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On Unitarity for $\mathfrak{sl}(m|n)$-Supermodules: Dirac Cohomology, Superdimension, Indices

Advisor:
Prof. Dr. Johannes Walcher
Prof. Dr. Rainer Weissauer

“I think nature’s imagination is so much greater than man’s, she’s never going to let us relax.”

(Richard P. Feynman)

Abstract

This thesis investigates unitarizable supermodules over special linear Lie superalgebras $\mathfrak{sl}(m|n)$ and their basic classical counterparts $A(m|n)$, denoted $\mathfrak{g}$, with a focus on their structure, classification, and applications in both mathematics and theoretical physics. It is structured in four main parts, each exploring a distinct but interrelated aspect of the theory.

The first part develops a general framework for understanding unitarity and provides a concise classification of unitarizable simple $\mathfrak{g}$-supermodules, derived using the Dirac inequality and decomposition under the even Lie subalgebra. The Dirac operator and its associated Dirac cohomology serve as central tools in this study, capturing essential aspects of unitarity. We demonstrate that Dirac cohomology can uniquely determine unitarizable supermodules, and compute it explicitly for of unitarizable simple supermodules. This leads to a refined characterization of unitarity, forming the basis for our novel classification of unitarizable simple supermodules. Furthermore, we establish a connection between Dirac cohomology and Kostant's cohomology of Lie superalgebras, derive a decomposition of formal characters, and introduce a Dirac index.

In the second part, we construct a formal superdimension for infinite-dimensional unitarizable supermodules, inspired by the theory of relative discrete series representations. We show that this superdimension vanishes for most simple supermodules but is non-trivial precisely when the infinitesimal character has maximal degree of atypicality. In particular, our result aligns with the Kac–Wakimoto conjecture for finite-dimensional supermodules.

The third part investigates applications to theoretical physics, focusing in particular on the so-called "superconformal index" – a character-valued invariant assigned by physicists to unitarizable supermodules of Lie superalgebras, such as $\mathfrak{su}(2,2|n)$, which appear in the context of certain quantum field theories. The index is computed as a supertrace over a Hilbert space and remains constant across families of representations that arise from varying physical parameters. This invariance is due to the fact that only "short" simple supermodules contribute to the index, making it stable under recombination phenomena occurring at the boundary of the unitarity region. We develop these notions for unitarizable supermodules over $\mathfrak{g}$. Along the way, we provide a precise dictionary between various notions from theoretical physics and mathematical terminology. Our final result is a kind of "index theorem" that relates the counting of atypical constituents in a general unitarizable $\mathfrak{g}$-supermodule to the character-valued Q-Witten index, expressed as a supertrace over the full supermodule. The formal superdimension of part 2 can also be formulated in this framework.

The final part is an addendum that extends the Dirac operator and cohomology to their cubic counterparts. We develop a theory of cubic Dirac operators associated to parabolic subalgebras and prove a super-analog of the Casselman–Osborne theorem. We show that Dirac cohomology is trivial unless for highest weight supermodules, and demonstrate, under suitable conditions, an embedding of Dirac cohomology into Kostant's (co)homology. This embedding becomes an isomorphism in the unitarizable case. We also provide complete computations of Dirac cohomology for finite-dimensional simple supermodules with typical highest weight and for supermodules in the parabolic BGG category.

Zusammenfassung

Diese Dissertation untersucht unitarisierbare Supermoduln spezieller linearer Lie-Superalgebren $\mathfrak{sl}(m|n)$ und ihrer einfachen klassischen Gegenstücken $A(m|n)$, bezeichnet mit $\mathfrak{g}$, mit einem Schwerpunkt auf deren Struktur, Klassifikation und Anwendungen in der Mathematik und theoretischen Physik. Sie ist in vier Hauptteile gegliedert, von denen jeder einen eigenen, aber miteinander verknüpften Aspekt der Theorie beleuchtet.

Der erste Teil entwickelt einen allgemeinen Rahmen zum Verständnis der Unitarität und liefert eine einfache Klassifikation unitarisierbarer einfacher $\mathfrak{g}$-Supermoduln, abgeleitet mithilfe der Dirac-Ungleichung und der Zerlegung bezüglich der geraden Lie-Unteralgebra. Der Dirac-Operator und die zugehörige Dirac-Kohomologie dienen in dieser Arbeit als zentrale Werkzeuge und erfassen wesentliche Aspekte der Unitarität. Wir zeigen, dass die Dirac-Kohomologie unitarisierbare Supermoduln eindeutig bestimmen kann, und berechnen sie explizit für unitarisierbare einfache Supermoduln. Dies führt zu einer verfeinerten Charakterisierung der Unitarität, die die Grundlage für unsere neuartige Klassifikation unitarisierbarer einfacher Supermoduln bildet. Darüber hinaus stellen wir eine Verbindung zwischen der Dirac-Kohomologie und der Kostant-Kohomologie von Lie-Superalgebren her, leiten eine Zerlegung formaler Charaktere ab und führen einen Dirac-Index ein.

Im zweiten Teil konstruieren wir eine formale Superdimension für unendlich-dimensionale unitarisierbare Supermoduln, inspiriert von der Theorie der relativen diskreten Serien-Darstellungen. Wir zeigen, dass diese Superdimension für die meisten einfachen Supermoduln verschwindet, aber genau dann nicht trivial ist, wenn der infinitesimale Charakter den maximalen Atypi-kalitätsgrad besitzt. Insbesondere steht unser Ergebnis im Einklang mit der Kac–Wakimoto-Vermutung für endlich-dimensionale Supermoduln.

Der dritte Teil untersucht Anwendungen in der theoretischen Physik, mit besonderem Fokus auf den sogenannten „superkonformen Index“ – eine charakterwertige Invariante, welche in der Physik unitarisierbaren Supermoduln von Lie-Superalgebren wie $\mathfrak{su}(2,2|n)$ zugeordnet wird, die im Kontext bestimmter Quantenfeldtheorien auftreten. Der Index wird als Superspur über einem Hilbertraum berechnet und bleibt konstant innerhalb von Darstellungsfamilien, die durch Variation physikalischer Parameter entstehen. Diese Invarianz beruht darauf, dass nur „kurze“ einfache Supermoduln zum Index beitragen, wodurch er gegenüber Rekombinationsphänomenen stabil bleibt, die am Rand der Unitaritätsregion auftreten. Wir entwickeln diese Konzepte für unitarisierbare Supermoduln über $\mathfrak{g}$. Dabei geben wir ein präzises Wörterbuch zwischen verschiedenen Begriffen der theoretischen Physik und mathematischer Terminologie an. Unser Hauptergebnis ist eine Art „Index-theorem“, das die Zählung atypischer Bestandteile in einem allgemeinen unitarisierbaren $\mathfrak{g}$-Supermodul mit dem charakterwerigen Q-Witten-Index in Beziehung setzt, welcher als Superspur über den gesamten Supermodul ausgedrückt ist. Die formale Superdimension aus Teil 2 lässt sich ebenfalls in diesem Rahmen formulieren.

Der abschließende Teil ist ein Zusatz, der den Dirac-Operator und die Dirac-Kohomologie auf ihre kubischen Gegenstücke erweitert. Wir entwickeln eine Theorie kubischer Dirac-

Operatoren, die mit parabolischen Unteralgebren assoziiert sind, und beweisen ein Super-Analogon des Satzes von Casselman–Osborne. Wir zeigen, dass die Dirac-Kohomologie nur für höchstgewichtige Supermoduln nicht trivial ist, und demonstrieren unter geeigneten Voraussetzungen eine Einbettung der Dirac-Kohomologie in die (Ko-)Homologie von Kostant. Diese Einbettung wird im unitarisierbaren Fall zu einem Isomorphismus. Darüber hinaus liefern wir vollständige Berechnungen der Dirac-Kohomologie für endlich-dimensionale einfache Supermoduln mit typischem höchsten Gewicht sowie für Supermoduln in der parabolischen BGG-Kategorie.

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Published contents

This thesis is based on results which appeared in the following articles:

[123] S. Schmidt. Dirac Cohomology and Unitarizable Supermodules over Lie Superalgebras of Type $A(m|n)$. (November 2024) arXiv:2411.09404.

[106] S. Noja, S. Schmidt and Raphael Senghaas. Cubic Dirac Operators and Dirac Cohomology for Basic Classical Lie Superalgebras. (March 2025) arXiv:2503.04187.

[126] S. Schmidt and J. Walcher. An index for unitarizable $sl(m|n)$-supermodules. (April 2025) arXiv:2504.09161.

[124] S. Schmidt. Classification of Unitarizable $\mathfrak{sl}(m|n)$-Supermodules. In preparation (2025).

[125] S. Schmidt. A Superdimension for Unitarizable $\mathfrak{sl}(m|n)$-Supermodules. In preparation (2025).

The presentation of the Dirac operator, Dirac cohomology, and unitarizable supermodules in Chapter 7 is adapted from [123]. The classification of unitarizable supermodules in Chapter 8, and the formal superdimension in Chapter 10, are based on the forthcoming works [124] and [125], respectively. Furthermore, the exposition in Part IV, which concerns indices and physics, is based on [126]. Finally, the discussion of cubic Dirac operators, their Dirac cohomology, and the representation theory of basic classical Lie superalgebras is adapted from [106].

Contents

1. Introduction

1.1. Vue d'ensemble

Representation theory is the study of actions of algebraic structures — such as groups, rings, or algebras — on vector spaces through linear transformations. In its most classical form, the aim is to study a group G by analyzing the structure and properties of homomorphisms from G into the general linear group $\mathrm{GL}(V)$ of a vector space V. These homomorphisms are known as *representations* of G.

In the setting of Lie groups, representation theory becomes intimately connected with geometry and analysis. A particularly important class of representations in this context are *unitary representations* – homomorphisms from a Lie group G to the group of unitary operators on a Hilbert space $\mathcal{H}$. These representations preserve inner products and thus provide a natural framework for harmonic analysis, probability theory, and quantum physics. The motivation to study unitary representations of Lie groups originates from two seminal developments in the twentieth century: Wigner's classification of elementary particles and Gelfand's program of abstract harmonic analysis.

The Wigner program arises from the foundational principles of quantum mechanics. In quantum theory, the states of a physical system are modeled by unit vectors in a Hilbert space, and symmetries of the system correspond to unitary (or anti-unitary) transformations that preserve transition probabilities. In a relativistic setting, the symmetry group of spacetime is the Poincaré group, and Wigner's groundbreaking result in 1939 demonstrated that elementary particles correspond to irreducible unitary representations of this group. More generally, physical systems are governed by symmetry groups, and their possible quantum states and physical observables are encoded in the unitary representation theory of these groups. This insight laid the foundation for the application of representation theory in high-energy physics and quantum field theory.

The Gelfand program seeks to generalize classical Fourier analysis from the abelian case (*e.g.*, the circle group or the real line) to the non-abelian case, such as compact Lie groups or semisimple Lie groups. Classical Fourier analysis can be interpreted as decomposing functions into irreducible representations of the additive group $\mathbb{R}$ or the circle group $\mathbb{T}$. In the non-abelian setting, one instead decomposes $L^2(G)$, the space of square-integrable functions on a group G, into irreducible unitary representations of G. This leads to deep questions in harmonic analysis, differential geometry, and number theory. The celebrated Peter–Weyl theorem, the Plancherel formula, and the theory of characters all emerge as extensions of Fourier analysis within this broader framework.

One of the central goals in representation theory is to classify and analyze all possible unitary representations of a given Lie group G, particularly those which are irreducible. An *irreducible representation* is one that admits no proper invariant closed subspaces under the action of G. These irreducible representations serve as the basic building blocks for all representations, much like prime numbers for the integers. The theory revolves around several interrelated problems, each of which remains at the core of current research:

1. *Classify all unitary irreducible representations of a given Lie group G.* This is known as the *unitary dual problem*, and its complete solution is known only for specific classes of Lie groups, such as compact Lie groups, nilpotent groups (via the Kirillov orbit method), and some real reductive groups.

2. *Explain how a given unitary representation can be decomposed into irreducible components.* This includes understanding the direct integral decomposition of unitary representations and the Plancherel theorem for non-compact groups.

3. *Given a subgroup $H \subset G$, determine the branching laws for restricting representations of G to H.* These describe how an irreducible representation of G decomposes when restricted to H, a problem that lies at the heart of reciprocity phenomena in mathematics and physics.

4. *Describe the structure of the category of unitary representations.* This involves tensor product decompositions, dualities, and character theory.

While the classification for compact and abelian groups is relatively well understood, the situation becomes substantially more complex for non-compact and non-abelian Lie groups. In particular, for real reductive Lie groups the unitary dual is intricate, and only partial results are known despite the deep work of Harish-Chandra, Langlands, Knapp–Zuckerman, and others. Current research is primarily focused on benchmark problems, specific examples, and extending the scope of representation theory to include supergroups, which plays a significant role in the field's advancement.

The notion of symmetry has evolved significantly with the advent of *supersymmetry*, a theoretical framework in quantum field theory developed in the 1960s and 1970s. Supersymmetry posits a duality between two fundamental types of particles: *bosons*, which mediate forces, and *fermions*, which constitute matter. Mathematically, this duality is captured by enriching the classical notion of a symmetry group to a $\mathbb{Z}/2\mathbb{Z}$-graded structure. In this setting, the Hilbert space $\mathcal{H}$ of a supersymmetric quantum system naturally decomposes into bosonic and fermionic subspaces:

$$\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}},$$

where $\mathcal{H}_{\bar{0}}$ corresponds to bosonic states and $\mathcal{H}_{\bar{1}}$ to fermionic states. Supersymmetry allows for transformations that exchange these two types of states, and such transformations are governed by Lie superalgebras and their global counterparts, Lie supergroups. The symmetry transformations of a supersymmetric theory are thus encoded in the unitary representation theory of Lie supergroups.

Lie supergroups are generalizations of Lie groups that arise naturally when attempting to incorporate supersymmetry into geometric and algebraic frameworks. They are formalized most effectively using the language of *super Harish-Chandra pairs*, which consist of a pair $(G, \mathfrak{g})$, where G is a Lie group and $\mathfrak{g}$ is a Lie superalgebra such that the two structures are compatible. This viewpoint reduces the study of representations of Lie supergroups to the combined representation theory of Lie groups and Lie superalgebras.

A remarkable feature of unitary representation theory for Lie supergroups is that, for a large class of such supergroups, the unitary representations are fully determined (up to equivalence) by the unitarizable representations of the underlying Lie superalgebra, referred to as unitarizable supermodules. This shifts the focus of the unitary representation theory of Lie supergroups to unitarizable supermodules over Lie superalgebras.

A central family of examples in this context — and the principal Lie superalgebras studied in this thesis — are the special linear Lie superalgebras $\mathfrak{sl}(m|n)$ and their simple counterparts, the Lie superalgebras of type $A(m|n)$, defined as follows:

$$A(m|n) := \begin{cases} \mathfrak{sl}(m+1|n+1), & \text{if } m \neq n,\ m,n \geq 0, \\ \mathfrak{sl}(n+1|n+1)/\mathbb{C}E_{n+1,n+1}, & \text{if } m = n,\ n > 0. \end{cases}$$

The representation theory of $\mathfrak{sl}(m|n)$ and $A(m-1|n-1)$ can be developed in a unified framework, using the notation $\mathfrak{g}$ to refer to either Lie superalgebra. Furthermore, these supetalgebras play a foundational role in the representation theory of Lie superalgebras due to their structural properties and their appearance in physical applications. In particular, real forms of these algebras include the *superconformal algebras* in $d = 1, 4$ spacetime dimensions, which appear prominently in superconformal quantum field theories, where superconformal Lie algebras serve as symmetry algebras and their representation theory enables exact analysis of these theories, often revealing hidden dualities, non-perturbative structures, and deep connections to enumerative geometry and category theory.

The notion of unitarizable $\mathfrak{g}$-supermodules depends on the choice of a conjugate-linear anti-involution ω, which corresponds bijectively to a real form of $\mathfrak{g}$. In [104], Neeb and Salmasian demonstrated that non-trivial unitarizable $\mathfrak{g}$-supermodules exist only when ω is associated with a real form of the type $\mathfrak{su}(p,q|n,0)$ or $\mathfrak{su}(p,q|0,n)$, where $p+q = m$. Furthermore, in [48], Furutsu and Nishiyama proved that any unitarizable $\mathfrak{sl}(m|n)$-supermodule with respect to such a real form must be either a highest weight or a lowest weight supermodule. If $p = 0$ or $q = 0$, then the unitarizable $\mathfrak{g}$-supermodules are finite-dimensional, whereas for $p, q \neq 0$, they are infinite-dimensional. In this work, we focus on unitarizable highest weight $\mathfrak{g}$-supermodules associated with the real form $\mathfrak{su}(p,q|0,n)$. These were classified in [73], and more recently revisited in [53]. The classification of unitarizable lowest weight supermodules proceeds analogously.

For this thesis, the ideas of representation theory serve as a guiding philosophy: symmetries govern structure. Understanding how symmetries act linearly provides insight into both qualitative and quantitative features of physical systems.

This thesis is divided into five parts, each addressing interconnected aspects of the theory of unitarizable supermodules over $\mathfrak{sl}(m|n)$ and basic classical Lie superalgebras of type $A(m|n)$. The central aim is to explore the structure, classification, and applications of these representations, bringing together algebraic, and physical perspectives. Topics range from foundational theory and classification via Dirac cohomology to the introduction of a new invariant — the formal superdimension — and applications in physics through the superconformal index. An addendum extends the theory to cubic Dirac operators, deepening the structural framework developed throughout.

1.2. Dirac operators and Dirac cohomology

Dirac operators are first-order differential operators whose square equals the Laplacian [46]. Their origin lies at the heart of quantum theory: in 1928, P. A. M. Dirac introduced the eponymous operator in a groundbreaking attempt to formulate a quantum theory of the electron that is compatible with the principles of special relativity [8]. The resulting Dirac equation not only successfully predicted the existence of antiparticles but also inaugurated the mathematical structure underlying spin-$\frac{1}{2}$ fermions in quantum field theory.

Since their inception, Dirac operators have evolved from physical constructs into central objects of study in mathematics. Their significance spans multiple domains, including differential geometry, global analysis, topology, and — most pertinently for this thesis — representation theory. In particular, the role of Dirac operators in the context of the Atiyah–Singer index theorem marks one of the most profound developments in 20th-century mathematics, offering a deep connection between analysis, topology, and geometry [92].

Within representation theory, Dirac operators provide deep and elegant tools to address some of its most fundamental questions:

a) Construct explicit representations of a Lie group or Lie algebra.

b) Determine criteria for the unitarizability of such representations.

c) Classify irreducible (and in particular, unitary) representations.

Dirac operators and their associated cohomology were first brought into the representation-theoretic spotlight by Parthasarathy and Vogan. Parthasarathy used Dirac operators to construct discrete series representations of semisimple Lie groups [3, 113]. Vogan later introduced *Dirac cohomology*, which provides a robust invariant of $(\mathfrak{g}, K)$-modules, revealing significant information about their infinitesimal characters. This led to Vogan's celebrated conjecture: the Dirac cohomology of a $(\mathfrak{g}, K)$-module determines its infinitesimal character. This conjecture was established by Huang and Pandžić in [65].

Dirac cohomology has since emerged as a powerful invariant with deep connections to other cohomological theories. It relates closely to $\mathfrak{n}$-cohomology for highest weight modules and to $(\mathfrak{g}, K)$-cohomology for Vogan–Zuckerman modules $A_{\mathfrak{q}}(\lambda)$ [62, 63, 109]. From a computational perspective, Dirac cohomology often yields a more tractable approach than $(\mathfrak{g}, K)$-cohomology, while still capturing rich structural information about a representation. Furthermore, it has proved instrumental in the classification of unitarizable highest weight modules [110].

A good overview of the history and significance of the Dirac operator in representation theory can be found in Figure 1.1

In [67], Huang and Pandžić extended the theory of Dirac operators and Dirac cohomology to Lie superalgebras of Riemannian type. A key assumption is the existence of a non-degenerate invariant supersymmetric bilinear form B on $\mathfrak{g}$, which allows for the decomposition $\mathfrak{g}_{\bar{1}} = \mathfrak{l}^- \oplus \mathfrak{l}^+$ into complementary Lagrangian subspaces. This structure gives rise to a natural definition of a Dirac operator D acting on tensor products of $\mathfrak{g}$-supermodules with the oscillator module. The square of D again encodes key representation-theoretic information, much as in the classical setting. Dirac cohomology in this setting is defined as

$$\mathrm{H_D}(M) := \ker \mathrm{D} / (\ker \mathrm{D} \cap \mathrm{Im}\, \mathrm{D}),$$

where M is a $\mathfrak{g}$-supermodule. A major result of Huang and Pandžić is a superalgebraic analog of Vogan's conjecture: if the Dirac cohomology of a $\mathfrak{g}$-supermodule is nonzero, then it determines the infinitesimal character of the module [67, Theorem 10.4.7]. This

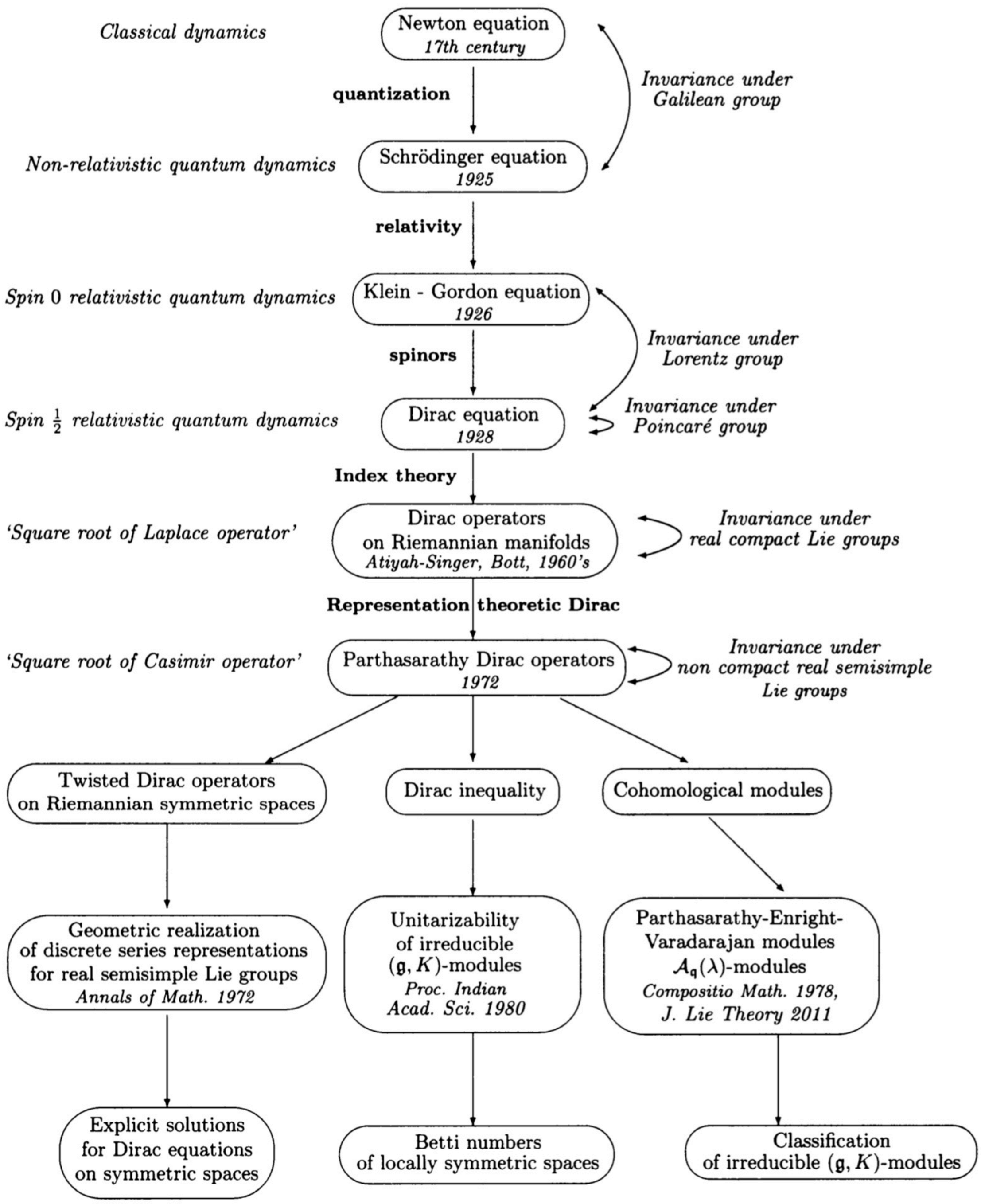


Figure 1.1.: Dirac operators and representation theory. Source: [94].

theorem suggests that Dirac cohomology could play an equally central role in the unitary representation theory of Lie superalgebras.

This thesis investigates the role of Dirac operators and Dirac cohomology in the representation theory of Lie superalgebras $\mathfrak{g}$, where $\mathfrak{g}$ denotes either $\mathfrak{sl}(m|n)$ or a Lie superalgebra of type $A(m|n)$. The focus lies on the classification and analysis of infinite-dimensional unitarizable modules. We aim to bridge the gap between the algebraic techniques used in the classification of ordinary Lie group representations and the more intricate structures encountered in the super setting. The results obtained provide new perspectives on the representation theory of basic classical Lie superalgebras and contribute to the ongoing development of geometric and cohomological methods in supersymmetric representation theory.

Our results are divided into two parts: the first explores the structural relationship between Dirac operators, Dirac cohomology and unitarity, while the second presents explicit extensions and perspectives, including a novel characterization of unitarity, connections to Kostant's cohomology, the formulation of infinitesimal characters, and a Dirac index.

We begin with the first part. A key observation is that, although the Dirac operator is defined independently of a basis, the choice of a positive system determines whether it is self-adjoint or anti-self-adjoint. We establish the existence of a basis that is compatible with the Riemannian structure of the Lie algebra and a fixed conjugate-linear anti-involution, ensuring compatibility with the real form. However, this basis does not induce a $\mathbb{Z}_2$-compatible $\mathbb{Z}$-grading but instead aligns with a maximal compact subalgebra $\mathfrak{k}$. Within this setup, we show that contravariance of a Hermitian form on a general supermodule and self-adjointness of the Dirac operator are equivalent (Proposition 7.2.4 and Theorem 7.2.5). This particularly implies a Dirac inequality, which is central to the further study of unitarizable supermodules.

Theorem 1.2.1. *Let $\langle\cdot,\cdot\rangle_M$ be a positive definite Hermitian form on a $\mathfrak{g}$-supermodule M with $\langle M_{\bar{0}}, M_{\bar{1}}\rangle_M = 0$. Then the following statements are equivalent*

a) $(M, \langle\cdot,\cdot\rangle_M)$ is a unitarizable $\mathfrak{g}$-supermodule.

b) The Dirac operator D is self-adjoint with respect to $\langle\cdot,\cdot\rangle_{M\otimes M(\mathfrak{g}_{\bar{1}})}$.

In particular, on unitarizable $\mathfrak{g}$-supermodules M a Dirac inequality holds:

$$\langle \mathrm{D}^2 v, v\rangle_{M\otimes M(\mathfrak{g}_{\bar{1}})} \geq 0$$

for all $v \in M \otimes M(\mathfrak{g}_{\bar{1}})$.

Turning to Dirac cohomology, we show that, for general supermodules, it lacks an adjoint functor but satisfies a six-term exact sequence. To remedy this, we construct an alternative Dirac cohomology that admits a right adjoint functor while coinciding with the standard Dirac cohomology on unitarizable supermodules.

The Dirac cohomology on unitarizable supermodules coincides with the kernel of the Dirac operator, *i.e.*, $\mathrm{H_D}(M) = \ker \mathrm{D}$, and the Dirac cohomology of simple unitarizable $\mathfrak{g}$-supermodules decomposes under $\mathfrak{g}_{\bar{0}}$ into a direct sum of unitarizable simple $\mathfrak{g}_{\bar{0}}$-supermodules. Applying the analog of Vogan's conjecture, we explicitly compute the Dirac cohomology of all unitarizable highest weight $\mathfrak{g}$-supermodules (Theorem 7.2.20 and Theorem 7.2.24):

Theorem 1.2.2. *The Dirac cohomology of a non-trivial unitarizable highest weight* $\mathfrak{g}$*-supermodule* $L(\Lambda)$ *with highest weight* Λ *is*

$$\mathrm{H_D}(L(\Lambda)) = L_0(\Lambda - \rho_{\bar{1}}).$$

In particular, two unitarizable $\mathfrak{g}$*-supermodules* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *are equivalent if and only if* $\mathrm{H_D}(\mathcal{H}_1) \cong \mathrm{H_D}(\mathcal{H}_2)$ *as* $\mathfrak{g}_{\bar{0}}$*-supermodules.*

We extend this result to the Dirac cohomology of Kac supermodules, which possess Jordan–-Hölder filtrations with unitarizable quotients.

The second part of Chapter 7 explores various perspectives on Dirac cohomology. Our first application provides a new description of unitarity for supermodules. This particularly addresses the following issue: a $\mathfrak{g}$-supermodule that is $\mathfrak{g}_{\bar{0}}$-semisimple such that every $\mathfrak{g}_{\bar{0}}$-constituent is unitarizable, need not itself be unitarizable. Conversely, the induction of a unitarizable $\mathfrak{g}_{\bar{0}}$-supermodule to $\mathfrak{g}$ does not have to be unitarizable. The Dirac inequality offers a necessary and sufficient condition for unitarity (Theorem 7.3.2).

Theorem 1.2.3. *Let* M *be a simple highest weight* $\mathfrak{g}$*-supermodule with highest weight* Λ *that is* $\mathfrak{g}_{\bar{0}}$*-semisimple. Then* M *is unitarizable if and only if the highest weight* $\mathfrak{g}_{\bar{0}}$*-supermodule* $L_0(\Lambda)$ *is unitarizable and the Dirac inequality holds strictly for each simple highest weight* $\mathfrak{g}_{\bar{0}}$*-constituent* $L_0(\mu) \subset M$ *with highest weight* $\mu \neq \Lambda$*, embedded in* $M \otimes M(\mathfrak{g}_{\bar{1}})$*,* i.e.,

$$(\mu + 2\rho, \mu) > (\Lambda + 2\rho, \Lambda).$$

This leads to an explicit decomposition of unitarizable simple $\mathfrak{g}$-supermodules under $\mathfrak{g}_{\bar{0}}$ in Theorem 7.3.4.

Next, we analyze the relationship between Dirac cohomology and Kostant's cohomology. The requirement that the Dirac operator be self-adjoint uniquely determines the choice of a positive system on $\mathfrak{g}_{\bar{1}}$. However, this choice introduces a challenge: Kostant's cohomology is naturally a module only over a maximal compact subalgebra $\mathfrak{k}$ of the even part of a real form of $\mathfrak{g}$ (Theorem 7.3.11).

Theorem 1.2.4. *For any unitarizable simple* $\mathfrak{g}$*-supermodule* $\mathcal{H}$*, there exists a* $\mathfrak{k}^{\mathbb{C}}$*-module isomorphism*

$$\mathrm{H_D}(\mathcal{H}) \cong \mathrm{H}^*(\mathfrak{g}_{+1}, \mathcal{H}) \otimes \mathbb{C}_{-\rho_{\bar{1}}}.$$

A natural question concerns the Euler characteristic of Dirac cohomology. To address this, we introduce the *Dirac index* for a general $\mathfrak{g}$-supermodule M, defined as the virtual $\mathfrak{g}_{\bar{0}}$-supermodule:

$$\mathrm{I}(M) := M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{0}} - M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{1}}.$$

The Dirac index coincides with the Euler characteristic of Dirac cohomology, meaning that $\mathrm{I}(M) = \mathrm{H}^+_{\mathrm{D}}(M) - \mathrm{H}^-_{\mathrm{D}}(M)$ (Proposition 7.3.12). Furthermore, $\mathrm{I}(\cdot)$ has nice categorical properties as it commutes with the functor associated with tensor multiplication by a finite-dimensional $\mathfrak{g}$-supermodule. The methods employed are analogous to those used in the case of reductive Lie algebras.

Furthermore, we derive two formulas for the formal $\mathfrak{k}^{\mathbb{C}}$-character of unitarizable simple supermodules using Kostant's cohomology and the Dirac index. Our main result can be summarized as follows (Theorem 7.3.15 and Theorem 7.3.18):

Theorem 1.2.5. *Let $\mathcal{H}$ be a unitarizable $\mathfrak{g}$-supermodules that admits an infinitesimal character, and let F^ν denote a simple $\mathfrak{k}^{\mathbb{C}}$-module of highest weight $\nu \in \mathfrak{h}^*$. Define $N(\mu) := \bigwedge \mathfrak{n}_{\bar{1}}^- \otimes F^\mu$.*

a) $\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\mathcal{H}) = \sum_\mu \sum_{k=0}^{\infty} (-1)^k [\mathrm{H}^k(\mathfrak{g}_{+1}, M) : F^\mu] \, \mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(N(\mu))$.

b) Assume $\mathrm{H}_{\mathrm{D}}^+(\mathcal{H}) = \sum_\mu F^\mu$ *and* $\mathrm{H}_{\mathrm{D}}^-(\mathcal{H}) = \sum_\nu F^\nu$. *Then*

$$\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\mathcal{H}) = \sum_\mu \mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(N(\mu + \rho_{\bar{1}})) - \sum_\nu \mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(N(\nu + \rho_{\bar{1}})).$$

The classification of unitarizable supermodules over Lie superalgebras remains poorly understood. Existing classifications for basic classical Lie superalgebras, such as those in [22, 48, 53, 73], rely on case-by-case analyses and intricate combinatorics, offering little in the way of a unifying theory or a geometric interpretation. Unlike the ordinary setting, where Enright–Howe–Wallach-type classifications of the unitary dual offer deep geometric insight [38], the unitary dual of basic classical Lie superalgebras exhibits a more elusive and irregular structure. Notably, Jakobsen's work [73] highlights a "zigzagging" phenomenon in the classification of unitarizable supermodules — an alternating pattern that lacks any clear analog in the representation theory of semisimple Lie groups. This suggests that new tools, such as Dirac cohomology, may be essential for uncovering the deeper structure of unitary supermodules. Chapter 8 presents our final result: a complete and novel classification of unitarizable simple supermodules that are finite-dimensional or have integral highest weight.

1.3. Cubic Dirac operators and Dirac cohomology

While classical Dirac operators have proven to be powerful tools in representation theory — particularly in the realization of discrete series and the study of Dirac cohomology — their original formulation is closely tied to the setting of symmetric spaces. To generalize these constructions beyond symmetric pairs, a fundamental modification is required.

In his influential work [89], Kostant introduced a new class of operators known as *cubic Dirac operators*, defined in the context of quadratic Lie algebras. These are Lie algebras equipped with a non-degenerate invariant bilinear form, allowing for a broader and more flexible framework. The defining feature of Kostant's construction is the inclusion of a cubic term, which plays a crucial role in preserving essential algebraic properties — most notably, the ability to encode representation-theoretic invariants such as infinitesimal characters.

The classical Dirac operator arises naturally when a Lie algebra $\mathfrak{g}$ admits a symmetric decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. In such cases, the Clifford algebra associated with $\mathfrak{p}$ serves as the domain for the spinor module on which the operator acts. However, when such a decomposition is not available, as in general quadratic Lie algebras, Kostant's construction circumvents this limitation by embedding the necessary structure algebraically into the operator itself via a cubic Clifford term. The resulting cubic Dirac operator retains compatibility with the adjoint action and exhibits a rich algebraic structure, enabling the analysis of representations in a more general setting.

More recently, analogs of Kostant's cubic Dirac operators have been developed for *quadratic Lie superalgebras* — superalgebras admitting an even, invariant, non-degenerate supersymmetric bilinear form. In this setting, Kang and Chen [79], as well as Meyer [95], constructed superalgebraic versions of cubic Dirac operators and extended many of the core algebraic properties known from the classical case. These developments open promising new avenues in the representation theory of Lie superalgebras, especially with respect to their unitarizable supermodules and geometric interpretation.

The aim of this part of the thesis is to study the Dirac cohomology associated with cubic Dirac operators, focusing on its applications to the representation theory of supermodules, where the formalism proves to be particularly powerful. Concretely, we investigate the structure and computation of Dirac cohomology in the setting of basic classical Lie superalgebras. Given a basic classical Lie superalgebra $\mathfrak{g}$ and a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, we consider the induced decomposition

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{s},$$

where $\mathfrak{l}$ is the Levi subalgebra of $\mathfrak{p}$ and $\mathfrak{s}$ its orthogonal complement with respect to a non-degenerate invariant supersymmetric bilinear form on $\mathfrak{g}$. This setup provides a natural framework for defining a Dirac operator associated with the pair $(\mathfrak{g}, \mathfrak{l})$ and studying its cohomology.

The Dirac cohomology $\mathrm{H}_{\mathrm{D}}(M)$ of a $\mathfrak{g}$-supermodule M demonstrates its full strength when M admits an infinitesimal character $\chi_\lambda : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$. That is, each central element $z \in \mathfrak{Z}(\mathfrak{g})$ acts on M as scalar multiplication by $\chi_\lambda(z)$. In this setting, we establish a super-analog of the Casselman–Osborne lemma (Theorem 13.2.27), which describes how the center of $\mathfrak{g}$ acts on the Dirac cohomology:

Theorem 1.3.1. *Let M be a $\mathfrak{g}$-supermodule with infinitesimal character χ_λ. Then the action of $z \in \mathfrak{Z}(\mathfrak{g})$ on $\mathrm{H}_{\mathrm{D}}(M)$ is given by*

$$z \cdot v = \eta_{\mathfrak{l}}(z) \cdot v$$

for a uniquely defined algebra homomorphism $\eta_{\mathfrak{l}} : \mathfrak{Z}(\mathfrak{g}) \to \mathfrak{Z}(\mathfrak{l})$. In particular, if $\mathrm{H}_{\mathrm{D}}(M)$ contains an $\mathfrak{l}$-submodule with infinitesimal character $\chi^{\mathfrak{l}}_\mu$, then

$$\chi_\lambda = \chi^{\mathfrak{l}}_\mu \circ \eta_{\mathfrak{l}}.$$

This result has powerful implications for the structure of highest weight supermodules. Indeed, we show that the Dirac cohomology of any highest weight $\mathfrak{g}$-supermodule is always non-trivial (Proposition 13.3.7):

Theorem 1.3.2. *Let M be a highest weight $\mathfrak{g}$-supermodule. Then its Dirac cohomology satisfies $\mathrm{H}_{\mathrm{D}}(M) \neq 0$.*

A further focus is placed on the explicit computation of Dirac cohomology for finite-dimensional supermodules. For Lie superalgebras of type I — namely, $\mathfrak{g} = \mathfrak{gl}(m|n)$, $A(m|n)$, or $C(n)$ — we obtain a concrete formula for the Dirac cohomology of finite-dimensional simple $(\mathfrak{g}, \mathfrak{l})$-supermodules with typical highest weights:

Theorem 1.3.3. *Let M be a finite-dimensional admissible simple $(\mathfrak{g}, \mathfrak{l})$-supermodule with typical highest weight Λ. Then*

$$\mathrm{H}_{\mathrm{D}}(M) = \bigoplus_{w \in W^{\mathfrak{l},1}_{\Lambda+\rho^{\mathfrak{u}}}} L_{\mathfrak{l}}(w(\Lambda + \rho) - \rho^{\mathfrak{l}}),$$

where ρ and $\rho^{\mathfrak{l}}$ are the Weyl vectors associated to $\mathfrak{g}$ and $\mathfrak{l}$, respectively.

In the context of the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$, we compute the Dirac cohomology for finite-dimensional simple objects (Theorem 13.3.11):

Theorem 1.3.4. *Let M be a finite-dimensional simple object in the parabolic category $\mathcal{O}^{\mathfrak{p}}$ with highest weight Λ. Then*

$$\mathrm{H_D}(M) = \bigoplus_{w \in W^{\mathfrak{l}_0,1}_{\Lambda+\rho^{\mathfrak{u}}}} L_{\mathfrak{l}_0}(w(\Lambda + \rho^{\mathfrak{u}} + \rho^{\mathfrak{l}_0}) - \rho^{\mathfrak{l}_0}),$$

where $\mathfrak{l}_0$ is the even part of the Levi subalgebra and $\rho^{\mathfrak{u}}$, $\rho^{\mathfrak{l}_0}$ denote the corresponding Weyl vectors.

Beyond structural results and computations, we also investigate the relationship between Dirac cohomology and Lie algebra cohomology. Specifically, we prove that Dirac cohomology embeds into both Lie algebra and Lie algebra homology of the nilpotent radical $\mathfrak{u}$, and that this embedding becomes an isomorphism under unitarity assumptions (Theorem 13.4.10):

Theorem 1.3.5. *Let M be an admissible simple $(\mathfrak{g}, \mathfrak{l})$-supermodule. Then there are injective morphisms of $\mathfrak{l}$-supermodules*

$$\mathrm{H_D}(M) \hookrightarrow H^*(\mathfrak{u}, M), \qquad \mathrm{H_D}(M) \hookrightarrow H_*(\bar{\mathfrak{u}}, M).$$

If M is unitarizable, then these maps are isomorphisms.

Furthermore, we prove that highest weight $\mathfrak{g}$-supermodules are precisely those supermodules with non-trivial Dirac cohomology (Theorem 13.4.20).

Theorem 1.3.6. *Let M be a simple weight $\mathfrak{g}$-supermodule. Then $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) = \{0\}$ unless M is a highest weight $\mathfrak{g}$-supermodule.*

These results suggest that, under suitable assumptions, the Dirac cohomology may fully determine the structure of a supermodule. This perspective offers a new lens through which to understand representation theory in the super context and motivates further investigation. Preliminary evidence indicates that similar phenomena extend to Lie superalgebras of type II, suggesting that Dirac cohomology has the potential to unify and clarify much of the representation theory of all basic classical Lie superalgebras.

1.4. Formal superdimension

A central goal in representation theory is the classification and understanding of representations through suitable invariants. For finite-dimensional supermodules of basic classical Lie superalgebras, one such fundamental invariant is the *superdimension*. Given a finite-dimensional $\mathfrak{g}$-supermodule $V = V_{\bar{0}} \oplus V_{\bar{1}}$, the superdimension is defined as

$$\mathrm{sdim}(V) = \dim(V_{\bar{0}}) - \dim(V_{\bar{1}}),$$

a natural extension of the usual notion of dimension that reflects the parity grading of V. If V is a simple $\mathfrak{g}$-supermodule, it is determined uniquely (up to isomorphism) by a highest weight λ. In 1994, Kac and Wakimoto conjectured that such a module has non-trivial superdimension if and only if λ has maximal atypicality, meaning its degree of atypicality equals the defect of $\mathfrak{g}$ [78]. This conjecture was later proven by Serganova in [128].

Nevertheless, no general formula for the superdimension in terms of the highest weight was given. In 2015, Heidersdorf and Weissauer derived such a formula for $\mathfrak{gl}(m|n)$ using the cup diagram language developed by Brundan and Stroppel [58]. Their derivation used the Duflo–Serganova functor (DS functor), a symmetric monoidal functor that preserves superdimension and plays a pivotal role in the structural analysis of $\mathfrak{g}$-supermodules.

While superdimension is well-defined for finite-dimensional supermodules, no such invariant exists for general infinite-dimensional supermodules, particularly for unitarizable ones. However, for a large class of infinite-dimensional unitarizable $\mathfrak{g}_{\bar{0}}$-modules, one can define an analog of a dimension, known as the *formal dimension*, originally studied in the context of Harish-Chandra modules and holomorphic discrete series. This suggests the possibility of extending the superdimension program to unitarizable highest weight supermodules over certain Lie superalgebras. The second part of this thesis introduces and develops a natural generalization of the superdimension theory to infinite-dimensional, unitarizable simple $\mathfrak{g}$-supermodules.

The proposal for the formal superdimension arises from a natural synthesis of several core ideas: the Harish-Chandra degree for $\mathfrak{g}_{\bar{0}}$-modules (see Section 5.4.2), the structure theory developed in the Dirac operator framework (see Section 7), and the parametrization of $\mathfrak{g}_{\bar{0}}$-constituents via highest weights (see Section 6.1.4).

To fix notation, let $\mathcal{H}$ be a non-trivial unitarizable simple highest weight $\mathfrak{g}$-supermodule with highest weight $\Lambda \in \mathfrak{h}^*$. Then:

a) $\mathcal{H}$ is simple and $\mathfrak{g}_{\bar{0}}$-semisimple. Each $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\Lambda_j)$ is a unitarizable highest weight supermodule over $\mathfrak{g}_{\bar{0}}$, and appears in $\mathcal{H}$ with finite multiplicity and fixed $\mathbb{Z}_2$-parity
$$p(\Lambda_j) = \sum_{k=1}^{n} (\Lambda_j - \Lambda, \delta_k) \mod 2,$$
computed relative to the highest weight vector of $\mathcal{H}$.

b) Each constituent $L_0(\Lambda_j)$ decomposes as a tensor product of the form
$$L_0(\Lambda_j) \cong L_0(\Lambda_j^{\mathfrak{L}}) \boxtimes L_0(\Lambda_j^{\mathfrak{R}}) \boxtimes \mathbb{C}_{\mu_j},$$
where $L_0(\Lambda_j^{\mathfrak{L}})$ is a unitarizable $\mathfrak{L} := \mathfrak{su}(p,q)$-module, $L_0(\Lambda_j^{\mathfrak{R}})$ a finite-dimensional $\mathfrak{R} := \mathfrak{su}(n)$-module, and $\mathbb{C}_{\mu_j}$ a one-dimensional module of the $\mathfrak{u}(1)$ factor. When $p = 0$ or $q = 0$, the $\mathfrak{su}(p,q)$-module is finite-dimensional.

c) When $p, q \neq 0$, a constituent $L_0(\Lambda_j)$ belongs to the *relative holomorphic discrete series* if it satisfies the *Harish-Chandra condition* (see Theorem 5.4.3),
$$(\Lambda_j + \rho_{\bar{0}}, \beta) < 0 \qquad \text{for all } \beta \in \Delta_n^+.$$
In this case, $L_0(\Lambda_j)$ admits a formal dimension $d(\Lambda_j) \in \mathbb{R}_+$ given by (Theorem 5.4.5)
$$d(\Lambda_j) = \prod_{\alpha \in \Delta_c^+} \frac{(\Lambda_j + \rho_c, \alpha)}{(\rho_c, \alpha)} \prod_{\beta \in \Delta_n^+} \frac{|(\Lambda_j + \rho_{\bar{0}}, \beta)|}{(\rho_{\bar{0}}, \beta)}.$$

This formal dimension serves as a substitute for the ordinary dimension in the infinite-dimensional case, and coincides with the dimension for finite-dimensional unitarizable modules.

When a constituent $L_0(\Lambda_j)$ fails to satisfy the Harish-Chandra condition, no meaningful notion of formal dimension exists. This motivates the following terminology: A unitarizable highest weight $\mathfrak{g}$-supermodule $\mathcal{H}$ is called a *(relative) holomorphic discrete series supermodule* if each of its $\mathfrak{g}_{\bar{0}}$-constituents are Harish-Chandra modules associated to a (relative) holomorphic discrete series representation of the universal cover of the underlying real Lie group.

With this terminology in place, we define the central invariant of this part of the thesis:

Definition 1.4.1. Let $\mathcal{H}$ be a unitarizable highest weight $\mathfrak{g}$-supermodule whose $\mathfrak{g}_{\bar{0}}$-constituents all belong to the relative holomorphic discrete series. Then the *formal superdimension* of $\mathcal{H}$ is defined as

$$\operatorname{sdim}(\mathcal{H}) := \sum_j \operatorname{sdim}(L_0(\Lambda_j)), \qquad \operatorname{sdim}(L_0(\Lambda_j)) := (-1)^{p(\mathcal{H})+p(\Lambda_j)} \cdot d(\Lambda_j),$$

where the sum runs over all $\mathfrak{g}_{\bar{0}}$-constituents $L_0(\Lambda_j)$ of $\mathcal{H}$.

This construction preserves several desirable properties: the formal superdimension is additive across short exact sequences and it behaves compatibly under tensoring with finite-dimensional modules.

We further prove a *generalized Kac–Wakimoto conjecture* for the formal superdimension (Theorem 10.2.4). The proof is in the special case $m = n$, though our method applies to all m, n.

Theorem 1.4.2. *Let $\mathcal{H}$ be a relative holomorphic discrete series $\mathfrak{g}$-supermodule. Then* $\operatorname{sdim}(\mathcal{H}) = 0$ *unless the highest weight of $\mathcal{H}$ is maximally atypical.*

Conversely any unitarizable maximally atypical supermodule consists only of $\mathfrak{g}_{\bar{0}}$-modules in the (limit of) relative holomorphic discrete series. This removes the limitations of the formal superdimension to some degree.

A part of this proof is based on the *Duflo–Serganova functor* (or *DS functor*), introduced in [34]. This functor plays a pivotal role in the study of the category of finite-dimensional $\mathfrak{g}$-supermodules, where it provides a powerful means of probing structure and invariants. The DS functor is symmetric monoidal and, crucially, it preserves the superdimension in the finite-dimensional setting. There exists no analog of this construction in the classical theory of Harish-Chandra modules, which makes it particularly well-suited for applications in the super setting.

The DS functor is defined with respect to an element x in the *self-commuting variety*

$$\mathcal{Y} := \{x \in \mathfrak{g}_{\bar{1}} : [x, x] = 0\}.$$

Given a $\mathfrak{g}$-supermodule M, such an x induces a nilpotent endomorphism $x_M \in \operatorname{End}_{\mathbb{C}}(M)$ satisfying $x_M^2 = 0$. This allows one to define a cohomological reduction of M by

$$DS_x(M) := M_x := \ker(x_M)/\operatorname{im}(x_M),$$

which is naturally a $\mathfrak{g}_x$-supermodule, where $\mathfrak{g}_x := \ker(\mathrm{ad}_x)/\mathrm{im}(\mathrm{ad}_x)$ inherits the structure of a Lie superalgebra. The assignment $M \mapsto DS_x(M)$ defines a functor from the category of $\mathfrak{g}$-supermodules to that of $\mathfrak{g}_x$-supermodules. In the present work, we extend the use of the DS functor to the setting of infinite-dimensional highest weight $\mathfrak{g}$-supermodules. However, the DS functor fails to preserve unitarity, necessitating a generalization of the construction outlined above. This issue will be addressed in the following section.

The superdimension developed here not only generalizes the known results for finite-dimensional supermodules but also has potential applications to mathematical physics. In particular, it appears closely related to the superconformal index in $4d$ $\mathcal{N} = 4$ superconformal field theory, which captures contributions only from certain unitarizable simple modules over $\mathfrak{psu}(2,2|4)$ with atypical highest weight. This suggests a deep mathematical structure behind such indices, traceable back to representation-theoretic invariants like the superdimension.

1.5. Indices

Indices play a central role in both mathematics and physics, capturing subtle topological information through analytic means. A prominent example is the *Fredholm index*, associated with bounded linear operators between infinite-dimensional Hilbert spaces. Given such an operator D with finite-dimensional kernel and cokernel, its index is defined by

$$\mathrm{ind}(D) = \dim\ker D - \dim\operatorname{coker} D.$$

This integer-valued invariant remains stable under compact perturbations and plays a central role in the study of elliptic differential operators, where it connects analytic properties of operators to topological invariants of the underlying space.

