

Dissertation

submitted to the

Combined Faculty of Mathematics, Engineering and Natural Sciences

of Heidelberg University, Germany

for the degree of

Doctor of Natural Sciences

Put forward by

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born in: **Hamburg, Germany**

Oral examination: **09.07.2025**

Strongly correlated quantum fields in and out of equilibrium

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This dissertation investigates the emergence of collective phenomena and universal patterns in strongly correlated quantum many-body systems, both in and out of equilibrium. We propose an asymptotically free scalar field theory in four dimensions and solve the coupling renormalization group flow to the strongly coupled infrared using a large- N resummation based on a global tensor symmetry. This provides a rare example of a non-perturbatively controlled interacting theory in four dimensions. We then demonstrate the formation of stable droplets in bosonic atom-molecule systems near narrow Feshbach resonances, stabilized by coherent interconversion dynamics, enabling a controlled study of the crossover from few-body clusters to many-body phases. Far from equilibrium, we analyze the emergence of universal scaling associated with nonthermal attractors in quantum systems ranging from cold atoms to the quark-gluon plasma. We demonstrate dynamical self-stabilization via a universal scaling instability, despite unstable perturbations from interactions between emergent quasiparticle states. We explain how systems approach these attractors through power-law resummation in time, which was recently confirmed in a cold atom experiment, and extend this so-called prescaling to gauge-invariant observables in Yang-Mills theories. Finally, we propose and demonstrate the emergence of hydrodynamic excitations near nonthermal attractors, characterized by time-dependent transport coefficients like shear viscosity.

Stark korrelierte Quantenfelder im und außerhalb des Gleichgewichts

Diese Dissertation untersucht die Entstehung kollektiver Phänomene und universeller Muster in stark korrelierten Quanten-Vielteilchensystemen, sowohl im als auch außerhalb des Gleichgewichts. Wir schlagen dafür eine asymptotisch freie skalare Feldtheorie in vier Dimensionen vor und lösen den Kopplungsrenormierungsfluss im stark gekoppelten Infrarotbereich mittels einer Groß- N -Resummation basierend auf globaler Tensorsymmetrie. Dies stellt ein seltenes Beispiel einer nichtperturbativ kontrollierten wechselwirkenden Theorie in vier Dimensionen dar. Anschließend demonstrieren wir die Bildung stabiler Tropfen in bosonischen Atom-Molekül-Systemen nahe einer schmalen Feshbach-Resonanz, stabilisiert durch kohärente Umwandlungsdynamik, was eine kontrollierte Untersuchung des Übergangs von wenigen Teilchenclustern zu kollektiven Vielteilchenphasen ermöglicht. Fern des Gleichgewichts analysieren wir das Auftreten universeller Skalierungsphänomene in Verbindung mit nichtthermischen Attraktoren in Quantensystemen von kalten Atomen bis zum Quark-Gluon-Plasma. Wir demonstrieren, dass dynamische Selbststabilisierung über eine universelle Skalierungsinstabilität stattfindet, trotz der Existenz instabiler Störungen durch Wechselwirkungen zwischen emergenten Quasiteilchenzuständen. Wir erläutern, wie Quantensysteme diese Attraktoren durch eine zeitliche Potenzreihensummation erreichen, was kürzlich in einem Kaltatomexperiment bestätigt wurde, und erweitern dieses sogenannte Pre-scaling auf eichinvariante Observablen in Yang-Mills-Theorien. Abschließend schlagen wir die Existenz hydrodynamischer Anregungen nahe nichtthermischer Attraktoren vor, charakterisiert durch zeitabhängige Transportkoeffizienten wie die Scherviskosität.

Für Evi und Helmut

List of publications

This cumulative dissertation is based on five peer-reviewed publications and two recently published preprints. The chapters present these works in a logical rather than chronological order. Below, the associated publications are listed in the order in which the corresponding research was conducted. Where applicable, main authorship is indicated by a superscript star, and alphabetical author listing – as is customary in high-energy physics – is noted.

Peer-reviewed

- [1] Thimo Preis*, Michal P. Heller, and Jürgen Berges. *Stable and unstable perturbations in universal scaling phenomena far from equilibrium*. **Phys. Rev. Lett.** 130, 031602. January 20, 2023. DOI: 10.1103/PhysRevLett.130.031602.
- [2] Jürgen Berges, Razvan Gurau, and Thimo Preis* (alphabetical). *Asymptotic freedom in a strongly interacting scalar quantum field theory in four Euclidean dimensions*. **Phys. Rev. D** 108, 016019. July 31, 2023. DOI: 10.1103/PhysRevD.108.016019.
- [3] Michal P. Heller, Aleksas Mazeliauskas, and Thimo Preis* (alphabetical). *Prescaling relaxation to nonthermal attractors*. **Phys. Rev. Lett.** 132, 071602. February 15, 2024. DOI: 10.1103/PhysRevLett.132.071602.
- [4] Jürgen Berges, Razvan Gurau, Hannes Keppler, and Thimo Preis* (alphabetical). *Coupling renormalization flow in the strongly interacting regime of an asymptotically free quantum field theory in four dimensions*. **Phys. Rev. D** 110, 036007. August 9, 2024. DOI: 10.1103/PhysRevD.110.036007.
- [5] Daniel Spitz*, Kirill Boguslavski, and Thimo Preis. *Condensation and prescaling in spatial Polyakov loop correlations far from equilibrium*. **Phys. Rev. D** 111, 096006. May 6, 2025. DOI: 10.1103/PhysRevD.111.096006.

Recent preprints

- [6] Ke Wang*, Thimo Preis, and Dam Thanh Son. *Droplets of Bosons at a Narrow Resonance*. July 3, 2024. DOI: 10.48550/arXiv.2407.03102.
- [7] Jürgen Berges, Gabriel S. Denicol, Michal P. Heller, and Thimo Preis* (alphabetical). *Far from equilibrium hydrodynamics of nonthermal fixed points*. April 26, 2025. DOI: 10.48550/arXiv.2504.18754.

Wherever content from these articles has been used in this thesis, it is clearly indicated at the beginning of each chapter, along with a summary of my contributions to the collaborative publications.

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1 Introduction and overview

Strongly correlated systems play a central role across a broad range of disciplines, from high-energy and condensed matter physics to statistical mechanics and ultracold atomic gases. On a macroscopic level, strong correlations can give rise to emergent collective phenomena, critical behavior, and universal scaling laws [8]. However, their microscopic origins can be diverse. These may include strong interactions between the dominant degrees of freedom [9], the absence of characteristic length scales – as seen in turbulent [10] and critical systems [11] – or the proliferation of excitations leading to large statistical fluctuations [12]. The study of such systems often presents formidable challenges, where in particular the absence of a perturbative expansion parameter can render first-principles approaches intractable. Using novel non-perturbative expansion techniques and large-scale numerical simulations, in this dissertation we study how such strong correlations and associated collective behaviors emerge in quantum dynamics from first principles.

In many strongly correlated quantum many-body systems, collective phenomena emerge that cannot be understood by analyzing individual constituents in isolation. Effects such as spontaneous symmetry breaking, phase transitions, and universal scaling behavior arise from the interplay of a large number of degrees of freedom and typically require a statistical or field-theoretic treatment. In equilibrium, these phenomena are well described within statistical ensemble frameworks, especially near critical points where fluctuations dominate. In such regimes, the macroscopic behavior departs qualitatively from the microscopic dynamics of the underlying constituents [13], reflecting the emergence of effective degrees of freedom and large-scale organizing principles. Similar mechanisms govern far-from-equilibrium systems, where self-organized attractor dynamics and scaling behavior can emerge without fine-tuning. Across both settings, a central challenge is to understand how universal collective behavior arises from microscopic quantum dynamics.

While strongly correlated quantum many-body systems are often difficult to study from first principles, the concept of universality offers a powerful framework for understanding their emergent macroscopic behavior. Distinct many-body systems, which may differ substantially at the microscopic level, can exhibit remarkably similar dynamics on large scales. One important example arises in regimes of asymptotically strong interactions, where universal behavior can emerge in the form of bounds on transport coefficients and thermalization timescales, as conjectured in holographic models [14, 15]. Notably, the holographic lower bound on the shear-viscosity-to-entropy-density ratio shows striking agreement with experimental observations in two seemingly unrelated systems: the quark-gluon plasma produced in relativistic heavy-ion collisions and the unitary Fermi gas near a Feshbach resonance [16, 17]. Despite differing in microscopic constituents and energy scales, both exhibit transport behavior near the holographic strong-coupling limit

– highlighting the unifying role of universality in describing strongly interacting matter. A paradigmatic equilibrium example is the divergence of the correlation length at a second-order phase transition, giving rise to universal critical phenomena [18]. Universality also extends far beyond equilibrium, with recent observations of universal scaling behavior in far-from-equilibrium systems that evolve toward nonthermal fixed points or nonthermal attractors [12]. These arise in systems driven far from equilibrium by large initial energy densities or quenches, and are characterized by enhanced statistical fluctuations and self-similar evolution in time. Such phenomena are relevant in a variety of physical contexts, including inflaton dynamics in the early universe, post-quench evolution in ultracold atomic gases, and the nonequilibrium dynamics of gauge fields in heavy-ion collisions. In contrast to universal scaling near thermal equilibrium – typically requiring fine-tuning of a relevant parameter, such as temperature or coupling strength – these nonequilibrium scaling phenomena have been observed to arise from a wide range of initial conditions, as illustrated by comparisons in recent experimental realizations [19–21]. In both cases, thermal and nonthermal scaling, correlation functions display power-law behavior characterized by universal scaling exponents and scaling functions that, aside from nonuniversal normalizations, are identical across microscopically distinct systems within the same universality class. This correspondence allows one to extract universal properties of an otherwise intractable system by studying a simpler or more experimentally accessible representative of the same universality class.

Establishing an explicit connection between microscopic dynamics and emergent macroscopic behavior remains a central challenge in theoretical physics. Even attempting to bridge this gap offers profound insight into the emergence of complexity from fundamentally simple laws. Understanding how physical properties evolve across different length, energy, or time scales constitutes therefore a fundamental problem in physics, and more broadly in the natural sciences. This conceptual framework is formalized by the renormalization group (RG). Given the microscopic description of a system at one scale, the RG provides a systematic procedure for determining how observables change under coarse-graining to other scales. Fixed points of the RG flow play a central role in this picture. At these points, a system becomes invariant under changes in the length scale at which it is probed, reflecting the absence of any intrinsic scale. A well-known instance is the second-order phase transition, where the divergence of the correlation length signals the onset of scale invariance and long-range correlations. Nonthermal fixed points offer a far-from-equilibrium perspective on the concept of RG fixed points, where time-evolving systems display universal scaling behavior during transient evolution before thermalization. Similar to equilibrium critical phenomena, correlation functions near a nonthermal fixed point evolve self-similarly and observables become insensitive to microscopic details. Physically, nonthermal attractors further provide insight into how macroscopic thermalization can emerge from unitary quantum evolution. While unitary quantum dynamics is fundamentally reversible, attractor behavior leads to an effective loss of memory of the initial state, driven by dynamical coarse-graining and, in the case of nonthermal attractors, by the proliferation of degrees of freedom. In both equilibrium and nonequilibrium contexts, microscopically distinct systems can evolve along RG trajectories that flow toward the same fixed point, illustrating how universality governs diverse physical settings.

This explains why systems as varied as spin chains, superfluids, and quark-gluon plasmas can exhibit the same universal behavior near strongly correlated regimes, despite differing dramatically in their underlying microscopic constituents.

1.1 Overview

This dissertation explores various manifestations of strongly correlated quantum many-body systems, both in and out of equilibrium. Each chapter focuses on a specific phenomenon, examining the emergence of collective behavior from first-principles approaches. We also provide an outlook on the experimental realizations of the strongly correlated phenomena discussed in this thesis, particularly through ultra-cold atom platforms, in Ch. 5.

Throughout this thesis, apart from Ch. 3, we use natural units where $\hbar = c = k_B = 1$ and a mostly-minus metric signature.

1.1.1 Strong interactions with tensor fields in four dimensions

A fundamental component of the Standard Model of particle physics is the class of Yang-Mills theories, which are quantum field theories characterized by local gauge symmetries. In non-Abelian gauge theories of this type, the gauge interactions become weaker at smaller distance scales or equivalently, at higher energies [22, 23], a phenomenon known as asymptotic freedom. Conversely, at larger distance scales or lower energies, asymptotic freedom implies an increase in interaction strength, resulting in the breakdown of perturbation theory. The paradigmatic example within the Standard Model is quantum chromodynamics (QCD), where asymptotic freedom leads to the confinement of quarks and gluons into hadrons at low energies. Asymptotic freedom is often posed to be a crucial property when discussing fundamental theories of nature, as the weakening of the interaction between its constituents at larger energies allows for such theories to be renormalized and well-defined over all distance scales. Asymptotic freedom has conventionally been understood to occur in four dimensions only in Yang-Mills theories [24], where, however, an analytically tractable resummation for the strongly coupled infrared regime is currently out of reach [25, 26]. As a consequence, a complete analytic understanding of confinement remains one of the central open problems in theoretical physics.

By contrast, scalar quantum field theories in four dimensions are typically believed to exhibit triviality [27, 28], whereby the interaction strength decreases toward lower energies, effectively rendering the theory noninteracting at large distances. In contrast to asymptotically free theories, theories exhibiting triviality – originally encountered for quantum electrodynamics due to its Abelian gauge symmetry – have historically been understood as deficient or, more recently, as effective field theories valid below a cutoff scale [29].

In the search for a strongly-interacting quantum field theory in four dimensions that is analytically tractable, in Ch. 2 we present a scalar quantum field theory in four dimensions

that exhibits asymptotic freedom and possesses a renormalization group flow that remains well-defined in the infrared. The theory is based on fields exhibiting a global tensor symmetry – a so-called tensor field theory [30]. This symmetry facilitates a non-perturbative but analytically tractable resummation [31, 32], that provides a novel setting in which the strongly coupled regime becomes analytically accessible from first principles. Based on [2, 4], in Sec. 2.2 and Sec. 2.3 we determine the coupling renormalization flow from the ultraviolet to the infrared. This construction offers a controlled framework for investigating strong coupling dynamics beyond the scope of perturbative methods and provides new insights into nonperturbative quantum field theory from first principles.

1.1.2 Self-bound droplets in cold atom systems

Ch. 3 investigates the formation and properties of self-bound droplets in bosonic systems. Droplets are dilute, macroscopically coherent clusters stabilized by interactions, yet capable of existing without external confinement. Recent theoretical and experimental developments have demonstrated in particular the realization of quantum droplets in ultracold Bose-Bose mixtures, which are stabilized by beyond-mean-field quantum fluctuations [33, 34]. This represents an example of nontrivial many-body behavior emerging in nonrelativistic quantum field theory, where interactions and fluctuations combine to support self-bound states without external trapping.

In Sec. 3.2, based on the work of [6], we analyze the formation of droplets in a different setting: bosonic atom-molecule mixtures near a narrow Feshbach resonance. Narrow resonances are particularly suited for exploring nonperturbative quantum many-body physics, as they allow controlled expansions in the resonance width [35] and access to regimes where standard weak-coupling techniques fail. These systems provide, for example, insight into the crossover between Bose-Einstein condensation (BEC) and Bardeen-Cooper-Schrieffer (BCS) superfluidity in fermionic gases, with the unitary Fermi gas at the core of this transition. As interactions approach the unitary point of divergent interaction strength, the unitary Fermi gas exhibits universal transport properties, such as shear viscosity, which approach bounds set by holographic theories [16, 17].

In our study, we consider bosons at fixed particle number and find that the interconversion between atoms and molecules – mediated by a magnetic Feshbach coupling – is sufficient to stabilize bound clusters even at the mean-field level. Unlike previous scenarios where quantum droplets rely on fluctuation-induced stabilization, the droplets here arise from coherent atom-molecule dynamics. This makes them analytically tractable and positions them as an ideal laboratory for exploring the transition from few-body clusters to self-bound many-body states in a controlled setting.

1.1.3 Large statistical fluctuations in nonthermal attractors

Strong correlations in quantum many-body systems can also emerge far from equilibrium, particularly in systems associated with nonthermal attractors. Following the proliferation of statistical fluctuations, these systems exhibit self-similar scaling in space and

time, characterized by universal scaling exponents and scaling functions [12]. These far-from-equilibrium attractors represent quasi-stationary states of the underlying quantum dynamics, providing a far-from-equilibrium mechanism for the loss of sensitivity on initial conditions in the process of thermalization. The works compiled in Ch. 4, based on [1, 3, 5, 7], study how and why such self-organized attractor behavior emerges in isolated quantum many-body systems ranging from ultracold atoms to the quark-gluon plasma.

Sec. 4.2 analyzes the stability properties of nonthermal fixed points under perturbations of the scaling solution. Despite the attractor nature of these states, we detect the presence of unstable perturbations that evolve away from the scaling solution. Nevertheless, through a complex interplay of emergent quasiparticles – not present in thermal equilibrium – and self-similar scaling behavior, the system remains dynamically attracted to the scaling solution. This demonstrates the robustness and self-organization of nonthermal attractors.

As a system enters the basin of attraction of a nonthermal fixed point, its correlation functions have been observed to assume the shape of the universal scaling functions, even while the scaling exponents remain time-dependent [36]. This intermediate regime, referred to as prescaling, is addressed in Sec. 4.3. There, we derive the analytic form of the time-dependent scaling exponents from the underlying dynamics and demonstrate the emergence of prescaling behavior in both ultracold atomic and high-energy systems. This analytical prescaling solution has recently been confirmed in experimental realization of nonthermal attractors with cold atoms [21]. Interpreting the nonthermal attractor solutions as dynamical analogs of RG fixed points, prescaling can be viewed as describing the approach along a universal flow trajectory in theory space.

We confirm the presence of prescaling in high-energy gauge theories in Sec. 4.4 by analyzing gauge-invariant correlation functions of global observables. Investigating these observables provides a crucial foundation for exploring non-Abelian gauge theories at large distances, particularly in the vicinity of nonthermal attractors, and for understanding the corresponding low-energy effective degrees of freedom. These results further substantiate the universality of prescaling phenomena across a wide class of far-from-equilibrium systems, reinforcing the robustness of the concept of universality in describing quantum dynamics.

Altogether, Secs. 4.2, 4.3, and 4.4 investigate how and why nonthermal attractors emerge in systems governed by dynamics that preserve the attractor's underlying symmetries. This leads to a natural, yet so far unanswered, question regarding the behavior of nonthermal attractors under external perturbations that break these symmetries. To address this, Sec. 4.5 introduces a novel far-from-equilibrium hydrodynamic framework that describes the evolution of spatial inhomogeneities and anisotropies around the otherwise spatially homogeneous and isotropic scaling background. The dynamics within this framework are characterized by time-dependent, far-from-equilibrium generalizations of transport coefficients – most notably, shear viscosity – providing universal ratios that enable meaningful comparisons with near-equilibrium hydrodynamics and thermal attractors.

2 Strong interactions in tensor field theories

In this chapter, we propose an asymptotically free scalar quantum field theory in four dimensions that admits an analytically tractable yet non-trivial large- N resummation, enabled by a global tensor symmetry. Sec. 2.1 introduces the foundational concepts of renormalization theory, the model, and its large- N limit. The model is formulated and analyzed perturbatively in Sec. 2.2, and subsequently solved across length scales – from ultraviolet to infrared – using the tensor large- N resummation in Sec. 2.3. Conclusions are provided in Sec. 2.4, with additional technical details included in App. A.

This chapter is based on the following publications:

- [2] Jürgen Berges, Razvan Gurau, and Thimo Preis* (alphabetical). *Asymptotic freedom in a strongly interacting scalar quantum field theory in four Euclidean dimensions*. **Phys. Rev. D** 108, 016019. July 31, 2023. DOI: 10.1103/PhysRevD.108.016019.

I was the main author of this work and initiated the idea of exploring asymptotic freedom in tensor field theories in 3+1 dimensions through the use of a complex coupling. While all co-authors contributed to the writing of the manuscript, I played a central role in structuring and phrasing the text, carried out the analysis of the perturbative beta functions, and created the figures.

- [4] Jürgen Berges, Razvan Gurau, Hannes Keppler, and Thimo Preis* (alphabetical). *Coupling renormalization flow in the strongly interacting regime of an asymptotically free quantum field theory in four dimensions*. **Phys. Rev. D** 110, 036007. August 9, 2024. DOI: 10.1103/PhysRevD.110.036007.

The original idea was developed jointly by J. Berges, R. Gurau, and myself. I wrote the initial version of the numerical code, which was later substantially extended by H. Keppler; this final version was used to solve the equations. H. Keppler and I contributed equally to the data analysis, and I created all figures for the publication. While sections V, VI, VII, and the Appendix were initially written by H. Keppler, I proposed and drafted sections I–IV, including the Introduction and Conclusions. All authors contributed to the manuscript through corrections and suggestions.

Additional material not included in [2, 4] is presented in App. A.

2.1 Theoretical background

In this section, we provide theoretical background for the discussion of renormalization group flow in a tensor field theory as presented in Sec. 2.2 and Sec. 2.3.

2.1.1 Renormalization theory

The modern understanding of renormalization is rooted in the insight that the physical behavior of a system fundamentally depends on the scale at which it is probed. Physical phenomena at different length or energy scales can exhibit qualitatively distinct behavior, even when arising from the same underlying microscopic dynamics.

The Wilsonian framework and effective field theories

In the Wilsonian framework, quantum field theories are regarded as effective descriptions of nature, valid up to a certain ultraviolet (UV) cutoff scale, denoted Λ . Beyond this scale, the field-theoretic description may cease to accurately capture the relevant degrees of freedom or interactions. The renormalization group (RG) provides a systematic procedure for understanding how the parameters of a theory, such as masses and couplings, evolve as the energy scale is varied from Λ down to lower energies. As one integrates out quantum fluctuations with momenta between Λ and some lower scale k , one obtains an effective action Γ_k that captures the physics with typical energies below k or typical distances larger than $1/k$. The Wilsonian RG thus describes a continuous interpolation between the microscopic theory defined at Λ – represented by the microscopic classical (bare) action $\Gamma_\Lambda \simeq S$ – and the long-distance, macroscopic physics described by $\Gamma_{k \rightarrow 0} = \Gamma$.

All operators consistent with the symmetries of the theory are generally included in the effective action Γ_k , even if they were not present at the UV scale. As the scale k is lowered, the couplings of these operators evolve according to the RG flow. Fixed points of the RG flow correspond to theories that are scale-invariant. At such points, the couplings do not change with scale. Near a fixed point, operators can be classified according to how their couplings behave under infinitesimal departures from scale invariance:

1. Relevant operators, whose couplings grow as the scale k is lowered, and therefore dominate the long-distance behavior of the theory.
2. Irrelevant operators, whose couplings decrease as k is lowered, and whose effects diminish at large distances.
3. Marginal operators, whose couplings remain approximately constant at the classical level but whose behavior may be altered by quantum corrections.

This classification can often be anticipated by simple dimensional analysis (power counting) based on the canonical mass dimension of the couplings. For example, in four-dimensional scalar field theory, the φ^4 interaction has a dimensionless coupling and is classically marginal, while operators such as φ^6 carry couplings with negative mass dimension and are considered irrelevant in $D = 4$.

In the Wilsonian framework, renormalizability is understood in terms of the number of relevant and marginal couplings that need to be specified in order to predict the low-energy behavior of the system. A theory is said to be renormalizable if only a finite number of parameters need to be fixed by experiment to describe its long-distance physics.

Renormalization group flows and fixed points

The RG flow describes how couplings evolve with energy scale. This can be quantified by the beta function $\beta(g)$, which encodes the scale dependence of a dimensionless (renormalized) coupling $g(\mu)$ as $\mu \partial_\mu g = \beta(g)$ with μ an energy scale.

If $\beta(g) > 0$, the coupling increases with energy. This can lead to a Landau pole where the coupling diverges at finite energy, such that only the free theory is consistent in the UV. This behavior is believed to occur in four-dimensional scalar φ^4 theory [28]. Conversely, if $\beta(g) < 0$, the coupling decreases with increasing energy. If $g(\mu) \rightarrow 0$ as $\mu \rightarrow \infty$, the theory is said to be asymptotically free. This property ensures that the theory remains weakly coupled at high energies, as in the case of non-Abelian gauge theories such as QCD [22, 23]. A zero of the beta function $\beta(g^*) = 0$ indicates a fixed point, which is called non-trivial if $g^* \neq 0$ and Gaussian if $g^* = 0$.

A Gaussian (i.e., asymptotic freedom) or non-trivial (i.e., asymptotic safety [37]) UV fixed point is essential for ensuring UV completeness, as it allows the RG flow to be consistently extended to arbitrarily high energies. Non-trivial fixed points in the IR, such as the Wilson-Fisher fixed point in $D = 3$ scalar field theory [38], describe scale-invariant behavior at long distances. The set of all RG trajectories that flow into a fixed point constitutes its universality class. Different microscopic theories may flow to the same IR fixed point, exhibiting universal behavior at long distances despite differing UV descriptions. As such, non-trivial IR fixed points are associated with critical phenomena in statistical systems and govern the universal properties of second-order phase transitions.

Near a fixed point, the RG flow can be linearized and described by the stability matrix B_{ij} , which characterizes the scaling of perturbations away from the fixed point. Couplings that grow as the energy scale decreases (i.e., relevant operators) require fine-tuning to remain near the critical surface of the fixed point. Irrelevant operators flow back towards the fixed point and become negligible at long distances. Marginal operators require higher-order analysis to determine their behavior.

2.1.2 Large- N expansions

The theoretical description of strongly correlated systems remains one of the major challenges in modern theoretical physics. Such systems often require non-perturbative methods or resummations to all orders, particularly in regimes where no weak coupling expansion is available or where large fluctuations dominate. In this context, large- N expansion methods have proven to be an effective analytical tool. Since their introduction, these methods have been applied in a wide range of problems in quantum field theory, statistical mechanics, and quantum gravity.

The central idea of large- N techniques is to consider a theory where fields transform under a global or gauge symmetry group of rank N and to study the limit $N \rightarrow \infty$. The parameter N typically corresponds to the number of field components, flavors, or colors

in a given theory. Depending on the symmetry group and the representation of the fields, the large- N limit leads to distinct diagrammatic expansions and physical phenomena.

The simplest class of models exhibiting a large- N limit are vector models, in which the fundamental fields carry a single index and transform under groups such as $O(N)$ or $U(N)$. The leading-order (LO) dynamics at large- N is in these models determined by the resummation of local so-called bubble (also daisy, snail, or cactus) diagrams. The resulting models are often exactly solvable in the large- N limit, with fluctuations around the saddle point suppressed by powers of $1/N$. Despite their simplicity, vector models have played an important role in the study of non-perturbative phenomena, particularly in critical phenomena and statistical mechanics [39–41].

A further level of complexity is introduced in matrix models, where fields carry two indices and typically transform under the adjoint representation of $U(N)$ or $SU(N)$. In these models, the large- N limit leads to a topological expansion, organized by the genus g of the Feynman diagrams. At leading order in $1/N$, planar diagrams (genus zero) dominate [25]. While exact resummations are generally not tractable, matrix models capture a wide range of non-trivial physical phenomena. One important example is the large- N_c expansion in the number of colors N_c of Yang-Mills theory $SU(N_c)$, which provides insight into the dynamics of confining gauge theories and qualitative understanding of QCD [42]. In the presence of dynamical fermions, the Veneziano limit [43] generalizes this expansion by taking $N_c \rightarrow \infty$ while keeping the ratio with the number of flavors N_f/N_c fixed, which has applications in both gauge theories and string theory.

A different class of models arises when considering fields with ≥ 3 indices, transforming under a product of symmetry groups. Surprisingly, these tensor models exhibit a large- N limit that is analytically tractable but non-trivial, which sets it apart from that of vector or matrix models. In these models, the leading-order contributions are dominated by melonic diagrams, which can be resummed exactly [31, 32]. Tensor models have attracted significant attention due to their relation to the Sachdev-Ye-Kitaev (SYK) model [44, 45], a quantum mechanical model of strongly interacting fermions. Tensor models provide disorder-free realizations of the same dynamics as the SYK model and are also of interest in the study of quantum gravity and holography [46, 47].

Examples of typical large- N diagrams for vector, matrix, and tensor models are shown in Fig. 2.1. Read from left to right, local insertions of tadpole diagrams dominate the large- N

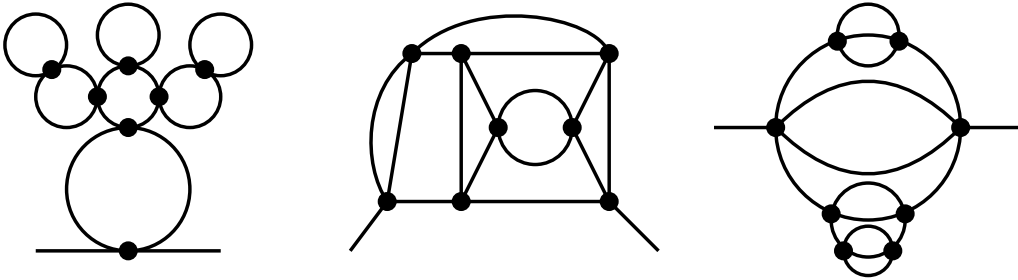


Figure 2.1: Typical diagrams contributing at leading-order large- N to vector (left), matrix (middle), and tensor models (right).

limit of vector models and lead to so-called cactus or daisy resummations. Matrix models are dominated by planar graphs, and tensor models are dominated by bi-local insertions of melonic or sunset diagrams.

2.1.3 The $O(N)^3$ model

In Sec. 2.2 and Sec 2.3, we discuss the RG flow of a tensor model with global $O(N)^3 = O(N) \otimes O(N) \otimes O(N)$ symmetry in four Euclidean dimensions. We consider a real tensor field of rank three transforming in the tri-fundamental representation of $O(N)^3$, $\bar{\varphi}_{\mathbf{a}}(x)$, where $\mathbf{a} = (a_1, a_2, a_3)$ and $a_{1,2,3} = 1, \dots, N$. The classical action reads in D Euclidean space-time dimensions [48]

$$S[\varphi] = \int d^D x \left\{ \frac{1}{2} \bar{\varphi}_{\mathbf{a}}(x) (\partial^2 + m^2) \bar{\varphi}_{\mathbf{a}}(x) + \left(\frac{\bar{g}_p}{4N^2} \delta_{\mathbf{ab};\mathbf{cd}}^p + \frac{\bar{g}_d}{4N^3} \delta_{\mathbf{ab};\mathbf{cd}}^d + \frac{\bar{g}}{4N^{3/2}} \delta_{\mathbf{abcd}}^t \right) \bar{\varphi}_{\mathbf{a}}(x) \bar{\varphi}_{\mathbf{b}}(x) \bar{\varphi}_{\mathbf{c}}(x) \bar{\varphi}_{\mathbf{d}}(x) \right\}, \quad (2.1)$$

where m , $\bar{g}_p$, $\bar{g}_d$, and $\bar{g}$ denote the real bare mass and coupling parameters, respectively. The literature on this model and its relation to other tensor field theories is discussed in VI of Sec. 2.3. $O(N)^3$ permits three invariant contraction patterns, respectively called pillow, double-trace, and tetrahedral. A graphical visualization is shown in Fig. 2.2, which can

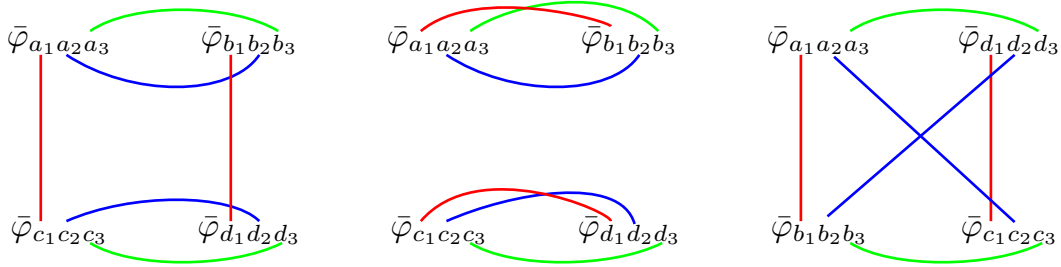


Figure 2.2: Graphical representation of the three $O(N)^3$ invariant contraction patterns. From left to right: Pillow, double-trace, and tetrahedral. Each $O(N)$ sector is represented by a distinct color connecting the corresponding indices.

be formalized with the following projection operators

$$\begin{aligned} \delta_{\mathbf{ab};\mathbf{cd}}^p &= \frac{1}{3} \sum_{i=1}^3 \delta_{a^i c^i} \delta_{b^i d^i} \prod_{j \neq i} \delta_{a^j b^j} \delta_{c^j d^j}, \\ \delta_{\mathbf{ab};\mathbf{cd}}^d &= \prod_{i=1}^3 \delta_{a^i b^i} \prod_{j=1}^3 \delta_{c^j d^j}, \\ \delta_{\mathbf{abcd}}^t &= \delta_{a^1 b^1} \delta_{c^1 d^1} \delta_{a^2 c^2} \delta_{b^2 d^2} \delta_{a^3 d^3} \delta_{b^3 c^3}. \end{aligned} \quad (2.2)$$

An equivalent and convenient parametrization for technical evaluations turns out to be given by a basis rotation with couplings $\bar{g}_1 = \bar{g}_p/3$ and $\bar{g}_2 = \bar{g}_p + \bar{g}_d$ and corresponding

projectors

$$P_{\mathbf{ab};\mathbf{cd}}^{(1)} = 3(\delta_{\mathbf{ab};\mathbf{cd}}^p - \delta_{\mathbf{ab};\mathbf{cd}}^d), \quad P_{\mathbf{ab};\mathbf{cd}}^{(2)} = \delta_{\mathbf{ab};\mathbf{cd}}^d. \quad (2.3)$$

In contrast to the pillow and double-trace projectors, the projectors of the $1/2$ -basis are orthonormal, see App. A.1 for further information. Additionally introducing normalized projectors in anticipation of the ensuing large- N expansion

$$\hat{\delta}_{\mathbf{ab};\mathbf{cd}}^p = \frac{\delta_{\mathbf{ab};\mathbf{cd}}^p}{N^2}, \quad \hat{\delta}_{\mathbf{ab};\mathbf{cd}}^d = \frac{\delta_{\mathbf{ab};\mathbf{cd}}^d}{N^3}, \quad \hat{\delta}_{\mathbf{abcd}}^t = \frac{\delta_{\mathbf{abcd}}^t}{N^{3/2}}, \quad (2.4)$$

an equivalent representation of the classical action is

$$S[\varphi] = \int d^D x \left\{ \frac{1}{2} \bar{\varphi}_{\mathbf{a}}(x) (\partial^2 + m^2) \bar{\varphi}_{\mathbf{a}}(x) + \frac{1}{4} \left(\bar{g}_1 \hat{P}_{\mathbf{ab};\mathbf{cd}}^{(1)} + \bar{g}_2 \hat{P}_{\mathbf{ab};\mathbf{cd}}^{(2)} + \bar{g} \hat{\delta}_{\mathbf{abcd}}^t \right) \bar{\varphi}_{\mathbf{a}}(x) \bar{\varphi}_{\mathbf{b}}(x) \bar{\varphi}_{\mathbf{c}}(x) \bar{\varphi}_{\mathbf{d}}(x) \right\}. \quad (2.5)$$

As will be presented in Sec. 2.2 and Sec. 2.3, choosing the tetrahedral coupling $\bar{g}$ to be purely imaginary leads to an asymptotically free theory that can be studied over all distance scales. Some of the following results are presented in the p/d -basis for visual illustration, but we will restrict technical calculations to the more efficient $1/2$ -basis.

We are interested in computing Euclidean correlation functions of the $O(N)^3$ model using a truncation to leading order (LO) in large- N . The melonic large- N limit of the quantum theory is efficiently resummed by the free-energy functional $\Gamma[G]$, which is the two-particle irreducible (2PI) effective action [49, 50]. Starting from the familiar Schwinger generating functional for the connected correlation functions in the presence of a local and bi-local source term:

$$W[J, R] = \ln \int \mathcal{D}\varphi e^{-S[\varphi] + \int d^D x J_{\mathbf{a}}(x) \varphi_{\mathbf{a}}(x) + \frac{1}{2} \int d^D x d^D y \varphi_{\mathbf{a}}(x) R_{\mathbf{ab}}(x, y) \varphi_{\mathbf{b}}(y)}, \quad (2.6)$$

the 2PI effective action follows through a double-Legendre transform with respect to the local and bi-local sources and can be parametrized as

$$\Gamma[\phi, G] = S[\phi] + \frac{1}{2} \text{tr} \ln G^{-1} + \frac{1}{2} \text{tr} G_0^{-1} G + \Gamma_2[\phi, G]. \quad (2.7)$$

$\Gamma[\phi, G]$ is a functional of the full propagator $G_{\mathbf{ab}}(x, y)$ and the one-point function $\phi_{\mathbf{a}}(x) = \langle \varphi_{\mathbf{a}}(x) \rangle$, where $G_{0,\mathbf{ab}}^{-1}(x, y) \equiv \delta^2 S / \delta \varphi_{\mathbf{a}}(x) \delta \varphi_{\mathbf{b}}(y)$ is the classical inverse propagator. Correlation functions are determined through functional differentiation for a given solution of the variational principles:

$$\frac{\delta \Gamma[\phi, G]}{\delta \phi_{\mathbf{a}}(x)} = 0, \quad \frac{\delta \Gamma[\phi, G]}{\delta G_{\mathbf{ab}}(x, y)} = 0, \quad (2.8)$$

and for a given truncation of Γ_2 , which contains all 2PI diagrams (those that are not disconnected by cutting two propagator lines).

We will, in the following, focus on the $O(N)^3$ -symmetric regime with $\phi_{\mathbf{a}}(x) \equiv 0$ and $G_{\mathbf{ab}}(x, y) = \delta_{\mathbf{ab}} G(x, y) = G(x, y) \prod_{i=1}^3 \delta_{a_i b_i}$. Up to NLO in $1/N$, the 2PI contributions

are then given by [51–53]

$$\begin{aligned} \Gamma^{\text{LO}}[G] = & \left[\frac{\bar{g}_p}{4N^2} \delta_{\mathbf{ab};\mathbf{cc}}^p + \frac{\bar{g}_d}{4N^3} \delta_{\mathbf{ab};\mathbf{cd}}^d \right] \int d^D x G_{\mathbf{ab}}(x, x) G_{\mathbf{cd}}(x, x) \\ & - \frac{\bar{g}^2}{8N^3} \int d^D x d^D y \delta_{\mathbf{abcd}}^t \delta_{\mathbf{a'b'c'd'}}^t G_{\mathbf{aa'}}(x, y) G_{\mathbf{bb'}}(x, y) G_{\mathbf{cc'}}(x, y) G_{\mathbf{dd'}}(x, y) \\ & G_{\mathbf{ab}} \equiv \delta_{\mathbf{ab}} G \quad N^3 \frac{\bar{g}_2}{4} \int d^D x G^2(x, x) - N^3 \frac{\bar{g}^2}{8} \int d^D x d^D y G^4(x, y) , \end{aligned} \quad (2.9a)$$

$$\begin{aligned} \Gamma^{\text{NLO}}[G] = & \frac{\bar{g}}{4N^{3/2}} \int d^D x \left(\delta_{\mathbf{abcd}}^t + \delta_{\mathbf{acbd}}^t + \delta_{\mathbf{acdb}}^t \right) G_{\mathbf{ab}}(x, x) G_{\mathbf{cd}}(x, x) \\ & G_{\mathbf{ab}} \equiv \delta_{\mathbf{ab}} G + \frac{3\bar{g}}{4} N^{5/2} \int d^D x G^2(x, x) . \end{aligned} \quad (2.9b)$$

The LO contribution scales as expected with $\sim N^3$, while the NLO term – comprising tadpole corrections in the tetrahedral coupling – is suppressed by a factor of $1/\sqrt{N}$.

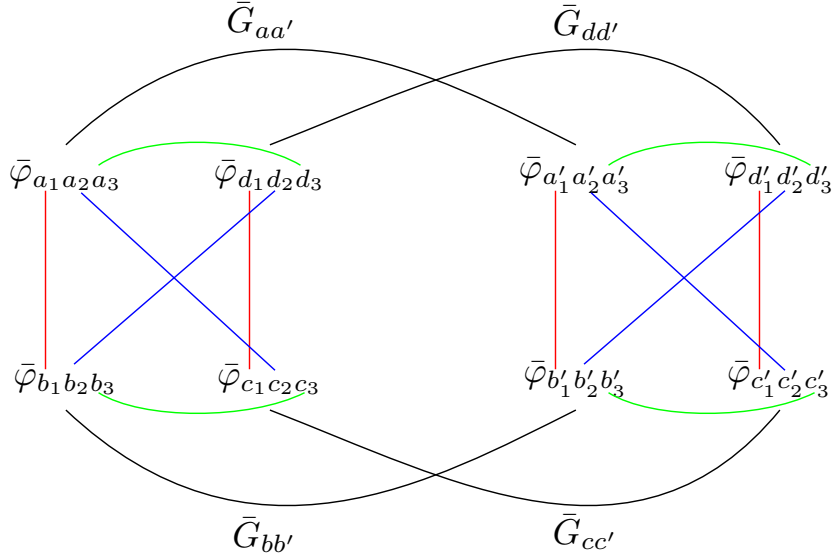


Figure 2.3: Full index structure of the 2PI melon graph with two tetrahedral interactions.

At LO in the large- N expansion, local tadpole contributions from the pillow and double-trace couplings $\bar{g}_p$ and $\bar{g}_d$ appear, as familiar from the LO behavior in each $O(N)$ vector model sector. However, the $O(N)^3$ tensor model also features a non-trivial LO diagram: the melon graph involving two tetrahedral interactions $\sim \bar{g}^2$. This diagram, including its full index structure, is depicted in Fig. 2.3.

The full propagator follows from the variational principle (2.8) as:

$$G_{\mathbf{ab}}^{-1}(x, y) = G_{0, \mathbf{ab}}^{-1}(x, y) - \Sigma_{\mathbf{ab}}(x, y) , \quad (2.10)$$

where the 1PI self-energy $\Sigma_{\mathbf{ab}}(x, y) = -2\delta\Gamma_2/\delta G_{\mathbf{ab}}(x, y)$ is given to NLO in large- N by:

$$\begin{aligned} \Sigma_{\mathbf{ab}}(x, y) = & - \left[\frac{\bar{g}_p}{N^2} \delta_{\mathbf{ab};\mathbf{cd}}^p + \frac{\bar{g}_d}{N^3} \delta_{\mathbf{ab};\mathbf{cd}}^d \right] \delta(x, y) G_{\mathbf{cd}}(x, x) \\ & + \frac{\bar{g}^2}{N^3} \delta_{\mathbf{ac}_1\mathbf{c}_2\mathbf{c}_3}^t \delta_{\mathbf{bd}_1\mathbf{d}_2\mathbf{d}_3}^t G_{\mathbf{c}_1\mathbf{d}_1}(x, y) G_{\mathbf{c}_2\mathbf{d}_2}(x, y) G_{\mathbf{c}_3\mathbf{d}_3}(x, y) \\ & - \frac{\bar{g}}{N^{3/2}} \delta_{xy} \left(\delta_{\mathbf{abcd}}^t + \delta_{\mathbf{acbd}}^t + \delta_{\mathbf{acdb}}^t \right) G_{\mathbf{cd}}(x, x) \end{aligned} \quad (2.11)$$

$$G_{\mathbf{ab}} \equiv \delta_{\mathbf{ab}} G - \left[g_2 + \frac{3g}{\sqrt{N}} \right] \delta(x, y) G(x, x) + \bar{g}^2 G^3(x, y) . \quad (2.12)$$

We chose the 1/2-basis in the last step in order to prepare the calculations presented in the following Sec. 2.2 and Sec. 2.3.

2.2

Asymptotic freedom in a strongly interacting scalar quantum field theory in four Euclidean dimensions

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(Received 27 January 2023; accepted 18 July 2023; published 31 July 2023)

We show that scalar quantum field theory in four Euclidean dimensions with global $O(N)^3$ symmetry and imaginary tetrahedral coupling is asymptotically free and bounded from below in the large- N limit. While the Hamiltonian is non-Hermitian, the full quantum effective action for the large- N theory only depends on the square of that coupling which is real. A perturbative analysis uncovers that the renormalization group flow of the quartic couplings connects a Gaussian ultraviolet fixed point to a strongly interacting theory in the infrared. This realizes a renormalizable field theory which exhibits nontrivial dynamics, such as direct scattering, while still being analytically tractable also nonperturbatively. Our findings open up a way to address outstanding problems in strongly coupled theories from first principles.

DOI: [10.1103/PhysRevD.108.016019](https://doi.org/10.1103/PhysRevD.108.016019)

I. INTRODUCTION

Asymptotic freedom in the theory of the strong interaction is a hallmark for the fundamental description of nature encoded in the Standard Model of particle physics [1,2]. The phenomenon describes the change of the interaction strength with the characteristic energy scale, such that the theory becomes noninteracting at high energy-momenta. In turn, interactions become strong at low energies with striking phenomenological implications such as the confinement of hadronic matter [3]. While dynamical strong-interaction phenomena, like quantum bounds on crucial observables such as viscosities [4] or relaxation times [5], have been suggested by holographic models, their analysis based directly on quantum theories with scale-dependent interactions still remains elusive.

Asymptotic freedom is linked to local space-time symmetries of non-Abelian gauge fields in the Standard Model. However, it can in principle be realized even for simple scalar field theories in four space-time dimensions if the condition of Hermiticity is relaxed [6]. A known example is scalar field theory with negative quartic self-interaction [7,8]. However, the energy of the system cannot be bounded from below and the theory is considered to be ill defined [6]. Nevertheless, it has been suggested that the parity and time-reversal

symmetric theory may still possess a real and positive eigenspectrum, obtained by an analytic continuation procedure from the corresponding bounded theory with positive interaction [8–11]. This fosters the discussion about the remarkable possibility of a larger class of well-defined asymptotically free theories without relying on Hermiticity, which is a sufficient but not necessary condition. The notion of asymptotically free scalar field theories may also provide an important ingredient in our understanding of the scalar Higgs particle with potential consequences also for physics beyond the Standard Model [12–14].

In this work, we show that both asymptotic freedom and boundedness from below can be realized in scalar quantum field theory in four Euclidean dimensions. We demonstrate our finding for a theory with global $O(N)^3$

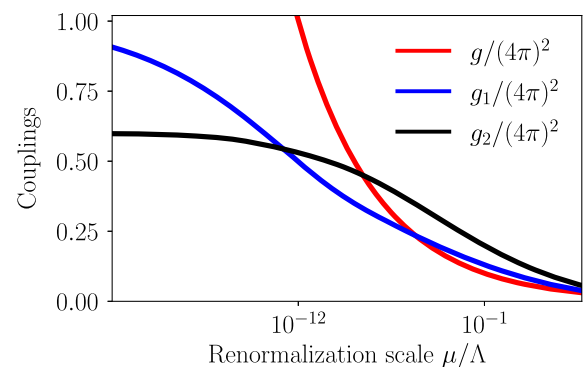


FIG. 1. Renormalization group flow of the couplings as a function of the renormalization scale μ/Λ . The flow connects the asymptotically free UV with the strongly interacting IR, corresponding to the upper (red) separatrices of Fig. 2.

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symmetry, where the dynamics can be studied via a systematic large- N limit in the number of field components $\varphi_{\mathbf{a}=(a^1,a^2,a^3)}$ with $a^{i=1,2,3} = 1, \dots, N$. To uncover its scale dependence, we analyze the renormalization group flow for the tensor field which features two real quartic couplings, g_1 and g_2 (“pillow and double-trace”), and an imaginary coupling, $i\bar{g}$ (“tetrahedral”), without the need for any analytic continuation. It turns out that all the couplings exhibit asymptotic freedom in a regime governed by the flow of the imaginary coupling, leading to a renormalizable field theory with a well-defined ultraviolet (UV) limit and a strongly coupled infrared (IR). The renormalization group flow of the three couplings in this case as a function of the renormalization scale μ with UV scale Λ is displayed in Fig. 1.

In contrast to the more conventional N -component vector field theories [15], such as describing the Higgs sector of the Standard Model for finite $N = 4$, the real bosonic tensor field $\varphi_{\mathbf{a}}$ exhibits nontrivial dynamics involving direct scattering already at leading order in the large- N expansion while still being analytically tractable [16,17]. The latter represents also a significant advantage compared to the large- N limit in matrix models involving all planar Feynman diagrams for quantum corrections [18,19]. It has been remarked [20] that tensor quantum mechanical models provide an alternative to the Sachdev-Ye-Kitaev model [21,22] without disorder. Subsequently tensor field theories have been shown to display strongly coupled infrared fixed points in lower Euclidean space-time dimensions $d < 4$ [23–26]. In the following, we reveal that this theory becomes highly nontrivial in $d = 4$.

II. TENSOR FIELD THEORY WITH IMAGINARY TETRAHEDRAL COUPLING

The scalar field $\varphi_{\mathbf{a}}(x)$, with x a Euclidean space-time point, is a real tensor of rank three transforming in the trifundamental representation of $O(N)^3$ [24]. The classical action of the theory reads in four dimensions:

$$S[\varphi] = \int d^4x \left\{ \frac{1}{2} \varphi_{\mathbf{a}}(x) (-\partial^2 + \bar{m}^2) \varphi_{\mathbf{a}}(x) + \frac{1}{4} \left(\bar{g}_1 \hat{P}_{\mathbf{ab};\mathbf{cd}}^{(1)} + \bar{g}_2 \hat{P}_{\mathbf{ab};\mathbf{cd}}^{(2)} + i\bar{g} \hat{\delta}_{\mathbf{abcd}}' \right) \times \varphi_{\mathbf{a}}(x) \varphi_{\mathbf{b}}(x) \varphi_{\mathbf{c}}(x) \varphi_{\mathbf{d}}(x) \right\}, \quad (1)$$

where $\bar{m}$ denotes the real mass parameter and we take the coupling parameters $\bar{g}_1$, $\bar{g}_2$ and $\bar{g}$ to be real such that $i\bar{g}$ appearing in Eq. (1) is purely imaginary. The corresponding three interaction terms in the action reflect the three $O(N)^3$ invariant contraction patterns (pillow, double-trace and tetrahedral):

$$\begin{aligned} \hat{\delta}_{\mathbf{ab};\mathbf{cd}}^p &= \frac{1}{3N^2} \sum_{i=1}^3 \delta_{a^i c^i} \delta_{b^i d^i} \prod_{j \neq i} \delta_{a^j b^j} \delta_{c^j d^j}, \\ \hat{\delta}_{\mathbf{ab};\mathbf{cd}}^d &= N^{-3} \prod_{i=1}^3 \delta_{a^i b^i} \prod_{j=1}^3 \delta_{c^j d^j}, \\ \hat{\delta}_{\mathbf{abcd}}' &= N^{-3/2} \delta_{a^1 b^1} \delta_{c^1 d^1} \delta_{a^2 c^2} \delta_{b^2 d^2} \delta_{a^3 d^3} \delta_{b^3 c^3}, \end{aligned} \quad (2)$$

which relate to the orthonormal projectors $\hat{P}^{(1)} = 3(\hat{\delta}^p - \hat{\delta}^d)$ and $\hat{P}^{(2)} = \hat{\delta}^d$. Their scaling with N is determined such that the theory has a well-defined large- N limit.

The full quantum theory is described in path-integral quantization by a functional integral whose norm,

$$\left| \int \mathcal{D}\varphi e^{-S[\varphi; \bar{g}_1, \bar{g}_2, i\bar{g}]} \right| \leq \int \mathcal{D}\varphi |e^{-S[\varphi; \bar{g}_1, \bar{g}_2, i\bar{g}]}| = \int \mathcal{D}\varphi e^{-S[\varphi; \bar{g}_1, \bar{g}_2, i\bar{g} \equiv 0]}, \quad (3)$$

is bounded by the corresponding model with real action $S[\varphi; \bar{g}_1, \bar{g}_2, i\bar{g} \equiv 0]$, since the couplings $\bar{g}_1$ and $\bar{g}_2$ are real. The theory is thus stable and bounded from below if $\bar{g}_1$ and $\bar{g}_2$ are positive.

The imaginary tetrahedral coupling parameter $i\bar{g}$ introduces an oscillatory term in the functional integral. This is difficult to compute with standard importance sampling techniques, where the situation is reminiscent of the notorious sign problem, as e.g. in quantum chromodynamics at nonzero baryon number density [27]. However, in our case it poses no particular problem for first-principles computations using large- N techniques.

The quantum theory in the large- N limit is conveniently described in terms of a free-energy functional $\Gamma[G]$, which is the two-particle irreducible (2PI) effective action [28,29]. It is a functional of the full propagator $G_{\mathbf{ab}}(x, y)$, which is determined self-consistently from a variational principle $\delta\Gamma[G]/\delta G_{\mathbf{ab}}(x, y) = 0$. The 2PI effective action can be written as

$$\Gamma[G] = \frac{1}{2} \text{Tr} \ln G^{-1} + \frac{1}{2} \text{Tr}[G_0^{-1} G] + \Gamma_2[G]. \quad (4)$$

Here the classical inverse propagator is given by $G_{0,\mathbf{ab}}^{-1}(x, y) \equiv \delta^2 S / \delta \varphi_{\mathbf{a}}(x) \delta \varphi_{\mathbf{b}}(y)$, and $\Gamma_2[G]$ contains all 2PI contributions (Feynman graphs that are not disconnected by cutting two propagator lines). Considering the $O(N)^3$ symmetric regime with $G_{\mathbf{ab}}(x, y) = G(x, y) \prod_{i=1}^3 \delta_{a^i b^i}$, we get at leading order large- N :

$$\frac{\Gamma_2[G]}{N^3} = \frac{\bar{g}_2}{4} \int d^4x G^2(x, x) + \frac{\bar{g}^2}{8} \int d^4x d^4y G^4(x, y). \quad (5)$$

The self-consistent propagator reads in Fourier space:

$$G^{-1}(p) = p^2 + \bar{m}^2 + \bar{g}_2 \int \frac{d^4 q}{(2\pi)^4} G(q) + \bar{g}^2 \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} G(q) G(k) G(p+q+k). \quad (6)$$

Apart from the different factors and couplings in front of the tadpole term $\sim \bar{g}_2$ and sunset (or melon) contribution $\sim \bar{g}^2$, this equation is of the same form as the one obtained in the $O(N)$ symmetric N -vector model from a 2PI two-loop expansion [30]. We emphasize that, while the corresponding equation is only valid at weak coupling in the N -vector model, Eq. (6) is exact in the large- N limit of the $O(N)^3$ tensor field theory we consider.

In the large- N limit, but at all orders in perturbation theory, the tetrahedral coupling enters the 2PI effective action (5), and consequentially the two- and four-point correlation functions, only quadratically (cf. also the discussion in Ref. [25]). Therefore, the purely imaginary tetrahedral coupling simply leads in the large- N quantum effective action to some sign changes with respect to the real case.

The next-to-leading-order contribution to the 2PI effective action is suppressed by $N^{-1/2}$ [29]. It consists of double tadpoles with only one tetrahedral vertex such that the next-to-leading contribution exhibits an imaginary term. Its presence leads to two classes of subleading imaginary contributions in correlation functions. First, one gets a shift in the bare mass. However, since the physical mass is fixed by a real infrared renormalization condition, in our case this is without consequence. Second, one obtains a subleading term in the four-point kernel corresponding to the tetrahedral pattern of identification of tensor indices. The latter backreacts on the two-loop β functions [cf. Eq. (2.40) of Ref. [31]] and brings in corrections $\sim iN^{-1/2}$. While imaginary contributions arise, the large- N behavior is not singular and the corrections with respect to the leading-order behavior are organized in powers of $N^{-1/2}$, such that the Euclidean model remains asymptotically free and stable: Although they acquire small imaginary parts, the couplings g_1, g_2 maintain a positive real part and the $N^{-1/2}$ corrections do not spoil the boundedness from below of the real part of the action. Therefore, the real effective action (5) is exact in the large- N limit which is well controlled.

III. RENORMALIZATION GROUP FLOW

To demonstrate that the theory realizes asymptotic freedom, we study the perturbative renormalization group flow of the quartic couplings. Their flow is described in terms of β functions, which encode how quantum corrections change the corresponding renormalized couplings g, g_1 , and g_2 with the renormalization scale μ : $\beta \equiv \mu \partial_\mu g(\mu)$

and analogously for $g_1(\mu)$ and $g_2(\mu)$. The corresponding system of β functions of the model has been investigated in $d < 4$; see Refs. [24,25] and subsequent work.

If one considers all couplings to be real, in $d = 4$ the theory exhibits triviality; i.e. it becomes noninteracting corresponding to a Gaussian fixed point in the IR as $\mu \rightarrow 0$. For the case of an imaginary tetrahedral coupling we are discussing in this work, we find a strikingly different behavior. We obtain the corresponding beta functions in the large- N limit and at two-loop order for the massless theory, which is sufficient for our considerations, from Ref. [24] as

$$\beta = -\frac{2g^3}{(4\pi)^4}, \quad (7a)$$

$$\beta_1 = 2 \left(\frac{g_1^2 - g^2}{(4\pi)^2} + \frac{g_1 g^2}{(4\pi)^4} \right), \quad (7b)$$

$$\beta_2 = 2 \left(\frac{g_2^2 - 3g^2}{(4\pi)^2} + \frac{5g_2 g^2}{(4\pi)^4} \right). \quad (7c)$$

Since the β function for the tetrahedral coupling (7a) is independent of the other couplings at this order, for given coupling value g_Λ at some UV scale Λ its flow is solved by

$$g^2(\mu) = \frac{g_\Lambda^2}{1 - \frac{4g_\Lambda^2}{(4\pi)^4} \ln(\Lambda/\mu)}. \quad (8)$$

This demonstrates that $g^2(\mu)$ approaches zero logarithmically as μ increases. Conversely, $g^2(\mu)$ grows toward the IR and, to lowest order in perturbation theory, the coupling diverges at $\mu_* = \Lambda \exp[-(4\pi)^4/(4g_\Lambda^2)]$. At two-loop order in the large- N limit we additionally infer that, corresponding to the negative sign of the β function for the tetrahedral coupling, the anomalous dimension [24] $\gamma_\phi = -g^2/(2(4\pi)^4)$ is also negative, similar to other theories with negative β functions [32,33].

The flow of the other two couplings, which depend on the values of g , is depicted in Fig. 2. The blue lines represent the values of $g_1(\mu)/(4\pi)^2$ (left graph) and $g_2(\mu)/(4\pi)^2$ (right graph) for corresponding values of $g(\mu)/(4\pi)^2$, where arrows point toward smaller (IR) values of μ . The (upper) red and (lower) black lines denote separatrices dividing the flow diagrams into distinct regions. For all $g > 0$ above the black separatrix, the theory exhibits a nontrivial IR behavior for g_1 and g_2 , with vanishing β functions for the fixed values $g_1/(4\pi)^2 \rightarrow 1$ and $g_2/(4\pi)^2 \rightarrow 3/5$. Importantly, in the sector comprised between the red and the black separatrices, the renormalization group flow connects an ultra-violet attractive Gaussian fixed point (red dot) to the nontrivial infrared theory. All quartic couplings g, g_1 , and g_2 are thus found to exhibit asymptotic freedom along these trajectories, which *a posteriori* justifies the validity of

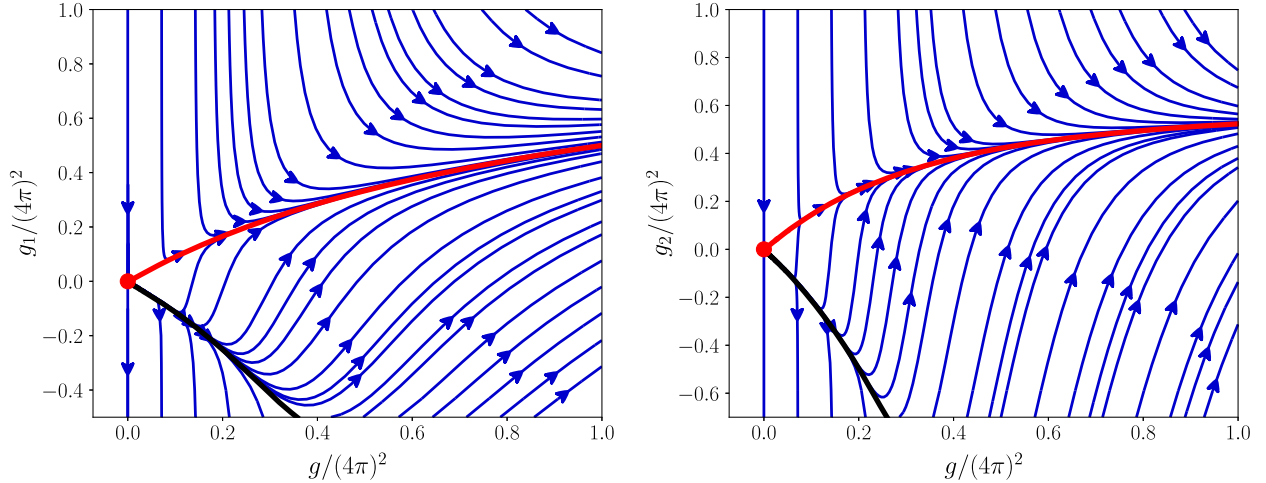


FIG. 2. Flow trajectories of g_1 (left) and g_2 (right) as obtained from Eq. (7), where arrows point in the direction of smaller scale μ . Bold upper (red) and lower (black) lines denote separatrices, such that all models with coupling values at and below the red separatrix exhibit asymptotic freedom, while for values above the red line a Landau pole is realized in the UV. All nonzero couplings above the black separatrix feature a strongly interacting IR with positive values.

our two-loop analysis of the corresponding β functions in this regime. In particular, along the red separatrix the couplings are positive at all scales. This realizes a renormalizable scalar field theory, where the nontetrahedral quartic couplings are real and positive, thereby ensuring that the corresponding path integral is bounded from below, with a well-defined UV limit and a strongly coupled IR. The corresponding flow of couplings as a function of the renormalization scale μ is given in Fig. 1.

IV. CONCLUSION AND OUTLOOK

Our findings establish asymptotic freedom in a well-defined strongly interacting scalar quantum field theory in four Euclidean dimensions. This is a remarkable result also in view of the fact that the tetrahedral coupling driving the renormalization group flow in this regime is purely imaginary. Despite this, the full quantum effective action for the theory in the large- N limit only depends on the square of that coupling which is real. Correspondingly, the renormalization group β functions we discussed are also real. These findings support the exciting possibility of an extended class of well-defined theories where the standard condition of Hermiticity is relaxed [34,35].

The large- N limit of the tensor field theory we considered is so interesting because it is already highly nontrivial, involving crucial processes such as direct scatterings for the dynamics, and analytically tractable also nonperturbatively. This is in contrast to the large- N limit in N -vector models, where collisional processes are absent; or in matrix models, where no general closed form for the nontrivial leading-order dynamics is known.

So far, our findings concern the Euclidean field theory, which is relevant for vacuum or equilibrium properties. It is an important open question to what extent our results can be taken over to Minkowski space-time, in particular, since our arguments on boundedness use well-defined properties of the Euclidean theory. This would make our theory a versatile theoretical laboratory to investigate crucial open dynamical questions, such as real-time dynamics in the strong-coupling regime. While holographic models can give very important insights at strong coupling [36,37], it is typically difficult to include the relevant changes of the interaction strength with the scale as is the case in an asymptotically free theory.

At lowest nontrivial order in perturbation theory, we found that the tetrahedral coupling grows without bound in the IR. An important question is whether the renormalization group flow is altered when the coupling becomes nonperturbatively large. This question can be answered from the full large- N result, which resums contributions from perturbation theory to all orders in the coupling. In order to derive the exact β functions in the large- N limit, one needs to deal appropriately with the wave function renormalization in conjunction with Eq. (6). Once this is done, the all-order running of the four-point couplings should be accessible by techniques similar to the ones employed in Ref. [25]. A striking possible outcome would be that the tetrahedral coupling g approaches a finite IR value in the nonperturbative regime. Together with our findings for g_1 and g_2 this would correspond to a proper infrared fixed point. Its universal properties then allow one to make very powerful statements about all possible strong-coupling theories in the same universality class, which only depends on general properties such as symmetries.

ACKNOWLEDGMENTS

This work is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC2181/1-390900948 (the Heidelberg STRUCTURES Excellence Cluster)

and the Collaborative Research Center, Project-ID No. 273811115, SFB 1225 ISOQUANT, as well as the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (Grant Agreement No. 818066).

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2.3

Coupling renormalization flow in the strongly interacting regime of an asymptotically free quantum field theory in four dimensions

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(Received 30 May 2024; accepted 5 July 2024; published 9 August 2024)

We consider a scalar quantum field theory with global $O(N)^3$ symmetry in four Euclidean dimensions and solve it numerically in closed form in the large- N limit. For imaginary tetrahedral coupling the theory is asymptotically free, with stable and real quantum effective action. We demonstrate the dynamical build-up of a strong interaction as the correlation length increases in a regime where the coupling renormalization flow remains well defined in the infrared. This is in contrast to perturbative results of asymptotically free theories, which predict that the coupling becomes ill defined in the infrared, like in quantum chromodynamics. These properties make the model an important laboratory for the study of strong-coupling phenomena in quantum field theory from first principles.

DOI: [10.1103/PhysRevD.110.036007](https://doi.org/10.1103/PhysRevD.110.036007)

I. INTRODUCTION AND OVERVIEW

Asymptotically free quantum field theories are a cornerstone in the fundamental description of nature. A prominent example is the theory of quantum chromodynamics (QCD) in the Standard Model of particle physics [1,2]. While the high-momentum (ultraviolet) behavior of the theory is perturbatively accessible, the scale-dependent (“running”) coupling increases toward low momenta and becomes ill defined in the infrared. The divergence of the coupling at a finite infrared momentum predicted by perturbation theory illustrates the dynamical generation of a nonperturbative scale by quantum fluctuations. Such behavior is characteristic for the perturbative analysis of asymptotically free theories, and it would be highly valuable to establish a nonperturbative example where the coupling is well-defined and can be followed all the way from the weakly coupled high-momentum regime to the strongly interacting infrared.

In this work we investigate the large- N limit of a four dimensional scalar quantum field theory with global $O(N)^3$ symmetry introduced in Ref. [3]. The model has three independent quartic couplings, whose perturbative renormalization flow, which encodes how the physical couplings change with the momentum scale due to quantum corrections, has been analyzed in Refs. [3–5]. In four Euclidean dimensions the couplings exhibit asymptotic freedom in a

regime governed by the flow of an imaginary tetrahedral coupling $ig(p)$ [3]. In turn, the tetrahedral coupling diverges in perturbation theory at a finite infrared momentum scale μ_{pert}^* . A corresponding perturbative behavior is found, in particular, also for the running coupling in quantum chromodynamics [1,2]. The perturbative scale-dependence of the squared coupling of our model is represented by the dashed curve in Fig. 1.

This is to be contrasted with the nonperturbative large- N renormalization flow of the coupling we obtain in this work, which is displayed by the solid (colored) curves for various values of the renormalized mass in Fig. 1. Contrary to the perturbative prediction, the full coupling is found to

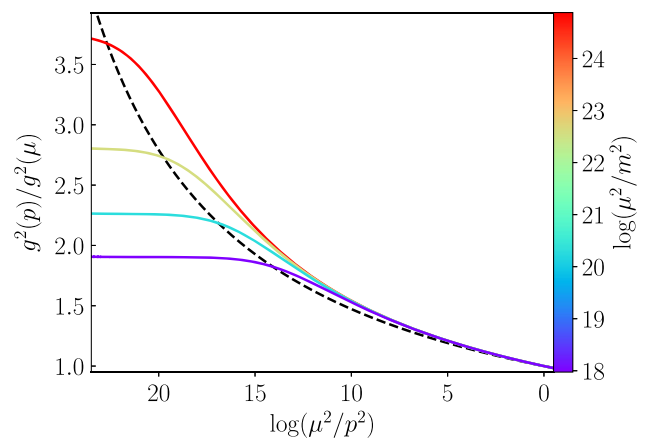


FIG. 1. Flow of the squared tetrahedral coupling g^2 with momentum p for various renormalized masses m^2 (color scale), with $g(\mu) = 20$ at the renormalization scale μ . The two-loop perturbative running is mass independent and represented as dashed.

