n}/2}{2}$ cliques on $2\sqrt{n}$ vertices each, where each of these cliques in $\mathcal{C}$ contains precisely one edge from each of the aforementioned cliques in $\mathcal{H}$. Hence any $\mathcal{C}$-free edge set has size at most $(\sqrt{n}/2)^2/2 = n/8$.

Note that the above example for the necessity of the third condition satisfies the fourth condition for all disjoint e, f. However for two distinct e, f with a vertex in A in their intersection, it fails to satisfy the fourth condition. In this spirit, we note that we can indeed omit the third condition if we make the fourth condition stronger so that $|N_{\mathcal{C}}^{(2)}(e) \cap N_{\mathcal{C}}^{(2)}(f)| \leq d^{1-6\varepsilon}$ holds for *all distinct* $e, f \in E(\mathcal{H})$. Basically, we can deduce from this stronger fourth condition that the number of edges e that fail to satisfy the third condition for some v is at most $d^{-\varepsilon}|\mathcal{H}|$. We can just delete all such edges and add a few auxiliary vertices and edges containing at least one auxiliary vertex to make $\mathcal{H}$ almost regular. Then it is easy to see that a $\mathcal{C}$-free almost perfect matching in this hypergraph yields a $\mathcal{C}$-free almost perfect matching in the original hypergraph.

2.5 Variations and extensions of the main theorem

In this section, we provide several variations of Theorem 1.1.2 that we prove in Section 2.10. In particular Theorem 1.1.2 is an immediate consequence of Theorem 2.5.1.

For a k-graph $\mathcal{H}$, we say that a (not necessarily uniform) hypergraph $\mathcal{C}$ with vertex set $\mathcal{H}$ whose edges have size at least 2, which may be used to encode the subsets which we wish to avoid as subsets of a matching as in Theorem 1.1.2, is a *conflict system* for $\mathcal{H}$. We call the edges of $\mathcal{C}$ *conflicts* of $\mathcal{C}$ and for $e \in \mathcal{H}$ and $C \in \mathcal{C}_e$, where $\mathcal{C}_e$ denotes the link of e in $\mathcal{C}$, we say that C is a *semiconflict*.

To obtain a $\mathcal{C}$-free matching $\mathcal{M} \subseteq \mathcal{H}$, it is crucial that $\mathcal{C}$ satisfies suitable boundedness conditions similarly as in Theorem 1.1.2, however slightly weaker conditions are sufficient. To this end, for integers $d \geq 1$ and $\ell \geq 2$ and reals $\Gamma \geq 0$ and $0 < \varepsilon < 1$, we say that $\mathcal{C}$ is $(d, \ell, \Gamma, \varepsilon)$-*bounded* if the following holds.

(C1) $2 \leq |C| \leq \ell$ for all $C \in \mathcal{C}$;

(C2) $\sum_{j\in[\ell]} \frac{\Delta(\mathcal{C}^{(j)})}{d^{j-1}} \leq \Gamma$ and $|\{j \in [\ell]_2 : \mathcal{C}^{(j)} \neq \emptyset\}| \leq \Gamma$;

(C3) $\Delta_{j'}(\mathcal{C}^{(j)}) \leq d^{j-j'-\varepsilon}$ for all $j \in [\ell]_2$ and $j' \in [j-1]_2$;

(C4) $|\{f \in N^{(2)}_{\mathcal{C}}(e) : v \in f\}| \leq d^{1-\varepsilon}$ for all $e \in \mathcal{H}$ and $v \in V(\mathcal{H})$;

(C5) $|N^{(2)}_{\mathcal{C}}(e) \cap N^{(2)}_{\mathcal{C}}(f)| \leq d^{1-\varepsilon}$ for all disjoint $e, f \in \mathcal{H}$.

Theorem 2.5.1. *For all $k \geq 2$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, there exists d_0 such that the following holds for all $d \geq d_0$. Suppose $\ell \geq 2$ is an integer and suppose $\Gamma \geq 1$ and $0 < \mu \leq 1/\ell$ are reals such that $1/\mu^{\Gamma\ell} \leq d^{\varepsilon^2}$. Suppose $\mathcal{H}$ is a k-graph on $n \leq \exp(d^{\varepsilon^2/\ell})$ vertices with $(1 - d^{-\varepsilon})d \leq \delta(\mathcal{H}) \leq \Delta(\mathcal{H}) \leq d$ and $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon}$ and suppose $\mathcal{C}$ is a $(d, \ell, \Gamma, \varepsilon)$-bounded conflict system for $\mathcal{H}$.*

Then, there exists a $\mathcal{C}$-free matching $\mathcal{M} \subseteq \mathcal{H}$ of size $(1 - \mu)n/k$.

Note that the somewhat unusual condition $1/\mu^{\Gamma\ell} \leq d^{\varepsilon^2}$ allows for various tradeoffs between the parameters. In particular, when ℓ and μ are constant, we can allow Γ to be of order $\log d$. Theorem 2.5.1 is an immediate consequence of a further extension, namely Theorem 2.5.2, where we also obtain a matching $\mathcal{M}$ that is almost-perfect. There, $\mathcal{M}$ additionally has properties which random edge sets that include the edges of $\mathcal{H}$ independently with probability $|\mathcal{M}|/|\mathcal{H}|$ typically exhibit. In more detail, we show that $\mathcal{M}$ can be chosen such that for a given sufficiently large edge set $\mathcal{Z} \subseteq \mathcal{H}$, we have $|\mathcal{Z} \cap \mathcal{M}| \approx |\mathcal{Z}| \cdot |\mathcal{M}|/|\mathcal{H}|$, which is what would be expected if the edges of $\mathcal{H}$ were included in $\mathcal{M}$ independently at random with probability $|\mathcal{M}|/|\mathcal{H}|$. Further, we show that an analogous statement holds for suitable sets $\mathcal{Z} \subseteq \binom{\mathcal{H}}{j}$ and we show that it can be satisfied for multiple $\mathcal{Z}$ simultaneously. Again, as for the conflict systems, it is convenient to interpret sets $\mathcal{Z} \in \binom{\mathcal{H}}{j}$ where $j \geq 1$ as edge sets of hypergraphs with vertex set $\mathcal{H}$. Thinking of these $\mathcal{Z}$ as a way to test $\mathcal{M}$ for properties that it satisfies, we say that a uniform hypergraph $\mathcal{Z}$ with vertex set $\mathcal{H}$ whose edges are matchings is a *test system* and we call edges of $\mathcal{Z}$ *tests*. Note that in particular, we allow 1-uniform test systems $\mathcal{Z}$, so edge sets $E \subseteq \mathcal{H}$ may be treated as test systems by considering the test system $\mathcal{Z}$ with edge set $\binom{E}{1}$. We can only allow test systems that are well behaved in the sense that we can keep track of their properties during the evolution of our random iterative matching construction such that in the end, we can guarantee that the matching behaves as expected with respect to $\mathcal{Z}$. To this end, for integers $j, d \geq 1$, a real $\varepsilon > 0$ and a conflict system $\mathcal{C}$ for $\mathcal{H}$, we say that a j-uniform test system $\mathcal{Z}$ for $\mathcal{H}$ is $(d, \varepsilon, \mathcal{C})$-*trackable* if the following holds.

(Z1) $|\mathcal{Z}| \geq d^{j+\varepsilon}$;

(Z2) $\Delta_{j'}(\mathcal{Z}) \leq |\mathcal{Z}|/d^{j'+\varepsilon}$ for all $j' \in [j-1]$;

(Z3) $|\mathcal{C}_e^{(j')} \cap \mathcal{C}_f^{(j')}| \leq d^{j'-\varepsilon}$ for all distinct $e, f \in \mathcal{H}$ with $d_{\mathcal{Z}}(ef) \geq 1$ and all $j' \in [\ell-1]$;

(Z4) Z is $\mathcal{C}$-free for all $Z \in \mathcal{Z}$.

Intuitively, again thinking about $\mathcal{M}$ as an edge set that behaves as if it was chosen uniformly at random among all subsets of $\mathcal{H}$ of the same size, (Z1) and (Z2) ensure that $|\{Z \in \mathcal{Z} : Z \subseteq \mathcal{M}\}|$ is close to its expectation and not dominated by rare events with large effects; observe that in this respect both conditions cannot be relaxed beyond omitting the d^{ε} factor. We require Condition (Z3) to guarantee that for all tests $Z \in \mathcal{Z}$, all edges $e \in Z$ enter $\mathcal{M}$ approximately independently. Indeed, omitting the $d^{-\varepsilon}$ factor in this condition would allow us to construct test systems $\mathcal{Z}$ where there are tests $Z \in \mathcal{Z}$ with edges $e, f \in Z$ such that whenever $\mathcal{M}$ cannot contain e due to conflicts it cannot contain f either. Condition (Z4) is also natural since tests Z which are not $\mathcal{C}$-free are never contained in $\mathcal{M}$. In fact, a $(d, \varepsilon, \mathcal{C})$-trackable test system has properties similar to those of a link of an edge in $\mathcal{C}$ provided that $\mathcal{C}$ is a suitable $(d, \ell, \Gamma, \varepsilon)$-bounded conflict system (see Lemma 2.8.1).

Theorem 2.5.2. *Assume the setup of Theorem 2.5.1 and suppose $\mathscr{Z}$ is a set of $(d, \varepsilon, \mathcal{C})$-trackable test systems for $\mathcal{H}$ of uniformity at most ℓ with $|\mathscr{Z}| \leq \exp(d^{\varepsilon^2/\ell})$. Then, there exists a $\mathcal{C}$-free matching $\mathcal{M} \subseteq \mathcal{H}$ of size $(1-\mu)n/k$ with $|\{Z \in \mathcal{Z} : Z \subseteq \mathcal{M}\}| = (1 \pm d^{-\varepsilon/900})(|\mathcal{M}|/|\mathcal{H}|)^j |\mathcal{Z}|$ for all j-uniform $\mathcal{Z} \in \mathscr{Z}$.*

We also provide a version of Theorem 2.5.2 that allows tracking of test weight functions instead of test systems. A *test function* for $\mathcal{H}$ is a function $w\colon \binom{\mathcal{H}}{j} \to [0,1]$ where $j \geq 1$ such that $w(E) = 0$ whenever $E \in \binom{\mathcal{H}}{j}$ is not a matching. We refer to j as the *uniformity* of w and we say that w is j-*uniform*. In general, for a function $w\colon A \to \mathbb{R}$ and a finite set $X \subseteq A$, we define $w(X) := \sum_{x \in X} w(x)$. If w is a j-uniform test function, we also use w to denote the extension of w to arbitrary subsets of $\mathcal{H}$ such that for all $E \subseteq \mathcal{H}$, we have $w(E) = w(\binom{E}{j})$. Note that for j-sets $E \subseteq \mathcal{H}$, there is no ambiguity since in this case, E is the only subset of E that has size j. Analogously to the definition of $(d, \varepsilon, \mathcal{C})$-trackability for test systems, we say that a j-uniform test function w for $\mathcal{H}$ is $(d, \varepsilon, \mathcal{C})$-*trackable* if the following holds.

(W1) $w(\mathcal{H}) \geq d^{j+\varepsilon}$;

(W2) $w(\{E \in \binom{\mathcal{H}}{j} : E' \subseteq E\}) \leq w(\mathcal{H})/d^{j'+\varepsilon}$ for all $j' \in [j-1]$ and $E' \in \binom{\mathcal{H}}{j'}$;

(W3) $|\mathcal{C}_e^{(j')} \cap \mathcal{C}_f^{(j')}| \leq d^{j'-\varepsilon}$ for all distinct $e, f \in \mathcal{H}$ with $w(\{E \in \binom{\mathcal{H}}{j} : e, f \in E\}) > 0$ and all $j' \in [\ell - 1]$;

(W4) $w(E) = 0$ for all $E \in \binom{\mathcal{H}}{j}$ that are not $\mathcal{C}$-free.

Theorem 2.5.3. *Assume the setup of Theorem 2.5.1 and suppose $\mathscr{W}$ is a set of $(d, \varepsilon, \mathcal{C})$-trackable test functions for $\mathcal{H}$ of uniformity at most ℓ with $|\mathscr{W}| \leq \exp(d^{\varepsilon^2/\ell})$. Then, there exists a $\mathcal{C}$-free matching $\mathcal{M} \subseteq \mathcal{H}$ of size $(1 - \mu)n/k$ with $w(\mathcal{M}) = (1 \pm d^{-\varepsilon/900})(|\mathcal{M}|/|\mathcal{H}|)^j w(\mathcal{H})$ for all j-uniform $w \in \mathscr{W}$.*

Furthermore, we deduce the following version of Theorem 2.5.3 that allows a constant relative deviation of the degrees of $\mathcal{H}$, but in turn also only yields a constant fraction of vertices that are not covered by the matching (in general, this cannot be avoided as one can see by considering a slightly unbalanced complete bipartite graph).

Theorem 2.5.4. *For all $k \geq 2$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, there exists d_0 such that the following holds for all $d \geq d_0$. Suppose $\ell \geq 2$ is an integer and suppose $\Gamma \geq 1$ is a real such that $1/\varepsilon^{\Gamma\ell} \leq d^{\varepsilon^2}$. Suppose $\mathcal{H}$ is a k-graph on $n \leq \exp(d^{\varepsilon^2/\ell})$ vertices with $(1-\varepsilon)d \leq \delta(\mathcal{H}) \leq \Delta(\mathcal{H}) \leq d$ and $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon}$, suppose $\mathcal{C}$ is a $(d, \ell, \Gamma, \varepsilon)$-bounded conflict system for $\mathcal{H}$ and suppose $\mathscr{W}$ is a set of $(d, \varepsilon, \mathcal{C})$-trackable test functions for $\mathcal{H}$ of uniformity at most $1/\varepsilon^{1/3}$ with $|\mathscr{W}| \leq \exp(d^{\varepsilon^2/\ell})$.*

Then, there exists a $\mathcal{C}$-free matching $\mathcal{M} \subseteq \mathcal{H}$ of size at least $(1 - \sqrt{\varepsilon})\frac{n}{k}$ with $w(\mathcal{M}) = (1 \pm \sqrt{\varepsilon})(|\mathcal{M}|/|\mathcal{H}|)^j w(\mathcal{H})$ for all j-uniform $w \in \mathscr{W}$.

We remark that further variations of our main theorem can be obtained building on Theorem 2.9.2. For example assume the setup of Theorem 2.5.1 and consider a j-graph $\mathcal{X}$ whose vertices are edges of $\mathcal{H}$ such that $\Delta_{j'}(\mathcal{X}) \leq d_0/d^{j'}$ for some d_0 and all $j' \in [j-1]_0$. Then, combining Theorem 2.9.2 with Lemma 2.8.5 shows that for all $\varepsilon > 0$, there exists an almost perfect matching with at most $\max\{d_0/d^{j-\varepsilon}, d^{\varepsilon}\}$ edges of $\mathcal{X}$ as subsets of the matching. In a sense, such a hypergraph $\mathcal{X}$ resembles a test system that only satisfies a weaker trackability condition which entails that we can only guarantee a crude upper bound on the number of edges of $\mathcal{X}$ that are subsets of the matching.

Finally, we also prove a counting version of Theorem 1.1.2. Before we present a formal statement that provides a lower bound for the number of large conflict-free matchings, let us consider some heuristic for how many almost-perfect $\mathcal{C}$-free matchings of size $m = (1-\mu)n/k$ one may expect at least in the setting of Theorem 2.5.1. Since we are interested in almost-perfect matchings, we assume that μ is sufficiently small, for example $\mu \leq d^{-\varepsilon^3}$. Suppose that $\mathcal{M}$ is chosen uniformly at random among all edge sets in $\binom{\mathcal{H}}{m}$. The edge set $\mathcal{M}$ is a matching if there is a set of km vertices of $\mathcal{H}$ that are all contained in exactly one edge $e \in \mathcal{M}$. Every edge of $\mathcal{H}$ is an edge of $\mathcal{M}$ with probability $m/|\mathcal{H}| \approx d^{-1}$. Hence, for a fixed vertex $v \in V(\mathcal{H})$, the expected number of edges containing v is approximately $d_{\mathcal{H}}(v)/d \approx 1$, so by the Poisson paradigm, we estimate that the probability of the event that v is contained in exactly one edge $e \in \mathcal{M}$ is approximately $1/\mathrm{e}$.

Thus, for a fixed $U \in \binom{V(\mathcal{H})}{km}$, we may expect all vertices $u \in U$ to be contained in exactly one edge $e \in \mathcal{M}$ with probability $\exp(-km)$ and hence we may expect $\mathcal{M}$ to be a matching with probability roughly $\binom{n}{km}\exp(-km) = (1\pm\sqrt{\mu})^{km}\exp(-km) \approx \exp(-km)$. Since there were $\binom{|\mathcal{H}|}{m} \approx (\mathrm{e}|\mathcal{H}|/m)^m \approx (\mathrm{e}d)^m$ choices for $\mathcal{M}$, this suggests that there are roughly $(\mathrm{e}d)^m \cdot \exp(-km) = (d/\exp(k-1))^m$ matchings of size m in $\mathcal{H}$.

To estimate the number of matchings of size m that are $\mathcal{C}$-free, we may again employ the Poisson paradigm. For all $j \in [\ell]_2$, the number of conflicts $C \in \mathcal{C}^{(j)}$ is $\sum_{e\in\mathcal{H}} d_{\mathcal{C}^{(j)}}(e)/j \leq |\mathcal{H}|\Delta(\mathcal{C}^{(j)})/j$. Hence, again using that every edge of $\mathcal{H}$ is an edge of $\mathcal{M}$ with probability roughly $m/|\mathcal{H}|$, the expected number of conflicts of arbitrary size that are a subset of $\mathcal{M}$ is heuristically at most $m\sum_{j\in[\ell]_2}\frac{m^{j-1}\Delta(\mathcal{C}^{(j)})}{j|\mathcal{H}|^{j-1}} \leq m\sum_{j\in[\ell]_2}\frac{\Delta(\mathcal{C}^{(j)})}{jd^{j-1}}$. Thus, the Poisson paradigm suggests that $\mathcal{M}$ is $\mathcal{C}$-free with probability at least $\exp\big(-m\sum_{j\in[\ell]_2}\frac{\Delta(\mathcal{C}^{(j)})}{jd^{j-1}}\big)$. Combining this with our estimation for the number of matchings of size m in $\mathcal{H}$, this yields

$$\left(\frac{d}{\exp\big(k-1+\sum_{j\in[\ell]_2}\frac{\Delta(\mathcal{C}^{(j)})}{jd^{j-1}}\big)}\right)^m$$

as an approximate lower bound for the number of $\mathcal{C}$-free matchings of size m in $\mathcal{H}$.

Theorem 2.5.5. *For all $k,\ell \geq 2$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, there exists d_0 such that the following holds for all $d \geq d_0$. Suppose $\mathcal{H}$ is a k-graph on $n \leq \exp(d^{\varepsilon^3})$ vertices with $(1-d^{-\varepsilon})d \leq \delta(\mathcal{H}) \leq \Delta(\mathcal{H}) \leq d$ and $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon}$ and suppose $\mathcal{C}$ is a $(d,\ell,\ell,\varepsilon)$-bounded conflict system for $\mathcal{H}$.*

Then, the number of $\mathcal{C}$-free matchings $\mathcal{M} \subseteq \mathcal{H}$ of size $m := (1-d^{-\varepsilon^3})n/k$ is at least

$$\left(\frac{(1-d^{-\varepsilon^4})d}{\exp\big(k-1+\sum_{j\in[\ell]_2}\frac{\Delta(\mathcal{C}^{(j)})}{jd^{j-1}}\big)}\right)^m.$$

It is known [88] that the number of perfect matchings of a d-regular k-graph on n vertices with small 2-degrees is at most $((1+o(1))d/\exp(k-1))^{n/k}$. This can be proved using the so-called entropy method. It would be interesting to find out whether this method can also be used to provide an upper bound on the number of conflict-free matchings, complementing our lower bound from Theorem 2.5.5. This seems challenging, even in the case of Steiner triple systems with girth at least 7, see the discussions in [18, 49, 80].

2.6 Constructing the matching

2.6.1 The setting

Let us now describe the setting for our main proof and the random greedy algorithm we analyze. Here and in the subsequent sections, we work with the following setup. Fix $k \geq 2$, $\varepsilon > 0$ that is sufficiently small in terms of $1/k$ and $d \geq 1$ that is sufficiently large in terms of $1/\varepsilon$ and k. Suppose $\ell \geq 2$ is an integer and suppose $\Gamma \geq 1$ and $0 < \mu \leq 1/\ell$ are reals such that

$$\frac{1}{\mu^{\Gamma\ell}} \leq d^{\varepsilon^{3/2}}. \tag{2.6.1}$$

Let $\mathcal{H}$ denote a k-graph on $n \leq \exp(d^{\varepsilon/(400\ell)})$ vertices such that $(1-d^{-\varepsilon})d \leq \delta(\mathcal{H}) \leq \Delta(\mathcal{H}) \leq d$ and $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon}$. Let $\mathcal{C}$ denote a $(d, \ell, \Gamma, \varepsilon)$-bounded[1] conflict system for $\mathcal{H}$ such that in addition to the $(d, \ell, \Gamma, \varepsilon)$-boundedness, the following conditions are satisfied.

(C6) $d^{j-1-\varepsilon/100} \leq (1-d^{-\varepsilon})\Delta(\mathcal{C}^{(j)}) \leq \delta(\mathcal{C}^{(j)})$ for all $j \in [\ell]_2$ with $\mathcal{C}^{(j)} \neq \emptyset$;

(C7) $|\mathcal{C}_e^{(j)} \cap \mathcal{C}_f^{(j)}| \leq d^{j-\varepsilon}$ for all disjoint $e, f \in \mathcal{H}$ with $\{e, f\} \notin \mathcal{C}^{(2)}$ and all $j \in [\ell-1]$;

(C8) C is a matching for all $C \in \mathcal{C}$;

(C9) $C_1 \not\subseteq C_2$ for all distinct $C_1, C_2 \in \mathcal{C}$.

Considering the links of the conflict graph, note that these are almost $(d, \varepsilon, \mathcal{C})$-trackable test systems in the following sense. Condition (C8) enforces that all semiconflicts are matchings, the bound $d^{j-1-\varepsilon/100} \leq \delta(\mathcal{C}^{(j)})$ that we impose for all $j \in [\ell]_2$ plays a similar role as (Z1), Condition (C3) translates to a property similar to (Z2), Conditions (C5), (C7) and (C9) together yield (Z3) and Condition (C9) corresponds to (Z4). Let $\mathscr{Z}_0$ denote a set of $(d, \varepsilon, \mathcal{C})$-trackable test systems for $\mathcal{H}$ such that $|\mathscr{Z}_0| \leq \exp(d^{\varepsilon/(400\ell)})$. Note that besides requiring that $\mathcal{C}$ additionally satisfies the conditions (C6)–(C9), the conditions that we impose here are weaker than those in Theorem 2.5.2. The weaker bound $1/\mu^{\Gamma\ell}$ here and the bound on the number of nonempty uniform subgraphs $\mathcal{C}^{(j)}$ in Theorem 2.5.2 however allow us to deduce this theorem from the analysis of the construction for conflict graphs additionally satisfying (C6)–(C9) that we introduce in Section 2.6.2. The fact that we also allow more vertices and test systems here is useful for proving Theorems 2.5.3 and 2.5.4.

With the setup for this section, which we also keep for the subsequent sections in mind, we conclude this sections with some further remarks. The bound (2.6.1), which may be thought of as a way to state that d is sufficiently large in terms of ℓ, Γ and $1/\mu$, is crucial for many bounds that we derive throughout the chapter without explicitly referring to it. More specifically, we frequently use it to bound terms that depend on ℓ, Γ or μ from above using powers of d with a suitably small fraction of ε as their exponent. Besides directly using $1/\mu^{\Gamma\ell} \leq d^{\varepsilon^{3/2}}$, we for example use that $\ell^\ell \leq d^{\varepsilon/1000}$ and $\exp(\Gamma) \leq d^{\varepsilon/1000}$. Moreover, we often use that for all $j \in [\ell]_2$, we have $\Delta(\mathcal{C}^{(j)}) \leq \Gamma d^{j-1}$ as an immediate consequence of (C2).

2.6.2 The algorithm

We construct random matchings $\emptyset = \mathcal{M}(0) \subseteq \mathcal{M}(1) \subseteq \ldots$ in $\mathcal{H}$ as follows. As an initialization during step 0, we set $\mathcal{M}(0) := \emptyset$. Then, we proceed iteratively where in every step $i \geq 1$ we obtain $\mathcal{M}(i)$ by adding an edge $e(i)$ to $\mathcal{M}(i-1)$ that is chosen uniformly at random among all edges that are *available* in the sense that they can be added without generating a subset of $\mathcal{M}(i)$ that is a conflict or a nonempty intersection of two distinct edges in $\mathcal{M}(i)$. If for some step $i \geq 0$ there are no available edges, we terminate the construction. For every step $i \geq 0$, we use $V(i) := V \setminus \bigcup_{e \in \mathcal{M}(i)} e$ to denote the set of vertices that are not covered by $\mathcal{M}(i)$. To keep track of the available edges, we define the random k-graphs $\mathcal{H} = \mathcal{H}(0) \supseteq \mathcal{H}(1) \supseteq \ldots$, where in every step $i \geq 0$, the

[1]In fact, when working with the setup provided here, we do not need that $|\{j \in [\ell]_2 : \mathcal{C}^{(j)} \neq \emptyset\}| \leq \Gamma$.

vertex set of $\mathcal{H}(i)$ is $V(i)$ and the edge set of $\mathcal{H}(i)$ is the set of edges that are available for addition to the matching $\mathcal{M}(i)$. For all $e \in \mathcal{H}(i)$, as a special case of a more general notation that we introduce in Section 2.7, we use $\mathcal{C}_e^{[1]}(i)$ to denote the random subgraph of $\mathcal{C}_e$ with vertex set $V(\mathcal{C}_e)$ and edge set

$$\mathcal{C}_e^{[1]}(i) = \{C \in \mathcal{C}_e : |C \cap \mathcal{H}(i)| = 1 \text{ and } |C \cap \mathcal{M}(i)| = |C| - 1\},$$

that is, the set of all semiconflicts C that stem from conflicts containing e where all edges in C except one belong to $\mathcal{M}(i)$ and where the remaining edge is available. Overall, we make our random choices according to Algorithm 2.6.1.

Algorithm 2.6.1: Construction of the matching

$\mathcal{M}(0) \leftarrow \emptyset$
$V(0) \leftarrow V(\mathcal{H})$
$\mathcal{H}(0) \leftarrow \mathcal{H}$
$i \leftarrow 1$
while $\mathcal{H}(i-1) \neq \emptyset$ **do**
    choose $e(i) \in \mathcal{H}(i-1)$ uniformly at random
    $\mathcal{M}(i) \leftarrow \mathcal{M}(i-1) \cup \{e(i)\}$
    $V(i) \leftarrow V(i-1) \setminus e(i)$
    $E_C(i) \leftarrow \{e \in \mathcal{H}(i-1) : \{e\} = C \setminus \mathcal{M}(i-1) \text{ for some } C \in \mathcal{C}_{e(i)}^{[1]}(i-1)\}$
    $\mathcal{H}_C(i) \leftarrow (V(i-1), \mathcal{H}(i-1) \setminus E_C(i))$
    $\mathcal{H}(i) \leftarrow \mathcal{H}_C(i)[V(i)]$
    $i \leftarrow i+1$
end

We refer to the assignments before the first iteration of the loop as step 0, for $i \geq 1$, we refer to the i-th iteration of the loop as step i and we use $\mathfrak{F}(i)$ to denote the i-th element of the natural filtration associated with this random process.

Let $m := (1-\mu)n/k$. As an immediate consequence of Theorem 2.9.2 in Section 2.9, we obtain the following statement.

Theorem 2.6.2. *With probability at least $1 - \exp(-d^{\varepsilon/(500\ell)})$, the following happens. Algorithm 2.6.1 runs for at least m steps and hence generates a matching of size at least m and additionally, we have*

$$|\{Z \in \mathcal{Z} : Z \subseteq \mathcal{M}(m)\}| = (1 \pm d^{-\varepsilon/75}) \left(\frac{mk}{dn}\right)^j |\mathcal{Z}|$$

for all $j \in [\ell]$ and all j-uniform $\mathcal{Z} \in \mathscr{Z}_0$.

In Section 2.10, we show that Theorems 2.5.2, 2.5.3 and 2.5.4 are consequences of Theorem 2.6.2.

2.7 Key random variables and trajectories

In this section, we define key random variables for the analysis of the conflict-free matching process described in Section 2.6. Additionally, we provide some intuition for their evolution during the process that leads to idealized trajectories that certain quantities typically follow.

2.7.1 Key random variables

The process increases the size of the matching as long as there are available edges, that is, as long as $|\mathcal{H}(i)| \geq 1$, so we are interested in analyzing the availability of edges. To account for the fact that an edge becomes unavailable for the matching when an edge containing one of its vertices is added to the matching, one set of key random variables that we wish to investigate are the random sets of edges of $\mathcal{H}(i)$ that contain a given vertex $v \in V(\mathcal{H})$. For $v \in V(\mathcal{H})$, we define

$$\mathcal{D}_v(i) := \{e \in \mathcal{H}(i) : v \in e\}.$$

To account for the $\mathcal{C}$-free condition, for all $e \in \mathcal{H}$, we are interested in tracking the number of semiconflicts in $\mathcal{C}_e$ that have already partially entered the matching in the sense that they contain a given number of edges in the matching. As tracking these is a special case of keeping track of how more general uniform hypergraphs whose vertex sets are subsets of $\mathcal{H}$ behave with respect to the matching, we can treat the collection of these sets as another set of test systems. We define

$$\mathscr{C} := \{\mathcal{C}_e^{(j)} : e \in \mathcal{H}, j \in [\ell-1], \mathcal{C}^{(j+1)} \neq \emptyset\} \quad \text{and} \quad \mathscr{Z} := \mathscr{Z}_0 \cup \mathscr{C}.$$

Observe that for all $\mathcal{Z} \in \mathscr{C}$, the pair (e, j) with $\mathcal{Z} = \mathcal{C}_e^{(j)}$ is unique. Indeed, as $d^{j-\varepsilon/100} \leq \delta(\mathcal{C}^{(j+1)}) \leq |\mathcal{C}_e^{(j)}|$, $\mathcal{Z}$ is not empty, so j is uniquely determined and we have $\{e\} = \mathcal{H} \setminus V(\mathcal{Z})$ determining e. Also observe that $\mathscr{C} \cap \mathscr{Z}_0 = \emptyset$ due to (C3) and (Z1), so there will never be confusion whether $\mathcal{Z} \in \mathscr{Z}$ is an element of $\mathscr{Z}_0$ or $\mathscr{C}$. These observations are convenient for the following considerations of test systems $\mathcal{Z} \in \mathscr{Z}$ that at some point become irrelevant for the process.

If an edge $e \in \mathcal{H}$ is not present in $\mathcal{H}(i)$ for some $i \geq 1$, it is no longer relevant for the process, so we no longer have to consider it in our analysis. Thus, for $i \geq 0$, we define

$$\mathscr{C}(i) := \{\mathcal{C}_e^{(j)} : e \in \mathcal{H}(i), j \in [\ell-1], \mathcal{C}^{(j+1)} \neq \emptyset\}, \quad \text{and} \quad \mathscr{Z}(i) := \mathscr{Z}_0 \cup \mathscr{C}(i).$$

For $\mathcal{Z} \in \mathscr{Z}$ and $e \in \mathcal{H}$, if $\mathcal{Z} = \mathcal{C}_f^{(j)}$ for some $j \in [\ell-1]$ and $f \in \mathcal{H}$ with $e \cap f \neq \emptyset$ or $\{e, f\} \in \mathcal{C}^{(2)}$, then e entering the matching in some step $i \geq 1$ in the sense that $e(i) = e$ entails $\mathcal{Z}$ becoming irrelevant for the process and hence not being present in $\mathscr{Z}(i)$, in a sense getting evicted from $\mathscr{Z}(i-1)$. In this case, that is, if $\mathcal{Z} = \mathcal{C}_f^{(j)}$ for some $j \in [\ell-1]$ and $f \in \mathcal{H}$ with $e \cap f \neq \emptyset$ or $\{e, f\} \in \mathcal{C}^{(2)}$, we say that $e \in \mathcal{H}$ is an *immediate evictor* for $\mathcal{Z} \in \mathscr{Z}$. We write $e \nearrow \mathcal{Z}$ to mean that e is an immediate evictor for $\mathcal{Z}$ and $e \not\nearrow \mathcal{Z}$ to mean that e is not an immediate evictor for $\mathcal{Z}$. Note that besides this immediate eviction $\mathcal{Z} = \mathcal{C}_f^{(j)}$ also becomes irrelevant whenever f and $e(i)$ are the only edges in a conflict of arbitrary size that are not in $\mathcal{M}(i-1)$. Finally, as the uniformity of a test system is crucial for its behavior, for $i \geq 0$ and $j \in [\ell]$, we define

$$\mathscr{Z}_0^{(j)} := \{\mathcal{Z} \in \mathscr{Z}_0 : \mathcal{Z} \text{ is a } j\text{-graph}\}, \quad \mathscr{C}^{(j)} := \{\mathcal{C}_e^{(j)} \in \mathscr{C} : e \in \mathcal{H}\}, \quad \mathscr{Z}^{(j)} := \mathscr{Z}_0^{(j)} \cup \mathscr{C}^{(j)}$$
$$\mathscr{C}^{(j)}(i) := \{\mathcal{C}_e^{(j)} \in \mathscr{C}(i) : e \in \mathcal{H}(i)\} \quad \text{and} \quad \mathscr{Z}^{(j)}(i) := \mathscr{Z}_0^{(j)} \cup \mathscr{C}^{(j)}(i).$$

We introduce the following notation. For a (not necessarily uniform) hypergraph $\mathcal{X}$ with $V(\mathcal{X}) \subseteq \mathcal{H}$ and integer $i, s \geq 0$, the *partially matched subgraph* $\mathcal{X}^{[s]}(i)$ of $\mathcal{X}$ with parameter s at step i is the random hypergraph with vertex set $V(\mathcal{X})$ and

$$\mathcal{X}^{[s]}(i) = \{X \in \mathcal{X} : |X \cap \mathcal{H}(i)| = s \text{ and } |X \cap \mathcal{M}(i)| = |X| - s\}.$$

Here, we use square brackets to avoid ambiguity regarding our notation $\mathcal{X}^{(j)}$ for the j-uniform subgraph of $\mathcal{X}$ with $V(\mathcal{X}^{(j)}) = V(\mathcal{X})$. Note that for all $e \in \mathcal{H}$ and $\mathcal{Z} = \mathcal{C}_e$ this definition of $\mathcal{Z}^{[1]}(i) = \mathcal{C}_e^{[1]}(i)$ coincides with that given in Section 2.6.

For $\mathcal{Z} \in \mathscr{Z}$, we are particularly interested in the random hypergraphs $\mathcal{Z}^{[s]}(i)$. Indeed, the random hypergraphs $\mathcal{C}_e^{[1]}(i)$ play a crucial role in Algorithm 2.6.1, for $\mathcal{Z} \in \mathscr{Z}_0$, we are interested in $\mathcal{Z}^{[0]}(m)$ and for all $j \in [\ell]$, $\mathcal{Z} \in \mathscr{Z}^{(j)}$, $i \geq 1$ and $s \in [j-1]_0$, the tests that enter when transitioning from $\mathcal{Z}^{[s]}(i-1)$ to $\mathcal{Z}^{[s]}(i)$ are the result of adding an edge $e \in \mathcal{H}$ to the matching that is an element of a test in $\mathcal{Z}^{[s+1]}(i-1)$.

Occasionally, we have to account for the fact that the test systems $\mathcal{C}_f^{(j)}$ in $\mathscr{C}$ are, in contrast to those in $\mathscr{Z}_0$, not $(d, \varepsilon, \mathcal{C})$-trackable as (C2) implies that they are too small to satisfy (Z1) at least by a factor of d^{ε}/Γ.

Following the intuition that every edge of $\mathcal{H}$ ends up in the matching roughly independently with probability d^{-1}, we estimate $|\mathcal{Z}^{[s]}(m)| \approx |\mathcal{Z}|/d^{j-s}$ for all j-uniform $\mathcal{Z} \in \mathscr{Z}$. Thus, if $|\mathcal{Z}|$ is not significantly larger than d^{j-s}, our analysis does not provide concentration around the expectation for $|\mathcal{Z}^{[s]}(m)| \approx |\mathcal{Z}|/d^{j-s}$ and hence the smaller size of the test systems $\mathcal{C}_f^{(j)} \in \mathscr{Z}$ with $e \in \mathcal{H}$ and $j \in [\ell-1]$ results in weaker tracking in the sense that we can only guarantee concentration for the random variables $|\mathcal{C}_f^{(j)[s]}(i)|$ with $s \in [\ell]$ and not for $s = 0$ (Note that whenever $s \geq j+1$, we trivially have $\mathcal{C}_f^{(j)[s]}(i) = \emptyset$ for all $i \geq 0$). However, this is sufficient for us because adding an edge of $\mathcal{H}$ that is an element of a semiconflict in $\mathcal{C}_f^{(j)[1]}$ to the matching makes f unavailable and hence all $\mathcal{C}_f^{(j)}$ with $j \in [\ell-1]$ irrelevant.

2.7.2 Intuition

Generally, if $\mathcal{X}(i)$ is a (random) hypergraph or a set, we define $|\mathcal{X}|(i) := |\mathcal{X}(i)|$ such that for a sequence $\mathcal{X}(0), \mathcal{X}(1), \ldots$, we have a notation for the elements of the associated sequence of sizes that uses a common symbol indexed with i. As before, we write such intuitively time-related indices that represent different stages in the evolution of a parameter or iterations in an algorithm as arguments instead of indices to distinguish them from other indices.

The heuristic arguments in this section start with the assumption that for all $i \geq 0$, edges of $\mathcal{H}$ are included in $\mathcal{M}(i)$ approximately independently with probability $i/|\mathcal{H}|$. With a similar intuition for $V(i)$, we obtain

$$\mathbb{P}[e \in \mathcal{M}(i)] \approx \frac{|\mathcal{M}|(i)}{|\mathcal{H}|(0)} \approx \frac{ik}{dn} =: \hat{p}_M(i) \quad \text{and} \quad \mathbb{P}[v \in V(i)] \approx \frac{|V(i)|}{|V(0)|} = 1 - \frac{ik}{n} =: \hat{p}_V(i)$$

for all $e \in \mathcal{H}$ and $v \in V(\mathcal{H})$. Since we wish to show that $\mathcal{H}(i-1)$ typically remains nonempty for all $i \in [m]$, we are interested in the size of $\mathcal{H}(i)$.

For $i \geq 0$, an edge $e \in \mathcal{H}$ is an edge of $\mathcal{H}(i)$ if and only if none of its vertices is covered by the matching $\mathcal{M}(i)$ and additionally, there is no conflict $C \in \mathcal{C}$ with $e \in C$ that forbids the addition of e to $\mathcal{M}(i)$ in the sense that $C \setminus \mathcal{M}(i) = \{e\}$. For all $j \in [\ell]_2$, there are approximately $\Delta(\mathcal{C}^{(j)})$ conflicts $C \in \mathcal{C}^{(j)}$ with $e \in C$ and for all conflicts $C \in \mathcal{C}$, we have $C \not\subseteq \mathcal{M}(i)$, so $C \setminus \mathcal{M}(i) = \{e\}$ happens if and only if $C \setminus \{e\} \subseteq \mathcal{M}(i)$. Hence, the expected number of such conflicts that forbid the addition of e during step i is

$$\mathbb{E}[|\{C \in \mathcal{C} : e \in C \text{ and } C \setminus \mathcal{M}(i) = \{e\}\}|] \approx \sum_{j \in [\ell]_2} \Delta(\mathcal{C}^{(j)}) \cdot \hat{p}_M(i)^{j-1} =: \hat{\Gamma}(i).$$

Thus, assuming approximate independence of relevant events the Poisson paradigm suggests

$$\mathbb{P}[e \in \mathcal{H}(i)] \approx \hat{p}_V(i)^k \cdot \exp(-\hat{\Gamma}(i)).$$

This yields the following idealized trajectories. For $|\mathcal{H}|(i)$, we estimate

$$\mathbb{E}[|\mathcal{H}|(i)] \approx \frac{dn}{k} \cdot \mathbb{P}[e \in \mathcal{H}(i)] \approx \frac{dn}{k} \cdot \hat{p}_V(i)^k \cdot \exp(-\hat{\Gamma}(i)) =: \hat{h}(i). \tag{2.7.1}$$

Note that $\hat{\Gamma}(0) \leq \ldots \leq \hat{\Gamma}(n/k)$. Hence, as we are interested in a setting where Algorithm 2.6.1 typically does not terminate too early, (2.7.1) shows the need for bounding $\hat{\Gamma}(n/k)$ from above and thus illustrates the importance of (C2), which provides Γ as an upper bound for $\hat{\Gamma}(n/k)$. Similarly as for $|\mathcal{H}|(i)$, for all $v \in V(\mathcal{H})$, where we only care about $\mathcal{D}_v(i)$ as long as $v \in V(i)$, we estimate

$$\begin{aligned}\mathbb{E}[|\mathcal{D}_v|(i) \mid v \in V(i)] &= \sum_{\substack{e \in \mathcal{H}: \\ v \in e}} \mathbb{P}[e \in \mathcal{H}(i) \mid v \in V(i)] \approx \frac{1}{\hat{p}_V(i)} \sum_{\substack{e \in \mathcal{H}: \\ v \in e}} \mathbb{P}[e \in \mathcal{H}(i)] \\ &\approx d \cdot \hat{p}_V(i)^{k-1} \cdot \exp(-\hat{\Gamma}(i)) =: \hat{d}(i).\end{aligned}$$

Finally, for all $j \in [\ell]$, $\mathcal{Z} \in \mathscr{Z}^{(j)}$ and $s \in [j]_0$ with $s \geq \mathbb{1}_{\mathscr{C}}(\mathcal{Z})$, where we only care about $\mathcal{Z}^{[s]}(i)$ as long as $\mathcal{Z} \in \mathscr{Z}(i)$, we estimate

$$\begin{aligned}\mathbb{E}[|\mathcal{Z}^{[s]}|(i) \mid \mathcal{Z} \in \mathscr{Z}(i)] &\approx \sum_{Z \in \mathcal{Z}} \sum_{X \in \binom{Z}{s}} \mathbb{P}\Big[\bigcap_{f \in X} \{f \in \mathcal{H}(i)\} \cap \bigcap_{f \in Z \setminus X} \{f \in \mathcal{M}(i)\}\Big] \\ &\approx |\mathcal{Z}| \cdot \binom{j}{s} \cdot \big(\hat{p}_V(i)^k \cdot \exp(-\hat{\Gamma}(i))\big)^s \cdot \hat{p}_M(i)^{j-s} = |\mathcal{Z}| \cdot \hat{z}_{j,s}(i),\end{aligned}$$

where

$$\hat{z}_{j,s}(i) := \binom{j}{s} \cdot \left(\hat{p}_V(i)^k \cdot \exp(-\hat{\Gamma}(i))\right)^s \cdot \hat{p}_M(i)^{j-s}.$$

Note that $\hat{z}_{j,0}(i) = \hat{p}_M(i)^j$ and hence $\hat{z}_{j,0}(m) = \left(\frac{mk}{dn}\right)^j$, which is the term we have in Theorem 2.6.2.

As a consequence of the construction of $E_C(i)$ in Algorithm 2.6.1, which ensures that the matchings $\mathcal{M}(0), \mathcal{M}(1), \dots$ are $\mathcal{C}$-free, random hypergraphs $\mathcal{C}_e^{[1]} = \bigcup_{j\in[\ell]} \mathcal{C}_e^{(j)[1]}$ with $e \in \mathcal{H}$ directly influence the construction of the matchings $\mathcal{M}(i)$ and hence the random hypergraphs $\mathcal{C}_e^{[1]}(i)$ are particularly important. Similarly as above, for all $e \in \mathcal{H}$, we obtain

$$\begin{aligned}\mathbb{E}[|\mathcal{C}_e^{[1]}|(i) \mid e \in \mathcal{H}(i)] &= \sum_{j\in[\ell-1]} \mathbb{E}[|\mathcal{C}_e^{(j)[1]}|(i) \mid e \in \mathcal{H}(i)] \approx \sum_{j\in[\ell-1]} d_{\mathcal{C}^{(j+1)}}(e) \cdot \hat{z}_{j,1}(i) \\ &\approx \sum_{j\in[\ell-1]} \Delta(\mathcal{C}^{(j+1)}) \cdot \hat{z}_{j,1}(i) =: \hat{c}(i).\end{aligned}$$

In Section 2.9, we formally prove that $|\mathcal{H}|(i)$, $|\mathcal{D}_v|(i)$, $|\mathcal{Z}^{[s]}|(i)$ and $|\mathcal{C}_e^{[1]}|(i)$ indeed typically follow the idealized trajectories $\hat{h}(i)$, $\hat{d}(i)$, $|\mathcal{Z}| \cdot \hat{z}_{j,s}(i)$ and $\hat{c}(i)$, respectively.

2.8 Bounding local interactions

In preparation for the proof of Theorem 2.9.2 (of which Theorem 2.6.2 is a consequence), we consider certain configurations that consist of one or two conflicts or tests. As these configurations mediate local interactions between edges of $\mathcal{H}$ concerning their availability, they are relevant for analyzing the impact a particular choice of $e(i)$ in some step i may have. Intuitively, our conditions for the conflict system $\mathcal{C}$ and the test systems $\mathcal{Z} \in \mathscr{Z}$ guarantee that initially, these configurations are spread out. We use this section to formally define what this means and we prove that this spreadness typically persists during the iterative construction of the matching.

Recall that as defined in the previous section, $\mathcal{C}$ is a conflict system for $\mathcal{H}$ and $\mathscr{Z}$ is the set of all test systems and all links of the uniform subgraphs $\mathcal{C}^{(2)}, \dots, \mathcal{C}^{(\ell)}$. For $\mathcal{Z} \in \mathscr{Z}$, $e, f \in \mathcal{H}$ and $v \in V(\mathcal{H})$, we consider the hypergraphs $\mathcal{Z}_v$, $\mathcal{Z}_e$, $\mathcal{Z}_{e,2}$, $\mathcal{Z}_2$, $\mathcal{C}_{e,2}$ and $\mathcal{C}^\star_{e,f,2}$ with vertex set $\mathcal{H}$ whose edges represent different types of *local interactions* in the sense that the following holds.

(i) $\mathcal{Z}_v = \{Z \in \mathcal{Z} : Z \cap \mathcal{D}_v(0) \neq \emptyset\}$;

(ii) $\mathcal{Z}_e = \{Z \setminus \{e\} : Z \in \mathcal{Z}, e \in Z\}$;

(iii) $\mathcal{Z}_{e,2} = \{Z \cup C : Z \in \mathcal{Z}, C \in \mathcal{C}_e, Z \cap C \neq \emptyset\}$;

(iv) $\mathcal{Z}_2 = \{Z \cup C : Z \in \mathcal{Z}, C \in \mathcal{C}, |Z \cap C| \geq 2, g \not\nearrow \mathcal{Z} \text{ for all } g \in C \setminus Z\}$;

(v) $\mathcal{C}_{e,2} = \{C_1 \cup C_2 : C_1, C_2 \in \mathcal{C}_e, C_1 \neq C_2, C_1 \cap C_2 \neq \emptyset\}$;

(vi) $\mathcal{C}^\star_{e,f,2} = \{C_f \in \mathcal{C}_f : |C_f| \geq 2, g \in C_f \text{ for some } \{g\} \in \mathcal{C}_e^{(1)}\}$.

Here, $\mathcal{Z}_e$ is again the link of e in $\mathcal{Z}$ and thus coincides with our previously introduced notation. Recall that on page 33 for a (not necessarily uniform) hypergraph $\mathcal{X}$

with $V(\mathcal{X}) \subseteq \mathcal{H}$ and integers $i, s \geq 0$, we introduced the random hypergraph $\mathcal{X}^{[s]}(i)$ with vertex set $V(\mathcal{X})$ and

$$\mathcal{X}^{[s]}(i) = \{X \in \mathcal{X} : |X \cap \mathcal{H}(i)| = s \text{ and } |X \cap \mathcal{M}(i)| = |X| - s\}.$$

For $\mathcal{Z} \in \mathscr{Z}$, $e, f \in \mathcal{H}$, $v \in V(\mathcal{H})$ $i \geq 0$, $j \in [2\ell]$ and $s \geq 0$, this yields random hypergraphs

$$\mathcal{Z}_v^{[s]}(i), \quad \mathcal{Z}_e^{[s]}(i), \quad \mathcal{Z}_{e,2}^{[s]}(i), \quad \mathcal{Z}_{e,2}^{(j)[s]}(i), \quad \mathcal{Z}_2^{[s]}(i), \quad \mathcal{Z}_2^{(j)[s]}(i),$$
$$\mathcal{C}_{e,2}^{[s]}(i), \quad \mathcal{C}_{e,2}^{(j)[s]}(i), \quad \mathcal{C}_{e,f,2}^{\star[s]}(i), \quad \mathcal{C}_{e,f,2}^{\star(j)[s]}(i).$$

For configurations that yield edges of these random hypergraphs that are particularly important in Section 2.9, see Figure 2.1.

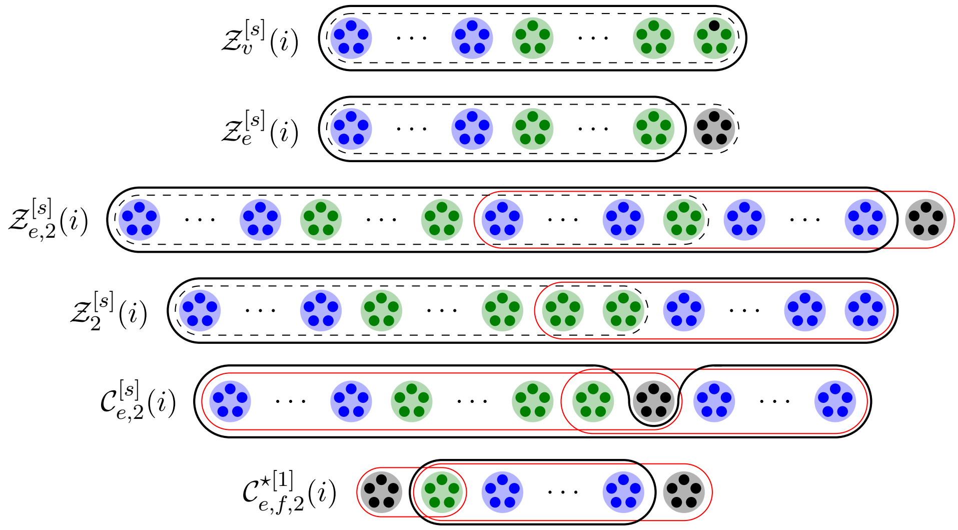


Figure 2.1: For $\mathcal{Z} \in \mathscr{Z}$, $v \in V(\mathcal{H})$, $e \in \mathcal{H}$, $i \geq 0$ and $s \in [\ell]$: Possible edges of the respective random hypergraphs are represented by a solid thick black outline. Dashed outlines represent tests in $\mathcal{Z}$ and solid red outlines represent conflicts. Blue dots and discs represent vertices covered by and edges in $\mathcal{M}(i)$, green dots and discs represent vertices and edges in $\mathcal{H}(i)$, black dots and grey discs represent vertices and edges that are fixed by the choice of e or v. The number of green discs is s.

The goal of this section is to show that during the first m steps of our construction, these random hypergraphs typically never have too many edges.

For $j \geq 1$, $d_0 \geq 0$ and $\delta \in [0,1]$, we say that a j-graph $\mathcal{X}$ is (d_0, δ)-*spread* if $\Delta_{j'}(\mathcal{X}) \leq \delta^{j'} d_0$ for all $j' \in [j-1]_0$. Our notions of $(d, \varepsilon, \mathcal{C})$-trackability and $(d, \ell, \Gamma, \varepsilon)$-boundedness were carefully chosen to imply (d_0, δ)-spreadness of the introduced hypergraphs as detailed in the following Lemma 2.8.3. Hence, ensuring that this lemma holds is one of the key motivations that leads to the conditions (C1)–(C5). To prove Lemma 2.8.3, let us first observe that the hypergraphs $\mathcal{Z} \in \mathscr{Z}$ share relevant properties that we collect in the following lemma. In particular, the elements of $\mathscr{Z}_0$ and $\mathscr{C}$ share these properties, which substantiates our approach to often treat them similarly.

Lemma 2.8.1. *Let $j \in [\ell]$ and $\mathcal{Z} \in \mathscr{Z}^{(j)}$. Then, the following holds.*
(i) $|\mathcal{Z}| \geq d^{j-\varepsilon/5}$;
(ii) $\Delta_{j'}(\mathcal{Z}) \leq |\mathcal{Z}|/d^{j'+4\varepsilon/5}$ *for all* $j' \in [j-1]$;
(iii) *if* $j = 1$, *then* $|\{\{e\} \in \mathcal{Z} : v \in e\}| \leq |\mathcal{Z}|/d^{4\varepsilon/5}$ *for all* $v \in V(\mathcal{H})$;
(iv) $|\mathcal{Z} \cap \mathcal{C}_e^{(j)}| \leq |\mathcal{Z}|/d^{4\varepsilon/5}$ *for all* $e \in \mathcal{H}$ *with* $e \not\nearrow \mathcal{Z}$;
(v) Z *is* $\mathcal{C}$*-free for all* $Z \in \mathcal{Z}$.

Proof. First, suppose that $\mathcal{Z} \in \mathscr{Z}_0$.

Then, (i) and (ii) are immediate from (Z1) and (Z2). To see (iii), note that if $j = 1$, then due to (Z1) we have

$$|\{\{e\} \in \mathcal{Z} : v \in e\}| \leq d \leq \frac{|\mathcal{Z}|}{d^{\varepsilon}}.$$

For (iv), we may again use (Z1) to conclude that for all $e \in \mathcal{H}$ we have

$$|\mathcal{Z} \cap \mathcal{C}_e^{(j)}| \leq \Delta(\mathcal{C}^{(j+1)}) \leq \Gamma d^j \leq \Gamma|\mathcal{Z}|/d^{\varepsilon} \leq |\mathcal{Z}|/d^{4\varepsilon/5}.$$

Finally, observe that (v) is immediate from (Z4).

Now, suppose that $\mathcal{Z} \in \mathscr{C}$. Then, $|\mathcal{Z}| \geq d^{j-\varepsilon/100}$ due to $\delta(\mathcal{C}^{(j+1)}) \geq d^{j-\varepsilon/100}$, which is a consequence of (C6). Hence, (i) holds and (C3) yields (ii). Furthermore, again due to $|\mathcal{Z}| \geq d^{j-\varepsilon/100}$, (iii) and (iv) follow from (C4) and (C7). Finally, (v) is a consequence of (C9). □

Before finally turning to Lemma 2.8.3, we prove the following statement that is helpful in the proof of Lemma 2.8.3.

Lemma 2.8.2. *Let $j \in [\ell]$, $\mathcal{Z} \in \mathscr{Z}^{(j)}$, $j' \in [2\ell]$ and $E \subseteq \mathcal{H}$ with $|E| \in [j'-1]_0$. If $E = \{e\}$ for some $e \in \mathcal{H}$, suppose that $e \not\nearrow \mathcal{Z}$. Then, the number of pairs $(Z, C) \in \mathcal{Z} \times \mathcal{C}$ with $Z \cap C \neq \emptyset$, $|Z \cup C| = j'$, $E \subseteq Z \cup C$ and $|C \cap (Z \cup E)| \geq 2$ is at most*

$$\frac{d^{j'-j-2\varepsilon/3}}{d^{|E|}}|\mathcal{Z}|.$$

Proof. Fix $i \in [j]$ and $E_Z, E_C \subseteq E$. We assume that there is at least one pair $(Z, C) \in \mathcal{Z} \times \mathcal{C}$ with

$$Z \cap C \neq \emptyset, \quad |Z \cup C| = j', \quad E \subseteq Z \cup C, \quad |C \cap (Z \cup E)| \geq 2,$$
$$|Z \cap C| = i, \quad E_Z = E \cap Z \quad \text{and} \quad E_C = E \cap C$$

and show that the number p of such pairs is at most

$$\frac{d^{j'-j-3\varepsilon/4}}{d^{|E|}}|\mathcal{Z}|.$$

As there were at most $j \cdot 2^{|E|} \cdot 2^{|E|} \leq \ell 16^{\ell} \leq d^{\varepsilon/12}$ choices for the parameters i, E_Z, E_C, this suffices to obtain the desired upper bound. For $I \subseteq \mathcal{H}$, and a hypergraph $\mathcal{X}$ with $V(\mathcal{X}) \subseteq \mathcal{H}$, we define

$$\mathcal{X}_I := \{X \in \mathcal{X} : I \subseteq X\}.$$

Note that here, in contrast to our notation for the link, the elements of $\mathcal{X}_I$ are edges of $\mathcal{X}$ as this is more convenient for this proof. For an integer j'', even if the set $\mathcal{X}_I$ is interpreted as a hypergraph with vertex set $V(\mathcal{X})$, there is no ambiguity when we write $\mathcal{X}_I^{(j'')}$ since $\{X \in \mathcal{X}^{(j'')} : I \subseteq X\} = \{X \in \mathcal{X}_I : |X| = j''\}$. Note that $|C| = j' - j + i$. If $1 \leq |E_Z| \leq j - 1$, then Lemma 2.8.1 (ii) with (C3) entails

$$p \leq \sum_{Z \in \mathcal{Z}_{E_Z}} \sum_{I \in \binom{Z}{i}} |\mathcal{C}_{I \cup E_C}^{(j'-j+i)}| \leq \Delta_{|E_Z|}(\mathcal{Z}) \cdot 2^\ell \cdot \Delta_{i+|E|-|E_Z|}(\mathcal{C}^{(j'-j+i)})$$
$$\leq \frac{|\mathcal{Z}|}{d^{|E_Z|+4\varepsilon/5}} \cdot 2^\ell \cdot d^{j'-j-|E|+|E_Z|} \leq \frac{d^{j'-j-3\varepsilon/4}}{d^{|E|}} |\mathcal{Z}|,$$

where we used that $|I \cup E_C| \geq i + |E_C \setminus Z| = i + |E \setminus Z| = i + |E| - |E_Z|$ for all $Z \in \mathcal{Z}_{E_Z}$ and $I \in \binom{Z}{i}$. If $2 \leq |E_C| \leq j' - j + i - 1$, then Lemma 2.8.1 (ii) with (C3) and additionally Lemma 2.8.1 (i) if $|E| - |E_C| + i = j$ entails

$$p \leq \sum_{C \in \mathcal{C}_{E_C}^{(j'-j+i)}} \sum_{I \in \binom{C}{i}} |\mathcal{Z}_{E_Z \cup I}| \leq \Delta_{|E_C|}(\mathcal{C}^{(j'-j+i)}) \cdot 2^\ell \cdot \Delta_{|E|-|E_C|+i}(\mathcal{Z})$$
$$\leq d^{j'-j+i-|E_C|-\varepsilon} \cdot 2^\ell \cdot \frac{|\mathcal{Z}|}{d^{|E|-|E_C|+i-\varepsilon/5}} \leq \frac{d^{j'-j-3\varepsilon/4}}{d^{|E|}} |\mathcal{Z}|.$$

It remains to consider the cases where $|E_Z| \in \{0, j\}$ and $|E_C| \in \{0, 1, j' - j + i\}$. Since we assume $p \geq 1$ and since for p, we only count pairs (Z, C) where $Z \cap C \neq \emptyset$ and $|C \cap (E \cup Z)| \geq 2$, we may exclude the case where $|E_Z| = 0$ and $|E_C| = j' - j + i$ and the case where $|E_Z| = j$ and $|E_C| \leq 1$. Indeed, if $|E_C| = j' - j + i = |C|$, then $C \subseteq E$ and hence $|E_Z| \geq |Z \cap C| = i \geq 1$, and if $|E_Z| = j = |Z|$, then $Z \subseteq E$ and hence $|E_C| = |C \cap (Z \cup E)| \geq 2$. Furthermore, since $|E| \leq j' - 1$, we may also exclude the case where $|E_Z| = j$ and $|E_C| = j' - j + i$. It remains to consider the case where $|E| = |E_Z| = |E_C| = 0$ and the case where $|E_Z| = 0$ and $|E_C| = 1$.

Suppose $|E| = |E_Z| = |E_C| = 0$. Since we assume $p \geq 1$ and only count pairs (Z, C) where $|C \cap (E \cup Z)| \geq 2$, we have $i \geq 2$. Furthermore, since all $Z \in \mathcal{Z}$ are $\mathcal{C}$-free by Lemma 2.8.1 (v), we also have $j' \geq j + 1$. From (C3), we obtain

$$p \leq \sum_{Z \in \mathcal{Z}_{E_Z}} \sum_{I \in \binom{Z}{i}} |\mathcal{C}_{I \cup E_C}^{(j'-j+i)}| \leq |\mathcal{Z}| \cdot 2^\ell \cdot \Delta_i(\mathcal{C}^{(j'-j+i)}) \leq |\mathcal{Z}| \cdot 2^\ell \cdot d^{j'-j-\varepsilon} \leq \frac{d^{j'-j-3\varepsilon/4}}{d^{|E|}} |\mathcal{Z}|.$$

Finally, suppose $|E_Z| = 0$ and $|E_C| = 1$. Here, we also have $j' \geq j + 1$. Note that by assumption, the single element e of E_C is not an immediate evictor for $\mathcal{Z}$. If $j' = j + 1$ and $i = j$, then for all the pairs (Z, C) that we count, we have $C = Z \cup \{e\}$ and hence Lemma 2.8.1 (iv) entails

$$p \leq |\mathcal{Z} \cap \mathcal{C}_e^{(j)}| \leq \frac{|\mathcal{Z}|}{d^{4\varepsilon/5}} \leq \frac{d^{j'-j-3\varepsilon/4}}{d^{|E|}} |\mathcal{Z}|.$$

If $j' = j+1$ and $i \leq j-1$, then Lemma 2.8.1 (ii) with (C2) entails

$$p \leq \sum_{C \in \mathcal{C}^{(j'-j+i)}_{E_C}} \sum_{I \in \binom{C}{i}} |\mathcal{Z}_{E_Z \cup I}| \leq \Delta(\mathcal{C}^{(j'-j+i)}) \cdot 2^\ell \cdot \Delta_i(\mathcal{Z})$$
$$\leq \Gamma d^{j'-j+i-1} \cdot 2^\ell \cdot \frac{|\mathcal{Z}|}{d^{i+4\varepsilon/5}} \leq \frac{d^{j'-j-3\varepsilon/4}}{d^{|E|}} |\mathcal{Z}|.$$

If $j' \geq j+2$, then, due to (C3),

$$p \leq \sum_{Z \in \mathcal{Z}_{E_Z}} \sum_{I \in \binom{Z}{i}} |\mathcal{C}^{(j'-j+i)}_{I \cup E_C}| \leq |\mathcal{Z}| \cdot 2^\ell \cdot \Delta_{i+1}(\mathcal{C}^{(j'-j+i)}) \leq |\mathcal{Z}| \cdot 2^\ell \cdot d^{j'-j-1-\varepsilon} \leq \frac{d^{j'-j-3\varepsilon/4}}{d^{|E|}} |\mathcal{Z}|,$$

which completes the proof. $\square$

Lemma 2.8.3. *Let $j \in [\ell]$, $\mathcal{Z} \in \mathscr{Z}^{(j)}$ and $j' \in [2\ell]$. Then, the following holds.*

(i) *$\mathcal{Z}_v$ is $(|\mathcal{Z}| d^{-\varepsilon/2}, d^{-1})$-spread for all $v \in V(\mathcal{H})$;*

(ii) *$\mathcal{Z}_e$ is $(|\mathcal{Z}| d^{-1-\varepsilon/2}, d^{-1})$-spread for all $e \in \mathcal{H}$;*

(iii) *$\mathcal{Z}^{(j')}_{e,2}$ is $(|\mathcal{Z}| d^{j'-j-\varepsilon/2}, d^{-1})$-spread for all $e \in \mathcal{H}$ with $e \not\nearrow \mathcal{Z}$*[2]*;*

(iv) *$\mathcal{Z}^{(j')}_2$ is $(|\mathcal{Z}| d^{j'-j-\varepsilon/2}, d^{-1})$-spread;*

(v) *$\mathcal{C}^{(j')}_{e,2}$ is $(d^{j'-\varepsilon/2}, d^{-1})$-spread for all $e \in \mathcal{H}$;*

(vi) *$\mathcal{C}^{\star(j')}_{e,f,2}$ is $(d^{j'-\varepsilon/2}, d^{-1})$-spread for all disjoint $e, f \in \mathcal{H}$.*

Proof. We prove (i)–(vi) individually.

(i) Fix $v \in V(\mathcal{H})$ and $E \subseteq \mathcal{H}$ with $|E| \in [j-1]_0$. Suppose first that $|E| = 0$. If $j = 1$, then as a consequence of Lemma 2.8.1 (iii) and otherwise as a consequence of $\Delta(\mathcal{H}) \leq d$ and Lemma 2.8.1 (ii), we obtain

$$d_{\mathcal{Z}_v}(E) \leq \frac{|\mathcal{Z}|}{d^{4\varepsilon/5}} \leq \frac{|\mathcal{Z}| d^{-\varepsilon/2}}{d^{|E|}}.$$

Suppose that $|E| \geq 1$. Then Lemma 2.8.1 (ii) yields

$$d_{\mathcal{Z}_v}(E) \leq d_{\mathcal{Z}}(E) \leq \Delta_{|E|}(\mathcal{Z}) \leq \frac{|\mathcal{Z}|}{d^{|E|+4\varepsilon/5}} \leq \frac{|\mathcal{Z}| d^{-\varepsilon/2}}{d^{|E|}}.$$

(ii) Fix $e \in \mathcal{H}$ and $E \subseteq \mathcal{H}$ with $|E| \in [j-2]_0$. If $e \in E$, then $d_{\mathcal{Z}_e}(E) = 0$. Hence, we may assume that $e \notin E$. Then, due to Lemma 2.8.1 (ii),

$$d_{\mathcal{Z}_e}(E) = d_{\mathcal{Z}}(E \cup \{e\}) \leq \Delta_{|E|+1}(\mathcal{Z}) \leq \frac{|\mathcal{Z}|}{d^{|E|+1+4\varepsilon/5}} \leq \frac{|\mathcal{Z}| d^{-1-\varepsilon/2}}{d^{|E|}}.$$

[2]That we need the additional assumption that e is not an immediate evictor for $\mathcal{Z}$ will not be an issue as we later circumvent this restriction (see Lemma 2.9.17).

(iii) Fix $e \in \mathcal{H}$ that is not an immediate evictor for $\mathcal{Z}$ and $E \subseteq \mathcal{H}$ with $|E| \in [j'-1]_0$. If $e \notin E$, then the number of pairs $(Z, C) \in \mathcal{Z} \times \mathcal{C}$ with $Z \cap C \neq \emptyset$, $|Z \cup C| \in \{j', j'+1\}$, $E \cup \{e\} \subseteq Z \cup C$ and $|C \cap (Z \cup E \cup \{e\})| \geq 2$ is an upper bound for $d_{\mathcal{Z}_{e,2}^{(j')}}(E)$ and so Lemma 2.8.2 with $E \cup \{e\}$ playing the role of E yields the desired bound.
If $e \in E$, then the number of $(Z, C) \in \mathcal{Z} \times \mathcal{C}$ with $Z \cap C \neq \emptyset$, $|Z \cup C| = j'$, $E \subseteq Z \cup C$ and $|C \cap (Z \cup E)| \geq 2$ is an upper bound for $d_{\mathcal{Z}_{e,2}^{(j')}}(E)$ and so also in this case the desired bound follows from Lemma 2.8.2.

(iv) Fix $E \subseteq \mathcal{H}$ with $|E| \in [j'-1]_0$. The number of pairs $(Z, C) \in \mathcal{Z} \times \mathcal{C}$ with $Z \cap C \neq \emptyset$, $|Z \cup C| = j'$, $E \subseteq Z \cup C$ and $|C \cap (Z \cup E)| \geq 2$ is an upper bound for $d_{\mathcal{Z}_2^{(j')}}(E)$ and so Lemma 2.8.2 yields the desired bound if E does not contain an immediate evictor for $\mathcal{Z}$. Due to (C9), no edge of $\mathcal{Z}_2$ contains an edge of $\mathcal{H}$ that is an immediate evictor for $\mathcal{Z}$. Hence, when considering a pair $(Z, C) \in \mathcal{Z} \times \mathcal{C}$ with $|Z \cap C| \geq 2$, $|Z \cup C| = j'$ and $g \not\nearrow \mathcal{Z}$ for all $g \in C \setminus Z$, the union $Z \cup C$ does not contain an immediate evictor for $\mathcal{Z}$. Thus, no edge of $\mathcal{Z}_2$ contains an immediate evictor for $\mathcal{Z}$, so whenever E contains an immediate evictor for $\mathcal{Z}$, we have $d_{\mathcal{Z}_2^{(j')}}(E) = 0$.

(v) Fix $e \in \mathcal{H}$ and $E \subseteq \mathcal{H}$ with $|E| \in [j'-1]_0$. The number of pairs $(C_1, C_2) \in \mathcal{C}_e \times \mathcal{C}_e$ with $C_1 \neq C_2$, $C_1 \cap C_2 \neq \emptyset$, $|C_1 \cup C_2| = j'$ and $E \subseteq C_1 \cup C_2$ is an upper bound for $d_{\mathcal{C}_{e,2}^{(j')}}(E)$. Note that for any such pair, the semiconflicts C_1 and C_2 both have size at least 2 as otherwise, one would be a subset of the other which, for distinct $C_1, C_2 \in \mathcal{C}_e$, contradicts (C9). Fix $j_1, j_2 \in [\ell]_2$ and let p denote the number of such pairs (C_1, C_2) with additionally $|C_1| = j_1$ and $|C_2| = j_2$. Since we have $\ell^2 \leq d^{\varepsilon/6}$, it suffices to show that

$$p \leq \frac{d^{j'-2\varepsilon/3}}{d^{|E|}}$$

whenever $p \geq 1$. Hence, assume that $p \geq 1$.
If $|E| = 0$, then, since we only count pairs (C_1, C_2) with $C_1 \neq C_2$, Condition (C9) implies $j_1 \leq j'-1$, hence we have $j_1 + j_2 - j' + 1 \leq j_2$, and so by (C2) and (C3), we obtain

$$\begin{aligned} p &\leq \sum_{C_1 \in \mathcal{C}_e^{(j_1)}} \sum_{I \in \binom{C_1}{j_1+j_2-j'}} |\mathcal{C}_{I \cup \{e\}}^{(j_2+1)}| \leq \Delta(\mathcal{C}^{(j_1+1)}) \cdot 2^\ell \cdot \Delta_{j_1+j_2-j'+1}(\mathcal{C}^{(j_2+1)}) \\ &\leq \Gamma d^{j_1} \cdot 2^\ell \cdot d^{j'-j_1-\varepsilon} \leq \frac{d^{j'-2\varepsilon/3}}{d^{|E|}}. \end{aligned}$$

No edge of $\mathcal{C}_{e,2}$ contains e, so the assumption $p \geq 1$ entails $e \notin E$. Hence, if $|E| \geq 1$, then the number of pairs $(C_1, C_2) \in \mathcal{C}_e^{(j_1)} \times \mathcal{C}^{(j_2+1)}$ with $C_1 \cap C_2 \neq \emptyset$, $|C_1 \cup C_2| = j'+1$, $E \cup \{e\} \subseteq C_1 \cup C_2$ and $|C_2 \cap (C_1 \cup E \cup \{e\})| \geq 2$ is an upper bound for p and so Lemma 2.8.2 with $\mathcal{C}_e^{(j_1)}$ playing the role of $\mathcal{Z}$ yields the desired bound.

(vi) Fix disjoint $e, f \in \mathcal{H}$ and $E \subseteq \mathcal{H}$ with $|E| \in [j'-1]_0$. If $|E| = 0$, then, due to (C2) and (C3),

$$d_{\mathcal{C}_{e,f,2}^{\star}}(E) \leq \sum_{C_1 \in \mathcal{C}_e^{(1)}} |\{C_2 \in \mathcal{C}_f^{(j')} : C_1 \subseteq C_2\}| \leq \Delta(\mathcal{C}^{(2)}) \cdot \Delta_2(\mathcal{C}^{(j'+1)})$$

$$\leq \Gamma d \cdot d^{j'-1-\varepsilon} \leq d^{j'-\varepsilon/2}.$$

Clearly, if $f \in E$, we have $d_{\mathcal{C}^\star_{e,f,2}}(E) = 0$. If $|E| \geq 1$ and $f \notin E$, then, by (C3),

$$d_{\mathcal{C}^\star_{e,f,2}}(E) \leq d_{\mathcal{C}(j')}(E \cup \{f\}) \leq \Delta_{|E|+1}(\mathcal{C}^{(j'+1)}) \leq d^{j'-|E|-\varepsilon},$$

which completes the proof. □

For an integer $i \geq 0$, let $\mathcal{A}(i)$ denote the *availability* event that we still have many edges available at the end of step i, or more precisely $|\mathcal{H}|(i) \geq d^{1-\varepsilon/(48\ell)}n/k$ (observe that this is rather a rough lower bound, which does not even depend on i). If $\mathcal{A}(i)$ occurs for some $i \geq 0$, then there were many choices for the randomly selected edges $e(1), \ldots, e(i+1)$. The following statement is a direct consequence of this observation. For an earlier derivation of such a statement using a similar argument see [15, Lemma 4.1].

Lemma 2.8.4. *Let $i \geq 1$ and $E \subseteq \mathcal{H}$. Then,*

$$\mathbb{P}[\mathcal{A}(i-1) \cap \{E \subseteq \mathcal{M}(i)\}] \leq \left(\frac{d^{\varepsilon/(48\ell)}}{d}\right)^{|E|}.$$

Proof. We employ a union bound over all choices of times at which the elements of E may be chosen as an element of $\mathcal{M}(i)$.

Let $j := |E|$, fix an injection $\sigma\colon E \hookrightarrow [i]$ and consider an ordering $i_1 < \ldots < i_j$ of the image of σ. For $j' \in [j]$, let $e_{j'} := \sigma^{-1}(i_{j'})$ and $\mathcal{E}(j') := \mathcal{A}(i_{j'}-1) \cap \{e(i_{j'}) = e_{j'}\}$. We obtain

$$\begin{aligned}
\mathbb{P}\Big[\mathcal{A}(i-1) \cap \bigcap_{j' \in [j]} \{e(i_{j'}) = e_{j'}\}\Big] &\leq \mathbb{P}\Big[\bigcap_{j' \in [j]} \mathcal{E}(j')\Big] = \prod_{j' \in [j]} \mathbb{P}\Big[\mathcal{E}(j') \;\Big|\; \bigcap_{j'' \in [j'-1]} \mathcal{E}(j'')\Big] \\
&\leq \prod_{j' \in [j]} \mathbb{P}\Big[e(i_{j'}) = e_{j'} \;\Big|\; \mathcal{A}(i_{j'}-1) \cap \bigcap_{j'' \in [j'-1]} \mathcal{E}(j'')\Big] \\
&\leq \left(\frac{k}{d^{1-\varepsilon/(48\ell)}n}\right)^j.
\end{aligned}$$

Since $i^j \leq (n/k)^j$, a union bound over all possible choices of σ completes the proof. □

Using the moment based approach used in [18, Proof of Theorem 3.5], Lemma 2.8.4 yields the following statement.

Lemma 2.8.5. *Let $j \in [2\ell]$. Suppose $\mathcal{X}$ is a (d_0, d^{-1})-spread j-graph with $V(\mathcal{X}) \subseteq \mathcal{H}$ and let $i \in [m]$ and $s \in [j]_0$. Then,*

$$\mathbb{P}\Big[\mathcal{A}(i-1) \cap \Big\{|\mathcal{X}^{[s]}|(i) \geq \frac{\max\{d_0, d^{j-s}\}}{d^{j-s-\varepsilon/12}}\Big\}\Big] \leq \exp(-d^{\varepsilon/(200\ell)}).$$

Here, note that $d_0 \geq d^{j-1}$ since $\mathcal{X}$ is (d_0, d^{-1})-spread.

Proof. The moments of the random variable $\mathbb{1}_{\mathcal{A}(i-1)}|\mathcal{X}^{[s]}|(i)$ depend on which unions of subsets of edges of $\mathcal{X}$ form a subset of $\mathcal{M}(i)$. Using Lemma 2.8.4 to bound the probability that such a union is a subset of $\mathcal{M}(i)$ and the spreadness of $\mathcal{X}$ to see that there are not too many small unions, for sufficiently large r, we obtain a suitable upper bound for the r-th moment of $\mathbb{1}_{\mathcal{A}(i-1)}|\mathcal{X}^{[s]}|(i)$. This then allows us to obtain the desired upper bound for

$$\mathbb{P}\left[\mathcal{A}(i-1)\cap\left\{|\mathcal{X}^{[s]}|(i)\geq\frac{\max\{d_0,d^{j-s}\}}{d^{j-s-\varepsilon/12}}\right\}\right]=\mathbb{P}\left[\mathbb{1}_{\mathcal{A}(i-1)}|\mathcal{X}^{[s]}|(i)\geq\frac{\max\{d_0,d^{j-s}\}}{d^{j-s-\varepsilon/12}}\right]$$

as a consequence of Markov's inequality.

Let us turn to the details. First, note the following. The (d_0,d^{-1})-spreadness of $\mathcal{X}$ guarantees $\Delta_{j'}(\mathcal{X})\leq d_0/d^{j'}$ for all $j'\in[j-1]_0$. Thus, for all $j'\in[j-s]_0$ we obtain

$$\Delta_{j'}(\mathcal{X})\leq\max\left\{\frac{d_0}{d^{j'}},1\right\}. \tag{2.8.1}$$

To handle the relevant intersections of edges of $\mathcal{X}$ with the matching $\mathcal{M}(i)$ we introduce

$$\mathcal{X}_{j-s}:=\left\{(X,M)\in\mathcal{X}\times\binom{\mathcal{H}}{j-s}:M\subseteq X\right\}.$$

If X_r for an integer $r\geq 1$ denotes a pair in $\mathcal{X}_{j-s}$, we use M_r to denote the second component of X_r. For all integers $r\geq 1$, Lemma 2.8.4 implies

$$\begin{aligned}\mathbb{E}[(\mathbb{1}_{\mathcal{A}(i-1)}|\mathcal{X}^{[s]}|(i))^r]&\leq\mathbb{E}\Big[\mathbb{1}_{\mathcal{A}(i-1)}\Big(\sum_{X_1\in\mathcal{X}_{j-s}}\mathbb{1}_{\{M_1\subseteq\mathcal{M}(i)\}}\Big)^r\Big]\\&=\sum_{X_1,\ldots,X_r\in\mathcal{X}_{j-s}}\mathbb{E}\Big[\mathbb{1}_{\mathcal{A}(i-1)}\prod_{r'\in[r]}\mathbb{1}_{\{M_{r'}\subseteq\mathcal{M}(i)\}}\Big]\\&\leq\sum_{X_1,\ldots,X_r\in\mathcal{X}_{j-s}}\frac{d^{\varepsilon r(j-s)/(48\ell)}}{d^{|\bigcup_{r'\in[r]}M_{r'}|}}\leq\sum_{X_1,\ldots,X_r\in\mathcal{X}_{j-s}}\frac{d^{\varepsilon r/24}}{d^{|\bigcup_{r'\in[r]}M_{r'}|}}.\end{aligned}\tag{2.8.2}$$

Using (2.8.1) with $j'=0$, we obtain

$$\sum_{X_1\in\mathcal{X}_{j-s}}\frac{1}{d^{|M_1|}}\leq\frac{\Delta_0(\mathcal{X})\cdot\binom{j}{j-s}}{d^{j-s}}\leq 4^\ell\frac{d_0}{d^{j-s}}. \tag{2.8.3}$$

Furthermore, for all integers $r\geq 2$, we have

$$\sum_{X_1,\ldots,X_r\in\mathcal{X}_{j-s}}\frac{1}{d^{|\bigcup_{r'\in[r]}M_{r'}|}}=\sum_{X_1,\ldots,X_{r-1}\in\mathcal{X}_{j-s}}\frac{1}{d^{|\bigcup_{r'\in[r-1]}M_{r'}|}}\sum_{X_r\in\mathcal{X}_{j-s}}\frac{d^{|M_r\cap\bigcup_{r'\in[r-1]}M_{r'}|}}{d^{j-s}}.$$

For all $X_1,\ldots,X_{r-1}\in\mathcal{X}_{j-s}$ and $M:=\bigcup_{r'\in[r-1]}M_{r'}$, exploiting again (2.8.1) for appropriate values of j', we obtain

$$\sum_{X_r\in\mathcal{X}_{j-s}}\frac{d^{|M_r\cap M|}}{d^{j-s}}\leq\sum_{\substack{N_r\subseteq M:\\|N_r|\leq j-s}}\sum_{\substack{X_r\in\mathcal{X}_{j-s}:\\N_r\subseteq M_r}}\frac{d^{|N_r|}}{d^{j-s}}\leq\sum_{\substack{N_r\subseteq M:\\|N_r|\leq j-s}}\frac{\Delta_{|N_r|}(\mathcal{X})\cdot\binom{j-|N_r|}{j-s-|N_r|}\cdot d^{|N_r|}}{d^{j-s}}$$

$$\leq \sum_{\substack{N_r \subseteq M: \\ |N_r| \leq j-s}} 4^\ell \frac{\max\{d_0, d^{|N_r|}\}}{d^{j-s}} \leq (4\ell r)^{2\ell} \max\left\{\frac{d_0}{d^{j-s}}, 1\right\}.$$

Thus, for all integers $r \geq 2$,

$$\sum_{X_1,\ldots,X_r \in \mathcal{X}_{j-s}} \frac{1}{d^{|\bigcup_{r' \in [r]} M_{r'}|}} \leq (4\ell r)^{2\ell} \max\left\{\frac{d_0}{d^{j-s}}, 1\right\} \sum_{X_1,\ldots,X_{r-1} \in \mathcal{X}_{j-s}} \frac{1}{d^{|\bigcup_{r' \in [r-1]} M_{r'}|}}. \quad (2.8.4)$$

Induction over r combining (2.8.3) and (2.8.4) shows that for all integers $r \geq 1$, we have

$$\sum_{X_1,\ldots,X_r \in \mathcal{X}_{j-s}} \frac{1}{d^{|\bigcup_{r' \in [r]} M_{r'}|}} \leq (4\ell r)^{2\ell r} \max\left\{\frac{d_0^r}{d^{r(j-s)}}, 1\right\}.$$

With (2.8.2), this yields

$$\mathbb{E}[(\mathbb{1}_{\mathcal{A}(i-1)}|\mathcal{X}^{[s]}|(i))^r] \leq (4\ell r)^{2\ell r} \frac{\max\{d_0^r, d^{r(j-s)}\}}{d^{r(j-s-\varepsilon/24)}}.$$

Markov's inequality entails

$$\begin{aligned}\mathbb{P}\left[\mathbb{1}_{\mathcal{A}(i-1)}|\mathcal{X}^{[s]}|(i) \geq \frac{\max\{d_0, d^{j-s}\}}{d^{j-s-\varepsilon/12}}\right] &= \mathbb{P}\left[(\mathbb{1}_{\mathcal{A}(i-1)}|\mathcal{X}^{[s]}|(i))^r \geq \frac{\max\{d_0^r, d^{r(j-s)}\}}{d^{r(j-s-\varepsilon/12)}}\right] \\ &\leq \frac{(4\ell r)^{2\ell r}}{d^{\varepsilon r/24}}\end{aligned}$$

and for $r = d^{\varepsilon/(200\ell)}$, we obtain

$$\frac{(4\ell r)^{2\ell r}}{d^{\varepsilon r/24}} = \left(\frac{(4\ell r)^{2\ell}}{d^{\varepsilon/24}}\right)^r \leq \exp(-r),$$

which completes the proof. □

For $i \geq 0$, we introduce a *spreadness* event $\mathcal{S}(i)$ that occurs whenever relevant configurations are spread out at the end of step i. More specifically, for $i \geq 0$, let denote the event that for all $j \in [\ell]$ and $\mathcal{Z} \in \mathscr{Z}^{(j)}$, the following holds.

(i) $|\mathcal{Z}_v^{[s]}|(i) \leq d^{s-j-\varepsilon/3}|\mathcal{Z}|$ for all $v \in V(\mathcal{H})$ and $s \in [\ell]$;
(ii) $|\mathcal{Z}_e^{[s]}|(i) \leq d^{s-j-\varepsilon/3}|\mathcal{Z}|$ for all $e \in \mathcal{H}$ and $s \in [\ell]_0$ with $s \geq \mathbb{1}_{\mathscr{C}}(\mathcal{Z})$;
(iii) $|\mathcal{Z}_{e,2}^{[s]}|(i) \leq d^{s-j-\varepsilon/3}|\mathcal{Z}|$ for all $e \in \mathcal{H}$ with $e \not\nearrow \mathcal{Z}$ and $s \in [\ell]$;
(iv) $|\mathcal{Z}_2^{[s]}|(i) \leq d^{s-j-\varepsilon/3}|\mathcal{Z}|$ for all $s \in [\ell]_2$;
(v) $|\mathcal{C}_{e,2}^{[s]}|(i) \leq d^{s-\varepsilon/3}$ for all $e \in \mathcal{H}$ and $s \in [\ell - 1]$;
(vi) $|\mathcal{C}_{e,f,2}^{\star[1]}|(i) \leq d^{1-\varepsilon/3}$ for all disjoint $e, f \in \mathcal{H}$.

Combining Lemmas 2.8.3 and 2.8.5, we conclude our observations in this section with the following statement showing that spreadness typically persists during the construction of the matching as long as many edges remain available.

Lemma 2.8.6. *We have* $\mathbb{P}[\mathcal{S}(0)^{\mathsf{c}} \cup \bigcup_{i\in[m]}(\mathcal{A}(i-1)\cap\mathcal{S}(i)^{\mathsf{c}})] \leq \exp(-d^{\varepsilon/400\ell})$.

Proof. Choose $i\in[m]$, $s\in[\ell]_0$ with $s\geq \mathbb{1}_{\mathscr{C}}(\mathcal{Z})$, $j\in[\ell]$, $\mathcal{Z}\in\mathscr{Z}^{(j)}$, $v\in V(\mathcal{H})$, $e,f\in\mathcal{H}$ and $\mathcal{X}\in\{\mathcal{Z}_v,\mathcal{Z}_e,\mathcal{Z}_{e,2},\mathcal{Z}_2,\mathcal{C}_{e,2},\mathcal{C}^\star_{e,f,2}\}$ such that e is not an immediate evictor for $\mathcal{Z}$ if $\mathcal{X}=\mathcal{Z}_{e,2}$. Note that there were at most

$$m\cdot(\ell+1)\cdot\ell\cdot(|\mathscr{Z}_0|+|\mathcal{H}|)\cdot n\cdot|\mathcal{H}|^2\cdot 6\leq 12\ell^2d^2n^4(|\mathscr{Z}_0|+\ell dn)\leq\exp(9d^{\varepsilon/(400\ell)})$$

possible choices for these parameters. If $\mathcal{X}\in\{\mathcal{Z}_v,\mathcal{Z}_e,\mathcal{Z}_{e,2},\mathcal{Z}_2\}$, let $a:=|\mathcal{Z}|$ and otherwise let $a:=d^j$. Lemma 2.8.3 shows that for all $j'\in[2\ell]$, the j'-graph $\mathcal{X}^{(j')}$ is $(ad^{j'-j-\varepsilon/2},d^{-1})$-spread[3] and thus Lemma 2.8.5 yields

$$\begin{aligned}\mathbb{P}[\mathcal{A}(i-1)\cap\{|\mathcal{X}^{[s]}|(i)\geq ad^{s-j-\varepsilon/3}\}] &\leq \mathbb{P}\Big[\bigcup_{j'\in[2\ell]}\mathcal{A}(i-1)\cap\Big\{|\mathcal{X}^{(j')[s]}|(i)\geq\frac{ad^{s-j-\varepsilon/3}}{2\ell}\Big\}\Big]\\ &\leq \mathbb{P}\Big[\bigcup_{j'\in[2\ell]}\mathcal{A}(i-1)\cap\Big\{|\mathcal{X}^{(j')[s]}|(i)\geq\frac{ad^{j'-j-\varepsilon/2}}{d^{j'-s-\varepsilon/12}}\Big\}\Big]\\ &\leq \exp(-d^{\varepsilon/(300\ell)}).\end{aligned}$$

Hence, considering the definition of the event $\mathcal{S}(i)$, with a suitable union bound over the at most $\exp(9d^{\varepsilon/(400\ell)})$ choices for the parameters, we obtain

$$\mathbb{P}\Big[\bigcup_{i\in[m]}\mathcal{A}(i-1)\cap\mathcal{S}(i)^{\mathsf{c}}\Big]\leq\exp(9d^{\varepsilon/(400\ell)})\cdot\exp(-d^{\varepsilon/(300\ell)})\leq\exp(-d^{\varepsilon/400\ell}).$$

As Lemma 2.8.3 shows that $\mathcal{S}(0)^{\mathsf{c}}=\emptyset$, this completes the proof. □

2.9 Tracking key random variables

In this section, our goal is to prove Theorem 2.6.2 by formally showing that the relevant quantities indeed typically follow the idealized trajectories given in Section 2.7.

To this end we extend the previously defined $\hat{p}_V(i)$, $\hat{p}_M(i)$, $\hat{\Gamma}(i)$, $\hat{h}(i)$, $\hat{d}(i)$, $\hat{z}_{j,s}(i)$ and $\hat{c}(i)$ to continuous trajectories by introducing the following functions. For $j\in[\ell]$ and $s\in[\ell]_0$, let $\hat{p}_V,\hat{p}_M,\hat{\Gamma},\hat{h},\hat{d},\hat{z}_{j,s},\hat{c}$ denote functions such that for all $x\in[0,n/k]$,

$$\hat{p}_V(x)=1-\frac{kx}{n},\quad \hat{p}_M(x)=\frac{kx}{dn},\quad \hat{\Gamma}(x)=\sum_{j\in[\ell]_2}\Delta(\mathcal{C}^{(j)})\cdot\hat{p}_M(x)^{j-1},$$
$$\hat{d}(x)=d\cdot\hat{p}_V(x)^{k-1}\cdot\exp(-\hat{\Gamma}(x)),\quad \hat{h}(x)=\frac{n}{k}\cdot\hat{p}_V(x)\cdot\hat{d}(x),$$
$$\hat{z}_{j,s}(x)=\binom{j}{s}\cdot\big(\hat{p}_V(x)^k\cdot\exp(-\hat{\Gamma}(x))\big)^s\cdot\hat{p}_M(x)^{j-s}$$

[3]Note here that for many values of j', the spreadness holds trivially since the respective j'-graph is empty. For example, when considering $\mathcal{Z}_e$, the only relevant case is when $j'=j-1$.

$$\text{and} \quad \hat{c}(x) = \sum_{j\in[\ell-1]} \Delta(\mathcal{C}^{(j+1)}) \cdot \hat{z}_{j,1}(x),$$

where we set $0^0 := 1$ and $\binom{j}{s} := 0$ whenever $s \notin [j]_0$. Furthermore, we introduce the following error functions. For $j \in [\ell]$ and $s \in [\ell]_0$, let $\xi, \delta, \eta, \zeta_{j,s}, \gamma$ denote functions such that for all $x \in [0, n/k]$,

$$\xi(x) = \left(\frac{1}{\hat{p}_V(x)}\right)^{300k\ell\Gamma} \cdot d^{-\varepsilon/32}, \quad \delta(x) = \xi(x) \cdot \hat{d}(x), \quad \eta(x) = \xi(x) \cdot \hat{h}(x),$$

$$\zeta_{j,s}(x) = \xi(x) \cdot \left(\hat{z}_{j,s}(x) + \binom{j}{s}\frac{\hat{d}(x)^s}{\Gamma\ell d^j}\right) \quad \text{and} \quad \gamma(x) = 2\sum_{j\in[\ell-1]} \Delta(\mathcal{C}^{(j+1)}) \cdot \zeta_{j,1}(x).$$

Let us gather some useful bounds for the trajectories and error functions.

Remark 2.9.1. For all $j \in [\ell]$, $s \in [\ell]_0$ and $x \in [0, m]$, we have

$$\begin{gathered} \mu \le \hat{p}_V(x) \le 1, \quad 0 \le \hat{p}_M(x) \le \frac{1}{d}, \quad 0 \le \hat{\Gamma}(x) \le \Gamma, \\ d^{1-\varepsilon/400} \le \hat{d}(x) \le d, \quad d^{1-\varepsilon/400} n \le \hat{h}(x) \le dn, \\ 0 \le \hat{z}_{j,s}(x) \le d^{s-j+\varepsilon/400}, \quad 0 \le \hat{c}(x) \le d^{1+\varepsilon/400}, \\ d^{-\varepsilon/32} \le \xi(x) \le d^{-\varepsilon/64}, \quad d^{1-\varepsilon/16} \le \delta(x) \le d^{1-\varepsilon/64}, \\ d^{1-\varepsilon/16} n \le \eta(x) \le d^{1-\varepsilon/64} n, \quad d^{s-j-\varepsilon/16} \le \zeta_{j,s}(x) \le d^{s-j-\varepsilon/100}. \end{gathered}$$

For $i \ge 0$, we introduce a *tracking* event $\mathcal{T}(i)$ that occurs whenever relevant quantities are close to the corresponding trajectories at the end of step i. More specifically, for $i \ge 0$, let $\mathcal{T}(i)$ denote the event that for all $v \in V(i)$, $e \in \mathcal{H}(i)$, $j \in [\ell]$, $\mathcal{Z} \in \mathscr{Z}^{(j)}(i)$ and $s \in [\ell]_0$ satisfying $s \ge \mathbb{1}_{\mathscr{C}}(\mathcal{Z})$, we have

$$\begin{gathered} |\mathcal{H}|(i) = \hat{h}(i) \pm \eta(i), \quad |\mathcal{D}_v|(i) = \hat{d}(i) \pm \delta(i), \\ |\mathcal{C}_e^{[1]}|(i) = \hat{c}(i) \pm \gamma(i) \quad \text{and} \quad |\mathcal{Z}^{[s]}|(i) = (\hat{z}_{j,s}(i) \pm \zeta_{j,s}(i))|\mathcal{Z}|. \end{gathered}$$

At the end of this section, we prove the following statement.

Theorem 2.9.2. *We have* $\mathbb{P}[\mathcal{T}(0) \cap \ldots \cap \mathcal{T}(m)] \ge 1 - \exp(-d^{\varepsilon/(500\ell)})$.

Note that since relevant values of the error functions are sufficiently small as detailed in Fact 2.9.1, Theorem 2.6.2 is a direct consequence of Theorem 2.9.2. Indeed, if $\mathcal{T}(m)$ occurs, then $|\mathcal{H}|(i) \ge |\mathcal{H}|(m) > 0$ for all $i \in [m]$ which entails $|\mathcal{M}|(m) = m$ and furthermore, for all $j \in [\ell]$ and $\mathcal{Z} \in \mathscr{Z}_0^{(j)}$, we have

$$|\{Z \in \mathcal{Z} : Z \subseteq \mathcal{M}(m)\}| = |\mathcal{Z}^{[0]}|(m) = (\hat{z}_{j,0}(m) \pm \zeta_{j,0}(m))|\mathcal{Z}| = (1 \pm d^{-\varepsilon/75})\left(\frac{km}{dn}\right)^j |\mathcal{Z}|.$$

We write $X =_{\mathcal{E}} Y$ for two expressions X and Y and an event $\mathcal{E}$, to express the statement that X and Y represent (possibly constant) random variables that are equal whenever $\mathcal{E}$

occurs, or equivalently, to express that $X \cdot \mathbb{1}_{\mathcal{E}} = Y \cdot \mathbb{1}_{\mathcal{E}}$. Similarly, we write $X \leq_{\mathcal{E}} Y$ to mean $X \cdot \mathbb{1}_{\mathcal{E}} \leq Y \cdot \mathbb{1}_{\mathcal{E}}$ and $X \geq_{\mathcal{E}} Y$ to mean $X \cdot \mathbb{1}_{\mathcal{E}} \geq Y \cdot \mathbb{1}_{\mathcal{E}}$. Hence, whenever we use $=_{\mathcal{E}}$, $\leq_{\mathcal{E}}$ or $\geq_{\mathcal{E}}$ to relate random variables, this allows us to assume that $\mathcal{E}$ occurred. Usually $\mathcal{E}$ will be an event with $\mathcal{E} \subseteq \mathcal{S}(i) \cup \mathcal{T}(i)$ for some $i \geq 0$ which then allows us to employ the properties used to define the events $\mathcal{S}(i)$ and $\mathcal{T}(i)$ on pages 43 and 45. Note that whenever $X =_{\mathcal{E}} Y$, then also for all events $\mathcal{E}' \subseteq \mathcal{E}$, we have $X =_{\mathcal{E}'} Y$ (and similarly for $\leq_{\mathcal{E}}$ and $\geq_{\mathcal{E}}$).

In this section we often encounter probabilities and expectations conditioned on the elements of the filtration $\mathfrak{F}(0), \mathfrak{F}(1), \ldots$, so for an event $\mathcal{E}$, a random variable X and $i \geq 0$, we introduce the shorthands $\mathbb{P}_i[\mathcal{E}] := \mathbb{P}[\mathcal{E} \mid \mathfrak{F}(i)]$ and $\mathbb{E}_i[X] := \mathbb{E}[X \mid \mathfrak{F}(i)]$.

To prove Theorem 2.9.2, first observe that it suffices to focus on $\mathcal{D}_v(i)$ and $\mathcal{Z}^{[s]}(i)$.

Lemma 2.9.3. *Let $i \geq 0$ and let $\mathcal{T}'(i)$ denote the event that for all $v \in V(i)$, $j \in [\ell]$, $\mathcal{Z} \in \mathscr{Z}^{(j)}(i)$ and $s \in [\ell]_0$ with $s \geq \mathbb{1}_{\mathscr{C}}(\mathcal{Z})$, we have*

$$|\mathcal{D}_v|(i) = \hat{d}(i) \pm \delta(i) \quad \textit{and} \quad |\mathcal{Z}^{[s]}|(i) = (\hat{z}_{j,s}(i) \pm \zeta_{j,s}(i))|\mathcal{Z}|.$$

Then, $\mathcal{T}'(i) = \mathcal{T}(i)$.

Proof. Obviously, we have $\mathcal{T}(i) \subseteq \mathcal{T}'(i)$, so it suffices to show that $\mathcal{T}'(i) \subseteq \mathcal{T}(i)$, that is that whenever $\mathcal{T}'(i)$ occurs, we have $|\mathcal{H}|(i) = \hat{h}(i) \pm \eta(i)$ and $|\mathcal{C}_e^{[1]}|(i) = \hat{c}(i) \pm \gamma(i)$ for all $e \in \mathcal{H}(i)$.

Let $\mathcal{X} := \mathcal{T}'(i)$. We have

$$|\mathcal{H}|(i) = \frac{1}{k} \sum_{v \in V(i)} |\mathcal{D}_v|(i) =_{\mathcal{X}} (\hat{d}(i) \pm \delta(i)) \frac{n - ki}{k} = (\hat{d}(i) \pm \delta(i)) \frac{n}{k} \hat{p}_V(i) = \hat{h}(i) \pm \eta(i).$$

Furthermore, for all $e \in \mathcal{H}(i)$ and $j \in [\ell - 1]$, using $\delta(\mathcal{C}^{(j+1)}) \geq (1 - d^{-\varepsilon})\Delta(\mathcal{C}^{(j+1)})$, we obtain

$$\begin{aligned} |\mathcal{C}_e^{(j)[1]}|(i) &=_{\mathcal{X}} (\hat{z}_{j,1}(i) \pm \zeta_{j,1}(i))|\mathcal{C}_e^{(j)}| = (\hat{z}_{j,1}(i) \pm \zeta_{j,1}(i))(1 \pm d^{-\varepsilon})\Delta(\mathcal{C}^{(j+1)}) \\ &= \left(\hat{z}_{j,1}(i) \pm \frac{\xi(i)\hat{z}_{j,1}(i)}{2} \pm \frac{3\zeta_{j,1}(i)}{2}\right)\Delta(\mathcal{C}^{(j+1)}) = (\hat{z}_{j,1}(i) \pm 2\zeta_{j,1}(i))\Delta(\mathcal{C}^{(j+1)}). \end{aligned}$$

Thus

$$|\mathcal{C}_e^{[1]}|(i) = \sum_{j \in [\ell-1]} |\mathcal{C}_e^{(j)[1]}|(i) =_{\mathcal{X}} \hat{c}(i) \pm \gamma(i),$$

which completes the proof. □

To control $|\mathcal{D}_v|(i)$ and $|\mathcal{Z}^{[s]}|(i)$, we employ the following version of Freedman's inequality for supermartingales.

Lemma 2.9.4 (Freedman's inequality for supermartingales [39]). *Suppose that the sequence $X(0), X(1), \ldots$ is a supermartingale with respect to a filtration $\mathfrak{X}(0), \mathfrak{X}(1), \ldots$ such that $|X(i+1) - X(i)| \leq a$ for all $i \geq 0$ and $\sum_{i \geq 0} \mathbb{E}[|X(i+1) - X(i)| \mid \mathfrak{X}(i)] \leq b$. Then, for all $t > 0$,*

$$\mathbb{P}[X(i) \geq X(0) + t \textit{ for some } i \geq 0] \leq \exp\left(-\frac{t^2}{2a(t+b)}\right).$$

To this end, for $i \geq 0$, $v \in V(\mathcal{H})$, $j \in [\ell]$, $\mathcal{Z} \in \mathscr{Z}^{(j)}$ and $s \in [\ell]_0$ with $s \geq \mathbb{1}_{\mathscr{C}}(\mathcal{Z})$, we define the differences

$$|\mathcal{D}_v|^+(i) := |\mathcal{D}_v|(i) - (\hat{d}(i) + \delta(i)), \quad |\mathcal{D}_v|^-(i) := (\hat{d}(i) - \delta(i)) - |\mathcal{D}_v|(i),$$
$$|\mathcal{Z}^{[s]}|^+(i) := |\mathcal{Z}^{[s]}|(i) - (\hat{z}_{j,s}(i) + \zeta_{j,s}(i))|\mathcal{Z}|$$
$$\text{and} \quad |\mathcal{Z}^{[s]}|^-(i) := (\hat{z}_{j,s}(i) - \zeta_{j,s}(i))|\mathcal{Z}| - |\mathcal{Z}^{[s]}|(i)$$

that measure by how much the respective random variable exceeds the permitted deviation from its idealized trajectory. Hence, we aim to show that these four quantities are negative. We wish to analyze the process only while it is well behaved. To this end, for $v \in V(\mathcal{H})$ and $\mathcal{Z} \in \mathscr{Z}$, we define the (random) freezing times

$$\tau_{\mathcal{S}} := \min\{i \geq 0 : \mathcal{S}(i)^{\mathsf{c}} \text{ occurs}\}, \quad \tau_{\mathcal{T}} := \min\{i \geq 0 : \mathcal{T}(i)^{\mathsf{c}} \text{ occurs}\},$$
$$\tau_v := \min\{i \geq 0 : v \notin V(i+1)\} \quad \text{and} \quad \tau_{\mathcal{Z}} := \min\{i \geq 0 : \mathcal{Z} \notin \mathscr{Z}(i+1)\})$$

where we set $\min \emptyset := \infty$. Note that $\tau_{\mathcal{S}}$ and $\tau_{\mathcal{T}}$ are stopping times with respect to the filtration $\mathfrak{F}(0), \mathfrak{F}(1), \ldots$, that is, we have $\{\tau_{\mathcal{S}} = i\}, \{\tau_{\mathcal{T}} = i\} \in \mathfrak{F}(i)$ for all $i \geq 0$, while τ_v and τ_e are not. As we do not use that these random variables are stopping times with respect to our filtration, we generally avoid the term stopping time and call them freezing times instead. Also note that $\tau_{\mathcal{Z}}$ is essentially only meaningful for $\mathcal{Z} \in \mathscr{C}$. Indeed, for $\mathcal{Z} \in \mathscr{Z}_0$, we have $\tau_{\mathcal{Z}} = \infty$ and for $\mathcal{Z} = \mathcal{C}_e^{(j)} \in \mathscr{C}$, we have $\tau_{\mathcal{Z}} = \min\{i \geq 0 : e \notin \mathcal{H}(i+1)\})$.

Using these freezing times, we define the following random variables forming processes that correspond to those introduced above and that freeze whenever something undesirable happens.

$$|\mathcal{D}_v|_{\mathrm{f}}^+(i) := |\mathcal{D}_v|^+(\min(\tau_{\mathcal{S}}, \tau_{\mathcal{T}}, \tau_v, m, i)), \quad |\mathcal{D}_v|_{\mathrm{f}}^-(i) := |\mathcal{D}_v|^-(\min(\tau_{\mathcal{S}}, \tau_{\mathcal{T}}, \tau_v, m, i)),$$
$$|\mathcal{Z}^{[s]}|_{\mathrm{f}}^+(i) := |\mathcal{Z}^{[s]}|^+(\min(\tau_{\mathcal{S}}, \tau_{\mathcal{T}}, \tau_{\mathcal{Z}}, m, i))$$
$$\text{and} \quad |\mathcal{Z}^{[s]}|_{\mathrm{f}}^-(i) := |\mathcal{Z}^{[s]}|^-(\min(\tau_{\mathcal{S}}, \tau_{\mathcal{T}}, \tau_{\mathcal{Z}}, m, i)).$$

Dedicating the following sections to the details, our argument that proves Theorem 2.9.2 goes as follows. Since for all $i \leq m$, we have

$$\hat{h}(i) - \eta(i) \geq \frac{dn\mu^k \exp(-\Gamma)}{2k} \geq \mu^{2k\Gamma} \frac{dn}{k} \geq \mu^{\Gamma\ell/(48\varepsilon^{1/2}\ell)} \frac{dn}{k} \geq \frac{d^{1-\varepsilon/(48\ell)}n}{k}, \tag{2.9.1}$$

the event $\mathcal{A}(i)$ occurs whenever $\mathcal{T}(i)$ occurs, so Lemma 2.8.6 shows that with high probability, spreadness is given at the start in the sense that $\mathcal{S}(0)$ occurs and that for all steps $i \in [m-1]_0$ where $\mathcal{T}(i)$ occurs, we have spreadness in the next step in the sense that $\mathcal{S}(i+1)$ occurs. In other words, Lemma 2.8.6 implies that $\tau_{\mathcal{S}} > \min(\tau_{\mathcal{T}}, m)$ happens with high probability. Investigations of the one-step changes of the processes

$$|\mathcal{D}_v|_{\mathrm{f}}^+(0), |\mathcal{D}_v|_{\mathrm{f}}^+(1), \ldots, \quad |\mathcal{D}_v|_{\mathrm{f}}^-(0), |\mathcal{D}_v|_{\mathrm{f}}^-(1), \ldots,$$
$$|\mathcal{Z}^{[s]}|_{\mathrm{f}}^+(0), |\mathcal{Z}^{[s]}|_{\mathrm{f}}^+(1), \ldots \quad \text{and} \quad |\mathcal{Z}^{[s]}|_{\mathrm{f}}^-(0), |\mathcal{Z}^{[s]}|_{\mathrm{f}}^-(1), \ldots$$

show that

$$|\mathcal{D}_v|_{\mathrm{f}}^{+}(m) \leq 0, \quad |\mathcal{D}_v|_{\mathrm{f}}^{-}(m) \leq 0, \quad |\mathcal{Z}^{[s]}|_{\mathrm{f}}^{+}(m) \leq 0 \quad \text{and} \quad |\mathcal{Z}^{[s]}|_{\mathrm{f}}^{-}(m) \leq 0 \tag{2.9.2}$$

happen with high probability as a consequence of Freedman's inequality for supermartingales. Note that Lemma 2.9.3 shows that whenever $\tau_{\mathcal{T}} \leq \min(\tau_{\mathcal{S}}, m)$, then there are $* \in \{+,-\}$ and $v \in V(\mathcal{H})$ with $|\mathcal{D}_v|_{\mathrm{f}}^{*}(\tau_{\mathcal{T}}) > 0$ or $* \in \{+,-\}$, $\mathcal{Z} \in \mathscr{Z}$ and $s \in [\ell]_0$ with $s \geq \mathbb{1}_{\mathscr{C}}(\mathcal{Z})$ and $|\mathcal{Z}^{[s]}|_{\mathrm{f}}^{*}(m) > 0$ which, due to the freezing, propagates to step m in the sense that (2.9.2) is violated. Hence, as this happens only with very low probability, we typically have $\tau_{\mathcal{T}} > \min(\tau_{\mathcal{S}}, m)$. Knowing that both $\tau_{\mathcal{S}} > \min(\tau_{\mathcal{T}}, m)$ and $\tau_{\mathcal{T}} > \min(\tau_{\mathcal{S}}, m)$ typically happen, we conclude that typically $\tau_{\mathcal{T}} > m$ holds and thus $\mathcal{T}(m)$ typically occurs as claimed in Theorem 2.9.2.