In mathematical physics, this notion finds a natural counterpart in supersymmetric theories through the *Witten index*, an analytic expression originally introduced in the context of Morse theory [139]. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space carrying a unitary $\mathbb{Z}_2$-graded representation of a Lie superalgebra. Let $(-1)^F \in \mathcal{L}(\mathcal{H})$ be a self-adjoint involution endowing $\mathcal{H}$ with a parity grading, and let $Q \in \mathcal{L}(\mathcal{H})$ be a closed, densely defined odd operator satisfying

$$(-1)^F Q = -Q(-1)^F, \qquad Q^2 = 0.$$

Denote by $Q^\dagger$ the Hilbert space adjoint of Q, and define the self-adjoint operator

$$\Xi := Q^\dagger Q + QQ^\dagger.$$

By von Neumann's theorem, Ξ is self-adjoint, non-negative, and commutes with Q. The *Witten index* of Q is defined by the supertrace

$$I^W_{\mathcal{H}}(Q;\beta) := \mathrm{tr}_{\mathcal{H}}\left((-1)^F e^{-\beta\Xi}\right),$$

for $\beta > 0$ such that $e^{-\beta\Xi}$ is trace-class. When Q satisfies the *Fredholm property* — that is, its image is closed and the kernels $\ker Q_+$ and $\ker Q_-$ are finite-dimensional — the Witten index becomes independent of β and is given by

$$I^W_{\mathcal{H}}(Q) = \dim\ker Q_+ - \dim\ker Q_- = \mathrm{tr}_{\ker Q\cap\ker Q^\dagger}(-1)^F,$$

where $Q_\pm = \frac{1}{2}(Q + Q^\dagger)(1 \pm (-1)^F)$ denote the chiral components of Q acting between the graded subspaces $\mathcal{H}_\pm = \frac{1}{2}(1 \pm (-1)^F)\mathcal{H}$. An important feature of the Witten index is its *topological invariance*:

$$I^W_{\mathcal{H}}(Q + \delta Q) = I^W_{\mathcal{H}}(Q),$$

for any relatively Q-compact perturbation δQ (i.e., such that $\delta Q(Q - i)^{-1}$ is compact). Thus, the Witten index is preserved under a large class of continuous deformations of the pair (Q, Ξ), making it stable under many quantum corrections.

In physical contexts, $(-1)^F$ corresponds to the fermion number operator, Q to a nilpotent complex supercharge, and Ξ to an energy operator (which may differ from the Hamiltonian by R-symmetry terms). In superconformal field theories (SCFTs), the Witten index specializes to the *superconformal index*, which counts protected BPS states in the Hilbert space of radial quantization. These states are annihilated by both Q and $Q^\dagger$, and their contributions are refined by their transformation properties under the centralizer of Q. In particular, only "short" representations contribute to the superconformal index, as long representations cancel due to a balance between bosonic and fermionic content.

One objective of this thesis is to develop a mathematical framework that captures the essence of physical indices (*e.g.* the Witten index) for unitarizable supermodules over the special linear Lie superalgebras $\mathfrak{sl}(m|n)$ with $m \geq 2$, $n \geq 1$, considered with respect to their real forms $\mathfrak{su}(p, q|n)$. While this does not encompass all superconformal algebras, it captures the key structural features, including the role of "atypical highest weight supermodules" as mathematical analogs of "short supermodules," and the nuanced criteria for unitarizability.

To formalize the notion of a physical index, several ingredients are required. We provide a detailed dictionary translating physical concepts — such as BPS states, short supermodules, and related phenomena — into their rigorous mathematical counterparts. In particular, we address the phenomenon of recombination and fragmentation, that is, the continuity of the fragmentation/recombination process of supermodules at the boundary of the unitarity region. This continuity is one of the most intriguing insights from physics that has, until now, lacked a clear mathematical analog. Physically, it refers to the smooth transition by which long supermodules decompose into short ones as unitarity bounds are saturated. Our approach remains somewhat ad hoc and provisional, but we believe it identifies core features relevant to the broader task of defining indices in the context of Lie superalgebras.

Having established a dictionary between the mathematical and physical settings, we give a precise definition of a general class of physical indices — the Kinney–Maldacena–Minwalla–Raju (KMMR) indices — which formalize and generalize the notion of the superconformal index introduced in [81] as an invariant count of short supermodules. These indices are defined so that they receive nontrivial contributions only from so-called short supermodules, which fail to recombine into long supermodules at the boundary of the unitarity region and thus encode BPS (Bogomol'nyi–Prasad–Sommerfield) states. These indices form a $\mathbb{Z}$-module, and a central problem is to understand its structure.

A principal example of a KMMR index is the *Q-Witten index*. In its original formulation, it was assumed for simplicity that $e^{-\beta\Xi}$ is of trace class. However, this assumption is generally too naive in infinite-dimensional settings. To address this issue, we introduce a refined version of the Witten index, which is character-valued and based on a generalized Duflo–Serganova (DS) cohomology theory. This approach clarifies the cohomological language commonly employed in physics when describing the Witten index.

The representation-theoretic framework we develop is based on a generalization of the classical DS functor, suitably modified to preserve unitarity, an essential feature not adequately captured by the traditional construction. Specifically, given an element $x \in \mathfrak{g}_{\overline{1}}$ such that $c := [x, x] \in \mathfrak{g}_{\overline{0}}$ is semisimple, we define the *rank variety*

$$\mathcal{Y}^{\text{hom}} := \{x \in \mathfrak{g}_{\overline{1}} : [x, x] \text{ is semisimple}\},$$

which is stable under the action of $G_{\overline{0}}$ but not closed in $\mathfrak{g}_{\overline{1}}$.

For a $\mathfrak{g}$-supermodule M, assuming that c acts semisimply, we define the generalized DS functor via the cohomology

$$DS_x(M) := (\ker(x|_{M^c}))/(\operatorname{im}(x|_{M^c})),$$

where M^c denotes the space of c-invariants in M. The functor DS_x maps $\mathfrak{g}$-supermodules to $\mathfrak{g}_x$-supermodules, where $\mathfrak{g}_x := DS_x(\mathfrak{g})$ forms a Lie superalgebra. Importantly, this functor preserves unitarity under the condition $\omega(x) = -x$ with respect to the anti-involution ω defining the real form $\mathfrak{g}^\omega$.

In physical models, we fix an element Q of the self-commuting variety $\mathcal{Y}$, acting on a non-trivial unitarizable $\mathfrak{g}$-supermodule $\mathcal{H}$. The adjoint $Q^\dagger$ of Q satisfies $Q^\dagger = \omega(Q)$, leading to the definition of an element $x := i(Q + Q^\dagger)$ with $\omega(x) = -x$ and $c := \frac{1}{2}[x, x]$ semisimple. Setting $\Xi := -c$, a positive-definite operator on $\mathcal{H}$, we interpret the positivity of Ξ as the *BPS bound* associated with Q. Physically, one aims to study the corresponding *BPS states* — elements of $\mathcal{H}(0)$ — by considering the supertrace of $e^{-\beta\Xi}$ for real $\beta > 0$.

Due to the infinite degeneracy of $\mathcal{H}(0)$, we refine the notion of the Witten index by introducing the *character-valued Witten index*. Viewing $\mathfrak{g}_x$ as a subalgebra of $\mathfrak{g}$, we consider the Cartan subalgebra $\mathfrak{t}_x \subset \mathfrak{g}_x$ induced from $\mathfrak{t} \subset \mathfrak{g}$, and denote by T_x the corresponding analytic subgroup. For any unitarizable highest weight $\mathfrak{g}_x$-supermodule V, we define its *supercharacter* as a function

$$\chi_V^x(e^X) := \operatorname{str}_V(e^X), \quad e^X \in T_x^{reg,+},$$

where $T_x^{reg,+}$ is the set of regular elements satisfying certain positivity conditions.

Using the decomposition $DS_x(M) = \bigoplus_i V_i$ into finitely many $\mathfrak{g}_x$-supermodules, we define:

Definition 1.5.1. Let M be a unitarizable highest weight $\mathfrak{g}$-supermodule, and let Q be an element of the self-commuting variety. The *Q-Witten index* of M is the $\mathfrak{g}_x$-supercharacter

$$I_M^W(Q, \cdot) := \sum_i \chi_{V_i}^x(\cdot) = \operatorname{str}_{DS_x(M)}(\cdot) \in X^*(T_x^{reg,+}).$$

We show that this construction coincides with the physical description of the Witten index and that for fixed $X \in \mathfrak{t}_x^{reg,+}$, the Q-Witten index defines a KMMR index. We further study the general algebraic properties of the Q-Witten index and establish that any KMMR index can be expressed as a linear combination of Witten indices.

Next, we motivate the search for a relation between the Q-Witten index and the formal superdimension — which itself defines a KMMR index for $m, n \geq 3$ — by considering the formula established in Proposition 12.2.5

$$I_{\mathcal{H}}^W(Q, X) = \operatorname{str}_{\mathcal{H}}\left(e^{-\beta\Xi + X}\right), \qquad (\text{where } Q \in \mathcal{Y},\ \Xi = [Q, Q^\dagger], \text{ and } X \in \mathfrak{t}_x^{\text{reg},+}),$$

which closely resembles the Weyl character formula for finite-dimensional representations of Lie groups, where the limit $X \to 0$ recovers the dimension of the representation. In the setting of infinite-dimensional representations, this analogy is best understood within the framework of Harish-Chandra characters and L-packets. Concretely, recalling

$$I_M^W(Q,X) = \sum_{i,j}(-1)^{\Lambda-\Lambda_{i;j}} \operatorname{tr}_{L_{0,x}(\Lambda_{i;j})}\left(e^X\right),$$

we may interpret each term $\operatorname{tr}_{L_{0,x}(\Lambda_{i;j})}\left(e^X\right)$ (in a distributional sense) as the Harish-Chandra character of the representation $\pi_{\Lambda_{i;j}}$.

To take the limit $X \to 0$, we associate to each $\mathfrak{g}_{x,\bar{0}}$-constituent its corresponding L-packet by summing over the Weyl group orbit. Specifically, we define

$$\widetilde{\Theta}_{\mathrm{DS}_x(M)} := \sum_{i,j}(-1)^{\Lambda-\Lambda_{i;j}} \sum_{w\in W_x/W_{x,c}} \Theta_{\pi_{w\Lambda_{i;j}}},$$

and set

$$\tilde{I}_M^W(Q,X) := \widetilde{\Theta}_{\mathrm{DS}_x(M)}\left(e^X\right).$$

Our final result relates the superdimension and the Q-Witten index.

Theorem 1.5.2. *Let M be a holomorphic discrete series $\mathfrak{g}$-supermodule, and let Q, x be as above. Then*

$$\operatorname{sdim}(\mathrm{DS}_x(M)) = \lim_{X\to 0} \tilde{I}_M^W(Q,X).$$

1.6. Conventions

Throughout this thesis, we work exclusively with vector spaces and algebras over the complex numbers $\mathbb{C}$. Any additional assumptions will be stated explicitly.

Definitions in this thesis generally follow the sign rule, *i.e.*, swapping two odd elements introduces a sign factor. This foundational principle, combined with the functorial nature of the constructions, ensures the consistency of all definitions.

Let $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ denote the ring of integers modulo 2. We write $\bar{0}$ and $\bar{1}$ for the residue classes of even and odd integers, respectively. For all $n \in \mathbb{Z}$, the expression $(-1)^n$ depends only on the residue class of n modulo 2. Hence, $(-1)^{\bar{n}}$ is well-defined for all $\bar{n} \in \mathbb{Z}_2$.

The $\mathbb{Z}_2$-grading (or "parity") of many constructions in supermathematics is often left implicit. When necessary, the induced sign (via the sign rule) is encoded as the "exponentiated Fermion number." On any super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, we define

$$(-1)^F := \mathrm{id}_{V_{\bar{0}}} \oplus (-\mathrm{id}_{V_{\bar{1}}}).$$

We denote the parity of a homogeneous element $v \in V$ by $p(v)$. Expressions involving terms like $(-1)^{p(v)p(w)}$ for general $v, w \in V$ are interpreted by first restricting to homogeneous elements, substituting their parities into the exponent, and then extending linearly.

1.7. Leitfaden

This thesis comprises five parts, with the last four exploring interconnected facets of the theory of unitarizable supermodules over Lie superalgebras. Its core objective is to investigate their structure, classification, and applications by integrating algebraic, and physical approaches.

Part I begins with a concise overview of the foundational concepts in the theory of Lie superalgebras and Lie supergroups, establishing the language and tools necessary for the developments in later chapters. Concretely, Chapter 2 presents the essential background on the theory of complex Lie superalgebras and their real forms. Furthermore, Lie supergroups and their formulation as super Harish-Chandra pairs are introduced.

Subsequently, in Chapter 3, we explore the representation theory of $\mathbb{Z}_2$-graded modules, referred to as supermodules, over Lie superalgebras and Lie supergroups. In particular, the notion of highest weight supermodules and their categorical treatment are discussed in detail.

Finally, in Chapter 4, we introduce superconformal Lie algebras, which serve as a motivating example for the remainder of the thesis. More precisely, when considering physical applications, the aim of this work is to derive statements about superconformal Lie algebras and their representation theory. Nonetheless, we primarily focus on general basic classical Lie superalgebras, which include the superconformal ones as special cases and share important properties with them. This approach enables us to discuss topics beyond superconformal theories.

In **Part II**, we turn to the structural theory and classification of unitarizable (highest weight) supermodules for $\mathfrak{sl}(m|n)$ and the special linear Lie superalgebras of type $A(m|n)$. Using Dirac operators and Dirac cohomology as guiding tools, we reveal the rich internal structure of these modules and the conditions under which unitarity can be achieved. This part is structured into three chapters, each building upon foundational and technical tools to arrive at a new classification result.

Chapter 5 begins with a review of unitary representation theory for real reductive Lie groups. We place particular emphasis on discrete series representations of semisimple Lie groups and their generalizations, the relative discrete series representations. This classical framework sets the stage for the subsequent extension to the superalgebraic context.

In Chapter 6, we introduce the concept of unitarity for Lie superalgebras and Lie supergroups, focusing on the case of $\mathfrak{sl}(m|n)$. We analyze the implications of unitarity in this setting and establish key structural properties of unitarizable supermodules.

A central tool throughout this part is the Dirac operator, introduced and studied in Chapter 7, where we present an overview of Dirac cohomology and construct a Dirac induction functor, left-adjoint to Dirac cohomology, tailored to unitarizable supermodules. Furthermore, in Section 7.2, we investigate the deep relationship between Dirac operators, Dirac cohomology, and unitarity. We demonstrate that the Dirac operator captures unitarity in a precise sense and compute the Dirac cohomology of unitarizable simple $\mathfrak{sl}(m|n)$-supermodules. These computations lay the groundwork for the classification result presented in the following chapter. The section concludes with a new formulation of the Dirac index and its connection to Kostant's cohomology and character theory. Section 7.3 extends the analysis by exploring additional consequences and applications of Dirac cohomology in the unitarizable setting. In particular, we derive a new characterization of unitarity, provide a decomposition of formal characters via nilpotent Lie superalgebra cohomology, and outline how these tools yield insight into the structure of the abelian category of representations.

Chapter 7 addresses two fundamental representation-theoretic problems: the classification and explicit decomposition of unitarizable $\mathfrak{sl}(m|n)$-supermodules. Existing classifications [22, 48, 53, 73] are computationally intensive and lack conceptual transparency. In Chapter 8, motivated by the approach of Enright–Howe–Wallach [38] for real Lie algebras, we develop a new classification scheme based on the theory of algebraic Dirac operators.

Part III introduces a new invariant — the *formal superdimension* — which extends the classical notion of dimension to a broader setting. Concretely, we associate a formal superdimension to a class of unitarizable supermodules over $\mathfrak{sl}(m|n)$ and basic classical Lie superalgebras of type $A(m|n)$ in Chapter 10. The construction is based on the formal dimension of relative discrete series representations of the underlying reductive Lie group, leading to relative holomorphic discrete $\mathfrak{g}$-supermodules. In this context, we introduce the Duflo–Serganova functor and translation functors in Chapter 9 to show that a relative holomorphic discrete series $\mathfrak{g}$-supermodule $\mathcal{H}$ has trivial superdimension unless the highest weight is maximally atypical. In the maximal atypical case, we classify the unitarizable supermodules belonging to holomorphic discrete series $\mathfrak{g}$-supermodules using Dirac's inequality and show that each is already a holomorphic discrete series, meaning the $\mathfrak{g}_{\bar{0}}$-constituents belong to the holomorphic discrete series of the underlying Lie group rather than the relative holomorphic discrete series of its universal cover. Moreover, we generalize the Kac–Wakimoto Theorem to holomorphic discrete series supermodules.

Part IV is devoted to developing a rigorous mathematical framework for understanding the superconformal index from a representation-theoretic perspective. Our aim is twofold: to promote a precise exchange of ideas between mathematics and physics, and to establish foundational tools that extend beyond physical applications.

Chapter 11 provides a detailed dictionary translating central physical notions — such as BPS states, short multiplets, and related phenomena — into their rigorous mathematical counterparts. In Section 11.1, we begin by outlining the technical assumptions underpinning our treatment of indices for unitarizable supermodules. While our approach is somewhat ad hoc and not fully comprehensive, we believe it captures essential features that illuminate the broader challenge of defining indices in the context of Lie superalgebras.

In Section 11.2, we develop a correspondence between physical terminology and key mathematical concepts, particularly atypicality and the Duflo–Serganova (DS) functor (see Section 9.1). This correspondence is motivated by a physical insight that, until now, lacked a precise mathematical formulation: the continuity of the fragmentation and recombination process at the boundary of the unitarity region. To address this, we provide a geometric description of the unitarity region in terms of slices of weight space, parametrized by dimension and R-charge. We give a detailed and rigorous account of the fragmentation/recombination phenomenon, phrased in terms of the decomposition behavior of Kac modules with atypical highest weights. This leads to a precise understanding of protected or short representations in physics, culminating in the central result that maximally atypical supermodules are absolutely protected.

Chapter 12 contains the core results of this part. We begin by formalizing the concept of a *counting index* for unitarizable $\mathfrak{g}$-supermodules, following the approach initiated by Kinney, Maldacena, Minwalla, and Raju [81]. We define the counting index as an additive and continuous function on the space of such supermodules.

Next, we introduce the Q-Witten index, defined as a supertrace valued in the supercharacters of the Duflo–Serganova twist of $\mathfrak{g}$, with respect to the operator $Q + Q^{\dagger}$. We then establish a real-linear equivalence between the KKMR index and the Q-Witten index, providing a bridge between algebraic and analytic viewpoints.

Finally, we interpret the formal superdimension of holomorphic discrete series $\mathfrak{g}$-supermodules as a real-valued instance of the KKMR index. We demonstrate how this quantity can be computed using Harish-Chandra character theory and relate it directly to the Q-Witten index. This yields a unified picture connecting geometric, cohomological, and analytic perspectives in the study of supersymmetric representation theory.

Part V, serving as an addendum, is devoted to the study of cubic Dirac operators and Dirac cohomology in the context of classical Lie superalgebras and is structured into four sections. In this part, we extend key ideas to the setting of *cubic Dirac operators* — generalizations inspired by Kostant's work and its superalgebraic analogues.

In Section 13.1, we begin with a concise overview of the foundational concepts relevant to the subsequent discussion. After introducing basic definitions and notations concerning classical Lie superalgebras $\mathfrak{g}$, we focus on their parabolic subalgebras $\mathfrak{p} \subset \mathfrak{g}$ and the associated decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{s}$, where $\mathfrak{l}$ denotes the Levi subalgebra and $\mathfrak{s}$ its orthogonal complement. We discuss the significance of this decomposition in the representation theory of Lie superalgebras, especially in the context of highest-weight modules. The section concludes with an introduction to Clifford and exterior superalgebras, culminating in the construction of an explicit embedding of $\mathfrak{l}$ into the Clifford superalgebra $C(\mathfrak{s})$. This construction forms a key ingredient in the development of the theory presented in the following sections.

Section 13.2 is devoted to the introduction and foundational analysis of the cubic Dirac operator D, the central object of study in this work. We explore its key properties and derive a decomposition into $\mathfrak{l}$-invariant summands, as formulated in Theorem 13.2.10. Subsequently, we define the Dirac cohomology $\mathrm{H}_{\mathrm{D}}(M)$ of a $\mathfrak{g}$-supermodule M, wherein the oscillator supermodule $\overline{M}(\mathfrak{s})$ plays a central role. We then turn our attention to supermodules admitting an infinitesimal character, a setting particularly well-suited for the application of Dirac cohomology. Within this framework, we establish a super-analog of the Casselman–Osborne lemma (Section 13.2.2). The section concludes with a brief discussion of homological properties associated with Dirac cohomology.

In Section 13.3, we specialize to highest-weight $\mathfrak{g}$-supermodules and examine their Dirac cohomology. We begin by proving that such supermodules always possess non-trivial Dirac cohomology (Proposition 13.3.7). We then refine our analysis to several key subclasses, including finite-dimensional supermodules for Lie superalgebras of type I with typical highest weight (Theorem 13.3.10), and simple objects in the category $\mathcal{O}^{\mathfrak{p}}$ (Theorem 13.3.11), for which we provide explicit computations.

Finally, Section 13.4 investigates the relationship between Dirac cohomology and Kostant's (co)homological functors. After reviewing the relevant notions of $\mathfrak{u}$-cohomology and $\bar{\mathfrak{u}}$-homology, we establish that Dirac cohomology naturally embeds into both (co)homologies, revealing a deep structural connection. In the second part of the section, we turn to unitarizable $\mathfrak{g}$-supermodules. For this class of representations, we demonstrate a Hodge-type decomposition of the cubic Dirac operator and prove that their Dirac cohomology is isomorphic, up to a twist, to their $\bar{\mathfrak{u}}$-cohomology as $\mathfrak{l}$-supermodules. We conclude by showing that simple weight $\mathfrak{g}$-supermodules have trivial Dirac cohomology unless they are of highest weight type—thus extending a classical result from the setting of reductive Lie algebras over $\mathbb{C}$ to the supergeometric context.

Part I.

Preliminaries: Mathematics and Physics

2. On Lie superalgebras and Lie supergroups

This section provides a brief introduction to the theory of Lie superalgebras and Lie supergroups, establishing the conventions and notation used throughout the thesis. All the definitions and statements are standard and can be found in the literature, for instance [12, 27, 129, 136].

2.1. Lie superalgebras

2.1.1. Super vector spaces

A super vector space is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Homogeneous elements of $V_{\bar{0}}$ are called *even*, while those in $V_{\bar{1}}$ are called *odd*. The parity or degree of any non-zero homogeneous element $v \in V$ is given by

$$p(v) := \begin{cases} 1 & \text{if } v \in V_{\bar{1}}, \\ 0 & \text{if } v \in V_{\bar{0}}. \end{cases}$$

The *superdimension* of a super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is $\operatorname{sdim} V = m - n$. To remember the even and odd dimension of V, we also set $\operatorname{sdim} V = (\dim V_{\bar{0}} | \dim V_{\bar{1}}) = (m|n)$ which is the element $m + n\epsilon$ in the ring $\mathbb{Z}[\epsilon]/(\epsilon^2 - 1)$, where $m := \dim(V_{\bar{0}})$ and $n := \dim(V_{\bar{1}})$. For any super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ of superdimension $(m|n)$, there exists a basis $\{e_1, \ldots, e_m\}$ of $V_{\bar{0}}$, and a basis $\{e'_1, \ldots, e'_n\}$ of $V_{\bar{1}}$, such that V is canonically isomorphic to the free $\mathbb{K}$-module $\mathbb{K}^{m|n}$ generated by $\{e_1, \ldots, e_m, e'_1, \ldots, e'_n\}$. This basis is called the *canonical basis of* V. In particular, $\mathbb{K}^{m|n}$ is the super vector space over $\mathbb{K}$, with:

$$\mathbb{K}^m \cong \mathbb{K}^{m|n}_{\bar{0}}, \quad \mathbb{K}^n \cong \mathbb{K}^{m|n}_{\bar{1}}.$$

A $\mathbb{K}$-linear map $f : V \to W$ between super vector spaces $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and $W = W_{\bar{0}} \oplus W_{\bar{1}}$ is called a *morphism of super vector spaces* if it preserves the $\mathbb{Z}_2$-grading, *i.e.*,

$$f(V_{\bar{i}}) \subset W_{\bar{i}}, \quad \bar{i} \in \mathbb{Z}_2.$$

A vector space isomorphism that preserves the $\mathbb{Z}_2$-grading is called an *isomorphism of super vector spaces*. The vector space of morphisms of super vector spaces V and W over $\mathbb{K}$ is denoted by $\operatorname{Hom}(V, W)$, and the category of super vector spaces is denoted by **sVect**. The category **sVect** is a tensor supercategory with inner Hom and dual.

The *inner Hom* $\underline{\operatorname{Hom}}(V, W)$ consists of all linear maps between the super vector spaces V and W. It is itself a super vector space with:

$$\begin{aligned} \underline{\operatorname{Hom}}_{\bar{0}}(V, W) &:= \{T : V \to W \ : \ T \text{ preserves parity}\} = \operatorname{Hom}(V, W), \\ \underline{\operatorname{Hom}}_{\bar{1}}(V, W) &:= \{T : V \to W \ : \ T \text{ reverses parity}\}. \end{aligned}$$

If $V = \mathbb{K}^{m|n}$ and $W = \mathbb{K}^{p|q}$, then in the canonical basis we have:

$$\underline{\mathrm{Hom}}_{\bar{0}}(V,W) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, \quad \underline{\mathrm{Hom}}_{\bar{1}}(V,W) = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\},$$

where A, B, C, D correspond to $(p \times m)$, $(p \times n)$, $(q \times m)$, and $(q \times n)$ matrices with entries in $\mathbb{K}$.

The dual V^* of a super vector space V is defined as $V^* := \underline{\mathrm{Hom}}(V, \mathbb{K})$, and the parity-reversing functor $\Pi : \mathbf{sVect} \to \mathbf{sVect}$ is defined by:

$$(\Pi V)_{\bar{0}} \cong V_{\bar{1}}, \quad (\Pi V)_{\bar{1}} \cong V_{\bar{0}}.$$

This makes **sVect** a supercategory. The supercategory **sVect** is a tensor category. For any two super vector spaces V and W, we define a $\mathbb{Z}_2$-graded tensor product $V \hat{\otimes} W$ as follows:

$$(V \hat{\otimes} W)_{\bar{0}} := (V_{\bar{0}} \otimes W_{\bar{0}}) \oplus (V_{\bar{1}} \otimes W_{\bar{1}}),$$

$$(V \hat{\otimes} W)_{\bar{1}} := (V_{\bar{0}} \otimes W_{\bar{1}}) \oplus (V_{\bar{1}} \otimes W_{\bar{0}}).$$

The assignment $(V, W) \mapsto V \hat{\otimes} W$ is additive and exact in each variable. Additionally, the tensor product $\hat{\otimes}$ is associative:

$$U \hat{\otimes} (V \hat{\otimes} W) \cong (U \hat{\otimes} V) \hat{\otimes} W,$$

and the map

$$C_{V,W} : V \hat{\otimes} W \to W \hat{\otimes} V, \quad v \otimes w \mapsto (-1)^{p(v)p(w)} w \otimes v,$$

is a natural isomorphism.

Next, we consider bilinear forms $B : V \times V \to \mathbb{K}$ on a super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. A bilinear form B on V is called *even* or *consistent* if: $B(V_{\bar{i}}, V_{\bar{j}}) = 0$ unless $\bar{i} + \bar{j} = \bar{0}$, and *odd* if $B(V_{\bar{i}}, V_{\bar{j}}) = 0$ unless $\bar{i} + \bar{j} = \bar{1}$, for $\bar{i}, \bar{j} \in \mathbb{Z}_2$. The form B is called *supersymmetric* if

$$B(v, w) = (-1)^{p(v)p(w)} B(w, v)$$

for all $v, w \in V$. We end this section with a particular important example of supersymmetric consistent bilinear form, the *supertrace*.

Example 2.1.1. Let V be a finite-dimensional super vector space. We identify $\underline{\mathrm{Hom}}(V, V) \cong V \otimes V^*$ by the canonical pairing

$$v \otimes f(w) := \langle f, w \rangle v, \quad v, w \in V, \ f \in V^*.$$

We define the *supertrace* $\mathrm{str} : \underline{\mathrm{Hom}}(V, V) \to \mathbb{K}$ as the composition $\mathrm{str} := \langle \cdot, \cdot \rangle \circ C_{V,V^*}$. In particular, if $V \cong \mathbb{K}^{m|n}$ and we express X in the canonical basis such that $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the supertrace is

$$\mathrm{str}(X) = \mathrm{tr}(A) - \mathrm{tr}(D),$$

where $\mathrm{tr}(\cdot)$ is the usual trace of a matrix. The supertrace induces a consistent supersymmetric bilinear form on $\underline{\mathrm{Hom}}(V, V)$ by defining

$$(X, Y) := \mathrm{str}(X \circ Y), \quad X, Y \in \underline{\mathrm{Hom}}(V, V).$$

2.1.2. Superalgebras and Lie superalgebras

A *superalgebra* is a super vector space $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ equipped with a bilinear multiplication $\tau : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ satisfying $\tau(\mathcal{A}_{\bar{i}}, \mathcal{A}_{\bar{j}}) \subset \mathcal{A}_{\overline{i+j}}$ for all $\bar{i}, \bar{j} \in \mathbb{Z}_2$. The superalgebra $\mathcal{A}$ is called *supercommutative* if $\tau \circ C_{\mathcal{A},\mathcal{A}} = \tau$ and *associative* if $\tau \circ (\tau \otimes \mathrm{id}) = \tau \circ (\mathrm{id} \otimes \tau)$ on $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$. Furthermore, $\mathcal{A}$ is said to have a *unit* if there exists an even element 1 such that $\tau(1 \otimes a) = \tau(a \otimes 1) = a$ for all $a \in \mathcal{A}$. A *supersubalgebra* of $\mathcal{A}$ is a superalgebra $\mathcal{B}$ such that $\mathcal{B} \subset \mathcal{A}$ as super vector spaces. An *ideal* of $\mathcal{A}$ is a supersubalgebra $\mathcal{I}$ such that, for any $i \in \mathcal{I}$, we have $\tau(i \otimes a) \in \mathcal{I}$ for all $a \in \mathcal{A}$. A superalgebra with no nontrivial ideals is called *simple*. In what follows, if the multiplication is clear, we simply write ab for $\tau(a \otimes b)$ for any $a, b \in \mathcal{A}$.

The superalgebras over $\mathbb{K}$ form a category, denoted by $\mathbf{sAlg}_{\mathbb{K}}$. Here, a *morphism of superalgebras* $\phi : \mathcal{A} \to \mathcal{B}$ is a morphism of the underlying super vector spaces such that $\phi(aa') = \phi(a)\phi(a')$ for all homogeneous $a, a' \in \mathcal{A}$. Moreover, the category of superalgebras has a tensor product $\mathcal{A} \hat{\otimes} \mathcal{B}$ with multiplication defined by

$$(a \hat{\otimes} b) \cdot (c \hat{\otimes} d) = (-1)^{p(b)p(c)} (ac \hat{\otimes} bd)$$

for all homogeneous $a \hat{\otimes} b$ and $c \hat{\otimes} d$ in $\mathcal{A} \hat{\otimes} \mathcal{B}$, and then extended linearly.

Example 2.1.2.

a) For any $V = \mathbb{K}^{m|n}$ and $W = \mathbb{K}^{p|q}$, the super vector spaces $\underline{\mathrm{Hom}}(V, W)$ form a superalgebra with product given by usual matrix multiplication.

b) *Tensor superalgebra*: Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a super vector space. We define the *tensor superalgebra* to be the super vector space

$$T(V) := \bigoplus_{n \geq 0} V^{\otimes n}, \quad T(V)_{\bar{0}} = \bigoplus_{n \text{ even}} V^{\otimes n}, \quad T(V)_{\bar{1}} = \bigoplus_{n \text{ odd}} V^{\otimes n}.$$

The product on the space $T(V)$ is given by the ordinary bilinear map $\Phi_{r,s} : V^{\otimes r} \times V^{\otimes s} \to V^{\otimes (r+s)}$, defined by

$$(v_1 \otimes \ldots \otimes v_r, w_1 \otimes \ldots \otimes w_s) \mapsto v_1 \otimes \ldots \otimes v_r \otimes w_1 \otimes \ldots \otimes w_s,$$

and then extended by linearity. The tensor superalgebra $T(V)$ is an associative superalgebra with unit, which is non-commutative except when V is even and one-dimensional.

c) *Symmetric superalgebra and exterior superalgebra:* Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a super vector space. We equip $T(V)$ with the natural $\mathbb{Z}_2$-grading coming from V. The symmetric superalgebra $S(V)$ is the quotient superalgebra $S(V) := T(V)/I$, where I is the homogeneous ideal generated by all elements of the form

$$v \otimes w - (-1)^{p(v)p(w)} w \otimes v, \qquad v, w \in V.$$

The exterior superalgebra $\bigwedge(V)$ on V is the quotient $\bigwedge(V) := T(V)/J$, where J is the homogeneous ideal in $T(V)$ generated by all elements of the form

$$v \otimes w + (-1)^{p(v)p(w)} w \otimes v, \qquad v, w \in V.$$

Here, both $S(V)$ and $\bigwedge(V)$ inherit the natural $\mathbb{Z}_2$-grading coming from V, as the ideals I, J are generated by homogeneous elements Moreover, we have as super vector spaces:

$$S(V) \cong S(V_{\bar{0}}) \otimes \bigwedge V_{\bar{1}}, \quad \bigwedge V \cong \bigwedge V_{\bar{0}} \otimes S(V_{\bar{1}}),$$

where $S(V_{\bar{0},\bar{1}})$ and $\bigwedge V_{\bar{0},\bar{1}}$ denote the usual symmetric and exterior algebra over the vector space $V_{\bar{0},\bar{1}}$. If V is an ordinary (even) vector space, then the symmetric and exterior superalgebras reduce to the symmetric and exterior algebra, respectively. Throughout this thesis, we use the same symbol to denote both the symmetric algebra and the symmetric superalgebra. The intended meaning will always be clear from the context.

Furthermore, the symmetric superalgebra $S(V)$ is also known *Grassmann superalgebra*: Let $V := \mathbb{K}^{m|n}$, and let $x_1, \ldots, x_m$ be a basis of $\mathbb{K}^{m|0}$, and $\theta_1, \ldots, \theta_n$ be a basis of $\mathbb{K}^{0|n}$. Then

$$S(V) = \mathbb{K}[x_1, \ldots, x_m, \theta_1, \ldots, \theta_n] := S(\mathbb{K}^{m|0}) \otimes_{\mathbb{K}} \bigwedge \mathbb{K}^{0|n}.$$

From now on, we will assume that all superalgebras are associative and with unit, unless otherwise specified.

Two other important examples of superalgebras are Lie superalgebras and their universal enveloping superalgebras.

Definition 2.1.3. A *Lie superalgebra* $\mathfrak{g}$ over $\mathbb{K}$ is an object in the category of super vector spaces together with a morphism $[\cdot,\cdot] : \mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g} \to \mathfrak{g}$, called the *super Lie bracket*, which satisfies the following two conditions for all homogeneous elements $x, y, z \in \mathfrak{g}$:

a) Skew-symmetry:
$$[x, y] = -(-1)^{p(x)p(y)}[y, x].$$

b) Super Jacobi identity:
$$(-1)^{p(x)p(z)}[x, [y, z]] + (-1)^{p(y)p(x)}[y, [z, x]] + (-1)^{p(z)p(y)}[z, [x, y]] = 0.$$

A *morphism* of Lie superalgebras is a $\mathbb{K}$-linear map between Lie superalgebras that preserves the superbrackets and grading. A *Lie superalgebra isomorphism* is a bijective Lie superalgebra morphism. The associated category of Lie superalgebras is denoted by $\mathbf{sLieAlg}_{\mathbb{K}}$.

A $\mathbb{Z}_2$-graded subspace $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ of a Lie superalgebra $(\mathfrak{g}, [\cdot,\cdot])$ is called a *Lie supersubalgebra* if $[x, y] \in \mathfrak{h}$ holds for all $x, y \in \mathfrak{h}$. An *ideal* is a Lie supersubalgebra $\mathfrak{I}$ of $\mathfrak{g}$ such that $[x, y] \in \mathfrak{I}$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{I}$. A Lie superalgebra $\mathfrak{g}$ is called *simple* if the only ideals are $\{0\}$ and $\mathfrak{g}$, and $\mathfrak{g}$ is not abelian, that is, there exists some $x, y \in \mathfrak{g}$ with $[x, y] \neq 0$.

Example 2.1.4. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a super vector space of superdimension $(m|n)$. The associative algebra $\mathrm{End}_{\mathbb{K}}(V) := \underline{\mathrm{Hom}}_{\mathbb{K}}(V, V)$ of all $\mathbb{K}$-linear transformations of V has a natural structure of a Lie superalgebra with $\mathbb{Z}_2$-grading given by

$$\mathrm{End}_{\mathbb{K}}(V)_{\bar{0}} = \underline{\mathrm{Hom}}(V, V)_{\bar{0}}, \quad \mathrm{End}_{\mathbb{K}}(V)_{\bar{1}} = \underline{\mathrm{Hom}}(V, V)_{\bar{1}}.$$

It becomes a Lie superalgebra if we define the superbracket as

$$[X, Y] = X \circ Y - (-1)^{p(X)p(Y)} Y \circ X,$$

where $X, Y \in \mathrm{End}_{\mathbb{K}}(V)$ are homogeneous elements.

If we choose a basis, we may identify $\mathrm{End}_{\mathbb{K}}(V)$ with block matrices such that

$$\mathrm{End}_{\mathbb{K}}(V)_{\bar{0}} \cong \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \;\middle|\; A \in \mathbb{K}^{m\times m},\ D \in \mathbb{K}^{n\times n} \right\},$$

$$\mathrm{End}_{\mathbb{K}}(V)_{\bar{1}} \cong \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \;\middle|\; B \in \mathbb{K}^{m\times n},\ C \in \mathbb{K}^{n\times m} \right\}.$$

In the rest of the thesis, we will use the notation

$$\mathfrak{gl}(m|n; \mathbb{K}) := \mathrm{End}_{\mathbb{K}}(V), \quad \mathfrak{gl}(m|n; \mathbb{K})_{\bar{i}} := \mathrm{End}_{\mathbb{K}}(V)_{\bar{i}}$$

for $\bar{i} \in \mathbb{Z}_2$. We call $\mathfrak{gl}(m|n; \mathbb{K})$ the *general linear Lie superalgebra* over $\mathbb{K}$, and we simply write $\mathfrak{gl}(m|n)$ if we work over $\mathbb{C}$. Note that $\mathfrak{gl}(m|n)$ has a natural $\mathbb{Z}$-grading $\mathfrak{gl}(m|n) = \mathfrak{gl}(m|n)_{-1} \oplus \mathfrak{gl}(m|n)_0 \oplus \mathfrak{gl}(m|n)_{+1}$ with $\mathfrak{gl}(m|n)_0 = \mathfrak{gl}(m|n)_{\bar{0}}$ and $\mathfrak{gl}(m|n)_{\bar{1}} = \mathfrak{gl}(m|n)_{-1} \oplus \mathfrak{gl}(m|n)_{+1}$, where

$$\mathfrak{gl}(m|n)_{+1} := \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} : B \in \mathbb{K}^{m\times n} \right\}, \quad \mathfrak{gl}(m|n)_{-1} := \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} : C \in \mathbb{K}^{n\times m} \right\}.$$

In particular, $\mathfrak{gl}(m|n)_{\pm 1}$ are $\mathfrak{g}_{\bar{0}}$-submodules which are abelian considered as Lie superalgebras, *i.e.*, $[\mathfrak{gl}(m|n)_{\pm 1}, \mathfrak{gl}(m|n)_{\pm 1}] = 0$.

Let $\mathfrak{g}$ be a Lie superalgebra, and let $T(\mathfrak{g})$ be the tensor superalgebra over the underlying super vector space $\mathfrak{g}$ with the natural inclusion $\iota : \mathfrak{g} \to T(\mathfrak{g})$. Let $\mathfrak{I} \subset T(\mathfrak{g})$ be the two-sided homogeneous ideal in $T(\mathfrak{g})$ generated by

$$\iota(X) \otimes \iota(Y) - (-1)^{p(X)p(Y)} \iota(Y) \otimes \iota(X) - \iota([X, Y]) \in T(\mathfrak{g})$$

for all homogeneous $X, Y \in \mathfrak{g}$. The *universal enveloping superalgebra* $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ is the quotient

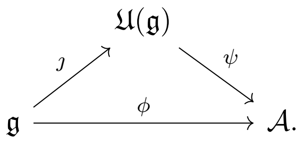
$$\mathfrak{U}(\mathfrak{g}) := T(\mathfrak{g})/\mathfrak{I}.$$

Note that $\mathfrak{U}(\mathfrak{g})$ inherits a $\mathbb{Z}_2$-grading from the $\mathbb{Z}_2$-grading of $T(\mathfrak{g})$, which extends the $\mathbb{Z}_2$-grading of $\mathfrak{g}$, as $\mathfrak{I}$ is generated by homogeneous elements. Let $\jmath : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ be the composition of the injective map $\iota : \mathfrak{g} \to T(\mathfrak{g})$ and the natural surjective superalgebra morphism $\pi : T(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$. Then $\jmath(\mathfrak{g})$ is injective, and it generates $\mathfrak{U}(\mathfrak{g})$ as an associative superalgebra, and in what follows, we denote the product of two elements $x, y \in \mathfrak{U}(\mathfrak{g})$ simply by xy. In particular, the universal enveloping superalgebra has a natural Lie superalgebra structure given by

$$[X, Y] := XY - (-1)^{p(X)p(Y)} YX$$

for all $X, Y \in \mathfrak{U}(\mathfrak{g})$. We collect important properties of $\mathfrak{U}(\mathfrak{g})$.

Proposition 2.1.5 ([12, Proposition 1.6.2]). *If $\mathcal{A}$ is an associative superalgebra and $\phi : \mathfrak{g} \to \mathcal{A}$ is a morphism of Lie superalgebras, then there exists a unique morphism $\psi : \mathfrak{U}(\mathfrak{g}) \to \mathcal{A}$ of associative superalgebras such that the following diagram commutes:*



Analogously to the Lie algebra case, a Poincaré–Birkhoff–Witt Theorem holds.

Theorem 2.1.6 ([12, Theorem 1.6.5]). *Let $\{X_1, \ldots, X_p\}$ be a basis for $\mathfrak{g}_{\bar{0}}$ and $\{Y_1, \ldots, Y_q\}$ a basis for $\mathfrak{g}_{\bar{1}}$. Then the set*

$$\left\{X_1^{r_1} X_2^{r_2} \cdots X_p^{r_p} Y_1^{s_1} Y_2^{s_2} \cdots Y_q^{s_q} \mid r_1, r_2, \ldots, r_p \in \mathbb{Z}_+,\ s_1, s_2, \ldots, s_q \in \{0, 1\}\right\}$$

is a basis for $\mathfrak{U}(\mathfrak{g})$. In particular, we have a linear isomorphism of super vector spaces

$$\mathfrak{U}(\mathfrak{g}) \cong \mathfrak{U}(\mathfrak{g}_{\bar{0}}) \otimes \bigwedge \mathfrak{g}_{\bar{1}},$$

where $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}_{\bar{0}}$ and $\bigwedge \mathfrak{g}_{\bar{1}}$ is the exterior algebra over the ordinary vector space $\mathfrak{g}_{\bar{1}}$.

2.1.3. Structure theory of Lie superalgebras

Basic classical Lie superalgebras

In the following, we consider (finite-dimensional) simple Lie superalgebras $\mathfrak{g}$ with $\mathfrak{g}_{\bar{0}} \neq \{0\}$ and $\mathfrak{g}_{\bar{1}} \neq \{0\}$, classified by Kac in [76]. A preliminary characterization of simple Lie superalgebras is provided by the following useful lemma.

Lemma 2.1.7 ([99, Lemma 1.2.1]). *Let $\mathfrak{g}$ be a Lie superalgebra with $\mathfrak{g}_{\bar{0}} \neq \{0\}$ and $\mathfrak{g}_{\bar{1}} \neq \{0\}$. Then $\mathfrak{g}$ is simple if and only if the following conditions hold:*

a) If $\mathfrak{a}$ is a non-zero $\mathfrak{g}_{\bar{0}}$-submodule of $\mathfrak{g}_{\bar{1}}$ such that $[\mathfrak{g}_{\bar{1}}, [\mathfrak{g}_{\bar{1}}, \mathfrak{a}]] \subset \mathfrak{a}$, then $[\mathfrak{g}_{\bar{1}}, \mathfrak{a}] = \mathfrak{g}_{\bar{0}}$.

b) $\mathfrak{g}_{\bar{1}}$ is a faithful $\mathfrak{g}_{\bar{0}}$-module under the adjoint action.

c) $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{1}}$.

Simple Lie superalgebras can be classified. Invariant bilinear forms $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ play a significant role in this classification. A bilinear form $B(\cdot, \cdot)$ is called *invariant* if it satisfies

$$B([x, y], z) = B(x, [y, z])$$

for all $x, y, z \in \mathfrak{g}$.

Proposition 2.1.8 ([99, Proposition 1.2.4]). *Let $\mathfrak{g}$ be a simple Lie superalgebra. Then the following assertions hold:*

a) Any invariant bilinear form on $\mathfrak{g}$ is either non-degenerate or identically zero.

b) Any invariant bilinear form on $\mathfrak{g}$ is supersymmetric.

c) Any two nonzero invariant bilinear forms on $\mathfrak{g}$ are proportional.

d) The invariant bilinear forms on $\mathfrak{g}$ are either all odd or all even.

In what follows, we assume that bilinear forms are even/consistent. In [76], Kac classified simple Lie superalgebras, which can be broadly split into the following types:

Simple Lie Superalgebras		
Classical		**Cartan**
Basic	**Strange**	
$A(m\|n)$, $B(m\|n)$, $C(n)$, $D(m\|n)$ $D(2.1;\alpha)$, $G(3)$, $F(4)$	$P(n)$, $Q(n)$	$W(n)$, $S(n)$, $\tilde{S}(n)$, $H(n)$

In this thesis, we are mainly interested in basic classical Lie superalgebras.

Definition 2.1.9. A simple Lie superalgebra $\mathfrak{g}$ is called *classical* if $\mathfrak{g}_{\bar{1}}$ is a completely reducible $\mathfrak{g}_{\bar{0}}$-module, where the action is given by the super Lie bracket. A Lie superalgebra is called *basic classical* if it is classical and admits a consistent, non-degenerate, invariant (even) bilinear form.

Remark 2.1.10. A simple Lie superalgebra $\mathfrak{g}$ is classical if and only if $\mathfrak{g}_{\bar{0}}$ is reductive.

Basic classical Lie superalgebras split into two types.

Lemma 2.1.11 ([129, Lemma 2])**.** *Let $\mathfrak{g}$ be a basic classical Lie superalgebra. Then one of the following two assertions holds:*

a) There is a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, such that $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\pm 1}$ are simple $\mathfrak{g}_{\bar{0}}$-modules.

b) The even part $\mathfrak{g}_{\bar{0}}$ is semisimple, and $\mathfrak{g}_{\bar{1}}$ is a simple $\mathfrak{g}_{\bar{0}}$-module.

The lemma leads to the following definition.

Definition 2.1.12. Let $\mathfrak{g}$ be a basic Lie superalgebra. We say that $\mathfrak{g}$ is *of type I* (respectively *type II*) if it satisfies a) (respectively b)) in Lemma 2.1.11.

For the basic classical Lie superalgebras $A(m|n)$ with $m \neq n$, $B(m|n)$, $C(n+1)$, $D(m|n)$ with $m \neq n+1$, $F(4)$, and $G(3)$, we use the *Killing form* as the non-degenerate, invariant, consistent bilinear form $(\cdot,\cdot)$ given by

$$(x, y) := \mathrm{str}(\mathrm{ad}_x \circ \mathrm{ad}_y), \qquad x, y \in \mathfrak{g},$$

where $\mathrm{ad}_z(\cdot) := [z, \cdot]$ is the adjoint representation of $\mathfrak{g}$, and $\mathrm{str}(\cdot)$ is the supertrace. For the remaining basic classical Lie superalgebras, the Killing form vanishes identically, and an alternative form can be constructed in an ad hoc manner [76].

This thesis primarily focuses on two specific examples: the special linear Lie superalgebras and the orthosymplectic Lie superalgebras.

Example 2.1.13 (Basic classical Lie superalgebras $A(m|n)$)**.** The *special linear Lie superalgebra* $\mathfrak{sl}(m|n)$ is the sub Lie superalgebra given by all matrices in $\mathfrak{gl}(m|n)$ with vanishing supertrace, that is,

$$\mathfrak{sl}(m|n) = \{X \in \mathfrak{gl}(m|n) : \mathrm{str}(X) = 0\}.$$

This is a codimension-1 ideal in $\mathfrak{gl}(m|n)$ with $[\mathfrak{gl}(m|n), \mathfrak{gl}(m|n)] = \mathfrak{sl}(m|n)$. Its even part $\mathfrak{g}_{\bar{0}}$ is naturally isomorphic to a direct sum of complex special linear Lie algebras and an abelian factor, $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathbb{C}E_{m|n}$ where $E_{m|n}$ denotes the even matrix with $A = nE_m$ and $D = mE_n$ in the representation above, and where E_m, E_n are unit matrices of the size indicated. When $m \neq n$ and $m + n \geq 2$, $\mathfrak{sl}(m|n)$ is simple, and the extension to $\mathfrak{gl}(m|n)$ splits (naturally, but non-canonically) via $\mathbb{C} \ni 1 \mapsto E_{m+n} \in \mathfrak{gl}(m|n)$. When $m = n$, we have $E_{2n} = \frac{1}{n}E_{n|n} \in \mathfrak{sl}(n|n)$. As a consequence, $\mathfrak{gl}(n|n)$ does not split, and $\mathfrak{sl}(n|n)$ becomes reducible, although it also does not split. The codimension-1 simple quotient is the *projective special linear Lie superalgebra* $\mathfrak{psl}(n|n) = \mathfrak{sl}(n|n)/\mathbb{C}E_{2n}$.

To address both cases, $m = n$ and $m \neq n$ simultaneously, we define the Lie superalgebras of type $A(m|n)$ as follows:

$$A(m|n) := \begin{cases} \mathfrak{sl}(m+1|n+1), & \text{for } m \neq n, \text{ and } m, n \geq 0, \\ \mathfrak{sl}(n+1|n+1)/\mathbb{C}E_{n+1,n+1}, & \text{for } m = n, \text{ and } n > 0, \end{cases}$$

The Lie superalgebras of type $A(m|n)$ are examples of Lie superalgebras of type 1, with the $\mathbb{Z}$-grading coming from that of $\mathfrak{gl}(m|n)$ in Example 2.1.4. It remains to construct the form $(\cdot,\cdot)$.

The general linear Lie superalgebra carries a natural even supersymmetric and invariant bilinear form $(\cdot,\cdot) : \mathfrak{gl}(m|n) \times \mathfrak{gl}(m|n) \longrightarrow \mathbb{C}$ defined by

$$(X, Y) := \mathrm{str}(XY)$$

for all $X, Y \in \mathfrak{gl}(m|n)$. Here, even supersymmetric means that $(\cdot,\cdot)$ is symmetric on $\mathfrak{gl}(m|n)_0$, skew-symmetric on $\mathfrak{gl}(m|n)_1$, and $\mathfrak{gl}(m|n)_0$ and $\mathfrak{gl}(m|n)_1$ are orthogonal to each other. Invariant means that $([X,Y],Z) = (X,[Y,Z])$ for all $X, Y, Z \in \mathfrak{gl}(m|n)$. The form $(\cdot,\cdot)$ is always non-degenerate on $\mathfrak{gl}(m|n)$. However, its restriction to $\mathfrak{sl}(m|n)$ is non-degenerate only as long as $m \neq n$. When $m = n$, the one-dimensional center of $\mathfrak{sl}(n|n)$ becomes the non-trivial radical of $(\cdot,\cdot)$.