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depend on the renormalized mass or, equivalently, the inverse correlation length. Our results show the generation of a strong interaction by quantum fluctuations, which builds up as the correlation length is increased. The running coupling remains well-defined and finite in the infrared for renormalized masses above a threshold value m^* . Approaching this threshold from above, the growth behavior of the running coupling defines a strong-interaction scale μ^* , which plays the role of a nonperturbative generalization of μ_{pert}^* . We find μ^* to be larger than the scale determined from two-loop perturbation theory. While masses above m^* allow us to investigate the running coupling in a controlled way for momenta even below μ^* , we emphasize that the two scales should not be identified, and we find m^* to be significantly smaller than μ^* . Disentangling these scales allows one to follow the coupling flow from an ultraviolet Gaussian fixed point to a strongly interacting but well-defined infrared in quantum field theory, opening up a new pathway for the investigation of strong-coupling phenomena in four dimensions from first principles.

II. $O(N)^3$ SYMMETRIC TENSOR FIELD THEORY

Following Ref. [3], we consider a scalar (under rotations) field $\bar{\varphi}_{\mathbf{a}=(a^1,a^2,a^3)}$ with $a^{i=1,2,3} = 1, \dots, N$ transforming as a real 3-index tensor in the trifundamental representation of $O(N)^3$ [4,6,7]. The model is defined by the bare (classical) action

$$S[\bar{\varphi}] = \int d^4x \left\{ \frac{1}{2} \bar{\varphi}_{\mathbf{a}}(x) (-\partial^2 + \bar{m}^2) \bar{\varphi}_{\mathbf{a}}(x) + \frac{1}{4} \left(\bar{g}_1 \hat{P}_{\mathbf{ab};\mathbf{cd}}^{(1)} + \bar{g}_2 \hat{P}_{\mathbf{ab};\mathbf{cd}}^{(2)} + i\bar{g} \hat{\delta}_{\mathbf{abcd}} \right) \times \bar{\varphi}_{\mathbf{a}}(x) \bar{\varphi}_{\mathbf{b}}(x) \bar{\varphi}_{\mathbf{c}}(x) \bar{\varphi}_{\mathbf{d}}(x) \right\} \quad (1)$$

in four Euclidean dimensions. Here $\bar{m}$ is the bare mass parameter, and we take the bare quartic couplings $\bar{g}_1$, $\bar{g}_2$ and $\bar{g}$ to be real such that $i\bar{g}$ in Eq. (1) is purely imaginary. The three interaction terms in the action stem from the three $O(N)^3$ invariant contraction patterns (“pillow,” “double-trace” and “tetrahedral”)

$$\begin{aligned} \hat{\delta}_{\mathbf{ab};\mathbf{cd}}^p &= \frac{1}{3N^2} \sum_{i=1}^3 \delta_{a^i c^i} \delta_{b^i d^i} \prod_{j \neq i} \delta_{a^j b^j} \delta_{c^j d^j}, \\ \hat{\delta}_{\mathbf{ab};\mathbf{cd}}^d &= N^{-3} \prod_{i=1}^3 \delta_{a^i b^i} \prod_{j=1}^3 \delta_{c^j d^j}, \\ \hat{\delta}_{\mathbf{abcd}} &= N^{-3/2} \delta_{a^1 b^1} \delta_{c^1 d^1} \delta_{a^2 c^2} \delta_{b^2 d^2} \delta_{a^3 d^3} \delta_{b^3 c^3}, \end{aligned} \quad (2)$$

which relate to the orthonormal projectors $\hat{P}^{(1)} = 3(\hat{\delta}^p - \hat{\delta}^d)$ and $\hat{P}^{(2)} = \hat{\delta}^d$ appearing in Eq. (1).

III. RENORMALIZED CORRELATION FUNCTIONS

Physical observables can be obtained from the renormalized large- N quantum field theory [8–15]. We aim to compute correlation functions representing expectation values of products of quantum fields, specifically the renormalized two-point correlation function or full propagator $G_{\mathbf{ab}}(x, y)$. In the tensor field theory this computation can be achieved in closed form in the large- N limit. By contrast, in asymptotically free theories like quantum chromodynamics a resummation of the large- N planar Feynman diagrams [16,17] is out of reach. This gives us unique access also to the nonperturbative infrared behavior of our large- N theory.

The renormalized field correlation functions are obtained after imposing renormalization conditions. Two of them concern the full propagator and we write

$$G^{-1}(0) = m^2, \quad \frac{G^{-1}(\mu) - G^{-1}(0)}{\mu^2} = 1. \quad (3)$$

The first condition fixes the renormalized mass m at zero momentum. The second one specifies the wave function renormalization

$$Z(p) = \frac{G^{-1}(p) - G^{-1}(0)}{p^2}, \quad (4)$$

at some high momentum scale μ (the renormalization scale) to $Z(\mu) = 1$. Three additional renormalization conditions fix the three couplings at the same renormalization scale μ ; in particular, the tetrahedral coupling is fixed to a given $g(\mu)$, and the results for the running coupling are presented as a ratio as in Fig. 1. In the large- N limit the running of the renormalized tetrahedral coupling is entirely encoded in the scale dependence of the wave function renormalization (4) as (see also the Appendix) [5]

$$g(p) = \frac{g(\mu)}{Z^2(p)}. \quad (5)$$

With these renormalization conditions the full large- N propagator in the $O(N)^3$ -symmetric regime, where $G_{\mathbf{ab}} = G\delta_{\mathbf{ab}}$, is determined in momentum space by

$$\begin{aligned} G^{-1}(p) &= p^2 + m^2 + g^2(\mu) \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} G(q)G(k) \\ &\times \left[G(p+q+k) - G(q+k) \right. \\ &\left. - \frac{p^2}{\mu^2} \left(G(\mu+q+k) - G(q+k) \right) \right]. \end{aligned} \quad (6)$$

The self-consistent solution of this equation may be viewed as resumming infinitely many perturbative contributions in

the quartic couplings. It is remarkable that Eq. (6) contains in closed form all the relevant phenomena in the asymptotically free regime and in the strongly interacting infrared we are addressing. The solution for the nonperturbative propagator $G(p)$ determines the wave function renormalization $Z(p)$ and the coupling $g(p)$ according to Eqs. (4) and (5). The behavior of the other couplings g_1 and g_2 does not enter the solution of Eq. (6). Their running is in turn dictated by the momentum dependence of $g(p)$ and has been shown to be perturbatively already well defined in the infrared in Ref. [3]. Even at next-to-leading order in the large- N expansion one only encounters further tadpole corrections which are compensated by the mass renormalization [3].

We iteratively solve Eq. (6) until convergence is observed (see details in the Appendix). We verified that the relevant physical results are insensitive to changes in the momentum discretization for the numerical parameters explored in this work.

IV. PERTURBATION THEORY

We first summarize the two-loop perturbative results as detailed in the Appendix, which predict [see Eq. (A10)]

$$g_{\text{pert}}^2(p) = \frac{g^2(\mu)}{1 + \frac{2g^2(\mu)}{(4\pi)^4} \log\left(\frac{p^2}{\mu^2}\right)}. \quad (7)$$

This perturbative result exhibits a pole at the finite infrared momentum scale

$$\mu_{\text{pert}}^* = \mu e^{-(4\pi)^4/(4g^2(\mu))}. \quad (8)$$

This behavior may be contrasted with the perturbative coupling flow of a familiar scalar quantum field theory with single-component field ϕ and quartic interaction term $\lambda\phi^4$ in four dimensions. In that case the renormalization flow is described by the beta-function $\beta_\lambda \sim \lambda^2$, leading to the scale-dependent coupling $\lambda(p) = \lambda(\mu)(1 - K\lambda(\mu) \log(p^2/\mu^2))^{-1}$ for some constant $K > 0$. Comparing to Eq. (7), one observes that a sign flip in the denominator transforms the UV Landau pole of the ϕ^4 model into an IR pole of an asymptotically free theory—a well known feature of the ϕ^4 -model with negative coupling, $\lambda < 0$ [18,19]. However, such a model with repulsive interaction has classically an unbounded spectrum and is therefore considered unstable. Recently this conclusion has been reinvestigated in the context of $\mathcal{PT}$ -symmetry [20–22]. In contrast, our theory is bounded from below due to the two additional (positive semidefinite) quartic couplings and, importantly, the two-loop beta-function $\beta_{g_{\text{real}}} \sim g_{\text{real}}^3$ changes sign for an imaginary coupling $g_{\text{real}} \rightarrow ig$, not a negative one. The beta-function of our tensor field theory starts at cubic order in the tetrahedral coupling because its flow is driven solely by the wavefunction renormalization.

V. NONPERTURBATIVE FLOW OF THE TETRAHEDRAL COUPLING

The running of the tetrahedral coupling is controlled by the wave function renormalization and, gathering Eqs. (4) and (5), we obtain

$$\frac{g^2(\mu)}{g^2(p)} = Z(p)^4 = \left[\frac{G^{-1}(p) - G^{-1}(0)}{p^2} \right]^4. \quad (9)$$

The full large- N result for the scale-dependent coupling is given for a wide range of renormalized masses (see color code) in Figs. 1 and 2. In Fig. 2, we plot its inverse on a logarithmic momentum scale for a broader range of masses to illustrate the deviation from the perturbative result (7), which is represented by a strictly straight dashed line. At large momenta, the perturbative and nonperturbative solutions agree increasingly well, consistent with the prediction of asymptotic freedom. However, the slope of the nonperturbative flow becomes steeper toward the IR such that the coupling grows faster than the perturbative one at intermediate scales.

While the perturbative result is insensitive to the renormalized mass, the full coupling is seen to depend on it. The coupling reaches a mass-dependent finite value in the deep infrared for masses above a threshold value. As we decrease the renormalized mass, the limiting value of the tetrahedral coupling grows, as depicted in Fig. 3 [23]. The observed behavior suggests that, for any given $g(\mu)$, there exist a finite mass m^* for which the tetrahedral coupling diverges at a finite IR momentum scale μ^* . We stress that the mass scale m^* should not be identified with the dynamically generated scale μ^* . We can estimate m^* from Fig. 3 and the momentum scale μ^* by extrapolating the envelope of the curves in Fig. 2. We find that the dynamically generated scale μ^* is significantly larger than the mass scale m^* , which in turn is much larger than the perturbative scale μ_{pert}^* .

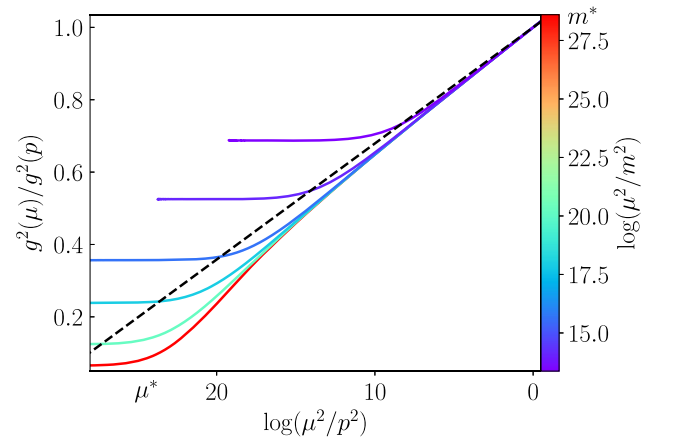


FIG. 2. Inverse of the squared tetrahedral coupling for different renormalized masses (color scale). The perturbative result is displayed as a black dashed line.

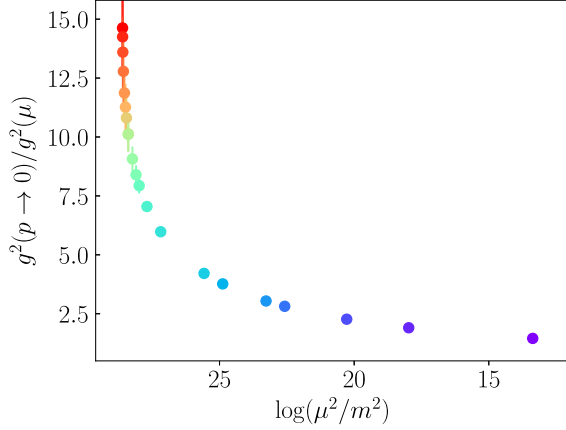


FIG. 3. Limiting value of the squared tetrahedral coupling in the infrared for varying mass.

[24]. These values of m^* and μ^* are highlighted in Fig. 2. For completeness, the corresponding results for the propagator are discussed in the Appendix.

VI. RELATION TO OTHER TENSOR FIELD THEORIES

In order to put our results into context, we compare to similar tensor models in the literature. First, one can consider the $O(N)^3$ model in $4 - \epsilon$ dimensions [4]. This model has a nontrivial (so-called “melonic”) fixed point with couplings of order $\sqrt{\epsilon}$. However, while the tetrahedral coupling is real, the pillow and double trace couplings are imaginary at the fixed point, and the resulting conformal field theory is unstable as it has a primary operator in the principal series [25,26]. Alternatively, one can consider a long-range version of the $O(N)^3$ model in $d < 4$ dimensions [5,27]. Picking the marginal scaling for the propagator and an imaginary tetrahedral coupling, one obtains a line of infrared fixed points (indexed by the exactly marginal tetrahedral coupling), which are stable and correspond to well-defined (and presumably unitary [28]) large- N conformal field theories. More generally, the renormalization group fixed points for tensor field theories give rise to a new family of conformal “melonic” field theories which can be studied analytically [29–36] (see also [37–41] for reviews and references therein) [42].

We stress that the behavior we encounter here is of a very different nature. The infrared regime we identify does not correspond to a renormalization group fixed point: although the (classically marginal) tetrahedral coupling flows to a fixed value, the renormalized mass is nonzero and larger than a threshold value.

VII. REAL TETRAHEDRAL COUPLING

If one considers the $O(N)^3$ model with a real tetrahedral coupling in exactly four dimensions, the melonic fixed

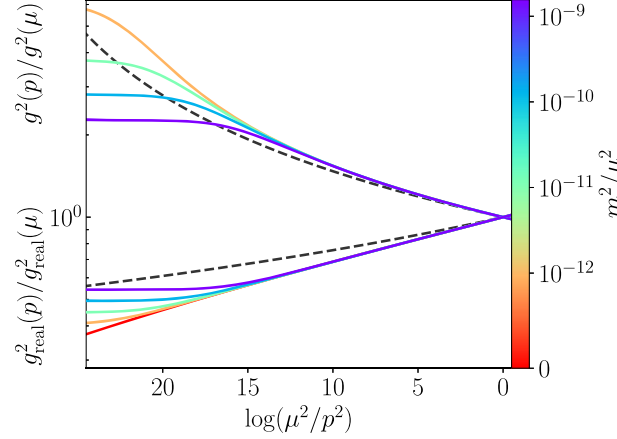


FIG. 4. Comparison of real vs imaginary tetrahedral coupling for different renormalized masses and $|g(\mu)| = |g_{\text{real}}(\mu)|$. The corresponding perturbative predictions are shown as dashed black lines. For real tetrahedral coupling also the solution at vanishing renormalized mass is shown (red curve).

point of order $\sqrt{\epsilon}$ coincides with the trivial Gaussian fixed point. The perturbative computation, Eq. (7) with the sign in the denominator flipped, shows that the tetrahedral coupling vanishes in the IR like $g^2(p) \sim \log(\mu^2/p^2)^{-1}$ and displays a UV Landau pole at a finite scale. It is not known whether this flow is completed by some nontrivial UV fixed point. In Fig. 4 we contrast the running in the real and imaginary case obtained by solving Eq. (6) with $g^2 \rightarrow -g_{\text{real}}^2$. The corresponding two-point functions are discussed in the Appendix. For a real tetrahedral coupling we are able to obtain a self-consistent solution for the propagator with vanishing renormalized mass. This solution exhibits a vanishing coupling for $p \rightarrow 0$ (red curve in Fig. 4). The self-consistent running decreases faster toward the IR than predicted by perturbation theory (dashed). Similar to the imaginary tetrahedral coupling case, the self-consistent solution with real tetrahedral coupling is sensitive to the presence of a nonvanishing renormalized mass, whereas the perturbative solution is not.

VIII. DISCUSSION AND OUTLOOK

Our results show that for an asymptotically free massive scalar field theory in four dimensions quantum fluctuations can generate a strong coupling that remains well defined in the infrared. The phenomenon is captured in closed form by the equation for the scalar field two-point correlator in the large- N limit. The wave function renormalization determines the growth of the running coupling toward the infrared. In contrast to the standard perturbative behavior, as long as the mass is above a threshold value, the full nonperturbative coupling remains finite in the infrared. By varying the mass over several orders of magnitude, we find that the infrared value of the coupling grows as the mass is decreased, and exceeds the perturbative estimate.

It is very interesting to consider these results in view of other asymptotically free theories such as QCD, where the nonperturbative generation of a strong-interaction scale by quantum fluctuations is known to have striking phenomenological consequences such as confinement [43]. The gluons in QCD are massless, which is protected by local gauge symmetry, and only gauge-invariant quantities are observable. In particular, in QCD the perturbative notion of a (gauge-variant) coupling ceases to hold in the infrared, where it diverges at the confinement scale Λ_{QCD} . Comparing this to the asymptotically free scalar field theory, the role of Λ_{QCD} is played by the scale μ_{pert}^* in the perturbative tensor model. However, the scalar field theory allows one to vary the mass scale and investigate the dynamical build-up of a strong-interaction as the correlation length increases in a regime where the nonperturbative coupling remains well defined and its infrared value determines the physical interaction strength.

ACKNOWLEDGMENTS

We thank J. Horak, I. Klebanov, T. Krajewski, T. Muller, J. Pawłowski, F. Popov, P. Romatschke, F. Sattler,

A. Tanasa, G. Tarnopolsky, R. Venugopalan, and J. Wessely for useful discussions and comments on the draft. The authors acknowledge support by the state of Baden-Württemberg through bwHPC and the German Research Foundation (DFG) through grant no INST 40/575-1 FUGG (JUSTUS 2 cluster), the Heidelberg STRUCTURES Excellence Cluster under Germany's Excellence Strategy EXC2181/1-390900948, the DFG under the Collaborative Research Center SFB 1225 ISOQUANT (Project-ID 273811115), and the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (Grant Agreement No. 818066). T.P. acknowledges partial support from the Simons Foundation under Award number 994318 (Simons Collaboration on Confinement and QCD Strings) and thanks Stony Brook University and Brookhaven National Laboratory for their hospitality.

APPENDIX

1. Renormalization and two-loop perturbation theory

The renormalized action of the model writes in terms of the renormalized field φ_a as:

$$S[\varphi] = \int d^4x \left\{ \frac{\tilde{Z}}{2} \varphi_a(x) (-\partial^2) \varphi_a(x) + \frac{1}{2} (m^2 + \delta m^2) \varphi_a(x) \varphi_a(x) + \frac{1}{4} \left((g_1 + \delta g_1) \hat{P}_{ab;cd}^{(1)} + (g_2 + \delta g_2) \hat{P}_{ab;cd}^{(2)} + i(g + \delta g) \hat{\delta}_{abcd}^t \right) \varphi_a(x) \varphi_b(x) \varphi_c(x) \varphi_d(x) \right\}, \quad (\text{A1})$$

where the wave function renormalization constant is $\tilde{Z} = 1 + \delta\tilde{Z}$. The bare and renormalized quantities are related by $\bar{\varphi}_a = \tilde{Z}^{1/2} \varphi_a$, $\bar{m}^2 = \tilde{Z}^{-1} (m^2 + \delta m^2)$, $\bar{g} = \tilde{Z}^{-2} (g + \delta g)$, $\bar{g}_{1,2} = \tilde{Z}^{-2} (g_{1,2} + \delta g_{1,2})$ and the counterterms $\delta\tilde{Z}$, δm^2 , δg and $\delta g_{1,2}$ ensure that the renormalized correlations are free of divergences. The counterterms are fixed by the renormalization conditions:

$$G^{-1}(0) = m^2, \quad Z(\mu) = \frac{G^{-1}(\mu) - G^{-1}(0)}{\mu^2} = 1, \quad \Gamma^{(4,t)}(p_1, p_2, p_3, p_4)|_{p_i^2=\mu^2} = \delta \left(\sum_{i=1}^4 p_i \right) g(\mu), \quad (\text{A2})$$

where $G_{ab}(x, y) = \langle \varphi_a(x) \varphi_b(y) \rangle = G(x, y) \delta_{ab}$ is the renormalized two-point function and $\Gamma^{(4,t)}$ is the tetrahedral channel of the renormalized four-point function. The four-point functions in the $\hat{P}^{(1)}$ and $\hat{P}^{(2)}$ channel are fixed similarly.

The function $Z(\mu)$ arising in the renormalization condition above is related to $\tilde{Z} = \tilde{Z}(\mu)$, the renormalization constant in the renormalized action. The renormalized propagator is $G^{-1}(p) = \tilde{Z}(\mu) G_b^{-1}(p)$ with G_b^{-1} the resummed

propagator computed in the bare theory, hence $Z(p) = \tilde{Z}(\mu) f_b(p)$ for f_b some function which depends parametrically on the bare parameters. This function is of course divergent, that is it exhibits $1/\epsilon$ poles in $d = 4 - \epsilon$, or logarithmic divergences with the ultraviolet momentum cutoff Λ at $d = 4$. Fixing $Z(\mu) = 1$ yields $\tilde{Z}(\mu) = 1/f_b(\mu)$ which in turn implies $Z(p) = \tilde{Z}(\mu)/\tilde{Z}(p)$. The renormalized two-point function at large- N respects the Schwinger-Dyson equation:

$$G^{-1}(p) = \tilde{Z} p^2 + m^2 + \delta m^2 + (g_2 + \delta g_2) \int \frac{d^4 q}{(2\pi)^4} G(q) + (g + \delta g)^2 \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} G(q) G(k) G(p + q + k). \quad (\text{A3})$$

Imposing the renormalization conditions and taking into account that in the large N limit $\delta g = 0$, see Ref. [5], we obtain Eq. (6) of the main text.

Keeping the renormalization scale μ fixed, we are interested in the momentum dependence of the physical couplings of the theory, that is the local parts of the effective action at a symmetric point

$$\Gamma^{(n)}(p_i)|_{p_i^2=p^2} = \delta\left(\sum_i p_i\right) Z(p)^{n/2} g^{(n)}(p). \quad (\text{A4})$$

As $\delta g = 0$ at large- N , the momentum dependent tetrahedral coupling is entirely driven by the wave function,

$Z^2(p)g(p) = g(\mu)$. The renormalized propagator $G(p)$ is determined self-consistently by Eq. (6), which in turn fixes the wave function renormalization $Z(p)$ and the running coupling $g(p)$. The running of g_1 and g_2 is dictated by the running of g and can be extracted from their corresponding Bethe-Salpeter equations, which we will address in future work.

Two loops. The renormalization group flow at two-loops is obtained using dimensional regularization and minimal subtraction in $d = 4 - \epsilon$ and setting $\epsilon = 0$. At two loops and after mass renormalization, Eq. (A3) reads (where we denote the renormalization scale with s here for convenience)

$$G_{\text{pert}}^{-1}(p) = \tilde{Z}p^2 + m^2 + g^2 s^{2\epsilon} \int \frac{d^d q}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{1}{(q^2 + m^2)(k^2 + m^2)} \left(\frac{1}{(p+q+k)^2 + m^2} - \frac{1}{(q+k)^2 + m^2} \right), \quad (\text{A5})$$

and the sunset (melon) diagram evaluates in a Laurent series in ϵ [44]

$$\begin{aligned} & g^2 s^{2\epsilon} \int \frac{d^d q}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{1}{(q^2 + m^2)(k^2 + m^2)((p+q+k)^2 + m^2)} \\ &= -\frac{g^2 m^2}{(4\pi)^4} \left\{ \frac{6}{\epsilon^2} + \frac{6}{\epsilon} \left[\frac{3}{2} - \gamma + \log\left(\frac{4\pi s^2}{m^2}\right) \right] + \frac{p^2}{2m^2\epsilon} + O(\epsilon^0) \right\}. \end{aligned} \quad (\text{A6})$$

Up to order $1/\epsilon$ the full propagator at two loops is $G_{\text{pert}}^{-1}(p) = \tilde{Z}p^2 + m^2 - p^2 \frac{g^2}{(4\pi)^4 2\epsilon}$. In the minimal subtraction scheme, renormalization is performed by requiring that at the renormalization scale s both the tetrahedral counterterm δg and the wave function counterterm $\delta \tilde{Z}$ are pure divergences. As $\delta g = 0$ we have

$$g(s) = g, \quad \tilde{Z}(s) = 1 + \frac{g^2}{(4\pi)^4 2\epsilon}, \quad (\text{A7})$$

where the first equation signifies that the coupling constant g in the renormalized action is exactly the physical four point function (in the tetrahedral channel) at the renormalization scale s .

We note that the minimal subtraction prescription differs from imposing the renormalization conditions in Eq. (3) by finite terms, but the beta functions up to two loops are prescription independent [45,46]. The bare tetrahedral coupling writes $\bar{g} = s^\epsilon g(s) \tilde{Z}^{-2}(s)$. Taking the s derivative at fixed $\bar{g}$, at two loops we obtain

$$\beta(g) = s \partial_s g(s) = -\epsilon g(s) - \frac{2g^3(s)}{(4\pi)^4}, \quad \eta = s \partial_s \log(\tilde{Z}(s)) = \frac{\beta(g) \partial_g \tilde{Z}(s)}{\tilde{Z}(s)} = -\frac{g^2(s)}{(4\pi)^4}. \quad (\text{A8})$$

We emphasize that we obtained a scale-dependent coupling despite $\delta g = 0$, as the flow is driven by the wavefunction renormalization $\tilde{Z}(s)$, which is nontrivial already at leading order in the large- N expansion. This features sets melonic tensor field theories, such as $O(N)^3$ at large- N , apart from the more standard $O(N)$ vector models at large- N . We set $\epsilon = 0$ and integrate the flow down from some reference scale μ in the UV to find

$$g_{\text{pert}}^2(s) = \frac{g^2(\mu)}{1 + \frac{2g^2(\mu)}{(4\pi)^4} \log\left(\frac{s^2}{\mu^2}\right)}, \quad \tilde{Z}_{\text{pert}}(s) = e^{-\int_{g(s)}^{g(\mu)} \frac{\eta(g') dg'}{\beta(g')}} \tilde{Z}(\mu) = \left[1 + \frac{2g^2(\mu)}{(4\pi)^4} \log\left(\frac{s^2}{\mu^2}\right) \right]^{-\frac{1}{4}} \tilde{Z}(\mu). \quad (\text{A9})$$

Taking into account the relation between $Z(p)$ and $\tilde{Z}(p)$, we get

$$Z_{\text{pert}}(p) = \left[1 + \frac{2g^2(\mu)}{(4\pi)^4} \log\left(\frac{p^2}{\mu^2}\right) \right]^{\frac{1}{4}}. \quad (\text{A10})$$

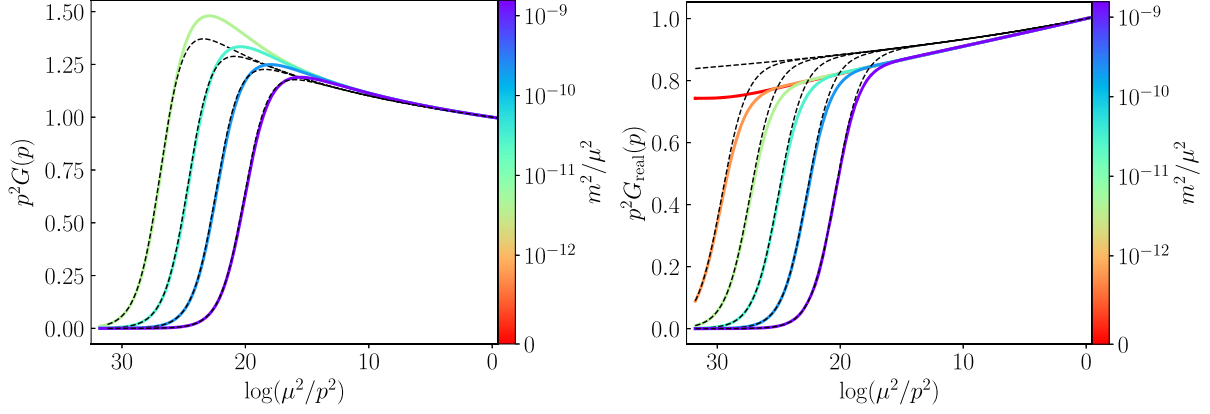


FIG. 5. Propagator with $1/p^2$ momentum dependence scaled out respectively for the case of imaginary tetrahedral coupling (left) and real tetrahedral coupling (right) displayed for various renormalized masses (color scale). The two loop results correspond to the black dashed lines.

2. Full propagator

In Fig. 5 we display the renormalized self-consistent propagator for different renormalized masses with the trivial $1/p^2$ momentum dependence scaled out. The flow of couplings discussed in Fig. 4 in the main text was obtained from the propagators depicted here.

The case with imaginary tetrahedral coupling is shown on the left. Due to the Gaussian UV fixed point, all the solutions asymptote to 1 at high momenta. As we decrease p toward the IR, $p^2 G(p)$ increases until the presence of a nonvanishing renormalized mass m^2 eventually suppresses the momentum dependence and $p^2 G(p) \sim p^2/m^2$ drops to zero. If we decrease m^2 , we extend the range of scales over which $p^2 G(p)$ grows and we observe an increasingly prominent bump at intermediate momentum scales with associated increasing deviations from the two-loop perturbation theory result

$$p^2 G_{\text{pert}}(p) \simeq \frac{p^2}{m^2 + p^2 \left[1 + \frac{2g^2(\mu)}{(4\pi)^4} \log\left(\frac{p^2}{\mu^2}\right) \right]^{\frac{1}{4}}}, \quad (\text{A11})$$

which is displayed in dashed black. Observe that Eq. (A11) is only valid for momenta $p^2 \geq (\mu_{\text{pert}}^*)^2$ with $\mu_{\text{pert}}^* = \mu \exp[-(4\pi)^4/(4g^2(\mu))]$ since the term in square brackets becomes negative for $p \leq \mu_{\text{pert}}^*$.

The case with real tetrahedral coupling is shown on the right in Fig. 5. In contrast to the imaginary case, the propagator is more suppressed for low momenta and we were able to obtain a solution for vanishing mass. In the perturbative two-loop result for the propagator in Eq. (A11) only the sign in front of $g^2(\mu)$ changes. This makes the perturbative result well defined for all momenta $p \leq \mu_{\text{pert}}^*$. These differences are not surprising as for real tetrahedral coupling the Gaussian fixed point is IR attractive. In contrast to the imaginary tetrahedral case, for real coupling it is not known whether there exists a nontrivial UV fixed point and consequently the theory might not exist without a UV cutoff.

3. Numerical implementation

Implementing the renormalization conditions (3) amounts to subtracting the sunset integral evaluated at zero external momentum respectively at the renormalization scale μ . The mass and wave function counterterms are

$$\delta m^2 = -(g_2 + \delta g_2) \int \frac{d^4 q}{(2\pi)^4} G(q) - g^2(\mu) \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} G(q) G(k) G(q+k), \quad (\text{A12})$$

$$\delta \tilde{Z} = -\frac{g^2(\mu)}{\mu^2} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} G(q) G(k) [G(\mu+q+k) - G(q+k)], \quad (\text{A13})$$

leading to the renormalized version of the Schwinger-Dyson equation in Eq. (6)

$$G^{-1}(p) = p^2(1 + \delta \tilde{Z}) + m^2 + g^2(\mu) I(G; p), \quad (\text{A14})$$

with

$$I(G; p) = \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} G(q) G(k) [G(p+q+k) - G(q+k)], \quad (\text{A15})$$

and $\delta \tilde{Z} = -g^2(\mu)/\mu^2 I(G; \mu)$.

We discretize by using a logarithmic p^2 grid with 7000 points and IR/UV cutoffs, respectively $\Lambda_{\text{IR}}/\mu = 1.25 \times 10^{-7}$ and $\Lambda/\mu = 1.25$, which regularize the integral.

a. Algorithm

We solve the renormalized Schwinger-Dyson equation for the propagator via fixed-point iteration. The algorithm proceeds as follows:

- (1) We initiate the solver by providing the arbitrary renormalization scale μ , chosen for convenience to lie in the UV with $\mu/\Lambda = 0.8$, and the renormalized parameters m^2 and $g(\mu)$.
- (2) Due to asymptotic freedom [3], we can and do choose the initial ansatz for the propagator to coincide with the classical one $[G^{-1}(p)]^{i=0} = p^2 + m^2$. Here the superscript i denotes the iteration step.
- (3) We calculate the integral $I([G]^i; p)$ in Eq. (A15). This requires some interpolation and extrapolation for p^2 values that are not elements of the grid (see next paragraph).
- (4) We determine $\delta\tilde{Z}$ from $I([G]^i; \mu)$.
- (5) We evaluate the right hand side of Eq. (A14) as $[\text{RHS}]^{i+1} = p^2(1 - g^2(\mu)/\mu^2 I([G]^i, \mu)) + m^2 + g^2(\mu)I([G]^i, p)$ and set $[G^{-1}]^{i+1} = \alpha[\text{RHS}]^{i+1} +$

$(1 - \alpha)[G^{-1}]^i$ with mixing parameter $\alpha = 0.2$ to improve the convergence of the algorithm.

- (6) We repeat steps 3–5 until apparent convergence is achieved. This is quantified by confirming that the grid-point-wise relative deviation of $[G^{-1}]^i$ and $[G^{-1}]^{i+1}$ averaged over all grid points is below a predefined threshold, 10^{-7} in our case.

We tested the insensitivity of all displayed results by varying the resolution of the grid and the cutoffs over four orders of magnitude.

b. Integration

We use hyperspherical coordinates $(r, \theta, \psi, \phi) \in [0, \infty) \times [0, \pi] \times [0, \pi] \times [0, 2\pi]$ and denote $z = \cos(\theta)$, $y = \cos(\psi)$, such that the integral measure on $\mathbb{R}^4$ can be written as $d^4p = \frac{1}{2}p^2\sqrt{1-z^2}d(p^2)dzdyd\phi$. Due to spherical symmetry all functions only depend on the invariant p^2 . We make use of the fact that the sunset integral:

$$M(p) = \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} G(q)G(k)G(p+q+k), \quad (\text{A16})$$

can be written as two nested convolutions. First, we define:

$$F(p) = \int \frac{d^4k}{(2\pi)^4} G(p+k)G(k) = \frac{1}{(2\pi)^3} \int_{\Lambda_{\text{IR}}^2}^{\Lambda^2} dk^2 \int_{-1}^1 dz k^2 \sqrt{1-z^2} G(p^2 + k^2 + |p||k|z)G(k^2), \quad (\text{A17})$$

and, second, we notice that $I(p) = M(p) - M(0)$ can be computed as:

$$I(p) = \int \frac{d^4q}{(2\pi)^4} (F(p+q) - F(q))G(q) = \int_{\Lambda_{\text{IR}}^2}^{\Lambda^2} \frac{dq^2}{(2\pi)^3} \int_{-1}^1 dz q^2 \sqrt{1-z^2} G(q^2)[F(p^2 + q^2 + |p||q|z) - F(q^2)]. \quad (\text{A18})$$

The angular z -integrals are performed via a Gauss-Chebyshev quadrature with 64 points and for the q^2 -integrals we use Gauss-Legendre quadrature [47] with 7000 points.

Momenta probed in the convolution range from 0 to 2Λ , such that $G^{-1}(p)$ and $F(p)$ need to be extrapolated. For $|p| < \Lambda_{\text{IR}}$ we set both functions to be equal to their values at Λ_{IR} . For $\Lambda < |p| < 2\Lambda$ we make use of asymptotic

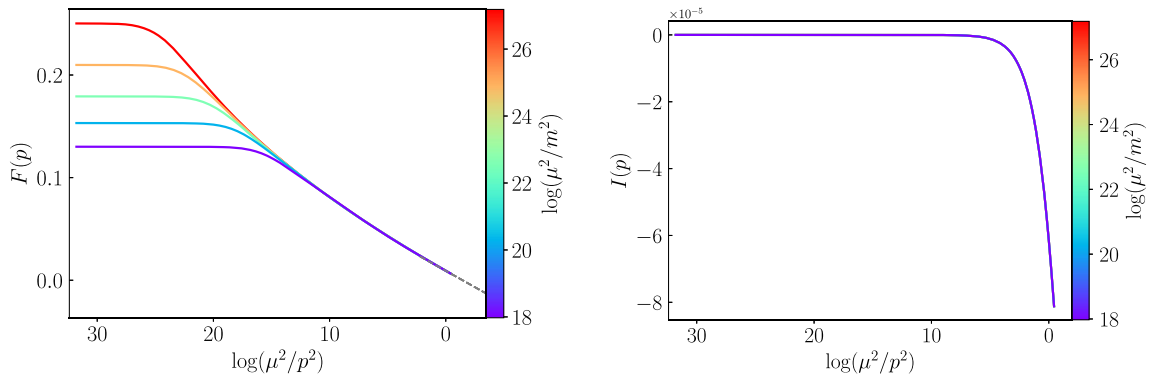


FIG. 6. Left: Numerical results for the convolution of two propagators $F(p) = \int \frac{d^4k}{(2\pi)^4} G(p+k)G(k)$ together with the prescribed extrapolation functions (A19). The extrapolation for $\Lambda < p \leq 2\Lambda$ is shown in dashed gray. Right: Numerical results for the melon integral with subtracted local part (A15). The individual lines for different masses are not distinguishable in this plot.

freedom and extrapolate with the momentum dependence inferred from perturbation theory [48]:

$$G_{\text{pert}}^{-1}(p) = m^2 + p^2 \left[1 + \frac{2g^2(\mu)}{(4\pi)^4} \log\left(\frac{p^2}{\mu^2}\right) \right]^{\frac{1}{4}},$$

$$F_{\text{pert}}(p) = F(\Lambda) \left[1 + \log\left(\frac{\Lambda^2}{p^2}\right) \right]. \quad (\text{A19})$$

The validity of these extrapolations is tested by varying the cutoffs and comparing to the numerical result at high and low momenta. The corresponding numerical result for the convolution $F(p) = \int \frac{d^4k}{(2\pi)^4} G(p+k)G(k)$ and the sunset integral with subtracted local part (A15) are shown in Fig. 6.

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2.4 Conclusions

We proposed an asymptotically free scalar quantum field theory in four dimensions, a setting in which scalar theories are typically weakly coupled [28]. Strongly coupled quantum field theories rarely admit non-perturbative analytical techniques and often rely on numerical methods such as importance sampling, which, however, break down outside of thermal equilibrium. Owing to a global tensor symmetry, the proposed theory offers a non-trivial yet analytically tractable large- N resummation. This enables a non-perturbative computation of both the resummed propagator and the renormalization group flow of the (square of the) tetrahedral coupling g across all distance scales. In Sec. 2.3, we observed that quantum fluctuations dynamically generate a strong coupling that remains finite in the infrared for masses above a threshold set by the strong coupling scale. This provides an analytically controlled framework for investigating the strongly coupled infrared regime of a quantum field theory in four dimensions – an open problem of central importance in theoretical high-energy physics.

As for the Euclidean theory, the non-perturbative flows of the couplings g_1 and g_2 remain to be studied by solving the corresponding Bethe-Salpeter equations, as for example given in Eq. (A.25). This would provide a non-perturbative investigation of the Separatrix discussed in Sec. 2.2, which defines the renormalization group trajectory along which the path integral remains bounded. Since the flows of g_1 and g_2 do not couple back to the flow of the tetrahedral coupling up to NNLO corrections in $1/N$ [51], this analysis will not alter the results obtained for the propagator and tetrahedral coupling g in Sec. 2.2 and Sec. 2.3. As the strong coupling scale is approached, the tetrahedral coupling exhibits unbounded growth. The resulting non-perturbative phenomena – such as possible bound state formation in the g_1 or g_2 channels – remain to be investigated.

The question of existence and stability of the theory in Minkowski space remains central to realizing a real-time description of a strongly-coupled quantum field theory from first principles. At LO in the large- N expansion, the diagrammatic structure suggests that the theory with real tetrahedral coupling thermalizes. In particular, the LO melon diagram matches the two-loop diagram of the quartic $\lambda\varphi^4$ theory, given in Eq. (4.20), which captures effects such as direct scattering and thermalization as has been extensively studied. For the asymptotically free theory with imaginary tetrahedral coupling, it remains to be determined whether the theory is stable and how unitarity is preserved in the presence of the unbounded growth of the tetrahedral coupling near the strong coupling scale. If the theory can be consistently formulated in Minkowski space, this would open the door to studying fundamental bounds on transport coefficients and thermalization times from first principles in the presence of a scale-dependent strong coupling.

A Combined Appendix on asymptotically free tensor field theory in four dimensions

In this Appendix, we provide further details on the $O(N)^3$ model and the analysis presented in Sec. 2.2 and Sec. 2.3.

A.1 Properties of projectors

We can transform to the $1/2$ -basis through the use of suitably defined projectors $\hat{P}^{(1)} = 3(\hat{\delta}^p - \hat{\delta}^d)$ and $\hat{P}^{(2)} = \hat{\delta}^d$, which is readily inverted to $\hat{\delta}^p = \hat{P}^{(1)}/3 + \hat{P}^{(2)}$.

They allow for $\hat{\delta}_{\mathbf{ab};\mathbf{cd}}^p = \hat{\delta}_{\mathbf{ba};\mathbf{cd}}^p = \hat{\delta}_{\mathbf{cd};\mathbf{ab}}^p$, $\hat{\delta}_{\mathbf{ab};\mathbf{cd}}^d = \hat{\delta}_{\mathbf{ba};\mathbf{cd}}^d = \hat{\delta}_{\mathbf{cd};\mathbf{ab}}^d$, $\hat{\delta}_{\mathbf{abcd}}^t = \hat{\delta}_{\mathbf{abdc}}^t$, and $\hat{\delta}_{\mathbf{a}^{\mathbf{b}}\mathbf{c}^{\mathbf{d}}\mathbf{e}^{\mathbf{f}}}^t = \hat{\delta}_{\mathbf{b}^{\mathbf{a}}\mathbf{c}^{\mathbf{d}}\mathbf{e}^{\mathbf{f}}}^t$. Accordingly, one also has $\hat{P}_{\mathbf{ab};\mathbf{cd}}^{(1/2)} = \hat{P}_{\mathbf{ba};\mathbf{cd}}^{(1/2)} = \hat{P}_{\mathbf{cd};\mathbf{ab}}^{(1/2)}$.

In the p/d -basis, the projectors are not orthonormal and one has mixing of channels as

$$\hat{\delta}^p \hat{\delta}^p = \frac{1}{3} \hat{\delta}^d + \frac{2}{3} \hat{\delta}^d, \quad (\text{A.1a})$$

$$\hat{\delta}^p \hat{\delta}^d = \hat{\delta}^d, \quad (\text{A.1b})$$

$$\hat{\delta}^d \hat{\delta}^d = \hat{\delta}^d. \quad (\text{A.1c})$$

In contrast, the projectors of the $1/2$ -basis are orthonormal

$$\hat{P}^{(1)} \hat{P}^{(1)} = 9 \left[\hat{\delta}^p \hat{\delta}^p - \hat{\delta}^p \hat{\delta}^d - \hat{\delta}^d \hat{\delta}^p + \hat{\delta}^d \hat{\delta}^d \right] = 3(\hat{\delta}^p - \hat{\delta}^d) = \hat{P}^{(1)}, \quad (\text{A.2a})$$

$$\hat{P}^{(1)} \hat{P}^{(2)} = 3(\hat{\delta}^d - \hat{\delta}^d) = 0 = \hat{P}^{(2)} \hat{P}^{(1)}, \quad (\text{A.2b})$$

$$\hat{P}^{(2)} \hat{P}^{(2)} = \hat{\delta}^d \hat{\delta}^d = \hat{\delta}^d = \hat{P}^{(1)}. \quad (\text{A.2c})$$

Upon contraction, one finds

$$\delta_{\mathbf{ac}_1\mathbf{c}_2\mathbf{c}_3}^t \delta_{\mathbf{bc}_1\mathbf{c}_2\mathbf{c}_3}^t = \delta_{\mathbf{ab}} N^3 \overset{\mathbf{a}=\mathbf{b}}{=} N^6, \quad (\text{A.3a})$$

$$\delta_{\mathbf{ac};\mathbf{bb}}^p = N^2 \delta_{\mathbf{ac}} \overset{\mathbf{a}=\mathbf{c}}{=} N^5, \quad (\text{A.3b})$$

$$\delta_{\mathbf{aa};\mathbf{bb}}^d = \delta_{\mathbf{aa}} \delta_{\mathbf{bb}} = N^6, \quad (\text{A.3c})$$

$$P_{\mathbf{aa};\mathbf{bb}}^{(1)} = 3(N^5 - N^6), \quad (\text{A.3d})$$

$$P_{\mathbf{aa};\mathbf{bb}}^{(2)} = N^6, \quad (\text{A.3e})$$

which is where the $\hat{\delta}$ -normalization originates.

In the symmetric regime $G_{\mathbf{ab}}(x, y) = \delta_{\mathbf{ab}} G(x, y)$, the following contraction patterns are

useful

$$\delta_{\mathbf{ab};\mathbf{c'd'}}^p \delta_{\mathbf{c'd'}} = \delta_{\mathbf{ab};\mathbf{cc}}^p, \quad (\text{A.4a})$$

$$= N^2 \delta_{\mathbf{ab}}, \quad (\text{A.4b})$$

$$\delta_{\mathbf{ab};\mathbf{c'd'}}^d \delta_{\mathbf{c'd'}} = \delta_{\mathbf{ab}} N^3, \quad (\text{A.4c})$$

$$\delta_{\mathbf{ac}_1\mathbf{c}_2\mathbf{c}_3}^t \delta_{\mathbf{bc}_1\mathbf{c}_2\mathbf{c}_3}^t = \delta_{\mathbf{ab}} N^3, \quad (\text{A.4d})$$

and finally

$$\delta_{\mathbf{acc}_2\mathbf{c}_3}^t \delta_{\mathbf{bdc}_2\mathbf{c}_3}^t + \delta_{\mathbf{ac}_1\mathbf{cc}_3}^t \delta_{\mathbf{bc}_1\mathbf{dc}_3}^t + \delta_{\mathbf{ac}_1\mathbf{c}_2\mathbf{c}}^t \delta_{\mathbf{bc}_1\mathbf{c}_2\mathbf{d}}^t = 3N \delta_{\mathbf{ab};\mathbf{cd}}^p. \quad (\text{A.5})$$

A.2 On the perturbative renormalization group flow

In this section, we provide further calculations and discussions on the perturbative beta functions given in Eq. (7) of Sec. 2.2.

We denote by g the renormalized tetrahedral coupling, by g_p the renormalized pillow coupling, and by g_d the renormalized double-trace coupling. Using dimensional regularization with $D = 4 - \varepsilon$ and $\varepsilon > 0$, we know from Eq. (3.12) of [48] up to two-loop order and in the large- N limit that

$$\beta = -\varepsilon g + 2g^3 = 2g^3, \quad (\text{A.6a})$$

$$\beta_p = -\varepsilon g_p + \left(6g^2 + \frac{2}{3}g_p^2\right) - 2g^2 g_p, \quad (\text{A.6b})$$

$$\beta_d = -\varepsilon g_d + \left(\frac{4}{3}g_p^2 + 4g_p g_d + 2g_d^2\right) - 2g^2 (4g_p + 5g_d). \quad (\text{A.6c})$$

This system of beta functions features a well-known large- N (“melonic”) fixed point [48] with couplings of order $\sqrt{\varepsilon}$. While the tetrahedral coupling takes real values at the fixed point, the pillow and double-trace couplings are imaginary, and the resulting conformal field theory is unstable [54, 55]. As discussed in the main text, we are, in contrast, interested in the case $\varepsilon = 0$.

With $g_1 = g_p/3$, $g_2 = g_p + g_d$, or inversely $g_p = 3g_1$, $g_d = g_2 - 3g_1$, we rewrite the beta functions into the 1/2-basis

$$\beta_1 = \frac{\beta_p}{3} = 2g^2 + 2g_1^2 - 2g^2 g_1, \quad (\text{A.7})$$

and

$$\beta_2 = \beta_p + \beta_d = 2 \left[g^2 (3 - 5g_2) + g_2^2 \right]. \quad (\text{A.8})$$

We arrive at the following system of beta functions in $D = 4$

$$\beta = 2g^3, \quad (\text{A.9a})$$

$$\beta_1 = 2 \left(g^2 + g_1^2 - g^2 g_1 \right), \quad (\text{A.9b})$$

$$\beta_2 = 2 \left[g^2(3 - 5g_2) + g_2^2 \right]. \quad (\text{A.9c})$$

The flow of the tetrahedral coupling is solved directly as

$$\frac{dg}{d \ln \mu} = 2g^3, \quad (\text{A.10})$$

$$\Leftrightarrow g^2(\mu) = \frac{g_\Lambda^2}{1 - 4g_\Lambda^2 \ln \left(\frac{\mu}{\Lambda} \right)}, \quad (\text{A.11})$$

which features a Landau pole at

$$\mu_* = \Lambda \exp \left(\frac{1}{4g_\Lambda^2} \right). \quad (\text{A.12})$$

Here, $g_\Lambda^2 \equiv g^2(\Lambda)$ with $\Lambda > \mu$ a UV scale. This behavior is familiar from QED, which also features a cubic beta function at lowest order with $\beta(e) = e^3/(12\pi)^2$. In practice, one considers

$$\alpha(\mu) = \frac{e^2(\mu)}{4\pi} = \frac{\alpha_\Lambda}{1 - \frac{2}{3\pi} \alpha_\Lambda \ln \left(\frac{\mu}{\Lambda} \right)}, \quad (\text{A.13})$$

which similarly exhibits a Landau pole at $\mu = \Lambda \exp \left(\frac{3\pi}{2\alpha_\Lambda} \right)$ with $\alpha_\Lambda \equiv \alpha(\Lambda)$.

The different sectors decouple at $g = 0$, where each sector respectively features a Landau pole as familiar from $O(N)$ -theories

$$\beta_1 = 2g_1^2, \quad (\text{A.14})$$

$$\beta_2 = 2g_2^2. \quad (\text{A.15})$$

This further illustrates how the tetrahedral coupling couples the different interaction channels at LO in large- N and drives the RG flow. A combined fixed point of all three real couplings in $D = 4$ is only given by the trivial one with $g = g_1 = g_2 = 0$.

As discussed in Sec. 4.3, the flows become asymptotically free if the tetrahedral coupling is chosen to be complex. Substituting g with ig above, where $g \in \mathbb{R}$, we recover Eq. (7) of Sec. 2.2.

A.3 Resummed renormalized perturbation theory

In this section, we provide details for the calculations of renormalized correlation functions in Sec. 2.2 and Sec. 2.3. The underlying self-consistent renormalization procedure in the Euclidean 2PI formalism has been discussed in [56–58].

Within renormalized perturbation theory, we introduce renormalized quantities as

$$\begin{aligned}\varphi_{\mathbf{a},R} &= Z^{-1/2}\bar{\varphi}_{\mathbf{a}} , \quad Z\bar{m}^2 = m^2 + \delta m^2 , \quad Z^2\bar{g}_\alpha = g_\alpha + \delta g_\alpha , \\ G(\bar{\phi}) &= ZG_R(\phi_R) , \quad Z = 1 + \delta Z ,\end{aligned}\tag{A.16a}$$

where α denotes tetrahedral (no subscript) as well as g_1 , and g_2 . The renormalized couplings are m, g, g_1, g_2 . For general correlation functions we have $Z^{n/2}\delta^n\Gamma/\delta\phi^n = \delta^n\Gamma/\delta\phi_R^n$. The renormalized 2PI effective action can be expressed in terms of the rescaled variables

$$\Gamma[G] = \Gamma_R[G_R] = \frac{1}{2}\text{Tr}\ln G_R^{-1} + \frac{1}{2}\text{Tr}ZG_0^{-1}G_R + \Gamma_2^R[G_R; \bar{g}_\alpha \rightarrow g_\alpha + \delta g_\alpha] .\tag{A.17}$$

As discussed in Sec. 2.3, the renormalized two-point function satisfies the following equation

$$\begin{aligned}G_R^{-1}(p) &= Zp^2 + m^2 + \delta m^2 + (g_2 + \delta g_2) \int_q G_R(q) \\ &+ g^2 \int_{q_1 q_2} G_R(q_1)G_R(q_2)G_R(p - q_1 - q_2) ,\end{aligned}\tag{A.18}$$

with short-hand notation for momentum integrals $\int_q \equiv \int d^D q/(2\pi)^D$. Besides the two-point function fixing the bare mass and wavefunction renormalization, we also need to provide four-point functions to renormalize the quartic couplings. Instead of $\Gamma^{(4)}$, it has been shown that one can equivalently use four-point functions obeying a Bethe-Salpether equation [56–59]

$$\begin{aligned}\bar{V}_{\mathbf{ab};\mathbf{cd}}(x, y; u, v) &= \bar{\Lambda}_{\mathbf{ab};\mathbf{cd}}(x, y; u, v) - \frac{1}{2} \int d^D w d^D w' d^D z d^D z' \bar{\Lambda}_{\mathbf{ab};\mathbf{ef}}(x, y; w, z) \\ &\times G_{\mathbf{ee}'}(w, w')G_{\mathbf{ff}'}(z, z')\bar{V}_{\mathbf{ef}';\mathbf{cd}}(w', z'; u, v) ,\end{aligned}\tag{A.19}$$

with four-point 2PI kernel to LO in $1/N$ in the $O(N)^3$ symmetric regime

$$\bar{\Lambda}_{\mathbf{ab};\mathbf{cd}}(x, y; z, w) = \frac{4\delta^2\Gamma_2}{\delta G_{\mathbf{ab}}(x, y)\delta G_{\mathbf{cd}}(z, w)}\tag{A.20}$$

$$\begin{aligned}&= \hat{P}_{\mathbf{ab};\mathbf{cd}}^{(1)} \left\{ 2\delta(x, y)\delta(x, u)\delta(x, v)\bar{g}_1 + 2\bar{g}^2 G^2(x, y)\delta(x, u)\delta(y, v) \right\} \\ &+ \hat{P}_{\mathbf{ab};\mathbf{cd}}^{(2)} \left\{ 2\delta(x, y)\delta(x, u)\delta(x, v)\bar{g}_2 + 6\bar{g}^2 G^2(x, y)\delta(x, u)\delta(y, v) \right\} .\end{aligned}\tag{A.21}$$

These can be calculated directly from the self-energy as

$$\frac{\delta\Sigma_{\mathbf{ab}}(x', y')}{\delta G_{\mathbf{cd}}(z, t)} = -2 \frac{\delta^2 \Gamma_2}{\delta G_{\mathbf{cd}}(z, t) \delta G_{\mathbf{ab}}(x', y')} \quad (\text{A.22})$$

$$\begin{aligned} &= - \left[\frac{\bar{g}_p}{N^2} \delta_{\mathbf{ab};\mathbf{cd}}^p + \frac{\bar{g}_d}{N^3} \delta_{\mathbf{ab};\mathbf{cd}}^d \right] \delta(x, y) \delta(x, z) \delta(x, t) \\ &+ \frac{\bar{g}^2}{2N^3} \delta_{\mathbf{ac}_1\mathbf{c}_2\mathbf{c}_3}^t \delta_{\mathbf{bd}_1\mathbf{d}_2\mathbf{d}_3}^t \sum_{i=1}^3 (\delta(x, z) \delta(y, t) \delta_{\mathbf{c}_i\mathbf{c}} \delta_{\mathbf{d}_i\mathbf{d}} + \delta(x, t) \delta(y, z) \delta_{\mathbf{c}_i\mathbf{d}} \delta_{\mathbf{d}_i\mathbf{c}}) \\ &\times \prod_{j \neq i} G_{\mathbf{c}_j\mathbf{d}_j}(x, y) \\ &G_{\mathbf{ab}} \equiv \delta_{\mathbf{ab}} G - \left[\bar{g}_p \hat{\delta}_{\mathbf{ab};\mathbf{cd}}^p + \bar{g}_d \hat{\delta}_{\mathbf{ab};\mathbf{cd}}^d \right] \delta(x, y) \delta(x, z) \delta(x, t) \\ &+ 3\bar{g}^2 G^2(x, y) \delta(x, z) \delta(y, t) \hat{\delta}_{\mathbf{ab};\mathbf{cd}}^p, \end{aligned} \quad (\text{A.23})$$

and are subsequently transformed to the 1/2-basis. As $\hat{P}^{(1)}$ and $\hat{P}^{(2)}$ are orthonormal projectors, the different channels do not mix in the four-point function $\bar{V}$ such that one can simply decompose

$$\bar{V}_{\mathbf{ab};\mathbf{cd}}(x, y; u, v) \equiv \hat{P}_{\mathbf{ab};\mathbf{cd}}^{(1)} \bar{V}^{(1)}(x, y; u, v) + \hat{P}_{\mathbf{ab};\mathbf{cd}}^{(2)} \bar{V}^{(2)}(x, y; u, v). \quad (\text{A.24})$$

Importantly, no tetrahedral channel is present such that the tetrahedral coupling does not receive quantum corrections, i.e. $\delta g = 0$, and its flow is dictated solely by the wavefunction renormalization.

Defining $\bar{V}_R^{(1,2)}(p) \equiv \bar{V}_R^{(1,2)}(p, -p; 0, 0)$, the respective Bethe-Salpeter equations read in a closed form in momentum space

$$\begin{aligned} \bar{V}_R^{(1)}(p) &= 2(g_1 + \delta g_1) + 2g^2 \int_q G_R(q) G_R(p - q) - (g_1 + \delta g_1) \int_q G_R^2(q) \bar{V}_R^{(1)}(q) \\ &- g^2 \int_{q_1 q_2} G_R(q_1) G_R^2(q_2) G_R(p - q_1 - q_2) \bar{V}_R^{(1)}(q_2), \end{aligned} \quad (\text{A.25a})$$

$$\begin{aligned} \bar{V}_R^{(2)}(p) &= 2(g_2 + \delta g_2) + 6g^2 \int_q G_R(q) G_R(p - q) - (g_2 + \delta g_2) \int_q G_R^2(q) \bar{V}_R^{(2)}(q) \\ &- 3g^2 \int_{q_1 q_2} G_R(q_1) G_R^2(q_2) G_R(p - q_1 - q_2) \bar{V}_R^{(2)}(q_2). \end{aligned} \quad (\text{A.25b})$$

One can determine the counterterms through the following renormalization conditions

$$Z\Gamma^{(2)}(p)|_{p=0} = G_R^{-1}(p^2)|_{p=0} = m^2, \quad (\text{A.26a})$$

$$Z \frac{d}{dp^2} \Gamma^{(2)}(p)|_{p=\mu} = \frac{dG_R^{-1}(p^2)}{dp^2}|_{p=\mu} = 1, \quad (\text{A.26b})$$

$$Z^2 \Gamma^{(4,\alpha)}|_{p_1=p_2=p_3=p_4=\mu} = Z^2 \bar{V}^{(\alpha)}|_{p_1=p_2=p_3=p_4=\mu} = 2g_\alpha, \quad (\text{A.26c})$$

where μ is an arbitrary renormalization scale. Beginning with the counterterms determined

from the two-point function, the wavefunction counterterm reads

$$\begin{aligned} \frac{d}{dp^2} G_R^{-1}(p)|_{p=\mu} &= 1 \\ \Leftrightarrow \quad \delta Z &= -g^2 \frac{d}{dp^2} \int_{q_1 q_2} G_R(q_1) G_R(q_2) G_R(p - q_1 - q_2) \Big|_{p=\mu}, \end{aligned} \quad (\text{A.27})$$

and the mass counterterm is calculated as

$$\begin{aligned} G_R^{-1}(p=0) &= m^2 \\ \Leftrightarrow \quad \delta m^2 &= -(g_2 + \delta g_2) \int_q G_R(q) - g^2 \int_{q_1 q_2} G_R(q_1) G_R(q_2) G_R(q_1 + q_2). \end{aligned} \quad (\text{A.28})$$

The coupling counterterms are determined from the respective Bethe-Salpeter equations as

$$\begin{aligned} \bar{V}_R^{(1)}(p_i = \mu) &= 2g_1 \\ \Leftrightarrow \quad \delta g_1 &= \left\{ -g^2 \int_q G_R(q) G_R(\mu - q) + \frac{g_1}{2} \int_q G_R^2(q) \bar{V}_R^{(1)}(q) \right. \\ &\quad \left. + \frac{g^2}{2} \int_{q_1 q_2} G_R(q_1) G_R^2(q_2) G_R(\mu - q_1 - q_2) \bar{V}_R^{(1)}(q_2) \right\} \\ &\quad \times \left\{ 1 - \frac{1}{2} \int_q G_R^2(q) \bar{V}_R^{(1)}(q) \right\}^{-1}, \end{aligned} \quad (\text{A.29})$$

and similarly

$$\begin{aligned} \delta g_2 &= \left\{ -3g^2 \int_q G_R(q) G_R(\mu - q) + \frac{g_2}{2} \int_q G_R^2(q) \bar{V}_R^{(2)}(q) \right. \\ &\quad \left. + \frac{3g^2}{2} \int_{q_1 q_2} G_R(q_1) G_R^2(q_2) G_R(\mu - q_1 - q_2) \bar{V}_R^{(2)}(q_2) \right\} \\ &\quad \times \left\{ 1 - \frac{1}{2} \int_q G_R^2(q) \bar{V}_R^{(2)}(q) \right\}^{-1}. \end{aligned} \quad (\text{A.30})$$

As discussed previously, there is no tetrahedral counterterm such that the tetrahedral flow is solely driven by the wavefunction renormalization $g = Z^2 \bar{g}$.

3 Self-bound droplets in ultracold gases

In this chapter, we demonstrate that self-bound droplets can emerge in bosonic systems near a narrow Feshbach resonance. The theoretical framework – including the model, the role of Feshbach resonances, and the characterization of phase transitions – is introduced in Sec. 3.1. In Sec. 3.2, we show that the droplets studied here are already stabilized at the mean-field level due to a coupling between atoms and molecules facilitated by the Feshbach resonance. Conclusions are presented in Sec. 3.3, and a potential experimental realization is discussed in Ch. 5.

This chapter is based on:

- [6] Ke Wang*, Thimo Preis, and Dam Thanh Son. *Droplets of Bosons at a Narrow Resonance*. July 3, 2024. DOI: 10.48550/arXiv.2407.03102.

The original idea was developed by K. Wang and D.T. Son. The wavefunction ansatz was proposed by D.T. Son and confirmed by K. Wang through imaginary-time simulations. I carried out the minimization and visualization of various bulk observables across a broad range of system parameters. K. Wang and I collaborated on the data visualization related to the minimization results and the phase diagram. All authors contributed to the writing of the manuscript.

3.1 Theoretical background

Ultracold atomic gases provide a versatile platform to study quantum many-body physics in controlled settings. In particular, they offer access to exotic quantum states of matter, including Bose-Einstein condensates (BECs) exhibiting superfluidity, as well as self-bound droplets. The focus of this chapter is on the formation and properties of self-bound bosonic droplets near a narrow magnetic Feshbach resonance.

Self-bound droplets are stable configurations of matter that do not require external confinement to remain localized. Unlike classical droplets bound by surface tension, these droplets emerge from a subtle balance between attractive and repulsive interactions, often stabilized by quantum fluctuations or interconversion between different particle species. Early examples include liquid ^4He clusters [60, 61], which are dense and strongly interacting. More recently, ultradilute quantum droplets have been observed in dipolar Bose gases [62] and in Bose-Bose mixtures [33, 34, 63–65], where stabilization arises from beyond-mean-field effects (as quantified by Lee-Huang-Yang corrections).

In contrast, the droplets studied in [6] and presented here are stabilized by the coherent exchange between atoms and molecules across a narrow Feshbach resonance. The associated hybrid condensates were previously explored in mean-field studies [66–70]. Our goal is to understand the non-trivial interplay between atomic and molecular components that results in droplet formation, and to characterize the resulting phases and transitions in the presence of tunable interactions.

3.1.1 The model

We consider a two-channel model that captures the essential physics of a narrow Feshbach resonance between atomic and molecular states (respectively referred to as open and closed channels). The system contains bosonic atoms (field ψ) and molecules (field ϕ), with coherent conversion enabled by a Feshbach coupling α and a tunable detuning ν . The Lagrangian reads [35, 67]

$$\begin{aligned} \mathcal{L} = & \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} \right) \psi + \phi^\dagger \left(i\partial_t + \frac{\nabla^2}{4m} \right) \phi - \nu \phi^\dagger \phi \\ & + \alpha \left(\psi^\dagger \psi^\dagger \phi + \phi^\dagger \psi \psi \right) - g_{11} \psi^\dagger \psi^\dagger \psi \psi - g_{12} \psi^\dagger \phi^\dagger \phi \psi - g_{22} \phi^\dagger \phi^\dagger \phi \phi . \end{aligned} \quad (3.1)$$

Here, g_{11} , g_{12} , and g_{22} represent background atom-atom, atom-molecule, and molecule-molecule interactions, respectively. We assume the molecular mass is $2m$, consistent with a diatomic structure, and neglect external trapping potentials for simplicity. The Feshbach term proportional to α encodes coherent interconversion: pairs of atoms can convert into a molecule and vice versa. The detuning ν acts as an effective mass term for the molecular field, controlling whether the closed-channel dimer is bound ($\nu < 0$), which favors molecule formation, or unbound ($\nu > 0$), which favors the atomic BEC.

In distinction to the relativistic setting in other chapters of this dissertation, we here consider non-relativistic units defined by $m = 1$. Accordingly, from Eq. (3.1) together with $[\nabla] = 1$, $[\partial_t] = 2$, and consequently $[\mathcal{L}] = d + 2$, we find that $[\psi] = [\phi] = d/2$. This allows us to discuss the relevance of the interaction terms through classical power-counting, as introduced in Sec. 2.1. We find that the quartic interactions are irrelevant in $d > 2$ with $[g_{ij}] = 2 - d$, whereas the detuning and the Feshbach coupling are relevant with $[\nu] = 2$ and $[\alpha] = 2 - d/2$ (relevant for $d < 4$). The theory given by Eq. (3.1) is thus superrenormalizable in $d = 3$.

We obtain effective equations of motions for the condensate fields $\phi = \langle \hat{\phi} \rangle$ and $\psi = \langle \hat{\psi} \rangle$ through a mean-field approximation as

$$i\partial_t \psi = -\frac{\nabla^2}{2m} \psi - 2\alpha \psi^\dagger \phi + 2g_{11} \psi^\dagger \psi \psi + g_{12} \phi^\dagger \phi \psi , \quad (3.2a)$$

$$i\partial_t \phi = -\frac{\nabla^2}{4m} \phi + \nu \phi - \alpha \psi \psi + g_{12} \psi^\dagger \phi \psi + 2g_{22} \phi^\dagger \phi \phi . \quad (3.2b)$$

These equations describe coupled dynamics of atomic and molecular condensates. One can recognize here the atomic $|\psi|^2$ and molecular $|\phi|^2$ condensate densities. $|\phi|^2$ contributes

twice to the overall particle number as the dimer molecule consists of two atoms, i.e. $N = \int d^d \mathbf{x} (|\psi|^2 + 2|\phi|^2)$. Note that the ϕ -field has a source term $\sim \alpha \psi^2$, which forces the molecular condensate to develop a nonzero expectation value whenever $\psi \neq 0$. The atom-molecule coupling creates a molecular condensate component in the presence of an atomic condensate. The interconversion between atoms and molecules facilitated by the $\sim \alpha$ term is the hallmark of a magnetic Feshbach resonance, where the atoms and molecules are coupled via an external magnetic field. The Feshbach coupling term $\sim -\alpha$ provides an effective attraction that lowers the system's energy when both components are present, thereby favoring the formation of a hybrid condensate and, ultimately, self-bound droplets.

3.1.2 Effective scattering parameters and Feshbach resonances

At low temperatures like in ultracold gases, most collisions happen with zero angular momentum – the so-called s -wave channel. The s -wave scattering length a is a key parameter that describes how two atoms interact at low energies. If $a > 0$ ($a < 0$), atoms effectively repel (attract) each other. If $|a|$ is very large, the atoms are on the verge of forming a bound state.