For our analysis in this section it is often crucial that the process is well behaved in step i for some $i \geq 0$ in the sense that $\mathcal{S}(i)$ and $\mathcal{T}(i)$ occurred. Hence, we define the *good* event $\mathcal{G}(i) := \mathcal{S}(i) \cap \mathcal{T}(i)$. Recall that configurations that yield edges of the random hypergraphs considered in the definition of $\mathcal{S}(i)$ and that are particularly important for many of the proofs in this section are visualized in Figure 2.1.

2.9.1 Derivatives and auxiliary bounds

Before we turn to the ingredients for the application of Freedman's inequality for supermartingales, let us state some further properties related to the derivatives of the trajectories and their corresponding error terms.

The main motivation for the choice of ξ in the definition of the error functions is the fact that $\xi'(x)/\xi(x)$ is a suitable multiple of the upper bound $\frac{2k\Gamma}{n\hat{p}_V(x)}$ in Lemma 2.9.5 and the factor $\frac{3k\ell\Gamma}{n\hat{p}_V(x)}$ in Lemma 2.9.6. This then yields the lower bounds for the derivatives of the error terms given in Fact 2.9.8 which in turn are crucial for proving that for $* \in \{+,-\}$, the processes

$$|\mathcal{D}_v|_{\mathrm{f}}^{*}(0), |\mathcal{D}_v|_{\mathrm{f}}^{*}(1), \ldots \quad \text{and} \quad |\mathcal{Z}^{[s]}|_{\mathrm{f}}^{*}(0), |\mathcal{Z}^{[s]}|_{\mathrm{f}}^{*}(1), \ldots$$

are supermartingales.

Lemma 2.9.5. *Let $x \in [0, m]$. Then,*

$$\frac{\hat{c}(x) + \hat{d}(x)}{\hat{h}(x)} \leq \frac{2k\Gamma}{n\hat{p}_V(x)}.$$

Proof. Recall that for all $j \in [\ell]_2$, the Binomial theorem implies that

$$\binom{j}{1}\left(1 - \frac{kx}{dn}\right)\left(\frac{kx}{dn}\right)^{j-1} \leq \sum_{s=0}^{j} \binom{j}{s}\left(\frac{kx}{n}\right)^{j-s}\left(1 - \frac{kx}{n}\right)^{s} = \left(\frac{kx}{n} + 1 - \frac{kx}{n}\right)^{j} = 1.$$

This yields

$$\frac{\hat{c}(x) + \hat{d}(x)}{\hat{h}(x)} = \frac{k}{n\hat{p}_V(x)} + \frac{k}{dn} \sum_{j \in [\ell-1]} \Delta(\mathcal{C}^{(j+1)}) \cdot \binom{j}{1} \cdot \hat{p}_M(x)^{j-1}$$

$$
\begin{aligned}
&= \frac{k}{n\hat{p}_V(x)} + \frac{k}{n\hat{p}_V(x)} \sum_{j\in[\ell-1]} \frac{\Delta(\mathcal{C}^{(j+1)})}{d^j} \cdot \binom{j}{1}\left(1-\frac{kx}{n}\right)\left(\frac{kx}{n}\right)^{j-1} \\
&\le \frac{k}{n\hat{p}_V(x)} + \frac{k\Gamma}{n\hat{p}_V(x)},
\end{aligned}
$$

which completes the proof. □

Lemma 2.9.6. *Let $j \in [\ell]$, $s \in [\ell]_0$ and $x \in [0,m]$. Then,*

$$(s+1)\frac{\zeta_{j,s+1}(x)}{\hat{h}(x)} \le \frac{3k\ell\Gamma}{n\hat{p}_V(x)}\zeta_{j,s}(x).$$

Proof. We assume that $s+1 \le j$, as otherwise $\zeta_{j,s+1}(x) = 0$. We show that $\zeta_{j,s+1}(x) \le \frac{3\ell\Gamma}{s+1}\hat{d}(x)\zeta_{j,s}(x)$ which is equivalent to the inequality in the statement. Recall that for all $s \in [j]_0$, we have $\zeta_{j,s}(x) = \xi(x)\hat{z}_{j,s}(x) + \xi(x)\binom{j}{s}\frac{\hat{d}(x)^s}{\Gamma\ell d^j}$. We bound each of $\xi(x)\hat{z}_{j,s+1}$ and $\xi(x)\binom{j}{s+1}\frac{\hat{d}(x)^{s+1}}{\Gamma\ell d^j}$ separately, where for the first one we use a multiple of $\xi(x)\binom{j}{s}\frac{\hat{d}(x)^s}{\Gamma\ell d^j}$ as an upper bound whenever x is small and a multiple of $\xi(x)\hat{z}_{j,s}(x)$ as an upper bound otherwise.

First, consider $\xi(x)\hat{z}_{j,s+1}(x)$. If $kx/n \le 1/2$, then

$$
\begin{aligned}
\xi(x)\hat{z}_{j,s+1}(x) &= \xi(x)\cdot\binom{j}{s+1}\cdot(\hat{p}_V(x)^k\cdot\exp(-\hat{\Gamma}(x)))^{s+1}\cdot\left(\frac{kx}{dn}\right)^{j-s-1} \\
&= (j-s)\left(\frac{kx}{n}\right)^{j-s-1}\hat{p}_V(x)^{s+1}\cdot\frac{\Gamma\ell}{s+1}\hat{d}(x)\cdot\xi(x)\binom{j}{s}\frac{\hat{d}(x)^s}{\Gamma\ell d^j} \\
&\le (j-s)\left(\frac{1}{2}\right)^{j-s-1}\hat{p}_V(x)^{s+1}\cdot\frac{\Gamma\ell}{s+1}\hat{d}(x)\cdot\zeta_{j,s}(x) \le \frac{\Gamma\ell}{s+1}\hat{d}(x)\cdot\zeta_{j,s}(x).
\end{aligned}
$$

If $kx/n \ge 1/2$, then

$$\xi(x)\hat{z}_{j,s+1}(x) = \frac{j-s}{s+1}\hat{p}_V(x)\hat{d}(x)\frac{n}{kx}\cdot\xi(x)\hat{z}_{j,s}(x) \le \frac{2\ell}{s+1}\hat{d}(x)\zeta_{j,s}(x).$$

Thus, for arbitrary x,

$$\xi(x)\hat{z}_{j,s+1}(x) \le \frac{2\Gamma\ell}{s+1}\hat{d}(x)\zeta_{j,s}(x). \tag{2.9.3}$$

Next, consider $\xi(x)\binom{j}{s+1}\frac{\hat{d}(x)^{s+1}}{\Gamma\ell d^j}$. We have

$$\xi(x)\binom{j}{s+1}\frac{\hat{d}(x)^{s+1}}{\Gamma\ell d^j} = \frac{j-s}{s+1}\hat{d}(x)\cdot\xi(x)\binom{j}{s}\frac{\hat{d}(x)^s}{\Gamma\ell d^j} \le \frac{\ell}{s+1}\hat{d}(x)\cdot\zeta_{j,s}(x).$$

Combining this with (2.9.3) yields

$$\zeta_{j,s+1}(x) = \xi(x)\hat{z}_{j,s+1}(x) + \xi(x)\binom{j}{s+1}\frac{\hat{d}(x)^{s+1}}{\Gamma\ell d^j} \le \frac{3\Gamma\ell}{s+1}\hat{d}(x)\zeta_{j,s}(x),$$

which completes the proof. □

Remark 2.9.7. Let $x \in [0, m]$, $j \in [\ell]$ and $s \in [\ell]_0$. Then,

$$\begin{aligned}
\hat{\Gamma}'(x) &= \sum_{j \in [\ell]_2} \frac{\Delta(\mathcal{C}^{(j)})}{d^{j-1}} \cdot (j-1) \cdot \left(\frac{k}{n}\right)^{j-1} x^{j-2} = \frac{\hat{c}(x)}{\hat{h}(x)}, \\
\xi'(x) &= \frac{300k^2\ell\Gamma}{n\hat{p}_V(x)} \xi(x), \\
\hat{d}'(x) &= -\left(\hat{\Gamma}'(x) + \frac{k(k-1)}{n\hat{p}_V(x)}\right)\hat{d}(x) = -\left(\frac{\hat{c}(x) + (k-1)\hat{d}(x)}{\hat{h}(x)}\right)\hat{d}(x), \\
\hat{z}'_{j,s}(x) &= \frac{s+1}{\hat{h}(x)} \hat{z}_{j,s+1}(x) - s\left(\hat{\Gamma}'(x) + \frac{k^2}{n\hat{p}_V(x)}\right)\hat{z}_{j,s}(x) \\
&= \frac{s+1}{\hat{h}(x)} \hat{z}_{j,s+1}(x) - s\frac{\hat{c}(x) + k\hat{d}(x)}{\hat{h}(x)} \hat{z}_{j,s}(x), \\
\delta'(x) &= \left(\frac{300k^2\ell\Gamma}{n\hat{p}_V(x)} - \hat{\Gamma}'(x) - \frac{k(k-1)}{n\hat{p}_V(x)}\right)\delta(x), \\
\zeta'_{j,s}(x) &= \left(\frac{300k^2\ell\Gamma}{n\hat{p}_V(x)} - s\hat{\Gamma}'(x) - \frac{k^2 s}{n\hat{p}_V(x)}\right)\zeta_{j,s}(x) \\
&\quad +(s+1)\frac{\xi(x)\hat{z}_{j,s+1}(x)}{\hat{h}(x)} + s\binom{j}{s}\frac{\xi(x)\hat{d}(x)^{s+1}}{\Gamma\ell d^j \hat{h}(x)}.
\end{aligned}$$

Let us provide some intuition for $\hat{d}'$ and $\hat{z}'_{j,s}$. For $i \in [m-1]_0$, consider the choice of $e(i+1)$ in step $i+1$ of Algorithm 2.6.1 assuming that $\mathcal{G}(i)$ occurred. For all $v \in V(i)$, all of the approximately $\hat{d}(i)$ edges $e \in \mathcal{D}_v(i)$ may become unavailable due to a conflict $C \in \mathcal{C}$ with $\{e, e(i+1)\} = C \setminus \mathcal{M}(i)$ or due to a nonempty intersection $e \cap e(i+1)$. Since for all conflicts $C \in \mathcal{C}$, all distinct edges $f, f' \in C$ are disjoint, e becomes unavailable either due to a conflict or a nonempty intersection, never both at once. For an edge $e \in \mathcal{H}(i)$ containing a vertex v that will not be covered by $\mathcal{M}(i+1)$ the number of possible choices for $e(i+1)$ that make e unavailable may be estimated as follows, where for the approximations we ignore some overcounting which is negligible due to $\mathcal{S}(i)$ occurring and $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon}$ as we show in the subsequent sections. The number of possible choices that make e unavailable due to conflicts is approximately the number of conflicts $C \in \mathcal{C}$ with $e \in C$ and $|C \setminus \mathcal{M}(i)| = 2$, so there are $|\mathcal{C}_e^{[1]}|(i) \approx \hat{c}(i)$ such choices. Since we assume that v will not be covered by $\mathcal{M}(i+1)$, the number of possible choices for $e(i+1)$ that make e unavailable due to a nonempty intersection is approximately $\sum_{u \in e \setminus \{v\}} |\mathcal{D}_u|(i) \approx (k-1)\hat{d}(i)$. Since for all $e \in \mathcal{H}(i)$, the probability that e is chosen to be $e(i+1)$ is $1/|\mathcal{H}|(i) \approx 1/\hat{h}(i)$, this suggests $\hat{d}'(i)$ for the one-step change $|\mathcal{D}_v(i+1)| - |\mathcal{D}_v(i)|$.

Similarly, for all $j \in [\ell]$, $\mathcal{Z} \in \mathscr{Z}^{(j)}(i)$ and $s \in [j]_0$, tests $Z \in \mathcal{Z}^{[s+1]}(i)$ where one of the $s+1$ available edges $e \in Z$ is chosen to be $e(i+1)$ will be present in $\mathcal{Z}^{[s]}(i)$ and tests $Z \in \mathcal{Z}^{[s]}(i)$ where one of the s available edges $e \in Z$ becomes unavailable (due to conflicts or intersections) will no longer be contained in $\mathcal{Z}^{[s]}(i+1)$. Again, for all $e \in \mathcal{H}(i)$, the probability that e is chosen to be $e(i+1)$ is approximately $1/\hat{h}(i)$ and similarly as above, now without the constraint that e contains some vertex v that will not be covered by $\mathcal{M}(i+1)$, the probability e becomes unavailable is approximately $(\hat{c}(i)+k\hat{d}(i))/\hat{h}(i)$. Hence, when transitioning from $\mathcal{Z}^{[s]}(i)$ to $\mathcal{Z}^{[s]}(i+1)$, we expect to gain approximately $(s+1)|\mathcal{Z}^{[s+1]}|(i)/\hat{h}(i) \approx (s+1)\hat{z}_{j,s+1}(i)|\mathcal{Z}|/\hat{h}(i)$ tests and to lose approximately $s(\hat{c}(i)+k\hat{d}(i))\hat{z}_{j,s}(i)|\mathcal{Z}|/\hat{h}(i)$ tests. This suggests $\hat{z}'_{j,s}(i)\cdot|\mathcal{Z}|$ for the one-step change $|\mathcal{Z}^{[s]}|(i+1)-|\mathcal{Z}^{[s]}|(i)$.

Remark 2.9.8. Let $x\in[0,m]$, $j\in[\ell]$ and $s\in[\ell]_0$. Then,

$$\hat{\Gamma}'(x)\le\frac{k\ell\Gamma}{n\hat{p}_V(x)},\quad \delta'(x)\ge\frac{200k^2\ell\Gamma}{n\hat{p}_V(x)}\delta(x)\ge\frac{d^{1-\varepsilon/2}}{n},$$
$$\zeta'_{j,s}(x)\ge\frac{200k^2\ell\Gamma}{n\hat{p}_V(x)}\zeta_{j,s}(x)\ge\mathbb{1}_{[j]_0}(s)\frac{d^{s-j-\varepsilon/2}}{n}.$$

We also use the following crude upper bounds concerning the derivatives.

Remark 2.9.9. Let $x\in[0,m]$, $j\in[\ell]$ and $s\in[\ell]_0$. Then,

$$|\hat{d}'(x)|\le\frac{d^{1+\varepsilon/32}}{n},\quad \frac{s+1}{\hat{h}(x)}\hat{z}_{j,s+1}(x)\le\frac{d^{s-j+\varepsilon/32}}{n},\quad s\frac{\hat{c}(x)+k\hat{d}(x)}{\hat{h}(x)}\hat{z}_{j,s}(x)\le\frac{d^{s-j+\varepsilon/32}}{n},$$
$$|\hat{z}'_{j,s}(x)|\le\frac{d^{s-j+\varepsilon/32}}{n},\quad |\delta'(x)|\le\frac{d}{n},\quad |\zeta'_{j,s}(x)|\le\frac{d^{s-j}}{n}.$$

To obtain the first order approximations of the one-step changes of the trajectories $\hat{d}$ and $\hat{z}_{j,s}$ as well as the error functions δ and $\zeta_{j,s}$ with $j\in[\ell]$ and $s\in[\ell]_0$ that are presented in Fact 2.9.13, we employ Taylor's theorem with remainder. More specifically, we use the following special case.

Lemma 2.9.10 (Taylor's theorem)**.** *Let $a<x<x+1<b$ and suppose $f\colon(a,b)\to\mathbb{R}$ is twice continuously differentiable. Then,*

$$f(x+1)=f(x)+f'(x)\pm\max_{\xi\in[x,x+1]}|f''(\xi)|.$$

To obtain the approximation errors given in Fact 2.9.13 we provide expressions for the second derivatives in Facts 2.9.11 and 2.9.12. To obtain these, note that $(1/\hat{h}(x))'=(\hat{c}(x)+k\hat{d}(x))/\hat{h}(x)^2$.

Remark 2.9.11. Let $x\in[0,m]$, $j\in[\ell]$ and $s\in[\ell]_0$. Then,

$$\begin{aligned}
\hat{\Gamma}''(x)&=\sum_{j\in[\ell]_3}\frac{\Delta(\mathcal{C}^{(j)})}{d^{j-1}}\cdot(j-1)(j-2)\cdot\left(\frac{k}{n}\right)^{j-1}x^{j-3},\\
\xi''(x)&=\frac{300k^3\ell\Gamma}{n^2\hat{p}_V(x)^2}\xi(x)+\frac{300k^2\ell\Gamma}{n\hat{p}_V(x)}\xi'(x),\\
\hat{d}''(x)&=-\left(\hat{\Gamma}''(x)+\frac{k^2(k-1)}{n^2\hat{p}_V(x)^2}\right)\hat{d}(x)-\left(\hat{\Gamma}'(x)-\frac{k(k-1)}{n\hat{p}_V(x)}\right)\hat{d}'(x),
\end{aligned}$$

$$\begin{aligned}
\hat{z}''_s(x) &= (s+1)\frac{\hat{c}(x)+k\hat{d}(x)}{\hat{h}(x)^2}\hat{z}_{j,s+1}(x) + \frac{s+1}{\hat{h}(x)}\hat{z}'_{j,s+1}(x) \\
&\quad -s\left(\hat{\Gamma}''(x) + \frac{k^3}{n^2\hat{p}_V(x)^2}\right)\hat{z}_{j,s}(x) - s\left(\hat{\Gamma}'(x) + \frac{k^2}{n\hat{p}_V(x)}\right)\hat{z}'_{j,s}(x), \\
\delta''(x) &= \left(\frac{300k^3\ell\Gamma}{n^2\hat{p}_V(x)^2} - \hat{\Gamma}''(x) - \frac{k^2(k-1)}{n^2\hat{p}_V(x)^2}\right)\delta(x) + \left(\frac{300k^2\ell\Gamma}{n\hat{p}_V(x)} - \hat{\Gamma}'(x) - \frac{k(k-1)}{n\hat{p}_V(x)}\right)\delta'(x), \\
\zeta''_{j,s}(x) &= \left(\frac{300k^3\ell\Gamma}{n^2\hat{p}_V(x)^2} - s\hat{\Gamma}''(x) - \frac{k^3 s}{n^2\hat{p}_V(x)^2}\right)\zeta_{j,s}(x) \\
&\quad +\left(\frac{300k^2\ell\Gamma}{n\hat{p}_V(x)} - s\hat{\Gamma}'(x) - \frac{k^2 s}{n\hat{p}_V(x)}\right)\zeta'_{j,s}(x) \\
&\quad +(s+1)\left(\frac{\hat{c}(x)+k\hat{d}(x)}{\hat{h}(x)^2}\zeta_{j,s+1}(x) + \frac{\zeta'_{j,s+1}(x)}{\hat{h}(x)}\right) \\
&\quad +(2s-j)\binom{j}{s}\left(\frac{\hat{c}(x)+k\hat{d}(x)}{\hat{h}(x)^2}\frac{\xi(x)\hat{d}(x)^{s+1}}{\Gamma\ell d^j} + \frac{300k^2\ell\Gamma}{n\hat{p}_V(x)}\frac{\xi(x)\hat{d}(x)^{s+1}}{\Gamma\ell d^j\hat{h}(x)}\right. \\
&\quad \left. +(s+1)\hat{d}'(x)\frac{\xi(x)\hat{d}(x)^s}{\Gamma\ell d^j\hat{h}(x)}\right).
\end{aligned}$$

Remark 2.9.12. Let $x \in [0,m]$, $j \in [\ell]$ and $s \in [\ell]_0$. Then,

$$|\hat{\Gamma}''(x)| \leq \frac{\ell^2k^2\Gamma}{n^2\mu^2}, \quad |\hat{d}''(x)| \leq \frac{d^{1+\varepsilon}}{n^2}, \quad |\hat{z}''_{j,s}(x)| \leq \frac{d^{s-j+\varepsilon}}{n^2},$$
$$|\delta''(x)| \leq \frac{d^{1+\varepsilon}}{n^2}, \quad |\zeta''_{j,s}(x)| \leq \frac{d^{s-j+\varepsilon}}{n^2}.$$

With these bounds for the second derivatives, using $n \geq d^{1/k}$, Taylor's theorem with remainder (Lemma 2.9.10) entails the following approximations.

Remark 2.9.13. Let $i \in [m-1]_0$, $j \in [\ell]$ and $s \in [\ell]_0$. Then,

$$\hat{d}(i+1) - \hat{d}(i) = \hat{d}'(i) \pm \frac{d^{1-\varepsilon}}{n}, \quad \hat{z}_{j,s}(i+1) - \hat{z}_{j,s}(i) = \hat{z}'_{j,s}(i) \pm \frac{d^{s-j-\varepsilon}}{n}$$
$$\delta(i+1) - \delta(i) = \delta'(i) \pm \frac{d^{1-\varepsilon}}{n}, \quad \text{and} \quad \zeta_{j,s}(i+1) - \zeta_{j,s}(i) = \zeta'_{j,s}(i) \pm \frac{d^{s-j-\varepsilon}}{n}.$$

2.9.2 Expected changes

In general, if $X(0), X(1), \dots$ is a sequence of numbers or random variables and $i \geq 0$, we define $\Delta X(i) := X(i+1) - X(i)$. In this section, we show that for all $v \in V(\mathcal{H})$, $\mathcal{Z} \in \mathscr{Z}$, $s \in [\ell]_0$ with $s \geq \mathbb{1}_{\mathscr{C}}(\mathcal{Z})$ and $* \in \{+,-\}$, the processes

$$|\mathcal{D}_v|^*_{\mathrm{f}}(0), |\mathcal{D}_v|^*_{\mathrm{f}}(1)\dots \quad \text{and} \quad |\mathcal{Z}^{[s]}|^*_{\mathrm{f}}(0), |\mathcal{Z}^{[s]}|^*_{\mathrm{f}}(1), \dots$$

are supermartingales with suitably bounded expected one-step changes in the sense that for all $i \geq 0$, the conditional expectations $\mathbb{E}_i[|\Delta|\mathcal{D}_v|^*_{\mathrm{f}}(i)|]$ and $\mathbb{E}_i[|\Delta|\mathcal{Z}^{[s]}|^*_{\mathrm{f}}(i)|]$ are never too large.

First, we present four statements where we bound probabilities related to the removal of edges. Given an edge $e \in \mathcal{H}$ that is available in some step i and $\emptyset \neq U \subseteq e$, the first result closely approximates the probability that in the next step, the following happens: the edge e becomes unavailable due to conflicts or due to an intersection of $e(i+1)$ with U.

Lemma 2.9.14. *Let $i \in [m-1]_0$, $e \in \mathcal{H}$ and $\emptyset \neq U \subseteq e$. Then,*

$$\mathbb{P}_i[e \notin \mathcal{H}_C(i+1) \text{ or } U \cap e(i+1) \neq \emptyset] =_{\mathcal{G}(i)\cap\{e\in\mathcal{H}(i)\}} (1 \pm 7\xi(i))\frac{\hat{c}(i) + |U|\hat{d}(i)}{\hat{h}(i)}.$$

Proof. Let $\mathcal{X} := \mathcal{G}(i) \cap \{e \in \mathcal{H}(i)\}$. We are only interested in the conditional probability if $\mathcal{G}(i)$ happened and in this case we have approximations for all key quantities in step i which makes it easy to obtain

$$\mathbb{P}_i[e \notin \mathcal{H}_C(i+1)] \cdot \mathbb{1}_{\mathcal{X}} \approx \frac{\hat{c}(i)}{\hat{h}(i)} \cdot \mathbb{1}_{\mathcal{X}}$$

as well as

$$\mathbb{P}_i[U \cap e(i+1) \neq \emptyset] \cdot \mathbb{1}_{\mathcal{X}} \approx \frac{|U|\hat{d}(i)}{\hat{h}(i)} \cdot \mathbb{1}_{\mathcal{X}}.$$

Quantifying the approximation error takes some additional care.

Let us turn to the details. Define the events $\mathcal{E}_{C,e} := \{e \notin \mathcal{H}_C(i+1)\}$ and $\mathcal{E}_U := \{U \cap e(i+1) \neq \emptyset\}$. The edges in any conflict are disjoint. Thus, we have $\mathcal{E}_{C,e} \cap \mathcal{E}_U = \emptyset$ and hence

$$\mathbb{P}_i[\mathcal{E}_{C,e} \cup \mathcal{E}_U] = \mathbb{P}_i[\mathcal{E}_{C,e}] + \mathbb{P}_i[\mathcal{E}_U].$$

Let us first consider $\mathbb{P}_i[\mathcal{E}_{C,e}]$. Using $X_e(i)$ to denote the (random) number of edges $f \in \mathcal{H}(i)$ with $\{f\} = C \setminus \mathcal{M}(i)$ for some $C \in \mathcal{C}_e$, we have $\mathbb{P}_i[\mathcal{E}_{C,e}] =_{\mathcal{X}} X_e(i)/|\mathcal{H}|(i)$. An upper bound for $X_e(i)$ is given by the (random) number of semiconflicts $C \in \mathcal{C}_e$ with $|C \cap \mathcal{H}(i)| = 1$ and $|C \cap \mathcal{M}(i)| = |C| - 1$. Thus, taking the definition of $\mathcal{T}(i)$ (see page 45) into account, we have

$$X_e(i) \leq |\mathcal{C}_e^{[1]}|(i) \leq_{\mathcal{X}} \hat{c}(i) + \gamma(i).$$

As for one edge $f \in \mathcal{H}(i)$, there may be two distinct $C, C' \in \mathcal{C}_e$ with $C \setminus \mathcal{M}(i) = C' \setminus \mathcal{M}(i) = \{f\}$, the random variable $|\mathcal{C}_e^{[1]}|(i)$ may be strictly larger than $X_e(i)$. For all such C, C', the union $C \cup C'$ is a set in $\mathcal{C}_{e,2}$ with $(C \cup C') \setminus \mathcal{M}(i) = \{f\}$ and for all $C_2 \in \mathcal{C}_{e,2}$, there are at most 4^ℓ pairs $(C, C') \in \mathcal{C}^2$ with $C \cup C' = C_2$. Thus, taking the definition of $\mathcal{S}(i)$ (see page 43) into account, we obtain

$$X_e(i) \geq |\mathcal{C}_e^{[1]}|(i) - 4^\ell |\mathcal{C}_{e,2}^{[1]}|(i) \geq_{\mathcal{X}} \hat{c}(i) - \gamma(i) - d^{1-\varepsilon/4} \geq \hat{c}(i) - 2\gamma(i).$$

Hence, we have

$$\frac{\hat{c}(i) - 2\gamma(i)}{\hat{h}(i) + \eta(i)} \leq_{\mathcal{X}} \mathbb{P}_i[\mathcal{E}_{C,e}] \leq_{\mathcal{X}} \frac{\hat{c}(i) + \gamma(i)}{\hat{h}(i) - \eta(i)}. \tag{2.9.4}$$

Let us now consider $\mathbb{P}_i[\mathcal{E}_U]$. Using a union bound, we have

$$\mathbb{P}_i[\mathcal{E}_U] \leq \sum_{u\in U} \mathbb{P}_i[u \in e(i+1)] = \sum_{u\in U} \frac{|\mathcal{D}_u|(i)}{|\mathcal{H}|(i)} \leq_{\mathcal{X}} \frac{|U|(\hat{d}(i)+\delta(i))}{\hat{h}(i)-\eta(i)}.$$

Furthermore, with another union bound over the pairwise intersections of the events $\{u \in e(i+1)\}$ with $u \in U$, the pair degree bound $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon}$ together with Fact 2.9.1 yields

$$\begin{aligned}\mathbb{P}_i[\mathcal{E}_U] &\geq \sum_{u\in U} \mathbb{P}_i[u \in e(i+1)] - \sum_{u,v\in U} \mathbb{P}_i[u,v \in e(i+1)] \\ &\geq_{\mathcal{X}} \left(\sum_{u\in U} \frac{|\mathcal{D}_u|(i)}{|\mathcal{H}|(i)}\right) - |U|^2 \frac{d^{1-\varepsilon}}{\hat{h}(i)-\eta(i)} \geq_{\mathcal{X}} \frac{|U|(\hat{d}(i)-2\delta(i))}{\hat{h}(i)+\eta(i)}.\end{aligned}$$

Hence, we have

$$\frac{|U|(\hat{d}(i)-2\delta(i))}{\hat{h}(i)+\eta(i)} \leq_{\mathcal{X}} \mathbb{P}_i[\mathcal{E}_U] \leq_{\mathcal{X}} \frac{|U|(\hat{d}(i)+\delta(i))}{\hat{h}(i)-\eta(i)}. \tag{2.9.5}$$

Combining (2.9.4) and (2.9.5) and using the upper bound

$$\begin{aligned}\gamma(i) &= 2\xi(i) \sum_{j\in[\ell-1]} \Delta(\mathcal{C}^{(j+1)}) \cdot \left(\hat{z}_{j,1}(i) + j\frac{\hat{d}(i)}{\Gamma \ell d^j}\right) \\ &\leq 2\xi(i)\hat{c}(i) + \frac{2\xi(i)\hat{d}(i)}{\Gamma} \sum_{j\in[\ell-1]} \frac{\Delta(\mathcal{C}^{(j+1)})}{d^j} \leq 2\xi(i)(\hat{c}(i)+\hat{d}(i)),\end{aligned}$$

we obtain

$$\begin{aligned}\mathbb{P}_i[\mathcal{E}_{C,e} \cup \mathcal{E}_U] &\leq_{\mathcal{X}} \frac{\hat{c}(i) + |U|\hat{d}(i) + \xi(i)(2\hat{c}(i) + 2\hat{d}(i) + |U|\hat{d}(i))}{\hat{h}(i)-\eta(i)} \\ &\leq (1+7\xi(i))\frac{\hat{c}(i)+|U|\hat{d}(i)}{\hat{h}(i)}\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}_i[\mathcal{E}_{C,e} \cup \mathcal{E}_U] &\geq_{\mathcal{X}} \frac{\hat{c}(i) + |U|\hat{d}(i) - \xi(i)(4\hat{c}(i) + 4\hat{d}(i) + 2|U|\hat{d}(i))}{\hat{h}(i)+\eta(i)} \\ &\geq (1-7\xi(i))\frac{\hat{c}(i)+|U|\hat{d}(i)}{\hat{h}(i)},\end{aligned}$$

which completes the proof. □

Whenever we are given two disjoint edges $e, f \in \mathcal{H}$ that are available at some step i, the next result provides a rough upper bound for the very small probability that in the next step, the following happens: the edge e becomes unavailable due to an intersection with $e(i+1)$ while f also becomes unavailable.

Lemma 2.9.15. *Let $e, f \in \mathcal{H}$ with $e \cap f = \emptyset$ and let $i \in [m-1]_0$. Then,*

$$\mathbb{P}_i[e \cap e(i+1) \neq \emptyset \text{ and } f \notin \mathcal{H}(i+1)] \leq_{\mathcal{G}(i)\cap\{f\in\mathcal{H}(i)\}} \frac{1}{d^{\varepsilon/4}n}.$$

Proof. Conceptually, this proof is similar to that of Lemma 2.9.14 except that we only care about an upper bound. Observe that there are two reasons why f might be unavailable in step $i+1$ if it was available in step i, namely f may become unavailable due to an intersection with $e(i+1)$ or f may become unavailable due to conflicts.

Define the events $\mathcal{X} := \mathcal{G}(i)\cap\{f \in \mathcal{H}(i)\}$, $\mathcal{E}_e := \{e\cap e(i+1) \neq \emptyset\}$, $\mathcal{E}_f := \{f\cap e(i+1) \neq \emptyset\}$ and $\mathcal{E}_{C,f} := \{f \notin \mathcal{H}_C(i+1)\}$. Then, we have

$$\mathbb{P}_i[\mathcal{E}_e \cap (\mathcal{E}_f \cup \mathcal{E}_{C,f})] \leq \mathbb{P}_i[\mathcal{E}_e \cap \mathcal{E}_f] + \mathbb{P}_i[\mathcal{E}_e \cap \mathcal{E}_{C,f}].$$

We bound the two summands separately.

First, note that taking the definition of $\mathcal{T}(i)$ (see page 45) into account, $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon}$ implies

$$\mathbb{P}_i[\mathcal{E}_e \cap \mathcal{E}_f] \leq \sum_{u\in e}\sum_{v\in f} \frac{d_{\mathcal{H}}(uv)}{|\mathcal{H}|(i)} \leq_{\mathcal{X}} \frac{k^2 d^{1-\varepsilon}}{\hat{h}(i)-\eta(i)} \leq \frac{2k^2 d^{1-\varepsilon}}{\hat{h}(i)} \leq \frac{1}{2d^{\varepsilon/4}n}.$$

Furthermore, considering the definitions of partially matched subgraphs and local interactions (see pages 33 and 35) as well as the definition of $\mathcal{S}(i)$ (see page 43), for all $v \in V(\mathcal{H})$, the (random) number of edges $g \in \mathcal{H}(i)$ with $v \in g$ and $\{g\} = C \setminus \mathcal{M}(i)$ for some $C \in \mathcal{C}_f$ is at most

$$\sum_{j\in[\ell-1]} |(\mathcal{C}_f^{(j)})_v^{[1]}|(i) \leq_{\mathcal{X}} \sum_{j\in[\ell-1]} |\mathcal{C}_f^{(j)}| d^{1-j-\varepsilon/3} \leq d^{1-\varepsilon/3} \sum_{j\in[\ell-1]} \frac{\Delta(\mathcal{C}^{(j+1)})}{d^j} \leq \Gamma d^{1-\varepsilon/3}$$

and thus we have

$$\mathbb{P}_i[\mathcal{E}_e \cap \mathcal{E}_{C,f}] \leq_{\mathcal{X}} \sum_{v\in e} \frac{\Gamma d^{1-\varepsilon/3}}{|\mathcal{H}|(i)} \leq_{\mathcal{X}} \frac{k\Gamma d^{1-\varepsilon/3}}{\hat{h}(i)-\eta(i)} \leq \frac{2k\Gamma d^{1-\varepsilon/3}}{\hat{h}(i)} \leq \frac{1}{2d^{\varepsilon/4}n}.$$

□

Whenever we are given two disjoint edges $e, f \in \mathcal{H}$ that are available at some step i and that do not form a conflict of size 2, the next result provides a rough upper bound for the very small probability that in the next step both edges become unavailable.

Lemma 2.9.16. *Let $e, f \in \mathcal{H}$ with $e \cap f = \emptyset$ and $\{e, f\} \notin \mathcal{C}^{(2)}$ and let $i \in [m-1]_0$. Then,*

$$\mathbb{P}_i[e, f \notin \mathcal{H}(i+1)] \leq_{\mathcal{G}(i)\cap\{e,f\in\mathcal{H}(i)\}} \frac{1}{d^{\varepsilon/5}n}.$$

Proof. This proof is an extension of the proof of Lemma 2.9.15 and conceptually similar. Let $\mathcal{X} := \mathcal{G}(i) \cap \{e, f \in \mathcal{H}(i)\}$ and for $g \in \{e, f\}$, define the events $\mathcal{E}_{C,g} := \{g \notin \mathcal{H}_C(i+1)\}$ and $\mathcal{E}_g := \{g \cap e(i+1) \neq \emptyset\}$. We obtain

$$\mathbb{P}_i[e, f \notin \mathcal{H}(i+1)] \leq \mathbb{P}_i[\mathcal{E}_e \cap \{f \notin \mathcal{H}(i+1)\}] + \mathbb{P}_i[\mathcal{E}_f \cap \{e \notin \mathcal{H}(i+1)\}] + \mathbb{P}_i[\mathcal{E}_{C,e} \cap \mathcal{E}_{C,f}].$$

Since Lemma 2.9.15 shows that $\frac{1}{d^{\varepsilon/4}n}$ is an upper bound for the first two summands, it suffices to obtain an appropriate upper bound for $\mathbb{P}_i[\mathcal{E}_{C,e} \cap \mathcal{E}_{C,f}]$.

For $j \in [\ell - 1]$, the edge f is not an immediate evictor for $\mathcal{C}_e^{(j)}$, and so, considering the definitions of partially matched subgraphs and local interactions (see pages 33 and 35) as well as the definition of $\mathcal{S}(i)$ (see page 43), the (random) number of edges $g \in \mathcal{H}(i)$ with $\{g\} = C_e \setminus \mathcal{M}(i) = C_f \setminus \mathcal{M}(i)$ for some $C_e \in \mathcal{C}_e$ and $C_f \in \mathcal{C}_f$ is at most

$$\begin{aligned}\sum_{j\in[\ell-1]} |(\mathcal{C}_e^{(j)})_{f,2}^{[1]}|(i) &\leq_{\mathcal{X}} \sum_{j\in[\ell-1]} |\mathcal{C}_e^{(j)}| d^{1-j-\varepsilon/3} \leq d^{1-\varepsilon/3} \sum_{j\in[\ell-1]} \frac{\Delta(\mathcal{C}^{(j+1)})}{d^j} \leq \Gamma d^{1-\varepsilon/3} \\ &\leq d^{1-\varepsilon/4}\end{aligned}$$

and thus using Fact 2.9.1, we obtain

$$\mathbb{P}_i[\mathcal{E}_{C,e} \cap \mathcal{E}_{C,f}] \leq_{\mathcal{X}} \frac{d^{1-\varepsilon/4}}{|\mathcal{H}|(i)} \leq_{\mathcal{X}} \frac{d^{1-\varepsilon/4}}{\hat{h}(i) - \eta(i)} \leq \frac{2d^{1-\varepsilon/4}}{\hat{h}(i)} \leq \frac{1}{2d^{\varepsilon/5}n},$$

which completes the proof. $\square$

For all $\mathcal{Z} \in \mathscr{Z}$ and $s \in [j]$, we use that the freezing has negligible impact on the expected one-step changes of the process $|\mathcal{Z}^{[s]}|_{\mathrm{f}}^{+}(0), |\mathcal{Z}^{[s]}|_{\mathrm{f}}^{+}(1), \ldots$ in the sense that for all $i \in [m-1]_0$, we have $\mathbb{E}_i[\Delta|\mathcal{Z}^{[s]}|_{\mathrm{f}}^{+}(i)] \approx \mathbb{E}_i[\Delta|\mathcal{Z}^{[s]}|^{+}(i)]$. Recall that freezing happens in particular if $\mathcal{Z} \notin \mathscr{Z}(i+1)$, which happens if and only if $\mathcal{Z} = \mathcal{C}_e^{(j)}$ for some $j \in [\ell - 1]$ and $e \in \mathcal{H}$ with $e \notin \mathcal{H}(i+1)$. The two expectations may differ due to events where freezing occurs as a consequence of $\mathcal{Z} \in \mathscr{Z}(i) \setminus \mathscr{Z}(i+1)$ while additionally, an available edge f^- that is contained in some test $Z \in \mathcal{Z}^{[s]}(i)$ becomes unavailable or an edge f^+ that is contained in some $Z \in \mathcal{Z}^{[s+1]}(i)$ is chosen to be $e(i+1)$. Because of such an edge becoming unavailable or chosen to be $e(i+1)$, these events may provide contributions to $\mathbb{E}_i[\Delta|\mathcal{Z}^{[s]}|^{+}(i)]$, but, due to the freezing triggered by $\mathcal{Z} \notin \mathscr{Z}(i+1)$, no contribution to $\mathbb{E}_i[\Delta|\mathcal{Z}^{[s]}|_{\mathrm{f}}^{+}(i)]$. To see that the contribution of those events where an edge f^+ that is contained in some $Z \in \mathcal{Z}^{[s+1]}(i)$ is chosen to be $e(i+1)$ is very small, we consider $\mathcal{C}_e^{[s+1]}$. To see the contribution of those events where an available edge f^- that is contained in some test $Z \in \mathcal{Z}^{[s]}(i)$ becomes unavailable is also very small, we employ the following lemma. If $\mathcal{Z}$ has uniformity at least 3, then this lemma follows from Lemma 2.9.16. However, Lemma 2.9.16 can only be applied for edges e, f with $\{e, f\} \notin \mathcal{C}^{(2)}$ which prevents us from using the same argument based on Lemma 2.9.16 if $\mathcal{Z}$ is 2-uniform. The exclusion of e, f with $\{e, f\} \in \mathcal{C}^{(2)}$ in Lemma 2.9.16 is a consequence of the restriction that e in Lemma 2.8.3 (iii) cannot be an immediate evictor for $\mathcal{Z}$. This can be traced back to excluding e, f with $\{e, f\} \in \mathcal{C}^{(2)}$ in Condition (C7). This exclusion is crucial for our approach to omitting such a condition entirely in Theorem 2.5.2. To circumvent the resulting restrictions, we need some additional arguments, in particular building on (C5).

Lemma 2.9.17. *Let $j \in [\ell]$, $\mathcal{Z} \in \mathscr{Z}^{(j)}$, $s \in [j]$ and $i \in [m-1]_0$. Then,*

$$\sum_{Z \in \mathcal{Z}^{[s]}(i)} \sum_{f \in Z \setminus \mathcal{M}(i)} \mathbb{P}_i[\mathcal{Z} \notin \mathscr{Z}(i+1) \text{ and } f \notin \mathcal{H}(i+1)] \leq_{\mathcal{G}(i) \cap \{\mathcal{Z} \in \mathscr{Z}(i)\}} \frac{d^{s-j-\varepsilon/6}}{n} |\mathcal{Z}|.$$

Proof. For $j \geq 2$, the statement is a consequence of Lemma 2.9.16, for $j = 1$ we additionally use condition (C5).

Let us now turn to the details. Note that for $\mathcal{Z} \in \mathscr{Z}_0$, we have $\{\mathcal{Z} \notin \mathscr{Z}(i+1)\} = \emptyset$, so we assume that $\mathcal{Z} = \mathcal{C}_e^{(j)}$ for some $e \in \mathcal{H}$. Let $\mathcal{X} := \mathcal{G}(i) \cap \{e \in \mathcal{H}(i)\}$ and for $f \in \mathcal{H}$, define the events $\mathcal{E}_f := \{e(i+1) \cap f \neq \emptyset\}$ and $\mathcal{E}_{C,f} := \{f \notin \mathcal{H}_C(i+1)\}$.

Recall that for all $Z \in \mathcal{Z}^{[s]}(i)$, all edges $f \in Z \setminus \mathcal{M}(i)$ are available in step i in the sense that $e \in \mathcal{H}(i)$. Furthermore, no conflict in $\mathcal{C}$ is a proper subset of another conflict by (C9) and all edges in a conflict are disjoint. So if $j \geq 2$, we may combine Fact 2.9.1 and Lemma 2.9.16 such that, taking the definition of $\mathcal{T}(i)$ (see page 45) into account, we obtain

$$\sum_{Z \in \mathcal{Z}^{[s]}(i)} \sum_{f \in Z \setminus \mathcal{M}(i)} \mathbb{P}_i[e, f \notin \mathcal{H}(i+1)] \leq_{\mathcal{X}} (\hat{z}_{j,s}(i) + \zeta_{j,s}(i))|\mathcal{Z}| \cdot s \cdot \frac{1}{d^{\varepsilon/5} n} \leq \frac{d^{s-j-\varepsilon/6}}{n} |\mathcal{Z}|.$$

Now consider the case where $j = 1$. In this case, we have $s = 1$. Hence

$$\sum_{Z \in \mathcal{Z}^{[s]}(i)} \sum_{f \in Z \setminus \mathcal{M}(i)} \mathbb{P}_i[e, f \notin \mathcal{H}(i+1)] = \sum_{\{f\} \in \mathcal{Z}^{[1]}(i)} \mathbb{P}_i[e, f \notin \mathcal{H}(i+1)].$$

We have

$$\begin{aligned}\sum_{\{f\} \in \mathcal{Z}^{[1]}(i)} \mathbb{P}_i[e, f \notin \mathcal{H}(i+1)] \leq & \sum_{\{f\} \in \mathcal{Z}^{[1]}(i)} \mathbb{P}_i[\mathcal{E}_e \cap \{f \notin \mathcal{H}(i+1)\}] \\ & + \mathbb{P}_i[\mathcal{E}_f \cap \{e \notin \mathcal{H}(i+1)\}] + \sum_{\{f\} \in \mathcal{Z}^{[1]}(i)} \mathbb{P}_i[\mathcal{E}_{C,e} \cap \mathcal{E}_{C,f}].\end{aligned}$$

Since combining Fact 2.9.1 and Lemma 2.9.15 and taking the definition of $\mathcal{T}(i)$ (see page 45) into account yields

$$\begin{aligned}\sum_{\{f\} \in \mathcal{Z}^{[1]}(i)} & \mathbb{P}_i[\mathcal{E}_e \cap \{f \notin \mathcal{H}(i+1)\}] + \mathbb{P}_i[\mathcal{E}_f \cap \{e \notin \mathcal{H}(i+1)\}] \\ & \leq_{\mathcal{X}} \frac{2|\mathcal{Z}^{[1]}|(i)}{d^{\varepsilon/4} n} \leq_{\mathcal{X}} 2\frac{\hat{z}_{j,s}(i) + \zeta_{j,s}(i)}{d^{\varepsilon/4} n} |\mathcal{Z}| \leq \frac{d^{s-j-\varepsilon/5}}{n} |\mathcal{Z}|,\end{aligned}$$

it suffices to find an appropriate upper bound for

$$\sum_{\{f\} \in \mathcal{Z}^{[1]}(i)} \mathbb{P}_i[\mathcal{E}_{C,e} \cap \mathcal{E}_{C,f}].$$

For all $f, g \in \mathcal{H}$ and $j' \in [\ell]_2$, let $I_{f,g}^{(j')}$ denote the indicator random variable of the event $\{C \setminus \mathcal{M}(i) = \{f, g\}$ for some $C \in \mathcal{C}^{(j')}\}$. Note that for all edges $f \in \mathcal{H}(i)$, the event $\mathcal{E}_{C,f}$ occurs if and only if there is a conflict $C \in \mathcal{C}$ with $C \setminus \mathcal{M}(i) = \{f, e(i+1)\}$. Hence,

$$\mathcal{E}_{C,e} \cap \mathcal{E}_{C,f} = \{I^{(j_1)}_{e,e(i+1)} = I^{(j_2)}_{f,e(i+1)} = 1 \text{ for some } j_1, j_2 \in [\ell]_2\}$$

and thus

$$\begin{aligned}\sum_{\{f\}\in\mathcal{Z}^{[1]}(i)} \mathbb{P}_i[\mathcal{E}_{C,e} \cap \mathcal{E}_{C,f}] &= \sum_{\{f\}\in\mathcal{Z}^{[1]}(i)} \sum_{g\in\mathcal{H}} \sum_{j_1,j_2\in[\ell]_2} I^{(j_1)}_{e,g} \cdot I^{(j_2)}_{f,g} \cdot \mathbb{P}_i[e(i+1) = g] \\ &\leq \frac{1}{|\mathcal{H}|(i)} \sum_{j_1,j_2\in[\ell]_2} \sum_{g\in\mathcal{H}\colon I^{(j_1)}_{e,g}=1} |\{\{f\} \in \mathcal{C}^{(1)[1]}_e(i) : I^{(j_2)}_{f,g} = 1\}|.\end{aligned}$$

First, let us bound the size of $F^{(j_2)}_g := \{\{f\} \in \mathcal{C}^{(1)[1]}_e(i) : I^{(j_2)}_{f,g} = 1\}$ for all $j_1, j_2 \in [\ell]_2$ and $g \in \mathcal{H}$ with $d_{\mathcal{C}^{(j_1)}}(eg) \geq 1$ (which is necessary for $I^{(j_1)}_{e,g} = 1$ to be possible).

Fix $j_1, j_2 \in [\ell]_2$ and $g \in \mathcal{H}$ with $d_{\mathcal{C}^{(j_1)}}(eg) \geq 1$. If $j_1 \geq 3$, the set $\{e, g\}$ is a proper subset of a conflict and hence not a conflict itself by (C9). If $j_2 = 2$, then $|F^{(j_2)}_g| \leq d^{1-\varepsilon}$ as a consequence of (C5) if $j_1 = 2$ and as a consequence of (C7) if $j_1 \geq 3$. If $j_2 \geq 3$, for all $\{f\} \in F^{(j_2)}_g$, assign an arbitrary semiconflict $C_f \in \mathcal{C}^{(j_2)}_g$ with $C_f \setminus \mathcal{M}(i) = \{f\}$ to f. Note that for distinct f, f', the assigned semiconflicts C_f and $C_{f'}$ are distinct. All assigned semiconflicts are elements of $\mathcal{C}^{\star[1]}_{e,g,2}(i)$ (see definitions of partially matched subgraphs and local interactions on pages 33 and 35) and so by definition of $\mathcal{S}(i)$ (see page 43), we obtain

$$|F^{(j_2)}_g| \leq |\mathcal{C}^{\star[1]}_{e,g,2}(i)| \leq_{\mathcal{X}} d^{1-\varepsilon/3}.$$

Taking the definition of $\mathcal{T}(i)$ (see page 45) into account, we use Fact 2.9.1 and $d^{1-\varepsilon/100} \leq \delta(\mathcal{C}^{(2)}) \leq |\mathcal{Z}|$ to conclude that

$$\begin{aligned}\sum_{\{f\}\in\mathcal{Z}^{[1]}} \mathbb{P}_i[\mathcal{E}_{C,e} \cap \mathcal{E}_{C,f}] &\leq_{\mathcal{X}} \frac{d^{1-\varepsilon/3}}{|\mathcal{H}|(i)} \sum_{j_1,j_2\in[\ell]_2} |\{g \in \mathcal{H}(i) : I^{(j_1)}_{e,g} = 1\}| \\ &\leq \frac{\ell d^{1-\varepsilon/3}}{|\mathcal{H}|(i)} \sum_{j_1\in[\ell-1]} |\mathcal{C}^{(j_1)[1]}_e|(i) \\ &\leq_{\mathcal{X}} \frac{\ell d^{1-\varepsilon/3}}{|\mathcal{H}|(i)} \sum_{j_1\in[\ell-1]} (\hat{z}_{j_1,1}(i) + \zeta_{j_1,1}(i))|\mathcal{C}^{(j_1)}_e| \\ &\leq_{\mathcal{X}} \frac{2\ell d^{1-\varepsilon/3}}{\hat{h}(i)} \sum_{j_1\in[\ell-1]} (\hat{z}_{j_1,1}(i) + \zeta_{j_1,1}(i))\Delta(\mathcal{C}^{(j_1+1)}) \\ &\leq \frac{2\ell d^{1-\varepsilon/3}}{\hat{h}(i)} \sum_{j_1\in[\ell-1]} d^{1-j_1+\varepsilon/12} \cdot \Gamma d^{j_1} \leq \frac{4\Gamma\ell^2 d^{2-\varepsilon/4}}{\hat{h}(i)} \\ &\leq \frac{d^{-\varepsilon/5}}{n} d^{1-\varepsilon/100} \leq \frac{d^{s-j-\varepsilon/5}}{n}|\mathcal{Z}|,\end{aligned}$$

which completes the proof. □

Lemma 2.9.18. *Let $v \in V(\mathcal{H})$ and $* \in \{+,-\}$. Then, the process $|\mathcal{D}_v|^*_{\mathrm{f}}(0), |\mathcal{D}_v|^*_{\mathrm{f}}(1), \dots$ is a supermartingale. Moreover, for all $i \geq 0$, we have*

$$\mathbb{E}_i[|\Delta|\mathcal{D}_v|^*_{\mathrm{f}}(i)|] \leq \frac{d^{1+\varepsilon/16}}{n}.$$

Proof. Only considering the frozen processes allows us to essentially assume that $\mathcal{G}(i)$ happened. As in the proofs of Lemmas 2.9.14 and 2.9.16, this provides approximations for all key quantities in step i which then makes it easy to obtain $\mathbb{E}_i[\Delta|\mathcal{D}_v|(i)] \approx \Delta\hat{d}(i)$. As the error function δ grows sufficiently fast, this yields the supermartingale property. On the contrary, for the boundedness of the expected one-step changes it is crucial that δ does not grow too fast.

Let us turn to the details. Fix $i \in [m-1]_0$. Let $\mathcal{X} := \mathcal{G}(i) \cap \{v \in V(i)\}$. We need to prove that $\mathbb{E}_i[\Delta|\mathcal{D}_v|^*_{\mathrm{f}}(i)] \leq 0$ and $\mathbb{E}_i[|\Delta|\mathcal{D}_v|^*_{\mathrm{f}}(i)|] \leq d^{1+\varepsilon/16}/n$. Due to $\Delta|\mathcal{D}_v|^*_{\mathrm{f}} =_{\mathcal{X}^c} 0$, both bounds follow if $\mathbb{E}_i[\Delta|\mathcal{D}_v|^*_{\mathrm{f}}(i)] \leq_{\mathcal{X}} 0$ and $\mathbb{E}_i[|\Delta|\mathcal{D}_v|^*_{\mathrm{f}}(i)|] \leq_{\mathcal{X}} d^{1+\varepsilon/16}/n$, so we aim to show these two bounds with $\leq_{\mathcal{X}}$ instead of $\leq$. Let $\mathcal{E}_v := \{v \in V(i+1)\}$ and for an edge $e \in \mathcal{H}$, define the events $\mathcal{E}_{C,e} := \{e \notin \mathcal{H}_C(i+1)\}$ and $\mathcal{E}_e := \{(e \setminus \{v\}) \cap e(i+1) \neq \emptyset\}$. Let us first argue why it suffices to obtain

$$\mathbb{E}_i[\mathbb{1}_{\mathcal{E}_v}\Delta|\mathcal{D}_v|(i)] =_{\mathcal{X}} \left(\hat{d}'(i) \pm \frac{1}{2}\delta'(i)\right)\mathbb{P}_i[\mathcal{E}_v]. \tag{2.9.6}$$

To this end, note that Facts 2.9.8 and 2.9.9 provide the bounds

$$|\hat{d}'(x)| \leq \frac{d^{1+\varepsilon/32}}{n} \quad \text{and} \quad \frac{d^{1-\varepsilon/2}}{n} \leq \delta'(x) \leq |\delta'(x)| \leq \frac{d}{n}. \tag{2.9.7}$$

Note that

$$\mathbb{E}_i[\Delta|\mathcal{D}_v|^+_{\mathrm{f}}(i)] = \mathbb{E}_i[\mathbb{1}_{\mathcal{E}_v}\Delta|\mathcal{D}_v|^+(i)] = \mathbb{E}_i[\mathbb{1}_{\mathcal{E}_v}\Delta|\mathcal{D}_v|(i)] - (\Delta\hat{d}(i) + \Delta\delta(i))\mathbb{P}_i[\mathcal{E}_v].$$

If (2.9.6) holds, then this together with Fact 2.9.13 yields

$$\left(-\frac{3}{2}\delta'(i) - \frac{2d^{1-\varepsilon}}{n}\right)\mathbb{P}_i[\mathcal{E}_v] \leq_{\mathcal{X}} \mathbb{E}_i[\Delta|\mathcal{D}_v|^+_{\mathrm{f}}(i)] \leq_{\mathcal{X}} \left(-\frac{1}{2}\delta'(i) + \frac{2d^{1-\varepsilon}}{n}\right)\mathbb{P}_i[\mathcal{E}_v]$$

and consequently the bound $\delta'(i) \geq d^{1-\varepsilon/2}/n$ in (2.9.7) then entails $\mathbb{E}_i[\Delta|\mathcal{D}_v|^+_{\mathrm{f}}(i)] \leq_{\mathcal{X}} 0$. Furthermore, observe that

$$\begin{aligned}\mathbb{E}_i[|\Delta|\mathcal{D}_v|^+_{\mathrm{f}}(i)|] &= \mathbb{E}_i[\mathbb{1}_{\mathcal{E}_v}|\Delta|\mathcal{D}_v|^+(i)|] \leq \mathbb{E}_i[\mathbb{1}_{\mathcal{E}_v}|\Delta|\mathcal{D}_v|(i)|] + (|\Delta\hat{d}(i)| + |\Delta\delta(i)|)\mathbb{P}_i[\mathcal{E}_v] \\ &= -\mathbb{E}_i[\mathbb{1}_{\mathcal{E}_v}\Delta|\mathcal{D}_v|(i)] + (|\Delta\hat{d}(i)| + |\Delta\delta(i)|)\mathbb{P}_i[\mathcal{E}_v].\end{aligned}$$

If (2.9.6) holds, then, again using Fact 2.9.13, this yields

$$\mathbb{E}_i[|\Delta|\mathcal{D}_v|^+_{\mathrm{f}}(i)|] \leq_{\mathcal{X}} 2|\hat{d}'(x)| + 2|\delta'(x)| + \frac{2d^{1-\varepsilon}}{n}$$

and consequently the two bounds $|\hat{d}'(x)| \leq d^{1+\varepsilon/32}/n$ and $|\delta'(x)| \leq d/n$ in (2.9.7) then entail $\mathbb{E}_i[|\Delta|\mathcal{D}_v|^+_{\mathrm{f}}(i)|] \leq_{\mathcal{X}} d^{1+\varepsilon/16}/n$. Similar arguments can be made to see that (2.9.6) implies $\mathbb{E}[\Delta|\mathcal{D}_v|_{\mathrm{f}}^{-}(i)] \leq_{\mathcal{X}} 0$ as well as $\mathbb{E}_i[|\Delta|\mathcal{D}_v|_{\mathrm{f}}^{-}(i)|] \leq_{\mathcal{X}} d^{1+\varepsilon/16}/n$. Hence, to obtain the claimed statement, it suffices to prove (2.9.6).

Before we continue with a proof of (2.9.6), note that taking the definition of $\mathcal{T}(i)$ (see page 45) into account, Fact 2.9.1 entails

$$\mathbb{P}_i[\mathcal{E}_v] =_{\mathcal{X}} 1 - \frac{|\mathcal{D}_v|(i)}{|\mathcal{H}|(i)} \geq_{\mathcal{X}} 1 - \frac{\hat{d}(i)+\delta(i)}{\hat{h}(i)-\eta(i)} \geq 1 - \frac{2k}{n\hat{p}_V(i)} \geq 1 - d^{-\varepsilon} \geq 1 - \xi(i). \quad (2.9.8)$$

The edges in any conflict are disjoint. Thus, we have $\mathcal{E}_{C,e} \cap \mathcal{E}_e = \emptyset$ and $\mathcal{E}_v{}^{\mathsf{c}} \cap \mathcal{E}_{C,e} = \emptyset$ for all $e \in \mathcal{D}_v(i)$ and hence

$$\begin{aligned}\mathbb{E}_i[\mathbb{1}_{\mathcal{E}_v}\Delta|\mathcal{D}_v|(i)] &= -\mathbb{E}_i\Big[\mathbb{1}_{\mathcal{E}_v}\sum_{e\in\mathcal{D}_v(i)}\mathbb{1}_{\mathcal{E}_{C,e}\cup\mathcal{E}_e}\Big] = -\sum_{e\in\mathcal{D}_v(i)}\mathbb{P}_i[\mathcal{E}_v \cap (\mathcal{E}_{C,e}\cup\mathcal{E}_e)] \\ &= -\sum_{e\in\mathcal{D}_v(i)}(\mathbb{P}_i[\mathcal{E}_{C,e}\cup\mathcal{E}_e] - \mathbb{P}_i[\mathcal{E}_v{}^{\mathsf{c}}\cap\mathcal{E}_e]).\end{aligned} \quad (2.9.9)$$

We employ Lemma 2.9.14 and use $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon}$ to obtain

$$\mathbb{E}_i[\mathbb{1}_{\mathcal{E}_v}\Delta|\mathcal{D}_v|(i)] =_{\mathcal{X}} -\sum_{e\in\mathcal{D}_v(i)}\left((1\pm 7\xi(i))\frac{\hat{c}(i)+(k-1)\hat{d}(i)}{\hat{h}(i)} \pm \frac{kd^{1-\varepsilon}}{\hat{h}(i)\pm\eta(i)}\right).$$

Using Fact 2.9.1 and (2.9.8), this yields

$$\begin{aligned}\mathbb{E}_i[\mathbb{1}_{\mathcal{E}_v}\Delta|\mathcal{D}_v|(i)] &=_{\mathcal{X}} -\sum_{e\in\mathcal{D}_v(i)}\left((1\pm 7\xi(i))\frac{\hat{c}(i)+(k-1)\hat{d}(i)}{\hat{h}(i)} \pm \frac{\xi(i)\hat{d}(i)}{\hat{h}(i)}\right) \\ &=_{\mathcal{X}} -(1\pm 10\xi(i))\cdot\hat{d}(i)\cdot\frac{\hat{c}(i)+(k-1)\hat{d}(i)}{\hat{h}(i)} \\ &=_{\mathcal{X}} -(1\pm 12\xi(i))\cdot\hat{d}(i)\cdot\frac{\hat{c}(i)+(k-1)\hat{d}(i)}{\hat{h}(i)}\cdot\mathbb{P}_i[\mathcal{E}_v].\end{aligned}$$

With Lemma 2.9.5 and the expression for $\hat{d}'(x)$ given in Fact 2.9.7 this implies

$$\mathbb{E}_i[\mathbb{1}_{\mathcal{E}_v}\Delta|\mathcal{D}_v|(i)] =_{\mathcal{X}} \left(\hat{d}'(i) \pm \frac{24k^2\Gamma}{n\hat{p}_V(i)}\delta(i)\right)\mathbb{P}_i[\mathcal{E}_v]$$

and thus, with the first lower bound for $\delta'(x)$ given in Fact 2.9.8, we conclude that (2.9.6) holds. □

The following statement is the analog of Lemma 2.9.18 where for $* \in \{+,-\}$, the process $|\mathcal{D}_v|_{\mathrm{f}}^{*}(0), |\mathcal{D}_v|_{\mathrm{f}}^{*}(1), \ldots$ is replaced by $|\mathcal{Z}^{[s]}|_{\mathrm{f}}^{*}(0), |\mathcal{Z}^{[s]}|_{\mathrm{f}}^{*}(1), \ldots$ with suitable choices for $\mathcal{Z}$ and s.

Lemma 2.9.19. *Let $j \in [\ell]$, $\mathcal{Z} \in \mathscr{Z}^{(j)}$ $s \in [\ell]_0$ with $s \geq \mathbb{1}_{\mathscr{C}}(\mathcal{Z})$ and $* \in \{+,-\}$. Then, the process $|\mathcal{Z}^{[s]}|_{\mathrm{f}}^{*}(0), |\mathcal{Z}^{[s]}|_{\mathrm{f}}^{*}(1), \ldots$ is a supermartingale. Moreover, for all $i \geq 0$, we have*

$$\mathbb{E}_i[|\Delta|\mathcal{Z}^{[s]}|_{\mathrm{f}}^{*}(i)|] \leq \frac{d^{s-j+\varepsilon/16}}{n}|\mathcal{Z}|.$$

Proof. Conceptually, this proof is similar to that of Lemma 2.9.18, but technically more involved.

We assume that $s \in [j]_0$ as otherwise $|\mathcal{Z}^{[s]}|_{\mathrm{f}}^{+}(i) = 0$ for all $i \geq 0$. Fix $i \in [m-1]_0$. Let $\mathcal{X} := \mathcal{G}(i) \cap \{\mathcal{Z} \in \mathscr{Z}(i)\}$ and $\mathcal{E}_{\mathcal{Z}} := \{\mathcal{Z} \in \mathscr{Z}(i+1)\}$.

When transitioning from step i to step $i+1$, some tests in $\mathcal{Z}^{[s+1]}(i)$ containing $e(i+1)$ may move to $\mathcal{Z}^{[s]}(i+1)$ while some tests in $\mathcal{Z}^{[s]}(i)$ may no longer be present in $\mathcal{Z}^{[s]}(i+1)$ due to the choice of $e(i+1)$.

Considering the expected gain E^+ and expected loss E^-, where

$$E^+ := \mathbb{E}_i[\mathbb{1}_{\mathcal{E}_{\mathcal{Z}}} |\mathcal{Z}^{[s]}(i+1) \setminus \mathcal{Z}^{[s]}(i)|] \quad \text{and} \quad E^- := \mathbb{E}_i[\mathbb{1}_{\mathcal{E}_{\mathcal{Z}}} |\mathcal{Z}^{[s]}(i) \setminus \mathcal{Z}^{[s]}(i+1)|],$$

we have $\mathbb{E}_i[\mathbb{1}_{\mathcal{E}_{\mathcal{Z}}} \Delta|\mathcal{Z}^{[s]}|(i)] = E^+ - E^-$. We bound E^+ and E^- separately. Reflecting this separation, we also split the value $\hat{z}'_{j,s}(i) = \hat{z}'^{+}_{j,s}(i) - \hat{z}'^{-}_{j,s}(i)$ of the derivative $\hat{z}'_{j,s}$ into a gain contribution $\hat{z}'^{+}_{j,s}(i)$ and a loss contribution $\hat{z}'^{-}_{j,s}(i)$, where

$$\hat{z}'^{+}_{j,s}(i) := \frac{s+1}{\hat{h}(i)} \hat{z}_{j,s+1}(i) \quad \text{and} \quad \hat{z}'^{-}_{j,s}(i) := s\frac{\hat{c}(i) + k\hat{d}(i)}{\hat{h}(i)} \hat{z}_{j,s}(i).$$

Recall that we already encountered this separation into gain and loss in the discussion of the derivative $\hat{z}'_{j,s}$ after Fact 2.9.7. Formally, gain and loss contribution correspond to E^+ and E^- in the sense that we will obtain

$$E^+ =_{\mathcal{X}} \left(\hat{z}'^{+}_{j,s}(i) \pm \frac{1}{4}\zeta'_{j,s}(i)\right)|\mathcal{Z}|\mathbb{P}_i[\mathcal{E}_{\mathcal{Z}}] \quad \text{and} \quad E^- =_{\mathcal{X}} \left(\hat{z}'^{-}_{j,s}(i) \pm \frac{1}{4}\zeta'_{j,s}(i)\right)|\mathcal{Z}|\mathbb{P}_i[\mathcal{E}_{\mathcal{Z}}]. \tag{2.9.10}$$

Let us first argue why it suffices to show that (2.9.10) holds. To this end, note that Facts 2.9.8 and 2.9.9 provide the bounds

$$\begin{gathered} |\hat{z}'^{+}_{j,s}(i)| = \hat{z}'^{+}_{j,s}(i) \leq \frac{d^{s-j+\varepsilon/32}}{n}, \quad |\hat{z}'^{-}_{j,s}(i)| = \hat{z}'^{-}_{j,s}(i) \leq \frac{d^{s-j+\varepsilon/32}}{n} \\ \text{and} \quad \frac{d^{s-j-\varepsilon/2}}{n} \leq \zeta'_{j,s}(i) = |\zeta'_{j,s}(i)| \leq \frac{d^{s-j}}{n}. \end{gathered} \tag{2.9.11}$$

The bounds in (2.9.10) imply

$$\mathbb{E}_i[\mathbb{1}_{\mathcal{E}_{\mathcal{Z}}} \Delta|\mathcal{Z}^{[s]}|(i)] = \left(\hat{z}'_{j,s}(i) \pm \frac{1}{2}\zeta'_{j,s}(i)\right)|\mathcal{Z}|\mathbb{P}_i[\mathcal{E}_{\mathcal{Z}}]. \tag{2.9.12}$$

Note that

$$\mathbb{E}_i[\Delta|\mathcal{Z}^{[s]}|_{\mathrm{f}}^{+}(i)] = \mathbb{E}_i[\mathbb{1}_{\mathcal{E}_{\mathcal{Z}}} \Delta|\mathcal{Z}^{[s]}|(i)] - (\Delta\hat{z}_{j,s}(i) + \Delta\zeta_{j,s}(i))|\mathcal{Z}|\mathbb{P}_i[\mathcal{E}_{\mathcal{Z}}].$$

If (2.9.12), which follows from (2.9.10), holds, then this together with Fact 2.9.13 yields

$$\left(-\frac{3}{2}\zeta'_{j,s}(i) - \frac{2d^{s-j-\varepsilon}}{n}\right)|\mathcal{Z}|\mathbb{P}_i[\mathcal{E}_{\mathcal{Z}}] \leq_{\mathcal{X}} \mathbb{E}_i[\Delta|\mathcal{Z}^{[s]}|_{\mathrm{f}}^{+}(i)] \leq_{\mathcal{X}} \left(-\frac{1}{2}\zeta'_{j,s}(i) + \frac{2d^{s-j-\varepsilon}}{n}\right)|\mathcal{Z}|\mathbb{P}_i[\mathcal{E}_{\mathcal{Z}}]$$

and consequently the bound $\zeta'_{j,s}(i) \geq d^{s-j-\varepsilon/2}/n$ in (2.9.11) then entails that we have $\mathbb{E}_i[\Delta|\mathcal{Z}^{[s]}|^+_{\mathrm{f}}(i)] \leq_{\mathcal{X}} 0$. Furthermore, observe that

$$\begin{aligned}\mathbb{E}_i[|\Delta|\mathcal{Z}^{[s]}|^+_{\mathrm{f}}(i)|] &\leq \mathbb{E}_i[|\mathbb{1}_{\mathcal{E}_{\mathcal{Z}}}|\mathcal{Z}^{[s]}(i+1)\setminus\mathcal{Z}^{[s]}(i)||] + \mathbb{E}_i[|\mathbb{1}_{\mathcal{E}_{\mathcal{Z}}}|\mathcal{Z}^{[s]}(i)\setminus\mathcal{Z}^{[s]}(i+1)||] \\ &\quad+(|\Delta\hat{z}_{j,s}(i)|+|\Delta\zeta_{j,s}(i)|)|\mathcal{Z}|\mathbb{P}_i[\mathcal{E}_{\mathcal{Z}}] \\ &= E^+ + E^- + (|\Delta\hat{z}_{j,s}(i)|+|\Delta\zeta_{j,s}(i)|)|\mathcal{Z}|\mathbb{P}_i[\mathcal{E}_{\mathcal{Z}}].\end{aligned}$$

If (2.9.10) holds, then this together with Fact 2.9.13 yields

$$\mathbb{E}_i[|\Delta|\mathcal{Z}^{[s]}|^+_{\mathrm{f}}(i)|] \leq_{\mathcal{X}} \left(\hat{z}'^{+}_{j,s}(i)+\hat{z}'^{-}_{j,s}(i)+|\hat{z}'_{j,s}(i)|+2|\zeta'_{j,s}(i)|+2\frac{d^{s-j-\varepsilon}}{n}\right)|\mathcal{Z}|\mathbb{P}_i[\mathcal{E}_{\mathcal{Z}}]$$

and consequently the bounds $|\hat{z}'^{+}_{j,s}(i)| \leq d^{s-j+\varepsilon/32}/n$, $|\hat{z}'^{-}_{j,s}(i)| \leq d^{s-j+\varepsilon/32}/n$ and $|\zeta'_{j,s}(i)| \leq d^{s-j}/n$ in (2.9.11) then entail that we have $\mathbb{E}_i[|\Delta|\mathcal{Z}^{[s]}|^+_{\mathrm{f}}(i)|] \leq_{\mathcal{X}} (d^{s-j+\varepsilon/16}/n)|\mathcal{Z}|$. Similar arguments can be made to see that (2.9.10) implies $\mathbb{E}_i[\Delta|\mathcal{Z}^{[s]}|^-_{\mathrm{f}}(i)] \leq_{\mathcal{X}} 0$ as well as $\mathbb{E}_i[|\Delta|\mathcal{Z}^{[s]}|^-_{\mathrm{f}}(i)|] \leq_{\mathcal{X}} (d^{s-j+\varepsilon/16}/n)|\mathcal{Z}|$. Hence, to obtain the claimed statement, it suffices to prove (2.9.10).

Before we continue with a proof for (2.9.10), note that if $\mathcal{Z} \in \mathscr{C}$, then $\mathcal{Z} = \mathcal{C}_e^{(j)}$ for some $e \in \mathcal{H}$. In this case, we have $\mathcal{E}_{\mathcal{Z}}{}^{\mathsf{c}} = \{e \notin \mathcal{H}(i+1)\}$. If $\mathcal{Z} \in \mathscr{Z}_0$, then we have $\mathcal{E}_{\mathcal{Z}}{}^{\mathsf{c}} = \emptyset$. Hence, Lemma 2.9.14 entails that in any case

$$\mathbb{P}_i[\mathcal{E}_{\mathcal{Z}}] \geq_{\mathcal{X}} 1-(1+7\xi(i))\frac{\hat{c}(i)+k\hat{d}(i)}{\hat{h}(i)}$$

and thus, as a consequence of Fact 2.9.1 and Lemma 2.9.5,

$$\mathbb{P}_i[\mathcal{E}_{\mathcal{Z}}] \geq_{\mathcal{X}} 1-2k\frac{\hat{c}(i)+\hat{d}(i)}{\hat{h}(i)} \geq 1-\frac{4k^2\Gamma}{n\hat{p}_V(i)} \geq 1-d^{-\varepsilon} \geq 1-\xi(i). \tag{2.9.13}$$

To prove the estimate for E^+ in (2.9.10), we first obtain a lower and an upper bound for E^+ which we subsequently combine to obtain the desired bounds for E^+ after some further analysis.

First, we consider an upper bound for E^+. The tests that, depending on the choice of $e(i+1)$, may enter the test system when transitioning from $\mathcal{Z}^{[s]}(i)$ to $\mathcal{Z}^{[s]}(i+1)$ are elements of $\mathcal{Z}^{[s+1]}(i)$. Every such test $Z \in \mathcal{Z}^{[s+1]}(i)$ is a test in $\mathcal{Z}^{[s]}(i+1)$ only if $e(i+1) \in Z \cap \mathcal{H}(i)$. Hence, $\mathbb{P}_i[Z \in \mathcal{Z}^{[s]}(i+1)] \leq (s+1)/|\mathcal{H}|(i)$, so we obtain

$$E^+ \leq \mathbb{E}_i[|\mathcal{Z}^{[s]}(i+1)\setminus\mathcal{Z}^{[s]}(i)|] \leq |\mathcal{Z}^{[s+1]}|(i)\frac{s+1}{|\mathcal{H}|(i)}. \tag{2.9.14}$$

For a lower bound, observe that a test $Z \in \mathcal{Z}^{[s+1]}(i)$ must enter the test system when transitioning from $\mathcal{Z}^{[s]}(i)$ to $\mathcal{Z}^{[s]}(i+1)$ if $e(i+1) \in Z$ unless there is a conflict $C \in \mathcal{C}$ containing two distinct available edges $f, g \in Z$ and $|C|-2$ edges in $\mathcal{M}(i)$, thus enforcing that g is unavailable in step $i+1$ if $e(i+1)$ is chosen to be f. Note that every test Z for which such a conflict exists is a subset of a set in $\mathcal{Z}_2^{[s+1]}(i)$ (see definitions of partially matched subgraphs and local interactions on pages 33 and 35). Every set in $\mathcal{Z}_2^{[s+1]}(i)$ has size at most 2ℓ and hence at most $2^{2\ell} = 4^\ell$ subsets. Thus, by definition of $\mathcal{S}(i)$ (see page 43) and Fact 2.9.1,

$$4^\ell |\mathcal{Z}_2^{[s+1]}|(i) \leq_{\mathcal{X}} 4^\ell d^{s-j+1-\varepsilon/3} |\mathcal{Z}| \leq \zeta_{j,s+1}(i)|\mathcal{Z}|$$

is an upper bound for the (random) number of tests $Z \in \mathcal{Z}^{[s+1]}(i)$ for which such a conflict exists. Furthermore, recall that as stated before, we either have $\mathcal{E}_{\mathcal{Z}}{}^{\mathsf{c}} = \emptyset$ or $\mathcal{Z} = \mathcal{C}_e^{(j)}$ and hence $\mathcal{E}_{\mathcal{Z}}{}^{\mathsf{c}} = \{e \notin \mathcal{H}(i+1)\}$ for some edge $e \in \mathcal{H}$ with $\{e \in \mathcal{H}(i)\} = \{\mathcal{Z} \in \mathscr{Z}(i)\}$. Similarly as above, observe the following. Considering the second case where $\mathcal{Z} = \mathcal{C}_e^{(j)}$, for $Z \in \mathcal{Z}^{[s+1]}(i)$ choosing $e(i+1)$ to be an element of Z while simultaneously making e unavailable in step $i+1$ is only possible if there is a conflict $C \in \mathcal{C}$ containing e, another available edge $f \in Z$ and $|C|-2$ edges in $\mathcal{M}(i)$, thus enforcing that e is unavailable in step $i+1$ if $e(i+1)$ is chosen to be f. Here, note that every test Z for which such a conflict exists is a subset of a set in $\mathcal{C}_{e,2}^{[s+1]}(i)$ (again, see definitions of partially matched subgraphs and local interactions on pages 33 and 35). Every set in $\mathcal{C}_{e,2}^{[s+1]}(i)$ has size at most 2ℓ and hence at most $2^{2\ell} = 4^\ell$ subsets. Thus, by definition of $\mathcal{S}(i)$ (see page 43), $d^{j-\varepsilon/100} \leq \delta(\mathcal{C}^{(j+1)}) \leq |\mathcal{Z}|$ and Fact 2.9.1,

$$4^\ell |\mathcal{C}_{e,2}^{[s+1]}|(i) \leq_{\mathcal{X}} 4^\ell d^{s+1-\varepsilon/3} \leq 4^\ell d^{s-j+1-\varepsilon/4} |\mathcal{Z}| \leq \zeta_{j,s+1}(i)|\mathcal{Z}|$$

is an upper bound for the (random) number of tests $Z \in \mathcal{Z}^{[s+1]}(i)$ for which such a conflict exists. In any case, we obtain

$$E^+ \geq_{\mathcal{X}} (|\mathcal{Z}^{[s+1]}|(i) - 2\zeta_{j,s+1}(i)|\mathcal{Z}|) \frac{s+1}{|\mathcal{H}|(i)}. \tag{2.9.15}$$

Combining (2.9.14) and (2.9.15) and taking the definition of $\mathcal{T}(i)$ (see page 45) into account, we obtain

$$\begin{aligned}
E^+ &=_{\mathcal{X}} (|\mathcal{Z}^{[s+1]}|(i) \pm 2\zeta_{j,s+1}(i)|\mathcal{Z}|) \frac{s+1}{|\mathcal{H}|(i)} =_{\mathcal{X}} (s+1) \frac{\hat{z}_{j,s+1}(i) \pm 3\zeta_{j,s+1}(i)}{\hat{h}(i) \pm \eta(i)} |\mathcal{Z}| \\
&= (s+1)(1 \pm 2\xi(i)) \frac{\hat{z}_{j,s+1}(i) \pm 3\zeta_{j,s+1}(i)}{\hat{h}(i)} |\mathcal{Z}| \\
&= \left((s+1) \frac{\hat{z}_{j,s+1}(i)}{\hat{h}(i)} \pm (s+1) \frac{2\xi(i)\hat{z}_{j,s+1}(i) + 4\zeta_{j,s+1}(i)}{\hat{h}(i)} \right) |\mathcal{Z}| \\
&= \left(\hat{z}'^{+}_{j,s}(i) \pm 6(s+1) \frac{\zeta_{j,s+1}(i)}{\hat{h}(i)} \right) |\mathcal{Z}|.
\end{aligned}$$

Using (2.9.13), this yields

$$E^+ =_{\mathcal{X}} \left(\hat{z}'^{+}_{j,s}(i) \pm 8(s+1) \frac{\zeta_{j,s+1}(i)}{\hat{h}(i)} \right) |\mathcal{Z}| \mathbb{P}_i[\mathcal{E}_Z].$$

With Lemma 2.9.6 and the first lower bound for $\zeta'_{j,s}(i)$ given in Fact 2.9.8, this entails the bounds for E^+ stated in (2.9.10).

It remains to prove the bounds for E^- given in (2.9.10). We proceed similarly to the approach we used for E^+. For all $e \in \mathcal{H}$, let $\mathcal{E}_e := \{e \notin \mathcal{H}(i+1)\}$. A test leaves the test system when transitioning from $\mathcal{Z}^{[s]}(i)$ to $\mathcal{Z}^{[s]}(i+1)$ if and only if one of its s available elements becomes unavailable due to the choice of $e(i+1)$, so we have

$$E^- = \sum_{Z \in \mathcal{Z}^{[s]}(i)} \mathbb{P}_i\Big[\bigcup_{e \in Z \setminus \mathcal{M}(i)} \mathcal{E}_{\mathcal{Z}} \cap \mathcal{E}_e\Big]. \tag{2.9.16}$$

For an upper bound, simply note that

$$E^- \leq \sum_{Z \in \mathcal{Z}^{[s]}(i)} \sum_{e \in Z \setminus \mathcal{M}(i)} \mathbb{P}_i[\mathcal{E}_e]. \tag{2.9.17}$$

For a lower bound, we employ Lemma 2.9.17 to obtain

$$\begin{aligned}
E^- &\geq \sum_{Z \in \mathcal{Z}^{[s]}(i)} \Big(\sum_{e \in Z \setminus \mathcal{M}(i)} \mathbb{P}_i[\mathcal{E}_{\mathcal{Z}} \cap \mathcal{E}_e] - \sum_{e,f \in Z \setminus \mathcal{M}(i) \colon e \neq f} \mathbb{P}_i[\mathcal{E}_e \cap \mathcal{E}_f] \Big) \\
&= \sum_{Z \in \mathcal{Z}^{[s]}(i)} \Big(\sum_{e \in Z \setminus \mathcal{M}(i)} \mathbb{P}_i[\mathcal{E}_e] - \sum_{e,f \in Z \setminus \mathcal{M}(i) \colon e \neq f} \mathbb{P}_i[\mathcal{E}_e \cap \mathcal{E}_f] - \sum_{e \in Z \setminus \mathcal{M}(i)} \mathbb{P}_i[\mathcal{E}_{\mathcal{Z}}{}^{\mathsf{c}} \cap \mathcal{E}_e] \Big) \\
&\geq_{\mathcal{X}} -\frac{d^{s-j-\varepsilon/6}}{n}|\mathcal{Z}| + \sum_{Z \in \mathcal{Z}^{[s]}(i)} \Big(\sum_{e \in Z \setminus \mathcal{M}(i)} \mathbb{P}_i[\mathcal{E}_e] - \sum_{e,f \in Z \setminus \mathcal{M}(i) \colon e \neq f} \mathbb{P}_i[\mathcal{E}_e \cap \mathcal{E}_f] \Big).
\end{aligned}$$

Due to Lemma 2.9.16 and the fact that all tests $Z \in \mathcal{Z}$ are $\mathcal{C}$-free, this yields

$$E^- \geq_{\mathcal{X}} -\frac{d^{s-j-\varepsilon/6}}{n}|\mathcal{Z}| + \sum_{Z \in \mathcal{Z}^{[s]}(i)} \Big(-\frac{s^2}{d^{\varepsilon/5}n} + \sum_{e \in Z \setminus \mathcal{M}(i)} \mathbb{P}_i[\mathcal{E}_e] \Big). \tag{2.9.18}$$

Combining (2.9.17) and (2.9.18), using Lemma 2.9.14 as well as Fact 2.9.1 and taking the definition of $\mathcal{T}(i)$ (see page 45) into account, we obtain

$$\begin{aligned}
E^- &=_{\mathcal{X}} (\hat{z}_{j,s}(i)) \pm \zeta_{j,s}(i)) \Big((1 \pm 7\xi(i)) s \frac{\hat{c}(i) + k\hat{d}(i)}{\hat{h}(i)} \pm \frac{s^2}{d^{\varepsilon/5}n} \Big) \pm \frac{d^{s-j-\varepsilon/6}}{n}|\mathcal{Z}| \\
&= (\hat{z}_{j,s}(i)) \pm \zeta_{j,s}(i)) \Big((1 \pm 7\xi(i)) s \frac{\hat{c}(i) + k\hat{d}(i)}{\hat{h}(i)} \pm s\frac{\xi(i)\hat{d}(i)}{\hat{h}(i)} \Big) \pm \frac{\zeta_{j,s}(i)\hat{d}(i)}{\hat{h}(i)}|\mathcal{Z}| \\
&= (\hat{z}_{j,s}(i)) \pm \zeta_{j,s}(i)) \Big((1 \pm 8\xi(i)) s \frac{\hat{c}(i) + k\hat{d}(i)}{\hat{h}(i)} \Big) |\mathcal{Z}| \pm \frac{\zeta_{j,s}(i)\hat{d}(i)}{\hat{h}(i)}|\mathcal{Z}| \\
&= \Big(s\frac{\hat{c}(i) + k\hat{d}(i)}{\hat{h}(i)} \zeta_{j,s}(i) \pm \ell \frac{\hat{c}(i) + k\hat{d}(i)}{\hat{h}(i)} (8\xi(i)\hat{z}_{j,s}(i) + 3\zeta_{j,s}(i)) \Big) |\mathcal{Z}| \\
&= \Big(\hat{z}'^{-}_{j,s}(i) \pm 11\ell \frac{\hat{c}(i) + k\hat{d}(i)}{\hat{h}(i)} \zeta_{j,s}(i) \Big) |\mathcal{Z}|.
\end{aligned}$$

Using (2.9.13), this yields

$$E^- =_{\mathcal{X}} \left(\hat{z}'^{-}_{j,s}(i) \pm 13\ell \frac{\hat{c}(i) + k\hat{d}(i)}{\hat{h}(i)} \zeta_{j,s}(i)\right) |\mathcal{Z}| \mathbb{P}_i[\mathcal{E}_Z].$$

With Lemma 2.9.5 and the first lower bound for $\zeta'_{j,s}(i)$ given in Fact 2.9.8, this entails the bounds for E^- stated in (2.9.10). □

2.9.3 Absolute changes

In this section, we show that for all $v \in V(\mathcal{H})$, $\mathcal{Z} \in \mathscr{X}$, $s \in [\ell]_0$ with $s \geq \mathbb{1}_{\mathscr{C}}(\mathcal{Z})$, $* \in \{+,-\}$ and $i \geq 0$, the absolute one-step changes

$$|\Delta|\mathcal{D}_v|^*_{\mathrm{f}}(i)| \quad \text{and} \quad |\Delta|\mathcal{Z}^{[s]}|^*_{\mathrm{f}}(i)|$$

are never too large. In the following two lemmas, we consider both quantities separately.

Lemma 2.9.20. *Let $v \in V(\mathcal{H})$, $i \geq 0$ and $* \in \{+,-\}$. Then,*

$$|\Delta|\mathcal{D}_v|^*_{\mathrm{f}}(i)| \leq d^{1-\varepsilon/4}.$$

Proof. On a high level, this is a consequence of $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon}$ as well as the fact that we freeze the process whenever $\mathcal{S}(i)^{\mathsf{c}}$ or $v \notin V(i+1)$ occurs.

Let $\mathcal{X} := \mathcal{G}(i) \cap \{v \in V(i+1)\}$. Due to $\Delta|\mathcal{D}_v|^*_{\mathrm{f}} =_{\mathcal{X}^{\mathsf{c}}} 0$, the desired bound follows if $|\Delta|\mathcal{D}_v|^*_{\mathrm{f}}(i)| \leq_{\mathcal{X}} d^{1-\varepsilon/4}$, so we aim to show the bound with $\leq_{\mathcal{X}}$ instead of $\leq$. Let us first argue why it suffices to obtain

$$|\Delta|\mathcal{D}_v|(i)| \leq_{\mathcal{X}} 2d^{1-7\varepsilon/24}. \tag{2.9.19}$$

Assuming (2.9.19), Fact 2.9.13 entails

$$|\Delta|\mathcal{D}_v|^+(i)| \leq |\Delta|\mathcal{D}_v|(i)| + |\Delta\hat{d}(i)| + |\Delta\delta(i)| \leq_{\mathcal{X}} 3d^{1-7\varepsilon/24} + |\hat{d}'(i)| + |\delta'(i)|$$

and thus using Fact 2.9.9, we conclude that $|\Delta|\mathcal{D}_v|^+(i)| \leq_{\mathcal{X}} d^{1-\varepsilon/4}$. Also starting with (2.9.19), a similar argument using Facts 2.9.13 and 2.9.9 shows that $|\Delta|\mathcal{D}_v|^-(i)| \leq_{\mathcal{X}} d^{1-\varepsilon/4}$.

Let us now prove (2.9.19). We have

$$|\Delta|\mathcal{D}_v|(i)| \leq |\mathcal{D}_v(i) \cap E_C(i+1)| + \sum_{w \in e(i+1)} |\mathcal{D}_v(i) \cap \mathcal{D}_w(i)|. \tag{2.9.20}$$

We bound the two summands separately.

Whenever e is an element of $\mathcal{D}_v(i) \cap E_C(i+1)$, then it is the single available element of an edge $C \in (\mathcal{C}^{(j)}_{e(i+1)})^{[1]}_v(i)$ for some $j \in [\ell-1]$ (see definitions of partially matched subgraphs and local interactions on pages 33 and 35). Thus, taking the definition of $\mathcal{S}(i)$ (see page 43) into account, we have

$$|\mathcal{D}_v(i) \cap E_C(i+1)| \leq \sum_{j\in[\ell-1]} |(\mathcal{C}^{(j)}_{e(i+1)})^{[1]}_v|(i) \leq_{\mathcal{X}} d^{1-\varepsilon/3} \sum_{j\in[\ell-1]} \frac{|\mathcal{C}^{(j)}_{e(i+1)}|}{d^j} \leq \Gamma d^{1-\varepsilon/3}$$
$$\leq d^{1-7\varepsilon/24}.$$

Furthermore, $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon}$ entails

$$\sum_{w\in e(i+1)} |\mathcal{D}_v(i) \cap \mathcal{D}_w(i)| \leq_{\mathcal{X}} k \cdot d^{1-\varepsilon} \leq d^{1-7\varepsilon/24},$$

which completes the proof. □

Lemma 2.9.21. *Let $j \in [\ell]$, $\mathcal{Z} \in \mathscr{Z}^{(j)}$, $s \in [\ell]_0$ with $s \geq \mathbb{1}_{\mathscr{C}}(\mathcal{Z})$, $i \geq 0$ and $* \in \{+,-\}$. Then,*

$$|\Delta|\mathcal{Z}^{[s]}|^*_{\mathrm{f}}(i)| \leq d^{s-j-\varepsilon/4}|\mathcal{Z}|.$$

Proof. This proof is conceptually very similar to that of Lemma 2.9.20. Let $\mathcal{X} := \mathcal{G}(i) \cap \{\mathcal{Z} \in \mathscr{Z}(i+1)\}$. Let us first argue why it suffices to obtain

$$|\Delta|\mathcal{Z}^{[s]}|(i)| \leq_{\mathcal{X}} 3d^{s-j-7\varepsilon/24}|\mathcal{Z}|. \tag{2.9.21}$$

Assuming (2.9.21), Fact 2.9.13 entails

$$|\Delta|\mathcal{Z}^{[s]}|^+(i)| \leq |\Delta|\mathcal{Z}^{[s]}|(i)| + (|\Delta\hat{z}_{j,s}(i)| + |\Delta\zeta_{j,s}(i)|)|\mathcal{Z}|$$
$$\leq_{\mathcal{X}} (4d^{s-j-7\varepsilon/24} + |\hat{z}'_{j,s}(i)| + |\zeta'_{j,s}(i)|)|\mathcal{Z}|$$

and thus using Fact 2.9.9, we conclude that $|\Delta|\mathcal{Z}^{[s]}|^+(i)| \leq_{\mathcal{X}} d^{s-j-\varepsilon/4}|\mathcal{Z}|$. Also starting with (2.9.21), a similar argument using Facts 2.9.13 and 2.9.9 shows that $|\Delta|\mathcal{Z}^{[s]}|^-(i)| \leq_{\mathcal{X}} d^{s-j-\varepsilon/4}|\mathcal{Z}|$.

Let us now prove (2.9.21). Observe that

$$|\Delta|\mathcal{Z}^{[s]}|(i)| \leq |\{Z \in \mathcal{Z}^{[s+1]}(i) : e(i+1) \in Z\}| + |\{Z \in \mathcal{Z}^{[s]}(i) : Z \cap E_C(i+1) \neq \emptyset\}|$$
$$+ \sum_{v\in e(i+1)} |\{Z \in \mathcal{Z}^{[s]}(i) : Z \cap \mathcal{D}_v(i) \neq \emptyset\}|.$$

We bound the three summands separately.

For every test $Z \in \mathcal{Z}^{[s+1]}(i)$ with $e(i+1) \in Z$, the set $Z' = Z \setminus \{e(i+1)\}$ is a set in $\mathcal{Z}^{[s]}_{e(i+1)}(i)$ (see definitions of partially matched subgraphs and local interactions on pages 33 and 35) and for distinct $Z_1, Z_2 \in \mathcal{Z}^{[s+1]}(i)$ with $e(i+1) \in Z_1, Z_2$, the sets Z'_1 and Z'_2 are distinct, so, taking the definition of $\mathcal{S}(i)$ (see page 43) into account, we have

$$|\{Z \in \mathcal{Z}^{[s+1]}(i) : e(i+1) \in Z\}| \leq |\mathcal{Z}^{[s]}_{e(i+1)}|(i) \leq_{\mathcal{X}} d^{s-j-\varepsilon/3}|\mathcal{Z}|.$$

It remains to bound the other two summands.

If $s = 0$, the two remaining summands are zero. Hence, suppose that $s \geq 1$. Whenever Z is a test in $\mathcal{Z}^{[s]}(i)$ with $Z \cap E_C(i+1) \neq \emptyset$, then Z is a subset of a set in $\mathcal{Z}^{[s]}_{e(i+1),2}(i)$ (again, see definitions of partially matched subgraphs and local interactions on pages 33 and 35) and every set in $\mathcal{Z}^{[s]}_{e(i+1),2}(i)$ has size at most 2ℓ and hence at most $2^{2\ell} = 4^\ell$ subsets. Furthermore, if $\mathcal{Z} \in \mathscr{Z}(i+1)$, then $e(i+1)$ is not an immediate evictor for $\mathcal{Z}$, so, again taking the definition of $\mathcal{S}(i)$ (see page 43) into account, we have

$$|\{Z \in \mathcal{Z}^{[s]}(i) : Z \cap E_C(i+1) \neq \emptyset\}| \leq 4^\ell |\mathcal{Z}^{[s]}_{e(i+1),2}|(i) \leq_{\mathcal{X}} 4^\ell d^{s-j-\varepsilon/3}|\mathcal{Z}|.$$

Whenever Z is a test in $\mathcal{Z}^{[s]}(i)$ with $Z \cap \mathcal{D}_v(i) \neq \emptyset$ for some $v \in V(\mathcal{H})$, then Z is a set in $\mathcal{Z}^{[s]}_v(i)$ (again, see definitions of partially matched subgraphs and local interactions on pages 33 and 35), so, again taking the definition of $\mathcal{S}(i)$ (see page 43) into account, we have

$$\sum_{v \in e(i+1)} |\{Z \in \mathcal{Z}^{[s]}(i) : Z \cap \mathcal{D}_v(i) \neq \emptyset\}| = \sum_{v \in e(i+1)} |\mathcal{Z}^{[s]}_v|(i) \leq_{\mathcal{X}} k \cdot d^{s-j-\varepsilon/3}|\mathcal{Z}|.$$

Combining those bounds, we conclude that

$$|\Delta|\mathcal{Z}^{[s]}|(i)| \leq_{\mathcal{X}} d^{s-j-\varepsilon/3}|\mathcal{Z}| + 4^\ell d^{s-j-\varepsilon/3}|\mathcal{Z}| + kd^{s-j-\varepsilon/3}|\mathcal{Z}|$$

and hence (2.9.21) holds.

□

2.9.4 Supermartingale argument

Using the results from the previous two sections, we immediately obtain the following two statements from Freedman's inequality for supermartingales.

Lemma 2.9.22. *For all $* \in \{+,-\}$, we have*

$$\mathbb{P}[|\mathcal{D}_v|^*_{\mathrm{f}}(i) > 0 \text{ for some } v \in V(\mathcal{H}),\ i \geq 0] \leq \exp(-d^{\varepsilon/32}).$$

Proof. As the results in the previous sections allow us to apply Freedman's inequality for supermartingales, this is a consequence of the fact that $|\mathcal{D}_v|^+_{\mathrm{f}}(0)$ and $|\mathcal{D}_v|^-_{\mathrm{f}}(0)$ are sufficiently small for all $v \in V(\mathcal{H})$.

Let $* \in \{+,-\}$ and $v \in V(\mathcal{H})$. First, note that

$$|\mathcal{D}_v|^*_{\mathrm{f}}(0) = \pm(|\mathcal{D}_v|(i) - \hat{d}(0)) - \delta(0) = \pm d^{1-\varepsilon} - d^{1-\varepsilon/32} \leq -\frac{d^{1-\varepsilon/32}}{2}.$$

Lemmas 2.9.18 and 2.9.20 allow us to apply Lemma 2.9.4 with $d^{1-\varepsilon/4}$, $d^{1+\varepsilon/16}$, $d^{1-\varepsilon/32}/2$ playing the roles of a, b, t to obtain

$$\mathbb{P}[|\mathcal{D}_v|^*_{\mathrm{f}}(i) > 0 \text{ for some } i \geq 0] \leq \mathbb{P}\left[|\mathcal{D}_v|^*_{\mathrm{f}}(i) \geq |\mathcal{D}_v|^*_{\mathrm{f}}(0) + \frac{d^{1-\varepsilon/32}}{2} \text{ for some } i \geq 0\right]$$

$$\leq \exp\left(-\frac{d^{2(1-\varepsilon/32)}}{8d^{1-\varepsilon/4}\cdot(d^{1-\varepsilon/32}+d^{1+\varepsilon/16})}\right)\leq \exp\left(-\frac{d^{\varepsilon/8}}{16}\right).$$

A union bound completes the proof. □

Lemma 2.9.23. *For all $* \in \{+,-\}$, we have*

$$\mathbb{P}[|\mathcal{Z}^{[s]}|^*_{\mathrm{f}}(i)>0 \textit{ for some } \mathcal{Z}\in\mathscr{Z},\ s\in[\ell]_0 \textit{ with } s\geq \mathbb{1}_{\mathscr{C}}(\mathcal{Z})]\leq \exp(-d^{\varepsilon/32}).$$

Proof. Let $* \in \{+,-\}$, $j\in[\ell]$, $\mathcal{Z}\in\mathscr{Z}^{(j)}$ and $s\in[j]_0$ with $s\geq \mathbb{1}_{\mathscr{C}}(\mathcal{Z})$. First, note that $|\mathcal{Z}^{[s]}|(0)=\hat{z}_{j,s}(0)|\mathcal{Z}|$. Hence,

$$\begin{aligned}|\mathcal{Z}^{[s]}|^*_{\mathrm{f}}(0)&=\pm(|\mathcal{Z}^{[s]}|(0)-\hat{z}_{j,s}(0)|\mathcal{Z}|)-\zeta_{j,s}(0)|\mathcal{Z}|=-\zeta_{j,s}(0)|\mathcal{Z}|\leq -\xi(0)\binom{j}{s}\frac{\hat{d}(0)^s}{\Gamma\ell d^j}|\mathcal{Z}|\\&\leq -d^{s-j-\varepsilon/16}|\mathcal{Z}|.\end{aligned}$$

Lemmas 2.9.19 and 2.9.21 allow us to apply Lemma 2.9.4 with $d^{s-j-\varepsilon/4}|\mathcal{Z}|$, $d^{s-j+\varepsilon/16}|\mathcal{Z}|$, $d^{s-j-\varepsilon/16}|\mathcal{Z}|$ playing the roles of a, b, t to obtain

$$\begin{aligned}\mathbb{P}[|\mathcal{Z}^{[s]}|^*_{\mathrm{f}}(i)>0 \text{ for some } i\geq 0]&\leq \mathbb{P}[|\mathcal{Z}^{[s]}|^*_{\mathrm{f}}(i)>|\mathcal{Z}^{[s]}|^*_{\mathrm{f}}(0)+d^{s-j-\varepsilon/16}|\mathcal{Z}| \text{ for some } i\geq 0]\\&\leq \exp\left(-\frac{d^{2(s-j-\varepsilon/16)}|\mathcal{Z}|^2}{2d^{s-j-\varepsilon/4}|\mathcal{Z}|\cdot(d^{s-j-\varepsilon/16}+d^{s-j+\varepsilon/16})|\mathcal{Z}|}\right)\\&\leq \exp\left(-\frac{d^{\varepsilon/16}}{4}\right).\end{aligned}$$

A union bound completes the proof. □

We are now ready to prove Theorem 2.9.2 which in turn yields Theorem 2.6.2.

Proof of Theorem 2.9.2. We argue as described towards the end of the introduction to this section. Due to (2.9.1), we have $\mathcal{T}(i)\subseteq\mathcal{A}(i)$ for all $i\leq m$ and thus Lemma 2.8.6 entails

$$\mathbb{P}[\tau_{\mathcal{S}}\leq\min(\tau_{\mathcal{T}},m)]\leq\exp(-d^{\varepsilon/(400\ell)}).$$

Lemma 2.9.3 shows that if the event $\{\tau_{\mathcal{T}}\leq\min(\tau_{\mathcal{S}},m)\}$ occurs, then at least one of the events

$$\begin{gathered}\{|\mathcal{D}_v|^+_{\mathrm{f}}(m)>0 \text{ for some } v\in V(\mathcal{H})\},\quad \{|\mathcal{D}_v|^-_{\mathrm{f}}(m)>0 \text{ for some } v\in V(\mathcal{H})\},\\ \{|\mathcal{Z}^{[s]}|^+_{\mathrm{f}}(m)>0 \text{ for some } \mathcal{Z}\in\mathscr{Z},\ s\in[\ell]_0 \text{ with } s\geq\mathbb{1}_{\mathscr{C}}(\mathcal{Z})\},\\ \text{and}\quad \{|\mathcal{Z}^{[s]}|^-_{\mathrm{f}}(m)>0 \text{ for some } \mathcal{Z}\in\mathscr{Z},\ s\in[\ell]_0 \text{ with } s\geq\mathbb{1}_{\mathscr{C}}(\mathcal{Z})\}\end{gathered}$$

occurs (see the discussion after Lemma 2.9.4). Hence, Lemmas 2.9.22 and 2.9.23 entail

$$\mathbb{P}[\tau_{\mathcal{T}}\leq\min(\tau_{\mathcal{S}},m)]\leq 4\exp(-d^{\varepsilon/32}).$$

This yields

$$\mathbb{P}\Big[\Big(\bigcap_{i\in[m]_0}\mathcal{T}(m)\Big)^{\mathsf{c}}\Big] = \mathbb{P}[\tau_{\mathcal{T}} \leq m] \leq \mathbb{P}[\tau_{\mathcal{T}} \leq m \text{ and } \tau_{\mathcal{T}} \leq \tau_{\mathcal{S}}] + \mathbb{P}[\tau_{\mathcal{T}} \leq m \text{ and } \tau_{\mathcal{S}} \leq \tau_{\mathcal{T}}]$$
$$\leq \mathbb{P}[\tau_{\mathcal{T}} \leq \min(\tau_{\mathcal{S}}, m)] + \mathbb{P}[\tau_{\mathcal{S}} \leq \min(\tau_{\mathcal{T}}, m)] \leq \exp(-d^{\varepsilon/(500\ell)}).$$

□

2.10 Proofs for the theorems in Section 2.5

In this section, we provide proofs for Theorems 2.5.2–2.5.4 by showing that they follow from Theorem 2.6.2. Here, we do not use the setup stated in the beginning of Section 2.6.

For the probabilistic constructions in this section, we need the following concentration inequalities.

Lemma 2.10.1 (Chernoff's inequality)**.** *Suppose $X_1, \dots, X_n$ are independent Bernoulli random variables and let $X := \sum_{i\in[n]} X_i$. Then, the following holds.*

(i) $\mathbb{P}[X \neq (1 \pm \delta)\mathbb{E}[X]] \leq 2\exp(-\delta^2\mathbb{E}[X]/3)$ *for all* $0 < \delta < 1$*;*

(ii) $\mathbb{P}[X \geq 2t] \leq \exp(-t/3)$ *for all positive* $t \geq \mathbb{E}[X]$*.*

Lemma 2.10.2 (McDiarmid's inequailty)**.** *Suppose $X_1, \dots, X_n$ are independent random variables and suppose $f\colon \mathrm{Im}(X_1)\times\cdots\times\mathrm{Im}(X_n) \to \mathbb{R}$ is a function such that for all $i \in [n]$, changing the i-th coordinate of $x \in \mathrm{dom}(f)$ changes $f(x)$ by at most $c_i > 0$. Then, for all $t > 0$,*

$$\mathbb{P}[f(X_1,\dots,X_n) - \mathbb{E}[f(X_1,\dots,X_n)] \geq t] \leq \exp\Big(-\frac{2t^2}{\sum_{i\in[n]} c_i^2}\Big).$$

2.10.1 Fractional degrees

In preparation for the proofs of the theorems in Section 2.5, consider the following situation. Suppose V is a finite set and $k \geq 1$ is an integer. Suppose that we are assigning a weight $w(e) \geq 0$ to every set $e \in \binom{V}{k}$ and suppose that for all $v \in V$, we are given a target value $d(v) \geq 0$ that we wish to realize as the total weight $\sum_{e\in\binom{V}{k}\colon v\in e} w(e)$ of all sets containing v. This can be interpreted as a fractional version of the problem of finding a k-graph $\mathcal{H}$ with vertex set V where $d_{\mathcal{H}}(v) = d(v)$ for all $v \in V$. In Lemma 2.10.4, which is a consequence of Lemma 2.10.3, we consider one possible assignment of weights that achieves the stated goal approximately.

Recall that for a function $f\colon A \to \mathbb{R}$ and a finite set $X \subseteq A$, we defined $w(X) := \sum_{x\in X} w(x)$.

Lemma 2.10.3. *Suppose $k \geq 1$ is an integer and A is a finite set. Let $f_{\max} > 0$ and suppose $f\colon A \to [0, f_{\max}]$ is a function with $f(A) > 0$. Then,*

$$\sum_{(a_1,\dots,a_k)\in A^{\underline{k}}} \prod_{i\in[k]} f(a_i) = \Big(1 \pm \frac{k^2 f_{\max}}{f(A)}\Big) f(A)^k.$$

Proof. The upper bound follows since

$$\sum_{(a_1,\ldots,a_k)\in A^{\underline{k}}} \prod_{i\in[k]} f(a_i) \le \sum_{a_1,\ldots,a_k\in A} \prod_{i\in[k]} f(a_i) = f(A)^k.$$

For the lower bound, observe that

$$\begin{aligned} f(A)^k - &\sum_{(a_1,\ldots,a_k)\in A^{\underline{k}}} \prod_{i\in[k]} f(a_i) \\ &\le \sum_{j\in[k]} \sum_{(a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_k)\in A^{k-1}} \sum_{a_j\in\{a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_k\}} \prod_{i\in[k]} f(a_i) \\ &\le k^2 f_{\max} \sum_{(a_1,\ldots,a_{k-1})\in A^{k-1}} \prod_{i\in[k-1]} f(a_i) = k^2 f_{\max} f(A)^{k-1}, \end{aligned}$$

which completes the proof. □

Lemma 2.10.4. *Suppose $k \ge 1$ is an integer and V is a finite set. Let $d_{\max} > 0$ and suppose $d\colon V \to [0, d_{\max}]$ is a function with $d(V) \ge 2kd_{\max}$. For $e \in \binom{V}{k}$, let*

$$w(e) := \frac{(k-1)!\prod_{v\in e} d(v)}{d(V)^{k-1}}.$$

Then, for all $j \in [k]_0$ and $U \in \binom{V}{j}$,

$$\sum_{e\in\binom{V}{k}\colon U\subseteq e} w(e) = \left(1 \pm \frac{4k^2 d_{\max}}{d(V)}\right) \frac{(k-1)!\prod_{u\in U} d(u)}{(k-j)!d(V)^{j-1}}.$$

In particular, for all $v \in V$,

$$\sum_{e\in\binom{V}{k}\colon v\in e} w(e) = \left(1 \pm \frac{4k^2 d_{\max}}{d(V)}\right) d(v).$$

Proof. Fix $j \in [k]_0$ and $U \in \binom{V}{j}$. Then, Lemma 2.10.3 entails

$$\begin{aligned} \sum_{e\in\binom{V}{k}\colon U\subseteq e} w(e) &= \frac{(k-1)!\prod_{u\in U} d(u)}{(k-j)!d(V)^{k-1}} \cdot \sum_{(v_{j+1},\ldots,v_k)\in(V\setminus U)^{\underline{k-j}}} \prod_{j'\in[k]_{j+1}} d(v_{j'}) \\ &= \left(1 \pm \frac{k^2 d_{\max}}{d(V\setminus U)}\right) \cdot \frac{d(V\setminus U)^{k-j}}{d(V)^{k-j}} \cdot \frac{(k-1)!\prod_{u\in U} d(u)}{(k-j)!d(V)^{j-1}}. \\ &= \left(1 \pm \frac{4k^2 d_{\max}}{d(V)}\right) \frac{(k-1)!\prod_{u\in U} d(u)}{(k-j)!d(V)^{j-1}}, \end{aligned}$$

where we used $d(V \setminus U) \ge d(V) - kd_{\max}$ and $d(V) \ge 2kd_{\max}$. □

2.10.2 Proof of Theorem 2.5.2

To prove Theorem 2.5.2, we employ Lemma 2.10.6 which shows that instead of directly working with a $(d, \ell, \Gamma, \varepsilon)$-bounded conflict system, we may transition to a more restrictive conflict system $\mathcal{C}$ that satisfies (C6)–(C9) while essentially retaining the boundedness properties and the conflict-freeness of the tests. In the proof of Lemma 2.10.6, we use Lemma 2.10.5 to establish (C7).