Example 2.1.14 (Orthosymplectic Lie superalgebras)**.** Let V be a super vector space with superdimension $\mathrm{sdim}\, V = (m|n)$, and let B be a non-degenerate, consistent, supersymmetric bilinear form on V. In particular, n must be even. The *orthosymplectic Lie superalgebra* is defined as $\mathfrak{osp}(V) := \mathfrak{osp}(V)_{\bar{0}} \oplus \mathfrak{osp}(V)_{\bar{1}}$, where

$$\mathfrak{osp}(V)_{\bar{s}} := \Big\{T \in \mathfrak{gl}(V)_{\bar{s}} : B(Tv, w) = -(-1)^{p(T)p(v)} B(v, Tw),\ v, w \in V \text{ homogeneous}\Big\}.$$

If we identify $V \cong \mathbb{K}^{m|n}$, we also write $\mathfrak{osp}(m|n;\mathbb{K})$ for $\mathfrak{osp}(V)$. Under this identification, we consider on V the canonical even, non-degenerate, supersymmetric bilinear form $B : V \times V \to \mathbb{K}$ given by

$$B(v, w) := v^T J_{(m|2k)} w, \qquad J_{(m|2k)} := \begin{pmatrix} E_m & 0 \\ 0 & \begin{pmatrix} 0 & E_k \\ -E_k & 0 \end{pmatrix} \end{pmatrix},$$

where $n = 2k$ is even. A general element of $\mathfrak{osp}(m|n;\mathbb{K})$ has the form

$$\begin{pmatrix} D & P & Q \\ -Q^T & A & B \\ P^T & C & -A^T \end{pmatrix},$$

where $D \in \mathfrak{so}(m,\mathbb{K})$, B, C are symmetric $k \times k$ matrices, $A \in \mathfrak{gl}(k,\mathbb{K})$, and P, Q are $k \times m$ matrices. The even subalgebra $\mathfrak{osp}(m|n;\mathbb{K})$ is isomorphic to $\mathfrak{so}(m,\mathbb{K}) \oplus \mathfrak{sp}(n,\mathbb{K})$.

We adopt Kac's notation and use the following conventions:

$$\begin{aligned} B(m|n) &:= \mathfrak{osp}(2m+1|2n), \quad && m \geq 0,\, n \geq 1, \\ D(m|n) &:= \mathfrak{osp}(2m|2n), && m \geq 2,\, n \geq 1, \\ C(n) &:= \mathfrak{osp}(2|2n-2), && n \geq 2. \end{aligned}$$

The Lie superalgebras $B(m|n)$ and $D(m|n)$ are of type 2, while $C(n)$ is of type 1.

Similarly, let C be a non-degenerate, odd, supersymmetric bilinear form on V. For $\bar{s} \in \mathbb{Z}_2$, we define $\mathfrak{spo}(V) := \mathfrak{spo}(V)_{\bar{0}} \oplus \mathfrak{spo}(V)_{\bar{1}}$, where

$$\mathfrak{spo}(V)_{\bar{s}} := \left\{ T \in \mathfrak{gl}(V)_{\bar{s}} : C(Tv, w) + (-1)^{\bar{s}p(T)} C(v, Tw) = 0,\ v, w \in V \text{ homogeneous} \right\}.$$

The superalgebra $\mathfrak{spo}(V)$ is a Lie superalgebra over $\mathbb{K}$, called the *orthosymplectic Lie superalgebra* with respect to C. To describe the relation to $\mathfrak{osp}(V)$, let Π denote the parity-reversing functor. On any super vector space V, a supersymmetric bilinear form B induces a skew-supersymmetric bilinear form C on $\Pi(V)$ defined by

$$C(\Pi(v), \Pi(w)) := B(v, w), \qquad v, w \in V.$$

The Lie superalgebras $\mathrm{End}(V)$ and $\mathrm{End}(\Pi(V))$ are isomorphic via the map

$$\Phi : \mathrm{End}(V) \to \mathrm{End}(\Pi(V)), \qquad T \mapsto \Pi \circ T \circ \Pi^{-1},$$

and its restriction $\Phi|_{\mathfrak{osp}(V)}$ maps $\mathfrak{osp}(V)$ to $\mathfrak{spo}(\Pi(V))$.

Structure Theory

Let $\mathfrak{g}$ be a basic classical Lie superalgebra. As $\mathfrak{g}_{\bar{0}}$ is reductive, it has a Cartan subalgebra $\mathfrak{h}$, which acts semisimply on $\mathfrak{g}_{\bar{0}}$. Recall that $\mathfrak{g}_{\bar{1}}$ is completely reducible as an $\mathfrak{g}_{\bar{0}}$-module under the adjoint action induced by the matrix supercommutator $[\cdot,\cdot]$. Thus, we obtain a root space decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}^\alpha, \quad \mathfrak{g}^\alpha := \{X \in \mathfrak{g} \ : \ [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}.$$

We call $\alpha \in \mathfrak{h}^*$ a *root* if $\alpha \neq 0$ and $\mathfrak{g}^\alpha \neq \{0\}$. Elements of $\mathfrak{g}^\alpha$ are called *root vectors*, and $\mathfrak{g}^\alpha$ is called the *root space* corresponding to the root α. Let Δ denote the set of all roots. Each root space $\mathfrak{g}^\alpha$ has either superdimension $(1|0)$ or $(0|1)$. A root $\alpha \in \Delta$ is called *even* if $\mathfrak{g}^\alpha \cap \mathfrak{g}_{\bar{0}} \neq \{0\}$ and *odd* if $\mathfrak{g}^\alpha \cap \mathfrak{g}_{\bar{1}} \neq \{0\}$. The associated sets of roots are denoted by $\Delta_{\bar{0}}$ and $\Delta_{\bar{1}}$, respectively. The following lemma is well-known.

Lemma 2.1.15 ([75, Proposition 1.3], [76, Proposition 2.5.5])**.** *The following assertions hold:*

a) If $\alpha \in \Delta$, then $-\alpha \in \Delta$. Moreover, $\Delta = -\Delta$, $\Delta_{\bar{0}} = -\Delta_{\bar{0}}$, and $\Delta_{\bar{1}} = -\Delta_{\bar{1}}$.

b) $\mathrm{sdim}(\mathfrak{g}^\alpha) = (1|0)$ *for all* $\alpha \in \Delta_{\bar{0}}$, *and* $\mathrm{sdim}(\mathfrak{g}^\alpha) = (0|1)$ *for all* $\alpha \in \Delta_{\bar{1}}$.

c) $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = 0$ *if and only if* $\alpha, \beta \in \Delta$ *and* $\alpha + \beta \notin \Delta$, *while* $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$ *for all* $\alpha, \beta \in \Delta$ *if* $\alpha + \beta \in \Delta$. *In particular,* $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ *is a one-dimensional subspace in* $\mathfrak{h}$ *for all* $\alpha \in \Delta$.

d) The restriction of the invariant form $(\cdot, \cdot)$ *on* $\mathfrak{h} \times \mathfrak{h}$ *is non-degenerate, and* $(\mathfrak{g}^\alpha, \mathfrak{g}^\beta) = 0$ *unless* $\alpha = -\beta \in \Delta$.

e) Fix some nonzero $e_\alpha \in \mathfrak{g}^\alpha$. *Then* $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha$, *where* h_α *is the* coroot *determined by*

$$(h_\alpha, h) = \alpha(h) \quad \text{for all } h \in \mathfrak{h}.$$

f) The bilinear form on $\mathfrak{h}^*$ *defined by* $(\lambda, \mu) := (h_\lambda, h_\mu)$ *is non-degenerate and invariant under the Weyl group of* $\mathfrak{g}_{\bar{0}}$.

g) Let $\lambda \in \Delta$. *Then* $k\lambda \in \Delta$ *for some integer* $k \neq \pm 1$ *if and only if* λ *is an odd root such that* $(\lambda, \lambda) \neq 0$*; in this case, we must have* $k = \pm 2$.

The root system is an example of a *generalized root system* (*cf.* [129]), that is:

a) If $\alpha \in \Delta$, then $-\alpha \in \Delta$.

b) If $\alpha, \beta \in \Delta$ and $(\alpha, \alpha) \neq 0$, then $k_{\alpha,\beta} := 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ and $r_\alpha(\beta) := \beta - k_{\alpha,\beta}\alpha \in \Delta$.

c) If $\alpha \in \Delta$ and $(\alpha, \alpha) = 0$, then there exists an invertible map $r_\alpha : \Delta \to \Delta$ such that

$$r_\alpha(\beta) = \begin{cases} \beta & \text{if } (\alpha, \beta) = 0, \\ \beta \pm \alpha & \text{if } (\alpha, \beta) \neq 0. \end{cases}$$

If $\alpha \in \Delta_{\bar{0}}$, then $r_\alpha(\cdot)$ is called an *even reflection*, whereas $r_\alpha(\cdot)$ is called an *odd reflection* if $(\alpha, \alpha) = 0$. Roots $\alpha \in \Delta$ satisfying $(\alpha, \alpha) = 0$ are referred to as *isotropic*. Note that isotropic roots are necessarily odd. If $\mathfrak{g}$ is of type 1, then all odd roots are isotropic. Further, note that 2α is not a root if α is an isotropic root.

The even reflections preserve $\Delta_{\bar{0}}$ and $\Delta_{\bar{1}}$, respectively. Indeed, they generate a group W, referred to as the *Weyl group* of $\mathfrak{g}$. Note that the Weyl group of $\mathfrak{g}$ coincides with that of the reductive Lie subalgebra $\mathfrak{g}_{\bar{0}}$. It acts on Δ by permutations.

Lemma 2.1.16 ([129]). *a) If* $\mathfrak{g}$ *is of type* 1*, the Weyl group has two orbits in* $\Delta_{\bar{1}}$*, namely the roots of* $\mathfrak{g}_{\pm 1}$.

b) If $\mathfrak{g}$ *is of type* 2*, the Weyl group acts transitively on the set of isotropic roots and the set of non-isotropic roots.*

We define the *dot action* of W on $\mathfrak{h}^*$ by

$$w \cdot \lambda := w(\lambda + \rho) - \rho$$

for any $\lambda \in \mathfrak{h}^*$ and $w \in W$. We say that $\lambda, \mu \in \mathfrak{h}^*$ are *W-linked* if there exists some $w \in W$ such that $\mu = w \cdot \lambda$. This gives an equivalence relation on $\mathfrak{h}^*$, and the Weyl orbit $\{w \cdot \lambda : w \in W\}$ of λ under the dot action is called the *W-linkage class* of λ.

Let E denote the real vector space spanned by Δ, such that $E \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{h}^*$. A *positive system* Δ^+ is a subset of Δ consisting of all roots $\alpha \in \Delta$ satisfying $\alpha > 0$ for some total ordering $\geq$ of E that is compatible with the real structure, *i.e.*, $v \geq w$ and $v' \geq w'$ imply $v + v' \geq w + w', -w \geq -v$, and $cv \geq cw$ for $c \in \mathbb{R}_{>0}$. Roots in Δ^+ are referred to as *positive*. By defining $\Delta^+_{\bar{i}} := \Delta^+ \cap \Delta_{\bar{i}}$ for $\bar{i} \in \mathbb{Z}_2$, we have

$$\Delta^+ = \Delta^+_{\bar{0}} \sqcup \Delta^+_{\bar{1}}.$$

To any choice Δ^+ of a positive system, we define the *Weyl vector* to be $\rho := \rho_{\bar{0}} - \rho_{\bar{1}}$ with

$$\rho_{\bar{0}} := \frac{1}{2} \sum_{\alpha \in \Delta^+_{\bar{0}}} \alpha, \qquad \rho_{\bar{1}} := \frac{1}{2} \sum_{\beta \in \Delta^+_{\bar{1}}} \beta.$$

Furthermore, we have a *triangular decomposition*

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

where $\mathfrak{n}^\pm$ are the $\mathrm{ad}(\mathfrak{h})$-stable supersubalgebras

$$\mathfrak{n}^+ := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha, \qquad \mathfrak{n}^- := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^{-\alpha}.$$

The Lie supersubalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is called a *Borel subalgebra*. For later applications, it is useful to decompose $\mathfrak{U}(\mathfrak{g})$ with respect to $\mathfrak{n}^\pm$ and $\mathfrak{b}$. The subsequent super vector space isomorphisms are straightforward:

$$\mathfrak{U}(\mathfrak{g}) \cong \mathfrak{U}(\mathfrak{n}^-) \otimes \mathfrak{U}(\mathfrak{b}) \cong \mathfrak{U}(\mathfrak{h}) \otimes \mathfrak{U}(\mathfrak{n}^- \mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{n}^+).$$

For a fixed positive system Δ^+, we define the *fundamental system* $\pi \subset \Delta^+$ to be the set of all $\alpha \in \Delta^+$ which cannot be written as the sum of two roots in Δ^+. Elements of π are called *simple*. For $\pi = \{\alpha_1, \ldots, \alpha_r\}$, any $\alpha \in \Delta$ can be uniquely represented as a linear combination

$$\alpha = \sum_{i=1}^{r} k_i \alpha_i,$$

where either all $k_i \in \mathbb{Z}_{\geq 0}$ or all $k_i \in \mathbb{Z}_{\leq 0}$.

To construct appropriate generators for $\mathfrak{n}^\pm$, the following lemma is central.

Lemma 2.1.17 ([75, Proposition 1.5])**.** *Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ be a Borel subalgebra of $\mathfrak{g}$, and let $\Pi := \{\alpha_1, \ldots, \alpha_r\}$ be a fundamental system.*

a) There exists elements $e_i \in \mathfrak{g}^{\alpha_i}, f_i \in \mathfrak{g}^{-\alpha_i}$ and $h_i \in \mathfrak{h}$ such that $\{e_i, f_i, h_i\}_{i=1,\ldots,r}$ is the system of generators of $\mathfrak{g}$ satisfying the relations:

$$[e_i, f_j] = \delta_{ij} h_i, \quad [h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j$$

for a non-singular matrix $A = (a_{ij})_{1 \leq i,j \leq r}$.

b) $\mathfrak{n}^\pm$ are generated by the elements e_i and f_i, respectively

The Weyl group W acts by permuting the elements of Δ, but not all fundamental systems can be transformed into one another through the action of W. For this, we need a sequence of *odd reflections*. Given an odd, isotropic simple root $\theta \in \pi$, we recall that an *odd reflection* satisfies:

$$r_\theta(\alpha) = \begin{cases} \alpha + \theta & \text{if } (\alpha, \theta) \neq 0, \\ \alpha & \text{if } (\alpha, \theta) = 0, \\ -\theta & \text{if } \alpha = \theta \end{cases} \tag{2.1.1}$$

for any $\alpha \in \pi$. Then, $\Delta_\theta^+ = -\theta \cup (\Delta^+ \setminus \theta)$ forms a new positive system with the fundamental system $\pi_\theta := r_\theta(\pi)$. The following lemma is immediate.

Lemma 2.1.18. *The following two assertions hold:*

a) If π and π' are two fundamental systems, then π' can be obtained form π by applying odd and even reflections.

b) If π and π' are two fundamental systems such that $\Delta_{\bar{0}}^+ = (\Delta')_{\bar{0}}^+$, then π' can be obtained from π by application of odd reflections.

The following lemma is straightforward but useful for computations.

Lemma 2.1.19. *Let Δ^+ be a positive system with fundamental system π. Let ρ denote the associated Weyl vector. Then*

$$(\rho, \beta) = \begin{cases} \frac{1}{2}(\beta, \beta) & \text{if } \beta \in \pi \cap \Delta_{\bar{0}}, \\ (\beta, \beta) & \text{if } \beta \in \pi \cap \Delta_{\bar{1}}. \end{cases}$$

For the rest of this subsection, we briefly summarize the structure theory of special linear Lie superalgebras.

Structure theory of $\mathfrak{sl}(m|n)$

The structure theory of $\mathfrak{sl}(m|n)$ is the same as the structure theory of $\mathfrak{gl}(m|n)$, where we realize $\mathfrak{sl}(m|n)$ as a Lie supersubalgebra, as in Example 2.1.13. We denote by $\mathfrak{d} := \{H = \operatorname{diag}(h_1, \ldots, h_{m+n})\}$ the abelian Lie subalgebra of diagonal matrices in $\mathfrak{gl}(m|n)$. We choose the subspace of diagonal matrices with vanishing supertrace, denoted by $\mathfrak{h}$, as Cartan subalgebra of $\mathfrak{g} = \mathfrak{sl}(m|n)$. The standard basis of the dual space $\mathfrak{d}^*$ of $\mathfrak{d}$ is $(\epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n)$ where $\epsilon_i(H) = h_i$, $\delta_k(H) = h_{k+m}$ for $i = 1, \ldots, m$ and $k = 1, \ldots, n$, for any $H \in \mathfrak{d}$. We (ab)use this basis also for the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$. Namely, we identify weights $\lambda \in \mathfrak{h}^*$ for $\mathfrak{g}$ with tuples $(\lambda^1, \ldots, \lambda^m | \mu^1, \ldots, \mu^n)$ via the expansion $\lambda = \lambda^1 \epsilon_1 + \ldots + \lambda^m \epsilon_m + \mu^1 \delta_1 + \ldots + \mu^n \delta_n$, keeping in mind that shifts by $(1, \ldots, 1 | -1, \ldots, -1)$ do not change the weight. The space of weights for $\mathfrak{psl}(n|n)$ is the subquotient of tuples with $\sum_{i=1}^n (\lambda^i + \mu^i) = 0$.

The set of roots is $\Delta = \Delta_{\bar{0}} \sqcup \Delta_{\bar{1}}$, where

$$\Delta_{\bar{0}} = \{\pm(\epsilon_i - \epsilon_j), \pm(\delta_k - \delta_l) \; : \; 1 \le i < j \le m, \; 1 \le k < l \le n\},$$
$$\Delta_{\bar{1}} = \{\pm(\epsilon_i - \delta_k) \; : \; 1 \le i \le m, \; 1 \le k \le n\},$$

are the *even* and *odd roots*, respectively. Note that each root space has superdimension $(1|0)$ or $(0|1)$, and $\Delta_{\bar{0}}$ is the disjoint union of root systems for $\mathfrak{sl}(m)$ and $\mathfrak{sl}(n)$. For the rest of the thesis, we fix the standard positive system on $\Delta_{\bar{0}}$:

$$\Delta_{\bar{0}}^{+} := \{\epsilon_i - \epsilon_j, \delta_k - \delta_l \ : \ 1 \leq i < j \leq m, \ 1 \leq k < l \leq n\}$$

such that the root vectors for $\epsilon_i - \epsilon_j$, $i < j$, are strictly upper triangular matrices of $\mathfrak{sl}(m)$, and the root vectors for $\delta_k - \delta_l$, $k < l$, are strictly upper triangular matrices of $\mathfrak{sl}(n)$, both diagonally embedded in $\mathfrak{g}_{\bar{0}}$. When extending this to the odd part, we usually make the *standard choice*:

$$\Delta_{\bar{1}}^{+} := \{\epsilon_i - \delta_k \ : \ 1 \leq i \leq m, \ 1 \leq k \leq n\}$$

such that the associated root vectors are the off-diagonal upper block matrices in $\mathfrak{g}_{\bar{1}}$. For the standard choice, $\mathfrak{n}^{+}$ is the space of strictly upper triangular matrices and $\mathfrak{n}^{-}$ the space of strictly lower triangular matrices. In particular, the simple roots are

$$\pi := (\epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n).$$

The space of real weights, defined as the $\mathbb{R}$-span of Δ inside of $\mathfrak{h}^*$, and denoted $\Delta_{\mathbb{R}}$ or $\mathfrak{h}^*_{\mathbb{R}}$, is a real vector space of dimension $n + m - 1$. Its dual inside of $\mathfrak{h}$, denoted $\mathfrak{h}_{\mathbb{R}}$, is the space of supertrace-less real diagonal matrices.

Moreover, the *Weyl vector* $\rho = \rho_{\bar{0}} - \rho_{\bar{1}}$ is

$$\rho = \frac{1}{2}\Big(\sum_{i=1}^{m}(m - n + 1 - 2i)\epsilon_i + \sum_{k=1}^{n}(m + n - 2k + 1)\delta_k\Big).$$

In calculations, however, it is simpler to use a shifted representative of the Weyl vector,

$$\begin{aligned}\rho &= \frac{1}{2}(m-n-1, \dots, -m-n-1|m+n-1, \dots, m-n-1) + \frac{m+n+1}{2}(1, \dots, 1|-1, \dots, -1) \\ &= (m, \dots, 2, 1|-1, -2, \dots, -n).\end{aligned}$$

We will also use a different positive system for $\Delta_{\bar{1}}$, which we refer to as the non-standard system. This system will be introduced in Section 6.1.2.

The *dot action* on weight vectors, defined for $w \in W$ and $\lambda \in \mathfrak{h}^*$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$ as well as constructs such as $(\lambda + \rho, \epsilon_i - \delta_k)$, which will be important below, are of course independent of the representative by construction.

The restriction of $(\cdot, \cdot)$ to $\mathfrak{d}$ is still non-degenerate, and identifies the subspace $\mathfrak{d}_{\mathbb{R}}$ of real diagonal matrices with the pseudo-orthogonal space $\mathbb{R}^{m,n}$. The bilinear form induced on the dual space is denoted by the same symbol. On the standard basis, we have for all $1 \leq i, j \leq m$ and $1 \leq k, l \leq n$

$$(\epsilon_i, \epsilon_j) = \delta_{ij}, \qquad (\delta_k, \delta_l) = -\delta_{kl}, \qquad (\epsilon_i, \delta_k) = 0. \tag{2.1.2}$$

When $n \neq m$, the restriction of $(\cdot, \cdot)$ to $\mathfrak{h}_{\mathbb{R}}$ is non-degenerate of signature $(m - 1, n)$ or $(m, n-1)$, depending on whether $m > n$ or $m < n$. We can then associate to any root $\alpha \in \Delta$ a unique element $h_\alpha \in \mathfrak{h}_{\mathbb{R}}$ through the condition $\alpha(H) = (H, h_\alpha)$ for all $H \in \mathfrak{h}$. The element h_α is called the *dual root* associated to α. When $n = m$, we have $\mathfrak{h}_{\mathbb{R}}/\mathbb{R}E_{2n} \cong \mathbb{R}^{m-1,n-1}$. We can fix the dual roots as elements of $\mathfrak{h}_{\mathbb{R}}$ by requiring $\alpha(H) = (H, h_\alpha)$ for all $H \in \mathfrak{d}$. However, the bilinear form on $\mathfrak{h}^*_{\mathbb{R}}$, which is equal to the linear extension of $(\alpha, \beta) := (h_\alpha, h_\beta)$ for $\alpha, \beta \in \Delta$, remains non-degenerate only when $n \neq m$. It follows from the above that the odd roots are all isotropic, *i.e.*, $(\alpha, \alpha) = 0$ for all $\alpha \in \Delta_{\bar{1}}$.

2.1.4. Real forms

We briefly summarize real forms of the basic classical Lie superalgebras $\mathfrak{g}$, which are central in the study of unitarizable supermodules. This subsection is mainly based on [21, 112, 131].

The classification of real forms of $\mathfrak{g}$ is governed by the classification of real forms of its underlying Lie algebra $\mathfrak{g}_{\bar{0}}$, which we briefly outline. For generality, let $\mathfrak{l}$ be a complex Lie algebra, and let $\mathfrak{l}_{\mathbb{R}}$ be a real Lie algebra. The complex Lie algebra $\mathfrak{l}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is called the *complexification* of $\mathfrak{l}_{\mathbb{R}}$, and $\mathfrak{l}_{\mathbb{R}}$ is called a *real form* of $\mathfrak{l}$ if there is an isomorphism of complex Lie algebras $\mathfrak{l}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{l}$. Classifying real forms of $\mathfrak{l}$ is equivalent to classifying conjugate-linear involutions on $\mathfrak{l}$, or equivalently, conjugate-linear anti-involutions. Here, a conjugate-linear involution is a real Lie algebra automorphism $\sigma : \mathfrak{l} \to \mathfrak{l}$ satisfying $\sigma(iX) = -i\sigma(X)$ for all $X \in \mathfrak{l}$ and $\sigma^2 = \mathrm{id}_{\mathfrak{l}}$. These maps are referred to as *conjugations*. Given a conjugation σ, the fixed-point set associated to σ, that is,

$$\mathfrak{l}^{\sigma} := \{X \in \mathfrak{l} : \sigma(X) = X\},$$

is a real form such that $\mathfrak{l}$ decomposes as a real Lie algebra as $\mathfrak{l} = \mathfrak{l}^{\sigma} \oplus i\mathfrak{l}^{\sigma}$. Conversely, every real form arises as the fixed-point set of a conjugation σ. Two real forms $\mathfrak{l}^{\sigma}, \mathfrak{l}^{\sigma'}$ are isomorphic if and only if the associated conjugations σ, σ' are *equivalent*, that is, there exists an automorphism $\varphi : \mathfrak{l} \to \mathfrak{l}$ such that $\sigma'\varphi = \varphi\sigma$.

Let $\mathfrak{g}$ be a (complex) basic classical Lie superalgebra. Analogously to the complex case, we define real basic classical Lie superalgebras. The relationship between real and complex basic classical Lie superalgebras is governed by their complexifications and conjugate-linear (anti-)involutions.

The *complexification* of a real (basic classical) Lie superalgebra $\mathfrak{g}_{\mathbb{R}}$ is $\mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, and $\mathfrak{g}_{\mathbb{R}}$ is called a *real form* of a complex (basic classical) Lie superalgebra $\mathfrak{g}$ if $\mathfrak{g} \cong \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Real forms naturally arise as fixed point sets of conjugate-linear involutions.

Definition 2.1.20. Let $\mathfrak{g}$ be a complex basic classical Lie superalgebra. An automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ of $\mathfrak{g}$, considered as a real Lie superalgebra, is called a *conjugate-linear involution* if the following two conditions hold:

a) $\sigma(\lambda x) = \overline{\lambda}\sigma(x)$ for all $x \in \mathfrak{g}$ $\lambda \in \mathbb{C}$.

b) $\sigma^2 = \mathrm{id}_{\mathfrak{g}}$

Conversely, we can work with conjugate-linear anti-involutions instead of conjugate-linear involutions. Indeed, to any conjugate-linear involution σ, we can assign a conjugate-linear anti-involution defined by $\omega(\cdot) := -i^{p(\cdot)}\sigma(\cdot)$. In this work, we use conjugate-linear involutions and conjugate-linear anti-involutions interchangeably, as both provide equivalent classifications of real forms.

Fix a basic classical Lie superalgebra $\mathfrak{g}$. Let σ be a conjugate-linear involution, and set

$$\mathfrak{g}^{\sigma} := \{x + \sigma(x) : x \in \mathfrak{g}\}.$$

Then $\mathfrak{g}^{\sigma}$ is a real basic classical Lie superalgebra [112, Proposition 1.4], *i.e.*, a *real form* of $\mathfrak{g}$. Two real forms $\mathfrak{g}^{\sigma}$ and $\mathfrak{g}^{\sigma'}$ with respect to two conjugate-linear involutions σ, σ' are isomorphic if and only if there exists an automorphism $\varphi : \mathfrak{g} \to \mathfrak{g}$ such that $\sigma'\varphi = \varphi\sigma$ [112, Proposition 1.6]. In this case, we say that σ and σ' are *equivalent*. Conversely, the complexification $\mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ of a real basic classical Lie superalgebra $\mathfrak{g}_{\mathbb{R}}$ is a complex basic classical Lie superalgebra, or the direct sum of two ideals which are basic classical [112, Proposition 1.5]. This leads to the following description of real basic classical Lie superalgebras.

Proposition 2.1.21 ([112]). *Let $\mathfrak{g}_\mathbb{R}$ be a real basic classical Lie superalgebra.*

a) If the complexification is simple, $\mathfrak{g}_\mathbb{R}$ is the subalgebra of fixed points of a conjugate-linear involution of the complexification.

b) If the complexification is not simple, $\mathfrak{g}$ is a complex basic classical Lie superalgebra considered as a real Lie superalgebra.

We conclude that the classification of real forms of a basic classical Lie superalgebra is equivalent to the classification of conjugate-linear involutions. It turns out that the classification of conjugate-linear involutions is already captured by the classification of conjugate-linear involutions for $\mathfrak{g}_{\bar{0}}$. To see this, given a conjugate-linear involution σ, we consider $\sigma_{\bar{0}} := \sigma\big|_{\mathfrak{g}_{\bar{0}}}$ and $\sigma_{\bar{1}} := \sigma\big|_{\mathfrak{g}_{\bar{1}}}$. Then $\sigma_{\bar{0}}$ is a conjugate-linear involution for $\mathfrak{g}_{\bar{0}}$, which determines a real form $\mathfrak{g}_{\bar{0}}^{\sigma_{\bar{0}}}$, up to isomorphism. Two conjugate-linear involutions σ and σ' are equivalent if and only if $\sigma_{\bar{0}}$ and $\sigma'_{\bar{0}}$ are equivalent [112, Theorem 2.5]. The proof of this result originates from the observation that any inner automorphism of $\mathfrak{g}_{\bar{0}}$ extends to an automorphism of $\mathfrak{g}$. Moreover, since $\mathfrak{g}$ is basic classical, there are only two possibilities, both resulting in isomorphic real forms. This leads to a complete classification of real forms of basic classical Lie superalgebras given in the table below:

$\mathfrak{g}$	$\mathfrak{g}_{\bar{0}}$	$\mathfrak{g}^\sigma$	$\mathfrak{g}_{\bar{0}}^\sigma$
$A(m-1\|n-1)$ $m>n>1$	$\mathfrak{sl}(m)\oplus\mathfrak{sl}(n)\oplus U(1)$	$\mathfrak{sl}(m\|n;\mathbb{R})$ $\mathfrak{su}^*(2p\|2q),\ m=2p,\ n=2q$ $\mathfrak{su}(p,m-p\|q,n-q)$	$\mathfrak{sl}(m,\mathbb{R})\oplus\mathfrak{sl}(n,\mathbb{R})\oplus\mathbb{R}$ $\mathfrak{su}^*(2p)\oplus\mathfrak{su}^*(2q)\oplus\mathbb{R}$ $\mathfrak{su}(p,m-p)\oplus\mathfrak{su}(q,n-q)\oplus i\mathbb{R}$
$A(n-1\|n-1)$ $n>1$	$\mathfrak{sl}(n)\oplus\mathfrak{sl}(n)$	$\mathfrak{psl}(n\|n;\mathbb{R})$ $\mathfrak{psu}^*(2p\|2q),\ m=2p,\ q=2n$ $\mathfrak{psu}(p,n-p\|q,n-q)$	$\mathfrak{sl}(n,\mathbb{R})\oplus\mathfrak{sl}(n,\mathbb{R})$ $\mathfrak{su}^*(2p)\oplus\mathfrak{su}^*(2q)$ $\mathfrak{su}(p,n-p)\oplus\mathfrak{su}(q,n-q)$
$B(m\|n)$ $B(0\|n)$	$\mathfrak{so}(2m+1)\oplus\mathfrak{sp}(2n)$ $\mathfrak{sp}(2n)$	$\mathfrak{osp}(p,2m+1-p\|2n;\mathbb{R})$ $\mathfrak{osp}(1\|2n;\mathbb{R})$	$\mathfrak{so}(p,2m+1-p)\oplus\mathfrak{sp}(2n,\mathbb{R})$ $\mathfrak{sp}(2n,\mathbb{R})$
$C(n+1)$	$\mathfrak{so}(2)\oplus\mathfrak{sp}(2n)$	$\mathfrak{osp}(2\|2n;\mathbb{R})$ $\mathfrak{osp}^*(2\|2q,2n-2q)$	$\mathfrak{so}(2)\oplus\mathfrak{sp}(2n,\mathbb{R})$ $\mathfrak{so}^*(2)\oplus\mathfrak{sp}(2q,2n-2q)$
$D(m\|n)$	$\mathfrak{so}(2m)\oplus\mathfrak{sp}(2n)$	$\mathfrak{osp}(p,2m-p\|2n;\mathbb{R})$ $\mathfrak{osp}^*(2m\|2q,2n-2q)$	$\mathfrak{so}(p,2m-p)\oplus\mathfrak{sp}(2n,\mathbb{R})$ $\mathfrak{so}^*(2m)\oplus\mathfrak{sp}(2q,2n-2q)$
$F(4)$	$\mathfrak{sl}(2)\oplus\mathfrak{so}(7)$	$F(4;0)$ $F(4;3)$ $F(4;2)$ $F(4;1)$	$\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{so}(7)$ $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{so}(1,6)$ $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{so}(2,5)$ $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{so}(3,4)$
$G(3)$	$\mathfrak{sl}(2)\oplus G_2$	$G(3;0)$ $G(3;1)$	$\mathfrak{sl}(2,\mathbb{R})\oplus G_{2,0}$ $\mathfrak{sl}(2,\mathbb{R})\oplus G_{2,2}$
$D(2,1;\alpha)$	$\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$	$D(2,1;\alpha;0)$ $D(2,1;\alpha;1)$ $D(2,1;\alpha;2)$	$\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ $\mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{sl}(2,\mathbb{R})$ $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$

Table 2.1.: Real forms of basic classical Lie superalgebras

As an example, we consider the real forms $\mathfrak{su}(p,q|r,s)$ of $\mathfrak{sl}(m|n)$ with a special focus on $\mathfrak{su}(p,q|n)$.

Special unitary Lie superalgebras $\mathfrak{su}(p,q|r,s)$

The *special unitary (indefinite) Lie superalgebras* $\mathfrak{su}(p,q|r,s)$ are real forms of the special linear Lie superalgebras $\mathfrak{sl}(m|n)$, where $p+q=m$ and $r+s=n$. These superalgebras play a central role in the theory of unitarizable supermodules over $\mathfrak{sl}(m|n)$.

We consider $V=\mathbb{C}^{m|n}$, *i.e.*, the complex super vector space of superdimension $(m|n)$. For any fixed $p,q,r,s\in\mathbb{Z}_+$ with $p+q=m$ and $r+s=n$, we equip V with the positive definite Hermitian form $\langle\cdot,\cdot\rangle : V\times V\to\mathbb{C}$ given by

$$\langle v,w\rangle := v^T J_{(p,q|r,s)}\overline{w}, \qquad J_{(p,q|r,s)} := \left(\begin{array}{c|c} I_{p,q} & 0 \\ 0 & I_{r,s}\end{array}\right).$$

Here, $\overline{\cdot}$ means complex conjugation and we consider any element of $\mathbb{C}^{m|n}$ as a column vector. The matrix $I_{k,l}$ is the diagonal matrix having the first k entries equal to 1 followed by the last l entries equal to -1. This Hermitian form is moreover *consistent*, that is, $\langle V_{\bar 0}, V_{\bar 1}\rangle = 0$.

We define the *(indefinite) unitary Lie superalgebra* $\mathfrak{u}(p,q|r,s)=\mathfrak{u}(p,q|r,s)_{\bar 0}\oplus\mathfrak{u}(p,q|r,s)_{\bar 1}$ as the Lie superalgebra which leaves the Hermitian form $\langle\cdot,\cdot\rangle$ invariant:

$$\mathfrak{u}(p,q|r,s)_{\bar k} := \{X\in\mathfrak{gl}(m|n)_{\bar k} : \langle Xv,w\rangle+\langle v,Xw\rangle = 0, \text{ for } v,w\in V\}.$$

The *(indefinite) special unitary Lie superalgebra* is defined as $\mathfrak{su}(p,q|r,s) := \mathfrak{u}(p,q|r,s)\cap\mathfrak{g}$. In cases where $m=n$, we simplify the notation by writing $\mathfrak{su}(p,q|r,s)$ instead of $\mathfrak{psu}(p,q|r,s)$ as it is common in literature. In our standard realization of $\mathfrak{g}$, $\mathfrak{su}(p,q|r,s)$ can be described explicitly as

$$\mathfrak{su}(p,q|r,s) = \left\{X\in\mathfrak{g} : J^{-1}_{(p,q|r,s)}X^\dagger J_{(p,q|r,s)} = -X\right\}.$$

The associated anti-linear anti-involutions on $\mathfrak{g}$, which are by definition anti-linear maps $\omega:\mathfrak{g}\to\mathfrak{g}$ satisfying $\omega^2=\mathrm{id}_{\mathfrak{g}}$ and $\omega([X,Y])=[\omega(Y),\omega(X)]$ for all $X,Y\in\mathfrak{g}$ which leads to the real forms $\mathfrak{su}(p,q|r,s)$ can be therefore written as

$$\omega(X) = J^{-1}_{(p,q|r,s)}X^\dagger J_{(p,q|r,s)}, \qquad X\in\mathfrak{g}.$$

Of special interest are the real forms $\mathfrak{su}(p,q|n,0)$ and $\mathfrak{su}(p,q|0,n)$, which are particularly isomorphic, as they are the only real forms of $\mathfrak{g}$ that admit non-trivial unitarizable supermodules.

Lie superalgebras $\mathfrak{su}(p,q|n,0)$and $\mathfrak{su}(p,q|0,n)$

We fix either $\mathfrak{su}(p,q|n,0)$ or $\mathfrak{su}(p,q|0,n)$ as a real form. In both cases, the even Lie subalgebra is $\mathfrak{su}(p,q)\oplus\mathfrak{su}(n)\oplus\mathfrak{u}(1)$ if $m\neq n$, and otherwise, $\mathfrak{su}(p,q)\oplus\mathfrak{su}(n)$, which is isomorphic to $\mathfrak{s}(\mathfrak{su}(p,q)\oplus\mathfrak{su}(n))$. Note that we can write $\mathfrak{g}_{\bar 0}\cong\mathfrak{s}(\mathfrak{su}(\mathfrak{p},\mathfrak{q})\oplus\mathfrak{su}(\mathfrak{n}))^{\mathbb{C}}$ to highlight the real form.

In $\mathfrak{su}(p,q|n,0)$, we fix the maximal compact subalgebra

$$\mathfrak{k} := \begin{cases}\mathfrak{su}(p)\oplus\mathfrak{su}(q)\oplus\mathfrak{u}(1)\oplus\mathfrak{su}(n)\oplus\mathfrak{u}(1), & \text{if } m\neq n;\\ \mathfrak{su}(p)\oplus\mathfrak{su}(q)\oplus\mathfrak{u}(1)\oplus\mathfrak{su}(n), & \text{if } m=n,\end{cases}$$

diagonally embedded in $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra for $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{g}_{\bar{0}}$, as $\mathfrak{k}^{\mathbb{C}}$ satisfies the *equal rank condition:*

$$\mathfrak{h} \subset \mathfrak{k}^{\mathbb{C}} \subset \mathfrak{g}_{\bar{0}} \subset \mathfrak{g}.$$

The root system Δ_c associated with $(\mathfrak{h}, \mathfrak{k}^{\mathbb{C}})$ is

$$\Delta_c := \{\pm(\epsilon_i - \epsilon_j) : 1 \leq i < j \leq p,\ p+1 \leq i < j \leq m\} \cup \{\pm(\delta_i - \delta_j) : 1 \leq i < j \leq n\},$$

which forms a subset of $\Delta_{\bar{0}}$. “Thus, a root $\alpha \in \Delta_{\bar{0}}$ is termed *compact* if $\alpha \in \Delta_c$, or equivalently, if the corresponding root vector belongs to $\mathfrak{k}^{\mathbb{C}}$; otherwise, it is called *non-compact*. The Weyl group associated to Δ_c will be denoted by W_c, which is indeed a subgroup of W. The set $\Delta_n := \Delta_{\bar{0}} \setminus \Delta_c$ will be called the set of non-compact roots, so that we have the decomposition:

$$\Delta_{\bar{0}} = \Delta_c \sqcup \Delta_n.$$

Moreover, we define the set of positive compact and non-compact roots by $\Delta^+_{c,n} := \Delta^+ \cap \Delta_{c,n}$. The associated Weyl elements are given by $\rho_{c,n} := \frac{1}{2}\sum_{\alpha \in \Delta^+_{c,n}} \alpha$, and with respect to Δ^+_c, an element $\lambda \in \mathfrak{h}^*$ is called Δ^+_c-*dominant* if it satisfies the following conditions:

$$\begin{cases} (\lambda + \rho_c, \alpha) \geq 0 & \text{for all } \alpha \in \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq p \text{ or } p+1 \leq i < j \leq m\}, \\ (\lambda + \rho_c, \alpha) \leq 0 & \text{for all } \alpha \in \{\delta_i - \delta_j : 1 \leq i < j \leq n\}. \end{cases}$$

Finally, for the sake of completeness, we give the explicit anti-linear anti-involutions leading to $\mathfrak{su}(p,q|n,0)$ and $\mathfrak{su}(p,q|0,n)$. In general, we distinguish between two cases.

First, assume that $p = 0$ and $q \neq 0$ or $p \neq 0$ and $q = 0$, in which $\mathfrak{k}^{\mathbb{C}} \cong \mathfrak{g}_{\bar{0}}$, and $\mathfrak{g}_{\pm 1}$ are simple $\mathfrak{k}^{\mathbb{C}}$-modules. In this case, the anti-linear anti-involutions leading to the real form are given in the following.

Lemma 2.1.22. *The real Lie superalgebra $\mathfrak{su}(m,0|n,0) = \mathfrak{su}(0,m|0,n)$ belongs to the anti-linear anti-involution ω_+, while the real Lie algebra $\mathfrak{su}(m,0|0,n) = \mathfrak{su}(0,m|n,0)$ belongs to the anti-linear anti-involution ω_-, where*

$$\omega_{\pm}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} A^\dagger & \pm C^\dagger \\ \hline \pm B^\dagger & D^\dagger \end{array}\right).$$

In the following, we will focus solely on the case ω_+ and denote the associated real form as $\mathfrak{su}(m|n)$. The case for ω_- is completely analogous. In particular, there is essentially one odd positive system, known as the *standard choice*:

$$\Delta^+_{\bar{1}} := \{\epsilon_i - \delta_k : 1 \leq i \leq m,\ 1 \leq k \leq n\},$$

such that the associated root vectors are the off-diagonal upper block matrices in $\mathfrak{g}_{\bar{1}}$, i.e., they belong to $\mathfrak{g}_{+1}$. We fix this positive system in the following when considering either $p = 0$ or $q = 0$.

Second, if $p, q \neq 0$, we decompose $\mathfrak{g}_{\bar{1}}$ as a direct sum of simple $\mathfrak{k}^{\mathbb{C}}$-modules, where the action is induced by the super commutator:

$$\mathfrak{g}_{\bar{1}} = \mathfrak{q}_1 \oplus \mathfrak{q}_2 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2.$$

Explicitly, we these are given by:

$$\mathfrak{p}_1 = \left\{ \left(\begin{array}{cc|c} 0 & 0 & P_1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) : P_1 \in \mathrm{Mat}(p,n;\mathbb{C}) \right\}, \quad \mathfrak{p}_2 = \left\{ \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & P_2 \\ \hline 0 & 0 & 0 \end{array}\right) : P_2 \in \mathrm{Mat}(q,n;\mathbb{C}) \right\},$$
$$\mathfrak{q}_1 = \left\{ \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline Q_1 & 0 & 0 \end{array}\right) : Q_1 \in \mathrm{Mat}(n,p;\mathbb{C}) \right\}, \quad \mathfrak{q}_2 = \left\{ \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & Q_2 & 0 \end{array}\right) : Q_2 \in \mathrm{Mat}(n,q;\mathbb{C}) \right\},$$

with $\mathrm{Mat}(k,l;\mathbb{C})$ denoting the space of complex $k\times l$-matrices. Based on the decomposition of $\mathfrak{g}_{\bar{1}}$, we can express a general element $X \in \mathfrak{g}$ as

$$X = \left(\begin{array}{cc|c} a & b & P_1 \\ c & d & P_2 \\ \hline Q_1 & Q_2 & E \end{array}\right),$$

where $P_i \in \mathfrak{p}_i$, $Q_j \in \mathfrak{q}_j$ for $1 \le i,j \le 2$, and $\mathrm{diag}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, E) \in \mathfrak{s}(\mathfrak{su}(p,q) \oplus \mathfrak{su}(n))^{\mathbb{C}}$.

Lemma 2.1.23 ([73, Lemma 5.1])**.** *There are exactly two anti-linear anti-involutions of $\mathfrak{g}$ compatible with the standard ordering, which produce the real form $\mathfrak{su}(p,q) \oplus \mathfrak{su}(n)$ on $\mathfrak{g}_{\bar{0}}$, namely*

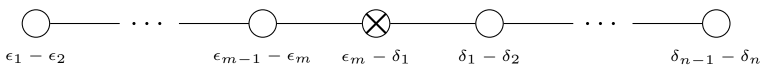
$$\omega_{(+,-)}(X) = \left(\begin{array}{cc|c} a^\dagger & -c^\dagger & Q_1^\dagger \\ -b^\dagger & d^\dagger & -Q_2^\dagger \\ \hline P_1^\dagger & -P_2^\dagger & E^\dagger \end{array}\right), \qquad \omega_{(-,+)}(X) = \left(\begin{array}{cc|c} a^\dagger & -c^\dagger & -Q_1^\dagger \\ -b^\dagger & d^\dagger & Q_2^\dagger \\ \hline -P_1^\dagger & P_2^\dagger & E^\dagger \end{array}\right).$$

The real form belonging to $\omega_{(+,-)}$ is $\mathfrak{su}(p,q|n,0)$, while the real form belonging to $\omega_{(-,+)}$ is $\mathfrak{su}(p,q|0,n)$.

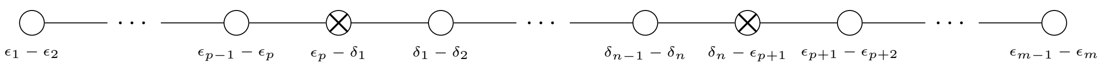
As a result, three odd positive systems are essentially relevant for the odd part, each compatible with the real forms $\mathfrak{su}(p,q|n,0) \cong \mathfrak{su}(p,q|0,n)$:

$$\mathfrak{n}^+_{\bar{1},\mathrm{st}} := \mathfrak{p}_1 \oplus \mathfrak{p}_2, \quad \mathfrak{n}^+_{\bar{1},\text{-st}} := \mathfrak{q}_1 \oplus \mathfrak{q}_2, \quad \mathfrak{n}^+_{\bar{1},\mathrm{nst}} := \mathfrak{p}_1 \oplus \mathfrak{q}_2,$$

referred to as *standard*, *minus standard*, and *non-standard*, respectively. For the sake of completeness, the Dynkin diagram for the standard choice is



whereas the Dynkin diagram for the non-standard choice is



As is customary, ◯ represents an even simple root, while ⊗ signifies a simple (isotropic) odd root.

The standard choice is distinguished by the fact that $\mathfrak{n}^+_{\bar{1},\mathrm{st}} = \mathfrak{g}_{+1}$, which means it is compatible with the $\mathbb{Z}_2$-compatible $\mathbb{Z}$-grading. On the other hand, the non-standard system is preferred in physics, as it coincides, after a natural embedding in the orthosymplectic Lie superalgebra (cf. [48]), with the standard positive root system of the orthosymplectic Lie superalgebra. For later calculations, the Weyl vectors for both systems are $\rho_{\mathrm{st}} = \rho_{\bar{0}} - \rho_{\bar{1},\mathrm{st}}$ and $\rho_{\mathrm{nst}} = \rho_{\bar{0}} - \rho_{\bar{1},\mathrm{nst}}$, where

$$\begin{aligned}\rho_{\bar{0}} &= \frac{1}{2}\left(\sum_{i=1}^{m}(m-2i+1)\epsilon_i + \sum_{j=1}^{n}(n-2j+1)\delta_j\right),\\ \rho_{\bar{1},\mathrm{st}} &= \frac{1}{2}\left(n\sum_{i=1}^{m}\epsilon_i - m\sum_{j=1}^{n}\delta_j\right),\\ \rho_{\bar{1},\mathrm{nst}} &= \frac{1}{2}\left(n\sum_{i=1}^{p}\epsilon_i - n\sum_{j=p+1}^{m}\epsilon_j + (q-p)\sum_{k=1}^{n}\delta_k\right).\end{aligned}$$

If the positive system is fixed, we simply write $\rho, \rho_{\bar{0}}$ and $\rho_{\bar{1}}$.

All results for the finite-dimensional case can then be derived by setting $p = 0$ or $q = 0$ and utilizing $\mathfrak{k}^{\mathbb{C}} = \mathfrak{g}_{\bar{0}}$.

2.2. Lie supergroups

2.2.1. Supermanifolds

In general, there are three main approaches to supermanifolds: the Rogers–DeWitt approach [28, 117], the Berezin–Kostant–Leites approach [6, 87], and the functor of points approach [120]. In this thesis, we focus on the algebro-geometric Berezin–Kostant–Leites approach, formulated in terms of super ringed spaces. This section is based on [12, Chapter 3].

Supermanifolds

A *super ringed space* $\mathcal{S} = (S, \mathcal{O}_{\mathcal{S}})$ is a topological space S endowed with a sheaf of supercommutative superrings[1] $\mathcal{O}_{\mathcal{S}}$, called the *structure sheaf* of $\mathcal{S}$.

A *superspace* is a super ringed space $\mathcal{S} = (S, \mathcal{O}_{\mathcal{S}})$ with the property that the stalk $\mathcal{O}_{\mathcal{S},x}$ is a local ring for all $x \in S$. Note that for any super ringed space $\mathcal{S} = (S, \mathcal{O}_{\mathcal{S}})$, the space $\mathcal{S}_{\bar{0}} := (S, \mathcal{O}_{\mathcal{S},\bar{0}})$ is an ordinary ringed space, where $\mathcal{O}_{\mathcal{S},\bar{0}}$ is a sheaf of ordinary rings on S.

A *morphism* of superspaces $\Phi : \mathcal{S} \to \mathcal{T}$ consists of a continuous map of the underlying topological spaces $\phi : S \to T$ together with a morphism of sheaves $\Phi^* : \mathcal{O}_{\mathcal{T}} \to \Phi_*(\mathcal{O}_{\mathcal{S}})$ such that

$$\Phi_x^*\left(\mathfrak{m}_{\mathcal{T},\phi(x)}\right) \subset \mathfrak{m}_{\mathcal{S},x}.$$

Here, $\mathfrak{m}_{\mathcal{S},x}$ is the maximal ideal in $\mathcal{O}_{\mathcal{S},x}$, $\mathfrak{m}_{\mathcal{T},\phi(x)}$ is the maximal ideal in $\mathcal{O}_{\mathcal{T},\phi(x)}$, $\Phi_*(\mathcal{O}_{\mathcal{S}})$ is the sheaf on T defined by $(\Phi_*\mathcal{O}_{\mathcal{S}})(U) := \mathcal{O}_{\mathcal{S}}(\phi^{-1}(U))$ for all open subsets $U \subset T$, and Φ_x^* is the induced map on stalks for $x \in S$.

Examples of superspaces include *smooth superdomains*. A *smooth superdomain* over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ of superdimension $(m|n)$, denoted by $\mathcal{U}^{m|n}$, is a super ringed space $(U, \mathcal{O}_{\mathcal{U}^{m|n}})$,

[1] A *superring* $\mathcal{A}$ is a $\mathbb{Z}_2$-graded ring $\mathcal{A} = \mathcal{A}_{\bar{0}} \cup \mathcal{A}_{\bar{1}}$, where the product map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfies $\mathcal{A}_{\bar{i}}\mathcal{A}_{\bar{j}} \subset \mathcal{A}_{\overline{i+j}}$. A superring $\mathcal{A}$ is called *supercommutative* if $ab - (-1)^{p(a)p(b)}ba = 0$ for all homogeneous $a, b \in \mathcal{A}$.

where U is an open subset of $\mathbb{K}^m$ and

$$\mathcal{O}_{\mathcal{U}^{m|n}}(V) = \mathcal{C}^\infty(V) \otimes \bigwedge(\theta^1, \ldots, \theta^n)$$

for every open subset $V \subset U$, with $\theta^1, \ldots, \theta^n$ being odd, linearly independent algebraic generators of the exterior algebra. We denote by $\mathcal{O}(\mathcal{U}^{m|n})$ the global sections of the sheaf $\mathcal{O}_{\mathcal{U}^{m|n}}$.