A magnetic Feshbach resonance enables experimental tuning of the s -wave scattering length a via an external magnetic field [71]. Imagine two atoms can interact via an open channel, the free atoms, and a closed channel, a molecular bound state. When the energy of the molecule matches the energy of the two atoms, the channels resonantly couple, allowing atoms to convert into molecules and back. The scattering length near such a resonance behaves as [72]

$$a(B) = a_{\text{bg}} \left(1 - \frac{\Delta}{B - B_0} \right), \quad (3.3)$$

where a_{bg} is the background scattering length, B is the magnetic field, B_0 is the resonance position, and Δ is the resonance width. As a consequence, near a resonance, the scattering length a diverges, with the resonance point being called unitarity, drastically enhancing interactions. How fast the interaction changes as one increases energy is encoded in the effective range r_0 , parametrizing the interaction potential. It serves as an effective parameter to regularize the short-distance physics. Narrow resonances lead to a weak coupling between atomic and molecular channels and a large, negative effective range $r_0 < 0$, resulting in long-lived molecular states [72]. By computing the atom-atom scattering amplitude without background interactions, one finds [35]

$$\alpha = \sqrt{\frac{4\pi}{-r_0}}, \quad \nu = -\frac{2}{(-r_0)a}. \quad (3.4)$$

The scattering length diverges as $\nu \rightarrow 0$, which corresponds to the molecular state becoming degenerate with the two-atom threshold. From Eq. (3.4), we can infer that narrower resonances have smaller α . Narrow resonances can be used to study non-perturbative physics by expanding in the dimensionless ratio of the width of the resonance to Fermi energy [35]. In contrast, in a wide Feshbach resonance, the coupling between atoms and

molecules is strong, the effective range r_0 is small, and the physics is dominated by the scattering length a alone.

3.1.3 Phase transitions

The interplay of atomic and molecular condensates gives rise to a rich phase structure. Phase transitions are qualitative changes in the state of a system as relevant external parameters, such as temperatures, detunings, chemical potential, or couplings, are varied. The liquid-gas phase transition in water is a familiar example, which are two phases that differ essentially by their density. In contrast to a gas that fills the available volume, a liquid forms a self-bound droplet with fixed density in a subvolume. Phase transitions are quantified using order parameters, where for example the peak density of a droplet in infinite volume takes a nonzero value in the liquid phase but vanishes for a gas. The transitions between different phases can be analyzed by minimizing a corresponding Landau free energy functional, where both atomic and molecular condensates will act here as order parameters in a given plane of relevant parameters. We will consider the dependence on detuning ν and chemical potential μ , which can be accounted for in the grand-canonical Hamiltonian or Lagrangian through a term $\sim \pm\mu(|\psi|^2 + 2|\phi|^2)$. This comes about since the total particle number is $N = \int d^d\mathbf{x}(|\psi|^2 + 2|\phi|^2)$. μ controls the population of the system. For example, if μ is large negative, the energy cost to add particles is too high and the system might prefer the vacuum, for suitable detuning. The structure and minima of this functional determine the ground state phases and their boundaries. Depending on the chemical potential μ and detuning ν , the system discussed here can reside in:

- A vacuum phase with no condensates;
- A molecular BEC phase ($\phi \neq 0, \psi = 0$);
- A hybrid phase with both atomic and molecular condensates ($\psi, \phi \neq 0$), where droplets can form.

Transitions between these phases can be first-order or second-order, depending on the behavior of the free energy and its derivatives. First-order transitions exhibit discontinuities in density, allowing for phase coexistence between a dense droplet and the vacuum – a hallmark of liquid formation. In contrast, second-order transitions involve continuous emergence of an order parameter and typically do not support phase coexistence. In particular, the first-order transition between vacuum and hybrid condensate is essential for droplet formation. Here, the system exhibits phase coexistence between a dense droplet and the vacuum, analogous to the liquid-gas transition. The phase diagram of Fig. 2 in Sec. 3.2 includes a first-order line separating the vacuum from the hybrid condensate phase, a second-order line separating the molecular BEC from the vacuum, a tricritical point, where these lines meet, and a critical end point shared by the three phases.

3.2

Droplets of Bosons at a Narrow Resonance

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(Dated: July 2024)

We consider bosons interacting through a narrow s -wave resonance. Such a resonance is characterized by an infinite scattering length and a large and negative effective range r_0 . We argue that any number $N \geq 3$ of bosons can form a self-bound cluster with the binding energy per particle increasing as N^2 for $1 \ll N \ll (-r_0/a_{\text{bg}})^{1/2}$, where a_{bg} is the background scattering length (between atoms and molecules). In the opposite limit $N \gg (-r_0/a_{\text{bg}})^{1/2}$, bosons form droplets with binding energy per particle saturating to a constant value independent of the particle number. The stability of clusters and droplets when the interaction is detuned from the resonance is also studied.

Introduction.—Dilute quantum droplets are particularly interesting physical systems that have recently become the subject of active theoretical and experimental study [1]. The oldest known quantum droplets are those formed by ^4He atoms [2, 3]. Such a droplet [4] exists for any number of helium atoms N , with binding energy per particle approaching the thermodynamic limit of 7.1 K when $N \rightarrow \infty$. It is notable that the approach to this asymptotic value is quite slow, and, in the intermediate regime $3 \leq N \lesssim 10$, the binding energy, instead of growing linearly, follows an approximate quadratic law of $(N - 2)^2$ [5]. It was suggested that the bosons with short-range attractive interaction can form a droplet in 3D [6] and 2D [7]. In 2015, Petrov [8] showed that in a bosonic mixture the collapse of the system can be prevented by quantum effects (see also Ref. [9]). Droplets stabilized by quantum effects have been realized experimentally [10, 11].

In this Letter, we study bound states of bosons which interact with each other through a narrow s -wave resonance. With ultracold atoms, narrow resonances are realized, for example, by cesium atoms in a magnetic field [12]. At such resonances, bosons can form both atomic and molecular condensates; the thermodynamics and dynamics of such a hybrid condensate have been studied [13–15]. In particular, it was found that such a system can form a “mutually trapped state” [16], in essence, a self-bound drop of a liquid phase that phase-coexists with the vacuum.

At the other end of the range of particle numbers, the three-body problem of bosons with narrow resonance has been solved [17]. The properties of the three-body system is completely characterized by the scattering length a and the effective range r_0 , both assumed to be much larger than any other length scales in the problem, and, in addition, r_0 is assumed to be negative. At $a = \infty$ there is an infinite tower of three-body Efimov states, with the energy of the ground state entirely determined by r_0 . It has been found that a stable three-body bound state exists only in a finite range of inverse scattering length $1/a$: when $1/a$ is too negative, the three-body bound state decays into free particles. Conversely, when

$1/a$ is too large and positive, the three-body bound state is unstable toward a decay into a particle and a bound dimer.

We will show that a system of a large number of bosons at a narrow s -wave resonance forms a bound cluster with binding energy that first increases as N^3 and whose size decreases as $1/N$ with increasing N . As N is increased further, one goes to the regime of large droplets with constant density and the core and binding energy increase as $\mathcal{O}(N)$ (i.e., constant binding energy per particle). The crossover between the two regimes happens at $N \sim (r_0/a_{\text{bg}})^{1/2}$, where a_{bg} is the characteristic scale of the background atom-atom, atom-molecule, and molecule-molecule scattering. Moreover, we show that the N -body bound state remains stable in a range of the detuning parameter $|r_0|/a$, which includes both positive and negative values of a . The width of this range scales as N^2 in the small cluster regime and is constant in the large droplet regime. Combining our results with the solution to the three-body problem [17], one comes to the conclusion that at a narrow resonance, a bound state of N bosons exists for any $N \geq 3$.

The model.—A narrow resonance is characterized by a large s -wave scattering length a and a large and negative effective range r_0 [18, 19]. The effective Hamiltonian describing this situation is (we set the mass of the atom $m = 1$, so the mass of the molecule is 2)

$$H = \int d\mathbf{x} \left[\frac{1}{2} |\nabla \psi|^2 + \frac{1}{4} |\nabla \phi|^2 - \alpha (\psi^\dagger \psi^\dagger \phi + \phi^\dagger \psi \psi) + \nu \phi^\dagger \phi \right]. \quad (1)$$

By computing the atom-atom scattering amplitude from (1), one can establish the connection between the parameters α and ν with the scattering length a and the effective range r_0 characterizing the low-energy interaction between the atoms,

$$\alpha = \sqrt{\frac{4\pi}{-r_0}}, \quad \nu = -\frac{2}{(-r_0)a}. \quad (2)$$

In particular, for negative detuning $\nu < 0$ the molecule is

bound in vacuum, while for $\nu > 0$ it is unbound. In the regime $a \ll r_0$, the binding energy of the dimer is $-\nu$.

As a nonrelativistic quantum field theory, the theory given by Eq. (1) is superrenormalizable. Indeed, the dimensions of both α and ν are positive: $[\alpha] = \frac{1}{2}$ and $[\nu] = 2$. Thus the theory (1) can be defined without an ultraviolet cutoff.

In the real world (1) is only an effective field theory. There exist irrelevant corrections to the Lagrangian with coefficients of natural (i.e., not finely tuned) magnitudes, and they limit the regime of validity of (1) in the ultraviolet. If we denote by a_{bg} the length scale associated with these terms, we can safely use Eq. (1) when the characteristic momentum is much smaller than $1/a_{\text{bg}}$. For now, let us assume that $a_{\text{bg}} = 0$, and check the effect of the irrelevant corrections to Eq. (1) later.

We will try to find the ground state of N bosons, with $N \gg 1$. First let us assume the resonance is at exact zero energy, i.e., the detuning parameter vanishes $\nu = 0$. We expect that the mean-field approximation works for $N \gg 1$ bosons. Thus the problem becomes that of minimizing the classical energy functional given by Eq. (1) with $\nu = 0$ under the constraint

$$\int d\mathbf{x} (\psi^\dagger \psi + 2\phi^\dagger \phi) = N. \quad (3)$$

There are two competing contributions to the energy of the droplet: the positive gradient energy and the negative attraction energy between the two condensates. To understand the interplay between these contributions, one can perform a simple variational calculation. Namely, we pick a profile function $f(r)$ which has a finite value at $r = 0$ and tends to zero exponentially at $r = 0$. For definiteness, we take

$$f(x) = \frac{1}{\cosh(x)}. \quad (4)$$

We then try the following ansatz for the condensates of atoms and molecules,

$$\psi(r) = \sqrt{\frac{cN}{4\pi I_2 R^3}} f\left(\frac{r}{R}\right), \quad (5a)$$

$$\phi(r) = \sqrt{\frac{(1-c)N}{8\pi I_2 R^3}} f\left(\frac{r}{R}\right). \quad (5b)$$

Here I_2 is a numerical constant that depends on the shape function $f(x)$,

$$I_2 = \int_0^\infty dx x^2 f^2(x), \quad (6)$$

with $I_2 \approx 0.822$ for the choice (4). The ansatz (5) corresponds to two clouds of atoms and molecules of the same shape and size. The variational parameter R controls the size of the droplet, and c denotes the fraction of free atoms not bound in molecules. The fraction of atoms

bound in molecules is $(1-c)$, and the total particle number is N .

In addition to $m = 1$ we will further set $-r_0 = 1$ for convenience. In particular, energy is measured in units of $\hbar^2/(mr_0^2)$. Inserting the ansatz (5) into the energy (1), one finds the variational energy

$$E(c, R) = a(c) \frac{N}{R^2} - b(c) \frac{N^{3/2}}{R^{3/2}}, \quad (7)$$

with

$$a(c) = \frac{3c+1}{8} \frac{K}{I_2}, \quad b(c) = c \sqrt{2(1-c)} \frac{I_3}{I_2^{3/2}}, \quad (8)$$

where we have defined two further characteristics of the shape

$$I_3 = \int_0^\infty dx x^2 f^3(x), \quad K = \int_0^\infty dx x^2 (f'(x))^2. \quad (9)$$

For the choice (4), $I_3 \approx 0.367$ and $K \approx 0.607$. The gradient terms in the energy lead to the $1/R^2$ contribution, which dominates at small R , and the Feshbach interaction [the terms proportional to α in Eq. (1)] leads to the $-1/R^{3/2}$ contribution, which dominates at large radii. At fixed c , the energy is minimized at radius

$$R = \frac{16a^2}{9b^2} \frac{1}{N}, \quad (10)$$

with the value at the minimum

$$E(c) = -\frac{27}{256} \frac{b^4}{a^3} N^3 = -A \frac{c^4(1-c)^2}{(c+\frac{1}{3})^3} N^3, \quad (11)$$

where $A = 8I_3^4/(I_2 K)^3 \approx 1.16$. Note that E_0 is zero both at $c = 0$ and $c = 1$: at these values one of the condensates vanishes and the coupling term $-\phi\psi^2$ does not give an attractive contribution to the energy. The minimal energy is achieved at the atomic fraction

$$c = \frac{\sqrt{17}-1}{6} \approx 0.521, \quad (12)$$

with the energy at the minimum $E_0 \approx -0.0316N^3$. Since this is only a variational calculation, this should be regarded as an upper bound on the energy. Indeed, a variational calculation based on the modified Woods-Saxon shape function $[\cosh(x/R) + \xi]^{-1}$, with five variational parameters corresponding to the parameters R and ξ of the atom and the molecular condensates, and the relative amplitude of the two condensates, yields

$$E_0 = -0.0347N^3 \frac{\hbar^2}{mr_0^2}, \quad (13)$$

where we have restored the unit of energy previously set to 1. Further minimization using the imaginary-time

Gross-Pitaevskii equation lowers the energy by less than the last digit given in Eq. (13) [20].

We now consider the fate of the N -boson cluster when the detuning parameter ν is nonzero. It was found in Ref. [17] that the three-body bound state (the trimer) exists only within a finite range of ν , $\nu_-(3) < \nu < \nu_+(3)$, where $\nu_+(3) \approx 8.72$ and $\nu_-(3) \approx -0.366$. For $\nu < \nu_-(3)$, the trimer decays into a dimer (molecule) and an atom, while for $\nu > \nu_+(3)$ the trimer is completely unbound. We now show that the N -body bound states at large N behave in a similar way: each of them exists in a finite range of the detuning parameter, $\nu_-(N) < \nu < \nu_+(N)$, but the range expands with increasing N : $\nu_{\pm}(N) \sim N^2$.

The qualitative behavior of the N -body cluster can again be investigated using the variational ansatz. The contribution of the detuning term to the energy is simply proportional to the number of molecules and independent of the cluster size R . Thus minimization over R proceeds as in the case of zero detuning and one obtains the energy as a function of atomic concentration c ,

$$E = -N^3 A \frac{c^4(1-c)^2}{(c + \frac{1}{3})^3} + \frac{\nu}{2} N(1-c). \quad (14)$$

The behavior of this function of c is controlled by $\tilde{\nu} = \nu/N^2$. There are two critical values, which in our variational calculation are $\tilde{\nu}_+ \approx 0.132A \approx 0.153$ and $\tilde{\nu}_- \approx -0.117A \approx -0.136$. For $\tilde{\nu}_- < \tilde{\nu} < \tilde{\nu}_+$, the minimum of $E(c)$ is located at a finite value of the atomic fraction c which changes from 0.667 at $\tilde{\nu} = \tilde{\nu}_+$ to 0.404 at $\tilde{\nu} = \tilde{\nu}_-$. A more accurate treatment [20] gives the values of the upper and lower ends of the interval, beyond which the N -boson bound state ceases to exist, as

$$\nu_+(N) \approx 0.191N^2, \quad \nu_-(N) \approx -0.140N^2. \quad (15)$$

If one fixes the value of the detuning parameter ν , parametrically larger than 1 (in the unit system $m = -r_0 = 1$), then a bound cluster can only be formed if the number of particles is larger than some $N_{\text{crit}} = (\nu/\tilde{\nu}_{\pm})^{1/2}$ for positive or negative detuning, respectively.

As we have seen, the size of the cluster decreases with increasing number of particles as $1/N$. The density at the center of the cluster scales like N^4 at large N . That means that at some value of N one can no longer ignore terms that were dropped when one writes down the energy functional. The leading irrelevant terms are

$$H_{\text{bg}} = \frac{1}{2} \int d\mathbf{x} (g_{11}|\psi|^4 + 2g_{12}|\phi|^2|\psi|^2 + g_{22}|\phi|^4), \quad (16)$$

where $g_{11} = 4\pi a_{11}/m$, $g_{12} = 3\pi a_{12}/m$, and $g_{22} = 2\pi a_{22}/m$, with a_{11} , a_{12} , and a_{22} being the background (i.e., the nonresonant part of) atom-atom, atom-molecule, and molecule-molecule scattering length. As the density of the droplet increases with increasing N , one expects that the four-point interaction, if it is repulsive (i.e., $g_{11} > 0$, $g_{22} > 0$, and $g_{11}g_{22} - g_{12}^2 > 0$), should stabilize the density at some finite value. This can be seen by minimizing the energy using the symmetrized Woods-Saxon

ansatz. At large N the density distribution flattens out in the center. At very large N , the droplet has a bulk of constant density surrounded by a thin wall across which the density drops to zero.

The situation here is simply the coexistence of two phases: the self-bound liquid and the vacuum. The Landau free energy density at chemical potential μ ,

$$V(\psi, \phi; \mu) = -\alpha(\psi^\dagger \psi^\dagger \phi + \phi^\dagger \psi \psi) + \frac{g_{11}}{2} |\psi|^4 + g_{12} |\psi|^2 |\phi|^2 + \frac{g_{22}}{2} |\phi|^4 - \mu(|\psi|^2 + 2|\phi|^2), \quad (17)$$

allows a first-order phase transition to occur at a negative value of μ where a nontrivial minimum of $\Omega(\psi, \phi)$ is degenerate with the trivial minimum at $\psi = \phi = 0$. This critical value of μ and the values of condensates in the liquid phase are the solution to the equations $\Omega = \partial_\psi \Omega(\psi_0, \phi_0) = \partial_\phi \Omega(\psi_0, \phi_0) = 0$. Parametrically, this occurs at

$$\mu \sim \frac{E}{N} \sim \frac{\hbar^2}{ma_{\text{bg}}|r_0|}. \quad (18)$$

The binding energy per particle has the same order of magnitude as μ . The density in the fluid phase is

$$n \sim \frac{1}{4\pi a_{\text{bg}}^2 |r_0|}. \quad (19)$$

At saturation density, the self-bound liquid is still dilute: $na_{\text{bg}}^3 \ll 1$, justifying the mean-field approximation. Also, provided that a_{bg} sets the magnitude of the coefficients of the additional terms involving higher powers of fields and higher derivatives, these can be ignored in Eq. (17).

In Fig. 1, we plot the binding energy per particle of a droplet as a function of N , showing a crossover between the $\mathcal{O}(N^2)$ behavior at small N to the $\mathcal{O}(N^0)$ behavior at large N .

Now we discuss the effect of detuning. We first consider how detuning affects the phase diagram of homogeneous matter. For this, one needs to investigate the behavior of the Landau functional (previously considered in Refs. [14, 15, 21–25])

$$V(\psi, \phi; \mu, \nu) = V(\psi, \phi; \mu) + \nu|\phi|^2. \quad (20)$$

A typical (μ, ν) phase diagram is shown in Fig. 2. One can understand the positive detuning ($\nu > 0$) region of the phase diagram by integrating out ϕ , assuming all fields are small. The minimum of ϕ is achieved at

$$\phi = \frac{\alpha}{\nu - 2\mu} \psi^2 + \mathcal{O}(\psi^4). \quad (21)$$

Substituting this into Eq. (20), we find the effective potential for ψ :

$$V_{\text{eff}}(\psi) = -\mu|\psi|^2 + \left(\frac{g_{11}}{2} - \frac{\alpha^2}{\nu - 2\mu} \right) |\psi|^4 + \frac{g_{12}\alpha^2}{(\nu - 2\mu)^2} |\psi|^6 + \mathcal{O}(|\psi|^8). \quad (22)$$

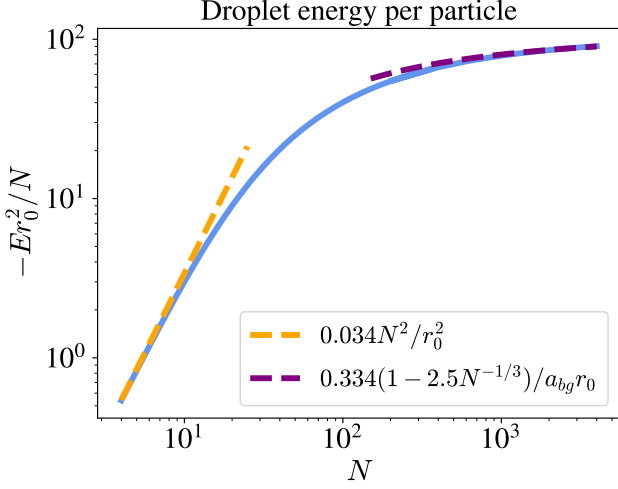


FIG. 1. Droplet binding energy per particle for $g_{11} = 4\pi a_{bg}$, $g_{22} = 2g_{11}/3$, $g_{12} = g_{22}/2$, and $a_{bg} = |r_0|/320$. The fit at large N is motivated by the picture of a droplet with finite surface tension.

We will limit ourselves to the case $g_{12} > 0$. Then at $\mu = 0$ and $\nu = \nu_+(\infty)$, with

$$\nu_+(\infty) = \frac{2\alpha^2}{g_{11}} = \frac{2\hbar^2}{ma_{11}|r_0|}, \quad (23)$$

the coefficients of both the $|\psi|^2$ and the $|\psi|^4$ term vanish. This point is the tricritical point T in Fig. 2. For $\nu > \nu_+(\infty)$, as one changes μ there is a phase transition at $\mu = 0$ from the vacuum to the Bose-Einstein condensate (BEC). For these values of ν the self-bound liquid does not exist. On the other hand, for $\nu < \nu_+(\infty)$ there is a first-order phase transition between the vacuum and the self-bound fluid. Thus, $\nu_+(\infty)$ is the $N \rightarrow \infty$ limit of the upper value detuning parameter for which the N -particle droplet exists.

At negative detuning, the line of vacuum-liquid first-order phase transition continues to exist at negative ν and meets the second-order phase transition line $\mu = \nu/2$ at a critical end point [26] (point S in Fig. 2) at

$$\begin{aligned} \nu_-(\infty) &= -\frac{2\alpha^2}{\sqrt{g_{11}g_{22}} + g_{12}} \\ &= -\frac{2\sqrt{2}\hbar^2}{m|r_0|(\sqrt{a_{11}a_{22}} + \frac{3}{\sqrt{2}}a_{12})}. \end{aligned} \quad (24)$$

For ν slightly smaller than $\nu_-(\infty)$, as one increases μ , the system goes through two phase transitions: first from the vacuum to a molecular BEC state with $\psi = 0, \phi \neq 0$, and then to a hybrid condensate phase with $\psi \neq 0, \phi \neq 0$. Since the hybrid condensate does not phase-coexist with the vacuum, there is no stable droplet consisting of any finite number of atoms. The first-order phase transition line between the hybrid condensate phase and the molecular BEC phase is a straight line that terminates at a $\mathbb{Z}_2$ tricritical point (point Z in Fig. 2) [20].

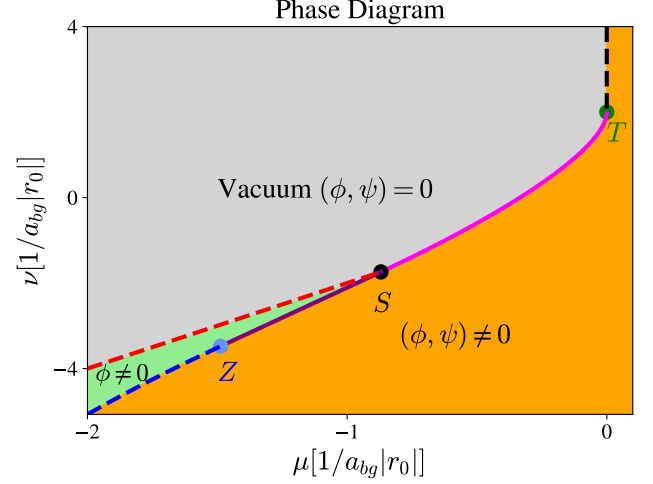


FIG. 2. Phase diagram in the plane of chemical potential μ and detuning ν in units of $1/(a_{bg}|r_0|)$ for $g_{11} = 4\pi a_{bg}$, $g_{22} = 2g_{11}/3$, and $g_{12} = g_{22}/2$. Solid and dashed lines represent first- and second-order transitions, respectively. The grey region represents the vacuum where the particle density is zero, the green region indicates the molecular BEC phase ($\psi = 0, \phi \neq 0$), and the orange region corresponds to the hybrid condensate phase ($\psi \neq 0, \phi \neq 0$). The tricritical points are $T = (0, \nu_+(\infty))$ and $Z = (\mu_{\mathbb{Z}_2}, \nu_{\mathbb{Z}_2})$ [20], while $S = (\nu_-(\infty)/2, \nu_-(\infty))$ is the critical end point shared by the three phases.

Conclusion.—In this Letter, we have considered the problem of a droplet of a finite but large number of bosonic atoms at a narrow Feshbach resonance. We find that the binding energy per particle should increase like N^2 and then flatten out to a constant.

In this work we have considered only the case when all atoms are identical bosons. It should be straightforward to extend the calculation to the case when the narrow resonance is formed from two nonidentical bosons.

The calculations in this paper are performed in the mean field approximation. For a small number of particles it will be important to compute quantum corrections to the energy. It would also be interesting if one could solve the four-body problem, e.g., by extending the zero-range calculations of Ref. [27] to an interaction with finite and negative effective range. For larger N , one may hope to find the binding energy through Monte Carlo methods [28].

Experimentally, one system where one may be able to create bound droplets is ultracold cesium at Feshbach resonance. For Cs the resonance at $B = 19.849(2)$ G has a very small width of $\Delta = 8.3(5)$ mG [29]. The effective range can be evaluated through [30]

$$r_0 \simeq -\frac{2\hbar^2}{m\Delta\delta\mu a_{bg}}. \quad (25)$$

Using $\delta\mu = 2\pi\hbar \times 0.76(3)$ MHz/G and $a_{bg} = 163(1)a_0$ (a_0 being the Bohr radius), we find $r_0/a_{bg} = -320(20)$.

The molecule-molecule s -wave scattering length is $a_{22} = 220(30)a_0$ [31], which corresponds to $g_{22} \approx \frac{2}{3}g_{11}$. The atom-molecule scattering is unknown, but if one assume that g_{12} is of the same order of magnitude as g_{11} and g_{22} , and that $g_{12}^2/(g_{11}g_{22}) < 1$, then one should expect the transition between the small cluster ($E_N \sim N^3$) regime to the large droplet ($E_N \sim N$) regime in the binding energy to occur at $N \sim (r_0/a_{\text{bg}})^{1/2} \sim 20$.

The authors thank Cheng Chin, Ubirajara van Kolck, Dmitry Petrov, Leo Radzihovsky, and Zhendong Zhang

for discussion. This work is supported, in part, by the U.S. DOE Grants No. DE-FG02-13ER41958 and by the Simons Collaboration on Ultra-Quantum Matter, which is a Grant from the Simons Foundation (No. 651440, DTS). T.P. acknowledges partial support by the DFG (EXC2181/1-390900948, 273811115) and thanks the University of Chicago for hospitality. KW acknowledges partial support from Kadanoff Center for Theoretical Physics and University of Chicago's Research Computing Center.

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— Supplemental Material —

Droplets of Bosons at a Narrow Resonance

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S1. THE SYMMETRIZED WOODS-SAXON VARIATIONAL ANSATZ

We introduce the symmetrized Woods-Saxon function

$$F(x; \xi, R) = \frac{1}{\cosh(x/R) + \xi}. \quad (\text{S1})$$

This function goes to zero exponentially as $x \rightarrow \infty$. It is an even function of x so its derivative at $x = 0$ vanishes.

We use the following five-parameter trial wave functions,

$$\psi(r) = \frac{N^2}{\sqrt{4\pi I_2}} F(Nr; \xi_1, R_1), \quad (\text{S2})$$

$$\phi(r) = \frac{N^2 b}{\sqrt{4\pi I_2}} F(Nr; \xi_2, R_2). \quad (\text{S3})$$

In order for the total number of particles to be equal to N , we require

$$I_2 = \int_0^\infty dx x^2 [F^2(x; \xi_1, R_1) + 2b^2 F^2(x; \xi_2, R_2)]. \quad (\text{S4})$$

Therefore I_2 is a function of the variational parameters $\xi_{1,2}$, $R_{1,2}$ and b .

The total energy is then

$$E = N^3 \left(\frac{K}{I_2} - \frac{2I_3}{I_2^{3/2}} \right), \quad (\text{S5})$$

where

$$K = \int_0^\infty dx \left[\frac{1}{2} F'^2(x; \xi_1, R_1) + \frac{1}{4} F'^2(x; \xi_2, R_2) \right], \quad (\text{S6})$$

$$I_3 = b \int_0^\infty dx F^2(x; \xi_1, R_1) F(x; \xi_2, R_2), \quad (\text{S7})$$

are both functions of the variational parameters. We now minimize the energy (S5) to find

$$E \approx -0.0347 N^3, \quad (\text{S8})$$

achieved at

$$\xi_1 \approx 0.167, \quad \xi_2 \approx 1.11, \quad R_1 \approx 1.53, \quad R_2 \approx 1.09, \quad b \approx 1.39. \quad (\text{S9})$$

Now we turn on detuning. Defining

$$I_{2m} = b^2 \int_0^\infty dx x^2 F^2(x; \xi_2, R_2), \quad (\text{S10})$$

the variational energy is now

$$\frac{E}{N^3} = \frac{K}{I_2} - \frac{2I_3}{I_2^{3/2}} - \frac{\nu}{N^2} \frac{I_{2m}}{I_2}. \quad (\text{S11})$$

For negative detuning, the droplet becomes unstable when the energy of the droplet crosses zero. This happens at $\tilde{\nu} \approx -0.191$, where $\xi_1 \approx -0.121$, $\xi_2 \approx 1.17$, $R_1 \approx 2.07$, $R_2 \approx 1.20$, $b \approx 1.52$. For positive detuning, the droplet becomes unstable when its energy is larger than the energy of $N/2$ dimers. This happens when $\tilde{\nu} \approx 0.140$, where $\xi_1 \approx 0.394$, $\xi_2 \approx 0.851$, $R_1 \approx 1.43$, $R_2 \approx 1.22$, $b \approx 1.19$.

Introducing new parameters into the ansatz (for example, by using the function $[\cosh(x/R) + \xi]^{-n}$ with n being an additional variational parameter, which can have different values for atoms and molecules) does not change the ground state energy at zero detuning and the values of $\tilde{\nu}_{\pm}$ appreciably.

S2. IMAGINARY-TIME GROSS-PITAEVSKII EQUATION

To numerically find the ground state wave function of the droplet, we consider the imaginary-time Gross-Pitaevskii evolution. For simplicity, we consider the case $g_{11} = g_{22}$ with $g_{12} = 0$,

$$-\partial_{\tau}\psi(r) = \left(-\frac{\partial_r^2 + 2r^{-1}\partial_r}{2m_1} - \mu_1\right)\psi(r) + g_{11}|\psi(r)|^2\psi(r) - 2\alpha\psi^*(r)\phi(r), \quad (\text{S12})$$

$$-\partial_{\tau}\phi(r) = \left(-\frac{\partial_r^2 + 2r^{-1}\partial_r}{2m_2} - \mu_2\right)\phi(r) + g_{22}|\phi(r)|^2\phi(r) - \alpha\psi(r)\psi(r). \quad (\text{S13})$$

The total particle number is required to be conserved during the evolution. Here, the chemical potential is evaluated self-consistently at each time step by $\mu_i = \partial E / \partial N_i$ and $N_{1/2}$ is the atomic/molecular particle number. We place the GPE on a discrete lattice with the lattice position $r(i) = d \times i$, where i the integer $0 \leq i \leq L$ and d is the lattice distance.

First, we use the imaginary-time Gross-Pitaevskii equation (itGPE) to confirm Eq. (13) in the main text: the droplet energy converges to $\approx -0.0347N^3/r_0^2$ when $g_{11} = g_{22} = 0$. We start from the symmetrized Woods-Saxon ansatz: the wave-functions are given by $\phi_1(r) = \sqrt{N/4I_2\pi}f(r)$ and $\phi_2(r) = \sqrt{N/4I_2\pi}g(r)$. Here I_2 is defined in Eq. (6) of the main text and the shape functions read

$$f(x) = \frac{1}{\exp(x + \xi_1)/R_1 + \exp(-(x - \xi_1)/R_1 + 1)}, \quad g(x) = \frac{b}{\exp(x + \xi_2)/R_2 + \exp(-(x - \xi_2)/R_2 + 1)}.$$

The minimization of the energy functional at $N = 100$ and $g_{11} = g_{22} = 0$ leads to the energy $E \approx -0.034666N^3/r_0^2$ with the following variational parameters: $\xi_1 \approx 0.0167r_0$, $R_1 \approx 0.01528r_0$, $\xi_2 \approx -0.008673r_0$, $R_2 \approx 0.01088r_0$, $b \approx 0.2094$.

Next, we simulate the itGPE with the ansatz above as the initial condition. Here we use a lattice with $d = 0.0005r_0$

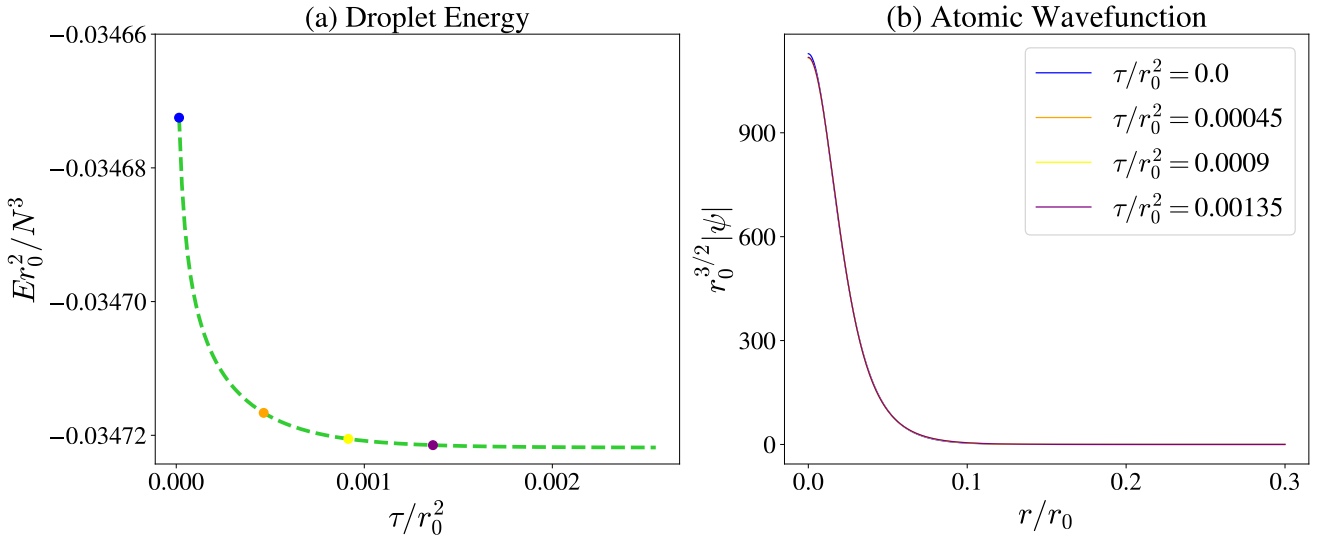


FIG. S1. Imaginary time Gross-Pitaevskii evolution of the Woods-Saxon ansatz. *Left*: Droplet energy versus the simulation time scale. We observe that the energy only decreases slightly and converges to $-0.03472N^3/r_0^2$. *Right*: Atomic wave function at four different times. The atomic wave function changes only very slightly near $r = 0$. This indicates that the Woods-Saxon ansatz is a very good approximation of the droplet's ground state.

and $L = 600$. Evaluation of the energy functional of the initial state *on* this lattice reads $\approx -0.034670N^3/r_0^2$. Compared to the coefficient -0.034666 from the continuous space, one may conclude that the error from the lattice discretization on the energy of this state is of the order of $\approx 5 \times 10^{-6}$. The simulation result is shown in Fig. S1: we find that the droplet energy converges to $\approx -0.03472N^3/r_0^2$. This confirms Eq. (13) in the main text.

Now we aim to check the first-order transition values of $\nu_{\pm}(N)$ in Eq. (15) of the main text. These two values are obtained by using the conditions $E_{\text{droplet}}(\nu_+) = 0$ and $E_{\text{droplet}}(\nu_-) = \nu N/2$. Here, E_{droplet} is the ground state energy of the droplet. Using the Woods-Saxon ansatz, the conditions above lead to $\nu_+ \approx 0.191N^2$ and $\nu_- \approx -0.140N^2$. We pick two ansatz wave functions ($\Psi_{\pm}$), which are obtained from the minimization at $\nu/N^2 = 0.1904$ and $\nu/N^2 = -0.14$. The variational parameters for $\nu/N^2 = 0.1904$ and $N = 100$ are: $\xi_1 \approx 0.11335r_0$, $R_1 \approx 0.019905r_0$, $\xi_2 \approx -0.01135r_0$, $R_2 \approx 0.01183r_0$, $b \approx 0.001824$. The variational parameters for $\nu/N^2 = -0.14$ and $N = 100$ are: $\xi_1 \approx 0.003406r_0$, $R_1 \approx 0.014305r_0$, $\xi_2 \approx -0.00649r_0$, $R_2 \approx 0.012214r_0$, $b \approx 0.55081$.

Then we use Ψ_+ , Ψ_- as the initial condition in the itGPE when ν/N^2 is varied around 0.191 and -0.14 . The energy of the converging state represents the droplet energy. Simulation results of itGPE are discussed below ($r_0 = 1$).

- For ν_+ , we find the energy of the droplet (converging) state at $\nu/N^2 = 0.1908, 0.1912, 0.1914$. The energy is $E \approx -6 \times 10^{-5}N^3$ at $\nu/N^2 = 0.1908$, while $E \approx 2.3 \times 10^{-5}N^3$ at $\nu/N^2 = 0.1914$. Therefore, ν_+/N^2 lies between 0.1908 and 0.1914. Furthermore, we show the imaginary time evolution in Fig. S2 at $\nu/N^2 = 0.1912$: the energy of the evolving state converges to a value around zero. Thus, we find $\nu_+/N^2 \approx 0.191$.
- For ν_- , we find the energy of the droplet state at $\nu/N^2 = -0.1399, -0.140, -0.1403$. The energy is $E - \nu N/2 \approx -2.4 \times 10^{-5}N^3$ at $\nu/N^2 = -0.1399$, while $E - \nu N/2 \approx 6 \times 10^{-5}N^3$ at $\nu/N^2 = -0.1403$. Therefore, ν_-/N^2 lies between -0.1399 and -0.1403 . The imaginary time evolution at $\nu/N^2 = -0.14$ is shown in Fig. S2: the energy of the evolving state converges to a value around zero. Thus, we find $\nu_-/N^2 \approx -0.140$.

According to these itGPE results, we confirm the first order transition values in Eq. (15) of the main text.

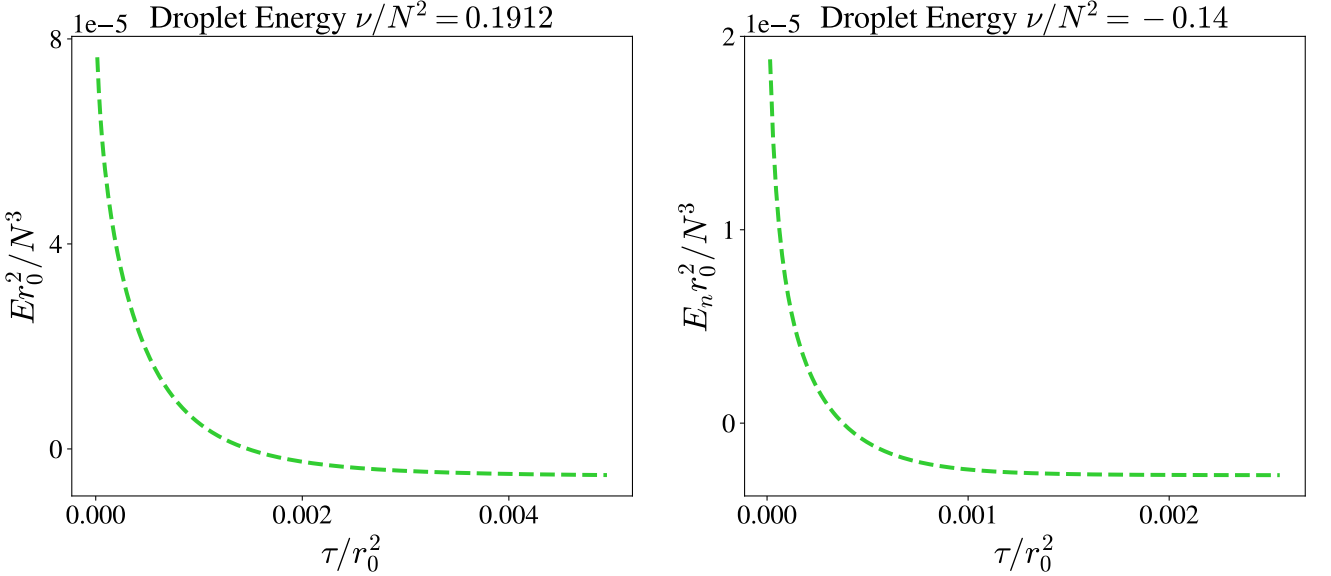


FIG. S2. Imaginary time Gross-Pitaevskii evolution. *Left*: Evolution of the ansatz Ψ_+ at $\nu = 0.1912N^2$. *Right*: Evolution of the ansatz Ψ_- at $\nu = -0.14N^2$. Here $E_n = E - \nu N/2$ is the energy measured from $\nu N/2$. Both energies decrease to values around zero, explicitly $\approx -5 \times 10^{-6}$ and $\approx -3 \times 10^{-6}$.

S3. THE HYBRID CONDENSATE-MOLECULAR BEC PHASE TRANSITION

From the energy given in Eq. (20), one finds that the phase diagram has a critical end point located at

$$\mu_{\text{cep}} = -\frac{\alpha^2}{\sqrt{g_{11}g_{22}} + g_{12}}, \quad (\text{S14a})$$

$$\nu_{\text{cep}} = -\frac{2\alpha^2}{\sqrt{g_{11}g_{22}} + g_{12}}, \quad (\text{S14b})$$

and a $\mathbb{Z}_2$ tricritical point located at

$$\mu_{\mathbb{Z}_2} = -\frac{2\alpha^2}{\sqrt{g_{11}g_{22}} + g_{12}} + \frac{g_{12}\alpha^2}{(\sqrt{g_{11}g_{22}} + g_{12})^2}, \quad (\text{S15a})$$

$$\nu_{\mathbb{Z}_2} = -\frac{4\alpha^2}{\sqrt{g_{11}g_{22}} + g_{12}} + \frac{(2g_{12} - g_{22})\alpha^2}{(\sqrt{g_{11}g_{22}} + g_{12})^2}. \quad (\text{S15b})$$

The line separating the molecular-BEC phase and the hybrid condensate phase is a straight line connecting $(\mu_{\text{cep}}, \nu_{\text{cep}})$ to $(\mu_{\mathbb{Z}_2}, \nu_{\mathbb{Z}_2})$:

$$\mu = -\frac{\alpha^2}{\sqrt{g_{11}g_{22}} + g_{12}} - x \frac{\sqrt{g_{11}g_{22}} \alpha^2}{(\sqrt{g_{11}g_{22}} + g_{12})^2}, \quad (\text{S16a})$$

$$\nu = -\frac{2\alpha^2}{\sqrt{g_{11}g_{22}} + g_{12}} - x \frac{(2\sqrt{g_{11}g_{22}} + g_{22})\alpha^2}{(\sqrt{g_{11}g_{22}} + g_{12})^2}, \quad (\text{S16b})$$

where x runs between 0 (the critical end point) to 1 (the $\mathbb{Z}_2$ tricritical point). On the molecular-BEC side of the phase transition, the fields are

$$\psi = 0, \quad (\text{S17a})$$

$$\phi = \frac{\alpha\sqrt{x}}{\sqrt{g_{11}g_{22}} + g_{12}}, \quad (\text{S17b})$$

and on the hybrid condensate side they are

$$\psi = \left(\frac{g_{22}}{g_{11}}\right)^{1/4} \frac{\alpha\sqrt{1-x}}{\sqrt{g_{11}g_{22}} + g_{12}}, \quad (\text{S18a})$$

$$\phi = \frac{\alpha}{\sqrt{g_{11}g_{22}} + g_{12}}. \quad (\text{S18b})$$

One can check that (S17) and (S18) are two local minima of the energy and these minima are degenerate in energy.

3.3 Conclusions

We investigated the formation of self-bound droplets composed of N bosonic atoms near a narrow Feshbach resonance. These droplets represent a novel phase of matter that exists without external confinement – stabilized not by quantum fluctuations as in previous ultracold realizations, but by coherent atom-molecule interactions already at the mean-field level.

We focused on the regime near unitarity, where the s -wave scattering length a diverges and the effective range r_0 is large and negative. In this limit, the interatomic interactions are strong and resonant, leading to scaling behavior. Remarkably, our analysis suggests that many-body bound states exist for all atom numbers N . As the number of atoms increases, the droplet becomes more compact and tightly bound, with its characteristic size decreasing as $R \sim 1/N$ and the binding energy increasing as $E \sim N^3$. This rapid scaling highlights the strongly attractive nature of the system near unitarity.

When the system is tuned slightly away from unitarity, the droplet remains bound only within a finite window of inverse scattering length. Specifically, for a given N , a bound state persists in the range $-0.191N^2 < 2|r_0|/a < 0.14N^2$. Outside this window, the interatomic attraction is either too weak to support a bound state or too strong to avoid collapse.

As N becomes large, the system undergoes a crossover from few-body clustering to a macroscopic droplet resembling a thermodynamic liquid. In this regime, the binding energy per particle saturates as $E \sim N$, and the droplet density approaches a finite value, indicating the emergence of bulk properties. Notably, even in this regime of large atom numbers, the droplets remain dilute with central densities $n \sim 1/(a_{ij}^2|r_0|) \ll 1/a_{ij}^3$, where a_{ij} represents the background scattering lengths between atoms and molecules. These features set the droplets apart from dense quantum liquids like helium clusters [60, 61].

A key outcome of this work is that droplet stabilization occurs already at the mean-field level due to the coherent Feshbach coupling. This stands in contrast to the stabilization mechanism in Bose-Bose mixtures [33, 34], where quantum droplets arise from a delicate balance between mean-field attraction and repulsive quantum fluctuations (Lee-Huang-Yang corrections). Whether quantum corrections reinforce or destabilize the droplets considered here remains an open question.

While our analysis focused primarily on the large- N limit, an important avenue for future work is the study of small clusters with $N = 3, 4, \dots$, where deviations from thermodynamic scaling are expected. Understanding the structure and binding of such few-body states would help bridge the gap between microscopic and macroscopic behavior. This could be pursued using exact diagonalization or quantum Monte Carlo simulations [73].

Another natural extension concerns the possibility of forming fermionic droplets. For instance, ^3He atoms are known to form droplets for $N \gtrsim N_{\text{crit}} \sim 20 - 40$ [74], although helium is far from unitarity. It is conceivable that the critical atom number required for

droplet formation in fermionic systems could be reduced by exploiting Feshbach resonances, potentially leading to the realization of fermionic droplets with far fewer particles.

Lastly, one can investigate the dynamical formation and collective excitations of droplets. Time-dependent Gross-Pitaevskii simulations could be used to study how droplets emerge from expanding clouds or under quench dynamics. The spectrum of low-lying excitations – such as breathing modes or surface oscillations – may reveal unique features of atom-molecule coherence and effective interactions, as seen in related works on dipolar or Bose-Bose droplets [75].

Overall, the narrow-resonance droplets explored here provide a controlled, tractable setting for studying the transition from few-body clusters to many-body quantum liquids, with promising connections to both cold atom experiments and broader questions in quantum many-body physics.

4 Universal scaling in nonthermal attractors

In this chapter, we explore various aspects of strong correlations near nonthermal attractors. Sec. 4.1 introduces the concept of nonthermal attractors and the framework of real-time nonequilibrium simulations used to study them. We then examine how and why nonthermal attractors emerge in unitary dynamics of isolated quantum many-body systems. As to why, in Sec. 4.2 we explore the stability of nonthermal attractors under external perturbations in the scaling regime. Details on the computation of far-from-equilibrium response rates, the identification of relevant scattering processes, and the scaling analysis are provided in the corresponding App. A. The question of how quantum systems approach nonthermal attractors is discussed in Sec. 4.3, where we study prescaling dynamics in systems ranging from cold atomic gases to the quark-gluon plasma. Supplementary material on prescaling in isotropic systems, along with novel results for longitudinally expanding systems relevant to the early stages of heavy-ion collisions, is presented in the corresponding App. B. Sec. 4.4 extends this discussion to composite correlation functions based on spatial Polyakov loops, revealing a similar prescaling behavior relevant to universal dynamics in both global and gauge-invariant observables of non-Abelian gauge theories. The question of how quantum systems approach nonthermal attractors is further explored in Sec. 4.5, where we broaden this discussion to the evolution of spatial inhomogeneities near nonthermal attractors, presenting a novel formulation of hydrodynamics around universal scaling. Conclusions to Secs. 4.2, 4.3, 4.4, and 4.5 are presented in Sec. 4.6. Further discussion on potential experimental signatures of the scaling instability and far-from-equilibrium quasi-particles (Sec. 4.2), prescaling breaking effects (Sec. 4.3), and far-from-equilibrium shear viscosity (Sec. 4.5) is provided in Ch. 5.

This chapter is based on:

- [1] Thimo Preis*, Michal P. Heller, and Jürgen Berges. *Stable and unstable perturbations in universal scaling phenomena far from equilibrium*. **Phys. Rev. Lett.** 130, 031602. January 20, 2023. DOI: 10.1103/PhysRevLett.130.031602.

I was the main author of this work and identified the tower of quasi-particle excitations far from equilibrium, building on insights from my Master’s Thesis, *Gapless modes in massive relativistic scalar field theory far from equilibrium*, completed at Heidelberg University under the supervision of J. Berges. This discovery led me to propose investigating the stability of non-thermal attractors by targeted perturbations. While all co-authors contributed to the writing of the manuscript, I performed the numerical calculations, generated the figures, and contributed significantly to the structuring of the text. Additional related material not included in [1] is provided in App. A.

- [3] Michal P. Heller, Aleksas Mazeliauskas, and Thimo Preis* (alphabetical). *Prescaling relaxation to nonthermal attractors*. **Phys. Rev. Lett.** 132, 071602. February 15, 2024. DOI: 10.1103/PhysRevLett.132.071602.

I was the main author of this work, deriving the prescaling equations and obtaining the analytical solution for prescaling in isotropic systems. I also carried out the numerical simulations to test prescaling, where the code used to test prescaling for the direct energy cascade in QCD kinetic theory was developed by A. Mazeliauskas. M.P. Heller and I collaborated on the analysis of the prescaling equation within the Fokker-Planck approximation for QCD kinetic theory, which I derived. I performed the scaling analyses and created the figures. All authors contributed to the writing of the manuscript. Further calculations and explanations not included in [3] are detailed in App. B.1. I present novel results on prescaling in longitudinally expanding systems relevant for the early stages of heavy-ion collisions in App. B.2, which were derived and analyzed by me.

- [5] Daniel Spitz*, Kirill Boguslavski, and Thimo Preis. *Condensation and prescaling in spatial Polyakov loop correlations far from equilibrium*. **Phys. Rev. D** 111, 096006. May 6, 2025. DOI: 10.1103/PhysRevD.111.096006.

D. Spitz was the main author of this work, carrying out the numerical simulations based on code developed by K. Boguslavski, and contributing significantly to the writing of the manuscript. I conducted the prescaling analysis, created the figures, and wrote the corresponding sections and appendices. All authors contributed to the manuscript through corrections and suggestions.

- [7] Jürgen Berges, Gabriel S. Denicol, Michal P. Heller, and Thimo Preis* (alphabetical). *Far from equilibrium hydrodynamics of nonthermal fixed points*. April 26, 2025. DOI: 10.48550/arXiv.2504.18754.

I was the main author of this work, and the original idea was developed jointly by G. S. Denicol, M. P. Heller, and myself. G. S. Denicol derived the shear stress evolution equation and the corresponding expressions for the transport coefficients. I performed the scaling analysis, conducted the numerical simulations, analyzed the data, and created the figures. While all authors participated in the writing of the manuscript, I contributed significantly to the structuring and phrasing of the text.

4.1 Theoretical background

In this section, we provide theoretical background to the ensuing discussion of nonthermal attractors in dynamical quantum many-body systems.

4.1.1 Attractors

The concept of attractors plays a central role in understanding how complex systems self-organize and thermalise from first principles. The emergence of thermalisation in quantum field theories in particular is an outstanding challenge since unitary quantum evolution admits no loss of information on a fundamental level. Thermal equilibrium is characterized by a few conserved quantities only, which implies a necessary loss of sensitivity on initial conditions in the process of thermalization. Nonthermal or thermal attractors provide such a mechanism in the course of the system dynamics, where the associated emergent universality implies insensitivity of the dynamics to details of initial conditions. Understanding how attractor behavior and an associated effective low-energy description emerges in the dynamics provides a central structuring pattern to explaining thermalisation for a wide range of dynamical systems.

Attractors in complex system dynamics constitute a set of states toward which a system evolves from a wide range of initial conditions. We will focus in particular on fixed-point attractors in the following. One example of an attractor central to this work is the nonthermal attractor or nonthermal fixed point. It emerges in weakly-coupled isolated quantum systems far from equilibrium under the proliferation of statistical fluctuations, where correlation functions exhibit spatio-temporal self-similar scaling behavior

$$f(t, \mathbf{p}) = s^{\alpha/\beta} f_S(s^{-1/\beta} t, s|\mathbf{p}|) \quad (4.1)$$

in a corresponding momentum regime. Here, the exemplary isotropic two-point function $f(t, \mathbf{p})$ describes the occupation number of constituents of the system with time t and momentum $\mathbf{p}$, which can be rescaled with universal scaling exponents (α, β) and a universal scaling function f_S through a dimensionless parameter $s \in \mathbb{R}_{>0}$. While Sec. 4.2 focuses on fixed-point scaling (4.1), in Sec. 4.3 and Sec. 4.4 we discuss the approach to fixed-point scaling as quantified by a more general prescaling ansatz

$$f(t, \mathbf{p}) = A(t) f_S(B(t)|\mathbf{p}|). \quad (4.2)$$

As we will show in Sec. 4.3, the prescaling factors $A(t)$ and $B(t)$ take into account non-universal and system-specific finite-time effects $\sim t_*$ in the approach to the fixed point through a power-law resummation in time [3]

$$A(t) = \left(\frac{t - t_*}{t_{\text{ref}}} \right)^\alpha, \quad B(t) = \left(\frac{t - t_*}{t_{\text{ref}}} \right)^\beta, \quad (4.3)$$

where t_{ref} is an arbitrary reference time. This self-similar prescaling evolution of the occupation of a quantum gas is visualized in Fig. 4.1, which was recently experimentally confirmed in [21].

This far-from-equilibrium scaling behavior has been observed in relativistic systems relevant to early universe cosmology [76–84] or heavy-ion physics [12, 36, 85–99] and non-relativistic cold atom systems [1, 3, 80, 100–107], where recent experimental cold atom observations include [19–21, 108–117]. Realized across a wide range of initial conditions

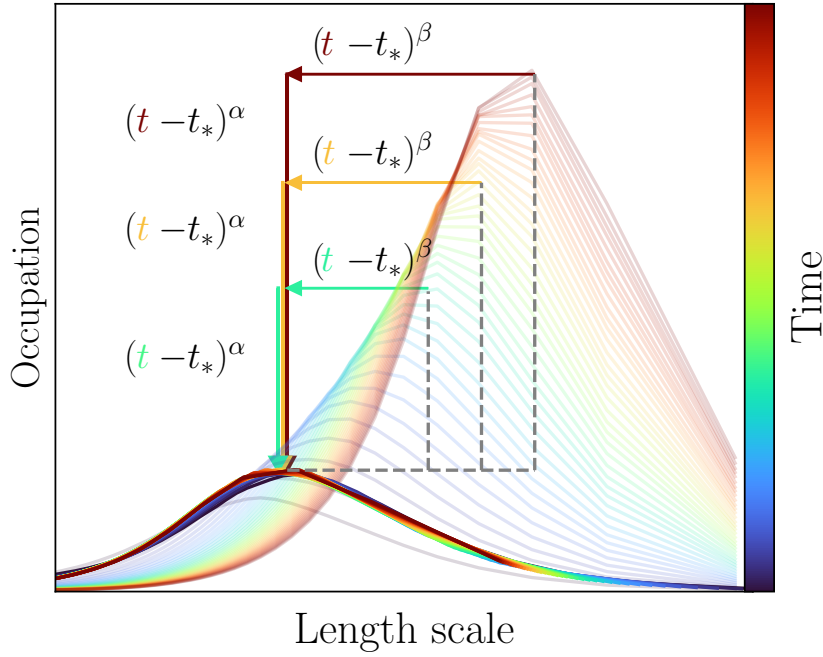


Figure 4.1: A quantum gas evolves self-similarly in time: Length scales can be transformed to time scales and vice versa as dictated by universal numbers shared between different physical quantum systems. The data is taken from Fig. 1 of Sec. 4.3.

and system parameters [19–21], such self-organized scaling behaviors emerge through a dynamical separation of scales [12, 80, 95, 118] associated with transport phenomena driving the underlying dynamics. In isotropic scalar systems, following an initial deposit of energy at a momentum scale Q , a double cascade occurs: energy is transported through a turbulent direct cascade to the UV with $\alpha, \beta < 0$, whereas particle number is transported through an inverse cascade to the IR with $\alpha, \beta > 0$. While nonthermal attractors have been realized for a wide range of systems with differing microscopic formulations, scaling exponents and scaling functions have been found to be universal [80, 83, 91]. While different universality classes have been observed, encompassing systems with different symmetries such as non-Abelian gauge and scalar $O(N)$ theories [91], relativistic and non-relativistic theories [80], or models with different number of field components [83, 107], it remains to be understood what determines the universality class of a given system. The question of universality class is intimately tied to that of the respective effective low-energy description. The emergent degrees of freedom and symmetries in these far-from-equilibrium effective descriptions need not be based on the thermal equilibrium or vacuum properties of the system under consideration. Addressing in particular the question of the approach to universality and the emergent low energy degrees of freedom will be central to this chapter.

Nonthermal attractors have also been observed in systems with momentum anisotropy. Together with so-called hydrodynamic attractors, they provide key elements to our first-principles thermalisation picture of the quark-gluon plasma in heavy-ion collisions. Our current understanding is based on the bottom-up (or BMSS) thermalisation scenario [12,

119–121]. The kinetic energy of the initially colliding relativistic nuclei is converted predominantly into the proliferation of gluons, while the running gauge coupling is small at sufficiently large energies. This results in a strongly correlated initial state for the subsequent evolution of the created quark-gluon plasma. The strong correlations lead to an effective loss of information about microscopic details of the underlying theory, such as gluon occupancy or the anisotropy of their distribution [122], and the emergence of universal scaling far from equilibrium, characteristic of the nonthermal attractor [90, 123]. As the system isotropizes in the last stage of the bottom-up thermalisation scenario [124, 125], the longitudinal expansion further leads to a decrease of occupancies whereas the running gauge coupling increases. While the system still deviates from thermal equilibrium, the surprising observation was that hydrodynamic descriptions can already be used to make phenomenological predictions due to an emergent hydrodynamic attractor [126–134]. This mechanism provides a possible explanation for the surprising success of hydrodynamics in modeling the later stages of heavy-ion collisions, see [135–138] for reviews. Note that one conventionally distinguishes between the hydrodynamic attractor (low occupancies, stronger couplings) at late times and the nonthermal attractor (high occupancies, weaker couplings) at early times, where their relation and the transition from one to the other remains subject of on-going investigation, see e.g. [139, 140].

While nonthermal attractors have already been observed in a variety of quantum many-body systems across energy scales, recent proposals suggest that hydrodynamic attractors might also be accessible in cold atom platforms [141, 142], pointing to deeper structural features of far-from-equilibrium evolution that transcend the details of microscopic interactions.

4.1.2 Emergence of complexity

The emergence of complexity in classical and quantum many-body systems is often accompanied by simple organizational principles, such as universal scaling laws. These laws describe the collective behavior of many interacting constituents and can apply broadly across systems with different microscopic details. A common feature of such systems is the appearance of power-law distributions in quantities like correlation functions, event sizes, or fluctuations. The absence of characteristic scales implied by these distributions is closely connected to universality, where similar macroscopic behavior arises in otherwise distinct systems.

There is a large abundance of scale-invariant dynamical systems that can be found in nature. Examples include the distribution of earthquakes [143–145], turbulence [10], or scaling relations in wildfire statistics [146, 147], financial markets [148] and in the spreading of diseases [149]. One conceptual framework that has been discussed as the underlying mechanism for the appearance of such scale-free organization in the dynamics is self-organized criticality [150]. In contrast to scaling behavior observed in thermal critical phenomena, complex systems exhibiting self-organized criticality evolve toward a stationary state in which spatio-temporal correlation functions display scale invariance without the fine-tuning of external parameters. The gap between the state of the system and

its critical far-from-equilibrium state self-tunes to zero by a system-inherent dynamical mechanism [112].

4.1.3 Real-time evolution of scalar fields

We want to study the real-time dynamics of quantum fields far from thermal equilibrium. Correlation functions of observables in thermal equilibrium have been successfully computed from first principles using importance sampling of the partition function [27, 151]. This approach has led to landmark achievements in lattice QCD, notably the *ab initio* calculation of baryon and meson masses [152–154]. However, such importance sampling methods fail out of equilibrium due to the notorious sign problem [155]. To address real-time dynamics in nonequilibrium quantum field theory, one typically resorts to truncated functional approaches, numerical simulations with limited ranges of validity, quantum simulation platforms, or effective descriptions such as kinetic theory and hydrodynamics.

In this section, we introduce a real-time framework for the nonequilibrium dynamics of real scalar fields, see [156, 157] for reviews. Based on resummed perturbation theory, this approach overcomes well-known challenges out of equilibrium, such as secularity [156, 158] and pinch singularities [159–162].

We consider a real scalar field operator $\Phi(x)$ in the Heisenberg picture, depending on the four-vector $x = (x^0, \mathbf{x})$ of time and space. Its dynamics are governed by the interacting Klein–Gordon equation

$$\left(\square_x + m^2\right) \Phi(x) + \frac{\lambda}{6} \Phi(x)^3 = 0 , \quad (4.4)$$

where m and λ denote the mass and interaction parameters, respectively. The field operator and its conjugate momentum $\Pi(x) = \partial_{x^0} \Phi(x)$ satisfy the equal-time commutation relation

$$[\Phi(x), \Pi(y)] \Big|_{x^0=y^0} = i\delta^{(d)}(\mathbf{x} - \mathbf{y}) . \quad (4.5)$$

We study correlation functions defined as expectation values of field products with respect to a density operator ϱ_0 at initial time t_0 :

$$\langle \Phi(x_1) \dots \Phi(x_n) \rangle \equiv \text{Tr} \left(\varrho_0 \Phi(x_1) \dots \Phi(x_n) \right) , \quad (4.6)$$

with the trace taken over a complete basis of quantum states.

Starting from the field equation (4.4), one can derive evolution equations for the n -point correlation functions by multiplying with additional field operators and taking the trace with ϱ_0 . In this work, we focus on the evolution equations for the two-point functions, namely the symmetrized and anti-symmetrized combinations of two field operators:

$$\left(\square_x + m^2\right) \langle \{\Phi(x), \Phi(y)\} \rangle + \frac{\lambda}{6} \langle \{\Phi(x)^3, \Phi(y)\} \rangle = 0 , \quad (4.7)$$

$$\left(\square_x + m^2\right) \langle [\Phi(x), \Phi(y)] \rangle + \frac{\lambda}{6} \langle [\Phi(x)^3, \Phi(y)] \rangle = 0 . \quad (4.8)$$

Equations (4.7) and (4.8) are exact but not closed: the two-point functions depend on four-point functions, which in turn depend on six-point functions, and so on. To obtain a tractable system, we consider first a self-consistent small- λ expansion and later a self-consistent large- N expansion. This requires taking into account contributions from both the interaction term $\sim \lambda$ in the Hamiltonian and, in general, from the initial density matrix ϱ_0 .

To incorporate initial correlations, we express ϱ_0 in a basis of eigenstates of the field operator at time t_0 :

$$\Phi(t_0, \mathbf{x})|\varphi\rangle = \varphi(\mathbf{x})|\varphi\rangle. \quad (4.9)$$

Inserting complete sets of field eigenstates $\{|\varphi^\pm\rangle\}$, the trace becomes

$$\text{Tr}(\varrho_0\{\Phi(x), \Phi(y)\}) = \int [d\varphi^+][d\varphi^-] \langle\varphi^+|\varrho_0|\varphi^-\rangle \langle\varphi^-|\{\Phi(x), \Phi(y)\}|\varphi^+\rangle, \quad (4.10)$$

with the functional integration measure given in continuum notation by

$$\int [d\varphi_\pm] = \int \prod_{\mathbf{x}} d\varphi_\pm(\mathbf{x}). \quad (4.11)$$

We parametrize the matrix elements of the initial density operator as a functional expansion:

$$\langle\varphi^+|\varrho_0|\varphi^-\rangle = \mathcal{N} \exp\left(\sum_{n=1}^{\infty} h_n[\varphi_\pm]\right), \quad (4.12)$$

where

$$h_n[\varphi_\pm] = \frac{1}{n!} \int_{\mathbf{x}_1 \dots \mathbf{x}_n} \sum_{i_1, \dots, i_n \in \{+, -\}} \alpha_{i_1 \dots i_n}^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n) \varphi_{i_1}(\mathbf{x}_1) \cdots \varphi_{i_n}(\mathbf{x}_n). \quad (4.13)$$

The coefficients $\alpha^{(n)}$ encode the initial n -point correlations and satisfy hermiticity conditions,

$$\langle\varphi^+|\varrho_0|\varphi^-\rangle = \langle\varphi^-|\varrho_0|\varphi^+\rangle^*, \quad (4.14)$$

implying that the complex conjugate of each $\alpha^{(n)}$ corresponds to reversing the order of its $\pm$ -indices.

We focus on the case $\langle\Phi(x)\rangle = 0$, in the absence of spontaneous symmetry breaking. As a consequence, all coefficient functions $\alpha^{(n)}$ with odd n can be taken to vanish identically. Additionally, we will consider Gaussian initial correlations only in the following. The underlying reason is that we are interested in attractor solutions, which the system approaches to within the basin of attraction also from Gaussian initial correlations. The density matrix (4.12) is approximated by its quadratic ‘‘Gaussian’’ part

$$\langle\varphi^+|\varrho_0|\varphi^-\rangle = \mathcal{N} e^{\frac{1}{2} \int_{\mathbf{x}_1 \mathbf{x}_2} \sum_{i_1, i_2} \alpha_{i_1 i_2}^{(2)}(\mathbf{x}_1, \mathbf{x}_2) \varphi_{i_1}(\mathbf{x}_1) \varphi_{i_2}(\mathbf{x}_2)}. \quad (4.15)$$

The contributions of ϱ_0 expanded to quadratic order can then conveniently be subsumed in the initial conditions, which are to be provided for the solution of the second-order

integro-partial differential equations of the anti-commutator or commutator. The choice of an initial condition does not represent an approximation to the dynamics and, in general, the time evolution will then build up even higher n -point correlations for times larger than t_0 .