Lemma 2.10.5. *For all $k \geq 2$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, there exists d_0 such that the following holds for all $d \geq d_0$. Suppose $\ell \geq 2$ is an integer and suppose $\Gamma \geq 1$ and $0 < \mu \leq 1/\ell$ are reals such that $1/\mu^{\Gamma\ell} \leq d^{\varepsilon^2}$. Suppose $\mathcal{H}$ is a k-graph, suppose $\mathcal{C}$ is a $(d, \ell, \Gamma, \varepsilon)$-bounded conflict system for $\mathcal{H}$ and suppose $\mathscr{Z}$ is a set of $(d, \varepsilon, \mathcal{C})$-trackable test systems for $\mathcal{H}$ of uniformity at most ℓ.*

Then, there exists a $(d, \ell, 2\Gamma, \varepsilon/3)$-bounded conflict system $\mathcal{C}'$ for $\mathcal{H}$ with $\mathcal{C} \subseteq \mathcal{C}'$ such that the following holds.

(i) $\Delta(\mathcal{C}'^{(2)}) \leq \Delta(\mathcal{C}^{(2)}) + d^{1-\varepsilon/3}$ *and* $\Delta(\mathcal{C}'^{(j)}) = \Delta(\mathcal{C}^{(j)})$ *for all* $j \in [\ell]_3$*;*

(ii) $|\mathcal{C}'^{(j)}_e \cap \mathcal{C}'^{(j)}_f| \leq d^{j-\varepsilon/3}$ *for all disjoint* $e, f \in \mathcal{H}$ *with* $\{e, f\} \notin \mathcal{C}'^{(2)}$ *and all* $j \in [\ell - 1]$*;*

(iii) *for all* $\mathcal{Z} \in \mathscr{Z}$*, all tests* $Z \in \mathcal{Z}$ *are* $\mathcal{C}'$*-free.*

Proof. For $j \in [\ell-1]_2$, we say that a pair (e, f) of disjoint edges $e, f \in \mathcal{H}$ with $\{e, f\} \notin \mathcal{C}^{(2)}$ is *j-bad* if $|\mathcal{C}^{(j)}_e \cap \mathcal{C}^{(j)}_f| \geq d^{j-\varepsilon/2}$, we say that it is *bad* if it is j-bad for some $j \in [\ell - 1]_2$ and we consider the conflict system $\mathcal{C}'$ with edge set

$$\mathcal{C} \cup \{\{e, f\} : (e, f) \text{ is bad}\}.$$

Due to (Z3), this construction preserves the conflict-freeness of all tests of test systems $\mathcal{Z} \in \mathscr{Z}$ in the sense that (iii) holds. Property (ii) follows by construction if $j \geq 2$ and if $j = 1$, then the bound follows if $\mathcal{C}'$ is $(d, \ell, 2\Gamma, \varepsilon/3)$-bounded, so it remains to show that the conflict system $\mathcal{C}'$ satisfies (i) and that it is $(d, \ell, 2\Gamma, \varepsilon/3)$-bounded.

To this end, we first bound the maximum degree of $\mathcal{C}'^{(2)}$. For all edges $e \in \mathcal{H}$ and all $j \in [\ell-1]_2$, there are at most $\Delta(\mathcal{C}^{(j+1)}) \cdot \Delta_j(\mathcal{C}^{(j+1)})$ pairs (C_e, C_f) of conflicts $C_e, C_f \in \mathcal{C}^{(j+1)}$ with $e \in C_e$ and

$$C_e \setminus \{e\} = C_f \setminus \{f\} \tag{2.10.1}$$

for some edge $f \in C_f$ and for all such pairs, the edge $f \in C_f$ with (2.10.1) is unique. Furthermore, for all edges $f \in \mathcal{H}$ such that (e, f) is j-bad, there are at least $d^{j-\varepsilon/2}$ pairs (C_e, C_f) of conflicts $C_e, C_f \in \mathcal{C}^{(j+1)}$ with $e \in C_e$ and (2.10.1). Using that for all $j \in [\ell]_2$, we have $\Delta(\mathcal{C}^{(j)}) \leq \Gamma d^{j-1}$ as a consequence of (C2) and additionally employing (C3), we conclude that for all edges $e \in \mathcal{H}$, the number of edges $f \in \mathcal{H}$ such that (e, f) is bad is at most

$$\sum_{j=2}^{\ell-1} \frac{\Delta(\mathcal{C}^{(j+1)}) \cdot \Delta_j(\mathcal{C}^{(j+1)})}{d^{j-\varepsilon/2}} \leq \ell \frac{\Gamma d^{j+1-\varepsilon}}{d^{j-\varepsilon/2}} \leq d^{1-2\varepsilon/5}.$$

Hence, for all $e \in \mathcal{H}$, there are at most $d^{1-2\varepsilon/5}$ conflicts $C \in \mathcal{C}'^{(2)}$ containing e that are not conflicts in $\mathcal{C}$. Since we only added conflicts of size 2 during the construction of $\mathcal{C}'$, this shows that (i) holds and that the $(d,\ell,2\Gamma,\varepsilon/3)$-boundedness of $\mathcal{C}'$ follows from the $(d,\ell,\Gamma,\varepsilon)$-boundedness of $\mathcal{C}$. $\square$

Lemma 2.10.6. *For all $k \geq 2$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, there exists d_0 such that the following holds for all $d \geq d_0$. Suppose $\ell \geq 2$ is an integer and suppose $\Gamma \geq 1$ and $0 < \mu \leq 1/\ell$ are reals such that $1/\mu^{\Gamma\ell} \leq d^{\varepsilon^2}$. Suppose $\mathcal{H}$ is a k-graph on $n \leq \exp(d^{\varepsilon})$ vertices with $(1-d^{-\varepsilon})d \leq \delta(\mathcal{H}) \leq \Delta(\mathcal{H}) \leq d$, suppose $\mathcal{C}$ is a $(d,\ell,\Gamma,\varepsilon)$-bounded conflict system for $\mathcal{H}$ and suppose $\mathscr{Z}$ is a set of $(d,\varepsilon,\mathcal{C})$-trackable test systems for $\mathcal{H}$ of uniformity at most ℓ with $|\mathscr{Z}| \leq \exp(d^{\varepsilon/(400\ell)})$*

Then, there exists a $(d,\ell,3\Gamma,\varepsilon/4)$-bounded conflict system $\mathcal{C}'$ for $\mathcal{H}$ with the following properties.

(i) $d_{\mathcal{C}'^{(j)}}(e) = (1 \pm d^{-\varepsilon/4})(1+d^{-\varepsilon/\ell})\max(d^{j-1-\varepsilon/600}, \Delta(\mathcal{C}^{(j)}))$ *for all $j \in [\ell]_2$ with $\mathcal{C}'^{(j)} \neq \emptyset$ and all $e \in \mathcal{H}$;*

(ii) $|\mathcal{C}'^{(j)}_e \cap \mathcal{C}'^{(j)}_f| \leq d^{j-\varepsilon/4}$ *for all disjoint $e,f \in \mathcal{H}$ with $\{e,f\} \notin \mathcal{C}'^{(2)}$ and all $j \in [\ell-1]$;*

(iii) *C is a matching for all $C \in \mathcal{C}'$;*

(iv) *$C \not\subseteq C'$ for all distinct $C, C' \in \mathcal{C}'$;*

(v) *every $\mathcal{C}'$-free subset of $\mathcal{H}$ is $\mathcal{C}$-free;*

(vi) *$|\{Z \in \mathcal{Z} : Z$ is not $\mathcal{C}'$-free$\}| \leq |\mathcal{Z}|/d^{\varepsilon}$ for all $\mathcal{Z} \in \mathscr{Z}$.*

Proof. Employing Lemma 2.10.5, we may assume that

$$|\mathcal{C}^{(j)}_e \cap \mathcal{C}^{(j)}_f| \leq d^{j-\varepsilon/3}$$

holds for all disjoint $e,f \in \mathcal{H}$ with $\{e,f\} \notin \mathcal{C}^{(2)}$ and all $j \in [\ell-1]$ at the cost of $\mathcal{C}$ being not necessarily $(d,\ell,\Gamma,\varepsilon)$-bounded, but still $(d,\ell,2\Gamma,\varepsilon/3)$-bounded, and then it suffices to show that there exists a $(d,\ell,3\Gamma,\varepsilon/4)$-bounded conflict system $\mathcal{C}'$ for $\mathcal{H}$ that satisfies

$$d_{\mathcal{C}'^{(j)}}(e) = (1 \pm d^{-\varepsilon/3})(1+d^{-\varepsilon/\ell})\max(d^{j-1-\varepsilon/600}, \Delta(\mathcal{C}^{(j)})) \tag{2.10.2}$$

for all $j \in [\ell]_2$ with $\mathcal{C}'^{(j)} \neq \emptyset$ and all $e \in \mathcal{H}$ as well as (ii)–(vi). To this end, we inductively show the existence of conflict systems $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell$ for $\mathcal{H}$ such that $\mathcal{C}_j$ with $j \in [\ell]$ is in a certain sense as desired up to and including uniformity j, which makes it an admissible step in our construction and in which case we call it j-admissible. In particular, $\mathcal{C}_\ell$ is then a conflict system with the desired properties.

Let us turn to the details. Formally, we define j-admissibility as follows. For $j \in [\ell]$, we say that a conflict system $\mathcal{C}_j$ for $\mathcal{H}$ is *j-admissible* if the following holds.

(A1) $\bigcup_{j' \in [j]_2} \mathcal{C}^{(j')}_j$ is $(d,\ell,3\Gamma,\varepsilon/4)$-bounded;

(A2) $\mathcal{C}^{(j')}_j = \emptyset$ for all $j' \in [j]$ with $\mathcal{C}^{(j')} = \emptyset$;

(A3) $\mathcal{C}^{(j')}_j \subseteq \mathcal{C}^{(j')}$ for all $j' \geq j+1$;

(A4) $d_{\mathcal{C}^{(j')}_j}(e) = (1 \pm d^{-\varepsilon/3})(1+d^{-\varepsilon/\ell})\max(d^{j-1-\varepsilon/600}, \Delta(\mathcal{C}^{(j')}))$ for all $j' \in [j]_2$ with $\mathcal{C}^{(j')} \neq \emptyset$ and all $e \in \mathcal{H}$;

(A5) $|(\mathcal{C}_j^{(j')})_e \cap (\mathcal{C}_j^{(j')})_f| \leq d^{j'-\varepsilon/4}$ for all disjoint $e, f \in \mathcal{H}$ with $\{e, f\} \notin \mathcal{C}_j^{(2)}$ and all $j' \in [j-1]$;

(A6) C is a matching for all $C \in \mathcal{C}_j$;

(A7) $C \not\subseteq C'$ for all distinct $C, C' \in \mathcal{C}_j$;

(A8) for all conflicts $C \in \mathcal{C}$, there is a subset $C' \subseteq C$ with $C' \in \mathcal{C}_j$;

(A9) $|\{Z \in \mathcal{Z} : Z \text{ is not } \mathcal{C}_j\text{-free}\}| \leq 4^j|\mathcal{Z}|/d^{2\varepsilon}$ for all $\mathcal{Z} \in \mathscr{Z}$.

For $j \in [\ell-1]$, we show that if there exists a j-admissible conflict system $\mathcal{C}_j$, then there also exists a $(j+1)$-admissible conflict system. Since

$$\mathcal{C}_1 := \{C \in \mathcal{C} : C \text{ is a matching with } C' \notin \mathcal{C} \text{ for all } C' \subsetneq C\}$$

is 1-admissible, this inductively proves that there is an ℓ-admissible conflict system $\mathcal{C}_\ell$. As every ℓ-admissible conflict system $\mathcal{C}_\ell$ is $(d, \ell, 3\Gamma, \varepsilon/4)$-bounded as a consequence of (A1) together with (A3), and additionally satisfies (2.10.2) and (ii)–(vi) due to (A4)–(A9), this then finishes the proof.

We proceed with the inductive proof for the existence of an ℓ-admissible conflict system. As in Sections 2.6–2.9 we frequently use the inequality $1/\mu^{\Gamma\ell} \leq d^{\varepsilon^2}$ to bound terms that depend on ℓ, Γ or μ from above using powers of d with a suitably small fraction of ε as their exponent. We also use that as an immediate consequence of the $(d, \ell, 2\Gamma, \varepsilon/3)$-boundedness of $\mathcal{C}$, we have $\Delta(\mathcal{C}^{(j)}) \leq 2\Gamma d^{j-1}$ for all $j \in [\ell]_2$. Additionally, to relate d and n, we often use that $d \leq n^k$.

Fix $j \in [\ell]_2$ and suppose $\mathcal{C}_{j-1}$ is a $(j-1)$-admissible conflict system. We show that a j-admissible conflict system $\mathcal{C}_j$ can be obtained by randomly adding conflicts of size j to $\mathcal{C}_{j-1}$ followed by deleting those conflicts of size at least $j+1$ that contain one of the added conflicts as a subset.

If $\mathcal{C}_{j-1}^{(j)} = \emptyset$, then $\mathcal{C}_{j-1}$ is also j-admissible and we do not add any conflicts. Now, assume $\mathcal{C}_{j-1}^{(j)} \neq \emptyset$. Define the target degree

$$d_{\mathrm{tar}} := (1 + d^{-\varepsilon/\ell}) \max(d^{j-1-\varepsilon/600}, \Delta(\mathcal{C}^{(j)})) \leq 4\Gamma d^{j-1}.$$

For all $e \in \mathcal{H}$, the target degree d_{tar} serves as a target value for $d_{\mathcal{C}_j^{(j)}}(e)$ that we aim for (but possibly only meet approximately). To this end, consider the function $d_{\mathrm{def}} \colon \mathcal{H} \to \mathbb{R}$, that for all $e \in \mathcal{H}$, maps e to the degree deficit $d_{\mathrm{def}}(e) := d_{\mathrm{tar}} - d_{\mathcal{C}_{j-1}^{(j)}}(e)$ and note that

$$\frac{d_{\mathrm{tar}}}{d^{2\varepsilon/\ell}} = d_{\mathrm{def}}(e) + d_{\mathcal{C}_{j-1}^{(j)}}(e) - (1 - d^{-2\varepsilon/\ell}) d_{\mathrm{tar}} \leq d_{\mathrm{def}}(e) + \Delta(\mathcal{C}_{j-1}^{(j)}) - \Delta(\mathcal{C}^{(j)}) \leq d_{\mathrm{def}}(e) \leq d_{\mathrm{tar}}, \tag{2.10.3}$$

so in particular

$$d^{j-1-2\varepsilon} \leq d_{\mathrm{def}}(e) \leq 4\Gamma d^{j-1}. \tag{2.10.4}$$

For $C \in \binom{\mathcal{H}}{j}$, motivated by Lemma 2.10.4, consider the weight

$$w_j(C) := \frac{(j-1)! \prod_{e \in C} d_{\mathrm{def}}(e)}{d_{\mathrm{def}}(\mathcal{H})^{j-1}} \leq \frac{(j-1)! d_{\mathrm{tar}}^j}{\left(\frac{d_{\mathrm{tar}}}{d^{2\varepsilon/\ell}} \cdot |\mathcal{H}|\right)^{j-1}} \leq \frac{2^\ell k^\ell \ell^\ell d^{2\varepsilon} d_{\mathrm{tar}}}{d^{j-1} n^{j-1}} \leq \frac{d^{3\varepsilon}}{n^{j-1}}. \tag{2.10.5}$$

In particular, note that $0 \leq w_j(C) \leq 1$.

Let $\mathcal{X}_j$ denote the (random) hypergraph with vertex set $\mathcal{H}$ whose edge set is obtained by including every matching $C \in \binom{\mathcal{H}}{j}$ satisfying $C' \not\subseteq C$ for all $C' \in \mathcal{C}_{j-1}$ independently at random with probability $w_j(C)$. Let $\mathcal{C}_j$ denote the (random) hypergraph obtained from $\mathcal{C}_{j-1} + \mathcal{X}_j$ by removing every edge $C' \in \mathcal{C}_{j-1}$ satisfying $C \subseteq C'$ for some $C \in \mathcal{X}_j$. Note that indeed, as claimed above, during this construction only edges of size j are added and only edges of size at least $j+1$ are removed.

The properties (A2), (A3) and (A6)–(A8) hold by construction, so it remains to consider the properties (A1), (A4), (A5) and (A9). To this end, let us argue why it suffices to show that the following holds with positive probability.

(I) $d_{\mathcal{C}_j^{(j)}}(e) = (1 \pm d^{-\varepsilon})d_{\mathrm{tar}}$ for all $e \in \mathcal{H}$;

(II) $\Delta_{j'}(\mathcal{C}_j^{(j)}) \leq d^{j-j'-\varepsilon/4}$ for all $j' \in [j-1]_2$;

(III) if $j=2$, then $|\{f \in N^{(2)}_{\mathcal{C}_j}(e) : v \in f\}| \leq d^{1-\varepsilon/4}$ for all $v \in V(\mathcal{H})$ and $e \in \mathcal{H}$;

(IV) if $j=2$, then $|N^{(2)}_{\mathcal{C}_j}(e) \cap N^{(2)}_{\mathcal{C}_j}(f)| \leq d^{1-\varepsilon/4}$ for all disjoint $e, f \in \mathcal{H}$.

(V) $|(\mathcal{C}_j)_e^{(j-1)} \cap (\mathcal{C}_j)_f^{(j-1)}| \leq d^{j-1-\varepsilon/4}$ for all disjoint $e, f \in \mathcal{H}$ with $\{e,f\} \notin \mathcal{C}_j^{(2)}$;

(VI) $|\{\mathcal{Z} \in \mathscr{Z} : \mathcal{Z} \text{ is not } \mathcal{C}_j\text{-free}\}| \leq 4^j|\mathcal{Z}|/d^{2\varepsilon}$ for all $\mathcal{Z} \in \mathscr{Z}$.

Since we only added conflicts of size j and only removed conflicts of size at least $j+1$ during the construction of $\mathcal{C}_j$, Properties (A4), (A5) and (A9) follow from the $(j-1)$-admissibility of $\mathcal{C}_{j-1}$ and (I), (V) and (VI), so let us turn to the $(d, \ell, 3\Gamma, \varepsilon/4)$-boundedness of $\bigcup_{j' \in [j]_2} \mathcal{C}_j^{(j')}$. For (C1), note that $2 \leq |C| \leq \ell$ holds for all $C \in \mathcal{C}_j$ by construction. For (C2) observe that (A2) yields $\{j' \in [\ell]_2 : \mathcal{C}_j^{(j')} \neq \emptyset\} \subseteq \{j' \in [\ell]_2 : \mathcal{C}^{(j')} \neq \emptyset\}$ and that with additionally (A4) and the $(d, \ell, 2\Gamma, \varepsilon/3)$-boundedness of $\mathcal{C}$, we obtain

$$\begin{aligned}\sum_{j' \in [j]_2} \frac{\Delta(\mathcal{C}_j^{(j')})}{d^{j'-1}} &\leq \frac{5}{4} \sum_{j' \in [j]_2 \colon \mathcal{C}_j^{(j')} \neq \emptyset} \frac{d^{j'-1-\varepsilon/600} + \Delta(\mathcal{C}^{(j')})}{d^{j'-1}} \\ &\leq \frac{5}{4} \sum_{j' \in [j]_2 \colon \mathcal{C}^{(j')} \neq \emptyset} d^{-\varepsilon/600} + \frac{\Delta(\mathcal{C}^{(j')})}{d^{j'-1}} \leq 3\Gamma.\end{aligned}$$

For (C3)–(C5) note that since we only add conflicts of size j during the construction of $\mathcal{C}_j$, the desired bounds follow from the $(j-1)$-admissibility of $\mathcal{C}_{j-1}$, in particular the $(d, \ell, 3\Gamma, \varepsilon/4)$-boundedness of $\bigcup_{j' \in [j-1]_2} \mathcal{C}_{j-1}^{(j')}$, and (II)–(IV).

We finish the proof by showing that (I)–(VI) hold with positive probability as a consequence of Chernoff's and McDiarmid's inequality (Lemmas 2.10.1 and 2.10.2). As every relevant random variable in (I)–(VI) is a sum or suitable function of independent Bernoulli random variables, it suffices to show that their expected values satisfy the desired bounds with some room for small relative errors.

For the degrees $d_{\mathcal{C}_j^{(j)}}(e)$ of the edges $e \in \mathcal{H}$ that are relevant for (I), this essentially follows from Lemma 2.10.4 which we apply with j, $\mathcal{H}$, d_{def} playing the roles of k, V, d.

For the random variables in (II)–(VI), the desired bounds simply follow from the upper bound on the weights $w_j(C)$ that we deduced in (2.10.5).

First, for (I) and (II), we consider the degrees. For all $e \in \mathcal{H}$, let $\mathcal{Y}_e$ denote the set of those j-sets of edges of $\mathcal{H}$ that we do not allow as candidates for randomly added sets containing e during the construction of $\mathcal{C}_j$, that is let $\mathcal{Y}_e$ denote the set of those sets $C \in \binom{\mathcal{H}}{j}$ containing e that are not matchings or that contain a conflict $C' \in \mathcal{C}_{j-1}$ as a proper subset.

We employ (2.10.5) to obtain

$$\mathbb{E}[d_{\mathcal{C}_j^{(j)}}(e)] = d_{\mathcal{C}_{j-1}^{(j)}}(e) + \Big(\sum_{C \in \binom{\mathcal{H}}{j}\colon e \in C} w_j(C) \Big) \pm |\mathcal{Y}_e| \frac{d^{3\varepsilon}}{n^{j-1}}.$$

Observe that (2.10.4) implies

$$\frac{4j^2 \max_{e \in \mathcal{H}} d_{\mathrm{def}}(e)}{d_{\mathrm{def}}(\mathcal{H})} \le \frac{4\ell^2 \cdot 4\Gamma d^{j-1}}{|\mathcal{H}| \cdot d^{j-1-2\varepsilon}} \le \frac{32k\ell^2\Gamma d^{2\varepsilon}}{dn} \le \frac{1}{d}. \tag{2.10.6}$$

Hence, Lemma 2.10.4 with j, $\mathcal{H}$, d_{def}, $\{e\}$ playing the roles of k, V, d, U yields

$$\mathbb{E}[d_{\mathcal{C}_j^{(j)}}(e)] = d_{\mathcal{C}_{j-1}^{(j)}}(e) + (1 \pm d^{-1}) d_{\mathrm{def}}(e) \pm |\mathcal{Y}_e| \frac{d^{3\varepsilon}}{n^{j-1}} = (1 \pm d^{-1}) d_{\mathrm{tar}} \pm |\mathcal{Y}_e| \frac{d^{3\varepsilon}}{n^{j-1}}.$$

We may bound $|\mathcal{Y}_e|$ as follows. There are at most

$$|\mathcal{H}|^{j-2} \cdot (j-1)k\Delta(\mathcal{H}) \le d^{j-1+\varepsilon} n^{j-2} \tag{2.10.7}$$

sets $C \in \binom{\mathcal{H}}{j}$ containing e that are not matchings. Furthermore, for all $j' \in [j]_2$, as a consequence of the $(j-1)$-admissibility of $\mathcal{C}_{j-1}$, more specifically (A4) and (A3) with $\mathcal{C}_{j-1}$ playing the role of $\mathcal{C}_j$, we have

$$\Delta(\mathcal{C}_{j-1}^{(j')}) \le 2\max(d^{j'-1}, \Delta(\mathcal{C}^{(j')})) \le 4\Gamma d^{j'-1}$$

and

$$|\mathcal{C}_{j-1}^{(j')}| \le |\mathcal{H}| \cdot \Delta(\mathcal{C}_{j-1}^{(j')}) \le \frac{dn}{k} \cdot 4\Gamma d^{j'-1} \le 4\Gamma d^{j'} n.$$

Thus, for all $e \in \mathcal{H}$, the number of sets $C \in \binom{\mathcal{H}}{j}$ with $e \in C$ and a subset $C' \subseteq C$ with $e \in C'$ and $C' \in \mathcal{C}_{j-1}$ is at most

$$\sum_{j' \in [j]_2} \Delta(\mathcal{C}_{j-1}^{(j')}) \cdot |\mathcal{H}|^{j-j'} \le \sum_{j' \in [j]_2} 4\Gamma d^{j'-1} \left(\frac{dn}{k}\right)^{j-j'} \le 4\Gamma\ell d^{j-1} n^{j-2} \le d^{j-1+\varepsilon} n^{j-2}$$

and the number of sets $C \in \binom{\mathcal{H}}{j}$ with $e \in C$ and a subset $C' \subseteq C$ with $e \notin C'$ and $C' \in \mathcal{C}_{j-1}$ is at most

$$\sum_{j' \in [j-1]_2} |\mathcal{C}_{j-1}^{(j')}| \cdot |\mathcal{H}|^{j-j'-1} \le \sum_{j' \in [j-1]_2} 4\Gamma d^{j'} n \left(\frac{dn}{k}\right)^{j-j'-1} \le 4\Gamma\ell d^{j-1} n^{j-2} \le d^{j-1+\varepsilon} n^{j-2}.$$

With (2.10.7), this yields $|\mathcal{Y}_e| \le 3d^{j-1+\varepsilon}n^{j-2}$ and hence

$$\mathbb{E}[d_{\mathcal{C}_j^{(j)}}(e)] = (1 \pm d^{-2\varepsilon})d_{\mathrm{tar}}.$$

For all edge sets $E \subseteq \mathcal{H}$ of size $j' \in [j-1]_2$, using (2.10.5), we obtain

$$\mathbb{E}[d_{\mathcal{C}_j^{(j)}}(E)] \le d_{\mathcal{C}^{(j)}}(E) + \left(\frac{dn}{k}\right)^{j-j'} \cdot \frac{d^{3\varepsilon}}{n^{j-1}} \le d^{j-j'-\varepsilon/3} + \frac{d^{j-j'+3\varepsilon}}{n} \le \frac{d^{j-j'-\varepsilon/4}}{2}.$$

It remains to consider (III)–(VI). Before dealing with the special case $j=2$ where (III) and (IV) are relevant, we turn to (V) and (VI). Note that all $e, f \in \mathcal{H}$ with $\{e,f\} \notin \mathcal{C}_j^{(2)}$ satisfy $\{e,f\} \notin \mathcal{C}_{j-1}^{(2)}$ because we did not remove any conflicts of size 2 during the construction of $\mathcal{C}_j$. For all disjoint $e, f \in \mathcal{H}$ with $\{e,f\} \notin \mathcal{C}_j^{(2)}$ and all $j \in [\ell]_2$, we use (2.10.5) to obtain

$$\begin{aligned}
\mathbb{E}[|(\mathcal{C}_j)_e^{(j-1)} \cap (\mathcal{C}_j)_f^{(j-1)}|] &\le |(\mathcal{C}_{j-1})_e^{(j-1)} \cap (\mathcal{C}_{j-1})_f^{(j-1)}| + \mathbb{E}[|(\mathcal{X}_j)_e^{(j-1)} \cap (\mathcal{C}_{j-1})_f^{(j-1)}|] \\
&\quad + \mathbb{E}[|(\mathcal{C}_{j-1})_e^{(j-1)} \cap (\mathcal{X}_j)_f^{(j-1)}|] + \mathbb{E}[|(\mathcal{X}_j)_e^{(j-1)} \cap (\mathcal{X}_j)_f^{(j-1)}|] \\
&\le d^{j-1-\varepsilon/3} + 4\max(d^{j-1}, \Delta(\mathcal{C}^{(j)})) \cdot \frac{d^{3\varepsilon}}{n^{j-1}} \\
&\quad + \left(\frac{dn}{k}\right)^{j-1} \cdot \left(\frac{d^{3\varepsilon}}{n^{j-1}}\right)^2 \\
&\le d^{j-1-\varepsilon/3} + \frac{8\Gamma d^{j-1+3\varepsilon}}{n} + \frac{d^{j-1+6\varepsilon}}{n} \le \frac{d^{j-1-\varepsilon/4}}{2}.
\end{aligned}$$

For all $\mathcal{Z} \in \mathscr{Z}$, we also use (2.10.5) to obtain

$$\begin{aligned}
\mathbb{E}[|\{Z \in \mathcal{Z} : Z \text{ is not } \mathcal{C}_j\text{-free}\}|] &\le |\{Z \in \mathcal{Z} : Z \text{ is not } \mathcal{C}_{j-1}\text{-free}\}| + \sum_{Z \in \mathcal{Z}} \sum_{Z' \in \binom{Z}{j}} w_j(Z') \\
&\le \frac{4^{j-1}|\mathcal{Z}|}{d^{2\varepsilon}} + \frac{2^\ell d^{3\varepsilon}|\mathcal{Z}|}{n^{j-1}} \le \frac{4^j|\mathcal{Z}|}{2d^{2\varepsilon}}.
\end{aligned}$$

If $j=2$, then, again using the upper bound (2.10.5), for all $v \in V(\mathcal{H})$ and $e \in \mathcal{H}$ we obtain

$$\mathbb{E}[|\{f \in N_{\mathcal{C}_j}^{(2)}(e) : v \in f\}|] \le |\{f \in N_{\mathcal{C}}^{(2)}(e) : v \in f\}| + d \cdot \frac{d^{3\varepsilon}}{n} \le d^{1-\varepsilon/3} + \frac{d^{1+3\varepsilon}}{n} \le \frac{d^{1-\varepsilon/4}}{2}$$

and for all disjoint $e, f \in \mathcal{H}$, we obtain

$$\begin{aligned}
\mathbb{E}[|N_{\mathcal{C}_j}^{(2)}(e) \cap N_{\mathcal{C}_j}^{(2)}(f)|] &\le |(\mathcal{C}_{j-1})_e^{(1)} \cap (\mathcal{C}_{j-1})_f^{(1)}| + \mathbb{E}[|(\mathcal{X}_j)_e^{(1)} \cap (\mathcal{C}_j)_f^{(1)}|] \\
&\quad + \mathbb{E}[|(\mathcal{C}_j)_e^{(1)} \cap (\mathcal{X}_j)_f^{(1)}|] + \mathbb{E}[|(\mathcal{X}_j)_e^{(1)} \cap (\mathcal{X}_j)_f^{(1)}|] \\
&\le d^{1-\varepsilon/3} + 4\max(d, \Delta(\mathcal{C}^{(2)})) \cdot \frac{d^{3\varepsilon}}{n^{j-1}} + \frac{dn}{k} \cdot \left(\frac{d^{3\varepsilon}}{n}\right)^2
\end{aligned}$$

$$\leq d^{1-\varepsilon/3} + \frac{8\Gamma d^{1+3\varepsilon}}{n} + \frac{d^{1+6\varepsilon}}{n} \leq \frac{d^{1-\varepsilon/4}}{2}.$$

With these bounds on the expected values, using Chernoff's inequality (Lemma 2.10.1) and a suitable union bound we conclude that with high probability (I)–(IV) hold. To see that (VI) also holds with high probability, we use McDiarmid's inequality (Lemma 2.10.2). For $\mathcal{Z} \in \mathscr{Z}$ and $C \in \binom{\mathcal{H}}{j}$, adding or removing C from $\mathcal{C}_j$ changes the number of tests $Z \in \mathcal{Z}$ that are not $\mathcal{C}_j$-free by at most $d_{\mathcal{Z}}(C)$ and we have

$$\sum_{C\in\binom{\mathcal{H}}{j}} d_{\mathcal{Z}}(C)^2 \leq \Delta_j(\mathcal{Z}) \cdot \sum_{C\in\binom{\mathcal{H}}{j}} d_{\mathcal{Z}}(C) \leq \frac{|\mathcal{Z}|}{d^{j+\varepsilon}} \cdot 2^{\ell}|\mathcal{Z}| \leq \frac{|\mathcal{Z}|^2}{d}.$$

Thus, since the expected number of tests $Z \in \mathcal{Z}$ that are not $\mathcal{C}_j$-free is at most $4^j|\mathcal{Z}|/(2d^{2\varepsilon})$, McDiarmid's inequality (Lemma 2.10.2) entails

$$\mathbb{P}\left[|\{Z \in \mathcal{Z} : Z \text{ ist not } \mathcal{C}_j\text{-free}\}| \geq \frac{4^j|\mathcal{Z}|}{d^{2\varepsilon}}\right] \leq \exp\left(-\frac{4^j|\mathcal{Z}|^2 d}{2d^{4\varepsilon}|\mathcal{Z}|^2}\right) \leq \exp(-d^{1/2}).$$

A suitable union bound completes the proof. □

Proof of Theorem 2.5.2. We deduce Theorem 2.5.2 from Theorem 2.6.2. In this proof, we actually allow more vertices in the sense that we only assume $n \leq \exp(d^{\varepsilon/(400\ell)})$ instead of $n \leq \exp(d^{\varepsilon^2/\ell})$, we only impose the weaker bound $1/\mu^{\Gamma\ell} \leq d^{\varepsilon^{5/3}}$ instead of $1/\mu^{\Gamma\ell} \leq d^{\varepsilon^2}$, we allow more test systems by only assuming $|\mathscr{Z}| \leq \exp(d^{\varepsilon/(400\ell)})$ instead of $|\mathscr{Z}| \leq \exp(d^{\varepsilon^2/\ell})$ and we obtain stronger bounds characterizing the properties of the matching. This will be convenient when proving Theorems 2.5.3 and 2.5.4.

Instead of $\mathcal{C}$, consider a conflict system $\mathcal{C}'$ as in Lemma 2.10.6, for $\mathcal{Z} \in \mathscr{Z}$, let $\mathcal{Z}' := \{Z \in \mathcal{Z} : Z \text{ is } \mathcal{C}'\text{-free}\}$ and define $\mathscr{Z}' := \{\mathcal{Z}' : \mathcal{Z} \in \mathscr{Z}\}$. Note that

$$\frac{1}{\mu^{3\Gamma\ell}} \leq d^{3\varepsilon^{5/3}} \leq d^{75(\varepsilon/5)^{5/3}} \leq d^{(\varepsilon/5)^{3/2}},$$

that for all $j \in [\ell]_2$, we have

$$d^{j-1-\varepsilon/500} \leq (1 - d^{-\varepsilon/5})\Delta(\mathcal{C}'^{(j)}) \leq \delta(\mathcal{C}'^{(j)}).$$

Furthermore, for all $\mathcal{Z} \in \mathscr{Z}$ of uniformity j, due to (Z1) we have

$$|\mathcal{Z}'| \geq (1 - d^{-\varepsilon})|\mathcal{Z}| \geq \frac{d^{j+\varepsilon}}{2} \geq d^{j+\varepsilon/5}$$

and since for all $e, f \in \mathcal{H}$ with $\{e, f\} \notin \mathcal{C}'^{(2)}$ and all $j \in [\ell-1]$ we have $|\mathcal{C}_e'^{(j)} \cap \mathcal{C}_f'^{(j)}| \leq d^{j-\varepsilon/5}$ by choice of $\mathcal{C}'$, the test systems $\mathcal{Z}' \in \mathscr{Z}'$ are $(d, \varepsilon/5, \mathcal{C}')$-trackable. Thus, Theorem 2.6.2 with $\varepsilon/5$, 3Γ, $\mathcal{C}'$, $\mathscr{Z}'$ playing the roles of ε, Γ, $\mathcal{C}$, $\mathscr{Z}$ yields a $\mathcal{C}'$-free matching $\mathcal{M} \subseteq \mathcal{H}$ with $|\mathcal{M}| = (1 - \mu)n/k$,

$$|\{Z \in \mathcal{Z} : Z \subseteq \mathcal{M}\}| \geq (1 - d^{-\varepsilon/375})\left(\frac{|\mathcal{M}|k}{dn}\right)^j |\mathcal{Z}'| \geq (1 - d^{-\varepsilon/375})\left((1 - d^{-\varepsilon})\frac{|\mathcal{M}|}{|\mathcal{H}|}\right)^j |\mathcal{Z}'|$$

$$\geq (1 - d^{-\varepsilon/375})(1 - \ell d^{-\varepsilon})\left(\frac{|\mathcal{M}|}{|\mathcal{H}|}\right)^j |\mathcal{Z}'| \geq (1 - d^{-\varepsilon/400})\left(\frac{|\mathcal{M}|}{|\mathcal{H}|}\right)^j |\mathcal{Z}|$$

and

$$|\{Z \in \mathcal{Z} : Z \subseteq \mathcal{M}\}| \leq (1 + d^{-\varepsilon/375})\left(\frac{|\mathcal{M}|k}{dn}\right)^j |\mathcal{Z}'| \leq (1 + d^{-\varepsilon/375})\left(\frac{|\mathcal{M}|}{|\mathcal{H}|}\right)^j |\mathcal{Z}|.$$

Furthermore, as the matching $\mathcal{M}$ is $\mathcal{C}'$-free, it is $\mathcal{C}$-free by the choice of $\mathcal{C}'$. □

2.10.3 Proof of Theorem 2.5.3

To obtain Theorem 2.5.3 as a consequence of Theorem 2.5.2, we employ the following lemma which allows us to approximate a test function w (or rather its extension to arbitrary edge sets) using test systems obtained as a series of samples. For a hypergraph $\mathcal{H}$, a finite sequence $\mathscr{Z} = \mathcal{Z}_1, \dots, \mathcal{Z}_z$ of j-uniform test systems for $\mathcal{H}$ and $E \subseteq \mathcal{H}$, we define the *total $\mathscr{Z}$-weight* $w_{\mathscr{Z}}(E) := \sum_{i\in[z]}|\mathcal{Z}_i \cap \binom{E}{j}|$.

Lemma 2.10.7. *For all $k \geq 2$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, there exists d_0 such that the following holds for all $d \geq d_0$. Suppose $\mathcal{H}$ is a k-graph on $n \leq \exp(d^{\varepsilon/600\ell})$ vertices and suppose $\mathcal{C}$ is a conflict system for $\mathcal{H}$. Suppose w is a j-uniform $(d, \varepsilon, \mathcal{C})$-trackable test function for $\mathcal{H}$ where $j \leq \log d$. Then, there exists a sequence $\mathscr{Z} = \mathcal{Z}_1, \dots, \mathcal{Z}_z$ of j-uniform $(d, \varepsilon/2, \mathcal{C})$-trackable test systems for $\mathcal{H}$ with $z = \exp(d^{\varepsilon/500\ell})$ and $|\mathcal{Z}_i| = (1 \pm d^{-1/2})w(\mathcal{H})$ for all $i \in [z]$ as well as*

$$w(E) = (1 \pm d^{-1})\frac{w_{\mathscr{Z}}(E)}{z} \tag{2.10.8}$$

for all $E \subseteq \mathcal{H}$ with $w_{\mathscr{Z}}(E) \geq z$.

Proof. To obtain the elements of the sequence $\mathscr{Z} = \mathcal{Z}_1, \dots, \mathcal{Z}_z$, we construct $\exp(d^{\varepsilon/500\ell})$ test systems $\mathcal{Z}$ by including every set $Z \in \binom{\mathcal{H}}{j}$ in $\mathcal{Z}$ independently at random with probability $w(Z)$. Then, Chernoff's inequality (Lemma 2.10.1) shows that the desired properties hold with positive probability.

Since w is $(d, \varepsilon, \mathcal{C})$-trackable, the following holds for all $\mathcal{Z} \in \{\mathcal{Z}_1, \dots, \mathcal{Z}_z\}$.

- $\mathbb{E}[|\mathcal{Z}|] = w(\mathcal{H}) \geq d^{j+\varepsilon}$;
- $\mathbb{E}[d_{\mathcal{Z}}(E')] = w(\{E \in \binom{\mathcal{H}}{j} : E' \subseteq E\}) \leq \mathbb{E}[|\mathcal{Z}|]/d^{j'+\varepsilon}$ for all $j' \in [j-1]$ and $E' \in \binom{\mathcal{H}}{j'}$;
- $\mathbb{P}[|\mathcal{C}_e^{(j')} \cap \mathcal{C}_f^{(j')}| \leq d^{j'-\varepsilon}$ for all $e, f \in \mathcal{H}$ with $d_{\mathcal{Z}}(ef) \geq 1$ and all $j' \in [\ell - 1]] = 1$;
- $\mathbb{P}[Z$ is a $\mathcal{C}$-free matching for all $Z \in \mathcal{Z}] = 1$.

Furthermore, for all $E \subseteq \mathcal{H}$, we have $\mathbb{E}[w_{\mathscr{Z}}(E)] = w(E)|\mathscr{Z}|$. Thus, with a suitable union bound, Chernoff's inequality (Lemma 2.10.1) shows that with positive probability, every (random) hypergraph $\mathcal{Z} \in \{\mathcal{Z}_1, \dots, \mathcal{Z}_z\}$ is a $(d, \varepsilon/2, \mathcal{C})$-trackable test system for $\mathcal{H}$ satisfying $|\mathcal{Z}| = (1 \pm d^{-1/2})w(\mathcal{H})$, $w_{\mathscr{Z}}(E) < z$ for all $E \subseteq \mathcal{H}$ with $w(E) \leq 1/2$ and

$$w_{\mathscr{Z}}(E) = \left(1 \pm \frac{d^{-1}}{2}\right)w(E)z$$

for all $E \subseteq \mathcal{H}$ with $w(E) \geq 1/2$ and thus (2.10.8) for all $E \in \mathcal{H}$ with $w_{\mathscr{Z}}(E) \geq z$. □

Proof of Theorem 2.5.3. In this proof, we actually allow more vertices and more test functions in the sense that we only assume $n \leq \exp(d^{\varepsilon/(600\ell)})$ and $|\mathscr{W}| \leq \exp(d^{\varepsilon/(600\ell)})$ instead of $n \leq \exp(d^{\varepsilon^2/\ell})$ and $|\mathcal{W}| \leq \exp(d^{\varepsilon^2/\ell})$. This will be convenient when proving Theorem 2.5.4.

Lemma 2.10.7 shows that for all j-uniform $w \in \mathscr{W}$, there exists a sequence $\mathcal{Z}_1^w, \ldots, \mathcal{Z}_z^w$ of j-uniform $(d, \varepsilon/2, \mathcal{C})$-trackable test systems for $\mathcal{H}$ with $z = \exp(d^{\varepsilon/(500\ell)})$,

$$|\mathcal{Z}| = (1 \pm d^{-1/2}) w(\mathcal{H}) \tag{2.10.9}$$

for all $\mathcal{Z} \in \{\mathcal{Z}_1^w, \ldots, \mathcal{Z}_z^w\}$ and

$$w(E) = (1 \pm d^{-1}) \frac{\sum_{i \in [z]} |\mathcal{Z}_i^w \cap \binom{E}{j}|}{z} \tag{2.10.10}$$

for all $E \subseteq \mathcal{H}$ with $\sum_{i \in [z]} |\mathcal{Z}_i^w \cap \binom{E}{j}| \geq z$. Let $\mathscr{Z} := \bigcup_{w \in \mathscr{W}} \{\mathcal{Z}_1^w, \ldots, \mathcal{Z}_z^w\}$. Note that

$$|\mathscr{Z}| \leq \exp(d^{\varepsilon/(600\ell)}) \cdot \exp(d^{\varepsilon/(500\ell)}) \leq \exp(d^{\varepsilon/(400\ell)})$$

and

$$\frac{1}{\mu^{\Gamma\ell}} \leq d^{\varepsilon^2} \leq d^{4(\varepsilon/2)^2} \leq d^{(\varepsilon/2)^{5/3}}.$$

Thus, an application of Theorem 2.5.2 with $\varepsilon/2$ playing the role of ε making use of the fact that we only worked with weaker assumptions while obtaining a slightly stronger output in the proof of Theorem 2.5.2, yields a $\mathcal{C}$-free matching $\mathcal{M} \subseteq \mathcal{H}$ with $|\mathcal{M}| = (1 - \mu)n/k$ and

$$\left| \mathcal{Z} \cap \binom{\mathcal{M}}{j} \right| = |\{Z \in \mathcal{Z} : Z \subseteq M\}| = (1 \pm d^{-\varepsilon/800}) \left(\frac{|\mathcal{M}|}{|\mathcal{H}|} \right)^j |\mathcal{Z}| \tag{2.10.11}$$

for all j-uniform $\mathcal{Z} \in \mathscr{Z}$. Fix a j-uniform $w \in \mathscr{W}$. In particular, (2.10.11) together with (Z1) implies $|\mathcal{Z} \cap \binom{\mathcal{M}}{j}| \geq d^{\varepsilon/2}/2^{j+1} \geq d^{\varepsilon/2}/4^\ell \geq 1$ for all $\mathcal{Z} \in \{\mathcal{Z}_1^w, \ldots, \mathcal{Z}_z^w\}$ and hence $\sum_{i \in [z]} |\mathcal{Z}_i^w \cap \binom{\mathcal{M}}{j}| \geq z$. This allows us to apply (2.10.10) such that combining it with (2.10.9) and (2.10.11), we conclude that

$$\begin{aligned} w(\mathcal{M}) &= (1 \pm d^{-1}) \frac{\sum_{i \in [z]} |\mathcal{Z}_i^w \cap \binom{\mathcal{M}}{j}|}{z} = (1 \pm d^{-\varepsilon/850}) \frac{\sum_{i \in [z]} (\frac{|\mathcal{M}|}{|\mathcal{H}|})^j |\mathcal{Z}|}{z} \\ &= (1 \pm d^{-\varepsilon/900}) \left(\frac{|\mathcal{M}|}{|\mathcal{H}|} \right)^j w(\mathcal{H}), \end{aligned}$$

which completes the proof. □

2.10.4 Proof of Theorem 2.5.4

To prove Theorem 2.5.4, we apply Theorem 2.5.3 with a suitable more regular k-graph $\mathcal{H}'$ where the given k-graph $\mathcal{H}$ is an induced subgraph of $\mathcal{H}'$. More specifically, we use the following lemma.

Lemma 2.10.8. *For all $k \geq 2$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, there exists d_0 such that the following holds for all $d \geq d_0$. Suppose $\mathcal{H}$ is a k-graph on $n \leq \exp(d^\varepsilon)$ vertices with $(1-\varepsilon)d \leq \delta(\mathcal{H}) \leq \Delta(\mathcal{H}) \leq d$ and $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon}$. Then, $\mathcal{H}$ is an induced subgraph of a k-graph $\mathcal{H}'$ on $3n$ vertices with $(1-d^{-\varepsilon})d \leq \delta(\mathcal{H}') \leq \Delta(\mathcal{H}') \leq d$, $\Delta_2(\mathcal{H}') \leq d^{1-\varepsilon}$ and*

$$|\{e \in \mathcal{H}' : 1 \leq |e \cap V(\mathcal{H})| \leq k-1\}| \leq 2\varepsilon dn. \tag{2.10.12}$$

Proof. After choosing an appropriate vertex set V' for $\mathcal{H}'$, we construct the edge set of $\mathcal{H}'$ by starting with $\mathcal{H}$ and adding sets $e \in \binom{V}{k}$ with $|e \cap V| \leq 1$ independently at random with suitable probabilities derived from Lemma 2.10.4.

In more detail, choose a set V' of size $3n$ with $V := V(\mathcal{H}) \subseteq V'$. Define the target degree $d_{\mathrm{tar}} := (1-d^{-2\varepsilon})d$ that for all $v \in V'$ serves as a target value for $d_{\mathcal{H}'}(v)$ that we aim for (but possibly only meet approximately). To this end consider the function $d_{\mathrm{def}} : V \to \mathbb{R}$, that for all $v \in V$, maps v to the degree deficit $d_{\mathrm{def}}(v) := \max(0, d_{\mathrm{tar}} - d_{\mathcal{H}}(v))$ and for all $e \in \binom{V'}{k}$ with $e \cap V = \{v\}$, consider the weight

$$w(e) := \frac{d_{\mathrm{def}}(v)}{\binom{2n}{k-1}} \leq \frac{\varepsilon d}{\binom{2n}{k-1}} \leq \frac{(1-\varepsilon)d}{\binom{n}{k-1}} \leq \frac{\delta(\mathcal{H})}{\binom{n}{k-1}} \leq 1.$$

When adding edges $e \in \binom{V'}{k}$ with $|e \cap V| = 1$ independently at random with probability $w(e)$, the expected contribution to the degrees of the vertices $u \in V^+ := V' \setminus V$ is

$$d(u) := \sum_{v \in V} \sum_{U \in \binom{V^+ \setminus \{u\}}{k-2}} w(\{u, v\} \cup U) \leq n \cdot \binom{2n-1}{k-2} \cdot \frac{\varepsilon d}{\binom{2n}{k-1}} = \frac{\varepsilon(k-1)dn}{2n} \leq \varepsilon k d_{\mathrm{tar}}.$$

Hence, extend the domain of d_{def} such that for all $u \in V^+$, d_{def} maps u to the degree deficit $d_{\mathrm{def}}(u) := d_{\mathrm{tar}} - d(u)$. For $e \in \binom{V^+}{k}$, motivated by Lemma 2.10.4, consider the weight

$$w(e) := \frac{(k-1)! \prod_{u \in e} d_{\mathrm{def}}(u)}{d_{\mathrm{def}}(V^+)^{k-1}} \leq \frac{(k-1)! d_{\mathrm{tar}}^k}{((1-\varepsilon k)d_{\mathrm{tar}} \cdot 2n)^{k-1}} \leq \frac{(1-\varepsilon)d}{\frac{n^{k-1}}{(k-1)!}} \leq \frac{\delta(\mathcal{H})}{\binom{n}{k-1}} \leq 1.$$

Let $\mathcal{H}'$ denote the (random) k-graph with vertex set V' whose edge set is obtained from $\mathcal{H}$ by adding every set $e \in \binom{V'}{k}$ with $|e \cap V| \leq 1$ independently at random with probability $w(e)$.

Let us investigate the expected degrees in $\mathcal{H}'$. Lemma 2.10.4 was the motivation for defining the weights $w(e)$ with $e \in \binom{V^+}{k}$, so first, we deduce a bound for the error term $4k^2 \max_{u \in V^+} d_{\mathrm{def}}(u)/d_{\mathrm{def}}(V^+)$. We have

$$\frac{4k^2 \max_{u \in V^+} d_{\mathrm{def}}(u)}{d_{\mathrm{def}}(V^+)} \leq \frac{4k^2 d_{\mathrm{tar}}}{(1-\varepsilon k)d_{\mathrm{tar}} \cdot 2n} \leq \frac{4k^2}{d^{1/k}} \leq d^{-2\varepsilon}.$$

For all $v \in V$ with $d_{\mathcal{H}}(v) \geq d_{\text{tar}}$, we have $d_{\mathcal{H}'}(v) = d_{\mathcal{H}}(v)$ and thus $d_{\text{tar}} \leq d_{\mathcal{H}'}(v) \leq d$ with probability 1. For all $v \in V$ with $d_{\mathcal{H}}(v) \leq d_{\text{tar}}$, we have

$$\mathbb{E}[d_{\mathcal{H}'}(v)] = d_{\mathcal{H}}(v) + \sum_{U \in \binom{V^+}{k-1}} w(\{v\} \cup U) = d_{\mathcal{H}}(v) + d_{\text{def}}(v) = d_{\text{tar}}.$$

For all $u \in V^+$, Lemma 2.10.4 yields

$$\mathbb{E}[d_{\mathcal{H}'}(u)] = d(u) + \sum_{e \in \binom{V^+}{k} \colon u \in e} w(e) = (1 \pm d^{-2\varepsilon}) d_{\text{tar}}.$$

Furthermore, for all $v_1, v_2 \in V$, we have $d_{\mathcal{H}'}(v_1 v_2) = d_{\mathcal{H}}(v_1 v_2) \leq d^{1-\varepsilon}$ with probability 1, for all $v \in V$ and $u \in V^+$, we have

$$\mathbb{E}[d_{\mathcal{H}'}(uv)] = \sum_{U \in \binom{V^+ \setminus \{u\}}{k-2}} w(\{u, v\} \cup U) \leq \binom{2n-1}{k-2} \cdot \frac{\varepsilon d}{\binom{2n}{k-1}} = \frac{\varepsilon(k-1)d}{2n} \leq \frac{d}{d^{1/k}} \leq d^{1-2\varepsilon}$$

and for all $u_1, u_2 \in V^+$, Lemma 2.10.4 yields

$$\begin{aligned}
\mathbb{E}[d_{\mathcal{H}'}(u_1 u_2)] &= \sum_{U \in \binom{V^+}{k-2}} w(\{u_1, u_2\} \cup U) + \sum_{v \in V} \sum_{U \in \binom{V^+}{k-3}} w(\{u_1, u_2, v\} \cup U) \\
&\leq (1 + d^{-2\varepsilon}) \frac{(k-1)! d_{\text{def}}(u_1) d_{\text{def}}(u_2)}{(k-2)! d_{\text{def}}(V^+)} + n \binom{2n}{k-3} \frac{d_{\text{def}}(v)}{\binom{2n}{k-1}} \\
&\leq \frac{k d_{\text{tar}}^2}{(1 - \varepsilon k) d_{\text{tar}} \cdot 2n} + \frac{\varepsilon k^2 d}{n} \leq \frac{2kd}{n} \leq \frac{2kd}{d^{1/k}} \leq d^{1-2\varepsilon}.
\end{aligned}$$

Finally, we obtain

$$\mathbb{E}[|\{e \in \mathcal{H}' : 1 \leq |e \cap V(\mathcal{H})| \leq k-1\}|] = \sum_{v \in V} \sum_{U \in \binom{V^+}{k-1}} w(\{v\} \cup U) = \sum_{v \in V} d_{\text{def}}(v) \leq \varepsilon dn.$$

With these bounds on the expected degrees, using Chernoff's inequality (Lemma 2.10.1) and a suitable union bound, we conclude that with high probability, we have $(1 - d^{-\varepsilon})d \leq d_{\mathcal{H}'}(v) \leq d$ for all $v \in V'$, $d_{\mathcal{H}'}(v_1 v_2) \leq d^{1-\varepsilon}$ for all $v_1, v_2 \in V'$ and (2.10.12). $\square$

Proof of Theorem 2.5.4. Lemma 2.10.8 shows that $\mathcal{H}$ is an induced subgraph of a k-graph $\mathcal{H}'$ on $3n \leq \exp(d^{\varepsilon/(600\ell)})$ vertices with $(1 - d^{-\varepsilon})d \leq \delta(\mathcal{H}') \leq \Delta(\mathcal{H}') \leq d$, $\Delta_2(\mathcal{H}') \leq d^{1-\varepsilon}$ and $|W| \leq 2\varepsilon dn$ where

$$W := \{e \in \mathcal{H}' : 1 \leq |e \cap V(\mathcal{H})| \leq k-1\}.$$

Note that $\mathcal{C}$ is a $(d, \ell, \Gamma, \varepsilon)$-bounded conflict system for $\mathcal{H}'$ and that every test function $w \in \mathscr{W}$ is a $(d, \varepsilon, \mathcal{C})$-trackable test function for $\mathcal{H}'$. Let $W' \subseteq \mathcal{H}'$ with $W \subseteq W'$ and $\varepsilon dn \leq$

$|W'| \leq 2\varepsilon dn$. To see that $w' := \mathbb{1}_{W'}$ is a $(d, \varepsilon, \mathcal{C})$-trackable test function for $\mathcal{H}'$, note that $|W'| \geq \varepsilon dn \geq d^{1+\varepsilon}$.

An application of Theorem 2.5.3 with ε, $\mathcal{H}'$, $\mathscr{W} \cup \{w'\}$ playing the roles of μ, $\mathcal{H}$, $\mathscr{W}$ making use of the fact that we allowed more vertices and test functions in the proof of Theorem 2.5.3 yields a $\mathcal{C}$-free matching $\mathcal{M}' \subseteq \mathcal{H}'$ with $|\mathcal{M}'| = 3(1-\varepsilon)n/k$ and

$$w(\mathcal{M}') = (1 \pm d^{-\varepsilon/900})\left(\frac{|\mathcal{M}'|}{|\mathcal{H}'|}\right)^j w(\mathcal{H}') \tag{2.10.13}$$

for all j-uniform $w \in \mathscr{W}$ and

$$|\mathcal{M}' \cap W| \leq w'(\mathcal{M}') \leq 2\frac{|\mathcal{M}'|}{|\mathcal{H}'|} w'(\mathcal{H}') \leq 4\frac{\frac{3n}{k}}{\frac{3(1-d^{-\varepsilon})dn}{k}} \varepsilon dn \leq 8\varepsilon n. \tag{2.10.14}$$

Let $V(\mathcal{M}') := \bigcup_{e \in \mathcal{M}'} e$ and $\mathcal{M} := \mathcal{M}' \cap \mathcal{H}$. Then, (2.10.14) entails

$$\begin{aligned}|\mathcal{M}| &\geq |\mathcal{M} \cup (\mathcal{M}' \cap W)| - 8\varepsilon n \geq \frac{|V(\mathcal{M}') \cap V(\mathcal{H})|}{k} - 8\varepsilon n \geq \frac{n - |V(\mathcal{H}') \setminus V(\mathcal{M}')|}{k} - 8\varepsilon n \\ &\geq (1 - \varepsilon^{6/7})\frac{n}{k}.\end{aligned}$$

This implies

$$\frac{|\mathcal{M}|}{|\mathcal{H}|} = \frac{(1 \pm \varepsilon^{6/7})\frac{n}{k}}{\frac{(1\pm\varepsilon)dn}{k}} = (1 \pm 2\varepsilon^{6/7})\frac{1}{d} = (1 \pm 3\varepsilon^{6/7})\frac{|\mathcal{M}'|}{|\mathcal{H}'|}$$

and thus, for all $j \in [1/\varepsilon^{1/3}]$ and all j-uniform $w \in \mathscr{W}$, from (2.10.13) we obtain

$$w(\mathcal{M}) = w(\mathcal{M}') = (1 \pm d^{-\varepsilon/900})\left(\frac{|\mathcal{M}'|}{|\mathcal{H}'|}\right)^j w(\mathcal{H}') = (1 \pm \sqrt{\varepsilon})\left(\frac{|\mathcal{M}|}{|\mathcal{H}|}\right)^j w(\mathcal{H}),$$

which completes the proof. □

2.10.5 Proof of Theorem 2.5.5

In this section, we prove Theorem 2.5.5. To this end, we first employ the same conflict regularization approach that we used in Section 2.10.2 and then we bound the number of possible choices in every step of Algorithm 2.6.1.

Proof of Theorem 2.5.5. Let $\mathcal{C}'$ denote a conflict system for $\mathcal{H}$ as in Lemma 2.10.6. Consider Algorithm 2.6.1 with $\mathcal{H}$ and $\mathcal{C}'$ playing the roles of the input parameters $\mathcal{H}$ and $\mathcal{C}$. For $i \in [m-1]_0$, let

$$\hat{p}_V(i) := 1 - \frac{ik}{n}, \quad \hat{\Gamma}_0(i) := \sum_{j \in [\ell]_2} \frac{\Delta(\mathcal{C}^{(j)})}{d^{j-1}}\left(\frac{ik}{n}\right)^{j-1}$$
$$\text{and} \quad \hat{\Gamma}(i) := \sum_{j \in [\ell]_2} \frac{\Delta(\mathcal{C}'^{(j)})}{d^{j-1}}\left(\frac{ik}{n}\right)^{j-1}.$$

Let $\mathscr{M}$ denote the set of all $\mathcal{C}$-free matchings $\mathcal{M} \subseteq \mathcal{H}$ with $|\mathcal{M}| = (1-\mu)n/k$. For all $\mathcal{M} \in \mathscr{M}$, let $\mathcal{E}_{\mathcal{M}} := \{\mathcal{M}(m) = \mathcal{M}\}$ and for $i \geq 0$, let $\mathcal{T}^*(i)$ denote the event that for all $i' \in [i-1]_0$, we have

$$|\mathcal{H}|(i') \geq (1-d^{-\varepsilon^2}) \cdot \frac{dn}{k} \cdot \hat{p}_V(i')^k \cdot \exp(-\hat{\Gamma}_0(i')) =: \hat{h}_0^+(i').$$

Then, since $\mathcal{T}^*(m) \subseteq \{\mathcal{M}(m) \in \mathscr{M}\}$, we have $\mathbb{P}[\mathcal{T}^*(m)] = \sum_{\mathcal{M}\in\mathscr{M}} \mathbb{P}[\mathcal{T}^*(m) \cap \mathcal{E}_{\mathcal{M}}]$ and thus

$$|\mathscr{M}| \geq \frac{\mathbb{P}[\mathcal{T}^*(m)]}{\max_{\mathcal{M}\in\mathscr{M}} \mathbb{P}[\mathcal{T}^*(m) \cap \mathcal{E}_{\mathcal{M}}]}. \tag{2.10.15}$$

Hence, we aim to find a suitable lower bound for $\mathbb{P}[\mathcal{T}^*(m)]$ and for all $\mathcal{M} \in \mathscr{M}$, a suitable upper bound for $\mathbb{P}[\mathcal{T}^*(m) \cap \mathcal{E}_{\mathcal{M}}]$. First, we consider $\mathbb{P}[\mathcal{T}^*(m)]$.

Theorem 2.9.2 together with Fact 2.9.1 implies that with probability at least $1 - \exp(-d^{\varepsilon^2})$, for all $i \in [m-1]_0$, we have

$$|\mathcal{H}|(i) \geq (1-d^{-2\varepsilon^2}) \cdot \frac{dn}{k} \cdot \hat{p}_V(i)^k \cdot \exp(-\hat{\Gamma}(i)) =: \hat{h}^+(i).$$

By choice of $\mathcal{C}'$, for all $i \in [m-1]_0$, we have

$$\hat{\Gamma}(i) \leq (1+d^{-3\varepsilon^2}) \sum_{j\in[\ell]_2} \frac{d^{j-1-\varepsilon/600} + \Delta(\mathcal{C}^{(j)})}{d^{j-1}} \left(\frac{ik}{n}\right)^{j-1} \leq \hat{\Gamma}_0(i) + d^{-2\varepsilon^2},$$

and hence

$$\exp(-\hat{\Gamma}(i)) \geq \exp(-d^{-2\varepsilon^2}) \exp(-\hat{\Gamma}_0(i)) \geq (1-d^{-2\varepsilon^2}) \exp(-\hat{\Gamma}_0(i)).$$

This shows $\hat{h}_0^+(i) \leq \hat{h}^+(i)$, so we obtain

$$\mathbb{P}[\mathcal{T}^*(m)] \geq 1 - \exp(-d^{\varepsilon^2}) \geq 1 - d^{-1}. \tag{2.10.16}$$

Next, we fix any $\mathcal{M} \in \mathscr{M}$ and consider $\mathbb{P}[\mathcal{T}^*(m) \cap \mathcal{E}_{\mathcal{M}}]$. Fix an ordering $e_1, \ldots, e_m$ of $\mathcal{M}$. We have

$$\begin{aligned}
\mathbb{P}\Big[\mathcal{T}^*(m) \cap \bigcap_{i\in[m]} \{e(i) = e_i\}\Big] &= \mathbb{P}\Big[\bigcap_{i\in[m]} (\{e(i) = e_i\} \cap \mathcal{T}^*(i))\Big] \\
&\leq \prod_{i\in[m]} \mathbb{P}\Big[e(i) = e_i \;\Big|\; \bigcap_{i'\in[i-1]} (\{e(i') = e_{i'}\} \cap \mathcal{T}^*(i'))\Big] \\
&\leq \prod_{i\in[m]} \frac{k \exp(\hat{\Gamma}_0(i-1))}{(1-d^{-\varepsilon^2}) \cdot dn \cdot \hat{p}_V(i-1)^k} \\
&= \frac{k^m \exp(\sum_{i\in[m-1]_0} \hat{\Gamma}_0(i))}{(1-d^{-\varepsilon^2})^m \cdot d^m n^m \cdot \prod_{i\in[m-1]_0} \hat{p}_V(i)^k}.
\end{aligned}$$

Note that

$$\sum_{i\in[m-1]_0}\hat{\Gamma}_0(i)=\sum_{j\in[\ell]_2}\frac{\Delta(\mathcal{C}^{(j)})}{d^{j-1}}\sum_{i\in[m-1]_0}\left(\frac{ik}{n}\right)^{j-1}\le\sum_{j\in[\ell]_2}\frac{\Delta(\mathcal{C}^{(j)})}{d^{j-1}}\sum_{i\in[m-1]_0}\left(\frac{i}{m}\right)^{j-1}$$
$$\le\sum_{j\in[\ell]_2}\frac{\Delta(\mathcal{C}^{(j)})}{d^{j-1}}\int_0^m\left(\frac{x}{m}\right)^{j-1}\mathrm{d}x=m\sum_{j\in[\ell]_2}\frac{\Delta(\mathcal{C}^{(j)})}{jd^{j-1}}$$

and

$$\prod_{i\in[m-1]_0}\hat{p}_V(i)=\frac{k^m}{n^m}\prod_{i\in[m-1]_0}\left(\frac{n}{k}-i\right)\ge\frac{k^m m!}{n^m}$$

as well as

$$\frac{k^m m!}{n^m}\ge\left(\frac{km}{\mathrm{e}n}\right)^m=(1-d^{-\varepsilon^3})^m\exp(-m).$$

Thus, since there were at most $m!$ choices for the ordering $e_1,\dots,e_m$, we obtain

$$\begin{aligned}\mathbb{P}[\mathcal{T}^*(m)\cap\mathcal{E}_{\mathcal{M}}]&\le m!\cdot\frac{k^m\exp\bigl(\sum_{j\in[\ell]_2}\frac{\Delta(\mathcal{C}^{(j)})}{jd^{j-1}}\bigr)^m}{(1-d^{-\varepsilon^2})^m\cdot d^m n^m\cdot\bigl(\frac{k^m m!}{n^m}\bigr)^k}\\&=\left(\frac{\exp\bigl(\sum_{j\in[\ell]_2}\frac{\Delta(\mathcal{C}^{(j)})}{jd^{j-1}}\bigr)}{(1-d^{-\varepsilon^2})\cdot d\cdot\bigl(\frac{k^m m!}{n^m}\bigr)^{(k-1)/m}}\right)^m\\&\le\left(\frac{\exp\bigl(k-1+\sum_{j\in[\ell]_2}\frac{\Delta(\mathcal{C}^{(j)})}{jd^{j-1}}\bigr)}{(1-d^{-2\varepsilon^4})d}\right)^m.\end{aligned}$$

Using (2.10.15) to combine this with (2.10.16) completes the proof. □

2.11 Sparse Steiner systems

In this section, we prove Theorem 1.1.1 and some variations. For a partial (m,s,t)-Steiner system $\mathcal{S}$, we use $\bigcup\mathcal{S}:=\bigcup_{S\in\mathcal{S}}S$ to denote the set of points that $\mathcal{S}$ spans. In [49] it was shown that if $\mathcal{S}$ is an (m,s,t)-Steiner system, that is, a partial (m,s,t)-Steiner system where every t-set $T\subseteq[m]$ is a subset of exactly one s-set $S\in\mathcal{S}$, then for all $j\in[|\mathcal{S}|]_2$, there is a collection $\mathcal{S}'\subseteq\mathcal{S}$ of size j that spans at most

$$\pi(j):=(s-t)j+t+1$$

points. Motivated by this we proceed as in [49] and introduce the notion that for an integer ℓ, a partial (m,s,t)-Steiner system $\mathcal{S}$ is *ℓ-sparse* if for all $j\in[\ell]_2$, every $\mathcal{S}'\in\binom{\mathcal{S}}{j}$ spans at least $\pi(j)$ points, or equivalently, if for all integers p with $2\le\kappa_{s,t}(p)+1\le\ell$, where

$$\kappa_{s,t}(p):=\left\lfloor\frac{p-t-1}{s-t}\right\rfloor,$$

every $\mathcal{S}' \subseteq \mathcal{S}$ that spans at most p points has size at most $\kappa_{s,t}(p)$.

In Theorem 2.11.2, we allow ℓ to grow with m, hence providing a lower bound for the maximum possible ℓ as a function of m. Lefmann, Phelps and Rödl [83] obtained the following upper bound.

Theorem 2.11.1 ([83]). *There exists $c > 0$ such that every Steiner triple system of order m contains a subset of size j where $4 \leq j \leq c \log m / \log\log m$ that spans at most $j + 2$ points.*

It would be interesting to close or significantly narrow the gap between these two bounds by determining more precisely how large ℓ may be chosen in terms of m.

Theorem 2.11.2. *For all $s > t \geq 2$, there exists m_0 such that for all $m \geq m_0$ and*

$$\ell := \frac{\log\log m}{3s \log\log\log m},$$

there exists an ℓ-sparse partial (m, s, t)-Steiner system $\mathcal{S}$ of size

$$(1 - \exp(-\sqrt{\log m}))\frac{\binom{m}{t}}{\binom{s}{t}}.$$

Proof. Fix $s > t \geq 2$, suppose that m is sufficiently large in terms of s and t and define ℓ as in the statement. Let $X := [m]$ and $k := \binom{s}{t}$. Consider the k-graph $\mathcal{H}$ with vertex set $\binom{X}{t}$ and edge set $\{\binom{S}{t} : S \in \binom{X}{s}\}$. With appropriately chosen parameters and conflicts enforcing ℓ-sparseness, we can apply Theorem 2.5.1 to obtain a matching $\mathcal{M} \subseteq \mathcal{H}$ that represents a partial (m, s, t)-Steiner system as desired.

First, let us introduce some further terminology and notation. As m, s and t are fixed throughout the proof, we call partial (m, s, t)-Steiner systems simply *partial Steiner systems*. We say that a partial Steiner system $\mathcal{S}$ is *forbidden* if it has size $j \in [\ell]_2$ and spans less than $\pi(j)$ points. Note that a partial Steiner system $\mathcal{S}$ is ℓ-sparse if and only if there is no forbidden partial Steiner system $\mathcal{S}' \subseteq \mathcal{S}$. Every edge $e \in \mathcal{H}$ is the set of all t-sets that are subsets of an s-set $S \subseteq X$, so given e, we may recover S by considering the set $\bigcup_{v \in e} v$ of points lying in one of the vertices in e. We can reverse this and obtain e by considering the set $\binom{S}{t}$ of all t-sets that are subsets of S. We extend these constructions to edge sets and collections of s-sets as follows. For an edge set $E \subseteq \mathcal{H}$ and for $\mathcal{S} \subseteq \binom{X}{s}$, we define

$$\mathcal{S}(E) := \Big\{\bigcup_{v \in e} v : e \in E\Big\} \quad \text{and} \quad E(\mathcal{S}) := \left\{\binom{S}{t} : S \in \mathcal{S}\right\}.$$

Furthermore, to provide access to the underlying subset of X, for $E \subseteq \mathcal{H}$, we define $X(E) := \bigcup \mathcal{S}(E)$ to be the set of points lying in vertices in edges of E and for $e \in \mathcal{H}$, we set $X(e) := X(\{e\})$. Note that mapping $E \subseteq \mathcal{H}$ to $\mathcal{S}(E)$ yields a size preserving bijection from the set of all matchings in $\mathcal{H}$ to the set of all partial Steiner systems and mapping $\mathcal{S}$ to $E(\mathcal{S})$ yields its inverse.

We are now ready to define appropriate parameters and conflicts. Let $\mathcal{C}$ denote the conflict system for $\mathcal{H}$ where a set $C \subseteq \mathcal{H}$ is a conflict in $\mathcal{C}$ if and only if $\mathcal{S}(C)$ is a forbidden partial Steiner system of size $j \in [\ell]_2$ such that there is no smaller forbidden partial Steiner system $\mathcal{S}' \subseteq \mathcal{S}$. Let

$$d := \binom{m-t}{s-t}, \quad \varepsilon := \frac{1}{s}, \quad \Gamma := (\ell s)^{\ell s+1}, \quad \text{and} \quad \mu := \exp(-\sqrt{\log m}).$$

The k-graph $\mathcal{H}$ is d-regular and for all distinct $u, v \in V(\mathcal{H})$, we have $|u \cup v| \geq t+1$ and thus

$$d_{\mathcal{H}}(uv) \leq \binom{m-t-1}{s-t-1} = \frac{s-t}{m-t} d \leq (s-t) d^{1-\frac{1}{s-t}} \leq d^{1-\varepsilon}.$$

Furthermore, we have $\Gamma \leq (\log m)^{2/5}$ and hence $1/\mu^{\Gamma\ell} \leq \exp((\log m)^{19/20}) \leq d^{\varepsilon^2}$. We show next that $\mathcal{C}$ is $(d, \ell, \Gamma, \varepsilon)$-bounded. Then, Theorem 2.5.1 yields a $\mathcal{C}$-free matching $\mathcal{M} \subseteq \mathcal{H}$ with size $(1-\mu)\binom{m}{t}/\binom{s}{t}$ and thus an ℓ-sparse partial Steiner system $\mathcal{S}(\mathcal{M})$ of the same size.

Condition (C1) holds by construction of $\mathcal{C}$, and since there is no forbidden partial Steiner system of size 2, conditions (C4) and (C5) are also trivially satisfied. It remains to check that (C2) and (C3) hold.

To this end, first note that removing any element of a forbidden partial Steiner system $\mathcal{S}$ of size $j \in [\ell]_3$ that spans less than $\pi(j-1)$ points yields a smaller forbidden partial Steiner system. This implies that all forbidden partial Steiner systems $\mathcal{S}$ of size $j \in [\ell]_2$ that do not contain a smaller forbidden partial Steiner system $\mathcal{S}' \subseteq \mathcal{S}$ span at least $\pi(j-1)$ points and hence we have $\pi(j-1) \leq |X(C)| \leq \pi(j) - 1$ for all $j \in [\ell]_2$ and $C \in \mathcal{C}^{(j)}$.

First, we aim to verify (C2). For all $j \in [\ell]_2$, $e \in \mathcal{H}$ and $p \in [\pi(j)-1]_{\pi(j-1)}$, we obtain

$$\begin{aligned}\sum_{P \in \binom{X}{p} \colon X(e) \subseteq P} |\{C \in \mathcal{C}^{(j)} : X(C) = P\}| &\leq \binom{m-s}{p-s}\binom{p}{s}^j \leq \frac{1}{(s!)^j} m^{p-s} p^{js} \leq \frac{(\ell s)^{\ell s}}{(s!)^j} m^{p-s} \\ &\leq \frac{(\ell s)^{\ell s}}{(s!)^j} m^{(s-t)(j-1)} \leq (\ell s)^{\ell s} d^{j-1}\end{aligned}$$

and thus

$$d_{\mathcal{C}^{(j)}}(e) \leq \sum_{p \in [\pi(j)-1]_{\pi(j-1)}} \sum_{P \in \binom{X}{p} \colon X(e) \subseteq P} |\{C \in \mathcal{C}^{(j)} : X(C) = P\}| \leq s(\ell s)^{\ell s} d^{j-1}.$$

This yields $\sum_{j \in [\ell]_2} \frac{\Delta(\mathcal{C}^{(j)})}{d^{j-1}} \leq \Gamma$. Clearly, $|\{j \in [\ell]_2 : \mathcal{C}^{(j)} \neq \emptyset\}| \leq \ell \leq \Gamma$, so (C2) holds.

For all $j \in [\ell]_2$, $j' \in [j-1]_2$ and $E \subseteq \mathcal{H}$ with $|E| = j'$ and $d_{\mathcal{C}^{(j)}}(E) \geq 1$, the partial Steiner system $\mathcal{S}(E)$ is not forbidden and thus spans at least $\pi(j')$ points. For all $p \in [\pi(j)-1]_{\pi(j-1)}$, this entails

$$\sum_{P \in \binom{X}{p} \colon X(E) \subseteq P} |\{C \in \mathcal{C}^{(j)} : X(C) = P\}| \leq \binom{m-(s-t)j'-t-1}{p-(s-t)j'-t-1}\binom{p}{s}^j$$

$$
\begin{aligned}
&\leq m^{p-(s-t)j'-t-1}p^{js} \\
&\leq (\ell s)^{\ell s}m^{p-(s-t)j'-t-1} \leq (\ell s)^{\ell s}m^{(s-t)(j-j')-1} \\
&= (\ell s)^{\ell s}\frac{m^{(s-t)(j-j'-1/s)}}{m^{t/s}} \leq \frac{d^{j-j'-\varepsilon}}{s}
\end{aligned}
$$

and thus

$$d_{\mathcal{C}^{(j)}}(E) \leq \sum_{p\in[\pi(j)-1]_{\pi(j-1)}} \sum_{P\in\binom{X}{p}:\, X(E)\subseteq P} |\{C\in\mathcal{C}^{(j)} : X(C)=P\}| \leq d^{j-j'-\varepsilon}.$$

This shows that (C3) holds and hence finishes the proof. □

Theorem 2.11.2 is a version of Theorem 1.1.1 where we allow ℓ to grow with m. Due to the growth of ℓ, we do not obtain the polynomially decreasing leftover fraction $m^{-\varepsilon}$ from Theorem 1.1.1, however, for fixed ℓ, it is straightforward to adapt the proof such that it yields a leftover fraction as in Theorem 1.1.1.

Additionally, using test systems, the proof can easily be extended to also provide control over the $(t-1)$-degrees of the leftover, that is, for all $(t-1)$-sets $Y \in \binom{[m]}{t-1}$, control over the number of t-sets T with $Y \subseteq T$ that are not subsets of an s-set $S \in \mathcal{S}$. Indeed, in the proof where we consider the $\binom{s}{t}$-graph $\mathcal{H}$ with vertex set $\binom{[m]}{t}$ and edge set $\left\{\binom{S}{t} : S \in \binom{[m]}{s}\right\}$, instead of Theorem 2.5.1, one may simply apply Theorem 2.5.2 using the sets $\{e \in \mathcal{H} : Y \subseteq \bigcup_{v\in e} v\}$ with $Y \in \binom{[m]}{t-1}$ as test systems.

Moreover, the same approach that we use in the proof can also be applied to prove Theorem 2.3.1 by considering the $\binom{s}{t}$-graph $\mathcal{H}$ whose vertices are the edges of $\mathcal{G}$ and whose edges are the edge sets of the cliques induced by the elements of $\mathcal{K}$. In particular, the arising conflict system is a subgraph of the conflict system $\mathcal{C}$ we analyzed in the proof of Theorem 2.11.2, so all the bounds still hold.

The following Lemma 2.11.4 allows us to obtain Theorem 2.3.2 as a consequence of Theorem 2.5.4. To prove this lemma, we use Hoeffding's inequality for sampling without replacement.

Lemma 2.11.3 (Hoeffding's inequality [55])**.** *Consider an n-set A and a function $f\colon A \to [0,1]$. Let $m \leq n$ and suppose that $X \in \binom{A}{m}$ is chosen uniformly at random. Let $f(X) := \sum_{x\in X} f(x)$. Then, for all $t > 0$,*

$$\mathbb{P}[|f(X) - \mathbb{E}[f(X)]| \geq t] \leq 2\exp\left(-\frac{2t^2}{m}\right).$$

Recall that for $2 \leq t < s$, we use $\delta_{t,s}$ to denote the fractional decomposition threshold for the t-uniform s-clique as defined in Section 2.3.2. Furthermore, for a t-graph $\mathcal{H}$, we use $\mathscr{K}_s(\mathcal{H})$ to denote the set of all s-cliques in $\mathcal{H}$.

Lemma 2.11.4. *For all $2 \leq t < s$ and $\varepsilon > 0$, there exists ζ_0 such that for all $0 < \zeta < \zeta_0$, there exists n_0 such that for all $n \geq n_0$ and all t-graphs $\mathcal{H}$ on n vertices with $\delta_{t-1}(\mathcal{H}) \geq (\delta_{t,s} + \varepsilon)n$, there exists $w\colon \mathscr{K}_s(\mathcal{H}) \to [0,1]$ such that the following holds.*

(i) $\sum_{\mathcal{K}\in\mathscr{K}_s(\mathcal{H})\colon e\in\mathcal{K}} w(\mathcal{K}) = 1 \pm \varepsilon$ *for all* $e \in \mathcal{H}$*;*

(ii) $\max\{w(\mathcal{K}) : \mathcal{K} \in \mathscr{K}_s(\mathcal{H})\} \leq 1/(\zeta n)^{s-t}$.

Proof. Suppose that $\varepsilon > 0$ is sufficiently small in terms of $1/s$, suppose that ζ is sufficiently small in terms of ε and suppose that n is sufficiently large in terms of $1/\zeta$. For $m := 1/(2\zeta)$, consider the collection

$$\mathcal{U} := \left\{U \in \binom{V(\mathcal{H})}{m} : \delta_{t-1}(\mathcal{H}[U]) \geq (\delta_{t,s} + \varepsilon^2)m\right\}.$$

For $U \in \mathcal{U}$ consider a fractional decomposition $w_U\colon \mathscr{K}_s(\mathcal{H}[U]) \to [0,1]$ which exists by definition of $\delta_{t,s}$. Then, since for all $e \in \mathcal{H}$, the collection $\mathcal{U}_e := \{U \in \mathcal{U} : e \subseteq U\}$ includes almost all m-sets $U \subseteq V(\mathcal{H})$ with $e \subseteq U$, the function w with

$$w(\mathcal{K}) = \binom{n-t}{m-t}^{-1} \sum_{U\in\mathcal{U}\colon \mathcal{K}\subseteq\mathcal{H}[U]} w_U(\mathcal{K})$$

for all $\mathcal{K} \in \mathscr{K}_s(\mathcal{H})$ satisfies (i) and (ii).

Let us turn to the details. The function w satisfies (ii), since for all $\mathcal{K} \in \mathscr{K}_s(\mathcal{H})$, we have

$$w(\mathcal{K}) \leq \binom{n-s}{m-s}\binom{n-t}{m-t}^{-1} \leq \frac{n^{m-s}}{(m-s)!} \cdot \frac{2(m-t)!}{n^{m-t}} \leq \left(\frac{2m}{n}\right)^{s-t}.$$

It remains to show that w also satisfies (i).

For all $e \in \mathcal{H}$, we have

$$\sum_{\substack{\mathcal{K}\in\mathscr{K}_s(\mathcal{H})\colon\\ e\in\mathcal{K}}} w(\mathcal{K}) = \binom{n-t}{m-t}^{-1} \sum_{U\in\mathcal{U}_e} \sum_{\substack{\mathcal{K}\in\mathscr{K}_s(\mathcal{H}[U])\colon\\ e\in\mathcal{K}}} w_U(\mathcal{K}) = \binom{n-t}{m-t}^{-1} |\mathcal{U}_e|.$$

Hence, it suffices to show that $|\mathcal{U}_e| \geq (1-\varepsilon)\binom{n-t}{m-t}$ or equivalently, that for $U_- \in \binom{V(\mathcal{H})\setminus e}{m-t}$ chosen uniformly at random and $U := U_- \cup e$, we have $\mathbb{P}[U \notin \mathcal{U}_e] \leq \varepsilon$. To this end, let $d := (\delta_{t,s} + \varepsilon^2)m$ and observe that

$$\mathbb{P}[U \notin \mathcal{U}_e] \leq \sum_{t'\in[t]_0} \sum_{\{v_1,\dots,v_{t'}\}\in\binom{e}{t'}} \sum_{\{v_{t'+1},\dots,v_t\}\in\binom{V(\mathcal{H})\setminus e}{t-t'}} \mathbb{P}[d_{\mathcal{H}[U]}(v_1 \dots v_t) \leq d \text{ and } v_{t'+1},\dots,v_t \in U_-], \tag{2.11.1}$$

so it suffices to obtain appropriate bounds for the probabilities on the right.

Fix $t' \in [t]_0$ and distinct $v_1,\dots,v_{t'} \in e$ and $v_{t'+1},\dots,v_t \in V(\mathcal{H}) \setminus e$. Lemma 2.11.3 entails

$$\begin{aligned}\mathbb{P}[d_{\mathcal{H}[U]}(v_1,\dots,v_t) \leq d \text{ and } v_{t'+1},\dots,v_t \in U_-]\\ &\leq \frac{2m^{t-t'}}{n^{t-t'}}\mathbb{P}[d_{\mathcal{H}[U]}(v_1,\dots,v_t) \leq d \mid v_{t'+1},\dots,v_t \in U_-]\\ &\leq \frac{2m^{t-t'}}{n^{t-t'}}\exp(-\varepsilon^3 m) \leq \frac{\varepsilon}{t\cdot t^t\cdot n^{t-t'}}.\end{aligned}$$

Using (2.11.1), this bound yields $\mathbb{P}[U \notin \mathcal{U}_e] \leq \varepsilon$. □

Using this lemma, we may again use a similar approach as in the proof of Theorem 2.11.2 to obtain Theorem 2.3.2. To this end, suppose that $\varepsilon > 0$ is sufficiently small in terms of $1/\ell$ and $1/s$, suppose that ζ is sufficiently small in terms of ε and suppose that m is sufficiently large in terms of $1/\zeta$. Then, for every t-graph $\mathcal{G}$ on m vertices with $\delta_{t-1}(\mathcal{G}) \geq (\delta_{t,s} + \varepsilon)m$, we may apply Lemma 2.11.4 with ε^4 playing the role of ε to obtain a suitable approximate fractional decomposition w. Let $k := \binom{s}{t}$ and consider the k-graph $\mathcal{H}$ with vertex set $\mathcal{G}$ where the edges are the edge sets of s-cliques in $\mathcal{G}$. This k-graph plays a similar role as $\mathcal{H}$ in the proof of Theorem 2.11.2 and we may define a similar conflict system $\mathcal{C}$ that captures the configurations that we wish to avoid as conflicts. Let $d := (\zeta n)^{s-t}$. Lemma 2.10.1 entails that with positive probability, for a random subgraph $\mathcal{H}'$ of $\mathcal{H}$ with the same vertex set where every edge set of an s-clique K is an edge of $\mathcal{H}'$ independently with probability $(1-\varepsilon^3)(\zeta n)^{s-t} w(K)$, we have $(1-\varepsilon^2)d \leq \delta(\mathcal{H}') \leq \Delta(\mathcal{H}') \leq d$. Suppose that $\mathcal{H}'$ is such a subgraph of $\mathcal{H}$.

Let $\Gamma := (\ell s)^{\ell s+1}$. Similar arguments as in the proof of Theorem 2.11.2 show that $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon^2}$ and $1/\varepsilon^{2\Gamma\ell} \leq d^{\varepsilon^4}$ hold and that $\mathcal{C}$ is $(d, \ell, \Gamma, \varepsilon^2)$-bounded. Hence, we may apply Theorem 2.5.4 with $\mathcal{H}'$ and ε^2 playing the roles of $\mathcal{H}$ and ε to obtain a $\mathcal{C}$-free matching $\mathcal{M} \subseteq \mathcal{H}$ with size $(1-\varepsilon)|\mathcal{G}|/\binom{s}{t}$ and thus a Steiner system as desired.

2.12 Further remarks

We already give a few applications of our main theorems in Section 2.3, but we believe that there are many more. In fact, since the publication of the research this chapter is based on, several such applications have been found, see [6, 11, 47, 52, 63, 81] and [24] which builds on a result from [47].

We end the chapter by repeating two open questions that we briefly mention in the chapter. Firstly, Theorem 2.5.5 yields a lower bound on the number of conflict-free almost-perfect matchings and we wonder if the bound in Theorem 2.5.5 is essentially tight. If there are no conflicts an upper bound can be derived with the so-called entropy method. Potentially this method can be adapted to also yield the corresponding upper bound in a setting with conflicts. Even in the case of Steiner triple systems with girth at least 7 this seems challenging (see the discussions in [18, 49, 80]).

Secondly, Theorem 2.11.2 shows that there are approximate Steiner systems on m points of girth $\Omega(\frac{\log\log m}{\log\log\log m})$ whereas Lefmann, Phelps and Rödl [83] proved that in general the girth cannot be larger than $O(\frac{\log m}{\log\log m})$. It remains an interesting question to determine the largest possible girth of (approximate) Steiner systems.

Chapter 3

The Hypergraph Removal Process

3.1 Introduction

In this chapter, we consider the random greedy hypergraph packing process as a removal process that iteratively removes edges of chosen copies as described in Chapter 1. To determine when the $\mathcal{F}$-removal process terminates, we crucially rely on closely tracking the evolution of the numbers of occurrences of certain key substructures within the random hypergraphs generated by the iterated removal of the randomly chosen copies of $\mathcal{F}$. We do this essentially all the way until we would expect no more remaining copies and this tracking constitutes the heart of our proof. The main obstacle here lies in selecting appropriate substructures that allow us to carry out such an analysis with sufficient precision for the necessary number of steps. When we finally arrive at a step where typically only few copies remain, the structural insights that the knowledge of these key quantities provide allow us to apply Theorem 1.1.7 or Theorem 1.1.9 to show that then, the $\mathcal{F}$-removal process typically quickly terminates such that the overall runtime is as expected. The proof of Theorem 1.1.7 and Theorem 1.1.9 relies on an argument that is separate from the analysis of the algorithm up to the point where typically only few copies remain and we present it at the end of the chapter starting in Section 3.9.

The number of copies of $\mathcal{F}$ still present in $\mathcal{H}$ is one obvious example for one of the aforementioned key quantities that is crucial for understanding the behavior and following the evolution of the process. Similarly as in Chapter 2, we employ supermartingale concentration techniques to show that the random processes given by the key quantities that we select typically closely follow deterministic trajectories that we deduce from heuristic considerations. Again, our approach resembles the differential equation method introduced by Wormald [113]. To maintain precise control over the key random processes in the sense that we can still guarantee that expected one-step changes are as suggested by intuition, we exploit a phenomenon that can be described as a self-correcting behavior certain key quantities inherently exhibit. Furthermore, we require precise estimates also for the quantities that determine the one-step changes of the key random processes, which often forces us to enlarge our collection.

More specifically, let $\mathcal{H}^*$ denote the $|\mathcal{F}|$-graph where the edges present at some step i, that is after i iterations, form the vertex set of $\mathcal{H}^*$ and where the edge sets of copies of $\mathcal{F}$ present at step i are the edges of $\mathcal{H}^*$. Let H^* denote the number of edges of $\mathcal{H}^*$, that is, the number of present copies of $\mathcal{F}$. Let $\mathfrak{F}(0), \mathfrak{F}(1), \dots$ denote the natural filtration associated with the $\mathcal{F}$-removal process and consider the following example. Assuming that for all distinct edges e and f, the number of copies of $\mathcal{F}$ that contain both e and f is negligible compared to the degrees $d_{\mathcal{H}^*}(e)$ and $d_{\mathcal{H}^*}(f)$, in expectation, the one-step change ΔH^* of the number of present copies when transitioning to the next step is

$$\mathbb{E}[\Delta H^* \mid \mathfrak{F}(i)] \approx - \sum_{\mathcal{F}' \in \mathcal{H}^*} \sum_{e \in \mathcal{F}'} \frac{d_{\mathcal{H}^*}(e)}{H^*}. \tag{3.1.1}$$

Note that here, the larger H^*, the larger the expected decrease (we divide by H^*, but the remaining copies are counted by both, the number of summands in the outer sum and the degrees). When considering the one-step changes of a process that measures the deviation of the number of remaining copies from an appropriate deterministic prediction, this causes a drift that, in expectation, steers the number of copies towards the prediction. Exploiting such self-correcting behavior turns out to be crucial for a precise analysis of the process. This leads to the critical interval method approach briefly mentioned in Chapter 1. Other applications of such an approach can be found in [13, 14, 16, 17, 34, 56, 108]. While in Chapter 2, we were only interested in a guarantee that at most a vanishing fraction of vertices remains uncovered by the matching we generate using the conflict-free matching process, in this chapter we specifically wish to quantify this fraction. Thus, while in this chapter we analyze a comparatively simpler special case of the ordinary random greedy hypergraph matching process without any modifications, the main challenge now which greatly complicates the analysis is that we wish to exploit such self-correcting behavior to enable a quantitatively much more precise analysis.

Another important observation is that (3.1.1) introduces the degrees $d_{\mathcal{H}^*}(e)$ of remaining edges e as further crucial quantities whose evolution we wish to follow using supermartingale concentration. As such an edge e itself could be removed during the next removal of a copy of $\mathcal{F}$, it is more convenient to instead consider the degree $d'_{\mathcal{H}^*}(e)$ of e in the hypergraph $\mathcal{H}^*_{[e]}$ obtained from $\mathcal{H}^*$ by adding e as a vertex and the edge sets of all copies $\mathcal{F}'$ of $\mathcal{F}$ where all edges $f \in \mathcal{F}' \setminus \{e\}$ are present as edges. Note that if $e \in \mathcal{H}^*$, then $\mathcal{H}^*_{[e]} = \mathcal{H}^*$ and $d'_{\mathcal{H}^*}(e) = d_{\mathcal{H}^*}(e)$. Since we again aim to rely on supermartingale concentration, for a remaining edge e, we are again interested in the one-step change $\Delta d'_{\mathcal{H}^*}(e)$ of $d'_{\mathcal{H}^*}(e)$ when transitioning to the next step.

Similarly as above, we estimate

$$\mathbb{E}[\Delta d'_{\mathcal{H}^*}(e) \mid \mathfrak{F}(i)] \approx - \sum_{\mathcal{F}' \in \mathcal{H}^*_{[e]} \colon e \in \mathcal{F}'} \sum_{f \in \mathcal{F}' \setminus \{e\}} \frac{d_{\mathcal{H}^*}(f)}{H^*}.$$

Since the degrees of remaining edges are included in our collection of key quantities, we have estimates available for the degrees that we could use to approximate the expected one-step changes of the degrees. This is a valid approach that leads to a natural barrier in the analysis, see [9,13]. However, due to undesirable accumulation of estimation errors, such an approach is insufficient for an analysis up to the point where we may apply Theorem 1.1.7 or Theorem 1.1.9.

Consider the following idea to circumvent this issue. If precise estimates for the number Φ_e of substructures within $\mathcal{H}^*_{[e]}$ that consist of two copies of $\mathcal{F}$ that share an edge $e' \neq e$ and where one copy contains e were available, we could rely on the identity

$$\mathbb{E}[\Delta d'_{\mathcal{H}^*}(e) \mid \mathfrak{F}(i)] = -\frac{\Phi_e}{H^*}.$$

However, if we now add the random variables Φ_e to our collection of tracked key quantities, we essentially only shift the problem to determining the one-step changes of these new random variables and similarly iterating the extension of the collection by adding further key quantities that count substructures consisting of more and more copies of $\mathcal{F}$ overlapping at edges quickly becomes unsustainable as the collection becomes too large.

The very high-level approach described so far, including the separation into an analysis of the early evolution and an analysis of the late evolution of the process, is essentially the same as in the analysis of the case where $\mathcal{F}$ is a triangle [14]. Consequently, the same obstacle mentioned above is encountered. To remedy this issue, Bohman, Frieze and Lubetzky [14] carefully control the extension of the collection of key quantities manually by giving explicit descriptions of the elements of a suitably chosen collection of structures of overlapping triangles using sequences of the symbols 0, 1 and $\mathtt{e}$. This collection is chosen roughly based on the above idea and its size grows with $1/\varepsilon$ to allow for sufficiently precise estimates, but at the cost of some however negligible precision, the collection is still sufficiently small to allow an analysis of the evolution of all the relevant random variables.

Explicitly describing the relevant substructures that facilitate such an analysis seems practically infeasible for hypergraphs or even graphs larger than the triangle. Instead, we implicitly choose our collection as a with respect to inclusion minimal collection of substructures that is closed under certain carefully chosen substructure transformations, where intuitively we still follow the above idea of considering substructures of overlapping copies. With this definition, we need to rely on a density argument to see that this even yields a finite collection. While the size of our collection grows with $1/\varepsilon$, we show that it is not too large and that, by choice of the transformations, it allows a precise analysis of the evolution of all key quantities related to the substructures in the collection. Due to the implicit nature of our collection, we have to make our arguments without concrete knowledge of the structures we consider and all properties need to be deduced from the fact that the collection is minimal among all collections that are closed under the aforementioned transformations. This often makes our arguments substantially more abstract. For example, for the analysis of the triangle case in [14], substructures called *fans* in [14] which essentially correspond to graphs that for some $\ell \geq 1$ consist of vertices $u, v_1, \ldots, v_\ell$ and the edges $\{u, v_i\}$ and $\{v_j, v_{j+1}\}$ where $1 \leq i \leq \ell$ and $1 \leq j \leq \ell-1$ play a key role. In our more general analysis, we instead work with maximizers of density based optimization problems that we consider without concrete knowledge of their structure.

A further obstacle that we overcome in our analysis is related to a possible lack of symmetry of $\mathcal{F}$ compared to a triangle. The structure of two overlapping copies of $\mathcal{F}$ depends not only on the size of the overlap but also on the specific choice of the shared part. This can cause transformations to switch between different non-interchangeable choices within copies of $\mathcal{F}$, which complicates the crucial part of the argument where estimation errors need to be calibrated such that the self-correcting behavior of the random processes remains mostly undisturbed by other quantities that also occur in the expressions for the expected one-step changes. We overcome this by considering our random processes in groups to restore symmetry in the sense that whenever we apply transformations to all members of a group simultaneously, we remain in a situation where all non-interchangeable choices within copies of $\mathcal{F}$ are represented if this was previously the case.

Finally, as mentioned above, to complete our argumentation it remains to prove Theorems 1.1.7 and 1.1.9. In our significantly more general setting, adapting the argument presented in [14] to obtain a similar statement for the triangle case requires additional insights for a sufficient understanding of the structure of the random hypergraphs typically encountered around the time when we would typically expect the process to terminate. While in the triangle case certain configurations formed by overlapping copies of $\mathcal{F}$ are impossible as the triangle is simply too small to allow such overlaps of distinct copies, arguments bounding the numbers of such configurations are non-trivial for larger hypergraphs or even graphs.

3.2 Organization of the chapter

Theorem 1.1.4 is an immediate consequence of Theorem 1.1.5. Furthermore, the upper bounds in Theorems 1.1.5 and 1.1.8 follow from Theorem 1.1.6. In the first part of the chapter, our goal is to analyze the removal process for a sufficient number of steps to see that with high probability, the process eventually generates a k-graph that is sufficiently sparse to confirm Theorem 1.1.6 and that satisfies the properties necessary for an application of Theorem 1.1.7 or Theorem 1.1.9 that then establishes the lower bound in Theorem 1.1.5 or Theorem 1.1.8. Subsequently, in the second part, we prove Theorems 1.1.7 and 1.1.9.

As mentioned in Section 3.1, our precise analysis of the process consists of closely tracking the evolution of the number of occurrences of certain key substructures within the random k-graphs generated by the process. We present this core of our proof as two closely related instances of a supermartingale concentration argument. Section 3.6 is dedicated to implicitly defining our carefully selected substructures and obtaining key insights concerning singular such substructures. In Section 3.7, we adjust our point of view and consider these structures in groups to establish symmetry, which is crucial for the careful calibration of estimation error needed to exploit self-correcting behavior. In Section 3.8, we show that, essentially as an immediate consequence of the insights gained in Sections 3.6 and 3.7, the conditions for the intended application of Theorems 1.1.7 and 1.1.9 are indeed satisfied.

As preparation for the argumentation in Sections 3.6 and 3.7, we first proceed as follows. We introduce the setup for the first part of the chapter and formally state the goal for this part in Section 3.3. Then, in Section 3.4, we describe the heuristics that lead to our choices of deterministic trajectories that we expect key quantities to follow. Furthermore, towards the end of Section 3.4, we formally describe how introducing appropriate stopping times allows us to present the aforementioned two instances mostly separately. As final preparations for Sections 3.6 and 3.7, in Section 3.5 we subsequently introduce notation and terminology specific to our situation, we define key stopping times and we gather some statements concerning key quantities defined up to this point.

For the second part of the chapter, where we prove Theorems 1.1.7 and 1.1.9, we first describe the setup for this part in Section 3.9. Starting in Section 3.9, we focus on the case where $\mathcal{F}$ has at least 3 edges since the case where $\mathcal{F}$ has only two edges, that is the case that is relevant for Theorem 1.1.9, requires some specific modifications in the argumentation that we discuss in detail in Section 3.14. In Section 3.10, we further investigate the structure of the hypergraphs generated towards the expected end of the process to deduce bounds that we subsequently rely on. Then, in Section 3.11, we present an extended tracking argument for the number of remaining copies which serves as further preparation for the arguments in Section 3.12 where we finally show that typically, sufficiently many edges remain when the process terminates. This completes the main part of our analysis and allows us to obtain proofs for our main theorems, see Sections 3.13 and 3.14.

In Sections 3.15–3.17, we provide proofs for three auxiliary statements that we relied on in the previous sections of the chapter. These proofs consist of further instances of a critical interval argument that are mostly separate from one another and the other parts of the chapter.

At the end of the chapter, in Section 3.18, we introduce two new conjectures concerning the $\mathcal{F}$-free and the $\mathcal{F}$-removal process that we obtain from heuristic considerations. Furthermore, we use Section 3.18 to briefly discuss possible applications of the results in this chapter.

3.3 Removal process

From now on, until the end of Section 3.8, we focus on the first part. To this end, in this section, we describe the removal process that we analyze in the subsequent sections. For now, we assume a slightly more general setup similar to the one in Theorem 1.1.6. In more detail, let $k \geq 2$ and fix a k-balanced k-graph $\mathcal{F}$ on m vertices with $|\mathcal{F}| \geq 2$ and k-density $\rho_{\mathcal{F}}$. Suppose that $0 < \varepsilon < 1$ is sufficiently small in terms of $1/m$, that $0 < \delta < 1$ is sufficiently small in terms of ε and that n is sufficiently large in terms of $1/\delta$. Suppose that $\mathcal{H}(0)$ is a $(\varepsilon^4, \delta, \rho_{\mathcal{F}})$-pseudorandom k-graph on n vertices where

$$\vartheta := \frac{k!\,|\mathcal{H}(0)|}{n^k} \geq n^{-1/\rho_{\mathcal{F}}+\varepsilon}$$

and let $\mathcal{H}^*(0)$ denote the $|\mathcal{F}|$-graph with vertex set $\mathcal{H}(0)$ whose edges are the edge sets of copies of $\mathcal{F}$ that are subgraphs of $\mathcal{H}(0)$. Consider the following random process.

Algorithm 3.3.1: Random $\mathcal{F}$-removal

$i \leftarrow 1$
while $\mathcal{H}^*(i-1) \neq \emptyset$ **do**
    choose $\mathcal{F}_0(i) \in \mathcal{H}^*(i-1)$ uniformly at random
    $\mathcal{H}^*(i) \leftarrow \mathcal{H}^*(i-1) - \mathcal{F}_0(i)$
    $i \leftarrow i+1$
end

If the process fails to execute step $i+1$ and instead terminates, that is if $\mathcal{H}^*(i) = \emptyset$, then, for $j \geq i+1$, let $\mathcal{H}^*(j) := \mathcal{H}^*(i)$. For $i \geq 1$, let $\mathcal{H}(i)$ denote the k-graph with vertex set $V_{\mathcal{H}} := V_{\mathcal{H}(0)}$ and edge set $V_{\mathcal{H}^*(i)}$. Furthermore, let

$$H^*(i) := |\mathcal{H}^*(i)| \quad \text{and} \quad H(i) = |\mathcal{H}(i)|.$$

Let $\mathfrak{F}(0), \mathfrak{F}(1), \ldots$ denote the natural filtration associated with the random process above. Finally, define the stopping time

$$\tau_\emptyset := \min\{i \geq 0 : \mathcal{H}^*(i) = \emptyset\}$$

that indicates when Algorithm 3.3.1 terminates in the sense that $\tau_\emptyset$ is the number of successfully executed steps and hence the number of copies that were removed until termination.

Since during every successful step of the process exactly $|\mathcal{F}|$ edges are removed, an analysis up to step

$$i^\star := \frac{(\vartheta - n^{-1/\rho_{\mathcal{F}}+\varepsilon})n^k}{|\mathcal{F}|k!}$$

is sufficient for our purpose. Specifically, in Section 3.8, we show that Theorem 3.3.2 below holds.

Theorem 3.3.2. *With the setup above, the following holds. With probability at least* $1 - \exp(-(\log n)^{4/3})$*, the* k*-graph* $\mathcal{H}(i^\star)$ *is* $(4m, n^\varepsilon)$*-bounded,* $\mathcal{F}$*-populated,* k'*-populated for all* $1 \leq k' \leq k - 1/\rho_{\mathcal{F}}$ *and has* $n^{k-1/\rho_{\mathcal{F}}+\varepsilon}/k!$ *edges.*

An application of Theorem 3.3.2 with ε^5 playing the role of ε immediately yields Theorem 1.1.6 and hence the upper bounds in Theorems 1.1.5 and 1.1.8. Additionally, in combination with Theorems 1.1.7 and 1.1.9, such an application of Theorem 3.3.2 also yields the lower bounds in Theorems 1.1.5 and 1.1.8. Thus, for the first part, it only remains to prove Theorem 3.3.2.

3.4 Trajectories

In every step of Algorithm 3.3.1, exactly $|\mathcal{F}|$ edges are removed. Hence, if $0 \leq i \leq \tau_{\emptyset}$, we have

$$H(i) = \frac{\vartheta n^k}{k!} - |\mathcal{F}|i.$$

The heuristic arguments in this section are based on the assumption that typically, for all $i \geq 0$, the edge set of $\mathcal{H}(i)$ behaves as if it was obtained by including every k-set $e \subseteq V_{\mathcal{H}(0)}$ independently at random with probability

$$\hat{p}(i) := \vartheta - \frac{|\mathcal{F}|k!\, i}{n^k}.$$

Note that $\hat{p}(i)$ is chosen such that when following the probabilistic construction above, the expected number of included edges is essentially the true number of edges in $\mathcal{H}(i)$.

Let $\mathrm{Aut}(\mathcal{F})$ denote the set of automorphisms of $\mathcal{F}$, that is the set of bijections $\varphi\colon V_{\mathcal{F}} \xrightarrow{\sim} V_{\mathcal{F}}$ with $\varphi(e), \varphi^{-1}(e) \in \mathcal{F}$ for all $e \in \mathcal{F}$ and let $\mathrm{aut}(\mathcal{F}) := |\mathrm{Aut}(\mathcal{F})|$. Based on the above assumption about the behavior of $\mathcal{H}(i)$, we estimate

$$\mathbb{E}[H^*(i)] \approx \frac{n^m \hat{p}(i)^{|\mathcal{F}|}}{\mathrm{aut}(\mathcal{F})} =: \hat{h}^*(i).$$

As outlined in Section 3.1, our precise analysis of the random removal process essentially consists of proving that the numbers of many carefully chosen additional substructures within $\mathcal{H}^*(i)$ are typically concentrated around a deterministic trajectory. More specifically, these substructures will be given by embeddings of templates. Recall that, as defined in Chapter 1, a k-template is a pair $(\mathcal{A}, I)$ of a k-graph $\mathcal{A}$ and a vertex set $I \subseteq V_{\mathcal{A}}$. For $i \geq 0$, a k-template $(\mathcal{A}, I)$ and an injection $\psi\colon I \hookrightarrow V_{\mathcal{H}(i)}$, which may be thought of as a partial localization of the template $(\mathcal{A}, I)$ within $\mathcal{H}(i)$, we are interested in the collection $\Phi^{\sim}_{\mathcal{A},\psi}(i)$ of embeddings of $\mathcal{A}$ into $\mathcal{H}(i)$ that extend ψ. Formally, we set

$$\Phi^{\sim}_{\mathcal{A},\psi}(i) := \{\varphi\colon V_{\mathcal{A}} \hookrightarrow V_{\mathcal{H}(i)} : \varphi|_I = \psi \text{ and } \varphi(e) \in \mathcal{H}(i) \text{ for all } e \in \mathcal{A} \setminus \mathcal{A}[I]\}.$$

For a template $(\mathcal{A}, I)$ and $\psi\colon I \hookrightarrow V_{\mathcal{H}(i)}$, we anticipate

$$\mathbb{E}[|\Phi^{\sim}_{\mathcal{A},\psi}(i)|] \approx n^{|V_{\mathcal{A}}|-|I|}\hat{p}(i)^{|\mathcal{A}|-|\mathcal{A}[I]|} =: \hat{\varphi}_{\mathcal{A},I}(i).$$

This final estimate is only valid if $(\mathcal{A}, I)$ has certain desirable properties that make it well-behaved and that we specify in Section 3.5. We ensure that all templates where we are interested in precise estimates for the number of embeddings satisfy these properties.