A *morphism* between two superdomains is a morphism of the underlying super ringed spaces. If $\{t^1, \ldots, t^m\}$ are coordinate functions in $\mathcal{C}^\infty(U)$ and $\{\theta^1, \ldots, \theta^n\}$ is a system of linearly independent algebraic generators of $\bigwedge(\theta^1, \ldots, \theta^n)$, the set

$$\{t^i, \theta^j \mid i = 1, \ldots, m,\ j = 1, \ldots, n\}$$

is called a *system of supercoordinates* on $\mathcal{U}^{m|n}$, and we write

$$\mathcal{O}_{\mathcal{U}^{m|n}}(U) = \mathcal{C}^\infty(t^1, \ldots, t^m)[\theta^1, \ldots, \theta^n].$$

The smooth superdomain $\mathcal{U}^{m|n}$, together with a system of supercoordinates, is called a *superchart*.

Definition 2.2.1. A superspace $\mathcal{M} = (M, \mathcal{O}_M)$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is called a *supermanifold* if

a) M is a (locally compact) second countable Hausdorff topological space, and

b) for each $x \in M$, there exists an open neighborhood U of x admitting an isomorphism of superspaces
$$\varphi_U : (U, \mathcal{O}_{\mathcal{M}}|_U) \longrightarrow \mathcal{U}^{m|n},$$
for fixed $(m|n)$, where $\mathcal{U}^{m|n}$ is a superdomain in $\mathbb{K}^{m|n}$.

A *morphism* between supermanifolds is a morphism between the corresponding superspaces. The pair $(m|n)$ is called the *superdimension* of $\mathcal{M}$, and (U, φ_U) is called a *superchart* around $x \in U$.

Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a supermanifold. If U is an open subset of M, the superspace $(U, \mathcal{O}_{\mathcal{M}}|_U)$ is again a supermanifold, called the *subsupermanifold* determined by U. Supermanifolds and morphisms of supermanifolds form a category, the *category of smooth supermanifolds* $\mathbf{sMan}_{\mathbb{K}}$.

Remark 2.2.2. Let $\mathcal{M}$ be a supermanifold, $U \subset M$ open, and fix a system of supercoordinates such that $\mathcal{O}_M(U) = \mathcal{C}^\infty(t^1, \ldots, t^m)[\theta^1, \ldots, \theta^n]$. Then, for any section f on U, there exist even functions $f_I \in \mathcal{C}^\infty(t^1, \ldots, t^m)$ such that

$$f(t, \theta) = f_0(t) + \sum_{|I|=1}^{n} f_I(t)\theta^I,$$

where $t = (t^1, \ldots, t^m)$, $\theta = (\theta^1, \ldots, \theta^n)$, and $I = \{i_1 < i_2 < \ldots < i_r\}_{r=1,\ldots,n}$. In particular, the notion of evaluating a function at a point is more subtle in the super context. However, any section f has a *value* at $x \in U$, namely the unique real/complex number c such that $f - c$ is not invertible in any neighborhood of x.

Associated to any supermanifold $\mathcal{M}$ is an ordinary (smooth) manifold. For each $x \in M$, we define the *evaluation map*

$$\mathrm{ev}_x : \mathcal{O}(M) \longrightarrow \mathbb{K}, \quad s \mapsto \widetilde{s}(x),$$

where $\widetilde{s}(x)$ is defined as the unique real/complex number such that $s - \widetilde{s}(x)$ is not invertible in any neighborhood of x. The *evaluation map* ev_x is a graded algebra morphism, and it descends to a morphism

$$\mathrm{ev}_x : \mathcal{O}(M)_x \longrightarrow \mathbb{K}.$$

In particular, $J_x := \mathrm{Ker}(\mathrm{ev}_x)$ is a graded ideal of $\mathcal{O}(M)$. We define, for each open set $U \subset M$,

$$\mathcal{O}_{\mathcal{M}}(U) \longrightarrow \mathcal{C}^\infty(U), \quad s \mapsto (x \mapsto \widetilde{s}(x)).$$

The kernel $J_{\mathcal{M}}(U)$ of this map is given by the nilpotent elements in $\mathcal{O}_{\mathcal{M}}(U)$. Since the ideal $J_{\mathcal{M}}(U)$ is locally generated by the odd elements $\mathcal{O}\mathcal{M}(U)\bar{1}$, it follows that

$$\mathcal{C}^\infty(U) \cong \mathcal{O}_{\mathcal{M}}(U)/J_{\mathcal{M}}(U).$$

Proposition 2.2.3 ([12, Proposition 4.2.13])**.** *Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a supermanifold of superdimension $(m|n)$, and let $U \subset M$ be an open subset. The assignment*

$$U \mapsto \mathcal{O}_{\mathcal{M}}(U)/J_{\mathcal{M}}(U)$$

is a sheaf on M locally isomorphic to $\mathcal{C}^\infty(\mathbb{K}^m)$. We call the manifold $(M, \mathcal{O}_{\mathcal{M}}/J_{\mathcal{M}})$ the reduced manifold *of $\mathcal{M}$.*

Let $\Phi = (\phi, \Phi^*)$ be a morphism of smooth supermanifolds. Then $\Phi^*(J_{\mathcal{N}}) \subset J_{\mathcal{M}}$. We define a corresponding morphism of the associated reduced manifolds by

$$\mathcal{C}^\infty(N) \cong \mathcal{O}(N)/J_{\mathcal{N}}(N) \longrightarrow \mathcal{O}(M)/\Phi^*(J_{\mathcal{N}}(N)) \longrightarrow \mathcal{O}(M)/J_{\mathcal{M}}(M) \cong \mathcal{C}^\infty(M).$$

This construction defines a functor from the category of smooth supermanifolds to the category of smooth manifolds:

$$\mathbf{sMan}_{\mathbb{K}} \longrightarrow \mathbf{Man}_{\mathbb{K}}.$$

Super tangent space

We fix a supermanifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A *super tangent vector* at $x \in M$ is a super derivation of the stalk $\mathcal{O}_{\mathcal{M},x}$, *i.e.*, a linear map $v : \mathcal{O}_{\mathcal{M},x} \to \mathbb{K}$ such that

$$v(f \cdot g) = v(f)g(x) + (-1)^{p(f)p(v)} f(x)v(g),$$

for all $f, g \in \mathcal{O}_{\mathcal{M},x}$. The *super tangent space* at a point $x \in M$ is the super vector space of all super tangent vectors at x. We denote the super tangent space at x by $T_x\mathcal{M}$.

Proposition 2.2.4 ([12, Proposition 4.3.9])**.** *Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a supermanifold, $x \in M$, and let t^i, θ^j be a supercoordinate system around x.*

a) Each $v \in T_x\mathcal{M}$ is completely determined by $v([t^i])$ and $v([\theta^j])$, where $[t^i]$ and $[\theta^j]$ denote the equivalence classes in $\mathcal{O}_{\mathcal{M},x}$.

b) The superderivations

$$\left\{ \frac{\partial}{\partial t^1}\bigg|_x, \dots, \frac{\partial}{\partial t^m}\bigg|_x \right\} \quad \textit{and} \quad \left\{ \frac{\partial}{\partial \theta^1}\bigg|_x, \dots, \frac{\partial}{\partial \theta^n}\bigg|_x \right\},$$

defined by

$$\frac{\partial}{\partial t^i}\bigg|_x ([t^k]) = \delta_{ik}, \quad \frac{\partial}{\partial t^i}\bigg|_x ([\theta^j]) = 0, \quad \frac{\partial}{\partial \theta^j}\bigg|_x ([t^k]) = 0, \quad \frac{\partial}{\partial \theta^j}\bigg|_x ([\theta^l]) = \delta_{jl},$$

form a basis of $T_x\mathcal{M}$.

Associated to a super tangent space, we have a notion of a differential. Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of supermanifolds. Then the even linear map

$$(\mathrm{d}\Phi)_x : T_x\mathcal{M} \longrightarrow T_{\phi(x)}\mathcal{N}, \quad v \mapsto v \circ \Phi_x^*,$$

is called the *differential* of Φ at x. Here, recall that any supermanifold morphism $\Phi : \mathcal{M} \to \mathcal{N}$ induces a stalk morphism

$$\Phi_x^* : \mathcal{O}_{\mathcal{N},\phi(x)} \longrightarrow \mathcal{O}_{\mathcal{M},x}.$$

2.2.2. Lie supergroups and super Harish-Chandra pairs

We introduce Lie supergroups and super Harish-Chandra pairs, which provide an equivalent characterization of Lie supergroups. This section follows closely [12, Chapter 7] and [27, 91].

Lie supergroups

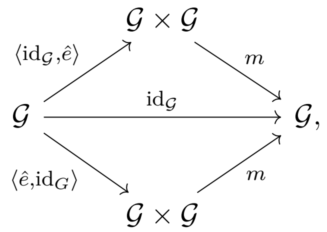
First, we define Lie supergroups and describe their relation to Lie superalgebras. Throughout this section, we work over the field $\mathbb{K}$ of complex or real numbers.

A *Lie supergroup* $\mathcal{G}$ is a group object in the category of smooth supermanifolds $\mathbf{sMan}_{\mathbb{K}}$; *i.e.*, $\mathcal{G}$ is a smooth supermanifold equipped with three morphisms $m : \mathcal{G}\times\mathcal{G} \to \mathcal{G}$, $i : \mathcal{G} \to \mathcal{G}$, and $e : \mathbb{R}^{0|0} \to \mathcal{G}$, called *multiplication, inverse*, and *identity*, respectively, such that the following diagrams commute:

a) Associativity of the multiplication m:

$$\begin{CD} \mathcal{G}\times\mathcal{G}\times\mathcal{G} @>{m\times \mathrm{id}}>> \mathcal{G}\times\mathcal{G} \\ @V{\mathrm{id}\times m}VV @VV{m}V \\ \mathcal{G}\times\mathcal{G} @>>{m}> \mathcal{G}. \end{CD}$$

b) Multiplication with the identity e:



where $\hat{e}$ denotes the composition of the identity $e : \mathbb{R}^{0|0} \to \mathcal{G}$ with the unique map $\mathcal{G} \to \mathbb{R}^{0|0}$, and $\langle \phi, \psi \rangle$ denotes the map $(\psi \times \phi) \circ \mathrm{diag}$, with $\mathrm{diag} : \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ being the canonical diagonal map.

c) Inverse property of the inverse i:

$$\begin{array}{ccccc} & & \mathcal{G} \times \mathcal{G} & & \\ & {\scriptstyle \langle \mathrm{id}_\mathcal{G}, i \rangle} \nearrow & & \searrow {\scriptstyle m} & \\ \mathcal{G} & & \xrightarrow{\quad \mathrm{id}_\mathcal{G} \quad} & & \mathcal{G}. \\ & {\scriptstyle \langle i, \mathrm{id}_\mathcal{G} \rangle} \searrow & & \nearrow {\scriptstyle m} & \\ & & \mathcal{G} \times \mathcal{G} & & \end{array}$$

A *morphism* $\Phi : \mathcal{G} \to \mathcal{H}$ of Lie supergroups is a morphism in $\mathbf{sMan}_\mathbb{K}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{G} & \xrightarrow{m_\mathcal{G}} & \mathcal{G} \\ {\scriptstyle \Phi \times \Phi} \downarrow & & \downarrow {\scriptstyle \Phi} \\ \mathcal{H} \times \mathcal{H} & \xrightarrow[m_\mathcal{H}]{} & \mathcal{H} \end{array}$$

where $m_\mathcal{H}$ and $m_\mathcal{G}$ denote the multiplication morphisms of $\mathcal{H}$ and $\mathcal{G}$, respectively. We denote the associated category of Lie supergroups by $\mathbf{SLG}_\mathbb{K}$.

Any Lie supergroup $\mathcal{G}$ has an underlying Lie group G. To define it, let G denote the reduced real smooth manifold of $\mathcal{G}$, and let m_{red}, i_{red}, and e_{red} denote the reduced maps of m, i, and e, respectively. Then, $(G, m_{\mathrm{red}}, i_{\mathrm{red}}, e_{\mathrm{red}})$ is a group object in the category of smooth manifolds $\mathbf{Man}_\mathbb{K}$, since reduction is functorial. Consequently, G is the underlying Lie group of $\mathcal{G}$, and the supermanifold associated with $\mathcal{G}$ is the pair $(G, \mathcal{O}_\mathcal{G})$.

Example 2.2.5. Let $\mathrm{Mat}(m|n) := \mathbb{K}^{m^2+n^2|2mn}$ be the superspace corresponding to the super vector space of $(m|n) \times (m|n)$-matrices with the direct product of $m \times m$ and $n \times n$-matrices, that is $\mathrm{Mat}(m, \mathbb{K}) \times \mathrm{Mat}(n, \mathbb{K})$, as the underlying topological space. As a super vector space, we have

$$\mathrm{Mat}(m|n) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\}, \quad \mathrm{Mat}(m|n)_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, \quad \mathrm{Mat}(m|n)_{\bar{1}} := \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\},$$

where A, B, C, D are respectively $m \times m$, $m \times n$, $n \times m$, and $n \times n$ matrices with entries in $\mathbb{K}$. Hence, as a superspace, $\mathrm{Mat}(m|n)$ has $m^2 + n^2$ even global coordinates t^{ij} for $1 \leq i, j \leq m$ or $m + 1 \leq i, j \leq m + n$ and $2mn$ odd global coordinates θ^{kl} where $1 \leq k \leq m$ and $m+1 \leq l \leq m+n$, or $m+1 \leq k \leq m+n$, $1 \leq l \leq m$. Additionally, $\mathrm{Mat}(m|n)$ is a smooth supermanifold with structural sheaf

$$\mathrm{Mat}(m, \mathbb{K}) \times \mathrm{Mat}(n, \mathbb{K}) \underset{\text{open}}{\supset} V \longrightarrow \mathcal{O}_{\mathrm{Mat}(m|n)}(V) := \mathcal{C}^\infty_{\mathrm{Mat}(m,\mathbb{K}) \times \mathrm{Mat}(n,\mathbb{K})}(V) \otimes \bigwedge \left(\theta^{kl} \right).$$

The supermanifold $\mathrm{Mat}(m|n)$ is, with standard multiplication, a Lie supergroup.

Let U be the subspace of $\mathrm{Mat}(m, \mathbb{K}) \times \mathrm{Mat}(n, \mathbb{K})$ consisting of the points for which $\det(t^{ij})_{1 \leq i,j \leq m} \neq 0$ and $\det(t^{ij})_{m+1 \leq i,j \leq m+n} \neq 0$. We define the superspace

$$\mathrm{GL}(m|n) := \left(U, \mathcal{O}_{\mathrm{Mat}(m|n)} \Big|_U \right)$$

to be the open subsuperspace of $\mathrm{Mat}(m|n)$ associated to the open set U. With matrix multiplication, $\mathrm{GL}(m|n)$ is a Lie supergroup, called the *general linear supergroup*. In U, we can consider the subspace U' of matrices X with

$$\mathrm{Ber}(X) = \det(A - BD^{-1}C)\det(D)^{-1} = 1, \quad X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in standard block form. Here, $\mathrm{Ber}(X)$ is called the *Berezinian* of X. We define the supermanifold

$$\mathrm{SL}(m|n) := \left(U', \mathcal{O}_{\mathrm{Mat}(m|n)}\Big|_{U'}\right)$$

to be the open supersubspace of $\mathrm{Mat}(m|n)$ associated to the open set U', which is with matrix multiplication a Lie supergroup. The Lie supergroup $\mathrm{SL}(m|n)$ is called the *special linear Lie supergroup*.

Finally, if $n = 2k$ is even, we can consider in U the open subspace U'' of matrices X such that

$$\begin{pmatrix} A^T & -C^T \\ B^T & D^T \end{pmatrix}\begin{pmatrix} I_m & 0 \\ 0 & J_n \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & J_n \end{pmatrix}, \quad J_n := \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}.$$

We define the supermanifold $\mathrm{OSp}(m|2k)$ as the open superspace of $\mathrm{Mat}(m|2k)$ associated with the open subspace U'', which, with matrix multiplication, forms a Lie supergroup.

Associated to any Lie supergroup $\mathcal{G} = (G, \mathcal{O}_{\mathcal{G}})$ is a Lie superalgebra $\mathfrak{g}$. To construct it, we introduce the notion of super vector field. A *super vector field* V on a Lie supergroup $\mathcal{G} = (G, \mathcal{O}_{\mathcal{G}})$ is a $\mathbb{K}$-linear *superderivation* of $\mathcal{O}_{\mathcal{G}}$, *i.e.*, a family of superderivations $V_U : \mathcal{O}_{\mathcal{G}}(U) \to \mathcal{O}_{\mathcal{G}}(U)$, $U \subset G$ open, which is compatible with restrictions. A super vector field V is called *left-invariant* if

$$(\mathbb{1} \otimes V) \circ m^* = m^* \circ V,$$

where $\mathbb{1}$ is the identity on the level of sheaf morphisms, and m^* is the sheaf morphism induced by the multiplication morphism. The super vector space $\mathfrak{g}$ of left-invariant super vector fields forms, together with the supercommutator

$$[V, W] := V \circ W - (-1)^{p(V)p(W)} W \circ V, \qquad V, W \in \mathfrak{g},$$

a Lie superalgebra, called the *Lie superalgebra of the Lie supergroup* $\mathcal{G}$.

Example 2.2.6. The Lie superalgebras $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$, and $\mathfrak{osp}(m|2k)$ introduced in Examples 2.1.4, 2.1.13, and 2.1.14 are the Lie superalgebras of the Lie supergroups $\mathrm{GL}(m|n)$, $\mathrm{SL}(m|n)$, and $\mathrm{OSp}(m|2k)$, respectively.

On the other hand, there is an isomorphism of super vector spaces [12, Proposition 7.2.3.]:

$$T_{e_G}\mathcal{G} \longrightarrow \mathfrak{g}, \qquad X_{e_G} \mapsto X := (\mathbb{1} \otimes X_{e_G})m^*,$$

where $e_G \in G$ denotes the identity element. Thus, any morphism $\Phi : \mathcal{G} \to \mathcal{H}$ of Lie supergroups induces a morphism of Lie superalgebras $(\mathrm{d}\phi)_{e_G} : \mathfrak{g} \to \mathfrak{h}$ and, in particular, the canonical inclusion $\iota : G \hookrightarrow \mathcal{G}$ gives us a canonical identification of the even part $\mathfrak{g}_{\bar{0}}$ of the Lie superalgebra $\mathfrak{g}$ and the Lie algebra $\mathrm{Lie}(G)$ of the Lie group G [12, Proposition 7.2.5.].

Super Harish-Chandra pairs

In this thesis, we prefer another approach to Lie supergroups, namely *super Harish-Chandra pairs*. In this language, we can study representations without any reference to the structural sheaf.

Definition 2.2.7. A *super Harish-Chandra pair* is a triple $(G, \mathfrak{g}, \pi)$ consisting of a Lie group G, a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, and a morphism of Lie supergroups $\pi : G \to \mathrm{GL}(\mathfrak{g})$ which satisfies the following:

a) $\mathfrak{g}_{\bar{0}}$ is the Lie algebra of G.

b) $\pi(G)\big|_{\mathfrak{g}_{\bar{0}}} = \mathrm{Ad}$ is the adjoint representation, and $(\mathrm{d}\pi)_e : \mathfrak{g}_{\bar{0}} \to \mathfrak{gl}(\mathfrak{g})$ is

$$\mathrm{d}\pi(X)(Y) = [X, Y], \qquad X \in \mathfrak{g}_{\bar{0}},\ Y \in \mathfrak{g}.$$

A *morphism* between super Harish-Chandra pairs $(G, \mathfrak{g}, \pi)$ and $(H, \mathfrak{h}, \sigma)$ is a pair (ψ, ρ^ψ) such that the following holds:

a) $\psi : G \to H$ is a homomorphism of Lie groups.

b) $\rho^\psi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie superalgebras.

c) ψ and ρ^ψ are compatible in the sense that

$$\rho^\psi\big|_{\mathfrak{g}_{\bar{0}}} = (\mathrm{d}\psi)_e,$$
$$\sigma(\psi(g)) \circ \rho^\psi = \rho^\psi \circ \pi(g)$$

for all $g \in G$.

We can assign to any Lie supergroup $\mathcal{G}$ a super Harish-Chandra pair $(G, \mathfrak{g}, \mathrm{ad})$, where G is the reduced Lie group, $\mathfrak{g}$ the Lie superalgebra associated to $\mathcal{G}$, and ad is the adjoint representation

$$\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g}),$$

induced from

$$\mathrm{ad}(g)(X) := (\mathrm{ev}_g \otimes X \otimes \mathrm{ev}_{g^{-1}})(\mathbb{1} \otimes m^*)m^*,$$

with $g \in G$, $X \in \mathfrak{g}$, and ev_g being the evaluation map at g. Conversely, we can assign to any super Harish-Chandra pair $(G, \mathfrak{g}, \sigma)$ a Lie supergroup $\mathcal{G}$. The following result is due to Kostant for real Lie supergroups and due to Vishnyakova for complex Lie supergroups.

Theorem 2.2.8 ([88, 137])**.** *For each super Harish-Chandra pair $(G, \mathfrak{g})$ there exists (up to isomorphy) a unique Lie supergroup $\mathcal{G}$ with underlying Lie group G and Lie superalgebra $\mathfrak{g}$. Furthermore, each morphism of super Harish-Chandra pairs induces up to isomorphism a unique morphism of the corresponding Lie supergroups. Abstractly, the category of Lie supergroups is equivalent to the category of super Harish-Chandra pairs.*

3. Supermodules over Lie superalgebras and Lie supergroups

We introduce supermodules over Lie superalgebras and Lie supergroups. For Lie supermodules, we study restriction to the even Lie subalgebra and induction from it, leading to the notion of Kac supermodules. Additionally, we introduce highest weight supermodules, and the infinitesimal character.

3.1. Supermodules over Lie superalgebras

In this section, let $\mathfrak{g}$ be a basic classical Lie superalgebra, and let $\mathfrak{U}(\mathfrak{g})$ denote its universal enveloping superalgebra. All definitions are standard, and our main references are [9, 12, 75, 76, 129].

3.1.1. Basic definitions

For any (complex) superalgebra $\mathcal{A}$, we define the notion of an $\mathcal{A}$-supermodule. A *left $\mathcal{A}$-supermodule* is a super vector space M together with a morphism of super vector spaces $\mathcal{A} \otimes M \to M,\ a \otimes m \mapsto am$, such that for all $a, b \in \mathcal{A}$ and $x, y \in M$, the following holds:

a) $a(x + y) = ax + ay$,

b) $(a + b)x = ax + bx$,

c) $(ab)x = a(bx)$,

d) $1_{\mathcal{A}}x = x$.

A *right $\mathcal{A}$-supermodule* is defined analogously. If $\mathcal{A}$ is supercommutative, the definitions of left and right $\mathcal{A}$-supermodules coincide when we set

$$ma := (-1)^{p(m)p(a)}am$$

for all $m \in M$ and $a \in \mathcal{A}$.

In the following, we always consider left $\mathcal{A}$-supermodules unless stated otherwise, and we simply write $\mathcal{A}$-supermodule instead of left $\mathcal{A}$-supermodule. A *morphism* $\phi : M_1 \to M_2$ of $\mathcal{A}$-supermodules is a super vector space morphism satisfying $\phi(am) = a\phi(m)$ for all $a \in \mathcal{A}$ and $m \in M$. In conclusion, we have a category of $\mathcal{A}$-supermodules, denoted by $\mathcal{A}$-**smod**.

We focus on $\mathfrak{g}$-supermodules, equivalently, $\mathbb{Z}_2$-graded representations of $\mathfrak{g}$, where $\mathfrak{g}$ is a basic classical Lie superalgebra. A *$\mathbb{Z}_2$-graded representation* (ρ, M) of $\mathfrak{g}$ consists of a super vector space $M = M_{\bar{0}} \oplus M_{\bar{1}}$ and a Lie superalgebra homomorphism $\rho : \mathfrak{g} \to \mathrm{End}(M)$. If M is finite-dimensional, the *dimension* (respectively, *superdimension*) of ρ is the dimension (respectively, superdimension) of the underlying vector space. Since ρ preserves the superbracket, it follows that it also preserves the Lie bracket on $\mathfrak{g}_{\bar{0}}$. Consequently, the restriction

$$\rho|_{\mathfrak{g}_{\bar{0}}} : \mathfrak{g}_{\bar{0}} \to \mathrm{End}(M)$$

defines a Lie algebra representation of $\mathfrak{g}_{\bar{0}}$ with representation space M.

Let (ρ, M) be a $\mathbb{Z}_2$-graded representation of $\mathfrak{g}$. Then the super vector space M inherits the structure of a $\mathfrak{g}$-supermodule via the action

$$Xv = X \cdot v := \rho(X)v, \qquad X \in \mathfrak{g},\ v \in M.$$

In other words, the super vector space $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is equipped with a graded linear left action of $\mathfrak{g}$ such that

$$[X, Y]v = X(Yv) - (-1)^{p(X)p(Y)} Y(Xv),$$

for all homogeneous elements $X, Y \in \mathfrak{g}$ and $v \in M$. Conversely, any $\mathfrak{g}$-supermodule M defines a $\mathbb{Z}_2$-graded representation of $\mathfrak{g}$ via $\rho(X)v := Xv$. Equivalently, one can define $\mathbb{Z}_2$-graded representations of $\mathfrak{U}(\mathfrak{g})$ or, equivalently, $\mathfrak{U}(\mathfrak{g})$-supermodules.

Throughout this thesis, we adopt the convention of working with left $\mathfrak{g}$-supermodules and left $\mathfrak{U}(\mathfrak{g})$-supermodules without further explicit mention.

Let (ρ, M) be a $\mathbb{Z}_2$-graded representation of $\mathfrak{g}$. Then the super vector space M inherits the structure of a $\mathfrak{g}$-supermodule via the action

$$Xv = X \cdot v := \rho(X)v, \qquad X \in \mathfrak{g},\ v \in M.$$

In other words, the super vector space $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is equipped with a graded linear left action of $\mathfrak{g}$ such that

$$[X, Y]v = X(Yv) - (-1)^{p(X)p(Y)} Y(Xv),$$

for all homogeneous $X, Y \in \mathfrak{g}$ and $v \in M$. Conversely, any $\mathfrak{g}$-supermodule M defines a $\mathbb{Z}_2$-graded representation of $\mathfrak{g}$ via $\rho(X)v := Xv$. Equivalently, we define $\mathbb{Z}_2$-graded representations of $\mathfrak{U}(\mathfrak{g})$ or $\mathfrak{U}(\mathfrak{g})$-supermodules.

Throughout this thesis, we work with the notion of left $\mathfrak{g}$-supermodules and left $\mathfrak{U}(\mathfrak{g})$-supermodules without further explicit mention.

By the universal property of the universal enveloping superalgebra $\mathfrak{U}(\mathfrak{g})$, we can work interchangeably with $\mathfrak{g}$-supermodules and $\mathfrak{U}(\mathfrak{g})$-supermodules.

Example 3.1.1. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie superalgebra. Then the map

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}), \qquad \mathrm{ad}(X)(Y) := [X, Y], \qquad \text{for all } X, Y \in \mathfrak{g},$$

defines a $\mathfrak{g}$-supermodule, called the *adjoint representation* of $\mathfrak{g}$.

Definition 3.1.2. Let M be a $\mathfrak{g}$-supermodule.

a) A $\mathfrak{g}$-*subsupermodule* is a super vector subspace $N \subset M$ that is preserved under the action of $\mathfrak{g}$, *i.e.*, $XN \subset N$ for all $X \in \mathfrak{g}$.

b) M is called *simple* if it has no $\mathfrak{g}$-subsupermodules other than $\{0\}$ and M.

c) M is called *completely reducible* if it decomposes as a direct sum of simple $\mathfrak{g}$-super - modules.

d) M is called *indecomposable* if it is nontrivial and cannot be written as the direct sum of two nonzero $\mathfrak{g}$-subsupermodules.

The category of $\mathfrak{g}$-supermodules forms a $\mathbb{C}$-linear abelian category, denoted by $\mathfrak{g}$**-smod**. The morphisms in this category are called *intertwining operators*.

Definition 3.1.3. Let (ρ, M) and (ρ', M') be two $\mathfrak{g}$-supermodules. A super vector space homomorphism $\varphi : M \to M'$ is called an *intertwining operator* if

$$\rho'(X) \circ \varphi = \varphi \circ \rho(X)$$

for all $X \in \mathfrak{g}$. The supermodules M and M' are called *equivalent* if there exists an intertwining operator $\varphi : M \to M'$ that is an isomorphism of super vector spaces.

We denote by $\mathfrak{g}_{\bar{0}}$**-mod** the category of (left) $\mathfrak{g}_{\bar{0}}$-modules. When $\mathfrak{g}_{\bar{0}}$ is considered as a purely even Lie superalgebra, the category $\mathfrak{g}$**-smod** is simply the direct sum of two copies of $\mathfrak{g}_{\bar{0}}$**-mod**. We view any $\mathfrak{g}_{\bar{0}}$-module as a $\mathfrak{g}_{\bar{0}}$-supermodule concentrated in a single parity. Additionally, if we disregard the parity, any $\mathfrak{g}$-supermodule M can be viewed as a $\mathfrak{g}_{\bar{0}}$-module, denoted by M_{ev}.

The abelian category $\mathfrak{g}$**-smod** is equipped with an endofunctor Π, the *parity-reversing functor*. The parity-reversing functor is defined by $\Pi(M)_{\bar{0}} = M_{\bar{1}}$ and $\Pi(M)_{\bar{1}} = M_{\bar{0}}$ for any $\mathfrak{g}$-supermodule M. Moreover, $\Pi(M)$ is viewed as a $\mathfrak{g}$-supermodule with the new action

$$X \cdot v := (-1)^{p(X)} X v$$

for any $X \in \mathfrak{g}$ and $v \in M$. In particular, a $\mathfrak{g}$-supermodule M is not necessarily isomorphic to $\Pi(M)$.

For any two $\mathfrak{g}$-supermodules M and M', we can construct new $\mathfrak{g}$-supermodules. The *direct sum* of $\mathfrak{g}$-supermodules $M \oplus M'$ is defined by

$$X(v \oplus v') := Xv \oplus Xv',$$

for $X \in \mathfrak{g}$ and $v \in M$, $v' \in M'$.

The *tensor product* of $\mathfrak{g}$-supermodules $M \otimes M'$ with representation space $M \hat{\otimes} M'$ is defined by

$$X(v \otimes v') := Xv \otimes v' + (-1)^{p(X)p(v)} v \otimes Xv',$$

where $X \in \mathfrak{g}$ and $v \in M$, $v' \in M'$.

3.1.2. Induction, coinduction and Kac supermodules

We describe the relationship between the category $\mathfrak{g}$**-smod** of $\mathfrak{g}$-supermodules and the category $\mathfrak{g}_{\bar{0}}$**-smod** of $\mathfrak{g}_{\bar{0}}$-supermodules. This naturally leads to the induction and coinduction functors. Furthermore, if $\mathfrak{g}$ is a Lie superalgebra of type 1, we introduce the Kac induction functor, which can be viewed as a generalized Verma supermodule (*cf.* Section 3.1.3). This section is based on [16, 50].

Induction and coinduction

We consider $\mathfrak{g}_{\bar{0}}$ as a purely even Lie superalgebra. Then, the category $\mathfrak{g}_{\bar{0}}$**-smod** of $\mathfrak{g}_{\bar{0}}$-supermodules is naturally equivalent to the direct sum of the even and odd copies of the category $\mathfrak{g}_{\bar{0}}$**-mod** of $\mathfrak{g}_{\bar{0}}$-modules, that is,

$$\mathfrak{g}_{\bar{0}}\textbf{-smod} \cong (\mathfrak{g}_{\bar{0}}\textbf{-mod})_{\bar{0}} \oplus (\mathfrak{g}_{\bar{0}}\textbf{-mod})_{\bar{1}}.$$

We denote the associated restriction functor by

$$\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}} : \mathfrak{g}\textbf{-smod} \to \mathfrak{g}_{\bar{0}}\textbf{-smod}.$$

The restriction functor $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}$ is exact and has both a left adjoint and a right adjoint functor. The left adjoint functor is the *induction functor*:

$$\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}} := \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} - : \mathfrak{g}_{\bar{0}}\textbf{-smod} \to \mathfrak{g}\textbf{-smod},$$

while the right adjoint is the *coinduction functor*:

$$\mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}} := \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})}(\mathfrak{U}(\mathfrak{g}), -) : \mathfrak{g}_{\bar{0}}\textbf{-smod} \to \mathfrak{g}\textbf{-smod}.$$

Here, $\mathfrak{U}(\mathfrak{g})$ is considered as a left $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$-module, and the $\mathfrak{U}(\mathfrak{g})$-supermodule structure on $\mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}(V)$ is

$$(Xf)(Y) := f(YX), \qquad f \in \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})}(\mathfrak{U}(\mathfrak{g}), V),\ X, Y \in \mathfrak{U}(\mathfrak{g}).$$

In particular, both $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}$ and $\mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}$ are exact functors. The following proposition collects important properties of the induction and coinduction functors.

Proposition 3.1.4 ([50])**.** *a)* $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}} \cong \Pi^{\dim \mathfrak{g}_{\bar{1}}} \circ \mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}$.

b) $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}} \mathfrak{U}(\mathfrak{g}_{\bar{0}}) \cong \mathfrak{U}(\mathfrak{g})$.

Kac supermodules

We introduce Kac supermodules for basic classical Lie superalgebras of type 1 (or $\mathfrak{gl}(m|n)$), describe their basic properties, and provide an elegant classification of simple supermodules following [16]. Recall that a basic classical Lie superalgebra of type 1 has the distinguishing property of admitting a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\pm 1}$ are simple $\mathfrak{g}_{\bar{0}}$-modules.

Kac supermodules can be described as generalized Verma supermodules with respect to $\mathfrak{g}_{\bar{0}}$ as a Levi subalgebra. Concretely, one starts by extending any $V \in \mathfrak{g}_{\bar{0}}$**-mod**, which is placed in either even or odd degree, trivially to a $\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{+1}$-supermodule. Then, the induced $\mathfrak{g}$-supermodule

$$K(V) := \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{+1})} V,$$

is called a *Kac supermodule*. The associated map $K(\cdot) : \mathfrak{g}_{\bar{0}}\textbf{-smod} \to \mathfrak{g}\textbf{-smod}$ defines an exact functor, the *Kac induction functor*. Note that, as super vector spaces, we have the following isomorphisms:

$$K(V) \cong \bigwedge\nolimits^{\bullet}(\mathfrak{g}_{\bar{1}}) \otimes V \cong \left(\bigoplus_{i=0}^{\dim(\mathfrak{g}_{+1})} \bigwedge\nolimits^{i}(\mathfrak{g}_{+1}) \right) \otimes V,$$

according to the PBW Theorem. The following adjunction property of the Kac induction functor is straightforward.

Lemma 3.1.5. *For a $\mathfrak{g}_{\bar{0}}$-module V, placed in either even or odd degree, the Kac supermodule $K(V)$ satisfies the following adjunction:*

$$\mathrm{Hom}_{\mathfrak{U}(\mathfrak{g})}(K(V), M) = \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})}(V, M^{\mathfrak{g}_{+1}}),$$

where $M^{\mathfrak{g}_{+1}} := \{m \in M : \mathfrak{g}_{+1}m = 0\}$.

In the following, we show that the Kac induction functor induces a bijection between the isomorphism classes of simple objects in $\mathfrak{g}_{\bar{0}}$**-mod** and the isomorphism classes of simple objects in $\mathfrak{g}$**-smod** [16]. To achieve this, we realize simple $\mathfrak{g}$-supermodules as the socle of suitable Kac supermodules, where the *socle* of a Kac supermodule $K(V)$ is the sum of all its simple subsupermodules, denoted by $\mathrm{soc}(K(V))$.

We fix a non-trivial $\mathfrak{g}_{\bar{0}}$-module V which is placed in either even or odd degree. Any non-zero $\mathfrak{g}_{-1}$-submodule of $K(V)$ has a non-trivial intersection with $\bigwedge^{\dim(\mathfrak{g}_{+1})}(\mathfrak{g}_{-1}) \otimes V$, and $K(V)^{\mathfrak{g}_{-1}} = \bigwedge^{\dim(\mathfrak{g}_{+1})}(\mathfrak{g}_{-1}) \otimes V$ [16, Lemma 3.1]. If V is simple, it is well-known that any $\mathfrak{g}$-subsupermodule of $K(V)$ contains $\bigwedge^{\dim(\mathfrak{g}_{+1})}(\mathfrak{g}_{-1}) \otimes V$. These observations lead to the following lemma.

Lemma 3.1.6 ([16])**.** *a) The socle of the Kac supermodule $K(V)$ is given by*

$$\mathrm{soc}(K(V)) = \mathfrak{U}(\mathfrak{g}) \left(\bigwedge^{\dim(\mathfrak{g}_{+1})}(\mathfrak{g}_{-1}) \otimes V \right).$$

In particular, $\mathrm{soc}(K(V))$ is a simple $\mathfrak{g}$-supermodule.

b) For any two simple $\mathfrak{g}_{\bar{0}}$-modules V and W, we have

$$\mathrm{soc}(K(V)) \cong \mathrm{soc}(K(W)) \iff V \cong W.$$

This leads to a realization of each simple $\mathfrak{g}$-supermodule as the socle of a Kac supermodule.

Lemma 3.1.7. *For any simple $\mathfrak{g}$-supermodule M, there exists a simple $\mathfrak{g}_{\bar{0}}$-module V such that*

$$M \cong \mathrm{soc}(K(V)).$$

Proof. We prove the lemma for simple $\mathfrak{g}$-supermodules M that are $\mathfrak{g}_{\bar{0}}$-semisimple. The general proof appears in [16].

We decompose M under $\mathfrak{g}_{\bar{0}}$ and select a simple $\mathfrak{g}_{\bar{0}}$-module V from this decomposition. Since M is a simple $\mathfrak{g}$-supermodule, we have the isomorphisms:

$$\begin{aligned} M &\cong \mathfrak{U}(\mathfrak{g})V \\ &\cong \mathfrak{U}(\mathfrak{g}) \left(\bigwedge^{\dim(\mathfrak{g}_{+1})}(\mathfrak{g}_{-1}) \otimes \bigwedge^{\dim(\mathfrak{g}_{+1})}(\mathfrak{g}_{-1}^*) \otimes V \right) \\ &\cong \mathrm{soc}\left(K\left(\bigwedge^{\dim(\mathfrak{g}_{+1})}(\mathfrak{g}_{-1}^*) \otimes V \right) \right), \end{aligned}$$

where we use the fact that $\bigwedge^{\dim(\mathfrak{g}_{+1})}(\mathfrak{g}_{-1}^*) \otimes \bigwedge^{\dim(\mathfrak{g}_{+1})}(\mathfrak{g}_{-1}) \cong \mathbb{C}$ is trivial and apply Lemma 3.1.6 to deduce the second and third isomorphism, respectively. Here, $\mathfrak{g}_{-1}^*$ denotes the dual super vector space of $\mathfrak{g}_{-1}$.

Since V is simple, $\bigwedge^{\dim(\mathfrak{g}_{+1})}(\mathfrak{g}_{-1}^*) \otimes V$ is a simple $\mathfrak{g}_{\bar{0}}$-module. This completes the proof. □

The *radical* of $K(V)$ is defined as the intersection of all maximal subsupermodules of $K(V)$, denoted by $\mathrm{Rad}(K(V))$. The radical is the smallest subsupermodule of $K(V)$ such that the quotient $K(V)/\mathrm{Rad}(K(V))$ is semisimple. This quotient is called the *head* or *top* of $K(V)$, denoted by $\mathrm{Hd}(K(V))$. The head $\mathrm{Hd}(K(V))$ of $K(V)$ is simple. To see this, let L be a simple quotient of $K(V)$. Then there exists a simple $\mathfrak{g}_{\bar{0}}$-module W such that $\mathrm{soc}(K(W)) \cong L$, and W is (up to isomorphism) uniquely determined by V. This leads to the following result:

Lemma 3.1.8 ([16], Lemma 4.4)**.** *For any simple $\mathfrak{g}_{\bar{0}}$-module V, the Kac supermodule $K(V)$ has a unique maximal subsupermodule. The unique simple top of $K(V)$ is denoted by $L(V)$.*

We show that any simple top $L(V)$ of a Kac supermodule $K(V)$ is $\mathbb{Z}$-gradable. To this end, we introduce appropriate *grading operators*. For $\mathfrak{g} := \mathfrak{gl}(m|n)$, we define the grading operator

$$d := \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & E_n \end{array}\right) \in \mathfrak{z}(\mathfrak{g}_{\bar{0}}),$$

while for $\mathfrak{g} = \mathfrak{osp}(2|2n)$, we define the grading operator

$$d := \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right) \in \mathfrak{z}(\mathfrak{g}_{\bar{0}}).$$

In both cases, the grading operator d acts on any simple $\mathfrak{g}_{\bar{0}}$-module V as a scalar $d_V \in \mathbb{C}$ [32, Proposition 2.6.8]. By construction, $K(V)$ is completely reducible as an $\mathfrak{g}_{\bar{0}}$-module, which yields a decomposition of $K(V)$ into d-eigenspaces:

$$K(V) = \bigoplus_{i=0}^{\dim(\mathfrak{g}_{+1})} K(V)^{d_V - i} \cong \bigoplus_{i=0}^{\dim(\mathfrak{g}_{+1})} \bigwedge^i(\mathfrak{g}_{-1}) \otimes V,$$

where $K(V)^{d_V-i}$ denotes the eigenspace corresponding to the eigenvalue $d_V - i$. Thus, we can define a $\mathbb{Z}$-grading on $K(V)$where the homogeneous components correspond to the eigenspaces of d with distinct eigenvalues.

If N is any subsupermodule of $K(V)$, then

$$N = \bigoplus_{k \geq 0} \left(N \cap \left(\bigwedge^k(\mathfrak{g}_{-1}) \otimes V \right) \right)$$

is the eigenspace decomposition with respect to the action of d. In particular, if we consider the standard $\mathbb{Z}$-grading on $K(V)$, all subsupermodules are $\mathbb{Z}$-graded.

Corollary 3.1.9. *Any simple $\mathfrak{g}$-supermodule is $\mathbb{Z}$-gradable.*

Proof. For any simple $\mathfrak{g}$-supermodule L, there exists a simple $\mathfrak{g}_{\bar{0}}$-module V such that $L \cong \mathrm{soc}(K(V))$ by Lemma 3.1.7. The Kac supermodule $K(V)$ has a $\mathbb{Z}$-grading induced by the grading operator d. In particular, any subsupermodule of $K(V)$ is $\mathbb{Z}$-gradable, and hence $\mathrm{soc}(K(V))$ is also $\mathbb{Z}$-gradable. This completes the proof. □

Theorem 3.1.10. *[16, Theorem 4.1] The map $V \mapsto L(V)$ gives rise to a bijection between the set of isomorphism classes of simple $\mathfrak{g}_{\bar{0}}$-modules and the set of isomorphism classes of simple $\mathfrak{g}$-supermodules.*

Proof. This proof is from [16]. We repeat it to accentuate the simplicity and beauty of the arguments. Let V be a simple $\mathfrak{g}_{\bar{0}}$-module. As above, fix a $\mathbb{Z}$-grading of $L(V)$ such that the top non-zero graded component is of degree 0. Note that the top non-zero graded component is isomorphic to V. If W denotes another simple $\mathfrak{g}_{\bar{0}}$-module, then any morphism $f : L(V) \to L(W)$ must preserve the gradings. In particular, this implies that the non-zero graded top components must be isomorphic, hence $V \cong W$.

Let L be a non-zero simple $\mathfrak{g}$-supermodule. By Corollary 3.1.9, L admits a $\mathbb{Z}$-grading such that all non-zero components have non-positive degree and the degree zero component V is non-trivial. In particular, $\mathfrak{g}_{+1}V = 0$, so V is a $\mathfrak{g}_{\bar{0}}$-module. Since L is simple and Kac induction is exact, V is also simple. Finally, using the adjunction between induction and restriction, we obtain:

$$0 \neq \mathrm{Hom}_{\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{+1}}(V, L^{\mathfrak{g}_{+1}}) = \mathrm{Hom}_{\mathfrak{g}}(L, \mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}}(V)),$$

i.e., there exists a non-zero homomorphism from $K(V)$ to L. □

3.1.3. Highest weight supermodules

In this subsection, we introduce an important class of supermodules over $\mathfrak{g}$: the highest weight supermodules, which include all finite-dimensional $\mathfrak{g}$-supermodules. After discussing their fundamental properties, we explicitly realize simple highest weight supermodules as unique simple quotients of Verma supermodules. Our main references are [99, 129].

Recall that for a fixed positive system Δ^+ of the root system Δ, the basic classical Lie superalgebra $\mathfrak{g}$ admits a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^\pm$ are the spans of the root vectors associated to the positive and negative roots, respectively. In particular, the Borel subalgebra is given by $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$. We begin by summarizing the key definitions.

Definition 3.1.11. Let M be a $\mathfrak{g}$-supermodule. A nonzero vector $v_\Lambda \in M$ is called *primitive* of weight $\Lambda \in \mathfrak{h}^*$ if it satisfies the following conditions:

a) $Hv_\Lambda = \Lambda(H)v_\Lambda$ for all $H \in \mathfrak{h}$.

b) $Xv_\Lambda = 0$ for all $X \in \mathfrak{n}^+$.

Definition 3.1.12. A $\mathfrak{g}$-supermodule M is called *highest weight $\mathfrak{g}$-supermodule of highest weight* $\Lambda \in \mathfrak{h}^*$ if there exists a primitive vector $v_\Lambda \in M$ of weight Λ such that $\mathfrak{U}(\mathfrak{g})v_\Lambda = M$. We call v_Λ *highest weight vector* of M, and Λ its *highest weight*.

We record some elementary algebraic properties that follow directly from the definition, or more concretely, from the universal property of Verma supermodules stated below.

Proposition 3.1.13. *Let M be a highest weight $\mathfrak{g}$-supermodule with highest weight Λ. Then the following assertions hold:*

a) M is a weight supermodule*, meaning that*

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda, \qquad M^\lambda := \{m \in M : Hm = \lambda(H)m \text{ for all } H \in \mathfrak{h}\}.$$

The space M^λ is called the weight space *of weight λ, and* $\dim(M^\lambda)$ *is referred to as its* multiplicity*.*

b) Let $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_l$ be an enumeration of the even and odd positive roots, respectively. Choose root vectors $X_i \in \mathfrak{g}^{-\alpha_i}$ and $Y_j \in \mathfrak{g}^{-\beta_j}$. Then M is spanned by the vectors

$$X_1^{r_1} \cdots X_k^{r_k} Y_1^{s_1} \cdots Y_l^{s_l} v_\Lambda,$$

where $r_1, \ldots, r_k \in \mathbb{Z}_+$ and $s_1, \ldots, s_l \in \{0, 1\}$, with weight

$$\Lambda - \sum_{i=1}^{k} r_i \alpha_i - \sum_{j=1}^{l} s_j \beta_j.$$

c) For all weights λ of M, we have $\dim(M^\lambda) < \infty$. In particular, $\dim(M^\Lambda) = 1$.

d) Each nonzero quotient of M is a highest weight supermodule of weight Λ.

e) M has a unique maximal subsupermodule and a unique simple quotient. In particular, M is indecomposable.

f) All simple highest weight supermodules with highest weight Λ are isomorphic.

Example 3.1.14. Let M be a finite-dimensional simple $\mathfrak{g}$-supermodule. Then M is also a finite-dimensional $\mathfrak{g}_{\bar{0}}$-module, and since $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}$, it follows that M is a weight supermodule. In particular, because $\mathfrak{n}^+$ raises weights and M is finite-dimensional, there exists a weight vector v_Λ of weight $\Lambda \in \mathfrak{h}^*$ such that $\mathfrak{n}^+ v_\Lambda = 0$. Consequently, v_Λ is a primitive vector. Furthermore, since M is simple, we have $M = \mathfrak{U}(\mathfrak{n}^-) v_\Lambda$, which shows that M is a highest weight $\mathfrak{g}$-supermodule.

We realize any simple highest weight $\mathfrak{g}$-supermodule as the unique simple quotient of certain universal highest weight supermodules, called *Verma supermodules*. First, note that $\mathfrak{U}(\mathfrak{g})$ is a right $\mathfrak{U}(\mathfrak{b})$-supermodule with respect to right multiplication. For each one-dimensional $\mathfrak{U}(\mathfrak{b})$-supermodule $\mathbb{C}_\Lambda$ defined by $\Lambda : \mathfrak{b} \to \mathbb{C}$, *i.e.*, $\Lambda \in \mathfrak{h}^*$, we define the *Verma supermodule*

$$M^{\mathfrak{b}}(\Lambda) := \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b})} \mathbb{C}_\Lambda.$$

We regard $M^{\mathfrak{b}}(\Lambda)$ as a $\mathfrak{U}(\mathfrak{g})$-supermodule that is the quotient of the tensor product $\mathfrak{U}(\mathfrak{g}) \otimes \mathbb{C}_\Lambda$, where $\mathbb{C}_\Lambda$ is treated as a trivial $\mathfrak{g}$-supermodule. This quotient is taken modulo the subspace generated by elements of the form $XH \otimes z - X \otimes \Lambda(H)z$, with $X \in \mathfrak{U}(\mathfrak{g})$, $H \in \mathfrak{b}$, and $z \in \mathbb{C}$. The superscript $\mathfrak{b}$ will be omitted when the Borel subalgebra is fixed. Then the supermodule $M(\Lambda)$ is a generalized highest weight $\mathfrak{g}$-supermodule with highest weight Λ, and highest weight vector $[1 \otimes 1]$. In particular, $\mathfrak{U}(\mathfrak{n}^-) v_\Lambda = M(\Lambda)$. Consequently, for each primitive vector v_Λ of weight Λ in a $\mathfrak{g}$-supermodule M, there exists a uniquely determined $\mathfrak{g}$-supermodule morphism $M(\Lambda) \to M$ sending $[1 \otimes 1]$ to v_Λ.

The well-known properties of Verma supermodules are summarized in the following proposition.