We will be interested in the quantum evolution equations of the real statistical and spectral functions:

$$F(x, y) = \frac{1}{2} \langle \{ \Phi(x), \Phi(y) \} \rangle , \quad (4.16a)$$

$$\rho(x, y) = i \langle [\Phi(x), \Phi(y)] \rangle , \quad (4.16b)$$

which then follow from Eq. (4.7) and Eq. (4.8) as

$$(\square_x + m^2)F(x, y) = -\frac{\lambda}{12} \langle \{ \Phi(x)^3, \Phi(y) \} \rangle \equiv -L^F(x, y) , \quad (4.17a)$$

$$(\square_x + m^2)\rho(x, y) = -\frac{\lambda i}{6} \langle [\Phi(x)^3, \Phi(y)] \rangle \equiv -L^\rho(x, y) , \quad (4.17b)$$

where we introduced statistical and spectral components of the four-point correlation function $L(x, y) \equiv L(x, x, x, y)$. $L(x, y)$ can be computed following standard procedures in self-consistently resummed perturbation theory through an expansion of contributions in the quartic interaction term. If we consider an expansion in a small quartic coupling λ , to lowest order we find disconnected contributions [163]

$$(\square_x + m^2)F(x, y) = -\frac{\lambda}{2} F(x, x)F(x, y) \equiv -L^{F,(\lambda)}(x, y) , \quad (4.18a)$$

$$(\square_x + m^2)\rho(x, y) = -\frac{\lambda}{2} F(x, x)\rho(x, y) \equiv -L^{\rho,(\lambda)}(x, y) . \quad (4.18b)$$

The resulting dynamics do not lead to thermalization and are entirely determined by the initial conditions, which can be chosen at will for F but are fixed by the equal-time commutation relations (4.5) for the spectral function as

$$\rho(x, y)|_{x^0=y^0=t_0} = 0 = \partial_{x^0}\partial_{y^0}\rho(x, y)|_{x^0=y^0=t_0}, \quad \partial_{x^0}\rho(x, y)|_{x^0=y^0=t_0} = \delta^{(d)}(\mathbf{x} - \mathbf{y}) . \quad (4.19)$$

The disconnected part of the four-point function simply leads to an in-medium mass shift of the dispersion relation of the propagating solutions. In a field theory language, it provides the leading local term of the self-energy $\Sigma^{(0)}(x) = \lambda F(x, x)/2$, which resums quantum corrections into the two-point functions. The self-energy diagram corresponds to local tadpole insertions $\sim \Sigma^{(0)}(x)$ as depicted on the left of Fig. 2.1 in Ch. 2. The first non-trivial contribution to the four-point functions, and correspondingly the leading non-local contribution to the self-energy $\sim \Sigma(x, y)$, is encountered at two-loop order $\mathcal{O}(\lambda^2)$. It corresponds to the melon diagram depicted on the right of Fig. 2.1 in Ch. 2 and the

corresponding four-point functions are given by [163]

$$L^{F,(\lambda^2)}(x, y) = \int_{t_0}^{x^0} dz^0 \int d^d \mathbf{z} \Sigma^{\rho,(\lambda^2)}(x, z) F(z, y) - \int_{t_0}^{y^0} dz^0 \int d^d \mathbf{z} \Sigma^{F,(\lambda^2)}(x, z) \rho(z, y) , \quad (4.20a)$$

$$L^{\rho,(\lambda^2)}(x, y) = \int_{y^0}^{x^0} dz^0 \int d^d \mathbf{z} \Sigma^{\rho,(\lambda^2)}(x, z) \rho(z, y) , \quad (4.20b)$$

where we defined statistical and spectral parts of the non-local self-energy as

$$\Sigma^{F,(\lambda^2)}(x, z) = -\frac{\lambda^2}{6} F(x, z) \left(F^2(x, z) - \frac{3}{4} \rho^2(x, z) \right) , \quad (4.21a)$$

$$\Sigma^{\rho,(\lambda^2)}(x, z) = -\frac{\lambda^2}{2} \rho(x, z) \left(F^2(x, z) - \frac{1}{12} \rho^2(x, z) \right) . \quad (4.21b)$$

The inclusion of non-local self-energies leads to the appearance of memory integrals $\sim \int dz^0$, which sum over the history of the evolution of the system. Already this quadratic contribution encodes direct scattering effects and is able to capture thermalization [163]. Conceptually, it is important to emphasize that equilibrium can not be reached globally by an isolated system under unitary evolution, as the equilibrium state has no memory of the past except for the few conserved quantities characterizing it. The memory integrals relate directly to the reversible unitary time evolution, as the history of the evolution from initial condition to the present is cumulatively encoded in the state of the system. As we discussed in Sec. 4.1.1, isolated quantum systems can nevertheless exhibit an effective loss of details about the initial state by approaching attractors in their dynamics, be they nonthermal or thermal. In particular, for the thermal attractor, an observable is said to thermalize when its evolution becomes indistinguishable from that of its thermal equilibrium value, which is characterized by a few conserved quantities only.

To describe the strong correlations arising near a nonthermal attractor, a non-perturbative resummation scheme is required. As discussed in Sec. 2.1.2, such a scheme is provided, for example, by multi-component scalar field theories with global $O(N)$ symmetry, which admit a systematic large- N expansion in the number of field components N . The dynamics of the vector of real scalar field operators $\Phi_{a=1,\dots,N}(x)$ in the Heisenberg picture is governed by

$$(\square_x + m^2) \Phi_a(x) + \frac{\lambda}{6} \Phi_b(x)^2 \Phi_a(x) = 0 , \quad (4.22)$$

and the corresponding evolution equations for the statistical and spectral two-point functions are

$$(\square_x + m^2) F_{ab}(x, y) = -\frac{\lambda}{12} \langle \{ \Phi_c(x)^2 \Phi_a(x), \Phi_b(y) \} \rangle \equiv -L_{ab}^F(x, y) , \quad (4.23a)$$

$$(\square_x + m^2) \rho_{ab}(x, y) = -\frac{\lambda i}{6} \langle [\Phi_c(x)^2 \Phi_a(x), \Phi_b(y)] \rangle \equiv -L_{ab}^\rho(x, y) . \quad (4.23b)$$

Contributions can then be organized by their scaling in N , making use of the $O(N)$ -invariant traces $\sim \text{Tr}_{O(N)}(F^n)$ and $\text{Tr}_{O(N)}(\rho^n)$ [40, 164]. At leading order (LO) in the

1/N expansion, only local self-energy contributions $\sim \Sigma_{ab}^{(0)}$ appear. These do not capture thermalization or non-trivial dynamics. To go beyond this, we consider the next-to-leading order (NLO), where the four-point functions take the form [164]:

$$\begin{aligned}
 -L_{ab}^F(x, y) = & -\frac{\lambda}{6N} [\delta_{ac} F_{dd}(x, x) + 2F_{ac}(x, x)] F_{cb}(x, y) \\
 & + \int_{t_0}^{x^0} dz^0 \int d^d \mathbf{z} \Sigma_{ac}^{\rho, \text{NLO}}(x, z) F_{cb}(z, y) \\
 & - \int_{t_0}^{y^0} dz^0 \int d^d \mathbf{z} \Sigma_{ac}^{F, \text{NLO}}(x, z) \rho_{cb}(z, y) ,
 \end{aligned} \tag{4.24a}$$

$$\begin{aligned}
 -L_{ab}^\rho(x, y) = & -\frac{\lambda}{6N} [\delta_{ac} F_{dd}(x, x) + 2F_{ac}(x, x)] \rho_{cb}(x, y) \\
 & - \int_{y^0}^{x^0} dz^0 \int d^d \mathbf{z} \Sigma_{ac}^{\rho, \text{NLO}}(x, z) \rho_{cb}(z, y) .
 \end{aligned} \tag{4.24b}$$

The non-local self-energies at NLO in 1/N are given by:

$$\Sigma_{ab}^{F, \text{NLO}}(x, y) = -\frac{\lambda}{3N} \left[I^F(x, y) F_{ab}(x, y) - \frac{1}{4} I^\rho(x, y) \rho_{ab}(x, y) \right] , \tag{4.25a}$$

$$\Sigma_{ab}^{\rho, \text{NLO}}(x, y) = -\frac{\lambda}{3N} \left[I^F(x, y) \rho_{ab}(x, y) + I^\rho(x, y) F_{ab}(x, y) \right] , \tag{4.25b}$$

where the auxiliary resummation functions $I^{F, \rho}(x, y)$ satisfy

$$I^F(x, y) = \Pi^F(x, y) - \int_{t_0}^{x^0} d^4 z I^\rho(x, z) \Pi^F(z, y) + \int_{t_0}^{y^0} d^4 z I^F(x, z) \Pi^\rho(z, y) , \tag{4.26a}$$

$$I^\rho(x, y) = \Pi^\rho(x, y) - \int_{y^0}^{x^0} d^4 z I^\rho(x, z) \Pi^\rho(z, y) , \tag{4.26b}$$

with one-loop terms

$$\Pi^F(x, y) = \frac{\lambda}{6N} \left\{ \text{Tr}_{O(N)}[F^2(x, y)] - \frac{1}{4} \text{Tr}_{O(N)}[\rho^2(x, y)] \right\} , \tag{4.27a}$$

$$\Pi^\rho(x, y) = \frac{\lambda}{3N} \text{Tr}_{O(N)}[F(x, y) \rho(x, y)] . \tag{4.27b}$$

The effective in-medium mass is defined via local self-energy insertions as

$$M_{ab}^2(x) = \delta_{ab} \left[m^2 + \frac{\lambda}{6N} F_{cc}(x, x) \right] + \frac{\lambda}{3N} F_{ab}(x, x) . \tag{4.28}$$

This finally yields the quantum evolution equations for Gaussian initial conditions:

$$\begin{aligned} \left[\delta_{ac} \square_x + M_{ac}^2(x) \right] F_{cb}(x, y) = & - \int_{t_0}^{x^0} dz^0 \int d^d \mathbf{z} \Sigma_{ac}^{\rho, \text{NLO}}(x, z) F_{cb}(z, y) \\ & + \int_{t_0}^{y^0} dz^0 \int d^d \mathbf{z} \Sigma_{ac}^{F, \text{NLO}}(x, z) \rho_{cb}(z, y) , \end{aligned} \quad (4.29a)$$

$$\left[\delta_{ac} \square_x + M_{ac}^2(x) \right] \rho_{cb}(x, y) = - \int_{y^0}^{x^0} dz^0 \int d^d \mathbf{z} \Sigma_{ac}^{\rho, \text{NLO}}(x, z) \rho_{cb}(z, y) . \quad (4.29b)$$

In the symmetric phase, where $\langle \Phi_a(x) \rangle = 0$, one can choose an $O(N)$ basis in which the two-point functions are diagonal: $F_{ab} = \delta_{ab} F$ and $\rho_{ab} = \delta_{ab} \rho$, with $\delta_{aa} = N$ and implicit summation over repeated indices.

In the following sections, we will investigate how and why nonthermal attractors emerge during the unitary evolution of isolated quantum many-body systems. This can be addressed using the real-time quantum evolution equations for the statistical and spectral functions derived above. Near nonthermal attractors, strong correlations arise from the proliferation of statistical fluctuations, characterized by the regime $F^2 \gg \rho^2$ [78]. In this regime, a self-consistent resummation to next-to-leading order (NLO) in $1/N$ is necessary, as the combination of parametrically large statistical correlators and suppressed vertices $\sim \lambda/N$ leads to contributions of order unity, $\mathcal{O}(1)$.

Universal scaling behavior manifests in the form of power laws in correlation functions, accurately captured by the non-local memory integrals appearing in the NLO equations. These equations have been shown to admit both nonthermal attractor solutions [78, 81, 82, 165] and thermal attractor solutions [163, 164, 166–168]. The discovery of nonthermal scaling in particular has provided an essential organizing principle for the challenging real-time description of quantum many-body systems far from equilibrium, particularly for weakly coupled systems with large classical fluctuations [78]. Extending the lifetime of nonthermal attractors in relativistic systems requires asymptotically weak coupling, such that sub-leading number-changing processes remain suppressed. At stronger couplings, sustaining a nonthermal attractor remains an open question, as intermediate-coupling studies have revealed a rapid onset of thermalization due to the growing significance of number-changing interactions [102]. Driving the system externally, injecting sufficient energy to maintain the double cascade while inhibiting thermalization, may be necessary to stabilize nonthermal scaling behavior at stronger couplings.

Finally, studies of isolated systems offer a valuable opportunity to understand the mechanisms behind the deviation from nonthermal attractors towards eventual thermalization within realizable simulation times. Through the inclusion of memory integrals in the evolution equations and corrections to fixed point scaling, the following sections will provide a possible point of departure for addressing the transition from nonthermal scaling to thermal equilibrium from first principles.

4.2

Stable and Unstable Perturbations in Universal Scaling Phenomena Far from Equilibrium

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(Received 5 October 2022; accepted 16 December 2022; published 20 January 2023)

We study the dynamics of perturbations around nonthermal fixed points associated with universal scaling phenomena in quantum many-body systems far from equilibrium. For an N -component scalar quantum field theory in $3 + 1$ space-time dimensions, we determine the stability scaling exponents using a self-consistent large- N expansion to next-to-leading order. Our analysis reveals the presence of both stable and unstable perturbations, the latter leading to quasiexponential deviations from the fixed point in the infrared. We identify a tower of far-from-equilibrium quasiparticle states and their dispersion relations by computing the spectral function. With the help of linear response theory, we demonstrate that unstable dynamics arises from a competition between elastic scattering processes among the quasiparticle states. What ultimately renders the fixed point dynamically attractive is the phenomenon of a “scaling instability,” which is the universal scaling of the unstable regime toward the infrared due to a self-similar quasiparticle cascade. Our results provide *ab initio* understanding of emergent stability properties in self-organized scaling phenomena.

DOI: [10.1103/PhysRevLett.130.031602](https://doi.org/10.1103/PhysRevLett.130.031602)

Introduction.—One of the greatest challenges in physics is to understand how emergent collective behavior arises from the underlying quantum dynamics of a system. A prominent example concerns universal scaling phenomena associated with phase transitions after fine-tuning of relevant parameters, such as adjustment to a critical temperature [1]. By contrast, scaling phenomena without fine-tuning of parameters can play an important role for the buildup of complex structures, in particular, far from equilibrium [2,3]. On a fundamental quantum level, important examples include universal scaling in quantum many-body systems associated with nonthermal fixed points, which exhibit attractor properties [4]. Here the applications range from relativistic collisions with heavy nuclei [5,6] and early Universe cosmology [7,8] to experiments with ultracold quantum gases, providing unprecedented data for universal scaling in isolated [9–11] and driven systems [12,13]. However, an understanding of the observation of attractor behavior for scaling solutions based directly on the underlying quantum dynamics is still missing.

In this Letter, we compute the stability properties of a nonthermal fixed point for an interacting scalar quantum field theory in $3 + 1$ space-time dimensions from first principles. While the approach to universal scaling behavior of this system is known to be observed from a wide

range of far-from-equilibrium initial conditions without fine-tuning [14–19], our analysis reveals the presence of both stable and unstable perturbations around the scaling solution. The unstable modes lead to quasiexponential deviations from the fixed point in the infrared. This is exemplified in Fig. 1, where the time evolution

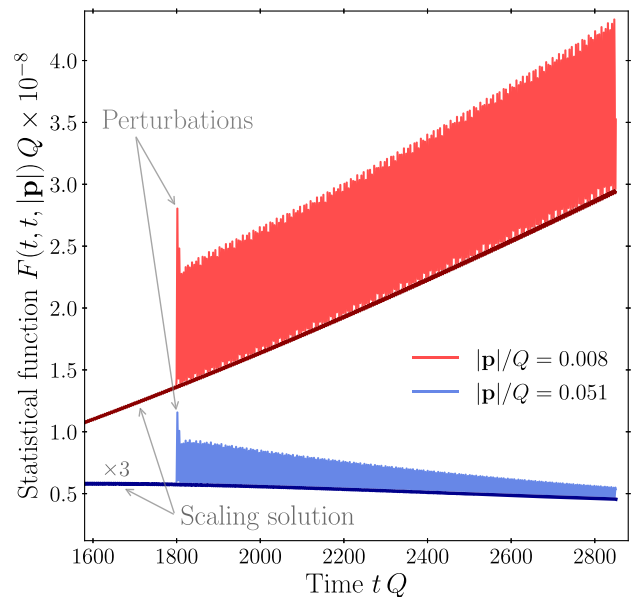


FIG. 1. Statistical function $F(t, t, |\mathbf{p}|)$ for two different values of the momentum $|\mathbf{p}|$ as a function of time. Shown are the unperturbed scaling solutions (solid bold lines) and the responses after a perturbation $\delta F(t_i, t_i, |\mathbf{p}|)$ at time $t_i Q = 1800$. We find that perturbations grow (red) at sufficiently low momenta and decay (blue) at higher momenta.

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of the statistical correlation function F (see below) for two different, unstable (red) and stable (blue), momenta is shown.

Our results are obtained from a systematic large- N expansion to next-to-leading order, where N is the number of components of the real field operator $\hat{\Phi}_{a=1,\dots,N}(t, \mathbf{x})$ as a function of time t and space $\mathbf{x}$ or, respectively, spatial Fourier momentum $\mathbf{p}$. The full unitary quantum dynamics of correlation functions at this order is obtained numerically. We uncover the underlying physical processes using a linear response analysis around the time-evolving scaling solution and explain the observations of attractive behavior at fixed momenta in the presence of both negative and positive stability exponents.

Far-from-equilibrium quantum fields.—We consider a relativistic quantum field theory described by the $O(N)$ -symmetric Hamilton operator

$$\hat{H}(t) = \int d^3x \left[\frac{1}{2} \hat{\Pi}_a^2(t, \mathbf{x}) + \frac{1}{2} (\nabla_{\mathbf{x}} \hat{\Phi}_a(t, \mathbf{x}))^2 + \frac{\lambda}{4!N} (\hat{\Phi}_a(t, \mathbf{x}) \hat{\Phi}_a(t, \mathbf{x}))^2 \right], \quad (1)$$

with $\hat{\Pi}_a(t, \mathbf{x}) = \partial_t \hat{\Phi}_a(t, \mathbf{x})$ and a summation over repeated indices is implied. While we specify values of the coupling λ for the massless fields below, the detailed values of these parameters do not enter universal quantities [15]. We solve for the nonequilibrium dynamics employing a self-consistent large- N expansion to next-to-leading order [20,21]. Diagrammatically, this takes into account an infinite series of quantum corrections, which is indicated in Fig. 2 for the expectation value of the Hamiltonian. More precisely, we solve the quantum evolution equations for the connected correlation functions, i.e., the expectation value of the anticommutator (statistical function)

$$F_{ab}(t, t', \mathbf{x} - \mathbf{x}') = \frac{1}{2} \langle \{ \hat{\Phi}_a(t, \mathbf{x}), \hat{\Phi}_b(t', \mathbf{x}') \} \rangle - \langle \hat{\Phi}_a(t, \mathbf{x}) \rangle \langle \hat{\Phi}_b(t', \mathbf{x}') \rangle \quad (2)$$

and of the commutator (spectral function)

$$\rho_{ab}(t, t', \mathbf{x} - \mathbf{x}') = i \langle [\hat{\Phi}_a(t, \mathbf{x}), \hat{\Phi}_b(t', \mathbf{x}')] \rangle \quad (3)$$

for spatially homogeneous and isotropic systems. The non-equilibrium time evolution equations are derived from the two-particle irreducible (2PI) effective action on the closed

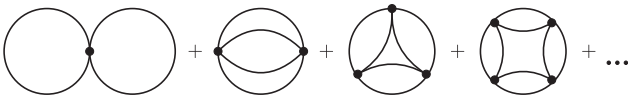


FIG. 2. Diagrammatic representation of the infinite series of contributions to $\langle \hat{H}(t) \rangle$ taken into account with the 2PI effective action to NLO in the large- N expansion. Lines correspond to self-consistently resummed propagators and circles to bare vertices.

time contour at next-to-leading order (NLO) in the resummed large- N expansion. By virtue of $O(N)$ symmetry, we consider $\langle \hat{\Phi}_a(t, \mathbf{x}) \rangle = 0$, such that the correlators are diagonal in field space with $F_{ab} = \delta_{ab} F$ and $\rho_{ab} = \delta_{ab} \rho$.

The spectral function is related to the retarded propagator as $G^R(t, t', \mathbf{x}) = \theta(t - t') \rho(t, t', \mathbf{x})$ and encodes the bosonic equal-time commutation relations $\rho(t, t', \mathbf{x})|_{t=t'} = 0$, $\partial_t \rho(t, t', \mathbf{x})|_{t=t'} = \delta^{(3)}(\mathbf{x})$. While the latter determine also ρ at some initial time t_0 , for F we consider the following class of initial conditions in spatial Fourier space:

$$F(t_0, t_0, \mathbf{p}) = \frac{1}{\sqrt{\mathbf{p}^2 + m^2}} \left[\left(\frac{N n_0}{\lambda} \right) \theta(Q - |\mathbf{p}|) + \frac{1}{2} \right] \quad (4)$$

in the limit $m^2 \rightarrow 0^+$. To start the evolution far from equilibrium, we choose $N n_0 / \lambda \gg 1$ up to the momentum scale Q determining the (conserved) energy density of the system.

Perturbations around the universal scaling solution.—The subsequent time evolution approaches self-similar scaling behavior described by

$$F(t, t, |\mathbf{p}|) = (t/t_{\text{ref}})^\alpha F_S[(t/t_{\text{ref}})^\beta |\mathbf{p}|], \quad (5)$$

corresponding to the solid lines exemplified in Fig. 1. The scaling exponents are universal, as well as the form of the scaling function F_S depending on the single scaling variable $(t/t_{\text{ref}})^\beta |\mathbf{p}|$, with $\beta \simeq 1/2$, $\alpha = d\beta$, and $d = 3$ in our case [15]. Here t_{ref} denotes some reference time, which does not affect universal properties and is described below.

The approach to (5) has been studied starting from a wide range of different initial conditions. Examples include parametric resonance initial conditions in a strong field regime [8,22] or large initial occupation numbers similar to (4) [14,15,17–19,23]. Corrections to the scaling exponents of the high-momentum cascade were recently explored via kinetic methods [24,25] in the context of the prescaling phenomenon [26]. Furthermore, nonthermal fixed points were realized in cold atom experiments, e.g., from sudden quenches across phase transitions [9–11]. While universal quantities like scaling exponents do not depend on microscopic details, such as specific values of λ or n_0 , the time for the approach to scaling is not a universal quantity. Moreover, scaling according to (5) is a transient phenomenon in an isolated system, with the system eventually approaching thermal equilibrium [22]. The latter can always be delayed by increasing $(N n_0) / \lambda$ entering the initial condition (4) such that (5) is accurately realized for suitable time ranges in units of Q .

For the numerical results presented in this Letter, we employ $\lambda = 0.01$, $n_0 = 25$, and $N = 4$, which, e.g., reflects the Higgs sector of the standard model of particle physics [27]. Starting from the initial conditions we consider, the universal scaling regime of interest emerges in the infrared after times of order $10^3/Q$ and our scaling solution results

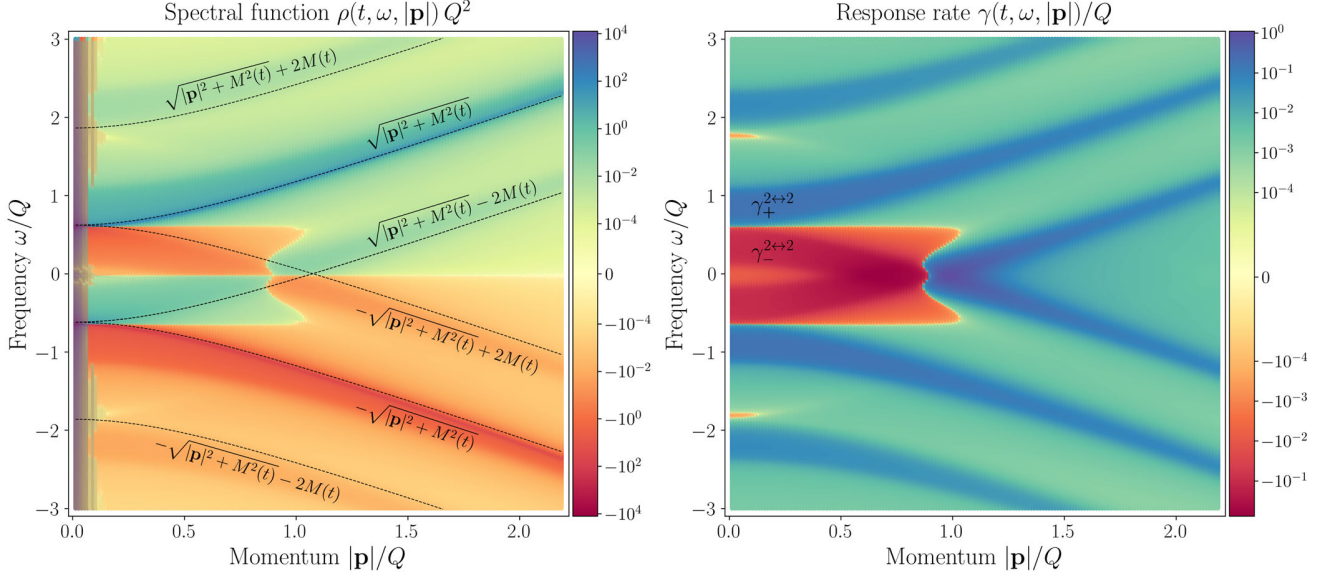


FIG. 3. Left: frequency and momentum dependence of the nonequilibrium spectral function $\rho(t, \omega, |\mathbf{p}|) = -\rho(t, -\omega, |\mathbf{p}|)$ at $tQ = 2400$. Apart from the quasiparticle peak with relativistic dispersion relation, we find a tower of additional quasiparticle peaks. Right: linear response rate $\gamma(t, \omega, |\mathbf{p}|) = \gamma(t, -\omega, |\mathbf{p}|)$ versus frequency and momentum at the same time. For low momenta, the on-shell response rate at $\omega \simeq M$ receives competing contributions from adjacent peaks and turns negative.

are in accordance with earlier studies of nonthermal fixed points.

After the behavior (5) is established, we consider perturbations around the scaling solution according to

$$F(t, t, |\mathbf{p}|) = (t/t_{\text{ref}})^\alpha F_S[(t/t_{\text{ref}})^\beta |\mathbf{p}|] + \delta F(t, t, |\mathbf{p}|) \quad (6)$$

by suddenly turning on $\delta F > 0$ at a given time t_i . Perturbations of two different momentum modes at $t_i Q = 1800$ are depicted in Fig. 1. In the following, we analyze the underlying processes that lead to the quasi-exponential growth of perturbations in time at low values of $|\mathbf{p}|$ and decay at higher momenta.

Far-from-equilibrium quasiparticles.—Since the universal exponent β is positive, the scaling behavior (5) describes an inverse cascade toward low momenta during which a macroscopically occupied zero-momentum mode emerges [15]. We demonstrate in the following that this gives rise to a tower of emergent quasiparticle excitations out of equilibrium. The spectrum is encoded in the spectral function ρ , which we access by a Wigner transform with respect to relative time $\Delta t = t - t'$. Expressed as a function of the central time $\tau = \frac{1}{2}(t + t')$, the spectral function is obtained via (a factor of i is included for the Wigner transformed ρ to ensure realness)

$$\rho(\tau, \omega, |\mathbf{p}|) = -i \int_{-2\tau}^{2\tau} d\Delta t e^{i\omega\Delta t} \rho\left(\tau + \frac{\Delta t}{2}, \tau - \frac{\Delta t}{2}, |\mathbf{p}|\right). \quad (7)$$

For notational simplicity, we write $\tau = t$ in the following, and the Wigner transformed spectral function is shown in the

left graph of Fig. 3 for $tQ = 2400$. One observes a pronounced quasiparticle peak obeying a relativistic dispersion relation $\omega(t, \mathbf{p}) \equiv \sqrt{\mathbf{p}^2 + M^2(t)}$ with in-medium mass $M(t)$. Its value is established quickly and approximately constant in the scaling regime with $M/Q \simeq 0.62$ for the parameters employed for Fig. 3. The associated widths scale to zero in the scaling regime as time progresses, leading to increasingly long-lived quasiparticles [18,19].

However, we also find additional quasiparticles to be present. Their dispersion relations are well described by

$$\omega^\pm(t, \mathbf{p}) = 2M(t) \pm \omega(t, \mathbf{p}). \quad (8)$$

This multitude of excitations out of equilibrium may be understood as phase fluctuations of a macroscopic zero mode rotating in N -component field space [19], here built up by the inverse cascade. Though the additional quasiparticle modes at $\omega^\pm(t, \mathbf{p})$ exhibit a much lower magnitude as compared to the pronounced peak at $\omega(t, \mathbf{p})$, their contributions play a leading role for the stability properties of the system at sufficiently low momenta, as we show next.

Linear response around the universal scaling solution.—To gain further analytical insights, we perform a linear response analysis around the nonequilibrium scaling solution, described by

$$\partial_t \delta F(t, \omega, |\mathbf{p}|) = -\gamma(t, \omega, |\mathbf{p}|) \delta F(t, \omega, |\mathbf{p}|). \quad (9)$$

Here $\gamma(t, \omega, |\mathbf{p}|)$ denotes the *time-dependent* response rate for the frequency- and momentum-resolved (Wigner transformed) perturbations of the statistical function.

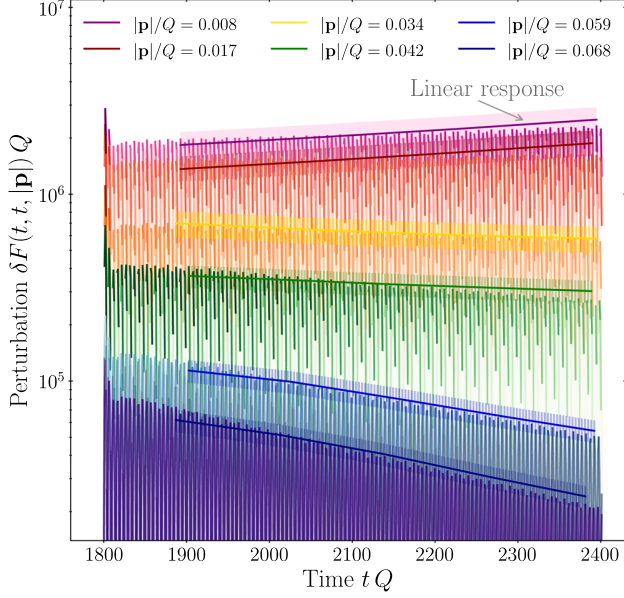


FIG. 4. The perturbation $\delta F(t, t, |\mathbf{p}|)$ around the universal scaling solution as a function of time for different momenta. The growth and decay of the envelopes for the oscillatory full solution of the quantum evolution equations is rather well described by the linear response result.

This response rate for the far-from-equilibrium state is time dependent, in contrast to conventional linear response estimates around equilibrium solutions. The dynamics of the perturbation

$$\delta F(t, \omega, |\mathbf{p}|) = \delta F(t_i, \omega, |\mathbf{p}|) \exp[-\Gamma(t, \omega, |\mathbf{p}|)] \quad (10)$$

is then determined by a rate integral Γ , with $\partial_t \Gamma = \gamma$.

We compute (9) from the quantum evolution equations at NLO large N using linearization [29,30] and a lowest-order gradient expansion [31,32]. The result for the nonequilibrium response rate is displayed in the right graph of Fig. 3 for the same time in the universal scaling regime as the spectral function on the left. One observes very similar pronounced structures, taking into account that the response rate is

symmetric in frequency with $\gamma(t, \omega, |\mathbf{p}|) = \gamma(t, -\omega, |\mathbf{p}|)$ rather than antisymmetric as the spectral function. Remarkably, for small momenta, the negative regions (red) around $\omega \simeq \pm M$ exhibit the largest values of $|\gamma|$. This has the striking consequence that in the infrared the dominant on-shell rate $\gamma(t, \omega \simeq M, |\mathbf{p}| \simeq 0)$ turns negative. According to (9) or (10), respectively, this gives rise to growing (unstable) perturbations around the universal scaling solution.

The linear response results for the on-shell rate can be compared to the time evolution of perturbations $\delta F(t, t, |\mathbf{p}|)$ obtained from the solution of the full quantum evolution equations at NLO large N . This is shown for various momenta in Fig. 4, after an initial “kick” $\delta F(t_i, t_i, |\mathbf{p}|) = 10^{-3} (t_i/t_{\text{ref}})^\alpha F_S[(t_i/t_{\text{ref}})^\beta |\mathbf{p}|]$ at $t_i Q = 1800$. For the comparison, the initial perturbation $\delta F(t_i, t_i, |\mathbf{p}|)$ was fitted as a momentum-dependent constant. One observes that the quasiexponential growth or decay of the envelopes of oscillations in time is rather well captured by linear response, where the error band indicates a residual mild dependence on the infrared cutoff for the low momenta considered. The rapid oscillation timescale of the full solution is set by the quasiparticle mass $M(t)$, which is not captured by the linear response analysis employing also a gradient expansion in time. We note that for small enough δF also from the full evolution dynamics an approximately linearly independent evolution of the different modes by considering also separately excited modes is observed, which corroborates a linear response approach.

The linear response analysis also allows us to uncover the underlying dynamical scattering processes of the various quasiparticles. This is conveniently described by introducing the (off-shell) distribution function $f_p(t)$ with four-momentum $p = (p^0, \mathbf{p})$ according to

$$F(t, p) = \left[f_p(t) + \frac{1}{2} \right] \rho(t, p). \quad (11)$$

The leading contribution to the on-shell linear response results arises from elastic (“two-to-two”) scatterings described by

$$\begin{aligned} \gamma^{2 \leftrightarrow 2}(t, p) = & -\frac{\lambda^2}{36Np^0} \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 l}{(2\pi)^3} \frac{d^3 r}{(2\pi)^3} (2\pi)^4 \int_0^\infty \frac{dq^0 dl^0 dr^0}{(2\pi)^3} \delta^{(4)}(p + l - q - r) \rho(t, q) \rho(t, l) \rho(t, r) \\ & \times [v_{\text{eff}}(t, p + l) + v_{\text{eff}}(t, p - q) + v_{\text{eff}}(t, p - r)] \{ [f_l(t) + 1] f_q(t) f_r(t) - f_l(t) [f_q(t) + 1] [f_r(t) + 1] \}. \end{aligned} \quad (12)$$

Here the effective vertex

$$v_{\text{eff}}(t, p) = \left| 1 + \frac{\lambda}{3} \int \frac{d^4 k}{(2\pi)^4} F(t, p - k) G^R(t, k) \right|^{-2} \quad (13)$$

resums the infinitely many s -channel scatterings taken into account at NLO in the large- N expansion depicted in Fig. 2 [32,33].

The on-shell response rate $\gamma^{2 \leftrightarrow 2}(t, p^0 \simeq M, |\mathbf{p}| \simeq 0)$ in the infrared exhibits two competing contributions of comparable magnitude: a positive (stable) contribution $\gamma_+^{2 \leftrightarrow 2}(t, \mathbf{p}) = \gamma^{2 \leftrightarrow 2}(t, p^0 = \omega(t, \mathbf{p}), \mathbf{p})$ and a negative (unstable) contribution $\gamma_-^{2 \leftrightarrow 2}(t, \mathbf{p}) = \gamma^{2 \leftrightarrow 2}(t, p^0 = \omega^-(t, \mathbf{p}), \mathbf{p})$, which are respectively denoted in the right graph of Fig. 3. The latter dominates in the scaling regime at

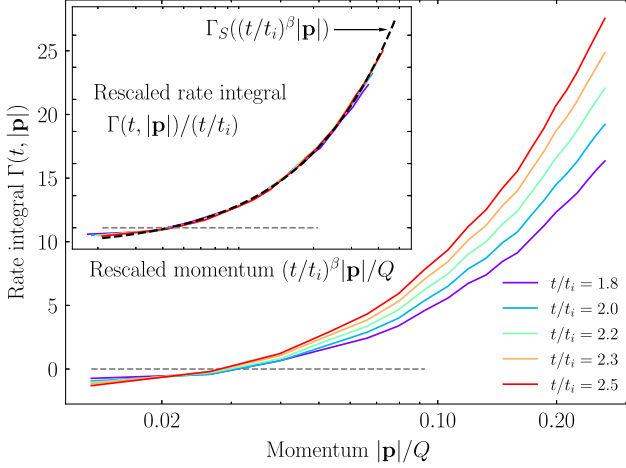


FIG. 5. Rate integral as a function of momentum for different times. The inset shows the same data but rescaled, such that all curves at different times collapse to a single scaling function, which is well described by (15) as displayed by the dashed black line in the inset.

$|\mathbf{p}| = 0$ for which $\omega(t, \mathbf{p} = 0) = \omega^-(t, \mathbf{p} = 0) = M(t)$. These competing processes can be related to elastic scatterings, where $p^0 + \omega(t, \mathbf{l}) = \omega(t, \mathbf{q}) + \omega(t, \mathbf{r})$, with contributions at $p^0 = \omega(t, \mathbf{p})$ and additionally at $p^0 = \omega^-(t, \mathbf{p})$. The latter can be associated with the production of particles with mass M at rest and thereby contribute to the buildup of the macroscopic zero mode.

We observe that the emergence of the competing contributions in this strongly correlated system hinges on the combination of a gapped dispersion relation and a steep power-law behavior of the distribution function for characteristic infrared momenta $f(t, \mathbf{p}) \sim |\mathbf{p}|^{-\kappa}$ for $\kappa \gtrsim 2$. In our case $\kappa \simeq 5$ and, in particular, the phenomenon is absent in thermal equilibrium [29,30,34].

Universal scaling of perturbations.—Since the linear response analysis describes the dynamics of perturbations in terms of the universal scaling solution itself, the response rate γ and the respective rate integral Γ can exhibit scaling. For the on-shell rate integral, we find

$$\Gamma(t, |\mathbf{p}|) = (t/t_{\text{ref}}) \Gamma_S[(t/t_{\text{ref}})^\beta |\mathbf{p}|], \quad (14)$$

with scaling function

$$\Gamma_S[(t/t_{\text{ref}})^\beta |\mathbf{p}|] = A(t/t_{\text{ref}})^\beta |\mathbf{p}|/Q - B, \quad (15)$$

corresponding to an on-shell response rate given by $\gamma(t, |\mathbf{p}|) = (\beta + 1)(A/t_{\text{ref}})(t/t_{\text{ref}})^\beta |\mathbf{p}|/Q - B/t_{\text{ref}}$.

One observes that the overall scaling exponent of time in (14) is unity, which follows analytically from a scaling analysis of (12) and (13). Similarly, it follows that scaling functions do not depend on time and momentum separately, but only on the product $t^\beta |\mathbf{p}|$ with universal exponent β . We confirm these results by analyzing also the numerical

solution of the full evolution equations at NLO large N in Fig. 5 [35]. While the main graph shows the unrescaled data, the inset demonstrates that the rescaled results all collapse to a single scaling function to very good accuracy.

The form of the scaling function (15) is universal up to an overall constant factor and normalization of the argument, which we subsume into the constants A and B . Their magnitudes, as well as the universal linear dependence of Γ_S on $t^\beta |\mathbf{p}|$, are obtained numerically via a fit of (15) to the rescaled rate integral, as displayed by the dashed black line in the inset of Fig. 5. We choose $t_i = t_{\text{ref}}$ and for $t \gg t_i$ we find that $A/(Qt_{\text{ref}})^{\beta+1} = (1.10 \pm 0.01) \times 10^{-3}$ and $B/(Qt_{\text{ref}}) = (1.41 \pm 0.05) \times 10^{-3}$ become independent of the choice of t_{ref} in units of Q , with fit errors indicated. While the values of A and B depend on the chosen microscopic parameters, such as coupling and initial conditions, the amplitude ratio $A/B^{\beta+1} = 20.8 \pm 1.2$ is universal.

From (15) we find that for small enough momenta the negative contributions $\sim Bt$ to the rate integral Γ always dominate. However, for given momentum $|\mathbf{p}|$ and as time progresses the positive (stable) term $\sim At^{\beta+1}|\mathbf{p}|$ will eventually outgrow the negative (unstable) contribution for all nonzero momenta in the scaling regime.

Conclusion.—Our results establish the stability properties of a nonthermal fixed point from first principles by solving for the dynamics of perturbations around it. The perturbations turn out to be captured well by a time- and momentum-dependent response rate γ . Our results demonstrate that γ exhibits universal properties and we determine the dynamical exponents, amplitude ratios, and scaling functions. We discover the phenomenon of a scaling instability. While the low-momentum response rate is negative leading to unstable perturbations and higher momenta are stable, γ is a self-similar scaling function of $\sim t^\beta |\mathbf{p}|$. As a consequence, the system shows attractor behavior after $t \sim 1/|\mathbf{p}|^{1/\beta}$ for any nonvanishing momentum $|\mathbf{p}|$. Since for a system with linear size L the smallest resolved momentum is $\sim 1/L$, measurements in finite systems will always detect attractor properties at late enough times.

Since the considered universal scaling is known to be present in a wide range of quantum and classical-statistical relativistic as well as nonrelativistic many-body systems, our results provide an important *ab initio* example of the emergence of attractor properties in dynamical scaling phenomena. Moreover, the example shows that such “self-organized” scaling, in which complexity emerges in a robust way that requires no particular fine-tuning, can be realized in the presence of both stable and unstable directions for the dynamics. This opens new perspectives on the underlying mechanisms and scope of models that have a critical point as an attractor. Moreover, present-day experimental platforms with ultracold quantum gases in the many-body regime can give direct access to these intriguing far-from-equilibrium phenomena.

We thank Gregor Fauth for his involvement during the initial stages of this work. The authors acknowledge support by the state of Baden-Württemberg through bwHPC and the German Research Foundation (DFG) through Grant No. INST 40/575-1 FUGG (JUSTUS 2 cluster). This work is supported by the DFG under the Collaborative Research Center SFB 1225 ISOQUANT (Project-ID 27381115) and the Heidelberg STRUCTURES Excellence Cluster under Germany's Excellence Strategy EXC2181/1-390900948.

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4.3

Prescaling Relaxation to Nonthermal Attractors

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(Received 30 August 2023; revised 21 December 2023; accepted 29 January 2024; published 15 February 2024)

We study how isotropic and homogeneous far-from-equilibrium quantum systems relax to nonthermal attractors, which are of interest for cold atoms and nuclear collisions. We demonstrate that a first-order ordinary differential equation governs the self-similar approach to nonthermal attractors, i.e., the prescaling. We also show that certain natural scaling-breaking terms induce logarithmically slow corrections that prevent the scaling exponents from reaching the constant values during the system's lifetime. We propose that, analogously to hydrodynamic attractors, the appropriate mathematical structure to describe such dynamics is the transseries. We verify our analytic predictions with state-of-the-art 2PI simulations of the large- N vector model and QCD kinetic theory.

DOI: [10.1103/PhysRevLett.132.071602](https://doi.org/10.1103/PhysRevLett.132.071602)

Introduction.—Thermalization of isolated quantum many-body systems is an important contemporary research problem of a broad scope. Its relevance ranges from cold atom systems, through QCD in ultrarelativistic nuclear collisions, all the way to gravity and black hole physics [1]. Given the complexity of modeling quantum many-body dynamics and the richness of nonequilibrium phenomena, emergent regularities that form a basis for a quantitative understanding are of particular interest.

In this work, we are concerned with an important instance of such an emergent regularity: far-from-equilibrium self-similar time evolution of nonthermal attractors, also known as nonthermal fixed points. These phenomena are transient stages in the thermalization dynamics, whose defining feature is self-similar scaling behavior in time. Consider a momentum distribution function $f(t, \mathbf{p})$ of a homogeneous and isotropic system, where t is time and $\mathbf{p}$ spatial momentum. The system reaches a nonthermal attractor, when f scales with time in a characteristic momentum range

$$f(t, \mathbf{p}) = (t/t_{\text{ref}})^{\alpha_{\infty}} f_S[(t/t_{\text{ref}})^{\beta_{\infty}} |\mathbf{p}|] \quad (1)$$

with *constant* scaling exponents α_{∞} and β_{∞} . Indeed, such behavior corresponds to a vast reduction in the complexity, as the knowledge of the distribution function at some time allows one to determine the distribution function at a different time by a simple rescaling.

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Nonthermal attractors appear in the studies of isolated quantum systems across a wide range of energy scales: ultracold quantum gases [2–11], ultrarelativistic nuclear collisions [12–14], and early Universe cosmology [15,16]. Despite significant interest in nonthermal attractors, a quantitative understanding of how a system approaches a nonthermal fixed point remains elusive [17–21].

In [17], it was proposed that even prior to reaching the nonthermal attractor (1) the system can exhibit prescaling, where f has already assumed the fixed-point shape f_S but continues to evolve with time-dependent scaling exponents $\alpha(t)$ and $\beta(t)$:

$$f(t, \mathbf{p}) = A(t) f_S[B(t) |\mathbf{p}|]. \quad (2)$$

The prescaling factor $A(t) = \exp[\int_{t_0}^t dt' \alpha(t')/t']$ reduces to the fixed-point scaling of Eq. (1) when $\alpha(t)$ approaches α_{∞} . The same holds for $B(t)$ in terms of $\beta(t)$.

Given that scaling (1) is an asymptotic late-time statement known to be reached slowly, the systems of interest might, in fact, spend a much greater fraction of their lifetime prescaling (2) rather than scaling (1). Therefore, a quantitative understanding of prescaling is as important as understanding scaling itself.

In our work, we develop a simple theoretical description of prescaling dynamics that uses the same assumptions as the ones used to derive scaling. We test our predictions using strongly correlated large- N vector model and weak coupling QCD kinetic theory simulations.

Scaling implies prescaling.—Understanding prescaling requires identifying laws governing time evolution of $A(t)$ and $B(t)$ [or, alternatively, $\alpha(t)$ and $\beta(t)$]. As we show, these laws have a surprisingly simple origin and form.

The key role in deriving scaling (1) is played by conserved quantities: particle number density $n = \int d^d \mathbf{p} f / (2\pi)^d$ or energy density $\mathcal{E} = \int d^d \mathbf{p} \omega_{\mathbf{p}} f / (2\pi)^d$, where d is the number of spatial dimensions and $\omega_{\mathbf{p}}$ is the dispersion relation of particles. We focus on $\omega_{\mathbf{p}} \sim |\mathbf{p}|^z$. Applying conservation of n or $\mathcal{E}$ in the momentum regime of interest imposes the relation $\alpha_{\infty} = \sigma \beta_{\infty}$ between the scaling exponents [22]. When $n = \text{const}$, then $\sigma = d$, while $\mathcal{E} = \text{const}$ gives $\sigma = (d + z)$. The conserved quantities are local in time, which means that they, in fact, constrain also prescaling exponents in exactly the same way: $\alpha(t) = \sigma \beta(t)$. Equivalently, $A(t) = B(t)^{\sigma}$. This implies that there is only one independent degree of freedom in the isotropic and homogeneous prescaling, which we will choose to be $B(t)$.

The time evolution for the independent prescaling factor $B(t)$ is still subject to the equation of motion for f . In the case of a kinetic theory, it is given by the Boltzmann equation with collision kernel $\mathcal{C}[f]$:

$$\partial_t f(t, \mathbf{p}) = \mathcal{C}[f](t, \mathbf{p}). \quad (3)$$

In the present section, we assume the collision kernel to be a homogeneous functional of particles momenta, i.e., to simply scale under Eq. (2) by $A(t)^{\mu_{\alpha}} B(t)^{\mu_{\beta}} \equiv B(t)^{\sigma \mu_{\alpha} + \mu_{\beta}}$ for some real numbers $\mu_{\alpha, \beta}$. This assumption applies to many (but not all) collision kernels describing nonthermal attractors (see Supplemental Material [23] for explicit examples). Typically, overoccupation singles out terms with the highest power of the distribution function, and associated matrix elements often happen to scale homogeneously under rescalings of momenta. For such collision kernels, we can separate time-dependent and (rescaled) momentum-dependent contributions by substituting the prescaling ansatz (2) in the Boltzmann equation (3) and using $A(t) = B(t)^{\sigma}$:

$$\frac{B(t)^{1-1/\beta_{\infty}}}{\partial_t B(t)} = \frac{1}{D_1} = \frac{[\sigma + \bar{\mathbf{p}} \cdot \partial_{\bar{\mathbf{p}}}] f_s(\bar{\mathbf{p}})}{\mathcal{C}[f_s](\bar{\mathbf{p}})}, \quad (4)$$

where $\bar{\mathbf{p}} = B(t) \mathbf{p}$ is the rescaled momentum, $1/\beta_{\infty} = (1 - \mu_{\alpha})\sigma - \mu_{\beta}$, and D_1 is a separation of variables constant. The intrinsic time dependence of our setup implies nonzero D_1 , which can be fixed at any time in the prescaling evolution, and we choose $D_1 = \beta(t_0)/t_0$.

The idea of separation of variables in the context of nonthermal attractors appeared already in [24], but for homogeneous $\mathcal{C}[f]$ only solutions with constant scaling exponents were considered. Our key observation here is that prescaling is encapsulated by the general solution

$$B(t) = \left(\frac{t - t_*}{t_{\text{ref}}} \right)^{\beta_{\infty}} \approx \left(\frac{t}{t_{\text{ref}}} \right)^{\beta_{\infty}} \left(1 - \beta_{\infty} \frac{t_*}{t} + \dots \right), \quad (5a)$$

$$\beta(t) = \beta_{\infty} \frac{t}{t - t_*} \approx \beta_{\infty} + \beta_{\infty} \frac{t_*}{t} + \dots \quad (5b)$$

From Eq. (5), it is clear that prescaling induces power-law corrections to scaling. The prescaling originates from the presence of nonzero $t_* = t_0(1 - \beta_{\infty}/\beta_0)$, where t_0 and $\beta_0 \equiv \beta(t_0)$ correspond to the initial data. Its appearance comes as no surprise: The dynamics of the system in question is time translationally invariant, and, therefore, t appearing in formulas needs to be measured with respect to some time t_* . The reason why it does not appear in Eq. (1) is because the exact scaling is an asymptotic late-time statement and dependence on t_* drops. Note that, while β_{∞} and α_{∞} are theory specific and independent of initial conditions, t_* will depend on a chosen initial state.

Before we move to testing Eq. (5) using *ab initio* solutions of quantum dynamics, let us reiterate that this result originates from the Boltzmann equation and pertinent conservation laws. These are exactly the same constraints as used in a conventional scaling analysis [22]. Prescaling in the case of collision kernels being homogeneous functionals of momenta can, therefore, be understood as a direct consequence of the existence of scaling.

Prescaling in large- N vector model.—We begin by testing Eq. (5) against the full quantum dynamics of a next-to-leading order (NLO) large- N vector model at small coupling λ . This model features a dual cascade [22,25] consisting of an inverse particle cascade in the IR and a direct energy cascade to the UV. We focus on the IR scaling, which is characterized by effective particle number conservation with $(\alpha_{\infty}, \beta_{\infty}) = (d/2, 1/2)$ and was also realized in cold quantum gases [2,4]. This fixed point arises for large occupations $f \sim 1/\lambda \gg 1$ such that its description requires going beyond a standard kinetic theory analysis. Large- N kinetic theory addresses this regime due to inclusion of relevant resummations [22,26]. The corresponding collision kernel scales homogeneously with $\mu_{\alpha} = 1$ and $\mu_{\beta} = -2$ [22].

We perform *ab initio* studies of this fixed point in $3 + 1$ dimensions using the 2PI formalism following [21] to validate Eq. (5) and the underlying assumptions. Below the mass gap, the equal-time statistical function $F(t, t, |\mathbf{p}|)$ reduces to $f(t, |\mathbf{p}|)$ [22], where the system is initialized with $f(t_0, |\mathbf{p}|) = f_0 \theta(Q - |\mathbf{p}|)$ with $f_0 = 100/\lambda$, $\lambda = 0.01$, and Q is the characteristic hard scale far from equilibrium. Following [17], we extract the prescaling factors $A(t)$ and $B(t)$ from the time evolution of integral moments of $F(t, t, |\mathbf{p}|)$ [23], e.g., $B(t) = n(t)\mathcal{E}(t_0)/[\mathcal{E}(t)n(t_0)]$ [27]. In the upper right panel in Fig. 1, we show how rescaling with $A(t)$ and $B(t)$ leads to an early collapse of distributions at different times, while a considerable spread remains when rescaling with the fixed-point exponents (left panel). The evolution of the extracted $B(t)$ is shown in the lower panel to be well described by Eq. (5a) already at early times (dashed black line) and asymptotes only to the corresponding fixed-point scaling behavior (1) (solid green line). The full NLO quantum dynamics are shown to be captured

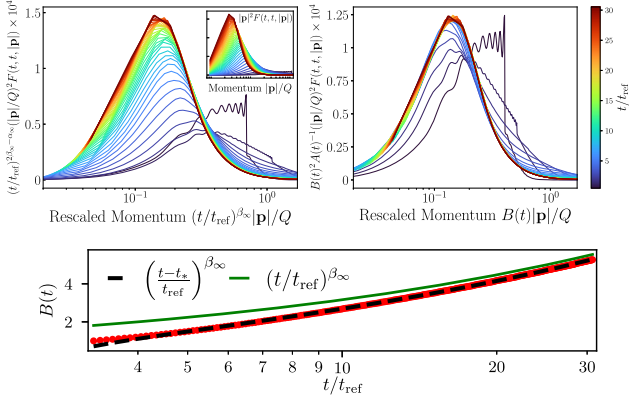


FIG. 1. Top: statistical function $|\mathbf{p}|^2 F(t, t, |\mathbf{p}|)$ from 2PI simulations of large- N vector model rescaled with (left) fixed-point exponents $(\alpha_\infty, \beta_\infty) = (3/2, 1/2)$ and (right) with prescaling factors $A(t)$ and $B(t)$, where $t_{\text{ref}} Q \sim 93$. The unrescaled distribution is shown in the inset. Bottom: extracted $B(t)$ compared with Eq. (5a) and asymptotic form.

remarkably well by our effective kinetic description of Eq. (5) already from times close to initialization.

Prescaling in isotropic QCD kinetic theory.—We move now to studying prescaling dynamics in QCD, whose nonthermal fixed point plays an important role in our understanding of thermalization dynamics in weakly coupled models of ultrarelativistic nuclear collisions [1]. We use QCD kinetic theory, where the evolution of the color and polarization averaged gluon distribution function $f(t, \mathbf{p})$ is described by $2 \leftrightarrow 2$ and $1 \leftrightarrow 2$ processes [28]: $\partial_t f(t, \mathbf{p}) = \mathcal{C}^{2 \leftrightarrow 2}[f](t, \mathbf{p}) + \mathcal{C}^{1 \leftrightarrow 2}[f](t, \mathbf{p})$; see [23, 28–30]. Nonthermal fixed points can be reached from a wide range of initial conditions including large occupation numbers [31–37], which we implement via $f(t_i, \mathbf{p}) = n_0/g^2 \exp[-\mathbf{p}^2/Q^2]$. Here, g^2 is the square of the coupling and n_0 is the initial occupation. We consider $n_0 = 1$ and $g^2 = 10^{-8}$. To obtain the precise late-time behavior, we initialize at $t_i Q = 0$ and evolve for very long times until $t_f Q = 10^8$. Results will be given in units of the characteristic energy scale Q as given by the maximum of $|\mathbf{p}|^2 f(t_i, \mathbf{p})$. We discuss explicitly only the pure gluon simulations where the scaling phenomenon is encountered after checking that our results do not change under the inclusion of quark or antiquark dynamics.

A scaling analysis for the vacuum QCD collision kernel together with energy conservation $\sigma = d + z = 4$ reveals the direct energy cascade fixed point $(\alpha_\infty, \beta_\infty) = (-4/7, -1/7)$; see [38–40]. However, the overall scaling of the elastic collision kernel is broken by the presence of the Debye mass $m_D(t)^2 \sim \int d^3 \mathbf{p} f(t, \mathbf{p})/p \sim A(t)B(t)^{-2} \bar{m}_D^2$ that regulates soft elastic scatterings where $\bar{m}_D^2 \equiv m_D(t_0)^2$ [23]. With $\alpha_\infty, \beta_\infty < 0$, the Debye mass decreases over time such that the violation of scaling leads only to a delay in the approach to the fixed point. This (diminishing) breaking of

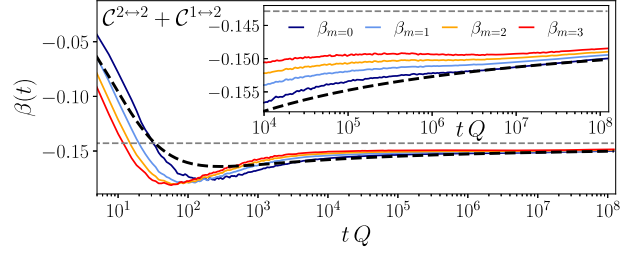


FIG. 2. Comparison between prescaling exponents extracted from QCD kinetic theory simulations with scaling-breaking elastic processes and the analytical prescaling expectation of Eq. (8) (dashed black line). Displayed are $\beta_{m=0,1,2,3}(t)$ computed from different moments of the distribution function [23].

scaling for QCD kinetic theory is demonstrated in Fig. 2, where we extract $\beta(t)$ (solid color lines) from different moments of the distribution function [23]. We do not display $\alpha(t)$ explicitly, as we find the scaling relation $\alpha(t)/\beta(t) = 4$ realized to a very good accuracy. $\beta(t)$ is observed to approach the fixed-point value $\beta_\infty = -1/7$ (dashed gray line), but a finite deviation remains due to $m_D(t)$ even after 8 orders of magnitude in simulation time. The decreasing but finite spread between different moments further demonstrates that different momenta of the distribution function approach the fixed point on different timescales. A fast convergence of the moments and collapse to β_∞ is, in contrast, found for simulations without the elastic collision kernel $\mathcal{C}^{2 \leftrightarrow 2}$ [23]. In the following, we will study these deviations analytically in a small-angle scattering approximation of $\mathcal{C}^{2 \leftrightarrow 2}$.

Effect of scaling-breaking terms in the Fokker-Planck approximation.—The breaking of scaling inhibits prescaling exponents extracted from different moments to share the same universal prescaling dynamics. Nevertheless, at qualitative level the scaling dynamics can be reasonably modeled via the Fokker-Planck (FP) approximation [41–43]. This approach assumes the dominance of small angle scatterings and has previously been used in the context of nonthermal attractors [44] and prescaling [19, 20]. We will compare our analytical results from the FP approximation to simulations using the full QCD collision kernel. The corresponding FP collision kernel allows us to factorize the scaling-breaking Coulomb logarithm, which involves the ratio of the UV scale, the characteristic gluon energy $\langle p \rangle$, and the IR scale, the Debye mass m_D :

$$\mathcal{C}^{\text{FP}}[f](t, \mathbf{p}) = \frac{A(t)^3}{B(t)} \log \left[\frac{\langle \bar{p} \rangle}{A(t)^{1/2} \bar{m}_D} \right] \tilde{\mathcal{C}}^{\text{FP}}[f_s](\bar{\mathbf{p}}), \quad (6)$$

where we display only terms relevant for the scaling analysis [23]. We note that the time dependence due to the Coulomb log can be identified with the background term in the formalism of [24]. The FP kernel does not scale homogeneously, and the solution (5) does not apply.

However, separation of variables can still be performed, as the necessary property is factorization in t and $\mathbf{p}$ dependencies. Upon separating variables with associated constant D_2 [see Eq. (4)] and relating $\alpha(t) = \sigma\beta(t)$ as before, we now obtain

$$\partial_t B(t) = \frac{\beta_0}{t_0} B(t)^{1-1/\beta_\infty} \left[1 - \frac{\sigma}{2} \frac{\log[B(t)]}{\log[\langle \bar{p} \rangle / \bar{m}_D]} \right]. \quad (7)$$

Equation (7) can be directly integrated, but then $B(t)$ appears as an argument of a nontrivial transcendental function. A more useful approach is to derive from Eq. (7) a second-order differential equation for $\beta(t)$:

$$\frac{\ddot{\beta}(t)}{\beta(t)} = \frac{\dot{\beta}(t)^2}{\beta(t)^2} + \frac{14\dot{\beta}}{t} - \frac{[7\beta(t) + 1]^2}{t^2}, \quad (8)$$

which is directly solved by $\beta(t) = \beta_\infty = -1/7$. The solution to Eq. (8) (dashed black line) is shown in Fig. 2 to capture well the evolution of $\beta(t)$ obtained from solving the full QCD kinetic theory collision kernel from times shortly after initialization of the system over more than 8 orders of magnitude. We obtain this result by solving Eq. (8) with initial conditions for $\dot{\beta}_0$ determined consistently from the full QCD kinetic theory data at t_0 [23]. In the inset, we show that solving Eq. (8) shortly after initialization describes qualitatively well the late-time dynamics.

With the applicability of Eq. (8) demonstrated in Fig. 2, we now use it to study the prescaling dynamics in the vicinity of the fixed point. We can linearize Eq. (8) in perturbations $\delta\beta(t)$ around the fixed-point value $\beta_\infty = -1/7$, which yields a power-law decay from below $\delta\beta(t) \sim -1/t$. The full solution to Eq. (8) at late times, however, turns out to be governed by logarithmic corrections not captured by this linearization procedure. We find that a consistent late-time solution to Eq. (8) is given by

$$\beta(t) \approx \beta_\infty + \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \beta_{m,n} \frac{\log[\log(tQ)]^n}{\log(tQ)^m} + \mathcal{O}\left(\frac{1}{t}\right) \quad (9a)$$

$$\approx \beta_\infty \left[1 + \frac{1}{\log(tQ)} \right], \quad (9b)$$

where we used Q as the reference scale but emphasize that the choice of a constant does not matter at late enough times. Similar late-time power-law [20] corrections from linearization and late-time logarithmic [19] corrections induced by the temporal evolution of the Coulomb logarithm were found in the FP approximation for the Baier-Mueller-Schiff-Son [12] fixed point in longitudinal expanding plasma.

The simple power-law approach to the fixed point found in the absence of scaling-breaking terms in Eq. (5b) is, therefore, enriched to involve both fast (power-law) and slow (inverse powers of logarithms and slower) behavior. This

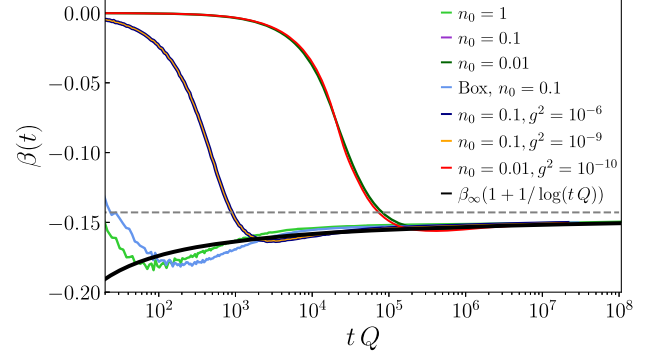


FIG. 3. Comparison of prescaling trajectories in QCD kinetic theory simulations. Solid lines correspond to prescaling exponents extracted from simulations with different initial conditions, where for visibility only results from the first moment are displayed.

discussion is reminiscent of the transseries form [45,46] for late-time dynamics of the energy-momentum tensor of matter undergoing longitudinal boost-invariant expansion [47–50]. There slow modes came from relativistic hydrodynamics and exponentially faster modes from transient excitations. Here, slow modes come from the Debye mass breaking the homogeneity of the collision kernel with respect to rescalings of momenta, and fast modes are the original prescaling excitations encountered already in (5b). Similarly to [47–50], it is not difficult to gather finite-order indications that the series containing slow modes (9a) has a vanishing radius of convergence with $\beta_{m+1,0}/\beta_{m,0} \sim m$ [23]. Curiously, the leading (at each m) doubly logarithmic term behaves geometrically: $\beta_{m+1,m}/\beta_{m,m-1} = -1$. It would be interesting to develop systematic understanding of this behavior, including resummations of the resulting transseries. A good starting point might be analysis of the Painlevé I equation in [51,52], which is also second order and exhibits expansion in three building blocks analogous to our t , $\log(tQ)$, and $\log[\log(tQ)]$.

In Fig. 3, we visualize the attractive nature of the prescaling dynamics by extracting prescaling exponents from different initial conditions. All simulations are initialized with variations in parameters of the class of initial condition used in this work apart from the data represented by light blue, which use box initial conditions $f(t_i, \mathbf{p}) = n_0/g^2\theta(Q - |\mathbf{p}|)$. The prescaling exponents extracted from different simulations are all found to converge to a universal late-time behavior, which we additionally show is well described by Eq. (9b) (solid black line). Furthermore, we want to emphasize the similarity between the behavior shown in Fig. 3 and hydrodynamic attractors, where different solutions converge to a single universal curve which at sufficiently late times is described by relativistic hydrodynamics [47,53,54].

The above analysis has an important bearing on the appearance of scaling. The regime when the highest-order

terms in the collision kernel dominate parametrically ends when the typical occupancy becomes of $\mathcal{O}(1)$. This is realized if $t_f Q \geq \alpha_s^{-7/4}$ [13]. At that time, we have a deviation of $\delta\beta(t)/\beta \sim 1/\log(\alpha_s^{-7/4}) \lesssim 0.03$ with $g^2 = 10^{-8}$. As a consequence of this, QCD kinetic theory will, therefore, still show percent deviations from the fixed-point values when the direct energy cascade ceases and ultraviolet modes $|\mathbf{p}|/Q \geq 1$ start to thermalize.

Conclusions.—We studied the approach of isotropic and spatially homogeneous quantum many-body systems to nonthermal attractors. Our results demonstrate that the prescaling is governed by a simple first-order ordinary differential equation obtained from the underlying dynamics via emergent conservation laws.

Our analytical prediction implies that prescaling entails infinitely many power-law corrections to constant scaling exponents. They conspire to a simple time offset in the fixed-point scaling. We have successfully tested our simple formula for prescaling against *ab initio* simulations of a relativistic vector model quantum field theory using 2PI formalism and QCD kinetic theory simulations. Our QCD kinetic theory simulations span 8 orders of magnitude in time and provide the most accurate extraction of scaling exponents to date.

The exact scaling associated with nonthermal attractors requires the collision term to be a homogeneous functional of particle momenta at large occupations. For QCD kinetic theory, this property is violated by the Debye mass term that regulates the Coulomb divergence in the elastic scattering matrix element. We demonstrate that exact scaling exponents are not reached during the lifetime of the system. Using the Fokker-Planck approximation to QCD kinetic theory, we show that the scaling-breaking Coulomb logarithm significantly enriches the prescaling dynamics. The late-time behavior is given by a factorial divergent series that includes inverse powers of logarithms and positive powers of double logarithms of time. This constitutes a striking structural similarity with theoretical descriptions of hydrodynamic attractors in the boost-invariant models of nuclear collisions.

Our work shows that prescaling is an unavoidable consequence of nonthermal attractors. Therefore, our analytical predictions for prescaling can be verified experimentally in cold atom systems. Furthermore, we uncovered that scaling-breaking terms generate rich prescaling dynamics that bears similarities to transseries in the context of hydrodynamic attractors. It would be fascinating to utilize the enormous degree of control in cold atom systems to induce scaling-breaking terms and experimentally discover the phenomenology of transseries.

Note added.—Recently, Ref. [55] appeared and reported the observation of the prescaling solutions that we derived in Eq. (5) in an ultracold atom experiment.

We thank I. Aniceto, J. Berges, K. Boguslavski, J. Brewer, M. Gałka, A. Kurkela, A. Mikheev, J. Noronha, B. Scheihsing-Hitschfeld, S. Schlichting, A. Serantes, R. Venugopalan, and Y. Yin for useful discussions and comments on the draft. The authors acknowledge support by the state of Baden-Württemberg through bwHPC and the German Research Foundation (DFG) through Grant No. INST 40/575-1 FUGG (JUSTUS 2 cluster), the DFG under the Collaborative Research Center SFB 1225 ISOQUANT (Project-ID No. 273811115), the Simons Foundation (Grant No. 994318, Simons Collaboration on Confinement and QCD Strings), and the Heidelberg STRUCTURES Excellence Cluster under Germany's Excellence Strategy EXC2181/1-390900948. The work of A. M. is funded by DFG—Project No. 496831614. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (Grant No. 101089093, project acronym High-TheQ). Views and opinions expressed are, however, those of the authors only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them. We thank KITP for its hospitality during the program “The Many Faces of Relativistic Fluid Dynamics” supported by the National Science Foundation under Grant No. NSF PHY-1748958.