Our organization of the proof that up to step $i^\star$, key quantities remain close to their trajectory with high probability is as follows. In the subsequent sections, we define stopping times $\tau_{\mathcal{H}^*}, \tau_{\mathscr{B}}, \tau_{\mathscr{B}'}, \tau_{\mathfrak{C}}, \tau_{\mathfrak{B}}$ that measure when key quantities significantly deviate from their trajectory. Then, to argue that

$$i^\star < \tau_{\mathcal{H}^*} \wedge \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'} \wedge \tau_{\mathfrak{C}} \wedge \tau_{\mathfrak{B}} =: \tau^\star$$

holds with high probability, we observe that

$$\{\tau^\star \le i^\star\} = \bigcup_{\tau \in \{\tau_{\mathcal{H}^*}, \tau_{\mathscr{B}}, \tau_{\mathscr{B}'}, \tau_{\mathfrak{C}}, \tau_{\mathfrak{B}}\}} \{\tau \le \tau^\star \wedge i^\star\}$$

and show that the probabilities for the five events on the right are small. For $\tau \in \{\tau_{\mathcal{H}^*}, \tau_{\mathscr{B}}, \tau_{\mathscr{B}'}\}$, a suitable bound for the probability of the corresponding event on the right may be obtained similarly as the analogous statements for the triangle case in [14] by employing standard critical interval arguments. New ideas that allow us to carry out an analysis of the hypergraph removal process in great generality are required for suitable bounds for the two remaining events, that is when $\tau \in \{\tau_{\mathfrak{C}}, \tau_{\mathfrak{B}}\}$. We dedicate Sections 3.6 and 3.7 to bounding the probabilities of these two events. Note that in fact, each of these five events occurs if and only if the corresponding inequality holds with equality.

3.5 Template embeddings and key stopping times

We introduce the following conventions and notations to simplify notation. In general, as in Chapter 2, if $X(0), X(1), \dots$ is a sequence of numbers or random variables and $i \ge 0$, we define $\Delta X(i) := X(i+1) - X(i)$. For an event $\mathcal{E}$, a random variable X and $i \ge 0$, we define $\mathbb{P}_i[\mathcal{E}] := \mathbb{P}[\mathcal{E} \mid \mathfrak{F}(i)]$ and $\mathbb{E}_i[X] := \mathbb{E}[X \mid \mathfrak{F}(i)]$. We again write $X =_{\mathcal{E}} Y$ for two expressions X and Y and an event $\mathcal{E}$, to express the statement that X and Y represent (possibly constant) random variables that are equal whenever $\mathcal{E}$ occurs, or equivalently, to express that $X \cdot \mathbb{1}_{\mathcal{E}} = Y \cdot \mathbb{1}_{\mathcal{E}}$. Similarly, we write $X \le_{\mathcal{E}} Y$ to mean $X \cdot \mathbb{1}_{\mathcal{E}} \le Y \cdot \mathbb{1}_{\mathcal{E}}$ and $X \ge_{\mathcal{E}} Y$ to mean $X \cdot \mathbb{1}_{\mathcal{E}} \ge Y \cdot \mathbb{1}_{\mathcal{E}}$. However, unlike in Chapter 2, we introduce the following convention to improve the clarity of the presentation of the sometimes longer and more involved expressions encountered here. To refer to a previously defined $X(i)$, we often only write X to mean $X(i)$, so for example when we only write H^*, this is meant to be replaced with $H^*(i)$. Note that this introduces no ambiguity concerning $V_{\mathcal{H}}$ since $V_{\mathcal{H}(i)}$ is the same for all $i \ge 0$.

Extending the terminology concerning templates that we introduce in Section 1.1, we say that a template $(\mathcal{A}, I)$ is a *copy* of a template $(\mathcal{B}, J)$ if there exists a bijection $\varphi\colon V_{\mathcal{A}} \xrightarrow{\sim} V_{\mathcal{B}}$ with $\varphi(e) \in \mathcal{B}$ for all $e \in \mathcal{A}$, $\varphi^{-1}(e) \in \mathcal{A}$ for all $e \in \mathcal{B}$ and $\varphi(I) = J$. We say that $(\mathcal{A}, I)$ is *balanced* if $\rho_{\mathcal{B},I} \le \rho_{\mathcal{A},I}$ for all $(\mathcal{B}, I) \subseteq (\mathcal{A}, I)$. Note that a k-graph $\mathcal{G}$ is k-balanced if and only if $(\mathcal{G}, e)$ is balanced for all $e \in \mathcal{G}$. For a template $(\mathcal{A}, I)$, $\psi\colon I \hookrightarrow V_{\mathcal{H}}$ and $i \ge 0$ let $\Phi_{\mathcal{A},\psi}(i) := |\Phi^{\sim}_{\mathcal{A},\psi}|$.

The definition of the stopping times mentioned in Section 3.4 depend on what it means to deviate significantly from a corresponding trajectory. The formal definition relies on appropriately chosen error terms that we define for the key quantities that we wish to track and that quantify the maximum deviation from the trajectory that we allow. Many of these error terms are expressed in terms of δ and $\zeta(i)$, where for $i \ge 0$, we set

$$\zeta(i) := \frac{n^{\varepsilon^2}}{n^{1/2}\hat{p}^{\rho_{\mathcal{F}}/2}}.$$

For $\alpha \geq 0$ and a template $(\mathcal{A}, I)$ let

$$i^{\alpha}_{\mathcal{A},I} := \min\{i \geq 0 : \hat{\varphi}_{\mathcal{A},I} \leq \zeta^{-\alpha}\},$$

where we set $\min \emptyset := \infty$. Note in particular, that $i^{0}_{\mathcal{A},I} = \min\{i \geq 0 : \hat{\varphi}_{\mathcal{A},I} \leq 1\}$.

We consider the families of templates

$$\begin{aligned}
\mathscr{F} &:= \{(\mathcal{F}, f) : f \in \mathcal{F}\},\\
\mathscr{B} &:= \{(\mathcal{A}, I) : (\mathcal{A}, I) \text{ is a balanced } k\text{-template with } |V_{\mathcal{A}}| \leq 1/\varepsilon^4 \text{ and } i^{\delta^{1/2}}_{\mathcal{A},I} \geq 1\},\\
\mathscr{B}' &:= \{(\mathcal{A}, I) : (\mathcal{A}, I) \text{ is a strictly balanced } k\text{-template with } |V_{\mathcal{A}}| \leq 1/\varepsilon^4 \text{ and } i^{0}_{\mathcal{A},I} \geq 1\}.
\end{aligned}$$

For $x \geq 0$, let

$$\alpha_x := 2^{x+1} - 2$$

and let $\alpha_{\mathcal{A},I} := \alpha_{|V_{\mathcal{A}}|-|I|}$. In the following observation, we briefly state the properties that motivate the choice of α_x and that we rely on for arguments further below.

Observation 3.5.1. *Let $x, y \geq 0$ and $z \geq 1$. Then,*

$$2\alpha_x + 2 \leq \alpha_{x+1}, \quad \alpha_x + \alpha_y \leq \alpha_{x+y}, \quad \alpha_z \geq 2.$$

We define the stopping times

$$\begin{aligned}
\tau_{\mathcal{H}^*} &:= \min\{i \geq 0 : H^* \neq (1 \pm \zeta^{1+\varepsilon^3})\hat{h}^*\},\\
\tau_{\mathscr{F}} &:= \min\{i \geq 0 : \Phi_{\mathcal{F},\psi} \neq (1 \pm \delta^{-1}\zeta)\hat{\varphi}_{\mathcal{F},f} \text{ for some } (\mathcal{F}, f) \in \mathscr{F}, \psi \colon f \hookrightarrow V_{\mathcal{H}}\},\\
\tau_{\mathscr{B}} &:= \min\left\{\begin{matrix} i \geq 0 : \Phi_{\mathcal{A},\psi} \neq (1 \pm \zeta^{\delta})\hat{\varphi}_{\mathcal{A},I} \text{ and } i \leq i^{\delta^{1/2}}_{\mathcal{A},I} \\ \text{for some } (\mathcal{A}, I) \in \mathscr{B}, \psi \colon I \hookrightarrow V_{\mathcal{H}} \end{matrix}\right\},\\
\tau_{\mathscr{B}'} &:= \min\left\{\begin{matrix} i \geq 0 : \Phi_{\mathcal{A},\psi} \neq (1 \pm (\log n)^{\alpha_{\mathcal{A},I}} \hat{\varphi}^{-\delta^{1/2}}_{\mathcal{A},I})\hat{\varphi}_{\mathcal{A},I} \text{ and } i^{\delta^{1/2}}_{\mathcal{A},I} \leq i \leq i^{0}_{\mathcal{A},I} \\ \text{for some } (\mathcal{A}, I) \in \mathscr{B}', \psi \colon I \hookrightarrow V_{\mathcal{H}} \end{matrix}\right\}.
\end{aligned}$$

Three of these stopping times are mentioned in Section 3.4. Since the precise definition of the other two stopping times $\tau_{\mathfrak{C}}$ and $\tau_{\mathfrak{B}}$ is not always relevant, we occasionally only work with the simpler stopping time $\tau_{\mathscr{F}}$ that satisfies $\tau_{\mathscr{F}} \geq \tau_{\mathfrak{C}}$ and we define

$$\tilde{\tau}^{\star} := \tau_{\mathcal{H}^*} \wedge \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'} \wedge \tau_{\mathscr{F}} \geq \tau^{\star}. \tag{3.5.1}$$

Observe that the relative error $\zeta^{1+\varepsilon^3}$ that we allow for H^* is significantly smaller than the relative error $\delta^{-1}\zeta$ that we allow for $\Phi_{\mathcal{F},f}$ where $f \in \mathcal{F}$. Furthermore, the relative error ζ^{δ} that we use for the number of embeddings $\Phi_{\mathcal{A},\psi}$ corresponding to a balanced extension $(\mathcal{A}, I) \in \mathscr{B}$ and $\psi \colon I \hookrightarrow V(\mathcal{H})$ is significantly larger than these two previous error terms. However, it is at most $n^{-\delta^2}$, reflecting the fact that we still expect tight concentration around the corresponding trajectory provided that we can still expect $\Phi_{\mathcal{A},\psi}$ to be sufficiently large in the sense that we are not beyond step $i^{\delta^{1/2}}_{\mathcal{A},I}$. Finally, concerning the fourth stopping time, we are only interested in the further evolution of the number of embeddings beyond step $i^{\delta^{1/2}}_{\mathcal{A},I}$, but still at most up to step $i^{0}_{\mathcal{A},I}$, if $(\mathcal{A}, I)$ is strictly balanced. For this further evolution, our relative error term is essentially potentially as large as $(\log n)^{\alpha_{\mathcal{A},I}}$. Note that all error terms are sensible in the sense that at least in the very beginning, before the removal of any copy, the corresponding random variables are within the margin of error as implied by Lemma 3.5.4. Before we turn to this lemma and its proof, we first state two useful Lemmas. Lemma 3.5.2 formulates a convenient fact concerning the trajectories corresponding to the numbers of embeddings of templates that we use below without explicitly referencing it. In Lemma 3.5.3, we consider a construction of strictly balanced templates within k-graphs. It is convenient to have Lemma 3.5.3 available for the proof of Lemma 3.5.4 and furthermore, the simple construction plays a crucial role in Section 3.6. Overall, the verification in Lemma 3.5.4 that the initial conditions are suitable and the following lemmas in Sections 3.5.1–3.5.4 play mostly an auxiliary role and the proofs rely on standard arguments and are not important for understanding the setup and argumentation in Sections 3.6 and 3.7 where we turn to the new ideas that allow us to analyze the $\mathcal{F}$-removal process in great generality. Hence, if the desire is to focus on these new contributions, one may skip these results and continue reading at the beginning of Section 3.5.5 where we make some final remarks concerning the overall setup as preparation for Sections 3.6 and 3.7.

Lemma 3.5.2. *Let $i \geq 0$. Suppose that $(\mathcal{A}, I)$ is a template and let $I \subseteq U \subseteq V_{\mathcal{A}}$. Then, $\hat{\varphi}_{\mathcal{A},I} = \hat{\varphi}_{\mathcal{A},U} \cdot \hat{\varphi}_{\mathcal{A}[U],I}$.*

Proof. We have

$$\hat{\varphi}_{\mathcal{A},I} = n^{|V_{\mathcal{A}}|-|I|}\hat{p}^{|\mathcal{A}|-|\mathcal{A}[I]|} = n^{|V_{\mathcal{A}}|-|U|}\hat{p}^{|\mathcal{A}|-|\mathcal{A}[U]|}n^{|U|-|I|}\hat{p}^{|\mathcal{A}[U]|-|\mathcal{A}[I]|} = \hat{\varphi}_{\mathcal{A},U}\hat{\varphi}_{\mathcal{A}[U],I},$$

which completes the proof. □

Lemma 3.5.3. *Suppose that $\mathcal{A}$ is a k-graph and let $\alpha \geq 0$ and $U \subseteq V_{\mathcal{A}}$. Suppose that among all subsets $U \subseteq I' \subsetneq V_{\mathcal{A}}$ with $\rho_{\mathcal{A},I'} \leq \alpha$, the set I has maximal size. Then, the template $(\mathcal{A}, I)$ is strictly balanced.*

Proof. Let $(\mathcal{B}, I) \subseteq (\mathcal{A}, I)$ with $I \neq V_{\mathcal{B}}$ and $\mathcal{B} \neq \mathcal{A}$. We show that $\rho_{\mathcal{B},I} < \rho_{\mathcal{A},I}$. We may assume that $\mathcal{B}$ is an induced subgraph of $\mathcal{A}$ and then we have $I \subsetneq V_{\mathcal{B}} \subsetneq V_{\mathcal{A}}$. By choice of I, we obtain $\rho_{\mathcal{A},V_{\mathcal{B}}} > \alpha \geq \rho_{\mathcal{A},I}$ and hence

$$\begin{aligned}\rho_{\mathcal{B},I} &= \frac{\rho_{\mathcal{A},I}(|V_{\mathcal{A}}|-|I|) - \rho_{\mathcal{A},V_{\mathcal{B}}}(|V_{\mathcal{A}}|-|V_{\mathcal{B}}|)}{|V_{\mathcal{B}}|-|I|} < \frac{\rho_{\mathcal{A},I}(|V_{\mathcal{A}}|-|I|) - \rho_{\mathcal{A},I}(|V_{\mathcal{A}}|-|V_{\mathcal{B}}|)}{|V_{\mathcal{B}}|-|I|} \\ &= \rho_{\mathcal{A},I},\end{aligned}$$

which completes the proof. □

Lemma 3.5.4. *Let $i := 0$. Suppose that $(\mathcal{A}, I)$ is a k-template with $|V_{\mathcal{A}}| \leq 1/\varepsilon^4$ and let $\psi\colon I \hookrightarrow V_{\mathcal{H}}$. Then, the following holds.*

(i) *If $\rho_{\mathcal{B},I} \leq \rho_{\mathcal{F}}$ for all $(\mathcal{B}, I) \subseteq (\mathcal{A}, I)$, then $\Phi_{\mathcal{A},\psi} = (1 \pm \zeta^{1+2\varepsilon^3})\hat{\varphi}_{\mathcal{A},I}$.*

(ii) *$H^* = (1 \pm \zeta^{1+2\varepsilon^3})\hat{h}^*$.*

(iii) *If $(\mathcal{A}, I) \in \mathscr{B}$, then $\Phi_{\mathcal{A},\psi} = (1 \pm \zeta^{\delta+\delta^2})\hat{\varphi}_{\mathcal{A},I}$.*

(iv) *If* $(\mathcal{A}, I) \in \mathscr{B}'$ *and* $i^{\delta^{1/2}}_{\mathcal{A},I} = 0$, *then* $\Phi_{\mathcal{A},\psi} = (1 \pm (\log n)^{\alpha_{\mathcal{A},I}-1/2} \hat{\varphi}^{-\delta^{1/2}}_{\mathcal{A},I})\hat{\varphi}_{\mathcal{A},I}$.

Proof. We obtain (ii) as an immediate consequence of (i) and we show that (i), (iii) and (iv) follow from the $(\varepsilon^4, \delta, \rho_{\mathcal{F}})$-pseudorandomness of $\mathcal{H}$. More specifically, while (iv) is a direct consequence of the pseudorandomness, for (i) and (iii), we deconstruct $(\mathcal{A}, I)$ into a series of strictly balanced templates to employ the pseudorandomness. Note that in the definition of $(\varepsilon^4, \delta, \rho_{\mathcal{F}})$-pseudorandomness, the fraction $\zeta_0 := n^\delta/(n\vartheta^{\rho_{\mathcal{F}}})^{1/2}$ played the role of ζ in the definition, however, here we have $\zeta = \zeta(0) = n^{\varepsilon^2}/(n\vartheta^{\rho_{\mathcal{F}}})^{1/2} = n^{\varepsilon^2}\zeta_0/n^\delta$. Choosing a larger ζ here and in the definitions of the key stopping times gives us additional room for errors that we exploit in the proof. In detail, we prove the four statements as follows.

(i) Suppose that $\rho_{\mathcal{B},I} \le \rho_{\mathcal{F}}$ holds for all $(\mathcal{B}, I) \subseteq (\mathcal{A}, I)$. We use induction on $|V_{\mathcal{A}}| - |I|$ to show that

$$\Phi_{\mathcal{A},\psi} = (1 \pm 2(|V_{\mathcal{A}}| - |I|)\zeta^{1+3\varepsilon^3})\hat{\varphi}_{\mathcal{A},I}. \tag{3.5.2}$$

Since $|V_{\mathcal{A}}| \le 1/\varepsilon^4$, this is sufficient.

Let us proceed with the proof by induction. If $|V_{\mathcal{A}}| - |I| = 0$, then $\Phi_{\mathcal{A},I} = 1 = \hat{\varphi}_{\mathcal{A},I}$. Let $\ell \ge 1$ and suppose that(3.5.2) holds if $|V_{\mathcal{A}}| - |I| \le \ell - 1$. Suppose that $|V_{\mathcal{A}}| - |I| = \ell$. Suppose that among all subsets $I \subseteq U' \subsetneq V_{\mathcal{A}}$ with $\rho_{\mathcal{A},U'} \le \rho_{\mathcal{F}}$, the set U has maximal size. By Lemma 3.5.3, the extension $(\mathcal{A}, U)$ is strictly balanced. We have

$$\Phi_{\mathcal{A},\psi} = \sum_{\varphi \in \Phi^{\sim}_{\mathcal{A}[U],\psi}} \Phi_{\mathcal{A},\varphi}. \tag{3.5.3}$$

We use the estimate for $\Phi_{\mathcal{A}[U],\psi}$ provided by the induction hypothesis and for $\varphi \in \Phi^{\sim}_{\mathcal{A}[U],\psi}$, we estimate $\Phi_{\mathcal{A},\varphi}$ using the pseudorandomness of $\mathcal{H}$.

Let us turn to the details. The template $(\mathcal{A}, U)$ is strictly balanced and satisfies $\rho_{\mathcal{A},U} \le \rho_{\mathcal{F}}$, so since $\mathcal{H}$ is $(\varepsilon^4, \delta, \rho_{\mathcal{F}})$-pseudorandom, for all $\varphi \in \Phi^{\sim}_{\mathcal{A}[U],\psi}$, we have

$$\Phi_{\mathcal{A},\varphi} = (1 \pm \zeta_0)\hat{\varphi}_{\mathcal{A},U} = \left(1 \pm \frac{n^\delta}{n^{\varepsilon^2}}\zeta\right)\hat{\varphi}_{\mathcal{A},U} = (1 \pm \zeta^{1+3\varepsilon^3})\hat{\varphi}_{\mathcal{A},U}.$$

Since by induction hypothesis, we have $\Phi_{\mathcal{A}[U],\psi} = (1 \pm 2(|U| - |I|)\zeta^{1+3\varepsilon^3})\hat{\varphi}_{\mathcal{A}[U],I}$, returning to (3.5.3), we conclude that

$$\Phi_{\mathcal{A},\psi} = (1 \pm 2(|U| - |I|)\zeta^{1+3\varepsilon^3})\hat{\varphi}_{\mathcal{A}[U],I} \cdot (1 \pm \zeta^{1+3\varepsilon^3})\hat{\varphi}_{\mathcal{A},U} = (1 \pm 2(|V_{\mathcal{A}}| - |I|)\zeta^{1+3\varepsilon^3})\hat{\varphi}_{\mathcal{A},I},$$

which completes the proof of (i).

(ii) This is a consequence of (i) and the fact that $\mathcal{F}$ is k-balanced. To see this, we argue as follows. Fix $f \in \mathcal{F}$ and let $\psi\colon \emptyset \to V_{\mathcal{H}}$. Then, we have

$$\begin{aligned} H^* = \frac{\Phi_{\mathcal{F},\psi}}{\mathrm{aut}(\mathcal{F})} &= \frac{1}{\mathrm{aut}(\mathcal{F})} \sum_{\varphi \in \Phi^{\sim}_{\mathcal{F}[f],\psi}} \Phi_{\mathcal{F},\varphi} = (1 \pm \zeta^{1+2\varepsilon^3})\frac{\hat{\varphi}_{\mathcal{F},f} \cdot \Phi_{\mathcal{F}[f],\psi}}{\mathrm{aut}(\mathcal{F})} \\ &= (1 \pm \zeta^{1+2\varepsilon^3})\frac{\hat{\varphi}_{\mathcal{F},f} \cdot k!\,H}{\mathrm{aut}(\mathcal{F})} = (1 \pm \zeta^{1+2\varepsilon^3})\frac{\hat{\varphi}_{\mathcal{F},f} \cdot \vartheta n^k}{\mathrm{aut}(\mathcal{F})} = (1 \pm \zeta^{1+2\varepsilon^3})\hat{h}^*, \end{aligned}$$

which completes the proof of (ii).

(iii) Suppose that $(\mathcal{A}, I)$ is balanced and that $\hat{\varphi}_{\mathcal{A},I} \geq \zeta^{-\delta^{4/7}(|V_\mathcal{A}|-|I|)}$. We argue similarly as in the proof of (i) and use induction on $|V_\mathcal{A}| - |I|$ to show that

$$\Phi_{\mathcal{A},\psi} = (1 \pm 2(|V_\mathcal{A}| - |I|)\zeta^{\delta+2\delta^2})\hat{\varphi}_{\mathcal{A},I}. \tag{3.5.4}$$

Since $|V_\mathcal{A}| \leq 1/\varepsilon^4$, this is sufficient.

Let us proceed with the proof by induction. If $|V_\mathcal{A}| - |I| = 0$, then $\Phi_{\mathcal{A},I} = 1 = \hat{\varphi}_{\mathcal{A},I}$. Let $\ell \geq 1$ and suppose that (3.5.4) holds if $|V_\mathcal{A}| - |I| \leq \ell - 1$. Suppose that $|V_\mathcal{A}| - |I| = \ell$. Suppose that among all subsets $I \subseteq U' \subsetneq V_\mathcal{A}$ with $\rho_{\mathcal{A},U'} \leq \rho_{\mathcal{A},I}$, the set U has maximal size. By Lemma 3.5.3, the extension $(\mathcal{A}, U)$ is strictly balanced. Due to $\vartheta \geq n^{-1/\rho_\mathcal{F}+\varepsilon}$, we have

$$\begin{aligned}\hat{\varphi}_{\mathcal{A},I} &\geq \zeta^{-\delta^{4/7}(|V_\mathcal{A}|-|I|)} \geq \left(\frac{n^{\varepsilon^2}}{n^\delta}\right)^{-\delta^{4/7}} \zeta_0^{-\delta^{4/7}} > \left(\frac{n^{\varepsilon\rho_\mathcal{F}/2}}{n^\delta}\right)^{-\delta^{4/7}/2} \zeta_0^{-\delta^{4/7}} \\ &\geq \left(\frac{n^{1/2}\vartheta^{\rho_\mathcal{F}/2}}{n^\delta}\right)^{-\delta^{4/7}/2} \zeta_0^{-\delta^{4/7}} = \zeta_0^{-\delta^{4/7}/2} \geq \zeta_0^{-\delta^{3/5}}.\end{aligned} \tag{3.5.5}$$

Hence, if $U = I$, then, since $(\mathcal{A}, U)$ is strictly balanced and since $\zeta^{\delta+\delta^2} \geq \zeta_0^\delta$, the desired estimate follows from the fact that $\mathcal{H}$ is $(\varepsilon^4, \delta, \rho_\mathcal{F})$-pseudorandom. Thus, we may assume that $U \neq I$. We have

$$\begin{aligned}\rho_{\mathcal{A}[U],I} &= \frac{\rho_{\mathcal{A},I}(|V_\mathcal{A}| - |I|) - \rho_{\mathcal{A},U}(|V_\mathcal{A}| - |U|)}{|U| - |I|} \geq \frac{\rho_{\mathcal{A},I}(|V_\mathcal{A}| - |I|) - \rho_{\mathcal{A},I}(|V_\mathcal{A}| - |U|)}{|U| - |I|} \\ &= \rho_{\mathcal{A},I}.\end{aligned}$$

Hence, since $(\mathcal{A}, I)$ is balanced, the template $(\mathcal{A}[U], I)$ has density $\rho_{\mathcal{A}[U],I} = \rho_{\mathcal{A},I}$ and is also balanced. Additionally, we have

$$\hat{\varphi}_{\mathcal{A}[U],I} = \hat{\varphi}_{\mathcal{A},I}^{(|U|-|I|)/(|V_\mathcal{A}|-|I|)} \geq \zeta^{-\delta^{4/7}(|U|-|I|)}$$

and (3.5.5) entails

$$\hat{\varphi}_{\mathcal{A},U} \geq \hat{\varphi}_{\mathcal{A},I}^{(|V_\mathcal{A}|-|U|)/(|V_\mathcal{A}|-|I|)} \geq \hat{\varphi}_{\mathcal{A},I}^{\varepsilon^4} \geq \zeta_0^{-\delta^{2/3}}.$$

We have

$$\Phi_{\mathcal{A},\psi} = \sum_{\varphi \in \Phi^\sim_{\mathcal{A}[U],\psi}} \Phi_{\mathcal{A},\varphi}. \tag{3.5.6}$$

We use the estimate for $\Phi_{\mathcal{A}[U],\psi}$ provided by the induction hypothesis and for $\varphi \in \Phi^\sim_{\mathcal{A}[U],\psi}$, we estimate $\Phi_{\mathcal{A},\varphi}$ using the pseudorandomness of $\mathcal{H}$.

Let us turn to the details. The template $(\mathcal{A}, U)$ is strictly balanced and we have $\hat{\varphi}_{\mathcal{A},U} \geq \zeta_0^{-\delta^{2/3}}$, so since $\mathcal{H}$ is $(\varepsilon^4, \delta, \rho_\mathcal{F})$-pseudorandom, for all $\varphi \in \Phi_{\mathcal{A}[U],\psi}$, we obtain

$$\Phi_{\mathcal{A},\varphi} = \left(1 \pm \left(\frac{n^\delta}{n^{\varepsilon^2}}\zeta\right)^\delta\right)\hat{\varphi}_{\mathcal{A},U} = (1 \pm \zeta^{\delta+2\delta^2})\hat{\varphi}_{\mathcal{A},U}.$$

Furthermore, the template $(\mathcal{A}[U], I)$ is balanced and we have $\hat{\varphi}_{\mathcal{A}[U],I} \geq \zeta^{-\delta^{4/7}(|U|-|I|)}$, so by induction hypothesis, we obtain

$$\Phi_{\mathcal{A},\psi} = (1 \pm 2(|U| - |I|)\zeta^{\delta+2\delta^2})\hat{\varphi}_{\mathcal{A}[U],I}.$$

Returning to (3.5.6), we conclude that

$$\Phi_{\mathcal{A},\psi} = (1 \pm 2(|U| - |I|)\zeta^{\delta+2\delta^2})\hat{\varphi}_{\mathcal{A}[U],I} \cdot (1 \pm \zeta^{\delta+2\delta^2})\hat{\varphi}_{\mathcal{A},U} = (1 \pm 2(|V_\mathcal{A}| - |I|)\zeta^{\delta+2\delta^2})\hat{\varphi}_{\mathcal{A},I},$$

which completes the proof of (iii).

(iv) Suppose that $(\mathcal{A}, I) \in \mathscr{B}'$ and $i^{\delta^{1/2}}_{\mathcal{A},I} = 0$. We may assume that $I \neq V_\mathcal{A}$. If $\hat{\varphi}_{\mathcal{A},I} \geq \zeta_0^{-\delta^{2/3}}$, then since $\mathcal{H}$ is $(\varepsilon^4, \delta, \rho_\mathcal{F})$-pseudorandom, using $\hat{\varphi}_{\mathcal{A},I} \leq \zeta^{-\delta^{1/2}}$, we have

$$\Phi_{\mathcal{A},I} = (1 \pm \zeta_0^\delta)\hat{\varphi}_{\mathcal{A},I} = (1 \pm \zeta^\delta)\hat{\varphi}_{\mathcal{A},I} = (1 \pm \hat{\varphi}_{\mathcal{A},I}^{-\delta^{1/2}})\hat{\varphi}_{\mathcal{A},I} = (1 \pm (\log n)^{\alpha_{\mathcal{A},I}-1/2}\hat{\varphi}_{\mathcal{A},I}^{-\delta^{1/2}})\hat{\varphi}_{\mathcal{A},I}.$$

If $\hat{\varphi}_{\mathcal{A},I} \leq \zeta_0^{-\delta^{2/3}}$, then again since $\mathcal{H}$ is $(\varepsilon^4, \delta, \rho_\mathcal{F})$-pseudorandom, we obtain

$$\Phi_{\mathcal{A},I} = (1 \pm (\log n)^{3(|V_\mathcal{A}|-|I|)/2}\hat{\varphi}_{\mathcal{A},I}^{-\delta^{1/2}})\hat{\varphi}_{\mathcal{A},I} = (1 \pm (\log n)^{\alpha_{\mathcal{A},I}-1/2}\hat{\varphi}_{\mathcal{A},I}^{-\delta^{1/2}})\hat{\varphi}_{\mathcal{A},I},$$

which completes the proof of (iv). □

3.5.1 Auxiliary results about key quantities

We gather some statements concerning the key quantities defined up to this point. Lemmas 3.5.5– 3.5.9 provide useful bounds concerning $\hat{p}$, ζ and H.

Lemma 3.5.5. *Let $0 \leq i \leq i^\star$. Then $n^{-1/\rho_\mathcal{F}+\varepsilon} \leq \hat{p} \leq 1$.*

Proof. We obviously have $\hat{p} \leq \vartheta \leq 1$ and furthermore $\hat{p} \geq \hat{p}(i^\star) = n^{-1/\rho_\mathcal{F}+\varepsilon}$. □

Lemma 3.5.6. *Let $0 \leq i \leq i^\star$. Then, $\hat{p}(i+1) \geq (1 - n^{-\varepsilon^2})\hat{p}$.*

Proof. Lemma 3.5.5 implies

$$\hat{p}(i+1) = \left(1 - \frac{|\mathcal{F}|k!}{n^k\hat{p}}\right)\hat{p} \geq \left(1 - \frac{2|\mathcal{F}|k!}{n^\varepsilon}\right)\hat{p} \geq (1 - n^{-\varepsilon^2})\hat{p},$$

which completes the proof. □

Lemma 3.5.7. *Let $0 \leq i \leq i^\star$ and $\mathcal{X} := \{i \leq \tau_\emptyset\}$. Then, $H =_\mathcal{X} n^k\hat{p}/k!$.*

Proof. We have

$$H =_\mathcal{X} \frac{\vartheta n^k}{k!} - |\mathcal{F}|i = \frac{n^k\hat{p}}{k!},$$

which completes the proof. □

Lemma 3.5.8. *Let $0 \leq i \leq i^\star$. Then, $n^{-1/2+\varepsilon^2} \leq \zeta \leq n^{-\varepsilon^2}$.*

Proof. Indeed, using Lemma 3.5.5, we obtain

$$n^{-1/2+\varepsilon^2} \le \frac{n^{\varepsilon^2}}{n^{1/2}\hat{p}^{\rho_{\mathcal{F}}/2}} = \zeta \le \frac{n^{\varepsilon^2}}{n^{1/2}\hat{p}(i^\star)^{\rho_{\mathcal{F}}/2}} = \frac{n^{\varepsilon^2}}{n^{1/2}n^{(-1+\varepsilon\rho_{\mathcal{F}})/2}} = \frac{n^{\varepsilon^2}}{n^{\varepsilon\rho_{\mathcal{F}}/2}} \le n^{-\varepsilon^2},$$

which completes the proof. □

Lemma 3.5.9. *Let $0 \le i \le i^\star$ and $\mathcal{X} := \{i \le \tau_\emptyset\}$. Then, $1/H \le_{\mathcal{X}} k!/(n\hat{p}^{\rho_{\mathcal{F}}}) \le \zeta^{2+2\varepsilon^2}$.*

Proof. Lemma 3.5.7 together with Lemma 3.5.5 entails

$$\frac{1}{H} =_{\mathcal{X}} \frac{k!}{n^k\hat{p}} \le \frac{k!}{(n\hat{p}^{\rho_{\mathcal{F}}})^k} \le \frac{k!}{n\hat{p}^{\rho_{\mathcal{F}}}}.$$

Furthermore, using Lemma 3.5.8, we obtain

$$\frac{k!}{n\hat{p}^{\rho_{\mathcal{F}}}} \le \frac{n^{\varepsilon^2}}{n\hat{p}^{\rho_{\mathcal{F}}}} = n^{-\varepsilon^2}\zeta^2 \le \zeta^{2+2\varepsilon^2},$$

which completes the proof. □

3.5.2 Deterministic changes

Next, we gather bounds mostly concerning the behavior of deterministic trajectories and their one-step changes. To this end, similarly as in Chapter 2, we use a consequence of Taylor's theorem (Lemma 2.9.10).

Observation 3.5.10. *Extend $\hat{p}$ and $\hat{\varphi}_{\mathcal{A},I}$ to continuous trajectories defined on the whole interval $[0, i^\star + 1]$ using the same expressions as above. Then, for $x \in [0, i^\star + 1]$,*

$$\hat{\varphi}'_{\mathcal{A},I}(x) = -(|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{A},I}(x)}{n^k\hat{p}(x)},$$
$$\hat{\varphi}''_{\mathcal{A},I}(x) = (|\mathcal{A}| - |\mathcal{A}[I]|)(|\mathcal{A}| - |\mathcal{A}[I]| - 1)\frac{|\mathcal{F}|^2(k!)^2\hat{\varphi}_{\mathcal{A},I}(x)}{n^{2k}\hat{p}(x)^2}.$$

Lemma 3.5.11. *Let $0 \le i \le i^\star$ and $\mathcal{X} := \{i \le \tau_\emptyset\}$. Suppose that $(\mathcal{A}, I)$ is a template. Then,*

$$\Delta\hat{\varphi}_{\mathcal{A},I} =_{\mathcal{X}} -(|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|\hat{\varphi}_{\mathcal{A},I}}{H} \pm \frac{\zeta^{2+\varepsilon^2}\hat{\varphi}_{\mathcal{A},I}}{H}.$$

Proof. This is a consequence of Taylor's theorem. In detail, we argue as follows.

Together with Observation 3.5.10, Lemma 2.9.10 yields

$$\Delta\hat{\varphi}_{\mathcal{A},I} = -(|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{A},I}}{n^k\hat{p}} \pm \max_{x\in[i,i+1]}(|\mathcal{A}| - |\mathcal{A}[I]|)(|\mathcal{A}| - |\mathcal{A}[I]| - 1)\frac{|\mathcal{F}|^2(k!)^2\hat{\varphi}_{\mathcal{A},I}(x)}{n^{2k}\hat{p}(x)^2}.$$

We investigate the first term and the maximum separately. Using Lemma 3.5.7, we have

$$-(|\mathcal{A}|-|\mathcal{A}[I]|)\frac{|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{A},I}}{n^k\hat{p}} =_{\mathcal{X}} -(|\mathcal{A}|-|\mathcal{A}[I]|)\frac{|\mathcal{F}|\hat{\varphi}_{\mathcal{A},I}}{H}.$$

If $\hat{\varphi}_{\mathcal{A},I}(x)/\hat{p}(x)^2$ is not decreasing in x for $x\in[i,i+1]$, then $|\mathcal{A}|-|\mathcal{A}[I]|=0$ or $|\mathcal{A}|-|\mathcal{A}[I]|-1=0$. Hence, Lemma 3.5.7 together with Lemma 3.5.9 yields

$$\begin{aligned}\max_{x\in[i,i+1]}(|\mathcal{A}|-|\mathcal{A}[I]|)(|\mathcal{A}|-|\mathcal{A}[I]|-1)\frac{|\mathcal{F}|^2(k!)^2\hat{\varphi}_{\mathcal{A},I}(x)}{n^{2k}\hat{p}(x)^2} &\le \frac{|\mathcal{A}|^2|\mathcal{F}|^2(k!)^2\hat{\varphi}_{\mathcal{A},I}}{n^{2k}\hat{p}^2}\\ &=_{\mathcal{X}} \frac{|\mathcal{A}|^2|\mathcal{F}|^2\hat{\varphi}_{\mathcal{A},I}}{H^2}\\ &\le \frac{\zeta^{2+2\varepsilon^2}|\mathcal{A}|^2|\mathcal{F}|^2\hat{\varphi}_{\mathcal{A},I}}{H}\\ &\le \frac{\zeta^{2+\varepsilon^2}\hat{\varphi}_{\mathcal{A},I}}{H},\end{aligned}$$

which completes the proof. □

Lemma 3.5.12. *Let $\alpha\ge 0$. Suppose that $(\mathcal{A},I)$ is a template with $|V_{\mathcal{A}}|\le 1/\varepsilon^4$ and $i^{\alpha}_{\mathcal{A},I}\ge 1$. Let $0\le i\le i^{\alpha}_{\mathcal{A},I}$. Then, $\hat{\varphi}_{\mathcal{A},I}\ge(1-n^{-\varepsilon^3})\zeta^{-\alpha}$.*

Proof. For $j\ge 0$, let $\hat{\psi}_{\mathcal{A},I}(j):=\zeta(j)^{\alpha}\hat{\varphi}_{\mathcal{A},I}(j)$. It suffices to show that $\hat{\psi}_{\mathcal{A},I}\ge(1-n^{-\varepsilon})$. Note that $\hat{\psi}_{\mathcal{A},I}(j)\ge 1$ for all $0\le j\le i^{\alpha}_{\mathcal{A},I}-1$. If $|\mathcal{A}|-|\mathcal{A}[I]|-\alpha\rho_{\mathcal{F}}/2\le 0$, then $\hat{\psi}_{\mathcal{A},I}\ge\hat{\psi}_{\mathcal{A},I}(0)\ge 1$. Otherwise, from Lemma 3.5.6, we obtain

$$\hat{\psi}_{\mathcal{A},I}\ge\hat{\psi}_{\mathcal{A},I}(i^{\alpha}_{\mathcal{A},I})\ge(1-n^{-\varepsilon^2})^{|\mathcal{A}|}\hat{\psi}_{\mathcal{A},I}(i^{\alpha}_{\mathcal{A},I}-1)\ge(1-n^{-\varepsilon^2})^{|\mathcal{A}|}\ge(1-n^{-\varepsilon^3}),$$

which completes the proof. □

Lemma 3.5.13. *Suppose that $(\mathcal{A},I)$ is a strictly balanced template with $|V_{\mathcal{A}}|\le 1/\varepsilon^4$. Let $i\ge 0$ and $\mathcal{X}:=\{i<\tau_{\mathscr{B}}\wedge\tau_{\mathscr{B}'}\}$. Let $\psi\colon V_{\mathcal{A}}\hookrightarrow V_{\mathcal{H}}$. Then, $\Phi_{\mathcal{A},\psi}\le_{\mathcal{X}}(1+\log n)^{\alpha_{\mathcal{A},I}}(1\vee\hat{\varphi}_{\mathcal{A},I})$.*

Proof. We may assume that $I\ne V_{\mathcal{A}}$. If $i^0_{\mathcal{A},I}=0$, then $\hat{\varphi}_{\mathcal{A},I}(0)\le 1$ and thus, since $\mathcal{H}$ is $(\varepsilon^4,\delta,\rho_{\mathcal{F}})$-pseudorandom, we have

$$\Phi_{\mathcal{A},I}\le\Phi_{\mathcal{A},I}(0)\le(\log n)^{3(|V_{\mathcal{A}}|-|I|)/2}\le(1+\log n)^{\alpha_{\mathcal{A},I}}.$$

Hence, we may also assume that $(\mathcal{A},I)\in\mathscr{B}'$. If $i\ge i^0_{\mathcal{A},I}$, then Lemma 3.5.12 entails

$$\Phi_{\mathcal{A},I}\le\Phi_{\mathcal{A},I}(i^0_{\mathcal{A},I})\le_{\mathcal{X}}(1+(\log n)^{\alpha_{\mathcal{A},I}}\hat{\varphi}_{\mathcal{A},I}(i^0_{\mathcal{A},I})^{-\delta^{1/2}})\hat{\varphi}_{\mathcal{A},I}(i^0_{\mathcal{A},I})\le(1+\log n)^{\alpha_{\mathcal{A},I}},$$

so we may additionally assume that $i<i^0_{\mathcal{A},I}$. If $i\ge i^{\delta^{1/2}}_{\mathcal{A},I}$, then

$$\Phi_{\mathcal{A},I}\le_{\mathcal{X}}(1+(\log n)^{\alpha_{\mathcal{A},I}}\hat{\varphi}_{\mathcal{A},I}^{-\delta^{1/2}})\hat{\varphi}_{\mathcal{A},I}\le(1+\log n)^{\alpha_{\mathcal{A},I}}\hat{\varphi}_{\mathcal{A},I}.$$

Hence, we may also additionally assume that $i<i^{\delta^{1/2}}_{\mathcal{A},I}$ and thus in particular $(\mathcal{A},I)\in\mathscr{B}$. Then,

$$\Phi_{\mathcal{A},I}\le_{\mathcal{X}}(1+\zeta^{\delta})\hat{\varphi}_{\mathcal{A},I}\le(1+\log n)^{\alpha_{\mathcal{A},I}}\hat{\varphi}_{\mathcal{A},I},$$

which completes the proof. □

3.5.3 Control over templates

Here, we present three statements that show that control over the numbers of balanced templates and strictly balanced templates also provides some control over the number of certain templates that are not necessarily balanced. Lemma 3.5.14 may be interpreted as a generalization of [14, Corollary 3.3] and, with respect to the main part of the analysis, plays a similar auxiliary role.

Lemma 3.5.14. *Let $i \geq 0$ and let $\mathcal{X} := \{i < \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'}\}$. Suppose that $(\mathcal{A}, I)$ is a template with $|V_{\mathcal{A}}| \leq 1/\varepsilon^4$ and let $\psi\colon I \hookrightarrow V_{\mathcal{H}}$. Then, the following holds.*

(i) *If $\hat{\varphi}_{\mathcal{B},I} \geq 1$ for all $(\mathcal{B}, I) \subseteq (\mathcal{A}, I)$, then $\Phi_{\mathcal{A},\psi} \leq_{\mathcal{X}} (1 + \log n)^{\alpha_{\mathcal{A},I}} \hat{\varphi}_{\mathcal{A},I}$;*

(ii) *If $\hat{\varphi}_{\mathcal{A},J} \leq 1$ for all $I \subseteq J \subseteq V_{\mathcal{A}}$, then $\Phi_{\mathcal{A},\psi} \leq_{\mathcal{X}} (1 + \log n)^{\alpha_{\mathcal{A},I}}$.*

Proof. We use induction on $|V_{\mathcal{A}}| - |I|$ to show that (i) and (ii) hold. If $|V_{\mathcal{A}}| - |I| = 0$, then $\Phi_{\mathcal{A},\psi} = 1 = \hat{\varphi}_{\mathcal{A},I}$ and hence (i) and (ii) are true.

Let $\ell \geq 1$ and suppose that both statements hold if $|V_{\mathcal{A}}| - |I| \leq \ell - 1$. Suppose that $|V_{\mathcal{A}}| - |I| = \ell$. First, suppose that there is an isolated vertex $v \notin I$ in $\mathcal{A}$. If $\hat{\varphi}_{\mathcal{B},I} \geq 1$ for all $(\mathcal{B}, I) \subseteq (\mathcal{A}, I)$, using the induction hypothesis, we obtain

$$\Phi_{\mathcal{A},\psi} = (n - |V_{\mathcal{A}}| + 1) \cdot \Phi_{\mathcal{A}-\{v\},\psi} \leq_{\mathcal{X}} (1 + \log n)^{\alpha_{\mathcal{A}-\{v\},I}} \hat{\varphi}_{\mathcal{A},I} \leq (1 + \log n)^{\alpha_{\mathcal{A},I}} \hat{\varphi}_{\mathcal{A},I},$$

so (i) holds. Furthermore, we have $\hat{\varphi}_{\mathcal{A},V_{\mathcal{A}}\setminus\{v\}} = n > 1$, so (ii) is vacuously true.

Hence, now suppose that there is no isolated vertex $v \notin I$ in $\mathcal{A}$. Let $I \subseteq U \subseteq V_{\mathcal{A}}$ such that $\rho_{\mathcal{A}[U],I}$ is maximal and subject to this, that $|U|$ is minimal. Then, $(\mathcal{A}[U], I)$ is strictly balanced. Furthermore, since there are no isolated vertices $v \notin I$ in $\mathcal{A}$, we have $\rho_{\mathcal{A}[U],I} \geq \rho_{\mathcal{A},I} > 0$ by choice of U and hence $U \neq I$. Note that

$$\Phi_{\mathcal{A},\psi} = \sum_{\varphi \in \Phi^{\sim}_{\mathcal{A}[U],\psi}} \Phi_{\mathcal{A},\varphi}. \tag{3.5.7}$$

To obtain (i) and (ii), we use the strict balancedness of $(\mathcal{A}[U], I)$ to bound $\Phi_{\mathcal{A}[U],\psi}$ and the induction hypothesis to bound $\Phi_{\mathcal{A},\varphi}$ for all $\varphi \in \Phi^{\sim}_{\mathcal{A}[U],\psi}$.

In more detail, for (i) we argue as follows. Suppose that $\hat{\varphi}_{\mathcal{B},I} \geq 1$ for all $(\mathcal{B}, I) \subseteq (\mathcal{A}, I)$. For all $(\mathcal{B}, U) \subseteq (\mathcal{A}, U)$ and $\mathcal{B}' := \mathcal{B} + \mathcal{A}[U]$, we have $\rho_{\mathcal{B}',I} \leq \rho_{\mathcal{A}[U],I}$ by choice of U. Thus, since $\mathcal{B}'[U] = \mathcal{A}[U]$ and $\mathcal{B}'[I] = \mathcal{A}[I]$, we obtain

$$\begin{aligned}
\hat{\varphi}_{\mathcal{B},U} &= \hat{\varphi}_{\mathcal{B}',U} = n^{|V_{\mathcal{B}'}|-|U|} \hat{p}^{|\mathcal{B}'|-|\mathcal{B}'[I]|-(|\mathcal{A}[U]|-|\mathcal{A}[I]|)} \\
&= n^{|V_{\mathcal{B}'}|-|U|} \hat{p}^{\rho_{\mathcal{B}',I}(|V_{\mathcal{B}'}|-|I|)-\rho_{\mathcal{A}[U],I}(|U|-|I|)} \geq n^{|V_{\mathcal{B}'}|-|U|} \hat{p}^{\rho_{\mathcal{B}',I}(|V_{\mathcal{B}'}|-|U|)} \\
&= \hat{\varphi}_{\mathcal{B}',I}^{(|V_{\mathcal{B}'}|-|U|)/(|V_{\mathcal{B}'}|-|I|)} \geq 1.
\end{aligned}$$

Hence, for all $\varphi \in \Phi^{\sim}_{\mathcal{A}[U],\psi}$, by induction hypothesis,

$$\Phi_{\mathcal{A},\varphi} \leq_{\mathcal{X}} (1 + \log n)^{\alpha_{\mathcal{A},U}} \hat{\varphi}_{\mathcal{A},U}. \tag{3.5.8}$$

The template $(\mathcal{A}[U], I) \subseteq (\mathcal{A}, I)$ is strictly balanced. Furthermore, since we suppose that $\hat{\varphi}_{\mathcal{B},I} \geq 1$ for all $(\mathcal{B}, I) \subseteq (\mathcal{A}, I)$, we have $\hat{\varphi}_{\mathcal{A}[U],I} \geq 1$. Thus, Lemma 3.5.13 entails

$$\Phi_{\mathcal{A}[U],\psi} \leq_{\mathcal{X}} (1 + \log n)^{\alpha_{\mathcal{A}[U],I}} \hat{\varphi}_{\mathcal{A}[U],I}. \tag{3.5.9}$$

Combining (3.5.8) and (3.5.9) with (3.5.7), we obtain

$$\Phi_{\mathcal{A},\psi} \leq_{\mathcal{X}} (1+\log n)^{\alpha_{\mathcal{A}[U],I}} \hat{\varphi}_{\mathcal{A}[U],I} \cdot (1+\log n)^{\alpha_{\mathcal{A},U}} \hat{\varphi}_{\mathcal{A},U}.$$

Hence, employing Observation 3.5.1 as well as Lemma 3.5.2 yields

$$\Phi_{\mathcal{A},\psi} \leq_{\mathcal{X}} (1+\log n)^{\alpha_{\mathcal{A},I}} \hat{\varphi}_{\mathcal{A},I}$$

and thus shows that (i) holds. Recall that, as mentioned above, when we use the fact expressed in Lemma 3.5.2, we will not always explicitly reference this lemma.

Let us turn to (ii). Now, no longer suppose that necessarily $\hat{\varphi}_{\mathcal{B},I} \geq 1$ for all $(\mathcal{B}, I) \subseteq (\mathcal{A}, I)$ and instead suppose that $\hat{\varphi}_{\mathcal{A},J} \leq 1$ for all $I \subseteq J \subseteq V_{\mathcal{A}}$. Then, in particular $\hat{\varphi}_{\mathcal{A},J} \leq 1$ for all $U \subseteq J \subseteq V_{\mathcal{A}}$. Hence, for all $\varphi \in \Phi^{\sim}_{\mathcal{A}[U],\psi}$, by induction hypothesis,

$$\Phi_{\mathcal{A},\varphi} \leq_{\mathcal{X}} (1+\log n)^{\alpha_{\mathcal{A},U}}. \tag{3.5.10}$$

The template $(\mathcal{A}[U], I) \subseteq (\mathcal{A}, I)$ is strictly balanced. Furthermore, since we suppose that $\hat{\varphi}_{\mathcal{A},J} \leq 1$ for all $I \subseteq J \subseteq V_{\mathcal{A}}$, so in particular $\hat{\varphi}_{\mathcal{A},I} \leq 1$, and since $\rho_{\mathcal{A},I} \leq \rho_{\mathcal{A}[U],I}$ by choice of U, we obtain

$$\hat{\varphi}_{\mathcal{A}[U],I} \leq n^{|U|-|I|} \hat{p}^{\rho_{\mathcal{A},I}(|U|-|I|)} = \hat{\varphi}_{\mathcal{A},I}^{(|U|-|I|)/(|V_{\mathcal{A}}|-|I|)} \leq 1.$$

Hence, Lemma 3.5.13 entails

$$\Phi_{\mathcal{A}[U],\psi} \leq_{\mathcal{X}} (1+\log n)^{\alpha_{\mathcal{A}[U],I}}. \tag{3.5.11}$$

Similarly as above, combining (3.5.10) and (3.5.11) with (3.5.7) and employing Observation 3.5.1 yields

$$\Phi_{\mathcal{A},\psi} \leq_{\mathcal{X}} (1+\log n)^{\alpha_{\mathcal{A}[U],I}} \cdot (1+\log n)^{\alpha_{\mathcal{A},U}} \leq (1+\log n)^{\alpha_{\mathcal{A},I}}$$

and hence shows that (ii) holds. □

Lemma 3.5.15. *Let $i \geq 0$ and $\mathcal{X} := \{i < \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'}\}$. Suppose that $(\mathcal{A}, I)$ is a template with $|V_{\mathcal{A}}| \leq 1/\varepsilon^4$, let $I \subsetneq J \subseteq V_{\mathcal{A}}$ and from all subtemplates $(\mathcal{B}', I) \subseteq (\mathcal{A}, I)$ with $J \subseteq V_{\mathcal{B}'}$, choose $(\mathcal{B}, I)$ such that $\hat{\varphi}_{\mathcal{B},I}$ is minimal. Let $\psi\colon J \hookrightarrow V_{\mathcal{H}}$. Then, $\Phi_{\mathcal{A},\psi} \leq_{\mathcal{X}} (1 + \log n)^{\alpha_{\mathcal{A},J}} \hat{\varphi}_{\mathcal{A},I}/\hat{\varphi}_{\mathcal{B},I}$.*

Proof. Since $|\mathcal{A}[V_{\mathcal{B}}]| - |\mathcal{A}[I]| \geq |\mathcal{B}| - |\mathcal{B}[I]|$ entails $\hat{\varphi}_{\mathcal{A}[V_{\mathcal{B}}],I} \leq \hat{\varphi}_{\mathcal{B},I}$, we may assume that $\mathcal{B}$ is an induced subgraph of $\mathcal{A}$. Indeed, by choice of $\mathcal{B}$, we obtain $\hat{\varphi}_{\mathcal{A}[V_{\mathcal{B}}],I} = \hat{\varphi}_{\mathcal{B},I}$, so we may replace $\mathcal{B}$ with $\mathcal{A}[V_{\mathcal{B}}]$ since the statement we wish to obtain only depends on $\hat{\varphi}_{\mathcal{B},I}$. Note that

$$\Phi_{\mathcal{A},\psi} = \sum_{\varphi \in \Phi^{\sim}_{\mathcal{B},\psi}} \Phi_{\mathcal{A},\varphi}. \tag{3.5.12}$$

We use Lemma 3.5.14 to bound $\Phi_{\mathcal{B},\psi}$ and $\Phi_{\mathcal{A},\varphi}$ for all $\varphi \in \Phi^{\sim}_{\mathcal{B},\psi}$.

In more detail, we argue as follows. Let $\varphi \in \Phi^{\sim}_{\mathcal{B},\psi}$ and consider a subtemplate $(\mathcal{C}, V_{\mathcal{B}}) \subseteq (\mathcal{A}, V_{\mathcal{B}})$. Then, for $\mathcal{C}' := \mathcal{C} + \mathcal{A}[V_{\mathcal{B}}]$, we have $\hat{\varphi}_{\mathcal{B},I} \leq \hat{\varphi}_{\mathcal{C}',I}$ by choice of $(\mathcal{B}, I)$ and hence

$$\hat{\varphi}_{\mathcal{C},V_{\mathcal{B}}} = \hat{\varphi}_{\mathcal{C}',V_{\mathcal{B}}} = \frac{\hat{\varphi}_{\mathcal{C}',I}}{\hat{\varphi}_{\mathcal{B},I}} \geq 1.$$

Thus, Lemma 3.5.14 (i) entails

$$\Phi_{\mathcal{A},\varphi} \leq_{\mathcal{X}} (1 + \log n)^{\alpha_{\mathcal{A},V_{\mathcal{B}}}} \hat{\varphi}_{\mathcal{A},V_{\mathcal{B}}}. \tag{3.5.13}$$

Next, in order so bound $\Phi_{\mathcal{B},\psi}$, suppose that $J \subseteq J' \subseteq V_{\mathcal{B}}$. Then, $\hat{\varphi}_{\mathcal{B},I} \leq \hat{\varphi}_{\mathcal{B}[J'],I}$ by choice of $(\mathcal{B}, I)$ and hence

$$\hat{\varphi}_{\mathcal{B},J'} = \frac{\hat{\varphi}_{\mathcal{B},I}}{\hat{\varphi}_{\mathcal{B}[J'],I}} \leq 1.$$

Thus, Lemma 3.5.14 (ii) entails

$$\Phi_{\mathcal{B},\psi} \leq_{\mathcal{X}} (1 + \log n)^{\alpha_{\mathcal{B},J}}. \tag{3.5.14}$$

Since $\mathcal{B}$ is an induced subgraph of $\mathcal{A}$, combining (3.5.13) and (3.5.14) with (3.5.12) and employing Observation 3.5.1 yields

$$\begin{aligned}\Phi_{\mathcal{A},\psi} &\leq_{\mathcal{X}} (1 + \log n)^{\alpha_{\mathcal{B},J}} \cdot (1 + \log n)^{\alpha_{\mathcal{A},V_{\mathcal{B}}}} \hat{\varphi}_{\mathcal{A},V_{\mathcal{B}}} \leq (1 + \log n)^{\alpha_{\mathcal{A},J}} \hat{\varphi}_{\mathcal{A},V_{\mathcal{B}}} \\ &= (1 + \log n)^{\alpha_{\mathcal{A},J}} \frac{\hat{\varphi}_{\mathcal{A},I}}{\hat{\varphi}_{\mathcal{B},I}},\end{aligned}$$

which completes the proof. $\square$

Lemma 3.5.16. *Let $i \geq 0$ and $\mathcal{X} := \{i < \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'}\}$. Suppose that $(\mathcal{A}, I)$ is a k-template with $|V_{\mathcal{A}}| \leq 1/\varepsilon^4$ and let $\psi \colon I \hookrightarrow V_{\mathcal{H}}$. Let $e \in \mathcal{A} \setminus \mathcal{A}[I]$ and from all subtemplates $(\mathcal{B}', I) \subseteq (\mathcal{A}, I)$ with $e \in \mathcal{B}'$, choose $(\mathcal{B}, I)$ such that $\hat{\varphi}_{\mathcal{B},I}$ is minimal. Then,*

$$|\{\varphi \in \Phi^{\sim}_{\mathcal{A},\psi} : \varphi(e) \in \mathcal{F}_0(i+1)\}| \leq_{\mathcal{X}} 2k!|\mathcal{F}|(\log n)^{\alpha_{\mathcal{A},I \cup e}} \frac{\hat{\varphi}_{\mathcal{A},I}}{\hat{\varphi}_{\mathcal{B},I}}.$$

Proof. Note that

$$|\{\varphi \in \Phi^{\sim}_{\mathcal{A},\psi} : \varphi(e) \in \mathcal{F}_0(i+1)\}| \leq \sum_{f \in \mathcal{F}_0(i+1)} |\{\varphi \in \Phi^{\sim}_{\mathcal{A},\psi} : \varphi(e) = f\}|,$$

so it suffices to obtain

$$|\{\varphi \in \Phi^{\sim}_{\mathcal{A},\psi} : \varphi(e) = f\}| \leq_{\mathcal{X}} 2k!\,(\log n)^{\alpha_{\mathcal{A},I \cup e}} \frac{\hat{\varphi}_{\mathcal{A},I}}{\hat{\varphi}_{\mathcal{B},I}}.$$

for all $f \in \mathcal{H}(0)$. This is a consequence of Lemma 3.5.15

In detail, we argue as follows. Fix $f \in \mathcal{H}(0)$. We have

$$|\{\varphi \in \Phi^{\sim}_{\mathcal{A},\psi} : \varphi(e) = f\}| \leq \sum_{\psi' \colon I \cup e \hookrightarrow \psi(I) \cup f \colon \psi'|_I = \psi} \Phi_{\mathcal{A},\psi'}. \tag{3.5.15}$$

For $\psi' \colon I \cup e \hookrightarrow \psi(I) \cup f$, Lemma 3.5.15 entails

$$\Phi_{\mathcal{A},\psi'} \leq_{\mathcal{X}} (1 + \log n)^{\alpha_{\mathcal{A},I \cup e}} \frac{\hat{\varphi}_{\mathcal{A},I}}{\hat{\varphi}_{\mathcal{B},I}} \leq 2(\log n)^{\alpha_{\mathcal{A},I \cup e}} \frac{\hat{\varphi}_{\mathcal{A},I}}{\hat{\varphi}_{\mathcal{B},I}}.$$

Combining this upper bound with (3.5.15) completes the proof. $\square$

3.5.4 Degrees

The numbers of embeddings of the templates $(\mathcal{F}, f)$ where $f \in \mathcal{F}$ play a special role since they are closely related to the degrees in $\mathcal{H}^*$.

Lemma 3.5.17. *Let $i \geq 0$ and $e \in \mathcal{H}$. Then,*

$$d_{\mathcal{H}^*}(e) = \frac{\sum_{f\in\mathcal{F}} \sum_{\psi\colon f \xrightarrow{\sim} e} \Phi_{\mathcal{F},\psi}}{\mathrm{aut}(\mathcal{F})}.$$

Proof. Let $\psi_0 \colon \emptyset \to V_{\mathcal{H}}$. We have

$$d_{\mathcal{H}^*}(e) = \frac{|\{\varphi \in \Phi^{\sim}_{\mathcal{F},\psi_0} : e \in \varphi(\mathcal{F})\}|}{\mathrm{aut}(\mathcal{F})} = \frac{\sum_{f\in\mathcal{F}} \sum_{\psi\colon f \xrightarrow{\sim} e} \Phi_{\mathcal{F},\psi}}{\mathrm{aut}(\mathcal{F})},$$

which completes the proof. $\square$

Lemma 3.5.18. *Let $0 \leq i \leq i^\star$ and $\mathcal{X} := \{i < \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'}\}$. Consider distinct $e_1, e_2 \in \mathcal{H}$ and $f \in \mathcal{F}$. Then, $d_{\mathcal{H}^*}(e_1, e_2) \leq_{\mathcal{X}} \zeta^{2+\varepsilon^2} \hat{\varphi}_{\mathcal{F},f}$.*

Proof. We have

$$d_{\mathcal{H}^*}(e_1, e_2) \leq \sum_{f_1, f_2 \in \mathcal{F}} \sum_{\psi\colon f_1\cup f_2 \xrightarrow{\sim} e_1\cup e_2} \Phi_{\mathcal{F},\psi}.$$

Fix distinct $f_1, f_2 \in \mathcal{F}$, $J := f_1 \cup f_2$ and $\psi \colon J \xrightarrow{\sim} e_1 \cup e_2$. We obtain a suitable upper bound for $\Phi_{\mathcal{F},\psi}$ from Lemma 3.5.15 as follows.

Since $(\mathcal{F}, f_1)$ is balanced, for all $(\mathcal{A}, f_1) \subseteq (\mathcal{F}, f_1)$ with $J \subseteq V_{\mathcal{A}}$, we have $\rho_{\mathcal{A},f_1} \leq \rho_{\mathcal{F}}$ and hence using Lemma 3.5.5, we obtain

$$\hat{\varphi}_{\mathcal{A},f_1} = (n\hat{p}^{\rho_{\mathcal{A},f_1}})^{|V_{\mathcal{A}}|-k} \geq (n\hat{p}^{\rho_{\mathcal{F}}})^{|V_{\mathcal{A}}|-k} \geq n\hat{p}^{\rho_{\mathcal{F}}}.$$

Thus, Lemma 3.5.15 together with Lemma 3.5.8 entails

$$\Phi_{\mathcal{F},\psi} \leq_{\mathcal{X}} n^{\varepsilon^2} \frac{\hat{\varphi}_{\mathcal{F},f_1}}{n\hat{p}^{\rho_{\mathcal{F}}}} = n^{-\varepsilon^2} \zeta^2 \hat{\varphi}_{\mathcal{F},f_1} \leq n^{-\varepsilon^2/2} \frac{\zeta^2 \hat{\varphi}_{\mathcal{F},f_1}}{|\mathcal{F}|^2 (2k)!} \leq \frac{\zeta^{2+\varepsilon^2} \hat{\varphi}_{\mathcal{F},f_1}}{|\mathcal{F}|^2 (2k)!},$$

which completes the proof. $\square$

3.5.5 Concentration of key quantities

Overall our proof relies on showing that key quantities that are crucial for our precise analysis of the process are typically concentrated around a deterministic trajectory. Establishing concentration for any of these quantities relies on the assumption that the other key quantities behave as expected. More specifically, for certain collections of key quantities, we show that it is unlikely that a key quantity from this collection is the first among all key quantities to significantly deviate from its corresponding trajectory as long as only steps $0 \leq i \leq i^\star$ are considered. Before we turn to the core of our argument that allows us to analyze the removal process in our very general setting, we end this section with Lemma 3.5.19 below that provides such statements for three collections of key quantities that correspond to stopping times defined above. Recall that as defined (3.5.1), the stopping time in $\tilde{\tau}^\star$ is the minimum of the four stopping times introduced in Section 3.5.

Lemma 3.5.19. (i) $\mathbb{P}[\tau_{\mathcal{H}^*} \le \tilde{\tau}^\star \wedge i^\star] \le \exp(-n^{\varepsilon^2})$.

(ii) $\mathbb{P}[\tau_{\mathscr{B}} \le \tilde{\tau}^\star \wedge i^\star] \le \exp(-n^{\delta^2})$.

(iii) $\mathbb{P}[\tau_{\mathscr{B}'} \le \tilde{\tau}^\star \wedge i^\star] \le \exp(-(\log n)^{3/2})$.

The three parts of Lemma 3.5.19 can be proved by standard applications of the critical interval method. Essentially, the argumentation for the analogous statements in the triangle case, see [14, Sections 2 and 3], can be adapted to the more general setting without encountering any major obstacles. We remark that for Lemma 3.5.19 (i), similarly as in [14], it is crucial to exploit that if for some $i \ge 0$, the hypergraph $\mathcal{H}^*$ is approximately vertex-regular and has negligible 2-degrees, we may approximate

$$\begin{aligned}\mathbb{E}_i[\Delta H^*] &\approx -\frac{1}{H^*}\sum_{\mathcal{F}'\in\mathcal{H}^*}\sum_{e\in\mathcal{F}'} d_{\mathcal{H}^*}(e) = -\frac{1}{H^*}\sum_{e\in\mathcal{H}^*} d_{\mathcal{H}^*}(e)^2 \approx -\frac{1}{H^*}\frac{(\sum_{e\in\mathcal{H}^*} d_{\mathcal{H}^*}(e))^2}{H}\\ &= -\frac{|\mathcal{F}|^2 H^*}{H}.\end{aligned}$$

Formally, one may rely on the following simple Lemma from [9] which we also apply further below.

Lemma 3.5.20 ([9, Lemma 3.1])**.** *Let $a, a_1, \dots, a_n$ and $b, b_1, \dots, b_n$ such that $|a_i - a| \le \alpha$ and $|b_i - b| \le \beta$ for all $i, j \in [n]$. Then,*

$$\sum_{1\le i\le n} a_i b_i = \frac{1}{n}\Big(\sum_{1\le i\le n} a_i\Big)\Big(\sum_{1\le i\le n} b_i\Big) \pm 2\alpha\beta n.$$

Proof. Note that

$$\sum_{1\le i\le n} a_i b_i - \frac{1}{n}\Big(\sum_{1\le i\le n} a_i\Big)\Big(\sum_{1\le i\le n} b_i\Big) = \sum_{1\le i\le n}(a_i - a)(b_i - b) - \frac{1}{n}\Big(\sum_{1\le i\le n}(a_i - a)\Big)\Big(\sum_{1\le i\le n}(b_i - b)\Big).$$

By the triangle inequality, we have

$$\Big|\sum_{1\le i\le n}(a_i - a)(b_i - b)\Big| \le \alpha\beta n \quad \text{and} \quad \Big|\Big(\sum_{1\le i\le n}(a_i - a)\Big)\Big(\sum_{1\le i\le n}(b_i - b)\Big)\Big| \le \alpha\beta n^2,$$

so the statement follows. □

Furthermore, when adapting the arguments from the triangle case, Lemma 3.5.18 replaces the trivial upper bound on the 2-degrees in $\mathcal{H}^*$ (given two edges, there is at most one triangle containing both). For completeness, we provide proofs for the three parts of Lemma 3.5.19 in Sections 3.15–3.17.

3.6 Chains

Our precise analysis of the hypergraph removal process crucially relies on precise estimates for the random variables $\Phi_{\mathcal{F},\psi}$ where $\psi\colon f \hookrightarrow V_{\mathcal{H}}$ for some $f \in \mathcal{F}$ that essentially correspond to the degrees in the random $|\mathcal{F}|$-graph $\mathcal{H}^*$ (see Lemma 3.5.17). More precisely, Lemma 3.5.19 provides estimates for key quantities at step i that hold with high probability only while $i < \tau_{\mathscr{F}}$. To complete our argument based on stopping times as outlined at the end of Section 3.4, we need to show that this typically holds if $i \leq i^\star$ provided that the key quantities analyzed in these previous sections behaved as expected up to this step.

The desire to control these numbers of embeddings motivates the definition of a collection $\mathfrak{C}$ of carefully chosen templates that includes the templates $(\mathcal{F}, f) \in \mathscr{F}$. Before providing formal definitions of the concepts involved in the definitions of these templates in Section 3.6.1, we first give some motivation and intuition where we omit some details.

We obtain the aforementioned templates from structures that we call *chains* and remark that in [14], substructures playing a similar role for the special case where $\mathcal{F}$ is a triangle are called *ladders*. Similarly as in [14], our choice of chains is based on the following idea. For a chain template $(\mathcal{C}, I)$, $\psi\colon I \to V_{\mathcal{H}}$ and $e \in \mathcal{C} \setminus \mathcal{C}[I]$, to estimate the number of embeddings $\varphi \in \Phi^{\sim}_{\mathcal{C},\psi}$ lost due to $\varphi(e) \notin \mathcal{H}(i+1)$, for an edge $f \in \mathcal{F}$ and a bijection $\beta\colon f \xrightarrow{\sim} e$, we are interested in the number

$$\sum_{\varphi \in \Phi^{\sim}_{\mathcal{C},\psi}} \Phi_{\mathcal{F},\varphi\circ\beta} \tag{3.6.1}$$

Simply obtaining an estimate for this number based on our estimates for $\Phi_{\mathcal{C},\psi}$ and $\Phi_{\mathcal{F},\psi'}$ where $\psi'\colon f \hookrightarrow V_{\mathcal{H}}$ would lead to an undesirable accumulation of errors. Instead, to achieve more precision that in the end allows us to closely follow the evolution of key quantities for a sufficient number of steps, the initial idea might be to include a chain in our collection $\mathfrak{C}$ that provides a template $(\mathcal{C}_+, I)$ where $\mathcal{C}_+$ is, in an intuitive sense, an *extension* of $\mathcal{C}$ obtained from $\mathcal{C}$ by gluing a copy $\mathcal{F}'$ of $\mathcal{F}$ onto $\mathcal{C}$ such that for all $v \in f$, the vertex v is identified with $\beta(v)$ while no other vertices outside e and f are identified with one another. Then, we could simply consider $\Phi_{\mathcal{C}_+,\psi}$. However, iterating this unrestricted extension approach yields a growing collection of chains that quickly becomes uncontrollable. To prevent this, we introduce another chain transformation that we call *reduction* that is meant to counterbalance the extension steps by potentially removing vertices from chains that grow due to extension such that in the end, up to being copies of one another, we only need a finite collection of chains. In particular, we are interested in a transformation $\mathcal{C}''$ of $\mathcal{C}_+$ that we call *branching* of $\mathcal{C}$ and that is obtained by combining an extension operation with a reduction operation. Formally, we define $\mathcal{C}''$ to be a suitable induced subgraph of $\mathcal{C}_+$. If for the vertex set V'' of the branching $\mathcal{C}''$, the embeddings of the template $(\mathcal{C}_+, V'')$ can be controlled based on our estimates for embeddings of balanced templates, then it could appear sensible to approximate the number in (3.6.1) as

$$\sum_{\varphi \in \Phi^{\sim}_{\mathcal{C},\psi}} \Phi_{\mathcal{F},\varphi\circ\beta} \approx \sum_{\varphi \in \Phi^{\sim}_{\mathcal{C}'',\psi}} \Phi_{\mathcal{C}_+,\varphi}. \tag{3.6.2}$$

Recall that our motivation was to analyze the one-step changes of $\Phi_{\mathcal{C},\psi}$ and that our goal is to exploit the self-correcting behavior of this number of embeddings in the following sense: If there are more embeddings than expected, then it is more likely that embeddings get destroyed hence providing a self-correcting drift (and similarly if there are fewer embeddings than expected). With the expression in (3.6.2) based only on the branching, this is hard to exploit directly since there is no explicit dependence on $\Phi_{\mathcal{C},\psi}$. To remedy this, we define another chain, which we call *support*, that is obtained from the branching through another transformation, which we call *truncation*. During truncation, we remove what remains of the vertices that were added when the copy $\mathcal{F}'$ was glued onto $\mathcal{C}$ and we choose the branching such that this truncation can be undone by again gluing the copy $\mathcal{F}'$ onto the support. This yields an induced subgraph $\mathcal{C}'$ of $\mathcal{C}$ which only depends on e and the original chain. We ensure that for the vertex set V' of the support, the embeddings of the template $(\mathcal{C}, V')$ can be controlled based on our estimates for embeddings of balanced templates. Then, Lemma 3.5.20 allows us approximate the number in (3.6.1) as

$$\begin{aligned}\sum_{\varphi\in\Phi^{\sim}_{\mathcal{C},\psi}} \Phi_{\mathcal{F},\varphi\circ\beta} &= \sum_{\psi'\in\Phi^{\sim}_{\mathcal{C}',\psi}} \Phi_{\mathcal{F},\psi'\circ\beta}\Phi_{\mathcal{C},\psi'} \\ &\approx \frac{\left(\sum_{\psi'\in\Phi^{\sim}_{\mathcal{C}',\psi}} \Phi_{\mathcal{F},\psi'\circ\beta}\right)\left(\sum_{\psi'\in\Phi^{\sim}_{\mathcal{C}',\psi}} \Phi_{\mathcal{C},\psi'}\right)}{\Phi_{\mathcal{C}',\psi}} \approx \frac{\Phi_{\mathcal{C}'',\psi}}{\Phi_{\mathcal{C}',\psi}}\Phi_{\mathcal{C},\psi}.\end{aligned} \tag{3.6.3}$$

The choice for our collection $\mathfrak{C}$ of chains is motivated by the fact that for such an argument, $\mathfrak{C}$ needs to be closed under taking branchings and supports of chains contained in $\mathfrak{C}$.

In Section 3.6.1, we formally define the terms *chain*, *extension*, *truncation*, *reduction*, *branching* and *support* and we fix our collection $\mathfrak{C}$. In Section 3.6.2, we turn the motivation outlined here into formal arguments to obtain a version of (3.6.3) with quantified errors. Our arguments that rely on the self-correcting behavior require a careful choice of error terms as well as a consideration of chains in groups that we call *branching families* to exploit symmetry that we discuss in Section 3.7.1. While we defer the analysis of branching families to Section 3.7, we define them in Section 3.6.3 and subsequently use them in a supermartingale argument based on the insight from Section 3.6.2 that ensures that the embeddings of chains are typically concentrated as desired.

3.6.1 Formal definition

Consider a sequence $A = \mathcal{A}_1, \dots, \mathcal{A}_\ell$ of k-graphs where $\ell \geq 0$ and for $0 \leq i \leq \ell$ define $q_i := 1 + \sum_{1\leq j\leq i}(|\mathcal{A}_j| - 1)$. We say that A is a *loose path* starting at a k-set I if there exists an ordering $e_1, \dots, e_{q_\ell}$ of $\mathcal{A}_1 + \dots + \mathcal{A}_\ell$ such that $e_1 = I$ and such that $\mathcal{A}_i = \{e_{q_{i-1}}, \dots, e_{q_i}\}$ for all $1 \leq i \leq \ell$. We call A *vertex-separated* if $V_{\mathcal{A}_1+\dots+\mathcal{A}_{i-1}} \cap V_{\mathcal{A}_i+\dots+\mathcal{A}_\ell} = e_{q_{i-1}}$ for all $2 \leq i \leq \ell$.

A triple $\mathfrak{c} = (F, V, I)$ where $F = \mathcal{F}_1, \dots, \mathcal{F}_\ell$ with $\ell \geq 0$ is a *chain* if F is the empty sequence and $V = I$ is a k-set or if F is a vertex-separated loose path of copies of $\mathcal{F}$ starting at I such that $I \subseteq V \subseteq V_{\mathcal{F}_1+\dots+\mathcal{F}_\ell} \subseteq V_{\mathcal{F}} \cup \mathbb{N}$. The choice of $\mathbb{N}$ here is essentially arbitrary and only serves to provide some infinite set of potential vertices, which is convenient when we want to consider the set of all chains. The *chain template* given by $\mathfrak{c}$ is the template $(\mathcal{C}_\mathfrak{c}, I)$ where $\mathcal{C}_\mathfrak{c}$ is the k-graph with vertex set I and edge set $\{I\}$ if $\ell = 0$ and where $\mathcal{C}_\mathfrak{c} = (\mathcal{F}_1 + \ldots + \mathcal{F}_\ell)[V]$ otherwise.

We now formally define the three basic transformations of chains mentioned in the beginning of this section: *extension*, *truncation* and *reduction*.

For all $\beta\colon f \xrightarrow{\sim} e$ where $f \in \mathcal{F}$ and $e \in \mathcal{C}_\mathfrak{c} \setminus \mathcal{C}_\mathfrak{c}[I]$ such that $e \notin \mathcal{F}_i$ for all $1 \leq i \leq \ell - 1$, fix an arbitrary copy $\mathcal{F}_\mathfrak{c}^\beta$ of $\mathcal{F}$ with vertex set $V_\mathfrak{c}^\beta \subseteq e \cup \mathbb{N}$ such that the following holds

(i) $e \in \mathcal{F}_\mathfrak{c}^\beta$;

(ii) $V_{\mathcal{F}_1+\ldots+\mathcal{F}_\ell} \cap V_\mathfrak{c}^\beta = e$;

(iii) there exists a bijection $\beta'\colon V_\mathcal{F} \xrightarrow{\sim} V_\mathfrak{c}^\beta$ with $\beta'(f') \in \mathcal{F}_\mathfrak{c}^\beta$ for all $f' \in \mathcal{F}$ and $\beta'|_f = \beta$;

(iv) $V_\mathfrak{c}^{\beta_1} \cap V_\mathfrak{c}^{\beta_2} = e$ for all distinct $\beta_1\colon f_1 \xrightarrow{\sim} e$ and $\beta_2\colon f_2 \xrightarrow{\sim} e$ with $f_1, f_2 \in \mathcal{F}$.

The β-*extension* of $\mathfrak{c}$ is the chain $\mathfrak{c}|\beta := (F', V', I)$ where

$$F' := \mathcal{F}_1, \ldots, \mathcal{F}_\ell, \mathcal{F}_\mathfrak{c}^\beta \quad \text{and} \quad V' := V \cup V_\mathfrak{c}^\beta.$$

For $0 \leq \ell' \leq \ell$, the ℓ'-*truncation* of $\mathfrak{c}$ is the chain $\mathfrak{c}|\ell' := (F', V', I)$ where F' is the empty sequence and $V' = I$ if $\ell' = 0$ and where

$$F' := \mathcal{F}_1, \ldots, \mathcal{F}_{\ell'} \quad \text{and} \quad V' := V \cap V_{\mathcal{F}_1+\ldots+\mathcal{F}_{\ell'}}$$

otherwise. For convenience, we set $\mathfrak{c}|- := \mathfrak{c}|\ell - 1$ if $\ell \geq 1$.

If $\ell = 0$, let $W_\mathfrak{c} := V$. If $\ell \geq 1$, then, among the vertex sets W with $(V_{\mathcal{F}_1} \cup V_{\mathcal{F}_\ell}) \cap V \subseteq W \subsetneq V$ and $\rho_{\mathcal{C}_\mathfrak{c}, W} \leq \rho_\mathcal{F} + \varepsilon^2$, choose $W_\mathfrak{c}$ such that $|W_\mathfrak{c}|$ is maximal if such a vertex set exists and choose $W_\mathfrak{c} = V$ otherwise. The *reduction* of $\mathfrak{c}$ is the chain $\mathfrak{c}|\mathtt{r}$ inductively defined as follows. If $W_\mathfrak{c} = V$, then $\mathfrak{c}|\mathtt{r} := \mathfrak{c}$. If $W_\mathfrak{c} \neq V$, then $\mathfrak{c}|\mathtt{r} := (F, W_\mathfrak{c}, I)|\mathtt{r}$. It is easy to see that this indeed provides a well-defined reduction for all chains. Crucially, Lemma 3.5.3 guarantees that each reduction step corresponds to a strictly balanced extension in the sense that if $W_\mathfrak{c} \neq V$, then $(\mathcal{C}_\mathfrak{c}, W_\mathfrak{c})$ is strictly balanced.

With these transformations, we can now formally define *branching* and *support*. Let $\beta\colon f \xrightarrow{\sim} e$ where $f \in \mathcal{F}$ and $e \in \mathcal{C}_\mathfrak{c} \setminus \mathcal{C}_\mathfrak{c}[I]$ and suppose that $\ell' \geq 0$ is minimal such that $e \in \mathcal{C}_{\mathfrak{c}|\ell'}$. We say that $\mathfrak{c}|[\beta] := \mathfrak{c}|\ell'|\beta|\mathtt{r}$ is the β-*branching* of $\mathfrak{c}$ and that the chain $\mathfrak{c}|e := \mathfrak{c}|[\beta]|-$, which only depends on e and $\mathfrak{c}$, is the e-*support* in $\mathfrak{c}$.

Suppose that $U \subseteq V$. For $\psi\colon U \hookrightarrow V_\mathcal{H}$ and $i \geq 0$, we set $\Phi^{\sim}_{\mathfrak{c},\psi}(i) := \Phi^{\sim}_{\mathcal{C}_\mathfrak{c},\psi}$ and $\Phi_{\mathfrak{c},\psi}(i) := |\Phi^{\sim}_{\mathfrak{c},\psi}|$. Furthermore, we set $\hat{\varphi}_{\mathfrak{c},U} := \hat{\varphi}_{\mathcal{C}_\mathfrak{c},U}$.

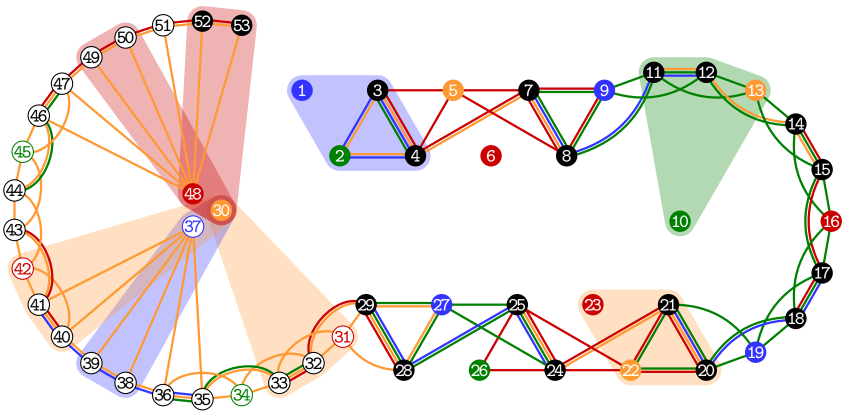
Finally, we choose the collection of chains $\mathfrak{c} = (F, V, I)$ where we are interested in $\Phi_{\mathfrak{c},\psi}$ for $\psi\colon I \hookrightarrow V_\mathcal{H}$. We call a collection $\mathfrak{C}'$ of chains *admissible* if it satisfies the following properties.

(i) $(\mathcal{F}, V_\mathcal{F}, f) \in \mathfrak{C}'$ for all $f \in \mathcal{F}$.

(ii) For all $\mathfrak{c} = (F, V, I) \in \mathfrak{C}'$ where F has length ℓ and all $1 \leq \ell' \leq \ell$, we have $\mathfrak{c}|\ell' \in \mathfrak{C}'$.

(iii) For all $\mathfrak{c} = (F, V, I) \in \mathfrak{C}'$ where $F = \mathcal{F}_1, \ldots, \mathcal{F}_\ell$, and all $\beta\colon f \hookrightarrow e$ where $f \in \mathcal{F}$ and $e \in \mathcal{C}_\mathfrak{c} \setminus \mathcal{C}_\mathfrak{c}[I]$ such that $e \notin \mathcal{F}_i$ for all $1 \leq i \leq \ell - 1$, we have $\mathfrak{c}|\beta|\mathtt{r} \in \mathfrak{C}'$.

Every arbitrary intersection of admissible collections of chains is also admissible. Hence, there exists an admissible collection of chains that is minimal with respect to inclusion.



1 2 3 4,	2 3 4 5,	3 4 5 6,	4 5 6 7,	5 6 7 8,	6 7 8 9,	7 8 9 10,	8 9 10 11,	9 10 11 12,
10 11 12 13,	10 12 13 14,	10 13 14 15,	10 14 15 16,	10 15 16 17,	10 16 17 18,	10 17 18 19,	10 18 19 20,	10 19 20 21,
10 20 21 22,	20 21 22 23,	21 22 23 24,	22 23 24 25,	23 24 25 26,	24 25 26 27,	25 26 27 28,	26 27 28 29,	27 28 29 30,
28 29 30 31,	29 30 31 32,	30 31 32 33,	30 32 33 34,	30 33 34 35,	30 34 35 36,	30 35 36 37,	30 36 37 38,	30 37 38 39,
30 37 39 40,	30 37 40 41,	30 40 41 42,	30 41 42 43,	30 42 43 44,	30 43 44 45,	30 44 45 46,	30 45 46 47,	30 46 47 48,
30 47 48 49,	30 48 49 50,	30 48 50 51,	30 48 51 52,	30 48 52 53				

Figure 3.1: A 3-uniform chain template $(\mathcal{C}, I)$ for the special case where $\mathcal{F} = K_4^{(3)}$. The chain template is given by a chain $\mathfrak{c} = (F, V, I)$ where $I = \{1, 2\}$ and where $F = \mathcal{F}_1, \dots, \mathcal{F}_{50}$ is a sequence of 50 copies of $\mathcal{F}$ whose vertices are elements of $\{1, \dots, 54\}$. The vertex sets of these copies are listed below the visualization of the chain template and for each copy $\mathcal{F}_i$ with $1 \leq i \leq 49$, the unique vertex of $\mathcal{F}_i$ that is not a vertex of $\mathcal{F}_{i+1}$ underlined. Instead of drawing the edges of $\mathcal{C}$, we instead draw edges of the links of selected colored, that is red, green blue or orange, vertices. Here, the *link* of a vertex $u \in V_{\mathcal{C}}$ is the 2-graph with vertex set $V_{\mathcal{C}}$ where $\{v, w\}$ is an edge if $\{u, v, w\} \in \mathcal{C}$. To distinguish more clearly between edges of $\mathcal{C}$ and edges of the links, here we call edges of $\mathcal{C}$ *faces*. For every face $f \in \mathcal{C}$, there exists a colored vertex $v \in f$ such that $f \setminus \{v\}$ is one of the edges of the link of v that is drawn in the same color as v. Hence, for a vertex u, incident faces are represented either by incident edges of a link of another vertex or as edges that have the same color as u. Not all edges of the link of a colored vertex are drawn. Every face is represented by exactly one drawn edge, so in particular, the number of faces is the number of drawn edges. Exactly two vertices of every copy in F are colored. Furthermore, the drawn edges are selected such that every copy $\mathcal{F}'$ in F corresponds to a monochromatic triangle together with a vertex of the same color in the following sense: the vertex together with the vertices of the triangle forms the vertex set of $\mathcal{F}'$ and the edges of the triangle together with an edge that has the same color as the unique colored vertex of the triangle represent the faces of $\mathcal{F}'$. Selected copies in F are highlighted using a colored background.
Suppose that $\varepsilon^2 = 1/10$. Then $W_{\mathfrak{c}} = \{1, \dots, 30, 48, 52, 53\}$ and the vertices outside this set are highlighted. Note that for the chain $\mathfrak{c}' := (F', V', I)$ with $F' = \mathcal{F}_1, \dots, \mathcal{F}_{49}$ and $V' = \{1, \dots, 52\}$, the reduction operation is trivial in the sense that $\mathfrak{c}'|\mathfrak{r} = \mathfrak{c}'$ due to $W_{\mathfrak{c}'} = V'$. Hence, an extension that transforms $\mathfrak{c}'$ into $\mathfrak{c}$ transforms a chain where reduction is trivial into a chain where this is not the case.

We choose the collection $\mathfrak{C}$ of chains $\mathfrak{c} = (F, V, I)$ with $F = \mathcal{F}_1, \dots, \mathcal{F}_\ell$ where we are interested in $\Phi_{\mathfrak{c},\psi}$ for all $\psi\colon I \to V_{\mathcal{H}}$ and $i \geq 0$ as this minimal admissible collection. For our arguments, it is crucial that when considering the chains $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$, the template $(\mathcal{C}, I)$ is not too large and that we do not end up with too many random processes $\Phi_{\mathfrak{c},\psi}(0), \Phi_{\mathfrak{c},\psi}(1), \dots$ where $\psi\colon I \hookrightarrow V_{\mathcal{H}}$ (note that we enforce no bound for the length of the sequence F). Lemma 3.6.3 below provides suitable bounds for the sizes of the vertex set V which in turn yields a suitable bound for the number of such random processes (see Lemma 3.6.17). Lemmas 3.6.4 and 3.6.5 state simple useful properties of chains $\mathfrak{c} \in \mathfrak{C}$ that are almost immediate from the definition of $\mathfrak{C}$.

Lemma 3.6.1. *Suppose that $\mathfrak{c} = (F, V, I)$ is a chain and let $(\mathcal{A}, I) \subseteq (\mathcal{C}_{\mathfrak{c}}, I)$. Then, $\rho_{\mathcal{A},I} \leq \rho_{\mathcal{F}}$.*

Proof. We may assume that F has length $\ell \geq 1$ and that $\mathcal{A}$ is an induced subgraph of $\mathcal{C}_{\mathfrak{c}}$. Suppose that $F = \mathcal{F}_1, \dots, \mathcal{F}_\ell$. For $1 \leq i \leq \ell$, let $V_i := V \cap V_{\mathcal{F}_i}$. Let $f_1 := I$ and for $2 \leq i \leq \ell$, let $f_i \in \mathcal{F}_{i-1} \cap \mathcal{F}_i$. For $1 \leq i \leq \ell$, let $U_i := (V_{\mathcal{A}} \cup f_i) \cap V_i$ and $\mathcal{A}_i := \mathcal{F}_i[U_i]$. Note that since $(\mathcal{F}_i, f_i)$ is balanced, we have $\rho_{\mathcal{A}_i,f_i} \leq \rho_{\mathcal{F}}$. Since F is a vertex-separated loose path, we have

$$V_{\mathcal{A}} \setminus I = \bigcup_{1\leq i\leq \ell} U_i \setminus f_i.$$

and $(U_i \setminus f_i) \cap (U_j \setminus f_j) = \emptyset$ for all $1 \leq i < j \leq \ell$. This entails $|V_{\mathcal{A}}| - |I| = \sum_{1\leq i\leq\ell} |U_i| - |f_i|$. Furthermore,

$$\mathcal{A} \setminus \mathcal{A}[I] = \bigcup_{1\leq i\leq \ell} \mathcal{F}_i[V_{\mathcal{A}} \cap V_i] \setminus \mathcal{F}_i[f_i] \subseteq \bigcup_{1\leq i\leq \ell} \mathcal{A}_i \setminus \mathcal{A}_i[f_i].$$

Similarly as above, since $(\mathcal{A}_i \setminus \mathcal{A}_i[f_i]) \cap (\mathcal{A}_j \setminus \mathcal{A}_j[f_j]) = \emptyset$ for all $1 \leq i < j \leq \ell$, this entails $|\mathcal{A}| - |\mathcal{A}[I]| \leq \sum_{1\leq i\leq\ell} |\mathcal{A}_i| - |\mathcal{A}_i[f_i]|$. Thus, we obtain

$$\rho_{\mathcal{A},I} \leq \frac{\sum_{1\leq i\leq\ell} |\mathcal{A}_i| - |\mathcal{A}_i[f_i]|}{\sum_{1\leq i\leq\ell} |U_i| - |f_i|} = \frac{\sum_{1\leq i\leq\ell} \rho_{\mathcal{A}_i,f_i}(|U_i| - |f_i|)}{\sum_{1\leq i\leq\ell} |U_i| - |f_i|} \leq \rho_{\mathcal{F}},$$

which completes the proof. □

Lemma 3.6.2. *Suppose that $\mathfrak{c} = (F, V, I)$ is a chain with $|V| \geq 1/\varepsilon^3$. Then, $W_{\mathfrak{c}} \neq V$.*

Proof. Suppose that $F = \mathcal{F}_1, \dots, \mathcal{F}_\ell$. We show that for $W := V_{\mathcal{F}_1 + \mathcal{F}_\ell}$, as a consequence of Lemma 3.6.1, we have $\rho_{\mathcal{C}_{\mathfrak{c}},W} \leq \rho_{\mathcal{F}} + \varepsilon^2$. Then, we obtain $W_{\mathfrak{c}} \neq V$ by choice of $W_{\mathfrak{c}}$.

Let us turn to the details. We have

$$|\mathcal{C}_{\mathfrak{c}}| - |\mathcal{C}_{\mathfrak{c}}[W]| \leq |\mathcal{C}_{\mathfrak{c}}| - |\mathcal{C}_{\mathfrak{c}}[I]|$$

and

$$\begin{aligned} |V| - |W| \geq |V| - |I| - 2m &= \left(1 - \frac{2m}{|V| - k}\right)(|V| - |I|) \geq \left(1 - \frac{2m}{\frac{1}{\varepsilon^3} - k}\right)(|V| - |I|) \\ &\geq \frac{|V| - |I|}{1 + \frac{\varepsilon^2}{\rho_{\mathcal{F}}}}. \end{aligned}$$

With Lemma 3.6.1, this yields

$$\rho_{\mathcal{C}_{\mathfrak{c}},W} \leq \left(1+\frac{\varepsilon^2}{\rho_{\mathcal{F}}}\right)\frac{|\mathcal{C}_{\mathfrak{c}}|-|\mathcal{C}_{\mathfrak{c}}[I]|}{|V|-|I|} \leq \left(1+\frac{\varepsilon^2}{\rho_{\mathcal{F}}}\right)\rho_{\mathcal{F}} = \rho_{\mathcal{F}}+\varepsilon^2,$$

which completes the proof. □

Lemma 3.6.3. *Let $(F,V,I) \in \mathfrak{C}$. Then, $|V| \leq 1/\varepsilon^3$.*

Proof. Consider the collection $\mathfrak{C}'$ of all chains (F,V,I) with $|V| \leq 1/\varepsilon^3$. As a consequence of Lemma 3.6.2, this collection is admissible, so we have $\mathfrak{C} \subseteq \mathfrak{C}'$. □

Lemma 3.6.4. *Let $\mathfrak{c} \in \mathfrak{C}$. Then, $\mathfrak{c} = \mathfrak{c}|\mathtt{r}$.*

Proof. Consider the collection $\mathfrak{C}'$ of all chains $\mathfrak{c}$ with $\mathfrak{c} = \mathfrak{c}|\mathtt{r}$. By choice of $\mathfrak{C}$, if $\mathfrak{C}'$ is admissible, then $\mathfrak{C} \subseteq \mathfrak{C}'$, so it suffices to show that $\mathfrak{C}'$ is admissible.

For all $f \in \mathcal{F}$ and $\mathfrak{c} := (\mathcal{F}, V_{\mathcal{F}}, f)$, we have $\mathfrak{c} = \mathfrak{c}|\mathtt{r}$. Consider $\mathfrak{c} = (F,V,I) \in \mathfrak{C}'$ where F has length ℓ and let $1 \leq \ell' \leq \ell$. Suppose that $\mathfrak{c}' = (F',V',I) = \mathfrak{c}|\ell'$, let $V'' := V_{\mathcal{F}_{\ell'+1}+\ldots+\mathcal{F}_\ell} \cap V$ and let $\mathcal{C} := \mathcal{C}_{\mathfrak{c}}$ and $\mathcal{C}' := \mathcal{C}_{\mathfrak{c}'}$. Since for all $(V_{\mathcal{F}_1} \cup V_{\mathcal{F}_{\ell'}}) \cap V \subseteq W \subseteq V'$, we have $\rho_{\mathcal{C}',W} = \rho_{\mathcal{C},W\cup V''}$, from $W_{\mathfrak{c}} = V$, we obtain $W_{\mathfrak{c}'} = V'$. Hence, we have $\mathfrak{c}'|\mathtt{r} = \mathfrak{c}'$ and thus $\mathfrak{c}' \in \mathfrak{C}'$. Finally, since for all chains $\mathfrak{c}$, we have $\mathfrak{c}|\mathtt{r} = \mathfrak{c}|\mathtt{r}|\mathtt{r}$, we conclude that $\mathfrak{C}'$ is admissible. □

Lemma 3.6.5. *Let $\mathfrak{c} = (F,V,I) \in \mathfrak{C}$ where $F = \mathcal{F}_1, \ldots, \mathcal{F}_\ell$. Then, $|\mathcal{C}_{\mathfrak{c}} \setminus \mathcal{C}_{\mathfrak{c}}[I]| \geq |\mathcal{F}| - 1$ and hence $\ell \geq 1$.*

Proof. Consider the collection $\mathfrak{C}'$ of all chains (F,V,I) where $F' = \mathcal{F}_1, \ldots, \mathcal{F}_\ell$ for some $\ell \geq 1$ such that $V_{\mathcal{F}_1} \subseteq V$. For all $\mathfrak{c}' = (F',V',I') \in \mathfrak{C}'$, we have $|\mathcal{C}_{\mathfrak{c}'} \setminus \mathcal{C}_{\mathfrak{c}'}[I']| \geq |\mathcal{F}| - 1$. Furthermore, $\mathfrak{C}'$ is admissible, so we have $\mathfrak{C} \subseteq \mathfrak{C}'$. □

3.6.2 Branching and support

In this section, we follow the argumentation in the beginning of Section 3.6 to obtain Lemma 3.6.15 where use the branching and support constructions to estimate the expected number of embeddings of a chain template lost when removing the next randomly chosen copy of $\mathcal{F}$. As preparation for the proof of Lemma 3.6.15 we first consider templates $(\mathcal{C}_{\mathfrak{c}}, V_{\mathcal{C}_{\mathfrak{c}'}})$ that correspond to truncation and reduction transformations introduced above in the sense that $\mathfrak{c}'$ is the transformation of the chain $\mathfrak{c}$. For such templates, we show that we can control the number of embeddings based on control over balanced extensions (see Lemma 3.6.11, Lemma 3.6.12 and Lemma 3.6.14). To this end, we first state Lemma 3.6.6 that quantifies the number of embeddings that avoid a given small subset of $V_{\mathcal{H}}$, which will be helpful in the following situations. Suppose that $(\mathcal{A}, I)$ is a template and that $J \subseteq I$ is a subset such that for all $e \in \mathcal{A}$ with $e \cap J \neq \emptyset$, we have $e \in \mathcal{A}[I]$ and suppose that $\psi\colon I \hookrightarrow V_{\mathcal{H}}$. Let $\psi' := \psi|_{I\setminus J}$. Then, the number $\Phi_{\mathcal{A},\psi}$ of embeddings of $(\mathcal{A}, I)$ that extend ψ is equal to the number of embeddings $\varphi \in \Phi^{\sim}_{\mathcal{A}-J,\psi'}$ of $(\mathcal{A}-J, I\setminus J)$ that extend ψ' and additionally avoid $\psi(J)$

in the sense that $\varphi(V_{\mathcal{A}-J}) \cap \psi(J) = \emptyset$. We introduce the following notation. For a template $(\mathcal{A}, I)$, $\psi\colon I \hookrightarrow V_{\mathcal{H}}$ and $W \subseteq V_{\mathcal{H}} \setminus \psi(I)$, let

$$\Phi^{\sim,W}_{\mathcal{A},I} := \{\varphi \in \Phi^{\sim}_{\mathcal{A},I} : \varphi(V_{\mathcal{A}}) \cap W = \emptyset\} \quad \text{and} \quad \Phi^{W}_{\mathcal{A},I} := |\Phi^{\sim,W}_{\mathcal{A},I}|.$$

Lemma 3.6.6. *Let $0 \le i \le i^\star$ and $\mathcal{X} := \{i < \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'}\}$. Suppose that $(\mathcal{A}, I)$ is a template with $|V_{\mathcal{A}}| \le 1/\varepsilon^4$ and $\rho_{\mathcal{B},I} \le \rho_{\mathcal{F}} + \varepsilon^2$ for all $(\mathcal{B}, I) \subseteq (\mathcal{A}, I)$. Let $\psi\colon I \hookrightarrow V_{\mathcal{H}}$ and $W \subseteq V_{\mathcal{H}} \setminus \psi(I)$ with $|W| \le 1/\varepsilon^3$. Then,*

$$\Phi_{\mathcal{A},\psi} - \Phi^{W}_{\mathcal{A},\psi} \le_{\mathcal{X}} \zeta^{3/2} \hat{\varphi}_{\mathcal{A},I}.$$

Proof. For $v \in V_{\mathcal{A}} \setminus I$ and $w \in W$, let $\psi^w_v\colon I \cup \{v\} \hookrightarrow \psi(I) \cup \{w\}$ with $\psi^w_v|_I = \psi$. We have

$$\Phi_{\mathcal{A},\psi} - \Phi^{W}_{\mathcal{A},\psi} \le \sum_{v \in V_{\mathcal{A}} \setminus I} \sum_{w \in W} |\{\varphi \in \Phi^{\sim}_{\mathcal{A},\psi} : \varphi(v) = w\}| = \sum_{v \in V_{\mathcal{A}} \setminus I} \sum_{w \in W} \Phi_{\mathcal{A},\psi^w_v}.$$

Hence, it suffices to show that for all $v \in V_{\mathcal{A}} \setminus I$ and $w \in W$, we have $\Phi_{\mathcal{A},\psi^w_v} \le \zeta^{5/3} \hat{\varphi}_{\mathcal{A},I}$. We show that this is a consequence of Lemma 3.5.15.

To this end, suppose that $v \in V_{\mathcal{A}} \setminus I$ and $w \in W$. For all subtemplates $(\mathcal{B}, I) \subseteq (\mathcal{A}, I)$ with $v \in V_{\mathcal{B}}$, using the fact that $\zeta^{-1} \le n^{1/2} \hat{p}^{\rho_{\mathcal{F}}/2}$ and Lemma 3.5.5, we have

$$\hat{\varphi}_{\mathcal{B},I} = (n\hat{p}^{\rho_{\mathcal{B},I}})^{|V_{\mathcal{B}}|-|I|} \ge (n\hat{p}^{\rho_{\mathcal{F}}+\varepsilon^2})^{|V_{\mathcal{B}}|-|I|} \ge n\hat{p}^{\rho_{\mathcal{F}}+\varepsilon^2} \ge (n\hat{p}^{\rho_{\mathcal{F}}+8\varepsilon^2})^{1/8} \zeta^{-7/4} \ge \zeta^{-7/4}.$$

Thus, Lemma 3.5.15 entails

$$\Phi_{\mathcal{A},\psi^w_v} \le_{\mathcal{X}} (1 + \log n)^{\alpha_{\mathcal{A},I \cup \{v\}}} \zeta^{7/4} \hat{\varphi}_{\mathcal{A},I} \le \zeta^{5/3} \hat{\varphi}_{\mathcal{A},I},$$

which completes the proof. □

Lemma 3.6.7. *Suppose that $\mathcal{A}_1, \mathcal{A}_2$ is a subsequence of a vertex-separated loose path. Then, there exist edges $e_1 \in \mathcal{A}_1$ and $e_2 \in \mathcal{A}_2$ with $V_{\mathcal{A}_1} \cap V_{\mathcal{A}_2} \subseteq e_1 \cap e_2$.*

Proof. Consider a vertex-separated loose path $B = \mathcal{B}_1, \dots, \mathcal{B}_\ell$ that has $\mathcal{A}_1, \mathcal{A}_2$ as a subsequence. Let $1 \le i < j \le \ell$ such that $\mathcal{B}_i = \mathcal{A}_1$ and $\mathcal{B}_j = \mathcal{A}_2$. Let e_1 denote the unique edge in $\mathcal{B}_i \cap \mathcal{B}_{i+1}$ and let e_2 denote the unique edge in $\mathcal{B}_{j-1} \cap \mathcal{B}_j$. Then,

$$V_{\mathcal{A}_1} \cap V_{\mathcal{A}_2} = V_{\mathcal{B}_i} \cap V_{\mathcal{B}_j} \subseteq V_{\mathcal{B}_1 + \ldots + \mathcal{B}_i} \cap V_{\mathcal{B}_{i+1} + \ldots + \mathcal{B}_\ell} = e_1$$

and similarly

$$V_{\mathcal{A}_1} \cap V_{\mathcal{A}_2} = V_{\mathcal{B}_i} \cap V_{\mathcal{B}_j} \subseteq V_{\mathcal{B}_1 + \ldots + \mathcal{B}_{j-1}} \cap V_{\mathcal{B}_j + \ldots + \mathcal{B}_\ell} = e_2,$$

which completes the proof. □

Lemma 3.6.8. *Suppose that $\mathcal{F}_1, \mathcal{F}_2$ is a subsequence of a vertex-separated loose path of copies of $\mathcal{F}$. Let $I := V_{\mathcal{F}_1} \cap V_{\mathcal{F}_2}$. Then, $|I| = k$ or $|I| \le k - 1/\rho_{\mathcal{F}}$. Hence, if $I \subseteq V_{\mathcal{A}}$ for some k-graph $\mathcal{A}$ that has exactly one edge and no isolated vertices, then $\rho_{\mathcal{A},I} \le \rho_{\mathcal{F}}$.*

Proof. If $\mathcal{F}_1 \cap \mathcal{F}_2 \neq \emptyset$, then $|I| = k$ and hence the statement follows. Thus, we may assume that $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$. Consider a vertex-separated loose path $F' = \mathcal{F}'_1, \ldots, \mathcal{F}'_\ell$ of copies of $\mathcal{F}$ that has $\mathcal{F}_1, \mathcal{F}_2$ as a subsequence. Let $1 \leq i < j \leq \ell$ such that $\mathcal{F}'_i = \mathcal{F}_1$ and $\mathcal{F}'_j = \mathcal{F}_2$. Since $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$, we have $j \geq i+2$. Choose $f_-, f_+ \in \mathcal{F}'_{i+1}$ such that f_- is the unique edge in $\mathcal{F}'_i \cap \mathcal{F}'_{i+1}$ and such that f_+ is the unique edge in $\mathcal{F}'_{i+1} \cap \mathcal{F}'_{i+2}$. Then,

$$V_{\mathcal{F}_1} \cap V_{\mathcal{F}_2} = V_{\mathcal{F}'_i} \cap V_{\mathcal{F}'_j} \subseteq V_{\mathcal{F}'_1+\ldots+\mathcal{F}'_i} \cap V_{\mathcal{F}'_{i+1}+\ldots+\mathcal{F}'_\ell} = f_-$$

and similarly

$$V_{\mathcal{F}_1} \cap V_{\mathcal{F}_2} = V_{\mathcal{F}'_i} \cap V_{\mathcal{F}'_j} \subseteq V_{\mathcal{F}'_1+\ldots+\mathcal{F}'_{i+1}} \cap V_{\mathcal{F}'_{i+2}+\ldots+\mathcal{F}'_\ell} = f_+.$$

Hence, $V_{\mathcal{F}_1} \cap V_{\mathcal{F}_2} \subseteq f_- \cap f_+$. Thus, it suffices to show that $|f_- \cap f_+| \leq k - 1/\rho_{\mathcal{F}}$. This follows from the fact that $(\mathcal{F}'_{i+1}, f_-)$ is balanced. To see this, consider the template $(\mathcal{F}'_{i+1}[f_- \cup f_+], f_-)$. Then,

$$\rho_{\mathcal{F}} \geq \rho_{\mathcal{F}'_{i+1}[f_- \cup f_+], f_-} \geq \frac{1}{|f_- \cup f_+| - |f_-|} = \frac{1}{k - |f_- \cap f_+|}$$

and hence $|f_- \cap f_+| \leq k - 1/\rho_{\mathcal{F}}$. □

Lemma 3.6.9. *Suppose that $\mathcal{F}_1, \mathcal{F}_2$ is a subsequence of a vertex-separated loose path of copies of $\mathcal{F}$. Suppose that $\mathcal{A}$ is a subgraph of $\mathcal{F}_1$ or $\mathcal{F}_2$. Let $I := V_{\mathcal{A}} \cap V_{\mathcal{F}_1} \cap V_{\mathcal{F}_2}$. Then, $\rho_{\mathcal{A},I} \leq \rho_{\mathcal{F}}$.*

Proof. Since $\mathcal{F}_2, \mathcal{F}_1$ is also a subsequence of a vertex-separated loose path of copies of $\mathcal{F}$, we may assume that $\mathcal{A}$ is a subgraph of $\mathcal{F}_1$. Furthermore, we may assume that $\mathcal{A}$ is an induced subgraph of $\mathcal{F}_1$. By Lemma 3.6.7, we may fix an edge $f_1 \in \mathcal{F}_1$ with $V_{\mathcal{F}_1} \cap V_{\mathcal{F}_2} \subseteq f_1$. If $f_1 \not\subseteq V_{\mathcal{A}}$, then $\mathcal{A}[I] = \emptyset$ and thus, using the fact that $(\mathcal{F}_1, f_1)$ is balanced, we obtain

$$\begin{aligned} |\mathcal{A}| - |\mathcal{A}[I]| &= |\mathcal{F}_1[V_{\mathcal{A}}]| \leq |\mathcal{F}_1[V_{\mathcal{A}} \cup f_1]| - |\mathcal{F}_1[f_1]| = \rho_{\mathcal{F}_1[V_{\mathcal{A}} \cup f_1], f_1}(|V_{\mathcal{A}} \cup f_1| - |f_1|) \\ &\leq \rho_{\mathcal{F}}(|V_{\mathcal{A}} \cup f_1| - |f_1|) = \rho_{\mathcal{F}}|V_{\mathcal{A}} \setminus f_1| \leq \rho_{\mathcal{F}}(|V_{\mathcal{A}}| - |I|). \end{aligned}$$

If $f_1 \subseteq V_{\mathcal{A}}$, then $I = V_{\mathcal{F}_1} \cap V_{\mathcal{F}_2}$, so using the fact that $(\mathcal{F}_1, f_1)$ is balanced and Lemma 3.6.8, we obtain

$$\begin{aligned} |\mathcal{A}| - |\mathcal{A}[I]| = |\mathcal{A}| - |\mathcal{A}[f_1]| + |\mathcal{F}_1[f_1]| - |\mathcal{F}_1[I]| &= \rho_{\mathcal{A},f_1}(|V_{\mathcal{A}}| - |f_1|) + \rho_{\mathcal{F}_1[f_1],I}(|f_1| - |I|) \\ &\leq \rho_{\mathcal{F}}(|V_{\mathcal{A}}| - |f_1|) + \rho_{\mathcal{F}}(|f_1| - |I|) = \rho_{\mathcal{F}}(|V_{\mathcal{A}}| - |I|), \end{aligned}$$

which completes the proof. □

Lemma 3.6.10. *Let $0 \leq i \leq i^\star$ and $\mathcal{X} := \{i < \tau_{\mathscr{B}}\}$. Suppose that $\mathcal{F}_1, \mathcal{F}_2$ is a subsequence of a vertex-separated loose path of copies of $\mathcal{F}$. Suppose that $\mathcal{A}$ is a subgraph of $\mathcal{F}_1$ or $\mathcal{F}_2$. Let $I := V_{\mathcal{A}} \cap V_{\mathcal{F}_1} \cap V_{\mathcal{F}_2}$. Let $\psi\colon I \hookrightarrow V_{\mathcal{H}}$. Then,*

$$\Phi_{\mathcal{A},\psi} =_{\mathcal{X}} (1 \pm \varepsilon^{-1}\zeta^{\delta})\hat{\varphi}_{\mathcal{A},I}.$$

Proof. We use induction on $|V_{\mathcal{A}}| - |I|$ to show that

$$\Phi_{\mathcal{A},\psi} =_{\mathcal{X}} (1 \pm 2(|V_{\mathcal{A}}| - |I|)\zeta^{\delta})\hat{\varphi}_{\mathcal{A},I}. \tag{3.6.4}$$

If $|V_{\mathcal{A}}| - |I| = 0$, then $\Phi_{\mathcal{A},\psi} = 1 = \hat{\varphi}_{\mathcal{A},I}$. Let $\ell \geq 1$ and suppose that (3.6.4) holds if $|V_{\mathcal{A}}| - |I| \leq \ell - 1$. Suppose that $|V_{\mathcal{A}}| - |I| = \ell$. From Lemma 3.6.9, we obtain $\rho_{\mathcal{A},I} \leq \rho_{\mathcal{F}}$. Suppose that among all subsets $I \subseteq U' \subsetneq V_{\mathcal{A}}$ with $\rho_{\mathcal{A},U'} \leq \rho_{\mathcal{F}}$, the set U has maximal size. By Lemma 3.5.3, the extension $(\mathcal{A}, U)$ is balanced. We have

$$\Phi_{\mathcal{A},\psi} = \sum_{\varphi \in \Phi^{\sim}_{\mathcal{A}[U],\psi}} \Phi_{\mathcal{A},\varphi}. \tag{3.6.5}$$

We use the estimate for $\Phi_{\mathcal{A}[U],\psi}$ provided by the induction hypothesis and for $\varphi \in \Phi^{\sim}_{\mathcal{A}[U],\psi}$, we estimate $\Phi_{\mathcal{A},\varphi}$ using the balancedness of $(\mathcal{A}, U)$ to conclude that $\Phi_{\mathcal{A},\psi}$ is bounded as desired.