Proposition 3.1.15 ([99, Chapter 8])**.** *a) The supermodule $M(\Lambda)$ is a generalized highest weight $\mathfrak{g}$-supermodule with highest weight Λ, and highest weight vector $[1 \otimes 1]$. In particular, $\mathfrak{U}(\mathfrak{n}^-) v_\Lambda = M(\Lambda)$.*

b) For each $\mathfrak{b}$-eigenvector v_Λ of weight Λ in a $\mathfrak{g}$-supermodule M, there exists a uniquely determined surjective $\mathfrak{g}$-supermodule morphism $M(\Lambda) \to M$ sending $[1 \otimes 1]$ to v_Λ.

c) $\mathrm{End}_{\mathfrak{g}}(M(\Lambda)) \cong \mathbb{C}$.

d) $M(\Lambda)$ *has a unique maximal subsupermodule and unique simple quotient, denoted by* $L(\Lambda)$*. In particular,* $M(\Lambda)$ *is indecomposable.*

e) $M(\Lambda)$ *has a finite Jordan–Hölder series.*

Finally, we state the relation of $M^{\mathfrak{b}}(\Lambda)$ to the classical Verma module $M^{\mathfrak{b}_{\bar{0}}}(\Lambda)$ in the case of type 1 basic classical Lie superalgebras $\mathfrak{g}$. We follow closely [99, Chapter 8]. Set $\mathfrak{p} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{+1}$, and define the nilpotent ideal $I := \mathfrak{g}_1\mathfrak{U}(\mathfrak{p}) = \mathfrak{U}(\mathfrak{p})\mathfrak{g}_1$ in $\mathfrak{U}(\mathfrak{p})$. Then $\mathfrak{U}(\mathfrak{g}_{\bar{0}}) \cong \mathfrak{U}(\mathfrak{p})/I$, and there is a surjective map of $\mathfrak{U}(\mathfrak{g})$-modules [99, Lemma 8.2.3]

$$\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} M^{\mathfrak{b}_{\bar{0}}}(\Lambda) \longrightarrow M^{\mathfrak{b}}(\Lambda).$$

However, if we regard $M^{\mathfrak{b}_{\bar{0}}}(\Lambda)$ as a $\mathfrak{U}(\mathfrak{p})$-module with I acting trivially, then [99, Lemma 8.2.4]

$$M^{\mathfrak{b}}(\Lambda) \cong \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} M_{\mathfrak{b}_{\bar{0}}}(\Lambda).$$

Consequently, as a $\mathfrak{U}(\mathfrak{g})$-module, $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} L^{\mathfrak{b}_{\bar{0}}}(\Lambda)$ has a unique simple factor module, which is isomorphic to $L^{\mathfrak{b}}(\Lambda)$. Conversely, as a $\mathfrak{U}(\mathfrak{p})$-module, $L^{\mathfrak{b}}(\Lambda)$ has a unique simple submodule, which is isomorphic to $L^{\mathfrak{b}_{\bar{0}}}(\Lambda)$.

Kac supermodules

If $\mathfrak{g}$ is a basic classical Lie superalgebra of type 1, we prefer to work with Kac supermodules instead of Verma supermodules. However, as mentioned in Section 3.1.2, Kac supermodules can be understood as generalized Verma supermodules.

Let M be a simple highest weight $\mathfrak{g}$-supermodule over a basic classical Lie superalgebra of type 1 with highest weight $\Lambda \in \mathfrak{h}^*$. We denote the $\mathbb{Z}$-grading of $\mathfrak{g}$ by $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$. By Theorem 3.1.10, there exists a simple $\mathfrak{g}_{\bar{0}}$-module V such that $M \cong L(V)$. We endow $L(V)$ with a $\mathbb{Z}$-grading where all nonzero components have non-positive degree and the degree zero component is V. Consequently, $L(V)^{\mathfrak{g}_{+1}} = V$, so in particular, the highest weight vector v_Λ belongs to V. Since V is a simple $\mathfrak{g}_{\bar{0}}$-module, we have $\mathfrak{U}(\mathfrak{g}_{\bar{0}})v_\Lambda = V$, *i.e.*, V is a highest weight $\mathfrak{g}_{\bar{0}}$-module. This yields the following theorem.

Theorem 3.1.16. *If V is a highest weight $\mathfrak{g}_{\bar{0}}$-module, then $K(V)$ and $L(V)$ are highest weight $\mathfrak{g}$-supermodules. Conversely, any simple highest weight $\mathfrak{g}$-supermodule is the unique simple quotient of the Kac induction from a simple highest weight $\mathfrak{g}_{\bar{0}}$-module of the same highest weight.*

Remark 3.1.17. If V is a simple highest weight $\mathfrak{g}_{\bar{0}}$-module with highest weight Λ, then $L(V)$ coincides with the simple quotient $L(\Lambda)$ of the Verma supermodule $M(\Lambda)$.

We denote by $L_0(\Lambda)$ the (up to isomorphism) unique simple highest weight $\mathfrak{g}_{\bar{0}}$-supermodule of highest weight $\Lambda \in \mathfrak{h}^*$. The Kac supermodule $K(\Lambda)$ is defined as $K(L_0(\Lambda))$, and $L(\Lambda)$ denotes its simple quotient $L(L_0(\Lambda))$. The parity of $L(\Lambda)$ will normally be clear from context, and otherwise is assumed to be *even.* In general, $K(\Lambda)$ is not equal to the Verma supermodule of highest weight Λ, except when $L_0(\Lambda)$ is isomorphic to the Verma supermodule over $\mathfrak{g}_{\bar{0}}$.

The following proposition is immediate but important.

Proposition 3.1.18. *Let $\Lambda \in \mathfrak{h}^*$. The following assertions are equivalent:*

a) $L(\Lambda)$ is finite-dimensional.

b) $L_0(\Lambda)$ is finite-dimensional.

c) $K(\Lambda)$ is finite-dimensional.

Proof. Regard $L(\Lambda)$ as a $\mathfrak{g}_{\bar{0}}$-module. Then $L_0(\Lambda)$ is a direct summand, so *a*) implies *b*). As a super vector space, the Kac supermodule $K(\Lambda)$ is isomorphic to $\bigwedge \mathfrak{g}_{-1} \otimes L_0(\Lambda)$, which shows that *c*) follows from *b*). Finally, since $L(\Lambda)$ is the unique (up to isomorphism) simple quotient of $K(\Lambda)$, we have a surjective map $K(\Lambda) \twoheadrightarrow L(\Lambda)$, such *a*) follows from *c*). □

Casimir element

We introduce the (quadratic) *Casimir element*, which is a specific element Ω in the center $\mathfrak{Z}(\mathfrak{g})$ of the universal enveloping superalgebra $\mathfrak{U}(\mathfrak{g})$, acting in a simple manner on highest weight $\mathfrak{g}$-supermodules. For details, we refer to [99, Section 8.5].

For the basic classical Lie superalgebra $\mathfrak{g}$, we fix two bases $x_1, \dots, x_n$ and $y_1, \dots, y_n$ such that $(x_i, y_j) = \delta_{ij}$, where x_i and y_i are homogeneous of the same degree. Then, the *(quadratic) Casimir* element is defined as

$$\Omega := \sum_{i=1}^{n} (-1)^{p(x_i)} x_i y_i \in \mathfrak{U}(\mathfrak{g}),$$

where the definition of Ω does not depend on the choice of basis. Furthermore, Ω is central in $\mathfrak{U}(\mathfrak{g})$, *i.e.*, it belongs to $\mathfrak{Z}(\mathfrak{g})$.

To describe Ω explicitly, fix a basis $h_1, \dots, h_m$ of $\mathfrak{h}$, with dual basis $k_1, \dots, k_m$. For each $\alpha \in \Delta^+$, choose elements $e_\alpha \in \mathfrak{g}^\alpha$ and $e_{-\alpha} \in \mathfrak{g}^{-\alpha}$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$. Then, we take the elements $\{x_i, y_i\}$ defined by

$$\{h_i, k_i\}, \ \{e_\alpha, e_{-\alpha}\}_{\alpha \in \Delta_{\bar{0}}}, \ \{e_\alpha, e_{-\alpha}\}_{\alpha \in \Delta_{\bar{1}}}, \ \{e_{-\alpha}, -e_\alpha\}_{\alpha \in \Delta_{\bar{1}}}.$$

In this basis, the Casimir element becomes

$$\Omega = \sum_{i=1}^{m} h_i k_i + \sum_{\alpha \in \Delta_{\bar{0}}^+} (e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha) + \sum_{\alpha \in \Delta_{\bar{1}}^+} (e_{-\alpha} e_\alpha - e_\alpha e_{-\alpha}).$$

A direct calculation leads to the following lemma.

Lemma 3.1.19 ([99, Lemma 8.5.3])**.** *Let M be a highest weight $\mathfrak{g}$-supermodule with highest weight Λ. Then the Casimir operator acts on M as a scalar multiple of the identity, given by*

$$(\Lambda + 2\rho, \Lambda).$$

3.1.4. Infinitesimal characters and (a)typicality

In this subsection, we introduce *infinitesimal characters*, which are algebra homomorphisms from the center $\mathfrak{Z}(\mathfrak{g})$ of the universal enveloping superalgebra $\mathfrak{U}(\mathfrak{g})$ to the complex numbers. An explicit description of infinitesimal characters will be provided using the Harish-Chandra homomorphism. This construction naturally leads to the concepts of *typicality* and *atypicality*, which play a central role in the representation theory of Lie superalgebras.

Infinitesimal characters

Infinitesimal characters are specific algebra homomorphisms $\chi : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$. Using the Harish-Chandra homomorphism for Lie superalgebras, which concerns $\mathfrak{Z}(\mathfrak{g})$, we can formulate these characters explicitly. To construct the Harish-Chandra homomorphism, recall the following decomposition of super vector spaces:

$$\mathfrak{U}(\mathfrak{g}) \cong \mathfrak{U}(\mathfrak{h}) \oplus (\mathfrak{n}^-\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{n}^+),$$

which is an immediate consequence of the PBW Theorem. The associated projection $p : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{h})$ is called the *Harish-Chandra projection*, and its restriction to $\mathfrak{Z}(\mathfrak{g})$ defines an algebra homomorphism:

$$p|_{\mathfrak{Z}(\mathfrak{g})} : \mathfrak{Z}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{h}) \cong \mathrm{S}(\mathfrak{h}),$$

where $\mathrm{S}(\mathfrak{h})$ denotes the symmetric algebra over $\mathfrak{h}$. More explicitly, since $\mathrm{ad}_{\mathfrak{h}}(z) = 0$ for all $z \in \mathfrak{Z}(\mathfrak{g})$ and $\mathfrak{h}$ preserves $\mathfrak{n}^\pm$, it follows from PBW that any element $z \in \mathfrak{Z}(\mathfrak{g})$ has a unique decomposition of the form:

$$z = h_z + \sum_i n_i^- h_i n_i^+, \qquad h_z, h_i \in \mathfrak{U}(\mathfrak{h}),\ n_i^\pm \in \mathfrak{n}^\pm\mathfrak{U}(\mathfrak{n}^\pm).$$

As a result, elements of $\mathfrak{Z}(\mathfrak{g})$ are necessarily of even parity, and $p|_{\mathfrak{Z}(\mathfrak{g})}(z) = h_z$. The *Harish-Chandra homomorphism* is then defined as

$$\mathrm{HC} := \zeta \circ p|_{\mathfrak{Z}(\mathfrak{g})} : \mathfrak{Z}(\mathfrak{g}) \to \mathrm{S}(\mathfrak{h}),$$

where $\zeta : \mathrm{S}(\mathfrak{h}) \to \mathrm{S}(\mathfrak{h})$ is a twist defined by

$$\lambda(\zeta(f)) := (\lambda - \rho)(f) \quad \text{for all } f \in \mathrm{S}(\mathfrak{h}),\ \lambda \in \mathfrak{h}^*.$$

Theorem 3.1.20 ([77, 132, 133])**.** *a) The Harish-Chandra homomorphism is an injective ring homomorphism.*

b) An element $\phi \in Sym(\mathfrak{h})$ belongs to the image of HC if and only if:

(i) $\phi(w(\lambda + \rho) - \rho) = \phi(\lambda)$ for any $\lambda \in \mathfrak{h}^*$ and $w \in W$, i.e., *$\phi \in S(\mathfrak{h})^W$.*

(ii) Let $S(\mathfrak{h})^W := \{f \in S(\mathfrak{h}) : w(\lambda)(f) = \lambda(f) \text{ for all } w \in W, \lambda \in \mathfrak{h}^*\}$ and, for any $\lambda \in \mathfrak{h}^*$, define

$$A_\lambda := \{\alpha \in \Delta_1^+ : (\lambda + \rho, \alpha) = 0\}.$$

Then, the image of HC is given by:

$$\mathrm{im}(HC) = \{f \in S(\mathfrak{h})^W : (\lambda + t\alpha)(f) = \lambda(f) \text{ for all } t \in \mathbb{C},\ \lambda \in \mathfrak{h}^*,\ \alpha \in A_{\lambda-\rho}\}.$$

Let $\lambda \in \mathfrak{h}^*$ be a linear functional on the Cartan subalgebra $\mathfrak{h}$. Then, the map

$$\chi_\lambda : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}, \qquad \chi_\lambda(z) := (\lambda + \rho)(\mathrm{HC}(z)),$$

defines an algebra homomorphism, *i.e.*, a one-dimensional representation of $\mathfrak{Z}(\mathfrak{g})$. This gives rise to the definition of an infinitesimal character for $\mathfrak{g}$-supermodules.

Definition 3.1.21. Let M be a $\mathfrak{g}$-supermodule. We say M has an *infinitesimal character* if there exists $\lambda \in \mathfrak{h}^*$ such that $\mathfrak{Z}(\mathfrak{g})$ acts on M via χ_λ. In this case, we call χ_λ the infinitesimal character of M.

Example 3.1.22. Let M be a highest weight supermodule with highest weight $\Lambda \in \mathfrak{h}^*$. Then M has infinitesimal character χ_Λ. Indeed, let v_Λ be the highest weight vector of M, and let $z \in \mathfrak{Z}(\mathfrak{g})$. Then, by Equation (3.1.4), we have

$$zv_\Lambda = \Lambda(h_z)v_\Lambda = (\Lambda + \rho)(\mathrm{HC}(z))v_\Lambda = \chi_\Lambda(z)v_\Lambda.$$

If $v \in M$ is a general element, there exists some $x \in \mathfrak{U}(\mathfrak{n}^- \oplus \mathfrak{h})$ such that $v = xv_\Lambda$. Since $z \in \mathfrak{Z}(\mathfrak{g})$ commutes with x, we have $zv = zxv_\Lambda = xzv_\Lambda = \chi_\Lambda(z)v$, which proves the claim.

Another approach is to observe that $\mathrm{End}_\mathfrak{g}(M) \cong \mathbb{C}$. Indeed, any endomorphism maps v_Λ to a scalar multiple of v_Λ as $\dim(M^\Lambda) = 1$. Since v_Λ generates M, we conclude $\mathrm{End}_\mathfrak{g}(M) = \mathbb{C}$. In particular, any $z \in \mathfrak{Z}(\mathfrak{g})$ acts as a scalar multiple of the identity, *i.e.*, M admits a central character.

As a direct consequence of Theorem 3.1.20, we obtain the following corollary.

Corollary 3.1.23. *Let $\lambda, \lambda' \in \mathfrak{h}^*$. Then $\chi_\lambda = \chi_{\lambda'}$ if and only if*

$$\lambda' = w\Big(\lambda + \rho + \sum_{i=1}^{k} t_i\alpha_i\Big) - \rho,$$

where $w \in W$, $t_i \in \mathbb{C}$, and $\alpha_1, \dots, \alpha_k \in A_\lambda = \{\alpha \in \Delta_{\bar 1}^+ : (\lambda + \rho, \alpha) = 0\}$ are linearly independent odd isotropic roots.

This leads to the definition of typicality and atypicality.

Definition 3.1.24. A weight $\Lambda \in \mathfrak{h}^*$ is called *typical* if $A_\Lambda = \emptyset$, *i.e.*, $(\Lambda + \rho, \alpha) \neq 0$ for all $\alpha \in \Delta_{\bar 1}^+$. Otherwise, Λ is called *atypical*. The *degree of atypicality* of Λ, denoted by $\mathrm{at}(\Lambda)$, is the maximal number of linearly independent mutually orthogonal positive odd (in particular, isotropic) roots $\alpha \in \Delta_{\bar 1}^+$ such that $(\Lambda + \rho, \alpha) = 0$. In brief, $\mathrm{at}(\Lambda)$ is the dimension of a maximal isotropic subspace of $\mathrm{Span}_\mathbb{C}(A_\Lambda) \subset \mathfrak{h}^*$. We call a highest weight $\mathfrak{g}$-supermodule M with highest weight Λ *typical* if $\mathrm{at}(\Lambda) = 0$, and otherwise *atypical*.

Remark 3.1.25. The degree of atypicality of a weight $\lambda \in \mathfrak{h}^*$ is invariant under both even and odd reflections (cf. Lemma 2.1.15), even though the set A_λ itself is not. Consequently, the degree of atypicality is independent of the choice of positive root system.

The *defect* of $\mathfrak{g}$, denoted by $\mathrm{def}(\mathfrak{g})$, is the dimension of a maximal isotropic subspace in the $\mathbb{R}$-span of Δ. For $\mathfrak{g}$ of type $A(m-1|n-1)$, $B(m|n)$, or $D(m|n)$, with $\Delta_\mathbb{R} \cong \mathbb{R}^{m,n}$, we have $\mathrm{def}(\mathfrak{g}) = \min(m, n)$. For $\mathfrak{g}$ of type $C(n)$, the defect is 1. Moreover, a simple Lie algebra or $\mathfrak{osp}(1|2n)$ has defect 0. In all cases, the degree of atypicality satisfies

$$0 \leq \mathrm{at}(\lambda) \leq \mathrm{def}(\mathfrak{g})$$

for any weight $\lambda \in \mathfrak{h}^*$.

3.2. Supermodules over Lie supergroups

We briefly summarize the definition of $\mathbb{Z}_2$-graded representations of Lie supergroups, called supermodules, following [2, 10]. In the subsequent, let $\mathcal{G}$ be a real or complex Lie supergroup.

First, we recall the classical definition. For a Lie group G and a topological vector space V, equipped with a continuous linear left action $G \times V \to V$ of G, we say that the induced map $\pi : G \to \mathrm{GL}(V)$ is a continuous representation of G on V. Here, $\mathrm{GL}(V)$ denotes the group of invertible continuous transformations on V with continuous inverse. If the topology on V is locally convex, we say a vector $v \in V$ is smooth if the orbit map $g \mapsto \pi(g)v$ is a smooth map. The space of all smooth vector will be denoted by V^∞. We set

$$\mathrm{d}\pi(X)v := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi(\exp(tX))v, \qquad X \in \mathrm{Lie}(G), v \in V^\infty.$$

which defines an action of the Lie algebra $\mathrm{Lie}(G)$ of G on the space V^∞ of all smooth vectors. We endow V^∞ with the coarsest locally convex topology such that for all $u \in \mathfrak{U}(\mathrm{Lie}(G))$ the linear map $\mathrm{d}\pi(u) : V^\infty \to V$ is continuous.

Definition 3.2.1. Let V be a topological super vector space and $\mathcal{G} = (G, \mathfrak{g})$ a Lie supergroup. Assume given a continuous representation π of G on $V_{\bar{0}}$ and a Lie superalgebra representation $\mathrm{d}\pi$ of $\mathfrak{g}$ on V^∞ such that the map $\mathfrak{g} \otimes V^\infty \to V^\infty$, $(X, v) \mapsto \mathrm{d}\pi(X)v$ is continuous. We say $(\pi, \mathrm{d}\pi)$ is a continuous $\mathcal{G}$-supermodule if $\mathrm{d}\pi$ is G-equivariant, *i.e.*,

$$\mathrm{d}\pi(\mathrm{Ad}(g)(X)) = \pi(g)\mathrm{d}\pi(X)\pi(g^{-1})$$

for all $X \in \mathfrak{g}, g \in G$. Here, Ad denotes the adjoint representation of $\mathcal{G}$ on $\mathfrak{g}$ defined in Section 2.2.2.

All concepts, such as subsupermodules, simplicity, or equivalence of supermodules over $\mathcal{G}$, are defined similarly to the corresponding notions for representations of Lie groups. For the sake of completeness, we provide an explicit formulation.

Definition 3.2.2. Let $\Pi = (\pi, \rho, V)$ and $\Pi' = (\pi', \rho', V')$ be two continuous $\mathcal{G} = (G, \mathfrak{g})$-supermodules. An *intertwining operator* $A : \Pi \to \Pi'$ is an even continuous operator from V to V' such that A intertwines π, ρ and π', ρ', respectively. If A is a topological isomorphism, we call A an *equivalence*. When such an A exists, we say that Π and Π' are equivalent.

Definition 3.2.3. Let $\Pi = (\pi, \rho, V)$ be a continuous $\mathcal{G}$-supermodule. A supermodule $\Pi' = (\pi', \rho', V')$ is called an $\mathcal{G}$-*subsupermodule* of Π if V' is a closed $\mathbb{Z}_2$-graded subspace of V that is invariant under π and ρ, and π' (respectively ρ') is the restriction of π (respectively ρ) to V' (respectively $V^\infty(\pi) \cap V'$).

In particular, having introduced the concept of $\mathcal{G}$-subsupermodules, we can define simple $\mathcal{G}$-supermodules. The subsupermodules given by the supermodule itself or $\{0\}$ are called *trivial*.

Definition 3.2.4. A continuous $\mathcal{G}$-supermodule $\Pi = (\pi, \rho, V)$ is called *simple* if the only $\mathcal{G}$-subsupermodules are the trivial ones.

4. Superconformal algebras

4.1. A word on the conformal group

In physics, fundamental principles are often expressed through the symmetries of physical models, which determine conservation laws and constraints. In conformal field theory, the relevant symmetries are conformal transformations. Geometrically, these transformations preserve angles and the local shape of infinitesimal structures while allowing changes in size or curvature. The set of all conformal transformations forms a group, referred to as the *conformal group*. This subsection serves as a brief introduction to the conformal group. We follow closely [127].

Let $(\mathbb{M} = \mathbb{R}^{1,d}, g_{1,d})$ be the Minkowski space for $d \geq 2$. For any two open subsets $U, V \subset \mathbb{M}$, a *conformal transformation* is a smooth map $\phi : U \to V$ such that $\phi^* g_{1,d} = \Omega^2 g_{1,d}$ for some smooth map $\Omega : U \to \mathbb{R}_{\geq 0}$. Here,

$$\phi^* g_{1,d}(X, Y) := g_{1,d}(T\phi(X), T\phi(Y)), \quad X, Y \in TU$$

and $T\phi : TU \to TV$ denotes the differential (tangent map) of ϕ. The map Ω is called the *conformal factor*.

Conformal transformations are classified using *conformal Killing fields*, that is, smooth vector fields X on an open subset $U \subset \mathbb{M}$ such that the associated local one-parameter group $(\phi_t^X)_{t\in\mathbb{R}}$ is a conformal transformation for all t in a neighborhood of 0. These satisfy the *Killing equation* [127, Theorem 1.4, Theorem 1.6], and solving the Killing equation leads to a classification of conformal transformations of $\mathbb{M}$.

Theorem 4.1.1 ([127, Theorem 1.9])**.** *Every conformal transformation $\phi : U \to \mathbb{M}$ on a connected open subset U of $\mathbb{M}$ is a composition of*

a) a translation $q \mapsto q + c$, $c \in \mathbb{M}$,

b) an orthogonal transformation $q \mapsto Oq$, $O \in O(1, d)$,

c) a dilatation $q \mapsto e^\lambda q$, $\lambda \in \mathbb{R}$ and

d) a special conformal transformation

$$q \mapsto \frac{q - g_{1,d}(q, q)b}{1 - 2g_{1,d}(q, b) + g_{1,d}(q, q)g_{1,d}(b, b)}, \quad b \in \mathbb{M}.$$

Furthermore, the conformal transformations form a group with respect to the composition, which is isomorphic to $O(2, d)/\{\pm I\}$.

For special conformal transformations, it becomes immediately clear that the action of the conformal group on Minkowski space is singular. This naturally leads to the conformal compactification of $\mathbb{M}$, achieved through a conformal rescaling of the metric via an isometric embedding into a compact domain of another pseudo-Riemannian manifold.

A *conformal compactification* of the connected semi-Riemannian manifold $\mathbb{M}$ is a compact semi-Riemannian manifold $\mathbb{N}$ together with a conformal embedding $\iota : \mathbb{M} \to \mathbb{N}$, *i.e.*,

a) $\iota(\mathbb{M})$ is dense in $\mathbb{N}$.

b) Every conformal transformation $\varphi : U \to \mathbb{M}$ on an open and connected subset $U \subset \mathbb{M}$, $U \neq \emptyset$, has a conformal continuation $\hat{\varphi} : \mathbb{N} \to \mathbb{N}$.

Such a conformal compactification is unique (up to a conformal diffeomorphism). We now describe a short and elegant conformal compactification $\mathbb{N}$ of $\mathbb{M}$. For that, let $\mathbb{P}_{d+1} = \mathbb{R}^{d+2}/\sim$ denote the smooth projective space. Then, it is well-known that the map

$$\iota : \mathbb{R}^{1,d} \to \mathbb{P}_{d+1}(\mathbb{R}), \;\; x := (x^1, \dots, x^{d+1}) \mapsto \left(\frac{1 - g_{1,d}(x,x)}{2} : x^1 : \dots : x^{d+1} : \frac{1 + g_{1,d}(x,x)}{2} \right)$$

defines a smooth embedding of $\mathbb{M}$ into $\mathbb{P}_{d+1}(\mathbb{R})$. Moreover, the closure of the image $\iota(M)$ is equal to the compact $d+1$-dimensional submanifold

$$\mathbb{N}^{1,d} := \{(x^0 : \dots : x^{d+2}) \in \mathbb{P}_{d+1}(\mathbb{R}) \mid g_{2,d}(x,x) = 0\} \subset \mathbb{P}_{d+1}(\mathbb{R}),$$

where $g_{2,d} = \mathrm{diag}(-1,-1,1,\dots,1)$. The quotient map $p : \mathbb{R}^{2,d} \to \mathbb{N}^{1,d}$, restricted to the product of the spheres $S^1 \times S^d$, defines a 2-to-1 covering. The metric on $S^1 \times S^d$, induced from $\mathbb{R}^{2,d}$, descends to $\mathbb{N}^{1,d}$ such that the covering becomes a (local) isometry.

Proposition 4.1.2 ([127, Proposition 2.5, Theorem 2.9])**.** *The compact pseudo-Riemannian manifold* $\mathbb{N}^{1,d}$ *is a conformal compactification of the Minkowski space* $\mathbb{M}$*.*

The *conformal group* $\mathrm{Conf}(\mathbb{M})$ is the connected component containing the identity in the group of conformal diffeomorphism of the conformal compactification of $\mathbb{M}$. The group of conformal diffeomorphism is considered as a topological group with the topology of compact convergence, that is, the topology of uniform convergence on the compact subsets.

Theorem 4.1.3 ([127, Theorem 2.9])**.** *The conformal group of the Minkowski space is either* $\mathrm{SO}^0(2,d)$ *or* $\mathrm{SO}^0(2,d)/\{\pm I\}$ *of* $-I$ *is in the connected component of* $\mathrm{O}(2,d)$ *containing the identity* I*. Here,* $\mathrm{SO}^0(2,d)$ *denotes the connected component of the identity in* $\mathrm{O}(2,d)$*.*

Example: $\mathbb{M} = \mathbb{R}^{1,3}$

As an example, we consider the four-dimensional Minkowski space $\mathbb{M} = \mathbb{R}^{1,3}$. We write $x = (t, x_1, x_2, x_3) \in \mathbb{M}$ and identify $\mathbb{M}$ with the space $H(2)$ of Hermitian 2×2-matrices

$$H(2) := \left\{ x = \begin{pmatrix} t + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & t - x_1 \end{pmatrix} \;:\; x \in \mathbb{M} \right\}.$$

Then

$$\det(x) = t^2 - x_1^2 - x_2^2 - x_3^2 = g_{1,3}(x,x).$$

A natural basis of $H(2)$ is given by the Pauli matrices

$$\sigma_0 = I_2, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

such that any $x \in \mathbb{M}$ can be written as $x = t\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$.

The connected component of $\mathrm{SO}(2,4) = \{X \in \mathrm{O}(2,4) : \det(X) = 1\}$ is

$$\mathrm{SO}^0(2,4) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SO}(2,4) \ : \ \det(A) > 0 \right\}.$$

In particular, it contains $\pm I$ and the conformal group is $\mathrm{SO}^0(2,4)/\{\pm I\}$.

In quantum theories, it is more convenient to work with the spin cover of $\mathrm{SO}^0(2,4)$, that is, $\mathrm{SU}(2,2)$. Under the identification above, $\mathrm{SU}(2,2)$ acts on $H(2)$ via

$$g \cdot x = (ax+b)(cx+d)^{-1}, \quad x \in H(2), \ g := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2,2),$$

which is well-defined except on a set of measure 0. This induces a continuous map into the identity component of the conformal group $\mathrm{SO}^0(2,4)$. Indeed, it defines a twofold cover, realizing $\mathrm{SU}(2,2)$ as the spin cover of the conformal group $\mathrm{SO}^0(2,4)$.

In the following, we are interested in conformal quantum field theories and call, by abuse of notation, $G := \mathrm{SU}(2,2)$ the *conformal group*. However, we emphasize that $\mathrm{SU}(2,2)$ is a fourfold cover of the original conformal group

$$\mathrm{Conf}(\mathbb{M}) \cong \mathrm{SO}^0(2,4)/\mathbb{Z}_2 \cong \mathrm{SU}(2,2)/\mathbb{Z}_4.$$

Note that $\pi_1(\mathrm{SU}(2,2)) = \mathbb{Z}$, *i.e.*, $\mathrm{Spin}(2,4) \cong \mathrm{SU}(2,2)$ is not the universal cover of $\mathrm{SO}^0(2,4)$. To realize the universal cover $\widetilde{G}$, we consider the action of G on $H(2)$ as above. For fixed $g \in G$, we define $\delta_g : H(2) \to \mathbb{R}$ via

$$\det(cx+d) = |\det(cx+d)| e^{i\delta_g(x)}.$$

δ_g is unique up addition of $2\pi n$, and we can define δ_g^n by imposing

$$2\pi n \le \delta_g^n < 2\pi(n+1), \quad n \in \mathbb{Z}.$$

Then the universal cover $\widetilde{G}$ of $G = \mathrm{SU}(2,2)$ is

$$\widetilde{G} = \{(g, \delta_g^n) \ : \ g \in G, n \in \mathbb{Z}\}$$

with group multiplication

$$(g, \delta_g^n) \cdot (h, \delta_h^m) = (gh, \delta_{gh}^n + \delta_h^m).$$

Indeed, the unit element is of the form $(I_4, 0)$ and its n-fold cover is $(I_4, 2\pi n)$ such that $\widetilde{G}$ fits in the short exact sequence

$$0 \to \mathbb{Z} = \pi_1(G) \to \widetilde{G} \to G \to 0.$$

The center of G is $\{\pm I_4, \pm i I_4\}$. The center Γ of $\widetilde{G}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}$ and has two generating elements $\gamma_1 = (-I_4, 0)$ and $\gamma_2 = (iI_4, 0)$ such that

$$\Gamma = \{\gamma_1^{n_1}\gamma_2^{n_2} \ : \ n_1 \in \{0,1\}, n_2 \in \{0, \pm 1, \pm 2, \dots\}\}.$$

Finally, we realize the conformal compactification of $\mathbb{M}$ as a homogeneous space and identify physically relevant subgroups. First, we note that the conformal group $\mathrm{SU}(2,2)/\mathbb{Z}_4$ acts by definition transitively on $\mathbb{M}$. As a marked point, we consider the zero matrix $0 \in \mathbb{M}$. The isotropy group of $0 \in \mathbb{M}$ is the 11-dimensional parabolic subgroup (in non-Segal realization)

$$P := \left\{ \begin{pmatrix} A & 0 \\ C & (A^\dagger)^{-1} \end{pmatrix} \in G \right\}$$

modulo $\mathbb{Z}_4$. P is the opposite of the scale-extended Poincaré group and isomorphic to $\mathrm{GL}(2,\mathbb{C}) \ltimes H(2)$. Hence, we can realize $\mathbb{M}$ as the homogeneous space

$$M \cong G/P.$$

Indeed, $\mathbb{M}$ is a dense open subset of $\mathbb{M}$. Therefore, we define for any $A \in \mathrm{GL}(2,\mathbb{C})$, and $x, y \in H(2)$

$$g(A) = \begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix}, \quad u(x) = \begin{pmatrix} I_2 & x \\ 0 & I_2 \end{pmatrix}, \quad v(y) := \begin{pmatrix} I_2 & 0 \\ y & I_2 \end{pmatrix}.$$

Then every element of P can be written uniquely as $v(y)g(A)$ and every element of G, except for a set of Haar measure 0, can be written as $g = v(y)g(A)u(x)$ for suitable $x, y \in H(2)$ and $A \in \mathrm{GL}(2,\mathbb{C})$. This decomposition gives an isomorphism between an open dense subset of $\mathbb{M}$ with $\mathbb{M}$.

Equivalently, $\mathbb{M}$ is a homogeneous space for $\widetilde{G}/\Gamma$, where Γ denotes the center of the universal covering group $\widetilde{G}$ above. Let $\widetilde{P}$ denote the universal cover of P. Then $\widetilde{P}/(\{\pm I\} \times \mathbb{Z})$ is the stabilizer subgroup of $0 \in \mathbb{M}$ in $\widetilde{G}/\Gamma$ and

$$M \cong \widetilde{G}/\widetilde{P}.$$

Here, note that the scale-extended Poincaré group is the 'opposite' of the Siegel parabolic subgroup, that is,

$$\mathrm{Iso}(1,3) := \left\{ \begin{pmatrix} A & B \\ 0 & (A^\dagger)^{-1} \end{pmatrix} \ : \ A \in \mathrm{GL}(2,\mathbb{C}), \ C \in \mathrm{Mat}(2,\mathbb{C}), \ B \in H(2) \right\}.$$

Both P and $\mathrm{Iso}(1,3)$ have Levi decompositions $P = L \ltimes N$ and $\mathrm{Iso}(1,3) = L \ltimes N'$, where

$$L = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix} \ : \ g \in \mathrm{GL}(2,\mathbb{C}), \ \det(g) \in \mathbb{R} \right\},$$

$$N = \left\{ \begin{pmatrix} E_2 & 0 \\ C & E_2 \end{pmatrix} \ : \ C \in H(2) \right\},$$

$$N' = \left\{ \begin{pmatrix} E_2 & B \\ 0 & E_2 \end{pmatrix} \ : \ B \in H(2) \right\}.$$

In addition, we have diffeomorphisms $P = MAN$ and $\mathrm{Iso}(1,3) = MAT$ where $\mathbb{M}$ is the group of Lorentz transformations, A the group of dilatations, N the group of special conformal transformations and T the group of translations.

a) *Lorentz transformations* $\mathbb{M}$

$$m = \begin{pmatrix} a & 0 \\ 0 & \overline{a}^{-1} \end{pmatrix}, \quad a \in \mathrm{SL}(2, \mathbb{C}).$$

The infinitesimal generators are

$$M_{\mu\nu} := \begin{pmatrix} \sigma_\mu \sigma_\nu & 0 \\ 0 & \sigma_\nu \sigma_\mu \end{pmatrix}$$

for $0 \leq \mu, \nu \leq 4$. The matrices σ_μ are the Pauli matrices.

b) *Dilatations* A

$$a = \begin{pmatrix} |a|^{\frac{1}{2}} E_2 & 0 \\ 0 & |a|^{-\frac{1}{2}} E_2 \end{pmatrix}, \quad |a| > 0.$$

The infinitesimal generator is

$$D := \begin{pmatrix} E_2 & 0 \\ 0 & -E_2 \end{pmatrix}.$$

c) *Special conformal transformations* N

$$\begin{pmatrix} E_2 & 0 \\ iN(x) & E_2 \end{pmatrix}, \quad N(x) := x_0 E_2 - \sum_{k=1}^{3} x_k \sigma_k, \ x \in \mathbb{R}^4.$$

The infinitesimal generators are

$$K^0 = \begin{pmatrix} 0 & 0 \\ iE_2 & 0 \end{pmatrix}, \quad K^k = \begin{pmatrix} 0 & 0 \\ -i\sigma_k & 0 \end{pmatrix}.$$

d) *Translations* T

$$\begin{pmatrix} E_2 & iT(x) \\ 0 & E_2 \end{pmatrix}, \quad T(x) := x_0 E_2 + \sum_{k=1}^{3} x_k \sigma_k, \ x \in \mathbb{R}^4.$$

The infinitesimal generators are

$$P^0 = \begin{pmatrix} 0 & iE_2 \\ 0 & 0 \end{pmatrix}, \quad P^k = \begin{pmatrix} 0 & i\sigma_k \\ 0 & 0 \end{pmatrix}, \ k = 1, 2, 3.$$

4.2. Superconformal algebra

Superconformal algebras can be understood as supersymmetric extensions of conformal algebras. They contain the Poincaré superalgebra as a subsuperalgebra and include the conformal algebra within their even (bosonic) part. These algebras are expected to satisfy the following two conditions:

- They act as infinitesimal automorphisms on Minkowski superspace, extending the natural action of the Poincaré superalgebra.
- When restricted to ordinary Minkowski space, the action of the even part extends the natural action of the conformal algebra.

Before providing a precise definition of superconformal algebras, we briefly introduce the necessary concepts.

Minkowski superspace / Superspacetime

Minkowski superspace, or *superspacetime*, is a supersymmetric extension of Minkowski space, which is modeled as an affine space underlying $(\mathbb{R}^{1,d-1}, g_{1,d-1})$. The construction presented here is based on [45, Lecture 3] and [136, Chapter 7].

We begin by fixing some data. Let M denote the real vector space $(\mathbb{R}^{1,d-1}, g_{1,d-1})$, and let $\mathbb{M}$ be the affine Minkowski space modeled on M. A vector $v \in M$ is called *timelike* if $g(v, v) < 0$. The set of all future-directed timelike vectors forms a cone, denoted by C.

The isometry group of M is the Poincaré group $\mathrm{ISO}(1, d-1) = \mathbb{R}^{1,d-1} \rtimes \mathrm{O}(1, d-1)$, with Lie algebra $\mathfrak{iso}(1, d-1) = \mathbb{R}^{1,d-1} \oplus \mathfrak{so}(1, d-1)$. The Lie group $\mathrm{O}(1, d-1)$ is called the *Lorentz group*, and its Lie algebra $\mathfrak{so}(1, d-1)$ is called the *Lorentz algebra*.

Let (κ, S) be a real spin representation of $\mathfrak{so}(1, d-1)$, not necessarily minimal. On S, we consider a symmetric bilinear form $\Gamma : S \otimes S \longrightarrow M$, as discussed in [15, 19, 136], which satisfies

$$\Gamma(\kappa(A)u, v) + \Gamma(u, \kappa(A)v) = A\Gamma(u, v)$$

for all $u, v \in S$ and $A \in \mathfrak{so}(1, d-1)$. We also require $\Gamma(\cdot, \cdot)$ to satisfy a *positivity condition*:

$$\Gamma(s, s) \in \overline{C}, \qquad \Gamma(s, s) = 0 \iff s = 0,$$

where $\overline{C}$ denotes the closure of C.

Superspacetime reflects the symmetry between vectors and spinors, viewed respectively as representations of bosons and fermions. This symmetry leads to the definition of a *translation superalgebra* $\mathfrak{l} := M \oplus S^*$, equipped with a Lie superalgebra structure. The Lie bracket $[\cdot, \cdot]$ is trivial on M and defined on S as follows:

$$[s_1, s_2] = -2\Gamma(s_1, s_2) \in M, \qquad s_1, s_2 \in S.$$

The *Minkowski superspace*, or *superspacetime*, is the supermanifold underlying the real Lie supergroup $\exp(\mathfrak{l})$; explicitly,

$$\mathcal{M} \cong \mathbb{M} \times \Pi(S).$$

The supersymmetry algebra of $\mathcal{M}$ is the Poincaré superalgebra. It contains both the translation superalgebra $\mathfrak{l}$ and the Poincaré algebra $\mathfrak{iso}(1, d-1)$. For the remainder of this section, we focus primarily on the associated complexifications.

Poincaré superalgebra

We construct the *complex Poincaré superalgebras* as described in [135]. Real Poincaré superalgebras then arise as specific real forms of their complex counterparts.

Let S be the complex spin representation of $\mathfrak{so}(d, \mathbb{C})$, the complex special orthogonal algebra in dimension d. A real form is the Lorentz algebra $\mathfrak{so}(1, d-1)$. If d is even, there are two inequivalent complex linear irreducible representations, S_1 and S_2, called *Weyl spinor representations*, such that $S = S_1 \oplus S_2$. If d is odd, the complex spin representation is irreducible.

The complex Poincaré superalgebra $\mathfrak{siso}(d,\mathbb{C})$ is defined by

$$\mathfrak{siso}(d,\mathbb{C})_{\bar{0}} := \mathfrak{so}(d,\mathbb{C}) \oplus \mathbb{C}^d,$$
$$\mathfrak{siso}(d,\mathbb{C})_{\bar{1}} := \begin{cases} S_1 \oplus S_2, & d \equiv 0,4 \mod 8, \\ S_1, & d \equiv 2 \mod 8, \\ S_1 \oplus S_1, & d \equiv 6 \mod 8, \\ S, & d \equiv 1,3 \mod 8, \\ S \oplus S, & d \equiv 5,7 \mod 8. \end{cases}$$

The bracket between two elements of $\mathfrak{siso}(d,\mathbb{C})_{\bar{0}}$ is the usual Lie bracket. The bracket between an element $(A,x) \in \mathfrak{siso}(d,\mathbb{C})_{\bar{0}}$ and $s \in \mathfrak{siso}(d,\mathbb{C})_{\bar{1}}$ is defined by

$$[(A,x),s] := \kappa(A)s,$$

where κ denotes the spin representation.

The bracket on $\mathfrak{siso}(d,\mathbb{C})_{\bar{1}}$ is defined via the non-zero symmetric bilinear form $\Gamma : S \otimes S \longrightarrow \mathbb{C}^d$, which is the complex version of Γ above. The bracket on the odd part, $\mathfrak{siso}(d,\mathbb{C})_{\bar{1}}$, is then defined as follows:

$$[u,v] := \Gamma(u,v), \qquad \begin{cases} u \in S_1,\ v \in S_2, & d \equiv 0,4 \mod 8, \\ u,v \in S_1, & d \equiv 2 \mod 8, \\ u,v \in S, & d \equiv 1,3 \mod 8, \end{cases}$$
$$[(u_1,u_2),(v_1,v_2)] := \Gamma(u_1,v_2) - \Gamma(u_2,v_1), \qquad \begin{cases} u_i,v_i \in S_1, & d \equiv 6 \mod 8, \\ u_i,v_i \in S, & d \equiv 5,7 \mod 8. \end{cases}$$

Superconformal algebras

We now provide a motivated definition of superconformal algebras. Let $\mathcal{M} = (M, \mathcal{O}_M)$ be the complex Minkowski superspace. A complex Lie superalgebra $\mathfrak{L} = \mathfrak{L}_{\bar{0}} \oplus \mathfrak{L}_{\bar{1}}$ satisfying the following three conditions is referred to as a *complex superconformal Lie algebra* (see [135]):

a) $\mathfrak{L}$ acts as infinitesimal automorphisms of $\mathcal{M}$; that is, there exists a Lie superalgebra homomorphism from $\mathfrak{L}$ into the Lie superalgebra of graded derivations of the ring of superfunctions on $\mathcal{M}$:
$$\mathfrak{L} \longrightarrow \mathrm{Der}(\mathcal{O}_M).$$

b) $\mathfrak{L}$ contains the Poincaré superalgebra as a Lie subsuperalgebra, and the action described in (a), when restricted to this subsuperalgebra, is compatible with the natural action of the Poincaré superalgebra.

c) The even part $\mathfrak{L}_{\bar{0}}$ contains the conformal algebra $\mathfrak{so}(2,d)$ as a subalgebra. When its action is restricted to ordinary Minkowski space, it is compatible with the natural action of the conformal algebra.

The classification of complex superconformal Lie algebras is due to Nahm [100] and Shnider [135]. Shnider's classification relies on the observation that any superconformal algebra is simple, and it builds upon Kac's classification of simple Lie superalgebras in [76].

More precisely, the classification proceeds in three steps. First, it is shown that $\mathfrak{L}$ is simple. Second, $\mathfrak{L}_{\bar{0}}$ is identified as the direct sum of $\mathfrak{so}(d+2,\mathbb{C})$ and a complementary ideal. Finally, Kac's classification theorem is applied to determine all such simple Lie superalgebras.

This analysis shows that for $d \geq 7$, any Lie superalgebra containing $\mathfrak{so}(d+2,\mathbb{C})$ in its even part fails to admit the spinor representation of $\mathfrak{so}(d,\mathbb{C})$ in the odd component. Therefore, no superconformal Lie algebra exists in dimension $d > 6$.

Furthermore, Kac's classification provides a complete description of the structure of superconformal Lie algebras.

Theorem 4.2.1 ([135]). *For a complex Minkowski superspace $\mathcal{M}$ of dimension $d \geq 7$, it is not possible to define a superconformal Lie algebra. In particular, complex superconformal extensions of the complex Poincaré superalgebra exist (apart from dimensions ≤ 2) in dimensions* 3, 4, 5, *and* 6, *as follows:*

d	***N***	***complex superconformal algebra***
3	$2k+1$	$\mathfrak{osp}(2k+1\|4)$
3	$2k$	$\mathfrak{osp}(2k\|4)$
4	$k+1$	$\mathfrak{sl}(4\|k+1)$
5	1	$F(4)$
6	k	$\mathfrak{osp}(8\|2k)$

Here, d denotes the spacetime dimension, N the number of supersymmetries, and k is a positive integer.

Remark 4.2.2. Superconformal algebras also exist for $d = 1, 2$. In two dimensions, they arise as subsuperalgebras of the Virasoro superalgebras.

Part II.

On Unitarizable Supermodules

5. The unitary dual and discrete series representations

In this chapter, unless otherwise stated, we consider a (linear) connected *reductive Lie group* G, that is, G is a closed connected group of real or complex matrices that is stable under conjugate transpose. If G has finite center, it is called *semisimple*. We are particularly interested in the semisimple Lie group $\mathrm{SU}(p,q)$, for which G is often used as a placeholder.

5.1. $\mathrm{SU}(p,q)$ and $\mathfrak{su}(p,q)$

We provide a brief introduction to $\mathrm{SU}(p,q)$, the indefinite special unitary group of signature $(p,-q)$, along with its Lie algebra $\mathfrak{su}(p,q)$ and associated structure theory.

A general element $g \in \mathrm{SU}(p,q)$ is of the form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad A^{\dagger} I_{p,q} A = I_{p,q}, \qquad \det(A) = 1,$$

where A is a $p \times p$ matrix, B a $p \times q$ matrix, C a $q \times p$ matrix, and D a $q \times q$ matrix. Moreover, $I_{p,q}$ is the diagonal matrix with 1 in the first p entries and -1 in the last q entries, and $(\cdot)^{\dagger}$ denotes the complex conjugate transpose. The group $\mathrm{SU}(p,q)$ is connected and non-compact.

The associated real semisimple Lie algebra of $\mathrm{SU}(p,q)$ is

$$\mathfrak{su}(p,q) = \left\{ A \in \mathrm{Mat}(p+q,\mathbb{C}) : A^{\dagger} I_{p,q} + I_{p,q} A = 0, \quad \mathrm{tr}(A) = 0 \right\},$$

equipped with the canonical matrix commutator. The (global) Cartan involution

$$\Theta : \mathrm{SU}(p,q) \longrightarrow \mathrm{SU}(p,q)$$

on $\mathrm{SU}(p,q)$, mapping A to $\left(A^{\dagger}\right)^{-1}$, induces an infinitesimal Cartan involution θ on $\mathfrak{su}(p,q)$, given by mapping an element $X \in \mathfrak{su}(p,q)$ to $-X^{\dagger}$. Then $\mathfrak{su}(p,q)$ has the *Cartan decomposition*

$$\mathfrak{su}(p,q) = \mathfrak{k} \oplus \mathfrak{p}$$

with $\mathfrak{k} = \{X \in \mathfrak{su}(p,q) : \theta(X) = X\}$ and $\mathfrak{p} = \{Y \in \mathfrak{su}(p,q) : \theta(Y) = -Y\}$. The Lie algebra $\mathfrak{k}$ is the compact part of $\mathfrak{su}(p,q)$, and $\mathfrak{p}$ is the non-compact part. The global Cartan decomposition is [83, Proposition 1.2]:

$$\mathrm{SU}(p,q) = K \cdot \exp(\mathfrak{p}).$$

Here, $K \cong \mathrm{S}\left(\mathrm{U}(p) \times \mathrm{U}(q)\right)$ is a maximal compact subgroup of $\mathrm{SU}(p,q)$ with Lie algebra $\mathfrak{k}$.

The complexification of $\mathfrak{su}(p,q)$ is the complex special linear Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ with $p+q=n$. Let $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{p}^{\mathbb{C}}$ denote the complexifications of $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Then the Cartan decomposition transfers to the complexifications

$$\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}.$$

In $\mathfrak{su}(p,q)$, we fix the (maximal compact) Cartan subalgebra $\mathfrak{t}$ of diagonal matrices, and we denote by $\mathfrak{t}^{\mathbb{C}}$ its dual space. The analytic subgroup of $\mathrm{SU}(p,q)$ corresponding to $\mathfrak{t}$ will be denoted by T; it is a maximal compact torus in K. The complexification $\mathfrak{t}^{\mathbb{C}}$ of $\mathfrak{t}$ gives us a Cartan subalgebra of $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{sl}(n,\mathbb{C})$, respectively. Thus, we can construct the root systems $\Delta := \Delta(\mathfrak{sl}(n,\mathbb{C});\mathfrak{t}^{\mathbb{C}})$ and $\Delta_c := \Delta(\mathfrak{k}^{\mathbb{C}};\mathfrak{t}^{\mathbb{C}})$. We equip $\mathfrak{t}^{\mathbb{C}}$ with the canonical basis given by the elementary matrices $\{E_{ij}\}$. The canonical dual basis takes the form $\{\epsilon_1,\ldots,\epsilon_n\}$ with $\epsilon_i(H)=h_i$ for $i=1,\ldots,n$ and any $H=\mathrm{diag}(h_1,\ldots,h_n)\in\mathfrak{t}$. In particular, any $\lambda\in(\mathfrak{t}^{\mathbb{C}})^*$ can be written as

$$\lambda = \sum_{i=1}^{n} \lambda_i\epsilon_i = (\lambda_1,\ldots,\lambda_n), \qquad \lambda_i\in\mathbb{C}.$$

The root system is $\Delta=\{\epsilon_i-\epsilon_j : i\neq j\}$ with root spaces $\mathfrak{g}^{\epsilon_i-\epsilon_j}=\mathbb{C}E_{ij}$, satisfying the relation $\sum_{i=1}^{n}\epsilon_i=0$. On the other hand, the compact root system is given by

$$\Delta_c = \{\pm(\epsilon_i-\epsilon_j), \pm(\epsilon_k-\epsilon_l) : 1\le i<j\le p,\ p+1\le k<l\le n\},$$

which we identify naturally with a subset of Δ. Consequently, the root system Δ can be decomposed into a set of compact roots Δ_c and a set of non-compact roots $\Delta_n := \Delta\setminus\Delta_c$.

We identify the Weyl group W of Δ with the symmetric group S_n of degree n acting on $\mathfrak{t}^{\mathbb{C}}$ by permutation of diagonal matrix elements. The compact Weyl group $W_c\subset W$ can be identified with the subgroup $S_p\times S_q$ in the canonical way.

On the set of compact roots, we fix for this thesis the positive system

$$\Delta_c^+ = \{\epsilon_i-\epsilon_j, \epsilon_k-\epsilon_l : 1\le i<j\le p,\ p+1\le k<l\le n\}.$$

Any set of positive roots $\Delta^+\subset\Delta$ must contain Δ_c^+. With respect to a fixed positive system Δ^+, we then define the associated Weyl elements

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha, \qquad \rho_c = \frac{1}{2}\sum_{\alpha\in\Delta_c^+}\alpha, \qquad \rho_n = \frac{1}{2}\sum_{\alpha\in\Delta_n^+}\alpha,$$

where $\Delta_n^+ := \Delta^+\cap\Delta_n$. Moreover, we define the standard Borel subalgebra of $\mathfrak{sl}(n,\mathbb{C})$ to be

$$\mathfrak{b} := \mathfrak{t}^{\mathbb{C}}\oplus\mathfrak{n}^+, \qquad \text{with } \mathfrak{n}^+ := \bigoplus_{\alpha\in\Delta^+}\mathfrak{g}^{\alpha}.$$

The Killing form on $\mathfrak{sl}(n,\mathbb{C})$, given by $(\cdot,\cdot):\mathfrak{sl}(n,\mathbb{C})\times\mathfrak{sl}(n,\mathbb{C})\to\mathbb{C}$ with $(X,Y):=\mathrm{tr}(XY)$ for any $X,Y\in\mathfrak{sl}(n,\mathbb{C})$, induces through restriction on $\mathfrak{t}^{\mathbb{C}}\times\mathfrak{t}^{\mathbb{C}}$ a W-invariant non-degenerate symmetric bilinear form $(\cdot,\cdot)$. Moreover, we obtain naturally a W-invariant non-degenerate symmetric bilinear form $(\cdot,\cdot)$ on the dual $(\mathfrak{t}^{\mathbb{C}})^*$.