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- [23] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.132.071602>. In Sec. I, we detail how to extract scaling exponents from the distribution function. In Secs. II and III, we present a prescaling analysis for QCD kinetic theory and the Fokker-Planck collision kernel. The breaking of scaling by the Debye mass is discussed in Sec. IV, details on the derivation of Eq. (8) of the main text are given in Sec. V, and the large order behavior of the transseries is discussed in Sec. VI.
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Supplemental Material: Prescaling relaxation to nonthermal attractors

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I. EXTRACTION OF SCALING EXPONENTS

Under the prescaling ansatz for an isotropic system, Eq. (2) in the main text, we introduce moments of the distribution function

$$n_m(t) = \int \frac{d^d \mathbf{p}}{(2\pi)^d} |\mathbf{p}|^m f(t, \mathbf{p}) = A(t) B(t)^{-(d+m)} \bar{n}_m, \quad (\text{A1})$$

whose evolutions are given by the dynamics of prescaling exponents [1]

$$\frac{d}{d \log t} \log(n_m(t)) = \alpha(t) - (d+m)\beta(t). \quad (\text{A2})$$

We can thus extract prescaling exponents from the evolution of the moments, for example,

$$\alpha_m(t) = (m+d)(m+1+d) \frac{d}{d \log t} \log \left[\frac{n_m(t)^{\frac{1}{m+d}}}{n_{m+1}(t)^{\frac{1}{m+1+d}}} \right], \quad \beta_m(t) = \frac{d}{d \log t} \log \left(\frac{n_m(t)}{n_{m+1}(t)} \right). \quad (\text{A3})$$

We can similarly obtain $A(t)$ and $B(t)$ from moments of the distribution function (or equivalently from moments of the equal-time statistical function $F(t, t, |\mathbf{p}|)$ for the discussion of the infrared fixed point) according to

$$A(t) = \left[\frac{n_m(t)^{1/(d+m)}}{n_{m+1}(t)^{1/(d+m+1)}} \frac{\bar{n}_{m+1}^{1/(d+m+1)}}{\bar{n}_m^{1/(d+m)}} \right]^{(d+m)(d+m+1)}, \quad B(t) = \frac{n_m(t)}{n_{m+1}(t)} \frac{\bar{n}_{m+1}}{\bar{n}_m}, \quad (\text{A4})$$

where $\bar{n}_m \equiv n_m(t_0)$ such that $B(t_0) = 1$.

II. QCD KINETIC THEORY AND PRESCALING

In this work we study QCD kinetic theory which includes $2 \leftrightarrow 2$ and $1 \leftrightarrow 2$ collinear scattering terms. We will give the corresponding equations for the gluon sector here and refer to [2, 3] for the complete expressions including other particle species. The gluon collision kernels are parametrized as

$$\begin{aligned} \mathcal{C}^{2 \leftrightarrow 2}[f](t, \mathbf{p}) = & -\frac{1}{8p(N_c^2 - 1)} \int \frac{d^3 \mathbf{k} d^3 \mathbf{p}' d^3 \mathbf{k}'}{(2\pi)^9 2k 2p' 2k'} |\mathcal{M}_{gg}^{gg}|^2(p, k, p', k') (2\pi)^4 \delta^{(4)}(p^\mu + k^\mu - p'^\mu - k'^\mu) \\ & \times [f_{\mathbf{p}} f_{\mathbf{k}} (1 + f_{\mathbf{k}'})(1 + f_{\mathbf{p}'}) - f_{\mathbf{p}'} f_{\mathbf{k}'} (1 + f_{\mathbf{p}})(1 + f_{\mathbf{k}})] \end{aligned} \quad (\text{A5a})$$

$$\begin{aligned} \mathcal{C}^{1 \leftrightarrow 2}[f](t, \mathbf{p}) = & -\frac{(2\pi)^3}{8p^2(N_c^2 - 1)} \int_0^\infty dp' dk' \left\{ \gamma_{gg}^g(p; p', k') \delta^{(1)}(p - p' - k') [f_{p\hat{\mathbf{p}}} (1 + f_{p'\hat{\mathbf{p}}}) (1 + f_{k'\hat{\mathbf{p}}}) - f_{p'\hat{\mathbf{p}}} f_{k'\hat{\mathbf{p}}} (1 + f_{p\hat{\mathbf{p}}})] \right. \\ & \left. - 2\gamma_{gg}^g(p'; p, k') \delta^{(1)}(p' - p - k') [f_{p'\hat{\mathbf{p}}} (1 + f_{p\hat{\mathbf{p}}}) (1 + f_{k'\hat{\mathbf{p}}}) - f_{p\hat{\mathbf{p}}} f_{k'\hat{\mathbf{p}}} (1 + f_{p'\hat{\mathbf{p}}})] \right\}, \end{aligned} \quad (\text{A5b})$$

where we used the abbreviations $f_{\mathbf{p}} = f(t, \mathbf{p})$, $p = |\mathbf{p}|$, and $\hat{\mathbf{p}} = \mathbf{p}/p$ is the direction of collinear splitting with corresponding rate γ_{gg}^g . $|\mathcal{M}_{gg}^{gg}|^2$ is the $2 \leftrightarrow 2$ scattering matrix element averaged over spin and color degrees of freedom. In vacuum, the corresponding expression [4]

$$|\mathcal{M}_{gg}^{gg}|^2(p, k, p', k') = 16(N_c^2 - 1)N_c^2 g^4 \left(3 - \frac{s_M t_M}{u_M^2} - \frac{s_M u_M}{t_M^2} - \frac{t_M u_M}{s_M^2} \right) \quad (\text{A6})$$

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contains an infrared divergent term $(u_M - s_M)/t_M \sim 1/q^2$ (also in the u_M -channel) with q the t_M -channel momentum transfer for a soft gluon exchange and u_M, s_M, t_M denote the usual Mandelstam variables. This is regulated by inclusion of necessary physical interactions with the medium, where the leading thermal corrections are obtained as [2, 5]

$$\frac{u_M - s_M}{t_M} \rightarrow \frac{u_M - s_M}{t_M} \frac{q^2}{q^2 + \xi_g^2 m_D(t)^2} = \frac{\bar{u}_M - \bar{s}_M}{\bar{t}_M} \frac{\bar{q}^2}{\bar{q}^2 + \xi_g^2 A(t) \bar{m}_D^2} \quad (\text{A7})$$

with $\xi_g = e^{5/6}/2$. The inclusion of screening effects however makes the scattering matrix element not invariant under rescaling as indicated due to different scaling of the Debye mass

$$m_D(t)^2 = 2g^2 N_c \int \frac{d^3 \mathbf{p}}{(2\pi)^3 p} f(t, \mathbf{p}) = A(t) B(t)^{-2} \bar{m}_D^2. \quad (\text{A8})$$

Moreover, the time-dependent Debye mass enters the scattering matrix element via Eq. (A7) in the denominator such that it can not be factored out thereby violating the assumption of overall scaling for the elastic collision kernel. The separation of variables in QCD kinetic theory can thus not generally be performed, as $m_D(t)$ leads to a mixing of time and (rescaled) momentum scales in the prescaling regime.

We now show that the inelastic collision kernel (A5b) scales with $\mu_\alpha = 3$ and $\mu_\beta = -1$ under prescaling in the non-expanding system (2). We consider only the first term since both have the same scaling behavior. Crucially, we need to know the scaling behavior of the splitting rate, which can be extracted from its two prevalent limiting regimes for soft gluon radiation $z = p'/p \ll 1$: the Bethe-Heitler (BH) limit in which interferences between successive scatterings are negligible and the Landau-Pomeranchuk-Migdal (LPM) limit in which successive scattering events by the medium interfere destructively. The rate in the respective limit reads

$$\gamma_{gg}^g(p; p', k')|_{\text{BH}}^{z \ll 1} \sim \frac{\hat{q}(\mu) p}{m_D(t)^2}|_{\mu=em_D(t)}, \quad \gamma_{gg}^g(p; p', k')|_{\text{LPM}}^{z \ll 1} \sim \sqrt{\hat{q}(\mu) p}, \quad (\text{A9})$$

where we only included those terms relevant for a scaling analysis with diffusion coefficient

$$\hat{q}(\mu) \sim \log \frac{\mu^2}{2m_D(t)^2} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} f_{\mathbf{p}} (1 + f_{\mathbf{p}}) \simeq \log \frac{\mu^2}{2A(t)B(t)^{-2}\bar{m}_D^2} A(t)^2 B(t)^{-3} \int \frac{d^3 \bar{\mathbf{p}}}{(2\pi)^3} f_{S, \bar{\mathbf{p}}} f_{S, \bar{\mathbf{p}}} \quad (\text{A10})$$

in this overoccupied scenario. For $\hat{q}(\mu)|_{\mu=em_D}$ in the BH limit, we thus have $\hat{q}(\mu) = A(t)^2 B(t)^{-3} \bar{\hat{q}}$ and

$$\gamma_{gg}^g(p; p', k')|_{\text{BH}}^{z \ll 1} \sim \frac{A(t)^2 B(t)^{-3} \bar{\hat{q}} B(t)^{-1} \bar{p}}{A(t) B(t)^{-2} \bar{m}_D^2} \sim A(t) B(t)^{-2} \bar{\gamma}_{gg}^g|_{\text{BH}}^{z \ll 1}. \quad (\text{A11})$$

For the LPM limit, μ is to next-to-leading-logarithmic order given self-consistently via

$$\mu^2 \sim \sqrt{\hat{q}(\mu) p} \sim \sqrt{A(t)^2 B(t)^{-3} \log \left(\frac{\mu^2}{2A(t)B(t)^{-2}\bar{m}_D^2} \right) B(t)^{-1}} \sim A(t) B(t)^{-2} \sqrt{\log \left(\frac{\mu^2}{2A(t)B(t)^{-2}\bar{m}_D^2} \right)}, \quad (\text{A12})$$

which show that μ scales self-consistently as $\mu^2 \sim A(t) B(t)^{-2}$ such that the time dependence drops out of $\log(\mu^2/(2m_D^2))$. Accordingly, for the LPM limit we thus find again

$$\gamma_{gg}^g(p; p', k')|_{\text{LPM}}^{z \ll 1} \sim \sqrt{\hat{q}(\mu) p} \sim \sqrt{A(t)^2 B(t)^{-3} \bar{\hat{q}}(\mu) B(t)^{-1} \bar{p}} \sim A(t) B(t)^{-2} \bar{\gamma}_{gg}^g|_{\text{LPM}}^{z \ll 1} \quad (\text{A13})$$

The rate therefore scales like the Debye mass in both limits and we will adopt this scaling for the complete rate $\gamma_{gg}^g \sim A(t) B(t)^{-2} \bar{\gamma}_{gg}^g$. This leads to the overall scaling prediction

$$\begin{aligned} \mathcal{C}^{1 \leftrightarrow 2}[f](t, \mathbf{p}) &\sim \frac{1}{B(t)^{-2} \bar{p}^2} \int_0^\infty B(t)^{-2} d\bar{p}' d\bar{k}' A(t) B(t)^{-2} \bar{\gamma}_{gg}^g(\bar{p}; \bar{p}', \bar{k}') B(t) \delta^{(1)}(\bar{p} - \bar{p}' - \bar{k}') \\ &\quad \times A(t)^2 [f_{S, \bar{\mathbf{p}}} (f_{S, \bar{\mathbf{p}}'} + f_{\bar{\mathbf{k}}'} - f_{S, \bar{\mathbf{p}}'} f_{S, \bar{\mathbf{k}}'}) - f_{S, \bar{\mathbf{p}}'} f_{S, \bar{\mathbf{k}}'}] \\ &\sim A(t)^3 B(t)^{-1} \mathcal{C}^{1 \leftrightarrow 2}[f_S](\bar{\mathbf{p}}) \end{aligned} \quad (\text{A14})$$

$$\sim A(t)^3 B(t)^{-1} \mathcal{C}^{1 \leftrightarrow 2}[f_S](\bar{\mathbf{p}}) \quad (\text{A15})$$

such that we can identify $\mu_\alpha = 3$ and $\mu_\beta = -1$ as anticipated. $\mathcal{C}^{1 \leftrightarrow 2}$ will therefore lead to the direct energy cascade fixed point $(\alpha_\infty, \beta_\infty) = (-4/7, -1/7)$.

The scaling analysis for the elastic collision kernel in the absence of the Debye mass has been performed in [6] and can simply be generalized to the prescaling case with again $\mu_\alpha = 3$ and $\mu_\beta = -1$. This scaling analysis however needs to be augmented by the inclusion of the Debye mass as we discussed above, which leads to an effective regulation of the divergent soft contributions to the elastic collision kernel and prevents one from extracting an overall scaling behavior thereof.

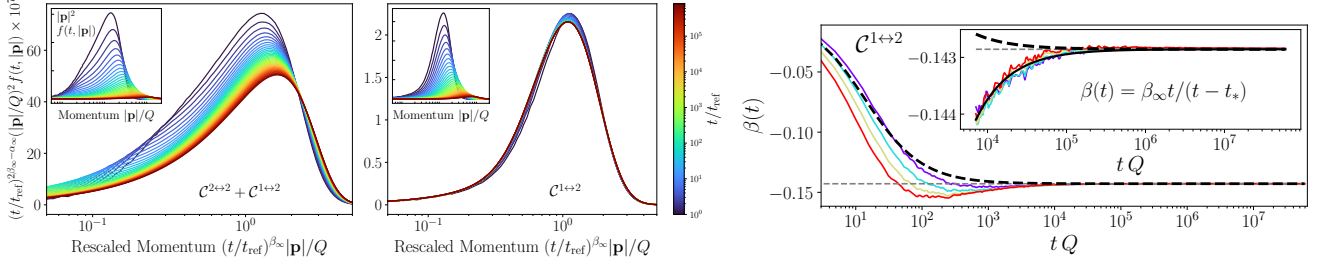


FIG. A1. (Left) Gluon distribution function $|\mathbf{p}|^2 f(t, |\mathbf{p}|)$ from kinetic theory simulation with (left) elastic and inelastic and (right) only inelastic processes rescaled with fixed point exponents $(\alpha_\infty, \beta_\infty) = (-4/7, -1/7)$, with $t_{\text{ref}} Q \sim 66$. The unrescaled distributions are shown in the insets. (Right) Comparison of prescaling exponents from inelastic-only simulations to analytical expectation (5) of the main text. We determine $t_* = t_0(1 - \beta_\infty/\beta_0)$ from the data at $t_0 Q \sim 2$ (dashed black line) and at $t_0 Q \sim 10^4$ (solid black line). Different colors represent prescaling exponents extracted from different integral moments, see Eq. (A3), where we depict $\beta_{m=0,1,2,3}$ with colors $(0, 1, 2, 3) \rightarrow (\text{violet}, \text{turquoise}, \text{yellow}, \text{red})$.

III. FOKKER-PLANCK APPROXIMATION

We assume the gluons to interact via elastic small-angle scatterings such that the collision kernel takes a FP form [3, 7–9]

$$\mathcal{C}^{\text{FP}}(t, \mathbf{p}) \sim \log \frac{\langle p \rangle(t)}{m_D(t)} \left[\int \frac{d^3 \mathbf{k}}{(2\pi)^3} f_{\mathbf{k}}(1 + f_{\mathbf{k}}) \partial_{\mathbf{p}}^2 f_{\mathbf{p}} + \int \frac{d^3 \mathbf{k}}{(2\pi)^3} 2f_{\mathbf{k}} \vec{\nabla}_{\mathbf{p}} \cdot \left(\frac{\mathbf{p}}{p} (1 + f_{\mathbf{p}}) f_{\mathbf{p}} \right) \right] \quad (\text{A16})$$

$$\equiv \log \frac{\langle p \rangle(t)}{m_D(t)} \tilde{\mathcal{C}}^{\text{FP}}[f](t, \mathbf{p}), \quad (\text{A17})$$

where we highlighted the contributions relevant for a scaling analysis and here $\langle \dots \rangle \equiv \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \dots f(t, \mathbf{p}) / \int \frac{d^3 \mathbf{p}}{(2\pi)^3} f(t, \mathbf{p})$. The advantage of the FP approximation for a prescaling analysis becomes apparent here, as the contribution due to the Debye mass in the elastic QCD scattering matrix element is simply factorized into a logarithm of the characteristic UV and IR scale. Plugging in the prescaling ansatz, we then find directly for the overoccupied system that

$$\mathcal{C}^{\text{FP}}[f](t, \mathbf{p}) = \log \left[\frac{\langle \bar{p} \rangle}{A(t)^{\frac{1}{2}} \bar{m}_D} \right] A(t)^3 B(t)^{-1} \tilde{\mathcal{C}}^{\text{FP}}[f_s](\bar{\mathbf{p}}). \quad (\text{A18})$$

IV. BREAKING OF SCALING

The breaking of scaling by the Debye mass in the elastic collision kernel discussed in the main text is visualized in the left and middle panel of Fig. A1, where we show $|\mathbf{p}|^2 f(t, |\mathbf{p}|)$ rescaled with fixed point exponents. The rescaling for simulations with only inelastic scatterings (middle) shows a very clear collapse of all curves from shortly after initialization over an evolution of six orders of magnitude, but a spread cannot be removed for all times even by time dependent rescalings if one includes elastic scatterings (top left) due to the presence of the Debye mass. In the right panel we extract prescaling exponents for only inelastic scatterings $C^{1 \leftrightarrow 2}$ from different moments of the distribution function (see Eq. (A3)). The resulting approach of $\beta(t)$ to the fixed point value $\beta_\infty = -1/7$ (dashed gray line) is demonstrated to be well-described over more than seven orders of magnitude in time by Eq. (5) of the main text (dashed black line) with $\mu_\alpha = 3$ and $\mu_\beta = -1$, see Eq. (A15), even at surprisingly early times shortly after initialization. In the inset, we show that this solution initialized at a very early time $t_0 Q \sim 2$ with t_* obtained from $\beta(t_0) \equiv \beta_0$ captures the evolution at intermediate times $tQ \sim 10^4$ only qualitatively. If we obtain t_* in the same way at a later time $t_0 Q \sim 10^4$, Eq. (5) describes the late-time evolution quantitatively well as given by the solid black line.

V. DERIVATION OF EQ. (8)

Here we give a brief account of how one derives Eq. (7) and Eq. (8) of the main text. As we demonstrated above, the FP collision kernel does not scale homogeneously under the prescaling ansatz and the corresponding prescaling

solutions are therefore not captured by Eq. (5) of the main text. The t and $\bar{\mathbf{p}}$ -dependent contributions of the FP collision kernel in Eq. (6) of the main text however factorize

$$\partial_t f(t, \mathbf{p}) = A(t) \left[\frac{\dot{A}(t)}{A(t)} + \frac{\dot{B}(t)}{B(t)} \bar{\mathbf{p}} \cdot \partial_{\bar{\mathbf{p}}} \right] f_S(\bar{\mathbf{p}}) \quad (\text{A19})$$

$$\stackrel{A=B^\sigma}{=} B(t)^{\sigma-1} \dot{B}(t) [\sigma + \bar{\mathbf{p}} \cdot \partial_{\bar{\mathbf{p}}}] f_S(\bar{\mathbf{p}}) \quad (\text{A20})$$

$$= \log \left[\frac{\langle \bar{p} \rangle}{B(t)^{\frac{\sigma}{2}} \bar{m}_D} \right] B(t)^{3\sigma-1} \tilde{C}^{\text{FP}}[f_S](\bar{\mathbf{p}}), \quad (\text{A21})$$

which is the necessary property to perform separation of variables. Similarly to the derivation of Eq. (4) one can now separate variables in the corresponding Boltzmann equation

$$\frac{B(t)^{-2\sigma} \dot{B}(t)}{\log \left[\frac{\langle \bar{p} \rangle}{B(t)^{\frac{\sigma}{2}} \bar{m}_D} \right]} = D_2 = \frac{\tilde{C}^{\text{FP}}[f_S](\bar{\mathbf{p}})}{[\sigma + \bar{\mathbf{p}} \cdot \partial_{\bar{\mathbf{p}}}] f_S(\bar{\mathbf{p}})} \quad (\text{A22})$$

with D_2 the associated separation of variables constant. Reorganizing the LHS

$$\partial_t B(t) = D_2 B(t)^{2\sigma} \left[\log \left(\frac{\langle \bar{p} \rangle}{\bar{m}_D} \right) - \frac{\sigma}{2} \log B(t) \right], \quad (\text{A23})$$

one can recognize Eq. (7) of the main text with D_2 fixed for convenience at time t_0 where $B(t_0) = 1$ such that

$$\partial_t B(t)|_{t_0} = \frac{\beta_0}{t_0} \quad (\text{A24})$$

$$= D_2 \log \left(\frac{\langle \bar{p} \rangle}{\bar{m}_D} \right) \quad (\text{A25})$$

$$\Leftrightarrow D_2 = \frac{\beta_0}{t_0 \log \left(\frac{\langle \bar{p} \rangle}{\bar{m}_D} \right)}. \quad (\text{A26})$$

We note that this is an intricate self-consistent equation as $\log[\langle \bar{p} \rangle / \bar{m}_D]$ depends implicitly on D_2 via f_S . We avoid the need to solve it as we utilize our simulations to obtain $\langle \bar{p} \rangle / \bar{m}_D$ as a function of β_0 / t_0 .

We now rewrite Eq. (A23) in terms of $\beta(t)$

$$\frac{\beta(t)}{t} = D_2 e^{(2\sigma-1) \int_{t_0}^t dt' \frac{\beta(t')}{t'}} \left[\log \frac{\langle \bar{p} \rangle}{\bar{m}_D} - \frac{\sigma}{2} \int_{t_0}^t dt' \frac{\beta(t')}{t'} \right] \quad (\text{A27})$$

and take two temporal derivatives to obtain Eq. (8). The first derivative yields

$$\partial_t \beta(t) = \frac{\beta(t)}{t} + (2\sigma-1) \frac{\beta(t)^2}{t} - 2D_2 \beta(t) e^{(2\sigma-1) \int_{t_0}^t dt' \frac{\beta(t')}{t'}} \quad (\text{A28})$$

$$\dot{\beta}(t_0) = \beta(t_0) \left[\frac{1}{t_0} + (2\sigma-1) \frac{\beta(t_0)}{t_0} - 2D_2 \right], \quad (\text{A29})$$

where for the second equation we evaluated the first equation at t_0 to obtain constraints on the initial data. A second temporal derivative then reproduces Eq. (8) of the main text, where we can use the first order equation $\dot{\beta}$ to get rid of the D_2 dependence

$$\frac{\ddot{\beta}(t)}{\beta(t)} = \frac{\dot{\beta}(t)^2}{\beta(t)^2} + \frac{14\dot{\beta}}{t} - \frac{(7\beta(t)+1)^2}{t^2}. \quad (\text{A30})$$

This equation is also subtle, as its second order character stays in contrast with the number of parameters needed to solve Eq. (A26), which requires specifying only β_0 at t_0 . Indeed, there is a nontrivial constraint on initial data for Eq. (A30) directly following from a derivative of Eq. (A26): $\dot{\beta}(t_0) \equiv \dot{\beta}_0 = \frac{\beta_0}{t_0} [1 - \beta_0/\beta_\infty - 1/\log \sqrt{\langle \bar{p} \rangle / \bar{m}_D}]$. Its complexity arises from the dependence of $\langle \bar{p} \rangle / \bar{m}_D$ on β_0 and t_0 . For the solution (dashed black line) in Fig. (2) of the main text, the initial conditions at t_0 for $\dot{\beta}_0$ are determined by the constraint and β_0 (as well as $\log \sqrt{\langle \bar{p} \rangle / \bar{m}_D}$) extracted from the data to consistently compare to the QCD kinetic theory simulations.

VI. LARGE ORDER BEHAVIOR OF TRANSERIES

We iteratively solved for the coefficients in Eq. (9a) of the main text in Mathematica. This allowed us to generate exactly within several hours the lowest 38 orders of the expansion in inverse powers of $\log(Qt)$. In Fig. A2 we show that the generated coefficients $\beta_{m0} \sim m!$ for m already as small as 10. Furthermore, we also found that the leading late time coefficient at each order obey $\beta_{m,m-1} = (-1)^m \frac{1}{7}$, i.e. they form a geometric series allowing to simply resum their contribution.

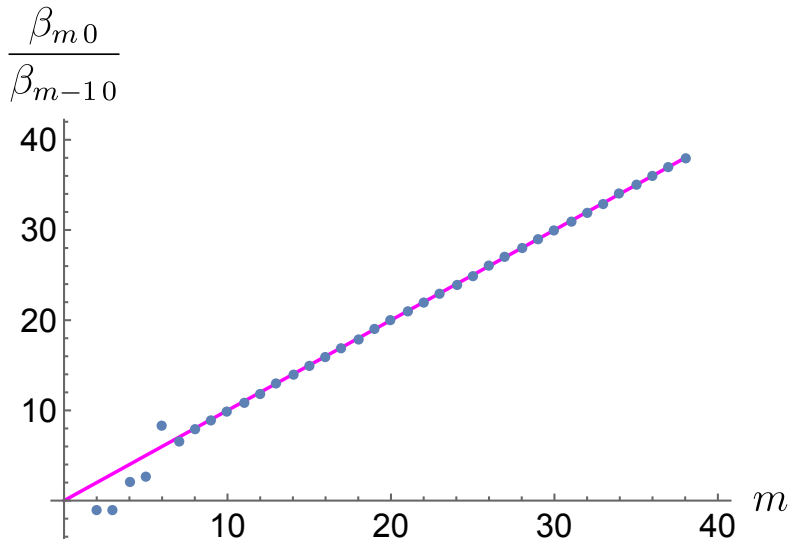


FIG. A2. Ratio test indication that the series (9a) has a vanishing radius of convergence

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4.4

Condensation and prescaling in spatial Polyakov loop correlations far from equilibrium

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(Received 17 November 2024; accepted 7 April 2025; published 6 May 2025)

The far-from-equilibrium dynamics of spatial Polyakov loop correlations, which provide gauge-invariant observables akin to effective particle numbers for gluon plasmas, are investigated within real-time $SU(N_c)$ lattice gauge theory at weak couplings and large gluon occupations. The momentum zero mode of these correlations reveals the dynamic formation of a condensate, while at nonzero momenta, energy is transported toward the ultraviolet. We demonstrate that the nonzero momentum dynamics is well described by a direct cascade in terms of gauge-invariant Polyakov loop excitations, exhibiting self-similar prescaling indicative of a nonthermal attractor. This behavior can be analytically understood through perturbation theory for the Polyakov loop correlations and the established dynamics of gauge field correlations. We perform simulations for both $SU(2)$ and $SU(3)$ gauge groups, providing further consistency checks on the N_c dependence of perturbative expectations. No evidence of an inverse cascade toward lower momenta is found for momenta above the electric screening scale.

DOI: [10.1103/PhysRevD.111.096006](https://doi.org/10.1103/PhysRevD.111.096006)

I. INTRODUCTION

Understanding the dynamics of strongly correlated nuclear matter, especially in high-energy heavy-ion collisions, poses a significant challenge to theoretical approaches based on first principles [1–3]. The quark-gluon plasma formed within the first yoctosecond of the collision leads to phenomena such as gluon saturation and overoccupation [4,5]. The subsequent evolution of the gluon-dominated plasma often exhibits universal dynamics, including universal time dependencies related to nonthermal fixed points [6–9], as well as hydrodynamic attractors [10–12].

Nonthermal fixed points occur far from equilibrium and are marked by self-similarity in the time evolution of correlations, characterized by universal scaling exponents and a universal shape of the distribution function within a specific momentum region. This allows systems to be grouped into far-from-equilibrium universality classes. In particular, relativistic and nonrelativistic scalars form one such class at low momenta based on equal-time distributions

[13], which can be refined further based on unequal-time and topological observables [14,15]. For longitudinally expanding systems, non-Abelian gauge fields and scalars constitute another class at high momenta [8]. Crucially, scalar $O(N)$ models exhibit a dual cascade near a nonthermal fixed point, which has been observed in a variety of experiments with cold quantum gases [16–19]. Here, energy is transported toward the ultraviolet in a direct cascade, while particle number is transported toward the infrared in an inverse cascade, dynamically generating a scalar condensate in the zero mode of the distribution function [13,20].

In the case of gluons, while a direct cascade is observed, to date there is no evidence of infrared dynamics that would lead to low-momentum correlations or condensate formation in gauge-fixed gluon distribution functions in terms of gauge field correlations [7,21–30] that had been previously proposed to emerge in overoccupied gluonic plasmas [31,32]. Defining a proper gauge-invariant condensate in gauge theory is, however, nontrivial. Recently, it has been found that out-of-equilibrium gluonic matter can generate a gauge-invariant condensate defined in terms of nonlocal order parameters, such as spatial Wilson loops or Polyakov loop correlations [33,34]. While Wilson loops have been used previously in studying topological transitions and string tension dynamics out of equilibrium [28,35], Polyakov loop traces provide a gauge-invariant (scalar) field whose correlations exhibit the emergence of a sizable condensate fraction. Furthermore, they can be used to

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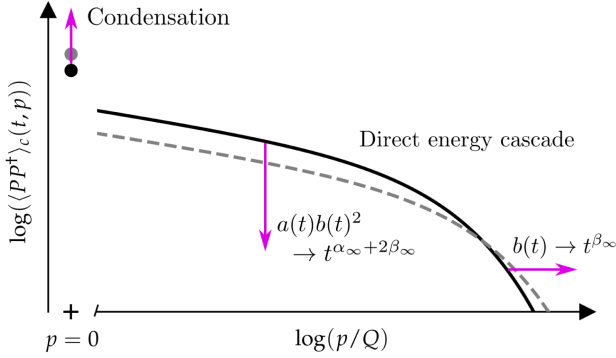


FIG. 1. Schematic picture of the evolution of Polyakov loop correlations, highlighting the buildup of a condensate in the zero mode and a direct cascade toward the ultraviolet exhibiting prescaling dynamics. There are no indications for an inverse cascade toward lower momenta in the investigated momentum regime. Pink arrows indicate dynamics for growing times. The prescaling factors $a(t)$, $b(t)$ and asymptotic scaling exponents $\alpha_\infty, \beta_\infty < 0$ are defined for gluon momentum distribution functions.

formulate a low-energy effective action [34]. Related gauge-invariant operators have also been used in contexts like the Abelian Higgs model and non-Abelian gauge theories on continuous toroidal space-times [36–38].

In this work, we extend the study of Polyakov loop correlations to momenta above the electric screening scale. Our study is based on real-time $SU(N_c)$ lattice gauge theory simulations in the classical-statistical regime of weak couplings and large gluon occupations, which describes well the early-time evolution of the quark-gluon plasma [5,6,23,26,39–43]. We carry out simulations for gauge groups $SU(2)$ and $SU(3)$ and find a relatively simple composition of the dynamics, as sketched in Fig. 1. It features the growth of the condensate fraction from early times, consistent with previous findings [34], while the (Polyakov loop correlator) distribution together with the energy of hard gluons is transported toward the ultraviolet in a direct, self-similar energy cascade. We find no evidence of an inverse cascade at low momenta that would populate the zero mode.

The description of the direct energy cascade requires time-dependent scaling exponents, which only asymptotically acquire their universal values. While these late-time values coincide with the universal scaling exponents of the original gluon distributions, their time-dependent deviations are a sign of prescaling. For a kinetic theory of quantum chromodynamics (QCD) [44] and for simulations of a three-dimensional Bose gas [45], prescaling regimes have been identified before. In the context of [44] that we follow in our work, prescaling occurs when the shape of correlations becomes universal much faster than the scaling exponents reach their universal values. It has been shown that prescaling provides a generic phenomenon characteristic of the approach to far-from-equilibrium attractors [46],

which has subsequently also been found experimentally [19].

Our numerical results are complemented by perturbative predictions for the Polyakov loop correlations, which provide analytic access to the asymptotic scaling behavior near the nonthermal attractor. We demonstrate that they agree well with our simulations and are based on the known behavior of gauge field correlations in this regime. Analytically, gauge invariance of the perturbative Polyakov loop correlations is preserved through the appearance of gauge transformation averages. As an additional validation, our simulations align with the nontrivial perturbative N_c dependence of Polyakov loop correlations.

This paper is structured as follows: Section II introduces the employed lattice setup. Subsequently, Sec. III presents results for the Polyakov loop correlations at zero and nonzero momenta. We discuss the formation of a condensate and the relationship between ultraviolet dynamics, energy transport, and prescaling. The section concludes with an analysis of the N_c dependence of the correlations. Section IV provides an analytic, perturbative derivation of the asymptotic scaling behavior of the Polyakov loop correlations and their N_c dependence, based on the known dynamics of gauge field correlations. Finally, in Sec. V, we conclude and give an outlook.

II. NUMERICAL LATTICE SETUP

We consider $SU(N_c)$ lattice gauge theories in $3+1$ space-time dimensions with the Minkowski metric for $N_c = 2$ and $N_c = 3$. Fields are discretized on a cubic spatial lattice with N_s^3 lattice sites for spatial lattice spacing a_s , so that the physical volume of the spatial lattice $\Lambda_s = \{1, \dots, N_s\}^3$ is $V = L_s^3 = a_s^3 N_s^3$. Spatially periodic boundary conditions are employed and the time evolution of the theory is simulated in time steps of width dt . We work in temporal-axial gauge, $A_0 \equiv 0$. The theory is formulated on the lattice in terms of $\mathfrak{su}(N_c)$ -valued (chromo)electric fields $E_i(t, \mathbf{x})$ and, to preserve gauge covariance, $SU(N_c)$ -valued link variables $U_i(t, \mathbf{x})$, with $i = 1, 2, 3$, $\mathbf{x} = (x_1, x_2, x_3)$. The latter are related to the gauge fields $A_i(t, \mathbf{x})$ via $U_i(t, \mathbf{x}) \approx \exp(-iga_s A_i(t, \mathbf{x}))$ up to lattice corrections, where g is the gauge coupling.

In this work, we employ highly occupied gluon initial conditions at weak couplings $g \ll 1$. At sufficiently early times, gluonic occupation numbers remain large, and the full quantum dynamics can be approximated well by a classical-statistical evolution [23,39–43,47]. Specifically, we sample over Gaussian initial conditions with variances

$$\langle AA \rangle(t=0, \mathbf{p}) = \frac{Q}{g^2 |\mathbf{p}|^2} \theta(Q - |\mathbf{p}|), \quad (1a)$$

$$\langle EE \rangle(t=0, \mathbf{p}) = \frac{Q}{g^2} \theta(Q - |\mathbf{p}|), \quad (1b)$$

where averages over transverse polarizations and color degrees are implied, and the momentum scale Q determines the width of the initial momentum space distribution. Such gluon overoccupation up to a gluon saturation scale is characteristic of a state shortly after the collision of heavy nuclei [3]. Gauss's law is fixed initially by means of a projection algorithm onto the constraint surface [48] and is preserved by the time evolution. Given the initial conditions, in the classical-statistical approximation, the time evolution of fields is computed for each initial configuration separately through the equations of motion that follow from the classical lattice Hamiltonian

$$H(t) = \frac{a_s^3}{g^2} \sum_{\mathbf{x} \in \Lambda_s} \left[\text{Tr}(\mathbf{E}(t, \mathbf{x})^2) + \frac{2}{a_s^4} \sum_{j>k} [N_c - \text{ReTr}(U_{jk}(t, \mathbf{x}))] \right], \quad (2)$$

with the lattice plaquette variables

$$U_{jk}(t, \mathbf{x}) = U_j(t, \mathbf{x}) U_k(t, \mathbf{x} + \hat{\mathbf{j}}) U_j^\dagger(t, \mathbf{x} + \hat{\mathbf{k}}) U_k^\dagger(t, \mathbf{x}). \quad (3)$$

Observables are then calculated by averaging over the resulting independent trajectories at the time of interest. More details regarding the initial conditions and the lattice setup can be found, e.g., in [7,29].

For our simulations, we choose $N_c = 2$ if not stated otherwise and $N_c = 3$ for comparison and set $N_s = 256$, $dt/a_s = 0.05$, and $Qa_s = 0.125$. We consider fields rescaled as $A \mapsto gA$ so that, with our initial conditions (1), the formulation becomes independent of the coupling, which is a property of the classical-statistical framework. We explicitly verified the insensitivity of our correlations at finite momentum to variations of the lattice parameters, specifically upon comparison with results for $Qa_s = 0.25$ with the same lattice volume and with results for a larger spatial lattice size $N_s = 512$. If not stated otherwise, the shown results are averaged over five simulations.

III. RESULTS FOR POLYAKOV LOOP CORRELATIONS

A. Correlations of spatial Polyakov loop traces

We define the spatial Polyakov loop trace as

$$P(t, \bar{\mathbf{x}}) := \frac{1}{N_c} \text{Tr} \mathcal{P} \prod_{x_2=1}^{N_s} U_2(t, \mathbf{x}), \quad (4)$$

where $\bar{\mathbf{x}} = (x_1, x_3)$ and $\mathcal{P}$ indicates path ordering along the positive 2-direction. Note that $P(t, \bar{\mathbf{x}})$ is real-valued for $N_c = 2$ but complex-valued for $N_c = 3$.¹ It is gauge-

¹We do not restrict to the real part of $P(t, \bar{\mathbf{x}})$ here in order to facilitate the perturbative description detailed in Sec. IV; see, in particular, Sec. IV A.

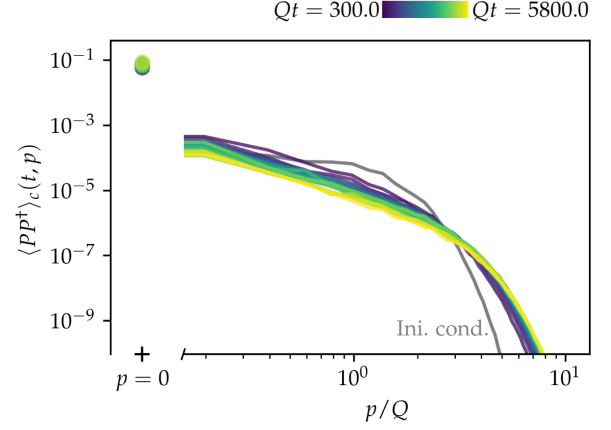


FIG. 2. Correlations $\langle PP^\dagger \rangle_c(t, p)$ versus momentum p for gauge group SU(2), shown for different times as indicated by the color bar. Absolute physical momenta in the 1-3 plane are employed: $p = |\bar{\mathbf{p}}|$. For the zero momentum correlations, the data at five consecutive readout times are averaged as detailed in the caption of Fig. 3.

invariant by construction and has only $(2+1)$ -dimensional space-time support since it describes a Wilson loop trace winding around the periodic 2-direction.

We focus on connected two-point correlations of the spatial Polyakov loop traces as provided by

$$\begin{aligned} \langle P(t, \bar{\mathbf{x}}) P^\dagger(t, \bar{\mathbf{y}}) \rangle_c \\ := \langle P(t, \bar{\mathbf{x}}) P^\dagger(t, \bar{\mathbf{y}}) \rangle - \langle P(t, \bar{\mathbf{x}}) \rangle \langle P^\dagger(t, \bar{\mathbf{y}}) \rangle. \end{aligned} \quad (5)$$

In spatially homogeneous settings, i.e., where the system does not depend on $\bar{\mathbf{x}} + \bar{\mathbf{y}}$ such as ours, these correlations are suitably described in momentum space,

$$\langle PP^\dagger \rangle_c(t, \bar{\mathbf{p}}) := a_s^2 \sum_{\Delta \bar{\mathbf{x}} \in \bar{\Lambda}_s} \langle P(t, \Delta \bar{\mathbf{x}}) P^\dagger(t, \bar{\mathbf{0}}) \rangle_c e^{-i \bar{\mathbf{p}} \Delta \bar{\mathbf{x}}}, \quad (6)$$

where $\bar{\mathbf{p}} = (p_1, p_3)$ is the physical momentum and $\tilde{\mathbf{p}} = (\tilde{p}_1, \tilde{p}_3)$ is the lattice momentum, $\tilde{p}_i \in 2\pi/(a_s N_s) \{-N_s/2, -N_s/2 + 1, \dots, N_s/2 - 1\}$. The physical momenta p_i are related to the lattice momenta via $p_i = (2/a_s) \sin(\tilde{p}_i a_s/2)$. For configurations, which are furthermore spatially isotropic in the 1-3 plane, the correlations (6) reduce to $\langle PP^\dagger \rangle_c(t, p) = \langle PP^\dagger \rangle_c(t, \bar{\mathbf{p}})$ for $p = |\bar{\mathbf{p}}|$ and are the main observables of interest in this work.

In Fig. 2 we display the Polyakov loop correlations $\langle PP^\dagger \rangle_c(t, p)$ for both the zero mode ($p = 0$) and the nonzero modes ($p > 0$) as a function of time, including the initial time. We find that the $p = 0$ correlation grows with time, whose relation to the dynamic formation of a gauge-invariant condensate in terms of Polyakov loops will be discussed in more detail in Sec. III B. For momenta $p > 0$, initial correlations spread at early times toward the ultraviolet. After some time, the shape of correlations

becomes universal, i.e., remains approximately invariant under time evolution, which is the temporal regime that we focus on. In this regime, the correlations decrease in overall value and shift further toward larger momenta with increasing time. For momenta $p/Q \lesssim 2.0$, the shape of the correlations can be well approximated by a power law with an exponent between -2.0 and -1.5 (not displayed). In terms of overall numbers, the zero mode correlation is much larger than those at individual nonzero momenta. Crucially, based on the Polyakov loop correlations analyzed, no inverse cascade—like those seen in scalar theories under weakly coupled, overoccupied conditions [13,20]—is observed within the investigated momentum region.

Interestingly, while it was proposed that a large initial density of gluons could facilitate the formation of a condensate [31,32], for the gluon distribution function below the Debye scale m_D , neither condensate formation nor an inverse cascade has been observed, which can be attributed to a nontrivial cancellation of elastic and inelastic processes at low momenta [6,23,26,27]. However, the perturbative relation between the gluon distribution function and the Polyakov loop correlator, which we derive in Eq. (20) below, does not apply at low momenta $p \lesssim m_D$, where occupation numbers can become large. Thus, we cannot exclude the presence of an inverse cascade in Polyakov loop correlations below the electric screening scale m_D . In our simulations, this scale decreases from $m_D/Q = 0.23$ at time $Qt = 300$ to $m_D/Q = 0.14$ at time $Qt = 5800$ and is therefore of the order of the smallest nonzero momenta resolved on our lattice.²

B. Infrared: Dynamic condensate formation

In order to further study the behavior of the system with regard to the zero mode of the Polyakov loop correlations, we consider the contributions of both the $p = 0$ mode and the $p > 0$ modes to the sum of all correlations. We define zero and nonzero mode Polyakov loop occupations,

$$N_0(t) := \frac{1}{(N_s a_s)^2} \langle PP^\dagger \rangle_c(t, \vec{p} = \vec{0}), \quad (7a)$$

$$N_{p>0}(t) := \frac{1}{(N_s a_s)^2} \sum_{\vec{p} \neq \vec{0}} \langle PP^\dagger \rangle_c(t, \vec{p}), \quad (7b)$$

along with their sum $N_{\text{tot}}(t) := N_0(t) + N_{p>0}(t)$. The sum in $N_{p>0}(t)$ and its prefactor correspond to a discretized version of $\int d^2 p / (2\pi)^2$. In Fig. 3 we show the two quotients $N_0(t)/N_{\text{tot}}(t)$ and $N_{p>0}(t)/N_{\text{tot}}(t)$ as functions of time. The variable $N_0(t)/N_{\text{tot}}(t)$ is analogous to the

²The Debye mass can be computed from the transversely polarized gauge field correlator $\langle AA \rangle(t, p)$ according to $m_D^2(t) = 4N_c \int d^3 p / (2\pi)^3 \langle AA \rangle(t, p)$ [29].

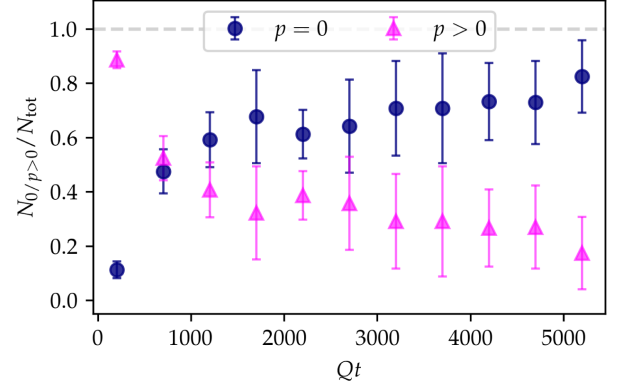


FIG. 3. The share of Polyakov loop correlations in the zero mode [condensate fraction $N_0(t)/N_{\text{tot}}(t)$, blue circles] versus the sum of higher modes [$N_{p>0}(t)/N_{\text{tot}}(t)$, pink triangles] as functions of time for an SU(2) plasma. The evolution highlights the buildup of a condensate through the redistribution of Polyakov loop correlations to the zero mode. Error bars are computed as standard errors of the mean over five simulation runs. Additionally, data points at time Qt are averaged over the readout times $Qt - 200$, $Qt - 100$, Qt , $Qt + 100$, $Qt + 200$ and the resulting uncertainty is computed via error propagation.

condensate fraction of excitations in the local Polyakov loop traces [34] and measures the amount of long-range coherence in the system [49].

At the initial time $t = 0$, the total amount of correlations is strongly dominated by nonzero momenta and the zero mode barely contributes, $N_{p>0}(0)/N_{\text{tot}}(0) \gg N_0(0)/N_{\text{tot}}(0)$. At early times, $N_0(t)/N_{\text{tot}}(t)$ starts to build up, indicating the dynamic formation of a Polyakov loop condensate. Simultaneously, the contribution of nonzero momentum modes to the total correlations steadily diminishes, though this trend gradually slows down over time.

To confirm the presence of a condensate, one would need to analyze the volume scaling of the time required for condensate formation. This was previously demonstrated for similar initial conditions in [33,34], where the condensate formation time was numerically shown to scale with an approximately universal power of volume, and the condensate buildup followed a universal, volume-independent form. We note that the Polyakov-loop-based condensate fraction defined in [34] from position space correlations is almost identical to N_0/N_{tot} as considered in this work.

C. Ultraviolet: Energy transport in a prescaling cascade

We turn to the dynamics of the correlations for nonzero momenta in more detail. As shown in Fig. 2, the shape of $\langle PP^\dagger \rangle_c(t, p)$ stays constant in time, up to statistical fluctuations. This can be characteristic of the dynamics of a self-similar cascade near a nonthermal fixed point. Self-similar scaling emerges already in the approach to such an attractor [44–46,50], a phenomenon called

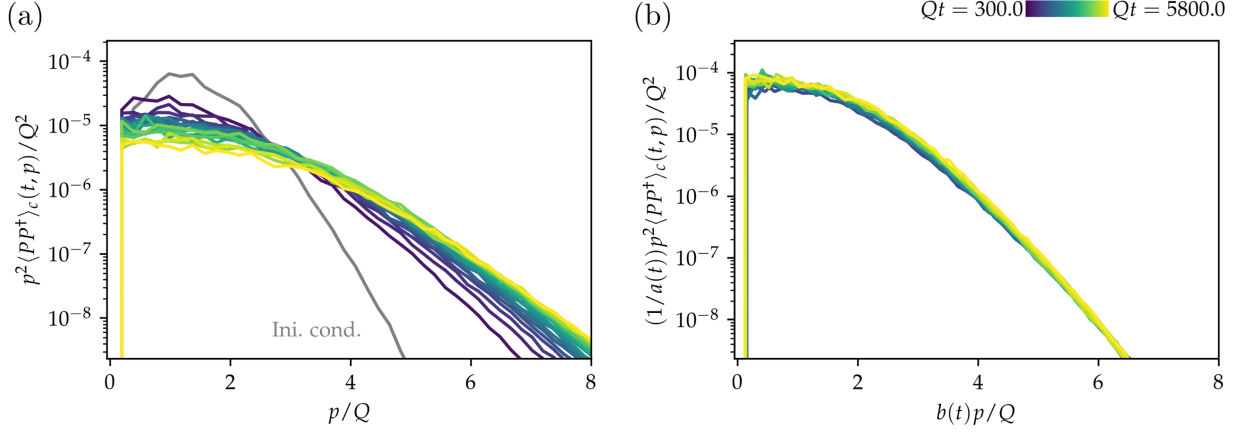


FIG. 4. (a) Correlations $p^2 \langle PP^\dagger \rangle_c(t, p)$ versus momentum p shown for different times as indicated by the color bar. (b) Corresponding rescaled correlations for the prescaling functions $a(t)$, $b(t)$ extracted as detailed in Sec. III C. The prefactor for the temporal scaling with the prescaling function $a(t)$ follows from Eq. (10) upon inclusion of the p^2 factor. Here, absolute physical momenta in the 1–3 plane are employed: $p = |\vec{p}|$.

prescaling [44]. Prescaling has been investigated so far only through distribution functions such as that of gluons [44,46], which is, up to momentum-dependent prefactors, proportional to the gauge-dependent gluon-gluon correlations $\langle AA \rangle(t, p)$. Specifically, near a nonthermal attractor, their time dependence can be suitably described by³

$$\langle AA \rangle(t, p) = a(t)b(t)\langle AA \rangle_s(b(t)p). \quad (8)$$

Here, $\langle AA \rangle_s(p)$ indicates the fixed-point shape of $\langle AA \rangle(t, p)$, and $a(t)$, $b(t)$ denote prescaling factors, which can equivalently be used to introduce time-dependent scaling exponents as $a(t) = \exp(\int_{t_0}^t dt' \alpha(t')/t')$ and $b(t) = \exp(\int_{t_0}^t dt' \beta(t')/t')$. The parameter t_0 specifies an initial timescale for the prescaling dynamics. For large times, if a nonthermal attractor is approached, $\alpha(t')$ and $\beta(t')$ converge to their asymptotic values α_∞ and β_∞ . In fact, much research on nonthermal fixed points has been based on these universal asymptotic values [3], but it has been shown in kinetic theory that prescaling appears generally in the vicinity of nonthermal attractors [46]. Usually, the instantaneous preservation of emergent conservation laws for the distribution function provides a link between $\alpha(t')$ and $\beta(t')$ during their evolution.

The prescaling behavior of the gluon correlations can be related to the Polyakov loop correlations by means of the leading-order perturbative expression

³Let $f(t, p)$ denote the gluon distribution function, which for momenta far above the thermal mass is related to transversely polarized gluon correlations via $f(t, p) \sim p \langle AA \rangle_c(t, p)$. The prescaling behavior $f(t, p) = a(t)f_s(b(t)p)$ then implies $\langle AA \rangle_c(t, p) = a(t)b(t)\langle AA \rangle_{c,s}(b(t)p)$, where $f_s(p)$ and $\langle AA \rangle_{c,s}(p)$, respectively, denote the fixed-point shapes of $f(t, p)$ and $\langle AA \rangle_c(t, p)$.

$$\begin{aligned} \langle PP^\dagger \rangle_c(t, p) &= \frac{g^2}{N_c} \int \mathcal{D}\mathcal{G} \int d^2(\vec{x} - \vec{y}) \oint_0^{L_s} dx_2 \oint_0^{L_s} dy_2 \\ &\times \text{Tr} \langle A_2(t, \mathbf{x}) A_2(t, \mathbf{y}) \rangle_c e^{-i\vec{p}(\vec{x} - \vec{y})} + \mathcal{O}(g^3), \end{aligned} \quad (9)$$

that we will derive for a continuous space-time with spatially periodic boundary conditions in Sec. IV, to which we also refer for notational details. Therefore, if the system shows prescaling in the gauge field correlator and the perturbative treatment is justified, the Polyakov loop correlations may inherit this property. This motivates the ansatz

$$\langle PP^\dagger \rangle_c(t, p) = a(t)b^2(t)\langle PP^\dagger \rangle_{c,s}(b(t)p), \quad (10)$$

where $\langle PP^\dagger \rangle_{c,s}(p)$ indicates the fixed-point shape of $\langle PP^\dagger \rangle_c(t, p)$.

Asymptotically, energy transport toward the ultraviolet for gluonic plasmas can proceed in a self-similar cascade with $\beta_\infty = -1/7$, which has been observed in the gluon distribution function [7,23,25,26,29] and in energy density correlations [51]. In physical terms, this implies that the hard scale that dominates the energy density grows with time asymptotically as $\sim Q(Qt)^{1/7}$. For relativistic gluons in three spatial dimensions, energy conservation yields $\alpha(t') = 4\beta(t')$, which generalizes the well-known relation for their asymptotic values ($\alpha_\infty = 4\beta_\infty$).

In order to investigate to what extent the ansatz (10) describes the Polyakov loop correlations well, in Fig. 4(a) we show $p^2 \langle PP^\dagger \rangle_c(t, p)$ for sufficiently late times when the correlations approximately maintain their shape. The depicted second moment of the correlations has proven particularly sensitive to self-similarity and thus suitable to visually verify the presence of prescaling [46]. Importantly,

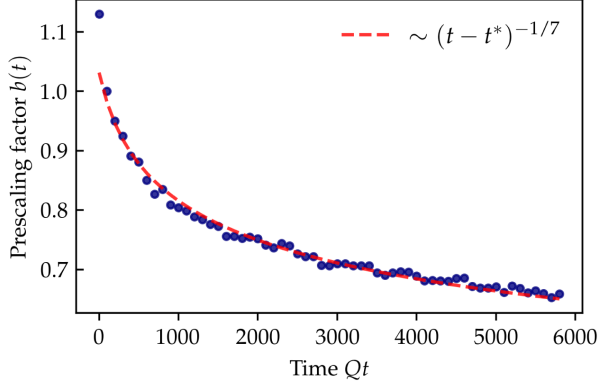


FIG. 5. The numerically extracted prescaling factor $b(t)$ shown as a function of time, similar to [46]. Prescaling factors extracted from different moments are averaged over, see the Appendix. The dashed red line corresponds to the prescaling solution in Eq. (11).

there are scaling functions $a(t)$ and $b(t)$ through which we can rescale this quantity so that the curves lie on top of each other up to numerical fluctuations for the shown time interval, see Fig. 4(b). This reveals the presence of a prescaling cascade in the Polyakov loop correlations.

The prescaling factors $a(t)$, $b(t)$ are extracted from different moments of the Polyakov loop correlations, reflecting that different momentum scales show the same scaling behavior in time. The time evolution of $b(t)$, in particular, is shown in Fig. 5 to be described well by the prescaling solution as derived in [46],

$$b(t) = ((t - t_*)/t_{\text{ref}})^{\beta_\infty}, \quad (11)$$

with observable- and initial condition-dependent parameters t_* and t_{ref} and with the universal exponent β_∞ . The timescale t_* is fixed by data at time t_0 and t_{ref} can be viewed as a normalization of the gluon distribution function. The function $a(t)$ numerically obeys $a(t) = b(t)^4$, or equivalently, $\alpha(t) = 4\beta(t)$, in agreement with energy conservation, see Fig. 8 in the Appendix. The extraction procedure and details on prescaling are described in the Appendix.

D. Dependence on N_c

We finally investigate the dependence of the Polyakov loop correlations on the number of colors N_c . We start with the condensate fraction that is shown in Fig. 6(a) for the gauge groups SU(2) and SU(3) as functions of time for the same initial conditions. The behavior of the condensate fraction computed for the gauge group SU(3) is very similar to SU(2) and the considerations of Sec. III B apply *ad verbatim*. Within numerical uncertainties, both dynamics are comparably fast and no N_c dependence is visible with the current level of accuracy. Because of the relatively large error bars, a decisive study on a possible N_c dependence of the condensate growth in Polyakov loop correlations would

require much better statistics and is beyond the scope of the present work.

In order to understand the behavior of the Polyakov loop correlations at nonzero momenta, we first examine a potential nontrivial N_c dependence of the general speed of the dynamics, which is based on the kinetic theory arguments outlined in [26]. For the times under investigation, the system can be well described within QCD effective kinetic theory [52] by means of a Boltzmann equation for the gluon distribution function $f(t, p)$,

$$\frac{df(t, p)}{dt} = -\mathcal{C}[f]. \quad (12)$$

The collision kernel $\mathcal{C}[f]$ consists of a sum of elastic and inelastic collision kernels, which we can consider for large gluon occupancies in the classical approximation. The leading N_c scaling comes from the elastic collision kernel $\mathcal{C}[f] \sim N_c^2$ so that Eq. (12) becomes approximately N_c independent if we consider all times rescaled as $N_c^2 t$ instead of t . The remaining N_c dependence lies in the inelastic collision kernel and in the soft screening scale m_D [26], which the numerical results seem not to be sensitive to at high momenta.

Based on this scaling argument, to properly compare the Polyakov loop correlations at nonzero momenta across different N_c for the same initial condition, we need to consider different times for the gauge groups SU(2) and SU(3). In Fig. 6(b) we show the correlations for the two gauge groups for times $Qt_{\text{SU}(2)} = 2200, \dots, 9400$ and $Qt_{\text{SU}(3)} = 1000, \dots, 4200$, all well within the prescaling regime discussed before. The inset shows the correlations for the times $Qt_{\text{SU}(3)}$ also in the SU(2) case and without any N_c -dependent rescalings. Clearly, the correlations exhibit a nontrivial N_c dependence, even though the shape of the SU(3) results is comparable to that of the gauge group SU(2). The main figure demonstrates that the correlations for the two gauge groups can be scaled on top of each other if, in addition to the mentioned time rescalings, correlations are multiplied by $N_c/(d_{N_c}(N_c^2 - 1))$. The variable d_{N_c} counts real degrees of freedom of the Polyakov loop trace and is defined as $d_{N_c} = 2$ for $N_c \geq 3$ and $d_2 = 1$. This overall rescaling factor will be derived from perturbation theory in Sec. IV B. The observation that, in the main panel of Fig. 6(b), all of the properly rescaled curves lie on top of each other at sufficiently high momenta is, together with the prescaling results, a strong indication for the applicability of the perturbative analysis of Sec. IV.

We emphasize that the perturbative arguments leading to the QCD kinetic theory of [52] are only well defined for sufficiently hard modes above the screening scale $p \gtrsim m_D$ as long as occupancies are not too large: $g^2 f \lesssim 1$. These conditions are not satisfied at low momenta, and the occupation number at zero momentum does not need to follow Eq. (12). The observed growth of the zero mode in

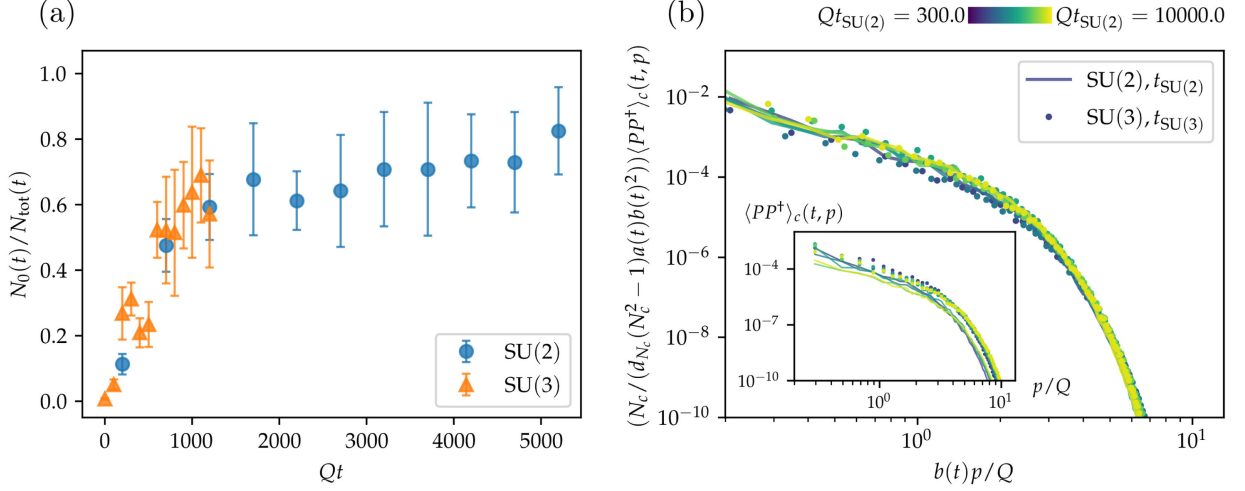


FIG. 6. N_c dependence of the Polyakov loop correlations. (a) Condensate fraction as a function of time for gauge groups SU(2) (blue disks) and SU(3) (orange triangles). Error bars are computed as averages over five simulation runs. For SU(2), the additional time average as detailed in the caption of Fig. 3 is used. (b) Connected Polyakov loop correlations versus physical momentum $p = |\vec{p}|$, shown for a range of times as indicated by the color bar for the SU(2) case [solid lines; provided are $Qt_{\text{SU}(2)} = 2200, 4000, 5800, 7600, 9400$]. In the case of gauge group SU(3), the corresponding times are given by $t_{\text{SU}(3)} = (2/3)^2 t_{\text{SU}(2)}$ in accordance with the considerations in the main text [points; displayed are $Qt_{\text{SU}(3)} = 1000, 1800, 2600, 3400, 4200$]. For both gauge groups, the employed prescaling factors are those for the times $Qt_{\text{SU}(2)}$. The inset in (b) shows the results without N_c and time rescalings between SU(2) and SU(3); the displayed times in the inset of (b) are the ones for SU(3). The results shown in (b) are consistent with the perturbative analytics provided in Sec. IV. They originate from a single simulation run due to longer available simulation times.

terms of Polyakov loop correlations is, indeed, inconsistent with the scaling ansatz (10) that would predict a decreasing trend instead, and its N_c -independent evolution does not seem to agree with predictions from the Boltzmann equation or the analysis of Sec. IV. These numerical indications confirm that, in contrast to the direct energy cascade, the condensate formation dynamics constitutes an intrinsically nonperturbative phenomenon.

IV. PERTURBATIVE EXPECTATIONS

Given the weak couplings of the simulations, it remains to investigate to what extent the observed self-similar prescaling cascade in Polyakov loop correlations and their N_c scaling can be understood analytically, based on perturbative expectations and known behavior of gauge field correlations. We first derive a leading-order expression for Polyakov loop correlations in terms of gauge fields, then deduce its perturbative dependence on N_c , and finally connect them to the asymptotic scaling exponents α_∞ and β_∞ .

A. Polyakov loop correlations from gauge field correlations

We carry out the perturbative derivation in the smooth setting, where the space-time is given by $\mathbb{R} \times T^3$, $\mathbb{R}$

denoting a continuous time direction and T^3 the spatial 3-torus $T^3 = (\mathbb{R} \bmod L_s)^3$. This implies spatially periodic boundary conditions as for the lattice setting, which are crucial for the construction of spatial Polyakov loops. We consider gauge groups $SU(N_c)$ for general N_c in this analysis and gauge fields without the map $A \mapsto gA$, which we applied for the simulations. Then the Polyakov loop trace reads

$$P(t, \vec{x}) = \frac{1}{N_c} \text{Tr} \mathcal{P} \exp \left\{ -ig \oint_0^{L_s} dx_2 A_2(t, \vec{x}) \right\}. \quad (13)$$

Its connected two-point correlations become

$$\begin{aligned} \langle P(t, \vec{x}) P^\dagger(t, \vec{y}) \rangle_c &= \frac{1}{N_c^2} \left\langle \text{Tr} \mathcal{P} \exp \left\{ -ig \oint_0^{L_s} dx_2 A_2(t, \vec{x}) \right\} \right. \\ &\quad \times \text{Tr} \mathcal{P}^{-1} \exp \left\{ ig \oint_0^{L_s} dy_2 A_2(t, \vec{y}) \right\} \Big\rangle \\ &\quad - \langle P(t, \vec{x}) \rangle \langle P^\dagger(t, \vec{y}) \rangle, \end{aligned} \quad (14)$$

where $\mathcal{P}^{-1}$ indicates path ordering along the negative 2-direction. We can rewrite the disconnected contributions to Eq. (14) using the cluster decomposition theorem [53,54],

$$\begin{aligned}\langle P(t, \bar{\mathbf{x}}) \rangle &= \frac{1}{N_c} \text{Tr} \exp \left\{ \sum_{n=1}^{\infty} \frac{(-ig)^n}{n!} \oint_0^{L_s} dx_2^{(1)} \cdots \oint_0^{L_s} dx_2^{(n)} \langle A_2(t, \mathbf{x}^{(1)}) \cdots A_2(t, \mathbf{x}^{(n)}) \rangle_c \right\} \\ &= 1 - \frac{g^2}{2N_c} \oint_0^{L_s} dx_2^{(1)} \oint_0^{L_s} dx_2^{(2)} \langle \text{Tr}(A_2(t, \mathbf{x}^{(1)}) A_2(t, \mathbf{x}^{(2)})) \rangle_c + \mathcal{O}(g^3),\end{aligned}\quad (15)$$

where $\mathbf{x}^{(i)} = (x_1, x_2^{(i)}, x_3)$ and the gauge field correlations are connected in the A_2 . In Eq. (15) we further assumed that $\langle A_2(t, \mathbf{x}) \rangle = 0$, motivated through Elitzur's theorem [55], which is consistent with our spatially homogeneous setup.

To further process the first term on the right-hand side of Eq. (14), we use a result from Lie group representation theory. Let $d\mu$ denote the Haar measure of $\text{SU}(N_c)$, normalized to $\mu(\text{SU}(N_c)) = 1$, and $f: \mathbb{C}^{N_c} \rightarrow \mathbb{C}^{N_c}$ be a linear map. Then,⁴

$$\int_{\text{SU}(N_c)} d\mu(V) V f V^\dagger = \frac{\text{Tr} f}{N_c} 1_{N_c \times N_c}, \quad (16)$$

where $1_{N_c \times N_c}$ denotes the $N_c \times N_c$ unit matrix. Using Eq. (16), we obtain

$$\begin{aligned}& \frac{1}{N_c^2} \left\langle \text{Tr} \mathcal{P} \exp \left\{ -ig \oint_0^{L_s} dx_2 A_2(t, \mathbf{x}) \right\} \text{Tr} \mathcal{P}^{-1} \exp \left\{ ig \oint_0^{L_s} dy_2 A_2(t, \mathbf{y}) \right\} \right\rangle \\ &= \frac{1}{N_c} \int_{\text{SU}(N_c)} d\mu(V) \int_{\text{SU}(N_c)} d\mu(W) \text{Tr} \left\langle V \mathcal{P}_{(t, \bar{\mathbf{x}}_0) \rightarrow (t, \bar{\mathbf{x}}_0)} \exp \left\{ -ig \oint_0^{L_s} dx_2 A_2(t, \mathbf{x}) \right\} V^\dagger \right. \\ &\quad \times \left. W \mathcal{P}_{(t, \bar{\mathbf{y}}_0) \rightarrow (t, \bar{\mathbf{y}}_0)}^{-1} \exp \left\{ ig \oint_0^{L_s} dy_2 A_2(t, \mathbf{y}) \right\} W^\dagger \right\rangle,\end{aligned}\quad (17)$$

where $\bar{\mathbf{x}}_0 = (x_1, 0, x_3)$, $\bar{\mathbf{y}}_0 = (y_1, 0, y_3)$, $\mathcal{P}_{(t, \bar{\mathbf{x}}_0) \rightarrow (t, \bar{\mathbf{x}}_0)}$ indicates path ordering along the positive 2-direction, starting and ending at $(t, \bar{\mathbf{x}}_0)$, and $\mathcal{P}_{(t, \bar{\mathbf{y}}_0) \rightarrow (t, \bar{\mathbf{y}}_0)}^{-1}$ indicates path ordering along the negative 2-direction, starting and ending at $(t, \bar{\mathbf{y}}_0)$. The choice of 0 in $\bar{\mathbf{x}}_0$ and $\bar{\mathbf{y}}_0$ is without loss of generality; independently, two other real numbers in the interval $[0, L_s)$ may equivalently be chosen.