Let us turn to the details. Since $\zeta^{-2} \leq n\hat{p}^{\rho_{\mathcal{F}}}$, for all $j \leq i$, we have

$$\hat{\varphi}_{\mathcal{A},U}(j) = (n\hat{p}(j)^{\rho_{\mathcal{A},U}})^{|V_{\mathcal{A}}|-|U|} \geq (n\hat{p}^{\rho_{\mathcal{F}}})^{|V_{\mathcal{A}}|-|U|} \geq \zeta^{-2(|V_{\mathcal{A}}|-|U|)} \geq \zeta^{-2} > \zeta^{-\delta^{1/2}}.$$

Hence $i < i^{\delta^{1/2}}_{\mathcal{A},U}$, and thus for all $\varphi \in \Phi^{\sim}_{\mathcal{A}[U],\psi}$, we have $\Phi_{\mathcal{A},\varphi} =_{\mathcal{X}} (1 \pm \zeta^{\delta})\hat{\varphi}_{\mathcal{A},U}$. Since by induction hypothesis, we have $\Phi_{\mathcal{A}[U],\psi} = (1 \pm 2(|U| - |I|)\zeta^{\delta})\hat{\varphi}_{\mathcal{A}[U],I}$, returning to (3.6.5), we conclude that

$$\Phi_{\mathcal{A},\psi} = (1 \pm 2(|U| - |I|)\zeta^{\delta})(1 \pm \zeta^{\delta})\hat{\varphi}_{\mathcal{A},U}\hat{\varphi}_{\mathcal{A}[U],I} = (1 \pm 2(|V_{\mathcal{A}}| - |I|)\zeta^{\delta})\hat{\varphi}_{\mathcal{A},I},$$

which completes the proof. □

Lemma 3.6.11. *Let $0 \leq i \leq i^{\star}$ and $\mathcal{X} := \{i < \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'}\}$. Suppose that $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$ is a chain where F has length ℓ. Let $0 \leq \ell' \leq \ell$ and suppose that $(F', V', I) = \mathfrak{c}|\ell'$. Let $\psi\colon V' \hookrightarrow V_{\mathcal{H}}$. Then, $\Phi_{\mathfrak{c},\psi} =_{\mathcal{X}} (1 \pm \varepsilon^{-5k}\zeta^{\delta})\hat{\varphi}_{\mathfrak{c},V'}$.*

Proof. For $0 \leq \ell_0 \leq \ell$, let

$$g_{\ell_0} := |\{\ell_0 \leq \ell_1 \leq \ell - 1 : \mathcal{C}_{\mathfrak{c}|\ell_1} \neq \mathcal{C}_{\mathfrak{c}|\ell_1+1}\}|.$$

We use induction on $\ell - \ell'$ to show that

$$\Phi_{\mathfrak{c},\psi} =_{\mathcal{X}} (1 \pm 4g_{\ell'}\varepsilon^{-1}\zeta^{\delta})\hat{\varphi}_{\mathcal{C},V'}. \tag{3.6.6}$$

By Lemma 3.6.3, we have $|V| \leq \varepsilon^{-3}$, hence $|\mathcal{C}_{\mathfrak{c}}| \leq \varepsilon^{-3k}$ and thus $g_{\ell'} \leq \varepsilon^{-3} + \varepsilon^{-3k} \leq \varepsilon^{-4k}$, so it suffices to obtain (3.6.6).

Let us proceed with the proof by induction. If $\ell - \ell' = 0$, then $\Phi_{\mathfrak{c},\psi} = 1 = \hat{\varphi}_{\mathcal{C},V'}$. Let $q \geq 1$ and suppose that (3.6.6) holds whenever $\ell - \ell' \leq q - 1$. Suppose that $\ell - \ell' = q$. Suppose that $\mathfrak{c}' = (F', V', I) = \mathfrak{c}|\ell'$ and $\mathfrak{c}'' = (F'', V'', I) = \mathfrak{c}|\ell'+1$. If $\mathcal{C}_{\mathfrak{c}''} = \mathcal{C}_{\mathfrak{c}'}$, then (3.6.6) follows by induction hypothesis, so we may assume $\mathcal{C}_{\mathfrak{c}''} \neq \mathcal{C}_{\mathfrak{c}'}$ and hence $g_{\ell'+1} = g_{\ell'} - 1$. We have

$$\Phi_{\mathfrak{c},\psi} = \sum_{\varphi \in \Phi^{\sim}_{\mathfrak{c}'',\psi}} \Phi_{\mathfrak{c},\varphi}. \tag{3.6.7}$$

We use Lemma 3.6.10 to estimate $\Phi_{\mathfrak{c}'',\psi}$ and for $\varphi \in \Phi^{\sim}_{\mathfrak{c}'',\psi}$, we use the estimate for $\Phi_{\mathfrak{c},\varphi}$ provided by the induction hypothesis to conclude that $\Phi_{\mathfrak{c},\psi}$ can be estimated as desired.

Let us turn to the details. Let $\mathcal{A} := \mathcal{F}_{\ell'+1}[V \cap V_{\mathcal{F}_{\ell'+1}}]$ and $J := V_{\mathcal{A}} \cap V_{\mathcal{F}_{\ell'}}$. Note that $\hat{\varphi}_{\mathcal{A},J} = \hat{\varphi}_{\mathcal{C}'',V'}$. Lemma 3.6.9 allows us to apply Lemma 3.6.6 such that using Lemma 3.6.10, we obtain

$$\Phi_{\mathfrak{c}'',\psi} = \Phi^{\psi(V'\setminus J)}_{\mathcal{A},\psi|_J} =_{\mathcal{X}} \Phi_{\mathcal{A},\psi|_J} \pm \zeta^{3/2}\hat{\varphi}_{\mathcal{A},J} =_{\mathcal{X}} (1 \pm 2\varepsilon^{-1}\zeta^{\delta})\hat{\varphi}_{\mathcal{A},J} = (1 \pm 2\varepsilon^{-1}\zeta^{\delta})\hat{\varphi}_{\mathcal{C}'',V'}.$$

Furthermore, by induction hypothesis, for all $\varphi \in \Phi^{\sim}_{\mathfrak{c}'',\psi}$, we have

$$\Phi_{\mathfrak{c},\varphi} =_{\mathcal{X}} (1 \pm 4g_{\ell'+1}\varepsilon^{-1}\zeta^{\delta})\hat{\varphi}_{\mathcal{C},V''} = (1 \pm 4(g_{\ell'} - 1)\varepsilon^{-1}\zeta^{\delta})\hat{\varphi}_{\mathcal{C},V''}.$$

Thus, returning to (3.6.7), we conclude that

$$\Phi_{\mathfrak{c},\psi} =_{\mathcal{X}} (1 \pm 2\varepsilon^{-1}\zeta^{\delta})\hat{\varphi}_{\mathcal{C}'',V'} \cdot (1 \pm 4(g_{\ell'} - 1)\varepsilon^{-1}\zeta^{\delta})\hat{\varphi}_{\mathcal{C},V''} = (1 \pm 4g_{\ell'}\varepsilon^{-1}\zeta^{\delta})\hat{\varphi}_{\mathcal{C},V'},$$

which completes the proof. □

Lemma 3.6.12. *Let $0 \leq i \leq i^\star$ and $\mathcal{X} := \{i < \tau_{\mathscr{B}}\}$. Suppose that $\mathfrak{c}$ is the β-extension of a chain in $\mathfrak{C}$ for some β and let $(F', V', I) = \mathfrak{c}|\mathfrak{r}$. Let $\psi\colon V' \hookrightarrow V_{\mathcal{H}}$. Then, $\Phi_{\mathfrak{c},\psi} = (1 \pm \varepsilon^{-4}\zeta^{\delta})\hat{\varphi}_{\mathfrak{c},V'}$.*

Proof. Suppose that $\mathfrak{c} = (F, V, I)$ where $F = \mathcal{F}_1, \ldots, \mathcal{F}_\ell$. By definition of $\mathfrak{c}' := (F', V', I)$, there exists a sequence of chains $\mathfrak{c} = (F, V_0, I), \ldots, (F, V_t, I) = \mathfrak{c}'$ with $V_0 \supseteq \ldots \supseteq V_t$ such that for all $1 \leq s \leq t$, the set V_s is a subset of V_{s-1} of maximal size chosen from all subsets $(V_{\mathcal{F}_1} \cup V_{\mathcal{F}_\ell}) \cap V_{s-1} \subseteq W \subsetneq V_{s-1}$ with $\rho_{\mathcal{C}_{(F,V_{s-1},I)},W} \leq \rho_{\mathcal{F}} + \varepsilon^2$.

For $0 \leq s \leq t$, let $\mathcal{C}_s := \mathcal{C}_{(F,V_s,I)}$. Using induction on s, we show that for all $0 \leq s \leq t$ and $\psi_s\colon V_s \hookrightarrow V_{\mathcal{H}}$, we have

$$\Phi_{\mathfrak{c},\psi_s} =_{\mathcal{X}} (1 \pm 2s\zeta^{\delta})\hat{\varphi}_{\mathfrak{c},V_s}. \tag{3.6.8}$$

By Lemma 3.6.3, we have $|V| \leq 2\varepsilon^{-3}$ and hence $2t \leq \varepsilon^{-4}$, so this is sufficient.

Let us proceed with the proof by induction. If $s = 0$, then, for all $\psi_s\colon V_s \hookrightarrow V_{\mathcal{H}}$, we have $\Phi_{\mathfrak{c},\psi_s} = 1 = \hat{\varphi}_{\mathfrak{c},V_s}$. Let $q \geq 1$ and suppose that (3.6.8) holds whenever $s \leq q-1$. Suppose that $s = q$ and let $\psi_s\colon V_s \hookrightarrow V_{\mathcal{H}}$. We have

$$\Phi_{\mathfrak{c},\psi_s} = \sum_{\varphi \in \Phi^{\sim}_{\mathcal{C}_{s-1},\psi_s}} \Phi_{\mathfrak{c},\varphi}. \tag{3.6.9}$$

By Lemma 3.5.3, the extension $(\mathcal{C}_{s-1}, V_s)$ is balanced, so we may estimate $\Phi_{\mathcal{C}_{s-1},\psi_s}$ based on balancedness, while for $\varphi \in \Phi^{\sim}_{\mathcal{C}_{s-1},\psi_s}$ the induction hypothesis provides an estimate for $\Phi_{\mathfrak{c},\varphi}$.

Let us turn to the details. Using the fact that $\zeta^{-1} \leq n^{1/2}\hat{p}^{\rho_{\mathcal{F}}/2}$ and Lemma 3.5.5, for all $j \leq i$, we obtain

$$\begin{aligned}\hat{\varphi}_{\mathcal{C}_{s-1},V_s}(j) &= (n\hat{p}(j)^{\rho_{\mathcal{C}_{s-1},V_s}})^{|V_{s-1}|-|V_s|} \geq (n\hat{p}^{\rho_{\mathcal{F}}+\varepsilon^2})^{|V_{s-1}|-|V_s|} \geq n\hat{p}^{\rho_{\mathcal{F}}+\varepsilon^2}\\ &\geq (n\hat{p}^{\rho_{\mathcal{F}}+2\varepsilon^2})^{1/2}\zeta^{-1} \geq \zeta^{-1} > \zeta^{-\delta^{1/2}}.\end{aligned}$$

Hence $i < i^{\delta^{1/2}}_{\mathcal{C}_{s-1},V_s}$ and thus $\Phi_{\mathcal{C}_{s-1},\psi_s} =_{\mathcal{X}} (1 \pm \zeta^\delta)\hat{\varphi}_{\mathcal{C}_{s-1},V_s}$. Furthermore, for all $\varphi \in \Phi^{\sim}_{\mathcal{C}_{s-1},\psi_s}$, by induction hypothesis we have $\Phi_{\mathfrak{c},\varphi} =_{\mathcal{X}} (1 \pm 2(s-1)\zeta^\delta)\hat{\varphi}_{\mathfrak{c},V_{s-1}}$, so returning to (3.6.9), we conclude that

$$\Phi_{\mathfrak{c},\psi_s} =_{\mathcal{X}} (1 \pm \zeta^\delta)\hat{\varphi}_{\mathcal{C}_{s-1},V_s} \cdot (1 \pm 2(s-1)\zeta^\delta)\hat{\varphi}_{\mathfrak{c},V_{s-1}} = (1 \pm 2s\zeta^\delta)\hat{\varphi}_{\mathfrak{c},V_s},$$

which completes the proof. □

Lemma 3.6.13. *Suppose that* $\mathfrak{c}$ *is the* β*-extension of a chain in* $\mathfrak{C}$ *for some* β *and let* $(F', V', I) = \mathfrak{c}|\mathtt{r}$. *Let* $(\mathcal{A}, V') \subseteq (\mathcal{C}_\mathfrak{c}, V')$ *Then,* $\rho_{\mathcal{A},V'} \le \rho_\mathcal{F} + \varepsilon^2$.

Proof. Suppose that $\mathfrak{c} = (F, V, I)$ where $F = \mathcal{F}_1, \dots, \mathcal{F}_\ell$. By definition of $\mathfrak{c}' := (F', V', I)$, there exists a sequence of chains $\mathfrak{c} = (F, V_0, I), \dots, (F, V_t, I) = \mathfrak{c}'$ with $V_0 \supseteq \dots \supseteq V_t$ such that for all $1 \le s \le t$, the set V_s is a subset of V_{s-1} of maximal size chosen from all subsets $(V_{\mathcal{F}_1} \cup V_{\mathcal{F}_\ell}) \cap V_{s-1} \subseteq W \subsetneq V_{s-1}$ with $\rho_{\mathcal{C}_{(F,V_{s-1},I)},W} \le \rho_\mathcal{F} + \varepsilon^2$.

For $0 \le s \le t$, let $\mathcal{C}_s := \mathcal{C}_{(F,V_s,I)}$ and $\mathcal{A}_s := \mathcal{A}[V_s \cap V_\mathcal{A}]$ and for $0 \le s \le t-1$, let $\mathcal{A}'_s := \mathcal{A}_s + \mathcal{C}_{s+1}$. For $0 \le s \le t-1$, consider the extensions $(\mathcal{A}_s, V_{\mathcal{A}_{s+1}})$ and $(\mathcal{A}'_s, V_{s+1})$. We have

$$V_{\mathcal{A}_s} \setminus V_{\mathcal{A}_{s+1}} = (V_s \cap V_\mathcal{A}) \setminus (V_{s+1} \cap V_\mathcal{A}) = (V_s \cap V_\mathcal{A}) \setminus V_{s+1} = (V_{\mathcal{A}_s} \cup V_{s+1}) \setminus V_{s+1} = V_{\mathcal{A}'_s} \setminus V_{s+1}$$

and hence $|V_{\mathcal{A}_s}| - |V_{\mathcal{A}_{s+1}}| = |V_{\mathcal{A}'_s}| - |V_{s+1}|$. Furthermore, we have $\mathcal{C}_{s+1} \cap \mathcal{A}_s = \mathcal{A}_s[V_{\mathcal{A}_{s+1}}]$ and $\mathcal{A}_s[V_{\mathcal{A}_{s+1}}] \cup \mathcal{C}_{s+1} = \mathcal{A}'_s[V_{s+1}]$, hence

$$\mathcal{A}_s \setminus \mathcal{A}_s[V_{\mathcal{A}_{s+1}}] = \mathcal{A}_s \setminus (\mathcal{A}_s[V_{\mathcal{A}_{s+1}}] \cup \mathcal{C}_{s+1}) = \mathcal{A}'_s \setminus (\mathcal{A}_s[V_{\mathcal{A}_{s+1}}] \cup \mathcal{C}_{s+1}) = \mathcal{A}'_s \setminus \mathcal{A}'_s[V_{s+1}]$$

and thus $|\mathcal{A}_s| - |\mathcal{A}_s[V_{\mathcal{A}_{s+1}}]| = |\mathcal{A}'_s| - |\mathcal{A}'_s[V_{s+1}]|$. In particular, this yields $\rho_{\mathcal{A}_s,V_{\mathcal{A}_{s+1}}} = \rho_{\mathcal{A}'_s,V_{s+1}}$. Since $\mathcal{A} \subseteq \mathcal{C}_\mathfrak{c}$ implies $\mathcal{A}_s \subseteq \mathcal{C}_\mathfrak{c}[V_s] = \mathcal{C}_s$, we have $(\mathcal{A}'_s, V_{s+1}) \subseteq (\mathcal{C}_s, V_{s+1})$. Using Lemma 3.5.3, this entails

$$\rho_{\mathcal{A}_s,V_{\mathcal{A}_{s+1}}} = \rho_{\mathcal{A}'_s,V_{s+1}} \le \rho_{\mathcal{C}_s,V_{s+1}} \le \rho_\mathcal{F} + \varepsilon^2.$$

We conclude that

$$\begin{aligned}|\mathcal{A}| - |\mathcal{A}[V']| &= \sum_{s=0}^{t-1} |\mathcal{A}_s| - |\mathcal{A}_{s+1}| = \sum_{s=0}^{t-1} \rho_{\mathcal{A}_s,V_{\mathcal{A}_{s+1}}}(|V_{\mathcal{A}_s}| - |V_{\mathcal{A}_{s+1}}|)\\ &\le (\rho_\mathcal{F} + \varepsilon^2)\sum_{s=0}^{t-1} |V_{\mathcal{A}_s}| - |V_{\mathcal{A}_{s+1}}| = (\rho_\mathcal{F} + \varepsilon^2)(|V_\mathcal{A}| - |V'|),\end{aligned}$$

which completes the proof. □

Lemma 3.6.14. *Let* $0 \le i \le i^\star$ *and* $\mathcal{X} := \{i < \tau_\mathscr{B} \wedge \tau_{\mathscr{B}'}\}$. *Let* $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$ *and* $e \in \mathcal{C}_\mathfrak{c} \setminus \mathcal{C}_\mathfrak{c}[I]$. *Suppose that* $\mathfrak{c}' = (F', V', I) = \mathfrak{c}|e$. *Let* $\psi\colon V' \hookrightarrow V_\mathcal{H}$. *Then,* $\Phi_{\mathfrak{c},\psi} = (1 \pm \varepsilon^{-6k}\zeta^\delta)\hat{\varphi}_{\mathfrak{c},V'}$.

Proof. Consider an arbitrary $\beta\colon f \overset{\sim}{\to} e$ where $f \in \mathcal{F}$. Suppose that $\mathfrak{c}'' = (F'', V'', I) = \mathfrak{c}|[\beta]$. Furthermore, suppose that $\ell \geq 1$ is minimal with $e \in \mathcal{C}_{\mathfrak{c}|\ell}$ and suppose that

$$\mathfrak{c}'_+ = (F'_+, V'_+, I) := \mathfrak{c}|\ell, \quad \mathfrak{c}''_+ = (F''_+, V''_+, I) := \mathfrak{c}|\ell|\beta.$$

We have

$$\Phi_{\mathfrak{c},\psi} = \sum_{\varphi \in \Phi^{\sim}_{\mathfrak{c}'_+,\psi}} \Phi_{\mathfrak{c},\varphi}. \tag{3.6.10}$$

We use Lemma 3.6.12 to estimate $\Phi_{\mathfrak{c}'_+,\psi}$ and for $\varphi \in \Phi^{\sim}_{\mathfrak{c}'_+,\psi}$, we use Lemma 3.6.11 to estimate $\Phi_{\mathfrak{c},\varphi}$.

Let us turn to the details. First, consider $\Phi_{\mathfrak{c}'_+,\psi}$. Choose an arbitrary injection $\psi'\colon V'' \hookrightarrow V_{\mathcal{H}}$ with $\psi'|_{V'} = \psi$. With Lemma 3.6.12, since $\hat{\varphi}_{\mathfrak{c}''_+,V''} = \hat{\varphi}_{\mathfrak{c}'_+,V'}$, we obtain

$$\begin{aligned}
\Phi_{\mathfrak{c}'_+,\psi} &= \Phi_{\mathfrak{c}''_+,\psi'} + \Phi_{\mathfrak{c}'_+,\psi} - \Phi^{\psi'(V''\setminus V')}_{\mathfrak{c}'_+,\psi} =_{\mathcal{X}} (1 \pm \varepsilon^{-4}\zeta^{\delta})\hat{\varphi}_{\mathfrak{c}''_+,V''} + \Phi_{\mathfrak{c}'_+,\psi} - \Phi^{\psi'(V''\setminus V')}_{\mathfrak{c}'_+,\psi} \\
&= (1 \pm \varepsilon^{-4}\zeta^{\delta})\hat{\varphi}_{\mathfrak{c}'_+,V'} + \Phi_{\mathfrak{c}'_+,\psi} - \Phi^{\psi'(V''\setminus V')}_{\mathfrak{c}'_+,\psi}.
\end{aligned}$$

To bound $\Phi_{\mathfrak{c}'_+,\psi} - \Phi^{\psi'(V''\setminus V')}_{\mathfrak{c}'_+,\psi}$, we employ Lemma 3.6.6 which we may apply as a consequence of Lemma 3.6.13. To this end, recall that in Section 3.6.1, to define the β-extension of $\mathfrak{c}$, we fixed a copy $\mathcal{F}^{\beta}_{\mathfrak{c}}$ of $\mathcal{F}$. For all $(\mathcal{A}, V') \subseteq (\mathcal{C}_{\mathfrak{c}'_+}, V')$ and $\mathcal{A}' := \mathcal{A} + \mathcal{F}^{\beta}_{\mathfrak{c}}$, the template $(\mathcal{A}', V'')$ is a subtemplate of $(\mathcal{C}_{\mathfrak{c}''_+}, V'')$ and we have $\rho_{\mathcal{A},V'} = \rho_{\mathcal{A}',V''}$, so Lemma 3.6.13 entails $\rho_{\mathcal{A},V'} \leq \rho_{\mathcal{F}} + \varepsilon^2$. Hence, we may apply Lemma 3.6.6 to obtain

$$\Phi_{\mathfrak{c}'_+,\psi} - \Phi^{\psi'(V''\setminus V')}_{\mathfrak{c}'_+,\psi} \leq_{\mathcal{X}} \zeta^{3/2}\hat{\varphi}_{\mathfrak{c}'_+,V'}.$$

Thus,

$$\Phi_{\mathfrak{c}'_+,\psi} =_{\mathcal{X}} (1 \pm \varepsilon^{-5}\zeta^{\delta})\hat{\varphi}_{\mathfrak{c}'_+,V'}.$$

Next, fix $\varphi \in \Phi^{\sim}_{\mathfrak{c}'_+,\psi}$ and consider $\Phi_{\mathfrak{c},\varphi}$. Then, Lemma 3.6.11 entails

$$\Phi_{\mathfrak{c},\varphi} =_{\mathcal{X}} (1 \pm \varepsilon^{-5k}\zeta^{\delta})\hat{\varphi}_{\mathfrak{c},V'_+}.$$

Thus, returning to (3.6.10), we conclude that

$$\Phi_{\mathfrak{c},\psi} =_{\mathcal{X}} (1 \pm \varepsilon^{-5}\zeta^{\delta})\hat{\varphi}_{\mathfrak{c}'_+,V'} \cdot (1 \pm \varepsilon^{-5k}\zeta^{\delta})\hat{\varphi}_{\mathfrak{c},V'_+} = (1 \pm \varepsilon^{-6k}\zeta^{\delta})\hat{\varphi}_{\mathfrak{c},V'},$$

which completes the proof. □

Lemma 3.6.15. *Let $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$ and let $e \in \mathcal{C}_{\mathfrak{c}} \setminus \mathcal{C}_{\mathfrak{c}}[I]$. Let $0 \leq i \leq i^{\star}$ and*

$$\mathcal{X} := \{i < \tau_{\mathcal{H}^*} \wedge \tau_{\mathscr{F}} \wedge \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'}\} \cap \{\Phi_{\mathfrak{c},\psi} \leq 2\hat{\varphi}_{\mathfrak{c},I}\} \cap \{\Phi_{\mathfrak{c}|e,\psi} \leq 2\hat{\varphi}_{\mathfrak{c}|e,I}\}.$$

Then,

$$\mathbb{E}_i[|\{\varphi \in \Phi^{\sim}_{\mathfrak{c},\psi} : \varphi(e) \in \mathcal{F}_0(i+1)\}|] =_{\mathcal{X}} \left(\sum_{f \in \mathcal{F}} \sum_{\beta\colon f \overset{\sim}{\to} e} \frac{\Phi_{\mathfrak{c}|[\beta],\psi}\Phi_{\mathfrak{c},\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}} \right) \pm \zeta^{1+\delta/2}\frac{\hat{\varphi}_{\mathfrak{c},I}}{H}.$$

Proof. Lemma 3.5.17 entails

$$\mathbb{E}_i[|\{\varphi \in \Phi^{\sim}_{\mathfrak{c},\psi} : \varphi(e) \in \mathcal{F}_0(i+1)\}|] = \sum_{\varphi \in \Phi^{\sim}_{\mathfrak{c},\psi}} \frac{d_{\mathcal{H}^*}(\varphi(e))}{H^*} = \sum_{f \in \mathcal{F}} \sum_{\beta\colon f \xrightarrow{\sim} e} \frac{\sum_{\varphi \in \Phi^{\sim}_{\mathfrak{c},\psi}} \Phi_{\mathcal{F},\varphi\circ\beta}}{\mathrm{aut}(\mathcal{F}) H^*}. \tag{3.6.11}$$

Suppose that $\mathfrak{c}|e = (F', V', I)$. For $f \in \mathcal{F}$ and $\beta\colon f \xrightarrow{\sim} e$, using Lemma 3.6.14 and the fact that $\Phi_{\mathcal{F},\varphi\circ\beta} =_{\mathcal{X}} (1 \pm \delta^{-1}\zeta)\hat{\varphi}_{\mathcal{F},f}$ holds for all $\varphi \in \Phi^{\sim}_{\mathfrak{c},\psi}$, Lemma 3.5.20 yields

$$\begin{aligned}
\sum_{\varphi \in \Phi^{\sim}_{\mathfrak{c},\psi}} \Phi_{\mathcal{F},\varphi\circ\beta} &= \sum_{\varphi \in \Phi^{\sim}_{\mathfrak{c}|e,\psi}} \Phi_{\mathcal{F},\varphi\circ\beta}\Phi_{\mathfrak{c},\varphi} \\
&=_{\mathcal{X}} \frac{1}{\Phi_{\mathfrak{c}|e,\psi}} \Big(\sum_{\varphi \in \Phi^{\sim}_{\mathfrak{c}|e,\psi}} \Phi_{\mathcal{F},\varphi\circ\beta} \Big)\Big(\sum_{\varphi \in \Phi^{\sim}_{\mathfrak{c}|e,\psi}} \Phi_{\mathfrak{c},\varphi} \Big) \pm \delta^{-2}\zeta^{1+\delta}\hat{\varphi}_{\mathcal{F},f}\hat{\varphi}_{\mathfrak{c},V'}\Phi_{\mathfrak{c}|e,\psi} \\
&= \frac{\Phi_{\mathfrak{c},\psi}}{\Phi_{\mathfrak{c}|e,\psi}} \Big(\sum_{\varphi \in \Phi^{\sim}_{\mathfrak{c}|e,\psi}} \Phi_{\mathcal{F},\varphi\circ\beta} \Big) \pm \delta^{-2}\zeta^{1+\delta}\hat{\varphi}_{\mathcal{F},f}\hat{\varphi}_{\mathfrak{c},V'}\Phi_{\mathfrak{c}|e,\psi}.
\end{aligned}$$

Since Lemma 3.6.6 entails

$$\sum_{\varphi \in \Phi^{\sim}_{\mathfrak{c}|e,\psi}} \Phi_{\mathcal{F},\varphi\circ\beta} =_{\mathcal{X}} \sum_{\varphi \in \Phi^{\sim}_{\mathfrak{c}|e,\psi}} (\Phi^{\varphi(V'\setminus e)}_{\mathcal{F},\varphi\circ\beta} \pm \zeta^{3/2}\hat{\varphi}_{\mathcal{F},f}) = \Phi_{\mathfrak{c}|[\beta],\psi} \pm \zeta^{3/2}\hat{\varphi}_{\mathcal{F},f}\Phi_{\mathfrak{c}|e,\psi},$$

we conclude that

$$\begin{aligned}
\sum_{\varphi \in \Phi^{\sim}_{\mathfrak{c},\psi}} \Phi_{\mathcal{F},\psi\circ\beta} &=_{\mathcal{X}} \frac{\Phi_{\mathfrak{c},\psi}\Phi_{\mathfrak{c}|[\beta],\psi}}{\Phi_{\mathfrak{c}|e,\psi}} \pm \delta^{-2}\zeta^{1+\delta}\hat{\varphi}_{\mathcal{F},f}\hat{\varphi}_{\mathfrak{c},V'}\Phi_{\mathfrak{c}|e,\psi} \pm \zeta^{3/2}\hat{\varphi}_{\mathcal{F},f}\Phi_{\mathfrak{c},\psi} \\
&=_{\mathcal{X}} \frac{\Phi_{\mathfrak{c},\psi}\Phi_{\mathfrak{c}|[\beta],\psi}}{\Phi_{\mathfrak{c}|e,\psi}} \pm \delta^{-3}\zeta^{1+\delta}\hat{\varphi}_{\mathcal{F},f}\hat{\varphi}_{\mathfrak{c},V'}\hat{\varphi}_{\mathfrak{c}|e,I} \pm \zeta^{4/3}\hat{\varphi}_{\mathcal{F},f}\hat{\varphi}_{\mathfrak{c},I} \\
&= \frac{\Phi_{\mathfrak{c},\psi}\Phi_{\mathfrak{c}|[\beta],\psi}}{\Phi_{\mathfrak{c}|e,\psi}} \pm \delta^{-4}\zeta^{1+\delta}\hat{\varphi}_{\mathcal{F},f}\hat{\varphi}_{\mathfrak{c},I}.
\end{aligned}$$

Combining this with (3.6.11), we obtain

$$\begin{aligned}
&\mathbb{E}_i[|\{\varphi \in \Phi^{\sim}_{\mathfrak{c},\psi} : \varphi(e) \in \mathcal{F}_0(i+1)\}|] \\
&\qquad =_{\mathcal{X}} \left(\sum_{f \in \mathcal{F}} \sum_{\beta\colon f \xrightarrow{\sim} e} \frac{\Phi_{\mathfrak{c}|[\beta],\psi}\Phi_{\mathfrak{c},\psi}}{\mathrm{aut}(\mathcal{F}) H^* \Phi_{\mathfrak{c}|e,\psi}} \right) \pm \frac{|\mathcal{F}| k!\, \zeta^{1+\delta}\hat{\varphi}_{\mathcal{F},f}\hat{\varphi}_{\mathfrak{c},I}}{\delta^4\, \mathrm{aut}(\mathcal{F}) H^*}.
\end{aligned}$$

Since Lemma 3.5.7 yields

$$\frac{|\mathcal{F}| k!\, \zeta^{1+\delta}\hat{\varphi}_{\mathcal{F},f}\hat{\varphi}_{\mathfrak{c},I}}{\delta^4\, \mathrm{aut}(\mathcal{F}) H^*} \leq_{\mathcal{X}} \frac{|\mathcal{F}| k!\, \zeta^{1+\delta}\hat{\varphi}_{\mathcal{F},f}\hat{\varphi}_{\mathfrak{c},I}}{\delta^5\, \mathrm{aut}(\mathcal{F}) \hat{h}^*} = \frac{|\mathcal{F}| k!\, \zeta^{1+\delta}\hat{\varphi}_{\mathfrak{c},I}}{\delta^5 n^k \hat{p}} \leq_{\mathcal{X}} \zeta^{1+\delta/2}\frac{\hat{\varphi}_{\mathfrak{c},I}}{H},$$

this completes the proof. □

3.6.3 Tracking chains

Suppose that $0 \leq i \leq i^\star$, consider a chain $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$ with $F = \mathcal{F}_1, \ldots, \mathcal{F}_\ell$ and let $\psi\colon I \hookrightarrow V_\mathcal{H}$. We do not directly show that the number of embeddings $\Phi_{\mathfrak{c},\psi}$ is typically close to a deterministic trajectory. Instead, we define

$$\mathcal{G}_\mathfrak{c} := \mathcal{F}_\ell[V \cap V_{\mathcal{F}_\ell}] \quad \text{and} \quad J_\mathfrak{c} := \begin{cases} I & \text{if } \ell = 1; \\ V_{\mathcal{F}_{\ell-1}} \cap V_{\mathcal{G}_\mathfrak{c}} & \text{if } \ell \geq 2 \end{cases}$$

and show that $\Phi_{\mathfrak{c},\psi}$ is typically close to $\hat{\varphi}_{\mathcal{G}_\mathfrak{c},J_\mathfrak{c}} \Phi_{\mathfrak{c}|-,\psi}$ which given $\Phi_{\mathfrak{c}|-,\psi}$ is the random quantity our deterministic heuristic estimates for embeddings suggest for

$$\sum_{\varphi \in \Phi^{\sim}_{\mathfrak{c}|-,\psi}} \Phi_{\mathcal{G}_\mathfrak{c},\varphi|_{J_\mathfrak{c}}} \approx \Phi_{\mathfrak{c},\psi}.$$

To this end, let

$$\hat{\Phi}_{\mathfrak{c},\psi}(i) := \hat{\varphi}_{\mathcal{G}_\mathfrak{c},J_\mathfrak{c}} \Phi_{\mathcal{C}_{\mathfrak{c}|-},\psi} \quad \text{and} \quad X_{\mathfrak{c},\psi}(i) := \Phi_{\mathcal{C}_\mathfrak{c},\psi} - \hat{\Phi}_{\mathfrak{c},\psi}.$$

Our analysis of $\Phi_{\mathfrak{c},\psi}$ crucially relies on Lemma 3.6.15. There, a sum of numbers of embeddings of branchings of $\mathfrak{c}$ is a key quantity which motivates the following definition. For $e \in \mathcal{C}_\mathfrak{c} \setminus \mathcal{C}_\mathfrak{c}[I]$, the *$e$-branching family* of $\mathfrak{c}$ is

$$\mathfrak{B}^e_\mathfrak{c} := \{\mathfrak{b} : \mathfrak{b} \text{ is the } \beta\text{-branching of } \mathfrak{c} \text{ for some } \beta\colon f \xrightarrow{\sim} e \text{ where } f \in \mathcal{F}\}.$$

We define the stopping times

$$\tau_\mathfrak{C} := \min\{i \geq 0 : \Phi_{\mathfrak{c},\psi} \neq \hat{\Phi}_{\mathfrak{c},\psi} \pm \delta^{-1}\zeta\hat{\varphi}_{\mathfrak{c},I} \text{ for some } \mathfrak{c} = (F, V, I) \in \mathfrak{C}, \psi\colon I \hookrightarrow V_\mathcal{H}\},$$
$$\tilde{\tau}_\mathfrak{B} := \min\left\{\begin{matrix} i \geq 0 : \sum_{\mathfrak{b}\in\mathfrak{B}^e_\mathfrak{c}} \Phi_{\mathfrak{b},\psi} \neq \sum_{\mathfrak{b}\in\mathfrak{B}^e_\mathfrak{c}} \hat{\Phi}_{\mathfrak{b},\psi} \pm \delta^{-1/2}\zeta\hat{\varphi}_{\mathfrak{b},I} \\ \text{for some } \mathfrak{c} = (F, V, I) \in \mathfrak{C}, e \in \mathcal{C}_\mathfrak{c} \setminus \mathcal{C}_\mathfrak{c}[I], \psi\colon I \hookrightarrow V_\mathcal{H} \end{matrix}\right\}.$$

The stopping time $\tau_\mathfrak{C}$ is the fourth stopping time mentioned in Section 3.4. Similarly as with the introduction of the stopping time $\tau_\mathscr{F} \geq \tau_\mathfrak{C}$ in Section 3.5, the precise definition of $\tau_\mathfrak{B}$ is not relevant in this section, so we instead work with the stopping time $\tilde{\tau}_\mathfrak{B}$ that satisfies $\tilde{\tau}_\mathfrak{B} \geq \tau_\mathfrak{B}$. We set

$$\tilde{\tau}^\star_\mathfrak{C} := \tau_{\mathcal{H}^*} \wedge \tau_\mathscr{B} \wedge \tau_{\mathscr{B}'} \wedge \tau_\mathfrak{C} \wedge \tilde{\tau}_\mathfrak{B} \geq \tau^\star.$$

We remark that whenever the aforementioned numbers of embeddings are close to their corresponding random trajectories, they are also close to a corresponding deterministic trajectory in the following sense.

Lemma 3.6.16. *Let $i \geq 0$ and $\mathcal{X} := \{i < \tau_\mathfrak{C}\}$. Let $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$ and $\psi\colon I \hookrightarrow V_\mathcal{H}$. Then, $\Phi_{\mathfrak{c},\psi} =_\mathcal{X} (1 \pm \delta^5)\hat{\varphi}_{\mathfrak{c},I}$.*

Proof. Similarly as in the proof of Lemma 3.6.11, for every chain $\mathfrak{c}' = (F', V', I)$ where F' has length ℓ', let

$$g_{\mathfrak{c}'} := |\{0 \leq \ell'' \leq \ell' - 1 : \mathcal{C}_{\mathfrak{c}'|\ell''} \neq \mathcal{C}_{\mathfrak{c}'|\ell''+1}\}|.$$

Suppose that $F = \mathcal{F}_1, \ldots, \mathcal{F}_\ell$. We use induction on ℓ to show that

$$\Phi_{\mathfrak{c},\psi} =_{\mathcal{X}} (1 \pm g_{\mathfrak{c}}\delta^{-1}\zeta)\hat{\varphi}_{\mathfrak{c},I}. \tag{3.6.12}$$

By Lemma 3.6.3, we have $|V| \leq \varepsilon^{-3}$, hence $|\mathcal{C}_{\mathfrak{c}}| \leq \varepsilon^{-3k}$ and thus $g_{\mathfrak{c}} \leq \varepsilon^{-3} + \varepsilon^{-3k}$, so this is sufficient.

Let us proceed with the proof by induction. If $\ell = 1$, then $g_{\mathfrak{c}} = 1$ by Lemma 3.6.5 and we have $\Phi_{\mathfrak{c},\psi} =_{\mathcal{X}} (1 \pm \delta^{-1}\zeta)\hat{\varphi}_{\mathfrak{c},I}$. Let $q \geq 2$ and suppose that (3.6.12) holds if $\ell \leq q - 1$. Suppose that $\ell = q$. If $\mathcal{C}_{\mathfrak{c}|-} = \mathcal{C}_{\mathfrak{c}}$, then (3.6.12) follows by induction hypothesis, so we may assume $\mathcal{C}_{\mathfrak{c}|-} \neq \mathcal{C}_{\mathfrak{c}}$ and hence $g_{\mathfrak{c}|-} = g_{\mathfrak{c}} - 1$. Then, by induction hypothesis we have

$$\Phi_{\mathfrak{c}|-,\psi} =_{\mathcal{X}} (1 \pm g_{\mathfrak{c}|-}\delta^{-1}\zeta)\hat{\varphi}_{\mathfrak{c}|-,I} = (1 \pm (g_{\mathfrak{c}} - 1)\delta^{-1}\zeta)\hat{\varphi}_{\mathfrak{c}|-,I}.$$

Since $\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}\hat{\varphi}_{\mathfrak{c}|-,I} = \hat{\varphi}_{\mathfrak{c},I}$, this yields

$$\begin{aligned}\Phi_{\mathfrak{c},\psi} &=_{\mathcal{X}} \hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}\Phi_{\mathfrak{c}|-,\psi} \pm \delta^{-1}\zeta\hat{\varphi}_{\mathfrak{c},I} =_{\mathcal{X}} (1 \pm (g_{\mathfrak{c}} - 1)\delta^{-1}\zeta)\hat{\varphi}_{\mathfrak{c},I} \pm \delta^{-1}\zeta\hat{\varphi}_{\mathfrak{c},I} \\ &= (1 \pm g_{\mathfrak{c}}\delta^{-1}\zeta)\hat{\varphi}_{\mathfrak{c},I},\end{aligned}$$

which completes the proof. □

In this section, we show that the probability that $\tau_{\mathfrak{C}} \leq \tilde{\tau}^{\star}_{\mathfrak{C}} \wedge i^{\star}$ is small. The collection $\mathfrak{C}$ is infinite, however, Lemma 3.6.17 shows that it suffices to consider a collection of chains of size at most $1/\delta$. By relying on a union bound argument, this allows us to essentially only consider one fixed chain $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$.

Lemma 3.6.17. *There exists a collection $\mathfrak{C}_0 \subseteq \mathfrak{C}$ with $|\mathfrak{C}_0| \leq 1/\delta$ such that for all $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$, there exists a chain $\mathfrak{c}_0 = (F_0, V_0, I_0) \in \mathfrak{C}_0$ such that $(\mathcal{C}_{\mathfrak{c}_0}, I_0)$ is a copy of $(\mathcal{C}_{\mathfrak{c}}, I)$ while $(\mathcal{C}_{\mathfrak{c}_0|-}, I_0)$ is a copy of $(\mathcal{C}_{\mathfrak{c}|-}, I)$.*

Proof. Consider the set $\mathscr{T}$ of all templates $(\mathcal{A}, I)$ where $V_{\mathcal{A}} \subseteq \{1, \ldots, 1/\varepsilon^3\}$. By Lemma 3.6.3, for all $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$, we may choose a template $\mathcal{T}_{\mathfrak{c}} \in \mathscr{T}$ that is a copy of $(\mathcal{C}_{\mathfrak{c}}, I)$. Let $\mathscr{T}_2 := \{(\mathcal{T}_{\mathfrak{c}}, \mathcal{T}_{\mathfrak{c}|-}) : \mathfrak{c} \in \mathfrak{C}\} \subseteq \mathscr{T}^2$ and for every pair $\mathscr{P} \in \mathscr{T}_2$, choose a chain $\mathfrak{c}_{\mathscr{P}} \in \mathfrak{C}$ with $\mathscr{P} = (\mathcal{T}_{\mathfrak{c}_{\mathscr{P}}}, \mathcal{T}_{\mathfrak{c}_{\mathscr{P}}|-})$. Then, $\{\mathfrak{c}_{\mathscr{P}} : \mathscr{P} \in \mathscr{T}_2\}$ is a collection as desired. □

Observation 3.6.18. *Suppose that $\mathfrak{C}_0 \subseteq \mathfrak{C}$ is a collection of chains as in Lemma 3.6.17. For $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$ and $\psi\colon I \hookrightarrow V_{\mathcal{H}}$, let*

$$\tau_{\mathfrak{c},\psi} := \min\{i \geq 0 : \Phi_{\mathfrak{c},\psi} \neq \hat{\Phi}_{\mathfrak{c},\psi} \pm \delta^{-1}\zeta\hat{\varphi}_{\mathfrak{c},I}\}.$$

Then,

$$\mathbb{P}[\tau_{\mathfrak{C}} \leq \tilde{\tau}^{\star}_{\mathfrak{C}} \wedge i^{\star}] \leq \sum_{\mathfrak{c}=(F,V,I)\in\mathfrak{C}_0, \psi\colon I\hookrightarrow V_{\mathcal{H}}:} \mathbb{P}[\tau_{\mathfrak{c},\psi} \leq \tilde{\tau}^{\star}_{\mathfrak{C}} \wedge i^{\star}].$$

Hence, fix $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$ where $F = \mathcal{F}_1, \dots, \mathcal{F}_\ell$ and furthermore fix $\psi \colon I \hookrightarrow V_{\mathcal{H}}$. Note that by Lemma 3.6.5, we have $\mathcal{C}_{\mathfrak{c}} \setminus \mathcal{C}_{\mathfrak{c}}[I] \neq \emptyset$ and $\ell \geq 1$. For $i \geq 0$, let $\xi_1(i)$ denote the corresponding absolute error appearing in the definition of $\tau_{\mathfrak{C}}$ and consider a slightly smaller error term $\xi_0(i)$, that is let

$$\xi_1(i) := \delta^{-1}\zeta\hat{\varphi}_{\mathfrak{c},I} \quad \text{and} \quad \xi_0(i) := (1-\delta)\xi_1(i)$$

and define the stopping time

$$\tau := \min\{i \geq 0 : \Phi_{\mathfrak{c},\psi} \neq \hat{\Phi}_{\mathfrak{c},\psi} \pm \xi_1\}.$$

Our goal is now to show that $\Phi_{\mathfrak{c},\psi}$ is typically in the interval $I_1(i) := [\hat{\Phi}_{\mathfrak{c},\psi} - \xi_1, \hat{\Phi}_{\mathfrak{c},\psi} + \xi_1]$ as long as other key quantities are as predicted. More formally, our goal is to show that the probability that $\tau \leq \tilde{\tau}_{\mathfrak{C}}^{\star} \wedge i^{\star}$ is sufficiently small. Define the "critical" intervals

$$I^-(i) := [\hat{\Phi}_{\mathfrak{c},\psi} - \xi_1, \hat{\Phi}_{\mathfrak{c},\psi} - \xi_0], \quad I^+(i) := [\hat{\Phi}_{\mathfrak{c},\psi} + \xi_0, \hat{\Phi}_{\mathfrak{c},\psi} + \xi_1].$$

As long as $\Phi_{\mathfrak{c},\psi}$ is not close to the boundary of I_1 in the sense that $\Phi_{\mathfrak{c},\psi}$ is in the interval $I_0(i) := [\hat{\Phi}_{\mathfrak{c},\psi} - \xi_0, \hat{\Phi}_{\mathfrak{c},\psi} + \xi_0]$, within the next few steps i, there is no danger that $\Phi_{\mathfrak{c},\psi}$ could be outside I_1 provided that we chose ξ_1 to be sufficiently large compared to ξ_0. The situation only becomes "critical" when $\Phi_{\mathfrak{c},\psi}$ is outside I_0, that is when $\Phi_{\mathfrak{c},\psi}$ enters the critical interval I^- or I^+. Exploiting the fact that whenever this is the case, the process exhibits self-correcting behavior in the sense that whenever this is the case, in expectation $\Phi_{\mathfrak{c},\psi}$ returns to values close to $\hat{\Phi}_{\mathfrak{c},\psi}$, we show that it is unlikely that $\Phi_{\mathfrak{c},\psi}$ ever fully crosses one of the critical intervals. Since, as we formally show later, $\Phi_{\mathfrak{c},\psi}$ cannot jump over one of the critical intervals in one step, it suffices to restrict our attention to the behavior of $\Phi_{\mathfrak{c},\psi}$ inside the critical intervals.

For $\divideontimes \in \{-, +\}$, consider the random variable

$$Y^{\divideontimes}(i) := \divideontimes X_{\mathfrak{c},\psi} - \xi_1$$

that measures by how much $\Phi_{\mathfrak{c},\psi}$ exceeds the permitted deviation ξ_1 from $\hat{\Phi}_{\mathfrak{c},\psi}$. Our goal is to show that $Y^{\divideontimes}$ is non-positive whenever $i \leq \tilde{\tau}_{\mathfrak{C}}^{\star}$. To show that this is the case, for all $i_0 \geq 0$, we consider an auxiliary random process $Z_{i_0}^{\divideontimes}(i_0), Z_{i_0}^{\divideontimes}(i_0+1), \dots$ that follows the evolution of $Y^{\divideontimes}(i_0), Y^{\divideontimes}(i_0+1), \dots$ as long as the situation is relevant for our analysis, that is until $\Phi_{\mathfrak{c},\psi}$ has left the critical interval $I^{\divideontimes}$ or until we are at step $\tilde{\tau}_{\mathfrak{C}}^{\star} \wedge i^{\star}$. In these cases, that is when $Z_{i_0}^{\divideontimes}$ no longer follows $Y^{\divideontimes}$, we simply define the auxiliary process to remain constant. Note in particular, that if a deviation of $\Phi_{\mathfrak{c},\psi}$ from $\hat{\Phi}_{\mathfrak{c},\psi}$ beyond ξ_1 caused the auxiliary process to no longer follows $Y^{\divideontimes}$, then the value of the auxiliary process at step $i^{\star}$ indicates this since the relevant value $Y^{\divideontimes}(\tilde{\tau}_{\mathfrak{C}}^{\star} \wedge i^{\star})$ is the last value captured. Formally, for $i_0 \geq 0$, we define the stopping time

$$\tau_{i_0}^{\divideontimes} := \min\{i \geq i_0 : \Phi_{\mathfrak{c},\psi} \notin I^{\divideontimes}\}$$

that measures when, starting at step i_0, the random variable $\Phi_{\mathfrak{c},\psi}$ is first outside the critical interval $I^{\divideontimes}$. Note that if $\Phi_{\mathfrak{c},\psi}(i_0) \notin I^{\divideontimes}$, then $\tau_{i_0}^{\divideontimes} = i_0$. For $i \geq i_0$, let

$$Z_{i_0}^{\divideontimes}(i) := Y^{\divideontimes}(i_0 \vee (i \wedge \tau_{i_0}^{\divideontimes} \wedge \tilde{\tau}_{\mathfrak{C}}^{\star} \wedge i^{\star})).$$

In fact, for our analysis it suffices to consider only the evolution of $Z^{\divideontimes}_{\sigma^{\divideontimes}}(\sigma^{\divideontimes}), Z^{\divideontimes}_{\sigma^{\divideontimes}}(\sigma^{\divideontimes}+1),\dots$ where

$$\sigma^{\divideontimes} := \min\{j \geq 0 : \divideontimes X_{\mathfrak{c},\psi} \geq \xi_0 \text{ for all } j \leq i < \tilde{\tau}^{\star}_{\mathfrak{C}} \wedge i^{\star}\} \leq \tilde{\tau}^{\star}_{\mathfrak{C}} \wedge i^{\star}$$

is the last step at which $\Phi_{\mathfrak{c},\psi}$ entered the critical interval $I^{\divideontimes}$ before step $\tilde{\tau}^{\star}_{\mathfrak{C}} \wedge i^{\star}$. Indeed, if $\tau \leq \tilde{\tau}^{\star}_{\mathfrak{C}} \wedge i^{\star}$, then, for some $\divideontimes \in \{+,-\}$, we have $\Phi_{\mathfrak{c},\psi} \in I^{\divideontimes}$ for all $\sigma^{\divideontimes} \leq i < \tilde{\tau}^{\star}_{\mathfrak{C}} \wedge i^{\star}$, hence $\tau^{\divideontimes}_{\sigma^{\divideontimes}} = \tilde{\tau}^{\star}_{\mathfrak{C}} \wedge i^{\star}$ and thus $Z^{\divideontimes}_{\sigma^{\divideontimes}}(i^{\star}) = Y^{\divideontimes}(\tilde{\tau}^{\star}_{\mathfrak{C}} \wedge i^{\star}) = Y^{\divideontimes}(\tau) > 0$. This reasoning leads to the following observation.

Observation 3.6.19. $\{\tau \leq \tilde{\tau}^{\star}_{\mathfrak{C}} \wedge i^{\star}\} \subseteq \{Z^{-}_{\sigma^{-}}(i^{\star}) > 0\} \cup \{Z^{+}_{\sigma^{+}}(i^{\star}) > 0\}$.

Similarly as in Chapter 2, we use Freedman's inequality for supermartingales to show that the probabilities of the events on the right in Observation 3.6.19 are sufficiently small.

We dedicate Sections 3.6.3 and 3.6.3 to proving that the auxiliary random processes satisfy the conditions that are necessary for an application of Lemma 2.9.4. The application itself is the topic of Section 3.6.3.

Trend

Here, we prove that for all $\divideontimes \in \{-,+\}$ and $i_0 \geq 0$, the expected one-step changes of the process $Z^{\divideontimes}_{i_0}(i_0), Z^{\divideontimes}_{i_0}(i_0+1),\dots$ are non-positive. In Lemma 3.6.21, we estimate the one-step changes of the error term that we use in this section. Then in Lemma 3.6.23, we state a precise estimate for the expected one-step change of the random process $X_{\mathfrak{c},\psi}(0), X_{\mathfrak{c},\psi}(1),\dots$ that measures the deviations from the random trajectory given by $\hat{\Phi}_{\mathfrak{c},\psi}(0), \hat{\Phi}_{\mathfrak{c},\psi}(1),\dots$. To obtain this precise estimate, which is the key argument in this section, we crucially rely on Lemma 3.6.15 and the even more precise control over branching families that we have in step i whenever $i < \tau_{\mathfrak{B}}$. Assuming such control over branching families in our arguments here serves to shift the main arguments based on the exploitation of self-correcting behavior to a slightly different setting, namely from individual chains to families, which turns out to be crucial for our argumentation (see Section 3.7). At the end of this section, we combine the previously collected estimates to conclude that $Z^{\divideontimes}_{i_0}(i_0), Z^{\divideontimes}_{i_0}(i_0+1),\dots$ is indeed a supermartingale for all $\divideontimes \in \{-,+\}$ and $i_0 \geq 0$ (see Lemma 3.6.24).

Observation 3.6.20. *Extend $\hat{p}$ and ξ_1 to continuous trajectories defined on the whole interval $[0, i^{\star}+1]$ using the same expression as above. Then, for $x \in [0, i^{\star}+1]$,*

$$\xi_1'(x) = -\left(|\mathcal{C}_{\mathfrak{c}}| - 1 - \frac{\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|k!\,\xi_1(x)}{n^k \hat{p}(x)},$$
$$\xi_1''(x) = -\left(|\mathcal{C}_{\mathfrak{c}}| - 1 - \frac{\rho_{\mathcal{F}}}{2}\right)\left(|\mathcal{C}_{\mathfrak{c}}| - 2 - \frac{\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|^2(k!)^2\xi_1(x)}{n^{2k}\hat{p}(x)^2}.$$

Lemma 3.6.21. *Let $0 \leq i \leq i^{\star}$ and $\mathcal{X} := \{i \leq \tau_{\emptyset}\}$. Then,*

$$\Delta\xi_1 =_{\mathcal{X}} -\left(|\mathcal{C}_{\mathfrak{c}}| - 1 - \frac{\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|\xi_1}{H} \pm \frac{\zeta^2\xi_1}{H}.$$

Proof. This is a consequence of Taylor's theorem. In detail, we argue as follows.

Together with Observation 3.6.20 and Lemma 3.6.3, Lemma 2.9.10 yields

$$\Delta\xi_1 = -\left(|\mathcal{C}_{\mathfrak{c}}| - 1 - \frac{\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|k!\,\xi_1}{n^k\hat{p}} \pm \max_{x\in[i,i+1]}\frac{\xi_1(x)}{\delta n^{2k}\hat{p}(x)^2}$$

We investigate the first term and the maximum separately. Using Lemma 3.5.7, we have

$$-\left(|\mathcal{C}_{\mathfrak{c}}| - 1 - \frac{\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|k!\,\xi_1}{n^k\hat{p}} =_{\varkappa} -\left(|\mathcal{C}_{\mathfrak{c}}| - 1 - \frac{\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|\xi_1}{H}.$$

Furthermore, using Lemma 3.5.6, Lemma 3.5.7 and Lemma 3.5.9 yields

$$\max_{x\in[i,i+1]}\frac{\xi_1(x)}{\delta n^{2k}\hat{p}(x)^2} \le \frac{\xi_1}{\delta n^{2k}\hat{p}(i+1)^2} \le \frac{\xi_1}{\delta^2 n^{2k}\hat{p}^2} \le_{\varkappa} \frac{\xi_1}{\delta^2 H^2} \le_{\varkappa} \frac{\zeta^{2+2\varepsilon^2}\xi_1}{\delta^2 H} \le \frac{\zeta^{2+\varepsilon^2}\xi_1}{H}.$$

Thus we obtain the desired expression for $\Delta\xi_1$. □

Lemma 3.6.22. *For all* $0 \le i \le i^\star$*, we have*

$$\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}(i+1) = (1\pm\zeta^2)\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}.$$

Proof. This follows from Lemma 3.5.11 and Lemma 3.5.9. □

In the next lemma, we state the expression for the expected one-step change $\mathbb{E}_i[\Delta X_{\mathfrak{c},\psi}]$ that we subsequently use to obtain the desired supermartingale property. In the proof, ignoring error terms, we essentially argue as follows. We have

$$\begin{aligned}\mathbb{E}_i[\Delta X_{\mathfrak{c},\psi}] &= \mathbb{E}_i[\Delta\Phi_{\mathfrak{c},\psi}] - \mathbb{E}_i[\Delta(\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}\Phi_{\mathfrak{c}|-,\psi})]\\ &= \mathbb{E}_i[\Delta\Phi_{\mathfrak{c},\psi}] - (\Delta\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}})\Phi_{\mathfrak{c}|-,\psi} - \hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}(i+1)\mathbb{E}_i[\Delta\Phi_{\mathfrak{c}|-,\psi}].\end{aligned} \tag{3.6.13}$$

Since $\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}(i+1) \approx \hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}$, this yields

$$\mathbb{E}_i[\Delta X_{\mathfrak{c},\psi}] \approx \mathbb{E}_i[\Delta\Phi_{\mathfrak{c},\psi}] - (\Delta\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}})\Phi_{\mathfrak{c}|-,\psi} - \hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}\mathbb{E}_i[\Delta\Phi_{\mathfrak{c}|-,\psi}]. \tag{3.6.14}$$

Contributions to $\Delta\Phi_{\mathfrak{c},\psi}$ come from the loss of edges $\varphi(e)$ where $\varphi \in \Phi^{\sim}_{\mathfrak{c},\psi}$ and $e \in \mathcal{C}_{\mathfrak{c}}\setminus\mathcal{C}_{\mathfrak{c}}[I]$. Note that if $e \in \mathcal{C}_{\mathfrak{c}|-}$, then for this loss of $\varphi(e)$, there is a corresponding contribution to $\Delta\Phi_{\mathfrak{c}|-,\psi}$. Otherwise, there is no corresponding contribution to $\Delta\Phi_{\mathfrak{c}|-,\psi}$, however, we find a corresponding contribution in

$$(\Delta\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}})\Phi_{\mathfrak{c}|-,\psi} \approx -|\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}|\frac{|\mathcal{F}|}{H}\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}\Phi_{\mathfrak{c}|-,\psi} = -|\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}|\frac{|\mathcal{F}|}{H}\hat{\Phi}_{\mathfrak{c},\psi}.$$

With this in mind, relying on Lemma 3.6.15 and Lemma 3.5.7, for $f \in \mathcal{F}$, we estimate

$$\begin{aligned}\mathbb{E}_i[\Delta\Phi_{\mathfrak{c},\psi}] &\approx -\sum_{e\in\mathcal{C}_{\mathfrak{c}}\setminus\mathcal{C}_{\mathfrak{c}}[I]}\sum_{f\in\mathcal{F}}\sum_{\beta\colon f\xrightarrow{\sim} e}\frac{\Phi_{\mathfrak{c}|[\beta],\psi}\Phi_{\mathfrak{c},\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}}\\ &\approx -\sum_{e\in\mathcal{C}_{\mathfrak{c}|-}\setminus\mathcal{C}_{\mathfrak{c}|-}[I]}\sum_{f\in\mathcal{F}}\sum_{\beta\colon f\xrightarrow{\sim} e}\frac{\hat{\varphi}_{\mathcal{F},f}\Phi_{\mathfrak{c},\psi}}{\mathrm{aut}(\mathcal{F})\hat{h}^*} - \sum_{e\in\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}}\sum_{f\in\mathcal{F}}\sum_{\beta\colon f\xrightarrow{\sim} e}\frac{\Phi_{\mathfrak{c},\psi}\Phi_{\mathfrak{c}|[\beta],\psi}}{\mathrm{aut}(\mathcal{F})\hat{h}^*\Phi_{\mathfrak{c}|e,\psi}}\\ &\approx -\frac{(|\mathcal{C}_{\mathfrak{c}|-}|-1)|\mathcal{F}|}{H}\Phi_{\mathfrak{c},\psi} - \sum_{e\in\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}}\sum_{f\in\mathcal{F}}\sum_{\beta\colon f\xrightarrow{\sim} e}\frac{\Phi_{\mathfrak{c},\psi}\Phi_{\mathfrak{c}|[\beta],\psi}}{k!\,H\hat{\varphi}_{\mathcal{F},f}\Phi_{\mathfrak{c}|e,\psi}}\end{aligned}$$

and similarly

$$\mathbb{E}_i[\Delta\Phi_{\mathfrak{c},\psi}] \approx -\sum_{e\in\mathcal{C}_{\mathfrak{c}|-}\setminus\mathcal{C}_{\mathfrak{c}|-}[I]}\sum_{f\in\mathcal{F}}\sum_{\beta\colon f\overset{\sim}{\rightarrow}e}\frac{\Phi_{\mathfrak{c}|[\beta],\psi}\Phi_{\mathfrak{c}|-,\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}} \approx -\frac{(|\mathcal{C}_{\mathfrak{c}|-}|-1)|\mathcal{F}|}{H}\Phi_{\mathfrak{c}|-,\psi}.$$

Combining the previous three estimates with (3.6.14), we obtain

$$\mathbb{E}_i[\Delta X_{\mathfrak{c},\psi}] \approx -\frac{(|\mathcal{C}_{\mathfrak{c}|-}|-1)|\mathcal{F}|}{H}X_{\mathfrak{c},\psi} - \sum_{e\in\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}}\sum_{f\in\mathcal{F}}\sum_{\beta\colon f\overset{\sim}{\rightarrow}e}\left(\frac{\Phi_{\mathfrak{c},\psi}\Phi_{\mathfrak{c}|[\beta],\psi}}{k!\,H\hat{\varphi}_{\mathcal{F},f}\Phi_{\mathfrak{c}|e,\psi}} - \frac{\hat{\Phi}_{\mathfrak{c},\psi}}{k!\,H}\right).$$

Let us investigate the innermost sum on the right. The branchings of $\mathfrak{c}$ are two extension transformations away from the chain $\mathfrak{c}|-$ that appears in the corresponding contributions. As our chain tracking only compares chains that are one extension step apart, we introduce the chain $\mathfrak{c}$ itself to compare the contributions in the sense that for $e\in\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}$, $f\in\mathcal{F}$ and $\beta\colon f\overset{\sim}{\rightarrow}e$, we write

$$\begin{aligned}
\frac{\Phi_{\mathfrak{c},\psi}\Phi_{\mathfrak{c}|[\beta],\psi}}{k!\,H\hat{\varphi}_{\mathcal{F},f}\Phi_{\mathfrak{c}|e,\psi}} - \frac{\hat{\Phi}_{\mathfrak{c},\psi}}{k!\,H} &= \frac{\Phi_{\mathfrak{c},\psi}X_{\mathfrak{c}|[\beta],\psi}}{k!\,H\hat{\varphi}_{\mathcal{F},f}\Phi_{\mathfrak{c}|e,\psi}} + \frac{\Phi_{\mathfrak{c},\psi}}{k!\,H} - \frac{\hat{\Phi}_{\mathfrak{c},\psi}}{k!\,H}\\
&= \frac{\Phi_{\mathfrak{c},\psi}}{k!\,H\hat{\varphi}_{\mathcal{F},f}\Phi_{\mathfrak{c}|e,\psi}}X_{\mathfrak{c}|[\beta],\psi} + \frac{1}{k!\,H}X_{\mathfrak{c},\psi}\\
&\approx \frac{\hat{\varphi}_{\mathfrak{c},I}}{k!\,H\hat{\varphi}_{\mathfrak{c}|[\beta],I}}X_{\mathfrak{c}|[\beta],\psi} + \frac{1}{k!\,H}X_{\mathfrak{c},\psi}
\end{aligned}$$

Overall, this leads to

$$\begin{aligned}
\mathbb{E}_i[\Delta X_{\mathfrak{c},\psi}] &\approx -\frac{(|\mathcal{C}_{\mathfrak{c}|-}|-1)|\mathcal{F}|}{H}X_{\mathfrak{c},\psi} - \frac{|\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}||\mathcal{F}|}{H}X_{\mathfrak{c},\psi} - \sum_{e\in\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}}\sum_{\mathfrak{b}\in\mathfrak{B}_{\mathfrak{c}}^e}\frac{\hat{\varphi}_{\mathfrak{c},I}}{k!\,H\hat{\varphi}_{\mathfrak{b},I}}X_{\mathfrak{b},\psi}\\
&= -\frac{(|\mathcal{C}_{\mathfrak{c}}|-1)|\mathcal{F}|}{H}X_{\mathfrak{c},\psi} - \sum_{e\in\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}}\sum_{\mathfrak{b}\in\mathfrak{B}_{\mathfrak{c}}^e}\frac{\hat{\varphi}_{\mathfrak{c},I}}{k!\,H\hat{\varphi}_{\mathfrak{b},I}}X_{\mathfrak{b},\psi}.
\end{aligned}$$

Lemma 3.6.23. *Let $0\le i\le i^\star$ and $\mathcal{X} := \{i<\tau_{\mathcal{H}^*}\wedge\tau_{\mathscr{B}}\wedge\tau_{\mathscr{B}'}\wedge\tau_{\mathfrak{C}}\}$. Then,*

$$\mathbb{E}_i[\Delta X_{\mathfrak{c},\psi}] =_{\mathcal{X}} -\frac{(|\mathcal{C}_{\mathfrak{c}}|-1)|\mathcal{F}|}{H}X_{\mathfrak{c},\psi} - \left(\sum_{e\in\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}}\sum_{\mathfrak{b}\in\mathfrak{B}_{\mathfrak{c}}^e}\frac{\hat{\varphi}_{\mathfrak{c},I}}{k!\,H\hat{\varphi}_{\mathfrak{b},I}}X_{\mathfrak{b},\psi}\right) \pm \delta^2\frac{\xi_1}{H}.$$

Proof. Similarly as in (3.6.13), we have

$$\Delta X_{\mathfrak{c},\psi} = (\Delta\Phi_{\mathfrak{c},\psi}) - \hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}(i+1)(\Delta\Phi_{\mathfrak{c}|-,\psi}) - (\Delta\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}})\Phi_{\mathfrak{c}|-,\psi}.$$

By Lemma 3.5.11 and Lemma 3.6.22, this entails

$$\begin{aligned}
\Delta X_{\mathfrak{c},\psi} &= (\Delta\Phi_{\mathfrak{c},\psi}) - (1\pm\zeta^2)\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}(\Delta\Phi_{\mathfrak{c}|-,\psi}) + (1\pm\zeta^2)\frac{|\mathcal{F}||\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}|}{H}\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}\Phi_{\mathfrak{c}|-,\psi}\\
&= (\Delta\Phi_{\mathfrak{c},\psi}) - \hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}(\Delta\Phi_{\mathfrak{c}|-,\psi}) + \frac{|\mathcal{F}||\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}|}{H}\hat{\Phi}_{\mathfrak{c},\psi} \pm \zeta^2\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}(\Delta\Phi_{\mathfrak{c}|-,\psi})\\
&\quad\pm\zeta^{3/2}\frac{\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}\Phi_{\mathfrak{c}|-,\psi}}{H}.
\end{aligned}$$

Since by Lemma 3.6.16 we have $\Phi_{\mathfrak{c}|-,\psi} =_{\mathcal{X}} (1 \pm \delta^5)\hat{\varphi}_{\mathfrak{c}|-,I}$, this yields

$$\Delta X_{\mathfrak{c},\psi} =_{\mathcal{X}} (\Delta\Phi_{\mathfrak{c},\psi}) - \hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}(\Delta\Phi_{\mathfrak{c}|-,\psi}) + \frac{|\mathcal{F}||\mathcal{G}_{\mathfrak{c}} \setminus \{J_{\mathfrak{c}}\}|}{H}\hat{\Phi}_{\mathfrak{c},\psi} \pm \zeta^2\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}(\Delta\Phi_{\mathfrak{c}|-,\psi}) \pm \zeta^{4/3}\frac{\hat{\varphi}_{\mathfrak{c},I}}{H}. \tag{3.6.15}$$

Using Lemma 3.6.15, we obtain

$$\begin{aligned}\mathbb{E}_i[\Delta\Phi_{\mathfrak{c},\psi}] &= -\sum_{e\in\mathcal{C}_{\mathfrak{c}}\setminus\mathcal{C}_{\mathfrak{c}}[I]} \mathbb{E}_i[|\{\varphi \in \Phi^{\sim}_{\mathfrak{c},\psi} : \varphi(e) \in \mathcal{F}_0(i+1)\}|] \\ &=_{\mathcal{X}} -\Big(\sum_{e\in\mathcal{C}_{\mathfrak{c}}\setminus\mathcal{C}_{\mathfrak{c}}[I]}\sum_{f\in\mathcal{F}}\sum_{\beta\colon f \xrightarrow{\sim} e}\frac{\Phi_{\mathfrak{c}|[\beta],\psi}\Phi_{\mathfrak{c},\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}}\Big) \pm \zeta^{1+\delta/3}\frac{\hat{\varphi}_{\mathfrak{c},I}}{H}.\end{aligned} \tag{3.6.16}$$

Note that for $e \in \mathcal{C}_{\mathfrak{c}|-}$, $f \in \mathcal{F}$ and $\beta\colon f \xrightarrow{\sim} e$, we have $\mathfrak{c}|-|e = \mathfrak{c}|e$ and $\mathfrak{c}|-|[\beta] = \mathfrak{c}|[\beta]$. Hence, again using Lemma 3.6.15, we similarly obtain

$$\begin{aligned}\mathbb{E}_i[\Delta\Phi_{\mathfrak{c}|-,\psi}] &= -\sum_{e\in\mathcal{C}_{\mathfrak{c}|-}\setminus\mathcal{C}_{\mathfrak{c}|-}[I]} \mathbb{E}_i[|\{\varphi \in \Phi^{\sim}_{\mathfrak{c}|-,\psi} : \varphi(e) \in \mathcal{F}_0(i+1)\}|] \\ &=_{\mathcal{X}} -\Big(\sum_{e\in\mathcal{C}_{\mathfrak{c}|-}\setminus\mathcal{C}_{\mathfrak{c}|-}[I]}\sum_{f\in\mathcal{F}}\sum_{\beta\colon f \xrightarrow{\sim} e}\frac{\Phi_{\mathfrak{c}|[\beta],\psi}\Phi_{\mathfrak{c}|-,\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}}\Big) \pm \zeta^{1+\delta/3}\frac{\hat{\varphi}_{\mathfrak{c}|-,I}}{H}.\end{aligned} \tag{3.6.17}$$

Furthermore, since by Lemma 3.6.16 we have $\Phi_{\mathfrak{c}|[\beta],\psi} =_{\mathcal{X}} (1 \pm \delta^5)\hat{\varphi}_{\mathfrak{c}|[\beta],I}$, $\Phi_{\mathfrak{c}|-,\psi} =_{\mathcal{X}} (1 \pm \delta^5)\hat{\varphi}_{\mathfrak{c}|-,I}$ and $\Phi_{\mathfrak{c}|e,\psi} =_{\mathcal{X}} (1 \pm \delta^5)\hat{\varphi}_{\mathfrak{c}|e,I}$, using Lemma 3.5.7, for $f \in \mathcal{F}$, this yields

$$|\mathbb{E}_i[\Delta\Phi_{\mathfrak{c}|-,\psi}]| \leq_{\mathcal{X}} \frac{2|\mathcal{C}_{\mathfrak{c}}||\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{F},f}\hat{\varphi}_{\mathfrak{c}|-,I}}{\mathrm{aut}(\mathcal{F})H^*} + \zeta^{1+\delta/3}\frac{\hat{\varphi}_{\mathfrak{c}|-,I}}{H} \leq_{\mathcal{X}} \frac{3|\mathcal{C}_{\mathfrak{c}}||\mathcal{F}|\hat{\varphi}_{\mathfrak{c}|-,I}}{H}. \tag{3.6.18}$$

From (3.6.15), using (3.6.16) and (3.6.17) as well as the fact that $\mathcal{C}_{\mathfrak{c}} \setminus \mathcal{C}_{\mathfrak{c}}[I] = (\mathcal{C}_{\mathfrak{c}|-} \setminus \mathcal{C}_{\mathfrak{c}|-}[I]) \cup (\mathcal{G}_{\mathfrak{c}} \setminus \{J_{\mathfrak{c}}\})$, we obtain

$$\begin{aligned}\mathbb{E}_i[\Delta X_{\mathfrak{c},\psi}] =_{\mathcal{X}} &-\Big(\sum_{e\in\mathcal{C}_{\mathfrak{c}|-}\setminus\mathcal{C}_{\mathfrak{c}|-}[I]}\sum_{f\in\mathcal{F}}\sum_{\beta\colon f \xrightarrow{\sim} e}\frac{\Phi_{\mathfrak{c}|[\beta],\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}}\Big)(\Phi_{\mathfrak{c},\psi} - \hat{\Phi}_{\mathfrak{c},\psi}) \\ &- \sum_{e\in\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}}\Big(\Big(\sum_{f\in\mathcal{F}}\sum_{\beta\colon f \xrightarrow{\sim} e}\frac{\Phi_{\mathfrak{c},\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}}\Phi_{\mathfrak{c}|[\beta],\psi}\Big) - \frac{|\mathcal{F}|}{H}\hat{\Phi}_{\mathfrak{c},\psi}\Big) \\ &\pm\zeta^2\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}\mathbb{E}_i[\Delta\Phi_{\mathfrak{c}|-,\psi}] \pm \zeta^{1+\delta/4}\frac{\hat{\varphi}_{\mathfrak{c},I}}{H}.\end{aligned}$$

Due to (3.6.18), this yields

$$\begin{aligned}\mathbb{E}_i[\Delta X_{\mathfrak{c},\psi}] =_{\mathcal{X}} &-\Big(\sum_{e\in\mathcal{C}_{\mathfrak{c}|-}\setminus\mathcal{C}_{\mathfrak{c}|-}[I]}\sum_{f\in\mathcal{F}}\sum_{\beta\colon f \xrightarrow{\sim} e}\frac{\Phi_{\mathfrak{c}|[\beta],\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}}\Big)(\Phi_{\mathfrak{c},\psi} - \hat{\Phi}_{\mathfrak{c},\psi}) \\ &- \sum_{e\in\mathcal{G}_{\mathfrak{c}}\setminus J_{\mathfrak{c}}}\Big(\Big(\sum_{f\in\mathcal{F}}\sum_{\beta\colon f \xrightarrow{\sim} e}\frac{\Phi_{\mathfrak{c},\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}}\Phi_{\mathfrak{c}|[\beta],\psi}\Big) - \frac{|\mathcal{F}|}{H}\hat{\Phi}_{\mathfrak{c},\psi}\Big) \\ &\pm\zeta^{1+\delta/5}\frac{\hat{\varphi}_{\mathfrak{c},I}}{H}.\end{aligned} \tag{3.6.19}$$

We investigate the first two terms of the sum on the right side separately.

First, note that for all $e \in \mathcal{C}_{\mathfrak{c}|-} \setminus \mathcal{C}_{\mathfrak{c}|-}[I]$, using Lemma 3.6.16 and Lemma 3.5.7, we obtain

$$\sum_{f\in\mathcal{F}}\sum_{\beta\colon f\overset{\sim}{\rightarrow} e}\frac{\Phi_{\mathfrak{c}|[\beta],\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}} =_{\mathcal{X}} (1\pm 4\delta^5)\frac{|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{F}_{\mathfrak{c}}^{\beta},e}}{\mathrm{aut}(\mathcal{F})H^*} =_{\mathcal{X}} (1\pm 5\delta^5)\frac{|\mathcal{F}|}{H}.$$

Thus, for the first term, using $X_{\mathfrak{c},\psi} \leq_{\mathcal{X}} \xi_1$ and Lemma 3.6.3, we obtain

$$\begin{aligned}-\Bigg(\sum_{e\in\mathcal{C}_{\mathfrak{c}|-}\setminus\mathcal{C}_{\mathfrak{c}|-}[I]}\sum_{f\in\mathcal{F}}\sum_{\beta\colon f\overset{\sim}{\rightarrow} e}\frac{\Phi_{\mathfrak{c}|[\beta],\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}}\Bigg)(\Phi_{\mathfrak{c},\psi}-\hat{\Phi}_{\mathfrak{c},\psi})&\\ =_{\mathcal{X}} -(1\pm 5\delta^5)\frac{(|\mathcal{C}_{\mathfrak{c}|-}|-1)|\mathcal{F}|}{H}X_{\mathfrak{c},\psi} =_{\mathcal{X}} -\frac{(|\mathcal{C}_{\mathfrak{c}|-}|-1)|\mathcal{F}|}{H}X_{\mathfrak{c},\psi}\pm\delta^4\frac{\xi_1}{H}.&\end{aligned}\tag{3.6.20}$$

Let us consider the second term. For all $e \in \mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}$, using the fact that for all $f,f'\in\mathcal{F}$ and $\beta\colon f \overset{\sim}{\rightarrow} e$, we have $\hat{\Phi}_{\mathfrak{c}|[\beta],\psi} = \hat{\varphi}_{\mathcal{F}_{\mathfrak{c}}^{\beta},e}\Phi_{\mathfrak{c}|[\beta]|-,\psi} = \hat{\varphi}_{\mathcal{F},f'}\Phi_{\mathfrak{c}|e,\psi}$, we obtain

$$\begin{aligned}&\Bigg(\sum_{f\in\mathcal{F}}\sum_{\beta\colon f\overset{\sim}{\rightarrow} e}\frac{\Phi_{\mathfrak{c},\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}}\Phi_{\mathfrak{c}|[\beta],\psi}\Bigg)-\frac{|\mathcal{F}|}{H}\hat{\Phi}_{\mathfrak{c},\psi}\\ &=\Bigg(\sum_{f\in\mathcal{F}}\sum_{\beta\colon f\overset{\sim}{\rightarrow} e}\frac{\Phi_{\mathfrak{c},\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}}X_{\mathfrak{c}|[\beta],\psi}\Bigg)+\Bigg(\sum_{f\in\mathcal{F}}\sum_{\beta\colon f\overset{\sim}{\rightarrow} e}\frac{\Phi_{\mathfrak{c},\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}}\hat{\Phi}_{\mathfrak{c}|[\beta],\psi}\Bigg)\\ &\qquad-\frac{|\mathcal{F}|}{H}\hat{\Phi}_{\mathfrak{c},\psi}\\ &=\Bigg(\sum_{f\in\mathcal{F}}\sum_{\beta\colon f\overset{\sim}{\rightarrow} e}\frac{\Phi_{\mathfrak{c},\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}}X_{\mathfrak{c}|[\beta],\psi}\Bigg)+\frac{|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{F},f'}}{\mathrm{aut}(\mathcal{F})H^*}\Phi_{\mathfrak{c},\psi}-\frac{|\mathcal{F}|}{H}\hat{\Phi}_{\mathfrak{c},\psi}\end{aligned}\tag{3.6.21}$$

Note that from Lemma 3.6.16 together with Lemma 3.5.7, for all $f\in\mathcal{F}$ and $\beta\colon f\overset{\sim}{\rightarrow} e$, we obtain

$$\begin{aligned}\frac{\Phi_{\mathfrak{c},\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}}X_{\mathfrak{c}|[\beta],\psi} &=_{\mathcal{X}} (1\pm 4\delta^5)\frac{\hat{\varphi}_{\mathfrak{c},I}}{\mathrm{aut}(\mathcal{F})H^*\hat{\varphi}_{\mathfrak{c}|e,I}}X_{\mathfrak{c}|[\beta],\psi}\\ &=_{\mathcal{X}} (1\pm 5\delta^5)\frac{\hat{\varphi}_{\mathfrak{c},I}}{k!\,H\hat{\varphi}_{\mathfrak{c}|[\beta],I}}X_{\mathfrak{c}|[\beta],\psi}\end{aligned}\tag{3.6.22}$$

and that again Lemma 3.5.7 together with Lemma 3.6.16 yields

$$\frac{|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{F},f'}}{\mathrm{aut}(\mathcal{F})H^*}\Phi_{\mathfrak{c},\psi}-\frac{|\mathcal{F}|}{H}\hat{\Phi}_{\mathfrak{c},\psi} =_{\mathcal{X}} (1\pm\zeta^{1+\varepsilon^4})\frac{|\mathcal{F}|}{H}\Phi_{\mathfrak{c},\psi}-\frac{|\mathcal{F}|}{H}\hat{\Phi}_{\mathfrak{c},\psi} =_{\mathcal{X}} \frac{|\mathcal{F}|}{H}X_{\mathfrak{c},\psi}\pm\zeta^{1+\varepsilon^5}\frac{\hat{\varphi}_{\mathfrak{c},I}}{H}.\tag{3.6.23}$$

From (3.6.21), using (3.6.22) and (3.6.23) as well as the fact that $X_{\mathfrak{c}|[\beta],\psi} \leq_{\mathcal{X}} \delta^{-1}\zeta\hat{\varphi}_{\mathfrak{c}|[\beta],I}$, we obtain

$$\begin{aligned}\Bigg(\sum_{f\in\mathcal{F}}\sum_{\beta\colon f\xrightarrow{\sim}e}&\frac{\Phi_{\mathfrak{c},\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}}\Phi_{\mathfrak{c}|[\beta],\psi}\Bigg)-\frac{|\mathcal{F}|}{H}\hat{\Phi}_{\mathfrak{c},\psi}\\&=_{\mathcal{X}}(1\pm 5\delta^5)\Bigg(\sum_{f\in\mathcal{F}}\sum_{\beta\colon f\xrightarrow{\sim}e}\frac{\hat{\varphi}_{\mathfrak{c},I}}{k!\,H\hat{\varphi}_{\mathfrak{c}|[\beta],I}}X_{\mathfrak{c}|[\beta],\psi}\Bigg)+\frac{|\mathcal{F}|}{H}X_{\mathfrak{c},\psi}\pm\zeta^{1+\varepsilon^5}\frac{\hat{\varphi}_{\mathfrak{c},I}}{H}\\&=_{\mathcal{X}}\Bigg(\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}}\frac{\hat{\varphi}_{\mathfrak{c},I}}{k!\,H\hat{\varphi}_{\mathfrak{b},I}}X_{\mathfrak{b},\psi}\Bigg)+\frac{|\mathcal{F}|}{H}X_{\mathfrak{c},\psi}\pm\delta^4\frac{\xi_1}{H}.\end{aligned}$$

Thus, for the second term we have

$$\begin{aligned}-\sum_{e\in\mathcal{G}_{\mathfrak{c}}\setminus J_{\mathfrak{c}}}&\Bigg(\Bigg(\sum_{f\in\mathcal{F}}\sum_{\beta\colon f\xrightarrow{\sim}e}\frac{\Phi_{\mathfrak{c},\psi}}{\mathrm{aut}(\mathcal{F})H^*\Phi_{\mathfrak{c}|e,\psi}}\Phi_{\mathfrak{c}|[\beta],\psi}\Bigg)-\frac{|\mathcal{F}|}{H}\hat{\Phi}_{\mathfrak{c},\psi}\Bigg)\\&=_{\mathcal{X}}-\frac{|\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}||\mathcal{F}|}{H}X_{\mathfrak{c},\psi}-\Bigg(\sum_{e\in\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}}\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}}\frac{\hat{\varphi}_{\mathfrak{c},I}}{k!\,H\hat{\varphi}_{\mathfrak{b},I}}X_{\mathfrak{b},\psi}\Bigg)\pm\delta^3\frac{\xi_1}{H}.\end{aligned}\tag{3.6.24}$$

Since $|\mathcal{C}_{\mathfrak{c}}|=|\mathcal{C}_{\mathfrak{c}|-}|+|\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}|$, combining (3.6.19) with (3.6.20) and (3.6.24) completes the proof. □

Lemma 3.6.24. *Let* $0\le i_0\le i$ *and* $\ast\in\{-,+\}$*. Then,* $\mathbb{E}_i[\Delta Z^{\ast}_{i_0}]\le 0$.

Proof. Suppose that $i<i^\star$ and let $\mathcal{X}:=\{i<\tau^{\ast}_{i_0}\wedge\tilde{\tau}^{\star}_{\mathfrak{C}}\}$. We have $\mathbb{E}_i[\Delta Z^{\ast}_{i_0}]=_{\mathcal{X}^{\mathrm{c}}}0$ and $\mathbb{E}_i[\Delta Z^{\ast}_{i_0}]=_{\mathcal{X}}\mathbb{E}_i[\Delta Y^{\ast}]$, so it suffices to obtain $\mathbb{E}_i[\Delta Y^{\ast}]\le_{\mathcal{X}}0$. From Lemma 3.6.23, we obtain

$$\begin{aligned}\mathbb{E}_i[\Delta(\ast X_{\mathfrak{c},\psi})]&\le_{\mathcal{X}}-\frac{(|\mathcal{C}_{\mathfrak{c}}|-1)|\mathcal{F}|}{H}(\ast X_{\mathfrak{c},\psi})-\Bigg(\sum_{e\in\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}}\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}}\frac{\hat{\varphi}_{\mathfrak{c},I}}{k!\,H\hat{\varphi}_{\mathfrak{b},I}}(\ast X_{\mathfrak{b},\psi})\Bigg)+\delta^2\frac{\xi_1}{H}\\&\le-\frac{|\mathcal{F}|}{H}\Bigg((|\mathcal{C}_{\mathfrak{c}}|-1)(\ast X_{\mathfrak{c},\psi})-\Bigg(\sum_{e\in\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}}\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}}\frac{\hat{\varphi}_{\mathfrak{c},I}}{|\mathcal{F}|k!\,\hat{\varphi}_{\mathfrak{b},I}}(\ast X_{\mathfrak{b},\psi})\Bigg)-\delta^2\xi_1\Bigg).\end{aligned}$$

Note that for all $e\in\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}$ and $\mathfrak{b}_1,\mathfrak{b}_2\in\mathfrak{B}^e_{\mathfrak{c}}$, we have $\hat{\varphi}_{\mathfrak{b}_1,I}=\hat{\varphi}_{\mathfrak{b}_2,I}$, so we may choose $\hat{\varphi}^e_{\mathfrak{c},I}$ such that $\hat{\varphi}^e_{\mathfrak{c},I}=\hat{\varphi}_{\mathfrak{b},I}$ for all $\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}$. With Lemma 3.6.3, using that $\ast X_{\mathfrak{c},\psi}\ge_{\mathcal{X}}(1-\delta)\xi_1$ as well as

$$\Big|\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}}X_{\mathfrak{b},\psi}\Big|\le_{\mathcal{X}}\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}}\delta^{-1/2}\zeta\hat{\varphi}_{\mathfrak{b},\psi},$$

we obtain

$$\begin{aligned}
\mathbb{E}_i[\Delta(\ast X_{\mathfrak{c},\psi})] &\le -\frac{|\mathcal{F}|}{H}\left((|\mathcal{C}_{\mathfrak{c}}|-1)(\ast X_{\mathfrak{c},\psi}) - \frac{1}{|\mathcal{F}|k!}\left(\sum_{e\in\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}} \frac{\hat\varphi_{\mathfrak{c},I}}{\hat\varphi^e_{\mathfrak{c},I}} \sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \ast X_{\mathfrak{b},\psi}\right) - \delta^2\xi_1\right)\\
&\le_{\mathcal{X}} -\frac{|\mathcal{F}|}{H}\Bigg((|\mathcal{C}_{\mathfrak{c}}|-1)(1-\delta)\xi_1\\
&\qquad - \frac{1}{|\mathcal{F}|k!}\left(\sum_{e\in\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}} \frac{\hat\varphi_{\mathfrak{c},I}}{\hat\varphi^e_{\mathfrak{c},I}} \sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \delta^{-1/2}\zeta\hat\varphi_{\mathfrak{b},I}\right) - \delta^2\xi_1\Bigg)\\
&\le -\frac{|\mathcal{F}|}{H}\left((|\mathcal{C}_{\mathfrak{c}}|-1)\xi_1 - \frac{1}{|\mathcal{F}|k!}\left(\sum_{e\in\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}} \sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \delta^{-1/2}\zeta\hat\varphi_{\mathfrak{c},I}\right) - \varepsilon\xi_1\right)\\
&= -\frac{|\mathcal{F}|}{H}((|\mathcal{C}_{\mathfrak{c}}|-1)\xi_1 - \delta^{1/2}|\mathcal{G}_{\mathfrak{c}}\setminus\{J_{\mathfrak{c}}\}|\xi_1 - \varepsilon\xi_1)\\
&\le -\frac{|\mathcal{F}|}{H}((|\mathcal{C}_{\mathfrak{c}}|-1)\xi_1 - \varepsilon^{1/2}\xi_1).
\end{aligned}$$

Thus, due to Lemma 3.6.21, we have

$$\mathbb{E}_i[\Delta Y^{\ast}] \le_{\mathcal{X}} -\frac{|\mathcal{F}|}{H}\left(\frac{\rho_{\mathcal{F}}}{2}\xi_1 - \varepsilon^{1/3}\xi_1\right) \le 0,$$

which completes the proof. □

Boundedness

Here, we first obtain suitable bounds for the absolute one-step changes of the processes $Y^{\ast}(0), Y^{\ast}(1), \dots$ and $Z^{\ast}_{i_0}(i_0), Z^{\ast}_{i_0}(i_0+1), \dots$ (see Lemmas 3.6.26 and 3.6.27) as well as for the expected absolute one-step changes of the second process (see Lemma 3.6.29).

Lemma 3.6.25. *Let* $0 \le i_0 \le i \le i^\star$, $\ast \in \{-,+\}$ *and* $\mathcal{X} := \{i < \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'} \wedge \tau_{\mathfrak{C}}\}$. *Then,*

$$|\Delta X_{\mathfrak{c},\psi}| \le_{\mathcal{X}} n^{\varepsilon^4} \frac{\hat\varphi_{\mathfrak{c},I}(i_0)}{n\hat p(i_0)^{\rho_{\mathcal{F}}}}.$$

Proof. For all $(\mathcal{A}, I) \subseteq (\mathcal{C}_{\mathfrak{c}}, I)$ with $V_{\mathcal{A}} \ne I$, Lemma 3.6.1 together with Lemma 3.5.5 implies

$$\hat\varphi_{\mathcal{A},I} \ge (n\hat p^{\rho_{\mathcal{F}}})^{|\mathcal{A}|-|\mathcal{A}[I]|} \ge n\hat p^{\rho_{\mathcal{F}}}.$$

Hence, due to Lemma 3.6.3, Lemma 3.5.16 together with Lemma 3.6.5 implies

$$|\Delta\Phi_{\mathfrak{c},\psi}| \le_{\mathcal{X}} |\mathcal{C}_{\mathfrak{c}}|\cdot 2k!|\mathcal{F}|(\log n)^{\alpha_{\mathcal{C}_{\mathfrak{c}},I}} \frac{\hat\varphi_{\mathfrak{c},I}}{n\hat p^{\rho_{\mathcal{F}}}} \le n^{\varepsilon^5}\frac{\hat\varphi_{\mathfrak{c},I}}{n\hat p^{\rho_{\mathcal{F}}}} \le n^{\varepsilon^5}\frac{\hat\varphi_{\mathfrak{c},I}(i_0)}{n\hat p(i_0)^{\rho_{\mathcal{F}}}}.$$

Similarly, we obtain

$$|\Delta\Phi_{\mathfrak{c}|-,\psi}| \le_{\mathcal{X}} n^{\varepsilon^5}\frac{\hat\varphi_{\mathfrak{c}|-,I}(i_0)}{n\hat p(i_0)^{\rho_{\mathcal{F}}}}.$$

With Lemma 3.5.11, Lemma 3.6.16 and Lemma 3.6.22, using Lemma 3.6.5, we conclude that

$$\begin{aligned}|\Delta X_{\mathfrak{c},\psi}| &\leq |\Delta\Phi_{\mathfrak{c},\psi}| + \hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}(i+1)|\Delta\Phi_{\mathfrak{c}|-,\psi}| + |\Delta\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}|\Phi_{\mathfrak{c}|-,\psi}\\ &\leq |\Delta\Phi_{\mathfrak{c},\psi}| + 2\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}|\Delta\Phi_{\mathfrak{c}|-,\psi}| + 2|\mathcal{F}|^2\frac{\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}\Phi_{\mathfrak{c}|-,\psi}}{H}\\ &\leq_{\mathcal{X}} n^{\varepsilon^5}\frac{\hat{\varphi}_{\mathfrak{c},I}(i_0)}{n\hat{p}(i_0)^{\rho_{\mathcal{F}}}} + 2n^{\varepsilon^5}\frac{\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}\hat{\varphi}_{\mathfrak{c}|-,I}(i_0)}{n\hat{p}(i_0)^{\rho_{\mathcal{F}}}} + 4|\mathcal{F}|^2\frac{\hat{\varphi}_{\mathfrak{c},I}}{H}\\ &\leq n^{\varepsilon^5}\frac{\hat{\varphi}_{\mathfrak{c},I}(i_0)}{n\hat{p}(i_0)^{\rho_{\mathcal{F}}}} + 2n^{\varepsilon^5}\frac{\hat{\varphi}_{\mathfrak{c},I}(i_0)}{n\hat{p}(i_0)^{\rho_{\mathcal{F}}}} + 4|\mathcal{F}|^2\frac{\hat{\varphi}_{\mathfrak{c},I}(i_0)}{H(i_0)}.\end{aligned}$$

With Lemma 3.5.9, this completes the proof. □

Lemma 3.6.26. *Let $0 \leq i_0 \leq i \leq i^\star$, $\ast \in \{-,+\}$ and $\mathcal{X} := \{i < \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'} \wedge \tau_{\mathfrak{C}}\}$. Then,*

$$|\Delta Y^\ast| \leq n^{\varepsilon^3}\frac{\hat{\varphi}_{\mathfrak{c},I}(i_0)}{n\hat{p}(i_0)^{\rho_{\mathcal{F}}}}.$$

Proof. Combining Lemma 3.6.21 and Lemma 3.6.25, using Lemma 3.6.5, we obtain

$$|\Delta Y^\ast| \leq |\Delta X_{\mathfrak{c},\psi}| + |\Delta\xi_1| \leq n^{\varepsilon^4}\frac{\hat{\varphi}_{\mathfrak{c},I}(i_0)}{n\hat{p}(i_0)^{\rho_{\mathcal{F}}}} + \frac{\hat{\varphi}_{\mathfrak{c},I}}{H} \leq n^{\varepsilon^4}\frac{\hat{\varphi}_{\mathfrak{c},I}(i_0)}{n\hat{p}(i_0)^{\rho_{\mathcal{F}}}} + \frac{\hat{\varphi}_{\mathfrak{c},I}(i_0)}{H(i_0)}.$$

With Lemma 3.5.9, this completes the proof. □

Lemma 3.6.27. *Let $0 \leq i_0 \leq i \leq i^\star$ and $\ast \in \{-,+\}$. Then,*

$$|\Delta Z^\ast_{i_0}| \leq n^{\varepsilon^3}\frac{\hat{\varphi}_{\mathfrak{c},I}(i_0)}{n\hat{p}(i_0)^{\rho_{\mathcal{F}}}}.$$

Proof. This is an immediate consequence of Lemma 3.6.26. □

Lemma 3.6.28. *Let $0 \leq i \leq i^\star$, $\ast \in \{-,+\}$ and $\mathcal{X} := \{i < \tau_{\mathcal{H}^*} \wedge \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'} \wedge \tau_{\mathfrak{C}}\}$. Then,*

$$\mathbb{E}_i[|\Delta X_{\mathfrak{c},\psi}|] \leq_{\mathcal{X}} n^{\varepsilon^4}\frac{\hat{\varphi}_{\mathfrak{c},I}}{n^k\hat{p}}.$$

Proof. With Lemma 3.5.11, Lemma 3.6.16 and Lemma 3.6.22, we obtain

$$\begin{aligned}\mathbb{E}_i[|\Delta X_{\mathfrak{c},\psi}|] &\leq \mathbb{E}_i[|\Delta\Phi_{\mathfrak{c},\psi}|] + \hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}(i+1)\mathbb{E}_i[|\Delta\Phi_{\mathfrak{c}|-,\psi}|] + |\Delta\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}|\Phi_{\mathfrak{c}|-,\psi}\\ &\leq \mathbb{E}_i[|\Delta\Phi_{\mathfrak{c},\psi}|] + 2\hat{\varphi}_{\mathcal{G}_{\mathfrak{c}},J_{\mathfrak{c}}}\mathbb{E}_i[|\Delta\Phi_{\mathfrak{c}|-,\psi}|] + 2|\mathcal{F}|^2\frac{\hat{\varphi}_{\mathfrak{c},I}}{H}\end{aligned}$$

Thus, due to Lemma 3.5.7, it suffices to obtain

$$\mathbb{E}_i[|\Delta\Phi_{\mathfrak{c},\psi}|] \leq_{\mathcal{X}} n^{\varepsilon^5}\frac{\hat{\varphi}_{\mathfrak{c},I}}{n^k\hat{p}} \quad\text{and}\quad \mathbb{E}_i[|\Delta\Phi_{\mathfrak{c}|-,\psi}|] \leq_{\mathcal{X}} n^{\varepsilon^5}\frac{\hat{\varphi}_{\mathfrak{c}|-,I}}{n^k\hat{p}}.$$

To this end, for $e \in \mathcal{C}_{\mathfrak{c}} \setminus \mathcal{C}_{\mathfrak{c}}[I]$, from all subtemplates $(\mathcal{A}, I) \subseteq (\mathcal{C}_{\mathfrak{c}}, I)$ with $e \in \mathcal{A}$, choose $(\mathcal{A}_e, I)$ such that $\hat{\varphi}_{\mathcal{A}_e,I}$ is minimal. Furthermore, for every subtemplate $(\mathcal{A}, I) \subseteq (\mathcal{C}_{\mathfrak{c}}, I)$, let

$$\Phi^e_{\mathcal{A},\psi} := |\{\varphi \in \Phi^{\sim}_{\mathcal{A},\psi} : \varphi(e) \in \mathcal{F}_0(i+1)\}|.$$

Then, due to Lemma 3.6.3, Lemma 3.5.16 yields

$$\Phi^e_{\mathcal{C}_\mathfrak{c},\psi} \leq_{\mathcal{X}} 2k!|\mathcal{F}|(\log n)^{\alpha_{\mathcal{C}_\mathfrak{c},I\cup e}} \frac{\hat{\varphi}_{\mathfrak{c},I}}{\hat{\varphi}_{\mathcal{A}_e,I}},$$

so we obtain

$$\begin{aligned}
|\Delta\Phi_{\mathfrak{c},\psi}| &\leq \sum_{e\in\mathcal{C}_\mathfrak{c}\setminus\mathcal{C}_\mathfrak{c}[I]} \Phi^e_{\mathcal{C}_\mathfrak{c},\psi} = \sum_{e\in\mathcal{C}_\mathfrak{c}\setminus\mathcal{C}_\mathfrak{c}[I]} \mathbb{1}_{\{\Phi^e_{\mathcal{C}_\mathfrak{c},\psi}\geq 1\}}\Phi^e_{\mathcal{C}_\mathfrak{c},\psi} \\
&\leq_{\mathcal{X}} 2k!|\mathcal{F}|(\log n)^{\alpha_{\mathcal{C}_\mathfrak{c},I\cup e}}\hat{\varphi}_{\mathfrak{c},I} \sum_{e\in\mathcal{C}_\mathfrak{c}\setminus\mathcal{C}_\mathfrak{c}[I]} \frac{\mathbb{1}_{\{\Phi^e_{\mathcal{C}_\mathfrak{c},\psi}\geq 1\}}}{\hat{\varphi}_{\mathcal{A}_e,I}} \leq n^{\varepsilon^6}\hat{\varphi}_{\mathfrak{c},I} \sum_{e\in\mathcal{C}_\mathfrak{c}\setminus\mathcal{C}_\mathfrak{c}[I]} \frac{\mathbb{1}_{\{\Phi^e_{\mathcal{C}_\mathfrak{c},\psi}\geq 1\}}}{\hat{\varphi}_{\mathcal{A}_e,I}} \\
&\leq n^{\varepsilon^6}\hat{\varphi}_{\mathfrak{c},I} \sum_{e\in\mathcal{C}_\mathfrak{c}\setminus\mathcal{C}_\mathfrak{c}[I]} \frac{\mathbb{1}_{\{\Phi^e_{\mathcal{A}_e,\psi}\geq 1\}}}{\hat{\varphi}_{\mathcal{A}_e,I}} \leq n^{\varepsilon^6}\hat{\varphi}_{\mathfrak{c},I} \sum_{e\in\mathcal{C}_\mathfrak{c}\setminus\mathcal{C}_\mathfrak{c}[I]} \sum_{\varphi\in\Phi^{\sim}_{\mathcal{A}_e,\psi}} \frac{\mathbb{1}_{\{\varphi(e)\in\mathcal{F}_0(i+1)\}}}{\hat{\varphi}_{\mathcal{A}_e,I}}.
\end{aligned} \tag{3.6.25}$$

For all $e\in\mathcal{H}$, $f\in\mathcal{F}$ and $\psi'\colon f \rightsquigarrow e$, we have $\Phi_{\mathcal{F},\psi'} =_{\mathcal{X}} (1\pm\delta^{-1}\zeta)\hat{\varphi}_{\mathcal{F},f}$. Furthermore, we have $H^* =_{\mathcal{X}} (1\pm\zeta^{1+\varepsilon^3})\hat{h}^*$. Thus, using Lemma 3.5.17, for all $e\in\mathcal{C}_\mathfrak{c}\setminus\mathcal{C}_\mathfrak{c}[I]$ and $\varphi\in\Phi^e_{\mathcal{A}_e,\psi}$, we obtain

$$\mathbb{P}_i[\varphi(e)\in\mathcal{F}_0(i+1)] = \frac{d_{\mathcal{H}^*}(\varphi(e))}{H^*} \leq_{\mathcal{X}} \frac{2|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{F},f}}{H^*} \leq_{\mathcal{X}} \frac{4|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{F},f}}{\hat{h}^*} \leq \frac{n^{\varepsilon^6}}{n^k\hat{p}}.$$

Combining this with (3.6.25) yields

$$\begin{aligned}
\mathbb{E}_i[|\Delta\Phi_{\mathfrak{c},\psi}|] &\leq_{\mathcal{X}} n^{\varepsilon^6}\hat{\varphi}_{\mathfrak{c},I} \sum_{e\in\mathcal{C}_\mathfrak{c}\setminus\mathcal{C}_\mathfrak{c}[I]} \sum_{\varphi\in\Phi^{\sim}_{\mathcal{A}_e,\psi}} \frac{\mathbb{P}_i[\varphi(e)\in\mathcal{F}_0(i+1)]}{\hat{\varphi}_{\mathcal{A}_e,I}} \\
&\leq_{\mathcal{X}} n^{2\varepsilon^6}\frac{\hat{\varphi}_{\mathfrak{c},I}}{n^k\hat{p}} \sum_{e\in\mathcal{C}_\mathfrak{c}\setminus\mathcal{C}_\mathfrak{c}[I]} \frac{\Phi_{\mathcal{A}_e,\psi}}{\hat{\varphi}_{\mathcal{A}_e,I}}.
\end{aligned}$$

For all $e\in\mathcal{C}_\mathfrak{c}\setminus\mathcal{C}_\mathfrak{c}[I]$ and $(\mathcal{B},I)\subseteq(\mathcal{A}_e,I)\subseteq(\mathcal{C}_\mathfrak{c},I)$, Lemma 3.6.1 together with Lemma 3.5.5 entails

$$\hat{\varphi}_{\mathcal{B},I} = (n\hat{p}^{\rho_{\mathcal{B},I}})^{|V_\mathcal{B}|-|I|} \geq (n\hat{p}^{\rho_\mathcal{F}})^{|V_\mathcal{B}|-|I|} \geq 1$$

and so Lemma 3.5.14 yields

$$\Phi_{\mathcal{A}_e,I} \leq_{\mathcal{X}} 2(\log n)^{\alpha_{\mathcal{A}_e,I}}\hat{\varphi}_{\mathcal{A}_e,I} \leq n^{\varepsilon^6}\hat{\varphi}_{\mathcal{A}_e,I}.$$

We conclude that

$$\mathbb{E}_i[|\Delta\Phi_{\mathfrak{c},\psi}|] \leq_{\mathcal{X}} n^{3\varepsilon^6}|\mathcal{C}_\mathfrak{c}\setminus\mathcal{C}_\mathfrak{c}[I]|\frac{\hat{\varphi}_{\mathfrak{c},I}}{n^k\hat{p}} \leq n^{4\varepsilon^6}\frac{\hat{\varphi}_{\mathfrak{c},I}}{n^k\hat{p}}.$$

Similarly, we obtain

$$\mathbb{E}_i[|\Delta\Phi_{\mathfrak{c}|-,\psi}|] \leq_{\mathcal{X}} n^{4\varepsilon^6}\frac{\hat{\varphi}_{\mathfrak{c}|-,I}}{n^k\hat{p}},$$

which completes the proof. □

Lemma 3.6.29. *Let* $0 \leq i_0 \leq i^\star$ *and* $\ast \in \{-,+\}$. *Then,* $\sum_{i \geq i_0} \mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] \leq n^{\varepsilon^3} \hat{\varphi}_{\mathfrak{c},I}(i_0)$.