A linear functional on $\mathfrak{t}^{\mathbb{C}}$ is called *analytically integral* if for any $H\in\mathfrak{t}$ with $\exp(H)=1$, we have $\lambda(H)\in 2\pi i\mathbb{Z}$. Any analytically integral linear functional on $\mathfrak{t}^{\mathbb{C}}$ satisfies

$$2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\in\mathbb{Z}$$

for all $\alpha\in\Delta$. A linear functional satisfying the latter condition is called *algebraically integral*, or simply *integral*.

5.2. The unitary dual

This section discusses the definition of the *unitary dual* of a real reductive Lie group G, which consists of all equivalence classes of irreducible unitary representations of G. We begin by recalling key definitions in Section 5.2.1 and the decomposition of general unitary representations in Section 5.2.2. While Section 5.2.3 focuses on the decomposition of unitary representations under a maximal compact subgroup, an algebraic approach to unitary irreducible representations is presented in Section 5.2.4. Our exposition follows [44, Chapter 3], [83, Chapters III and IX], and [103].

5.2.1. Basic definitions

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a (complex) Hilbert space. A *unitary operator* U is an invertible continuous transformation with a continuous inverse such that $U^\dagger = U^{-1}$, where $U^\dagger$ denotes the Hilbert space adjoint of U. The set of all unitary operators on $\mathcal{H}$ forms a group, called the *unitary group* $U(\mathcal{H})$. Throughout this thesis, we consider $U(\mathcal{H})$ as a topological group equipped with the *weak operator topology*, which is the coarsest topology for which all functions

$$f_{v,w} : U(\mathcal{H}) \to \mathbb{C}, \qquad U \mapsto \langle Uv, w\rangle, \qquad v, w \in \mathcal{H}$$

are continuous. This topology coincides with the *strong operator topology*, which is the coarsest topology for which all maps

$$U(\mathcal{H}) \to \mathcal{H}, \qquad U \mapsto Uv, \quad v \in \mathcal{H}$$

are continuous.

Definition 5.2.1. A *unitary representation* of G is a pair $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space and $\pi : G \to U(\mathcal{H})$ is a continuous group homomorphism.

Note that the continuity of a unitary representation $(\pi, \mathcal{H})$ is equivalent to the continuity of all *matrix coefficients*

$$\pi_{v,w} : G \to \mathbb{C}, \qquad \pi_{v,w}(g) := \langle \pi(g)v, w\rangle.$$

Here, it is enough to check continuity for a dense subspace of $\mathcal{H}$.

Example 5.2.2. Fix a Haar measure $\mathrm{d}g$ on G, and consider the Hilbert space $L^2(G) := L^2(G, \mathrm{d}g)$ of square-integrable functions, equipped with the inner product

$$\langle f_1, f_2\rangle_{L^2(G)} := \int_G f_1(g)\overline{f_2(g)}\,\mathrm{d}g, \qquad f_1, f_2 \in L^2(G).$$

The left translation action of G on itself induces the *left regular representation* of G on $L^2(G)$, given by

$$[\pi_L(x)f](y) = f(x^{-1}y), \qquad f \in L^2(G), \quad x, y \in G.$$

This defines a unitary representation.

Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be two unitary representations of G. A continuous linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ is called an intertwining operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ if T satisfies

$$T \circ \rho_1(g) = \rho_2(g) \circ T, \qquad \forall g \in G.$$

If T is an isomorphism, we say that $\mathcal{H}_1$ and $\mathcal{H}_2$ *equivalent*. The set of all intertwining operators is denoted by $\mathcal{C}(\pi_1, \pi_2)$. In addition, if T is unitary, we call $\mathcal{H}_1$ and $\mathcal{H}_2$ *unitarily equivalent*. However, as $\mathcal{H}_1, \mathcal{H}_2$ are unitary representations, equivalence and unitary equivalence coincide.

A closed subspace $V \subset \mathcal{H}$ of a unitary representation $(\pi, \mathcal{H})$ is called *G-invariant* if $\pi(g)V \subset V$ for all $g \in G$. If V is G-invariant, then $(\pi|_V, V)$ is a unitary representation of G, called a *subrepresentation* of $(\pi, \mathcal{H})$. A unitary representation $(\pi, \mathcal{H})$ is called *irreducible* if there are no non-trivial closed G-invariant subspaces, i.e., $\{0\}$ and $\mathcal{H}$ are the only G-invariant subspaces.

Definition 5.2.3. If $(\pi, \mathcal{H})$ is an irreducible unitary representation of G, we write $[\pi]$ for its unitary equivalence class. The set of all equivalence classes of irreducible unitary representations is denoted by $\widehat{G}$ and is called the *unitary dual* of G.

A central problem of representation theory is to describe the unitary dual $\widehat{G}$ of a given Lie group G, which provides building blocks for unitary representations as we shall now see.

5.2.2. Decomposing unitary representations

We decompose unitary representations $(\pi, \mathcal{H})$ into simpler pieces, known as *cyclic subspaces*. Furthermore, we decompose $\mathcal{H}$ into discrete and continuous parts, where the discrete part is identified as the closed subspace generated by all irreducible subrepresentations.

The first basic observation is that for any closed invariant subspace, its orthogonal complement is also invariant. This establishes subrepresentations as summands in direct sums.

Proposition 5.2.4. *Let $(\pi, \mathcal{H})$ be a unitary representation of G. If V is a G-invariant closed subspace, then $V^\perp := \{v \in \mathcal{H} : \langle v, w\rangle = 0,$ for all $w \in V\}$ is a closed G-invariant subspace. In particular, if π has a non-trivial closed G-invariant subspace V, then $\mathcal{H} = V \oplus V^\perp$.*

The proof of this proposition is straightforward and will be omitted. Iterating this process leads to a decomposition of $\mathcal{H}$ into smaller pieces called *cyclic subrepresentations*. For any $v \in \mathcal{H}$, the closed linear span $\mathcal{H}_v$ of $\{\pi(g)v : g \in G\}$ in $\mathcal{H}$ is called the *cyclic subspace* generated by v. If $\mathcal{H} = \mathcal{H}_v$, the vector v is called a *cyclic vector* for π, and π is called a *cyclic representation*. Combining Proposition 5.2.4 and Zorn's lemma, the following theorem follows immediately.

Theorem 5.2.5 ([44, Proposition 3.3])**.** *Any unitary representation of G is a direct sum of cyclic representations.*

The direct sum is possibly infinite, and we have to clarify what we mean by a *Hilbert space direct sum*. For a countable family $(\mathcal{H}_i)_{i\in I}$ of Hilbert spaces, we define the Hilbert space direct sum by

$$\bigoplus_{i\in I}\mathcal{H}_i := \{(v_i)_{i\in I} \in \prod_{i\in I}\mathcal{H}_i : \sum_{i\in I}\langle v_i, v_i\rangle < \infty\},$$

which is a Hilbert space with inner product

$$\langle (v_i)_{i\in I}, (w_i)_{i\in I}\rangle := \sum_{i\in I}\langle v_i, w_i\rangle_{\mathcal{H}_i}.$$

The Hilbert space $\mathcal{H}_i$ can be identified with the subspace $\{(v_i)_{i\in I} : v_j = 0 \text{ for all } j \neq i\}$.

If K is a compact group, every unitary representation is indeed a direct sum of irreducible subrepresentations. This is the famous Peter–Weyl theorem.

Theorem 5.2.6 ([44, Theorem 5.12])**.** *Let $(\pi, \mathcal{H})$ be a continuous representation of a compact Lie group K. Then $(\pi, \mathcal{H})$ is a direct sum of irreducible representations. Moreover, any irreducible representation of K is finite-dimensional.*

The irreducibility of a given unitary representation $(\pi, \mathcal{H})$ can be investigated with $\mathcal{C}(\pi) := \mathcal{C}(\pi, \pi)$, i.e., the space of bounded operators on $\mathcal{H}$ that commute with $\pi(g)$ for any $g \in G$. We call $\mathcal{C}(\pi)$ the *commutant* of π. The central result is *Schur's lemma.*

Lemma 5.2.7 ([44, Lemma 3.5])**.** *a) A unitary representation $(\pi, \mathcal{H})$ of G is irreducible if and only if $\mathcal{C}(\pi)$ consists solely of scalar multiples of the identity.*

b) Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be two irreducible unitary representations of G. If π_1 and π_2 are equivalent, then $\mathcal{C}(\pi_1, \pi_2)$ is one-dimensional; otherwise, $\mathcal{C}(\pi_1, \pi_2) = \{0\}$.

Corollary 5.2.8. *Let $(\pi, \mathcal{H})$ be a unitary representation of G, and let $(\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2)$ be two non-equivalent irreducible subrepresentations. Then $\mathcal{H}_1 \perp \mathcal{H}_2$.*

We close this subsection by observing that any unitary representation $(\pi, \mathcal{H})$ of G can naturally be decomposed into a discrete part $(\pi_d, \mathcal{H}_d)$ and a continuous part $(\pi_c, \mathcal{H}_c)$. The discrete part consists of a direct sum of irreducible representations, while the continuous part contains no irreducible subrepresentations.

Proposition 5.2.9 ([103, Proposition 2.2.5])**.** *Let $(\pi, \mathcal{H})$ be a unitary representation of G. Let $\mathcal{H}_d \subset \mathcal{H}$ be the closed subspace generated by all irreducible subrepresentations. Then the following assertions hold:*

a) $\mathcal{H}_d$ is G-invariant, and the representation $(\pi_d, \mathcal{H}_d)$ is a direct sum of irreducible representations.

b) The orthogonal complement $\mathcal{H}_c := \mathcal{H}_d^\perp$ carries a representation $(\pi_c, \mathcal{H}_c)$ of G that does not contain any irreducible subrepresentations.

This leads to the notion of *isotypic components* of $\mathcal{H}$. For an equivalence class $[\pi] \in \widehat{G}$, we write $\mathcal{H}_{[\pi]} \subset \mathcal{H}$ for the closed subspace generated by all irreducible subrepresentations of type $[\pi]$. The discrete part of $(\pi, \mathcal{H})$ is the orthogonal direct sum

$$\mathcal{H}_d = \bigoplus_{[\pi]\in\widehat{G}} \mathcal{H}_{[\pi]},$$

and the subspaces $\mathcal{H}_{[\pi]}$ are known as *isotypic components* of $\mathcal{H}$.

5.2.3. Unitary representations are admissible

The following subsection examines a given (unitary) representation $(\pi, \mathcal{H})$ of G as a (unitary) representation of an underlying maximal compact subgroup K. We follow [83, Chapter VIII].

Let $(\pi, \mathcal{H})$ be a unitary representation of G. Then $(\pi|_K, \mathcal{H})$ is a unitary representation of K, and by the Peter–Weyl Theorem, $(\pi|_K, \mathcal{H})$ decomposes into an orthogonal sum of spaces on which $\pi|_K$ is irreducible. Specifically, we have

$$\pi|_K \cong \sum_{[\rho] \in \widehat{K}} m_\rho \rho,$$

where $\widehat{K}$ denotes the set of equivalence classes of irreducible representations of K and $m_\rho \in \mathbb{N} \cup \{\infty\}$ denotes the multiplicity. The equivalence classes $[\rho]$ occurring $\pi|_K$ with positive multiplicity are called the K-types of π. A unitary representation is called *admissible* if every K-type occurs with finite multiplicity.

Theorem 5.2.10 ([83, Theorem 8.1])**.** *Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G. Then the multiplicities of the K-types in $\pi|_K$ satisfy*

$$m_\rho \leq \dim \rho, \qquad [\rho] \in \widehat{K}.$$

In particular, irreducible unitary representations of G are admissible.

The theorem introduces the notion of K-finite vectors. A vector $v \in \mathcal{H}$, in a unitary representation $(\pi, \mathcal{H})$, is called *K-finite* if $\pi(K)v$ spans a finite-dimensional space. We denote the space of all K-finite vectors by $\mathcal{H}^K$. If $(\pi, \mathcal{H})$ is irreducible, the space $\mathcal{H}^K$ is dense in $\mathcal{H}$ by the Peter–Weyl Theorem.

5.2.4. Harish-Chandra modules and infinitesimal unitary equivalence

We fix an admissible unitary representation $(\pi, \mathcal{H})$ of G. Let $\mathfrak{g}$ denote the Lie algebra of G. Associated to $(\pi, \mathcal{H})$ is a (unitarizable) representation of $\mathfrak{g}$, referred to as the *derived representation*. We follow [83, Chapter III, Chapter VIII] and [4].

To define this representation, we consider the subspace $\mathcal{H}^\infty \subset \mathcal{H}$ of smooth vectors. A vector $v \in \mathcal{H}$ is called *smooth* if the map

$$G \to \mathcal{H}, \qquad g \mapsto \pi(g)v$$

is infinitely differentiable. The space $\mathcal{H}^\infty$ consists of all such smooth vectors, and is known to be dense in $\mathcal{H}$ with respect to the Hilbert space topology [83, Theorem 3.15].

The Lie algebra $\mathfrak{g}$ acts naturally on the space of smooth vectors, yielding the *derived representation*

$$d\pi : \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty), \qquad d\pi(X)v := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi(\exp(tX))v.$$

A direct calculation confirms that for all $X \in \mathfrak{g}$, the operator $d\pi(X)$ stabilizes $\mathcal{H}^\infty$, and furthermore, the assignment $X \mapsto d\pi(X)$ respects the Lie bracket. This implies that $\mathcal{H}^\infty$ naturally carries the structure of a module over the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ [83, Proposition 3.9]. Moreover, the conjugation action of G on $\mathrm{End}(\mathcal{H})$ induces an automorphism of $\mathcal{U}(\mathfrak{g})$, which is preserved under π:

$$\pi(g)d\pi(X)\pi(g)^{-1} = d\pi(\mathrm{Ad}(g)(X)), \qquad \forall g \in G, \quad X \in \mathfrak{g}.$$

Despite the natural analytical structure of $\mathcal{H}^\infty$, it is often too large to be effectively used in algebraic considerations. Therefore, a smaller, more manageable subspace is considered. Specifically, we focus on the subspace of K-finite vectors, denoted by $\mathcal{H}^K$, where K is a maximal compact subgroup of G.

Every K-finite vector is smooth, and it follows from [83, Proposition 8.5] that the subspace $\mathcal{H}^K$ is stable under the action of $d\pi(\mathfrak{g})$. Consequently, any admissible unitary representation $(\pi, \mathcal{H})$ of G defines a well-behaved representation of $\mathfrak{g}$ on $\mathcal{H}^K$, allowing us to compare admissible unitary representations at the level of Lie algebra representations.

A key observation is that the pair $(d\pi, \mathcal{H}^K)$ is itself *unitarizable*, meaning that the space $\mathcal{H}^K$ of K-finite vectors admits an inner product with respect to which $\mathrm{d}\pi(\mathfrak{g})$ acts by skew-Hermitian operators. If $(\pi, \mathcal{H})$ is irreducible, this inner product is unique up to a constant.

The space $\mathcal{H}^K$ serves simultaneously as a $\mathcal{U}(\mathfrak{g})$-module and as a representation of K, where the action of K preserves the $\mathcal{U}(\mathfrak{g})$-module structure. In other words, $\mathcal{H}^K$ forms a $(\mathfrak{g}, K)$*-module.* If π is irreducible, then each K-isotypic component $\mathcal{H}_\sigma$, corresponding to an irreducible representation σ of K, is finite-dimensional. In this case, $\mathcal{H}^K$ is called the *Harish-Chandra module* associated with $(\pi, \mathcal{H})$. Two representations are called *infinitesimally equivalent* if their Harish-Chandra modules are algebraic equivalent. Notably, Harish-Chandra modules completely classify unitary irreducible representations, meaning that the unitary representation $(\pi, \mathcal{H})$ is uniquely determined by its associated Harish-Chandra module.

Proposition 5.2.11 ([4, Theorem 5]). *Let $(\pi, \mathcal{H})$ be a unitary representation, and let $(d\pi, \mathcal{H}^K)$ be the corresponding $(\mathfrak{g}, K)$-module. Then there is a one-to-one correspondence*

$$\{\textit{closed } G\textit{-invariant subspaces of } \mathcal{H}\} \longleftrightarrow \{(\mathfrak{g}, K)\textit{-invariant subspaces of } \mathcal{H}^K\}$$

given by

$$U \mapsto U \cap \mathcal{H}^K, \qquad \overline{W} \mapsto W,$$

where $\overline{W}$ denotes the Hilbert space closure of W in $\mathcal{H}$. In particular, two unitary irreducible representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are unitarily equivalent if and only if their corresponding Harish-Chandra modules $(d\pi_1, \mathcal{H}_1^K)$ and $(d\pi_2, \mathcal{H}_2^K)$ are isomorphic.

We record that the Harish-Chandra module of an irreducible unitary representation $(\pi, \mathcal{H})$ consists entirely of analytic vectors. Here, a smooth vector $v \in \mathcal{H}^\infty$ is called *analytic* if there exists an $r > 0$ such that the power series

$$f_v : B_r \to \mathcal{H}, \qquad f_v(X) = \sum_{k=0}^{\infty} \frac{1}{k!}\rho(X)^k v \tag{5.2.1}$$

defines a holomorphic function on $B_r := \{X \in \mathfrak{g} : \|X\| < r\}$, where $\|\cdot\|$ denotes an $\mathrm{Ad}(T)$-invariant norm on $\mathfrak{g}$, T being the Cartan subgroup associated to the Cartan subalgebra $\mathfrak{t}$, and we have extended the derived representation to a representation of the complexification $\mathfrak{g}$ on the complex vector space $\mathcal{H}^K$, denoted by the same symbol. For every $r > 0$, let $\mathcal{H}^{a,r}$

denote the set of all analytic vectors for which (5.2.1) converges on B_r. These spaces are dense in $\mathcal{H}$. An important class of unitary G-representations consists of the *highest weight representations*, which are characterized by the property that $\mathcal{H}^K$ is a highest weight module for the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. In this setting, the derived representation $d\pi$ extends naturally to a representation of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ on the complex vector space $\mathcal{H}^\infty$, which we continue to denote by the same symbol.

Definition 5.2.12. A unitarizable highest weight $\mathfrak{g}$-module $L(\Lambda)$ is called a *highest weight representation for* G if there exists a unitary representation $(\pi, \mathcal{H})$ of G such that $L(\Lambda)$ is isomorphic to the Harish-Chandra module of $(\pi, \mathcal{H})$.

Proposition 5.2.13 ([4, Theorem 7], [102, Theorem X.2.8])**.** *A simple highest weight $\mathfrak{g}$-module $L(\Lambda)$ is* unitarizable *if and only if it is a highest weight representation for the universal cover $\widetilde{G}$ of G.*

In the case of highest weight representations, unitarizability imposes additional constraints on the possible highest weights Λ, typically requiring them to be in a certain real form and satisfying positivity conditions. The classification of unitarizable highest weight modules is a central problem in representation theory and has been extensively studied in various contexts.

5.3. Discrete series representations

The motivation for discrete series representations of semisimple Lie groups naturally arises from Plancherel's theorem, which describes the decomposition of the regular representation of a unimodular locally compact group G on $L^2(G)$ in terms of its unitary dual. The Plancherel Theorem is a generalization of the Peter–Weyl Theorem for compact groups. The Peter–Weyl Theorem asserts that the regular representation $L^2(K)$ of a compact group K decomposes in a countable direct sum of irreducibles and each irreducible appears with finite multiplicity (see Theorem 5.2.6. For a semisimple Lie group G, Harish-Chandra's work in [55] and [56] reveal the discrete part of $L^2(G)$. The irreducible unitary representations which enter here are the discrete series representations of G.

This section provides a brief introduction to discrete series representations of a general semisimple Lie group G with Lie algebra $\mathfrak{g}$. First, we define the discrete series of G and state general properties. Next, we give an equivalent characterization using Harish-Chandra characters, and consider the holomorphic discrete series more in detail. Subsequently, we introduce the limits of discrete series via Zuckerman functors. For both discrete series and their limits, we describe the K-types and relate the associated Harish-Chandra modules. The last part is devoted to the Blattner formula, which describes the decomposition of unitary irreducible representations under a maximal compact subgroup.

5.3.1. Discrete series

The left regular representation $L^2(G)$ (see Example 5.2.2) admits a direct integral decomposition known as the *Plancherel decomposition* (*cf.* [59]). Harish-Chandra's foundational work in [55, 56] identifies the discrete component of $L^2(G)$. The irreducible unitary representations contributing to this discrete part are precisely the *discrete series representations* of G.

Definition 5.3.1. A unitary irreducible representation $(\pi, \mathcal{H})$ of G is called a *discrete series representation* of G if π appears as a direct summand of the left regular representation $(\pi_L, L^2(G))$. The set of isomorphism classes of discrete series representations is referred to as the *discrete series* of G.

More precisely, the *Plancherel decomposition* of the left regular representation of G takes the form [31, 59]:

$$L^2(G) \cong \int_{\widehat{G}}^{\oplus} \mathcal{H}_j \hat{\otimes} \mathcal{H}_j^* \, \mathrm{d}\mu(j),$$

where $\mathrm{d}\mu$ is a positive Plancherel measure on the unitary dual $\widehat{G}$, $\mathcal{H}_j$ denotes an irreducible unitary representation indexed by $j \in \widehat{G}$, and $\mathcal{H}_j \hat{\otimes} \mathcal{H}_j^*$ is the Hilbert space tensor product of $\mathcal{H}_j$ and its dual. The symbol $\int^{\oplus}$ denotes the direct integral decomposition, for which we refer to [44] for details. The discrete series representations of G correspond precisely to those representations for which $\mathrm{d}\mu(j_0) > 0$ [31, Chapter 18.8].

Equivalently, discrete series representations are exactly those unitary irreducible representations $(\pi, \mathcal{H})$ whose K-finite matrix coefficients [84, Proposition 9.6]

$$\phi_{v,w} : G \longrightarrow \mathbb{C}, \quad g \mapsto \phi_{v,w}(g) := \langle v, \pi(g)w \rangle, \qquad v, w \in \mathcal{H}^K,$$

belong to $L^2(G)$.

Now that we have established the notion of discrete series representations, natural questions arise regarding their existence and classification, both of which Harish-Chandra addressed in [55, 56]. We begin with the existence criterion.

Theorem 5.3.2 ([84, Theorem 12.20])**.** *A semisimple Lie group G admits nontrivial discrete series representations if and only if its rank coincides with that of a maximal compact subgroup K,* i.e., $\mathrm{rk}(G) = \mathrm{rk}(K)$.

A real Lie group G obtained from a complex semisimple Lie group by forgetting the complex structure never has discrete series representations, as $\mathrm{rk}(G) = 2\mathrm{rk}(K)$ for any maximal compact K. As an illustration, we summarize the existence of discrete series representations for certain non-exceptional Lie groups in Table 5.1.

G	**K**	**rk(G)**	**rk(K)**	**Discrete series?**
$\mathrm{SL}(n, \mathbb{C})$	$\mathrm{SU}(n)$	$2n-2$	$n-1$	No
$\mathrm{SL}(n, \mathbb{R})$	$\mathrm{SO}(n)$	$n-1$	$\lfloor \frac{n}{2} \rfloor$	If $n = 2$
$\mathrm{SU}(p, q)$	$\mathrm{S}(U(p) \times U(q))$	$p+q-1$	$p+q-1$	Yes
$\mathrm{SO}(n, \mathbb{C})$	$\mathrm{SO}(n)$	$2\lfloor \frac{n}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor$	No
$\mathrm{SO}(p, q)$	$\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$	$\lfloor \frac{p+q}{2} \rfloor$	$\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor$	If p, q even
$\mathrm{Sp}(n, \mathbb{R})$	$\mathrm{U}(n)$	n	n	Yes

Table 5.1.: Existence of discrete series representations for some non-exceptional Lie groups.

The classification of discrete series representations was established by Harish-Chandra in [56]. We state the main result following the notation in [84] and of Section 5.1.

Theorem 5.3.3 ([84, Theorem 9.20 and Theorem 12.21])**.** *Let $\lambda \in i\mathfrak{t}^*$ be non-singular relative to the root system Δ, and define the set of positive roots as*

$$\Delta^+ := \{\alpha \in \Delta \mid (\lambda, \alpha) > 0\}.$$

If $\lambda + \rho$ is analytically integral, then there exists a discrete series representation π_λ of G with the following properties:

a) π_λ has infinitesimal character χ_λ.

b) The restriction $\pi_\lambda|_K$ contains, with multiplicity one, the K-type with highest weight

$$\Lambda = \lambda + \rho - 2\rho_c,$$

where ρ and ρ_c are defined relative to Δ^+.

c) If Λ' is the highest weight of a K-type occurring in $\pi_\lambda|_K$, then it is of the form

$$\Lambda' = \Lambda + \sum_{\alpha \in \Delta^+} n_\alpha \alpha, \quad n_\alpha \in \mathbb{Z}_+.$$

Moreover, two such representations π_λ are equivalent if and only if their parameters λ are conjugate under W_c, the Weyl group associated with Δ_c. The parameter λ is called the Harish-Chandra parameter *of the discrete series π_λ, while the K-type parameter Λ is known as the* Blattner parameter. *Furthermore, the discrete series representations of G are precisely the representations π_λ, up to equivalence.*

Remark 5.3.4. a) The number of mutually inequivalent discrete series representations of G is given by $|W|/|W_c|$.

b) Let ω denote the conjugate-linear anti-involution associated to the real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$. Then the parameters $\lambda \in i\mathfrak{t}^*$ are precisely those satisfying $\overline{\lambda(\cdot)} = \lambda(\omega(\cdot))$. In particular, we may talk about Harish-Chandra parameters $\Lambda \in (\mathfrak{t}^{\mathbb{C}})^*$.

c) Discrete series representations are uniquely determined by their minimal K-type [115]. The *minimal K-type* of an admissible representation of G on a Hilbert space $\mathcal{H}$ is the K-type $\tau_{\Lambda'}$, among all K-types τ_Λ occurring in π, for which

$$|\Lambda + 2\rho_c|^2$$

is minimized by $\Lambda' = \Lambda$. Clearly, for discrete series representations, the minimal K-type coincides with the Blattner parameter.

Example 5.3.5. We consider the conformal group $\mathrm{SU}(2,2)$ as an example (see Section 4.1). Its structure theory is discussed in 5.1. As a $\mathfrak{t}$-basis, we use the elements

$$H_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad H_3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

such that we can rewrite the set of roots as

$$\Delta = \{(\pm 2, 0, 0), (0, \pm 2, 0), (\pm 1, \pm 1, \pm 2)\},$$

where $\alpha = (m, n, l)$, given by $m = \alpha(H_1)$, $n = \alpha(H_2)$, and $l = \alpha(H_3)$. The compact system is

$$\Delta_c = \{\pm(e_1 - e_2), \pm(e_3 - e_4)\} = \{(\pm 2, 0, 0), (0, \pm 2, 0)\},$$

and as a positive system, we choose

$$\Delta^+ = \{e_i - e_j \mid i < j\} = \{(2,0,0),(0,2,0),(1,1,2),(1,-1,2),(-1,-1,2),(-1,1,2)\},$$
$$\Delta_c^+ = \{e_1 - e_2, e_3 - e_4\} = \{(2,0,0),(0,2,0)\}.$$

Hence, the associated Weyl elements are

$$\rho_n = e_1 + e_2 - e_3 - e_4 = (1,0,4),$$
$$\rho_c = \frac{1}{2}(e_1 - e_2 + e_3 - e_4) = (1,1,0).$$

Obviously, $\ker(\exp(\mathfrak{t}^{\mathbb{C}})) = \{2\pi i n I \mid n \in \mathbb{Z}\}$ and $\rho_{\mathrm{SU}(2,2)}(2\pi i n I) = 0$ for all $n \in \mathbb{Z}$, *i.e.*, the Weyl element is analytically integral, and $\lambda + \rho_{\mathrm{SU}(2,2)}$ is analytically integral if and only if the Harish-Chandra parameter is analytically integral. The Harish-Chandra parameters are the non-singular integral linear functionals on $i\mathfrak{t}^{\mathbb{C}}$, *i.e.*, if we write a linear functional on $i\mathfrak{t}^{\mathbb{C}}$ as

$$\sum_{i=1}^{4} c_i e_i \mod \mathbb{C}(e_1 + e_2 + e_3 + e_4), \qquad c_i \in \mathbb{C},$$

they satisfy $c_i - c_j \in \mathbb{Z}$ (integral condition) and $c_i \neq c_j$ for $i \neq j$ (non-singular condition). The interchange of indices 1 and 2, or indices 3 and 4, or both, results in distinct equivalence classes of the corresponding representations.

There are exactly six positive systems of the absolute roots containing compact positive roots Δ_c^+:

$$\Delta_I^+ = \{(2,0,0),(1,-1,2),(1,1,2),(-1,-1,2),(-1,1,2),(0,2,0)\},$$
$$\Delta_{II}^+ = \{(1,-1,2),(2,0,0),(1,1,2),(1,1,-2),(0,2,0),(-1,1,2)\},$$
$$\Delta_{III}^+ = \{(1,-1,2),(1,1,2),(2,0,0),(0,2,0),(1,1,-2),(-1,-1,-2)\},$$
$$\Delta_{IV}^+ = \{(-1,1,-2),(1,1,-2),(0,2,0),(2,0,0),(1,1,2),(-1,1,2)\},$$
$$\Delta_V^+ = \{(-1,1,-2),(0,2,0),(1,1,-2),(1,1,2),(2,0,0),(1,-1,-2)\},$$
$$\Delta_{VI}^+ = \{(0,2,0),(-1,1,2),(1,1,-2),(-1,-1,-2),(1,-1,-2),(2,0,0)\}.$$

For each $J \in \{I, II, III, IV, V, VI\}$, we write $\Delta_{J,n}^+ := \Delta_J^+ \setminus \Delta_c^+$ for the set of non-compact positive roots. Further, we define the subset of dominant weights by

$$\Xi_J := \{\Lambda = [\Lambda_1, \Lambda_2, \Lambda_3] \mid \Lambda \text{ is } \Delta_{J,n}^+ - \text{dominant}\},$$

the space gives the Harish-Chandra parametrization of the discrete series representations of $\mathrm{SU}(2,2)$. Let π_λ be a discrete series representation of $\mathrm{SU}(2,2)$ with Harish-Chandra parameter $\lambda \in \Xi_J$. Then the Blattner parameter of π_λ is given by $\Lambda = \lambda - \rho_c + \rho_{J,n}$, where $\rho_{J,n}$ is the half-sum of non-compact positive roots in $\Delta_{J,n}^+$. Note that π_λ has minimal K-type whose highest weight is Λ. For each J, Λ is given as

$$\Lambda = \begin{cases} \lambda + (-1,-1,4) & \lambda \in \Xi_I, \\ \lambda + (0,0,2) & \lambda \in \Xi_{II}, \\ \lambda + (1,-1,0) & \lambda \in \Xi_{III}, \\ \lambda + (-1,1,0) & \lambda \in \Xi_{IV}, \\ \lambda + (0,0,-2) & \lambda \in \Xi_V, \\ \lambda + (-1,-1,-4) & \lambda \in \Xi_{VI}. \end{cases}$$

The representations with the Harish-Chandra parameter $\lambda \in \Xi_{II} \cup \Xi_V$ are called *large discrete series*. Those with Harish-Chandra parameter $\lambda \in \Xi_I$ (resp. Ξ_{VI}) are called holomorphic (resp. anti-holomorphic) discrete series. The discrete series representations with Harish-Chandra parameter $\lambda \in \Xi_{III} \cup \Xi_{IV}$ are called *middle discrete series representations*.

5.3.2. Harish-Chandra characters

Discrete series representations of G are uniquely determined by their Harish-Chandra characters, which we now introduce following [83, Chapter X] and [85].

To that end, fix a discrete series representation $(\pi_\Lambda, \mathcal{H}_\Lambda)$ of G, and recall the definition of a trace class operator:

Definition 5.3.6. A *trace class operator* L on $(\mathcal{H}_\Lambda, \langle\cdot,\cdot\rangle_{\mathcal{H}_\Lambda})$ is a bounded linear operator for which

$$\sum_i \left|\langle B^{-1}LBe_i, e_i\rangle_{\mathcal{H}_\Lambda}\right| < \infty$$

holds for every orthonormal basis $\{e_i\}$ of $\mathcal{H}_\Lambda$ and every bounded operator $B : \mathcal{H}_\Lambda \to \mathcal{H}_\Lambda$ with bounded inverse. In this case, the sum $\sum_i |\langle B^{-1}LBe_i, e_i\rangle_{\mathcal{H}_\Lambda}|$ is independent of B and is called the *trace* of L.

The Harish-Chandra character is a natural analog of the character of a finite-dimensional representation, which is defined as the trace of its matrix coefficients. However, in general, $\pi_\Lambda(x)$ is not trace class for all $x \in G$, so the trace of matrix coefficients is not well-defined. To address this, Harish-Chandra introduced in [55] the notion of characters as distributions on G.

Let $\mathcal{C}_c^\infty(G)$ denote the space of smooth functions with compact support on G. For $f \in \mathcal{C}_c^\infty(G)$ and $v \in \mathcal{H}_\Lambda$, define

$$\pi_\Lambda(f)v := \int_G f(x)\pi_\Lambda(x)v \, dx.$$

Using Theorem 5.2.10, one can show that $\pi_\Lambda(f)$ is trace-class for any $f \in \mathcal{C}_c^\infty(G)$. We set $\Theta(f) := \operatorname{tr} \pi(f)$, which is invariant under conjugation. This leads to the following definition:

Definition 5.3.7. The distribution Θ on G is called the *Harish-Chandra character* of $(\pi_\Lambda, \mathcal{H}_\Lambda)$.

We next show that the Harish-Chandra character uniquely determines discrete series representations. We begin with the following proposition.

Proposition 5.3.8. *Let $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ be two infinitesimally equivalent discrete series representations of G, with Harish-Chandra characters Θ and Θ', respectively. Then $\Theta = \Theta'$.*

Proof. By Theorem 5.2.10, we can decompose $\mathcal{H}$ and $\mathcal{H}'$ into their K-types. Fix an orthonormal basis of $\mathcal{H}$ compatible with this decomposition. For each K-type ρ, let $\mathcal{H}_\rho \subset \mathcal{H}$ denote the subspace transforming according to ρ, and let P_ρ denote the corresponding orthogonal projection. Then, for any $f \in \mathcal{C}_c^\infty(G)$, we have

$$\operatorname{tr}(\pi(f)) = \sum_{\rho \in \widehat{K}} \int_G f(x) \operatorname{tr}(P_\rho \pi(x) P_\rho) \, dx.$$

The integrands $\mathrm{tr}(P_\rho \pi(x) P_\rho)$ are built from K-finite matrix coefficients and are invariant under infinitesimal equivalence. This proves the claim. $\square$

Furthermore, inequivalent discrete series representations of G have linearly independent Harish-Chandra characters [83, Theorem 10.6]. Consequently, if the global characters of two discrete series representations of G are equal, then the representations are infinitesimally equivalent. Altogether, we obtain the following theorem.

Theorem 5.3.9. *Discrete series representations are (up to unitary equivalence) uniquely determined by their Harish-Chandra characters.*

In [55, 56, 57], Harish-Chandra derived an explicit formula for the Harish-Chandra character. His proof is based on the observation that Θ is an eigendistribution of the center $\mathfrak{Z}(\mathfrak{g})$ of the universal enveloping algebra, with eigenvalue equal to the infinitesimal character of the representation. That is,

$$z \cdot \Theta = \chi(z)\Theta,$$

where $z \in \mathfrak{Z}(\mathfrak{g})$ is considered as a left-invariant differential operator.

To state his result, fix a compact Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$, with associated compact torus T, and let K be a maximal compact subgroup of G. An element $g \in G$ is called *regular* if the dimension of its centralizer in G equals the dimension of $\mathfrak{t}$. Let $G_{\mathrm{reg}} \subset G$ denote the set of regular elements. We can now summarize Harish-Chandra's result as follows:

Theorem 5.3.10 ([55, 56, 57])**.** *a) There exists a locally integrable function $\Theta_\Lambda : G \to \mathbb{C}$ such that*

$$\Theta(f) = \int_G \Theta_\Lambda(g) f(g)\, \mathrm{d}g, \qquad \forall f \in \mathcal{C}_c^\infty(G).$$

b) For all regular elements $g \in T$, the Harish-Chandra character is given by

$$\Theta_\Lambda(g) = (-1)^{\frac{\dim(G/K)}{2}} \left(\frac{\sum_{w \in W_c} \mathit{sign}(w) e^{w\Lambda}}{e^{\rho_{\bar{0}}} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})} \right)(g),$$

where $\mathit{sign}(w)$ denotes the signature of $w \in W_c$.

When referring to the calculation of a representation's character, we typically mean the evaluation of Θ_Λ on G_{reg}.

5.3.3. Holomorphic discrete series representations

We are interested in a special type of discrete series representations: the *holomorphic discrete series*. These unitary irreducible representations of G are distinguished as the associated Harish-Chandra modules are of highest weight type. We give a brief introduction to holomorphic discrete series, explain their name, and provide a classification. The standard literature is [83, Chapter VI] and [55].

We start with a standard construction that explains the name. Let $\mu \in i\mathfrak{t}^*$ be a dominant integral weight with respect to Δ_c^+, and let (σ, V_σ) denote the associated finite-dimensional irreducible representation of K. Then K acts freely and properly on the product $G \times V_\sigma$ by

$$p \cdot (g, v) := (gp^{-1}, \sigma(p)v)$$

for all $(g, v) \in G \times V_\sigma$ and $p \in K$. We denote the associated quotient space by $G \times_K V_\sigma$. It has a unique structure of a smooth manifold, making the map $G \times V_\sigma \to G \times_K V_\sigma$ into a principal fiber bundle with structure group K. Let

$$\mathrm{pr}_1 : G \times_K V_\sigma \to G/K, \qquad (g, v) \mapsto gK$$

be the projection on the first coordinate. Then pr_1 is a fiber bundle. We require that for any $g \in G$ the bijection

$$\varphi_g : V_\sigma \to (\mathrm{pr}_1)^{-1}(gK), \qquad v \mapsto (g, v)$$

is linear, so that pr_1 indeed defines a vector bundle over G/K. We call this vector bundle associated to (σ, V_σ), and denote it by $\mathcal{V}_\sigma := (G \times_K V_\sigma, \mathrm{pr}_1)$. The group G acts smoothly by left multiplication on the first coordinate on $\mathcal{V}_\sigma$. Thus, $\mathcal{V}_\sigma$ can be viewed as a homogeneous vector bundle over G/K, *i.e.*, the following diagram commutes

$$\begin{array}{ccc} \mathcal{V}_\sigma & \xrightarrow{g\cdot} & \mathcal{V}_\sigma \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle pr_1} \\ G/K & \xrightarrow[s]{} & G/K. \end{array}$$

In particular, $g\cdot$ maps each fiber $(\mathrm{pr}_1)^{-1}(xK) = (\mathcal{V}_\sigma)_x$ onto the fiber $(\mathcal{V}_\sigma)_{gx}$ and for each $x \in G/K$, the map $g\cdot : (\mathcal{V}_\sigma)_x \to (\mathcal{V}_\sigma)_{gx}$ is linear. Indeed, the category of continuous finite-dimensional representations of K is equivalent to the category of G-homogeneous vector bundles on G/K.

Both G/K and $\mathcal{V}_\sigma$ have a G-invariant holomorphic structure. For this purpose, we assume the complexification $G^{\mathbb{C}}$ of G is simply connected. We denote the associated Lie algebra by $\mathfrak{g}^{\mathbb{C}}$, the complexification of $\mathfrak{g}$. We consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ from Section 5.1, and define $\mathfrak{p}^{\pm} := \sum_{\alpha \in \Delta_n^+} \mathfrak{g}^{\pm\alpha}$. Then $[\mathfrak{k}, \mathfrak{p}^{\pm}] \subset \mathfrak{p}^{\pm}$, and a direct calculation shows that $\mathfrak{p}^{\pm}$ are abelian Lie algebras. We denote by K, $K^{\mathbb{C}}$, P^+, and P^- the analytic subgroups corresponding to $\mathfrak{k}$, $\mathfrak{k}^{\mathbb{C}}$, $\mathfrak{p}^+$, and $\mathfrak{p}^-$.

By the Harish-Chandra decomposition [83, Theorem 6.3], the space $P^+ K^{\mathbb{C}} P^-$ is open in $G^{\mathbb{C}}$. Moreover, the product map from $P^+ \times K^{\mathbb{C}} \times P^-$ to $G^{\mathbb{C}}$ is a holomorphic diffeomorphism. In the complex analytic manifold $G^{\mathbb{C}}/K^{\mathbb{C}}P^+$, we consider the orbit of the base point under G. This yields an embedding of G/K into $G^{\mathbb{C}}/K^{\mathbb{C}}P^+$ as a complex open submanifold, using the fact that $K = K^{\mathbb{C}}P^+ \cap G$. This is known as the *Borel embedding*.

To obtain a holomorphic structure on $\mathcal{V}_\sigma$, we lift the representation σ uniquely to a holomorphic representation of $K^{\mathbb{C}}P^+$ so that $\sigma(P^+) = 1$. Let $\mathcal{V}_\sigma \to G^{\mathbb{C}}/K^{\mathbb{C}}P^+$ be the associated holomorphic bundle arising from the holomorphic fibration $K^{\mathbb{C}}P^+ \to G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}P^+$. Then, the pullback of the bundle $\mathcal{V}_\sigma \to G^{\mathbb{C}}/K^{\mathbb{C}}P^+$ under the Borel embedding restricts to exactly $\mathcal{V}_\sigma$ over G/K. Thus, $\mathcal{V}_\sigma$ has a holomorphic structure.

Let $d\overline{k}$ be the unique (up to a constant multiple) G-invariant Haar measure on G/K. We define

$$\mathcal{H}_\mu := \left\{ s : G/K \to \mathcal{V}_\mu \ : \ s \text{ is a holomorphic section, } \int_{G/K} ||s(gK)||^2 \, \mathrm{d}\overline{k} < \infty \right\},$$

where the norm is defined with respect to the unique (up to a constant) Hermitian form $\langle \cdot, \cdot \rangle_{V_\sigma}$ on V_σ. In particular, $\mathcal{H}_\mu$ has a definite inner product

$$\langle s, s' \rangle := \int_{G/K} \langle s(gK), s'(gK) \rangle_{V_\sigma} \, \mathrm{d}\overline{k},$$

and G acts unitarily on $\mathcal{H}_\mu$ via

$$(gs)(x) = g \cdot s(g^{-1} \cdot x)$$

for $g \in G, s \in \mathcal{H}_\mu$ and $x \in G/K$.

Theorem 5.3.11 ([55, Theorem 4])**.** *$\mathcal{H}_\mu$ is a Hilbert space and $\mathcal{H}_\mu \neq \{0\}$ if and only if $(\mu + \rho, \alpha) < 0$ for any $\alpha \in \Delta_n^+$. Moreover, if $\mathcal{H}_\mu \neq \{0\}$, then $\mathcal{H}_\mu$ is an irreducible unitary representation of G and has square-integrable matrix coefficients.*

We say that the discrete series representations π_λ are *holomorphic* if λ is Δ_c^+-dominant integral and satisfies the Harish-Chandra condition $(\lambda + \rho, \alpha) < 0$ for all $\alpha \in \Delta_n^+$. This is with respect to the choice of positive root system Δ^+ and the corresponding G-invariant complex structure on G/K.

Theorem 5.3.11 allows us to parametrize the holomorphic discrete series in terms of the character group $X^*(T^{\mathbb{C}})$ of the complexification $T^{\mathbb{C}}$ of T and Weyl chambers. The coset $X^*(T^{\mathbb{C}}) + \rho$ in $X^*(T^{\mathbb{C}}) \otimes \mathbb{R}$ is independent of the choice of positive roots. We call an element λ in $X^*(T^{\mathbb{C}}) \otimes \mathbb{R}$ *regular* if no dual root is orthogonal to λ; otherwise, λ is called *singular*. Let $(X^*(T^{\mathbb{C}}) \otimes \mathbb{R})^{reg}$ denote the set of regular elements in $X^*(T^{\mathbb{C}}) \otimes \mathbb{R}$, and set $(X^*(T^{\mathbb{C}}) + \rho)^{reg} := (X^*(T^{\mathbb{C}}) + \rho) \cap (X^*(T^{\mathbb{C}}) \otimes \mathbb{R})^{reg}$. A *Weyl chamber* of G is a connected component of $(X^*(T^{\mathbb{C}}) \otimes \mathbb{R})^{reg}$. The Weyl chambers are in one-to-one correspondence with the systems of positive roots for $(\mathfrak{g}, \mathfrak{t})$. We define a Weyl chamber C to be *holomorphic* if for any λ in the interior of C we have $(\lambda, \alpha) < 0$ for all $\alpha \in \Delta_n^+$. Then there are exactly $|W_c|$ holomorphic Weyl chambers, forming a single orbit for the action of W_c. A *holomorphic Harish-Chandra parameter* is then a pair $(\lambda - \rho, C)$, where C is a holomorphic Weyl chamber and $\lambda \in (X^*(T^{\mathbb{C}}) + \rho)^{reg} \cap C$, *i.e.*, there exists a bijection between the set of isomorphism classes of holomorphic discrete series representations and $(X^*(T^{\mathbb{C}}) + \rho)^{reg}/W_c$, where the Weyl group W_c acts naturally.

Corollary 5.3.12. *With respect to the standard positive system Δ^+, the holomorphic discrete series π_λ of $\mathrm{SU}(p, q)$ are exactly those (up to action of W_c) with Harish-Chandra parameter $\Lambda = (\lambda_1, \ldots, \lambda_p, \lambda_{p+1}, \ldots, \lambda_{p+q}) = \sum_{i=1}^{p+q} \lambda_i \epsilon_i$ such that*

$$\lambda_p \leq \lambda_{p-1} \leq \cdots \leq \lambda_1 \leq \lambda_{p+q} \leq \cdots \leq \lambda_{p+1}.$$

The holomorphic discrete series representations are examples of highest weight representations for G [4, Theorem 7], [102, Theorem X.2.8], that is, the Harish-Chandra modules $\mathcal{H}^K$ of holomorphic discrete series representations are highest weight $\mathfrak{g}$-modules. Note that a unitarizable highest weight module can be integrated to a unitary irreducible representation of G only if the highest weight is integral, as all weights are obtained by differentiation of a representation of a maximal torus in G.

5.3.4. Limits of discrete series representations

A unitary highest weight representation $(\pi_\Lambda, \mathcal{H}_\Lambda)$ belongs to the holomorphic discrete series if $(\Lambda + \rho, \beta) < 0$ for any $\beta \in \Delta_n^+$, according to Harish-Chandra's condition. Broadly speaking, $(\pi_\Lambda, \mathcal{H}_\Lambda)$ belongs to the limit of the holomorphic discrete series if $(\Lambda + \rho, \beta) = 0$ for some $\beta \in \Delta_n^+$. Unfortunately, there is no intrinsic definition of limits of holomorphic discrete series. Explicit constructions are provided through Harish-Chandra's character formula and Zuckerman–Jantzen tensoring.

An *irregular holomorphic Harish-Chandra parameter* is a pair (λ, C), where C is a holomorphic Weyl chamber and $\lambda \in (X^*(T^{\mathbb{C}}) + \rho_{\bar{0}}) \cap C$, such that λ is not orthogonal to any simple compact dual root. A unitary highest weight representation $(\pi_\Lambda, \mathcal{H}_\Lambda)$, $\Lambda = \lambda - \rho$, of G belongs to the limit of the holomorphic discrete series if there exists a holomorphic Weyl chamber C such that (Λ, C) is an irregular holomorphic Harish-Chandra parameter. Consequently, there is a bijection between the set of isomorphism classes of limits of holomorphic discrete series representations and the W_c-orbits of irregular Harish-Chandra parameters.

For completeness, we briefly outline the construction of limits of discrete series representations of G following [84], since the idea parallels that of translation functors in Section 9.2. We work in the category of $(\mathfrak{g}, K)$-modules, denoted by $\mathbf{Mod}_{(\mathfrak{g},K)}$ (see Section 5.2.4).

Fix a $(\mathfrak{g}, K)$-module (π, V). Let $\mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$ be the center of the universal enveloping algebra of the complexification $\mathfrak{g}^{\mathbb{C}}$. Then (π, V) decomposes into generalized eigenspaces under $\mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$ [84, Proposition 10.41]; that is, there exist linear functionals $\lambda_1, \dots, \lambda_n$ on $\mathfrak{t}^{\mathbb{C}}$, $\mathfrak{g}$-invariant subspaces $V_1, \dots, V_n$ of V, and some positive integer $d \in \mathbb{Z}_+$ such that:

a) $\lambda_1, \dots, \lambda_n$ are mutually inequivalent under W.

b) $V = V_1 \oplus \cdots \oplus V_n$.

c) $(z - \chi_{\lambda_j}(z))^d$ acts as the zero operator on V_j for all $z \in \mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$.

d) For all $1 \leq j \leq n$, there exists a $z_j \in \mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$ such that V_j is the image of the action on V by
$$\prod_{i \neq j} (z_j - \chi_{\lambda_i}(z_j))^d.$$

These properties uniquely determine $V_1, \dots, V_n$ and determine $\lambda_1, \dots, \lambda_n$ up to the action of W. Moreover, the closures $\overline{V}_j$ of V_j are invariant under G. We say the representations $\overline{V}_j$ have infinitesimal character λ_j. The generalized eigenspaces under $\mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$ do not depend on $\mathfrak{t}$.

For any object V in $\mathbf{Mod}_{(\mathfrak{g},K)}$, let p'_λ denote the projection onto the generalized eigenspace associated with the infinitesimal character λ. This induces an exact functor (see [84, Proposition 10.43]):
$$\mathrm{p}_\lambda : \mathbf{Mod}_{(\mathfrak{g},K)} \to \mathbf{Mod}_{(\mathfrak{g},K)}, \quad V \mapsto p_\lambda(V) := \overline{\mathrm{p}'_\lambda(V)}.$$

With respect to a fixed Cartan subalgebra $\mathfrak{t}$, we say $\lambda \in \mathfrak{t}^{\mathbb{C}}$ is real if $\lambda|_{\mathfrak{t}\cap\mathfrak{p}}$ is real, and $\lambda|_{\mathfrak{t}\cap\mathfrak{k}}$ is imaginary. We then decompose λ as $\lambda = \mathrm{Re}(\lambda) + i\mathrm{Im}(\lambda)$. Now, choose a positive system Δ^+ such that $\mathrm{Re}(\lambda)$ is dominant. Let F^ν and $F_{-\nu}$ denote the finite-dimensional irreducible representations of G with highest weight ν and lowest weight $-\nu$. Assume that both $\mathrm{Re}(\lambda)$ and ν are dominant with respect to Δ^+. Note that $\mathbf{Mod}_{(\mathfrak{g},K)}$ is closed under tensoring with finite-dimensional $\mathfrak{g}$-modules. Then the *Zuckerman tensoring functors* φ and ψ are defined by

$$\varphi^\lambda_{\lambda+\nu} := p_{\lambda+\nu}\left[(\cdot) \otimes F^\nu\right] p_\lambda,$$
$$\psi^{\lambda+\nu}_\lambda := p_\lambda\left[(\cdot) \otimes F_{-\nu}\right] p_{\lambda+\nu}$$

for suitable domains. Both functors carry short exact sequences in short exact sequences and are independent of the choice of $\mathfrak{t}$ and of Δ^+.