Assuming $\bar{\mathbf{x}} \neq \bar{\mathbf{y}}$, the $\text{SU}(N_c)$ integrals in Eq. (17) can be viewed as an average over all gauge transformations of the quantity

$$\frac{1}{N_c} \text{Tr} \left\langle \mathcal{P}_{(t, \bar{\mathbf{x}}_0) \rightarrow (t, \bar{\mathbf{x}}_0)} \exp \left\{ -ig \oint_0^{L_s} dx_2 A_2(t, \mathbf{x}) \right\} \mathcal{P}_{(t, \bar{\mathbf{y}}_0) \rightarrow (t, \bar{\mathbf{y}}_0)}^{-1} \exp \left\{ ig \oint_0^{L_s} dy_2 A_2(t, \mathbf{y}) \right\} \right\rangle. \quad (18)$$

We can therefore rewrite Eq. (17) as

$$\frac{1}{N_c} \int \mathcal{D}\mathcal{G} \text{Tr} \left\langle \mathcal{P}'_{t, \bar{\mathbf{x}}_0, \bar{\mathbf{y}}_0} \exp \left\{ -ig \oint_0^{L_s} dx_2 A_2(t, \mathbf{x}) + ig \oint_0^{L_s} dy_2 A_2(t, \mathbf{y}) \right\} \right\rangle, \quad (19)$$

where $\mathcal{D}\mathcal{G}$ indicates a functional integral over all $\text{SU}(N_c)$ -valued gauge transformations, (formally) normalized to $\int \mathcal{D}\mathcal{G} = 1$. $\mathcal{P}'_{t, \bar{\mathbf{x}}_0, \bar{\mathbf{y}}_0}$ denotes path ordering along two disconnected loops: starting at $(t, \bar{\mathbf{y}}_0)$, winding around the negative 2-direction to $(t, \bar{\mathbf{y}}_0)$, then starting the second loop

at $(t, \bar{\mathbf{x}}_0)$, winding around the positive 2-direction and ending at $(t, \bar{\mathbf{x}}_0)$. By cyclicity of the trace, Eq. (19) is independent of the ordering among the two disconnected loops. In writing the two exponentials in Eq. (17) as a single one in Eq. (19), we further employed that the equal-time gauge fields commute at different spatial positions. If we had defined $P(t, \bar{\mathbf{x}})$ using its real part only, then the right-hand side of Eq. (17) would include taking real parts as well, which would impede this rewriting as a single exponential. Employing again the cluster decomposition theorem [53,54], Eq. (19) reads to lowest order in the coupling g ,

⁴This follows, for instance, from Lemma 4.42 in [56], which states that for G a compact finite-dimensional Lie group, $d\nu$ its Haar measure, ρ an irreducible finite-dimensional complex representation of G , and $f: \rho \rightarrow \rho$ a linear map: $\int_G d\nu(g) \rho(g) f \rho(g)^{-1} = (\text{Tr} f / \dim \rho) \text{id}_\rho$. In our case, $G = \text{SU}(N_c)$ and ρ is given by the fundamental representation of $\text{SU}(N_c)$, which is indeed irreducible.

$$1 - \frac{g^2}{2N_c} \int \mathcal{D}\mathcal{G} \left[\oint_0^{L_s} dx_2^{(1)} \oint_0^{L_s} dx_2^{(2)} \langle \text{Tr}(A_2(t, \mathbf{x}^{(1)}) A_2(t, \mathbf{x}^{(2)})) \rangle_c - 2 \oint_0^{L_s} dx_2 \oint_0^{L_s} dy_2 \langle \text{Tr}(A_2(t, \mathbf{x}) A_2(t, \mathbf{y})) \rangle_c \right. \\ \left. + \oint_0^{L_s} dy_2^{(1)} \oint_0^{L_s} dy_2^{(2)} \langle \text{Tr}(A_2(t, \mathbf{y}^{(1)}) A_2(t, \mathbf{y}^{(2)})) \rangle_c \right] + \mathcal{O}(g^3). \quad (20)$$

Since the Polyakov loop trace expectation value (15) is gauge invariant, taking its gauge transformation average $\int \mathcal{D}\mathcal{G}$ before a perturbative expansion is an identity map. Therefore, inserting Eqs. (15) and (20) into Eq. (14), we finally obtain the leading-order perturbative expression

$$\langle P(t, \bar{\mathbf{x}}) P^\dagger(t, \bar{\mathbf{y}}) \rangle_c = \frac{g^2}{N_c} \int \mathcal{D}\mathcal{G} \oint_0^{L_s} dx_2 \oint_0^{L_s} dy_2 \text{Tr} \langle A_2(t, \mathbf{x}) A_2(t, \mathbf{y}) \rangle_c + \mathcal{O}(g^3). \quad (21)$$

We emphasize that, without the gauge averaging $\int \mathcal{D}\mathcal{G}$, the perturbative approximation would not be gauge invariant.

B. Perturbative N_c dependence

From Eq. (21), we can deduce the perturbative leading-order N_c scaling of the connected Polyakov loop correlations for nonzero momenta. Apart from the explicit prefactor of $1/N_c$, we have

$$\text{Tr} \langle A_2(t, \mathbf{x}) A_2(t, \mathbf{y}) \rangle_c = \frac{1}{2} \sum_{a=1}^{N_c^2-1} \langle A_2^a(t, \mathbf{x}) A_2^a(t, \mathbf{y}) \rangle_c \\ \sim N_c^2 - 1, \quad (22)$$

where we used that the color-specific gauge field correlations $\langle A_2^a(t, \mathbf{x}) A_2^a(t, \mathbf{y}) \rangle_c$ contribute equally and are insensitive to N_c . Moreover, while traces over $\text{SU}(2)$ elements are always real valued, for $N_c \geq 3$ traces over $\text{SU}(N_c)$ elements can also have an imaginary part. Since the perturbative derivation leading to Eq. (21) is based on $P(t, \bar{\mathbf{x}})$, which is not restricted to the real part, the gauge average $\int \mathcal{D}\mathcal{G}$ results in a combinatorial prefactor of $d_{N_c} = 2$ for $N_c \geq 3$ and $d_2 = 1$. Putting these prefactors together, we obtain

$$\langle P(t, \bar{\mathbf{x}}) P^\dagger(t, \bar{\mathbf{y}}) \rangle_c \sim d_{N_c} \frac{N_c^2 - 1}{N_c}. \quad (23)$$

This is indeed consistent with the findings of Fig. 6(b).

C. Self-similar scaling

For the chosen initial conditions with large gluonic overoccupation in the infrared, gauge field distributions and correlations approach self-similar (pre)scaling dynamics in time, which constitutes the direct energy cascade. In spatially homogeneous scenarios such as ours, we can write for the gluon correlations

$$\text{Tr} \langle A_2 A_2 \rangle_c(t, \Delta \mathbf{x}) = \text{Tr} \langle A_2(t, \mathbf{x}) A_2(t, \mathbf{y}) \rangle_c, \quad (24)$$

where $\Delta \mathbf{x} = \mathbf{x} - \mathbf{y}$. Focusing on self-similar dynamics at asymptotically late times, these can then exhibit the

following scaling behavior in the vicinity of the nonthermal fixed point⁵:

$$\text{Tr} \langle A_2 A_2 \rangle_c(t, \Delta \mathbf{x}) = t^{\alpha_\infty - 2\beta_\infty} \text{Tr} \langle A_2 A_2 \rangle_{c,S}(t^{-\beta_\infty} \Delta \mathbf{x}). \quad (25)$$

Here, $\text{Tr} \langle A_2 A_2 \rangle_{c,S}(\Delta \mathbf{x})$ indicates the fixed-point shape of $\text{Tr} \langle A_2 A_2 \rangle_c(t, \Delta \mathbf{x})$. A reference timescale for the temporal power law has been absorbed into the former.

Insertion of Eq. (25) into Eq. (21) yields the self-similar scaling behavior

$$\langle PP^\dagger \rangle_c(t, \Delta \bar{\mathbf{x}}) = t^{\alpha_\infty} \langle PP^\dagger \rangle_{c,S}(t^{-\beta_\infty} \Delta \bar{\mathbf{x}}), \quad (26)$$

where periodicity of the loop integrals in Eq. (21) has been employed and $\langle PP^\dagger \rangle_{c,S}$ denotes again the fixed-point shape. Equation (26) can now be conveniently Fourier transformed to momentum space, yielding

$$\langle PP^\dagger \rangle_c(t, \bar{\mathbf{p}}) = \int_{\Delta \bar{\mathbf{x}}} \langle PP^\dagger \rangle_c(t, \Delta \bar{\mathbf{x}}) e^{-i\bar{\mathbf{p}} \Delta \bar{\mathbf{x}}} \\ = t^{\alpha_\infty + 2\beta_\infty} \langle PP^\dagger \rangle_{c,S}(t^{\beta_\infty} \bar{\mathbf{p}}). \quad (27)$$

For the energy cascade in pure Yang-Mills theory out of equilibrium, one has $\alpha_\infty = 4\beta_\infty$ and $\beta_\infty = -1/7$ [7,23,25,26,29]. The scaling behavior of Eq. (27) is consistent with our results discussed in Sec. III C for asymptotically late times. The prescaling ansatz generalizes these results by substituting $t^{\alpha_\infty} \mapsto a(t)$ and $t^{\beta_\infty} \mapsto b(t)$.

In the derivation, we have implicitly assumed that the scaling behavior (25) for the gauge field correlations holds in arbitrary gauge. While no studies of the nonequilibrium gluon plasma are known to us that employ gauges other than the temporal-axial and spatial Coulomb-type gauges for the considered system, this assumption is *a posteriori* justified by the accurate (pre)scaling description of the

⁵This specific form of self-similar scaling follows from the prescaling ansatz (8) for the gluon correlations in momentum space, a Fourier transform, and the behavior of $a(t)$ and $b(t)$ at asymptotically late times.

gauge-invariant Polyakov loop correlations and their perturbative relation to the gauge field correlations.

V. CONCLUSIONS

In this work, we considered real-time $SU(N_c)$ lattice gauge theory in the classical-statistical regime governed by weak couplings and large gluon occupations. We focused on the connected two-point correlations of spatial Polyakov loop traces, which are gauge-invariant observables related to gluon occupation numbers but distinct from gauge-dependent gluon distribution functions. Our results confirm that the zero mode of the Polyakov loop correlations exhibits the dynamic formation of a gauge-invariant condensate. For nonzero momenta, we observed that the correlations are governed by a self-similar direct cascade consistent with energy conservation. We demonstrated their dynamics to be accurately described by prescaling, where scaling exponents exhibit nontrivial time dependencies and only asymptotically converge to their universal fixed values that agree with the universal exponents from gauge field correlations. This leads to a relatively simple composition of Polyakov loop correlation dynamics: the zero mode condenses while nonzero modes transport energy toward the ultraviolet in a direct prescaling cascade, with the condensate fraction growing to dominate the total Polyakov loop occupations over time.

To better understand the prescaling and, in particular, the asymptotic values of the scaling functions, we carried out a detailed perturbative analysis of Polyakov loop correlations. Utilizing basic results from Lie group representation theory, the perturbative expressions featured averages over gauge transformations to maintain gauge invariance. We derived the leading-order contributions of gauge field correlations to the Polyakov loop correlations. Based on the known scaling behavior of the former, this allowed us to consistently describe the asymptotic self-similar behavior of the Polyakov loop correlations and motivated a working ansatz for their prescaling dynamics.

We also examined the N_c dependence of Polyakov loop correlations by comparing results for gauge groups $SU(2)$ and $SU(3)$. In the temporal regime studied, dynamics progressed by a factor $(3/2)^2$ faster for $SU(3)$ than for $SU(2)$, as predicted by kinetic theory in the classical approximation [26]. The nonzero momentum modes scaled with a nontrivial N_c -dependent factor, consistent with our perturbative analysis.

Other than for scalar theories, where condensation, as captured by the buildup of zero mode correlations, is often accompanied by an inverse cascade transporting particle number toward the infrared [13,20], we found no evidence for the existence of a similar inverse cascade in the investigated non-Abelian gauge theories for the chosen lattice discretizations. However, our numerical study does not rule out the possibility of an inverse cascade below the electric screening scale, warranting further studies to clarify

whether an inverse cascade exists and, more generally, what the dynamical condensation mechanism is.

We anticipate that spatial Wilson loop correlations would exhibit dynamics similar to those for Polyakov loop correlations due to existing similarities between spatial Polyakov and Wilson loops in far-from-equilibrium conditions, see, e.g., [33–35]. Furthermore, gauge-invariant condensation, as captured by Wilson loops, bears similarities to scalar $O(N)$ models near a nonthermal fixed point [33]. Investigating Polyakov loop correlations in anisotropic scenarios may also yield insights relevant to the successful description of collisions of atomic nuclei [3,57]. For instance, for relativistic scalars, the dynamic condensate formation can be similar for both isotropic and expanding anisotropic systems [58]. Finally, our results may provide crucial indications for novel effective or kinetic gauge-invariant descriptions of gluon dynamics based on spatial Polyakov loop correlations.

In conclusion, by employing gauge-invariant observables, our work provides an important step toward a better understanding of the nonequilibrium dynamics in gluonic systems and their role in the thermalization of the quark-gluon plasma. Despite similarities with scalar theories, the simple picture of condensation and ultraviolet cascade observed here for $SU(N_c)$ gauge theories using Polyakov loop correlations may distinguish the gluon dynamics from relativistic scalar field theories, which typically exhibit a pronounced inverse cascade driving condensate formation. Further studies are needed to confirm or refute this intriguing picture.

ACKNOWLEDGMENTS

We thank J. Berges, T. Butler, L. de Bruin, J.M. Pawłowski, and P. Radpay for valuable discussions and collaboration on related studies. This work is part of and funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2181/1–390900948 (the Heidelberg STRUCTURES Excellence Cluster) and the Collaborative Research Centre, Project-ID No. 273811115, SFB 1225 ISOQUANT. This research was funded in part by the Austrian Science Fund (FWF) [Grant DOI: 10.55776/P34455]. We acknowledge support by the Interdisciplinary Center for Scientific Computing (IWR) at Heidelberg University, where part of the numerical work has been carried out.

DATA AVAILABILITY

The data that support the findings of this article are openly available [59].

APPENDIX: PRESCALING FUNCTIONS AND THEIR EXTRACTION

The prescaling functions $a(t)$, $b(t)$ are extracted according to the protocol presented in [46]. Starting from

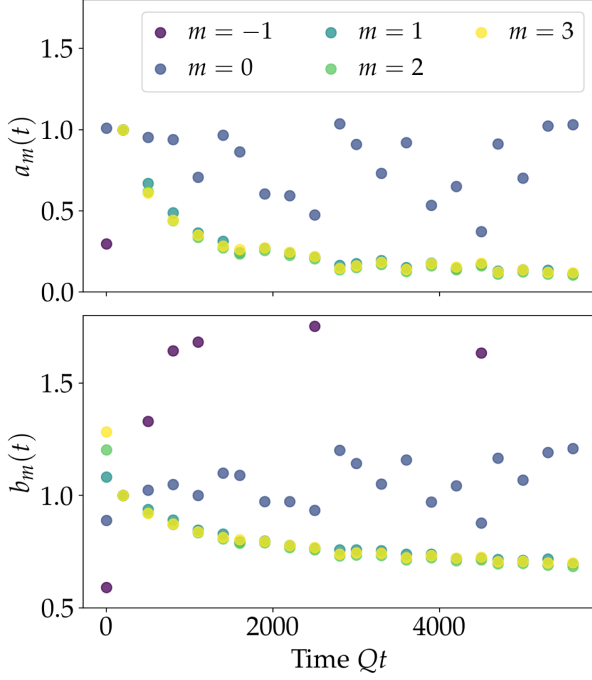


FIG. 7. Prescaling factors $a_m(t)$, $b_m(t)$ extracted from different moments $n_m(t)$ of the Polyakov correlations. The higher moments $m \geq 1$ probe the momentum range of the direct cascade and are observed to converge quickly in time, reflecting prescaling.

Eq. (10), we introduce moments of the Polyakov loop correlations

$$n_m(t) = \int \frac{d^2\vec{p}}{(2\pi)^2} p^m \langle PP^\dagger \rangle_c(t, p) \quad (\text{A1})$$

$$= a(t)b^{-m}(t)n_m(t_{q0}). \quad (\text{A2})$$

We then extract the prescaling factors $a(t)$, $b(t)$ from combinations of the moments according to

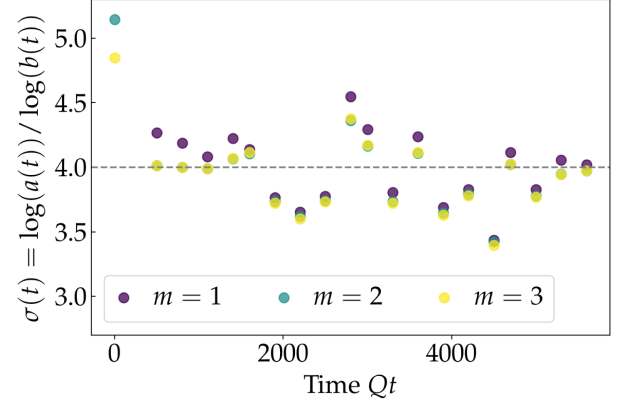


FIG. 8. Logarithmic ratio of the prescaling factors $a(t)$, $b(t)$ extracted from different moments $n_m(t)$ of the Polyakov correlations. The logarithmic ratio $\sigma(t) = 4$ (horizontal dashed line) reflects mode energy conservation of relativistic gluons in three spatial dimensions.

$$a_m(t) = \left[\frac{n_m^{1/m}(t)}{n_{m+1}^{1/(m+1)}(t)} \frac{n_{m+1}^{1/(m+1)}(t_0)}{n_m^{1/m}(t_0)} \right]^{m(m+1)}, \quad (\text{A3a})$$

$$b_m(t) = \frac{n_m(t)}{n_{m+1}(t)} \frac{n_{m+1}(t_0)}{n_m(t_0)}, \quad (\text{A3b})$$

with $a_m(t_0) = b_m(t_0) = 1$, which are shown in Fig. 7. While lower moments mix infrared and ultraviolet dynamics, prescaling factors extracted from $m \geq 1$ converge quickly in time, reflecting the emergence of prescaling for the direct cascade in the ultraviolet. For the prescaling figures with $a(t)$, $b(t)$ in the main text, we average the prescaling factors obtained from $m = 1, 2, 3$.

The direct cascade encodes energy transport of relativistic gluons in three spatial dimensions, which is reflected in $a(t) = b(t)^4$ as is visible in Fig. 8.

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Far from equilibrium hydrodynamics of nonthermal fixed points

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Nonthermal fixed points are paradigmatic far-from-equilibrium phenomena of relevance to high-energy physics, cosmology, and cold atomic gases. We propose that, despite their intrinsically nonequilibrium nature, nonthermal fixed points give rise to hydrodynamic excitations otherwise known in the vicinity of thermal equilibrium. As a result, nonthermal fixed points can also be characterized by transport coefficients, such as a far-from-equilibrium, and therefore manifestly time-dependent, incarnation of shear viscosity. We corroborate our proposal with explicit studies using relativistic kinetic theory with binary collisions of massless particles in the 14-moment approximation and comparisons to QCD kinetic theory simulations.

I. INTRODUCTION

Our understanding of nonequilibrium physics is chiefly rooted in studying small departures from equilibria. This is in particular the origin of fluidity that gives rise to a wealth of phenomena across scales, from tiny droplets of quark-gluon plasma at RHIC and LHC, through cold atomic gases, many macroscopic phenomena on Earth, all the way to modeling astrophysical objects, and even to cosmological scenarios, see [1–4] for a broad exposition of these topics. Indeed, in its textbook form, fluid mechanics relies on a derivative expansion in gradients of fluid velocity and thermodynamic variables such as temperature or pressure around local thermal equilibrium [5].

In recent years, initially motivated by the physics of the quark-gluon plasma at RHIC and LHC [6, 7] and now also by cold atomic gases [8, 9], significant efforts have been invested into understanding far from equilibrium generalizations of fluid mechanics. This ongoing research program, reviewed in [10, 11], has been concerned with particular resummations of hydrodynamic derivative expansion with the aim of extending its validity deeper into a nonequilibrium regime.

In the present work, inspired by these ongoing efforts to push hydrodynamics far from equilibrium, we propose that fluid mechanics naturally emerges as an effective description of the evolution of spatial inhomogeneities around nonthermal fixed points. Nonthermal fixed points are intrinsically far from equilibrium phenomena, whose defining feature is self-similarity of the distribution of particles in time [12–15]. They are of broad theoretical relevance from cosmology, through thermalization in QCD all the way to cold atomic gases, see [16, 17] for reviews. Furthermore, in the latter case, nonthermal fixed points have been realized experimentally in several complementary cold atomic setups [18–29]. This adds a layer

of observational relevance to the novel manifestation of fluidity that we propose.

Our proposal that nonthermal fixed points support fluid mechanics should come across as a conservative one. One way to motivate standard, near-equilibrium hydrodynamics, at least at a linearized level, is the attractive nature of equilibrium combined with the presence of conservation laws. Local excesses of energy, momentum, or other conserved charges cannot dissipate arbitrarily quickly as their dynamics are constrained by the microscopic conservation law. Instead, equilibrium is approached via the transport of conserved quantities across the system, which is the defining feature of hydrodynamics. Similar assumptions hold around (at least) isotropic nonthermal fixed points: ab initio simulations, experimental observations, and recent results on the spectrum of linearized perturbations [30, 31] indicate their stability and local conservation laws hold.

In practical terms, nonthermal fixed points studied to date were (on average) spatially homogeneous. The core of our idea is that spatial inhomogeneity around a nonthermal fixed point relaxes according to hydrodynamic equations of motion with transport coefficients, the most important of which is the shear viscosity, being intrinsically time-dependent. Note that ultimately this notion of far from equilibrium hydrodynamics is novel with respect to the efforts that motivated us. The reason for it is that already our point of departure for constructing a hydrodynamic expansion is far from equilibrium and this is ultimately the source of time-dependent transport coefficients, rather than a resummation of the near-equilibrium derivative expansion.

Our work naturally connects with two major research directions of the past two decades: studying patterns of real-time response of quantum field theories near equilibrium in different parametric regimes [32–39] and, related to it, comparing shear viscosity of different physical systems that led to consideration of its ratio with entropy density [40]. The novel aspect of our results in this context stems from the starting point of our analysis, the underlying nonthermal fixed point, being intrinsically time dependent and far from equilibrium. In particular,

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one resulting problem that we address is how to compare time-dependent shear viscosity between nonthermal fixed points or with a shear viscosity of the same underlying system near equilibrium, given that its ratio with statistical entropy turns out to retain time dependence. The latter obstructs the usage of the shear viscosity to entropy density ratio in this context.

While our idea about far from equilibrium fluidity around nonthermal fixed points is general, the realization we pursue is in kinetic theory for binary collisions of particles and comparisons of our predictions with QCD kinetic theory simulations [15, 41–43]. In our studies, we will rely on the 14-moment approximation that has previously been used to construct near-equilibrium hydrodynamic equations of motion from kinetic theory, see [44, 45] for a review, and Ref. [46] for an initial discussion of far from equilibrium hydrodynamics in kinetic theory.

Note about conventions and notation: In our paper, we use natural units $k_B = \hbar = c = 1$ and adopt the mostly minus metric signature convention. We mostly specialize to near-isotropic scale-invariant (massless) systems in three spatial dimensions, for which the background energy density $\mathcal{E}$ and pressure $\mathcal{P}$ are related as $\mathcal{E} = 3\mathcal{P}$ and the bulk viscosity vanishes. Further notation is specified in Appendix A.

II. HYDRODYNAMICS OF NONTHERMAL FIXED POINTS

A. Universal scaling far from equilibrium

We will consider perturbations of isotropic (in the local rest frame of the system) nonthermal fixed points, where correlation functions exhibit characteristic self-similar behavior in a corresponding momentum regime

$$f(t, \mathbf{k}) = A(t)f_S(B(t)|\mathbf{k}|). \quad (1)$$

In the equation above, $f(t, \mathbf{k})$ is the one particle distribution function subject to the spatially homogeneous Boltzmann equation. The factors $A(t)$ and $B(t)$ are power laws in time

$$A(t) = \left(\frac{t - t_*}{t_{\text{ref}}}\right)^\alpha, \quad B(t) = \left(\frac{t - t_*}{t_{\text{ref}}}\right)^\beta, \quad (2)$$

whose exponents together with the scaling function f_S constitute universal scaling properties far from equilibrium. The time offset t_* is needed for the system to dynamically account for the notion of the origin of time, which plays a special role for scaling phenomena [28, 47].

Through the lenses of Eq. (1), nonthermal fixed points are self-similar turbulent cascades transporting conserved quantities in the given momentum range. We will focus on nonthermal attractors characterized by local mode energy conservation, i.e., where $\mathcal{E} = \int d^d\mathbf{k}/(2\pi)^d \omega_{\mathbf{k}} f(t, \mathbf{k})$

is constant in the momentum range. Using Eq. (1), this implies the relations

$$\sigma \equiv \log(A(t))/\log(B(t)) = d + z = \alpha/\beta, \quad (3)$$

where d is the dimensionality of space (for us, $d = 3$) and z the dynamical scaling exponent that can be deduced from the underlying (quasi-)particle dispersion relation $\omega_{\mathbf{k}} \sim |\mathbf{k}|^z$ (for us, $z = 1$).

B. Hydrodynamics

The main equations governing the dynamics of a relativistic fluid emerge from the conservation laws for energy and momentum [48],

$$\partial_\mu T^{\mu\nu} = 0, \quad (4)$$

with $T^{\mu\nu}$ being the energy-momentum tensor of the fluid. It is convenient to decompose $T^{\mu\nu}$ in terms of the fluid's four-velocity, u^μ ,

$$T^{\mu\nu} = \mathcal{E}u^\mu u^\nu - \mathcal{P}\Delta^{\mu\nu} + \pi^{\mu\nu}, \quad (5)$$

where $\mathcal{E}$ is the energy density in the local rest frame of the fluid, $\mathcal{P}$ is the thermodynamic pressure, and $\pi^{\mu\nu}$ is the shear stress tensor. The focus of this paper will be conformal systems, leading to the relation $\mathcal{P} = \mathcal{E}/3$ regardless of whether the system is close to equilibrium. We note that in the tensor decomposition above, u^μ is defined according to the Landau picture [49] as the unit-normalized ($u_\mu u^\mu = 1$) timelike eigenvector of the energy-momentum tensor, satisfying $T^\mu_\nu u^\nu = \mathcal{E}u^\mu$.

Naturally, the continuity equations describing energy-momentum conservation are not sufficient to completely determine the dynamics of the energy-momentum tensor – which contains a total of 9 independent degrees of freedom since $\mathcal{E} = 3\mathcal{P}$. In order to close this system of equations one must provide additional constraints for the shear stress tensor. In the relativistic regime, causality and linear stability around global equilibrium dictate that the shear stress tensor cannot be simply determined by constitutive relations, as is often the case in the non-relativistic regime [49], satisfying instead dynamical equations of motion. The most widespread theory of relativistic hydrodynamics was proposed by Israel and Stewart in the 1970's [50]. This formulation of relativistic hydrodynamics is widely employed in simulations of ultrarelativistic heavy ion collisions to describe the time evolution of the quark-gluon plasma [51, 52]. It has also been systematically tested in the domain of kinetic theory [44].

In the Israel and Stewart approach, the shear stress tensor $\pi^{\mu\nu}$ obeys relaxation-type equations of motion

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} = -\pi^{\mu\nu} + 2\eta\sigma^{\mu\nu} + \dots, \quad (6)$$

with the shear tensor $\sigma^{\mu\nu}$ defined by

$$\sigma^{\mu\nu} \equiv \partial^{\langle\mu} u^{\nu\rangle}, \quad (7)$$

and the ellipsis denoting possible nonlinear terms. The brackets denote the double, symmetric, and traceless projection of the tensor, as defined in Appendix A. In Eq. (6), η is the shear viscosity, which also appears in Navier-Stokes theory, and τ_π is the relaxation time, a novel parameter that appears in relativistic hydrodynamic equations of motion and quantifies the time it takes for the shear stress tensor to relax back to equilibrium. Linear causality and stability around equilibrium for massless particles (conformal system) dictate that it has a lower bound [53]

$$\frac{\eta}{\tau_\pi(\mathcal{E} + \mathcal{P})} \leq \frac{1}{2}. \quad (8)$$

The combination appearing on the left-hand side of the above inequality will play an important role later in the paper.

The equations of motion for relativistic hydrodynamics of the type (6) can be derived from the Boltzmann equation [54], which provides a microscopic description of dilute gases in terms of the single particle momentum distribution function, $f(x, k) \equiv f_{\mathbf{k}}$. The Boltzmann equation is a partial integro-differential equation that determines the time evolution of $f_{\mathbf{k}}$,

$$k^\mu \partial_\mu f_{\mathbf{k}} = \mathcal{C}[f], \quad (9)$$

with $k^\mu = (\omega_{\mathbf{k}}, \mathbf{k})$ being the on-shell four-momentum and $\mathcal{C}[f]$ the collision kernel encoding microscopic interactions. Here we consider a collision term with elastic two-to-two collisions in the overoccupied limit, $f_{\mathbf{k}} \gg 1$,

$$\begin{aligned} \mathcal{C}[f] \approx & \frac{1}{2} \int dK' dP dP' W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'} \\ & \times [(f_{\mathbf{p}} f_{\mathbf{p}'} (f_{\mathbf{k}} + f_{\mathbf{k}'} - f_{\mathbf{k}} f_{\mathbf{k}'} (f_{\mathbf{p}} + f_{\mathbf{p}'}))], \end{aligned} \quad (10)$$

where $W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'}$ is the Lorentz-invariant transition rate and $dK = d^3\mathbf{k}/[(2\pi)^3 \omega_{\mathbf{k}}]$ is the Lorentz-invariant phase volume. We will focus on nonthermal attractors associated with the turbulent transport of mode energy to smaller distance scales through a direct energy cascade. The underlying physics is captured using weakly-interacting relativistic massless scalars, where the corresponding transition element for example in the $O(N)$ model reads [55]

$$\begin{aligned} W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'} = & \lambda^2 \frac{N+2}{48N^2} \\ & \times (2\pi)^{d+1} \delta^{(d+1)}(k^\mu + k'^\mu - p^\mu - p'^\mu), \end{aligned} \quad (11)$$

with $\omega_{\mathbf{k}} = |\mathbf{k}|$.

By taking appropriate moments of this equation, one recovers the conservation laws in Eq. (4) and identifies the following microscopic expression for $T^{\mu\nu}$

$$T^{\mu\nu} = \int dK k^\mu k^\nu f_{\mathbf{k}}. \quad (12)$$

The local energy density and the shear stress tensor can then be identified as the following moments of $f_{\mathbf{k}}$,

$$\mathcal{E} = \int dK (u_\mu k^\mu)^2 f_{\mathbf{k}}, \quad (13a)$$

$$\pi^{\mu\nu} = \int dK k^{\langle\mu} k^{\nu\rangle} f_{\mathbf{k}}. \quad (13b)$$

We note that $\pi^{\mu\nu}$ vanishes when integrated over any isotropic momentum distribution and, hence, will also vanish when integrated over the isotropic nonthermal fixed point distribution of Eq. (1). Perturbations around the nonthermal fixed point can then lead to finite values of the shear stress tensor that, as we shall demonstrate, can be described using hydrodynamic-like equations.

The fundamental step in the derivation of a fluid-dynamical theory is the expansion of the distribution function about some background state,

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}}, \quad (14)$$

and then deriving an approximate expression for the correction $\delta f_{\mathbf{k}}$. Since the fluid-dynamical regime is traditionally expected to emerge if the system is sufficiently close to equilibrium, $f_{0\mathbf{k}}$ is usually assumed to be the local equilibrium distribution function and $\delta f_{\mathbf{k}}$ the corresponding nonequilibrium correction. One then applies the 14-moment approximation [50] and expresses the correction $\delta f_{\mathbf{k}}$ solely in terms of the shear stress tensor (and other moments of $f_{\mathbf{k}}$, when applicable), effectively closing the hydrodynamic equations of motion. In this work, we extend this procedure and take $f_{0\mathbf{k}}$ to be an isotropic nonthermal fixed point distribution as in Eq. (1). We do so by noting that the traditional 14-moment approximation does not necessarily rely on the assumption that $f_{0\mathbf{k}}$ is the local equilibrium distribution: it can be directly applied to expansions around any *isotropic* momentum distribution function. One then obtains the direct extension of the 14-moment approximation [46],

$$\delta f_{\mathbf{k}} = f_{0\mathbf{k}} \frac{2p^\mu p^\nu}{15N_4(t)} \pi_{\mu\nu}(t), \quad (15)$$

with

$$N_4(t) = \int dK |\mathbf{k}|^4 f_{0\mathbf{k}}, \quad (16)$$

and $f_{0\mathbf{k}}$ being the isotropic nonthermal fixed point distribution from Eq. (1). More details can be found in Appendix B.

Next, following the procedure outlined in [46, 56], we obtain an equation of motion for $\pi^{\mu\nu}$ by taking the comoving derivative of its kinetic expression (13b),

$$\dot{\pi}^{\langle\mu\nu\rangle} = \Delta_{\alpha\beta}^{\mu\nu} \frac{d}{d\tau} \int dK k^{\langle\alpha} k^{\beta\rangle} f_{\mathbf{k}}, \quad (17)$$

and using the Boltzmann equation to replace the comoving derivatives of $f_{\mathbf{k}}$, together with the 14-moment approximation (15). Considering this more general application of the method of moments, one can generalize the

derivation of fluid dynamics proposed by Israel and Stewart to a broader class of systems that are not necessarily close to equilibrium. In this work, we perform this task for systems that are in the vicinity of a nonthermal attractor, leading to a fluid-dynamical description of the fluctuations around this scaling background.

The evolution equation of the shear stress becomes,

$$\begin{aligned} \tau_\pi(t) \dot{\pi}^{(\mu\nu)} = & -\pi^{\mu\nu} + 2\eta(t)\sigma^{\mu\nu} \\ & + \tau_\pi(t) \left[-\frac{4}{3}\pi^{\mu\nu}\theta - \frac{10}{7}\pi^{\lambda(\mu}\sigma_{\lambda}^{\nu)} - 2\pi^{\lambda(\mu}\omega_{\lambda}^{\nu)} \right], \end{aligned} \quad (18)$$

where the expansion rate θ , the vorticity tensor $\omega^{\alpha\beta}$, and details of this derivation are given in Appendix B. In direct analogy with the thermal case (6), we identified the shear viscosity and relaxation time transport coefficients as

$$\eta(t) = \frac{8}{15^2} \frac{\mathcal{E} N_4(t)}{\delta\mathcal{C}(t)}, \quad (19a)$$

$$\tau_\pi(t) = \frac{2}{15} \frac{N_4(t)}{\delta\mathcal{C}(t)}. \quad (19b)$$

The quantity $\delta\mathcal{C}(t)$ emerges from the linearized collision kernel and encodes all information about the microscopic interactions; an exact expression is given in Eq. (B13) of Appendix B. Crucially, in comparison with its counterpart (6) derived in an expansion around local thermal equilibrium, the transport coefficients following from an expansion around the scaling background display an intrinsic time dependence. We note that Eq. (18) also contains the nonlinear terms omitted from (6), as well as their corresponding transport coefficients.

C. Comparisons with equilibrium

Note that, without dwelling into details of Eq. (19), we can already conclude that the ratio of the shear viscosity to the relaxation time is time-independent far from equilibrium. This is important, as it allows for making a comparison with equilibrium hydrodynamics across microscopic models. A powerful notion considered before near equilibrium is the dimensionless ratio of the shear viscosity to the entropy density [40]. Its importance stems from connecting a very basic hydrodynamic notion with the microscopic interaction strength in a way that allows for a straightforward distinction between small and large couplings. However, as one can convince oneself, the η/s ratio evaluated on a nonthermal fixed point exhibits an unilluminating time dependence, which renders it useless, at least for near-equilibrium comparisons.

What eliminates the time dependence is precisely the ratio η/τ_π . Its drawback, however, is that it is dimensionful. To make it dimensionless, we can use the energy density or pressure, but it is more insightful to consider the combination of $\mathcal{E} + \mathcal{P}$, as we did in (8). The resulting

quantity is

$$\frac{\eta(t)}{\tau_\pi(t)(\mathcal{E} + \mathcal{P})} = \frac{1}{5}. \quad (20)$$

Since the derivation behind (19) relied only on the isotropy of $f_{0\mathbf{k}}$, the expression (20) applies also to the standard near-equilibrium hydrodynamics. This indicates a degree of universality between the equilibrium and nonthermal fixed point hydrodynamics for the same underlying weakly-coupled quantum field theory.

In holography at infinite coupling, the shear viscosity is universally given by $\eta = \frac{1}{4\pi} \frac{\mathcal{E} + \mathcal{P}}{T}$, where we used thermodynamic identities to rephrase the entropy density in terms of the variables used in (20). On the other hand, the relaxation time is simply the gap in the spectrum of quasinormal modes associated with the energy-momentum tensor sector [32]. This gap changes once the conformal symmetry is relaxed [57, 58], but in the absence of chemical potential these changes are of limited capacity when expressed in the units of the temperature. From the results of [32], we take $\tau_\pi T \approx 1.3 \times 2\pi$. This leads to

$$\frac{\eta}{\tau_\pi(\mathcal{E} + \mathcal{P})} \approx 0.65. \quad (21)$$

Interestingly, this is the same order of magnitude as the weak coupling result (20) and goes hand-in-hand with earlier attempts to compare weak and strong coupling dynamics by normalizing time by η/s and making it dimensionless using temperature, as in [59, 60]. This is in stark contrast with η/s , which by construction is parametrically different at weak and strong coupling.

D. Scaling analysis

The kinetic theory origin of nonthermal fixed points can be traced to being able to factor out the time dependence from the collision kernel when evaluated on the overoccupied distribution function of the scaling form (1) [47]

$$\mathcal{C}[f](t, \mathbf{p}) = B^{1-1/\beta}(t) \mathcal{C}[f_S](B(t)|\mathbf{p}|). \quad (22)$$

Using a perturbative transition matrix element $W \sim \delta^{(d+1)}(k^\mu + k'^\mu - p^\mu - p'^\mu)$ like in (11), one can perform a general scaling analysis of the time-dependent transport coefficients

$$\eta(t) = B^{4-4z+1/\beta}(t) \bar{\eta}, \quad (23)$$

with $\tau_\pi(t)$ scaling in the same way as dictated by Eq. (20). To make the notation uniform, we will write

$$\tau_\pi(t) = B^{4-4z+1/\beta}(t) \bar{\tau}_\pi. \quad (24)$$

The dimensionful constants $\bar{\eta}$ and, via Eq. (20), $\bar{\tau}_\pi = \frac{15}{4} \bar{\eta}/\mathcal{E}$ arise from evaluating Eqs. (19) on the scaling function f_S . In the following, we will be primarily interested in the scaling properties associated with the factors of $B^{4-4z+1/\beta}(t)$ and their implications for the dynamics, rather than in the precise values of $\bar{\eta}$ and $\bar{\tau}_\pi$.

III. THE RELATIVISTIC DIRECT ENERGY CASCADE

We will now focus on the dynamics of hydrodynamic perturbations around the nonthermal energy attractor in relativistic systems, which is of importance to early-universe cosmology [12, 13, 61] and quark-gluon plasma physics [47, 62–64].

Starting from a representative of the corresponding universality class, such as e.g. relativistic weakly-coupled scalars with transition matrix element as in Eq. (11), the analysis of the Boltzmann equation leads to the following scaling exponents

$$\alpha = -\frac{4}{7} \quad \text{and} \quad \beta = -\frac{1}{7}, \quad (25)$$

which were originally introduced in Eq. (1). We note that the same scaling exponents arise for the QCD effective kinetic description [47, 63–65].

Utilizing the values (25), we determine the underlying time dependence of the transport coefficients,

$$\eta(t) = B^{-7}(t) \bar{\eta} = \left(\frac{t - t_*}{t_{\text{ref}}} \right) \bar{\eta}, \quad (26a)$$

$$\tau_\pi(t) = B^{-7}(t) \bar{\tau}_\pi = \left(\frac{t - t_*}{t_{\text{ref}}} \right) \bar{\tau}_\pi. \quad (26b)$$

This implies that both the shear viscosity and the relaxation time grow linearly with time. This behavior has an important consequence for the decay to a nonthermal fixed point, which we will explore next.

A. Homogeneous isotropization dynamics

The far-from-equilibrium hydrodynamic equations derived in the previous section from kinetic theory (18) have a particularly simple sector associated with spatial homogeneity, in which case they reduce to

$$\tau_\pi(t) \dot{\pi}^{(\mu\nu)} + \pi^{\mu\nu} = 0. \quad (27)$$

Let us emphasize that this equation does not describe hydrodynamic excitations, but instead is a consequence of the natural framework we applied, that approximates kinetic theory in a way encapsulating leading hydrodynamic effects.

Building on the earlier studies in [66–69], we adopt the ansatz

$$\pi^{\mu\nu} = \text{diag} \left(0, \frac{1}{3} \Delta \mathcal{P}(t), \frac{1}{3} \Delta \mathcal{P}(t), -\frac{2}{3} \Delta \mathcal{P}(t) \right), \quad (28)$$

where $\Delta \mathcal{P}(t)$ characterizes the time-dependent pressure anisotropy. Eq. (27) is then solved by

$$\begin{aligned} \Delta \mathcal{P}(t) &= \Delta \mathcal{P}(t_0) \exp \left[- \int_{t_0}^t \frac{dt'}{\tau_\pi(t')} \right] \\ &= \Delta \mathcal{P}(t_0) \left(\frac{t_0 - t_*}{t - t_*} \right)^{\frac{t_{\text{ref}}}{\bar{\tau}_\pi}}, \end{aligned} \quad (29)$$

where, in the last step, we used the relaxation time calculated assuming an energy cascade from Eq. (26). This is to be compared with the thermal case in which τ_π is simply a constant determined from the background temperature and the pressure anisotropy relaxes exponentially to zero,

$$\Delta \mathcal{P}(t) \Big|_{\text{thermal}} = \Delta \mathcal{P}(t_0) \exp \left(- \frac{t - t_0}{\tau_\pi} \right). \quad (30)$$

Similar exponential relaxation is seen in various causal formulations of relativistic hydrodynamics and in holography [70]. Indeed, the reason behind the sub-exponential isotropization around the nonthermal fixed point is the effective relaxation time that increases with time. Note that this is in line with recent results [31] about *isotropic* relaxation to the same non-thermal fixed point going beyond Eq. (18), which also exhibits a power-law rather than exponential approach. Finally, let us emphasize that Eq. (18) is an effective approximation to the full kinetic theory dynamics and we do not stipulate that the kinetic theory in question does have precisely an excitation described by (29). Indeed, we expect the nonhydrodynamic sector of kinetic theory to be much richer based on earlier near-equilibrium [34, 36–39] and nonthermal fixed point studies [31].

The result (29) will serve as a basis for a qualitative comparison with the QCD kinetic theory in Sec. IV.

B. Hydrodynamic excitations and linear response theory

At a linearized level around equilibrium, hydrodynamic excitations manifest themselves as gapless eigenmodes of the equilibrium counterpart of equations like Eq. (18). Both equilibrium and known nonthermal fixed points are homogeneous, as a result of which it is natural to seek for perturbations in a Fourier space. Due to isotropy, we can align the spatial momentum with any of the directions, e.g. with x^3 : $e^{i q x^3}$. Note that q is the spatial momentum of perturbations associated with macroscopic inhomogeneity in space, not to be confused with the momentum of microscopic particles.

Now, in equilibrium it is natural to consider the time-dependence in the standard Fourier form, $e^{-i \Omega(q)(t-t_0)}$. However, since both the shear viscosity and the relaxation time are time-dependent, we will instead consider a more general expression, $e^{-i \int_{t_0}^t dt' \Omega(t', q)}$, which in the absence of time dependence in $\Omega(t, q)$ reduces to the usual Fourier analyses performed for equilibrium states. The statement of gaplessness of hydrodynamics is simply $\Omega(t, q) \rightarrow 0$ as $q \rightarrow 0$, i.e. perturbations of arbitrarily small spatial momentum become arbitrarily long-lived and slowly evolving. The origin of this lies in the conservation equations, for us Eq. (4), in which all terms contain first-order derivatives.

There is one more standard technicality to take into account. The perturbations we consider are tensor per-

turbations and the choice of the spatial momentum gives rise to a residual $SO(2)$ symmetry. As a result, different irreducible representations of this $SO(2)$ decouple in their dynamics, giving rise to sound wave perturbations and shear modes, see e.g. [32, 71] for discussions.

Starting with the unperturbed nonthermal fixed point having $\mathcal{E}$ and $u^0 = 1$, $u^{1,2,3} = 0$, $\pi^{\mu\nu} = 0$, the sound channel corresponds to $\delta\mathcal{E}$, δu^3 , $\delta\pi^{33}$ and $\delta\pi^{11} = \delta\pi^{22} = -\frac{1}{2}\delta\pi^{33}$. The shear channel is then encapsulated by δu^1 and $\delta\pi^{13}$ with other possible components obtainable by a rotation around the x^3 axis.

It turns out that, after defining

$$\bar{\Omega}(\bar{q}) = \Omega(t, q) \times \left(\frac{t - t_*}{t_{\text{ref}}} \right), \quad (31a)$$

$$\bar{q} = q \times \left(\frac{t - t_*}{t_{\text{ref}}} \right), \quad (31b)$$

the dispersion relations $\bar{\Omega}(\bar{q})$ become not only time-independent, but also become identical to their near-equilibrium counterparts [71]. This implies that at least some of the lessons learned from these equations will generalize to our far from equilibrium system.

For the shear mode, the dispersion relation solves

$$\bar{\Omega}(\bar{q}) + i \frac{3\bar{\eta}}{4\mathcal{E}} \bar{q}^2 - i \bar{\tau}_\pi \bar{\Omega}(\bar{q})^2 = 0. \quad (32)$$

There are two solutions of this equation. The first one is hydrodynamic and describes diffusion at low q

$$\bar{\Omega}_{\text{shear}}(\bar{q}) \approx -i \frac{3\bar{\eta}}{4\mathcal{E}} \bar{q}^2, \quad (33)$$

with the combination $\frac{3\bar{\eta}}{4\mathcal{E}}$ playing the role of the diffusivity constant. Note that the quantity that ultimately dictates the time-dependence is $e^{-i \int_{t_0}^t dt' \Omega(t', q)}$, which means that at low- q the shear diffusion far from equilibrium will proceed at late times as $e^{-\frac{3\bar{\eta}}{4\mathcal{E}} q^2 \left(\frac{t-t_*}{t_{\text{ref}}} \right)^2}$, whereas near-equilibrium the behavior is $e^{-\frac{3\eta}{4\mathcal{E}} q^2 t}$. Despite this qualitative change of the behavior with respect to near-equilibrium dynamics, the far from equilibrium diffusive mode still gets arbitrarily long-lived as $q \rightarrow 0$. The other excitation associated with (32) should be viewed as transient, which at low q behaves as $\left(\frac{t-t_*}{t_{\text{ref}}} \right)^{-\frac{\bar{\tau}_\pi}{t_{\text{ref}}}}$. Note that the behavior of both modes is consistent with the earlier studies in [31], which, for the same nonthermal fixed point at $q = 0$, reported the presence of a zero mode and a tower of transient decaying modes as power-law excitations.

For the sound channel, the equation specifying the dispersion relations reads

$$+i \frac{\bar{\eta}}{\mathcal{E}} \bar{q}^2 \bar{\Omega}(\bar{q}) + \frac{1}{3} i \bar{\tau}_\pi \bar{q}^2 \bar{\Omega}(\bar{q}) - i \bar{\tau}_\pi \bar{\Omega}(\bar{q})^3 + \bar{\Omega}(\bar{q})^2 - \frac{1}{3} \bar{q}^2 = 0. \quad (34)$$

There are now three solutions, two of which correspond to sound waves

$$\bar{\Omega}_{\text{sound}}(\bar{q}) \approx \pm \frac{1}{\sqrt{3}} \bar{q}. \quad (35)$$

Unsurprisingly, for sound waves at the leading order the behavior is $e^{\pm i \frac{1}{\sqrt{3}} q t}$, which, when combined with $e^{i q x^3}$, gives rise to the sound wave propagation familiar from equilibrium. The remaining solution of Eq. (34) is another transient, which at $q = 0$ has exactly the same form as the shear. This transient is nothing else than the homogeneous isotropization we considered in Sec. III A.

The last thing that we want to highlight regarding this novel instance of hydrodynamics is lurking in the rescalings (31). As one sees, the large time limit does not commute with the small spatial momentum limit. As a result, as time progresses, at fixed q higher order terms become gradually more relevant and the full form of the dispersion relation becomes important. Since our derivation is based on a truncation of the kinetic theory description, ideally one should consider the dispersion relations in the full kinetic theory. On the other hand, from the point of view of cold atomic gases experiments that have a finite lifetime, as long as one has a sufficient spatial volume at disposal, one can make the momentum of the inhomogeneity small enough so that the breakdown is not observed. On the other hand, the breakdown we predict is a robust consequence of the shear viscosity and relaxation time increasing as the system evolves, so it is an interesting phenomenon on its own.

IV. APPLICATION TO QCD KINETIC THEORY

We want to test the robustness of our derivation of far from equilibrium hydrodynamics around nonthermal fixed points against ab initio simulations. The most satisfactory solution would be to break homogeneity around a nonthermal fixed point and compare the results of an ab initio simulation with the predictions of Eq. (18). This is numerically costly.

Instead, we will proceed with a different, qualitative route that at the same time indicates our results are more broadly applicable. To this end, we will study isotropic nonthermal fixed points in QCD kinetic theory that also give rise to the scaling (25), but incorporate both binary and one-to-two collision processes of gluons. Because of the latter, the results of the previous section strictly speaking do not apply. Nevertheless, we will see that, at a qualitative level, our predictions are consistent with the QCD kinetic theory. In practical terms, we will start from an overoccupied initial condition $f(t_i, \mathbf{p}) = n_0/g^2 \exp[-\mathbf{p}^2/Q^2]$, with t_i being the initial time of the simulation, g^2 the square of the coupling and n_0 the initial occupation. We consider $n_0 = 1$ and $g^2 = 10^{-8}$ and use the code adopted earlier in, e.g., [47]. Results will be given in units of the characteristic energy scale Q as given by the maximum of $|\mathbf{p}|^2 f(t_i, \mathbf{p})$.

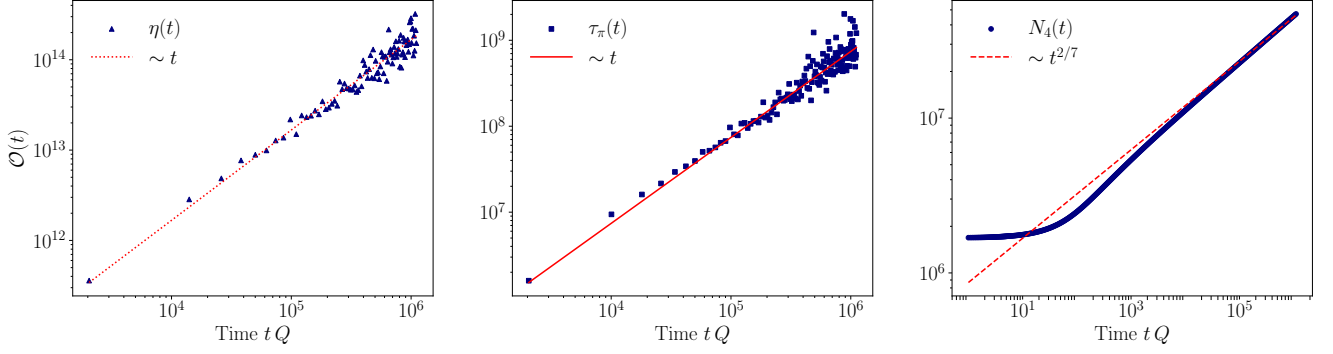


FIG. 1. Time evolution of shear viscosity (*Left*), relaxation time (*Center*), and $N_4(t)$ (*Right*) extracted from a simulation with $g^2 = 10^{-8}$, $N_c = 3$, $n_0 = 1$. $\delta\mathcal{C}$ is evaluated using 2×10^9 Monte Carlo samples at every 100th time step ($Qdt = 0.1$, or every $tQ = 10$ steps), and then averaged over 40 neighboring points to further reduce Monte Carlo fluctuations.

The first test we ran is concerned with the scaling of different contributions to the expressions (19). Fig. 1 shows strong indications that, given a scaling function of QCD, the transport coefficients as well as ingredients in their calculation scale appropriately. We note that, while the evolution involves also inelastic processes, we here compute only the linearized elastic collision kernel $\delta\mathcal{C}(t)$ defined in Eq. (B13) with the gluon-gluon transition matrix element of QCD kinetic theory, provided in Appendix C 1. The evaluation of the collision kernel $\delta\mathcal{C}$ uses Monte Carlo sampling, which leads to a spread in the uncertainty at later times. We checked that this spread can be systematically improved by increasing the number of Monte Carlo samples used in the evaluation of $\delta\mathcal{C}$. The transport coefficients are significantly affected by this spread since $\delta\mathcal{C} \sim 1/t^{5/7}$, see Eq. (C12) in the Appendix, which enters the expressions for the transport coefficients in the denominator. To better capture the overall scaling evolution, we average $\delta\mathcal{C}$ over 40 neighboring points. We note that such an averaging procedure is not required for calculating $N_4(t)$ as it follows from a simple momentum integral of the distribution function. We then compare to the asymptotic fixed point scaling, where we neglect the non-universal prescaling effects including $\sim t_*$ and extract the amplitude from the bulk observables at an arbitrary reference point $\sim \mathcal{O}(t_{\text{ref}})$. Overall, we observe that the scaling of different bulk observables is accurately captured by Eq. (26).

The second ab initio result we obtained concerns homogeneous isotropization, discussed in Sec. III A. To this end, we add an anisotropic perturbation to the gluon distribution function in the vicinity of the scaling regime:

$$\delta f_{\mathbf{p}} = f_0(t_0, \mathbf{p}) \frac{A}{3} [-(p^1)^2 - (p^2)^2 + 2(p^3)^2], \quad (36)$$

and we follow its relaxation as witnessed by the pressure anisotropy. We apply the perturbation at a time $t_0 Q = 2 \times 10^3$, when the distribution function and bulk observables, such as those in Fig. 1, show self-similar scaling evolution. The amplitude is set to $A = 12$, though we verified that the results are independent of variations in

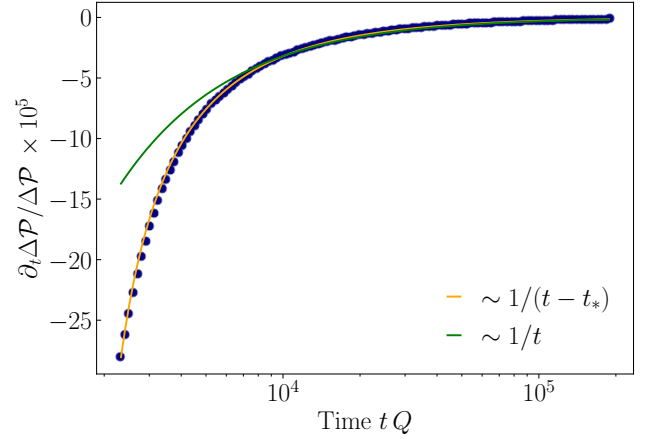


FIG. 2. Evolution of the pressure anisotropy after an anisotropic perturbation has been applied to the isotropic gluonic distribution function in the vicinity of the direct energy cascade. We compare with fixed-point scaling (green) and prescaling (orange), where the amplitude and finite-time scale $t_* Q \simeq 1.3 \times 10^3$ are obtained at an arbitrary reference time.

A. The resulting pressure anisotropy evolution is compared to the power-law behavior of Eq. (29) in Fig. 2. Similar to the evolution of bulk quantities in Fig. 1, the late-time behavior follows fixed-point scaling $\sim 1/t$. When finite-time corrections are included via the non-universal scale t_* , the pressure anisotropy is accurately described even for times close to the perturbation.

V. THE NONRELATIVISTIC DIRECT ENERGY CASCADE

The phenomenology of the direct energy cascade, discussed for relativistic systems in Sec. III, can also be found at lower energies in non-relativistic cold atom systems. Such nonthermal attractors were recently realized in cold atom platforms quenched far from equilib-

rium [20, 25, 28, 72]. While this attractor is also driven by a direct energy transport to smaller distance scales, the underlying quasi-particles are free non-relativistic waves with $z = 2$, i.e., $\omega_{\mathbf{k}} \sim |\mathbf{k}|^2$.

The self-similar dynamics are governed by (perturbative) non-relativistic elastic four-wave interactions, with collision kernel [55]

$$\mathcal{C}_{\text{non-rel}}^{2\leftrightarrow 2}[f] = \frac{1}{2} \int \frac{d^d \mathbf{k}'}{(2\pi)^d} \frac{d^d \mathbf{p}}{(2\pi)^d} \frac{d^d \mathbf{p}'}{(2\pi)^d} W_{\mathbf{k}\mathbf{k}' \leftrightarrow \mathbf{p}\mathbf{p}'; \text{U}(1)}^{\text{non-rel}} \times [(f_{\mathbf{p}} f_{\mathbf{p}'} (f_{\mathbf{k}} + f_{\mathbf{k}'})) - f_{\mathbf{k}} f_{\mathbf{k}'} (f_{\mathbf{p}} + f_{\mathbf{p}'})], \quad (37)$$

and scattering matrix element

$$W_{\mathbf{k}\mathbf{k}' \leftrightarrow \mathbf{p}\mathbf{p}'; \text{U}(1)}^{\text{non-rel}} = 4g^2 (2\pi)^{d+1} \delta^{(1)}(\omega_{\mathbf{k}} + \omega_{\mathbf{k}'} - \omega_{\mathbf{p}} - \omega_{\mathbf{p}'}) \times \delta^{(d)}(\mathbf{k} + \mathbf{k}' - \mathbf{p} - \mathbf{p}'), \quad (38)$$

where g is the non-relativistic quartic coupling. A scaling analysis of this collision kernel reveals the scaling exponents

$$\alpha = (d + z)\beta, \quad \beta = -\frac{1}{3z} = -\frac{1}{6}, \quad (39)$$

with $d = 3$ and $z = 2$ for us.

For general spatial dimension d , this implies the following scaling behavior of the transport coefficients

$$\eta(t) = B^{-6}(t) \bar{\eta} = \left(\frac{t - t_*}{t_{\text{ref}}} \right) \bar{\eta}, \quad (40a)$$

$$\tau_{\pi}(t) = B^{-6}(t) \bar{\tau}_{\pi} = \left(\frac{t - t_*}{t_{\text{ref}}} \right) \bar{\tau}_{\pi}. \quad (40b)$$

Although the β and z exponents disagree between the relativistic and non-relativistic direct energy cascade, the transport coefficients are interestingly observed to exhibit the same scaling: they grow linearly with time. Consequently, the time evolution of the low-energy hydrodynamic excitations in the non-relativistic energy cascade aligns with those found for the relativistic case in Sec. III B.

Since the non-relativistic energy cascade has already been realized in well-controlled cold atom platforms [20, 25, 28, 72], this opens up the possibility of experimentally studying the here proposed hydrodynamic signatures near nonthermal attractors. In particular, the evolution of the pressure anisotropy, as described in Eq. (29) together with Eq. (40), can be explored following an anisotropic perturbation in the atom or spin density. The time evolution of the shear viscosity, with $\eta(t) \sim t$ from Eq. (40a), is also promising, as shear viscosity is commonly measured in cold atom systems near equilibrium [2, 73].

VI. CONCLUSIONS

In the present work, we proposed that long wavelength perturbations of nonthermal fixed points are hydrodynamic in nature. This insight provides a novel extension

of fluid mechanics to phenomena that are intrinsically far from equilibrium in the sense that hydrodynamic relaxation leads not to a local equilibrium, but to a local nonthermal fixed point.

We envision that bridging nonthermal fixed points and hydrodynamics will allow one to adopt ideas and techniques from the latter, venerable field into a genuinely new context of far from equilibrium evolutions. For example, the breakdown of the derivative expansion with time we encountered is reminiscent of the phenomena associated with the radius of convergence of relativistic hydrodynamics in momentum space discussed in [74–76]. It would be very interesting to understand this connection more thoroughly.

Our studies open new avenues for experimental studies of nonthermal fixed points in cold atom experiments, as they predict intricate dynamics associated with local conservation laws when spatial homogeneity is broken.

The results we presented call also for new theoretical developments and analyses. The most pressing one would be going beyond the 14-moment approximation and understanding the hydrodynamic dispersion relations in full kinetic theory models. This can be achieved, e.g., by incorporating momentum dependence in the studies of [31].

Furthermore, it would be very interesting to make a qualitative analysis of the applicability of far from equilibrium hydrodynamics using ab initio simulations with broken homogeneity. Here a good point of departure could be ideas and implementations underlying the KØMPØST framework [77–79] (see also [80]), originally devised to deal with transverse inhomogeneity in nuclear collisions treated as a small perturbation of highly symmetric far from equilibrium boost-invariant dynamics [81].

Finally and in the same context, nuclear collision dynamics gives rise to an anisotropic nonthermal fixed point studied in [14, 43, 65, 82]. It would be very interesting to understand if hydrodynamic perturbations around this nonthermal fixed point can be framed in the language of anisotropic hydrodynamics (aHydro) that was born to deal precisely with such situations [83–85].

ACKNOWLEDGMENTS

We thank Kirill Boguslavski, Aleksi Kurkela, Matisse De Lescluze, and Aleksas Mazeliauskas for useful discussions and comments on the draft. The authors acknowledge support by the state of Baden-Württemberg through bwHPC and the German Research Foundation (DFG) through grant no INST 40/575-1 FUGG (JUSTUS 2 cluster), the DFG under the Collaborative Research Center SFB 1225 ISOQUANT (Project-ID 273811115) and the Heidelberg STRUCTURES Excellence Cluster under Germany's Excellence Strategy EXC2181/1-390900948. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and inno-

vation programme (grant number: 101089093 / project acronym: High-TheQ). Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them. G.S.D. acknowledges support from CNPq and Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPERJ), grant No. E-26/202.747/2018.

We would like to thank for hospitality KITP during the program “The Many Faces of Relativistic Fluid Dynamics” supported by the National Science Foundation under Grant No. NSF PHY-1748958, where this work was initiated, and INFN Galileo Galilei Institute for Theoretical Physics during the program “Foundations and Applications of Relativistic Hydrodynamics”, where it was presented and completed.

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Appendix A: Notation

We define the double symmetric, traceless projection of a tensor as

$$A^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu\alpha\beta} A_{\alpha\beta}, \quad (\text{A1})$$

with

$$\Delta^{\mu\nu\alpha\beta} = \frac{1}{2} \left(\Delta^{\mu\alpha} \Delta^{\nu\beta} + \Delta^{\nu\alpha} \Delta^{\mu\beta} - \frac{2}{3} \Delta^{\mu\nu} \Delta^{\alpha\beta} \right). \quad (\text{A2})$$

We further define the projection tensor onto the d -space orthogonal to the four-velocity,

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu, \quad (\text{A3})$$

in terms of fluid four-velocity u^μ and the Minkowski spacetime metric $g_{\mu\nu}$. Finally, we denote the comoving time derivative as

$$\dot{A} \equiv u^\mu \partial_\mu A \equiv \frac{d}{d\tau} A, \quad (\text{A4})$$

where τ is the proper time along a flow line.

Appendix B: Derivation of Hydrodynamic equations around a nonthermal fixed point

In this Appendix, we discuss the derivation of the hydrodynamic equations of the type (6) around a nonthermal fixed point. The starting point is the relativistic Boltzmann equation,

$$k^\mu \partial_\mu f_{\mathbf{k}} = \mathcal{C}[f], \quad (\text{B1})$$

with $k^\mu = (\omega_{\mathbf{k}}, \mathbf{k})$ being the on-shell four-momentum and $\mathcal{C}[f]$ the collision kernel. The shear stress tensor is determined as the following moment of $f_{\mathbf{k}}$,

$$\pi^{\mu\nu} = \int dK k^{\langle\mu} k^{\nu\rangle} f_{\mathbf{k}}, \quad (\text{B2})$$

where $dK = d^3\mathbf{k}/[(2\pi)^3 \omega_{\mathbf{k}}]$ is the Lorentz-invariant phase volume. We pursue the procedure outlined in [46, 56] and obtain an equation of motion for $\pi^{\mu\nu}$ by directly taking the comoving derivative of (B2),

$$\dot{\pi}^{\langle\mu\nu\rangle} = \Delta_{\alpha\beta}^{\mu\nu} \frac{d}{d\tau} \int dK k^{\langle\alpha} k^{\beta\rangle} f_{\mathbf{k}}, \quad (\text{B3})$$

and using the Boltzmann equation in the form

$$u^\mu \partial_\mu f_{\mathbf{k}} = -\frac{1}{u_\alpha k^\alpha} k^\mu \nabla_\mu f_{\mathbf{k}} + \frac{1}{u_\alpha k^\alpha} \mathcal{C}[f]. \quad (\text{B4})$$

This leads to the exact equation of motion,

$$\dot{\pi}^{\langle\mu\nu\rangle} = \mathcal{C}_{-1}^{\mu\nu} + \frac{8}{15} \mathcal{E} \sigma^{\mu\nu} - 2\pi^{\lambda\langle\mu} \omega_{\lambda}^{\nu\rangle} - \frac{10}{7} \sigma_{\lambda}^{\langle\mu} \pi^{\nu\rangle\lambda} - \frac{4}{3} \pi^{\mu\nu} \theta - \rho_{-2}^{\mu\nu\alpha\beta} \sigma_{\alpha\beta} - \Delta_{\mu_1\mu_2}^{\mu\nu} \nabla_{\mu_3} \rho_{-1}^{\mu_1\mu_2\mu_3}, \quad (\text{B5})$$

where we defined the space-like gradient $\nabla_\mu \equiv \Delta_\mu^\nu \partial_\nu$, the expansion rate as the four-divergence of the four-velocity $\theta = \nabla_\mu u^\mu$, and the fluid vorticity tensor $\omega^{\mu\nu} = (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$. Explicit details for the derivation can be found in [56]. We neglect the contributions related to heat flow, since we mostly focus on the physics of shear viscosity. Eq. (B5) is not closed since it depends on the collision integral $\mathcal{C}_{-1}^{\mu\nu} = \int dK k^{\langle\mu} k^{\nu\rangle} \mathcal{C}[f]/(u_\mu k^\mu)$ as well as on the non-hydrodynamic fields $\rho_{-1}^{\mu\nu\alpha} = \int dK k^{\langle\mu} k^{\nu} k^{\alpha\rangle} f_{\mathbf{k}}/(u_\mu k^\mu)$ and $\rho_{-2}^{\mu\nu\alpha\beta} = \int dK k^{\langle\mu} k^{\nu} k^{\alpha} k^{\beta\rangle} f_{\mathbf{k}}/(u_\mu k^\mu)^2$. The brackets denote the symmetric, traceless projection of the tensors orthogonal to the four-velocity as defined in Ref. [86]. One can close this equation by approximating these quantities, which is traditionally achieved using the 14-moment approximation [50].

As stated in the main text of the paper, such an approach usually commences by expanding the distribution function about some background state,

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}}, \quad (\text{B6})$$

and then deriving an approximate expression for the correction $\delta f_{\mathbf{k}}$. In the 14-moment approximation, the correction $\delta f_{\mathbf{k}}$ is expanded in momentum space to second order in tensors of the particle four-momentum k^μ , i.e., $1, k^\mu, k^\mu k^\nu$, and truncated at second order,

$$\delta f_{\mathbf{k}} \simeq f_{0\mathbf{k}} (\epsilon + k^\mu \epsilon_\mu + k^\mu k^\nu \epsilon_{\mu\nu}), \quad (\text{B7})$$

leading to fourteen unknown expansion coefficients, namely ϵ, ϵ_μ , and $\epsilon_{\mu\nu}$. Without loss of generality, $\epsilon_{\mu\nu}$ is assumed to be traceless $\epsilon^\mu_\mu = 0$ (can be incorporated in ϵ) and symmetric. In the presence of a particle or net-charge current, N^μ , the conserved currents N^μ and $T^{\mu\nu}$ contain a total of 14 independent degrees of freedom that can be directly matched to the expansion coefficients. The scalar coefficient ϵ and the vector coefficient ϵ^μ can be directly related to the bulk viscous pressure and heat flow dissipative contributions, respectively [44, 50, 54], and, thus, will not be relevant in this analysis. On the other hand, the tensor coefficient $\epsilon^{\mu\nu}$ is directly matched to the shear stress tensor of the fluid, $\pi^{\mu\nu}$, and contains the dominant contribution to the hydrodynamic equations. This matching procedure can be solved directly following the procedure outlined in [46, 86, 87], which can be applied without *any* modifications even if the background distribution $f_{0\mathbf{k}}$ is not the local equilibrium distribution function – the fundamental assumption employed in this method is that $f_{0\mathbf{k}}$ must be isotropic. Thus, for any isotropic $f_{0\mathbf{k}}$, one can demonstrate that

$$\delta f_{\mathbf{k}} = f_{0\mathbf{k}} \phi_{\mathbf{k}} = f_{0\mathbf{k}} \frac{2p^\mu p^\nu}{15N_4(t)} \pi_{\mu\nu}(t), \quad (\text{B8})$$

with $N_4(t) = \int dK |\mathbf{k}|^4 f_{0\mathbf{k}}$. We note that the inclusion of additional terms in the moment expansion $\sim k^\mu k^\nu k^\rho, \dots$ will increase the precision, but their contribution is suppressed in thermal hydrodynamics as small momentum scales dominate the physics due to the underlying separation of scales.