Proof. Suppose that $i_0 \leq i < i^\star$ and let $\mathcal{X} := \{i < \tau_{\mathcal{H}^*} \wedge \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'} \wedge \tau_{\mathfrak{C}}\}$. We have $\mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] =_{\mathcal{X}^c} 0$ and with Lemma 3.6.21, Lemma 3.6.28 and Lemma 3.5.7, using Lemma 3.6.5, we obtain

$$\mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] \leq \mathbb{E}_i[|\Delta Y^{\ast}|] \leq \mathbb{E}_i[|\Delta X_{\mathfrak{c},\psi}|] + |\Delta \xi_1| \leq_{\mathcal{X}} n^{\varepsilon^4} \frac{\hat{\varphi}_{\mathfrak{c},I}}{n^k \hat{p}} + \frac{\hat{\varphi}_{\mathfrak{c},I}}{H} \leq_{\mathcal{X}} n^{\varepsilon^3} \frac{\hat{\varphi}_{\mathfrak{c},I}}{n^k \hat{p}}$$
$$\leq n^{\varepsilon^3} \frac{\hat{\varphi}_{\mathfrak{c},I}(i_0)}{n^k \hat{p}(i_0)}.$$

Thus,

$$\sum_{i \geq i_0} \mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] = \sum_{i_0 \leq i \leq i^\star - 1} \mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] \leq (i^\star - i_0) \frac{n^{\varepsilon^3} \hat{\varphi}_{\mathfrak{c},I}(i_0)}{n^k \hat{p}(i_0)}.$$

Since

$$i^\star - i_0 \leq \frac{\vartheta n^k}{|\mathcal{F}| k!} - i_0 = \frac{n^k \hat{p}(i_0)}{|\mathcal{F}| k!} \leq n^k \hat{p}(i_0),$$

this completes the proof. □

Supermartingale argument

In this section, we obtain the final ingredient for our application of Lemma 2.9.4 and subsequently show that the probabilities of the events on the right in Observation 3.6.19 are indeed small.

In more detail, we first prove Lemma 3.6.30 that states that for all $\ast \in \{-,+\}$, at time $i = \sigma^{\ast}$ where the process $\Phi_{\mathfrak{c},\psi}(0), \Phi_{\mathfrak{c},\psi}(1), \ldots$ just left the non-critical interval between the critical intervals, it cannot have jumped over the critical interval $I^{\ast}$. Then, we combine this insight with the results form the previous two sections to apply Lemma 2.9.4 in the proof of Lemma 3.6.31.

Lemma 3.6.30. *Let* $\ast \in \{-,+\}$. *Then,* $Z^{\ast}_{\sigma^{\ast}}(\sigma^{\ast}) \leq -\delta^2 \xi_1(\sigma^{\ast})$.

Proof. Together with Lemma 3.6.1, Lemma 3.5.4 implies $\tilde{\tau}^\star_{\mathfrak{C}} \geq 1$ and $\ast X_{\mathfrak{c},\psi}(0) < \xi_0(0)$, so we have $\sigma^{\ast} \geq 1$. Thus, by definition of $\sigma^{\ast}$, for $i := \sigma^{\ast} - 1$, we have $\ast X_{\mathfrak{c},\psi} \leq \xi_0$ and thus

$$Z^{\ast}_i = \ast X_{\mathfrak{c},\psi} - \xi_1 \leq -\delta \xi_1.$$

Furthermore, since $\sigma^{\ast} \leq \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'} \wedge \tau_{\mathfrak{C}}$, we may apply Lemma 3.6.26 to obtain

$$Z^{\ast}_{\sigma^{\ast}}(\sigma^{\ast}) = Z^{\ast}_i + \Delta Y^{\ast} \leq Z^{\ast}_i + \delta^2 \xi_1 \leq -\delta \xi_1 + \delta^2 \xi_1 \leq -\delta^2 \xi_1.$$

Since Lemma 3.6.5 entails $\Delta \xi_1 \leq 0$, this completes the proof. □

Lemma 3.6.31. $\mathbb{P}[\tau_{\mathfrak{C}} \leq \tilde{\tau}^\star_{\mathfrak{C}} \wedge i^\star] \leq \exp(-n^{\varepsilon^3})$.

Proof. Considering Observation 3.6.18, it suffices to show that

$$\mathbb{P}[\tau \leq \tilde{\tau}^{\star}_{\mathfrak{C}} \wedge i^\star] \leq \exp(-n^{2\varepsilon^3}).$$

Hence, by Observation 3.6.19, is suffices to show that for $\ast \in \{-,+\}$, we have

$$\mathbb{P}[Z^{\ast}_{\sigma^{\ast}}(i^\star) > 0] \leq \exp(-n^{3\varepsilon^3}).$$

Due to Lemma 3.6.30, we have

$$\mathbb{P}[Z^{\ast}_{\sigma^{\ast}}(i^\star) > 0] \leq \mathbb{P}[Z^{\ast}_{\sigma^{\ast}}(i^\star) - Z^{\ast}_{\sigma^{\ast}}(\sigma^{\ast}) > \delta^2\xi_1(\sigma^{\ast})] \leq \sum_{0\leq i\leq i^\star} \mathbb{P}[Z^{\ast}_i(i^\star) - Z^{\ast}_i > \delta^2\xi_1].$$

Thus, for $0 \leq i \leq i^\star$, it suffices to obtain

$$\mathbb{P}[Z^{\ast}_i(i^\star) - Z^{\ast}_i > \delta^2\xi_1] \leq \exp(-n^{4\varepsilon^3}).$$

We show that this bound is a consequence of We show that this bound is a consequence of Freedman's inequality for supermartingales.

Let us turn to the details. Lemma 3.6.24 shows that $Z^{\ast}_i(i), Z^{\ast}_i(i+1), \dots$ is a supermartingale, while Lemma 3.6.27 provides the bound $|\Delta Z^{\ast}_i(j)| \leq n^{\varepsilon^3}\hat{\varphi}_{\mathfrak{c},I}/(n\hat{p}^{\rho_{\mathcal{F}}})$ for all $j \geq i$ and Lemma 3.6.29 provides the bound $\sum_{j\geq i}\mathbb{E}_j[|\Delta Z^{\ast}_i(j)|] \leq n^{\varepsilon^3}\hat{\varphi}_{\mathfrak{c},I}$. Hence, we may apply Lemma 2.9.4 to obtain

$$\begin{aligned}\mathbb{P}[Z^{\ast}_i(i^\star) - Z^{\ast}_i > \delta^2\xi_1] &\leq \exp\left(-\frac{\delta^4\xi_1^2}{2n^{\varepsilon^3}\frac{\hat{\varphi}_{\mathfrak{c},I}}{n\hat{p}^{\rho_{\mathcal{F}}}}(\delta^2\xi_1 + n^{\varepsilon^3}\hat{\varphi}_{\mathfrak{c},I})}\right) \leq \exp\left(-\frac{\delta^4\xi_1^2 n\hat{p}^{\rho_{\mathcal{F}}}}{4n^{2\varepsilon^3}\hat{\varphi}^2_{\mathfrak{c},I}}\right)\\ &= \exp\left(-\frac{\delta^2 n^{2\varepsilon^2}}{4n^{2\varepsilon^3}}\right) \leq \exp(-n^{4\varepsilon^3}),\end{aligned}$$

which completes the proof. □

3.7 Branching families

This section is dedicated to introducing and analyzing the special setup based on branching families that we rely on for exploiting the self-correcting behavior of the process. Suppose that $0 \leq i \leq i^\star$, consider a chain $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$ and $\psi\colon I \hookrightarrow V_{\mathcal{H}}$. As suggested by our definition of $\tilde{\tau}_{\mathfrak{B}}$, we wish to show that $\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}}\Phi_{\mathcal{C}_{\mathfrak{b}},\psi}$ is typically close to $\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}}\hat{\Phi}_{\mathfrak{b},\psi}$, however, instead of choosing $\delta^{-1/2}\zeta\hat{\varphi}_{\mathcal{C}_{\mathfrak{b}},I}$ as the error term that quantifies the deviation that we allow, we use $\varepsilon^{-\chi_{\mathfrak{B}^e_{\mathfrak{c}}}}\zeta\hat{\varphi}_{\mathcal{C}_{\mathfrak{b}},I}$ for a carefully chosen *error parameter* $\chi_{\mathfrak{B}^e_{\mathfrak{c}}}$ that crucially depends on the branching family $\mathfrak{B}^e_{\mathfrak{c}}$.

Considering branching families instead of individual chains and using different error terms for different branching families allows us to overcome the following obstacles that we encounter when attempting to exploit self-correcting behavior. When we analyze the expected one-step changes of $\Phi_{\mathfrak{c},\psi}$ for a chain $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$ and $\psi\colon I \hookrightarrow V_{\mathcal{H}}$ using Lemma 3.6.15, different chains besides $\mathfrak{c}$ itself play an important role and their behavior could undermine the self-correcting drift that would naturally steer $\Phi_{\mathfrak{c},\psi}$ closer to the anticipated trajectory whenever it deviates. In an attempt to control this we might want to allow only significantly smaller deviations for these other chains such that the self-correcting drift still dominates. This approach leads to the desire to implement a hierarchy of error terms such that the error terms of other chains that appear as transformations of $\mathfrak{c}$ are negligible. If $\mathcal{F}$ is not symmetric, on the level of individual chains, necessary negligibility may form cyclic dependencies that make it impossible to find such a hierarchy. However, since relevant other chains that appear as transformations always appear in groups, analyzing these groups instead allows us to reduce the aforementioned directed cyclic structures to loops such that on the level of branching families, such a hierarchic approach is feasible.

In Section 3.7.1, we discuss the careful choice of error parameters. In Section 3.7.2, we subsequently employ supermartingale concentration techniques that exploit the self-correcting behavior to show that branching families typically behave as expected such that our dependence on the stopping time $\tilde{\tau}_{\mathfrak{B}}$ in Section 3.6.3 is justified.

3.7.1 Error parameter

This section is dedicated to providing and analyzing appropriate choices for the error parameters mentioned in the beginning of Section 3.7. To this end, we introduce the following concepts. For a sequence $F = \mathcal{F}_1, \dots, \mathcal{F}_\ell$ of copies of $\mathcal{F}$, we define

$$\chi_F := -\varepsilon^{-5k(k+1)} \sum_{1 \le i \le \ell-1} \varepsilon^{5k|V_{\mathcal{F}_i} \cap V_{\mathcal{F}_{i+1}}|}.$$

For a chain $\mathfrak{c} = (F, V, I)$, we say that a subsequence $F' = \mathcal{F}_1, \dots, \mathcal{F}_\ell$ of F is *$\mathfrak{c}$-sufficient* if $(\mathcal{F}_1 + \dots + \mathcal{F}_\ell)[V] = \mathcal{C}_{\mathfrak{c}}$ and we say that F' is *minimally $\mathfrak{c}$-sufficient* if F' is $\mathfrak{c}$-sufficient while no proper subsequence of F' is $\mathfrak{c}$-sufficient. The *error parameter* of $\mathfrak{c}$ is

$$\chi_{\mathfrak{c}} := |V| + \min_{F' \colon F' \text{ is minimally } \mathfrak{c}\text{-sufficient}} \chi_{F'}.$$

We observe that for all $e \in \mathcal{C}_{\mathfrak{c}} \setminus \mathcal{C}_{\mathfrak{c}}[I]$, all error parameters of branchings $\mathfrak{b} \in \mathfrak{B}_{\mathfrak{c}}^e$ are equal (see Lemma 3.7.2), which we obtain as a consequence of the following observation.

Observation 3.7.1. *Suppose that $\mathfrak{c} = (F, V, I)$ is a chain and suppose that $e \in \mathcal{C}_{\mathfrak{c}} \setminus \mathcal{C}_{\mathfrak{c}}[I]$. Let $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}_{\mathfrak{c}}^e$. Suppose that $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ is $\mathfrak{b}$-sufficient and that $\mathcal{F}'_\ell$ is the last element in the first component of $\mathfrak{b}'$. Then, $\mathcal{F}_1, \dots, \mathcal{F}_{\ell-1}, \mathcal{F}'_\ell$ is $\mathfrak{b}'$ sufficient.*

Lemma 3.7.2. *Suppose that $\mathfrak{c} = (F, V, I)$ is a chain and suppose that $e \in \mathcal{C}_{\mathfrak{c}} \setminus \mathcal{C}_{\mathfrak{c}}[I]$. Let $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}_{\mathfrak{c}}^e$. Then, $\chi_{\mathfrak{b}} = \chi_{\mathfrak{b}'}$.*

Proof. Suppose that $F = \mathcal{F}_1, \dots, \mathcal{F}_\ell$ is minimally $\mathfrak{b}$-sufficient. Due to symmetry, it suffices to show that there exists a minimally $\mathfrak{b}'$-sufficient sequence F' with $\chi_{F'} = \chi_F$. Suppose that $\mathcal{F}'$ is the last element in the first component of $\mathfrak{b}'$ and let $F' := \mathcal{F}_1, \dots, \mathcal{F}_{\ell-1}, \mathcal{F}'$. By Observation 3.7.1, the sequence F' is $\mathfrak{b}'$-sufficient. Furthermore, for every $\mathfrak{b}'$-sufficient subsequence of F', replacing the last element with $\mathcal{F}_\ell$ yields a subsequence of F which again by Lemma 3.7.1 is $\mathfrak{b}$-sufficient. Hence, since F is minimally $\mathfrak{b}$-sufficient, the sequence F' is minimally $\mathfrak{b}'$-sufficient. Furthermore, we have

$$V_{\mathcal{F}_{\ell-1}} \cap V_{\mathcal{F}_\ell} = V_{\mathcal{F}_{\ell-1}} \cap e = V_{\mathcal{F}_{\ell-1}} \cap V_{\mathcal{F}'}$$

and thus $\chi_{F'} = \chi_F$. $\square$

For a chain $\mathfrak{c} = (F, V, I)$ and $e \in \mathcal{C}_\mathfrak{c} \setminus \mathcal{C}_\mathfrak{c}[I]$, this allows us to choose the error parameter $\chi_{\mathfrak{B}^e_\mathfrak{c}}$ of $\mathfrak{B}^e_\mathfrak{c}$ such that $\chi_{\mathfrak{B}^e_\mathfrak{c}} = \chi_\mathfrak{b}$ for all $\mathfrak{b} \in \mathfrak{B}^e_\mathfrak{c}$. The key property of our error parameters that we formally state in Lemma 3.7.8 is that whenever we consider the branching $\mathfrak{b}'$ of a branching $\mathfrak{b}$ of a chain $\mathfrak{c} \in \mathfrak{C}$, then $\chi_{\mathfrak{b}'} \leq \chi_\mathfrak{b} - 1$ or we are in a situation where the branching families of $\mathfrak{c}$ and $\mathfrak{b}$ are essentially the same.

To formally state the close relationship between branching families that we encounter whenever the branching of a branching has the same error parameter, we introduce the following term. For two chains $\mathfrak{c} = (F, V, I)$ and $\mathfrak{c}' = (F', V', I')$ and edges $e \in \mathcal{C}_\mathfrak{c} \setminus \mathcal{C}_\mathfrak{c}[I]$ and $e' \in \mathcal{C}_{\mathfrak{c}'} \setminus \mathcal{C}_{\mathfrak{c}'}[I']$, we say that the branching families $\mathfrak{B}^e_\mathfrak{c}$ and $\mathfrak{B}^{e'}_{\mathfrak{c}'}$ are *template equivalent* if there exists a bijection $\gamma\colon \mathfrak{B}^e_\mathfrak{c} \xrightarrow{\sim} \mathfrak{B}^{e'}_{\mathfrak{c}'}$ such that for all $\mathfrak{b} \in \mathfrak{B}^e_\mathfrak{c}$, the chain template $(\mathcal{C}_\mathfrak{b}, I)$ is a copy of $(\mathcal{C}_{\gamma(\mathfrak{b})}, I')$ while $(\mathcal{C}_{\mathfrak{b}|-}, I)$ is a copy of $(\mathcal{C}_{\gamma(\mathfrak{b})|-}, I')$. We encounter such a close relationship between branching families for example when comparing the branching family of a chain and the branching family of the corresponding support (see Lemma 3.7.3).

To show that we have template equivalence of relevant branching families, we argue based on a refined notion of copy for templates. More specifically, for two templates $(\mathcal{A}, I)$ and $(\mathcal{B}, J)$ and $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we say that $(\mathcal{B}, J)$ is a copy of $(\mathcal{A}, I)$ *with b playing the role of a* if there exists a bijection $\varphi\colon V_\mathcal{A} \xrightarrow{\sim} V_\mathcal{B}$ with $\varphi(e) \in \mathcal{B}$ for all $e \in \mathcal{A}$, $\varphi^{-1}(e) \in \mathcal{A}$ for all $e \in \mathcal{B}$, $\varphi(I) = J$ and $\varphi(a) = b$. Lemma 3.7.4 states the connection between this notion of copy and template equivalence that we rely on.

Lemmas 3.7.5–3.7.7 serve as further preparation for the proof of Lemma 3.7.8.

Lemma 3.7.3. *Suppose that $\mathfrak{s}$ is the e-support of a chain $\mathfrak{c}$. Then, $\mathfrak{B}^e_\mathfrak{c}$ and $\mathfrak{B}^e_\mathfrak{s}$ are template equivalent.*

Proof. Suppose that $\mathfrak{c} = (F, V, I)$ where F has length ℓ. Let $\beta\colon f \xrightarrow{\sim} e$ where $f \in \mathcal{F}$. We have $\mathfrak{s} = \mathfrak{c}|\beta|\mathtt{r}|-$, so the chain template given by $\mathfrak{s}|\beta$ is a copy of the chain template given by $\mathfrak{c}|\beta|\mathtt{r}$. Since $\mathfrak{s}$ is the e-support of $\mathfrak{c}$, we have $\mathfrak{s}|\beta|\mathtt{r} = \mathfrak{s}|\beta$. Thus, the chain template given by $\mathfrak{s}|\beta|\mathtt{r}$ is a copy of the chain template given by $\mathfrak{c}|\beta|\mathtt{r}$. Furthermore, we additionally have $\mathfrak{c}|\beta|\mathtt{r}|- = \mathfrak{s} = \mathfrak{s}|\beta|\mathtt{r}|-$ so a bijection $\gamma\colon \mathfrak{B}^e_\mathfrak{c} \xrightarrow{\sim} \mathfrak{B}^e_\mathfrak{s}$ as in the definition of template equivalence exists. $\square$

Lemma 3.7.4. *Suppose that $\mathfrak{c} = (F, V, I)$ is the e-support of a chain. Suppose that $\mathfrak{c}' = (F', V', I')$ is a chain such that for some $e' \in \mathcal{C}_{\mathfrak{c}'} \setminus \mathcal{C}_{\mathfrak{c}'}[I']$, the template $(\mathcal{C}_{\mathfrak{c}'}, I')$ is a copy of $(\mathcal{C}_\mathfrak{c}, I)$ with e' playing the role of e. Then, $\mathfrak{B}^e_\mathfrak{c}$ and $\mathfrak{B}^{e'}_{\mathfrak{c}'}$ are template equivalent.*

Proof. Suppose that $\varphi\colon V \xrightarrow{\sim} V'$ is a bijection with $\varphi(e) \in \mathcal{C}_{\mathfrak{c}'}$ for all $e \in \mathcal{C}_\mathfrak{c}$ and $\varphi^{-1}(e) \in \mathcal{C}_{\mathfrak{c}'}$ for all $e \in \mathcal{C}_\mathfrak{c}$, $\varphi(I) = I'$ and $\varphi(e) = e'$. Suppose that $\mathfrak{b} \in \mathfrak{B}^e_\mathfrak{c}$ where $\mathfrak{b} = \mathfrak{c}|\beta|\mathtt{r}$ for some $\beta\colon f \xrightarrow{\sim} e$ where $f \in \mathcal{F}$. Let $\beta' := \varphi \circ \beta$ and $\mathfrak{b}' := \mathfrak{c}'|\beta'|\mathtt{r}$. To see that assigning $\mathfrak{b}'$

as the image of $\mathfrak{b}$ under a map $\gamma\colon \mathfrak{B}^e_{\mathfrak{c}} \to \mathfrak{B}^{e'}_{\mathfrak{c}'}$ yields a bijection as desired, it suffices to show that $(\mathcal{C}_{\mathfrak{b}}, I)$ is a copy of $(\mathcal{C}_{\mathfrak{b}'}, I')$ while $(\mathcal{C}_{\mathfrak{b}|-}, I)$ is a copy of $(\mathcal{C}_{\mathfrak{b}'|-}, I')$.

First, observe that there exists a bijection

$$\varphi_+\colon V \cup V_{\mathcal{F}^\beta_{\mathfrak{c}}} \xrightarrow{\sim} V' \cup V_{\mathcal{F}^{\beta'}_{\mathfrak{c}'}}$$

with $\varphi_+|_V = \varphi$ such that $\varphi_+(e) \in \mathcal{C}_{\mathfrak{b}'}$ for all $e \in \mathcal{C}_{\mathfrak{b}}$ and $\varphi_+^{-1}(e) \in \mathcal{C}_{\mathfrak{b}'}$ for all $e \in \mathcal{C}_{\mathfrak{b}}$. Hence, $(\mathcal{C}_{\mathfrak{c}|\beta}, I)$ is a copy of $(\mathcal{C}_{\mathfrak{c}'|\beta'}, I')$. Since $\mathfrak{c}$ is the e-support of a chain, we have $\mathfrak{c}|\beta|\mathtt{r} = \mathfrak{c}|\beta$ and thus $\mathfrak{c}'|\beta'|\mathtt{r} = \mathfrak{c}'|\beta'$; so $(\mathcal{C}_{\mathfrak{b}}, I)$ is a copy of $(\mathcal{C}_{\mathfrak{b}'}, I')$. Furthermore, we obtain $\mathfrak{b}|- = \mathfrak{c}$ and $\mathfrak{b}'|- = \mathfrak{c}'$, which completes the proof. □

Lemma 3.7.5. *Suppose that $\mathfrak{c} = (I, F, V)$ is a chain with $F = \mathcal{F}_1, \ldots, \mathcal{F}_\ell$. For $1 \le i \le \ell$, let $V_i := V_{\mathcal{F}_i}$. Let $1 \le i \le i' \le j' \le j \le \ell$. Then, $V_i \cap V_j \subseteq V_{i'} \cap V_{j'}$.*

Proof. Since F is a vertex-separated loose path, we have $V_i \cap V_j \subseteq V_{i'} \cap V_j$ and $V_i \cap V_j \subseteq V_{j'} \cap V_j$. Thus,

$$V_i \cap V_j \subseteq V_{i'} \cap V_{j'} \cap V_j \subseteq V_{i'} \cap V_{j'},$$

which completes the proof. □

Lemma 3.7.6. *Suppose that $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$ is a chain and that $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ is minimally $\mathfrak{c}$-sufficient. For $1 \le i \le \ell$, let $V_i := V_{\mathcal{F}_i}$. Then, for $1 \le i \le j \le \ell$ where $i \le j - 2$, we have*

$$|V_i \cap V_j| \le \min_{i \le i' \le j-1} |V_{i'} \cap V_{i'+1}| - 1.$$

Proof. Let $i + 1 \le i'' \le j - 1$ such that

$$\min_{i \le i' \le j-1} |V_{i'} \cap V_{i'+1}| = \min_{i''-1 \le i' \le i''} |V_{i'} \cap V_{i'+1}|.$$

Then, since Lemma 3.7.5 entails $|V_i \cap V_j| \le |V_{i''-1} \cap V_{i''+1}|$, it suffices to show that

$$|V_{i''-1} \cap V_{i''+1}| \le \min_{i''-1 \le i' \le i''} |V_{i'} \cap V_{i'+1}| - 1.$$

To prove this, we use contraposition and argue as follows. Suppose now that $\mathfrak{c} = (F_0, V, I)$ is a chain and that $F = \mathcal{F}_1, \ldots, \mathcal{F}_\ell$ is minimally $\mathfrak{c}$-sufficient. Let $\mathcal{C} := \mathcal{C}_{\mathfrak{c}}$. For $1 \le i \le \ell$, let $V_i := V_{\mathcal{F}_i}$. Suppose that there exists $2 \le i' \le \ell - 1$ with

$$|V_{i'-1} \cap V_{i'+1}| \ge |V_{i'-1} \cap V_{i'}| \quad \text{or} \quad |V_{i'-1} \cap V_{i'+1}| \ge |V_{i'} \cap V_{i'+1}|.$$

We show that then, for

$$U := \Big(\bigcup_{1 \le i \le \ell\colon i \ne i'} V_i \Big) \cap V, \quad J := U \cap V_{i'},$$

we have $U \ne V$ and that furthermore, as a consequence of Lemma 3.6.9, we have

$$\rho_{\mathcal{C},U} = \rho_{\mathcal{F}_{i'}[V_i \cap V],J} \le \rho_{\mathcal{F}}.$$

This implies that $W_{\mathfrak{c}} \neq V$ and hence $\mathfrak{c} \neq \mathfrak{c}|\mathbf{r}$. With Lemma 3.6.4, this yields $\mathfrak{c} \notin \mathfrak{C}$ and thus completes the proof by contraposition.

Let us turn to the details. First, note that by choice of i', Lemma 3.7.5 entails that we have

$$V_{i'-1} \cap V_{i'+1} = V_{i'-1} \cap V_{i'} \quad \text{or} \quad V_{i'-1} \cap V_{i'+1} = V_{i'} \cap V_{i'+1}.$$

If $V_{i'-1} \cap V_{i'+1} = V_{i'-1} \cap V_{i'}$, then

$$V_{i'-1} \cap V_{i'} = V_{i'-1} \cap V_{i'+1} \cap V_{i'} \subseteq V_{i'} \cap V_{i'+1}.$$

Similarly, if $V_{i'-1} \cap V_{i'+1} = V_{i'} \cap V_{i'+1}$, then

$$V_{i'} \cap V_{i'+1} = V_{i'-1} \cap V_{i'+1} \cap V_{i'} \subseteq V_{i'-1} \cap V_{i'}.$$

Hence, in particular we have

$$V_{i'-1} \cap V_{i'} \subseteq V_{i'} \cap V_{i'+1} \quad \text{or} \quad V_{i'} \cap V_{i'+1} \subseteq V_{i'-1} \cap V_{i'}.$$

Since Lemma 3.7.5 implies

$$J = \Big(\bigcup_{1 \leq i \leq \ell \colon i \neq i'} V_{i'} \cap V_i \Big) \cap V = ((V_{i'-1} \cap V_{i'}) \cup (V_{i'} \cap V_{i'+1})) \cap V,$$

this yields

$$J = V_{i'-1} \cap V_{i'} \cap V \quad \text{or} \quad J = V_{i'} \cap V_{i'+1} \cap V. \tag{3.7.1}$$

To see that $U \neq V$, we argue as follows. Since $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ is minimally $\mathfrak{c}$-sufficient, for

$$\mathcal{S} := \mathcal{F}_1 + \ldots + \mathcal{F}_\ell \quad \text{and} \quad \mathcal{S}_{i'} := \mathcal{F}_1 + \ldots + \mathcal{F}_{i'-1} + \mathcal{F}_{i'+1} + \ldots + \mathcal{F}_\ell,$$

we obtain $\mathcal{S}_{i'}[U] \neq \mathcal{S}[V]$. If there exists a vertex $v \in V \setminus U$, then $U \neq V$. Thus, for our proof that $U \neq V$, we may assume that there exists an edge $e \in \mathcal{S}[V] \setminus \mathcal{S}_{i'}[U] \subseteq \mathcal{F}_{i'}[V \cap V_{i'}] \setminus \mathcal{S}_{i'}[U]$. If $|J| \leq k-1$, then $\mathcal{F}_{i'}[J] = \emptyset$ and if $|J| \geq k$, then, since F is a subsequence of a vertex-separated loose path, due to (3.7.1), we have $|J| = k$ and furthermore $\mathcal{F}_{i'}[J] \subseteq \mathcal{F}_{i'-1}[V \cap V_{i'-1}]$ or $\mathcal{F}_{i'}[J] \subseteq \mathcal{F}_{i'+1}[V \cap V_{i'+1}]$. Hence, in any case, we have $\mathcal{F}_{i'}[J] \subseteq \mathcal{S}_{i'}[U]$ and thus $e \in \mathcal{F}_{i'}[V \cap V_{i'}] \setminus \mathcal{F}_{i'}[J]$. This implies that there exists

$$v \in e \setminus J \subseteq (V \cap V_{i'}) \setminus J = (V \cap V_{i'}) \setminus U \subseteq V \setminus U,$$

so we have $U \neq V$.

It remains to prove that $\rho_{\mathcal{C},U} \leq \rho_{\mathcal{F}}$. To this end, let $\mathcal{A} := \mathcal{F}_{i'}[V \cap V_{i'}]$ and note that for all $1 \leq i \leq \ell$ with $i \neq i'$ and $f \in \mathcal{F}_i[V \cap V_i]$, we have $f \subseteq U$ and hence $f \in \mathcal{C}[U]$. Thus,

$$\mathcal{C} \setminus \mathcal{C}[U] = \Big(\bigcup_{1 \leq i \leq \ell} \mathcal{F}_i[V \cap V_i] \Big) \setminus \mathcal{C}[U] = \mathcal{A} \setminus \mathcal{C}[U] = \mathcal{A} \setminus (\mathcal{C}[U] \cap \mathcal{A}) = \mathcal{A} \setminus \mathcal{A}[J].$$

Furthermore, for all $1 \leq i \leq \ell$ with $i \neq i'$ and $v \in V \cap V_i$, we have $v \in U$, so we also have

$$V \setminus U = \Big(\bigcup_{1 \leq i \leq \ell} V \cap V_i \Big) \setminus U = (V \cap V_i) \setminus U = (V \cap V_i) \setminus J.$$

Thus, $\rho_{\mathcal{C},U} = \rho_{\mathcal{A},J}$. Hence, since (3.7.1) states that we have $J = V_{\mathcal{A}} \cap V_{i'-1} \cap V_{i'}$ or $J = V_{\mathcal{A}} \cap V_{i'} \cap V_{i'+1}$, Lemma 3.6.9 entails $\rho_{\mathcal{C},U} \leq \rho_{\mathcal{F}}$, which completes the proof as explained above. $\square$

Lemma 3.7.7. *Suppose that $F = \mathcal{F}_1, \dots, \mathcal{F}_\ell$ is minimally $\mathfrak{c}$-sufficient for some chain $\mathfrak{c} \in \mathfrak{C}$. Then, $\ell \le 1/\varepsilon^{4k}$.*

Proof. Suppose that $\mathfrak{c} = (V, F, I)$. By Lemma 3.6.3, we have $|V| \le \varepsilon^{-3}$, hence $|\mathcal{C}_\mathfrak{c}| \le \varepsilon^{-3k}$ and thus, since F is minimally $\mathfrak{c}$-sufficient, $\ell \le \varepsilon^{-3} + \varepsilon^{-3k}$. □

Lemma 3.7.8. *Suppose that $\mathfrak{c} \in \mathfrak{C}$. Let $e \in \mathcal{C}_\mathfrak{c} \setminus \mathcal{C}_\mathfrak{c}[I]$ and $\beta\colon f \overset{\sim}{\rightarrow} e$ where $f \in \mathcal{F}$. Suppose that $\mathfrak{b}$ is the β-branching of $\mathfrak{c}$. Let $e' \in \mathcal{F}^\beta_\mathfrak{c} \setminus \{e\}$ and $\beta'\colon f' \overset{\sim}{\rightarrow} e'$ where $f' \in \mathcal{F}$. Suppose that $\mathfrak{b}'$ is the β'-branching of $\mathfrak{b}$. Then,*

$$\chi_{\mathfrak{b}'} \le \chi_\mathfrak{b} - 1 \quad \text{or} \quad \chi_{\mathfrak{b}'} = \chi_\mathfrak{b}.$$

Furthermore, if $\chi_{\mathfrak{b}'} = \chi_\mathfrak{b}$, then $\mathfrak{B}^e_\mathfrak{c}$ and $\mathfrak{B}^{e'}_\mathfrak{b}$ are template equivalent.

Proof. Suppose that $\mathfrak{b} = (F, V, I)$ and $\mathfrak{b}' = (F', V', I)$. From all minimally $\mathfrak{b}$-sufficient sequences, choose $\mathcal{F}_1, \dots, \mathcal{F}_{\ell-1}$ such that $\chi_{\mathcal{F}_1,\dots,\mathcal{F}_{\ell-1}}$ is minimal. Let $\mathcal{F}_\ell := \mathcal{F}^{\beta'}_\mathfrak{b}$. For $1 \le i \le \ell$, let $V_i := V_{\mathcal{F}_i}$. Observe that the sequence $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ is $\mathfrak{b}'$-sufficient. Consider a minimally $\mathfrak{b}'$-sufficient subsequence $\mathcal{F}_{i_1}, \dots, \mathcal{F}_{i_{\ell'}}$ of $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ with $i_1 = 1$. Note that $i_{\ell'} = \ell$. To shorten notation, for $1 \le i, j \le \ell$, we set

$$f(i,j) := \varepsilon^{5k|V_i \cap V_j|}.$$

For all $1 \le j \le \ell' - 1$ with $i_{j+1} = i_j + 1$, we vacuously have

$$\sum_{i_j \le i \le i_{j+1}-1} f(i, i+1) = f(i_j, i_{j+1})$$

and for all $1 \le j \le \ell' - 2$ with $i_{j+1} \ge i_j + 2$, Lemma 3.7.6 together with Lemma 3.7.7 implies

$$\begin{aligned}\sum_{i_j \le i \le i_{j+1}-1} f(i, i+1) &\le (i_{j+1} - i_j)\varepsilon^{5k} f(i_j, i_{j+1}) \le \varepsilon f(i_j, i_{j+1}) \\ &= f(i_j, i_{j+1}) - (1-\varepsilon) f(i_j, i_{j+1}) \le f(i_j, i_{j+1}) - \frac{\varepsilon^{5k^2}}{2}.\end{aligned}$$

For

$$\Lambda := \begin{cases} 1 & \text{if } i_1, \dots, i_{\ell'-1} \ne 1, \dots, \ell - 2; \\ 0 & \text{otherwise,} \end{cases}$$

using that V' is the disjoint union of $V' \setminus V$ and $V \cap V' = V \setminus (V \setminus V')$, this yields

$$\begin{aligned}\chi_{\mathfrak{b}'} &\le |V'| - \varepsilon^{-5k(k+1)} \Big(\sum_{1 \le j \le \ell'-2} f(i_j, i_{j+1}) \Big) - \varepsilon^{-5k(k+1)} f(i_{\ell'-1}, \ell) \\ &\le |V'| - \varepsilon^{-5k(k+1)} \Big(\frac{\varepsilon^{5k^2}}{2}\Lambda + \sum_{1 \le j \le \ell'-2} \sum_{i_j \le i \le i_{j+1}-1} f(i, i+1) \Big) - \varepsilon^{-5k(k+1)} f(i_{\ell'-1}, \ell)\end{aligned}$$

$$
\begin{aligned}
&= |V'| - \varepsilon^{-5k(k+1)}\Big(\sum_{1\le i\le i_{\ell'-1}-1} f(i,i+1)\Big) - \frac{\varepsilon^{-5k}}{2}\Lambda - \varepsilon^{-5k(k+1)}f(i_{\ell'-1},\ell)\\
&= |V| + m - k - |V\setminus V'| - \varepsilon^{-5k(k+1)}\Big(\sum_{1\le i\le \ell-2} f(i,i+1)\Big)\\
&\qquad + \varepsilon^{-5k(k+1)}\Big(\sum_{i_{\ell'-1}\le i\le \ell-2} f(i,i+1)\Big) - \frac{\varepsilon^{-5k}}{2}\Lambda - \varepsilon^{-5k(k+1)}f(i_{\ell'-1},\ell)\\
&= \chi_{\mathfrak{b}} + m - k - |V\setminus V'| + \varepsilon^{-5k(k+1)}\Big(\sum_{i_{\ell'-1}\le i\le \ell-2} f(i,i+1)\Big)\\
&\qquad - \varepsilon^{-5k(k+1)}f(i_{\ell'-1},\ell) - \frac{\varepsilon^{-5k}}{2}\Lambda.
\end{aligned}
$$

Note that $i_{\ell'-1} \le \ell-3$, $i_{\ell'-1} = \ell-2$ or $i_{\ell'-1} = \ell-1$. We investigate the three cases separately.

First, suppose that $i_{\ell'-1} \le \ell-3$. Then using Lemma 3.7.6, Lemma 3.7.7 and Lemma 3.7.5, we obtain

$$
\begin{aligned}
\sum_{i_{\ell'-1}\le i\le \ell-2} f(i,i+1) &\le (\ell - i_{\ell'-1} - 1)\varepsilon^{5k} f(i_{\ell'-1},\ell-1) \le \varepsilon f(i_{\ell'-1},\ell-1)\\
&= f(i_{\ell'-1},\ell-1) - (1-\varepsilon)f(i_{\ell'-1},\ell-1) \le f(i_{\ell'-1},\ell-1) - \frac{\varepsilon^{5k^2}}{2}\\
&\le f(i_{\ell'-1},\ell) - \frac{\varepsilon^{5k^2}}{2}.
\end{aligned}
$$

Hence, if $i_{\ell'-1} \le \ell-3$, then

$$\chi_{\mathfrak{b}'} \le \chi_{\mathfrak{b}} + m - k - |V\setminus V'| - \frac{\varepsilon^{-5k}}{2} \le \chi_{\mathfrak{b}} - 1.$$

Next, suppose that $i_{\ell'-1} = \ell-2$. If

$$|V_{i_{\ell'-1}} \cap V_{\ell-1}| \ge |V_{i_{\ell'-1}} \cap V_{\ell}| + 1,$$

then

$$
\begin{aligned}
\sum_{i_{\ell'-1}\le i\le \ell-2} f(i,i+1) &= f(i_{\ell'-1},\ell-1) \le \varepsilon^{5k} f(i_{\ell'-1},\ell)\\
&= f(i_{\ell'-1},\ell) - (1-\varepsilon^{5k})f(i_{\ell'-1},\ell) \le f(i_{\ell'-1},\ell) - \frac{\varepsilon^{5k^2}}{2}
\end{aligned}
$$

and thus

$$\chi_{\mathfrak{b}'} \le \chi_{\mathfrak{b}} + m - k - |V\setminus V'| - \frac{\varepsilon^{-5k}}{2} \le \chi_{\mathfrak{b}} - 1.$$

If

$$|V_{i_{\ell'-1}} \cap V_{\ell-1}| \le |V_{i_{\ell'-1}} \cap V_{\ell}|, \tag{3.7.2}$$

then Lemma 3.7.5 entails

$$V_{i_{\ell'-1}} \cap V_{\ell-1} = V_{i_{\ell'-1}} \cap V_\ell, \tag{3.7.3}$$

and thus

$$\sum_{i_{\ell'-1} \leq i \leq \ell-2} f(i, i+1) = f(i_{\ell'-1}, \ell-1) = f(i_{\ell'-1}, \ell).$$

Due to Lemma 3.7.5, a consequence of (3.7.3) is

$$V_{i_{\ell'-1}} \cap V_{\ell-1} \subseteq V_{\ell-1} \cap V_\ell.$$

Since we assume that $i_{\ell'-1} = \ell - 2$, this yields

$$V_{\ell-1} \cap V' \subseteq (V_{i_{\ell'-1}} \cup V_\ell) \cap V_{\ell-1} \subseteq V_\ell \cap V_{\ell-1} = e' \tag{3.7.4}$$

and so in particular $|V_{\ell-1} \cap V'| \leq k$ and thus $|V \setminus V'| \geq m - k$. Hence, if (3.7.2) holds, then

$$\chi_{\mathfrak{b}'} \leq \chi_{\mathfrak{b}} + m - k - |V \setminus V'| - \frac{\varepsilon^{-5k}}{2}\Lambda \leq \chi_{\mathfrak{b}} - \frac{\varepsilon^{-5k}}{2}\Lambda \tag{3.7.5}$$

and thus $\chi_{\mathfrak{b}'} \leq \chi_{\mathfrak{b}} - 1$ or $\chi_{\mathfrak{b}'} = \chi_{\mathfrak{b}}$.

Finally, suppose that $i_{\ell'-1} = \ell - 1$. Then,

$$\chi_{\mathfrak{b}'} \leq \chi_{\mathfrak{b}} + m - k - |V \setminus V'| - \varepsilon^{-5k(k+1)} f(i_{\ell'-1}, \ell) \leq \chi_{\mathfrak{b}} + m - k - |V \setminus V'| - \varepsilon^{-5k} \leq \chi_{\mathfrak{b}} - 1.$$

This finishes the analysis of the three cases and the proof that we have $\chi_{\mathfrak{b}'} \leq \chi_{\mathfrak{b}} - 1$ or $\chi_{\mathfrak{b}'} = \chi_{\mathfrak{b}}$.

It remains to further investigate the case where $\chi_{\mathfrak{b}'} = \chi_{\mathfrak{b}}$. Suppose that $\chi_{\mathfrak{b}'} = \chi_{\mathfrak{b}}$. Note that by Lemma 3.7.3, it suffices to obtain that $\mathfrak{B}^{e}_{\mathfrak{b}|-}$ and $\mathfrak{B}^{e'}_{\mathfrak{b}'|-}$ are template equivalent, so due to Lemma 3.7.4, it suffices to show that $(\mathcal{C}_{\mathfrak{b}'|-}, I)$ is a copy of $(\mathcal{C}_{\mathfrak{b}|-}, I)$ with e' playing the role of e. Our analysis of the three cases above shows that $\chi_{\mathfrak{b}'} = \chi_{\mathfrak{b}}$ is only possible if $i_{\ell'-1} = \ell - 2$, (3.7.3), (3.7.4) and (3.7.5) hold. Revisiting the first inequality in (3.7.5), we see that $\Lambda = 0$ and $|V \setminus V'| = m - k$ necessarily hold. Let $\mathcal{S} := \mathcal{F}_1 + \ldots + \mathcal{F}_{\ell-2}$, let $\mathcal{E}$ denote the k-graph with vertex set e and edge set $\{e\}$ and let $\mathcal{E}'$ denote the k-graph with vertex set e' and edge set $\{e'\}$. Note that $\mathcal{C}_{\mathfrak{b}|-} = \mathcal{S}[V \cap V_{\mathcal{S}}] + \mathcal{E}$ and that as a consequence of $\Lambda = 0$, we have $\mathcal{C}_{\mathfrak{b}'|-} = \mathcal{S}[V' \cap V_{\mathcal{S}}] + \mathcal{E}'$. Thus, to see that $(\mathcal{C}_{\mathfrak{b}'|-}, I)$ is a copy of $(\mathcal{C}_{\mathfrak{b}|-}, I)$ with e' playing the role of e, it suffices to obtain $V \cap V_{\mathcal{S}} = V' \cap V_{\mathcal{S}}$ and additionally $e \cap V \cap V_{\mathcal{S}} = e' \cap V' \cap V_{\mathcal{S}}$. Since (3.7.4) entails

$$V_{\ell-1} \setminus e' \subseteq V_{\ell-1} \setminus (V_{\ell-1} \cap V') = V_{\ell-1} \setminus V' \subseteq V \setminus V',$$

from $|V \setminus V'| = m - k$, we obtain $V \setminus V' = V_{\ell-1} \setminus e'$ and thus using (3.7.3), we have

$$\begin{aligned}(V \cap V_{\mathcal{S}}) \setminus (V' \cap V_{\mathcal{S}}) &= (V \setminus V') \cap V_{\mathcal{S}} = (V_{\ell-1} \setminus e') \cap V_{\mathcal{S}} = (V_{\ell-1} \cap V_{\mathcal{S}}) \setminus e' \\ &= (V_{\ell-1} \cap V_{\ell-2}) \setminus e' = (V_\ell \cap V_{\ell-1} \cap V_{\ell-2}) \setminus e' = \emptyset.\end{aligned}$$

Since $V' \cap V_{\mathcal{S}} \subseteq V \cap V_{\mathcal{S}}$, this yields

$$V \cap V_{\mathcal{S}} = V' \cap V_{\mathcal{S}}. \tag{3.7.6}$$

Furthermore, again using (3.7.3), we obtain

$$e \cap V_{\ell-2} = V_{\ell-1} \cap V_{\ell-2} = V_\ell \cap V_{\ell-1} \cap V_{\ell-2} = e' \cap V_{\ell-2}.$$

Combining this with (3.7.6) yields

$$\begin{aligned} e \cap V \cap V_{\mathcal{S}} &= (e \cap V_{\ell-2}) \cap V \cap V_{\mathcal{S}} = (e' \cap V_{\ell-2}) \cap V \cap V_{\mathcal{S}} = (e' \cap V_{\ell-2}) \cap V' \cap V_{\mathcal{S}} \\ &= e' \cap V_{\ell-1} \cap V_{\ell-2} \cap V' \cap V_{\mathcal{S}}. \end{aligned}$$

Since Lemma 3.7.5 entails $V_{\ell-1} \cap V_i \subseteq V_{\ell-1} \cap V_{\ell-2}$ for all $1 \le i \le \ell - 2$, thus $V_{\ell-1} \cap V_{\mathcal{S}} \subseteq V_{\ell-1} \cap V_{\ell-2}$ and hence $V_{\ell-1} \cap V_{\mathcal{S}} = V_{\ell-1} \cap V_{\ell-2} \cap V_{\mathcal{S}}$, this yields

$$e \cap V \cap V_{\mathcal{S}} = e' \cap V_{\ell-1} \cap V' \cap V_{\mathcal{S}} = e' \cap V' \cap V_{\mathcal{S}},$$

which completes the proof. □

3.7.2 Tracking branching families

Suppose that $0 \le i \le i^\star$, consider a chain $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$ and let $\psi\colon I \hookrightarrow V_{\mathcal{H}}$. Let $e \in \mathcal{C}_{\mathfrak{c}} \setminus \mathcal{C}_{\mathfrak{c}}[I]$. Similarly as in Section 3.6.3, we show that $\sum_{\mathfrak{b} \in \mathfrak{B}_{\mathfrak{c}}^e} \Phi_{\mathcal{C}_{\mathfrak{b}},\psi}$ is typically close to $\sum_{\mathfrak{b} \in \mathfrak{B}_{\mathfrak{c}}^e} \hat{\Phi}_{\mathfrak{b},\psi}$, that is that

$$X_{\mathfrak{c},\psi}^e := \sum_{\mathfrak{b} \in \mathfrak{B}_{\mathfrak{c}}^e} \Phi_{\mathcal{C}_{\mathfrak{b}},\psi} - \sum_{\mathfrak{b} \in \mathfrak{B}_{\mathfrak{c}}^e} \hat{\Phi}_{\mathfrak{b},\psi} = \sum_{\mathfrak{b} \in \mathfrak{B}_{\mathfrak{c}}^e} X_{\mathfrak{b},\psi}$$

is typically small, where the quantification of the deviation we allow crucially relies on the insights from Section 3.7.1. Formally, we finally define the fifth stopping time mentioned in Section 3.4 as

$$\tau_{\mathfrak{B}} := \min \left\{ \begin{array}{l} i \ge 0 : \sum_{\mathfrak{b} \in \mathfrak{B}_{\mathfrak{c}}^e} \Phi_{\mathcal{C}_{\mathfrak{b}},\psi} \ne \sum_{\mathfrak{b} \in \mathfrak{B}_{\mathfrak{c}}^e} \hat{\Phi}_{\mathfrak{b},\psi} \pm \varepsilon^{-\chi_{\mathfrak{B}_{\mathfrak{c}}^e}} \zeta \hat{\varphi}_{\mathfrak{b},I} \\ \qquad \text{for some } \mathfrak{c} = (F, V, I) \in \mathfrak{C}, e \in \mathcal{C}_{\mathfrak{c}} \setminus \mathcal{C}_{\mathfrak{c}}[I], \psi\colon I \hookrightarrow V_{\mathcal{H}} \end{array} \right\}$$

and we show that the probability that $\tau_{\mathfrak{B}} \le \tau^\star \wedge i^\star$ is small. The following Lemma 3.7.9 shows that indeed $\tilde{\tau}_{\mathfrak{B}} \ge \tau_{\mathfrak{B}}$. Similarly as in Section 3.6.3, Lemma 3.7.11 shows that it suffices to consider a collection of branching families that has size at most $1/\delta$, which in turn allows us to restrict our attention to only one fixed branching family. To prove Lemma 3.7.11, we observe that there are only finitely many relevant error parameters (see Lemma 3.7.10).

Lemma 3.7.9. *Let* $\mathfrak{c} \in \mathfrak{C}$. *Then,* $\delta^{1/2} \le \varepsilon^{-\chi_{\mathfrak{c}}} \le \delta^{-1/2}$.

Proof. Suppose that $\mathfrak{c} = (F, V, I)$. From Lemma 3.6.3, we obtain $\chi_{\mathfrak{c}} \le |V| \le \varepsilon^{-3}$ and from Lemma 3.7.7, we obtain $\chi_{\mathfrak{c}} \ge -\varepsilon^{-5k(k+1)} \cdot \varepsilon^{-4k^2}$, so the statement follows. □

Lemma 3.7.10. *The set* $\{\chi_{\mathfrak{c}} : \mathfrak{c} \in \mathfrak{C}\}$ *is finite.*

Proof. As a consequence of Lemma 3.6.3, it suffices to show that

$$X := \{\chi_F : \ F \text{ is minimally } \mathfrak{c}\text{-sufficient for some } \mathfrak{c} \in \mathfrak{C}\ \}$$

is finite. By Lemma 3.7.7, every sequence that is minimally $\mathfrak{c}$-sufficient for some $\mathfrak{c} \in \mathfrak{C}$ has length at most ε^{-4k}, which entails that X is indeed finite. $\square$

Lemma 3.7.11. *There exists a collection $\mathfrak{C}_0 \subseteq \mathfrak{C}$ with $|\mathfrak{C}_0| \leq 1/\delta$ such that for all $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$ and $e \in \mathcal{C}_\mathfrak{c} \setminus \mathcal{C}_\mathfrak{c}[I]$, there exist $\mathfrak{c}_0 = (F_0, V_0, I_0) \in \mathfrak{C}_0$ and $e_0 \in \mathcal{C}_{\mathfrak{c}_0} \setminus \mathcal{C}_{\mathfrak{c}_0}[I]$ such that $\mathfrak{B}^{e_0}_{\mathfrak{c}_0}$ and $\mathfrak{B}^e_\mathfrak{c}$ are template equivalent with $\chi_{\mathfrak{B}^{e_0}_{\mathfrak{c}_0}} \leq \chi_{\mathfrak{B}^e_\mathfrak{c}}$.*

Proof. Similarly as in the proof of Lemma 3.6.17, consider the set $\mathscr{T}$ of all templates $(\mathcal{A}, I)$ where $V_\mathcal{A} \subseteq \{1, \dots, 1/\varepsilon^3\}$. By Lemma 3.6.3, for all $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$, we may choose a template $\mathcal{T}_\mathfrak{c} \in \mathscr{T}$ that is a copy of $(\mathcal{C}_\mathfrak{c}, I)$. For every chain $\mathfrak{c} = (F, V, I)$ and $e \in \mathcal{C}_\mathfrak{c} \setminus \mathcal{C}_\mathfrak{c}[I]$, we may consider the unordered $(|\mathcal{F}|k!)$-tuple $((\mathcal{T}_\mathfrak{b}, \mathcal{T}_{\mathfrak{b}|-}) : \mathfrak{b} \in \mathfrak{B}^e_\mathfrak{c})$ whose components are the pairs $(\mathcal{T}_\mathfrak{b}, \mathcal{T}_{\mathfrak{b}|-})$ of templates where $\mathfrak{b} \in \mathfrak{B}^e_\mathfrak{c}$. We use $\mathscr{T}_2$ to denote the set of such unordered tuples, that is we set

$$\mathscr{T}_2 := \{((\mathcal{T}_\mathfrak{b}, \mathcal{T}_{\mathfrak{b}|-}) : \mathfrak{b} \in \mathfrak{B}^e_\mathfrak{c}) : \mathfrak{c} = (F, V, I) \in \mathfrak{C}, e \in \mathcal{C}_\mathfrak{c} \setminus \mathcal{C}_\mathfrak{c}[I]\}.$$

Note that $|\mathscr{T}_2| \leq |\mathscr{T}|^{2|\mathcal{F}|k!}$. Consider an unordered tuple $\mathscr{P} \in \mathscr{T}_2$. As a consequence of Lemma 3.7.10, among all pairs $(\mathfrak{c}, e)$ where $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$ and $e \in \mathcal{C}_\mathfrak{c} \setminus \mathcal{C}_\mathfrak{c}[I]$ such that $\mathscr{P} = ((\mathcal{T}_\mathfrak{b}, \mathcal{T}_{\mathfrak{b}|-}) : \mathfrak{b} \in \mathfrak{B}^e_\mathfrak{c})$, we may choose a pair $(\mathfrak{c}_\mathscr{P}, e_\mathscr{P})$ such that $\chi_{\mathfrak{B}^{e_\mathscr{P}}_{\mathfrak{c}_\mathscr{P}}}$ is minimal. Then, $\{\mathfrak{c}_\mathscr{P} : \mathscr{P} \in \mathscr{T}_2\}$ is a collection as desired. $\square$

Observation 3.7.12. *Suppose that $\mathfrak{C}_0 \subseteq \mathfrak{C}$ is a collection of chains as in Lemma 3.7.11. For $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$, $e \in \mathcal{C}_\mathfrak{c} \setminus \mathcal{C}_\mathfrak{c}[I]$ and $\psi \colon I \hookrightarrow V_\mathcal{H}$, let*

$$\tau^e_{\mathfrak{c},\psi} := \min\Big\{ i \geq 0 : \sum_{\mathfrak{b} \in \mathfrak{B}^e_\mathfrak{c}} \Phi_{\mathfrak{b},\psi} \neq \sum_{\mathfrak{b} \in \mathfrak{B}^e_\mathfrak{c}} \hat{\Phi}_{\mathfrak{b},\psi} \pm \varepsilon^{-\chi_{\mathfrak{B}^e_\mathfrak{c}}} \zeta \hat{\varphi}_{\mathfrak{b},I} \Big\}.$$

Then,

$$\mathbb{P}[\tau_\mathfrak{B} \leq \tau^\star \wedge i^\star] \leq \sum_{\substack{\mathfrak{c}=(F,V,I)\in\mathfrak{C}_0,\\ e\in\mathcal{C}_\mathfrak{c}\setminus\mathcal{C}_\mathfrak{c}[I], \psi\colon I\hookrightarrow V_\mathcal{H}}} \mathbb{P}[\tau^e_{\mathfrak{c},\psi} \leq \tau^\star \wedge i^\star].$$

Hence, fix $\mathfrak{c} = (F, V, I) \in \mathfrak{C}$, $e \in \mathcal{C}_\mathfrak{c} \setminus \mathcal{C}_\mathfrak{c}[I]$ and $\psi \colon I \hookrightarrow V_\mathcal{H}$ and let $\chi := \chi_{\mathfrak{B}^e_\mathfrak{c}}$. Besides $\mathfrak{c}$ and ψ, we redefine several other symbols from Section 3.6, for example ξ_0, ξ_1 and τ. However, we still use some symbols from previous sections that we do not redefine. Whenever we use a symbol, its most recent definition applies. For $i \geq 0$, let

$$\xi_1(i) := \sum_{\mathfrak{b} \in \mathfrak{B}^e_\mathfrak{c}} \varepsilon^{-\chi} \zeta \hat{\varphi}_{\mathfrak{b},I}, \quad \xi_0(i) := (1 - \delta)\xi_1$$

and define the stopping time

$$\tau := \min\Big\{ i \geq 0 : \sum_{\mathfrak{b} \in \mathfrak{B}^e_\mathfrak{c}} \Phi_{\mathcal{C}_\mathfrak{b},\psi} \neq \Big(\sum_{\mathfrak{b} \in \mathfrak{B}^e_\mathfrak{c}} \hat{\Phi}_{\mathfrak{b},\psi} \Big) \pm \xi_1 \Big\}.$$

Define the critical intervals

$$I^-(i) := \Big[\Big(\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\Phi}_{\mathfrak{b},\psi}\Big) - \xi_1, \Big(\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\Phi}_{\mathfrak{b},\psi}\Big) - \xi_0\Big],$$
$$I^+(i) := \Big[\Big(\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\Phi}_{\mathfrak{b},\psi}\Big) + \xi_0, \Big(\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\Phi}_{\mathfrak{b},\psi}\Big) + \xi_1\Big].$$

For $\divideontimes \in \{-,+\}$, let

$$Y^{\divideontimes}(i) := \divideontimes X^e_{\mathfrak{c},\psi} - \xi_1.$$

For $i_0 \geq 0$, define the stopping time

$$\tau^{\divideontimes}_{i_0} := \min\Big\{i \geq i_0 : \sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \Phi_{\mathcal{C}_{\mathfrak{c}},\psi} \notin I^{\divideontimes}\Big\}$$

and for $i \geq i_0$, let

$$Z^{\divideontimes}_{i_0}(i) := Y^{\divideontimes}(i_0 \vee (i \wedge \tau^{\divideontimes}_{i_0} \wedge \tau^\star \wedge i^\star)).$$

Let

$$\sigma^{\divideontimes} := \min\{j \geq 0 : \divideontimes X^e_{\mathfrak{c},\psi} \geq \xi_0 \text{ for all } j \leq i < \tau^\star \wedge i^\star\} \leq \tau^\star \wedge i^\star.$$

With this setup, similarly as in Section 3.6.3, it in fact suffices to consider the evolution of $Z^{\divideontimes}_{\sigma^{\divideontimes}}(\sigma^{\divideontimes}), Z^{\divideontimes}_{\sigma^{\divideontimes}}(\sigma^{\divideontimes}+1), \ldots$.

Observation 3.7.13. $\{\tau \leq \tau^\star \wedge i^\star\} \subseteq \{Z^-_{\sigma^-}(i^\star) > 0\} \cup \{Z^+_{\sigma^+}(i^\star) > 0\}$.

We again use Lemma 2.9.4 to show that the probabilities of the events on the right in Observation 3.7.13 are sufficiently small.

Trend

Here, we prove that for all $\divideontimes \in \{-,+\}$ and $i_0 \geq 0$, the expected one-step changes of the process $Z^{\divideontimes}_{i_0}(i_0), Z^{\divideontimes}_{i_0}(i_0+1), \ldots$ are non-positive. Branching families are closely related to individual chains, so we may use statements from Section 3.6.3 as a starting point for our arguments here. As a consequence of Lemma 3.6.21, we obtain Lemma 3.7.14 where we state estimates for the one-step changes of the error term that we use in this section. Using these estimates, we turn to proving that the process we consider here is indeed a supermartingale (see Lemma 3.7.18). We prove this by revisiting the expression for individual chains stated in Lemma 3.6.28 where, since we are now in the setting of branching families, we may now exploit that one step-changes depend on branching families. This allows us to no longer differentiate between the different branchings as they always appear in complete families. This ultimately enables us to identify self-correcting behavior as desired as a consequence of our careful choice of error parameters crucially relying on the insights from Section 3.7.1.

Note that for $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}^e_{\mathfrak{c}}$, we have $|\mathcal{C}_{\mathfrak{b}}| = |\mathcal{C}_{\mathfrak{b}'}|$. Hence, we may choose b such that $b = |\mathcal{C}_{\mathfrak{b}}|$ for all $\mathfrak{b} \in \mathfrak{B}^e_{\mathfrak{c}}$.

Lemma 3.7.14. *Let* $0 \le i \le i^\star$ *and* $\mathcal{X} := \{i \le \tau_\emptyset\}$. *Then,*

$$\Delta\xi_1 =_{\mathcal{X}} -\left(b - 1 - \frac{\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|\xi_1}{H} \pm \frac{\zeta^2\xi_1}{H}.$$

Proof. For $\mathfrak{b} \in \mathfrak{B}$, we may apply Lemma 3.6.21 with $\mathfrak{b}$ playing the role of $\mathfrak{c}$ to obtain

$$\Delta(\delta^{-1}\zeta\hat{\varphi}_{\mathfrak{b},I}) = -\left(|\mathcal{C}_{\mathfrak{b}}| - 1 - \frac{\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|\delta^{-1}\zeta\hat{\varphi}_{\mathfrak{b},I}}{H} \pm \frac{\zeta^2\delta^{-1}\zeta\hat{\varphi}_{\mathfrak{b},I}}{H}.$$

This yields

$$\Delta\xi_1 = \sum_{\mathfrak{b}\in\mathfrak{B}} \varepsilon^{-\chi}\delta\Delta(\delta^{-1}\zeta\hat{\varphi}_{\mathfrak{b},I}) =_{\mathcal{X}} -\left(b - 1 - \frac{\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|\sum_{\mathfrak{b}\in\mathfrak{B}}\varepsilon^{-\chi}\zeta\hat{\varphi}_{\mathfrak{b},I}}{H} \pm \frac{\zeta^2\sum_{\mathfrak{b}\in\mathfrak{B}}\varepsilon^{-\chi}\zeta\hat{\varphi}_{\mathfrak{b},I}}{H},$$

which completes the proof. □

Lemma 3.7.15. *Let* $0 \le i_0 \le i$ *and* $\divideontimes \in \{-,+\}$. *Then,* $\mathbb{E}_i[\Delta Z^{\divideontimes}_{i_0}] \le 0$.

Proof. Suppose that $i < i^\star$ and let $\mathcal{X} := \{i < \tau^{\divideontimes}_{i_0} \wedge \tau^\star\}$. We have $\mathbb{E}_i[\Delta Z^{\divideontimes}_{i_0}] =_{\mathcal{X}^{\mathrm{c}}} 0$ and $\mathbb{E}_i[\Delta Z^{\divideontimes}_{i_0}] =_{\mathcal{X}} \mathbb{E}_i[\Delta Y^{\divideontimes}]$, so it suffices to obtain $\mathbb{E}_i[\Delta Y^{\divideontimes}] \le_{\mathcal{X}} 0$. From Lemma 3.6.23, using Lemma 3.7.9, we obtain

$$\begin{aligned}
&\mathbb{E}_i[\Delta(\divideontimes X^e_{\mathfrak{c},\psi})]\\
&\quad= \divideontimes \sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \mathbb{E}_i[\Delta X_{\mathfrak{b},\psi}]\\
&\quad\le_{\mathcal{X}} \divideontimes \sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}}\left(-\frac{(|\mathcal{C}_{\mathfrak{b}}|-1)|\mathcal{F}|}{H}X_{\mathfrak{b},\psi} - \left(\sum_{e'\in\mathcal{G}_{\mathfrak{b}}\setminus\{J_{\mathfrak{b}}\}}\sum_{\mathfrak{b}'\in\mathfrak{B}^{e'}_{\mathfrak{b}}}\frac{\hat{\varphi}_{\mathfrak{b},I}}{k!\,H\hat{\varphi}_{\mathfrak{b}',I}}X_{\mathfrak{b}',\psi}\right) + \delta^2\frac{\delta^{-1}\zeta\hat{\varphi}_{\mathfrak{b},I}}{H}\right)\\
&\quad\le -\frac{|\mathcal{F}|}{H}\left((b-1)(\divideontimes X^e_{\mathfrak{c},\psi}) + \frac{1}{|\mathcal{F}|k!}\left(\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}}\sum_{e'\in\mathcal{G}_{\mathfrak{b}}\setminus\{J_{\mathfrak{b}}\}}\sum_{\mathfrak{b}'\in\mathfrak{B}^{e'}_{\mathfrak{b}}}\frac{\hat{\varphi}_{\mathfrak{b},I}}{\hat{\varphi}_{\mathfrak{b}',I}}(\divideontimes X_{\mathfrak{b}',\psi})\right) - \varepsilon\xi_1\right)
\end{aligned}$$

Note that for all $\mathfrak{b} \in \mathfrak{B}^e_{\mathfrak{c}}$, $e' \in \mathcal{G}_{\mathfrak{b}} \setminus \{J_{\mathfrak{b}}\}$ and $\mathfrak{b}'_1, \mathfrak{b}'_2 \in \mathfrak{B}^{e'}_{\mathfrak{b}}$, we have $\hat{\varphi}_{\mathfrak{b}'_1,I} = \hat{\varphi}_{\mathfrak{b}'_2,I}$, so we may choose $\hat{\varphi}^{e'}_{\mathfrak{b},I}$ such that $\hat{\varphi}^{e'}_{\mathfrak{b},I} = \hat{\varphi}_{\mathfrak{b}',I}$ for all $\mathfrak{b}' \in \mathfrak{B}^{e'}_{\mathfrak{b}}$. With Lemma 3.6.3, we obtain

$$\begin{aligned}
&\mathbb{E}_i[\Delta(\divideontimes X^e_{\mathfrak{c},\psi})]\\
&\qquad\le_{\mathcal{X}} -\frac{|\mathcal{F}|}{H}\left((b-1)(\divideontimes X^e_{\mathfrak{c},\psi}) + \frac{1}{|\mathcal{F}|k!}\left(\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}}\sum_{e'\in\mathcal{G}_{\mathfrak{b}}\setminus\{J_{\mathfrak{b}}\}}\frac{\hat{\varphi}_{\mathfrak{b},I}}{\hat{\varphi}^{e'}_{\mathfrak{b},I}}\sum_{\mathfrak{b}'\in\mathfrak{B}^{e'}_{\mathfrak{b}}}\divideontimes X_{\mathfrak{b}',\psi}\right) - \varepsilon\xi_1\right)\\
&\qquad\le_{\mathcal{X}} -\frac{|\mathcal{F}|}{H}\left((b-1)(1-\delta)\xi_1 + \frac{1}{|\mathcal{F}|k!}\left(\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}}\sum_{e'\in\mathcal{G}_{\mathfrak{b}}\setminus\{J_{\mathfrak{b}}\}}\frac{\hat{\varphi}_{\mathfrak{b},I}}{\hat{\varphi}^{e'}_{\mathfrak{b},I}}(\divideontimes X^{e'}_{\mathfrak{b},\psi})\right) - \varepsilon\xi_1\right)\\
&\qquad\le -\frac{|\mathcal{F}|}{H}\left((b-1)\xi_1 + \frac{1}{|\mathcal{F}|k!}\left(\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}}\sum_{e'\in\mathcal{G}_{\mathfrak{b}}\setminus\{J_{\mathfrak{b}}\}}\frac{\hat{\varphi}_{\mathfrak{b},I}}{\hat{\varphi}^{e'}_{\mathfrak{b},I}}(\divideontimes X^{e'}_{\mathfrak{b},\psi})\right) - \varepsilon^{1/2}\xi_1\right).
\end{aligned}$$

Thus, due to Lemma 3.7.14, we have

$$\mathbb{E}_i[\Delta Y^{\ast}] \leq_{\mathcal{X}} -\frac{|\mathcal{F}|}{H}\left(\frac{\rho_{\mathcal{F}}}{2}\xi_1 + \frac{1}{|\mathcal{F}|k!}\left(\sum_{\mathfrak{b}\in\mathfrak{B}_{\mathfrak{c}}^{e}}\sum_{e'\in\mathcal{G}_{\mathfrak{b}}\setminus\{J_{\mathfrak{b}}\}}\frac{\hat{\varphi}_{\mathfrak{b},I}}{\hat{\varphi}_{\mathfrak{b},I}^{e'}}(\ast X_{\mathfrak{b},\psi}^{e'})\right) - \varepsilon^{1/3}\xi_1\right). \tag{3.7.7}$$

Note that for all $\mathfrak{b}\in\mathfrak{B}_{\mathfrak{c}}^{e}$ and $e'\in\mathcal{G}_{\mathfrak{b}}\setminus\{J_{\mathfrak{b}}\}$, if $\mathfrak{B}_{\mathfrak{c}}^{e}$ and $\mathfrak{B}_{\mathfrak{b}}^{e'}$ are template equivalent, then

$$\ast X_{\mathfrak{b},\psi}^{e'} = \ast X_{\mathfrak{c},\psi}^{e} \geq_{\mathcal{X}} \xi_0 \geq 0$$

and otherwise, Lemma 3.7.8 implies

$$|\ast X_{\mathfrak{b},\psi}^{e'}| \leq_{\mathcal{X}} \sum_{\mathfrak{b}'\in\mathfrak{B}_{\mathfrak{b}}^{e'}} \varepsilon^{-\chi_{\mathfrak{b}'}}\zeta\hat{\varphi}_{\mathfrak{b}',I} \leq \varepsilon \sum_{\mathfrak{b}'\in\mathfrak{B}_{\mathfrak{b}}^{e'}} \varepsilon^{-\chi_{\mathfrak{b}}}\zeta\hat{\varphi}_{\mathfrak{b}',I} = \varepsilon|\mathcal{F}|k!\cdot\varepsilon^{-\chi}\zeta\hat{\varphi}_{\mathfrak{b},I}^{e'}.$$

Hence, in any case,

$$\begin{aligned}\sum_{\mathfrak{b}\in\mathfrak{B}_{\mathfrak{c}}^{e}}\sum_{e'\in\mathcal{G}_{\mathfrak{b}}\setminus\{J_{\mathfrak{b}}\}}\frac{\hat{\varphi}_{\mathfrak{b},I}}{\hat{\varphi}_{\mathfrak{b},I}^{e'}}(\ast X_{\mathfrak{b},\psi}^{e'}) &\geq_{\mathcal{X}} -\varepsilon|\mathcal{F}|k!\sum_{\mathfrak{b}\in\mathfrak{B}_{\mathfrak{c}}^{e}}\sum_{e'\in\mathcal{G}_{\mathfrak{b}}\setminus\{J_{\mathfrak{b}}\}}\varepsilon^{-\chi}\zeta\hat{\varphi}_{\mathfrak{b},I} = -\varepsilon|\mathcal{F}|k!\,|\mathcal{G}_{\mathfrak{b}}\setminus\{J_{\mathfrak{b}}\}|\xi_1\\ &\geq -\varepsilon^{1/2}\xi_1.\end{aligned}$$

Consequently, returning to (3.7.7), we obtain

$$\mathbb{E}_i[\Delta Y^{\ast}] \leq_{\mathcal{X}} -\frac{|\mathcal{F}|}{H}\left(\frac{\rho_{\mathcal{F}}}{2}\xi_1 - \varepsilon^{1/2}\xi_1 - \varepsilon^{1/3}\xi_1\right) \leq 0,$$

which completes the proof. $\square$

Boundedness

Here, we transfer the relevant results from Section 3.6.3 for individual chains, namely Lemma 3.6.26, Lemma 3.6.27 and Lemma 3.6.29, to branching families.

Lemma 3.7.16. *Let $0\leq i_0\leq i\leq i^\star$, $\ast\in\{-,+\}$ and $\mathcal{X}:=\{i<\tau_{\mathscr{B}}\wedge\tau_{\mathscr{B}'}\wedge\tau_{\mathfrak{C}}\}$. Then,*

$$|\Delta Y^{\ast}| \leq n^{\varepsilon^3}\frac{\sum_{\mathfrak{b}\in\mathfrak{B}_{\mathfrak{c}}^{e}}\hat{\varphi}_{\mathfrak{b},I}(i_0)}{n\hat{p}(i_0)^{\rho_{\mathcal{F}}}}.$$

Proof. Combining Lemma 3.7.14 and Lemma 3.6.25, we obtain

$$\begin{aligned}|\Delta Y^{\ast}| &\leq \Big(\sum_{\mathfrak{b}\in\mathfrak{B}_{\mathfrak{c}}^{e}}|\Delta X_{\mathfrak{b},\psi}|\Big) + |\Delta\xi_1| \leq_{\mathcal{X}} n^{\varepsilon^4}\frac{\sum_{\mathfrak{b}\in\mathfrak{B}_{\mathfrak{c}}^{e}}\hat{\varphi}_{\mathfrak{b},I}(i_0)}{n\hat{p}(i_0)^{\rho_{\mathcal{F}}}} + \frac{\sum_{\mathfrak{b}\in\mathfrak{B}_{\mathfrak{c}}^{e}}\hat{\varphi}_{\mathfrak{b},I}}{H}\\ &\leq n^{\varepsilon^4}\frac{\sum_{\mathfrak{b}\in\mathfrak{B}_{\mathfrak{c}}^{e}}\hat{\varphi}_{\mathfrak{b},I}(i_0)}{n\hat{p}(i_0)^{\rho_{\mathcal{F}}}} + \frac{\sum_{\mathfrak{b}\in\mathfrak{B}_{\mathfrak{c}}^{e}}\hat{\varphi}_{\mathfrak{b},I}(i_0)}{H(i_0)}\end{aligned}$$

With Lemma 3.5.9, this completes the proof. $\square$

Lemma 3.7.17. *Let* $0 \leq i_0 \leq i \leq i^\star$ *and* $\ast \in \{-,+\}$. *Then,*

$$|\Delta Z^{\ast}_{i_0}| \leq n^{\varepsilon^3} \frac{\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\varphi}_{\mathfrak{b},I}(i_0)}{n\hat{p}(i_0)^{\rho_{\mathcal{F}}}}.$$

Proof. This is an immediate consequence of Lemma 3.7.16. □

Lemma 3.7.18. *Let* $0 \leq i_0 \leq i^\star$ *and* $\ast \in \{-,+\}$. *Then,*

$$\sum_{i\geq i_0} \mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] \leq n^{\varepsilon^3} \sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\varphi}_{\mathfrak{b},I}.$$

Proof. Suppose that $i_0 \leq i < i^\star$ and let $\mathcal{X} := \{i < \tau_{\mathcal{H}^*} \wedge \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'} \wedge \tau_{\mathfrak{C}}\}$. We have $\mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] =_{\mathcal{X}^c} 0$ and with Lemma 3.7.14, Lemma 3.6.28 and Lemma 3.5.7, we obtain

$$\begin{aligned}\mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] \leq \mathbb{E}_i[|\Delta Y^{\ast}|] &\leq \Big(\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \mathbb{E}_i[|\Delta X_{\mathfrak{b},\psi}|]\Big) + |\Delta\xi_1| \leq_{\mathcal{X}} n^{\varepsilon^4} \frac{\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\varphi}_{\mathfrak{b},I}}{n^k\hat{p}} + \frac{\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\varphi}_{\mathfrak{b},I}}{H}\\ &\leq_{\mathcal{X}} n^{\varepsilon^3} \frac{\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\varphi}_{\mathfrak{b},I}}{n^k\hat{p}} \leq n^{\varepsilon^3} \frac{\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\varphi}_{\mathfrak{b},I}(i_0)}{n^k\hat{p}(i_0)}.\end{aligned}$$

Thus,

$$\sum_{i\geq i_0} \mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] = \sum_{i_0\leq i\leq i^\star-1} \mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] \leq (i^\star - i_0) n^{\varepsilon^3} \frac{\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\varphi}_{\mathfrak{b},I}(i_0)}{n^k\hat{p}(i_0)}.$$

Since

$$i^\star - i_0 \leq \frac{\vartheta n^k}{|\mathcal{F}|k!} - i_0 = \frac{n^k\hat{p}(i_0)}{|\mathcal{F}|k!} \leq n^k\hat{p}(i_0),$$

this completes the proof. □

Supermartingale argument

This section follows a similar structure as Section 3.6.3. Lemma 3.7.19 is the final ingredient that we use for our application of Lemma 2.9.4 in the proof of Lemma 3.7.20 where we show that the probabilities of the events on the right in Observation 3.7.13 are indeed small.

Lemma 3.7.19. *Let* $\ast \in \{-,+\}$. *Then,* $Z^{\ast}_{\sigma^{\ast}}(\sigma^{\ast}) \leq -\delta^2\xi_1(\sigma^{\ast})$.

Proof. Together with Lemma 3.6.1, Lemma 3.5.4 implies $\tau^\star \geq 1$ and $\ast X^e_{\mathfrak{c},\psi}(0) < \xi_0(0)$, so we have $\sigma^{\ast} \geq 1$. Thus, by definition of $\sigma^{\ast}$, for $i := \sigma^{\ast} - 1$, we have $\ast X^e_{\mathfrak{c},\psi} \leq \xi_0$ and thus

$$Z^{\ast}_i = \ast X^e_{\mathfrak{c},\psi} - \xi_1 \leq -\delta\xi_1.$$

Furthermore, since $\sigma^{\ast} \leq \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'} \wedge \tau_{\mathfrak{C}}$, we may apply Lemma 3.7.16 to obtain

$$Z^{\ast}_{\sigma^{\ast}}(\sigma^{\ast}) = Z^{\ast}_i + \Delta Y^{\ast} \leq Z^{\ast}_i + \delta^2\xi_1 \leq -\delta\xi_1 + \delta^2\xi_1 \leq -\delta^2\xi_1.$$

Since Lemma 3.6.5 entails $\Delta\xi_1 \leq 0$, this completes the proof. □

Lemma 3.7.20. $\mathbb{P}[\tau_{\mathfrak{B}} \leq \tau^\star \wedge i^\star] \leq \exp(-n^{\varepsilon^3})$.

Proof. Considering Observation 3.7.12, it suffices to show that

$$\mathbb{P}[\tau \leq \tau^\star \wedge i^\star] \leq \exp(-n^{2\varepsilon^3}).$$

Hence, by Observation 3.7.13, is suffices to show that for $\divideontimes \in \{-,+\}$, we have

$$\mathbb{P}[Z^{\divideontimes}_{\sigma^{\divideontimes}}(i^\star) > 0] \leq \exp(-n^{3\varepsilon^3}).$$

Due to Lemma 3.6.30, we have

$$\mathbb{P}[Z^{\divideontimes}_{\sigma^{\divideontimes}}(i^\star) > 0] \leq \mathbb{P}[Z^{\divideontimes}_{\sigma^{\divideontimes}}(i^\star) - Z^{\divideontimes}_{\sigma^{\divideontimes}}(\sigma^{\divideontimes}) > \delta^2\xi_1(\sigma^{\divideontimes})] \leq \sum_{0 \leq i \leq i^\star} \mathbb{P}[Z^{\divideontimes}_i(i^\star) - Z^{\divideontimes}_i > \delta^2\xi_1].$$

Thus, for $0 \leq i \leq i^\star$, it suffices to obtain

$$\mathbb{P}[Z^{\divideontimes}_i(i^\star) - Z^{\divideontimes}_i > \delta^2\xi_1] \leq \exp(-n^{4\varepsilon^3}).$$

We show that this bound is a consequence of Freedman's inequality for supermartingales.

Let us turn to the details. Lemma 3.7.15 shows that $Z^{\divideontimes}_i(i), Z^{\divideontimes}_i(i+1), \ldots$ is a supermartingale, while Lemma 3.7.17 provides the bound $|\Delta Z^{\divideontimes}_i(j)| \leq n^{\varepsilon^3}(\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\varphi}_{\mathfrak{b},I})/(n\hat{p}^{\rho_{\mathcal{F}}})$ for all $j \geq i$ and Lemma 3.7.18 provides the bound $\sum_{j\geq i} \mathbb{E}_j[|\Delta Z^{\divideontimes}_i(j)|] \leq n^{\varepsilon^3} \sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\varphi}_{\mathfrak{b},I}$. Hence, we may apply Lemma 2.9.4 such that using Lemma 3.7.9, we obtain

$$\begin{aligned}\mathbb{P}_i[Z^{\divideontimes}_i(i^\star) > 0] &\leq_{\mathcal{X}} \exp\left(-\frac{\delta^4\xi_1^2}{2n^{\varepsilon^3}\frac{\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\varphi}_{\mathfrak{b},I}}{n\hat{p}^{\rho_{\mathcal{F}}}}(\xi_1 + n^{\varepsilon^3}\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\varphi}_{\mathfrak{b},I})}\right)\\ &\leq \exp\left(-\frac{\delta^4\xi_1^2 n\hat{p}^{\rho_{\mathcal{F}}}}{4n^{2\varepsilon^3}(\sum_{\mathfrak{b}\in\mathfrak{B}^e_{\mathfrak{c}}} \hat{\varphi}_{\mathfrak{b},I})^2}\right) \leq \exp\left(-\frac{\delta^5 n^{2\varepsilon^2}}{4n^{2\varepsilon^3}}\right) \leq \exp(-n^{4\varepsilon^3}),\end{aligned}$$

which completes the proof. □

3.8 Proof of Theorem 3.3.2

In this section, we combine Lemma 3.5.19 with Lemma 3.6.31 and Lemma 3.7.20 to conclude that typically, we have $i^\star < \tau^\star$, see Lemma 3.8.1, which in turn yields a proof for Theorem 3.3.2.

Lemma 3.8.1. $\mathbb{P}[\tau^\star \leq i^\star] \leq \exp(-\log n)^{4/3})$.

Proof. Using Lemma 3.5.19, Lemma 3.6.31 and Lemma 3.7.20, we obtain

$$\begin{aligned}\mathbb{P}[\tau^\star \leq i^\star] &\leq \sum_{\tau\in\{\tau_{\mathcal{H}^*},\tau_{\mathscr{B}},\tau_{\mathscr{B}'},\tau_{\mathfrak{C}},\tau_{\mathfrak{B}}\}} \mathbb{P}[\tau \leq \tau^\star \wedge i^\star]\\ &\leq \Big(\sum_{\tau\in\{\tau_{\mathcal{H}^*},\tau_{\mathscr{B}},\tau_{\mathscr{B}'}\}} \mathbb{P}[\tau \leq \tilde{\tau}^\star \wedge i^\star]\Big) + \mathbb{P}[\tau_{\mathfrak{C}} \leq \tilde{\tau}^\star_{\mathfrak{C}} \wedge i^\star] + \mathbb{P}[\tau_{\mathfrak{B}} \leq \tau^\star \wedge i^\star]\\ &\leq 5\exp(-(\log n)^{3/2}),\end{aligned}$$

which completes the proof. □

Proof of Theorem 3.3.2. Let $\mathcal{X} := \{i^\star < \tau^\star\}$, $i := i^\star$ and $\vartheta^\star := \hat{p}$. By Lemma 3.8.1, it suffices to show that if $\mathcal{X}$ occurs, then $\mathcal{H}$ is $(4m, n^\varepsilon)$-bounded, $\mathcal{F}$-populated, k'-populated for all $1 \le k' \le k - 1/\rho_{\mathcal{F}}$ and has $n^{k-1/\rho_{\mathcal{F}}+\varepsilon}/k!$ edges.

Due to $\mathcal{X} \subseteq \{i^\star < \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'}\}$, for all strictly balanced k-templates $(\mathcal{A}, I)$ with $|V_{\mathcal{A}}| \le 1/\varepsilon^4$ and all $\psi \colon I \hookrightarrow V_{\mathcal{H}}$, Lemma 3.5.13 yields

$$\Phi_{\mathcal{A},I} \le_{\mathcal{X}} (1+\log n)^{\alpha_{\mathcal{A},I}} \max\{1, \hat{\varphi}_{\mathcal{A},I}\} \le n^\varepsilon \max\{1, n^{|V_{\mathcal{A}}|-|I|}(\vartheta^\star)^{|\mathcal{A}|-|\mathcal{A}[I]|}\}$$

Thus, H is $(4m, n^\varepsilon)$-bounded if $\mathcal{X}$ occurs.

Furthermore, due to $\mathcal{X} \subseteq \{i^\star < \tau_{\mathscr{F}}\}$, for all $e \in \mathcal{H}$, Lemma 3.5.17 entails

$$d_{\mathcal{H}^*}(e) \ge_{\mathcal{X}} \frac{|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{F},f}}{2\,\mathrm{aut}(\mathcal{F})} = \frac{|\mathcal{F}|k!\,n^{\varepsilon(|\mathcal{F}|-1)}}{2\,\mathrm{aut}(\mathcal{F})} \ge n^{\varepsilon^2},$$

which shows that $\mathcal{H}$ is $\mathcal{F}$-populated if $\mathcal{X}$ occurs.

Let $1 \le k' \le k - 1/\rho_{\mathcal{F}}$ and let $(\mathcal{A}, I)$ denote a k-template with $|V_{\mathcal{A}}| = k$, $|\mathcal{A}| = 1$ and $|I| = k'$. Fix a k'-set $U \subseteq V_{\mathcal{H}}$ and $\psi \colon I \hookrightarrow U$. We have $\rho_{\mathcal{A},I} \le \rho_{\mathcal{F}}$, so for all $j \le i$, Lemma 3.5.8 implies

$$\hat{\varphi}_{\mathcal{A},I}(j) \ge n^{k-k'}\hat{p}^{\rho_{\mathcal{F}}(k-k')} = n^{\varepsilon\rho_{\mathcal{F}}(k-k')} \ge n^{\varepsilon^2} > \zeta^{-\delta^{1/2}}$$

and hence $i^\star < i^{\delta^{1/2}}_{\mathcal{A},I}$. Thus, due to $\mathcal{X} \subseteq \{i^\star < \tau_{\mathscr{B}}\}$, we obtain

$$d_{\mathcal{H}}(U) = \frac{\Phi_{\mathcal{A},\psi}}{(k-k')!} \ge_{\mathcal{X}} \varepsilon\hat{\varphi}_{\mathcal{A},I} \ge n^{\varepsilon^2},$$

which shows that $\mathcal{H}$ is k'-populated if $\mathcal{X}$-occurs.

Finally, since $\mathcal{X} \subseteq \{i^\star < \tau_\emptyset\}$, Lemma 3.5.7 yields $H =_{\mathcal{X}} \vartheta^\star n^k/k! = n^{k-1/\rho_{\mathcal{F}}+\varepsilon}/k!$. □

3.9 The sparse setting

The first part of our argumentation is now complete and as mentioned in Section 3.1, we now focus on the second part. We first describe the setting for this section and subsequent sections and remark that from now on, we redefine some symbols that appeared in the first part. Let $k \ge 2$ and fix a k-graph $\mathcal{F}$ on m vertices with $|\mathcal{F}| \ge 2$ and k-density $\rho_{\mathcal{F}}$ that is not a matching such that $(\mathcal{F}, f)$ is strictly balanced for all $f \in \mathcal{F}$. Suppose that $0 < \varepsilon < 1$ is sufficiently small in terms of $1/m$ and that n is sufficiently large in terms of $1/\varepsilon$. Suppose that $\mathcal{H}(0)$ is a k-graph on n vertices with $n^{k-1/\rho_{\mathcal{F}}-\varepsilon^4} \le |\mathcal{H}(0)| \le n^{k-1/\rho_{\mathcal{F}}+\varepsilon^4}$ that is $(4m, n^{\varepsilon^4})$-bounded[1].

For the second part, that is for the proof of Theorems 1.1.7 and 1.1.9, one key idea is the identification of substructures in $\mathcal{H}(0)$ whose existence enforces the existence of edges that are no longer contained in a copy of $\mathcal{F}$ with a substantial probability. We

[1]Note that for $\mathcal{F}$, besides strictly k-balanced k-graphs, this setup also allows k-graphs as in Theorem 1.1.9. We choose this slightly more general setting as this makes many of the results we present available for a proof of Theorem 1.1.9 while only requiring very minor adaptations.

show that there is a sufficiently large subset of these substructures whose members are far apart from each other and hence act, to a large extent, independently. We employ a concentration inequality to verify that a substantial number of these substructures indeed give rise to edges that are no longer contained in a copy of $\mathcal{F}$ and hence remain until the termination of the process.

On a very high level, similar ideas have also been utilized by Bohman, Frieze and Lubetzky for determining the number of remaining edges in the triangle-removal process (starting at K_n), see [14, Section 6]. In our significantly more general setting however, we require additional insights concerning the distribution of copies of $\mathcal{F}$ in $\mathcal{H}(0)$. Notably, while in the special case where $\mathcal{F}$ is a triangle, two distinct copies of $\mathcal{F}$ that both contain an edge e cannot overlap outside e, such overlaps can exist in general. However, since $(\mathcal{F}, f)$ is strictly balanced for all $f \in \mathcal{F}$, if two copies of $\mathcal{F}$, both containing an edge e, overlap outside e, then their union forms a k-graph with k-density greater than $\rho_{\mathcal{F}}$. As a crucial step in our proof, we utilize this to show that certain substructures consisting of copies of $\mathcal{F}$ barely exist in the sense that we obtain a strong upper bound on the number of such structures.

The remainder of the chapter is organized as follows. In Section 3.10, we prove several structural results which are important for the following parts. This includes properties of the aforementioned substructures that yield the edges that still remain at the end of the process. In Section 3.11, we obtain an upper bound on the number of remaining copies that holds well beyond the point where we would expect the process to terminate (this general idea is taken from [14]). To this end, we again employ an approach that resembles the differential equation method or more specifically the critical interval method.

Combining the structural results from Section 3.10 and the upper bound on the number of edges at a very late time in the process obtained in Section 3.11, we finally prove Theorem 1.1.7 in Section 3.12. As mentioned above, here the idea is to identify certain configurations that have to appear frequently before the process terminates and that with sufficiently large probability lead to edges that remain in the hypergraph until termination. Compared to the (in spirit) similar argument in [14, Section 6] here the (involved) insights from Section 3.10 replace properties that are obvious in the triangle case.

For Theorem 1.1.9, one may argue very similarly, however, the structures that in the end enforce the existence of edges that remain until termination are different. In more detail, to obtain Theorem 1.1.9, parts of the argumentation in Section 3.10 and the key structures considered in Section 3.12 need to be replaced but the results from Section 3.11 remain valid and the high level structure of the proof remains the same. For completeness, we provide a full proof of Theorem 1.1.9 in Section 3.14.

3.10 Unions of strictly balanced hypergraphs

In this section, as preparation for the arguments in subsequent sections, we gather some lemmas that provide further insight into the distribution of the copies of $\mathcal{F}$ in $\mathcal{H}$. First, we state several lemmas concerning the densities of substructures obtained as unions of k-balanced k-graphs (see Lemmas 3.10.2–3.10.5). In particular, we are interested in structures that are in a sense cyclic, where formally for $\ell \geq 2$, we say that a sequence $\mathcal{A}_1, \dots, \mathcal{A}_\ell$ of distinct k-graphs forms a *self-avoiding cyclic walk* if there exist distinct $e_1, \dots, e_\ell$ such that $e_i \in \mathcal{A}_i \cap \mathcal{A}_{i+1}$ for all $1 \leq i \leq \ell$ with indices taken modulo ℓ.

From the $(4m, n^{\varepsilon^4})$-boundedness of $\mathcal{H}(0)$, we then deduce Lemma 3.10.6 where for all k-graphs $\mathcal{A}$ that satisfy a suitable density property, we bound the number $\Phi_{\mathcal{A}}$ of injections $\varphi\colon V_{\mathcal{A}} \hookrightarrow V_{\mathcal{H}}$ with $\varphi(e) \in \mathcal{H}(0)$ for all $e \in \mathcal{A}$ where we set $V_{\mathcal{H}} := V_{\mathcal{H}(0)}$.

Using $\rho_{\mathcal{F}} \geq 1/(k-1)$ (see Lemma 3.10.7), the aforementioned density observations allow us to apply Lemma 3.10.6 to then obtain Lemma 3.10.8 as an intermediate result and subsequently Lemma 3.10.9 which states that $\mathcal{H}(0)$ contains only few cyclic structures formed by copies of $\mathcal{F}$. This turns out to be a crucial observation concerning the structure of $\mathcal{H}(0)$ that we require in two separate places in our argumentation (namely in the proofs of Lemma 3.11.11 and Lemma 3.12.3).

As these objects frequently appear in our proofs, we generalize the notation $\Phi_{\mathcal{A}}$ as follows. For a template $(\mathcal{A}, I)$ and $\psi\colon I \hookrightarrow V_{\mathcal{H}}$, we use $\Phi^{\sim}_{\mathcal{A},\psi}$ to denote the set of injections $\varphi\colon V_{\mathcal{A}} \hookrightarrow V_{\mathcal{H}}$ with $\varphi|_I = \psi$ and $\varphi(e) \in \mathcal{H}(0)$ for all $e \in \mathcal{A} \setminus \mathcal{A}[I]$ and we set $\Phi_{\mathcal{A},\psi} := |\Phi^{\sim}_{\mathcal{A},\psi}|$. Additionally, we define $\Phi^{\sim}_{\mathcal{A}} := \Phi^{\sim}_{\mathcal{A},\psi}$ where ψ denotes the unique function from $\emptyset$ to $V_{\mathcal{H}}$. Note that $\Phi_{\mathcal{A}} = |\Phi^{\sim}_{\mathcal{A}}|$.

The bounds on $|\mathcal{H}(0)|$ and the numbers of embeddings of strictly balanced templates into $\mathcal{H}(0)$ yield the following lemma.

Lemma 3.10.1. *Suppose that $(\mathcal{A}, I)$ is a strictly balanced k-template with $|V_{\mathcal{A}}| \leq 4m$ and let $\psi\colon I \hookrightarrow V_{\mathcal{H}}$. Then, $\Phi_{\mathcal{A},\psi} \leq n^{\varepsilon^3} \cdot \max\{1, n^{|V_{\mathcal{A}}|-|I|-(|\mathcal{A}|-|\mathcal{A}[I]|)/\rho_{\mathcal{F}}}\}$.*

Proof. We have $|\mathcal{H}(0)| \leq n^{-1/\rho_{\mathcal{F}}+2\varepsilon^4} \cdot n^k/k!$, so since $\mathcal{H}(0)$ is $(4m, n^{\varepsilon^4})$-bounded, we obtain

$$\begin{aligned}\Phi_{\mathcal{A},\psi} &\leq n^{\varepsilon^4} \cdot \max\{1, n^{|V_{\mathcal{A}}|-|I|} n^{(-1/\rho_{\mathcal{F}}+2\varepsilon^4)(|\mathcal{A}|-|\mathcal{A}[I]|)}\} \\ &\leq n^{\varepsilon^3} \cdot \max\{1, n^{|V_{\mathcal{A}}|-|I|-(|\mathcal{A}|-|\mathcal{A}[I]|)/\rho_{\mathcal{F}}}\},\end{aligned}$$

which completes the proof. □

Lemma 3.10.2. *Let $\ell \geq 1$. Suppose that $\mathcal{A}_1, \dots, \mathcal{A}_\ell$ is a sequence of k-balanced k-graphs with k-density at least ρ. For $1 \leq i \leq \ell$, let $\mathcal{S}_i := \mathcal{A}_1 + \dots + \mathcal{A}_i$. Suppose that for all $2 \leq i \leq \ell$, we have $\mathcal{S}_{i-1} \cap \mathcal{A}_i \neq \emptyset$. Let $\mathcal{S} := \mathcal{S}_\ell$ and $J \subsetneq V_{\mathcal{S}}$ with $\mathcal{S}[J] \neq \emptyset$. Then, $\rho_{\mathcal{S},J} \geq \rho$.*

Proof. By rearranging the elements of $\mathcal{A}_1, \dots, \mathcal{A}_\ell$ if necessary, we may assume that we have $\mathcal{A}_1[J] \neq \emptyset$. For $1 \leq i \leq \ell$, let

$$\begin{aligned}&U := V_{\mathcal{S}} \setminus J, \quad E := \mathcal{S} \setminus \mathcal{S}[J], \quad W_{i-1} := V_{\mathcal{A}_1} \cup \dots \cup V_{\mathcal{A}_{i-1}}, \\ &J_i := (J \cup W_{i-1}) \cap V_{\mathcal{A}_i}, \quad U_i := V_{\mathcal{A}_i} \setminus J_i, \quad E_i := \mathcal{A}_i \setminus \mathcal{A}_i[J_i].\end{aligned}$$

Note that $U = \bigcup_{1 \leq i \leq \ell} U_i$ and $U_i \cap U_j = \emptyset$ for all $1 \leq i < j \leq \ell$. Hence, $|U| = \sum_{1 \leq i \leq \ell} |U_i|$. Similarly, we have $E \supseteq \bigcup_{1 \leq i \leq \ell} E_i$ and $E_i \cap E_j = \emptyset$ for all $1 \leq i < j \leq \ell$ and thus $|E| \geq \sum_{1 \leq i \leq \ell} |E_i|$. This yields

$$\rho_{\mathcal{S},J} = \frac{|E|}{|U|} \geq \frac{\sum_{1 \leq i \leq \ell} |E_i|}{\sum_{1 \leq i \leq \ell} |U_i|}.$$

Let $e_1 \in \mathcal{A}_1[J]$ and for $2 \leq i \leq \ell$, let $e_i \in \mathcal{A}_i \cap \mathcal{S}_{i-1}$. For all $1 \leq i \leq \ell$, the extension $(\mathcal{A}_i, e_i)$ is balanced and has density at least ρ, so due to $e_i \subseteq J_i$, we obtain

$$\begin{aligned}|E_i| &= \rho_{\mathcal{A}_i,e_i}(|V_{\mathcal{A}_i}| - k) - \rho_{\mathcal{A}_i[J_i],e_i}(|J_i| - k) \geq \rho_{\mathcal{A}_i,e_i}(|V_{\mathcal{A}_i}| - k) - \rho_{\mathcal{A}_i,e_i}(|J_i| - k)\\ &= \rho_{\mathcal{A}_i,e_i}|U_i| \geq \rho|U_i|.\end{aligned}$$

Hence, we obtain

$$\rho_{\mathcal{S},J} \geq \frac{\sum_{1\leq i\leq \ell} \rho|U_i|}{\sum_{1\leq i\leq \ell}|U_i|} = \rho,$$

which completes the proof. □

Lemma 3.10.3. *Let $\ell \geq 1$. Suppose that $\mathcal{A}_1, \dots, \mathcal{A}_\ell$ is a sequence of k-balanced k-graphs with k-density at least ρ. For $1 \leq i \leq \ell$, let $\mathcal{S}_i := \mathcal{A}_1 + \dots + \mathcal{A}_i$. Suppose that for all $2 \leq i \leq \ell$, we have $\mathcal{A}_i \cap \mathcal{S}_{i-1} \neq \emptyset$. Let $\mathcal{S} := \mathcal{S}_\ell$. Then, $\max_{\mathcal{B}\subseteq\mathcal{S}} \rho_{\mathcal{B},\emptyset} \geq \rho$ or $(\mathcal{S}, \emptyset)$ is strictly balanced.*

Proof. Suppose that $\max_{\mathcal{B}\subseteq\mathcal{S}} \rho_{\mathcal{B},\emptyset} < \rho$. We show that then $(\mathcal{S}, \emptyset)$ is strictly balanced. To this end, consider $(\mathcal{C}, \emptyset) \subseteq (\mathcal{S}, \emptyset)$ with $V_\mathcal{C} \neq \emptyset$ and $\mathcal{C} \neq \mathcal{S}$. It suffices to show that $\rho_{\mathcal{C},\emptyset} < \rho_{\mathcal{S},\emptyset}$.

First, note that we may assume that $\mathcal{C}$ is an induced subgraph of $\mathcal{S}$ with non-empty edge set. By Lemma 3.10.2, we have $\rho_{\mathcal{S},V_\mathcal{C}} \geq \rho$ and due to $\max_{\mathcal{B}\subseteq\mathcal{S}} \rho_{\mathcal{B},\emptyset} < \rho$ furthermore $\rho_{\mathcal{C},\emptyset} < \rho$. Hence $\rho_{\mathcal{S},V_\mathcal{C}} > \rho_{\mathcal{C},\emptyset}$. Thus,

$$\begin{aligned}\rho_{\mathcal{S},\emptyset} &= \frac{|\mathcal{S}| - |\mathcal{S}[V_\mathcal{C}]| + |\mathcal{C}|}{|V_\mathcal{S}|} = \frac{\rho_{\mathcal{S},V_\mathcal{C}}(|V_\mathcal{S}| - |V_\mathcal{C}|) + \rho_{\mathcal{C},\emptyset}|V_\mathcal{C}|}{|V_\mathcal{S}|} > \frac{\rho_{\mathcal{C},\emptyset}(|V_\mathcal{S}| - |V_\mathcal{C}|) + \rho_{\mathcal{C},\emptyset}|V_\mathcal{C}|}{|V_\mathcal{S}|}\\ &= \rho_{\mathcal{C},\emptyset},\end{aligned}$$

which completes the proof. □

Lemma 3.10.4. *Suppose that $\mathcal{A}_1, \dots, \mathcal{A}_\ell$ is a sequence of strictly k-balanced k-graphs with k-density ρ that forms a self-avoiding cyclic walk such that no proper subsequence forms a self-avoiding cyclic walk. Let $\mathcal{S} := \mathcal{A}_1 + \dots + \mathcal{A}_\ell$. Then, there exists $e \in \mathcal{S}$ such that $\rho_{\mathcal{S},e} > \rho$.*

Proof. First note that since no proper subsequence of $\mathcal{A}_1, \dots, \mathcal{A}_\ell$ forms a self-avoiding cyclic walk, we have $\mathcal{A}_\ell \cap \mathcal{A}_i = \emptyset$ for all $2 \leq i \leq \ell - 2$. Furthermore if $\ell \geq 3$, then again since no proper subsequence of $\mathcal{A}_1, \dots, \mathcal{A}_\ell$ forms a self-avoiding cyclic walk, for all $1 \leq i \leq \ell$, we have $|\mathcal{A}_i \cap \mathcal{A}_{i+1}| = 1$ with indices taken modulo ℓ (otherwise $\mathcal{A}_i, \mathcal{A}_{i+1}$ forms a self-avoiding cyclic walk). Hence if $\ell \geq 3$, then $|\mathcal{A}_{\ell-1} \cap \mathcal{A}_\ell| = |\mathcal{A}_\ell \cap \mathcal{A}_1| = 1$.

For $1 \leq i \leq \ell$, let $\mathcal{S}_i := \mathcal{A}_1 + \dots + \mathcal{A}_i$. If $\ell \geq 3$, then, as a consequence of the above observations, due to $|\mathcal{A}_1|, |\mathcal{A}_\ell| \geq 3$, we have $\mathcal{A}_\ell \setminus \mathcal{S}_{\ell-1} \neq \emptyset$ as well as $\mathcal{A}_1 \setminus \mathcal{A}_\ell \neq \emptyset$. If $\ell = 2$, then $\mathcal{A}_1 \not\subseteq \mathcal{A}_2$ and $\mathcal{A}_2 \not\subseteq \mathcal{A}_1$ and hence $\mathcal{A}_\ell \setminus \mathcal{S}_{\ell-1} \neq \emptyset$ and $\mathcal{A}_1 \setminus \mathcal{A}_\ell \neq \emptyset$ follow from the fact that $\mathcal{A}_1$ and $\mathcal{A}_2$ are distinct strictly k-balanced k-graphs with the same k-density. Let $e \in \mathcal{A}_1 \setminus \mathcal{A}_\ell$. As a consequence of Lemma 3.10.2, we have $\rho_{\mathcal{S}_{\ell-1},e} \geq \rho$. Hence,

$$\rho_{\mathcal{S},e} = \frac{\rho_{\mathcal{S}_{\ell-1},e}(|V_{\mathcal{S}_{\ell-1}}| - k) + |\mathcal{A}_\ell \setminus \mathcal{S}_{\ell-1}|}{|V_{\mathcal{S}_{\ell-1}}| - k + |V_{\mathcal{A}_\ell} \setminus V_{\mathcal{S}_{\ell-1}}|} \geq \frac{\rho(|V_{\mathcal{S}_{\ell-1}}| - k) + |\mathcal{A}_\ell \setminus \mathcal{S}_{\ell-1}|}{|V_{\mathcal{S}_{\ell-1}}| - k + |V_{\mathcal{A}_\ell} \setminus V_{\mathcal{S}_{\ell-1}}|}.$$

Thus, it suffices to show that $|\mathcal{A}_\ell \setminus \mathcal{S}_{\ell-1}| > \rho|V_{\mathcal{A}_\ell} \setminus V_{\mathcal{S}_{\ell-1}}|$. Due to $\mathcal{A}_\ell \setminus \mathcal{S}_{\ell-1} \neq \emptyset$, the inequality holds if $|V_{\mathcal{A}_\ell} \setminus V_{\mathcal{S}_{\ell-1}}| = 0$, so we may assume that $V_{\mathcal{A}_\ell} \setminus V_{\mathcal{S}_{\ell-1}} \neq \emptyset$. Since $\mathcal{A}_1, \dots, \mathcal{A}_\ell$ forms a self-avoiding cyclic walk, there exist distinct $e_1, e_2 \in \mathcal{S}_{\ell-1} \cap \mathcal{A}_\ell$, so in particular, we have $e_1 \subsetneq V_{\mathcal{S}_{\ell-1}} \cap V_{\mathcal{A}_\ell} \subsetneq V_{\mathcal{A}_\ell}$. The template $(\mathcal{A}_\ell, e_1)$ is strictly balanced, so we obtain

$$\begin{aligned}|\mathcal{A}_\ell \setminus \mathcal{S}_{\ell-1}| \geq |\mathcal{A}_\ell \setminus \mathcal{A}_\ell[V_{\mathcal{S}_{\ell-1}} \cap V_{\mathcal{A}_\ell}]| &= \rho(|V_{\mathcal{A}_\ell}| - k) - \rho_{\mathcal{A}_\ell[V_{\mathcal{S}_{\ell-1}} \cap V_{\mathcal{A}_\ell}], e_1}(|V_{\mathcal{S}_{\ell-1}} \cap V_{\mathcal{A}_\ell}| - k)\\ &> \rho(|V_{\mathcal{A}_\ell}| - k) - \rho(|V_{\mathcal{S}_{\ell-1}} \cap V_{\mathcal{A}_\ell}| - k) = \rho|V_{\mathcal{A}_\ell} \setminus V_{\mathcal{S}_{\ell-1}}|,\end{aligned}$$

which completes the proof. □

Lemma 3.10.5. *Suppose that $\mathcal{A}_1, \dots, \mathcal{A}_\ell$ is a sequence of strictly k-balanced k-graphs with k-density ρ that forms a self-avoiding cyclic walk. Let $\mathcal{S} := \mathcal{A}_1 + \dots + \mathcal{A}_\ell$. Then, there exists $e \in \mathcal{S}$ such that $\rho_{\mathcal{S},e} > \rho$.*

Proof. Consider a subsequence $\mathcal{A}_{i_1}, \dots, \mathcal{A}_{i_{\ell'}}$ of $\mathcal{A}_1, \dots, \mathcal{A}_\ell$ that forms a self-avoiding cyclic walk such that no proper subsequence forms a self-avoiding cyclic walk. Let $\mathcal{S}' := \mathcal{A}_{i_1} + \dots + \mathcal{A}_{i_{\ell'}}$. By Lemma 3.10.4, there exists $e \in \mathcal{S}'$ such that $\rho_{\mathcal{S}',e} > \rho$ and by Lemma 3.10.2, if $V_{\mathcal{S}'} \subsetneq V_{\mathcal{S}}$, then $\rho_{\mathcal{S},V_{\mathcal{S}'}} \geq \rho$. This yields

$$\begin{aligned}\rho_{\mathcal{S},e} &= \frac{\rho_{\mathcal{S}',e}(|V_{\mathcal{S}'}| - k) + |\mathcal{S} \setminus \mathcal{S}'|}{|V_{\mathcal{S}}| - k} \geq \frac{\rho_{\mathcal{S}',e}(|V_{\mathcal{S}'}| - k) + \rho_{\mathcal{S},V_{\mathcal{S}'}}(|V_{\mathcal{S}}| - |V_{\mathcal{S}'}|)}{|V_{\mathcal{S}}| - k}\\ &> \frac{\rho(|V_{\mathcal{S}'}| - k) + \rho(|V_{\mathcal{S}}| - |V_{\mathcal{S}'}|)}{|V_{\mathcal{S}}| - k} = \rho,\end{aligned}$$

which completes the proof. □

Lemma 3.10.6. *Suppose that $(\mathcal{A}, I)$ is a k-template with $|V_{\mathcal{A}}| \leq 1/\varepsilon$ and $\rho_{\mathcal{A},J} \geq \rho_{\mathcal{F}}$ for all $I \subseteq J \subsetneq V_{\mathcal{A}}$. Let $\psi\colon I \hookrightarrow V_{\mathcal{H}}$. Then, $\Phi_{\mathcal{A},\psi} \leq n^{\varepsilon^2}$.*

Proof. We use induction on $|V_{\mathcal{A}}| - |I|$ to show that

$$\Phi_{\mathcal{A},\psi} \leq n^{\varepsilon^3(|V_{\mathcal{A}}| - |I|)}. \tag{3.10.1}$$

Then, since $|V_{\mathcal{A}}| \leq 1/\varepsilon$, the statement follows.

If $|V_{\mathcal{A}}| - |I| = 0$, then $\Phi_{\mathcal{A},\psi} = 1 = \hat{\varphi}_{\mathcal{A},I}$. Let $\ell \geq 1$ and suppose that (3.10.1) holds if $|V_{\mathcal{A}}| - |I| \leq \ell - 1$. Suppose that $|V_{\mathcal{A}}| - |I| = \ell$. Let $I \subseteq U \subseteq V_{\mathcal{A}}$ such that $\rho_{\mathcal{A}[U],I}$ is maximal and subject to this, that $|U|$ is minimal. Then, $(\mathcal{A}[U], I)$ is strictly balanced. Furthermore, we have $\rho_{\mathcal{A}[U],I} \geq \rho_{\mathcal{A},I} \geq \rho_{\mathcal{F}} > 0$ and hence $U \neq I$. Note that

$$\Phi_{\mathcal{A},\psi} = \sum_{\varphi \in \Phi^{\sim}_{\mathcal{A}[U],\psi}} \Phi_{\mathcal{A},\varphi}. \tag{3.10.2}$$

We exploit the strict balancedness of $(\mathcal{A}[U], I)$ to bound $\Phi_{\mathcal{A}[U],\psi}$ and the induction hypothesis to bound $\Phi_{\mathcal{A},\varphi}$ for all $\varphi \in \Phi^{\sim}_{\mathcal{A}[U],\psi}$.