Proposition 5.3.13. *[84, Section 10] For a given positive system Δ^+, let λ and ν be linear functionals on $\mathfrak{t}^{\mathbb{C}}$ such that $Re(\lambda)$ and ν are dominant and ν is integral. Let W_λ and $W_{\lambda+\nu}$ be the subgroups of the Weyl group leaving λ, respectively $\lambda+\nu$, fixed.*

a) $\varphi^{\lambda}_{\lambda+\nu}$ carries any irreducible admissible representation of infinitesimal character λ to an irreducible representation of infinitesimal character $\lambda+\nu$.

b) $\psi^{\lambda+\nu}_{\lambda}$ carries any irreducible admissible representation with infinitesimal character $\lambda+\nu$ to an irreducible admissible representation of infinitesimal character λ.

c) If V is an irreducible admissible representation with infinitesimal character λ, then $\psi^{\lambda+\nu}_{\lambda}\varphi^{\lambda}_{\lambda+\nu}V$ is infinitesimally equivalent to V.

d) If V is an irreducible admissible representation with infinitesimal character $\lambda+\nu$, then $\varphi^{\lambda}_{\lambda+\nu}\psi^{\lambda+\nu}_{\lambda}V$ is infinitesimally equivalent to V.

Assume λ is a singular integral form on $\mathfrak{t}^{\mathbb{C}}$, *i.e.*, $(\lambda+\rho,\alpha)=0$ for some root α. Fix a positive system Δ^+ for Δ that makes λ dominant. Now, fix a dominant integral form μ such that $\lambda+\mu$ is non-singular. Using Zuckerman tensoring, we obtain an admissible representation

$$\pi(\lambda,\Delta^+)=\psi^{\lambda+\mu}_{\lambda}(\pi_{\lambda+\mu}),$$

called the *limit of discrete series.*

Proposition 5.3.14 ([84, Corollary 12.27])**.** *Let λ be dominant with respect to the choice of a positive system Δ^+. If $\lambda+\rho$ is analytically integral and $(\lambda,\alpha)\neq 0$ for all compact positive roots $\alpha\in\Delta^+_c$, then $\pi(\lambda,\Delta^+)$ is an irreducible unitary representation of G. Here, distinct choices of μ lead to infinitesimally equivalent versions of $\pi(\lambda,\Delta^+)$.*

Analogously to discrete series representations, we can decompose any limit of discrete series representations in K-types and all K-types are of the form [86, Proposition 11.29]

$$\lambda-\rho_c+\rho_n+\sum_{\alpha\in\Delta^+} n_\alpha\alpha, \qquad n_\alpha\in\mathbb{Z}_+.$$

In particular, the K-type $\Lambda:=\lambda-\rho_c+\rho_n$ has multiplicity one and is called the *Blattner parameter* of $\pi(\lambda,\Delta^+)$.

5.3.5. Blattner's formula

For a general connected semisimple Lie group G and a maximal compact subgroup $K\subset G$, *Blattner's formula* addresses the classical branching problem of determining the restriction of a discrete series representation of G to K. Specifically, it provides a closed formula for the multiplicities of each possible K-type. The foundational result is presented in [122], with further generalizations found in [3] and Theorem 8.29 together with Proposition 11.129 in [86]. In the following, we briefly state Blattner's formula for the holomorphic discrete series of G and their limits. We follow [122].

Without loss of generality, we consider $G:=\mathrm{SU}(p,q)$ and adopt the notations of Section 5.1. Moreover, let Ξ denote the lattice in $i\mathfrak{t}^*$ generated by the root system Δ.

Theorem 5.3.15 ([122])**.** *Let π_λ be a discrete series representation with Harish-Chandra parameter λ. If $\mu \in \Xi$ is dominant with respect to Δ_c^+, the irreducible K-module of highest weight μ occurs in the discrete series representation π_λ with multiplicity*

$$\sum_{w \in W} \epsilon(w) Q(w(\mu + \rho_c) - \lambda - \rho_n),$$

where $\epsilon(w)$ is the sign of the Weyl group element $w \in W$, and $Q(\alpha)$ denotes the number of distinct ways in which $\alpha \in \Xi$ can be written as a sum of positive, non-compact roots.

In [122], this result is established only for discrete series representations. However, the same formula holds for limits of discrete series, as shown in [86]. Indeed, for a singular λ choose the dominant integral element $\mu := \rho$ such that $\lambda + \mu$ is non-singular, and such that the limit of discrete series is $\pi_\lambda = \Psi_\lambda^{\lambda+\mu}(\pi_{\lambda+\mu})$ where Ψ denotes the usual Jantzen–Zuckerman translation functor. Let $F_{-\mu}$ be an irreducible finite-dimensional representation of G with lowest weight $-\mu$. Decompose $\pi_{\lambda-\mu}$ and $F_{-\mu}$ under K and take the tensor product $\pi_{\lambda+\mu} \otimes F_{-\mu}$. The K-types of the tensor product are $\lambda' + \nu$ where λ' is a K-type of $\pi_{\lambda+\mu}$ and ν is a weight of $F_{-\mu}$, or more precisely

$$\lambda' = \lambda + \mu + \sum_{\alpha \in \Delta^+} m_\alpha \alpha, \qquad \nu = -\mu + \sum_{\alpha \in \Delta^+} n_\alpha \alpha.$$

The Blattner parameter associated to the Harish-Chandra parameter λ is

$$\Lambda = \lambda - \rho_c + \rho_n = \lambda + \rho_{\Delta(\mathfrak{su}(p,q)^{\mathbb{C}}, \mathfrak{t})} - 2\rho_c.$$

In particular, we have the following corollary.

Corollary 5.3.16. *Let $L(\Lambda)$ be a unitarizable highest weight $\mathfrak{g}$-module. Then the irreducible K-module of τ_μ of highest weight μ occurs in $L(\Lambda)$ with multiplicity*

$$\sum_{w \in W} \epsilon(w) Q(w(\mu + \rho_c) - \Lambda - \rho_c).$$

5.4. Relative holomorphic discrete series representations

Relative holomorphic discrete series representations are generalizations of holomorphic discrete series representations of a semisimple Lie groups G to their simply connected covering group $\widetilde{G}$, which has infinite center. That is, they are unitary highest weight representations whose matrix coefficients are square-integrable modulo the infinite center, satisfying Harish-Chandra's condition on the highest weight. The *Harish-Chandra condition*, in particular, distinguishes them from other unitary irreducible highest weight representations. In this section, we define and characterize relative holomorphic discrete series representations following [101]. In particular, we assign any relative discrete series representation a formal dimension, which will play a central role in our construction of a formal superdimension in Chapter 10.

In this section, $\mathbf{G}$ denotes either a semisimple Lie group G or its universal cover $\widetilde{G}$. Let $\mathrm{d}g$ denote the Haar measure on $\mathbf{G}$ and note that $\mathbf{G}$ is unimodular. We denote the associated Lie algebra by $\mathfrak{g}$. Within $\mathfrak{g}$, we fix a maximal compact subalgebra $\mathfrak{k}$, as introduced in Section 2.1.4. The center in $\mathfrak{g}$ of the center of $\mathfrak{k}$ coincides with $\mathfrak{k}$, meaning that $\mathfrak{g}$ is *quasihermitian*. Moreover, we write $T, \widetilde{T}, \mathbf{T}$ and $K, \widetilde{K}, \mathbf{K}$ for the analytic subgroups corresponding to $\mathfrak{t}$ and $\mathfrak{k}$ in $G, \widetilde{G}$ and $\mathbf{G}$, respectively.

5.4.1. Definition and classification

Let $L(\Lambda)$ be a unitarizable highest weight $\mathfrak{g}$-module, such that $L(\Lambda)$ is a highest weight representation for $\mathbf{G}$. Let $(\pi_\Lambda, \mathcal{H}_\Lambda)$ denote the unique unitary irreducible representation of $\mathbf{G}$ such that its Harish-Chandra module $\mathcal{H}^K$ is isomorphic to $L(\Lambda)$. The representation $(\pi_\Lambda, \mathcal{H}_\Lambda)$ can be realized geometrically. Since the center $\mathbf{Z} \subset \mathbf{G}$ is a normal abelian subgroup, Schur's Lemma implies that each $z \in \mathbf{Z}$ acts on $\mathcal{H}_\Lambda$ as a scalar multiple of the identity. This induces a group homomorphism $\chi_{\pi_\Lambda} : \mathbf{Z} \to \mathbb{C}$, called the *central character* of $(\pi_\Lambda, \mathcal{H}_\Lambda)$, such that $\pi_\Lambda(z)v = \chi_{\pi_\Lambda}(z)v$ for any $z \in \mathbf{Z}$ and $v \in \mathcal{H}_\Lambda$. Let $M := \mathbf{Z}\backslash\mathbf{G}$ be the right $\mathbf{G}$-space of right cosets $\mathbf{Z}g$ for $g \in \mathbf{G}$. Note that $\mathbf{Z}\backslash\mathbf{G} = \mathbf{G}/\mathbf{Z}$, since $\mathbb{Z}$ is central.

We define $E_{\pi_\Lambda} := \mathbf{G} \times_{\mathbf{Z}} \mathbb{C}$ to be the *set of* $\mathbf{Z}$*-orbits*, where $z.(g, z') = (zg, \chi_{\pi_\Lambda}(z)z')$, and the *orbit* of (g, z) by $[g, z] := Z.(g, z)$. On E_{π_Λ}, the group $\mathbf{G}$ acts from the right by $[g', z].g := [g'g, z]$ for $g, g' \in \mathbf{G}$ and $z \in \mathbb{C}$. The projection map $p : E_{\pi_\Lambda} \to M$ given by $[g, z] \mapsto \mathbf{Z}g$ defines a complex line bundle over M. The vector space of continuous sections of E_{π_Λ}, denoted by $\Gamma(M, E_{\pi_\Lambda})$, is a representation of $\mathbf{G}$ with action given by $(g.s)(\mathbf{Z}x) := s(\mathbf{Z}xg).g^{-1}$. This representation is equivalent to the representation

$$\Gamma_{\mathbf{G}}(E_{\pi_\Lambda}) := \{f \in \mathcal{C}(\mathbf{G}, \mathbb{C}) \;:\; f(zg) = \chi_{\pi_\Lambda}(z)f(g) \text{ for all } z \in \mathbf{Z}\},$$

with action $(g \cdot f)(x) = f(xg)$. Indeed, if $f \in \Gamma_{\mathbf{G}}(E_{\pi_\Lambda})$, then $\sigma_f(\mathbf{Z}g) := [g, f(g)]$ defines a continuous section. Conversely, if $s \in \Gamma(M, E_{\pi_\Lambda})$ is a section, there must be a continuous function $f : \mathbf{G} \to \mathbb{C}$ such that $s(\mathbf{Z}g) = [g, f(g)] = [zg, \chi_{\pi_\Lambda}(z)f(g)]$, *i.e.*, $f \in \Gamma_{\mathbf{G}}(E_{\pi_\Lambda})$. The compatibility of the $\mathbf{G}$-actions follows from a straightforward calculation.

To construct a unitary representation, we consider the square-integrable sections. Since the group $\mathbf{G}$ is unimodular, we can fix a $\mathbf{G}$-invariant Haar measure $\mathrm{d}g$ on $\mathbf{G}$. This induces a $\mathbf{G}$-invariant measure $\mathrm{d}\mu_M$ on the quotient space $M := \mathbf{Z}\backslash\mathbf{G}$ [101, Proof of Proposition 2.2]. The space $\Gamma^2_{\mathbf{G}}(E_{\pi_\Lambda})$ of square-integrable elements,

$$\Gamma^2_{\mathbf{G}}(E_{\pi_\Lambda}) := \{f \in \Gamma_{\mathbf{G}}(E_{\pi_\Lambda}) \;:\; \int_M |f(g)|^2 \; \mathrm{d}\mu_M(\mathbf{Z}g) < \infty\},$$

carries a natural Hermitian form,

$$\langle f, h\rangle := \int_M \overline{f(g)}h(g) \; \mathrm{d}\mu_M(\mathbf{Z}g).$$

We continue to denote its Hilbert space completion by the same symbol. The action of $\mathbf{G}$ on $\Gamma^2_{\mathbf{G}}(E_{\pi_\Lambda})$ then defines a unitary representation of $\mathbf{G}$.

To compare $(\pi_\Lambda, \mathcal{H}_\Lambda)$ with $\Gamma^2_{\mathbf{G}}(E\pi_\Lambda)$, we examine the matrix coefficients of π_Λ, which define $\mathbf{G}$-equivariant maps:

$$\Psi : \mathcal{H}_\Lambda \to \Gamma_{\mathbf{G}}(E_{\pi_\Lambda}), \qquad \Psi(w)(g) := \langle w, \pi_\Lambda(g^{-1})v\rangle$$

for some fixed $v \in \mathcal{H}_\Lambda$. Then, $(\pi_\Lambda, \mathcal{H}_\Lambda)$ and $\Gamma^2_{\mathbf{G}}(E_{\pi_\Lambda})$ can be compared if and only if the matrix coefficients are square-integrable, in which case $(\pi_\Lambda, \mathcal{H}_\Lambda)$ can be considered a direct summand in $\Gamma^2_{\mathbf{G}}(E_{\pi_\Lambda})$. The representations that belong to the *relative holomorphic discrete series* are precisely those unitary representations that can be embedded in $\Gamma^2_{\mathbf{G}}(E_{\pi_\Lambda})$.

Definition 5.4.1 ([101, Definition 2.1])**.** A unitary irreducible highest weight representation $(\pi_\Lambda, \mathcal{H}_\Lambda)$ of $\mathbf{G}$ is said to belong to the *relative holomorphic discrete series* if there exist $v, w \in \mathcal{H}_\Lambda \setminus \{0\}$ such that the function defined by

$$\mathbf{Z}\backslash\mathbf{G} \to \mathbb{C}, \qquad \mathbf{Z}g \mapsto |\langle\pi_\Lambda(g)v, w\rangle|$$

is square-integrable. In this case, we also say $(\pi_\Lambda, \mathcal{H}_\Lambda)$ is a *relative holomorphic discrete series representation* of **G**.

Theorem 5.4.2 ([101, Theorem 2.1, Proposition 2.2]). *Let $(\pi_\Lambda, \mathcal{H}_\Lambda)$ be a relative holomorphic discrete series representation of $\boldsymbol{G}$, and let $\Psi(u)(g) := \langle u, \pi(g^{-1})v\rangle$ for a fixed element $0 \neq v \in \mathcal{H}_\Lambda$. Then there exists a constant $d(\pi_\Lambda) > 0$ such that $\sqrt{d(\pi_\Lambda)}\Psi$ is an isometric $\boldsymbol{G}$-equivariant embedding $\mathcal{H}_\Lambda \hookrightarrow \Gamma^2_{\boldsymbol{G}}(E_{\pi_\Lambda})$.*

Harish-Chandra classified in [54] all unitary highest weight representations of **G** that belong to the relative holomorphic discrete series via explicit evaluation of the integral

$$\int_M |\langle \pi_\Lambda(g)v_\Lambda, v_\Lambda\rangle|^2 \, \mathrm{d}\mu_M(\mathbf{Z}g),$$

where v_Λ denotes the highest weight vector of the underlying Harish-Chandra module $\mathcal{H}_\Lambda^K$.

Theorem 5.4.3 ([54, Lemma 27, Lemma 29]). *A unitary highest weight representation of $\boldsymbol{G}$ with highest weight Λ belongs to the relative holomorphic discrete series if and only if, for all $\beta \in \Delta_n^+$, we have*

$$(\Lambda + \rho, \beta) < 0.$$

5.4.2. Formal dimension

The positive constant $d(\pi_\Lambda)$ in the isometric **G**-equivariant embedding of a relative holomorphic discrete series $(\pi_\Lambda, \mathcal{H}_\Lambda)$ in $\Gamma^2_{\mathbf{G}}(E_{\pi_\Lambda})$ (see Theorem 5.4.2) is called the *formal dimension* or *formal degree* of $(\pi_\Lambda, \mathcal{H}_\Lambda)$. By Schur's Lemma, this constant appears in the orthogonality relations, which justifies its name.

Proposition 5.4.4 ([101, Proposition 2.3]). *Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be two relative holomorphic discrete series representations with the same central character χ. Then the* Godement–Harish-Chandra orthogonality relations *hold:*

$$\int_M \langle \pi_1(g)v, w\rangle \overline{\langle \pi_2(g)v', w'\rangle} \, \mathrm{d}\mu_M(\boldsymbol{Z}g) = \begin{cases} \frac{1}{d(\pi_1)}\langle w', w\rangle\langle v, v'\rangle & \text{if } \pi_1 \cong \pi_2, \\ 0 & \text{if } \pi_1 \ncong \pi_2. \end{cases}$$

The formal dimension depends on the normalization of the Haar measure on M. However, once a normalization is fixed, it can be regarded as a function of π_Λ. Evaluating the integral above allows one to determine a natural normalization of the measure.

Theorem 5.4.5 ([101, Theorem 3.17]). *With respect to a suitable normalization of the measure $\mathrm{d}\mu_M$ on M, the formal dimension $d(\pi_\Lambda)$ of a relative holomorphic discrete series representation $(\pi_\Lambda, \mathcal{H}_\Lambda)$ is*

$$d(\pi_\Lambda) = \prod_{\alpha \in \Delta_c^+} \frac{(\Lambda + \rho_c, \alpha)}{(\rho_c, \alpha)} \prod_{\beta \in \Delta_n^+} \frac{|(\Lambda + \rho, \beta)|}{(\rho, \beta)}.$$

Remark 5.4.6. The positive integer $\prod_{\alpha \in \Delta_c^+} \frac{(\Lambda+\rho_c,\alpha)}{(\rho_c,\alpha)}$ is precisely the dimension of the irreducible **K**-representation with highest weight Λ.

5.4.3. Harish-Chandra parameterization

Let $\mathbf{T}$ be the analytic subgroup of $\mathbf{G}$ corresponding to the Cartan subalgebra $\mathfrak{t}$, which is compact if we consider G. However, we have $\mathbf{Z} \subset \mathbf{T}$, and $\mathbf{T}/\mathbf{Z}$ is compact since $\mathrm{Ad}(\mathbf{G})$ of a reductive Lie group is closed. This establishes a relationship between relative holomorphic discrete series representations and square-integrable representations [101, Definition 3.4] of $\mathbf{G}$.

Lemma 5.4.7 ([101, Remark 3.5])**.** *The following assertions are equivalent:*

a) $(\pi_\Lambda, \mathcal{H}_\Lambda)$ *is a relative holomorphic discrete series representation.*

b) $(\pi_\Lambda, \mathcal{H}_\Lambda)$ *is a square-integrable representation,* i.e.*, there exists a non-trivial vector* $v_\Lambda \in \mathcal{H}_\Lambda^{\boldsymbol{K}}$ *such that* $\mathrm{d}\pi_\Lambda(X)v_\Lambda = 0$ *for any* $X \in \mathfrak{n}^+$*, and the function* $g\boldsymbol{T} \mapsto |\langle \pi(g)v_\Lambda, v_\Lambda\rangle|$ *belongs to* $L^2(\boldsymbol{G}/\boldsymbol{T})$*.*

Remark 5.4.8. If we focus explicitly on G and its universal cover $p : \tilde{G} \to G$, the following equivalence holds:

a) $(\pi_\Lambda, \mathcal{H}_\Lambda)$ is a square-integrable representation of G.

b) $(\pi_\Lambda \circ p, \mathcal{H}_\Lambda)$ is a square-integrable representation of $\tilde{G}$.

The lemma allows us to parametrize the relative holomorphic discrete series in terms of the character group $X^*(\mathbf{T}^\mathbb{C})$ of the complexification $\mathbf{T}^\mathbb{C}$ of $\mathbf{T}$ and Weyl chambers. The coset $X^*(\mathbf{T}^\mathbb{C}) + \rho$ of $X^*(\mathbf{T}^\mathbb{C}) \otimes \mathbb{R}$ is independent of the choice of positive roots. We call an element λ in $X^*(\mathbf{T}^\mathbb{C}) \otimes \mathbb{R}$ *regular* if no dual root is orthogonal to λ; otherwise, λ is called *singular*. Let $(X^*(\mathbf{T}^\mathbb{C}) \otimes \mathbb{R})^{reg}$ denote the set of regular elements in $X^*(\mathbf{T}^\mathbb{C}) \otimes \mathbb{R}$, and set $(X^*(\mathbf{T}^\mathbb{C}) + \rho)^{reg} := (X^*(\mathbf{T}^\mathbb{C}) + \rho) \cap (X^*(\mathbf{T}^\mathbb{C}) \otimes \mathbb{R})^{reg}$. A *Weyl chamber* of $\mathbf{G}$ is a connected component of $(X^*(\mathbf{T}^\mathbb{C}) \otimes \mathbb{R})^{reg}$. The Weyl chambers are in one-to-one correspondence with the systems of positive roots for $(\mathfrak{g}, \mathfrak{t})$. We define a Weyl chamber C to be *holomorphic* if for any λ in the interior of C we have $(\lambda, \alpha) < 0$ for all $\alpha \in \Delta_n^+$. There are exactly $|W_c|$ holomorphic Weyl chambers, forming a single orbit for the action of W_c. A *holomorphic Harish-Chandra parameter* is then a pair $(\lambda - \rho, C)$, where C is a holomorphic Weyl chamber and $\lambda \in (X^*(\mathbf{T}^\mathbb{C}) + \rho)^{reg} \cap C$, *i.e.*, there exists a bijection between the set of isomorphism classes of relative holomorphic discrete series representations and $(X^*(\mathbf{T}^\mathbb{C}) + \rho)^{reg}/W_c$, where the Weyl group W_c acts naturally.

If we fix the standard positive system for $\mathrm{SU}(p, q)$, and consequently the standard Weyl chamber C_{st}, any holomorphic Harish-Chandra parameter Λ satisfies

$$\lambda_p \leq \lambda_{p-1} \leq \cdots \leq \lambda_1 \leq \lambda_{p+q} \leq \cdots \leq \lambda_{p+1}.$$

Limit of relative holomorphic discrete series

A unitary highest weight representation $(\pi_\Lambda, \mathcal{H}_\Lambda)$ belongs to the relative holomorphic discrete series if $(\Lambda + \rho, \beta) < 0$ for any $\beta \in \Delta_n^+$, according to Harish-Chandra's condition. Broadly speaking, $(\pi_\Lambda, \mathcal{H}_\Lambda)$ belongs to the limit of the relative holomorphic discrete series if $(\Lambda + \rho, \beta) = 0$ for some $\beta \in \Delta_n^+$. Unfortunately, unlike the case of the relative holomorphic discrete series, there is no simple intrinsic definition for limits of relative holomorphic discrete series. Explicit constructions are provided by Harish-Chandra's character formula and via Zuckerman–Jantzen tensoring as above.

An *irregular holomorphic Harish-Chandra parameter* is a pair $(\lambda - \rho, C)$, where C is a holomorphic Weyl chamber and $\lambda \in (X^*(\mathbf{T}^{\mathbb{C}}) + \rho) \cap C$, such that λ is not regular and not orthogonal to any simple compact dual root. A unitary highest weight representation $(\pi_\Lambda, \mathcal{H}_\Lambda)$ of $\mathbf{G}$ belongs to the limit of the relative holomorphic discrete series if there exists a holomorphic Weyl chamber C such that $(\lambda - \rho, C)$ is an irregular holomorphic Harish-Chandra parameter. Consequently, there is a bijection between the set of isomorphism classes of limits of relative holomorphic discrete series representations and the W_c-orbits of irregular Harish-Chandra parameters.

5.4.4. L-packets and formal dimension

Up to this point, we have discussed the relative holomorphic discrete series of $\mathbf{G}$, which denotes either G or its simply connected cover $\tilde{G}$. If we consider solely G, then the relative holomorphic discrete series are exactly the *holomorphic discrete series representations* of G (see Section 5.3.1). In the following, we are solely interested in holomorphic discrete series representations of G, or rather their formal dimension. Concretely, we give a character interpretation of the formal dimension.

As in the case of finite-dimensional representations of G, where the Harish-Chandra character reduces to the Weyl character, it is natural to expect that taking the limit $g \to e_G$ of the Harish-Chandra character recovers the formal dimension. However, if one attempts to take the limit $g \to e_G$ in Harish-Chandra's character formula for π_Λ, the formal degree is not obtained, as the Harish-Chandra character is not continuous at e_G. This issue is resolved by considering the *L-packet* associated with π_Λ. The L-packet of π_Λ is the collection of (isomorphism classes of) all discrete series representations with the same infinitesimal character. By Section 5.4.3, the L-packet associated with π_Λ is

$$\tilde{\Theta} = \sum_{w \in W/W_c} \Theta_{w\Lambda}.$$

Proposition 5.4.9 ([60]). *The following assertion holds:*

$$d(\pi_\Lambda) = \lim_{g \to e_G,\ g \in T \cap G_{reg}} \tilde{\Theta}(g).$$

Moreover, $d(\pi_\Lambda) = d(\pi_{w\lambda})$ *for all* $w \in W/W_c$.

6. Unitarity for Lie superalgebras and Lie supergroups

This chapter introduces unitarizable supermodules over Lie superalgebras in Section 6.1 and unitary representations of Lie supergroups in Section 6.2. Specifically, we focus on unitarizable supermodules over $\mathfrak{sl}(m|n)$ and Lie superalgebras of type $A(m|n)$. We explore the implications of unitarity, provide a parameterization of the simple unitarizable supermodules, and establish that they are Harish-Chandra supermodules. Furthermore, we construct a contravariant, non-degenerate form on simple highest weight supermodules, known as the Shapovalov form. A central question is whether this form is positive definite, which is necessary to establish unitarity.

Our main result concerning unitary representations of Lie supergroups $\mathcal{G} = (G, \mathfrak{g})$ is that, when $\mathfrak{g}$ is of type $A(m|n)$ or $\mathfrak{sl}(m|n)$, these representations are completely determined by the unitarizable highest weight supermodules over $\mathfrak{g}$.

6.1. Unitarizable supermodules over Lie superalgebras

6.1.1. Basic definitions

To formalize the notion of unitarizable supermodules, we first introduce the concept of a *pre-super Hilbert space* which provides the foundational structure on which unitarizable supermodules are built. We follow the conventions outlined in [27, Part 1]. Our exposition is based on [73, 123, 126].

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a complex super vector space. A *super Hermitian form* ψ on V is a complex-valued sesquilinear form $\psi : V \times V \to \mathbb{C}$, which is linear in the first variable and conjugate-linear in the second, such that for all $v, w \in V$, the following holds:

$$\psi(v, w) = \begin{cases} (-1)^{p(v)p(w)}\overline{\psi(w, v)} & \text{if } p(v) = p(w), \\ 0 & \text{if } p(v) \neq p(w). \end{cases}$$

Here, $\overline{\cdot}$ denotes complex conjugation in $\mathbb{C}$. The pair (V, ψ) is referred to as a *Hermitian superspace*. A super Hermitian form ψ naturally decomposes as $\psi = \psi_{\bar{0}} + i\psi_{\bar{1}}$, where

$$\psi_{\bar{z}} := (-1)^{p(z)} \psi|_{V_{\bar{z}} \times V_{\bar{z}}}, \quad \bar{z} \in \mathbb{Z}_2,$$

such that $\psi_{\bar{0}}$ and $\psi_{\bar{1}}$ are Hermitian forms on the complex vector spaces $V_{\bar{0}}$ and $V_{\bar{1}}$, respectively. The super Hermitian form ψ is called *non-degenerate* if $\psi_{\bar{0}}$ and $\psi_{\bar{1}}$ are non-degenerate. Furthermore, ψ is called *super positive definite* if $\psi_{\bar{0}}$ is positive definite and $\psi_{\bar{1}}$ is negative definite, *i.e.*, ψ is positive definite on $V_{\bar{0}}$, and $-i$-times positive definite on $V_{\bar{1}}$. A super positive definite Hermitian form ψ on a complex super vector space V is called a *Hermitian product*.

For a Hermitian superspace (V, ψ) the amendment

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}, \qquad \langle v, w \rangle := \begin{cases} -i\psi(v,w) & \text{if } v, w \in V_{\bar{1}}, \\ \psi(v,w) & \text{if } v, w \in V_{\bar{0}}, \\ 0 & \text{else} \end{cases}$$

defines a Hermitian form on V, which is positive definite if ψ is super positive definite. Conversely, if $\langle \cdot, \cdot \rangle$ is a (positive definite) Hermitian form on V such that $\langle V_{\bar{0}}, V_{\bar{1}} \rangle = 0$, the amendment above associates a (super positive definite) super Hermitian form ψ to $\langle \cdot, \cdot \rangle$. We conclude with the following lemma.

Lemma 6.1.1. *For a complex super vector space V, the following two assertions are equivalent:*

a) V carries a super positive definite super Hermitian form ψ.

b) V is a $\mathbb{Z}_2$-graded complex pre-Hilbert space with a positive definite Hermitian form $\langle \cdot, \cdot \rangle$ such that $V_{\bar{0}}$ and $V_{\bar{1}}$ are mutually orthogonal subspaces of V.

This leads to the definition of a *super pre-Hilbert space*.

Definition 6.1.2. A complex super vector space $\mathcal{H}$ is called a *super pre-Hilbert space* if $\mathcal{H}$ satisfies either of the two equivalent properties of Lemma 6.1.1. In this case, write $\mathcal{H} = (\mathcal{H}, \psi, \langle \cdot, \cdot \rangle)$.

Fix a super pre-Hilbert space $\mathcal{H} = (\mathcal{H}, \psi, \langle \cdot, \cdot \rangle)$. On $\mathcal{H}$, any endomorphism T has two adjoints: one relative to ψ and one relative to $\langle \cdot, \cdot \rangle$. We define the adjoint T^* of T with respect to ψ by

$$\psi(Tv, w) = (-1)^{p(T)p(v)} \psi(v, T^* w)$$

for all $v, w \in \mathcal{H}$, and the adjoint $T^\dagger$ with respect to $\langle \cdot, \cdot \rangle$ by

$$\langle Tv, w \rangle = \langle v, T^\dagger w \rangle.$$

These are related by $T^* = T^\dagger$ when T is even and $T^* = iT^\dagger$ when T is odd. We call T^* the *super-adjoint* of T and $T^\dagger$ the *adjoint* of T. The adjoint $(\cdot)^\dagger$ satisfies $(ST)^\dagger = T^\dagger S^\dagger$, independent of the parity of $S, T \in \operatorname{End}(\mathcal{H})$. The superadjoint $(\cdot)^*$ satisfies $(ST)^* = (-1)^{p(S)p(T)} T^* S^*$. Namely, $(\cdot)^\dagger$ defines a conjugate-linear anti-involution on $\operatorname{End}(\mathcal{H})$ while $(\cdot)^*$ defines a conjugate-linear *graded* anti-involution. In what follows, we denote the conjugate-linear graded anti-involution $(\cdot)^*$ on $\operatorname{End}(\mathcal{H})$ also by σ.

The concept of unitarity for supermodules over $\mathfrak{g}$ is defined relative to real forms $\mathfrak{g}^{\mathbb{R}}$ of $\mathfrak{g}$, *i.e.*, fixed point sets of conjugate-linear involutions, as established in Section 2.1.4.

Definition 6.1.3. Let $\omega \in \overline{\operatorname{aut}}_{2,2}(\mathfrak{g})$ be a conjugate-linear involution of $\mathfrak{g}$, and let (M, ρ) be a $\mathfrak{g}$-supermodule. Then M is called ω-*unitarizable* if M is a super pre-Hilbert space $(M, \psi, \langle \cdot, \cdot \rangle)$ such that one of the following two equivalent conditions hold:

a) $\rho \circ \omega = \sigma \circ \rho$, that is, $\psi(\rho(X)v, w) = -(-1)^{p(X)p(v)} \psi(v, \rho(\omega(X))w)$ for all homogeneous $v, w \in M$ and $X \in \mathfrak{g}$.

b) $\langle \rho(X)v, w \rangle = \langle v, -i^{p(X)} \rho(\omega(X))w \rangle$ for all homogeneous $v, w \in M$ and $X \in \mathfrak{g}$.

If a) or b) is satisfied, we say that ψ or $\langle\cdot,\cdot\rangle$ is *contravariant.*

Remark 6.1.4. For a ω-unitarizable $\mathfrak{g}$-supermodule (M,ρ), the following holds:

$$\rho(X)^* = \begin{cases} -\rho(\omega(X)) & p(X)=0, \\ +\rho(\omega(X)) & p(X)=1, \end{cases} \qquad \rho(X)^\dagger = \begin{cases} -\rho(\omega(X)) & p(X)=0, \\ -i\rho(\omega(X)) & p(X)=1. \end{cases}$$

Consequently, (M,ρ) is unitarizable if and only if M is a super pre-Hilbert space and the following two conditions are satisfied:

a) For all $X\in\mathfrak{g}_{\bar{0}}$, the operator $i\rho(\omega(X))$ is symmetric on M.

b) For all $X\in\mathfrak{g}_{\bar{1}}$, the operator $e^{-i\frac{\pi}{4}}\rho(\omega(X))$ is symmetric on M.

Remark 6.1.5. We can use conjugate-linear anti-involutions instead of conjugate-linear involutions. Indeed, to any $\omega\in\overline{\mathrm{aut}}_{2,2}(\mathfrak{g})$, we can assign a conjugate-linear anti-involution defined by $\omega'(\cdot):=-i^{p(\cdot)}\omega(\cdot)$. The following two assertions are then equivalent:

a) $(M,\rho,\psi,\langle\cdot,\cdot\rangle)$ is ω-unitarizable.

b) $(M,\rho,\langle\cdot,\cdot\rangle)$ is a super pre-Hilbert space such that

$$\langle\rho(X)v,w\rangle = \langle v,\rho(\omega'(X))w\rangle.$$

In this thesis, we use the definition with respect to conjugate-linear involutions or conjugate-linear anti-involutions interchangeably.

For clarity, the definition is formulated in terms of the $\mathbb{Z}_2$-graded representation $\rho:\mathfrak{g}\to\mathrm{End}(M)$. In what follows, however, we will continue to work with $\mathfrak{g}$-supermodules suppressing the explicit Lie superalgebra homomorphism ρ. We will also think of the data as defining a "unitary representation" of the real Lie superalgebra $\mathfrak{g}^\omega\to\mathfrak{gl}(\mathcal{H})^\sigma$, although we will not be concerned with integrability to Lie supergroups.

In the context of $\mathfrak{U}(\mathfrak{g})$-supermodules, we extend ω to $\mathfrak{U}(\mathfrak{g})$ in the obvious way. In this context, a $\mathfrak{g}$-supermodule $\mathcal{H}$ is unitarizable if and only if it is a Hermitian representation of the supermodule $\mathcal{H}$ over $(\mathfrak{U}(\mathfrak{g}),\omega)$, meaning that $\langle Xv,w\rangle=\langle v,\omega(X)w\rangle$ holds for all $v,w\in\mathcal{H}$ and $X\in\mathfrak{U}(\mathfrak{g})$. When ω is understood from context, we just say "unitarizable".

Definition 6.1.6. Two unitarizable $\mathfrak{g}$-supermodules $\mathcal{H}_1$ and $\mathcal{H}_2$ are called *equivalent* if there exists an isomorphism $f:\mathcal{H}_1\to\mathcal{H}_2$ in **$\mathfrak{g}$-smod** that is compatible with the (super) Hermitian forms.

Note that in the definition of equivalence of unitarizable $\mathfrak{g}$-supermodules, intertwining operators are assumed to preserve the grading. This means that a unitarizable $\mathfrak{g}$-supermodule is not equivalent to its parity-changed counterpart.

Example 6.1.7. We consider the symmetry algebra of superconformal quantum mechanics [47, 36], that is $\mathfrak{su}(1,1|1)$. A fundamental example of such a unitarizable supermodule over $\mathfrak{su}(1,1|1)$ is the *oscillator supermodule*, $(\mathfrak{O},\langle\cdot,\cdot\rangle)$, also known as "singleton supermodule". Its construction depends on the isomorphism $\mathfrak{spo}(2|2)\cong\mathfrak{sl}(2|1)$. Algebraically, $\mathfrak{O}$ is the natural simple supermodule over the Weyl-Clifford algebra $\mathrm{WCl}(W)$ attached to a $(2|2)$-dimensional supersymplectic vector space W. Specifically, let $V=\mathbb{C}^{1|1}$, and $W=V\oplus V^*$.

The formula $\Omega(a,b) := f(w) - (-1)^{ab} g(v)$ for all homogeneous $a = (v,f)$ and $b = (w,g)$ in W defines an even supersymplectic form $\Omega : W \times W \to \mathbb{C}$. This is anti-symmetric on the even, and symmetric on the odd part of W. The natural action of W on the supersymmetric algebra $\mathrm{S}(V)$, defined via $a = (v,f) \mapsto v \cdot + \iota_f \in \mathrm{End}(\mathrm{S}(V))$, satisfies $[a,b] = \Omega(a,b)$ with respect to the canonical superbracket on $\mathrm{End}(\mathrm{S}(V))$. The image of W in $\mathrm{End}(\mathrm{S}(V))$ generates the Weyl-Clifford algebra $\mathrm{WCl}(W)$, and it is immediate to see that $\mathrm{S}(V)$ is a simple module over it. The oscillator supermodule $\mathfrak{O}$ is $\mathrm{S}(V)$ equipped with the restriction of $\mathrm{WCl}(W)$ to the orthosymplectic Lie superalgebra $\mathfrak{spo}(2|2) \cong \mathrm{S}^2(W) \subset \mathrm{End}(\mathrm{S}(V))$. With respect to a homogeneous basis (x,η) of $V = \mathbb{C}^{1|1}$, we can think of $\mathfrak{sl}(2|1)$ as the Lie superalgebra spanned by the following set of super-differential operators acting on $\mathbb{C}[x,\eta]$

$$E = -\tfrac{1}{2}\partial_x^2,\ H = -x\partial_x - \tfrac{1}{2},\ F = \tfrac{1}{2}x^2,\ J = \eta\partial_\eta - \tfrac{1}{2}$$
$$Q = \eta x,\ \bar{Q} = \partial_\eta x,\ S = -\eta\partial_x,\ \bar{S} = \partial_\eta\partial_x$$

which one easily verifies satisfy the above superbrackets.

As an $\mathfrak{sl}(2|1)$ supermodule, $\mathfrak{O}$ decomposes into two simple supermodules $\mathfrak{O} = \mathfrak{O}^+ \oplus \mathfrak{O}^-$, according to the total polynomial degree,

$$\mathfrak{O}^+ = \mathbb{C}[x^2, x\eta], \qquad \mathfrak{O}^- = \eta\mathbb{C}[x^2] + x\mathbb{C}[x^2]$$

Both are unitarizable highest weight modules. The Hermitian form on the supermodule $\mathfrak{O} \cong \mathbb{C}[x,\eta]$ is the sesquilinear extension of the formulas

$$\langle x^n, x^m\rangle = \begin{cases} n! & \text{if } n = m \\ 0 & \text{otherwise} \end{cases} \qquad \langle \eta,\eta\rangle = 1\,, \qquad \langle x^n, \eta\rangle = 0.$$

6.1.2. Implications of unitarity

We record some basic properties of ω-unitarizable $\mathfrak{g}$-supermodules, especially if $\mathfrak{g}$ is $\mathfrak{sl}(m|n)$ or of type $A(m|n)$. For the real forms of $\mathfrak{sl}(m|n)$ and $A(m|n)$, we refer to Section 2.1.4. The first result holds for any Lie superalgebra and follows by a standard argument.

Proposition 6.1.8. *Let $\mathcal{H}$ be an ω-unitarizable $\mathfrak{g}$-supermodule. Then*

a) $\mathcal{H}$ is completely reducible; that is, for any invariant subspace, its orthogonal complement is also an invariant subspace.

b) $\mathcal{H}$ is completely reducible as a $\mathfrak{g}_{\bar{0}}$-supermodule.

c) $\mathcal{H}_{\mathrm{ev}}$ (which we recall is the $\mathfrak{g}_{\bar{0}}$-module obtained by restriction and forgetting the $\mathbb{Z}_2$ grading) is a unitarizable $\mathfrak{g}_{\bar{0}}$-module with respect to the real form $\mathfrak{g}^\omega$. In particular, $\mathcal{H}_{\mathrm{ev}}$ is completely reducible as a unitarizable $\mathfrak{g}_{\bar{0}}$-module.

Remark 6.1.9. Complete reducibility of unitarizable supermodules is an important property that even finite-dimensional supermodules over $\mathfrak{g}$ do not necessarily share. Indeed, by the classical Weyl Theorem, all finite-dimensional representations of a complex semisimple Lie algebra are completely reducible; however, this is no longer true for basic classical Lie superalgebras by the Djoković–Hochschild Theorem (see [61, 107] and [121, Page 239]), which states that the only Lie superalgebras for which all finite-dimensional representations are completely reducible are the direct products of $\mathfrak{osp}(1|2n)$ superalgebras and semisimple Lie algebras. Hence, unitarity is a natural criterion for complete reducibility and for studying the category $\mathfrak{g}$-**smod**.

Moreover, unitarizable supermodules over basic classical Lie superalgebras are rare. For Lie superalgebras of type A, B, C, and D, we have the following result, combining [104, Theorem 6.2.1] with the classification of real forms in Table 2.1. The result for orthosymplectic Lie superalgebras is also due to Nishiyama in [49].

Theorem 6.1.10. *a) The special linear Lie algebra $\mathfrak{sl}(m|n)$ for $m \neq n$ and $\mathfrak{psl}(n|n)$ for $m = n$ has no non-trivial unitarizable supermodules unless the real form is either $\mathfrak{su}(p,q|n)$ or $\mathfrak{su}(m|r,s)$ and $\mathfrak{psu}(p,q|n)$ or $\mathfrak{psu}(m|r,s)$ for $p+q=m$ and $r+s=n$, respectively.*

b) The orthosymplectic Lie superalgebra $\mathfrak{osp}(m|2n;\mathbb{C})$ has no non-trivial unitarizable supermodules unless the real form is precisely $\mathfrak{osp}(m|2n,\mathbb{R})$.

In the remainder of this section, we discuss unitarizable supermodules over $\mathfrak{g}$, where $\mathfrak{g}$ denotes either $\mathfrak{sl}(m|n)$ or a basic classical Lie superalgebra of type $A(m|n)$. From this point forward, unless otherwise stated, we assume $m \leq n$ and $p, q \neq 0$. All results for the finite-dimensional case can then be derived by setting $p = 0$ or $q = 0$.

The unitarity condition imposes a specific relation on the weights of a unitarizable supermodule (as discussed below), implying that all unitarizable supermodules over $\mathfrak{g}$ are either of highest or lowest weight type.

Theorem 6.1.11 ([48, 104])**.** *The special linear Lie superalgebra $\mathfrak{g} = \mathfrak{sl}(m|n)$ admits non-trivial ω-unitarizable supermodules if and only if the anti-linear anti-involution ω corresponds to the real forms $\mathfrak{su}(p,q|n,0)$ or $\mathfrak{su}(p,q|0,n)$, where $p+q=m$. In particular, any ω-unitarizable simple $\mathfrak{g}$-supermodule must be either a highest or lowest weight module.*

Without loss of generality, we focus exclusively on ω-unitarizable *highest* weight $\mathfrak{g}$-supermodules, setting $\omega := \omega_{(-,+)}$. The study of unitarizable lowest weight $\mathfrak{g}$-supermodules is analogous, though it may require swapping $\omega_{(-,+)}$ with $\omega_{(+,-)}$ when $p, q \neq 0$.

Lemma 6.1.12. *If $p, q \neq 0$ the following assertions hold:*

a) There are no non-trivial $\omega_{(+,-)}$-unitarizable highest weight $\mathfrak{g}$-supermodules with respect to either $\mathfrak{n}^+_{\bar{1},st}$ or $\mathfrak{n}^+_{\bar{1},nst}$.

b) There are no non-trivial $\omega_{(-,+)}$-unitarizable lowest weight $\mathfrak{g}$-supermodules with respect to either $\mathfrak{n}^-_{\bar{1},st}$ or $\mathfrak{n}^-_{\bar{1},nst}$.

Proof. We provide a brief proof of assertion a); the remaining cases follow analogously. Assume $\mathcal{H}$ is a unitarizable simple highest weight $\mathfrak{g}$-supermodule with highest weight $\Lambda = (\lambda_1, \ldots, \lambda_m | \lambda'_1, \ldots, \lambda'_n) \in \mathfrak{h}^*$. By Proposition 6.1.8, $\mathcal{H}$ decomposes into a finite direct sum of unitarizable simple highest weight $\mathfrak{g}_{\bar{0}}$-modules. Moreover, the highest weight vector of $\mathcal{H}$ generates a unitarizable simple highest weight $\mathfrak{g}_{\bar{0}}$-module $L_0(\Lambda)$ with highest weight Λ. Since $L_0(\Lambda)$ is unitarizable, the highest weight must satisfy the conditions from [38, 72]:

$$\lambda_p \leq \ldots \leq \lambda_1 \leq \lambda_m \leq \ldots \leq \lambda_{p+1}, \qquad \lambda'_1 \geq \ldots \geq \lambda'_n.$$

For the standard positive system, fix two odd positive roots $\alpha, \beta \in \Delta^+_{\bar{1}}$, say $\alpha = \epsilon_i - \delta_j$ and $\beta = \epsilon_k - \delta_l$ for $1 \leq i \leq p < k \leq m$ and $1 \leq j, l \leq n$. The root spaces are $\mathfrak{g}^\alpha = \mathbb{C}E_{ij}$ and $\mathfrak{g}^\beta = \mathbb{C}E_{kl}$; we denote the associated basis by $X_\alpha := E_{ij} \in \mathfrak{g}^\alpha$ and $X_\beta := E_{kl} \in \mathfrak{g}^\beta$, respectively, with dual basis $X_{-\alpha} = E_{ji} \in \mathfrak{g}^{-\alpha} = \mathbb{C}E_{ji}$ and $X_{-\beta} = E_{lk} \in \mathfrak{g}^{-\beta} = \mathbb{C}E_{lk}$. Let v_Λ be the highest weight vector of $\mathcal{H}$ (or $L_0(\Lambda)$), and let $\langle \cdot, \cdot \rangle$ denote the Hermitian form. The relations

$$\langle X_{-\alpha}v_\Lambda, X_{-\alpha}v_\Lambda\rangle = \langle \omega_{(+,-)}(X_{-\alpha})X_{-\alpha}v_\Lambda\rangle = \Lambda([X_\alpha, X_{-\alpha}])\langle v_\Lambda, v_\Lambda\rangle,$$
$$\langle X_{-\beta}v_\Lambda, X_{-\beta}v_\Lambda\rangle = \langle \omega_{(+,-)}(X_{-\beta})X_{-\beta}v_\Lambda\rangle = -\Lambda([X_\beta, X_{-\beta}])\langle v_\Lambda, v_\Lambda\rangle,$$

imply that

$$0 \leq \Lambda([X_\alpha, X_{-\alpha}]) = \lambda_i + \lambda'_j, \qquad 0 \leq -\Lambda([X_\beta, X_{-\beta}]) = -\lambda_k - \lambda'_l.$$

We conclude $\lambda_k \leq \lambda_i$ for $i \leq p < k$, which leads to a contradiction.

For the non-standard positive system, we consider $\alpha = \epsilon_i - \delta_j$ and $\beta = -\epsilon_k + \delta_l$ with $1 \leq i \leq p < k \leq m$ and $1 \leq j, l \leq n$. Using the relation $[X_\beta, X_{-\beta}] = [X_{-\beta}, X_\beta]$, the statement follows by the same reasoning as in the standard system. □

The proof of lemma 6.1.12 yields the subsequent lemma.

Lemma 6.1.13. *The highest weight Λ of a unitarizable simple highest weight $\mathfrak{g}$-supermodule satisfies the following unitarity relations with respect to $\mathfrak{n}^+_{\bar{1},st}$ and $\mathfrak{n}^+_{\bar{1},nst}$:*

$$\lambda_{p+1} \geq \ldots \geq \lambda_m \geq -\lambda'_n \geq \ldots \geq -\lambda'_1 \geq \lambda_1 \geq \ldots \geq \lambda_p.$$

Proof. It is sufficient to focus on $\mathfrak{n}^+_{\bar{1},\mathrm{nst}}$. The root space for $\alpha = -\epsilon_i + \delta_j$, where $p+1 \leq i \leq m$ and $1 \leq j \leq n$, is given by $\mathfrak{g}^\alpha = \mathbb{C}E_{ji}$. Therefore, if we set $X = E_{ji} \in \mathfrak{g}^{-\alpha}$, its dual root vector is $Y = E_{ji} \in \mathfrak{g}^{-\alpha}$. We then conclude as in the proof of Lemma 6.1.12, by observing that $[X, Y] = [Y, X] = E_{ii} + E_{jj}$. □

Thus, we restrict our analysis to $\mathfrak{su}(p, q|0, n)$, which is associated with $\omega_{(-,+)}$ and the non-standard positive system (by Lemma 6.1.14), and we write $\mathfrak{su}(p, q|n)$ in place of $\mathfrak{su}(p, q|0, n)$. We implicitly regard $\mathfrak{g}$ as the complexification of $\mathfrak{su}(p, q|n)$ and denote the system of odd positive roots by $\Delta^+_{\bar{1}}$ and the associated Weyl element by ρ.

Odd reflections naturally relate both systems under consideration, *i.e.*, Δ^+_{st} and Δ^+_{nst}, as described in Section 2.1.3. Importantly, the effect of changing the positive system on highest weight supermodules is captured by the following lemma.

Lemma 6.1.14 ([17, Lemma 1.40])**.** *Let $L(\Lambda; \Delta^+)$ be a simple highest weight $\mathfrak{g}$-supermodule with highest weight Λ and highest weight vector v_Λ with respect to a positive system Δ^+. Let θ be an odd simple root.*

a) If $(\Lambda, \theta) = 0$, then $L(\Lambda; \Delta^+) = L(\Lambda; \Delta^+_\theta)$.

b) If $(\Lambda, \theta) \neq 0$, then $L(\Lambda, \Delta^+) = \Pi L(\Lambda - \theta; \Delta^+_\theta)$.

6.1.3. Shapovalov form

The basic point, which is familiar in physics since the early days of quantum mechanics [43], is that the Verma supermodule $M(\Lambda)$ (or Kac supermodule $K(\Lambda)$) has a unique (up to a real scalar) contravariant Hermitian form, the *Shapovalov* form (or *Kac–Shapovalov form*). This form induces a non-degenerate form on $L(\Lambda)$. The form on $L(\Lambda)$ is generally not positive definite, but if it is, then $L(\Lambda)$ is unitarizable. Namely, the classification of simple unitarizable highest weight $\mathfrak{g}$-supermodules reduces to the determination of all $\Lambda \in \mathfrak{h}^*$ with the property that the Shapovalov form on $L(\Lambda)$ is positive definite.

Verma supermodules

We begin by considering Verma supermodules. All constructions and results below remain valid for Kac supermodules $K(\lambda)$. Since our focus is on unitarizable supermodules, we fix some $\omega \in \overline{\mathrm{aut}}_{2,2}(\mathfrak{g})$. A Verma supermodule carries a natural contravariant Hermitian form with respect to ω, if the highest weight $\Lambda \in \mathfrak{h}^*$ satisfies $\overline{\Lambda(\omega(\cdot))} = \Lambda(\cdot)$. An element $\Lambda \in \mathfrak{h}^*$ that meets this condition is called *symmetric*. Notably, all highest weights of unitarizable highest weight $\mathfrak{g}$-supermodules are symmetric.