Considering this more general application of the method of moments, one can generalize the derivation of fluid dynamics proposed by Israel and Stewart to a more general class of systems that are not necessarily close to equilibrium. In this work, we perform this task for systems that are in the vicinity of a nonthermal attractor, which arises in weakly-coupled systems that are over-occupied, $f_{\mathbf{k}} \gg 1$. The dynamics are usually dominated by elastic two-to-two collisions, which can be captured by a collision term of the form [55]

$$C[f] = \frac{1}{2} \int dK' dP dP' W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'} I[f](\mathbf{k}, \mathbf{k}', \mathbf{p}, \mathbf{p}'), \quad (\text{B9})$$

where $W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'}$ is the Lorentz-invariant transition rate. For an over-occupied system, the gain-loss contribution can be approximated by its classical-wave limit

$$I[f](\mathbf{k}, \mathbf{k}', \mathbf{p}, \mathbf{p}') = f_{\mathbf{p}} f_{\mathbf{p}'} [1 + f_{\mathbf{k}}][1 + f_{\mathbf{k}'}] - f_{\mathbf{k}} f_{\mathbf{k}'} [1 + f_{\mathbf{p}}][1 + f_{\mathbf{p}'}] \quad (\text{B10})$$

$$\approx f_{\mathbf{p}} f_{\mathbf{p}'} (f_{\mathbf{k}} + f_{\mathbf{k}'} - f_{\mathbf{k}} f_{\mathbf{k}'} (f_{\mathbf{p}} + f_{\mathbf{p}'})). \quad (\text{B11})$$

As stated in the main text, we focus on nonthermal attractors associated with the turbulent transport of mode energy to smaller distance scales through a direct energy cascade, described by weakly-interacting relativistic massless scalars with a transition element in the $O(N)$ model [55], see Eq. (11) in the main text of the paper.

In a next step, we plug the expansion of Eq. (B6) together with the explicit perturbation of Eq. (B8) into the right-hand side of Eq. (B5), which will allow us to close the equation as the higher-order collision kernel and hydrodynamic fields will be expressed in terms of $\pi^{\mu\nu}$. The collision kernel contribution $\mathcal{C}_{-1}^{\mu\nu}$ linearized in the perturbation is then expressed in terms of $\pi^{\mu\nu}$ as

$$\mathcal{C}_{-1}^{\mu\nu} = -\delta\mathcal{C} \frac{15}{2N_4(t)} \pi^{\mu\nu}, \quad (\text{B12})$$

with the linearized collision kernel being defined as

$$\begin{aligned} \delta\mathcal{C} = & \int dK dK' dP dP' \frac{1}{u_\mu k^\mu} k_{\langle\alpha} k_{\beta\rangle} W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'} \left[-\frac{1}{10} f_{0\mathbf{p}} f_{0\mathbf{p}'} f_{0\mathbf{k}} \left(k^{\langle\alpha} k^{\beta\rangle} + 2p^{\langle\alpha} p^{\beta\rangle} \right) \right. \\ & \left. -\frac{1}{10} f_{0\mathbf{p}} f_{0\mathbf{p}'} f_{0\mathbf{k}'} \left(k'^{\langle\alpha} k'^{\beta\rangle} + 2p^{\langle\alpha} p^{\beta\rangle} \right) + \frac{1}{5} f_{0\mathbf{k}} f_{0\mathbf{k}'} f_{0\mathbf{p}} \left(k^{\langle\alpha} k^{\beta\rangle} + k'^{\langle\alpha} k'^{\beta\rangle} + p^{\langle\alpha} p^{\beta\rangle} \right) \right]. \end{aligned} \quad (\text{B13})$$

We note that since $\mathcal{C}_{-1}^{\mu\nu}$ is a symmetric, traceless rank-2 tensor orthogonal to the four-velocity field, it must vanish when integrated with any isotropic momentum distribution function, even though the collision term itself does not vanish in such a non-equilibrium state. Thus, the leading contribution to this collision integral will indeed be the dissipative contribution shown above, that has the same structure of the collision integral calculated around local equilibrium. Nevertheless, the coefficient $\sim \delta\mathcal{C}$ contains information about the nonthermal attractor of the background state and acts as a generalized transport coefficient.

The evolution equation of the hydrodynamic shear stress follows then directly from Eq. (B5)

$$\tau_\pi(t)\dot{\pi}^{\langle\mu\nu\rangle} = -\pi^{\mu\nu} + 2\eta(t)\sigma^{\mu\nu} + \tau_\pi(t) \left[-\frac{4}{3}\pi^{\mu\nu}\theta - \frac{10}{7}\pi^{\lambda\langle\mu}\sigma_{\lambda}^{\nu\rangle} - 2\pi^{\lambda\langle\mu}\omega_{\lambda}^{\nu\rangle} \right], \quad (\text{B14})$$

where we identified the shear viscosity and relaxation time transport coefficient by direct comparison with the thermal case (6),

$$\eta(t) = \frac{8}{15^2} \frac{\mathcal{E}N_4(t)}{\delta\mathcal{C}(t)}, \quad (\text{B15a})$$

$$\tau_\pi(t) = \frac{2}{15} \frac{N_4(t)}{\delta\mathcal{C}(t)}. \quad (\text{B15b})$$

Appendix C: Scaling analysis

In this Appendix, we analyse the scaling of bulk observables under the scaling ansatz (1). We use the notation $\bar{\mathbf{p}} = B(t)\mathbf{p}$ and the scaling relation $A(t) = B^\sigma(t)$ with $\sigma = d + z$ for energy number conservation.

The energy and fourth moment of the distribution function follow as

$$\mathcal{E}(t) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \omega_{\mathbf{p}} f(t, \mathbf{p}) = A(t)B^{-(d+z)}(t) \int \frac{d^3\bar{\mathbf{p}}}{(2\pi)^3} \omega_{\bar{\mathbf{p}}} f_S(\bar{\mathbf{p}}) \equiv B^{\sigma-(d+z)}(t) \bar{\mathcal{E}}, \quad (\text{C1a})$$

$$N_4(t) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 \omega_{\mathbf{p}}} |\mathbf{p}|^4 f(t, \mathbf{p}) = A(t)B^{-(d+4-z)}(t) \int \frac{d^3\bar{\mathbf{p}}}{(2\pi)^3 \omega_{\bar{\mathbf{p}}}} |\bar{\mathbf{p}}|^4 f_S(\bar{\mathbf{p}}) \equiv B^{\sigma-(d+4-z)}(t) \bar{N}_4, \quad (\text{C1b})$$

in the vicinity of the fixed point. To derive the scaling of transport coefficients from Eq. (19), we need to analyze the time evolution of the linearized collision kernel (B13). We will discuss this scaling analysis for the relativistic and non-relativistic energy cascade in turn.

1. Relativistic energy cascade

We consider first the relativistic energy cascade with the application of QCD kinetic theory in Sec. IV in mind. Starting from Eq. (B13), we evaluate the expressions as

$$\begin{aligned} \delta\mathcal{C}^{\text{rel,UV}}(t) = & \int \frac{d^3\mathbf{k}}{(2\pi)^3 \omega_{\mathbf{k}}} \frac{d^3\mathbf{k}'}{(2\pi)^3 \omega_{\mathbf{k}'}} \frac{d^3\mathbf{p}}{(2\pi)^3 \omega_{\mathbf{p}}} \frac{d^3\mathbf{p}'}{(2\pi)^3 \omega_{\mathbf{p}'}} \frac{1}{\omega_{\mathbf{k}}} \mathbf{k}^2 W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'} \left[-\frac{1}{10} f_{0\mathbf{p}} f_{0\mathbf{p}'} f_{0\mathbf{k}} \left(\mathbf{k}^2 \frac{2}{3} + 2\mathbf{p}^2 (\mu_{\mathbf{k}\mathbf{p}}^2 - 1/3) \right) \right. \\ & - \frac{1}{10} f_{0\mathbf{p}} f_{0\mathbf{p}'} f_{0\mathbf{k}'} (\mathbf{k}'^2 (\mu_{\mathbf{k}\mathbf{k}'}^2 - 1/3) + 2\mathbf{p}^2 (\mu_{\mathbf{k}\mathbf{p}}^2 - 1/3)) \\ & \left. + \frac{1}{5} f_{0\mathbf{k}} f_{0\mathbf{k}'} f_{0\mathbf{p}} \left(\frac{2}{3} \mathbf{k}^2 + \mathbf{k}'^2 (\mu_{\mathbf{k},\mathbf{k}'}^2 - 1/3) + \mathbf{p}^2 (\mu_{\mathbf{k}\mathbf{p}}^2 - 1/3) \right) \right], \end{aligned} \quad (\text{C2})$$

where we used the tensor contractions

$$k_{\langle\alpha} k_{\beta\rangle} p^{\langle\alpha} p^{\beta\rangle} = \Delta_{\alpha\beta\mu\nu} k^\mu k^\nu \Delta^{\alpha\beta\rho\sigma} p_\rho p_\sigma = \Delta_{\mu\nu}^{\rho\sigma} k^\mu k^\nu p_\rho p_\sigma \quad (\text{C3})$$

$$= \frac{1}{2} \left[2(k \cdot p - \omega_{\mathbf{k}} \omega_{\mathbf{p}})^2 - \frac{2}{3} (k^2 - \omega_{\mathbf{k}}^2) (p^2 - \omega_{\mathbf{p}}^2) \right] \quad (\text{C4})$$

$$= \mathbf{k}^2 \mathbf{p}^2 \left(\mu_{\mathbf{k}\mathbf{p}}^2 - \frac{1}{3} \right), \quad (\text{C5})$$

together with

$$k^2 - \omega_{\mathbf{k}}^2 = \omega_{\mathbf{k}}^2 - \mathbf{k}^2 - \omega_{\mathbf{k}}^2 = -\mathbf{k}^2, \quad (\text{C6a})$$

$$k \cdot p - \omega_{\mathbf{k}} \omega_{\mathbf{p}} = -\mathbf{k} \cdot \mathbf{p} = -|\mathbf{k}| |\mathbf{p}| \mu_{\mathbf{k}\mathbf{p}}, \quad (\text{C6b})$$

$$k_{\langle\alpha} k_{\beta\rangle} k^{\langle\alpha} k^{\beta\rangle} = \frac{2}{3} \mathbf{k}^4, \quad (\text{C6c})$$

$$k_{\langle\alpha} k_{\beta\rangle} k'^{\langle\alpha} k'^{\beta\rangle} = \mathbf{k}^2 \mathbf{k}'^2 \left(\mu_{\mathbf{k},\mathbf{k}'}^2 - \frac{1}{3} \right), \quad (\text{C6d})$$

$\mu_{\mathbf{k}\mathbf{p}} \equiv \cos(\theta_{\mathbf{k},\mathbf{p}})$, and $\theta \triangleleft (\mathbf{k}, \mathbf{p})$. The transition matrix element for the energy cascade is generally (for the relativistic and the non-relativistic energy cascade) given by

$$W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'} = |\mathcal{M}|_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'}^2 (2\pi)^{d+1} \delta^{(1)}(\omega_{\mathbf{k}} + \omega_{\mathbf{k}'} - \omega_{\mathbf{p}} - \omega_{\mathbf{p}'}) \delta^{(d)}(\mathbf{k} + \mathbf{k}' - \mathbf{p} - \mathbf{p}'), \quad (\text{C7})$$

with theory-specific scattering matrix element $|\mathcal{M}|^2$. For the relativistic $O(N)$ model and for QCD, we have

$$|\mathcal{M}_{O(N)}|^2 = \lambda^2 \frac{N+2}{6N^2}, \quad (\text{C8a})$$

$$|\mathcal{M}_{gg}^{gg}|^2(k, k', p, p') = 16(N_c^2 - 1)N_c^2 g^4 \left(3 - \frac{s_M t_M}{u_M^2} - \frac{s_M u_M}{t_M^2} - \frac{t_M u_M}{s_M^2} \right), \quad (\text{C8b})$$

where N_c denotes the number of colors and u_M, s_M, t_M the usual Mandelstam variables. Here we only give the vacuum expression for the $2 \leftrightarrow 2$ gluon scattering matrix element averaged over spin and color degrees of freedom. In practice, $|\mathcal{M}_{gg}^{gg}|$ has an infrared divergence which is regulated by the medium Debye mass. We will neglect the corresponding sub-leading logarithmic correction, see [47, 88], for the purposes of the scaling analysis. The leading order QCD collision kernel receives, besides $\mathcal{C}^{2 \leftrightarrow 2}$, a contribution from effective $1 \leftrightarrow 2$ scatterings, which we will also drop in the scaling analysis as it scales like the $2 \leftrightarrow 2$ kernel, see [47]. Parametrizing the scaling behavior of the collision kernel as

$$\mathcal{C}[f](t, \mathbf{p}) = A^{\mu_\alpha}(t) B^{\mu_\beta}(t) \mathcal{C}[f_S](\bar{\mathbf{p}}) = B^{\sigma\mu_\alpha + \mu_\beta}(t) \mathcal{C}[f_S](\bar{\mathbf{p}}), \quad (\text{C9})$$

where $1/\beta = (1 - \mu_\alpha)\sigma - \mu_\beta$, for both collision kernels we identify $\mu_\alpha = 3$ and $\mu_\beta = -2d + 5z$. The scaling of the linearized collision kernel contribution for the relativistic energy cascade can then be written as

$$\delta \mathcal{C}^{\text{rel,UV}}(t) \equiv B^{3\sigma - 4d + 5z - 4 + (d+z)}(t) \delta \bar{\mathcal{C}}^{\text{rel,UV}}(t) = B^{-1/\beta + 4z - 6}(t) \delta \bar{\mathcal{C}}_{-1, \text{UV}} \quad (\text{C10})$$

$$= B^5(t) \delta \bar{\mathcal{C}}^{\text{rel,UV}} \quad (\text{C11})$$

$$= \left(\frac{t - t_*}{t_{\text{ref}}} \right)^{-\frac{5}{7}} \delta \bar{\mathcal{C}}^{\text{rel,UV}}. \quad (\text{C12})$$

Altogether, using Eq. (19), this implies for the transport coefficients that

$$\eta^{\text{rel,UV}}(t) = \frac{8}{15^2} \frac{B^{\sigma - (d+z)}(t) \bar{\mathcal{E}} B^{\sigma - (d+4-z)}(t) \bar{N}_4}{B^{9z-4}(t) \delta \bar{\mathcal{C}}^{\text{rel,UV}}(t)} \equiv B^{-7z}(t) \bar{\eta}^{\text{rel,UV}} \quad (\text{C13})$$

$$= B^{2(1-z) + 1/\beta}(t) \bar{\eta}^{\text{rel,UV}} \quad (\text{C14})$$

$$\tau_\pi^{\text{rel,UV}}(t) = \frac{2}{15} \frac{B^{\sigma - (d+4-z)}(t) \bar{N}^4}{B^{9z-4}(t) \delta \bar{\mathcal{C}}^{\text{rel,UV}}(t)} \equiv B^{-7z}(t) \bar{\tau}_\pi^{\text{rel,UV}} \quad (\text{C15})$$

$$= B^{2(1-z) + 1/\beta}(t) \bar{\tau}_\pi^{\text{rel,UV}}. \quad (\text{C16})$$

This reproduces the scaling in Eq. (26) for $z = 1$ and $\beta = -1/7$.

2. Non-relativistic energy cascade

The kinetic description of a non-relativistic system differs to that of a relativistic one in the appearance of $\omega_{\mathbf{k}}$ factors in the Boltzmann equation and in the momentum integration measures, see Eq. (37) and Eq. (38). We can then

obtain the non-relativistic energy cascade scaling from the corresponding relativistic expressions by dropping the $\omega_{\mathbf{k}}$ factors

$$N_4(t) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathbf{p}^4 f(t, \mathbf{p}) = A(t) B^{-(d+4)}(t) \int \frac{d^3\bar{\mathbf{p}}}{(2\pi)^3} \bar{p}^4 f_S(\bar{\mathbf{p}}) \equiv B^{\sigma-(d+4)}(t) \bar{N}_4, \quad (\text{C17a})$$

$$\delta C^{\text{non-rel,UV}}(t) = B^{-4d+3\sigma-4+d+z}(t) \delta \bar{C}^{\text{non-rel,UV}} = B^{4(z-1)}(t) \delta \bar{C}^{\text{non-rel,UV}}, \quad (\text{C17b})$$

where we used that the non-relativistic collision kernel scales with $\mu_\alpha = 3$ and $\mu_\beta = d + z - 3d$. This implies the following scaling of transport coefficients

$$\eta^{\text{non-rel,UV}}(t) = B^{\sigma-(d+z)+\sigma-(d+4)-(4z-4)}(t) \bar{\eta}^{\text{non-rel,UV}} = B^{-3z}(t) \bar{\eta}^{\text{non-rel,UV}}, \quad (\text{C18a})$$

$$\tau_\pi^{\text{non-rel,UV}}(t) = B^{\sigma-(d+4)-(4z-4)}(t) \bar{\tau}_\pi^{\text{non-rel,UV}} = B^{-3z}(t) \bar{\tau}_\pi^{\text{non-rel,UV}}, \quad (\text{C18b})$$

which reproduces Eq. (40) with $z = 2$ and $\beta = -1/6$.

4.6 Conclusions

In this chapter, we examined how and why nonthermal attractor behavior arises in the unitary dynamics of isolated quantum many-body systems far from equilibrium. The following sections summarize the main conclusions and outline potential directions for future research involving nonthermal attractors. Possible experimental signatures – such as the scaling instability and the tower of far-from-equilibrium quasiparticles discussed in Sec. 4.2, prescaling breaking effects from Sec. 4.3, and far-from-equilibrium hydrodynamic properties such as shear viscosity from Sec. 4.5 – are considered in the context of ultracold atom systems in Ch. 5.

4.6.1 Stability properties of nonthermal attractors

Towards addressing the question of why, in Sec. 4.2 we studied the dynamics of perturbations around nonthermal fixed points. These universal scaling phenomena far from equilibrium have been studied across a wide range of initial conditions, system parameters and energy scales. This is a testament to their attractor features as a quasi-stationary state evolved to through an effective loss of sensitivity on initial conditions. The observed self-organized scaling is in stark contrast to the presence of both stable and unstable perturbations around universal scaling detected in [1]. Far from equilibrium linear response theory demonstrates that the unstable dynamics arises from a competition between elastic scattering processes among different emergent quasi-particle states, which are absent in thermal equilibrium. The resolving explanation is provided by the scaling background itself: the scaling instability renders the nonthermal fixed point dynamically attractive after $t \sim 1/|\mathbf{p}|^{1/\beta}$. This dynamical mechanism demonstrates how attractor behavior can be realized from unitary dynamics in the presence of unstable directions without the necessity for fine-tuning. This further bears a stark resemblance to the self-organization observed in complex system dynamics, compare Sec. 4.1.2.

The scaling instability provides for a possible mechanism underlying the abundant experimental detection of universal scaling in ultra-cold quantum gases in the many-body regime [19–21, 108–117]. Although the results of Sec. 4.2 have been obtained for relativistic Bose fields, the universal infrared properties, such as scaling exponents and the self-similar scaling form of the response function, are expected to be observable also in nonrelativistic theories due to the underlying common universality class [80]. On the theoretical side, extending the study of Sec. 4.2 to nonrelativistic systems includes adjusting the initial conditions to realistic experimental scenarios and extrapolating the results obtained at NLO in large- N to small N . Stability properties in the approach to the fixed point can be studied in nonrelativistic theories using classical-statistical simulations [169–171] (or truncated Wigner approximations [172, 173]), which will also give insight into the role of topological configurations appearing for small $N < 3$. While scaling exponents and scaling functions are expected to be universal across relativistic and nonrelativistic realizations of the infrared fixed point, the emergent quasi-particle pictures might differ. This poses

an interesting question to the underlying effective low-energy theory of the infrared fixed point.

Finally, the time-dependent stability properties not only encode how the system approaches the universal scaling behavior, but also how it deviates from it toward a subsequent thermalization process. The dynamical mechanism underlying the deviation from universal scaling remains an important open question, which might be related to the predicted unstable modes or to an increase in dominance of inelastic scatterings with an associated decrease in particle occupations. The real-time quantum evolution equations (4.29) employed in Ch. 4 provide an important tool to investigate the corresponding late-time behavior, as the included memory integrals are crucial to accurately capture power-law scaling solution. Connecting the approach to self-organized nonthermal attractors to the subsequent deviation towards thermal equilibrium will provide an important step into our understanding of the emergence of (nonthermal or thermal) attractor behavior in unitary dynamics and the associated effective loss of sensitivity on the initial conditions.

4.6.2 Prescaling relaxation to nonthermal attractors

Turning to the question of as to how, in Sec. 4.3 we investigated the approach to nonthermal fixed points. In its basin of attraction, particle distribution functions were observed in QCD kinetic theory to exhibit the scaling form already at early times, albeit with time-dependent scaling exponents, a phenomenon coined prescaling [36]. In Sec. 4.3, we derived the general analytical solution for the time-dependent scaling exponents, or equivalently the prescaling factors, in isotropic systems. These are given by a resummation of power-law contributions that are subleading to the fixed-point behavior at late times. This solution reveals that prescaling can be understood as a simple shift in time of fixed-point scaling (4.1), where the time shift is determined by non-universal scales $\sim t_*$ accounting for the distance of the system under consideration from the nonthermal attractor. As a result, prescaling effects remain relevant in any finite-time observation, since universal fixed-point scaling only emerges asymptotically. We demonstrated that the prescaling solution provides a quantitative description of the approach to universal scaling at nonthermal fixed points relevant for cold atom systems and early-universe cosmology. This prediction was recently confirmed experimentally in [21], where accounting for these finite-time prescaling effects led to agreement between extracted and theoretically predicted scaling exponents for both inverse and direct cascades in an ultracold atomic gas in $2 + 1$ dimensions.

We also derived the corresponding prescaling solution for the quark-gluon plasma in an isotropic geometry. Using kinetic theory in a small-angle approximation, we showed that the approach to the fixed point involves logarithmic corrections that dominate at late times. These arise from a medium-dependent regulator in the elastic gluon collision kernel. This highlights a competition between different time scales in the evolution toward nonthermal fixed points. One can further differentiate between the power-law or logarithmic prescaling relaxation; the exponential (power-law) corrections detected in the stability analysis of the infrared (direct) cascade in Sec. 4.2 (in [174]); and the exponential as well as sub-exponential decay following hydrodynamic perturbations, as observed in Sec. 4.5.

Understanding how these competing time scales of perturbations around nonthermal attractors interact provides a promising future direction for improving our knowledge of how nonthermal attractors emerge in quantum many-body systems.

Beyond isotropic geometries, nonthermal (and hydrodynamic) attractors play a central role in connecting first-principles descriptions of thermalization with emergent effective theories in heavy-ion collisions, as discussed in Sec. 4.1.1. In particular, the nonthermal attractor is relevant for the early-time, weak-coupling evolution of the quark-gluon plasma. In Sec. B.2, we presented novel results on prescaling in longitudinally expanding systems, extending the analysis of QCD kinetic theory from isotropic systems (Sec. 4.3) to Bjorken flow. Owing to the strong momentum anisotropy in this setting, we were able to derive an explicit analytical solution for the prescaling factors. This solution incorporates a resummation of both power-law and logarithmic corrections and aligns with QCD kinetic theory simulations in a longitudinally expanding background. Nonthermal attractors are relevant to the first stage of the bottom-up thermalization scenario. The derived prescaling solution allows one to quantify the value of scaling exponents for more realistic moderate coupling values, where time-dependent exponents still show percent-level deviations from their asymptotic bottom-up fixed-point values. As in the case of ultracold atom experiments [21], including prescaling corrections significantly improves the quantitative accuracy of the scaling description.

As the coupling strength grows, the system increasingly deviates from the fixed point and begins to thermalize. The combination of isotropization and expansion in the following last stage of the bottom-up thermalization scenario leads to a low-occupancy regime, where deviations from universal scaling are substantial and phenomenological predictions using hydrodynamic attractors become relevant. The nonthermal attractor in the early-time, weak-coupling regime at high occupancies is conventionally distinguished from the late-time hydrodynamic attractor at lower occupancies and stronger couplings. The quark-gluon-plasma created in heavy-ion collisions is however expected to be neither extremely weakly nor extremely strongly interacting, which renders addressing the relation between nonthermal and hydrodynamic attractors of particular relevance for phenomenological questions. The transseries structure and competition of different time-scales discovered in Sec. 4.3 provides a stark resemblance to the analysis of hydrodynamic attractors investigated in related holographic theories. This motivates investigations into their relation, where additionally the derived formalisms of prescaling (Sec. 4.3), stability analysis (Sec. 4.2) and far-from-equilibrium hydrodynamics (Sec. 4.5) may provide valuable tools for understanding the transition from nonthermal to hydrodynamic behavior.

4.6.3 Prescaling in gauge-invariant correlation functions

Regarding the question of ubiquity and universality of nonthermal attractors, universal far-from-equilibrium scaling has also been observed for gauge fields in gauge-fixed [12, 36, 86–92, 94, 97, 99, 122, 123, 175] and gauge-invariant correlation functions [95, 96, 98]. In particular, [98] studied fixed-point scaling using gauge-invariant correlation functions of spatial Polyakov loops. While [98] focused on the one-point function and the zero-mode of the two-point function to study gauge-invariant condensation, in Sec. 4.4 we extended

that study to non-vanishing momentum scales above the electric screening (Debye mass) scale and considered prescaling effects. Within the investigated momentum regime, we found that the gauge-invariant spatial Polyakov loop two-point function exhibits prescaling behavior characterized by the universal scaling exponents of the direct energy cascade, when accounting for prescaling effects using the framework developed in Sec. 4.3. This agrees with observations in gauge-fixed distribution functions of the fundamental gauge fields based on perturbative estimates applicable in this high-momentum regime. The detected prescaling is well-described by power-law prescaling. Longer simulation times will be necessary to account for possible logarithmic corrections, which were observed in long-time QCD kinetic theory simulations in Sec. 4.3. We further compared results for $N_c = 2$ and $N_c = 3$, demonstrating another aspect of universality if one accounts for N_c -dependent prefactors in the gauge-invariant spatial Polyakov loop correlation functions.

The study of such local gauge-invariant two-point correlation functions is crucial to investigate the non-perturbative infrared behavior of gauge theories far from equilibrium. Gauge-invariant condensation has been observed in one-point functions and zero-modes of the two-point functions of Wilson and Polyakov loops [5, 95, 96, 98], but the underlying mechanism remains to be investigated. For scalar theories, condensation far from equilibrium is driven by an inverse particle cascade [80]. Such a dynamical mechanism has not been observed in Sec. 4.4, where further studies of momentum scales below the electric screening scale will however be crucial. The associated thermalisation dynamics below this Debye mass scale remains an active topic of investigation. Important examples include the value of scaling exponents which depend on the treatment of infrared effects [36, 122, 176], or the effect of collective infrared phenomena such as pre-equilibrium condensation. The latter effect could be related to the phenomenologically observed deviations in low-momentum (pion) yields between data and state-of-the-art model predictions [177, 178]. Alongside extending the study of Sec. 4.4 to momentum scales below the Debye mass, addressing this question then also involves the development of an effective low-energy description of the infrared in terms of gauge-invariant observables and possible condensates or order parameter fields. A possible starting point has been discussed in [98] and similar frameworks are well-understood in thermal equilibrium [18, 179].

Such studies also open up the possibility of studying infrared perturbations of gauge fields far from equilibrium, where we observed an intriguing (in-)stability structure with a tower of emergent excitations in scalar fields in Sec. 4.2.

4.6.4 Hydrodynamics fluctuations of nonthermal attractors

The prescaling framework discussed in Sec. 4.3 and Sec. 4.4 provides insight into how systems approach their corresponding nonthermal attractors under dynamics that preserve the symmetries of the attractor. Similarly, the stability analysis of nonthermal attractors in Sec. 4.2 considered perturbations that were symmetry-preserving. This raises a natural and previously unexplored question: how do perturbations that break the underlying symmetries of a nonthermal attractor evolve with respect to the scaling background? We addressed this question in Sec. 4.5 by studying the evolution of spatial inhomogeneities and anisotropies around otherwise homogeneous and isotropic nonthermal attractors.

The dynamics of such perturbations are described within a novel framework of far-from-equilibrium hydrodynamics around nonthermal attractors, which involves generalizations of transport coefficients, such as shear viscosity. While hydrodynamic descriptions near thermal equilibrium rely on a separation of scales [180], where large-distance and long-time fluctuations dominate, a similar albeit dynamical separation of scales also emerges near nonthermal attractors [12, 80, 95, 118]. However, the space-time scaling in these systems requires generalizing the hydrodynamic expansion to account for the time-dependent scaling background.

In Sec. 4.5, we observed hydrodynamic excitations arising from the evolution of spatial inhomogeneities around nonthermal fixed points. Unlike hydrodynamic expansions around thermal equilibrium, the flow in such systems is characterized by time-dependent transport coefficients, including a far-from-equilibrium generalization of shear viscosity. The time-dependent background leads consequently to a modification of hydrodynamic relaxation dynamics as familiar from expansions around thermal equilibrium. For example, anisotropic perturbations exhibit sub-exponential isotropization and hydrodynamic shear diffuses faster around the scaling background. Although our hydrodynamic framework was developed for kinetic systems with binary collisions, we also demonstrated its ability to accurately capture the evolution of pressure anisotropy following anisotropic perturbations of the isotropic energy cascade in QCD kinetic theory involving also inelastic scatterings, highlighting its versatility.

The success of the 14-moment expansion in describing spatial inhomogeneities around the nonthermal fixed point raises important questions. Notably, why do higher moments of the distribution function carry redundant information? The inclusion of up to second moments of the distribution function with $\delta f \sim \pi^{\mu\nu}$ in Sec. 4.5 was already sufficient for a successful description of hydrodynamic excitations in QCD kinetic theory, similar to expansions around thermal equilibrium where higher moments are suppressed [181]. The 14-moment expansion around the time-dependent scaling background further has a dynamical radius of convergence due to the time-evolving characteristic length scales, in contrast to typical expansions around thermal equilibrium. It remains to be investigated whether higher k^μ moments, such as $k^\mu k^\nu k^\rho$, will have to be included for a successful description of the direct energy cascade at asymptotically late times. Since nonthermal attractors are transient scaling phenomena with finite lifetimes, we expect our 14-moment description to already capture the evolution of hydrodynamic signatures around nonthermal fixed points in cold atom systems with sufficient spatial volume.

Building on this, we anticipate that the 14-moment expansion will provide an increasingly accurate description of hydrodynamic evolution around the inverse particle cascade, whose stability properties and prescaling evolution we explored in Sec. 4.2 and Sec. 4.3, respectively. This attractor has been realized extensively in cold atom experiments [19–21, 108–116], where particles are transported to larger distances with $\alpha, \beta > 0$, leading to the formation of a far-from-equilibrium quasi-condensate [80, 100]. In this system, which is dilute but strongly correlated in the infrared with $f \sim 1/\lambda \gg 1$, the kinetic description requires an infinite resummation of perturbative diagrams. The resulting kinetic theory [79, 80, 102, 104] includes a momentum-dependent vertex, with the corresponding fixed point

characterized by $\beta \simeq 1/2$ and $\alpha = d\beta$. The framework developed in Sec. 4.5 serves as an ideal starting point for investigating hydrodynamic excitations around the inverse particle cascade. We anticipate the accuracy of the 14-moment expansion to improve over time as typical momenta scale to zero, with $|\mathbf{p}| \sim t^{-\beta}$. The build-up of a quasi-condensate, coupled with strong correlations in the system, suggests that shear viscosity will be small, though it may be time-dependent. Notably, the dynamics of a Bose-Einstein condensate in thermal equilibrium, with negligible quantum depletion, is described by the incompressible Euler equation – equivalent to a superfluid with vanishing shear viscosity [182].

Analyzing transport coefficients within the large- N kinetic theory underlying the successful description of the inverse particle cascade is also of interest, as the effective scattering vertex is momentum-dependent and transport coefficients are typically determined from leading-order kinetic theory without a running coupling [183]. Furthermore, the linear response investigations of Sec. 4.2 around the inverse particle cascade could enable the definition of far-from-equilibrium Kubo relations, which would provide a complementary approach for defining transport coefficients in this regime, based on correlation functions of the underlying quantum field theory [183].

Another nonthermal attractor of interest is the anisotropic bottom-up nonthermal attractor, relevant to the early stages of heavy-ion collisions, discussed in Sec. 4.1.1 and App. B.2. Extending our framework to incorporate the underlying longitudinal expansion associated with Bjorken flow would provide insight into the evolution of spatial inhomogeneities during the early stages of quark-gluon plasma evolution. Understanding the relationship between this description and anisotropic hydrodynamics [184–189] would further enrich our understanding of the hydrodynamic description of the early stages of heavy-ion collisions. We emphasize, in this context, that the framework of Sec. 4.5 has to be distinguished from the hydrodynamic attractor, which in contrast extends the applicability of near-equilibrium hydrodynamics through a resummation of the gradient expansion [130, 131]. Furthermore, applying the hydrodynamic framework of Sec. 4.5 to the bottom-up nonthermal attractor provides a complementary perspective on related prescaling phenomena, see App. B.2 and [36]. Slow corrections to bottom-up fixed-point scaling have been interpreted using an effective hydrodynamic theory [36], where the time-dependent scaling exponents serve as hydrodynamic degrees of freedom around the anisotropic scaling background. These are however restricted to Bjorken flow anisotropies, whereas the hydrodynamic framework of Sec. 4.5 allows for a general study of the dynamics of spatial inhomogeneities around the scaling background in terms of hydrodynamic degrees of freedom.

A Appendix for Stable and Unstable Perturbations in Universal Scaling Phenomena Far from Equilibrium

In this Appendix, we provide supplementary calculations and analysis on the work presented in Sec. 4.2. In Sec. A.1, we derive the out-of-equilibrium linear response rate from first principles and from kinetic theory, which will provide insight into the interpretation in terms of scattering processes alluded to in Sec. 4.2. The scaling of this rate and the corresponding rate integral is derived subsequently in Sec. A.2.

A.1 Analytical derivation of the response rate

In this section, we want to derive the time-dependent linear response rate appearing in Eq. (9) and Eq. (12) of Sec. 4.2. We will present two complementary derivations.

Starting from the transport equations

As described below Eq. (10) of Sec. 4.2, one can obtain the linear response rate from the time evolution of the connected two-point correlation functions in a slowly evolving regime from Eq. (4.29). Introducing center and relative coordinates as $2X = x + y$ and $s = x - y$, we consider the real-valued Wigner transformed two-point functions

$$F(X, p) = \int_{-2\tau}^{2\tau} d\Delta t \int d^3\mathbf{s} e^{i(p^0 \Delta t - \mathbf{s} \cdot \mathbf{p})} F\left(X + \frac{s}{2}, X - \frac{s}{2}\right), \quad (\text{A.1a})$$

$$\tilde{\rho}(X, p) = -i \int_{-2\tau}^{2\tau} d\Delta t \int d^3\mathbf{s} e^{i(p^0 \Delta t - \mathbf{s} \cdot \mathbf{p})} \rho\left(X + \frac{s}{2}, X - \frac{s}{2}\right), \quad (\text{A.1b})$$

with $X^0 = \tau$ and $s^0 = \Delta t$. Using an expansion in center gradients to lowest order $\sim \mathcal{O}(\partial_X)$, by considering differences of the equations of motion (4.29) one can derive the following transport equations [79]

$$2p^0 \partial_\tau F(\tau, p) = \tilde{\Sigma}^\rho(\tau, p) F(\tau, p) - \Sigma^F(\tau, p) \tilde{\rho}(\tau, p), \quad (\text{A.2a})$$

$$2p^0 \partial_\tau \tilde{\rho}(\tau, p) = 0, \quad (\text{A.2b})$$

where we restricted to systems with spatial homogeneity for simplicity. The spectral function in particular becomes central time-independent in this slowly evolving regime.

We are now interested in studying perturbations around a scaling solution and make the ansatz

$$F(\tau, p) = F_{\text{scl}}(\tau, p) + \delta F(\tau, p), \quad \tilde{\rho}(\tau, p) = \tilde{\rho}_{\text{scl}}(\tau, p), \quad (\text{A.3})$$

with δF small in comparison to the scaling solution F_{scl} . Linearizing Eq. (A.2a) in δF then allows one to identify a stability matrix

$$2p^0 \partial_\tau \delta F(\tau, p) \simeq \tilde{\Sigma}_{\text{scl}}^\rho(\tau, p) \delta F(\tau, p) + \int_q \left[\frac{\delta \tilde{\Sigma}^\rho(\tau, p)}{\delta F(\tau, q)} F_{\text{scl}}(\tau, p) - \frac{\delta \tilde{\Sigma}^F(\tau, p)}{\delta F(\tau, q)} \tilde{\rho}_{\text{scl}}(\tau, p) \right]_{F=F_{\text{scl}}, \tilde{\rho}=\tilde{\rho}_{\text{scl}}} \delta F(\tau, q) \quad (\text{A.4})$$

$$\equiv - \int_q S(p, q) \delta F(\tau, q) , \quad (\text{A.5})$$

where we use the shorthand notation $\int_q \equiv \int d^4 q / (2\pi)^4$. If we neglect the off-diagonal contribution, we can identify the response rate from the diagonal contribution $S(p, q) \sim \delta(p - q)$ as

$$\gamma(\tau, p) = - \frac{\tilde{\Sigma}_{\text{scl}}^\rho(\tau, p)}{2p^0} . \quad (\text{A.6})$$

Evaluated on-shell $p^0 = \omega_{\mathbf{p}}$, the same rate is familiar for linear response analysis around thermal equilibrium [168, 190–192], which is commonly used to determine transport properties like relaxation, charge conductivity, or viscosity. Its range of validity is however not restricted to the central time-independent thermal equilibrium background as we demonstrated in Sec. 4.2.

Also expanding the self-energy (4.25b) to leading order in central gradients, we have [156]

$$\tilde{\Sigma}^\rho(\tau, p) = - \frac{\lambda}{3N} \int_q v_{\text{eff}}(\tau, p - q) \left[\Pi^F(\tau, p - q) \tilde{\rho}(\tau, q) + \tilde{\Pi}^\rho(\tau, p - q) F(\tau, q) \right] , \quad (\text{A.7})$$

with effective momentum-dependent vertex

$$v_{\text{eff}}(\tau, p) = \left| 1 + \Pi^R(\tau, p) \right|^{-2} = \left| 1 + \frac{\lambda}{3} \int_k F(\tau, p - k) G^R(\tau, k) \right|^{-2} . \quad (\text{A.8})$$

We now want to derive Eq. (12) of Sec. 4.2, which requires us to isolate the different

scattering contributions to the rate. We start by splitting frequency contributions as

$$\begin{aligned}
 \tilde{\Sigma}^\rho(\tau, p) &= -\frac{\lambda^2}{18N} \int_{qlr} (2\pi)^4 \delta^{(4)}(p - q - l - r) v_{\text{eff}}(\tau, p - q) \\
 &\quad \times \left[F_q F_r \tilde{\rho}_l + \tilde{\rho}_q F_r F_l + \tilde{\rho}_r F_q F_l + \frac{1}{4} \tilde{\rho}_q \tilde{\rho}_r \tilde{\rho}_l \right] \\
 &= -\frac{\lambda^2}{18N} \int_{\mathbf{qlr}} \int_0^\infty \frac{dq^0 dl^0 dr^0}{(2\pi)^3} (2\pi)^4 \\
 &\quad \left\{ +\delta^{(4)}(p - q - l - r) v_{\text{eff}}(\tau, p - q) \left[F_q F_r \tilde{\rho}_l + \tilde{\rho}_q F_r F_l + \tilde{\rho}_r F_q F_l + \frac{1}{4} \tilde{\rho}_q \tilde{\rho}_r \tilde{\rho}_l \right] \right. \\
 &\quad +\delta^{(4)}(p - q - l + r) v_{\text{eff}}(\tau, p - q) \left[F_q F_r \tilde{\rho}_l + \tilde{\rho}_q F_r F_l - \tilde{\rho}_r F_q F_l - \frac{1}{4} \tilde{\rho}_q \tilde{\rho}_r \tilde{\rho}_l \right] \\
 &\quad +\delta^{(4)}(p - q + l - r) v_{\text{eff}}(\tau, p - q) \left[-F_q F_r \tilde{\rho}_l + \tilde{\rho}_q F_r F_l + \tilde{\rho}_r F_q F_l - \frac{1}{4} \tilde{\rho}_q \tilde{\rho}_r \tilde{\rho}_l \right] \\
 &\quad +\delta^{(4)}(p - q + l + r) v_{\text{eff}}(\tau, p - q) \left[-F_q F_r \tilde{\rho}_l + \tilde{\rho}_q F_r F_l - \tilde{\rho}_r F_q F_l + \frac{1}{4} \tilde{\rho}_q \tilde{\rho}_r \tilde{\rho}_l \right] \\
 &\quad +\delta^{(4)}(p + q - l - r) v_{\text{eff}}(\tau, p + q) \left[F_q F_r \tilde{\rho}_l - \tilde{\rho}_q F_r F_l + \tilde{\rho}_r F_q F_l - \frac{1}{4} \tilde{\rho}_q \tilde{\rho}_r \tilde{\rho}_l \right] \\
 &\quad +\delta^{(4)}(p + q - l + r) v_{\text{eff}}(\tau, p + q) \left[F_q F_r \tilde{\rho}_l - \tilde{\rho}_q F_r F_l - \tilde{\rho}_r F_q F_l + \frac{1}{4} \tilde{\rho}_q \tilde{\rho}_r \tilde{\rho}_l \right] \\
 &\quad +\delta^{(4)}(p + q + l - r) v_{\text{eff}}(\tau, p + q) \left[-F_q F_r \tilde{\rho}_l - \tilde{\rho}_q F_r F_l + \tilde{\rho}_r F_q F_l + \frac{1}{4} \tilde{\rho}_q \tilde{\rho}_r \tilde{\rho}_l \right] \\
 &\quad \left. +\delta^{(4)}(p + q + l + r) v_{\text{eff}}(\tau, p + q) \left[-F_q F_r \tilde{\rho}_l - \tilde{\rho}_q F_r F_l - \tilde{\rho}_r F_q F_l - \frac{1}{4} \tilde{\rho}_q \tilde{\rho}_r \tilde{\rho}_l \right] \right\} , \tag{A.9}
 \end{aligned}$$

where we suppressed the time dependence for notational ease as $F_p \equiv F(\tau, p)$. Relabeling internal momenta, this allows us to isolate the different scattering contributions as (where $F_{\text{scl}} \equiv F$ for simplicity)

$$\begin{aligned}
 \gamma^{2 \leftrightarrow 2}(\tau, p) &= +\frac{\lambda^2}{36Np^0} \int_{\mathbf{qlr}} \int_0^\infty \frac{dq^0 dl^0 dr^0}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(p + l - q - r) \\
 &\quad \times \left\{ v_{\text{eff}}(\tau, p - r) \left[F_r F_l \tilde{\rho}_q + \tilde{\rho}_r F_l F_q - \tilde{\rho}_l F_r F_q - \frac{1}{4} \tilde{\rho}_r \tilde{\rho}_l \tilde{\rho}_q \right] \right. \\
 &\quad +v_{\text{eff}}(\tau, p - q) \left[-F_q F_r \tilde{\rho}_l + \tilde{\rho}_q F_r F_l + \tilde{\rho}_r F_q F_l - \frac{1}{4} \tilde{\rho}_q \tilde{\rho}_r \tilde{\rho}_l \right] \\
 &\quad \left. +v_{\text{eff}}(\tau, p + l) \left[F_l F_r \tilde{\rho}_q - \tilde{\rho}_l F_r F_q + \tilde{\rho}_r F_l F_q - \frac{1}{4} \tilde{\rho}_l \tilde{\rho}_r \tilde{\rho}_q \right] \right\} , \tag{A.10a}
 \end{aligned}$$

$$\begin{aligned}
\gamma_{(a)}^{1\leftrightarrow 3}(\tau, p) = & + \frac{\lambda^2}{36Np^0} \int_{\mathbf{qlr}} \int_0^\infty \frac{dq^0 dl^0 dr^0}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(p + l + q - r) \\
& \times \left\{ v_{\text{eff}}(\tau, p - r) \left[-F_r F_q \tilde{\rho}_l + \tilde{\rho}_r F_q F_l - \tilde{\rho}_q F_r F_l + \frac{1}{4} \tilde{\rho}_r \tilde{\rho}_q \tilde{\rho}_l \right] \right. \\
& + v_{\text{eff}}(\tau, p + l) \left[F_l F_q \tilde{\rho}_r - \tilde{\rho}_l F_q F_r - \tilde{\rho}_q F_l F_r + \frac{1}{4} \tilde{\rho}_l \tilde{\rho}_q \tilde{\rho}_r \right] \\
& \left. + v_{\text{eff}}(\tau, p + q) \left[-F_q F_r \tilde{\rho}_l - \tilde{\rho}_q F_r F_l + \tilde{\rho}_r F_q F_l + \frac{1}{4} \tilde{\rho}_q \tilde{\rho}_r \tilde{\rho}_l \right] \right\}, \quad (\text{A.10b})
\end{aligned}$$

$$\begin{aligned}
\gamma_{(b)}^{1\leftrightarrow 3}(\tau, p) = & \frac{\lambda^2}{36Np^0} \int_{\mathbf{qlr}} \int_0^\infty \frac{dq^0 dl^0 dr^0}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(p - l - q - r) \\
& \times v_{\text{eff}}(\tau, p - l) \left[F_l F_r \tilde{\rho}_q + \tilde{\rho}_l F_r F_q + \tilde{\rho}_r F_l F_q + \frac{1}{4} \tilde{\rho}_l \tilde{\rho}_r \tilde{\rho}_q \right], \quad (\text{A.10c})
\end{aligned}$$

and

$$\begin{aligned}
\gamma^{0\leftrightarrow 4}(\tau, p) = & - \frac{\lambda^2}{36Np^0} \int_{\mathbf{qlr}} \int_0^\infty \frac{dq^0 dl^0 dr^0}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(p + l + q + r) \\
& \times v_{\text{eff}}(\tau, p + l) \left[F_l F_r \tilde{\rho}_q + \tilde{\rho}_l F_r F_q + \tilde{\rho}_r F_l F_q + \frac{1}{4} \tilde{\rho}_l \tilde{\rho}_r \tilde{\rho}_q \right]. \quad (\text{A.10d})
\end{aligned}$$

As we will show in the following, Eq. (A.10a) corresponds to Eq. (12) of Sec. 4.2 if one introduces an off-shell distribution function as

$$F(\tau, p) = \left[f(\tau, p) + \frac{1}{2} \right] \tilde{\rho}(\tau, p). \quad (\text{A.11})$$

The identification of the respective contributions with different scattering processes will become evident in the complementary derivation from kinetic theory.

Besides the elastic scattering contributions of Eq. (A.10a), we also identified contributions from inelastic $1 \leftrightarrow 3$ and $0 \leftrightarrow 4$ processes. The $1 \leftrightarrow 3$ processes are also visible in Fig. 3 of Sec. 4.2, compare the branch starting above $\omega/Q = 2$ and growing for larger $|\mathbf{p}|/Q$. Since the weight of the main quasi-particle branch in $\tilde{\rho}$ with $p^0 = \omega(\tau, \mathbf{p})$ is the largest, effectively only those branches near $p^0 = \omega(\tau, \mathbf{p})$ contribute to equal-time observables as they are integrated over all frequencies.

We analyze the scattering contributions to the response rate Eq. (A.10) further and focus on positive energies $p^0 \geq 0$. $2 \leftrightarrow 2$ was already discussed in Sec. 4.2. The corresponding energy-momentum delta function $\delta(p^0 + \omega(\tau, \mathbf{l}) - \omega(\tau, \mathbf{q}) - \omega(\tau, \mathbf{r}))$ has two solutions, namely elastic scattering among the standard quasi-particles $p^0 = \omega(\tau, \mathbf{p})$ and $p^0 = \omega^-(\tau, \mathbf{p}) = 2M - \omega(\tau, \mathbf{p})$ associated with the production of particles with mass M at rest. Inelastic $1 \leftrightarrow 3$ scatterings are represented by $\delta(p^0 - \omega(\tau, \mathbf{l}) - \omega(\tau, \mathbf{q}) - \omega(\tau, \mathbf{r}))$, with solution $p^0 = \omega^+(\tau, \mathbf{p}) = 2M + \omega(\tau, \mathbf{p})$. In comparison to elastic processes, this corresponds to the annihilation of particles with mass M at rest and therefore contributes to the depletion of the macroscopic zero mode. This process is suppressed in comparison to both elastic

scatterings, which provides support to the build up of a macroscopic zero mode far from equilibrium [80, 100].

Starting from the kinetic equation

Introducing occupation numbers as in Eq. (A.11), we can define an off-shell collision kernel

$$C(\tau, p) = \tilde{\Sigma}^p(\tau, p)F(\tau, p) - \Sigma^F(\tau, p)\tilde{\rho}(\tau, p) \quad (\text{A.12})$$

$$= -\frac{\lambda^2}{18N} \int_{qlr} (2\pi)^4 \delta^{(4)}(p - q - l - r) v_{\text{eff}}(\tau, p - q) \tilde{\rho}_p \tilde{\rho}_l \tilde{\rho}_q \tilde{\rho}_r \\ \times \{ [f_q + 1][f_l + 1][f_r + 1] f_p - f_q f_l f_r [f_p + 1] \} , \quad (\text{A.13})$$

where the collision kernel familiar from kinetic theory follows here as

$$C(\tau, \mathbf{p}) = \int_0^\infty \frac{dp^0}{2\pi} C(\tau, p) . \quad (\text{A.14})$$

As before, we now consider perturbations around the scaling solution

$$f(\tau, p) = f^{\text{scl}}(\tau, p) + \delta f(\tau, p) , \quad \tilde{\rho}(\tau, p) = \tilde{\rho}^{\text{scl}}(\tau, p) , \quad (\text{A.15})$$

and linearize the off-shell collision kernel in δf . Keeping only the diagonal part $\sim \delta(p - q)$, we can identify the kinetic linear response rate as

$$\delta C(\tau, p) \equiv -\delta f_p \frac{\lambda^2}{18N} \int_{qlr} (2\pi)^4 \delta^{(4)}(p - l - q - r) v_{\text{eff}}(\tau, p - q) \tilde{\rho}_p \tilde{\rho}_l \tilde{\rho}_q \tilde{\rho}_r \\ \times \left\{ [f_q^{\text{scl}} + 1][f_l^{\text{scl}} + 1][f_r^{\text{scl}} + 1] - f_q^{\text{scl}} f_l^{\text{scl}} f_r^{\text{scl}} \right\} \quad (\text{A.16})$$

$$\equiv -\bar{\gamma}(\tau, p) \delta f(\tau, p) . \quad (\text{A.17})$$

We again isolate scattering processes by splitting the frequency integrals ($f^{\text{scl}} \equiv f$ in the following for notational ease)

$$\bar{\gamma}(\tau, p) = \frac{\lambda^2}{18N} \int_{\mathbf{qlr}} \int_0^\infty \frac{dq^0 dl^0 dr^0}{(2\pi)^3} (2\pi)^4 \tilde{\rho}_p \quad (\text{A.18})$$

$$\times \left\{ \delta^{(4)}(p - q - l - r) v_{\text{eff}}(\tau, p - q) \tilde{\rho}_q \tilde{\rho}_l \tilde{\rho}_r [[f_q + 1][f_l + 1][f_r + 1] - f_q f_l f_r] \right. \\ + \delta^{(4)}(p - q - l + r) v_{\text{eff}}(\tau, p - q) \tilde{\rho}_q \tilde{\rho}_l \tilde{\rho}_{-r} [[f_q + 1][f_l + 1][f_{-r} + 1] - f_q f_l f_{-r}] \\ + \delta^{(4)}(p - q + l - r) v_{\text{eff}}(\tau, p - q) \tilde{\rho}_q \tilde{\rho}_{-l} \tilde{\rho}_r [[f_q + 1][f_{-l} + 1][f_r + 1] - f_q f_{-l} f_r] \\ + \delta^{(4)}(p - q + l + r) v_{\text{eff}}(\tau, p - q) \tilde{\rho}_q \tilde{\rho}_{-l} \tilde{\rho}_{-r} [[f_q + 1][f_{-l} + 1][f_{-r} + 1] - f_q f_{-l} f_{-r}] \\ + \delta^{(4)}(p + q - l - r) v_{\text{eff}}(\tau, p + q) \tilde{\rho}_{-q} \tilde{\rho}_l \tilde{\rho}_r [[f_{-q} + 1][f_l + 1][f_r + 1] - f_{-q} f_l f_r] \\ + \delta^{(4)}(p + q - l + r) v_{\text{eff}}(\tau, p + q) \tilde{\rho}_{-q} \tilde{\rho}_l \tilde{\rho}_{-r} [[f_{-q} + 1][f_l + 1][f_{-r} + 1] - f_{-q} f_l f_{-r}] \\ + \delta^{(4)}(p + q + l - r) v_{\text{eff}}(\tau, p + q) \tilde{\rho}_{-q} \tilde{\rho}_{-l} \tilde{\rho}_r [[f_{-q} + 1][f_{-l} + 1][f_r + 1] - f_{-q} f_{-l} f_r] \\ \left. + \delta^{(4)}(p + q + l + r) v_{\text{eff}}(\tau, p + q) \tilde{\rho}_{-q} \tilde{\rho}_{-l} \tilde{\rho}_{-r} [[f_{-q} + 1][f_{-l} + 1][f_{-r} + 1] - f_{-q} f_{-l} f_{-r}] \right\} .$$

Using $f_{-p} = -[f_p + 1]$ and $\tilde{\rho}_{-p} = -\tilde{\rho}_p$, we find

$$\bar{\gamma}(\tau, p) = \frac{\lambda^2}{18N} \int_{\mathbf{qlr}} \int_0^\infty \frac{dq^0 dl^0 dr^0}{(2\pi)^3} (2\pi)^4 \tilde{\rho}_p \tilde{\rho}_q \tilde{\rho}_l \tilde{\rho}_r \quad (\text{A.19})$$

$$\begin{aligned} & \left\{ \delta^{(4)}(p - q - l - r) v_{\text{eff}}(\tau, p - q) [[f_q + 1][f_l + 1][f_r + 1] - f_r f_q f_l] \right. \\ & + \delta^{(4)}(p - q - l + r) v_{\text{eff}}(\tau, p - q) [[f_q + 1][f_l + 1]f_r - f_q f_l [f_r + 1]] \\ & + \delta^{(4)}(p - q + l - r) v_{\text{eff}}(\tau, p - q) [[f_q + 1]f_l [f_r + 1] - f_q [f_l + 1]f_r] \\ & + \delta^{(4)}(p - q + l + r) v_{\text{eff}}(\tau, p - q) [[f_q + 1]f_l f_r - f_q [f_l + 1][f_r + 1]] \\ & + \delta^{(4)}(p + q - l - r) v_{\text{eff}}(\tau, p + q) [f_q [f_l + 1][f_r + 1] - [f_q + 1]f_l f_r] \\ & + \delta^{(4)}(p + q - l + r) v_{\text{eff}}(\tau, p + q) [f_q [f_l + 1]f_r - [f_q + 1]f_l [f_r + 1]] \\ & + \delta^{(4)}(p + q + l - r) v_{\text{eff}}(\tau, p + q) [f_q f_l [f_r + 1] - [f_q + 1][f_l + 1]f_r] \\ & \left. + \delta^{(4)}(p + q + l + r) v_{\text{eff}}(\tau, p + q) [f_q f_l f_r - [f_q + 1][f_l + 1][f_r + 1]] \right\} . \quad (\text{A.20}) \end{aligned}$$

Relabeling internal momenta, we can identify the different scattering contributions explicitly as

$$\begin{aligned} \bar{\gamma}(\tau, p) = & \int d\Omega^{2 \leftrightarrow 2} [f](p, l, q, r) [[f_l + 1]f_q f_r - f_l [f_q + 1][f_r + 1]] \\ & + \int d\Omega_{(a)}^{1 \leftrightarrow 3} [f](p, l, q, r) [[f_l + 1][f_q + 1]f_r - f_l f_q [f_r + 1]] \\ & + \int d\Omega_{(b)}^{1 \leftrightarrow 3} [f](p, l, q, r) [f_l f_q f_r - [f_l + 1][f_q + 1][f_r + 1]] \\ & + \int d\Omega^{0 \leftrightarrow 4} [f](p, l, q, r) [[f_l + 1][f_q + 1][f_r + 1] - f_l f_q f_r] , \quad (\text{A.21}) \end{aligned}$$

with

$$\begin{aligned} \int d\Omega^{2 \leftrightarrow 2} [f](p, l, q, r) = & -\frac{\lambda^2}{18N} \int_{\mathbf{qlr}} \int_0^\infty \frac{dq^0 dl^0 dr^0}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(p + l - q - r) \\ & \times \tilde{\rho}_p \tilde{\rho}_q \tilde{\rho}_l \tilde{\rho}_r [v_{\text{eff}}(\tau, p - q) + v_{\text{eff}}(\tau, p - r) + v_{\text{eff}}(\tau, p + l)] , \quad (\text{A.22a}) \end{aligned}$$

$$\begin{aligned} \int d\Omega_{(a)}^{1 \leftrightarrow 3} [f](p, l, q, r) = & -\frac{\lambda^2}{18N} \int_{\mathbf{qlr}} \int_0^\infty \frac{dq^0 dl^0 dr^0}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(p + l + q - r) \\ & \times \tilde{\rho}_p \tilde{\rho}_l \tilde{\rho}_q \tilde{\rho}_r [v_{\text{eff}}(\tau, p - r) + v_{\text{eff}}(\tau, p + q) + v_{\text{eff}}(\tau, p + l)] , \quad (\text{A.22b}) \end{aligned}$$

$$\begin{aligned} \int d\Omega_{(b)}^{1 \leftrightarrow 3} [f](p, l, q, r) = & -\frac{\lambda^2}{18N} \int_{\mathbf{qlr}} \int_0^\infty \frac{dq^0 dl^0 dr^0}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(p - q - l - r) \\ & \times \tilde{\rho}_p \tilde{\rho}_l \tilde{\rho}_q \tilde{\rho}_r v_{\text{eff}}(\tau, p - q) , \quad (\text{A.22c}) \end{aligned}$$

$$\begin{aligned} \int d\Omega^{0 \leftrightarrow 4} [f](p, l, q, r) = & -\frac{\lambda^2}{18N} \int_{\mathbf{qlr}} \int_0^\infty \frac{dq^0 dl^0 dr^0}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(p + l + q + r) \\ & \times \tilde{\rho}_p \tilde{\rho}_l \tilde{\rho}_q \tilde{\rho}_r v_{\text{eff}}(\tau, p + q) . \quad (\text{A.22d}) \end{aligned}$$

Here, the interpretation in terms of scattering processes becomes transparent, as the analysis begins directly from the collision kernel. By combining the energy- and momentum-conserving delta functions with the gain and loss terms, the underlying scattering processes

can be identified – provided one correctly accounts for the absence of f_p and $[f_p + 1]$ factors, which is a consequence of the linearization procedure. Alternatively, one may first derive the full kinetic collision kernel and linearize it at a later stage, yielding the same result.

To verify the equivalence of the rate expressions (A.10) and (A.21), one can explicitly introduce occupation numbers into Eq. (A.10) using the definition given in Eq. (A.11). The comparison at the level of the linearized transport (or kinetic) equation is completed by identifying

$$\delta F(\tau, p) = \delta f(\tau, p) \tilde{\rho}(\tau, p) , \quad (\text{A.23})$$

which follows from inserting the ansätze (A.3) and (A.15) into Eq. (A.11).

A.2 Scaling analysis of correlation functions in Wigner space

In this section, we derive the scaling prediction for the on-shell rate integral presented in Eq. (14) of Sec. 4.2. To this end, we analyze the scaling behavior of correlation functions in Wigner space near the infrared nonthermal fixed point associated with an inverse particle cascade. The analysis is restricted to the regime close to the fixed point, where prescaling corrections can be neglected. A corresponding analysis including prescaling effects proceeds analogously.

We start with the following scaling ansätze for the fundamental correlators [80]

$$F(\tau, p^0, \mathbf{p}) = s^{\alpha/\beta+z} F_S(s^{-1/\beta}\tau, s^z p^0, s\mathbf{p}) , \quad (\text{A.24a})$$

$$\rho(\tau, p^0, \mathbf{p}) = s^{2-\eta} \rho_S(s^{-1/\beta}\tau, s^z p^0, s\mathbf{p}) , \quad (\text{A.24b})$$

$$\omega_{\mathbf{p}} = s^{-z} \omega_{s\mathbf{p}} . \quad (\text{A.24c})$$

$s \in \mathbb{R}_{>0}$ is a general scaling parameter that can be chosen to conveniently reduce the dimension of the scaling function by one. E.g., one chooses $s\mathbf{p} = \mathbf{p}_{\text{ref}}$, $s^z p^0 = p_{\text{ref}}^0$, or $s^{-1/\beta}\tau = \tau_{\text{ref}}$ such that

$$F(\tau, p^0, \mathbf{p}) = \left(\frac{\tau}{\tau_{\text{ref}}} \right)^{\alpha+\beta z} F_{S,\tau} \left[\tau_{\text{ref}}, \left(\frac{\tau}{\tau_{\text{ref}}} \right)^{\beta z} \omega, \left(\frac{\tau}{\tau_{\text{ref}}} \right)^{\beta} \mathbf{p} \right] , \quad (\text{A.25a})$$

$$= \left(\frac{p^0}{p_{\text{ref}}^0} \right)^{-\alpha/(\beta z)-1} F_{S,p^0} \left[\left(\frac{p^0}{p_{\text{ref}}^0} \right)^{1/(\beta z)} \tau, p_{\text{ref}}^0, \left(\frac{p^0}{p_{\text{ref}}^0} \right)^{-1/z} \mathbf{p} \right] . \quad (\text{A.25b})$$

Toward Eq. (14) of Sec. 4.2, we will consider the choice $s = (\tau/\tau_{\text{ref}})^{\beta}$ when necessary.

The scaling of a correlation function in transformed Wigner space differs from the scaling

in coordinate Wigner space by an intermediate z factor as

$$\mathcal{O}(\tau, \Delta t, \mathbf{p}) = \int \frac{dp^0}{2\pi} e^{-ip^0 \Delta t} s^{\alpha \mathcal{O}} \mathcal{O}_S(s^{-1/\beta} \tau, s^z p^0, s\mathbf{p}) \quad (\text{A.26})$$

$$= s^{\alpha \mathcal{O} - z} \int \frac{d(s^z p^0)}{2\pi} e^{-is^z p^0 (s^{-z} \Delta t)} \mathcal{O}_S(s^{-1/\beta} \tau, s^z p^0, s\mathbf{p}) \quad (\text{A.27})$$

$$\equiv s^{\alpha \mathcal{O} - z} \mathcal{O}_S(s^{-1/\beta} \tau, s^{-z} \Delta t, s\mathbf{p}) . \quad (\text{A.28})$$

Accordingly, the scaling ansatz in coordinate Wigner space reads

$$F(\tau, \Delta t, \mathbf{p}) = s^{\alpha/\beta} F(s^{-1/\beta} \tau, s^{-z} \Delta t, s\mathbf{p}) , \quad (\text{A.29})$$

where equal-time scaling follows for $\Delta t \equiv 0$.

Scaling of one-loop terms

We will restrict our analysis to the infrared nonthermal fixed point in the following, where $F^2 \gg \rho^2$ and more generally in the comparison of statistical and spectral components of two-point correlation functions.