In detail, we argue as follows. Due to Lemma 3.10.1, we have

$$\Phi_{\mathcal{A}[U],\psi} \leq n^{\varepsilon^3} \cdot \max\{1, n^{(1-\rho_{\mathcal{A}[U],I}/\rho_{\mathcal{F}})(|U|-|I|)}\} = n^{\varepsilon^3}.$$

Furthermore, for all $\varphi \in \Phi^{\sim}_{\mathcal{A}[U],\psi}$, by induction hypothesis, we obtain

$$\Phi_{\mathcal{A},\varphi} \leq n^{\varepsilon^3(|V_{\mathcal{A}}|-|U|)}.$$

Combining this with (3.10.2) yields

$$\Phi_{\mathcal{A}[U],\psi} \leq n^{\varepsilon^3} \cdot n^{\varepsilon^3(|V_{\mathcal{A}}|-|U|)} \leq n^{\varepsilon^3(|V_{\mathcal{A}}|-|I|)},$$

which completes the proof. □

Lemma 3.10.7. $\rho_{\mathcal{F}} \geq 1/(k-1)$.

Proof. Since $\mathcal{F}$ is not a matching, there exist edges $e_1, e_2 \in \mathcal{F}$ with $e_1 \cap e_2 \neq \emptyset$. Let $\mathcal{A}$ denote the k-graph with vertex-set $e_1 \cup e_2$ and edge-set $\{e_1, e_2\}$. Since $(\mathcal{F}, e_1)$ is strictly balanced, we have

$$\rho_{\mathcal{F}} \geq \rho_{\mathcal{A},e_1} \geq \frac{1}{k-1},$$

which completes the proof □

Lemma 3.10.8. *Let $\ell \leq 4$ and suppose that $\mathcal{A}_1, \ldots, \mathcal{A}_\ell$ is a sequence of k-balanced k-graphs with k-density at least $\rho_{\mathcal{F}}$ and at most m vertices each that forms a self-avoiding cyclic walk. Let $\mathcal{S} := \mathcal{A}_1 + \ldots + \mathcal{A}_\ell$ and suppose that there exists $e \in \mathcal{S}$ with $\rho_{\mathcal{S},e} > \rho_{\mathcal{F}}$. Then $\Phi_{\mathcal{S}} \leq n^{k-1/\rho_{\mathcal{F}}-\varepsilon^{1/7}}$.*

Proof. Based on Lemma 3.10.3, we distinguish two cases: The first case, where we have $\max_{\mathcal{B}\subseteq\mathcal{S}} \rho_{\mathcal{B},\emptyset} \geq \rho_{\mathcal{F}}$ and the second case where $(\mathcal{S}, \emptyset)$ is strictly balanced.

First, suppose that $\max_{\mathcal{B}\subseteq\mathcal{S}} \rho_{\mathcal{B},\emptyset} \geq \rho_{\mathcal{F}}$. From all $(\mathcal{B}', \emptyset) \subseteq (\mathcal{S}, \emptyset)$ choose $(\mathcal{B}, \emptyset)$ such that $\rho_{\mathcal{B},\emptyset}$ is maximal and subject to this, that $|V_{\mathcal{B}}|$ is minimal. Then, $(\mathcal{B}, \emptyset)$ is strictly balanced and we have $\rho_{\mathcal{B},\emptyset} \geq \rho_{\mathcal{F}}$. Furthermore, we have

$$\Phi_{\mathcal{S}} = \sum_{\varphi \in \Phi^{\sim}_{\mathcal{B}}} \Phi_{\mathcal{S},\varphi}. \tag{3.10.3}$$

For all $\varphi \in \Phi^{\sim}_{\mathcal{B}}$, due to Lemma 3.10.2, we may apply Lemma 3.10.6 to obtain $\Phi_{\mathcal{S},\varphi} \leq n^{\varepsilon^2}$. Furthermore, due to Lemma 3.10.1, we have

$$\Phi_{\mathcal{B}} \leq n^{\varepsilon^3} \cdot \max\{1, n^{(1-\rho_{\mathcal{B},\emptyset}/\rho_{\mathcal{F}})|V_{\mathcal{B}}|}\} = n^{\varepsilon^3}.$$

Returning to (3.10.3), due to Lemma 3.10.7, this yields

$$\Phi_{\mathcal{S},\psi} \leq_{\mathcal{X}} n^{\varepsilon^2} \cdot n^{\varepsilon^3} \leq n^{k-1/\rho_{\mathcal{F}}-\varepsilon^{1/7}},$$

and hence completes our analysis of the first case.

We proceed with the second case. Hence, assume that $(\mathcal{S},\emptyset)$ is strictly balanced and that $\rho_{\mathcal{S},\emptyset}=\max_{\mathcal{B}\subseteq\mathcal{S}}\rho_{\mathcal{B},\emptyset}<\rho_{\mathcal{F}}$. Then

$$n^{|V_{\mathcal{S}}|-|\mathcal{S}|/\rho_{\mathcal{F}}}=n^{(1-\rho_{\mathcal{S},\emptyset}/\rho_{\mathcal{F}})|V_{\mathcal{S}}|}\geq 1.$$

Thus, Lemma 3.10.1 entails

$$\Phi_{\mathcal{S}}\leq n^{|V_{\mathcal{S}}|-|\mathcal{S}|/\rho_{\mathcal{F}}+\varepsilon^3}=n^{k-1/\rho_{\mathcal{F}}+\varepsilon^3}\cdot n^{|V_{\mathcal{S}}|-k-(|\mathcal{S}|-1)/\rho_{\mathcal{F}}}.$$

If there exists $e\in\mathcal{S}$ with $\rho_{\mathcal{S},e}>\rho_{\mathcal{F}}$, then since $\ell\leq 4$ and $|V_{\mathcal{A}_i}|\leq m$ for all $1\leq i\leq\ell$ we have

$$\rho_{\mathcal{F}}+\varepsilon^{1/8}<\rho_{\mathcal{S},e}=\frac{|\mathcal{S}|-1}{|V_{\mathcal{S}}|-k},$$

so we then obtain

$$\Phi_{\mathcal{S}}\leq n^{k-1/\rho_{\mathcal{F}}+\varepsilon^3}\cdot n^{|V_{\mathcal{S}}|-k-(\rho_{\mathcal{F}}+\varepsilon^{1/8})(|V_{\mathcal{S}}|-k)/\rho_{\mathcal{F}}}<n^{k-1/\rho_{\mathcal{F}}-\varepsilon^{1/7}},$$

which completes the proof. □

Lemma 3.10.9. *Let $\ell\leq 4$ and suppose that $\mathcal{F}_1,\ldots,\mathcal{F}_\ell$ is a sequence of copies of $\mathcal{F}$ that forms a self-avoiding cyclic walk. Let $\mathcal{S}:=\mathcal{A}_1+\ldots+\mathcal{A}_\ell$. If $|\mathcal{F}|\geq 3$, that is if $\mathcal{F}$ is strictly k-balanced, then $\Phi_{\mathcal{S}}\leq n^{k-1/\rho_{\mathcal{F}}-\varepsilon^{1/7}}$.*

Proof. Due to Lemma 3.10.5, this follows from Lemma 3.10.8. □

3.11 Bounding the number of copies of $\mathcal{F}$

We assume the setup described in Section 3.9 and, similarly as in Section 3.3, we define $\mathcal{H}^*(0)$ to be the $|\mathcal{F}|$-graph with vertex set $\mathcal{H}(0)$ whose edges are the edge sets of copies of $\mathcal{F}$ that are subgraphs of $\mathcal{H}(0)$. We now begin to analyze the $\mathcal{F}$-removal process formally again given by Algorithm 3.3.1. Again, if the process fails to execute step $i+1$ and instead terminates, that is if $\mathcal{H}^*(i)=\emptyset$, then, for $j\geq i+1$, we set $\mathcal{H}^*(j):=\mathcal{H}^*(i)$. For $i\geq 1$, we define $\mathcal{H}(i)$, $H^*(i)$, $H(i)$ and the filtration $\mathfrak{F}(0),\mathfrak{F}(1),\ldots$ as in Section 3.3. We again define the stopping time

$$\tau_\emptyset:=\min\{i\geq 0:\mathcal{H}^*(i)=\emptyset\}.$$

To prove Theorem 1.1.7, in Section 3.12, we show that the following theorem holds.

Theorem 3.11.1. *If $|\mathcal{F}|\geq 3$, then $\mathbb{P}[H(\tau_\emptyset)\leq n^{k-1/\rho-\varepsilon}]\leq\exp(-n^{1/4})$.*

For our proof of Theorem 1.1.7, in addition to the structural insights about configurations consisting of copies that we may encounter in $\mathcal{H}(0)$, we crucially rely on an upper bound for the number of copies of $\mathcal{F}$ present in $\mathcal{H}(i)$ for $i\geq 0$, which is the focus of this section. First, note that initially, we may bound the number of copies as follows. Let $\vartheta:=k!\,H(0)/n^k$.

Lemma 3.11.2. *Let $i \geq 0$ and $e \in \mathcal{H}$. Then, $d_{\mathcal{H}^*}(e) \leq n^{m-k+\varepsilon^{7/2}}\vartheta^{|\mathcal{F}|-1} \leq n^{\varepsilon^3}$.*

Proof. By our assumptions on $\mathcal{H}(0)$, we have $n^{-1/\rho_{\mathcal{F}}-\varepsilon^4} \leq \vartheta \leq n^{-1/\rho_{\mathcal{F}}+2\varepsilon^4}$. Hence, arguing similarly as in the proof of Lemma 3.10.1, we obtain

$$\begin{aligned} d_{\mathcal{H}^*}(e) \leq d_{\mathcal{H}^*(0)}(e) &\leq \sum_{f\in\mathcal{F}}\sum_{\psi\colon f\hookrightarrow e} \Phi_{\mathcal{F},\psi} \leq |\mathcal{F}|k!\cdot n^{\varepsilon^4}\cdot \max\{1, n^{m-k}n^{(-1/\rho_{\mathcal{F}}+2\varepsilon^4)(|\mathcal{F}|-1)}\} \\ &= |\mathcal{F}|k!\cdot n^{\varepsilon^4}\cdot n^{m-k}n^{(-1/\rho_{\mathcal{F}}-\varepsilon^4)(|\mathcal{F}|-1)}\cdot n^{3\varepsilon^4(|\mathcal{F}|-1)} \leq n^{m-k+\varepsilon^{7/2}}\vartheta^{|\mathcal{F}|-1}. \end{aligned}$$

Furthermore, again using $\vartheta \leq n^{-1/\rho_{\mathcal{F}}+2\varepsilon^4}$, we obtain

$$n^{m-k+\varepsilon^{7/2}}\vartheta^{|\mathcal{F}|-1} \leq n^{\varepsilon^{7/2}}\cdot n^{2\varepsilon^4(|\mathcal{F}|-1)} \leq n^{\varepsilon^3},$$

which completes the proof. □

Lemma 3.11.3. $H^*(0) \leq n^{m+\varepsilon^{7/2}}\vartheta^{|\mathcal{F}|}$.

Proof. Using Lemma 3.11.2, we obtain

$$H^*(0) = \frac{1}{|\mathcal{F}|}\sum_{e\in\mathcal{H}(0)} d_{\mathcal{H}^*}(e) \leq \frac{\vartheta n^k}{|\mathcal{F}|k!}\cdot n^{m-k+\varepsilon^{7/2}}\vartheta^{|\mathcal{F}|-1} \leq n^{m+\varepsilon^{7/2}}\vartheta^{|\mathcal{F}|},$$

which completes the proof. □

3.11.1 Heuristics

With the same justification as in Section 3.4, we again assume that typically, for all $i \geq 0$, the edge set of $\mathcal{H}$ behaves essentially as if it was obtained by including every k-set $e \subseteq V_{\mathcal{H}}$ independently at random with probability

$$\hat{p}(i) := \vartheta - \frac{|\mathcal{F}|k!\,i}{n^k}.$$

We may guess deterministic upper bounds for these numbers of copies that we expect to typically hold as follows by considering the expected one-step changes of these numbers. Lemma 3.10.9 in particular shows that for almost all distinct edges $e, f \in \mathcal{H}(0)$, there exists at most one copy $\mathcal{F}' \subseteq \mathcal{H}(0)$ of $\mathcal{F}$ with $e, f \in \mathcal{F}'$. Thus, for $i \geq 0$, for the one-step change ΔH^*, we estimate

$$\begin{aligned} \mathbb{E}_i[\Delta H^*] &= -\sum_{\mathcal{F}'\in\mathcal{H}^*} \mathbb{P}[\mathcal{F}' \notin \mathcal{H}^*(i+1)] \approx -\sum_{\mathcal{F}'\in\mathcal{H}^*} \frac{\left(\sum_{f\in\mathcal{F}'} d_{\mathcal{H}^*}(f)\right) - |\mathcal{F}| + 1}{H^*} \\ &= -\frac{\sum_{e\in\mathcal{H}} d_{\mathcal{H}^*}(e)^2}{H^*} + |\mathcal{F}| - 1. \end{aligned}$$

Using convexity and $H = \hat{p}n^k/k!$, this leads us to expect

$$\begin{aligned} \mathbb{E}_i[\Delta H^*] &\leq -\frac{\left(\sum_{e\in\mathcal{H}} d_{\mathcal{H}^*}(e)\right)^2}{H\cdot H^*} + |\mathcal{F}| - 1 = -\frac{|\mathcal{F}|^2 H^*}{H} + |\mathcal{F}| - 1 \\ &= -\frac{|\mathcal{F}|^2 k!\, H^*}{n^k\hat{p}} + |\mathcal{F}| - 1. \end{aligned}$$

Motivated by this, we aim to choose our deterministic upper bounds $\hat{h}^*(0), \hat{h}^*(1), \dots$ for the random variables $H^*(0), H^*(1), \dots$ such that, with some room to spare for estimation errors, they approximately satisfy

$$\Delta \hat{h}^* \geq -\frac{|\mathcal{F}|^2 k!\, \hat{h}^*}{n^k \hat{p}} + |\mathcal{F}| - 1.$$

By Lemma 3.11.3, initially, that is for $i = 0$, there are at most $n^{m+\varepsilon^3} \hat{p}^{|\mathcal{F}|}$ copies of $\mathcal{F}$ in $\mathcal{H}$. With this initial condition, guided by the above intuition, for $i \geq 0$, we set

$$\hat{h}^*(i) := n^{m+\varepsilon^3} \hat{p}^{|\mathcal{F}|-\varepsilon^3} + \frac{(|\mathcal{F}|-1) n^k \hat{p}}{|\mathcal{F}|(|\mathcal{F}|-1-\varepsilon^3)k!}.$$

Observe that this expression is the sum of two parts where the second part is negligible up to step i where $\hat{p} \approx n^{-(m-k+\varepsilon^3)/(|\mathcal{F}|-1-\varepsilon^3)}$ and where then, the first part becomes negligible. For our argumentation, we focus our attention on the evolution of the process up to step $i^\star$, where

$$i^\star := \frac{(\vartheta - n^{-1/\rho_{\mathcal{F}} - \varepsilon^2}) n^k}{|\mathcal{F}| k!}.$$

Note that following the above heuristic, for all $i \geq 0$ and $e \in \mathcal{H}$, up to constant factors, we would expect approximately $n^{m-k} \hat{p}(i)^{|\mathcal{F}|-1}$ copies of $\mathcal{F}$ in $\mathcal{H}$ that contain e, which suggests that the process should terminate around the step i where $\hat{p} \approx n^{-1/\rho_{\mathcal{F}}}$. Since $i^\star$ lies beyond this step, an analysis up to step $i^\star$ should suffice.

3.11.2 Formal setup

Formally, we argue similarly as in Sections 3.6.3 and 3.7.2 and phrase our statement about the boundedness of H from above for $0 \leq i \leq i^\star$ in terms of the stopping time

$$\tau^\star := \min\{i \geq 0 : H^* \geq \hat{h}^*\}.$$

Our goal is to show that typically, $i^\star < \tau^\star$. To this end, for a similar argumentation as in the aforementioned sections, for $i \geq 0$, define the critical interval

$$I(i) := [(1-\varepsilon^4)\hat{h}^*, \hat{h}^*].$$

For $i \geq 0$, let

$$Y(i) := H^* - \hat{h}^*.$$

For $i_0 \geq 0$, define the stopping time

$$\tau_{i_0} := \min\{i \geq i_0 : H^* \notin I\}$$

and for $i \geq i_0$, let

$$Z_{i_0}(i) := Y(i_0 \vee (i \wedge \tau_{i_0} \wedge i^\star)).$$

Let

$$\sigma := \min\{j \geq 0 : H^* \geq (1-\varepsilon^4)\hat{h}^* \text{ for all } j \leq i < \tau^\star \wedge i^\star\} \leq \tau^\star \wedge i^\star.$$

With this setup, similarly as in Sections 3.6.3 and 3.7.2, it in fact suffices to consider the evolution of $Z_\sigma(\sigma), Z_\sigma(\sigma+1), \dots$.

Observation 3.11.4. $\{\tau^\star \le i^\star\} \subseteq \{Z_\sigma(i^\star) > 0\}$.

We use Azuma's inequality below to show that the probability of the event on the right in Observation 3.11.4 is sufficiently small.

Lemma 3.11.5 (Azuma's inequality)**.** *Suppose that $X(0), X(1), \dots$ is a supermartingale with $|X(i+1) - X(i)| \le a_i$ for all $i \ge 0$. Then, for all $i \ge 0$ and $t > 0$,*

$$\mathbb{P}[X(i) - X(0) \ge t] \le \exp\biggl(-\frac{t^2}{2\sum_{0\le j\le i-1} a_j^2}\biggr).$$

Before we turn to verifying that the conditions for an application of Azuma's inequality in Sections 3.11.3 and 3.11.4 and applying the inequality in Section 3.11.5, similarly as in Section 3.5 however now for the sparse setting that we consider since Section 3.9, we gather some useful facts concerning key quantities defined up to this point.

Lemma 3.11.6. *Let $0 \le i \le i^\star$. Then, $n^{1-k-\varepsilon^2} \le n^{-1/\rho_{\mathcal{F}}-\varepsilon^2} \le \hat{p} \le n^{-1/\rho_{\mathcal{F}}+\varepsilon^3}$*

Proof. We have $n^{-1/\rho_{\mathcal{F}}-\varepsilon^2} = \hat{p}(i^\star) \le \hat{p} \le \hat{p}(0) = \vartheta \le n^{-1/\rho_{\mathcal{F}}+\varepsilon^3}$. With Lemma 3.10.7, this completes the proof. □

Lemma 3.11.7. *Let $0 \le i \le i^\star$. Then, $\hat{p}(i+1) \ge (1 - n^{-1/2})\hat{p}$.*

Proof. Lemma 3.11.6 implies

$$\hat{p}(i+1) = \biggl(1 - \frac{|\mathcal{F}|k!}{n^k\hat{p}}\biggr)\hat{p} \ge \biggl(1 - \frac{|\mathcal{F}|k!}{n^{1-\varepsilon^2}}\biggr)\hat{p} \ge (1 - n^{-1/2})\hat{p},$$

which completes the proof. □

Lemma 3.11.8. *Let $0 \le i \le i^\star$ and $\mathcal{X} := \{i \le \tau_\emptyset\}$. Then, $n^{1/2} \le n^k\hat{p}/k! \le H =_{\mathcal{X}} n^k\hat{p}/k!$.*

Proof. Indeed, we have $H \ge \vartheta n^k/k! - |\mathcal{F}|i =_{\mathcal{X}} H$ and $\vartheta n^k/k! - |\mathcal{F}|i = n^k\hat{p}/k!$, so Lemma 3.11.6 completes the proof. □

3.11.3 Trend

Here, essentially following the argumentation in Section 3.11.1, we prove that for all $i_0 \ge 0$, the expected one-step changes of the process $Z_{i_0}(i_0), Z_{i_0}(i_0+1), \dots$ are non-positive. We bound the one-step changes of $\hat{h}^*$ in Lemma 3.11.10, then we turn to the non-deterministic one-step changes of H^*. Crucially, to see that for $0 \le i \le i^\star$, the expected one-step changes of H^* are at most those of $\hat{h}^*$, which justifies our choice of $\hat{h}^*$, we employ Lemma 3.10.9 in the proof of Lemma 3.11.11.

Observation 3.11.9. *Extend $\hat{p}$ and $\hat{h}^*$ to continuous trajectories defined on the whole interval $[0, i^\star + 1]$ using the same expressions as above. Then, for $x \in [0, i^\star + 1]$,*

$$(\hat{h}^*)'(x) = -\frac{|\mathcal{F}|(|\mathcal{F}| - \varepsilon^3)k!\,\hat{h}^*}{n^k\hat{p}} + |\mathcal{F}| - 1,$$

$$(\hat{h}^*)''(x) = \frac{|\mathcal{F}|^2(|\mathcal{F}| - \varepsilon^3)(|\mathcal{F}| - 1 - \varepsilon^3)(k!)^2 n^{m+\varepsilon^3}\hat{p}^{|\mathcal{F}|-\varepsilon^3}}{(n^k\hat{p})^2}.$$

Lemma 3.11.10. *Let $0 \leq i \leq i^\star$ and $\mathcal{X} := \{i \leq \tau_\emptyset\}$. Then,*

$$\Delta \hat{h}^* \geq_{\mathcal{X}} -\frac{|\mathcal{F}|(|\mathcal{F}| - \varepsilon^3)\hat{h}^*}{H} + |\mathcal{F}| - 1 - \frac{n^{-\varepsilon}\hat{h}^*}{H}.$$

Proof. This is a consequence of Taylor's theorem.

In detail, we argue as follows. Together with Observation 3.11.9, Lemma 2.9.10 yields

$$\Delta \hat{h}^* = -\frac{|\mathcal{F}|k!\,(|\mathcal{F}| - \varepsilon^3)\hat{h}^*}{n^k \hat{p}} + |\mathcal{F}| - 1 \pm \max_{x \in [i,i+1]} \frac{|\mathcal{F}|^4 (k!)^2 n^{m+\varepsilon^3} \hat{p}(x)^{|\mathcal{F}|-\varepsilon^3}}{(n^k \hat{p}(x))^2}.$$

We investigate the first term and the maximum separately. Lemma 3.11.8 yields

$$\frac{|\mathcal{F}|k!\,(|\mathcal{F}| - \varepsilon^3)\hat{h}^*}{n^k \hat{p}} =_{\mathcal{X}} \frac{|\mathcal{F}|(|\mathcal{F}| - \varepsilon^3)\hat{h}^*}{H}.$$

Furthermore, using Lemma 3.11.7 and Lemma 3.11.8, we obtain

$$\begin{aligned}\max_{x \in [i,i+1]} \frac{|\mathcal{F}|^4 (k!)^2 n^{m+\varepsilon^3} \hat{p}(x)^{|\mathcal{F}|-\varepsilon^3}}{(n^k \hat{p}(x))^2} &\leq \max_{x \in [i,i+1]} \frac{|\mathcal{F}|^4 (k!)^2 \hat{h}^*(x)}{(n^k \hat{p}(x))^2} \leq \frac{2|\mathcal{F}|^4 (k!)^2 \hat{h}^*}{(n^k \hat{p})^2} \\ &=_{\mathcal{X}} \frac{2|\mathcal{F}|^4 \hat{h}^*}{H^2} \leq \frac{n^{-\varepsilon}\hat{h}^*}{H},\end{aligned}$$

which completes the proof. □

Lemma 3.11.11. *Let $0 \leq i \leq i^\star$. Let $\mathcal{X} := \{i < \tau_i\}$. Then,*

$$\mathbb{E}_i[\Delta H^*] \leq_{\mathcal{X}} -\frac{|\mathcal{F}|^2 H^*}{H} + |\mathcal{F}| - 1 + \frac{n^{-\varepsilon}\hat{h}^*_+}{H}.$$

Proof. Let $\mathscr{F}_2$ denote a collection of k-graphs $\mathcal{G}$ with $V_{\mathcal{G}} \subseteq \{1, \ldots, 2m\}$ such that for all copies $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{F}$ with $2 \leq |\mathcal{F}_1 \cap \mathcal{F}_2| \leq |\mathcal{F}| - 1$, the collection $\mathscr{F}_2$ contains a copy of $\mathcal{F}_1 + \mathcal{F}_2$ and that only contains copies of such k-graphs. We have

$$\begin{aligned}\mathbb{E}_i[\Delta H^*] &\leq -\frac{1}{H^*} \sum_{\mathcal{F}' \in \mathcal{H}^*} \Big(1 + \sum_{e \in \mathcal{F}'} (d_{\mathcal{H}^*}(e) - 1) - \sum_{\substack{e,f \in \mathcal{F}' \\ e \neq f}} (d_{\mathcal{H}^*}(ef) - 1)\Big) \\ &= -\Big(\frac{1}{H^*} \sum_{e \in \mathcal{H}} d_{\mathcal{H}^*}(e)^2\Big) + \Big(\frac{1}{H^*} \sum_{\mathcal{F}' \in \mathcal{H}^*} \sum_{\substack{e,f \in \mathcal{F}' : \\ e \neq f}} \sum_{\substack{\mathcal{F}'' \in \mathcal{H}^* \setminus \{\mathcal{F}'\} : \\ e,f \in \mathcal{F}''}} 1\Big) + |\mathcal{F}| - 1 \\ &\leq -\Big(\frac{1}{H^*} \sum_{e \in \mathcal{H}} d_{\mathcal{H}^*}(e)^2\Big) + \Big(\frac{2|\mathcal{F}|^2}{H^*} \sum_{\mathcal{G} \in \mathscr{F}_2} \Phi_{\mathcal{G}}\Big) + |\mathcal{F}| - 1.\end{aligned} \tag{3.11.1}$$

We investigate the first two terms separately.

For the first term, using convexity, we obtain

$$\frac{1}{H^*} \sum_{e \in \mathcal{H}} d_{\mathcal{H}^*}(e)^2 \geq \frac{1}{HH^*} \Big(\sum_{e \in \mathcal{H}} d_{\mathcal{H}^*}(e)\Big)^2 = \frac{|\mathcal{F}|^2 H^*}{H}. \tag{3.11.2}$$

Let us now consider the second term. If $|\mathcal{F}| = 2$, then $\mathscr{F}_2 = \emptyset$ and otherwise, for all $\mathcal{G} \in \mathscr{F}_2$, Lemma 3.10.9 together with Lemma 3.11.6 and Lemma 3.11.8 entails

$$\begin{aligned}\Phi_{\mathcal{G}} &\leq n^{k-1/\rho_{\mathcal{F}}-\varepsilon^{1/7}} \leq n^{k-1/\rho_{\mathcal{F}}-\varepsilon^{1/6}}(n^{1/\rho_{\mathcal{F}}}\hat{p})^{2(|\mathcal{F}|-1)} \\ &\leq n^{-\varepsilon^{1/5}} \cdot n^k\hat{p} \cdot n^{2(m-k)}\hat{p}^{2(|\mathcal{F}|-1)} \leq \frac{n^{-\varepsilon^{1/4}}(\hat{h}^*)^2}{n^k\hat{p}} \leq_{\mathcal{X}} \frac{n^{-\varepsilon^{1/3}}(\hat{h}^*)^2}{H} \leq_{\mathcal{X}} \frac{n^{-\varepsilon^{1/2}}H^*\hat{h}^*}{H}.\end{aligned}$$

Thus,

$$\frac{2|\mathcal{F}|^2}{H^*}\sum_{\mathcal{G}\in\mathscr{F}_2}\Phi_{\mathcal{G}} \leq \frac{n^{-\varepsilon}\hat{h}^*}{H}. \tag{3.11.3}$$

Combining (3.11.2) and (3.11.3) with (3.11.1) yields the desired upper bound for $\mathbb{E}_i[\Delta H^*]$. □

Lemma 3.11.12. *Let $0 \leq i_0 \leq i$. Then, $\mathbb{E}_i[\Delta Z_{i_0}] \leq 0$.*

Proof. Suppose that $i < i^\star$ and let $\mathcal{X} := \{i < \tau_{i_0}\}$. We have $\mathbb{E}_i[\Delta Z_{i_0}] =_{\mathcal{X}^c} 0$ and $\mathbb{E}_i[\Delta Z_{i_0}] =_{\mathcal{X}} \mathbb{E}_i[\Delta Y]$, so it suffices to obtain $\mathbb{E}_i[\Delta Y] \leq_{\mathcal{X}} 0$. Combining Lemma 3.11.10 with Lemma 3.11.11, we have

$$\begin{aligned}\mathbb{E}_i[\Delta Y] &\leq_{\mathcal{X}} -\frac{|\mathcal{F}|}{H}(|\mathcal{F}|H^* - (|\mathcal{F}| - \varepsilon^3)\hat{h}^*) + \frac{2n^{-\varepsilon}\hat{h}^*}{H} \\ &\leq_{\mathcal{X}} -\frac{|\mathcal{F}|}{H}(|\mathcal{F}|(1-\varepsilon^4)\hat{h}^* - (|\mathcal{F}| - \varepsilon^3)\hat{h}^*) + \frac{2n^{-\varepsilon}\hat{h}^*}{H} \leq -\frac{\varepsilon^4|\mathcal{F}|\hat{h}^*}{H} + \frac{2n^{-\varepsilon}\hat{h}^*}{H} \leq 0,\end{aligned}$$

which completes the proof. □

3.11.4 Boundedness

For our application of Azuma's inequality, it suffices to obtain suitable bounds for the absolute one-step changes of the processes $Y(0), Y(1), \ldots$ and $Z_{i_0}(i_0), Z_{i_0}(i_0+1), \ldots$. Furthermore, crude upper bounds that we obtain as an immediate consequence of the previously gained insights concerning the distribution of the copies of $\mathcal{F}$ within $\mathcal{H}(0)$ suffice.

Lemma 3.11.13. *Let $0 \leq i \leq i^\star$. Then, $|\Delta Y| \leq n^{\varepsilon}$.*

Proof. From Lemma 3.11.2, Lemma 3.11.10, Lemma 3.11.6, Lemma 3.11.8 and the second inequality in Lemma 3.11.2, we obtain

$$\begin{aligned}|\Delta Y| &\leq |\Delta H^*| + |\Delta\hat{h}^*| \leq \Big(\sum_{e\in\mathcal{F}_0(i+1)} d_{\mathcal{H}^*}(e)\Big) - \Delta\hat{h}^* \leq |\mathcal{F}|n^{\varepsilon^3} + |\mathcal{F}| + \frac{2|\mathcal{F}|^2\hat{h}^*}{H} \\ &\leq 2|\mathcal{F}|n^{\varepsilon^3} + 2|\mathcal{F}|^2k!\,n^{\varepsilon^2} \cdot n^{m-k}\vartheta^{|\mathcal{F}|-1} \leq n^{\varepsilon}.\end{aligned}$$

which completes the proof. □

Lemma 3.11.14. *Let $0 \leq i_0 \leq i \leq i^\star$. Then, $|\Delta Z_{i_0}| \leq n^{\varepsilon}$.*

Proof. This is an immediate consequence of Lemma 3.11.13. □

3.11.5 Supermartingale argument

Lemma 3.11.15 is the final ingredient that we use for our application of Azuma's inequality in the proof of Lemma 3.11.16 where we show that the probabilities of the events on the right in Observation 3.11.4 are indeed small.

Lemma 3.11.15. *$Z_\sigma(\sigma) \le -\varepsilon^5 \hat{h}^*(\sigma)$.*

Proof. Lemma 3.11.3 implies $\tau^\star \ge 1$ and $H^*(0) < (1-\varepsilon^4)\hat{h}^*(0)$, so we have $\sigma \ge 1$. Thus, by definition of σ, for $i := \sigma - 1$, we have $H^* \le (1-\varepsilon^4)\hat{h}^*$ and thus

$$Z_i = H^* - \hat{h}^* \le -\varepsilon^4 \hat{h}^*.$$

With Lemma 3.11.13 and Lemma 3.11.6, this then yields

$$Z_\sigma(\sigma) \le Z_i + \Delta Y \le -\varepsilon^4 \hat{h}^* + n^\varepsilon \le -\varepsilon^4 \hat{h}^* + n^{-2\varepsilon} n^k \hat{p} \le -\varepsilon^4 \hat{h}^* + n^{-\varepsilon} \hat{h}^* \le -\varepsilon^5 \hat{h}^*.$$

Since $\Delta \hat{h}^* \le 0$, this completes the proof. □

Lemma 3.11.16. *$\mathbb{P}[\tau^\star \le i^\star] \le \exp(-n^{1/3})$.*

Proof. Considering Observation 3.11.4, it suffices to show that

$$\mathbb{P}[Z_\sigma(i^\star) > 0] \le \exp(-n^{1/3}).$$

Due to Lemma 3.11.15, we have

$$\mathbb{P}[Z_\sigma(i^\star) > 0] \le \mathbb{P}[Z_\sigma(i^\star) - Z_\sigma(\sigma) > \varepsilon^5 \hat{h}^*] \le \sum_{0 \le i \le i^\star} \mathbb{P}[Z_i(i^\star) - Z_i > \varepsilon^5 \hat{h}^*].$$

Thus it suffices to show that for $0 \le i \le i^\star$, we have

$$\mathbb{P}[Z_i(i^\star) - Z_i > \varepsilon^5 \hat{h}^*] \le \exp(-n^{1/2}).$$

We show that this bound is a consequence of Azuma's inequality.

Let us turn to the details. Lemma 3.11.11 shows that $Z_i(i), Z_i(i+1), \ldots$ is a supermartingale, while Lemma 3.11.14 provides the bound $|\Delta Z_i(j)| \le n^\varepsilon$ for all $j \ge i$. Hence, we may apply Lemma 3.11.5 to obtain

$$\mathbb{P}[Z_i(i^\star) - Z_i > \varepsilon^5 \hat{h}^*] \le \exp\left(-\frac{\varepsilon^{10}(\hat{h}^*)^2}{2(i^\star - i)n^{2\varepsilon}}\right).$$

Since

$$i^\star - i \le \frac{\vartheta n^k}{|\mathcal{F}|k!} - i = \frac{n^k \hat{p}}{|\mathcal{F}|k!},$$

with Lemma 3.11.6, this yields

$$\begin{aligned}\mathbb{P}[Z_i(i^\star) - Z_i > \varepsilon^5 \hat{h}^*] &\le \exp\left(-\frac{\varepsilon^{11}(\hat{h}^*)^2}{n^{k+2\varepsilon}\hat{p}}\right) \le \exp(-\varepsilon^{11} n^{k-3\varepsilon}\hat{p} \cdot n^{2(m-k)} \hat{p}^{2(|\mathcal{F}|-1)}) \\ &\le \exp(-\varepsilon^{11} n^{k-3\varepsilon-2\varepsilon^2(|\mathcal{F}|-1)}\hat{p}) \le \exp(-n^{1/2}),\end{aligned}$$

which completes the proof. □

3.12 The isolation argument

In this section, we show that $H(\tau_\emptyset) \geq n^{k-1/\rho-\varepsilon}$ with high probability if $\mathcal{F}$ is strictly k-balanced. For this section, in addition to the setup described in Section 3.10, assume that $\mathcal{F}$ is strictly k-balanced, so in particular that $|\mathcal{F}| \geq 3$, and that $\mathcal{H}$ is $\mathcal{F}$-populated. Overall, our approach is inspired by [14, Proof of Theorem 6.1]; however, whenever $\mathcal{F}$ is not a triangle, copies of $\mathcal{F}$ can form substructures that may prevent a direct translation of the argument. For our more general setting, we rely on the insights gained in Section 3.10 to control these substructures in our analysis.

3.12.1 Overview

Instead of choosing the edge sets of copies $\mathcal{F}_0(i)$ with $i \geq 1$ uniformly at random in Algorithm 3.3.1, we may assume that during the initialization, a linear order $\preccurlyeq$ on $\mathcal{H}^*$ is chosen uniformly at random and that for all $i \geq 1$, the edge set $\mathcal{F}_0(i)$ is the minimum of $\mathcal{H}^*(i-1)$. Clearly, this yields the same random process.

Our argument that typically, sufficiently many edges of $\mathcal{H}(0)$ remain when Algorithm 3.3.1 terminates may be summarized as follows. We crucially rely on identifying edges of $\mathcal{H}(0)$ that for some $i \geq 0$ become isolated vertices of $\mathcal{H}^*$ and hence remain at the end of the process. We say that *almost-isolation* occurs at a copy $\mathcal{F}' \in \mathcal{H}^*(0)$ if for some edge $e \in \mathcal{F}'$ at some step, the copy $\mathcal{F}'$ is the only remaining copy that contains e and we say that *isolation* occurs at $\mathcal{F}'$ if additionally at a later step, a copy $\mathcal{F}'' \neq \mathcal{F}'$ with $e \notin \mathcal{F}' \cap \mathcal{F}'' \neq \emptyset$ is selected for removal hence causing e to become an isolated vertex in $\mathcal{H}^*$.

Initially, that is at step $i = 0$, for every edge $e \in \mathcal{H}$, there exist at least two copies of $\mathcal{F}$ that have e as one of their edges. If at step $i = i^\star$ we do not already have sufficiently many edges of $\mathcal{H}$ that are isolated vertices of $\mathcal{H}^*$, then since by Lemma 3.11.16 we may assume that there is essentially not more than one copy of $\mathcal{F}$ for every $|\mathcal{F}|$ edges that remain, we are in a situation where most of the remaining copies form a matching within $\mathcal{H}^*$. Thus, almost-isolation must have occurred many times.

If it is the removal of $\mathcal{F}_0$ during step i that causes almost-isolation at a copy $\mathcal{F}'$, then before this removal, for all edges $e \in \mathcal{F}'$, there was a copy $\mathcal{F}'' \neq \mathcal{F}'$ with $e \in \mathcal{F}''$ and hence as a consequence of Lemma 3.10.9, it only rarely happens that the removal of $\mathcal{F}_0$ destroys all copies $\mathcal{F}'' \neq \mathcal{F}'$ that previously shared an edge with $\mathcal{F}'$. Thus, in almost all cases where almost-isolation occurs, it is possible that isolation occurs. Furthermore, it turns out that the probability that this happens is not too small.

We ensure that the copies at which we look for almost-isolation are spaced out as this allows us to assume that at these copies, almost-isolation turns into isolation independently of the development at the other copies.

3.12.2 Formal setup

Formally, our setup is as follows. For $\ell \geq 1$, a hypergraph $\mathcal{A}$ and $e \in \mathcal{A}$, inductively define $\mathcal{N}^\ell_{\mathcal{A}}(e)$ as follows. Let $\mathcal{N}^1_{\mathcal{A}}(e) := \{f \in \mathcal{A} : e \cap f \neq \emptyset\}$ denote the set of edges of $\mathcal{A}$

that intersect with e and for $\ell \geq 2$, let

$$\mathcal{N}_{\mathcal{A}}^{\ell}(e) := \bigcup_{f \in N_{\mathcal{A}}^{\ell-1}(e)} \mathcal{N}_{\mathcal{A}}^{1}(f).$$

For $\ell \geq 1$, let $N_{\mathcal{A}}^{\ell}(e) := |\mathcal{N}_{\mathcal{A}}^{\ell}(e)|$. During the random removal process, we additionally construct random subsets $\emptyset =: \mathcal{R}(0) \subseteq \ldots \subseteq \mathcal{R}(i^\star) \subseteq \mathcal{H}^*(0)$ where we collect copies of $\mathcal{F}$ at which almost-isolation occurs. We inductively define $\mathcal{R}(i)$ with $1 \leq i \leq i^\star$ as described by the following procedure.

Algorithm 3.12.1: Construction of $\mathcal{R}(i)$.

```
R(i) ← R(i − 1)
consider an arbitrary ordering F_1, ..., F_ℓ of H*(i)
for ℓ' ← 1 to ℓ do
    if i = min{j ≥ 0 : d_{H*(j)}(e) = 1 for some e ∈ F_ℓ'} and
     N^4_{H*(0)}(F_ℓ') ∩ R(i) = ∅ then
        R(i) ← R(i) ∪ {F_ℓ'}
    end
end
```

To exclude the copies at which almost-isolation occurs without the option that isolation occurs, we define subsets $\mathcal{R}'(i) \subseteq \mathcal{R}(i)$ as follows. For $\mathcal{F}' \in \mathcal{R}(i^\star)$, let

$$i_{\mathcal{F}'} := \min\{i \geq 0 : \mathcal{F}' \in \mathcal{R}(i)\}$$

be the step where $\mathcal{F}'$ is added as an element for the eventually generated $\mathcal{R}(i^\star)$ and for $i \geq 0$, let

$$\mathcal{R}'(i) := \{\mathcal{F}' \in \mathcal{R}(i) : \mathcal{N}_{\mathcal{H}^*(i_{\mathcal{F}'})}^{1}(\mathcal{F}') \neq \{\mathcal{F}'\}\}$$

be the elements $\mathcal{F}' \in \mathcal{R}(i)$ where at step $i_{\mathcal{F}'}$, the copy $\mathcal{F}'$ shared at least one edge with another copy of $\mathcal{F}$. Finally, we define events that entail almost-isolation becoming isolation. For $\mathcal{F}' \in \mathcal{R}'(i)$, fix an arbitrary $\mathcal{G}_{\mathcal{F}'} \in \mathcal{N}_{\mathcal{H}^*(i_{\mathcal{F}'})}^{1}(\mathcal{F}') \setminus \{\mathcal{F}'\}$ and let

$$\mathcal{E}_{\mathcal{F}'} := \{\mathcal{G}_{\mathcal{F}'} \preccurlyeq \mathcal{G} \text{ for all } \mathcal{G} \in \mathcal{N}_{\mathcal{H}^*(0)}^{1}(\mathcal{G}_{\mathcal{F}'})\}.$$

3.12.3 Proof of Theorem 3.11.1

Since every almost-isolation that turns into isolation causes an edge of $\mathcal{H}(0)$ to become an isolated vertex of $\mathcal{H}^*$ for some $i \geq 0$ and hence an edge that remains at the end of the removal process, we obtain the following statement.

Observation 3.12.2. $H(\tau_\emptyset) \geq \sum_{\mathcal{F}' \in \mathcal{R}'(i^\star)} \mathbb{1}_{\mathcal{E}_{\mathcal{F}'}}$.

We organize the formal presentation of the arguments outlined above in two lemmas. At the end of the section, using the above observation together with these two lemmas, we prove Theorem 3.11.1.

Define the event

$$\mathcal{E}_0 := \{|\{e \in \mathcal{H}(i^\star) : d_{\mathcal{H}^*(i^\star)}(e) = 0\}| < \varepsilon H(i^\star)\}$$

that occurs if and only if the number of isolated vertices of $\mathcal{H}^*(i^\star)$ is only a small fraction of all present vertices.

Lemma 3.12.3. *Let $\mathcal{X} := \{i^\star < \tau^\star\} \cap \mathcal{E}_0$. Then, $|\mathcal{R}'(i^\star)| \geq_{\mathcal{X}} n^{k-1/\rho-4\varepsilon^2}$.*

Proof. Let $i := i^\star$ and consider the set

$$\mathcal{I}^* := \{\mathcal{F}' \in \mathcal{H}^* : \mathcal{N}^1_{\mathcal{H}^*}(\mathcal{F}') = \{\mathcal{F}'\}\}$$

of edge sets of copies of $\mathcal{F}$ in $\mathcal{H}$ that are isolated in the sense that they do not share an edge with another copy of $\mathcal{F}$. Since $\mathcal{H}(0)$ is $\mathcal{F}$-populated, by construction of $\mathcal{R}$, for every $\mathcal{F}' \in \mathcal{I}^*$, either $\mathcal{F}'$ itself is an element of $\mathcal{R}$ or there exists some $\mathcal{F}'' \in \mathcal{N}^4_{\mathcal{H}^*(0)}(\mathcal{F}') \cap \mathcal{R}$ that prevented the inclusion of $\mathcal{F}'$ in $\mathcal{R}$. Hence, there exists a function $\pi\colon \mathcal{I}^* \to \mathcal{R}$ that for every $\mathcal{F}' \in \mathcal{I}^*$ chooses a witness $\pi(\mathcal{F}')$ with $\pi(\mathcal{F}') \in \mathcal{N}^4_{\mathcal{H}^*(0)}(\mathcal{F}')$ or equivalently $\mathcal{F}' \in \mathcal{N}^4_{\mathcal{H}^*(0)}(\pi(\mathcal{F}'))$. If $\mathcal{F}' \in \mathcal{R}$ and $\mathcal{F}'' \in \pi^{-1}(\mathcal{F}')$, we have $\mathcal{F}'' \in \mathcal{N}^4_{\mathcal{H}^*(0)}(\mathcal{F}')$ and hence $\pi^{-1}(\mathcal{F}') \subseteq \mathcal{N}^4_{\mathcal{H}^*(0)}(\mathcal{F}')$. Thus, Lemma 3.11.2 entails $|\pi^{-1}(\mathcal{F}')| \leq N^4_{\mathcal{H}^*(0)}(\mathcal{F}') \leq n^{\varepsilon^2}$ and so we have

$$|\mathcal{I}^*| \leq \sum_{\mathcal{F}' \in \mathcal{R}} |\pi^{-1}(\mathcal{F}')| \leq |\mathcal{R}| n^{\varepsilon^2}. \tag{3.12.1}$$

First, we obtain a suitable lower bound for $|\mathcal{I}^*|$ which, by the above inequality, yields a lower bound for $|\mathcal{R}|$, then we show that $|\mathcal{R}|$ is essentially as large as $|\mathcal{R}'|$.

Let us proceed with the first step. Using Lemma 3.11.8, we have

$$\begin{aligned} H^* \leq_{\mathcal{X}} \hat{h}^* &\leq \left(\frac{|\mathcal{F}|-1}{|\mathcal{F}|-1-\varepsilon^2} + n^{m-k+2\varepsilon^3}\hat{p}^{|\mathcal{F}|-1-\varepsilon^3}\right)\frac{n^k\hat{p}}{|\mathcal{F}|k!} \\ &\leq (1+\varepsilon^{3/2} + n^{2\varepsilon^3-\varepsilon^2(|\mathcal{F}|-1-\varepsilon^3)+\varepsilon^3/\rho_{\mathcal{F}}})\frac{n^k\hat{p}}{|\mathcal{F}|k!} \leq (1+\varepsilon)\frac{H}{|\mathcal{F}|}. \end{aligned} \tag{3.12.2}$$

From this, we obtain

$$\begin{aligned} H &= |\{e \in \mathcal{H} : d_{\mathcal{H}^*}(e) = 0\}| + \sum_{\mathcal{F}' \in \mathcal{H}^*}\sum_{e \in \mathcal{F}'} \frac{1}{d_{\mathcal{H}^*}(e)} \leq_{\mathcal{X}} \varepsilon H + |\mathcal{F}||\mathcal{I}^*| + \left(|\mathcal{F}| - \frac{1}{2}\right)|\mathcal{H}^* \setminus \mathcal{I}^*| \\ &= \varepsilon H + \left(|\mathcal{F}| - \frac{1}{2}\right)H^* + \frac{1}{2}|\mathcal{I}^*| \leq_{\mathcal{X}} \varepsilon H + \left(|\mathcal{F}| - \frac{1}{2}\right)(1+\varepsilon)\frac{H}{|\mathcal{F}|} + \frac{1}{2}|\mathcal{I}^*| \\ &= H - \frac{1+\varepsilon-4\varepsilon|\mathcal{F}|}{2|\mathcal{F}|}H + \frac{1}{2}|\mathcal{I}^*| \leq H - \frac{1}{4|\mathcal{F}|}H + \frac{1}{2}|\mathcal{I}^*|. \end{aligned} \tag{3.12.3}$$

With Lemma 3.11.8, this implies

$$|\mathcal{I}^*| \geq_{\mathcal{X}} \frac{1}{2|\mathcal{F}|}H \geq \frac{n^k\hat{p}}{2|\mathcal{F}|k!} \geq n^{k-1/\rho_{\mathcal{F}}-2\varepsilon^2}. \tag{3.12.4}$$

Combining this with (3.12.1), we conclude that $|\mathcal{R}| \geq n^{k-1/\rho_{\mathcal{F}}-3\varepsilon^2}$, which completes the first step.

Consider a copy $\mathcal{F}'$ of $\mathcal{F}$ with $\mathcal{F}' \in \mathcal{R}\backslash\mathcal{R}'$. Let $e_1 \in \mathcal{F}'$. There exists a copy $\mathcal{F}_1 \neq \mathcal{F}'$ of $\mathcal{F}$ with $\mathcal{F}_1 \in \mathcal{H}^*(i_{\mathcal{F}'}-1)$ such that $e_1 \in \mathcal{F}_1$. Furthermore, there exists an edge $e_2 \in \mathcal{F}' \setminus \mathcal{F}_1$ and a copy $\mathcal{F}_2 \neq \mathcal{F}'$ of $\mathcal{F}$ with $\mathcal{F}_2 \in \mathcal{H}^*(i_{\mathcal{F}'}-1)$ such that $e_2 \in \mathcal{F}_2$. By choice of $\mathcal{F}'$ and $i_{\mathcal{F}'}$, both copies $\mathcal{F}_1$ and $\mathcal{F}_2$ have an edge that is contained in $\mathcal{F}_0(i_{\mathcal{F}'})$. Hence, if $\mathcal{F}_1, \mathcal{F}', \mathcal{F}_2$ does not form a self-avoiding cyclic walk, then, using $\mathcal{F}''$ to denote the copy of $\mathcal{F}$ with edge set $\mathcal{F}_0(i_{\mathcal{F}'})$, the sequence $\mathcal{F}_1, \mathcal{F}', \mathcal{F}_2, \mathcal{F}''$ forms a self-avoiding cyclic walk. Thus, for every copy $\mathcal{F}'$ of $\mathcal{F}$ with $\mathcal{F}' \in \mathcal{R} \setminus \mathcal{R}'$, there exist copies of $\mathcal{F}$ whose edge sets are elements of $\mathcal{H}^*(0)$ and that together with $\mathcal{F}'$ form a self-avoiding cyclic walk of length 3 or 4.

Let $\mathscr{F}_4$ denote a collection of k-graphs $\mathcal{G}$ with $V_{\mathcal{G}} \subseteq \{1, \ldots, 4m\}$ that for every self-avoiding walk $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ of copies of $\mathcal{F}$ with $3 \leq \ell \leq 4$ contains a copy of $\mathcal{F}_1 + \ldots + \mathcal{F}_\ell$ and that only contains copies of such k-graphs. Then, we have $|\mathcal{R}'| \geq |\mathcal{R}| - \sum_{\mathcal{G} \in \mathscr{F}_4} 4\Phi_{\mathcal{G}}$, so it suffices to show that $\Phi_{\mathcal{G}} \leq n^{k-1/\rho_{\mathcal{F}}-4\varepsilon^2}$ for all $\mathcal{G} \in \mathscr{F}_4$. This is a consequence of Lemma 3.10.9. □

Lemma 3.12.4. *Suppose that X is a binomial random variable with parameters $n^{k-1/\rho_{\mathcal{F}}-4\varepsilon^2}$ and $n^{-\varepsilon^2}$ and let $Y := (n^{k-1/\rho_{\mathcal{F}}-4\varepsilon^2} - |\mathcal{R}'(i^\star)|) \vee 0$. Let*

$$Z := Y + \sum_{\mathcal{F}' \in \mathcal{R}'(i^\star)} \mathbb{1}_{\mathcal{E}_{\mathcal{F}'}}.$$

Then, Z stochastically dominates X.

Proof. First, observe that by Lemma 3.11.2, whenever $\mathcal{F}' \in \mathcal{R}'(i^\star)$, for $i := 0$, we have

$$N^1_{\mathcal{H}^*}(\mathcal{G}_{\mathcal{F}'}) \leq \sum_{f \in \mathcal{G}_{\mathcal{F}'}} d_{\mathcal{H}^*}(f) \leq n^{\varepsilon^2}. \tag{3.12.5}$$

Consider distinct $\mathcal{F}', \mathcal{F}'' \in \mathcal{H}^*(0)$. By construction of $\mathcal{R}(i^\star)$, whenever $\mathcal{F}', \mathcal{F}'' \in \mathcal{R}(i^\star)$, then, for all $\mathcal{G}' \in \mathcal{N}^1_{\mathcal{H}^*(i_{\mathcal{F}'})}(\mathcal{F}')$ and $\mathcal{G}'' \in \mathcal{N}^1_{\mathcal{H}^*(i_{\mathcal{F}''})}(\mathcal{F}'')$, we have

$$\mathcal{N}^1_{\mathcal{H}^*(0)}(\mathcal{G}') \cap \mathcal{N}^1_{\mathcal{H}^*(0)}(\mathcal{G}'') = \emptyset.$$

Thus, for all distinct $\mathcal{F}_1, \ldots, \mathcal{F}_\ell \in \mathcal{R}'(i^\star_+)$ and all $z_1, \ldots, z_{\ell-1} \in \{0, 1\}$, from (3.12.5), we obtain

$$\mathbb{P}[\mathbb{1}_{\mathcal{E}_{\mathcal{F}_\ell}} = 1 \mid \mathbb{1}_{\mathcal{E}_{\mathcal{F}_{\ell'}}} = z_{\ell'} \text{ for all } 1 \leq \ell' < \ell] = \mathbb{P}[\mathcal{E}_{\mathcal{F}_\ell}] \geq n^{-\varepsilon^2},$$

which completes the proof. □

We use the following version of Chernoff's inequality which is slightly different compared to the version in Chapter 2.

Lemma 3.12.5 (Chernoff's inequality)**.** *Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables and let $X := \sum_{1 \leq i \leq n} X_i$. Then, for all $0 < \delta < 1$,*

$$\mathbb{P}[X \neq (1 \pm \delta)\mathbb{E}[X]] \leq 2 \exp\left(-\frac{\delta^2 \mathbb{E}[X]}{3}\right).$$

Proof of Theorem 3.11.1. Define the events

$$\mathcal{B} := \{H(\tau_\emptyset) \le n^{k-1/\rho_{\mathcal{F}}-\varepsilon}\} \quad \text{and} \quad \mathcal{X} := \{i^\star < \tau^\star\} \cap \mathcal{E}_0.$$

We need to show that $\mathbb{P}[\mathcal{B}]$ is sufficiently small. Choose X, Y and Z as in Lemma 3.12.4. Lemma 3.12.3 entails $\mathcal{X} \subseteq \{Y = 0\}$ and hence $\{Y \ne 0\} \subseteq \mathcal{X}^{\mathsf{c}}$. Thus, from Observation 3.12.2 and Lemma 3.12.4, we obtain

$$\begin{aligned}\mathcal{B} &= \Big\{ \sum_{\mathcal{F}' \in \mathcal{R}'(i^\star)} \mathbb{1}_{\mathcal{E}_{\mathcal{F}'}} \le n^{k-1/\rho_{\mathcal{F}}-\varepsilon} \Big\} \cap \mathcal{B} \subseteq (\{Z \le n^{k-1/\rho_{\mathcal{F}}-\varepsilon}\} \cup \{Y \ne 0\}) \cap \mathcal{B} \\ &\subseteq \{Z \le n^{k-1/\rho_{\mathcal{F}}-\varepsilon}\} \cup (\mathcal{X}^{\mathsf{c}} \cap \mathcal{B}) \subseteq \{Z \le n^{k-1/\rho_{\mathcal{F}}-\varepsilon}\} \cup \{\tau^\star \le i^\star\} \cup (\mathcal{E}_0^{\mathsf{c}} \cap \mathcal{B}).\end{aligned}$$

By Lemma 3.11.8, we have

$$H(\tau_\emptyset) \ge_{\mathcal{E}_0^{\mathsf{c}}} \varepsilon H(i^\star) \ge \varepsilon^2 n^k \hat{p}(i^\star) \ge n^{k-1/\rho_{\mathcal{F}}-2\varepsilon^2}$$

and hence $\mathcal{E}_0^{\mathsf{c}} \cap \mathcal{B} = \emptyset$. Thus, using Lemma 3.11.16, we obtain

$$\mathbb{P}[\mathcal{B}] \le \mathbb{P}[Z \le n^{k-1/\rho_{\mathcal{F}}-\varepsilon}] + \exp(-n^{1/3}).$$

With Lemma 3.12.4 and Chernoff's inequality (see Lemma 3.12.5), this completes the proof. □

3.13 Proofs for the main theorems

In this section, we show how to obtain Theorems 1.1.4–1.1.7 from Theorems 3.3.2 and 3.11.1. Proofs for Theorems 1.1.8 and 1.1.9 can be found in Section 3.14.

Proof of Theorem 1.1.6. This is an immediate consequence of Theorem 3.3.2. □

Proof of Theorem 1.1.7. By definition of $\tau_\emptyset$ in Section 3.11, this is an immediate consequence of Theorem 3.11.1. □

Proof of Theorem 1.1.5. Let $m := |V_{\mathcal{F}}|$. Suppose that $0 < \varepsilon < 1$ is sufficiently small in terms of $1/m$, that $0 < \delta < 1$ is sufficiently small in terms of ε and that n is sufficiently large in terms of $1/\delta$. Suppose that $\mathcal{H}$ is an $(\varepsilon^{20}, \delta, \rho)$-pseudorandom k-graph on n vertices with $|\mathcal{H}| \ge n^{k-1/\rho+\varepsilon^5}$. Let

$$\vartheta := \frac{k!\,|\mathcal{H}|}{n^k} \ge n^{-1/\rho+\varepsilon^5}.$$

We consider the $\mathcal{F}$-removal process starting at $\mathcal{H}$ where we assume the generated hypergraphs to remain constant if the process normally terminated due to the absence of copies of $\mathcal{F}$. Let $\mathcal{H}'$ denote the k-graph generated after $i^\star$ iterations, where

$$i^\star := \frac{(\vartheta - n^{-1/\rho+\varepsilon^5})n^k}{|\mathcal{F}|k!}.$$

Let $\mathcal{H}''$ denote the k-graph eventually generated by the process that contains no copies of $\mathcal{F}$ as subgraphs. Let $\mathcal{X}'$ denote the event that $\mathcal{H}'$ is $(4m, n^{\varepsilon^4})$-bounded, $\mathcal{F}$-populated and has $n^{k-1/\rho+\varepsilon^5}/k!$ edges. Let

$$\mathcal{X}'' := \{|\mathcal{H}''| \leq n^{k-1/\rho+\varepsilon}\} \quad \text{and} \quad \mathcal{Y}'' := \{n^{k-1/\rho-\varepsilon} \leq |\mathcal{H}''|\}.$$

We need to show that

$$\mathbb{P}[\mathcal{X}'' \cap \mathcal{Y}''] \geq 1 - \exp(-(\log n)^{5/4}).$$

Since $\mathcal{X}' \subseteq \mathcal{X}''$, we have $\mathbb{P}[\mathcal{X}'' \cap \mathcal{Y}''] \geq \mathbb{P}[\mathcal{X}' \cap \mathcal{Y}'']$, so it suffices to obtain sufficiently large lower bounds for $\mathbb{P}[\mathcal{X}']$ and $\mathbb{P}[\mathcal{Y}'']$. We may apply Theorem 3.3.2 with ε^5 playing the role of ε to obtain $\mathbb{P}[\mathcal{X}'] \geq 1 - \exp(-(\log n)^{4/3})$ and Theorem 3.11.1 shows that $\mathbb{P}[\mathcal{Y}'' \mid \mathcal{X}'] \geq 1 - \exp(-n^{1/4})$. Using $\mathbb{P}[\mathcal{Y}''] = \mathbb{P}[\mathcal{Y}'' \mid \mathcal{X}']\mathbb{P}[\mathcal{X}']$, this yields suitable lower bounds for $\mathbb{P}[\mathcal{X}']$ and $\mathbb{P}[\mathcal{Y}'']$. □

Proof of Theorem 1.1.4. This is an immediate consequence of Theorem 1.1.5. □

3.14 Cherries

In this section, we prove Theorems 1.1.8 and 1.1.9. We argue similarly as for Theorem 1.1.5 and 1.1.7 in the sense that we obtain Theorem 1.1.9 as a consequence of Theorem 3.14.1 below which plays a similar role as Theorem 3.11.1 and which we then apply together with Theorem 3.3.2 to obtain Theorem 1.1.8, see Section 3.14.4. To state Theorem 3.14.1, we assume the setup described in Section 3.9 and again consider the $\mathcal{F}$-removal process formally given by Algorithm 3.3.1 as in Section 3.11. In particular, we define $\mathcal{F}_0(i)$, $\mathcal{H}(i)$, $H(i)$, $\mathcal{H}^*(i)$ and $H^*(i)$ for $i \geq 0$ as well as $\tau_\emptyset$ as in Section 3.11. Furthermore, we introduce the following terminology. For a k-graph $\mathcal{A}$ and $1 \leq k' \leq k-1$, we say that $\mathcal{A}$ is a *k'-cherry* if $\mathcal{A}$ has no isolated vertices and exactly two edges such that the two edges of $\mathcal{A}$ share k' vertices. We say that $\mathcal{A}$ is a *cherry* if $\mathcal{A}$ is a k'-cherry for some $1 \leq k' \leq k-1$.

Theorem 3.14.1. *If $\mathcal{F}$ is a cherry, then* $\mathbb{P}[H(\tau_\emptyset) \leq n^{k-1/\rho_{\mathcal{F}}-\varepsilon}] \leq \exp(-n^{1/4})$.

3.14.1 Unions of cherries

To prove Theorem 3.14.1, we argue similarly as for Theorem 3.11.1. However, some of the key results in Section 3.10 only hold for hypergraphs with at least three edges since self-avoiding cyclic walks of cherries can form stars, that is hypergraphs where the intersection of any distinct edges is the same vertex set. This forces us to slightly adapt the corresponding arguments for the cherry case. More specifically, we employ the following two results that replace Lemma 3.10.5 and Lemma 3.10.9.

For $\ell \geq 2$, we say that a sequence $e_1, \dots, e_\ell$ of distinct k-sets forms a *k'-tight self-avoiding cyclic walk* if there exist distinct k'-sets $U_1, \dots, U_\ell$ with $U_i \subseteq e_i \cap e_{i+1}$ for all $1 \leq i \leq \ell$ with indices taken modulo ℓ. Note that the k-graph $\mathcal{S}$ with no isolated vertices and edge set $\{e_1, \dots, e_\ell\}$ is a union of cherries. Indeed, for $1 \le i \le \ell$, the k-graph $\mathcal{A}_i$ with no isolated vertices and edge set $\{e_i, e_{i+1}\}$ with indices taken modulo ℓ is a k''-cherry for some $k'' \ge k'$ and we have $\mathcal{S} = \mathcal{A}_1 + \dots + \mathcal{A}_\ell$. Furthermore, the k-graphs $\mathcal{A}_1, \dots, \mathcal{A}_\ell$ form a self-avoiding cyclic walk as defined in Section 3.10.

Lemma 3.14.2. *Let $1 \le k' \le k-1$. Suppose that $e_1, \dots, e_\ell$ forms a k'-tight self-avoiding cyclic walk and let $\mathcal{S}$ denote the k-graph without isolated vertices and edge set $\{e_1, \dots, e_\ell\}$. Then, there exists $e \in \mathcal{S}$ such that $\rho_{\mathcal{S},e} > 1/(k-k')$.*

Proof. For $0 \le i \le \ell$, let $V_i := e_1 \cup \dots \cup e_i$ and for $1 \le i \le \ell$, let $W_i := e_i \setminus V_{i-1}$. Note that $V_{\mathcal{S}} = \bigcup_{1 \le i \le \ell} W_i$ and that for all $1 \le i < j \le \ell$, we have $W_i \cap W_j = \emptyset$. Hence, $|V_{\mathcal{S}}| = \sum_{1 \le i \le \ell} |W_i|$. Since $e_1, \dots, e_\ell$ forms a k'-tight self-avoiding cyclic walk, there exist distinct k'-sets $U_1, \dots, U_\ell$ with $U_i \subseteq e_i \cap e_{i+1}$ for all $1 \le i \le \ell$ with indices taken modulo ℓ. Hence, for all $2 \le i \le \ell$, we have $|e_{i-1} \cap e_i| \ge k'$ and thus $|W_i| \le k - k'$. Furthermore, we have

$$|(e_1 \cup e_{\ell-1}) \cap e_\ell| \ge |U_{\ell-1} \cup U_\ell| \ge k' + 1$$

and thus $|W_\ell| \le k - k' - 1$. We conclude that

$$\rho_{\mathcal{S},e_1} = \frac{\ell - 1}{(\sum_{1 \le i \le \ell} |W_i|) - k} \ge \frac{\ell - 1}{(\ell - 2)(k - k') + k - k' - 1} > \frac{\ell - 1}{(\ell - 1)(k - k')} = \frac{1}{k - k'},$$

which completes the proof. □

Lemma 3.14.3. *Let $1 \le k' \le k-1$ and $\ell \le 4$ and suppose that $e_1, \dots, e_\ell$ forms a k'-tight self-avoiding cyclic walk. Let $\mathcal{S}$ denote the k-graph without isolated vertices and edge set $\{e_1, \dots, e_\ell\}$. If $\mathcal{F}$ is a k'-cherry, then $\Phi_{\mathcal{S}} \le n^{k-1/\rho_{\mathcal{F}} - \varepsilon^{1/7}}$.*

Proof. Suppose that $\mathcal{F}$ is a k'-cherry. For $1 \le i \le \ell$, let $\mathcal{A}_i$ denote the k-graph with no isolated vertices and edge set $\{e_i, e_{i+1}\}$ with indices taken modulo ℓ. Then $\mathcal{A}_i$ has k-density at least $\rho_{\mathcal{F}}$ and $\mathcal{A}_i$ is strictly k-balanced. Furthermore, $\mathcal{A}_1, \dots, \mathcal{A}_\ell$ forms a self-avoiding cyclic walk and we have $\mathcal{S} = \mathcal{A}_1 + \dots + \mathcal{A}_\ell$. Hence, due to Lemma 3.14.2, the statement follows from Lemma 3.10.8. □

3.14.2 Overview of the argument

In this section, we show that $H(\tau_\emptyset) \ge n^{k-1/\rho_{\mathcal{F}} - \varepsilon}$ with high probability if $\mathcal{F}$ is a cherry. To this end, from now on, for this section, in addition to the setup described in Section 3.9, we assume that $\mathcal{F}$ is a k'-cherry for some $1 \le k' \le k-1$ and that $\mathcal{H}$ is k'-populated. Furthermore, we define $i^\star$, $\tau^\star$ and $V_{\mathcal{H}}$ as in Section 3.11. Overall, we argue similarly as in Section 3.12 based on isolation, however, the structures we focus on here are different.

Still, instead of choosing the edge sets $\mathcal{F}_0(i)$ of copies with $i \ge 1$ uniformly at random in Algorithm 3.3.1, we again assume that during the initialization, a linear order $\preceq$ on $\mathcal{H}^*$ is chosen uniformly at random and that for all $i \ge 1$, the edge set $\mathcal{F}_0(i)$ is the minimum of $\mathcal{H}^*(i-1)$.

For a k'-set $U \subseteq V_{\mathcal{H}}$ and $i \geq 0$, we use

$$\mathcal{D}_{\mathcal{H}(i)}(U) := \{e \in \mathcal{H} : U \subseteq e\}$$

to denote the set of edges $e \in \mathcal{H}$ that contain U as a subset and we use

$$\mathcal{D}^*_{\mathcal{H}(i)}(U) := \{e \in \mathcal{D}_{\mathcal{H}}(U) : |e \cap f| = k' \text{ for some } f \in \mathcal{H} \setminus \mathcal{D}_{\mathcal{H}}(U)\}$$

to denote the set of edges $e \in \mathcal{H}$ that contain U as a subset and that are an edge of a k'-cherry where not both edges contain U as a subset. Note that $d_{\mathcal{H}}(U) = |\mathcal{D}_{\mathcal{H}}(U)|$. We set $d^*_{\mathcal{H}}(U) := |\mathcal{D}^*_{\mathcal{H}}(U)|$. We say that a k'-set $U \subseteq V_{\mathcal{H}}$ is *suitable* if there exists no sequence $e_1, \ldots, e_\ell$ of edges of $\mathcal{H}(0)$ that forms a k'-tight self-avoiding cyclic walk with $2 \leq \ell \leq 4$ such that $U \subseteq e_1$. We use $\mathcal{U}$ to denote the set of suitable k'-sets. Density considerations show that $\mathcal{U}$ includes almost all k'-sets $U \subseteq V_{\mathcal{H}}$. We say that *almost-isolation* occurs at $U \in \mathcal{U}$ if at some step $i \geq 0$, we have $1 \leq d^*_{\mathcal{H}}(U) \leq 2$ and $d_{\mathcal{H}}(U) \geq d^*_{\mathcal{H}}(U) + 1$. We say that *isolation* occurs at U if additionally at a later step $j > i$, we have $d^*_{\mathcal{H}(j)}(U) = 0$ while $d_{\mathcal{H}(j)}(U)$ is odd hence causing at least one of the edges $e \in \mathcal{H}(j)$ to eventually become an isolated vertex of $\mathcal{H}^*(j')$ for some $j' \geq j$.

If at step $i = i^\star$, we do not already have sufficiently many edges of $\mathcal{H}$ that are isolated vertices of $\mathcal{H}^*$, then by Lemma 3.11.16, we may assume that there is essentially not more than one copy of $\mathcal{F}$ for every $|\mathcal{F}|$ edges that remain. Hence, we are then in a situation where most of the remaining copies form a matching within $\mathcal{H}^*$. We claim that for these copies that form a matching, almost-isolation must have occurred at the set U of vertices that both edges of the copy share if $U \in \mathcal{U}$. This follows from Lemma 3.14.4 below. Indeed, the lemma guarantees that for such U, there exists $0 \leq i \leq i^\star$ with $d^*_{\mathcal{H}}(U) = 1$ or there exists $0 \leq i \leq i^\star - 1$ with $d^*_{\mathcal{H}}(U) = 2$, $d^*_{\mathcal{H}(i+1)}(U) = 0$ and $d_{\mathcal{H}}(U) - d_{\mathcal{H}(i+1)}(U) \geq 1$. Almost-isolation at U occurs in both cases.

Lemma 3.14.4. *Let $U \in \mathcal{U}$ and $0 \leq i \leq i^\star$. Then $\Delta d^*_{\mathcal{H}}(U) := d^*_{\mathcal{H}}(U) - d^*_{\mathcal{H}(i+1)}(U) \leq 2$. Furthermore, if $\Delta d^*_{\mathcal{H}}(U) = 2$, then $U \subseteq f$ for all $f \in \mathcal{F}_0$.*

Proof. We only assume that $U \subseteq V_{\mathcal{H}}$ is a k'-set and show that $\Delta d^*_{\mathcal{H}}(U) \geq 3$ entails $U \notin \mathcal{U}$ and furthermore that if $\Delta d^*_{\mathcal{H}}(U) = 2$ and $U \nsubseteq f$ for some $f \in \mathcal{F}_0$, then again $U \notin \mathcal{U}$. We distinguish three cases.

For the first case, assume that $U \subseteq f$ for all $f \in \mathcal{F}_0$. Then, only the edges of $\mathcal{F}_0$ can potentially be elements in $\Delta\mathcal{D}^* := \mathcal{D}^*_{\mathcal{H}(i-1)}(U) \setminus \mathcal{D}^*_{\mathcal{H}}(U)$, so we have $|\Delta\mathcal{D}^*| \leq 2$.

For the second case, assume that there is exactly one $f \in \mathcal{F}_0$ with $U \subseteq f$. Then if $|\Delta\mathcal{D}^*| \geq 2$, there exists $e \in \Delta\mathcal{D}^* \setminus \mathcal{F}_0$. For e to be in $\Delta\mathcal{D}^*$, it is necessary that there exists $f \in \mathcal{F}_0$ with $U \nsubseteq f$ and $|e \cap f| = k'$. There is only one possible choice for f and for this edge f, using f' to denote the other edge in $\mathcal{F}_0$, if e, f' does not form a k'-tight self-avoiding cyclic walk, then e, f, f' forms a k'-tight self-avoiding cyclic walk.

For the third case, assume that $U \nsubseteq f$ for all $f \in \mathcal{F}_0$. Then if $|\Delta\mathcal{D}^*| \geq 2$, there exist distinct $e_1, e_2 \in \Delta\mathcal{D}^* \setminus \mathcal{F}_0$ such that for $e \in \{e_1, e_2\}$, there exists $f \in \mathcal{F}_0$ with $|e \cap f| = k'$. If e_1, e_2 does not form a k'-tight self-avoiding cyclic walk and if for all $f \in \mathcal{F}_0$, the sequence e_1, e_2, f does not form a k'-tight self-avoiding cyclic walk, then, using f and f'

to denote the edges of $\mathcal{F}_0$, the sequence e_1, e_2, f, f' forms a k'-tight self-avoiding cyclic walk.

Furthermore, our above arguments show that if $\Delta d^*_{\mathcal{H}}(U) = 2$ and $U \not\subseteq f$ for some $f \in \mathcal{F}_0$, then $U \notin \mathcal{U}$. □

Overall, our argument shows that, if eventually most of the remaining copies form a matching within $\mathcal{H}^*$, almost-isolation must have occurred many times. In all cases where almost-isolation occurs, it is possible that this turns into isolation and the probability that this happens is not too small. We ensure that the k'-sets at which we look for almost isolation are spaced out as this allows us to assume that at these sets, almost-isolation turns into isolation independently of the development at the other sets.

3.14.3 Formal setup

Formally, our setup is as follows. For $\ell \geq 1$, a k-graph $\mathcal{A}$ and a k'-set $U \subseteq V_{\mathcal{A}}$, we inductively define $\mathcal{W}^{\ell}_{\mathcal{A}}(U)$ as follows. Let

$$\mathcal{W}^1_{\mathcal{A}}(U) := \left\{ U' \in \binom{V_{\mathcal{A}}}{k'} : d_{\mathcal{A}}(U \cup U') \geq 1 \right\}$$

and for $\ell \geq 2$, let

$$\mathcal{W}^{\ell}_{\mathcal{A}}(U) := \bigcup_{U' \in \mathcal{W}^{\ell-1}_{\mathcal{A}}(U)} \mathcal{W}^1_{\mathcal{A}}(U').$$

For $\ell \geq 1$, let $W^{\ell}_{\mathcal{A}}(U) := |\mathcal{W}^{\ell}_{\mathcal{A}}(U)|$. Similarly as in Section 3.12.2, during the random removal process, starting at step $i^\star$, we additionally construct random subsets $\emptyset =: \mathcal{R}(0) \subseteq \ldots \subseteq \mathcal{R}(i^\star) \subseteq \mathcal{U}$ where we collect k'-sets at which almost isolation occurs. We inductively define $\mathcal{R}(i)$ with $1 \leq i \leq i^\star$ as described by the following procedure.

Algorithm 3.14.5: Construction of $\mathcal{R}(i)$.

$\mathcal{R}(i) \leftarrow \mathcal{R}(i-1)$
consider an arbitrary ordering $U_1, \ldots, U_\ell$ of $\mathcal{U}$
for $\ell' \leftarrow 1$ **to** ℓ **do**
  if $i = \min\{j \geq 0 : 1 \leq d^*_{\mathcal{H}(j)}(U_{\ell'}) \leq 2$ *and* $d_{\mathcal{H}(j)}(U_{\ell'}) \geq d^*_{\mathcal{H}(j)}(U_{\ell'}) + 1\}$ *and* $W^4_{\mathcal{H}(0)}(U_{\ell'}) \cap \mathcal{R}(i) = \emptyset$ **then**
    $\mathcal{R}(i) \leftarrow \mathcal{R}(i) \cup \{U_{\ell'}\}$
  end
end

For $U \in \mathcal{R}(i^\star)$, let $i_U := \min\{i \geq 0 : U \in \mathcal{R}(i)\}$. To define events that entail almost-isolation becoming isolation, for $U \in \mathcal{R}(i^\star)$ choose possibly non-distinct copies $\mathcal{G}_U, \mathcal{G}'_U \in \mathcal{H}^*(i_U)$ of $\mathcal{F}$ whose vertex sets contain U as a subset as follows.

(i) If $d^*_{\mathcal{H}(i_U)}(U) = 1$ and $d_{\mathcal{H}(i_U)}(U)$ is even, choose $\mathcal{G}_U = \mathcal{G}'_U$ such that one edge of $\mathcal{G}_U$ is in $\mathcal{D}^*_{\mathcal{H}(i_U)}(U)$ while the other edge of $\mathcal{G}_U$ is not in $\mathcal{D}_{\mathcal{H}(i_U)}(U)$.

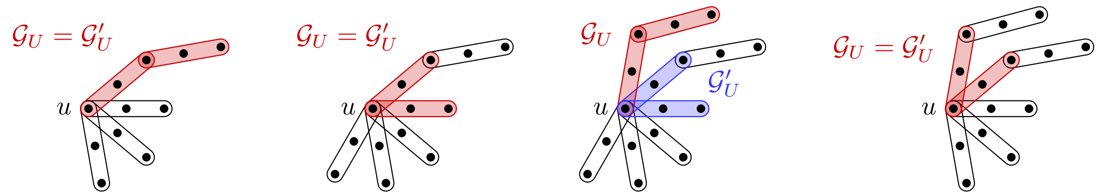


Figure 3.2: Examples for choices of the copies $\mathcal{G}_U$ and $\mathcal{G}'_U$ for the special case where $\mathcal{F}$ is a 3-uniform 1-cherry. Each example shows the situation of the edges containing $U = \{u\}$ as a subset at step i_U.

(ii) If $d^*_{\mathcal{H}(i_U)}(U) = 1$ and $d_{\mathcal{H}(i_U)}(U)$ is odd, choose $\mathcal{G}_U = \mathcal{G}'_U$ such that one edge of $\mathcal{G}_U$ is in $\mathcal{D}^*_{\mathcal{H}(i_U)}(U)$ while the other edge of $\mathcal{G}_U$ is in $\mathcal{D}_{\mathcal{H}(i_U)}(U)$.

(iii) If $d^*_{\mathcal{H}(j)}(U) = 2$ and $d_{\mathcal{H}(i_U)}(U)$ is even, choose $\mathcal{G}_U \neq \mathcal{G}'_U$ with $\mathcal{G}_U \cap \mathcal{G}'_U = \emptyset$ such that one edge of $\mathcal{G}_U$ is in $\mathcal{D}^*_{\mathcal{H}(i_U)}(U)$ while the other edge of $\mathcal{G}_U$ is not in $\mathcal{D}_{\mathcal{H}(i_U)}(U)$ and such that one edge of $\mathcal{G}'_U$ is in $\mathcal{D}^*_{\mathcal{H}(i_U)}(U)$ while the other edge of $\mathcal{G}'_U$ is in $\mathcal{D}_{\mathcal{H}(i_U)}(U)$

(iv) If $d^*_{\mathcal{H}(j)}(U) = 2$ and $d_{\mathcal{H}(i_U)}(U)$ is odd, choose $\mathcal{G}_U = \mathcal{G}'_U$ such that both edges of $\mathcal{G}_U$ are in $\mathcal{D}^*_{\mathcal{H}(i_U)}$.

Let

$$\mathcal{E}_U := \{\mathcal{G}_U \preceq \mathcal{G} \text{ for all } \mathcal{G} \in \mathcal{N}^1_{\mathcal{H}^*(0)}(\mathcal{G}_U) \text{ and } \mathcal{G}'_U \preceq \mathcal{G} \text{ for all } \mathcal{G} \in \mathcal{N}^1_{\mathcal{H}^*(0)}(\mathcal{G}'_U)\}.$$

3.14.4 Proof of Theorem 3.14.1

As in Section 3.12.3, since every almost-isolation that turns into isolation causes an edge of $\mathcal{H}(0)$ to become an isolated vertex of $\mathcal{H}^*$ at some step $i \geq 0$ and hence an edge that remains at the end of the removal process, we obtain the following statement.

Observation 3.14.6. $H(\tau_\emptyset) \geq \sum_{U \in \mathcal{R}(i^\star)} \mathbb{1}_{\mathcal{E}_U}$.

We again organize the formal presentation of the arguments outlined above into suitable lemmas. Some of these are similar to those in Section 3.12.3. Combining the lemmas with the above observation, we then obtain Theorem 3.14.1. We define the event $\mathcal{E}_0$ as in Section 3.12.3.

Lemma 3.14.7. *Let $\mathcal{X} := \{i^\star < \tau^\star\} \cap \mathcal{E}_0$. Then, $|\mathcal{R}(i^\star)| \geq_{\mathcal{X}} n^{k-1/\rho_{\mathcal{F}}-5\varepsilon^2}$.*

Proof. We argue similarly as in the proof of Lemma 3.12.3. Let $\mathcal{A}$ denote the k-graph with no isolated vertices and exactly one edge and fix a k'-set $I \subseteq V_{\mathcal{A}}$. Consider a k'-set $U \subseteq V_{\mathcal{H}}$ and $\psi\colon I \hookrightarrow U$. Combining the fact that $\mathcal{H}(0)$ is k'-populated and Lemma 3.10.1, we have

$$2 \leq d_{\mathcal{H}(0)}(U) \leq \Phi_{\mathcal{A},\psi} \leq n^{\varepsilon^3}. \tag{3.14.1}$$

Let $i := i^\star$ and consider the set

$$\mathcal{I}^* := \{\mathcal{F}' \in \mathcal{H}^* : \mathcal{N}^1_{\mathcal{H}^*}(\mathcal{F}') = \{\mathcal{F}'\}\}$$

of edge sets of copies of $\mathcal{F}$ in $\mathcal{H}$ that are isolated in the sense that they do not share an edge with another copy of $\mathcal{F}$. Let $\iota\colon \mathcal{I}^* \to 2^{V_{\mathcal{H}}}$ denote the function such that $\iota(\mathcal{F}')$ is the intersection of the two edges of $\mathcal{F}'$ for all $\mathcal{F}' \in \mathcal{I}^*$. As a consequence of the lower bound in (3.14.1), for all $U \in \mathcal{J} := \iota(\mathcal{I}^*) \cap \mathcal{U}$ almost isolation must have occurred at U due to Lemma 3.14.4 (see discussion in the paragraph before Lemma 3.14.4). Thus, either U itself is an element of $\mathcal{R}$ or there exists some $U' \in \mathcal{W}^4_{\mathcal{H}^*(0)}(U') \cap \mathcal{R}$ that prevented the inclusion of U in $\mathcal{R}$. Hence, we may choose a function $\pi\colon \mathcal{J} \to \mathcal{R}$ that for every $U \in \mathcal{J}$ chooses a witness $\pi(U)$ with $\pi(U) \in \mathcal{W}^4_{\mathcal{H}^*(0)}(U)$ or equivalently $U \in \mathcal{W}^4_{\mathcal{H}^*(0)}(\pi(U))$. If $U \in \mathcal{R}$ and $U' \in \pi^{-1}(U)$, we have $U' \in \mathcal{W}^4_{\mathcal{H}^*(0)}(U)$ and hence $\pi^{-1}(U) \subseteq \mathcal{W}^4_{\mathcal{H}^*(0)}(U)$. The upper bound in (3.14.1) entails $W^1_{\mathcal{H}^*(0)}(U) \le n^{\varepsilon^3} \cdot k^{k'}$ and for all $\ell \ge 1$ furthermore $W^{\ell+1}_{\mathcal{H}^*(0)}(U) \le W^{\ell}_{\mathcal{H}^*(0)}(U) \cdot n^{\varepsilon^3} \cdot k^{k'}$. Hence, we inductively obtain $W^{\ell}_{\mathcal{H}^*(0)}(U) \le k^{\ell k'} n^{\ell\varepsilon^3}$ and in particular $W^4_{\mathcal{H}^*(0)}(U) \le n^{\varepsilon^2}$. Thus,

$$|\mathcal{J}| \le \sum_{U \in \mathcal{R}} |\pi^{-1}(U)| \le |\mathcal{R}| n^{\varepsilon^2}.$$

As a consequence of Lemma 3.14.3, the number of k'-sets $U \subseteq V_{\mathcal{H}}$ that are not suitable is at most $3 \cdot n^{k-1/\rho_{\mathcal{F}}-\varepsilon^{1/7}} \cdot (4k)^{k'} \le n^{k-1/\rho_{\mathcal{F}}-\varepsilon^{1/6}}$. Hence, $|\mathcal{J}| \ge |\iota(\mathcal{I}^*)| - n^{k-1/\rho_{\mathcal{F}}-\varepsilon^{1/6}}$ and thus

$$|\mathcal{R}| \ge n^{-\varepsilon^2}(|\iota(\mathcal{I}^*)| - n^{k-1/\rho_{\mathcal{F}}-\varepsilon^{1/6}}).$$

Furthermore, if $U \subseteq V_{\mathcal{H}}$ is a k'-set and $e \in \mathcal{F}'$ for some $\mathcal{F}' \in \iota^{-1}(U)$, then for all $\mathcal{F}'' \in \iota^{-1}(U) \setminus \{\mathcal{F}'\}$ and $f \in \mathcal{F}''$, we have $|e \cap f| \ge k' + 1$ and for all distinct $\mathcal{F}'', \mathcal{F}''' \in \iota^{-1}(U) \setminus \{\mathcal{F}'\}$, $f \in \mathcal{F}''$ and $g \in \mathcal{F}'''$, we have $f \ne g$. Thus, there exists $k'+1 \le k'' \le k$ and at least $(|\iota^{-1}(U)| - 1)/k$ distinct edges $f_1, \ldots, f_\ell \in \mathcal{H}$ with $|e \cap f_{\ell'}| = k''$ for all $1 \le \ell' \le \ell$. Now, let $\mathcal{A}$ denote a k''-cherry, let $I \in \mathcal{A}$ and fix $\psi\colon I \hookrightarrow e$. By Lemma 3.10.1, we have

$$\frac{|\iota^{-1}(U)| - 1}{k} \le \ell \le \Phi_{\mathcal{A},\psi} \le n^{\varepsilon^3}$$

and thus

$$|\mathcal{I}^*| \le \sum_{U \in \iota(\mathcal{I}^*)} |\iota^{-1}(U)| \le |\iota(\mathcal{I}^*)| n^{\varepsilon^2}.$$

Overall, this yields

$$|\mathcal{R}| \ge n^{-\varepsilon^2}(n^{-\varepsilon^2}|\mathcal{I}^*| - n^{k-1/\rho_{\mathcal{F}}-\varepsilon^{1/6}}), \tag{3.14.2}$$

so it suffices to find an appropriate lower bound for $\mathcal{I}^*$. Similarly as in the proof of Lemma 3.12.3, we may again rely on Lemma 3.11.8 to obtain $H^* \le_{\mathcal{X}} (1+\varepsilon)H/|\mathcal{F}|$ precisely as in (3.12.2) and then

$$H \le_{\mathcal{X}} H - \frac{1}{4|\mathcal{F}|} H + \frac{1}{2}|\mathcal{I}^*|$$

precisely as in (3.12.3). With Lemma 3.11.8, precisely as in (3.12.4), this implies $|\mathcal{I}^*| \ge_{\mathcal{X}} n^{k-1/\rho_{\mathcal{F}}-2\varepsilon^2}$. Combining this with (3.14.2) yields $|\mathcal{R}| \ge_{\mathcal{X}} n^{k-1/\rho_{\mathcal{F}}-5\varepsilon^2}$. □

Lemma 3.14.8. *Suppose that X is a binomial random variable with parameters $n^{k-1/\rho_{\mathcal{F}}-5\varepsilon^2}$ and $n^{-2\varepsilon^2}$ and let $Y := (n^{k-1/\rho_{\mathcal{F}}-5\varepsilon^2} - |\mathcal{R}(i^\star)|) \vee 0$. Let*

$$Z := Y + \sum_{U \in \mathcal{R}(i^\star)} \mathbb{1}_{\mathcal{E}_U}.$$

Then, Z stochastically dominates X.

Proof. We argue similarly as in the proof of Lemma 3.12.4. First, observe that by Lemma 3.11.2, whenever $U \in \mathcal{R}(i^\star)$, for $i := 0$, we have

$$N^1_{\mathcal{H}^*}(\mathcal{G}_U) \leq \sum_{f \in \mathcal{G}_U} d_{\mathcal{H}^*}(f) \leq n^{\varepsilon^2}$$

and thus

$$|\mathcal{N}^1_{\mathcal{H}^*}(\mathcal{G}_U) \cup \mathcal{N}^1_{\mathcal{H}^*}(\mathcal{G}'_U)| \leq n^{2\varepsilon^2}. \tag{3.14.3}$$

Consider distinct k'-sets $U, U' \subseteq V_{\mathcal{H}}$. By construction of $\mathcal{R}(i^\star)$, whenever $U, U' \in \mathcal{R}(i^\star)$, then

$$(\mathcal{N}^1_{\mathcal{H}(0)}(\mathcal{G}_U) \cup \mathcal{N}^1_{\mathcal{H}(0)}(\mathcal{G}'_U)) \cap (\mathcal{N}^1_{\mathcal{H}(0)}(\mathcal{G}_{U'}) \cup \mathcal{N}^1_{\mathcal{H}(0)}(\mathcal{G}'_{U'})) = \emptyset.$$

Thus, for all distinct $U_1, \ldots, U_\ell \in \mathcal{R}(i^\star)$ and all $z_1, \ldots, z_{\ell-1} \in \{0,1\}$, from (3.14.3), we obtain

$$\mathbb{P}[\mathbb{1}_{\mathcal{E}_{U_\ell}} = 1 \mid \mathbb{1}_{\mathcal{E}_{U_{\ell'}}} = z_{\ell'} \text{ for all } 1 \leq \ell' \leq \ell] = \mathbb{P}[\mathcal{E}_{U_\ell}] \geq n^{-2\varepsilon^2},$$

which completes the proof. $\square$

Proof of Theorem 3.14.1. The proof is almost exactly the same as for Theorem 3.11.1 with the key difference that we replace objects and references with the appropriate analogous constructions and arguments form this section. Define the events

$$\mathcal{B} := \{H(\tau_\emptyset) \leq n^{k-1/\rho_{\mathcal{F}}-\varepsilon}\} \quad \text{and} \quad \mathcal{X} := \{i^\star < \tau^\star\} \cap \mathcal{E}_0.$$

We need to show that $\mathbb{P}[\mathcal{B}]$ is sufficiently small. Choose X, Y and Z as in Lemma 3.14.8. Lemma 3.14.7 entails $\mathcal{X} \subseteq \{Y = 0\}$ and hence $\{Y \neq 0\} \subseteq \mathcal{X}^{\mathsf{c}}$. Thus, from Observation 3.14.6 and Lemma 3.14.8, we obtain

$$\begin{aligned}\mathcal{B} &= \Big\{\sum_{U \in \mathcal{R}(i^\star)} \mathbb{1}_{\mathcal{E}_U} \leq n^{k-1/\rho_{\mathcal{F}}-\varepsilon}\Big\} \cap \mathcal{B} \subseteq (\{Z \leq n^{k-1/\rho_{\mathcal{F}}-\varepsilon}\} \cup \{Y \neq 0\}) \cap \mathcal{B} \\ &\subseteq \{Z \leq n^{k-1/\rho_{\mathcal{F}}-\varepsilon}\} \cup (\mathcal{X}^{\mathsf{c}} \cap \mathcal{B}) \subseteq \{Z \leq n^{k-1/\rho_{\mathcal{F}}-\varepsilon}\} \cup \{\tau^\star \leq i^\star\} \cup (\mathcal{E}_0^{\mathsf{c}} \cap \mathcal{B}).\end{aligned}$$

By Lemma 3.11.8, we have

$$H(\tau_\emptyset) \geq_{\mathcal{E}_0^{\mathsf{c}}} \varepsilon H(i^\star) \geq \varepsilon^2 n^k \hat{p}(i^\star) \geq n^{k-1/\rho_{\mathcal{F}}-2\varepsilon^2}$$

and hence $\mathcal{E}_0^{\mathsf{c}} \cap \mathcal{B} = \emptyset$. Thus, using Lemma 3.11.16, we obtain

$$\mathbb{P}[\mathcal{B}] \leq \mathbb{P}[Z \leq n^{k-1/\rho_{\mathcal{F}}-\varepsilon}] + \exp(-n^{1/3}).$$

With Lemma 3.14.8 and Chernoff's inequality (see Lemma 3.12.5), this completes the proof. $\square$

3.14.5 Proofs for Theorems 1.1.8 and 1.1.9

In this section, we show how to obtain Theorems 1.1.8 and 1.1.9 from Theorems 3.3.2 and 3.14.1.

Proof of Theorem 1.1.9. By definition of $\tau_\emptyset$ in Section 3.11, this is an immediate consequence of Theorem 3.14.1. □

Proof of Theorem 1.1.8. The argumentation is essentially the same as in the proof of Theorem 1.1.5 except that we use Theorem 3.14.1 instead of Theorem 3.11.1.

We define the constants m, ε, δ, n and the k-graphs $\mathcal{H}$ $\mathcal{H}'$ and $\mathcal{H}''$ precisely as in the proof of Theorem 1.1.5. Let $\mathcal{X}'$ denote the event that $\mathcal{H}'$ is $(4m, n^{\varepsilon^4})$-bounded, k'-populated and has $n^{k-1/\rho+\varepsilon^5}/k!$ edges and let $\mathcal{X}''$ denote the event that

$$\mathcal{X}'' := \{|\mathcal{H}''| \leq n^{k-1/\rho+\varepsilon}\} \quad \text{and} \quad \mathcal{Y}'' := \{n^{k-1/\rho-\varepsilon} \leq |\mathcal{H}''|\}.$$

We need to show that

$$\mathbb{P}[\mathcal{X}'' \cap \mathcal{Y}''] \geq 1 - \exp(-(\log n)^{5/4}).$$

Since $\mathcal{X}' \subseteq \mathcal{X}''$, we have $\mathbb{P}[\mathcal{X}'' \cap \mathcal{Y}''] \geq \mathbb{P}[\mathcal{X}' \cap \mathcal{Y}'']$, so it suffices to obtain sufficiently large lower bounds for $\mathbb{P}[\mathcal{X}']$ and $\mathbb{P}[\mathcal{Y}'']$. Due to $k' = k - 1/\rho$, we may apply Theorem 3.3.2 with ε^5 playing the role of ε to obtain $\mathbb{P}[\mathcal{X}'] \geq 1 - \exp(-(\log n)^{4/3})$ and Theorem 3.14.1 shows that $\mathbb{P}[\mathcal{Y}'' \mid \mathcal{X}'] \geq 1 - \exp(-n^{1/4})$. Using $\mathbb{P}[\mathcal{Y}''] = \mathbb{P}[\mathcal{Y}'' \mid \mathcal{X}']\mathbb{P}[\mathcal{X}']$, this yields suitable lower bounds for $\mathbb{P}[\mathcal{X}']$ and $\mathbb{P}[\mathcal{Y}'']$. □

3.15 Counting copies of $\mathcal{F}$

In this section, our goal is to prove Lemma 3.5.19 (i). Hence, for this section, we assume the setup that we used in Section 3.5 to state Lemma 3.5.19. Our approach is similar as in Sections 3.6.3 and 3.7.2.

For $i \geq 0$, let

$$\eta_1(i) := \zeta^{1+\varepsilon^3}\hat{h}^* \quad \text{and} \quad \eta_0(i) := (1-\varepsilon)\eta_1(i).$$

Define the critical intervals

$$I^-(i) := [\hat{h}^* - \eta_1, \hat{h}^* - \eta_0] \quad \text{and} \quad I^+(i) := [\hat{h}^* + \eta_0, \hat{h}^* + \eta_1].$$

For $\ast \in \{-,+\}$, let

$$Y^\ast(i) := \ast(H^* - \hat{h}^*) - \eta_1$$

For $i_0 \geq 0$, define the stopping time

$$\tau^\ast_{i_0} := \min\{i \geq i_0 : H^* \notin I^\ast\}$$

and for $i \geq i_0$, let

$$Z^\ast_{i_0}(i) := Y^\ast(i_0 \vee (i \wedge \tau^\ast_{i_0} \wedge \tilde{\tau}^\star \wedge i^\star)).$$

Let

$$\sigma^{\divideontimes} := \min\{j \geq 0 : \divideontimes(H^* - \hat{h}^*) \geq \eta_0 \text{ for all } j \leq i < \tilde{\tau}^\star \wedge i^\star\} \leq \tilde{\tau}^\star \wedge i^\star$$

With this setup, similarly as in Sections 3.6.3 and 3.7.2, it in fact suffices to consider the evolution of $Z^{\divideontimes}_{\sigma^{\divideontimes}}(\sigma^{\divideontimes}), Z^{\divideontimes}_{\sigma^{\divideontimes}}(\sigma^{\divideontimes}+1), \ldots$.

Observation 3.15.1. $\{\tau_{\mathcal{H}^*} \leq \tilde{\tau}^\star \wedge i^\star\} \subseteq \{Z^-_{\sigma^-}(i^\star) > 0\} \cup \{Z^+_{\sigma^+}(i^\star) > 0\}$.

We again use supermartingale concentration techniques to show that the probabilities of the events on the right in Observation 3.15.1 are sufficiently small. However, instead of relying on Freedman's inequality, here, similarly as in Section 3.11, we instead use Azuma's inequality.

3.15.1 Trend

Here, we prove that for all $\divideontimes \in \{-,+\}$ and $i_0 \geq 0$, the expected one-step changes of the process $Z^{\divideontimes}_{i_0}(i_0), Z^{\divideontimes}_{i_0}(i_0+1), \ldots$ are non-positive. We begin with estimating the one-step changes of the deterministic parts of this random process in Lemma 3.15.3. Using Lemma 3.5.20, we obtain Lemma 3.15.4 where we provide a precise estimate for the expected one-step change of the non-deterministic part that holds whenever the removal process was well-behaved up to the step we consider. Finally, we combine our estimates for the deterministic and non-deterministic parts to see that the above process is indeed a supermartingale (see Lemma 3.15.5).

Observation 3.15.2. *Extend $\hat{p}$, $\hat{h}^*$ and η_1 to continuous trajectories defined on the whole interval $[0, i^\star+1]$ using the same expressions as above. Then, for $x \in [0, i^\star+1]$,*

$$(\hat{h}^*)'(x) = -\frac{|\mathcal{F}|^2 k!\, \hat{h}^*(x)}{n^k \hat{p}(x)}, \quad (\hat{h}^*)''(x) = \frac{|\mathcal{F}|^3(|\mathcal{F}|-1)(k!)^2 \hat{h}^*(x)}{n^{2k}\hat{p}(x)^2},$$

$$\eta_1'(x) = -\frac{\big(|\mathcal{F}| - \frac{(1+\varepsilon^3)\rho_{\mathcal{F}}}{2}\big)|\mathcal{F}|k!\,\eta_1(x)}{n^k\hat{p}(x)},$$

$$\eta_1''(x) = -\frac{\big(|\mathcal{F}| - \frac{(1+\varepsilon^3)\rho_{\mathcal{F}}}{2}\big)\big(|\mathcal{F}| - \frac{(1+\varepsilon^3)\rho_{\mathcal{F}}}{2} - 1\big)|\mathcal{F}|^2(k!)^2\eta_1(x)}{n^{2k}\hat{p}(x)^2}.$$

Lemma 3.15.3. *Let $0 \leq i \leq i^\star$ and $\mathcal{X} := \{i \leq \tau_\emptyset\}$. Then,*

$$\Delta\hat{h}^* =_{\mathcal{X}} -\frac{|\mathcal{F}|^2\hat{h}^*}{H} \pm \frac{\zeta^{2+\varepsilon^2}\hat{h}^*}{H}, \quad \Delta\eta_1 =_{\mathcal{X}} -\left(|\mathcal{F}| - \frac{(1+\varepsilon^3)\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|\eta_1}{H} \pm \frac{\zeta^{2+\varepsilon^2}\eta_1}{H}.$$

Proof. This is a consequence of Taylor's theorem. In detail, we argue as follows.

Together with Observation 3.15.2, Lemma 2.9.10 yields

$$\Delta\hat{h}^* = -\frac{|\mathcal{F}|^2 k!\,\hat{h}^*}{n^k\hat{p}} \pm \max_{x\in[i,i+1]} \frac{\hat{h}^*(x)}{\delta n^{2k}\hat{p}(x)^2}.$$

We investigate the first term and the maximum separately. Using Lemma 3.5.7, we have

$$-\frac{|\mathcal{F}|^2 k!\,\hat{h}^*}{n^k\hat{p}} =_{\mathcal{X}} -\frac{|\mathcal{F}|^2\hat{h}^*}{H}.$$

Furthermore, since $\hat{h}^*(x)/\hat{p}(x)^2$ is non-decreasing in x for $x \in [i, i+1]$, Lemma 3.5.7 together with Lemma 3.5.9 yields

$$\max_{x\in[i,i+1]} \frac{\hat{h}^*(x)}{\delta n^{2k}\hat{p}(x)^2} \le \frac{\hat{h}^*}{\delta n^{2k}\hat{p}^2} \le_{\mathcal{X}} \frac{\hat{h}^*}{\delta H^2} \le \frac{\zeta^{2+2\varepsilon^2}\hat{h}^*}{\delta H} \le \frac{\zeta^{2+\varepsilon^2}\hat{h}^*}{H}.$$

Thus we obtain the desired expression for $\Delta\hat{h}^*$.

We argue similarly for $\Delta\eta_1$. Again together with Observation 3.15.2, Lemma 2.9.10 yields

$$\Delta\eta_1 = -\left(|\mathcal{F}| - \frac{(1+\varepsilon^3)\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|k!\,\eta_1}{n^k\hat{p}} \pm \max_{x\in[i,i+1]}\frac{\eta_1(x)}{\delta n^{2k}\hat{p}(x)^2}.$$

We again investigate the first term and the maximum separately. Using Lemma 3.5.7, we have

$$-\left(|\mathcal{F}| - \frac{(1+\varepsilon^3)\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|k!\,\eta_1}{n^k\hat{p}} =_{\mathcal{X}} -\left(|\mathcal{F}| - \frac{(1+\varepsilon^3)\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|\eta_1}{H}.$$

Furthermore, using Lemma 3.5.6, Lemma 3.5.7 and Lemma 3.5.9 yields

$$\max_{x\in[i,i+1]}\frac{\eta_1(x)}{\delta n^{2k}\hat{p}(x)^2} \le \frac{\eta_1}{\delta n^{2k}\hat{p}(i+1)^2} \le \frac{\eta_1}{\delta^2 n^{2k}\hat{p}^2} \le_{\mathcal{X}} \frac{\eta_1}{\delta^2 H^2} \le \frac{\zeta^{2+2\varepsilon^3}\eta_1}{\delta^2 H} \le \frac{\zeta^{2+\varepsilon^3}\eta_1}{H}.$$

Thus we also obtain the desired expression for $\Delta\eta_1$. □

Lemma 3.15.4. *Let $0 \le i \le i^\star$ and $\mathcal{X} := \{i < \tilde{\tau}^\star\}$. Then,*

$$\mathbb{E}_i[\Delta H^*] =_{\mathcal{X}} -\frac{|\mathcal{F}|^2 H^*}{H} \pm \frac{\zeta^2\hat{h}^*}{\delta^5 H}.$$

Proof. Fix $f \in \mathcal{F}$. Lemma 3.5.18 entails

$$\begin{aligned}\mathbb{E}_i[\Delta H^*] &=_{\mathcal{X}} -\frac{1}{H^*}\sum_{\mathcal{F}'\in\mathcal{H}^*}\left(\left(\sum_{e\in\mathcal{F}'} d_{\mathcal{H}^*}(e)\right) \pm |\mathcal{F}|^2\zeta^{2+\varepsilon^2}\hat{\varphi}_{\mathcal{F},f}\right)\\ &= -\frac{1}{H^*}\left(\sum_{e\in\mathcal{H}} d_{\mathcal{H}^*}(e)^2\right) \pm |\mathcal{F}|^2\zeta^{2+\varepsilon^2}\hat{\varphi}_{\mathcal{F},f}.\end{aligned}$$

For all $e \in \mathcal{H}$, from Lemma 3.5.17, we obtain

$$d_{\mathcal{H}^*}(e) =_{\mathcal{X}} \frac{|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{F},f}}{\mathrm{aut}(\mathcal{F})} \pm \frac{1}{\delta}|\mathcal{F}|k!\,\zeta\hat{\varphi}_{\mathcal{F},f}.$$

Thus, Lemma 3.5.20 yields

$$\mathbb{E}_i[\Delta H^*] =_{\mathcal{X}} -\frac{1}{H^*}\frac{(\sum_{e\in\mathcal{H}} d_{\mathcal{H}^*}(e))^2}{H} \pm \frac{2|\mathcal{F}|^2(k!)^2\zeta^2\hat{\varphi}^2_{\mathcal{F},f}H}{\delta^2 H^*} \pm |\mathcal{F}|^2\zeta^{2+\varepsilon^2}\hat{\varphi}_{\mathcal{F},f}$$

$$= -\frac{|\mathcal{F}|^2 H^*}{H} \pm \frac{\hat{\varphi}_{\mathcal{F},f} H}{H^*} \frac{\zeta^2 \hat{\varphi}_{\mathcal{F},f} H}{\delta^3 H} \pm \frac{\zeta^{2+\varepsilon^2} \hat{\varphi}_{\mathcal{F},f} H}{\delta H}.$$

Since Lemma 3.5.7 implies $\hat{\varphi}_{\mathcal{F},f} H \leq \hat{h}^*/\varepsilon$, we obtain

$$\mathbb{E}_i[\Delta H^*] =_{\mathcal{X}} -\frac{|\mathcal{F}|^2 H^*}{H} \pm \frac{\zeta^2 \hat{h}^*}{\delta^5 H},$$

which completes the proof. □

Lemma 3.15.5. *Let $0 \leq i_0 \leq i$ and $\divideontimes \in \{-,+\}$. Then, $\mathbb{E}_i[\Delta Z_{i_0}^{\divideontimes}] \leq 0$.*

Proof. Suppose that $i < i^\star$ and let $\mathcal{X} := \{i < \tau_{i_0}^{\divideontimes} \wedge \tilde{\tau}^\star\}$. We have $\mathbb{E}_i[\Delta Z_{i_0}^{\divideontimes}] =_{\mathcal{X}^c} 0$ and $\mathbb{E}_i[\Delta Z_{i_0}^{\divideontimes}] =_{\mathcal{X}} \mathbb{E}_i[\Delta Y^{\divideontimes}]$, so it suffices to obtain $\mathbb{E}_i[\Delta Y^{\divideontimes}] \leq_{\mathcal{X}} 0$. Combining Lemma 3.15.3 with Lemma 3.15.4, we obtain

$$\begin{aligned}
\mathbb{E}_i[\Delta Y^{\divideontimes}] &= \divideontimes(\mathbb{E}_i[\Delta H^*] - \Delta\hat{h}^*) - \Delta\eta_1 \\
&\leq_{\mathcal{X}} \divideontimes\left(-\frac{|\mathcal{F}|^2}{H} H^* + \frac{|\mathcal{F}|^2}{H}\hat{h}^*\right) + \left(|\mathcal{F}| - \frac{(1+\varepsilon^3)\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|}{H}\eta_1 \\
&\quad + \frac{\zeta^2}{\delta^5 H}\hat{h}^* + \frac{2\zeta^{2+\varepsilon^2}}{H}\hat{h}^* \\
&\leq -\frac{|\mathcal{F}|}{H}\left(\divideontimes|\mathcal{F}|(H^* - \hat{h}^*) - \left(|\mathcal{F}| - \frac{\rho_{\mathcal{F}}}{2}\right)\eta_1 - \varepsilon^2\eta_1\right) \\
&\leq_{\mathcal{X}} -\frac{|\mathcal{F}|}{H}\left(|\mathcal{F}|(1-\varepsilon)\eta_1 - \left(|\mathcal{F}| - \frac{\rho_{\mathcal{F}}}{2}\right)\eta_1 - \varepsilon^2\eta_1\right) \\
&= -\frac{|\mathcal{F}|\eta_1}{H}\left(\frac{\rho_{\mathcal{F}}}{2} - \varepsilon|\mathcal{F}| - \varepsilon^2\right) \leq 0,
\end{aligned}$$

which completes the proof. □

3.15.2 Boundedness

As we intend to apply Azuma's inequality, it suffices to obtain suitable bounds for the absolute one-step changes of the processes $Y^{\divideontimes}(0), Y^{\divideontimes}(1), \ldots$ and $Z_{i_0}^{\divideontimes}(i_0), Z_{i_0}^{\divideontimes}(i_0+1), \ldots$.