We begin with the scaling of the one-loop components of Eq. (4.27)

$$\begin{aligned} \Pi^F(\tau, p^0, \mathbf{p}) &= \frac{\lambda}{6} \int_q \left[F(\tau, p - q) F(\tau, q) - \frac{1}{4} \rho(\tau, p - q) \rho(\tau, q) \right] \\ &\simeq \frac{\lambda}{6} \int_q F(\tau, p - q) F(\tau, q) \\ &= \frac{\lambda}{6} \int_q s^{z+\alpha/\beta} F_S(s^{-1/\beta} \tau, s^z(p^0 - q^0), s(\mathbf{p} - \mathbf{q})) s^{z+\alpha/\beta} F_S(s^{-1/\beta} \tau, s^z q^0, s\mathbf{q}) \\ &= s^{2z+2\alpha/\beta-z-d} \frac{\lambda}{6} \int_q F_S(s^{-1/\beta} \tau, s^z p^0 - q^0, s\mathbf{p} - \mathbf{q}) F_S(s^{-1/\beta} \tau, q^0, \mathbf{q}) \\ &= s^{z+2\alpha/\beta-d} \Pi_S^F(s^{-1/\beta} \tau, s^z p^0, s\mathbf{p}) \equiv s^{\alpha_{\Pi^F}} \Pi_S^F(s^{-1/\beta} \tau, s^z p^0, s\mathbf{p}) , \end{aligned} \quad (\text{A.30})$$

where one rescales the integration variable as $(q^0, \mathbf{q}) \rightarrow (s^{-z} q^0, s^{-1} \mathbf{q})$ and thus $dq^0 d^d q \rightarrow dq^0 d^d q s^{-z-d}$. Analogously,

$$\begin{aligned} \Pi^\rho(\tau, p^0, \mathbf{p}) &= \frac{\lambda}{3} \int_q F(\tau, p - q) \rho(\tau, q) \\ &= \frac{\lambda}{3} \int_q s^{z+\alpha/\beta} F_S(s^{-1/\beta} \tau, s^z(p^0 - q^0), s(\mathbf{p} - \mathbf{q})) s^{2-\eta} \rho_S(s^{-1/\beta} \tau, s^z q^0, s\mathbf{q}) \\ &= s^{z+\alpha/\beta+2-\eta-z-d} \frac{\lambda}{3} \int_q F_S(s^{-1/\beta} \tau, s^z p^0 - q^0, s\mathbf{p} - \mathbf{q}) \rho_S(s^{-1/\beta} \tau, q^0, \mathbf{q}) \\ &= s^{2-\eta+\alpha/\beta-d} \Pi_S^\rho(s^{-1/\beta} \tau, s^z p^0, s\mathbf{p}) \equiv s^{\alpha_{\Pi^\rho}} \Pi_S^\rho(s^{-1/\beta} \tau, s^z p^0, s\mathbf{p}) . \end{aligned} \quad (\text{A.31})$$

Scaling of effective vertex

We need the scaling of the retarded one-loop bubble to extract the scaling of the momentum-dependent effective vertex (A.8). The scaling of retarded and advanced functions is found from the scaling of the corresponding spectral component through

$$G_{\mathcal{O}}^R(x, y) = \theta(x^0 - y^0) \mathcal{O}^\rho(x, y) , \quad (\text{A.32})$$

which in Wigner space becomes

$$G_{\mathcal{O}}^R(X, p^0, \mathbf{p}) = \int d^4s e^{ips} \theta(x^0 - y^0) \mathcal{O}^\rho(x, y) = \lim_{\epsilon \rightarrow 0^+} \int \frac{d\omega}{2\pi} \frac{i \mathcal{O}^\rho(X, \omega, \mathbf{p})}{p^0 - \omega + i\epsilon} , \quad (\text{A.33})$$

by virtue of the Wigner space representation of the theta function

$$\theta(x^0 - y^0) = \lim_{\epsilon \rightarrow 0^+} \int \frac{d\omega}{2\pi} e^{-i\omega(x^0 - y^0)} \frac{i}{\omega + i\epsilon} . \quad (\text{A.34})$$

Note that $G_{\mathcal{O}}^A$ follows analogously by flipping the sign $G_{\mathcal{O}}^R : +i\epsilon$ to $G_{\mathcal{O}}^A : -i\epsilon$, where the poles are shifted from $\text{Im}\omega > 0$ for the retarded contour to $\text{Im}\omega < 0$ for the advanced one. Consequently

$$\begin{aligned} \Pi^{R/A}(\tau, p^0, \mathbf{p}) &= \lim_{\epsilon \rightarrow 0^+} \int \frac{d\omega}{2\pi} \frac{i \Pi^\rho(\tau, \omega, \mathbf{p})}{p^0 - \omega \pm i\epsilon} \\ &= \lim_{\epsilon \rightarrow 0^+} s^{\alpha_{\Pi\rho}} \int \frac{d\omega}{2\pi} \frac{i \Pi^\rho(s^{-1/\beta} \tau, s^z \omega, s\mathbf{p})}{p^0 - \omega \pm i\epsilon} \\ &= \lim_{\epsilon \rightarrow 0^+} s^{\alpha_{\Pi\rho}} \int \frac{s^{-z} d\omega}{2\pi} \frac{\Pi_\rho(s^{-1/\beta} \tau, \omega, s\mathbf{p})}{s^{-z}(s^z p^0 - \omega \pm s^z i\epsilon)} \\ &= s^{\alpha_{\Pi\rho}} \Pi_S^{R/A}(s^{-1/\beta} \tau, s^z p^0, s\mathbf{p}) , \end{aligned} \quad (\text{A.35})$$

where one rescales $\tilde{\epsilon} = s^z \epsilon \in \mathbb{R}_{>0}$ as $s \in \mathbb{R}_{>0}$. Therefore, the scaling exponents of the retarded and advanced propagator both agree with that of the respective spectral component

$$\alpha_{\Pi^R} = \alpha_{\Pi^A} = \alpha_{\Pi\rho} = 2 - \eta - d + \frac{\alpha}{\beta} . \quad (\text{A.36})$$

For the infrared nonthermal fixed point, $F^2 \gg \rho^2$ implies then

$$\begin{aligned} v_{\text{eff}}(\tau, p^0, \mathbf{p}) &\simeq \left| \Pi^R(\tau, p^0, \mathbf{p}) \right|^{-2} \\ &= s^{-2\alpha_{\Pi\rho}} v_{\text{eff},S}(s^{-1/\beta} \tau, s^z p^0, s\mathbf{p}) \\ &= s^{-2(2-\eta-d+\alpha/\beta)} v_{\text{eff},S}(s^{-1/\beta} \tau, s^z p^0, s\mathbf{p}) . \end{aligned} \quad (\text{A.37})$$

Scaling of resummation functions

Using again a LO gradient expansion, we can relate the $I^{F/\rho}$ resummation functions of Eq. (4.26) to the one-loop bubble functions of Eq. (4.27) through the effective vertex of

Eq. (A.8) as [80]

$$I^F(\tau, p) = v_{\text{eff}}(\tau, p) \Pi^F(\tau, p) , \quad (\text{A.38a})$$

$$I^\rho(\tau, p) = v_{\text{eff}}(\tau, p) \Pi^\rho(\tau, p) . \quad (\text{A.38b})$$

The scaling of the I resummation functions then follows directly as

$$I^F(\tau, p^0, \mathbf{p}) = s^{d+z-2(2-\eta)} I_S^F(s^{-1/\beta} \tau, s^z \omega, s\mathbf{p}) , \quad (\text{A.39a})$$

$$I^\rho(\tau, p^0, \mathbf{p}) = s^{-2+\eta+d-\alpha/\beta} I_S^\rho(s^{-1/\beta} \tau, s^z \omega, s\mathbf{p}) . \quad (\text{A.39b})$$

Scaling of self-energies

Starting from the expression for the self-energy components expanded to NLO in $1/N$ in Eq. (4.25) and using the above results, we then obtain

$$\Sigma^F(\tau, p^0, \mathbf{p}) = s^{\alpha/\beta+z-2(2-\eta)} \Sigma_S^F(s^{-1/\beta} \tau, s^z p^0, s\mathbf{p}) , \quad (\text{A.40a})$$

$$\Sigma^\rho(\tau, p^0, \mathbf{p}) = s^{-(2-\eta)} \Sigma_S^\rho(s^{-1/\beta} \tau, s^z p^0, s\mathbf{p}) . \quad (\text{A.40b})$$

Scaling of response rate

With Eq. (A.6), the scaling of the rate then follows directly from that of $\tilde{\Sigma}^\rho$ as

$$\gamma(\tau, p^0, \mathbf{p}) = - \frac{s^{-(2-\eta)} \tilde{\Sigma}_S^\rho(s^{-1/\beta} \tau, s^z p^0, s\mathbf{p})}{2s^{-z} [s^z p^0]} \quad (\text{A.41})$$

$$= s^{\eta+(z-2)} \gamma_S(s^{-1/\beta} \tau, s^z p^0, s\mathbf{p}) . \quad (\text{A.42})$$

The infrared nonthermal fixed point is dominated by nonrelativistic quasi-particles with $z = 2$ and $\eta \simeq 0$, such that one concludes

$$\gamma(\tau, p^0, \mathbf{p}) \simeq \gamma_S(s^{-1/\beta} \tau, s^z p^0, s\mathbf{p}) . \quad (\text{A.43})$$

Scaling of rate integral

Finally, the scaling of the rate integral stated already in Eq. (12) of Sec. 4.2 is confirmed by explicit integration of the rate in time

$$\Gamma(\tau, p^0, \mathbf{p}) = \int_{\tau_i}^{\tau} dt \gamma(\tau, p^0, \mathbf{p}) \quad (\text{A.44})$$

$$\simeq \int_{\tau_i}^{\tau} dt \gamma_S(s^{-1/\beta} t, s^z p^0, s\mathbf{p}), \quad t' = s^{-1/\beta} t \quad (\text{A.45})$$

$$= \int_{s^{-1/\beta} \tau_i}^{s^{-1/\beta} \tau} dt' s^{1/\beta} \gamma_S(t', s^z p^0, s\mathbf{p}) \quad (\text{A.46})$$

$$\simeq s^{1/\beta} \Gamma_S(s^{-1/\beta} \tau, s^z p^0, s\mathbf{p}) , \quad (\text{A.47})$$

where we consider times much larger than the time of the perturbation $\tau \gg \tau_i$. For $s = (\tau/\tau_{\text{ref}})^\beta$, we thus recover Eq. (12) of Sec. 4.2 as

$$\Gamma(\tau, p^0, \mathbf{p}) \simeq \frac{\tau}{\tau_{\text{ref}}} \Gamma_S(\tau_{\text{ref}}, \tau^{\beta z} p^0, \tau^\beta \mathbf{p}) . \quad (\text{A.48})$$

B Appendix for Prescaling Relaxation to Nonthermal Attractors

In addition to the material discussed in the Supplemental Information of Sec. 4.3, Sec. B.1 provides further calculations and detailed analysis of the results presented for prescaling in isotropic systems. Unless otherwise indicated, equations cited below without roman section references refer to those appearing in Sec. 4.3.

A natural extension of this analysis is the study of prescaling in longitudinally expanding systems, which is particularly relevant for the early-time dynamics of heavy-ion collisions. Corresponding novel results, not previously discussed, are presented in Sec. B.2.

B.1 Further results and discussions on prescaling in isotropic systems

Relationship between β_0 , t_0 and $D_1 = \beta_\infty/t_{\text{ref}}$

Here we discuss the different non-universal constants appearing in the prescaling analysis.

The scaling exponent $\beta(t_0) = \beta_0$ with t_0 the integration constant from the lower time boundary t_0 is obtained by solving the LHS of Eq. (4). It is related to the value of the distribution function at t_0 . Without loss of generality, for convenience we chose the lower time boundary of the integration of Eq. (4) to coincide with the lower time boundary of the prescaling factors, i.e. $B(t) = \exp\left[\int_{t_0}^t dt' \beta(t')/t'\right]$. From a more general derivation with t_{ini} the initial time of integration of Eq. (4), one obtains Eq. (5) with

$$t_* = t_{\text{ini}} - \frac{\beta_\infty}{D_1} B(t_{\text{ini}})^{1/\beta_\infty} = t_{\text{ini}} - t_0 \frac{\beta_\infty}{\beta_0} \exp\left[\frac{1}{\beta_\infty} \int_{t_0}^{t_{\text{ini}}} dt' \frac{\beta(t')}{t'}\right], \quad (\text{B.1a})$$

$$t_{\text{ref}} = \frac{\beta_\infty}{D_1} = t_0 \frac{\beta_\infty}{\beta_0}. \quad (\text{B.1b})$$

If $t_{\text{ini}} = t_0$, $B(t_0) = 1$ and this reduces to $t_* = t_0(1 - \beta_\infty/\beta_0)$. This does not alter the results presented in Sec. 4.3 as t_* is a non-universal initial condition-dependent quantity.

One can identify D_1 at any point in time due to the scaling. Asymptotically late times can be chosen for convenience as $\dot{\beta}$ vanishes and one has a simple expression. This can be seen directly by considering the asymptotic fixed point limit of the LHS of Eq. (4)

$$\partial_t \left(\frac{t}{t_{\text{ref}}}\right)^{\beta_\infty} = \frac{\beta_\infty}{t_{\text{ref}}} \left(\frac{t}{t_{\text{ref}}}\right)^{\beta_\infty-1} \quad (\text{B.2})$$

$$= D_1 \left(\frac{t}{t_{\text{ref}}}\right)^{\beta_\infty-1} \quad (\text{B.3})$$

$$\Leftrightarrow D_1 = \frac{\beta_\infty}{t_{\text{ref}}}. \quad (\text{B.4})$$

One can however also fix the reference point at t_0 , where we for definiteness consider now Eq. (4) in terms of $\beta(t)$

$$\frac{\beta(t_0)}{t_0} = D_1 \exp \left[-\frac{1}{\beta_0} \int_{t_0}^{t_0} dt' \frac{\beta(t')}{t'} \right] \quad (\text{B.5})$$

$$= D_1 . \quad (\text{B.6})$$

How to compute t_*

The non-universal time-shift scale t_* is obtained as an integration constant corresponding to the initial data that is in general to be provided from the experiment. This can be seen explicitly by starting from Eq. (4), where the solution for $B(t)$ integrated from some t_0 to t reads

$$B(t) = \left[\frac{D_1}{\beta_\infty} (t - t_0) + B(t_0)^{1/\beta_\infty} \right]^{\beta_\infty} \quad (\text{B.7})$$

$$\equiv \left(\frac{t - t_*}{t_{\text{ref}}} \right)^{\beta_\infty} . \quad (\text{B.8})$$

We identified the reference time

$$t_* = t_0 - B(t_0)^{1/\beta_\infty} t_{\text{ref}} \quad \text{with} \quad t_{\text{ref}} = \frac{\beta_\infty}{D_1} , \quad (\text{B.9})$$

and D_1 can be fixed at any point in the evolution.

One could then experimentally extract t_* , e.g., from the zero mode of the distribution function (corresponding to the quasi-condensate for the infrared attractor in ultra-cold atom experiments) as

$$f(t_0, p=0) = \left(\frac{t_0 - t_*}{t_{\text{ref}}} \right)^{\alpha_\infty} f_S(0) \quad (\text{B.10})$$

$$\Leftrightarrow t_* = t_0 - \frac{t_{\text{ref}}}{f_S(0)^{1/\alpha_\infty}} f(t_0, p=0)^{1/\alpha_\infty} . \quad (\text{B.11})$$

For general p , knowledge of the scaling function $f_s((t_0 - t_*)^{\beta_\infty} p / t_{\text{ref}}^{\beta_\infty})$ is required as it depends on t_* . For the zero mode, $f_S(0)$ can simply be absorbed into the arbitrary reference scale $t'_{\text{ref}} = t_{\text{ref}} / f_S(0)^{1/\alpha_\infty}$. A similar protocol has been used in a recent experimental detection of prescaling [21], where t_* is extracted from the zero intercept defined by $f(t_*, p=0) = 0$, or more generally $f(t_*, p) = 0$ (compare Fig. 2d in [21]). In [21], the quasi-condensate occupation $f(t, p=0)$ is considered to avoid the need of specifying the scaling function and then plot $f(t, p=0)^{1/\alpha'}$ for different α' against time. They observe a straight line if $\alpha' = \alpha_\infty$, see Fig. 2c therein, corresponding to agreement with our prediction in Eq. (5), where t_* is obtained from the zero intercept $f(t_*, p=0) = 0$. The extracted t_* is positive or negative depending on the initial condition, see Fig. 2d.

Comparison of different time scales and their extraction

Finally, we provide a compilation of the different time scales appearing in the prescaling and scaling analysis. We start with the scaling ansatz

$$f(t, \mathbf{p}) = (t/t_{\text{ref},1})^{\beta_\infty} f_S((t/t_{\text{ref},1})^{\beta_\infty} p, t_{\text{ref},1}) , \quad (\text{B.12})$$

as well as the prescaling ansatz

$$f(t, \mathbf{p}) = A(t, t_{\text{ref},2}) f_S(B(t, t_{\text{ref},2}) p, t_{\text{ref},2}) , \quad (\text{B.13})$$

where $B(t, t_{\text{ref},2}) = \exp \left[\int_{t_{\text{ref},2}}^t dt' \beta(t')/t' \right]$. From the scaling ansatz, the prescaling equation gives us

$$\partial_t (t/t_{\text{ref},1})^{\beta_\infty} = D_1 (t/t_{\text{ref},1})^{\beta_\infty(1-1/\beta_\infty)} \quad (\text{B.14})$$

$$\Leftrightarrow D_1 = \frac{\beta_\infty}{t_{\text{ref},1}} . \quad (\text{B.15})$$

On the other hand, plugging in the prescaling ansatz we get

$$D_1(t - t_0) = \beta_\infty B^{1/\beta_\infty} \Big|_{B_0 \equiv B(t_0)}^{B(t)} \quad (\text{B.16})$$

$$\Leftrightarrow B(t) = \left[\frac{t - (t_0 - \beta_\infty B_0^{1/\beta_\infty} / D_1)}{\beta_\infty / D_1} \right]^{\beta_\infty} \quad (\text{B.17})$$

$$\equiv \left[\frac{t - t_*}{t_{\text{ref},3}} \right]^{\beta_\infty} , \quad (\text{B.18})$$

where

$$t_{\text{ref},3} = \beta_\infty / D_1 , \quad (\text{B.19a})$$

$$t_* = t_0 - \beta_\infty B_0^{1/\beta_\infty} / D_1 . \quad (\text{B.19b})$$

From the prescaling equation, we also extract for the prescaling ansatz

$$D_1 = \frac{\beta(t_{\text{ref},2})}{t_{\text{ref},2}} = \frac{\beta_{\text{ref},2}}{t_{\text{ref},2}} . \quad (\text{B.20})$$

Collecting, we have that

$$t_{\text{ref},3} = t_{\text{ref},2} \frac{\beta_\infty}{\beta_{\text{ref},2}} , \quad (\text{B.21})$$

and

$$D_1 = \frac{\beta_{\text{ref},2}}{t_{\text{ref},2}} = \frac{\beta_\infty}{t_{\text{ref},1}} \quad \Leftrightarrow \quad t_{\text{ref},1} = t_{\text{ref},2} \frac{\beta_\infty}{\beta_{\text{ref},2}} . \quad (\text{B.22})$$

Importantly, when doing an analysis, the reference time entering the extraction of the prescaling factors, e.g. $B(t, t_{\text{ref},2})$ with

$$B_m(t, t_{\text{ref},2}) = \frac{n_m(t)}{n_{m+1}(t)} \frac{n_{m+1}(t_{\text{ref},2})}{n_m(t_{\text{ref},2})}, \quad (\text{B.23})$$

is a different reference time than what enters in the rescaling plots, which is $t_{\text{ref},1}$. As above, these two are related by

$$t_{\text{ref},1} = t_{\text{ref},2} \frac{\beta_\infty}{\beta_{\text{ref},2}}. \quad (\text{B.24})$$

A more general solution avoiding the use of conservation laws

In the main text in Sec. 4.3, we assumed conservation laws to establish one scaling relation $A(t) = B(t)^\sigma$. In this section, we want to explore whether it is possible to solve for the prescaling factors without resorting to such an assumption.

We start from the Boltzmann equation together with the prescaling ansatz

$$\partial_t f(t, \mathbf{p}) = A(t) \left[\frac{\dot{A}(t)}{A(t)} + \frac{\dot{B}(t)}{B(t)} \bar{\mathbf{p}} \cdot \partial_{\bar{\mathbf{p}}} \right] f_S(\bar{\mathbf{p}}) \quad (\text{B.25})$$

$$= A(t)^{\mu_\alpha} B(t)^{\mu_\beta} \mathcal{C}[f_S](\bar{\mathbf{p}}) \quad (\text{B.26})$$

$$\Leftrightarrow \mathcal{C}[f_S](\bar{\mathbf{p}}) = \left[\frac{\dot{A}(t)}{A(t)^{\mu_\alpha} B(t)^{\mu_\beta}} + \frac{\dot{B}(t)}{A(t)^{\mu_\alpha-1} B(t)^{\mu_\beta+1}} \bar{\mathbf{p}} \cdot \partial_{\bar{\mathbf{p}}} \right] f_S(\bar{\mathbf{p}}). \quad (\text{B.27})$$

Instead of performing a separation of variables for one prescaling factor, we can also demand each time-dependent contribution separately to be constant

$$F_1 = \frac{\dot{A}(t)}{A(t)^{\mu_\alpha} B(t)^{\mu_\beta}}, \quad (\text{B.28a})$$

$$F_2 = \frac{\dot{B}(t)}{A(t)^{\mu_\alpha-1} B(t)^{\mu_\beta+1}}. \quad (\text{B.28b})$$

These coupled ordinary differential equations can be solved directly. Assuming a conservation law in the form of $A(t) = B(t)^\sigma$ as in Sec. 4.3, we immediately recognize σ in the ratio

$$\frac{F_1}{F_2} = \frac{\dot{A}(t) B(t)}{\dot{B}(t) A(t)} \quad (\text{B.29})$$

$$= \sigma. \quad (\text{B.30})$$

This demonstrates how an emergent conservation law is manifested in the time-independence of the fixed point equation as represented by the factors F_1 and F_2 .

Perturbations around the prescaling solution

We can give another perspective on perturbations around the prescaling solution presented in Sec. 4.3. Starting with the prescaling equations and their respective solutions

$$\partial_t B = D_1 B(t)^{1-1/\beta_\infty} \quad (\text{B.31a})$$

$$\frac{\beta}{t} = D_1 \exp \left[-\frac{1}{\beta_\infty} \int_{t_0}^t dt' \frac{\beta(t')}{t'} \right] \quad (\text{B.31b})$$

$$B(t) = \left(\frac{t - t_*}{t_{\text{ref}}} \right)^{\beta_\infty} \quad (\text{B.31c})$$

$$\beta(t) = \beta_\infty \frac{t}{t - t_*}, \quad (\text{B.31d})$$

we study linear response corrections around the prescaling solutions as $\beta' = \beta + \delta\beta$. Then

$$\beta + \delta\beta = D_1 t \exp \left[-\frac{1}{\beta_\infty} \int_{t_0}^t dt' \frac{\beta(t') + \delta\beta(t')}{t'} \right] \quad (\text{B.32})$$

$$\simeq D_1 t \exp \left[-\frac{1}{\beta_\infty} \int_{t_0}^t dt' \frac{\beta(t')}{t'} \right] \left(1 - \frac{1}{\beta_\infty} \int_{t_0}^t dt' \frac{\delta\beta(t')}{t'} + \dots \right) \quad (\text{B.33})$$

$$\Leftrightarrow \delta\beta(t) = -\frac{D_1}{\beta_\infty} t \exp \left[-\frac{1}{\beta_\infty} \int_{t_0}^t dt' \frac{\beta(t')}{t'} \right] \int_{t_0}^t dt' \frac{\delta\beta(t')}{t'}. \quad (\text{B.34})$$

Plugging in the prescaling solution, we can integrate directly

$$\exp \left[-\frac{1}{\beta_\infty} \int_{t_0}^t dt' \frac{\beta(t')}{t'} \right] = \exp \left[-\int_{t_0}^t dt' \frac{1}{t' - t_*} \right] \quad (\text{B.35})$$

$$= \frac{t_0 - t_*}{t - t_*} \quad (\text{B.36})$$

to write

$$\delta\beta(t) = -\frac{D_1}{\beta_\infty} t \frac{t_0 - t_*}{t - t_*} \int_{t_0}^t dt' \frac{\delta\beta(t')}{t'}. \quad (\text{B.37})$$

The solution follows analytically using $\delta b = \delta\beta/t$ with

$$\delta b = -\frac{D_1}{\beta_\infty} \frac{t_0 - t_*}{t - t_*} \int_{t_0}^t dt' \delta b(t') \quad (\text{B.38})$$

$$\partial_t \delta b = -\frac{\delta b}{t - t_*} - \frac{D_1}{\beta_\infty} \frac{t_0 - t_*}{t - t_*} \delta b(t) \quad (\text{B.39})$$

$$= -\frac{\delta b(t)}{t - t_*} \left(1 + \frac{D_1}{\beta_\infty} (t_0 - t_*) \right) \quad (\text{B.40})$$

as

$$\delta b(t) = \frac{c_1}{(t - t_*)^{1 + \frac{D_1}{\beta_\infty} (t_0 - t_*)}}, \quad (\text{B.41})$$

where c_1 is an integration constant. Note that, with $D_1 = \beta_0/t_0$ and $t_* = t_0(1 - \beta_\infty/\beta_0)$, we have

$$1 + \frac{D_1}{\beta_\infty}(t_0 - t_*) = 1 + \frac{\beta_0}{\beta_\infty t_0} t_0 \frac{\beta_\infty}{\beta_0} = 2. \quad (\text{B.42})$$

Accordingly

$$\delta\beta(t) = \frac{c_1 t}{(t - t_*)^2}. \quad (\text{B.43})$$

This presents a stable perturbation around the prescaling solution for late times as $\delta\beta \sim 1/t$ for $t \gg t_*$.

B.2 Prescaling with longitudinal expansion

We will now present novel results on the prescaling analysis for QCD kinetic theory in a system with boost-invariant longitudinal expansion, also known as the Bjorken flow [193]. The kinetic term in the Boltzmann equation takes now the form $\partial_t f \rightarrow (\partial_\tau - p_z/\tau \partial_{p_z})f$, where the clock variable τ is the proper time $\tau = \sqrt{t^2 - z^2}$. Longitudinal and transverse momenta now scale in general differently

$$f(\tau, \mathbf{p}_\perp, p_z) = A(\tau) f_S(B(\tau) \mathbf{p}_\perp, G(\tau) p_z), \quad (\text{B.44})$$

where $G(\tau) = \exp\left[\int_{\tau_{\text{ref}}}^\tau d\tau' \gamma(\tau')/\tau'\right]$ parameterizes the evolution of the longitudinal exponent γ . The exponents are expected to converge to the fixed-point values $\alpha_\infty = -2/3$, $\beta_\infty = 0$ and $\gamma_\infty = 1/3$ according to the first stage of the bottom-up thermalization scenario [119], which has been substantiated by classical-statistical simulations [90, 122]. For analytical insight, we will again resort to the Fokker-Planck approximation already employed in the isotropic case of Sec. 4.3 to derive prescaling dynamics in the presence of the scaling-breaking Debye mass.

Starting from the prescaling ansatz (B.44), we introduce moments of the distribution function

$$\begin{aligned} n_{m,n}(\tau) &= \int \frac{d^{d-1} \mathbf{p}_\perp dp_z}{(2\pi)^d} |\mathbf{p}_\perp|^m |p_z|^n f(\tau, \mathbf{p}_\perp, p_z) \\ &= A(\tau) B(\tau)^{-(m+d-1)} G(\tau)^{-(n+1)} \bar{n}_{m,n}. \end{aligned} \quad (\text{B.45})$$

Using combinations of

$$\frac{d}{d \log \tau} \log(n_{m,n}(\tau)) = \alpha(\tau) - (m + d - 1)\beta(\tau) - (n + 1)\gamma(\tau) \quad (\text{B.46})$$

for different moments with $d = 3$, we can then extract prescaling exponents as

$$\alpha_{m,n}(\tau) = (m+2)(m+1+2)(n+1)(n+1+1) \times \frac{d}{d \log \tau} \log \left[\frac{n_{m,n}^{\frac{1}{(m+2)(n+1)}}(\tau)}{n_{m+1,n}^{\frac{1}{(m+1+2)(n+1)}}(\tau)} \frac{n_{m+1,n+1}^{\frac{1}{(m+1+2)(n+1+1)}}(\tau)}{n_{m,n+1}^{\frac{1}{(m+2)(n+1+1)}}(\tau)} \right], \quad (\text{B.47a})$$

$$\beta_{m,n}(\tau) = \frac{d}{d \log \tau} \log \left(\frac{n_{m,n}(\tau)}{n_{m+1,n}(\tau)} \right), \quad (\text{B.47b})$$

$$\gamma_{m,n}(\tau) = \frac{d}{d \log \tau} \log \left(\frac{n_{m,n}(\tau)}{n_{m,n+1}(\tau)} \right). \quad (\text{B.47c})$$

Prescaling of free-streaming

We will begin with the prescaling analysis of free-streaming, i.e. Bjorken-expansion in the absence of scatterings. Plugging in the prescaling ansatz of Eq. (B.44), we immediately find

$$0 = \left[\partial_\tau - \frac{p_z}{\tau} \partial_{p_z} \right] f(\tau, \mathbf{p}_\perp, p_z) \quad (\text{B.48})$$

$$= A(\tau) \left\{ \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \bar{\mathbf{p}}_\perp \cdot \partial_{\bar{\mathbf{p}}_\perp} + \left(\frac{\dot{G}}{G} - \frac{1}{\tau} \right) \bar{p}_z \partial_{\bar{p}_z} \right\} f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z). \quad (\text{B.49})$$

This equation is solved by

$$f(\tau, \mathbf{p}_\perp, p_z) = f_S \left(\mathbf{p}_\perp, \tilde{p}_z = \frac{\tau}{\tau_0} p_z \right), \quad (\text{B.50})$$

which is the prescaling ansatz with $A(\tau) = 1 = B(\tau)$ and $G(\tau) = \tau/\tau_0$. This can be seen directly as then

$$0 = \left[\partial_\tau - \frac{p_z}{\tau} \partial_{p_z} \right] f(\tau, \mathbf{p}_\perp, p_z) \quad (\text{B.51})$$

$$= \left[\frac{\partial \tilde{p}_z}{\partial \tau} \partial_{\tilde{p}_z} - \frac{p_z}{\tau} \frac{\partial \tilde{p}_z}{\partial p_z} \partial_{\tilde{p}_z} \right] f_S(\mathbf{p}_\perp, \tilde{p}_z) \quad (\text{B.52})$$

$$= \left[\frac{p_z}{\tau_0} - \frac{p_z}{\tau_0} \right] \partial_{\tilde{p}_z} f_S(\mathbf{p}_\perp, \tilde{p}_z). \quad (\text{B.53})$$

Conservation laws

The additional prescaling factor $G(\tau)$ requires us to provide in total three scaling relations to close the equations. In the expanding case, one scaling relation is obtained from particle number conservation, which holds exactly in the small angle elastic scattering assumption underlying the elastic FP collision kernel. Particle number decreases as $n \sim 1/\tau$ due to

the dilution of the system, which allows us to extract

$$\tau n_{0,0}(\tau) = \tau A(\tau) B(\tau)^{-2} G(\tau)^{-1} \bar{n} \stackrel{!}{=} \text{const} \quad \Leftrightarrow \quad 0 = \alpha(\tau) - 2\beta(\tau) - \gamma(\tau) + 1. \quad (\text{B.54})$$

A second scaling relation follows from energy conservation. The time evolution of energy density due to longitudinal Bjorken-expansion follows as

$$\partial_\tau(\tau \mathcal{E}) = -\mathcal{P}_L \simeq 0, \quad (\text{B.55})$$

where we assumed the anisotropic limit $\mathcal{P}_L/\mathcal{P}_\perp \ll 1$ typical for the longitudinally expanding quark-gluon plasma in the early stages of a heavy-ion collision. This allows us to obtain another constraint equation for the prescaling exponents

$$\tau \mathcal{E}(\tau) = \tau A(\tau) B(\tau)^{-3} G(\tau)^{-1} \bar{\mathcal{E}} \stackrel{!}{=} \text{const} \quad \Leftrightarrow \quad \alpha(\tau) - 3\beta(\tau) - \gamma(\tau) + 1 = 0. \quad (\text{B.56})$$

The anisotropic limit, which effectively corresponds to neglecting p_z contributions in favor of $|\mathbf{p}_\perp|$ contributions, turns out to be crucial to isolate a definite solution for the prescaling factors and thus for extracting one overall scaling in the first place. Otherwise, already when computing the energy density, one has two contributions under a square root in $\omega_{\mathbf{p}}$ scaling a priori differently, which does not allow for one simple overall rescaling. We will use the anisotropic limit in the following and discuss a more general prescaling analysis without resorting to the anisotropic limit in Sec. B.2.

Combining the scaling relations from particle number (B.54) and energy conservation (B.56) only leaves one independent prescaling exponent with

$$\beta(\tau) = 0, \quad \alpha(\tau) = \gamma(\tau) - 1. \quad (\text{B.57})$$

In terms of prescaling factors, we can redo this analysis as follows. Energy conservation yields

$$\tau \mathcal{E}(\tau) = \text{const} \quad (\text{B.58})$$

$$= \tau A(\tau) B^{-3}(\tau) G^{-1}(\tau) \bar{\mathcal{E}} \quad (\text{B.59})$$

$$\Leftrightarrow \mathcal{E}_0 \equiv \tau A(\tau) B^{-3}(\tau) G^{-1}(\tau). \quad (\text{B.60})$$

Particle number conservation reads

$$\tau n(\tau) = \text{const} \quad (\text{B.61})$$

$$= \tau A(\tau) B^{-2}(\tau) G^{-1}(\tau) \bar{n} \quad (\text{B.62})$$

$$\Leftrightarrow n_0 \equiv \tau A(\tau) B^{-2}(\tau) G^{-1}(\tau). \quad (\text{B.63})$$

We can combine these to identify

$$\tau A(\tau) G^{-1}(\tau) = \mathcal{E}_0 B^3(\tau) \quad (\text{B.64})$$

$$= n_0 B^2(\tau) \quad (\text{B.65})$$

$$\Leftrightarrow B(\tau) = \frac{n_0}{\mathcal{E}_0} = \text{const}. \quad (\text{B.66})$$

This implies

$$\dot{B} = 0. \quad (\text{B.67})$$

Additionally, we can then identify

$$n_0 = \tau A(\tau) G^{-1}(\tau) \left(\frac{\mathcal{E}_0}{n_0} \right)^2 \quad (\text{B.68})$$

$$\Leftrightarrow G(\tau) = \tau A(\tau) \frac{\mathcal{E}_0^2}{n_0^3}, \quad (\text{B.69})$$

which implies

$$\dot{G} = A(\tau) \frac{\mathcal{E}_0^2}{n_0^3} + \tau \dot{A} \frac{\mathcal{E}_0^2}{n_0^3} \quad (\text{B.70})$$

$$\frac{\dot{G}}{G} = \frac{1}{\tau} + \frac{\dot{A}}{A}. \quad (\text{B.71})$$

We will use both perspectives in the upcoming derivation.

Prescaling of Fokker-Planck collision kernel with longitudinal expansion

We assume the gluons to interact via elastic small-angle scatterings such that the collision kernel takes a FP form [12, 121, 194, 195]

$$C^{\text{FP}}[f](\tau, \mathbf{p}_\perp, p_z) = \lambda_0 \log \left[\frac{\sqrt{\langle \mathbf{p}_\perp^2 \rangle}(\tau)}{m_D(\tau)} \right] \left[I_a[f](\tau, \mathbf{p}_\perp, p_z) \partial_{p_z}^2 f(\tau, \mathbf{p}_\perp, p_z) \right. \\ \left. + I_b[f](\tau, \mathbf{p}_\perp, p_z) \nabla_{\mathbf{p}} \cdot \left(\frac{\mathbf{p}}{p} [1 + f(\tau, \mathbf{p}_\perp, p_z)] f(\tau, \mathbf{p}_\perp, p_z) \right) \right]. \quad (\text{B.72})$$

The Debye mass prescales as

$$m_D^2(\tau) = 2g^2 N_c \int \frac{d^2 \mathbf{p}_\perp dp_z}{(2\pi)^3 p} f(\tau, \mathbf{p}_\perp, p_z) \\ \stackrel{p_\perp \gg p_z}{\simeq} 2g^2 N_c \int \frac{d^2 \mathbf{p}_\perp dp_z}{(2\pi)^3 p_\perp} f(\tau, \mathbf{p}_\perp, p_z) \quad (\text{B.73})$$

$$= A(\tau) B(\tau)^{-1} G(\tau)^{-1} \bar{m}_D^2, \quad (\text{B.74})$$

where we used the anisotropic limit in the second step. The first term prescales with

$$I_a[f](\tau, \mathbf{p}_\perp, p_z) = \int_{\mathbf{p}} f(\tau, \mathbf{p}_\perp, p_z) [1 + f(\tau, \mathbf{p}_\perp, p_z)] \quad (\text{B.75})$$

$$= A^2(\tau) B^{-2}(\tau) G^{-1}(\tau) \bar{I}_a(\bar{\mathbf{p}}_\perp, \bar{p}_z) \quad (\text{B.76})$$

as

$$I_a[f](\tau, \mathbf{p}_\perp, p_z) \partial_z^2 f(\tau, \mathbf{p}_\perp, p_z) = A(\tau)^{2+1} B(\tau)^{-2} G(\tau)^{-1+2} \bar{I}_a(\bar{\mathbf{p}}_\perp, \bar{p}_z) \partial_{\bar{p}_z}^2 f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z). \quad (\text{B.77})$$

Using

$$I_b[f](\tau, \mathbf{p}_\perp, p_z) = \int \frac{2}{p} f(\tau, \mathbf{p}_\perp, p_z) \quad (\text{B.78})$$

$$\stackrel{p_\perp \gg p_z}{\simeq} \int_{\mathbf{p}} \frac{2}{p_\perp} f(\tau, \mathbf{p}_\perp, p_z) \quad (\text{B.79})$$

$$= A(\tau) B(\tau)^{-1} G(\tau)^{-1} \bar{I}_b(\bar{\mathbf{p}}_\perp, \bar{p}_z) \quad (\text{B.80})$$

and

$$\begin{aligned} & \nabla_{\mathbf{p}} \cdot \left(\frac{\mathbf{p}}{p} [1 + f(\tau, \mathbf{p}_\perp, p_z)] f(\tau, \mathbf{p}_\perp, p_z) \right) \\ & \stackrel{f \gg 1}{\simeq} \nabla_{\mathbf{p}} \cdot \left(\frac{\mathbf{p}}{p} f^2(\tau, \mathbf{p}_\perp, p_z) \right) \end{aligned} \quad (\text{B.81})$$

$$= \frac{2f}{p} [f + p_\perp \partial_{p_\perp} f + p_z \partial_{p_z} f] \quad (\text{B.82})$$

$$\stackrel{p_\perp \gg p_z}{\simeq} \frac{2f}{p_\perp} [f + p_\perp \partial_{p_\perp} f + p_z \partial_{p_z} f] \quad (\text{B.83})$$

$$= A(\tau)^2 B(\tau) \frac{2f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z)}{\bar{\mathbf{p}}_\perp} [f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z) + \bar{\mathbf{p}}_\perp \partial_{\bar{\mathbf{p}}_\perp} f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z) + \bar{p}_z \partial_{\bar{p}_z} f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z)] , \quad (\text{B.84})$$

the second term prescales as

$$\begin{aligned} & I_b[f](\tau, \mathbf{p}_\perp, p_z) \nabla_{\mathbf{p}} \cdot \left(\frac{\mathbf{p}}{p} [1 + f(\tau, \mathbf{p}_\perp, p_z)] f(\tau, \mathbf{p}_\perp, p_z) \right) \\ & = A(\tau)^{1+2} B(\tau)^{-1+1} G(\tau)^{-1} \bar{I}_b(\bar{\mathbf{p}}_\perp, \bar{p}_z) \frac{2f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z)}{\bar{\mathbf{p}}_\perp} \\ & \times [f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z) + \bar{\mathbf{p}}_\perp \partial_{\bar{\mathbf{p}}_\perp} f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z) + \bar{p}_z \partial_{\bar{p}_z} f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z)] . \end{aligned} \quad (\text{B.85})$$

Together with

$$\langle p_\perp^2 \rangle(\tau) = \frac{\int_{\mathbf{p}} p_\perp^2 f(\tau, \mathbf{p}_\perp, p_z)}{\int_{\mathbf{p}} f(\tau, \mathbf{p}_\perp, p_z)} = B(\tau)^{-2} \langle \bar{\mathbf{p}}_\perp^2 \rangle , \quad (\text{B.86})$$

the Coulomb logarithm behaves as follows under the prescaling ansatz

$$\log \left[\frac{\sqrt{\langle \mathbf{p}_\perp^2 \rangle(\tau)}}{m_D(\tau)} \right] = \log \left[\sqrt{\frac{B(\tau)^{-2}}{A(\tau) B(\tau)^{-1} G(\tau)^{-1}}} \frac{\sqrt{\langle \bar{\mathbf{p}}_\perp^2 \rangle}}{\bar{m}_D} \right] \quad (\text{B.87})$$

$$= \log \left[A(\tau)^{-\frac{1}{2}} B(\tau)^{-\frac{1}{2}} G(\tau)^{\frac{1}{2}} \frac{\sqrt{\langle \bar{\mathbf{p}}_\perp^2 \rangle}}{\bar{m}_D} \right] . \quad (\text{B.88})$$

Altogether, the FP collision kernel exhibits the following time dependence under the prescaling ansatz

$$\begin{aligned}
 C^{\text{FP}}(\tau, \mathbf{p}_\perp, p_z) &= \lambda_0 \log \left[A(\tau)^{-\frac{1}{2}} B(\tau)^{-\frac{1}{2}} G(\tau)^{\frac{1}{2}} \frac{\sqrt{\langle \bar{\mathbf{p}}_\perp^2 \rangle}}{\bar{m}_D} \right] \\
 &\times \left\{ A(\tau)^{2+1} B(\tau)^{-2} G(\tau)^{-1+2} \bar{I}_a(\bar{\mathbf{p}}_\perp, \bar{p}_z) \partial_{\bar{p}_z}^2 f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z) \right. \\
 &+ A(\tau)^{1+2} B(\tau)^{-1+1} G(\tau)^{-1} \bar{I}_b(\bar{\mathbf{p}}_\perp, \bar{p}_z) \frac{2f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z)}{\bar{\mathbf{p}}_\perp} \\
 &\left. \times [1 + \bar{\mathbf{p}}_\perp \partial_{\bar{\mathbf{p}}_\perp} + \bar{p}_z \partial_{\bar{p}_z}] f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z) \right\}, \tag{B.89}
 \end{aligned}$$

where the two contributions are found to scale differently. In contrast, in the isotropic case they scale the same, compare with the scaling analysis in the Supplemental Material of Sec. 4.3. Due to the strong momentum anisotropy, longitudinal gradients are dominant and we can approximate

$$\begin{aligned}
 C^{\text{FP}}(\tau, \mathbf{p}_\perp, p_z) &\simeq A(\tau)^3 B(\tau)^{-2} G(\tau) \log \left[A(\tau)^{-\frac{1}{2}} B(\tau)^{-\frac{1}{2}} G(\tau)^{\frac{1}{2}} \frac{\sqrt{\langle \bar{\mathbf{p}}_\perp^2 \rangle}}{\bar{m}_D} \right] \\
 &\times \lambda_0 \bar{I}_a(\bar{\mathbf{p}}_\perp, \bar{p}_z) \partial_{\bar{p}_z}^2 f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z) \tag{B.90}
 \end{aligned}$$

$$\begin{aligned}
 &\equiv A(\tau)^3 B(\tau)^{-2} G(\tau) \log \left[A(\tau)^{-\frac{1}{2}} B(\tau)^{-\frac{1}{2}} G(\tau)^{\frac{1}{2}} \frac{\sqrt{\langle \bar{\mathbf{p}}_\perp^2 \rangle}}{\bar{m}_D} \right] \\
 &\times \tilde{C}^{\text{FP}}[f_S](\bar{\mathbf{p}}_\perp, \bar{p}_z). \tag{B.91}
 \end{aligned}$$

We note here that if one considers the scaling of the collision kernel due to $\sim I_a$ instead for dilute systems with $f \ll 1$, one recovers from the Boltzmann equation the dilute fixed point with $(\alpha_\infty, \beta_\infty, \gamma_\infty) = (-1, 0, 0)$ discussed in [195].

We can continue from here to derive a third scaling relation from the Boltzmann equation, which will allow us to solve for the different prescaling factors.

Boltzmann equation

We now consider the Boltzmann equation. On the left-hand side, we have

$$\begin{aligned}
 &\left[\partial_\tau - \frac{p_z}{\tau} \partial_{p_z} \right] A(\tau) f_S(B(\tau) \mathbf{p}_\perp = \bar{\mathbf{p}}_\perp, G(\tau) p_z = \bar{p}_z) \\
 &= A(\tau) \left\{ \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \bar{\mathbf{p}}_\perp \cdot \partial_{\bar{\mathbf{p}}_\perp} + \left(\frac{\dot{G}}{G} - \frac{1}{\tau} \right) \bar{p}_z \partial_{\bar{p}_z} \right\} f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z). \tag{B.92}
 \end{aligned}$$

Using the identities we derived from conservation laws in B.2, we can simplify the left-hand side to

$$= A(\tau) \left\{ \frac{\dot{A}}{A} + \left(\frac{1}{\tau} + \frac{\dot{A}}{A} - \frac{1}{\tau} \right) \bar{p}_z \partial_{\bar{p}_z} \right\} f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z) \quad (\text{B.93})$$

$$= \dot{A}(\tau) \{1 + \bar{p}_z \partial_{\bar{p}_z}\} f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z) \quad (\text{B.94})$$

$$= C^{\text{FP}}[f](\tau, \mathbf{p}_\perp, p_z) . \quad (\text{B.95})$$

Together with the scaling of the Fokker-Planck collision kernel of (B.91), we find

$$\begin{aligned} \dot{A}(\tau) \{1 + \bar{p}_z \partial_{\bar{p}_z}\} f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z) &= A^3(\tau) B^{-2}(\tau) G(\tau) \\ &\times \log \left[A^{-\frac{1}{2}} B^{-\frac{1}{2}} G^{\frac{1}{2}} \frac{\sqrt{\langle \bar{\mathbf{p}}_\perp^2 \rangle}}{\bar{m}_D} \right] \tilde{C}_S^{\text{FP}}[f_S](\bar{\mathbf{p}}_\perp, \bar{p}_z) . \end{aligned} \quad (\text{B.96})$$

Separating variables as in the isotropic case of Sec. 4.3, we have

$$\frac{1}{D_3} = \frac{[1 + \bar{p}_z \partial_{\bar{p}_z}] f_S(\bar{\mathbf{p}}_\perp, \bar{p}_z)}{\tilde{C}_S^{\text{FP}}[f_S](\bar{\mathbf{p}}_\perp, \bar{p}_z)} \quad (\text{B.97})$$

$$= \frac{A^3(\tau) B^{-2}(\tau) G(\tau) \log \left[A^{-\frac{1}{2}} B^{-\frac{1}{2}} G^{\frac{1}{2}} \frac{\sqrt{\langle \bar{\mathbf{p}}_\perp^2 \rangle}}{\bar{m}_D} \right]}{\dot{A}(\tau)} . \quad (\text{B.98})$$

We can fix D_3 by evaluating at $\tau = \tau_{\text{ref}}$

$$D_3 = \frac{\alpha_{\text{ref}}}{\tau_{\text{ref}} \log \left(\sqrt{\langle \bar{\mathbf{p}}_\perp^2 \rangle} / \bar{m}_D \right)} , \quad (\text{B.99})$$

where $A(\tau_{\text{ref}}) = 1 = B(\tau_{\text{ref}}) = G(\tau_{\text{ref}})$ and $\dot{A}(\tau_{\text{ref}}) = \alpha_{\text{ref}} / \tau_{\text{ref}}$. This allows us to isolate the third scaling relation as

$$\partial_\tau A(\tau) = D_3 A^3(\tau) B^{-2}(\tau) G(\tau) \log \left[A^{-\frac{1}{2}} B^{-\frac{1}{2}} G^{\frac{1}{2}} \frac{\sqrt{\langle \bar{\mathbf{p}}_\perp^2 \rangle}}{\bar{m}_D} \right] \quad (\text{B.100})$$

$$\begin{aligned} &= \frac{\alpha_{\text{ref}}}{\tau_{\text{ref}}} A^3(\tau) B^{-2}(\tau) G(\tau) \left[\frac{\log \left(A^{-\frac{1}{2}} B^{-\frac{1}{2}} G^{\frac{1}{2}} \right)}{\log \left(\sqrt{\langle \bar{\mathbf{p}}_\perp^2 \rangle} / \bar{m}_D \right)} + 1 \right] \\ &= \frac{\alpha_{\text{ref}}}{\tau_{\text{ref}}} A(\tau)^3 B(\tau)^{-2} G(\tau) \left[1 - \frac{\log \left(A(\tau) B(\tau) G(\tau)^{-1} \right)}{\log \left(\langle \bar{\mathbf{p}}_\perp^2 \rangle / \bar{m}_D^2 \right)} \right] . \end{aligned} \quad (\text{B.101})$$

Using the identities from (B.2), this can be simplified into an equation for $A(\tau)$ or $G(\tau)$ only such that

$$\partial_\tau A(\tau) = \frac{1}{2} D_3 A^4(\tau) \tau \frac{\mathcal{E}_0^4}{n_0^5} \log \left[\tau \frac{\mathcal{E}_0^3}{n_0^4} \frac{\langle \bar{\mathbf{p}}_\perp^2 \rangle}{\bar{m}_D^2} \right] \quad (\text{B.102})$$

$$\equiv D_{3,1} A^4(\tau) \tau \log [\tau D_{3,2}] , \quad (\text{B.103})$$

where we abbreviated

$$D_{3,1} = \frac{1}{2} D_3 \frac{\mathcal{E}_0^4}{n_0^5} , \quad D_{3,2} = \frac{\mathcal{E}_0^3}{n_0^4} \frac{\langle \bar{\mathbf{p}}_\perp^2 \rangle}{\bar{m}_D^2} . \quad (\text{B.104})$$

Equivalently, we can use $\beta(\tau) = 0$ and $\alpha(\tau) = \gamma(\tau) - 1$ from Eq. (B.57). This implies $B(\tau) = 1$ and $G(\tau) = A(\tau) \tau / \tau_{\text{ref}}$ and we find again Eq. (B.103), with

$$D_{3,1} = \frac{D_3}{2\tau_{\text{ref}}} , \quad D_{3,2} = \frac{\langle \bar{\mathbf{p}}_\perp^2 \rangle}{\tau_{\text{ref}} \bar{m}_D^2} . \quad (\text{B.105})$$

Eq. (B.103) has an exact solution (two further imaginary solutions are excluded)

$$A(\tau)^3 = \frac{2\alpha_\infty}{4A_1 - D_{3,1}\tau^2 \left[1 - 2\log(\tau D_{3,2}) \right]} , \quad (\text{B.106a})$$

$$\alpha(\tau) = \frac{2\alpha_\infty D_{3,1} \tau^2 \log(\tau D_{3,2})}{4A_1 - D_{3,1}\tau^2 \left[1 - 2\log(\tau D_{3,2}) \right]} , \quad (\text{B.106b})$$

where A_1 is an integration constant. At asymptotically late times, Eq. (B.106b) reduces to α_∞ , i.e. $\lim_{\tau \rightarrow \infty} \alpha(\tau) = \alpha_\infty$. We can extract the late-time behavior in the vicinity of the fixed point by expanding around asymptotically large but finite times

$$\lim_{\tau \rightarrow \infty} \alpha(\tau) \simeq \alpha_\infty \left[1 + \frac{1}{-1 + 2\log(\tau D_{3,2})} \right] . \quad (\text{B.107})$$

We thus find asymptotically a logarithmic approach to the fixed point value $\alpha_\infty = -2/3$, where faster power-law corrections have already decayed. This is consistent with what we observed in the isotropic case of Sec. 4.3. A similar logarithmic late-time correction was found for $\gamma(\tau) = 1 + \alpha(\tau)$ in [195].

As in the non-expanding case, this demonstrates that the system will only converge onto the fixed point at asymptotically late times. We note here that the first stage of bottom-up thermalization ends parametrically when $\tau_f Q \geq \alpha_S^{-3/2}$ [119].

Finally, as in the isotropic case, we will compare the prescaling solutions

$$\beta(\tau) = 0 , \quad \gamma(\tau) = \alpha(\tau) + 1 , \quad \alpha(\tau) = \frac{2\alpha_\infty D_{3,1} \tau^2 \log(\tau D_{3,2})}{4A_1 - D_{3,1}\tau^2 \left[1 - 2\log(\tau D_{3,2}) \right]} , \quad (\text{B.108})$$

derived using the Fokker-Planck collision kernel and the anisotropic limit, to QCD kinetic

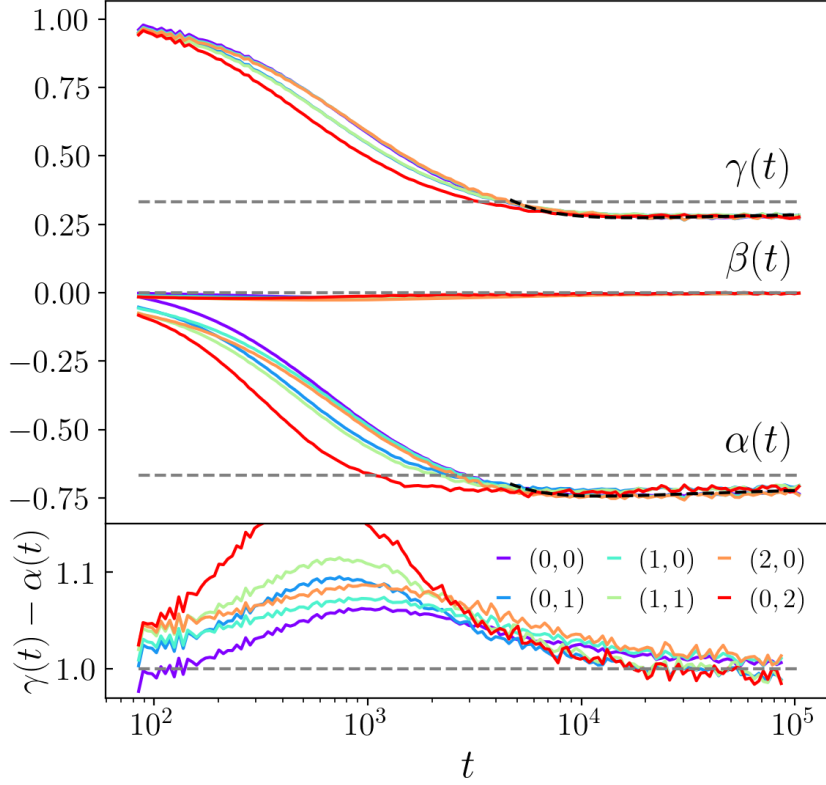


Figure B.1: *Top:* Prescaling exponents in color extracted from different moments according to Eq. (B.47) are compared to Eq. (B.106b) as represented by the dashed black lines. *Bottom:* Violation of scaling relation $1 = \gamma(\tau) - \alpha(\tau)$ for all considered moments over the majority of the time evolution. Results are shown for $n_0 = 0.1$, $g^2 = 10^{-6}$, and $\xi = 2$. The constants A_1 , $D_{3,1}$, and $D_{3,2}$ are adjusted through a least-squares minimization. The time is in units of Q .

theory. Starting from the same initial conditions and system parameters as in Sec. 4.3, we solve the Boltzmann equation with the full QCD kinetic theory kernel and longitudinal expansion. The time-dependent prescaling exponents can then be extracted from moments of the distribution function according to Eq. (B.47). The result of such a comparison can be found in Fig. B.1. We find that prescaling exponents from different moments converge considerably leading up to $tQ \simeq 5 \times 10^3$, after which they collectively approach their respective fixed point values. At finer resolution scale, a finite spread between exponents from different moments remains however, similar to the non-expanding case. We compare this dynamics to the predictions from Eq. (B.106b) as represented by the dashed black lines and find that they capture the late-time evolution well. In contrast to the non-expanding case, the early-time dynamics of a moment specific exponent is only captured qualitatively. This can be understood by the violation of the assumed scaling relations as represented by the time evolution of $\beta(\tau) \neq 0$ in the upper panel and the deviation of $\gamma(\tau) - \alpha(\tau)$ from unity in the lower panel for a majority of the time evolution. More explicitly, we find that $\tau n(\tau) \simeq \text{const}$ is violated by $\lesssim 5\%$ and $\tau \mathcal{E}(\tau) \simeq \text{const}$ is violated by $\lesssim 10\%$ from initial time to the time of convergence $tQ \simeq 5 \times 10^3$ (we emphasize that this is not the exact energy conservation equation, which is satisfied throughout). The

violations decrease below sub-percent for later times, where our solution Eq. (B.106b) also turns out to work well.

Avoiding the anisotropic limit

If we don't use the anisotropic limit, energy conservation equation provides us with a more general scaling relation

$$0 = \partial_\tau(\tau\mathcal{E}) + T^{zz} \quad (\text{B.109})$$

$$\begin{aligned} 0 = & \bar{\mathcal{E}}A(\tau)B^{-3}(\tau)G^{-1}(\tau) \left[1 + \tau \frac{\dot{A}}{A} - 3\tau \frac{\dot{B}}{B} - \tau \frac{\dot{G}}{G} \right] \\ & + A(\tau)B^{-1}(\tau)G^{-3}(\tau)\bar{T}^{zz} \end{aligned} \quad (\text{B.110})$$

$$\Leftrightarrow \quad 0 = \frac{\bar{\mathcal{E}}}{\bar{T}^{zz}}B^{-2}(\tau)G^2(\tau) \left[1 + \tau \left(\frac{\dot{A}}{A} - 3\frac{\dot{B}}{B} - \frac{\dot{G}}{G} \right) \right] + 1. \quad (\text{B.111})$$

In contrast to the case employing the anisotropic limit, we obtain another differential equation besides the one resulting from the Boltzmann equation. As a result, one is required to solve two coupled differential equations (using particle number conservation as before) and a simple analytical solution as we found using the anisotropic limit falls out of reach. Whether such a more general system of equations admits solutions besides the known bottom-up fixed point remains to be investigated.

5 Outlook

This thesis has explored various facets of strongly correlated quantum many-body systems, both in and out of thermal equilibrium, with a focus on emergent phenomena such as strong interactions, self-bound droplets, and nonthermal attractors. These phenomena arise from the intricate interplay between the constituent components of such systems. By approaching the dynamics of these systems from first principles, we have made substantial progress in understanding how collective behavior emerges from fundamental quantum dynamics, particularly in regimes where perturbative methods fail. To access these strongly correlated regimes, we employed non-perturbative large- N expansion techniques together with large-scale numerical simulations.

Universality has emerged as a central guiding principle in our investigation of strong correlations, both in and away from equilibrium. In Ch. 2, we employed novel large- N expansion techniques to study the emergence of strong interactions in four-dimensional tensor field theories from first principles. These theories offer a promising framework to investigate strong interactions in thermal equilibrium and potentially also far from equilibrium. The regime of asymptotically strong yet scale-dependent couplings offers universal insights into gap masses and possible bound states in equilibrium, as well as bounds on transport coefficients and thermalization times out of equilibrium – paralleling what is observed in holographic models with scale-independent couplings [14, 15]. These universal bounds play a central role across a broad range of strongly correlated systems, including the unitary Fermi gas and the quark-gluon plasma [16, 17], both of which exhibit a shear-viscosity-to-entropy-density ratio near the holographic strong-coupling bound [14], despite their distinct microscopics and energy scales. In Ch. 4, we further demonstrated the universal emergence of prescaling across a variety of universality classes, the self-organized attractor mechanism associated with the inverse particle cascade, and universal ratios of far-from-equilibrium time-dependent transport coefficients characterizing hydrodynamic excitations around nonthermal attractors. As nonthermal attractors become increasingly accessible in experimental platforms such as cold atoms, our understanding of what constitutes a universality class continues to refine. Identifying the minimal ingredients defining such classes allows us to explore the relevant low-energy degrees of freedom, crucial for analyzing otherwise intractable members of the same class – ranging from cold atomic gases to the quark-gluon plasma [91]. These findings highlight that universality is not only a powerful tool for understanding critical phenomena and transport in equilibrium, but also a conceptual framework for describing dynamics far from equilibrium.

Looking ahead, continued advances in experimental techniques across ultracold atomic systems, heavy-ion collisions, and quantum simulation will further expand the landscape of accessible strongly correlated quantum systems. These developments will enable deeper

investigations of regimes beyond the reach of traditional perturbative approaches. The non-perturbative frameworks developed in this thesis – such as large- N expansions, hydrodynamics, and universal scaling laws – offer a robust foundation for future studies in this area. In particular, ultracold atomic gases have emerged as well-controlled platforms for probing many-body quantum dynamics, opening up rich avenues for exploring the strongly correlated phenomena studied herein.

Future directions include the experimental realization of bosonic droplets, as explored in Ch. 3, particularly in ultracold cesium atoms, which exhibit an extremely narrow Feshbach resonance near 19.85 G [196, 197]. The associated large negative effective range and narrow resonance width render them ideal candidates for realizing self-bound droplets. Recent experiments have already begun probing few-body bound states and collective effects in this regime [198], providing a gateway to study the many-body crossover and nontrivial phase structure predicted by our theoretical model. While our analysis focused on droplets in free space, understanding their behavior under external confinement – such as harmonic traps or optical lattices – remains an important open challenge, as stability criteria can differ substantially from the homogeneous case.

Nonthermal fixed points have now been observed in several ultracold atom platforms operating in the many-body regime [19–21, 108–117]. These systems typically exhibit universal scaling only over finite time windows, as effects like atom loss, three-body scattering, or environmental interactions drive thermalization. Initially, values of scaling exponents observed in cold atom systems quenched far from equilibrium deviated from theory predictions. It was however recently demonstrated in $2 + 1d$ in [21], that the theoretically predicted scaling exponents and scaling relations can indeed be observed in the experimentally accessible finite-time dynamics if one accounts for the initial-state dependent prescaling effects derived in Sec. 4.3. These achievements further establish cold atomic gases as well-controlled platforms for studying universal scaling. They also motivate experiments involving scaling-breaking perturbations, such as time-dependent tuning of interactions via Feshbach resonances.

Progress toward observing the scaling instability and emergent quasiparticle structure described in Sec. 4.2 hinges on accessing the spectral function, which encodes the excitation spectrum and stability properties of the system. This requires one to study the response of the far-from-equilibrium system to external perturbations. Response measurements are an established tool in thermal equilibrium to extract transport properties such as shear viscosity through Kubo relations [199], or spectral properties such as dispersion relations or life times of excitations through the measurement of spectral functions [200]. Spectral properties then also provide access to statistical properties such as particle number distributions through the fluctuation-dissipation relation [201]. As demonstrated in Sec. 4.2, linear response measurements out of equilibrium are also possible, similarly providing information on now time-dependent spectral properties by studying the response to a perturbation around some nonequilibrium state. However, out of equilibrium, thermal concepts such as the fluctuation-dissipation relation and hydrodynamic constitutive relations have to be adjusted. On the experimental side, the scaling instability crucially remains manifest only in rescaled coordinates $\sim t^\beta |\mathbf{p}|$, whereas it scales to larger distances with increasing times in space-time coordinates. To investigate this phenomenon for nonrelativistic systems,

experimental realizations with sufficiently large spatial volumes will be necessary.

One can further study far-from-equilibrium perturbations that break the symmetry of the nonthermal attractor. As proposed in Sec. 4.5, the evolution of perturbations that break spatial homogeneity or isotropy in particular give rise to a far-from-equilibrium hydrodynamic evolution around the corresponding nonthermal attractor. Such perturbations offer access to the dynamical evolution of bulk properties, including pressure and the far-from-equilibrium generalization of transport coefficients, that could be probed in cold atom experiments. Of particular interest is the time evolution of shear viscosity far from equilibrium, which we proposed to exhibit a scaling behavior $\eta(t) \sim t$ in the non-relativistic energy cascade – a nonthermal attractor that has recently been realized experimentally in [20, 21, 114, 117]. Investigating this behavior experimentally may be feasible by adapting the established protocols for measuring shear viscosity in near-equilibrium ultracold atom systems [16, 17].

The discovery of nonthermal scaling has provided an essential organizing principle for the challenging real-time description of quantum many-body systems far from equilibrium, applicable to weakly coupled systems with large classical fluctuations (where anti-commutators dominate). A key unresolved challenge in quantum physics is the real-time description of strongly coupled systems dominated by quantum fluctuations (or commutators), where no comparable framework has yet been established. With the recent advent of quantum many-body simulators that operate beyond the classical regime [202–204], such understanding has become increasingly urgent, with potential applications ranging from quantum information and condensed matter to high-energy physics. Nonthermal scaling was first identified [78] for $O(N)$ or $U(N)$ models at next-to-leading order in large- N as possible solutions to the real-time evolution equations of correlation functions for large classical fluctuations. One can similarly conjecture that quantum scaling solutions exist in the regime of strong couplings and large quantum fluctuations. Promising early steps have recently been taken through studies of the quantum Kibble-Zurek mechanism in Rydberg atom arrays [205–207], but the broader landscape of strongly coupled quantum systems far from equilibrium remains largely unexplored. The discovery of such universal quantum scaling behavior could provide a similarly unifying perspective as that offered by nonthermal attractors in the classical regime.

Altogether, the frameworks developed and explored in this thesis not only pave the way for a deeper theoretical understanding of strongly correlated quantum many-body systems in and out of equilibrium, but also open new possibilities for experimental investigation. As theory and experiment continue to converge, the path forward promises a richer, more unified picture of quantum many-body physics under extreme conditions.

6 Acknowledgements

At this point, I would like to express my heartfelt gratitude to all those who have supported and accompanied me throughout the years of my doctoral studies.

Zu Beginn möchte ich mich bei meinem Doktorvater Jürgen Berges bedanken. Vielen Dank für die vielen inspirierenden Gespräche, dein Vertrauen und deinen Zuspruch, für deine richtungsweisenden Worte in der Wissenschaft und darüber hinaus, sowie für deine stetige Unterstützung. Ich bin dankbar für die von dir ermöglichte Freiheit, meine wissenschaftliche Identität zu entfalten, und für die umfassende Unterstützung, verschiedenste Forschungsperspektiven und -kooperationen wahrzunehmen. Deine wissenschaftliche Neugier hat mich stets motiviert und inspiriert. Vielen Dank für deine Freundschaft.

Mein besonderer Dank gilt auch Tilman Enss dafür, dass er sich bereit erklärt hat, als Zweitgutachter meiner Dissertation zu fungieren.

I would also like to thank Michal P. Heller for many inspiring and insightful conversations, for his hospitality during various visits, and for our ongoing research collaboration. I greatly appreciate your guidance throughout my doctoral journey. Furthermore, I am thankful to Razvan Gurau and Aleksas Mazeliauskas for our joint work and their valuable mentorship.

During my studies, I had the privilege of undertaking extended research visits abroad, for which I am truly grateful. In particular, I thank Raju Venugopalan for inviting me to Brookhaven National Laboratory (BNL), for his generous hospitality, stimulating discussions, and his continued support and encouragement. I am also grateful to the nuclear physics groups at BNL and Stony Brook University for welcoming me and for many enlightening conversations – especially João Barata, Shohini Bhattacharya, Adrien Florio, Sebastian Griener, Dima Kharzeev, Rob Pisarski, Himanshu Raj, Rajeev Singh, and Derek Teaney.

I also wish to thank the Simons Collaboration on Confinement and QCD Strings, whose support enabled my stay at BNL and participation in their stimulating workshops at the University of Michigan, Princeton University, and New York University. I appreciate the engaging scientific environment and discussions with collaborators including Sergei Dubovsky, Victor Gorbenko, Igor Klebanov, Fedor Popov, and Grisha Tarnopolsky.

I am further grateful for the many invitations to present my research and engage with local scientific communities. Beyond those already mentioned, I would like to thank Zohreh Davoudi and the University of Maryland, Paul Romatschke and the University of Colorado Boulder, Krishna Rajagopal and the Center of Theoretical Physics at MIT, and Dam

Thanh Son and the Kadanoff Center at the University of Chicago. I also extend my thanks to the organizers of the Rising Researchers Seminar for their hospitality.

My thanks also go to the Kavli Institute for Theoretical Physics and the organizers of the workshop *The Many Faces of Relativistic Fluid Dynamics* – particularly Michal P. Heller, Pavel Kovtun, and Jorge Noronha – for their extended hospitality, as well as for the many fascinating discussions during the program and the opportunity to present my work.

I am also grateful to Princeton University, the Institute for Advanced Study, and the Simons Collaboration on Confinement and QCD Strings for their invitation and support to attend *Understanding Confinement* as part of the Principles in Theoretical Physics program. Furthermore, I would like to thank the InQubator for Quantum Simulation at the University of Washington – especially Niklas Müller – for their hospitality during the workshop *Thermalization, from Cold Atoms to Hot Quantum Chromodynamics*. I also want to thank Ghent University and again Michal P. Heller for their warm welcome during my visits.

Finally, I would like to express my sincere appreciation to Dam Thanh Son and Vincenzo Vitelli for the opportunity to visit the Kadanoff Center at the University of Chicago for an extended stay and to explore common research directions. I am very thankful for this experience and the many stimulating discussions within this vibrant research community. In particular, I would like to mention Subham Chowdhury, Luca Delacretaz, Mikhail Stephanov, and Ke Wang.

Beyond those mentioned above, I thank the scientific and administrative community at Heidelberg University and beyond for their collaboration, discussions, support, and friendship. I am especially grateful to Laura Battini, Kirill Boguslavski, Aleksandr Chatrchyan, Alexander Schuckert, Gregor Fauth, Jannis Gebhard, Sonali Gera, Michal Heinrich, Louis Jussios, Tina Kuka, Jendrik Marijan, Aleksas Mazeliauskas, Viktoria Noel, Robert Ott, Asier Piñeiro Orioli, Daniel Spitz, Finn Schmutte, Adam Takacs, Torsten Zache, and Fabian Zhou.

I am especially thankful to Laura Battini, Michal Heinrich, Michal P. Heller, Razvan Gurau, and Daniel Spitz for reading and providing valuable feedback on parts of this thesis. Any remaining mistakes are, of course, entirely my own.

My doctoral research and the opportunities it afforded were made possible in part by the generous support of several institutions and programs. I gratefully acknowledge funding from the German Research Foundation (DFG) through the Collaborative Research Center SFB 1225 ISOQUANT (Project-ID 273811115) and the Heidelberg STRUCTURES Excellence Cluster under Germanys Excellence Strategy EXC2181/1390900948. Computational resources were provided by bwHPC under grant no. INST 40/575-1 FUGG (JUSTUS 2 cluster) with support from the state of Baden-Württemberg. I also acknowledge support from the German Academic Exchange Service (DAAD) through a research-abroad scholarship for doctoral students. Finally, I am deeply thankful for the longstanding support from the Studienstiftung des deutschen Volkes, which has enabled me to pursue my academic path.

Zum Schluss möchte ich mich bei meiner Familie, insbesondere meinen Eltern Martina und Volkmar, meinen Freund*innen und meiner Partnerin für ihre fortwährende Unterstützung bedanken. Besonders möchte ich Evi und Helmut danken: Danke, dass ihr meine Begeisterung für die Natur gefördert und mich immer bedingungslos unterstützt habt.

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