Lemma 3.15.6. *Let $0 \leq i_0 \leq i \leq i^\star$, $\divideontimes \in \{-,+\}$, $f \in \mathcal{F}$ and $\mathcal{X} := \{i < \tau_{\mathscr{F}}\}$. Then, $|\Delta Y^{\divideontimes}| \leq_{\mathcal{X}} \hat{\varphi}_{\mathcal{F},f}(i_0)/\delta$.*

Proof. From Lemma 3.5.17, we obtain

$$|\Delta H^*| \leq \sum_{e \in \mathcal{F}_0(i+1)} d_{\mathcal{H}^*}(e) \leq \sum_{e \in \mathcal{F}_0(i+1)} \sum_{f' \in \mathcal{F}} \sum_{\psi\colon f' \xrightarrow{\sim} e} \Phi_{\mathcal{F},\psi} \leq_{\mathcal{X}} 2|\mathcal{F}|^2 k!\, \hat{\varphi}_{\mathcal{F},f}.$$

Hence, using Lemma 3.15.3, we have

$$|\Delta Y^{\divideontimes}| \leq |\Delta H^*| + |\Delta\hat{h}^*| + |\Delta\eta_1| \leq_{\mathcal{X}} 2|\mathcal{F}|^2 k!\, \hat{\varphi}_{\mathcal{F},f} + \frac{2|\mathcal{F}|^2\hat{h}^*}{H} + \frac{2|\mathcal{F}|^2\eta_1}{H}.$$

With Lemma 3.5.7 and $\hat{\varphi}_{\mathcal{F},f} \leq \hat{\varphi}_{\mathcal{F},f}(i_0)$, this completes the proof. □

Lemma 3.15.7. *Let* $0 \leq i_0 \leq i$, $\ast \in \{-,+\}$ *and* $f \in \mathcal{F}$. *Then,* $|\Delta Z_{i_0}^{\ast}| \leq \hat{\varphi}_{\mathcal{F},f}(i_0)/\delta$.

Proof. This is an immediate consequence of Lemma 3.15.6. □

3.15.3 Supermartingale concentration

This section follows a similar structure as Sections 3.6.3 and 3.7.2. Lemma 3.15.8 is the final ingredient that we use for our application of Azuma's inequality in the proof of Lemma 3.15.9 where we show that the probabilities of the events on the right in Observation 3.15.1 are indeed small.

Lemma 3.15.8. *Let* $\ast \in \{-,+\}$. *Then,* $Z_{\sigma^{\ast}}^{\ast}(\sigma^{\ast}) \leq -\varepsilon^2\eta_1(\sigma^{\ast})$.

Proof. Lemma 3.5.4 implies $\tilde{\tau}^{\star} \geq 1$ and $\ast(H^*(0) - \hat{h}^*(0)) < \eta_0(0)$, so we have $\sigma^{\ast} \geq 1$. Thus, by definition of $\sigma^{\ast}$, for $i := \sigma^{\ast} - 1$, we have $\ast(H^* - \hat{h}^*) \leq \eta_0$ and thus

$$Z_i^{\ast} = \ast(H^* - \hat{h}^*) - \eta_1 \leq -\varepsilon\eta_1.$$

Furthermore, since $\sigma^{\ast} \leq \tau_{\mathscr{F}}$, we may apply Lemma 3.15.6 such that with Lemma 3.5.7 and Lemma 3.5.9, for $f \in \mathcal{F}$, we obtain

$$Z_{\sigma^{\ast}}^{\ast}(\sigma^{\ast}) = Z_i^{\ast} + \Delta Y^{\ast} \leq -\varepsilon\eta_1 + \frac{\hat{\varphi}_{\mathcal{F},f}}{\delta} \leq -\varepsilon\eta_1 + \frac{\hat{h}^*}{\delta^2 H} \leq -\varepsilon\eta_1 + \zeta^{2+\varepsilon^2}\hat{h}^* \leq -\varepsilon^2\eta_1.$$

Since $\Delta\eta_1 \leq 0$, this completes the proof. □

Lemma 3.15.9. $\mathbb{P}[\tau_{\mathcal{H}^*} \leq \tilde{\tau}^{\star} \wedge i^{\star}] \leq \exp(-n^{\varepsilon^2})$.

Proof. Fix $\ast \in \{-,+\}$. By Observation 3.15.1, is suffices to show that

$$\mathbb{P}[Z_{\sigma^{\ast}}^{\ast}(i^{\star}) > 0] \leq \exp(-n^{2\varepsilon^2}).$$

Due to Lemma 3.15.8, we have

$$\mathbb{P}[Z_{\sigma^{\ast}}^{\ast}(i^{\star}) > 0] \leq \mathbb{P}[Z_{\sigma^{\ast}}^{\ast}(i^{\star}) - Z_{\sigma^{\ast}}^{\ast}(\sigma^{\ast}) \geq \varepsilon^2\eta_1(\sigma^{\ast})] \leq \sum_{0 \leq i \leq i^{\star}} \mathbb{P}[Z_i^{\ast}(i^{\star}) - Z_i^{\ast} \geq \varepsilon^2\eta_1].$$

Thus, for $0 \leq i \leq i^{\star}$, it suffices to obtain

$$\mathbb{P}[Z_i^{\ast}(i^{\star}) - Z_i^{\ast} \geq \varepsilon^2\eta_1] \leq \exp(-n^{3\varepsilon^2}).$$

We show that this bound is a consequence of Azuma's inequality.

Fix $f \in \mathcal{F}$. Lemma 3.15.4 shows that $Z_i^{\ast}(i), Z_i^{\ast}(i+1), \ldots$ is a supermartingale, while Lemma 3.15.7 provides the bound $|\Delta Z_i^{\ast}(j)| \leq \hat{\varphi}_{\mathcal{F},f}/\delta$ for all $j \geq i$. Hence, we may apply Lemma 3.11.5 to obtain

$$\mathbb{P}[Z_i^{\ast}(i^{\star}) - Z_i^{\ast} \geq \varepsilon^2\eta_1] \leq \exp\left(-\frac{\varepsilon^4\delta^2\eta_1^2}{2(i^{\star} - i)\hat{\varphi}_{\mathcal{F},f}^2}\right).$$

Since

$$i^\star - i \leq \frac{\vartheta n^k}{|\mathcal{F}|k!} - i = \frac{n^k \hat{p}}{|\mathcal{F}|k!},$$

this yields

$$\begin{aligned}\mathbb{P}[Z_i^{\times}(i^\star) - Z_i^{\times} \geq \varepsilon^2\eta_1] &\leq \exp\Big(-\frac{\varepsilon^5\delta^2\eta_1^2}{n^k\hat{p}\hat{\varphi}_{\mathcal{F},f}^2}\Big) = \exp\Big(-\frac{\varepsilon^5\delta^2\zeta^{2+2\varepsilon^3}(\hat{h}^*)^2}{n^k\hat{p}\hat{\varphi}_{\mathcal{F},f}^2}\Big)\\ &\leq \exp(-\delta^3\zeta^{2+2\varepsilon^3}n^k\hat{p}) \leq \exp(-\delta^3\zeta^{2+2\varepsilon^3}(n\hat{p}^{\rho_{\mathcal{F}}})^k)\\ &= \exp(-\delta^3 n^{2k\varepsilon^2}\zeta^{2+2\varepsilon^3-2k}) \leq \exp(-n^{4\varepsilon^2}),\end{aligned}$$

which completes the proof. □

3.16 Counting balanced templates

In this section, our goal is to prove Lemma 3.5.19 (ii). Hence, for this section, we assume the setup that we used in Section 3.5 to state Lemma 3.5.19. Similarly as in Sections 3.6.3 and 3.7.2, this requires us to consider several balanced templates, however, it again suffices to essentially only consider a fixed balanced template $(\mathcal{A}, I)$, see Observation 3.16.1 below. Moreover, we may assume that $\mathcal{A} \setminus \mathcal{A}[I] \neq \emptyset$ as otherwise, for all $\psi\colon I \hookrightarrow V_{\mathcal{H}}$ and $0 \leq i \leq i^\star$, we have $\Phi_{\mathcal{A},\psi} = (1 \pm \zeta^\delta)\hat{\varphi}_{\mathcal{A},I}$ as a consequence of Lemma 3.5.8. Overall, our approach is similar as in Sections 3.6.3 and 3.7.2.

Observation 3.16.1. *For $(\mathcal{A}, I) \in \mathscr{B}$ and $\psi\colon I \hookrightarrow V_{\mathcal{H}}$, let*

$$\tau_{\mathcal{A},\psi} := \min\{i \geq 0 : \Phi_{\mathcal{A},\psi} \neq (1 \pm \zeta^\delta)\hat{\varphi}_{\mathcal{A},I}\}.$$

Then,

$$\mathbb{P}[\tau_{\mathscr{B}} \leq \tilde{\tau}^\star \wedge i^\star] \leq \sum_{\substack{(\mathcal{A},I)\in\mathscr{B}\colon \mathcal{A}\setminus\mathcal{A}[I]\neq\emptyset,\\ \psi\colon I\hookrightarrow V_{\mathcal{H}}}} \mathbb{P}[\tau_{\mathcal{A},\psi} \leq \tilde{\tau}^\star \wedge i_{\mathcal{A},I}^{\delta^{1/2}} \wedge i^\star].$$

Fix $(\mathcal{A}, I) \in \mathscr{B}$ with $\mathcal{A} \setminus \mathcal{A}[I] \neq \emptyset$ and $\psi\colon I \hookrightarrow V_{\mathcal{H}}$ and for $i \geq 0$, let

$$\xi_1(i) := \zeta^\delta\hat{\varphi}_{\mathcal{A},I} \quad \text{and} \quad \xi_0(i) := (1-\delta^2)\xi_1$$

and define the stopping time

$$\tau := \min\{i \geq 0 : \Phi_{\mathcal{A},\psi} \neq \hat{\varphi}_{\mathcal{A},I} \pm \xi_1\}.$$

We only expect tight concentration of $\Phi_{\mathcal{A},\psi}$ around $\hat{\varphi}_{\mathcal{A},I}$ as long as we expect $\Phi_{\mathcal{A},\psi}$ to be sufficiently large, that is up to step $i_{\mathcal{A},I}^{\delta^{1/2}}$. Formally, in this section it is our goal to obtain an upper bound for the probability that $\tau \leq \tilde{\tau}^\star \wedge i_{\mathcal{A},I}^{\delta^{1/2}} \wedge i^\star$ and hence the minimum $i_{\mathcal{A},I}^{\delta^{1/2}} \wedge i^\star$ often plays the role that $i^\star$ plays in Sections 3.6.3 and 3.7.2.

Define the critical intervals

$$I^-(i) := [\hat{\varphi}_{\mathcal{A},I} - \xi_1, \hat{\varphi}_{\mathcal{A},I} - \xi_0] \quad \text{and} \quad I^+(i) := [\hat{\varphi}_{\mathcal{A},I} + \xi_0, \hat{\varphi}_{\mathcal{A},I} + \xi_1].$$

For $\divideontimes \in \{-,+\}$, let

$$Y^{\divideontimes}(i) := \divideontimes(\Phi_{\mathcal{A},\psi} - \hat{\varphi}_{\mathcal{A},I}) - \xi_1.$$

For $i_0 \geq 0$ define the stopping time

$$\tau_{i_0}^{\divideontimes} := \min\{i \geq i_0 : \Phi_{\mathcal{A},\psi} \notin I^{\divideontimes}\}$$

and for $i \geq i_0$, let

$$Z_{i_0}^{\divideontimes}(i) := Y^{\divideontimes}(i_0 \vee (i \wedge \tau_{i_0}^{\divideontimes} \wedge \tilde{\tau}^{\star} \wedge i_{\mathcal{A},I}^{\delta^{1/2}} \wedge i^{\star})).$$

Let

$$\sigma^{\divideontimes} := \min\{j \geq 0 : \divideontimes(\Phi_{\mathcal{A},\psi} - \hat{\varphi}_{\mathcal{A},I}) \geq \xi_0 \text{ for all } j \leq i < \tilde{\tau}^{\star} \wedge i_{\mathcal{A},I}^{\delta^{1/2}} \wedge i^{\star}\} \leq \tilde{\tau}^{\star} \wedge i_{\mathcal{A},I}^{\delta^{1/2}} \wedge i^{\star}.$$

With this setup, similarly as in Sections 3.6.3 and 3.7.2, it in fact suffices to consider the evolution of $Z_{\sigma^{\divideontimes}}^{\divideontimes}(\sigma^{\divideontimes}), Z_{\sigma^{\divideontimes}}^{\divideontimes}(\sigma^{\divideontimes}+1), \ldots$.

Observation 3.16.2. $\{\tau \leq \tilde{\tau}^{\star} \wedge i_{\mathcal{A},I}^{\delta^{1/2}} \wedge i^{\star}\} \subseteq \{Z_{\sigma^-}^{-}(i^{\star}) > 0\} \cup \{Z_{\sigma^+}^{+}(i^{\star}) > 0\}$.

We again use supermartingale concentration techniques to show that the probabilities of the events on the right in Observation 3.16.2 are sufficiently small. More specifically, for this section, we use Lemma 2.9.4.

3.16.1 Trend

Here, we prove that for all $\divideontimes \in \{-,+\}$ and $i_0 \geq 0$, the expected one-step changes of the process $Z_{i_0}^{\divideontimes}(i_0), Z_{i_0}^{\divideontimes}(i_0+1), \ldots$ are non-positive. Lemma 3.5.11 already yields estimates for the one-step changes of the relevant deterministic trajectory, in Lemma 3.16.4 we estimate the one-step changes of the error term that we use in this section. Then we state Lemma 3.16.5 where we provide a precise estimate for the expected one-step change of the non-deterministic part that holds whenever the removal process was well-behaved up to the step we consider. Finally, combining these estimates shows that the above process is indeed a supermartingale (see Lemma 3.16.6).

Observation 3.16.3. *Extend $\hat{p}$ and ξ_1 to continuous trajectories defined on the whole interval $[0, i^{\star}+1]$ using the same expressions as above. Then, for $x \in [0, i^{\star}+1]$,*

$$\xi_1'(x) = -\frac{(|\mathcal{A}| - |\mathcal{A}[I]| - \frac{\delta\rho_{\mathcal{F}}}{2})|\mathcal{F}|k!\,\xi_1(x)}{n^k \hat{p}(x)},$$

$$\xi_1''(x) = -\frac{(|\mathcal{A}| - |\mathcal{A}[I]| - \frac{\delta\rho_{\mathcal{F}}}{2})(|\mathcal{A}| - |\mathcal{A}[I]| - \frac{\delta\rho_{\mathcal{F}}}{2} - 1)|\mathcal{F}|^2(k!)^2\xi_1(x)}{n^{2k}\hat{p}(x)^2}.$$

Lemma 3.16.4. *Let $0 \leq i \leq i^{\star}$ and $\mathcal{X} := \{i \leq \tau_{\emptyset}\}$. Then,*

$$\Delta\xi_1 =_{\mathcal{X}} -\left(|\mathcal{A}| - |\mathcal{A}[I]| - \frac{\delta\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|\xi_1}{H} \pm \frac{\zeta\xi_1}{H}.$$

Proof. This is a consequence of Taylor's theorem. In detail, we argue as follows.

Together with Observation 3.16.3, Lemma 2.9.10 yields

$$\Delta\xi_1 = -\left(|\mathcal{A}| - |\mathcal{A}[I]| - \frac{\delta\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|k!\,\xi_1}{n^k\hat{p}} \pm \max_{x\in[i,i+1]}\frac{\xi_1(x)}{\delta n^{2k}\hat{p}(x)^2}.$$

We investigate the first term and the maximum separately. Using Lemma 3.5.7, we have

$$-\left(|\mathcal{A}| - |\mathcal{A}[I]| - \frac{\delta\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|k!\,\xi_1}{n^k\hat{p}} =_{\mathcal{X}} -\left(|\mathcal{A}| - |\mathcal{A}[I]| - \frac{\delta\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|\xi_1}{H}.$$

Furthermore, using Lemma 3.5.6, Lemma 3.5.7 and Lemma 3.5.9 yields

$$\max_{x\in[i,i+1]}\frac{\xi_1(x)}{\delta n^{2k}\hat{p}(x)^2} \le \frac{\xi_1}{\delta n^{2k}\hat{p}(i+1)^2} \le \frac{\xi_1}{\delta^2 n^{2k}\hat{p}^2} \le_{\mathcal{X}} \frac{\xi_1}{\delta^2 H^2} \le \frac{\zeta^{2+2\varepsilon^2}\xi_1}{\delta^2 H} \le \frac{\zeta^{2+\varepsilon^2}\xi_1}{H}.$$

Thus we obtain the desired expression for $\Delta\xi_1$. $\square$

Lemma 3.16.5. *Let $0 \le i \le i^{\delta^{1/2}}_{\mathcal{A},I} \wedge i^\star$ and $\mathcal{X} := \{i < \tilde{\tau}^\star\}$. Then,*

$$\mathbb{E}_i[\Delta\Phi_{\mathcal{A},\psi}] =_{\mathcal{X}} -(|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|}{H}\Phi_{\mathcal{A},\psi} \pm \frac{\zeta^{1/2}\hat{\varphi}_{\mathcal{A},I}}{H}.$$

Proof. Fix $f \in \mathcal{F}$. Lemma 3.5.18 entails

$$\mathbb{E}_i[\Delta\Phi_{\mathcal{A},\psi}] =_{\mathcal{X}} -\frac{1}{H^*}\sum_{\varphi\in\Phi^{\sim}_{\mathcal{A},\psi}}\left(\Big(\sum_{e\in\mathcal{A}\setminus\mathcal{A}[I]} d_{\mathcal{H}^*}(\varphi(e))\Big) \pm |\mathcal{A}|^2\zeta\hat{\varphi}_{\mathcal{F},f}\right).$$

From Lemma 3.5.17, for all $e \in \mathcal{H}$, we obtain

$$d_{\mathcal{H}^*}(e) =_{\mathcal{X}} \frac{|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{F},f}}{\mathrm{aut}(\mathcal{F})} \pm \frac{1}{\delta}|\mathcal{F}|k!\,\zeta\hat{\varphi}_{\mathcal{F},f}.$$

Thus, due to Lemma 3.5.7, we have

$$\begin{aligned}
\mathbb{E}_i[\Delta\Phi_{\mathcal{A},\psi}] &=_{\mathcal{X}} -\frac{1}{H^*}\Phi_{\mathcal{A},\psi}\left((|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{F},f}}{\mathrm{aut}(\mathcal{F})} \pm \frac{1}{\delta^2}\zeta\hat{\varphi}_{\mathcal{F},f}\right)\\
&= -\frac{|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{F},f}}{\mathrm{aut}(\mathcal{F})H^*}\left((|\mathcal{A}| - |\mathcal{A}[I]|)\Phi_{\mathcal{A},\psi} \pm \frac{1}{\delta^3}\zeta\Phi_{\mathcal{A},\psi}\right)\\
&=_{\mathcal{X}} -(1\pm\zeta^{1+\varepsilon^4})\frac{|\mathcal{F}|}{H}\left((|\mathcal{A}| - |\mathcal{A}[I]|)\Phi_{\mathcal{A},\psi} \pm \frac{1}{\delta^3}\zeta\Phi_{\mathcal{A},\psi}\right)\\
&= -(|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|}{H}\Phi_{\mathcal{A},\psi} \pm \frac{\zeta\Phi_{\mathcal{A},\psi}}{\delta^4 H} =_{\mathcal{X}} -(|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|}{H}\Phi_{\mathcal{A},\psi} \pm \frac{\zeta^{1/2}\hat{\varphi}_{\mathcal{A},I}}{H},
\end{aligned}$$

which completes the proof. $\square$

Lemma 3.16.6. *Let $0 \le i_0 \le i$ and $\ast \in \{-,+\}$. Then, $\mathbb{E}_i[\Delta Z^{\ast}_{i_0}] \le 0$.*

Proof. Suppose that $i \leq i^{\delta^{1/2}}_{\mathcal{A},I} \wedge i^\star$ and let $\mathcal{X} := \{i < \tau^{\ast}_{i_0} \wedge \tilde{\tau}^\star\}$. We have $\mathbb{E}_i[\Delta Z^{\ast}_{i_0}] =_{\mathcal{X}^c} 0$ and $\mathbb{E}_i[\Delta Z^{\ast}_{i_0}] =_{\mathcal{X}} \mathbb{E}_i[\Delta Y^{\ast}]$, so it suffices to obtain $\mathbb{E}_i[\Delta Y^{\ast}] \leq_{\mathcal{X}} 0$. Combining Lemma 3.5.11, Lemma 3.16.4 and Lemma 3.16.5, we obtain

$$\begin{aligned}
\mathbb{E}_i[\Delta Y^{\ast}] &= \ast(\mathbb{E}_i[\Delta\Phi_{\mathcal{A},\psi}] - \Delta\hat{\varphi}_{\mathcal{A},I}) - \Delta\xi_1 \\
&\leq_{\mathcal{X}} \ast\left(-(|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|}{H}\Phi_{\mathcal{A},\psi} + (|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|}{H}\hat{\varphi}_{\mathcal{A},I}\right) \\
&\qquad + \left(|\mathcal{A}| - |\mathcal{A}[I]| - \frac{\delta\rho_{\mathcal{F}}}{2}\right)\frac{|\mathcal{F}|\xi_1}{H} + \frac{\zeta^{1/3}\hat{\varphi}_{\mathcal{A},I}}{H} \\
&\leq -\frac{|\mathcal{F}|}{H}\Bigg(\ast(|\mathcal{A}| - |\mathcal{A}[I]|)(\Phi_{\mathcal{A},\psi} - \hat{\varphi}_{\mathcal{A},I}) - \left(|\mathcal{A}| - |\mathcal{A}[I]| - \frac{\delta\rho_{\mathcal{F}}}{2}\right)\zeta^\delta\hat{\varphi}_{\mathcal{A},I} \\
&\qquad\qquad - \zeta^{1/3}\hat{\varphi}_{\mathcal{A},I}\Bigg) \\
&\leq_{\mathcal{X}} -\frac{|\mathcal{F}|\hat{\varphi}_{\mathcal{A},I}}{H}\left((|\mathcal{A}| - |\mathcal{A}[I]|)(1-\delta^2)\zeta^\delta - \left(|\mathcal{A}| - |\mathcal{A}[I]| - \frac{\delta\rho_{\mathcal{F}}}{2}\right)\zeta^\delta - \zeta^{1/3}\right) \\
&= -\frac{|\mathcal{F}|\hat{\varphi}_{\mathcal{A},I}}{H}\left(-(|\mathcal{A}| - |\mathcal{A}[I]|)\delta^2\zeta^\delta + \frac{\delta\rho_{\mathcal{F}}}{2}\zeta^\delta - \zeta^{1/3}\right) \leq 0,
\end{aligned}$$

which completes the proof. □

3.16.2 Boundedness

Here, similarly as in Sections 3.6.3 and 3.7.2, we obtain suitable bounds for the absolute one-step changes of the processes $Y^{\ast}(0), Y^{\ast}(1), \ldots$ and $Z^{\ast}_{i_0}(i_0), Z^{\ast}_{i_0}(i_0+1), \ldots$ (see Lemmas 3.16.7 and 3.16.8) as well as for the expected absolute one-step changes of these processes (see Lemma 3.16.9 and Lemma 3.16.10). To obtain these bounds, we argue similarly as in Section 3.6.3.

Lemma 3.16.7. *Let* $0 \leq i_0 \leq i \leq i^{\delta^{1/2}}_{\mathcal{A},I} \wedge i^\star$, $\ast \in \{-,+\}$ *and* $\mathcal{X} := \{i < \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'}\}$. *Then,* $|\Delta Y^{\ast}| \leq_{\mathcal{X}} \zeta(i_0)^{8\delta}\hat{\varphi}_{\mathcal{A},I}(i_0)$.

Proof. From Lemma 3.5.11 and Lemma 3.16.4, we obtain

$$|\Delta Y^{\ast}| \leq |\Delta\Phi_{\mathcal{A},\psi}| + |\Delta\hat{\varphi}_{\mathcal{A},I}| + |\Delta\xi_1| \leq |\Delta\Phi_{\mathcal{A},\psi}| + 2\frac{|\mathcal{A}||\mathcal{F}|\hat{\varphi}_{\mathcal{A},I}}{H} + 2\frac{|\mathcal{A}||\mathcal{F}|\xi_1}{H}.$$

Hence, since $\mathcal{A}\setminus\mathcal{A}[I] \neq \emptyset$ implies $\zeta^{8\delta}\hat{\varphi}_{\mathcal{A},I} \leq \zeta(i_0)^{8\delta}\hat{\varphi}_{\mathcal{A},I}(i_0)$, by Lemma 3.5.9 it suffices to show that

$$|\Delta\Phi_{\mathcal{A},\psi}| \leq_{\mathcal{X}} \zeta^{9\delta}\hat{\varphi}_{\mathcal{A},I},$$

which we obtain as a consequence of Lemma 3.5.16.

To this end, note that for all $(\mathcal{B}, I) \subseteq (\mathcal{A}, I)$ with $V_{\mathcal{B}} \neq I$, since $(\mathcal{A}, I)$ is balanced, we have $\rho_{\mathcal{B},I} \leq \rho_{\mathcal{A},I}$ and thus using Lemma 3.5.12, we obtain

$$\hat{\varphi}_{\mathcal{B},I} = (n\hat{p}^{\rho_{\mathcal{B},I}})^{|V_{\mathcal{B}}|-|I|} \geq (n\hat{p}^{\rho_{\mathcal{A},I}})^{|V_{\mathcal{B}}|-|I|} = \hat{\varphi}_{\mathcal{A},I}^{(|V_{\mathcal{B}}|-|I|)/(|V_{\mathcal{A}}|-|I|)} \geq \hat{\varphi}_{\mathcal{A},I}^{10\delta^{1/2}} \geq \frac{\zeta^{-10\delta}}{2}.$$

Hence, Lemma 3.5.16 implies

$$\begin{aligned}|\Delta\Phi_{\mathcal{A},\psi}| &\le \sum_{e\in\mathcal{A}\setminus\mathcal{A}[I]} |\{\varphi\in\Phi^{\sim}_{\mathcal{A},\psi} : \varphi(e)\in\mathcal{F}_0(i+1)\}| \le_{\mathcal{X}} |\mathcal{A}|\cdot 4k!\,|\mathcal{F}|(\log n)^{\alpha_{\mathcal{A},I}}\zeta^{10\delta}\hat{\varphi}_{\mathcal{A},I}\\ &\le \zeta^{9\delta}\hat{\varphi}_{\mathcal{A},I},\end{aligned}$$

which completes the proof. □

Lemma 3.16.8. *Let $0\le i_0\le i$ and $\divideontimes\in\{-,+\}$. Then, $|\Delta Z^{\divideontimes}_{i_0}|\le\zeta(i_0)^{8\delta}\hat{\varphi}_{\mathcal{A},I}(i_0)$.*

Proof. This is an immediate consequence of Lemma 3.16.7. □

Lemma 3.16.9. *Let $0\le i\le i^{\delta^{1/2}}_{\mathcal{A},I}\wedge i^\star$, $\divideontimes\in\{-,+\}$ and $\mathcal{X}:=\{i<\tilde{\tau}^\star\}$. Then,*

$$\mathbb{E}_i[|\Delta Y^{\divideontimes}|]\le_{\mathcal{X}}\frac{\hat{\varphi}_{\mathcal{A},I}}{\zeta^{5\delta}n^k\hat{p}}.$$

Proof. From Lemma 3.5.11 and Lemma 3.16.4, we obtain

$$\mathbb{E}_i[|\Delta Y^{\divideontimes}|]\le\mathbb{E}_i[|\Delta\Phi_{\mathcal{A},I}|]+|\Delta\hat{\varphi}_{\mathcal{A},I}|+|\Delta\xi_1|\le\mathbb{E}_i[|\Delta\Phi_{\mathcal{A},I}|]+2\frac{|\mathcal{A}||\mathcal{F}|\hat{\varphi}_{\mathcal{A},I}}{H}+2\frac{|\mathcal{A}||\mathcal{F}|\xi_1}{H}.$$

Hence, since $\mathcal{A}\setminus\mathcal{A}[I]\neq\emptyset$ implies

$$\frac{\hat{\varphi}_{\mathcal{A},I}}{\zeta^{5\delta}n^k\hat{p}}\le\frac{\hat{\varphi}_{\mathcal{A},I}(i_0)}{\zeta(i_0)^{5\delta}n^k\hat{p}(i_0)},$$

by Lemma 3.5.7 implies that it suffices to show that

$$\mathbb{E}_i[|\Delta\Phi_{\mathcal{A},I}|]\le\frac{\hat{\varphi}_{\mathcal{A},I}}{\zeta^{4\delta}n^k\hat{p}}.$$

We obtain this as a consequence of Lemma 3.5.14 and Lemma 3.5.16.

We argue similarly as in the proof of Lemma 3.6.28. For $e\in\mathcal{A}\setminus\mathcal{A}[I]$, from all subtemplates $(\mathcal{B},I)\subseteq(\mathcal{A},I)$ with $e\in\mathcal{B}$ choose $(\mathcal{B}_e,I)$ such that $\hat{\varphi}_{\mathcal{B}_e,I}$ is minimal. Furthermore, for every subtemplate $(\mathcal{B},I)\subseteq(\mathcal{A},I)$, let

$$\Phi^e_{\mathcal{B},\psi}:=|\{\varphi\in\Phi^{\sim}_{\mathcal{B},\psi}:\varphi(e)\in\mathcal{F}_0(i+1)\}|.$$

Lemma 3.5.16 yields

$$\Phi^e_{\mathcal{A},I}\le_{\mathcal{X}}2k!\,|\mathcal{F}|(\log n)^{\alpha_{\mathcal{A},I\cup e}}\frac{\hat{\varphi}_{\mathcal{A},I}}{\hat{\varphi}_{\mathcal{B}_e,I}},$$

so we obtain

$$\begin{aligned}|\Delta\Phi_{\mathcal{A},\psi}|&\le\sum_{e\in\mathcal{A}\setminus\mathcal{A}[I]}\Phi^e_{\mathcal{A},\psi}=\sum_{e\in\mathcal{A}\setminus\mathcal{A}[I]}\mathbb{1}_{\{\Phi^e_{\mathcal{A},\psi}\ge1\}}\Phi^e_{\mathcal{A},\psi}\\ &\le_{\mathcal{X}}2k!\,|\mathcal{F}|(\log n)^{\alpha_{\mathcal{A},I\cup e}}\hat{\varphi}_{\mathcal{A},I}\sum_{e\in\mathcal{A}\setminus\mathcal{A}[I]}\frac{\mathbb{1}_{\{\Phi^e_{\mathcal{A},\psi}\ge1\}}}{\hat{\varphi}_{\mathcal{B}_e,I}}\le\frac{\hat{\varphi}_{\mathcal{A},I}}{\zeta^\delta}\sum_{e\in\mathcal{A}\setminus\mathcal{A}[I]}\frac{\mathbb{1}_{\{\Phi^e_{\mathcal{A},\psi}\ge1\}}}{\hat{\varphi}_{\mathcal{B}_e,I}}\\ &\le\frac{\hat{\varphi}_{\mathcal{A},I}}{\zeta^\delta}\sum_{e\in\mathcal{A}\setminus\mathcal{A}[I]}\frac{\mathbb{1}_{\{\Phi^e_{\mathcal{B}_e,\psi}\ge1\}}}{\hat{\varphi}_{\mathcal{B}_e,I}}\le\frac{\hat{\varphi}_{\mathcal{A},I}}{\zeta^\delta}\sum_{e\in\mathcal{A}\setminus\mathcal{A}[I]}\sum_{\varphi\in\Phi^{\sim}_{\mathcal{B}_e,\psi}}\frac{\mathbb{1}_{\{\varphi(e)\in\mathcal{F}_0(i+1)\}}}{\hat{\varphi}_{\mathcal{B}_e,I}}.\end{aligned}\tag{3.16.1}$$

For all $e \in \mathcal{H}$, $f \in \mathcal{F}$ and $\psi' \colon f \xrightarrow{\sim} e$, we have $\Phi_{\mathcal{F},\psi'} =_{\mathcal{X}} (1 \pm \delta^{-1}\zeta)\hat{\varphi}_{\mathcal{F},f}$. Furthermore, we have $H^* =_{\mathcal{X}} (1 \pm \zeta^{1+\varepsilon^3})\hat{h}^*$. Thus, using Lemma 3.5.17, for all $e \in \mathcal{A} \setminus \mathcal{A}[I]$ and $\varphi \in \Phi^{\sim}_{\mathcal{B}_e,\psi}$, we obtain

$$\mathbb{P}_i[\varphi(e) \in \mathcal{F}_0(i+1)] = \frac{d_{\mathcal{H}^*}(\varphi(e))}{H^*} \leq_{\mathcal{X}} \frac{2|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{F},f}}{H^*} \leq_{\mathcal{X}} \frac{4|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{F},f}}{\hat{h}^*} \leq \frac{1}{\zeta^\delta n^k \hat{p}}.$$

Combining this with (3.16.1) yields

$$\mathbb{E}_i[|\Delta\Phi_{\mathcal{A},I}|] \leq_{\mathcal{X}} \frac{\hat{\varphi}_{\mathcal{A},I}}{\zeta^\delta} \sum_{e \in \mathcal{A}\setminus\mathcal{A}[I]} \sum_{\varphi \in \Phi^{\sim}_{\mathcal{B}_e,I}} \frac{\mathbb{P}_i[\varphi(e) \in \mathcal{F}_0(i+1)]}{\hat{\varphi}_{\mathcal{B}_e,I}} \leq_{\mathcal{X}} \frac{\hat{\varphi}_{\mathcal{A},I}}{\zeta^{2\delta} n^k \hat{p}} \sum_{e \in \mathcal{A}\setminus\mathcal{A}[I]} \frac{\Phi_{\mathcal{B}_e,I}}{\hat{\varphi}_{\mathcal{B}_e,I}}.$$

This shows that it suffices to prove that $\Phi_{\mathcal{B}_e,I} \leq_{\mathcal{X}} \hat{\varphi}_{\mathcal{B}_e,I}/\zeta^\delta$, which we obtain as a consequence of Lemma 3.5.14 as follows. First, note that since $(\mathcal{A}, I)$ is balanced, for all $e \in \mathcal{A} \setminus \mathcal{A}[I]$ and $(\mathcal{C}, I) \subseteq (\mathcal{B}_e, I) \subseteq (\mathcal{A}, I)$, we have $\rho_{\mathcal{C},I} \leq \rho_{\mathcal{A},I}$ and thus

$$\hat{\varphi}_{\mathcal{C},I} = (n\hat{p}^{\rho_{\mathcal{C},I}})^{|V_\mathcal{C}|-|I|} \geq (n\hat{p}^{\rho_{\mathcal{A},I}})^{|V_\mathcal{C}|-|I|} = \hat{\varphi}_{\mathcal{A},I}^{(|V_\mathcal{C}|-|I|)/(|V_\mathcal{A}|-|I|)}.$$

As Lemma 3.5.12 implies $\hat{\varphi}_{\mathcal{A},I} \geq (1 - n^{-\varepsilon^3})\zeta^{-\delta^{1/2}} \geq 1$, this entails $\hat{\varphi}_{\mathcal{C},I} \geq 1$ and so Lemma 3.5.14 indeed yields

$$\Phi_{\mathcal{B}_e,I} \leq_{\mathcal{X}} 2(\log n)^{\alpha_{\mathcal{B}_e,I}} \hat{\varphi}_{\mathcal{B}_e,I} \leq \frac{1}{\zeta^\delta}\hat{\varphi}_{\mathcal{B}_e,I},$$

which completes the proof. □

Lemma 3.16.10. *Let* $0 \leq i_0 \leq i^\star$ *and* $\ast \in \{-, +\}$. *Then,* $\sum_{i \geq i_0} \mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] \leq \hat{\varphi}_{\mathcal{A},I}(i_0)/\zeta(i_0)^{5\delta}$.

Proof. Lemma 3.16.9 entails

$$\sum_{i \geq i_0} \mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] = \sum_{i_0 \leq i \leq i^\star - 1} \mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] \leq (i^\star - i_0)\frac{\hat{\varphi}_{\mathcal{A},I}(i_0)}{\zeta(i_0)^{5\delta} n^k \hat{p}(i_0)}.$$

Since

$$i^\star - i_0 \leq \frac{\vartheta n^k}{|\mathcal{F}|k!} - i_0 = \frac{n^k \hat{p}(i_0)}{|\mathcal{F}|k!} \leq n^k \hat{p}(i_0),$$

this completes the proof. □

3.16.3 Supermartingale concentration

This section follows a similar structure as Sections 3.6.3 and 3.7.2. Lemma 3.16.11 is the final ingredient that we use for our application of Lemma 2.9.4 in the proof of Lemma 3.16.12 where we show that the probabilities of the events on the right in Observation 3.16.2 are indeed small.

Lemma 3.16.11. *Let* $\ast \in \{-, +\}$. *Then,* $Z^{\ast}_{\sigma^\ast}(\sigma^\ast) \leq -\delta^3\xi_1(\sigma^\ast)$.

Proof. Lemma 3.5.4 implies $\tilde{\tau}^\star \geq 1$ and we have $i_{\mathcal{A},I}^{\delta^{1/2}} \geq 1$. Hence, we have $\tilde{\tau}^\star \wedge i_{\mathcal{A},I}^{\delta^{1/2}} \wedge i^\star \geq 1$ and since for $i := 0$, Lemma 3.5.4 also implies $\ast(\Phi_{\mathcal{A},\psi} - \hat{\varphi}_{\mathcal{A},I}) \leq \xi_0$, we have $\sigma^\ast \geq 1$. Thus, by definition of $\sigma^\ast$, for $i := \sigma^\ast - 1$, we have $\ast(\Phi_{\mathcal{A},\psi} - \hat{\varphi}_{\mathcal{A},I}) \leq \xi_0$ and thus

$$Z_i^\ast = \ast(\Phi_{\mathcal{A},\psi} - \hat{\varphi}_{\mathcal{A},I}) - \xi_1 \leq -\delta^2\xi_1.$$

Furthermore, since $\sigma^\ast \leq \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'}$, we may apply Lemma 3.16.7 to obtain

$$Z_{\sigma^\ast}^\ast(\sigma^\ast) = Z_i^\ast + \Delta Y^\ast \leq Z_i^\ast + \zeta^{8\delta}\hat{\varphi}_{\mathcal{A},I} \leq -\delta^2\xi_1 + \zeta^{8\delta}\hat{\varphi}_{\mathcal{A},I} \leq -\delta^3\xi_1.$$

Since $\Delta\xi_1 \leq 0$, this completes the proof. □

Lemma 3.16.12. $\mathbb{P}[\tau_{\mathscr{B}} \leq \tilde{\tau}^\star \wedge i^\star] \leq \exp(-n^{\delta^2})$.

Proof. Considering Observation 3.16.1, it suffices to show that

$$\mathbb{P}[\tau \leq \tilde{\tau}^\star \wedge i_{\mathcal{A},I}^{\delta^{1/2}} \wedge i^\star] \leq \exp(-n^{2\delta^2}).$$

Hence, by Observation 3.16.2, it suffices to show that for $\ast \in \{-,+\}$, we have

$$\mathbb{P}[Z_{\sigma^\ast}^\ast(i^\star) > 0] \leq \exp(-n^{3\delta^2}).$$

Due to Lemma 3.16.11, we have

$$\mathbb{P}[Z_{\sigma^\ast}^\ast(i^\star) > 0] \leq \mathbb{P}[Z_{\sigma^\ast}^\ast(i^\star) - Z_{\sigma^\ast}^\ast(\sigma^\ast) \geq \delta^3\xi_1(\sigma^\ast)] \leq \sum_{0\leq i\leq i^\star} \mathbb{P}[Z_i^\ast(i^\star) - Z_i^\ast \geq \delta^3\xi_1].$$

Thus, for $0 \leq i \leq i^\star$, it suffices to obtain

$$\mathbb{P}[Z_i^\ast(i^\star) - Z_i^\ast \geq \delta^3\xi_1] \leq \exp(-n^{4\delta^2}).$$

We show that this bound is a consequence of Lemma 2.9.4.

Let us turn to the details. Lemma 3.16.6 shows that $Z_i^\ast(i), Z_i^\ast(i+1), \ldots$ is a supermartingale, while Lemma 3.16.8 provides the bound $|\Delta Z_i^\ast(j)| \leq \zeta^{8\delta}\hat{\varphi}_{\mathcal{A},I}$ for all $j \geq i$ and Lemma 3.16.10 provides the bound $\sum_{j\geq i}\mathbb{E}_j[|\Delta Z_i^\ast(j)|] \leq \zeta^{-5\delta}\hat{\varphi}_{\mathcal{A},I}$. Hence, we may apply Lemma 2.9.4 such that using Lemma 3.5.8, we obtain

$$\begin{aligned}\mathbb{P}_i[Z_i^\ast(i^\star) - Z_i^\ast > \delta^3\xi_1] &\leq \exp\left(-\frac{\delta^6\zeta^{2\delta}\hat{\varphi}_{\mathcal{A},I}^2}{2\zeta^{8\delta}\hat{\varphi}_{\mathcal{A},I}(\delta^3\zeta^\delta\hat{\varphi}_{\mathcal{A},I} + \zeta^{-5\delta}\hat{\varphi}_{\mathcal{A},I})}\right) \leq \exp(\delta^7\zeta^{-\delta}) \\ &\leq \exp(-n^{4\delta^2}),\end{aligned}$$

which completes the proof. □

3.17 Counting strictly balanced templates

Lemma 3.5.19 (ii) states that for a balanced template $(\mathcal{A}, I) \in \mathscr{B}$ and $0 \le i \le i^\star$, the number $\Phi_{\mathcal{A},I}$ behaves as expected as long as the corresponding trajectory still suggests a significant number of embeddings in the sense that $i \le i^{\delta^{1/2}}_{\mathcal{A},I}$. In this section, our goal is to extend this guarantee that the number of embeddings is typically concentrated around the trajectory also beyond step $i^{\delta^{1/2}}_{\mathcal{A},I}$ up to step $i^0_{\mathcal{A},I}$ and also if $i^{\delta^{1/2}}_{\mathcal{A},I} = 0$ subject to the following two restrictions. First, we obtain this guarantee only for strictly balanced templates $(\mathcal{A}, I)$ with $i^0_{\mathcal{A},I} \ge 1$ and second, we allow larger relative deviations from the trajectory compared to Lemma 3.5.19 (ii). Formally, for this section, we assume the setup that we used in Section 3.5 to state Lemma 3.5.19 and show that the probability that $\tau_{\mathscr{B}'} \le \tilde{\tau}^\star \wedge i^\star$ is small. Similarly as in Sections 3.6.3 and 3.7.2 we may again restrict our attention to only one fixed strictly balanced template $(\mathcal{A}, I)$ with $I \ne V_\mathcal{A}$ and $i^{\delta^{1/2}}_{\mathcal{A},I} \le i^\star$, see Observation 3.17.1. Note that $I \ne V_\mathcal{A}$ together with $i^{\delta^{1/2}}_{\mathcal{A},I} \le i^\star$ in particular entails $\mathcal{A} \setminus \mathcal{A}[I] \ne \emptyset$. Overall, our approach is similar as in Sections 3.6.3 and 3.7.2, however, the fact that here we are only interested in steps $i \ge i^{\delta^{1/2}}_{\mathcal{A},I}$ leads to a slightly different setup where we intuitively shift the beginning of our considerations from step 0 to step $i^{\delta^{1/2}}_{\mathcal{A},I}$. To control the initial situation at this shifted start, we rely on Lemma 3.5.19 (ii).

Observation 3.17.1. *For $(\mathcal{A}, I) \in \mathscr{B}'$ and $\psi\colon I \hookrightarrow V_\mathcal{H}$, let*

$$\tau_{\mathcal{A},\psi} := \min\{i \ge i^{\delta^{1/2}}_{\mathcal{A},I} : \Phi_{\mathcal{A},\psi} \ne (1 \pm (\log n)^{\alpha_{\mathcal{A},I}} \hat{\varphi}^{-\delta^{1/2}}_{\mathcal{A},I})\hat{\varphi}_{\mathcal{A},I}\}.$$

Then,

$$\mathbb{P}[\tau_{\mathscr{B}'} \le \tilde{\tau}^\star \wedge i^\star] \le \sum_{\substack{(\mathcal{A},I)\in\mathscr{B}'\colon I\ne V_\mathcal{A} \text{ and } i^{\delta^{1/2}}_{\mathcal{A},I} \le i^\star \\ \psi\colon I\hookrightarrow V_\mathcal{H}}} \mathbb{P}[\tau_{\mathcal{A},\psi} \le \tilde{\tau}^\star \wedge i^0_{\mathcal{A},I} \wedge i^\star].$$

Fix $(\mathcal{A}, I) \in \mathscr{B}'$ with $I \ne V_\mathcal{A}$ and $i^{\delta^{1/2}}_{\mathcal{A},I} \le i^\star$ and hence $\mathcal{A} \setminus \mathcal{A}[I] \ne \emptyset$. Let $\psi\colon I \hookrightarrow V_\mathcal{H}$ and for $i \ge 0$, let

$$\xi_1(i) := (\log n)^{\alpha_{\mathcal{A},I}} \hat{\varphi}^{1-\delta^{1/2}}_{\mathcal{A},I}, \quad \xi_0(i) := (1-\delta)\xi_1$$

and define the stopping time

$$\tau := \min\{i \ge i^{\delta^{1/2}}_{\mathcal{A},I} : \Phi_{\mathcal{A},\psi} \ne \hat{\varphi}_{\mathcal{A},I} \pm \xi_1\}.$$

Define the critical intervals

$$I^-(i) := [\hat{\varphi}_{\mathcal{A},I} - \xi_1, \hat{\varphi}_{\mathcal{A},I} - \xi_0], \quad I^+(i) := [\hat{\varphi}_{\mathcal{A},I} + \xi_0, \hat{\varphi}_{\mathcal{A},I} + \xi_1].$$

For $\ast \in \{-, +\}$, let

$$Y^\ast(i) := \ast(\Phi_{\mathcal{A},\psi} - \hat{\varphi}_{\mathcal{A},I}) - \xi_1.$$

For $i_0 \ge i^{\delta^{1/2}}_{\mathcal{A},I}$, define the stopping time

$$\tau^\ast_{i_0} := \min\{i \ge i_0 : \Phi_{\mathcal{A},\psi} \notin I^\ast\}$$

and for $i \geq i_0$, let

$$Z_{i_0}^{\ast}(i) := \mathbb{1}_{\{i_{\mathcal{A},I}^{\delta^{1/2}} < \tau_{\mathscr{B}}\}} Y^{\ast}(i_0 \vee (i \wedge \tau_{i_0}^{\ast} \wedge \tilde{\tau}^{\star} \wedge i_{\mathcal{A},I}^{0} \wedge i^{\star})).$$

Let

$$\sigma^{\ast} := \min\{j \geq i_{\mathcal{A},I}^{\delta^{1/2}} : \ast(\Phi_{\mathcal{A},\psi} - \hat{\varphi}_{\mathcal{A},I}) \geq \xi_0 \text{ for all } j \leq i < \tilde{\tau}^{\star} \wedge i_{\mathcal{A},I}^{0} \wedge i^{\star}\} \leq \tilde{\tau}^{\star} \wedge i_{\mathcal{A},I}^{0} \wedge i^{\star}.$$

With this setup, similarly as in Section 3.6.3 and Section 3.7.2, it in fact again suffices to consider the evolution of $Z_{\sigma^{\ast}}^{\ast}(\sigma^{\ast}), Z_{\sigma^{\ast}}^{\ast}(\sigma^{\ast}+1), \ldots$. Indeed, we have

$$\begin{aligned}
\{\tau \leq \tilde{\tau}^{\star} \wedge i_{\mathcal{A},I}^{0} \wedge i^{\star}\} &\subseteq \{\tau_{\mathscr{B}} \leq i_{\mathcal{A},I}^{\delta^{1/2}} \text{ and } \tau \leq \tilde{\tau}^{\star} \wedge i_{\mathcal{A},I}^{0} \wedge i^{\star}\} \\
&\qquad \cup \{Z_{\sigma^-}^{-}(i^{\star}) > 0\} \cup \{Z_{\sigma^+}^{+}(i^{\star}) > 0\} \\
&\subseteq \{i_{\mathcal{A},I}^{\delta^{1/2}} \leq \tau \leq \tilde{\tau}^{\star} \leq \tau_{\mathscr{B}} \leq i_{\mathcal{A},I}^{\delta^{1/2}}\} \cup \{Z_{\sigma^-}^{-}(i^{\star}) > 0\} \cup \{Z_{\sigma^+}^{+}(i^{\star}) > 0\} \\
&\subseteq \{\tau_{\mathscr{B}} \leq \tilde{\tau}^{\star} \wedge i_{\mathcal{A},I}^{\delta^{1/2}}\} \cup \{Z_{\sigma^-}^{-}(i^{\star}) > 0\} \cup \{Z_{\sigma^+}^{+}(i^{\star}) > 0\}
\end{aligned}$$

and due to $i_{\mathcal{A},I}^{\delta^{1/2}} \leq i^{\star}$, this leads to the following observation.

Observation 3.17.2. $\{\tau \leq \tilde{\tau}^{\star} \wedge i_{\mathcal{A},I}^{0} \wedge i^{\star}\} \subseteq \{\tau_{\mathscr{B}} \leq \tilde{\tau}^{\star} \wedge i^{\star}\} \cup \{Z_{\sigma^-}^{-}(i^{\star}) > 0\} \cup \{Z_{\sigma^+}^{+}(i^{\star}) > 0\}$.

Lemma 3.5.19 (ii) shows that the probability of the first event on the right in Observation 3.17.2 is sufficiently small and we again use supermartingale concentration techniques to show that the probabilities of the other two events are also sufficiently small. More specifically, for this section, we use Lemma 2.9.4.

3.17.1 Trend

Here, we prove that for all $\ast \in \{-,+\}$ and $i_0 \geq i_{\mathcal{A},I}^{\delta^{1/2}}$, the expected one-step changes of the process $Z_{i_0}^{\ast}(i_0), Z_{i_0}^{\ast}(i_0+1), \ldots$ are non-positive. Lemma 3.5.11 already yields estimates for the one-step changes of the relevant deterministic trajectory, in Lemma 3.17.4 we estimate the one-step changes of the error term that we use in this section. Furthermore, Lemma 3.16.5 provides a precise estimate for the expected one-step change of the non-deterministic part of the random process. Combining these estimates shows that the process indeed is a supermartingale (see Lemma 3.17.5).

Observation 3.17.3. *Extend $\hat{p}$ and ξ_1 to continuous trajectories defined on the whole interval $[0, i^{\star}+1]$ using the same expressions as above. Then, for $x \in [0, i^{\star}+1]$,*

$$\xi_1'(x) = -\frac{(1-\delta^{1/2})(|\mathcal{A}| - |\mathcal{A}[I]|)|\mathcal{F}|k!\,\xi_1(x)}{n^k \hat{p}(x)},$$

$$\xi_1''(x) = -\frac{(1-\delta^{1/2})(|\mathcal{A}| - |\mathcal{A}[I]|)((1-\delta^{1/2})(|\mathcal{A}| - |\mathcal{A}[I]|) - 1)|\mathcal{F}|^2(k!)^2\xi_1(x)}{n^{2k}\hat{p}(x)^2},$$

Lemma 3.17.4. *Let* $0 \leq i \leq i^\star$ *and* $\mathcal{X} := \{i \leq \tau_\emptyset\}$. *Then,*

$$\Delta\xi_1 =_{\mathcal{X}} -(1-\delta^{1/2})(|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|\xi_1}{H} \pm \frac{\hat{\varphi}_{\mathcal{A},I}^{1-\delta^{1/2}}}{H}.$$

Proof. This is a consequence of Taylor's theorem. In detail, we argue as follows.
Together with Observation 3.17.3, Lemma 2.9.10 yields

$$\Delta\xi_1 = -\frac{(1-\delta^{1/2})(|\mathcal{A}| - |\mathcal{A}[I]|)|\mathcal{F}|k!\,\xi_1}{n^k\hat{p}} \pm \max_{x\in[i,i+1]} \frac{\xi_1(x)}{\delta n^{2k}\hat{p}(x)^2}.$$

We investigate the first term and the maximum separately. Using Lemma 3.5.7, we have

$$-\frac{(1-\delta^{1/2})(|\mathcal{A}| - |\mathcal{A}[I]|)|\mathcal{F}|k!\,\xi_1}{n^k\hat{p}} =_{\mathcal{X}} -(1-\delta^{1/2})(|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|\xi_1}{H}.$$

Furthermore, precisely as at the end of the proof of Lemma 3.16.4, we obtain

$$\max_{x\in[i,i+1]} \frac{\xi_1(x)}{\delta n^{2k}\hat{p}(x)^2} \leq_{\mathcal{X}} \frac{\zeta^{2+\varepsilon^2}\xi_1}{H}.$$

With Lemma 3.5.8, this completes the proof. □

Lemma 3.17.5. *Let* $i_{\mathcal{A},I}^{\delta^{1/2}} \leq i_0 \leq i$ *and* $\ast \in \{-,+\}$. *Then,* $\mathbb{E}_i[\Delta Z_{i_0}^{\ast}] \leq 0$.

Proof. Suppose that $i \leq i^\star$ and let $\mathcal{X} := \{i < \tau_{i_0}^{\ast} \wedge \tilde{\tau}^\star\}$. We have $\mathbb{E}_i[\Delta Z_{i_0}^{\ast}] =_{\mathcal{X}^{c}} 0$ and $\mathbb{E}_i[\Delta Z_{i_0}^{\ast}] =_{\mathcal{X}} \mathbb{E}_i[\Delta Y^{\ast}]$, so it suffices to obtain $\mathbb{E}_i[\Delta Y^{\ast}] \leq_{\mathcal{X}} 0$. Due to Lemma 3.5.8, we have $\zeta^{1/2} \leq n^{-\delta^{1/2}|V_{\mathcal{A}}|} \leq \hat{\varphi}_{\mathcal{A},I}^{-\delta^{1/2}}$. Hence, Lemma 3.5.11 yields (with room to spare)

$$\Delta\hat{\varphi}_{\mathcal{A},I} = -(|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|\hat{\varphi}_{\mathcal{A},I}}{H} \pm \frac{\hat{\varphi}_{\mathcal{A},I}^{1-\delta^{1/2}}}{H}.$$

Arguing precisely as in the proof of Lemma 3.16.5 for the first equality, we obtain

$$\mathbb{E}_i[\Delta\Phi_{\mathcal{A},\psi}] =_{\mathcal{X}} -(|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|}{H}\Phi_{\mathcal{A},\psi} \pm \frac{\zeta^{1/2}\hat{\varphi}_{\mathcal{A},I}}{H} = -(|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|}{H}\Phi_{\mathcal{A},\psi} \pm \frac{\hat{\varphi}_{\mathcal{A},I}^{1-\delta^{1/2}}}{H}.$$

Combining these two estimates with Lemma 3.17.4, we obtain

$$\begin{aligned}
\mathbb{E}_i[\Delta Y^{\ast}] &= \ast(\mathbb{E}_i[\Delta\Phi_{\mathcal{A},\psi}] - \Delta\hat{\varphi}_{\mathcal{A},I}) - \Delta\xi_1 \\
&\leq_{\mathcal{X}} \ast\left(-(|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|}{H}\Phi_{\mathcal{A},\psi} + (|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|}{H}\hat{\varphi}_{\mathcal{A},I}\right) \\
&\qquad +(1-\delta^{1/2})(|\mathcal{A}| - |\mathcal{A}[I]|)\frac{|\mathcal{F}|\xi_1}{H} + \frac{3\hat{\varphi}_{\mathcal{A},I}^{1-\delta^{1/2}}}{H} \\
&\leq -\frac{|\mathcal{F}|(|\mathcal{A}| - |\mathcal{A}[I]|)}{H}(\ast(\Phi_{\mathcal{A},\psi} - \hat{\varphi}_{\mathcal{A},I}) - (1-\delta^{1/2})\xi_1 - 3\hat{\varphi}_{\mathcal{A},I}^{1-\delta^{1/2}}) \\
&\leq_{\mathcal{X}} -\frac{|\mathcal{F}|(|\mathcal{A}| - |\mathcal{A}[I]|)\xi_1}{H}((1-\delta) - (1-\delta^{1/2}) - \delta) \leq 0,
\end{aligned}$$

which completes the proof. □

3.17.2 Boundedness

Here, similarly as in Sections 3.6.3 and 3.7.2, we obtain suitable bounds for the absolute one-step changes of the processes $Y^{\ast}(0), Y^{\ast}(1), \dots$ and $Z^{\ast}_{i_0}(i_0), Z^{\ast}_{i_0}(i_0+1), \dots$ (see Lemmas 3.17.7 and 3.17.8) as well as the expected absolute one-step changes of these processes (see Lemma 3.17.9 and Lemma 3.17.10). The fact that we analyze the evolution potentially even until $\hat{\varphi}_{\mathcal{A},I}$ is essentially 1 often plays an important role in this section. Furthermore, we crucially exploit that $(\mathcal{A}, I)$ is strictly balanced and not just balanced.

Lemma 3.17.6. *Let $i^{\delta^{1/2}}_{\mathcal{A},I} \le i \le i^\star$. Fix $e \in \mathcal{A} \setminus \mathcal{A}[I]$ and $(\mathcal{B}, I) \subseteq (\mathcal{A}, I)$ with $e \in \mathcal{B}$. Then, $\hat{\varphi}_{\mathcal{A},I} \le \hat{\varphi}_{\mathcal{B},I}$.*

Proof. If $\hat{\varphi}_{\mathcal{A},I} \le 1$, then

$$\hat{\varphi}_{\mathcal{A},I} \le \hat{\varphi}_{\mathcal{A},I}^{(|V_\mathcal{B}|-|I|)/(|V_\mathcal{A}|-|I|)} = (n\hat{p}^{\rho_{\mathcal{A},I}})^{|V_\mathcal{B}|-|I|} \le (n\hat{p}^{\rho_{\mathcal{B},I}})^{|V_\mathcal{B}|-|I|} = \hat{\varphi}_{\mathcal{B},I}.$$

Hence, we may assume $\hat{\varphi}_{\mathcal{A},I} \ge 1$. Furthermore, we may assume that $\mathcal{B} \neq \mathcal{A}$.

Since $(\mathcal{A}, I)$ is strictly balanced, we have $\rho_{\mathcal{B},I} + \delta^{1/4} \le \rho_{\mathcal{A},I}$. This allows us to obtain

$$\begin{aligned}\hat{\varphi}_{\mathcal{A},I} &= (n\hat{p}^{\rho_{\mathcal{A},I}})^{|V_\mathcal{B}|-|I|}(n\hat{p}^{\rho_{\mathcal{A},I}})^{|V_\mathcal{A}|-|V_\mathcal{B}|} = (n\hat{p}^{\rho_{\mathcal{A},I}})^{|V_\mathcal{B}|-|I|}\hat{\varphi}_{\mathcal{A},I}^{(|V_\mathcal{A}|-|V_\mathcal{B}|)/(|V_\mathcal{A}|-|I|)} \\ &\le (n\hat{p}^{\rho_{\mathcal{B},I}+\delta^{1/4}})^{|V_\mathcal{B}|-|I|}\hat{\varphi}_{\mathcal{A},I} \le \hat{\varphi}_{\mathcal{B},I} \cdot \hat{p}^{\delta^{1/4}}\hat{\varphi}_{\mathcal{A},I}.\end{aligned}$$

Hence, it suffices to show that $\hat{p}^{\delta^{1/4}} \le 1/\hat{\varphi}_{\mathcal{A},I}$. Indeed, using Lemma 3.5.8 and the fact that $\hat{\varphi}_{\mathcal{A},I} \le \zeta^{-\delta^{1/2}}$, we obtain

$$\begin{aligned}\hat{p}^{\delta^{1/4}} &\le \hat{p}^{\delta^{1/3}(|\mathcal{A}|-|\mathcal{A}[I]|)} = n^{-\delta^{1/3}(|V_\mathcal{A}|-|I|)}\hat{\varphi}_{\mathcal{A},I}^{\delta^{1/3}} \le n^{-\delta^{1/3}}\hat{\varphi}_{\mathcal{A},I} \le \zeta^{\delta^{1/3}}\hat{\varphi}_{\mathcal{A},I} \\ &= (\zeta^{\delta^{1/2}})^{\delta^{-1/6}}\hat{\varphi}_{\mathcal{A},I} \le \hat{\varphi}_{\mathcal{A},I}^{1-\delta^{-1/6}} \le \hat{\varphi}_{\mathcal{A},I}^{-1},\end{aligned}$$

which completes the proof. □

Lemma 3.17.7. *Let $i^{\delta^{1/2}}_{\mathcal{A},I} \le i \le i^\star$, $\ast \in \{-,+\}$ and $\mathcal{X} := \{i < \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'}\}$. Then,*

$$|\Delta Y^\ast| \le_{\mathcal{X}} \frac{(\log n)^{\alpha_{\mathcal{A},I}/2}}{\delta^2 \log n}.$$

Proof. From Lemma 3.5.11 and Lemma 3.17.4, using the fact that $\hat{\varphi}_{\mathcal{A},I} \le \zeta^{-\delta^{1/2}}$, we obtain

$$\begin{aligned}|\Delta Y^\ast| &\le |\Delta\Phi_{\mathcal{A},\psi}| + |\Delta\hat{\varphi}_{\mathcal{A},I}| + |\Delta\xi_1| \le |\Delta\Phi_{\mathcal{A},\psi}| + 2\frac{|\mathcal{A}||\mathcal{F}|\hat{\varphi}_{\mathcal{A},I}}{H} + 2\frac{|\mathcal{A}||\mathcal{F}|\xi_1}{H} \\ &\le |\Delta\Phi_{\mathcal{A},\psi}| + 2\frac{|\mathcal{A}||\mathcal{F}|\zeta^{-\delta^{1/2}}}{H} + 2\frac{|\mathcal{A}||\mathcal{F}|(\log n)^{\alpha_{\mathcal{A},I}}\zeta^{-\delta^{1/2}(1-\delta^{1/2})}}{H} \\ &\le |\Delta\Phi_{\mathcal{A},\psi}| + 3\frac{|\mathcal{A}||\mathcal{F}|}{\zeta^{\delta^{1/2}}H}.\end{aligned}$$

Hence, Lemma 3.5.9 implies that it suffices to show that

$$|\Delta\Phi_{\mathcal{A},\psi}| \leq_{\mathcal{X}} \frac{(\log n)^{\alpha_{\mathcal{A},I}/2}}{\delta \log n}$$

which we obtain as a consequence of Lemma 3.5.16 and Lemma 3.17.6. Indeed, these two lemmas together with Observation 3.5.1 imply

$$\begin{aligned}|\Delta\Phi_{\mathcal{A},\psi}| &\leq \sum_{e\in\mathcal{A}\setminus\mathcal{A}[I]} |\{\varphi \in \Phi^{\sim}_{\mathcal{A},\psi} : \varphi(e) \in \mathcal{F}_0(i+1)\}| \leq_{\mathcal{X}} |\mathcal{A}| \cdot 2k!\, |\mathcal{F}|(\log n)^{\alpha_{\mathcal{A},I\cup e}} \\ &\leq \frac{(\log n)^{\alpha_{\mathcal{A},I}/2}}{\delta \log n},\end{aligned}$$

which completes the proof. □

Lemma 3.17.8. *Let $i^{\delta^{1/2}}_{\mathcal{A},I} \leq i_0 \leq i$ and $\ast \in \{-,+\}$. Then,*

$$|\Delta Z^{\ast}_{i_0}| \leq \frac{(\log n)^{\alpha_{\mathcal{A},I}/2}}{\delta^2 \log n}.$$

Proof. This is an immediate consequence of Lemma 3.17.7. □

Lemma 3.17.9. *Let $i^{\delta^{1/2}}_{\mathcal{A},I} \leq i_0 \leq i \leq i^0_{\mathcal{A},I} \wedge i^\star$, $\ast \in \{-,+\}$ and $\mathcal{X} := \{i < \tilde{\tau}^\star\}$. Then,*

$$\mathbb{E}_i[|\Delta Y^{\ast}|] \leq_{\mathcal{X}} \frac{(\log n)^{3\alpha_{\mathcal{A},I}/2}\hat{\varphi}_{\mathcal{A},I}(i_0)}{\delta^5 n^k \hat{p}(i_0) \log n}.$$

Proof. From Lemma 3.5.11 and Lemma 3.17.4, we obtain

$$\mathbb{E}_i[|\Delta Y^{\ast}|] \leq \mathbb{E}_i[|\Delta\Phi_{\mathcal{A},I}|] + |\Delta\hat{\varphi}_{\mathcal{A},I}| + |\Delta\xi_1| \leq \mathbb{E}_i[|\Delta\Phi_{\mathcal{A},I}|] + 2\frac{|\mathcal{A}||\mathcal{F}|\hat{\varphi}_{\mathcal{A},I}}{H} + 2\frac{|\mathcal{A}||\mathcal{F}|\xi_1}{H}.$$

Since $\mathcal{A}\setminus\mathcal{A}[I] \neq \emptyset$ implies $\hat{\varphi}_{\mathcal{A},I}/\hat{p} \leq \hat{\varphi}_{\mathcal{A},I}(i_0)/\hat{p}(i_0)$, due to Lemma 3.5.12 and Lemma 3.5.7, it suffices to show that

$$\mathbb{E}_i[|\Delta\Phi_{\mathcal{A},I}|] \leq \frac{(\log n)^{3\alpha_{\mathcal{A},I}/2}\hat{\varphi}_{\mathcal{A},I}}{\delta^4 n^k \hat{p} \log n}.$$

Arguing similarly as in the proof of 3.6.28, we obtain this as a consequence of Lemma 3.5.16 and Lemma 3.17.6.

To this end, for $e \in \mathcal{A} \setminus \mathcal{A}[I]$, let

$$\Phi^e_{\mathcal{A},I} := |\{\varphi \in \Phi^{\sim}_{\mathcal{A},I} : \varphi(e) \in \mathcal{F}_0(i+1)\}|.$$

Using Observation 3.5.1, Lemma 3.5.16 together with Lemma 3.17.6 yields

$$\Phi^e_{\mathcal{A},I} \leq_{\mathcal{X}} 2k!\, |\mathcal{F}|(\log n)^{\alpha_{\mathcal{A},I\cup e}} \leq \frac{(\log n)^{\alpha_{\mathcal{A},I}/2}}{\delta \log n},$$

so we obtain

$$
\begin{aligned}
|\Delta\Phi_{\mathcal{A},I}| &\le \sum_{e\in\mathcal{A}\setminus\mathcal{A}[I]} \Phi^e_{\mathcal{A},I} = \sum_{e\in\mathcal{A}\setminus\mathcal{A}[I]} \mathbb{1}_{\{\Phi^e_{\mathcal{A},I}\ge 1\}}\Phi^e_{\mathcal{A},I} \le_{\mathcal{X}} \frac{(\log n)^{\alpha_{\mathcal{A},I}/2}}{\delta\log n}\sum_{e\in\mathcal{A}\setminus\mathcal{A}[I]} \mathbb{1}_{\{\Phi^e_{\mathcal{A},I}\ge 1\}} \\
&\le \frac{(\log n)^{\alpha_{\mathcal{A},I}/2}}{\delta\log n}\sum_{e\in\mathcal{A}\setminus\mathcal{A}[I]}\sum_{\varphi\in\Phi^{\sim}_{\mathcal{A},I}} \mathbb{1}_{\{\varphi(e)\in\mathcal{F}_0(i+1)\}}.
\end{aligned}
\tag{3.17.1}
$$

For all $e\in\mathcal{H}$, $f\in\mathcal{F}$ and $\psi'\colon f\hookrightarrow e$, we have $\Phi_{\mathcal{F},\psi'} =_{\mathcal{X}} (1\pm\delta^{-1}\zeta)\hat{\varphi}_{\mathcal{F},f}$. Furthermore, we have $H^* =_{\mathcal{X}} (1\pm\zeta^{1+\varepsilon^3})\hat{h}^*$. Thus, using Lemma 3.5.17, for all $e\in\mathcal{A}\setminus\mathcal{A}[I]$ and $\varphi\in\Phi^{\sim}_{\mathcal{A},I}$, we obtain

$$
\mathbb{P}_i[\varphi(e)\in\mathcal{F}_0(i+1)] = \frac{d_{\mathcal{H}^*}(\varphi(e))}{H^*} \le_{\mathcal{X}} \frac{2|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{F},f}}{H^*} \le_{\mathcal{X}} \frac{4|\mathcal{F}|k!\,\hat{\varphi}_{\mathcal{F},f}}{\hat{h}^*} \le \frac{1}{\delta n^k\hat{p}}.
$$

Combining this with (3.17.1) and using the fact that $\Phi_{\mathcal{A},I} =_{\mathcal{X}} (1\pm(\log n)^{\alpha_{\mathcal{A},I}}\hat{\varphi}_{\mathcal{A},I}^{-\delta^{1/2}})\hat{\varphi}_{\mathcal{A},I}$ as well as Lemma 3.5.12 yields

$$
\begin{aligned}
\mathbb{E}_i[|\Delta\Phi_{\mathcal{A},I}|] &\le_{\mathcal{X}} \frac{(\log n)^{\alpha_{\mathcal{A},I}/2}}{\delta^2 n^k\hat{p}\log n}\sum_{e\in\mathcal{A}\setminus\mathcal{A}[I]}\Phi_{\mathcal{A},I} \le \frac{(\log n)^{\alpha_{\mathcal{A},I}/2}}{\delta^3 n^k\hat{p}\log n}\Phi_{\mathcal{A},I} \\
&\le_{\mathcal{X}} \frac{(\log n)^{\alpha_{\mathcal{A},I}/2}}{\delta^3 n^k\hat{p}\log n}(1+(\log n)^{\alpha_{\mathcal{A},I}}\hat{\varphi}_{\mathcal{A},I}^{-\delta^{1/2}})\hat{\varphi}_{\mathcal{A},I} \\
&\le \frac{(\log n)^{\alpha_{\mathcal{A},I}/2}}{\delta^3 n^k\hat{p}\log n}(1+2(\log n)^{\alpha_{\mathcal{A},I}})\hat{\varphi}_{\mathcal{A},I} \\
&\le \frac{(\log n)^{3\alpha_{\mathcal{A},I}/2}\hat{\varphi}_{\mathcal{A},I}}{\delta^4 n^k\hat{p}\log n},
\end{aligned}
$$

which completes the proof. □

Lemma 3.17.10. *Let $i^{\delta^{1/2}}_{\mathcal{A},I} \le i_0 \le i^\star$ and $\ast\in\{-,+\}$. Then,*

$$
\sum_{i\ge i_0}\mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] \le \frac{(\log n)^{3\alpha_{\mathcal{A},I}/2}\hat{\varphi}_{\mathcal{A},I}(i_0)}{\delta^5\log n}.
$$

Proof. Lemma 3.17.9 entails

$$
\sum_{i\ge i_0}\mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] = \sum_{i_0\le i\le i^\star-1}\mathbb{E}_i[|\Delta Z^{\ast}_{i_0}|] \le (i^\star-i_0)\frac{(\log n)^{3\alpha_{\mathcal{A},I}/2}\hat{\varphi}_{\mathcal{A},I}(i_0)}{\delta^5 n^k\hat{p}(i_0)\log n}.
$$

Since

$$
i^\star - i_0 \le \frac{\vartheta n^k}{|\mathcal{F}|k!} - i_0 = \frac{n^k\hat{p}(i_0)}{|\mathcal{F}|k!} \le n^k\hat{p}(i_0),
$$

this completes the proof. □

3.17.3 Supermartingale concentration

This section follows a similar structure as Sections 3.6.3 and 3.7.2. Lemma 3.17.11 is the final ingredient that we use for our application of Lemma 2.9.4 in the proof of Lemma 3.17.12 where we show that the probabilities of the events on the right in Observation 3.17.2 are indeed small. One notable difference compared to the aforementioned sections is the fact that here, our analysis does not start at step 0 but instead at step $i^{\delta^{1/2}}_{\mathcal{A},I}$.

Lemma 3.17.11. *Let* $\divideontimes \in \{-,+\}$ *and* $\mathcal{X} := \{i^{\delta^{1/2}}_{\mathcal{A},I} < \tau_{\mathscr{B}}\}$. *Then,* $Z^{\divideontimes}_{\sigma^{\divideontimes}}(\sigma^{\divideontimes}) \leq_{\mathcal{X}} -\delta^2\xi_1(\sigma^{\divideontimes})$.

Proof. If $i = i^{\delta^{1/2}}_{\mathcal{A},I} = 0$, then Lemma 3.5.4 implies $\divideontimes(\Phi_{\mathcal{A},\psi} - \hat{\varphi}_{\mathcal{A},I}) \leq_{\mathcal{X}} \xi_0$. If $i = i^{\delta^{1/2}}_{\mathcal{A},I} \geq 1$, then due to $\hat{\varphi}_{\mathcal{A},I} \leq \zeta^{-\delta^{1/2}}$, we have

$$\divideontimes(\Phi_{\mathcal{A},I} - \hat{\varphi}_{\mathcal{A},I}) \leq_{\mathcal{X}} \zeta^{\delta}\hat{\varphi}_{\mathcal{A},I} \leq \hat{\varphi}^{1-\delta^{1/2}}_{\mathcal{A},I} \leq \xi_0$$

Hence, if $\sigma^{\divideontimes} = i^{\delta^{1/2}}_{\mathcal{A},I}$, then $Z^{\divideontimes}_{\sigma^{\divideontimes}} \leq_{\mathcal{X}} \xi_0(\sigma^{\divideontimes}) - \xi_1(\sigma^{\divideontimes}) = -\delta\xi_1(\sigma^{\divideontimes})$, so we may assume $\sigma^{\divideontimes} \geq i^{\delta^{1/2}}_{\mathcal{A},I} + 1$. Then, by definition of $\sigma^{\divideontimes}$, for $i := \sigma^{\divideontimes} - 1$, we have $\divideontimes(\Phi_{\mathcal{A},\psi} - \hat{\varphi}_{\mathcal{A},I}) \leq \xi_0$ and thus

$$Z^{\divideontimes}_i = \divideontimes(\Phi_{\mathcal{A},\psi} - \hat{\varphi}_{\mathcal{A},I}) - \xi_1 \leq -\delta\xi_1.$$

Furthermore, since $\sigma^{\divideontimes} \leq \tau_{\mathscr{B}} \wedge \tau_{\mathscr{B}'} \wedge i^0_{\mathcal{A},I}$, we may apply Lemma 3.17.7 and Lemma 3.5.12 to obtain

$$Z^{\divideontimes}_{\sigma^{\divideontimes}}(\sigma^{\divideontimes}) = Z^{\divideontimes}_i + \Delta Y^{\divideontimes} \leq Z^{\divideontimes}_i + \frac{(\log n)^{\alpha_{\mathcal{A},I}/2}}{\delta^2 \log n} \leq -\delta\xi_1 + \frac{2(\log n)^{\alpha_{\mathcal{A},I}/2}\hat{\varphi}^{1-\delta^{1/2}}_{\mathcal{A},I}}{\delta^2 \log n} \leq -\delta^2\xi_1.$$

Since $\Delta\xi_1 \leq 0$, this completes the proof. □

Lemma 3.17.12. $\mathbb{P}[\tau_{\mathscr{B}'} \leq \tilde{\tau}^{\star} \wedge i^{\star}] \leq \exp(-(\log n)^{3/2})$.

Proof. Considering Observation 3.17.1, it suffices to obtain

$$\mathbb{P}[\tau \leq \tilde{\tau}^{\star} \wedge i^0_{\mathcal{A},I} \wedge i^{\star}] \leq \exp(-(\log n)^{5/3}).$$

Hence, by Observation 3.17.2 and Lemma 3.5.19 (ii), it suffices to show that for $\divideontimes \in \{-,+\}$, we have

$$\mathbb{P}[Z^{\divideontimes}_{\sigma^{\divideontimes}}(i^{\star}) > 0] \leq \exp(-(\log n)^{7/4}).$$

Using Lemma 3.17.11, we obtain

$$\mathbb{P}[Z^{\divideontimes}_{\sigma^{\divideontimes}}(i^{\star}) > 0] \leq \mathbb{P}[Z^{\divideontimes}_{\sigma^{\divideontimes}}(i^{\star}) - Z^{\divideontimes}_{\sigma^{\divideontimes}}(\sigma^{\divideontimes}) \geq \delta^2\xi_1(\sigma^{\divideontimes})] \leq \sum_{i^{\delta^{1/2}}_{\mathcal{A},I} \leq i \leq i^{\star}} \mathbb{P}[Z^{\divideontimes}_i(i^{\star}) - Z^{\divideontimes}_i \geq \delta^2\xi_1].$$

Thus, for $i^{\delta^{1/2}}_{\mathcal{A},I} \leq i \leq i^{\star}$, it suffices to obtain

$$\mathbb{P}[Z^{\divideontimes}_i(i^{\star}) - Z^{\divideontimes}_i \geq \delta^2\xi_1] \leq \exp(-(\log n)^{9/5}).$$

We show that this bound is a consequence of Lemma 2.9.4.

Lemma 3.17.5 shows that $Z_i^{\divideontimes}(i), Z_i^{\divideontimes}(i+1), \dots$ is a supermartingale, while Lemma 3.17.8 provides the bound

$$|\Delta Z_i^{\divideontimes}(j)| \le \frac{(\log n)^{\alpha_{\mathcal{A},I}/2}}{\delta^2 \log n}$$

for all $j \ge i$ and Lemma 3.17.10 provides the bound

$$\sum_{j\ge 0} \mathbb{E}_j[|\Delta Z_i^{\divideontimes}(j)|] \le \frac{(\log n)^{3\alpha_{\mathcal{A},I}/2}\hat{\varphi}_{\mathcal{A},I}}{\delta^5 \log n}.$$

Observe that due to Lemma 3.5.12, we have

$$\frac{(\log n)^{3\alpha_{\mathcal{A},I}/2}\hat{\varphi}_{\mathcal{A},I}}{\delta^5 \log n} + \delta^2\xi_1 \le \frac{(\log n)^{3\alpha_{\mathcal{A},I}/2}\hat{\varphi}_{\mathcal{A},I}}{\delta^5 \log n} + (\log n)^{\alpha_{\mathcal{A},I}}\hat{\varphi}_{\mathcal{A},I} \le \frac{(\log n)^{3\alpha_{\mathcal{A},I}/2}\hat{\varphi}_{\mathcal{A},I}}{\delta^6 \log n}.$$

Hence, we may apply Lemma 2.9.4 to obtain

$$\begin{aligned}\mathbb{P}[Z_i^{\divideontimes}(i^\star) - Z_i^{\divideontimes} \ge \delta^2\xi_1] &\le \exp\left(-\frac{\delta^4(\log n)^{2\alpha_{\mathcal{A},I}}\hat{\varphi}_{\mathcal{A},I}^{2-2\delta^{1/2}}}{2\delta^{-2}(\log n)^{\alpha_{\mathcal{A},I}/2-1}\cdot\delta^{-6}(\log n)^{3\alpha_{\mathcal{A},I}/2-1}\hat{\varphi}_{\mathcal{A},I}}\right)\\ &= \exp(-\delta^{13}(\log n)^2\hat{\varphi}_{\mathcal{A},I}^{1-2\delta^{1/2}}).\end{aligned}$$

Another application of Lemma 3.5.12 shows that $\hat{\varphi}_{\mathcal{A},I}^{1-2\delta^{1/2}} \ge 1/2$ and hence completes the proof. □

3.18 Further remarks

For both, the $\mathcal{F}$-free process and the $\mathcal{F}$-removal process, the number of edges present at step i of the process, that is, after i iterations, is a deterministic quantity. Heuristically, intuition suggests that the set of edges present at step i behaves as if it was obtained by including every k-set of vertices independently at random with an appropriate probability p.

For the $\mathcal{F}$-free process on n vertices, we have $p \approx k!\,i/n^k$. There are approximately $(1-p)n^k/k!$ potential edges that are not yet present. Using $\operatorname{aut}(\mathcal{F})$ to denote the number of automorphisms of $\mathcal{F}$, following the above heuristic, for every such edge e, the expected number of copies of $\mathcal{F}$ that would be generated by adding e is $|\mathcal{F}|k!\,n^{|V_\mathcal{F}|-k}p^{|\mathcal{F}|-1}/\operatorname{aut}(\mathcal{F})$. Hence, the Poisson paradigm suggests that the number of potential edges that are available for addition in a later step is approximately

$$(1-p)\exp\left(-\frac{|\mathcal{F}|k!\,n^{|V_\mathcal{F}|-k}p^{|\mathcal{F}|-1}}{\operatorname{aut}(\mathcal{F})}\right)\frac{n^k}{k!}.$$

This number becomes negligible compared to the approximate number $n^kp/k!$ of present edges when

$$p = \left(\left(\frac{\operatorname{aut}(\mathcal{F})(|V_\mathcal{F}|-k)}{|\mathcal{F}|(|\mathcal{F}|-1)k!}\right)^{\frac{1}{|\mathcal{F}|-1}} \pm o(1)\right)(\log n)^{\frac{1}{|\mathcal{F}|-1}} n^{-\frac{|V_\mathcal{F}|-k}{|\mathcal{F}|-1}}.$$

Hence, we conjecture the following.

Conjecture 3.18.1. *Let $k \geq 2$ and consider a strictly k-balanced k-graph $\mathcal{F}$ with k-density ρ. Then, for all $\varepsilon > 0$, there exists $n_0 \geq 0$ such that for all $n \geq n_0$, with probability at least $1 - \varepsilon$, we have*

$$F(n, \mathcal{F}) = \left(\frac{1}{k!} \left(\frac{\mathrm{aut}(\mathcal{F})(|V_{\mathcal{F}}| - k)}{|\mathcal{F}|(|\mathcal{F}| - 1)k!} \right)^{\frac{1}{|\mathcal{F}|-1}} \pm \varepsilon \right) (\log n)^{\frac{1}{|\mathcal{F}|-1}} n^{k - \frac{|V_{\mathcal{F}}|-k}{|\mathcal{F}|-1}}.$$

The known bounds for the case where $\mathcal{F}$ is a triangle, see [16,34], match this prediction and it would be interesting to further investigate other cases. Conjecture 3.18.1 is closely related to [15, Conjecture 13.1].

Again following the above heuristic, for the $\mathcal{F}$-removal process we have $p \approx 1 - |\mathcal{F}| k!\, i/n^k$ such that again, there are approximately $n^k p/k!$ edges present. Let $\mathcal{H}^*$ denote the auxiliary hypergraph where the present edges are the vertices and where the edges sets of present copies are the edges. Let H^* denote the number of edges of $\mathcal{H}^*$, that is the number of remaining copies of $\mathcal{F}$. We expect the 2-degrees in $\mathcal{H}^*$, that is the number of edges in $\mathcal{H}^*$ that contain two fixed vertices of $\mathcal{H}$, to be generally negligible compared to the vertex degrees in $\mathcal{H}^*$. Hence for the probability that a fixed present copy $\mathcal{F}'$ of $\mathcal{F}$ is no longer present in the next step, we estimate

$$\frac{\left(\sum_{e \in \mathcal{F}'} d_{\mathcal{H}^*}(e)\right) - |\mathcal{F}| + 1}{H^*}.$$

Then, using $\mathfrak{F}_0, \mathfrak{F}_1, \dots$ to denote the natural filtration associated with the process, for the expected one-step change $\mathbb{E}[\Delta H^* \mid \mathfrak{F}_i]$ of H^*, we obtain

$$\mathbb{E}[\Delta H^* \mid \mathfrak{F}_i] \approx - \sum_{\mathcal{F}' \in \mathcal{H}^*} \frac{\left(\sum_{e \in \mathcal{F}'} d_{\mathcal{H}^*}(e)\right) - |\mathcal{F}| + 1}{H^*} = -\frac{1}{H^*} \Big(\sum_{d \geq 0} d_{\mathcal{H}^*}(e)^2 \Big) + |\mathcal{F}| - 1.$$

We expect the degrees in $\mathcal{H}^*$ to be Poisson distributed and mutually independent. Thus, since the average vertex degree in $\mathcal{H}^*$ is approximately $\lambda := |\mathcal{F}| k!\, H^*/(n^k p)$, we expect that for all $d \geq 0$, the random variable $|\{e \in \mathcal{H} : d_{\mathcal{H}^*}(e) = d\}|$ is concentrated around

$$\frac{n^k p}{k!} \cdot \frac{\lambda^d \exp(-\lambda)}{d!}.$$

Thus, we estimate

$$\begin{aligned}
\mathbb{E}[\Delta H^* \mid \mathfrak{F}_i] &\approx -\frac{1}{H^*} \Big(\sum_{d \geq 0} d^2 |\{e \in \mathcal{H} : d_{\mathcal{H}^*}(e) = d\}| \Big) + |\mathcal{F}| - 1 \\
&\approx -\frac{n^k p}{k!\, H^*} \Big(\sum_{d \geq 0} d^2 \cdot \frac{\lambda^d \exp(-\lambda)}{d!} \Big) + |\mathcal{F}| - 1 = -\frac{n^k p}{k!\, H^*} (\lambda^2 + \lambda) + |\mathcal{F}| - 1 \\
&= -\frac{|\mathcal{F}|^2 k! H^*}{n^k p} - 1.
\end{aligned}$$

We expect the number of present copies to typically closely follow a deterministic trajectory $\hat{h}_0^*, \hat{h}_1^*, \dots$ which by our above argument should satisfy

$$\hat{h}_{i+1}^* - \hat{h}_i^* \approx -\frac{|\mathcal{F}|^2 k! \hat{h}_i^*}{n^k p} - 1.$$

Guided by this intuition, for $i \geq 0$, we obtain an expression for $\hat{h}_i^*$ by solving the corresponding differential equation. Specifically, since initially the number of copies of $\mathcal{F}$ in $K_n^{(n)}$ is approximately $n^{|V_\mathcal{F}|}/\operatorname{aut}(\mathcal{F})$, we set

$$\hat{h}_i^* := \frac{n^{|V_\mathcal{F}|} p^{|\mathcal{F}|}}{\operatorname{aut}(\mathcal{F})} - \frac{n^k p}{|\mathcal{F}|(|\mathcal{F}|-1)k!}.$$

This quantity becomes zero when

$$p = \left(\left(\frac{\operatorname{aut}(\mathcal{F})}{|\mathcal{F}|(|\mathcal{F}|-1)k!} \right)^{\frac{1}{|\mathcal{F}|-1}} \pm o(1) \right) n^{-\frac{|V_\mathcal{F}|-k}{|\mathcal{F}|-1}}.$$

Hence, for the $\mathcal{F}$-removal process, we conjecture the following.

Conjecture 3.18.2. *Let $k \geq 2$ and consider a strictly k-balanced k-graph $\mathcal{F}$ with k-density ρ. Then, for all $\varepsilon > 0$, there exists $n_0 \geq 0$ such that for all $n \geq n_0$, with probability at least $1 - \varepsilon$, we have*

$$R(n, \mathcal{F}) = \left(\frac{1}{k!} \left(\frac{\operatorname{aut}(\mathcal{F})}{|\mathcal{F}|(|\mathcal{F}|-1)k!} \right)^{\frac{1}{|\mathcal{F}|-1}} \pm \varepsilon \right) n^{k - \frac{|V_\mathcal{F}|-k}{|\mathcal{F}|-1}}.$$

Theorem 1.1.4 confirms the order of magnitude in this conjecture whenever $\mathcal{H}$ is strictly k-balanced. It would be interesting to obtain more precise results and to confirm the asymptotic value of the constant factor.

The $\mathcal{F}$-free process where $\mathcal{F}$ is a diamond, which is a graph that is not strictly 2-balanced, typically terminates with a final number of edges that has a different exponent for the logarithmic factor compared to Conjecture 3.18.1, see [92]. Hence, for the $\mathcal{F}$-free process as well as the $\mathcal{F}$-removal process, it could be interesting to further investigate the situation for graphs or hypergraphs that are not (strictly) balanced.

In terms of applications, the conjectures above suggest that the $\mathcal{F}$-free process is more suitable for generating dense $\mathcal{F}$-free graphs, however, the $\mathcal{F}$-removal process might prove to be a useful tool for decomposition and packing problems since it carefully constructs a maximal collection of edge-disjoint copies of $\mathcal{F}$. For such applications, we believe that the fact that we do not require the initial hypergraph to be complete might be crucial.

Additionally, as we believe that such an extension could be useful for applications, we remark that directly using Lemma 3.8.1 instead of one of the theorems makes it possible to easily amend our analysis as follows if the goal is to show that the random hypergraphs generated by the process typically exhibit further properties that we did not consider in our analysis. Similarly to how we organized our analysis by using stopping times, one may define a stopping time τ that measures when the desired property is first violated. Then for $\tau^\star$ and $i^\star$ as defined in Lemma 3.8.1, it suffices show that $\mathbb{P}[\tau \leq \tau^\star \wedge i^\star]$ is small as this entails that $\mathbb{P}[\tau \wedge \tau^\star \leq i^\star]$ is small and hence that the process typically runs for at least $i^\star$ steps while maintaining the desired property. For example, it is easy to see that in fact, typically a more precise estimate for the number of copies of $\mathcal{F}$ in every step holds provided that the guarantees concerning the initial hypergraph are more precise. This might be useful for counting the number of choices available for every deletion which can in turn be useful for counting the number of $\mathcal{F}$-packings in a large complete hypergraph. Specifically, instead of only obtaining $\hat{h}^*(i) \pm \zeta(i)^{1+\varepsilon^3}$ as an estimate for the number of copies present after i deletions as in our first part of the proof, it is possible to instead obtain $\hat{h}^*(i) \pm \delta^{-6}\zeta(i)^2$ if a slightly more precise estimate holds for $i = 0$. To obtain this refinement following an approach as mentioned above, it suffices use the same argumentation that proves Lemma 3.5.19 (i) with only minor adaptations.

Declaration of contributions

This thesis consists of the results of two out of the five projects [46, 47, 60–62] I worked on during my time as a PhD student. Chapter 2 closely corresponds to the article [46] about finding matchings that avoid certain edge-sets and the research behind this was conducted in collaboration with Stefan Glock, Felix Joos, Jaehoon Kim and Lyuben Lichev. In many ways, I contributed proof ideas for achieving the goals of the project, I chose the structure of the argumentation, prepared the details and I formulated and wrote the core parts of the manuscript the chapter is based on.

Chapter 3 closely corresponds to the article [61] about the hypergraph removal process and the research behind this was conducted in collaboration with Felix Joos, who suggested working on such random processes. I chose the direction of the research, conceived the ideas for the proofs, chose the argumentation and I formulated and wrote a complete draft including the details of the proofs. This draft eventually evolved into the manuscript the chapter is based on and I again formulated and wrote the core parts of this manuscript.

Chapter 1 also contains several parts and paragraphs that are taken from one of the two articles [46, 61].

Bibliography

[1] M. Ajtai, J. Komlós, J. Pintz, J. Spencer, and E. Szemerédi, *Extremal uncrowded hypergraphs*, J. Combin. Theory Ser. A **32** (1982), 321–335.

[2] N. Alon, J.-H. Kim, and J. Spencer, *Nearly perfect matchings in regular simple hypergraphs*, Israel J. Math. **100** (1997), 171–187.

[3] N. Alon and B. Sudakov, *Disjoint systems*, Random Structures Algorithms **6** (1995), 13–20.

[4] N. Alon and R. Yuster, *On a hypergraph matching problem*, Graphs Combin. **21** (2005), 377–384.

[5] M. Axenovich, Z. Füredi, and D. Mubayi, *On generalized Ramsey theory: the bipartite case*, J. Combin. Theory Ser. B **79** (2000), 66–86.

[6] D. Bal, P. Bennett, E. Heath, and S. Zerbib, *Generalized ramsey numbers of cycles, paths, and hypergraphs*, arXiv:2405.15904 (2024).

[7] B. Barber, D. Kühn, A. Lo, R. Montgomery, and D. Osthus, *Fractional clique decompositions of dense graphs and hypergraphs*, J. Combin. Theory Ser. B **127** (2017), 148–186.

[8] P. Bennett and T. Bohman, *A note on the random greedy independent set algorithm*, Random Structures Algorithms **49** (2016), 479–502.

[9] P. Bennett and T. Bohman, *A natural barrier in random greedy hypergraph matching*, Combin. Probab. Comput. **28** (2019), 816–825.

[10] P. Bennett, R. Cushman, A. Dudek, and P. Prałat, *The Erdős-Gyárfás function* $f(n,4,5) = \frac{5}{6}n + o(n)$*—so Gyárfás was right*, J. Combin. Theory Ser. B **169** (2024), 253–297.

[11] P. Bennett, E. Heath, and S. Zerbib, *Edge-coloring a graph g so that every copy of a graph h has an odd color class*, arXiv:2307.01314 (2023).

[12] T. Bohman, *The triangle-free process*, Adv. Math. **221** (2009), 1653–1677.

[13] T. Bohman, A. Frieze, and E. Lubetzky, *A note on the random greedy triangle-packing algorithm*, J. Comb. **1** (2010), 477–488.

[14] T. Bohman, A. Frieze, and E. Lubetzky, *Random triangle removal*, Adv. Math. **280** (2015), 379–438.

[15] T. Bohman and P. Keevash, *The early evolution of the H-free process*, Invent. Math. **181** (2010), 291–336.

[16] T. Bohman and P. Keevash, *Dynamic concentration of the triangle-free process*, Random Structures Algorithms **58** (2021), 221–293.

[17] T. Bohman and M. Picollelli, *SIR epidemics on random graphs with a fixed degree sequence*, Random Structures Algorithms **41** (2012), 179–214.

[18] T. Bohman and L. Warnke, *Large girth approximate Steiner triple systems*, J. Lond. Math. Soc. (2) **100** (2019), 895–913.

[19] B. Bollobás, *To prove and conjecture: Paul Erdős and his mathematics*, Amer. Math. Monthly **105** (1998), 209–237.

[20] B. Bollobás, *Random graphs*, vol. 73 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, second ed. (2001).

[21] B. Bollobás and O. Riordan, *Constrained graph processes*, Electron. J. Combin. **7** (2000), Research Paper 18, 20.

[22] W. G. Brown, P. Erdős, and V. T. Sós, *Some extremal problems on r-graphs*, in *New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich., 1971)*, Academic Press, New York-London (1973), 53–63.

[23] M. Delcourt and L. Postle, *Finding an almost perfect matching in a hypergraph avoiding forbidden submatchings*, arXiv:2204.08981 (2024).

[24] M. Delcourt and L. Postle, *The limit in the $(k+2,k)$-problem of Brown, Erdős and Sós exists for all $k \geq 2$*, Proc. Amer. Math. Soc. **152** (2024), 1881–1891.

[25] M. Delcourt and L. Postle, *Proof of the high girth existence conjecture via refined absorption*, arXiv:2402.17856 (2024).

[26] M. Delcourt and L. Postle, *Refined absorption: A new proof of the existence conjecture*, arXiv:2402.17855 (2024).

[27] S. Ehard, S. Glock, and F. Joos, *Pseudorandom hypergraph matchings*, Combin. Probab. Comput. **29** (2020), 868–885.

[28] D. Ellis and N. Linial, *On regular hypergraphs of high girth*, Electron. J. Combin. **21** (2014), Paper 1.54, 17.

[29] P. Erdős, *A problem on independent r-tuples*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **8** (1965), 93–95.

[30] P. Erdős, *Problems and results in combinatorial analysis*, in *Colloquio Internazionale sulle Teorie Combinatorie*, Accad. Naz. Lincei, Rome (1976), 3–17.

[31] P. Erdős and H. Hanani, *On a limit theorem in combinatorial analysis*, Publ. Math. Debrecen **10** (1963), 10–13.

[32] P. Erdős, S. Suen, and P. Winkler, *On the size of a random maximal graph*, in *Proceedings of the Sixth International Seminar on Random Graphs and Probabilistic Methods in Combinatorics and Computer Science, "Random Graphs '93" (Poznań, 1993)*, vol. 6 (1995), 309–318.

[33] P. Erdős and A. Gyárfás, *A variant of the classical Ramsey problem*, Combinatorica **17** (1997), 459–467.

[34] G. Fiz Pontiveros, S. Griffiths, and R. Morris, *The triangle-free process and the Ramsey number $R(3,k)$*, Mem. Amer. Math. Soc. **263** (2020), v+125.

[35] K. Ford, B. Green, S. Konyagin, J. Maynard, and T. Tao, *Long gaps between primes*, J. Amer. Math. Soc. **31** (2018), 65–105.

[36] P. Frankl and Z. Füredi, *Colored packing of sets*, in *Combinatorial design theory*, vol. 149 of *North-Holland Math. Stud.*, North-Holland, Amsterdam (1987), 165–177.

[37] P. Frankl and V. Rödl, *Near perfect coverings in graphs and hypergraphs*, European J. Combin. **6** (1985), 317–326.

[38] P. Frankl and N. Tokushige, *Invitation to intersection problems for finite sets*, J. Combin. Theory Ser. A **144** (2016), 157–211.

[39] D. A. Freedman, *On tail probabilities for martingales*, Ann. Probability **3** (1975), 100–118.

[40] A. Frieze and M. Karoński, *Introduction to random graphs*, Cambridge University Press, Cambridge (2016).

[41] Z. Füredi, *2-cancellative hypergraphs and codes*, Combin. Probab. Comput. **21** (2012), 159–177.

[42] Z. Füredi and M. Ruszinkó, *Uniform hypergraphs containing no grids*, Adv. Math. **240** (2013), 302–324.

[43] S. Gerke and T. Makai, *No dense subgraphs appear in the triangle-free graph process*, Electron. J. Combin. **18** (2011), Paper 168, 7.

[44] L. Gishboliner and A. Shapira, *Constructing dense grid-free linear 3-graphs*, Proc. Amer. Math. Soc. **150** (2022), 69–74.

[45] S. Glock, *Triple systems with no three triples spanning at most five points*, Bull. Lond. Math. Soc. **51** (2019), 230–236.

[46] S. Glock, F. Joos, J. Kim, M. Kühn, and L. Lichev, *Conflict-free hypergraph matchings*, J. Lond. Math. Soc. (2) **109** (2024), Paper No. e12899, 78.

[47] S. Glock, F. Joos, J. Kim, M. Kühn, L. Lichev, and O. Pikhurko, *On the* $(6,4)$*-problem of Brown, Erdős, and Sós*, Proc. Amer. Math. Soc. Ser. B **11** (2024), 173–186.

[48] S. Glock, J. Kim, L. Lichev, O. Pikhurko, and S. Sun, *On the* $(k+2,k)$*-problem of Brown, Erdős and Sós for* $k=5,6,7$, Canad. J. Math. (2025), 1–43.

[49] S. Glock, D. Kühn, A. Lo, and D. Osthus, *On a conjecture of Erdős on locally sparse Steiner triple systems*, Combinatorica **40** (2020), 363–403.

[50] S. Glock, D. Kühn, A. Lo, and D. Osthus, *The existence of designs via iterative absorption: hypergraph* F*-designs for arbitrary* F, Mem. Amer. Math. Soc. **284** (2023), v+131.

[51] S. Glock, D. Kühn, and D. Osthus, *Extremal aspects of graph and hypergraph decomposition problems*, in *Surveys in combinatorics 2021*, vol. 470 of *London Math. Soc. Lecture Note Ser.*, Cambridge Univ. Press, Cambridge (2021), 235–265.

[52] E. Gomez-Leos, E. Heath, A. Parker, C. Schwieder, and S. Zerbib, *New bounds on the generalized ramsey number* $f(n,5,8)$, arXiv:2308.16365 (2024).

[53] D. A. Grable, *On random greedy triangle packing*, Electron. J. Combin. **4** (1997), Research Paper 11, 19.

[54] P. Haxell, *Independent transversals and hypergraph matchings—an elementary approach*, in *Recent trends in combinatorics*, vol. 159 of *IMA Vol. Math. Appl.*, Springer, Cham (2016), 215–233.

[55] W. Hoeffding, *Probability inequalities for sums of bounded random variables*, J. Amer. Statist. Assoc. **58** (1963), 13–30.

[56] J. Hofstad, *Behaviour of the minimum degree throughout the d-process*, Combin. Probab. Comput. **33** (2024), 564–582.

[57] S. Janson, *Poisson approximation for large deviations*, Random Structures Algorithms **1** (1990), 221–229.

[58] S. Janson, T. Łuczak, and A. Rucinski, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York (2000).

[59] S. Janson and A. Ruciński, *The deletion method for upper tail estimates*, Combinatorica **24** (2004), 615–640.

[60] F. Joos and M. Kühn, *Fractional cycle decompositions in hypergraphs*, Random Structures Algorithms **61** (2022), 425–443.

[61] F. Joos and M. Kühn, *The hypergraph removal process*, arXiv:2412.15039 (2024).

[62] F. Joos, M. Kühn, and B. Schülke, *Decomposing hypergraphs into cycle factors*, J. Combin. Theory Ser. B **158** (2023), 136–175.

[63] F. Joos and D. Mubayi, *Ramsey theory constructions from hypergraph matchings*, Proc. Amer. Math. Soc. (to appear).

[64] D. Y. Kang, T. Kelly, D. Kühn, A. Methuku, and D. Osthus, *Graph and hypergraph colouring via nibble methods: a survey*, in *European Congress of Mathematics*, EMS Press, Berlin (2023), 771–823.

[65] D. Y. Kang, T. Kelly, D. Kühn, A. Methuku, and D. Osthus, *A proof of the Erdős-Faber-Lovász conjecture*, Ann. of Math. (2) **198** (2023), 537–618.

[66] D. Y. Kang, D. Kühn, A. Methuku, and D. Osthus, *New bounds on the size of nearly perfect matchings in almost regular hypergraphs*, J. Lond. Math. Soc. (2) **108** (2023), 1701–1746.

[67] R. M. Karp, *Reducibility among combinatorial problems*, in *Complexity of computer computations (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972)*, The IBM Research Symposia Series, Plenum, New York-London (1972), 85–103.

[68] A. D. Keedwell and J. Dénes, *Latin squares and their applications*, Elsevier/North-Holland, Amsterdam, second ed. (2015), with a foreword to the previous edition by Paul Erdős.

[69] P. Keevash, *The existence of designs*, arXiv:1401.3665 (2014).

[70] P. Keevash, *Hypergraph matchings and designs*, in *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. IV. Invited lectures*, World Sci. Publ., Hackensack, NJ (2018), 3113–3135.

[71] P. Keevash, *A short proof of the existence of designs*, arXiv:2411.18291 (2024).

[72] P. Keevash and J. Long, *The Brown–Erdős–Sós conjecture for hypergraphs of large uniformity*, Proc. Amer. Math. Soc. (to appear).

[73] P. Keevash and K. Staden, *Ringel's tree packing conjecture in quasirandom graphs*, J. Eur. Math. Soc. (to appear).

[74] T. P. Kirkman, *On a problem in combinatorics*, Cambridge and Dublin Math. J. **2** (1847), 191–204.

[75] M. Krivelevich, M. Kwan, P.-S. Loh, and B. Sudakov, *The random k-matching-free process*, Random Structures Algorithms **53** (2018), 692–716.

[76] D. Kühn and D. Osthus, *Embedding large subgraphs into dense graphs*, in *Surveys in combinatorics 2009*, vol. 365 of *London Math. Soc. Lecture Note Ser.*, Cambridge Univ. Press, Cambridge (2009), 137–167.

[77] D. Kühn and D. Osthus, *Hamilton cycles in graphs and hypergraphs: an extremal perspective*, in *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. IV*, Kyung Moon Sa, Seoul (2014), 381–406.

[78] D. Kühn, D. Osthus, and A. Taylor, *On the random greedy F-free hypergraph process*, SIAM J. Discrete Math. **30** (2016), 1343–1350.

[79] M. Kwan, A. Sah, M. Sawhney, and M. Simkin, *Substructures in Latin squares*, Israel J. Math. **256** (2023), 363–416.

[80] M. Kwan, A. Sah, M. Sawhney, and M. Simkin, *High-girth Steiner triple systems*, Ann. of Math. (2) **200** (2024), 1059–1156.

[81] A. Lane and N. Morrison, *Generalized ramsey numbers via conflict-free hypergraph matchings*, arXiv:2405.16653 (2024).

[82] L. C. Lau, R. Ravi, and M. Singh, *Iterative methods in combinatorial optimization*, Cambridge Texts in Applied Mathematics, Cambridge University Press, New York (2011).

[83] H. Lefmann, K. T. Phelps, and V. Rödl, *Extremal problems for triple systems*, J. Combin. Des. **1** (1993), 379–394.

[84] N. Linial and Z. Luria, *An upper bound on the number of high-dimensional permutations*, Combinatorica **34** (2014), 471–486.

[85] N. Linial and Z. Luria, *Discrepancy of high-dimensional permutations*, Discrete Anal. (2016), Paper No. 11, 8.

[86] N. Linial and M. Simkin, *Monotone subsequences in high-dimensional permutations*, Combin. Probab. Comput. **27** (2018), 69–83.

[87] L. Lovász and M. D. Plummer, *Matching theory*, AMS Chelsea Publishing, Providence, RI (2009), corrected reprint of the 1986 original.

[88] Z. Luria, *New bounds on the number of n-queens configurations*, arXiv:1705.05225 (2017).

[89] C. S. J. Nash-Williams, *An unsolved problem concerning decomposition of graphs into triangles*, in P. Erdős, A. Rényi, and V. Sós, eds., *Combinatorial Theory and its Applications III*, North Holland (1970), 1179–1183.

[90] D. Osthus and A. Taraz, *Random maximal H-free graphs*, Random Structures Algorithms **18** (2001), 61–82.

[91] M. E. Picollelli, *The final size of the C_4-free process*, Combin. Probab. Comput. **20** (2011), 939–955.

[92] M. E. Picollelli, *The diamond-free process*, Random Structures Algorithms **45** (2014), 513–551.

[93] M. E. Picollelli, *The final size of the C_ℓ-free process*, SIAM J. Discrete Math. **28** (2014), 1276–1305.

[94] N. Pippenger and J. Spencer, *Asymptotic behavior of the chromatic index for hypergraphs*, J. Combin. Theory Ser. A **51** (1989), 24–42.

[95] F. P. Ramsey, *On a Problem of Formal Logic*, Proc. London Math. Soc. (2) **30** (1929), 264–286.

[96] V. Rödl, *On a packing and covering problem*, European J. Combin. **6** (1985), 69–78.

[97] V. Rödl and A. Ruciński, *Dirac-type questions for hypergraphs—a survey (or more problems for Endre to solve)*, in *An irregular mind*, vol. 21 of *Bolyai Soc. Math. Stud.*, János Bolyai Math. Soc., Budapest (2010), 561–590.

[98] V. Rödl and L. Thoma, *Asymptotic packing and the random greedy algorithm*, Random Structures Algorithms **8** (1996), 161–177.

[99] A. Ruciński and N. C. Wormald, *Random graph processes with degree restrictions*, Combin. Probab. Comput. **1** (1992), 169–180.

[100] H. J. Ryser, *Neuere Probleme der Kombinatorik*, in *Vorträge über Kombinatorik*, Oberwolfach, MFO, Germany (1967), 69–91.

[101] A. Sah, M. Sawhney, and M. Simkin, *Threshold for Steiner triple systems*, Geom. Funct. Anal. **33** (2023), 1141–1172.

[102] A. Schrijver, *Combinatorial optimization: Polyhedra and efficiency*, vol. 24 of *Algorithms and Combinatorics*, Springer-Verlag, Berlin (2003).

[103] C. Shangguan and I. Tamo, *Degenerate Turán densities of sparse hypergraphs*, J. Combin. Theory Ser. A **173** (2020), 105228, 25.

[104] A. Sidorenko, *Approximate Steiner $(r-1, r, n)$-systems without three blocks on $r+2$ points*, J. Combin. Des. **28** (2020), 144–148.

[105] J. Spencer, *Asymptotic packing via a branching process*, Random Structures Algorithms **7** (1995), 167–172.

[106] J. Spencer, *Maximal trianglefree graphs and Ramsey $R(3, k)$* (1995), unpublished manuscript.

[107] J. Spencer, *The strange logic of random graphs*, vol. 22 of *Algorithms and Combinatorics*, Springer-Verlag, Berlin (2001).

[108] A. Telcs, N. Wormald, and S. Zhou, *Hamiltonicity of random graphs produced by 2-processes*, Random Structures Algorithms **31** (2007), 450–481.

[109] V. H. Vu, *New bounds on nearly perfect matchings in hypergraphs: higher codegrees do help*, Random Structures Algorithms **17** (2000), 29–63.

[110] L. Warnke, *Dense subgraphs in the H-free process*, Discrete Math. **311** (2011), 2703–2707.

[111] L. Warnke, *The C_ℓ-free process*, Random Structures Algorithms **44** (2014), 490–526.

[112] L. Warnke, *When does the K_4-free process stop?*, Random Structures Algorithms **44** (2014), 355–397.

[113] N. C. Wormald, *The differential equation method for random graph processes and greedy algorithms*, In M. Karoński and H. Prömel, Eds., Lectures on Approximation and Randomized Algorithms (1999), 73–155.

[114] R. Yuster, *Combinatorial and computational aspects of graph packing and graph decomposition*, Computer Science Review **1** (2007), 12–26.

[115] Y. Zhao, *Recent advances on Dirac-type problems for hypergraphs*, in *Recent trends in combinatorics*, vol. 159 of *IMA Vol. Math. Appl.*, Springer, Cham (2016), 145–165.

[116] Y. Zhao, *Extremal regular graphs: independent sets and graph homomorphisms*, Amer. Math. Monthly **124** (2017), 827–843.