Let $\chi_\Lambda : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$ be the central character of $M(\Lambda)$. We define, with respect to χ_Λ, a map:

$$\widetilde{\chi}_\Lambda : \mathfrak{U}(\mathfrak{g}) \to \mathbb{C}, \qquad X \mapsto \chi_\Lambda(\mathrm{pr}(X)),$$

where $\mathrm{pr} : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{h})$ is the *Harish-Chandra projection* on the first summand of the direct decomposition

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{h}) \oplus (\mathfrak{n}^- \mathfrak{U}(\mathfrak{g}) \oplus \mathfrak{U}(\mathfrak{g})\mathfrak{n}^+).$$

This decomposition is stable under ω, naturally extended to $\mathfrak{U}(\mathfrak{g})$. Moreover, as $\overline{\Lambda(H)} = \Lambda(\omega(H))$ for any $H \in \mathfrak{U}(\mathfrak{h})$, we obtain a Hermitian symmetric form

$$(X, Y)_\Lambda := \widetilde{\chi}_\Lambda(X\omega(Y))$$

for all $X, Y \in \mathfrak{U}(\mathfrak{g})$, which is, in particular, contravariant, *i.e.*, $(ZX, Y)_\Lambda = (X, \omega(Z)Y)_\Lambda$ for any $X, Y, Z \in \mathfrak{U}(\mathfrak{g})$. As $M(\Lambda) = \mathfrak{U}(\mathfrak{g})[1 \otimes 1]$, the form $(\cdot, \cdot)_\Lambda$ induces a well-defined contravariant Hermitian form on $M(\Lambda)$, known as the *Shapovalov form*, denoted by $\langle \cdot, \cdot \rangle_{M(\Lambda)}$.

Proposition 6.1.15. *If $\Lambda \in \mathfrak{h}^*$ is symmetric, then there exists a unique contravariant Hermitian form on $M(\Lambda)$ satisfying $\langle [1 \otimes 1], [1 \otimes 1] \rangle_{M(\Lambda)} = 1$ and $\langle M(\Lambda)^\mu, M(\Lambda)^\nu \rangle = 0$ whenever $\mu \neq \nu$. All other contravariant Hermitian forms are real multiples of this form.*

Proof. It remains to prove uniqueness. Thus, it suffices to show that any contravariant Hermitian form $\langle \cdot, \cdot \rangle$ on $M(\Lambda)$, for which $\langle [1 \otimes 1], [1 \otimes 1] \rangle = 0$, must vanish. To see this, consider the following calculation:

$$\begin{aligned}\langle M(\Lambda), M(\Lambda) \rangle &= \langle \mathfrak{U}(\mathfrak{g})[1 \otimes 1], \mathfrak{U}(\mathfrak{g})[1 \otimes 1] \rangle = \langle [1 \otimes 1], \mathfrak{U}(\mathfrak{g})[1 \otimes 1] \rangle \\ &= \langle [1 \otimes 1], \mathfrak{U}(\mathfrak{b}^\omega)[1 \otimes 1] \rangle = \langle \mathfrak{U}(\mathfrak{b})[1 \otimes 1], [1 \otimes 1] \rangle \\ &\subset \mathbb{C}\langle [1 \otimes 1], [1 \otimes 1] \rangle = 0.\end{aligned}$$

□

By the construction of the Shapovalov form, weight spaces of different weights with respect to $\mathfrak{h}$ are orthogonal. Therefore, we can conclude the following corollary.

Corollary 6.1.16. *If Λ is symmetric, the maximal proper subsupermodule of $M(\Lambda)$ coincides with the radical R of the Shapovalov form $\langle \cdot, \cdot \rangle_{M(\Lambda)}$.*

Thus, a Verma (Kac) supermodule is simple if and only if the radical of $\langle \cdot, \cdot \rangle$ is trivial. The size of the radical is related to the degree of atypicality of $M(\Lambda)$ ($K(\Lambda)$) by construction of the Shapovalov form. This is essentially a version of Kac's criterion (*cf.* [16, Theorem 4.12]), according to which a highest weight Verma (Kac) supermodule $M(\Lambda)$ ($K(\Lambda)$) is simple if and only if Λ is typical. Concerning Kac supermodules, the essence of Kac induction is that it elucidates the structure of the radical in terms of the constituents in the decomposition of $K(\Lambda)$ as a $\mathfrak{g}_{\bar{0}}$-module. This decomposition will be achieved in subsection 7.3.1.

Unitarizable highest weight $\mathfrak{g}$-supermodules

Let $\mathcal{H}$ be a unitarizable highest weight $\mathfrak{g}$-supermodule with highest weight Λ and highest weight vector v_Λ. By Proposition 3.1.15, there is a $\mathfrak{g}$-supermodule homomorphism $q : M(\Lambda) \to \mathcal{H}$ that maps $[1 \otimes 1]$ to v_Λ. Since the highest weight Λ is symmetric, the maximal super submodule of $M(\Lambda)$ is the same as the radical R of the Shapovalov form. As a result, the kernel of q must lie within R. This leads us to the following proposition.

Proposition 6.1.17. *$\mathcal{H}$ carries a non-zero unique contravariant consistent Hermitian form $\langle \cdot, \cdot \rangle$ such that $\langle v_\Lambda, v_\Lambda \rangle = 1$. The form is non-degenerate if and only if $\mathcal{H}$ is simple, in which case $\mathcal{H} \cong L(\Lambda) := M(\Lambda)/R$, and the space of $\mathfrak{b}$-eigenvectors of weight Λ in $L(\Lambda)$ is one-dimensional.*

Proof. It remains to show that the form is non-degenerate if and only if $\mathcal{H}$ is simple. It is clear that the Shapovalov form induces a non-degenerate form on $L(\Lambda) = M(\Lambda)/R$.

If, conversely, $\mathcal{H}$ carries a non-degenerate Hermitian form, then $\ker(q)$ cannot be properly contained in R. Consequently, $R = \ker(q)$ and $\mathcal{H} = L(\Lambda)$. □

By abuse of notation, we call the unique form on $\mathcal{H}$ also the *Shapovalov form*. Moreover, we identify $\mathcal{H} \cong L(\Lambda)$ whenever $\mathcal{H}$ is simple. We also consider $L(\Lambda)$ as a $\mathfrak{g}_{\bar{0}}$-module by restriction, and we denote the associated $\mathfrak{g}_{\bar{0}}$-module by $L(\Lambda)_{\mathrm{ev}}$, where parity is neglected. The restriction $L(\Lambda)_{\mathrm{ev}}$ decomposes into a finite direct sum of unitarizable $\mathfrak{g}_{\bar{0}}$-supermodules by Proposition 6.1.8. However, since $L(\Lambda)$ is a highest weight $\mathfrak{g}$-supermodule, any $\mathfrak{g}_{\bar{0}}$-constituent of $L(\Lambda)_{\mathrm{ev}}$ must be a unitarizable highest weight $\mathfrak{g}_{\bar{0}}$-module of the form $L_0(\mu)$ for some $\mu \in \mathfrak{h}^*$. The question then arises as to which $\mathfrak{g}_{\bar{0}}$-constituents appear in $L(\Lambda)_{\mathrm{ev}}$. For this purpose, let Γ be the sum of distinct odd positive roots, and for each $\gamma \in \Gamma$, let $p(\gamma)$ denote the number of distinct partitions of γ into odd positive roots. Additionally, let S be a subset of $\Delta_{\bar{1}}^+$, and define $\Gamma_S = \sum_{\gamma \in S} \gamma$. Set $X_{-S} := \prod_{\alpha \in S} X_{-\alpha}$, where $X_{-\alpha}$ is the root vector corresponding to the root $-\alpha \in \Delta_{\bar{1}}^-$, with $\Delta_{\bar{1}}^- := -\Delta_{\bar{1}}^+$. Fix an ordering $S_1, \ldots, S_N$ on subsets of $\Delta_{\bar{1}}^+$ such that if $\Gamma_{S_i} < \Gamma_{S_j}$, then $i < j$, and define

$$M_k := \bigoplus_{1 \leq j \leq k} \mathfrak{U}(\mathfrak{n}_{\bar{0}}^-) X_{-S_j}[1 \otimes 1].$$

Proposition 6.1.18 ([99, Theorem 10.4.5])**.** *The Verma supermodule $M(\Lambda)$ has a filtration as a $\mathfrak{g}_{\bar{0}}$-supermodule:*

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_r = M(\Lambda),$$

such that each factor M_{i+1}/M_i is isomorphic to a Verma module $M^0(\Lambda - \gamma)$, where $\gamma \in \Gamma$. This module appears with multiplicity $p(\gamma)$ in the filtration.

If $\mathcal{H}$ is a unitarizable simple highest weight $\mathfrak{g}$-supermodule with highest weight Λ, then by Proposition 6.1.17 and the universal property of the Verma supermodule (Proposition 3.1.15), $\mathcal{H}$ satisfies $\mathcal{H} \cong L(\Lambda) \cong M(\Lambda)/R$, where R is the radical of the Shapovalov form. Modding suitably out by R provides a composition series for $\mathcal{H}$ which gives us a direct sum decomposition as $\mathcal{H}$ is unitarizable.

Corollary 6.1.19. *Let $\mathcal{H}$ be a unitarizable simple $\mathfrak{g}$-supermodule with highest weight Λ. Then, as a $\mathfrak{g}_{\bar{0}}$-supermodule, $\mathcal{H}$ decomposes as*

$$\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_r$$

such that each $\mathfrak{g}_{\bar{0}}$-constituent $\mathcal{H}_i$ is isomorphic to a unitarizable highest weight $\mathfrak{g}_{\bar{0}}$-supermodule $L_0(\Lambda - \Gamma_S)$ for some subset $S \subset \Delta_{\bar{1}}^+$.

The question which $\mathfrak{g}_{\bar{0}}$-constituents appear exactly, will be addressed in Section 7.3.1.

6.1.4. Parameterization of unitarizable $\mathfrak{g}_{\bar{0}}$-supermodules

We describe the subset of the space $\mathfrak{h}^*$ of weights that correspond to unitarizable highest weight $\mathfrak{g}_{\bar{0}}$-modules. It follows from the definition of (Kac) induction and Proposition 6.1.8 that a necessary condition for $L(\Lambda)$ to be unitarizable as a $\mathfrak{g}$-supermodule is that $L_0(\Lambda)$ be unitarizable as a $\mathfrak{g}_{\bar{0}}$-module, where we write

$$\mathfrak{g}_{\bar{0}} = \begin{cases} \mathfrak{su}(p,q)^{\mathbb{C}} \oplus \mathfrak{su}(n)^{\mathbb{C}} \oplus \mathfrak{u}(1)^{\mathbb{C}} & \text{if } m \neq n, \\ \mathfrak{su}(p,q)^{\mathbb{C}} \oplus \mathfrak{su}(n)^{\mathbb{C}} & \text{if } m = n. \end{cases}$$

to emphasize the real form. This imposes a classical sequence of standard conditions on the highest weight, which we recall is parameterized in terms of the standard coordinates on $\mathfrak{h}^*$ as

$$\Lambda = (\lambda_1, \ldots, \lambda_m | \lambda_1', \ldots, \lambda_n'),$$

modulo shifts by $(1, \ldots, 1|-1, \ldots, -1)$.

First, we consider the restriction to the maximal compact subalgebra $\mathfrak{k}$. If $L_0(\Lambda)$ is unitarizable as a $\mathfrak{g}_{\bar{0}}$-module, then as a $\mathfrak{k}^{\mathbb{C}}$-module it is semisimple with finite multiplicities. In particular, Λ is the highest weight of a unitarizable simple (hence finite-dimensional) $\mathfrak{k}^{\mathbb{C}}$-module, which appears with multiplicity one. Namely, Λ must be integral and dominant with respect to the positive system induced from $\mathfrak{g}$. On the simple $\mathfrak{k}^{\mathbb{C}}$-roots this means

$$\begin{aligned} (\Lambda, \epsilon_i - \epsilon_{i+1}) &= \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0} && \text{for } i = 1, 2, \ldots, p-1 \\ (\Lambda, \epsilon_j - \epsilon_{j+1}) &= \lambda_j - \lambda_{j+1} \in \mathbb{Z}_{\geq 0} && \text{for } j = p+1, \ldots, m-1 \\ -(\Lambda, \delta_k - \delta_{k+1}) &= \lambda_k' - \lambda_{k+1}' \in \mathbb{Z}_{\geq 0} && \text{for } k = 1, \ldots, n-1 \end{aligned}$$

To deduce further conditions, we decompose appropriately $L_0(\Lambda)$ as a $\mathfrak{g}_{\bar{0}}$-module.

Lemma 6.1.20. *Let $L_0(\lambda)$ and $L_0(\mu)$ be two unitarizable simple highest weight $\mathfrak{g}_{\bar{0}}$-modules with highest weight vectors v_λ and v_μ, respectively. Define $V := L_0(\lambda) \otimes L_0(\mu)$. Then the following assertions hold:*

a) The tensor product of the Shapovalov forms on both factors defines a non-degenerate contravariant Hermitian form on V.

b) $v_\lambda \otimes v_\mu$ is a primitive element in V, that is, a non-zero $\mathfrak{b}_{\bar{0}}$-eigenvector.

c) $V \cong L_0(\lambda + \mu)$ if and only if $v_\lambda \otimes v_\mu$ is cyclic in V.

Proof. We denote the Shapovalov forms on $L_0(\lambda)$ and $L_0(\mu)$ by $\langle \cdot, \cdot \rangle_\lambda$ and $\langle \cdot, \cdot \rangle_\mu$, respectively. Then V has a contravariant Hermitian form

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \langle v_1, v_2 \rangle_\lambda \langle w_1, w_2 \rangle_\mu.$$

To see that $\langle \cdot, \cdot \rangle$ is non-degenerate, consider $x := \sum_{j=1}^k v_j \otimes w_j$ with $\langle x, y \rangle = 0$ for all $y \in V$. Without loss of generality, assume that $v_1, \ldots, v_k$ are linearly independent. Since $L_0(\lambda)$

is simple, $\langle \cdot, \cdot \rangle_\lambda$ is non-degenerate, and we find for each $1 \leq j \leq k$ an element $y_j \in L_0(\lambda)$ such that $\langle v_i, y_j \rangle_\lambda = \delta_{ij}$. Consequently, $\langle x, y_j \otimes w \rangle = \langle w_j, w \rangle_\mu = 0$, so $w_j = 0$ as $\langle \cdot, \cdot \rangle_\mu$ is non-degenerate. This concludes the proof for assertion a).

For assertions b) and c), note that the set of weights of V, denoted by P_V, equals the sum of the sets of weights for $L_0(\lambda)$ and $L_0(\mu)$, *i.e.*, $P_V \subset \lambda + \mu - \mathbb{Z}_+[\Delta_{\bar{0}}^+]$ with $\dim V^{\lambda+\mu} = 1$. Assertion b) is now clear, while assertion c) follows directly from a). □

Proposition 6.1.21. *Let $\mathfrak{g}_{\bar{0}} := \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the direct sum of Lie algebras with root space decompositions. Let $\mathfrak{h} := \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_1$ and $\mathfrak{h}_2$ denote the Cartan subalgebras of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. Identify the root systems Δ_1 and Δ_2 with subsets of $\Delta_{\bar{0}}$, and let $\lambda = (\lambda_1, \lambda_2) \in \mathfrak{h}^* \cong \mathfrak{h}_1^* \times \mathfrak{h}_2^*$. Then*

$$L_0(\lambda) \cong L_0(\lambda_1; \mathfrak{g}_1) \boxtimes L_0(\lambda_2; \mathfrak{g}_2).$$

Proof. The positive systems for $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are $\Delta_1^+ := \Delta_1 \cap \Delta_{\bar{0}}^+$ and $\Delta_2^+ := \Delta_2 \cap \Delta_{\bar{0}}^+$. The projections $p_j : \mathfrak{g}_{\bar{0}} \to \mathfrak{g}_j$ are Lie algebra morphisms, allowing us to identify $L_0(\lambda_j; \mathfrak{g}_j)$ with $L_0(\lambda_j)$. Define $v_\lambda := v_{\lambda_1} \otimes v_{\lambda_2}$, where v_{λ_j} are the highest weight vectors for $L_0(\lambda_j)$. We compute

$$\mathfrak{U}(\mathfrak{g})v_\lambda = (\mathfrak{U}(\mathfrak{g}_1)v_{\lambda_1}) \otimes (\mathfrak{U}(\mathfrak{g}_2)v_{\lambda_2}) = L(\lambda_1; \mathfrak{g}_1) \boxtimes L(\lambda_2; \mathfrak{g}_2),$$

i.e., v_λ is cyclic, and the statement follows with Lemma 6.1.20. □

In our situation, we have the direct sum of Lie algebras $\mathfrak{g}_{\bar{0}} := \mathfrak{L} \oplus \mathfrak{R} \oplus \mathfrak{u}(1)^{\mathbb{C}}$, where $\mathfrak{L} := \mathfrak{su}(p,q)^{\mathbb{C}}$, $\mathfrak{R} := \mathfrak{su}(n)^{\mathbb{C}}$, and the $\mathfrak{u}(1)^{\mathbb{C}}$-part is only present if $m \neq n$. We denote the associated root systems of $(\mathfrak{L}, \mathfrak{h}|_{\mathfrak{L}})$ and $(\mathfrak{R}, \mathfrak{h}|_{\mathfrak{R}})$ by $\Delta_{\mathfrak{L}}$ and $\Delta_{\mathfrak{R}}$, respectively. In particular, $\Delta_{\bar{0}} = \Delta_{\mathfrak{L}} \sqcup \Delta_{\mathfrak{R}}$. The positive systems are $\Delta_{\mathfrak{L}}^+ := \Delta_{\bar{0}}^+ \cap \Delta_{\mathfrak{L}}$ and $\Delta_{\mathfrak{R}}^+ := \Delta_{\bar{0}}^+ \cap \Delta_{\mathfrak{R}}$. The highest weight of a unitarizable simple $\mathfrak{g}$-supermodule is then of the form $\mu = (\mu^{\mathfrak{L}} | \mu^{\mathfrak{R}})$ with $\mu^{\mathfrak{L}} = (\mu_1, \ldots, \mu_m)$ and $\mu^{\mathfrak{R}} = (\mu'_1, \ldots, \mu'_n)$, such that $\mu^{\mathfrak{L}}$ is the highest weight of a Harish-Chandra module of $\mathfrak{L}$, while $\mu^{\mathfrak{R}}$ is the highest weight of a finite-dimensional simple $\mathfrak{R}$-module.

Corollary 6.1.22. *Any unitarizable simple highest weight $\mathfrak{g}_{\bar{0}}$-module $L_0(\mu)$ is given by the outer tensor product of a unitarizable simple highest weight $\mathfrak{su}(p,q)^{\mathbb{C}}, \mathfrak{su}(n)^{\mathbb{C}}$ and $\mathfrak{u}(1)^{\mathbb{C}}$-module, respectively,* i.e.,

$$L_0(\mu) \cong L_0(\mu^{\mathfrak{L}}; \mathfrak{L}) \boxtimes L_0(\mu^{\mathfrak{R}}; \mathfrak{R}) \boxtimes \mathbb{C}_\mu.$$

We list all these modules, yielding a parameterization of the highest weights of unitarizable highest weight $\mathfrak{g}_{\bar{0}}$-modules. First, $\mathfrak{u}(1)$ is an abelian Lie algebra, so by Schur's Lemma, any simple module is one-dimensional. Unitarity requires that any such simple module is uniquely determined by a positive real number α. The list of all simple $\mathfrak{u}(1)^{\mathbb{C}}$-modules is parameterized by

$$\frac{\alpha}{2}(1, \ldots, 1 | 1, \ldots, 1) \in \mathfrak{h}^*, \qquad \alpha \in \mathbb{R}.$$

Second, $\mathfrak{su}(n)$ is a compact Lie algebra, meaning all simple modules are finite-dimensional and unitarizable. By the highest weight theorem, every simple module is uniquely characterized by a dominant integral highest weight. Therefore, the list of all unitarizable simple $\mathfrak{su}(n)^{\mathbb{C}}$-modules is parameterized by

$$(0, \ldots, 0 | b_1, \ldots, b_n) \in \mathfrak{h}^*,$$

where $b_1, \dots, b_n$ are positive integers satisfying $b_1 \geq b_2 \geq \dots \geq b_{n-1} \geq b_n = 0$.

Third, $\mathfrak{su}(p,q)$ is a non-compact semisimple Lie algebra if and only if $p, q \neq 0$. If $p = 0$ or $q = 0$, the list of all simple $\mathfrak{su}(m)^{\mathbb{C}}$-modules is parameterized as above, *i.e.*,

$$(a_1, \dots, a_m | 0, \dots, 0)$$

with $a_1, \dots, a_m$ positive integers satisfying $a_1 \geq \dots \geq a_m = 0$. These modules are finite-dimensional. However, if $p, q \neq 0$, the non-trivial unitarizable simple highest weight modules are infinite-dimensional, and a complete classification is given in [38, 72]. We briefly state the result. The complete list with highest weights of unitarizable simple highest weight $\mathfrak{su}(p,q)$-modules is given by

$$\left(\frac{\lambda}{2} - a_1, \frac{\lambda}{2} - a_2, \dots, \frac{\lambda}{2} - a_p, -\frac{\lambda}{2} + a_{p+1}, \dots, -\frac{\lambda}{2} + a_{p+q} | 0, \dots, 0\right),$$

where $a_1, \dots, a_p$ and $a_{p+1}, \dots, a_{p+q}$ are positive integers such that $0 \leq a_2 \leq \dots \leq a_p$ and $a_{p+1} \geq a_{p+2} \geq \dots \geq a_{p+q-1} \geq a_{p+q} = 0$, and λ belongs to the set

$$(-\infty, -m + x - (r-1)) \cup \{-m + x - (r-1), -m + x - (r-1) + 1, \dots, -m + x\}.$$

Here, $m := p + q$, $r := \min(i_0, j_0)$ and $x = i_0 + j_0$, where i_0 is the biggest index for which $a_i = 0$, and j_0 is the smallest index for which $a_{p+q-j_0} \neq 0$ (if $a_{p+1} = 0$ then $j_0 = q$). The values i_0 and j_0 are part of the Δ_c^+-dominance of Λ, and can be deduced from the *length* of the following two Young diagrams:

$$Y_1(\lambda) := (\lambda_1 - \lambda_p, \dots, \lambda_1 - \lambda_2, 0),$$
$$Y_2(\lambda) := (\lambda_{p+1} - \lambda_m, \dots, \lambda_{m-1} - \lambda_m, 0).$$

Indeed, if $\mathrm{len}_i(\lambda) := \mathrm{length}(Y_1(\lambda))$, we have $i_0 = \mathrm{len}_1(\lambda)$ and $j_0 = m - \mathrm{len}_2(\lambda)$.

Altogether, we parameterize the solution to these constraints by writing

$$\Lambda = (0, a_2, \dots, a_{m-1}, 0 | b_1, \dots, b_{n-1}, 0) + \frac{\lambda}{2}(1, \dots, 1, -1, \dots, -1 | 0, \dots, 0) + \frac{\alpha}{2}(1, \dots, 1 | 1, \dots, 1),$$

with integers a_i, a_j satisfying $a_{p+1} \geq \cdots a_{m-1} \geq 0 \geq a_2 \geq \cdots \geq a_p$, integers b_k satisfying $b_1 \geq \cdots \geq b_{n-1}$ and real numbers α and λ, the second of which is non-positive.

6.1.5. Harish-Chandra supermodules

We consider $\mathfrak{g}$, where $\mathfrak{g}$ denotes either $\mathfrak{sl}(m|n)$ or a basic classical Lie superalgebra of type $A(m|n)$. Unitarizable simple highest weight $\mathfrak{g}$-supermodules are examples of *Harish-Chandra supermodules*, allowing us to view them as modules over the complexification of the fixed maximal compact subalgebra $\mathfrak{k}^{\mathbb{C}}$. Recall that $\mathfrak{k}^{\mathbb{C}}$ satisfies the equal rank condition $\mathfrak{k}^{\mathbb{C}} \subset \mathfrak{g}_{\bar{0}} \subset \mathfrak{g}$. This implies that any $\mathfrak{g}$-supermodule M can be viewed as a $\mathfrak{g}_{\bar{0}}$-supermodule, and conversely, any $\mathfrak{g}_{\bar{0}}$-supermodule can be regarded as a $\mathfrak{k}^{\mathbb{C}}$-supermodule. If $\mathcal{H}$ is unitarizable, the action of $\mathfrak{k}^{\mathbb{C}}$ is contravariant, leading to a decomposition of $\mathcal{H}$ and any of its $\mathfrak{g}_{\bar{0}}$-constituents into $\mathfrak{k}^{\mathbb{C}}$-types. Explicitly, $\mathcal{H}$ decomposes as

$$\mathcal{H} = \sum_{[\mu] \in \widehat{\mathfrak{k}^{\mathbb{C}}}} F^{[\mu]},$$

where the sum is both algebraic and direct. Here, $\widehat{\mathfrak{k}^{\mathbb{C}}}$ denotes the set of equivalence classes of finite-dimensional simple $\mathfrak{k}^{\mathbb{C}}$-supermodules, and $F^{[\nu]}$ represents the sum of all supermodules occurring in $\mathcal{H}$ that belong to the class $[\nu] \in \widehat{\mathfrak{k}^{\mathbb{C}}}$. We say that $\mathcal{H}$ is a $(\mathfrak{g}, \mathfrak{k}^{\mathbb{C}})$-supermodule, and each $\mathfrak{g}_{\bar{0}}$-constituent is a $(\mathfrak{g}_{\bar{0}}, \mathfrak{k}^{\mathbb{C}})$-supermodule, meaning they are $\mathfrak{k}^{\mathbb{C}}$-semisimple.

An important class of $(\mathfrak{g}_{\bar{0}}, \mathfrak{k}^{\mathbb{C}})$-supermodules is formed by those that originate from unitary irreducible representations of the simply connected Lie group associated with the Lie algebra $\mathfrak{su}(p, q|n)_{\bar{0}}$. These modules possess an additional property, leading to the definition of Harish-Chandra (super)modules.

Definition 6.1.23. A complex $(\mathfrak{g}, \mathfrak{k}^{\mathbb{C}})$-supermodule ($(\mathfrak{g}_{\bar{0}}, \mathfrak{k}^{\mathbb{C}})$-module) is called *Harish-Chandra supermodule (Harish-Chandra module)* if it is finitely generated and locally finite as a $\mathfrak{k}^{\mathbb{C}}$-module.

The Harish-Chandra supermodules naturally form a category, denoted by $\mathcal{M}(\mathfrak{g}, \mathfrak{k})$. We show that unitarizable simple supermodules belong to $\mathcal{M}(\mathfrak{g}, \mathfrak{k})$. Firstly, unitarizable simple $\mathfrak{g}$-supermodules are highest weight supermodules, where the conditions of local finiteness as $\mathfrak{k}^{\mathbb{C}}$-modules and finite generation are redundant in the definition above.

Proposition 6.1.24 ([13, Proposition 2.8])**.** *Let M be a highest weight $\mathfrak{g}$-supermodule with highest weight Λ and highest weight vector v_Λ. The following assertions are equivalent:*

a) $\dim(\mathfrak{U}(\mathfrak{k}^{\mathbb{C}})v_\Lambda) < \infty$.

b) M is a $(\mathfrak{g}, \mathfrak{k}^{\mathbb{C}})$-supermodule.

c) M is a Harish-Chandra supermodule.

If these assertions hold, then $\mathfrak{U}(\mathfrak{k}^{\mathbb{C}})v_\Lambda$ is a simple $\mathfrak{k}^{\mathbb{C}}$-supermodule.

Any unitarizable highest weight $\mathfrak{g}$-supermodule is $\mathfrak{g}_{\bar{0}}$-semisimple, and each $\mathfrak{g}_{\bar{0}}$-constituent is a unitarizable highest weight $\mathfrak{g}_{\bar{0}}$-supermodule, for which we have the following well-known lemma.

Lemma 6.1.25 ([102, Lemma IX.3.10])**.** *Any unitarizable highest weight $\mathfrak{g}_{\bar{0}}$-supermodule is a Harish-Chandra module.*

Conversely, the following lemma holds.

Lemma 6.1.26. *Let $\mathcal{H}$ be a finitely generated $\mathfrak{g}$-supermodule. If $\mathcal{H}$ is a $(\mathfrak{g}_{\bar{0}}, \mathfrak{k}^{\mathbb{C}})$-supermodule, then $\mathcal{H}$ is a $(\mathfrak{g}, \mathfrak{k}^{\mathbb{C}})$-supermodule. In particular, $\mathcal{M}(\mathfrak{g}, \mathfrak{k})$ is closed under tensoring with finite-dimensional $\mathfrak{g}$-supermodules.*

Proof. The statement is immediate, as $\mathfrak{U}(\mathfrak{g})$ is a finitely generated $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$-module. □

Thus, we establish the following assertion.

Proposition 6.1.27. *Let $\mathcal{H}$ be a unitarizable highest weight $\mathfrak{g}$-supermodule. Then $\mathcal{H}$ is a Harish-Chandra supermodule.*

For completeness, we note that the Kac induction of a $(\mathfrak{g}_{\bar{0}}, \mathfrak{k}^{\mathbb{C}})$-module yields a $(\mathfrak{g}, \mathfrak{k}^{\mathbb{C}})$-supermodule, as shown in [14, Proposition 2.8]. By Theorem 3.1.16, this leads to another proof of the proposition.

6.2. Unitary supermodules over Lie supergroups

We briefly summarize the classification of unitary representations of Lie supergroups $\mathcal{G}$ that are associated with basic classical Lie superalgebras, following [11, 104]. The main result states that unitary representations of $\mathcal{G} = (G, \mathfrak{g})$ are classified by unitarizable supermodules over $\mathfrak{g}$. Thus, we focus on studying unitarizable supermodules over basic classical Lie superalgebras.

We identify a Lie supergroup $\mathcal{G}$ with its super Harish-Chandra pair $(G, \mathfrak{g})$ (see Section 2.2.2), following standard conventions. Recall that a representation of $\mathcal{G}$ is given by a triple $(\pi, \rho^\pi, \mathcal{H})$, where π is an even representation of the Lie group G in a $\mathbb{Z}_2$-graded Banach space (or $\mathbb{Z}_2$-graded Fréchet space) $\mathcal{H}$, and ρ^π is a $\mathbb{Z}_2$-graded representation of the Lie superalgebra $\mathfrak{g}$ in $\mathcal{H}$, compatible with π (see Section 3.2).

In the unitary setting, we consider a super Hilbert space $\mathcal{H}$, a unitary even representation π, and require ρ^π to satisfy unitarity at the infinitesimal level. Moreover, since elements in $\mathfrak{g}_{\bar{0}}$ are typically unbounded, appropriate domain conditions must be imposed.

We start with the definition of a *super Hilbert space*.

Definition 6.2.1. A *super Hilbert space* $\mathcal{H}$ is a $\mathbb{Z}_2$-graded complex Hilbert space, $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$, equipped with a positive definite Hermitian form $\langle \cdot, \cdot \rangle$, where $\mathcal{H}_{\bar{0}}$ and $\mathcal{H}_{\bar{1}}$ are mutually orthogonal closed subspaces.

Before discussing general infinite-dimensional unitary representations, we first consider the finite-dimensional ones. Recall that a finite-dimensional representation of a Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$ consist a triple (π, ρ^π, V), where π is an even representation of G on a super vector space V of finite dimension over $\mathbb{C}$, and ρ^π is a representation of the Lie superalgebra $\mathfrak{g}$ on V, such that

a) $\pi\big|_{\mathfrak{g}_{\bar{0}}} = \mathrm{d}\pi$ and

b) $\mathrm{d}\pi(\mathrm{ad}(g)(X)) = \pi(g)\mathrm{d}\pi(X)\pi(g^{-1})$ for all $g \in G$ and $X \in \mathfrak{g}$.

The representation (π, V) of G is called *unitary* if there exists a positive definite Hermitian form $\langle \cdot, \cdot \rangle$ on V such that

$$\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle, \qquad \text{for all } v, w \in V \text{ and } g \in G.$$

The derived representation $\mathrm{d}\pi$ then satisfies

$$\langle \mathrm{d}\pi(X)v, w \rangle + \langle v, \mathrm{d}\pi(X)w \rangle = 0,$$

for all $X \in \mathrm{Lie}(G)$, which implies that $\rho(X)^\dagger = -\rho(X)$. Generalizing this to the super setting, we define:

Definition 6.2.2. A finite-dimensional representation $(\pi, \rho^\pi, \mathcal{H})$ of a Lie supergroup $(G, \mathfrak{g})$ is called *unitary* if $\mathcal{H}$ is a super Hilbert space and $\rho^\pi(X)^\dagger = -\rho^\pi(X)$ for all $X \in \mathfrak{g}$.

The condition for unitarity is equivalent to stating that π is a unitary representation of G, and that $\rho^\pi(X)^\dagger = -i\rho^\pi(X)$ for all $X \in \mathfrak{g}_{\bar{1}}$. Consequently, a finite-dimensional unitary representation of a Lie supergroup $(G, \mathfrak{g})$ consists of a triple $(\pi, \rho^\pi, \mathcal{H})$, where

a) π is a unitary representation of G on a finite-dimensional super Hilbert space $\mathcal{H}$,

b) ρ^π is a linear map of $\mathfrak{g}_{\bar{1}}$ into the space $\mathfrak{gl}(\mathcal{H})_{\bar{1}}$ of odd endomorphisms of $\mathcal{H}$ such that $\rho^\pi(X)^\dagger = -i\rho^\pi(X)$ for all $X \in \mathfrak{g}_{\bar{1}}$,

c) $\mathrm{d}\pi([X,Y]) = \rho^\pi(X)\rho^\pi(Y) + \rho^\pi(Y)\rho^\pi(X)$ for $X, Y \in \mathfrak{g}_{\bar{1}}$,

d) $\rho^\pi(gX) = \pi(g)\rho^\pi(X)\pi(g)^{-1}$ for $X \in \mathfrak{g}_{\bar{1}}$ and $g \in G$.

If we replace $\rho^\pi(X)$ by $\rho(X) := e^{-i\frac{\pi}{4}}\rho^\pi(X)$ for $X \in \mathfrak{g}_{\bar{1}}$, then condition b) implies that $\rho(X)$ is self-adjoint for all $X \in \mathfrak{g}_{\bar{1}}$, while condition c) transforms into

$$-i\mathrm{d}\pi([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X), \qquad X, Y \in \mathfrak{g}_{\bar{1}}.$$

We now extend this definition to the infinite-dimensional theory. First, condition c) implies

$$\mathrm{d}\pi([X,X]) = 2\rho^\pi(X)^2, \qquad X \in \mathfrak{g}_{\bar{1}}.$$

Moreover, since $\mathrm{d}\pi$ maps elements of $\mathfrak{g}_{\bar{0}}$ to unbounded operators, $\rho^\pi(X)$ is generally unbounded. Furthermore, in the infinite-dimensional setting, the differential of a representation of a Lie group is typically not defined on the entire representation space. Thus, as in the classical setting, we consider a certain invariant dense subspace: the space of smooth vectors (*cf.* Section 5.2).

We will denote the space of smooth vectors of a unitary representation $(\pi, \mathcal{H})$ by $\mathcal{H}^\infty$. Focusing solely on irreducible unitary representations of Lie supergroups allows us to assume, without loss of generality, that $\mathcal{H}$ is separable.

Definition 6.2.3. A *unitary representation* of a Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$ is a triple $\Pi := (\pi, \rho^\pi, \mathcal{H})$ satisfying the following conditions:

a) $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$ is a super Hilbert space.

b) $(\pi, \mathcal{H})$ is a unitary representation of G, where $\pi(g) \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})_{\bar{0}}$ for all $g \in G$.

c) $\rho^\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathcal{H}^\infty)$ is a $\mathbb{R}$-linear $\mathbb{Z}_2$-graded map. Furthermore, for all $X, Y \in \mathfrak{g}_{\bar{1}}$, we have:
$$\rho^\pi(X)\rho^\pi(Y) - \rho^\pi(Y)\rho^\pi(X) = -i\rho^\pi([X,Y]).$$

d) $\rho^\pi(X) = \mathrm{d}\pi(X)\big|_{\mathcal{H}^\infty}$ for all $X \in \mathfrak{g}_{\bar{0}}$.

e) The operator $\rho^\pi(X)$ with domain $\mathcal{H}^\infty$ is symmetric for all $X \in \mathfrak{g}_{\bar{1}}$.

f) For all $g \in G$ and $X \in \mathfrak{g}$, the following compatibility relation holds:
$$\rho^\pi(\mathrm{ad}(g)(X)) = \pi(g)\rho^\pi(X)\pi(g)^{-1}.$$

The following lemma summarizes some immediate consequences of the definition.

Lemma 6.2.4 ([10, Chapter 7.2.3], [104])**.** *Let $(\pi, \rho^\pi, \mathcal{H})$ be a unitary representation of the Lie supergroup $\mathcal{G}$.*

a) The operator $\rho^\pi : \mathcal{H}^\infty \to \mathcal{H}^\infty$ is continuous with respect to the Fréchet topology on $\mathcal{H}^\infty$. In addition, the bilinear map

$$\mathfrak{g} \times \mathcal{H}^\infty \to \mathcal{H}^\infty, \quad (X, v) \mapsto \rho^\pi(X)v$$

is continuous.

b) *For any $X \in \mathfrak{g}_{\bar{0}}$, the operator $\mathrm{i}\mathrm{d}\pi(X)$ is essentially self-adjoint on $\mathcal{H}^\infty$, and*

$$\mathrm{d}\pi : \mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \to \mathrm{End}(\mathcal{H}^\infty), \quad X_0 + X_1 \mapsto \mathrm{d}\pi(X_0) + e^{i\frac{\pi}{4}}\rho^\pi(X_1)$$

is a representation of $\mathfrak{g}$ in $\mathcal{H}^\infty$.

c) *For all $X \in \mathfrak{g}_{\bar{1}}$, the operator $\rho^\pi(X)$ is essentially self-adjoint on the $\mathcal{H}^\infty$.*

All concepts from the ordinary setting, such as subrepresentations, irreducibility, or unitary equivalence of unitary representations of Lie supergroups, are defined similarly to those for representations of Lie groups.

Moreover, the following proposition follows from standard arguments.

Proposition 6.2.5. *Let Π be a unitary representation of a Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$.*

a) *Π is completely reducible; that is, for any closed invariant subspace, its orthogonal complement is also a closed invariant subspace.*

b) *Π is irreducible if and only if $\mathrm{Hom}_{\mathcal{G}}(\Pi, \Pi) = \mathbb{C}$.*

For a proof of b), known as Schur's lemma for Lie supergroups, see [10, Lemma 15].

The complete classification of irreducible unitary representations of Lie supergroups $\mathcal{G} = (G, \mathfrak{g})$ where $\mathfrak{g}$ is basic classical is due to Neeb and Salmasian in [104]. They study more generally $\star$*-reduced Lie supergroups*, that is, a Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$ such that for every non-zero $X \in \mathfrak{g}$ there exists a unitary representation $(\pi, \rho^\pi, \mathcal{H})$ of $\mathcal{G}$ with $\rho^\pi(X) \neq 0$. If $\mathfrak{g}$ is simple, then a Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$ is $\star$-reduced if and only if it has a non-trivial unitary representation.

We fix some notation that will appear in the classification theorem. An element $X_0 \in \mathfrak{g}$ is called *regular* if the space

$$\mathcal{N}_{\mathfrak{g}}(X_0) = \bigcup_n \ker\left(\mathrm{ad}^n_{X_0}\right)$$

has minimal dimension. For any regular element X_0, the space $\mathfrak{h} := \mathcal{N}_{\mathfrak{g}}(X_0)$ is a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{h}_{\bar{0}}$ is compactly embedded [104, Lemma 5.2.1]. A Lie subalgebra $\mathfrak{h}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$ is called *compactly embedded* in $\mathfrak{g}$ if the closure of the subgroup generated by $e^{\mathrm{ad}(\mathfrak{h}_{\bar{0}})}$ in the automorphism group of $\mathfrak{g}$ is compact. Moreover, for a Lie supergroup $\mathcal{G}$, the *cone* $\mathrm{Cone}(G, \mathfrak{g})$ is defined as the space generated by elements of the form $[X, X]$ for $X \in \mathfrak{g}_{\bar{1}}$. The interior of the cone is denoted by $\mathrm{Int}(\mathrm{Cone}(\mathcal{G}))$.

Theorem 6.2.6 ([104, Theorem 7.3.2])**.** *Let $\mathcal{G} = (G, \mathfrak{g})$ be a $\star$-reduced Lie supergroup satisfying $\mathfrak{g}_{\bar{0}} = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$. Let $(\pi, \rho^\pi, \mathcal{H})$ be a unitary irreducible representation of $\mathcal{G}$. Choose a regular element $X_0 \in \mathrm{Int}(\mathrm{Cone}(\mathcal{G}))$, and let $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ be the corresponding Cartan subalgebra of $\mathfrak{g}$. Suppose that no roots vanish on X_0. Then the following assertions hold:*

a) *$\mathfrak{h}_{\bar{0}}$ is compactly embedded. Moreover, if we set $\Delta^+ := \{\alpha \in \Delta \mid \alpha(X_0) > 0\}$, then $\Delta \setminus \{0\} = \Delta^+ \sqcup -\Delta^+$.*

b) *The space $\mathcal{H}^{\mathfrak{h}}$ of $\mathfrak{h}$-finite elements in $\mathcal{H}^\infty$ is an irreducible $\mathfrak{g}$-module, which is a $\mathfrak{h}_{\bar{0}}$-weight module and dense in $\mathcal{H}$.*

c) *The maximal eigenspace* $\mathcal{V}$ *of* $i\rho^{\pi}(X_0)$ *is an irreducible finite-dimensional* $\mathfrak{h}$*-module on which* $\mathfrak{h}_{\bar{0}}$ *acts by some weight* $\lambda \in \mathfrak{h}_{\bar{0}}^*$*. It generates the* $\mathfrak{g}$*-supermodule* $\mathcal{H}^{\mathfrak{h}}$ *and all other* $\mathfrak{h}_{\bar{0}}$*-weights in this space take the form*

$$\lambda - m_1\alpha_1 - \cdots - m_k\alpha_k, \quad \alpha_j \in \Delta^+, \quad k \in \mathbb{Z}_{\geq 1}, \quad m_1, \ldots, m_k \in \mathbb{Z}_+.$$

d) *Two unitary representations* $(\pi, \rho^{\pi}, \mathcal{H})$ *and* $(\pi', \rho^{\pi'}, \mathcal{H}')$ *of* $(G, \mathfrak{g})$ *are isomorphic if and only if the corresponding* $\mathfrak{h}$*-representations,* $\mathcal{V}$ *and* $\mathcal{V}'$*, are isomorphic.*

As a consequence, unitarity of Lie supergroups is entirely determined by unitarizable (highest weight supermodules) of the associated Lie superalgebras. For the remainder of this thesis, we focus exclusively on unitarizable supermodules.

7. Dirac operator, Dirac cohomology and unitarizable supermodules

Two fundamental questions in representation theory involve classifying and explicitly decomposing unitarizable supermodules over $\mathfrak{g}$ ($\mathfrak{sl}(m|n)$ or $A(m|n)$), noting that all such modules are either of highest or lowest weight type. The existing classification [22, 48, 53, 73] relies on complex combinatorial methods and extensive calculations, lacking structural clarity and geometric intuition. A classification akin to the Enright–Howe–Wallach approach [38] for real simple Lie algebras remains undeveloped.

We resolve this problem by studying an analogue of algebraic Dirac operators and Dirac cohomology, introduced by Parthasarathy and Vogan [26, 114] for discrete series representations, in the context of Lie superalgebras of Riemannian type, as developed by Huang and Pandžić in [66]. For these Lie superalgebras, including $\mathfrak{g}$, there exists a non-degenerate supersymmetric invariant bilinear form B that restricts to a symplectic form on the odd part $\mathfrak{g}_{\bar{1}}$. The Dirac operator D is associated with the symplectic and the corresponding Weyl algebra over $\mathfrak{g}_{\bar{1}}$. The Dirac cohomology of a supermodule M is $\mathrm{H_D}(M) := \ker D/ \ker D \cap \mathrm{im}\, D$.

The first part of the work concerns the relationship among the Dirac operator, Dirac cohomology, and unitarity for supermodules. As a result, we prove that the Dirac operator uniquely captures unitarity, and unitarizable supermodules are determined by their Dirac cohomology. Explicitly, We compute the Dirac cohomology of unitarizable simple supermodules.

Furthermore, the algebraic Dirac operator can be used to give a novel classification of unitarity, which leads to a simple classification of unitarizable supermodules. This particularly addresses the following issue: a $\mathfrak{g}_{\bar{0}}$-semisimple module, for which every $\mathfrak{g}_{\bar{0}}$-constituent is unitarizable, need not itself be unitarizable. Conversely, the induction of a unitarizable $\mathfrak{g}_{\bar{0}}$-module to $\mathfrak{g}$ does not have to be unitarizable. The Dirac inequality offers a necessary and sufficient condition for unitarity. We obtain an explicit $\mathfrak{g}_{\bar{0}}$-decomposition for any unitarizable $\mathfrak{g}$-supermodule $\mathcal{H}$. Additionally, we relate Dirac cohomology to nilpotent Lie superalgebra cohomology. This results in a decomposition of formal characters, and it provides new methods for studying unitarizable supermodules and reveals initial insights into the structure of the abelian representation category.

In this chapter, we fix the non-standard positive system $\Delta^+ := \Delta^+_{\mathrm{nst}}$ for $\mathfrak{g}$, as introduced in Section 2.1.4. For convenience, we set

$$\mathfrak{n}_{\bar{1}}^{+} := \bigoplus_{\alpha\in\Delta_{\bar{1}}^{+}} \mathfrak{g}^{\alpha}, \qquad \mathfrak{n}_{\bar{1}}^{-} := \bigoplus_{\alpha\in\Delta_{\bar{1}}^{+}} \mathfrak{g}^{-\alpha},$$

which induces the decomposition of super vector spaces

$$\mathfrak{g}_{\bar{1}} = \mathfrak{n}_{\bar{1}}^{-} \oplus \mathfrak{n}_{\bar{1}}^{+}, \qquad \mathfrak{g} = \mathfrak{n}_{\bar{1}}^{-} \oplus \mathfrak{g}_{\bar{0}} \oplus \mathfrak{n}_{\bar{1}}^{+}.$$

However, $[\mathfrak{n}_{\bar{1}}^{\pm}, \mathfrak{n}_{\bar{1}}^{\pm}] \neq \{0\}$, indicating that this $\mathbb{Z}$-grading is not compatible with the $\mathbb{Z}_2$-grading, as $\mathfrak{n}_{\bar{1}}^{\pm}$ are invariant under $\mathfrak{k}^{\mathbb{C}}$, but not under $\mathfrak{g}_{\bar{0}}$.

The following lemma summarizes the relevant commutation relations.

Lemma 7.0.1. *The following commutation relations hold in* $\mathfrak{g}$*, for all* $1 \leq k, l \leq 2$*:*

$$[\mathfrak{k}, \mathfrak{p}_k] \subset \mathfrak{p}_k, \qquad [\mathfrak{k}, \mathfrak{q}_l] \subset \mathfrak{q}_l, \qquad [\mathfrak{n}^+_{\bar{1}}, \mathfrak{n}^+_{\bar{1}}] \subset \mathfrak{n}^+_{\bar{0},n}.$$

For the remainder of this chapter, unless otherwise stated, we assume $m \leq n$ and $p, q \neq 0$. All results for the finite-dimensional case can then be derived by setting $p = 0$ or $q = 0$ and utilizing $\mathfrak{k}^{\mathbb{C}} = \mathfrak{g}_{\bar{0}}$.

This chapter is based on the author's work in [123].

7.1. Dirac operator and Dirac cohomology

We introduce the *Dirac operator* and *Dirac cohomology* for $\mathfrak{g}$, based on the approach outlined in [67, Chapter 10]. Additionally, we generalize these results by examining functorial properties of Dirac cohomology.

7.1.1. Dirac operator

A Dirac operator is generally defined on a Lie superalgebra of *Riemannian type*, meaning there exists a non-degenerate supersymmetric invariant bilinear form B. The Lie superalgebra $\mathfrak{g}$ is of Riemannian type with respect to the supertrace form $(\cdot, \cdot)$, which we modify for convenience as $B(\cdot, \cdot) := \frac{1}{2}(\cdot, \cdot)$. The restriction $B|_{\mathfrak{g}_{\bar{1}}}(\cdot, \cdot)$ to $\mathfrak{g}_{\bar{1}}$ defines a symplectic form on $\mathfrak{g}_{\bar{1}}$, which naturally leads to the construction of the *Weyl algebra*.

Weyl algebra

We define the *Weyl algebra* over $\mathfrak{g}_{\bar{1}}$ and describe an embedding of $\mathfrak{g}_{\bar{0}}$ into the Weyl algebra. To achieve this, we fix two special Lagrangian subspaces of $(\mathfrak{g}_{\bar{1}}, B(\cdot, \cdot))$, spanned by $\{x_1, \ldots, x_{mn}\}$ and $\{\partial_1, \ldots, \partial_{mn}\}$, which are compatible with the conjugate-linear anti-involution ω. Specifically, $\omega(x_k) = -\partial_k$ and $B(\partial_k, x_l) = \frac{1}{2}\delta_{kl}$ for $1 \leq k, l \leq mn$. In accordance with the chosen positive system, we fix:

$$\partial_{(l-1)n+(k-m)} = \begin{cases} E_{lk} & \text{for } 1 \leq l \leq p, \ m+1 \leq k \leq m+n, \\ E_{kl} & \text{for } p+1 \leq l \leq m, \ m+1 \leq k \leq m+n, \end{cases}$$

$$x_{(l-1)n+(k-m)} = \begin{cases} E_{kl} & \text{for } 1 \leq l \leq p, \ m+1 \leq k \leq m+n, \\ -E_{lk} & \text{for } p+1 \leq l \leq m, \ m+1 \leq k \leq m+n. \end{cases}$$

Here, the x_k's span $\mathfrak{n}^-_{\bar{1}} = \mathfrak{p}_2 \oplus \mathfrak{q}_1$ and the ∂_k's span $\mathfrak{n}^+_{\bar{1}} = \mathfrak{p}_1 \oplus \mathfrak{q}_2$.

Remark 7.1.1. For the Lagrangian subspaces $\mathfrak{l}^+ := \mathfrak{p}_1 \oplus \mathfrak{p}_2$ and $\mathfrak{l}^- := \mathfrak{q}_1 \oplus \mathfrak{q}_2$, there does not exist a basis $x_1, \ldots, x_{mn}$ of $\mathfrak{l}^-$ and $\partial_1, \ldots, \partial_{mn}$ of $\mathfrak{l}^+$ that simultaneously satisfies $B(\partial_k, x_l) = \frac{1}{2}\delta_{kl}$ and $\omega(x_k) = -\partial_k$.

Let $T(\mathfrak{g}_{\bar{1}})$ denote the *tensor algebra* over the vector space $\mathfrak{g}_{\bar{1}}$, considered with natural $\mathbb{Z}$-grading $T(\mathfrak{g}_{\bar{1}}) = \oplus_{n\geq 0} T^n(\mathfrak{g}_{\bar{1}})$. The *Weyl algebra* $\mathscr{W}(\mathfrak{g}_{\bar{1}})$ is defined as the quotient $T(\mathfrak{g}_{\bar{1}})/I$, where I is the two-sided ideal generated by all elements of the form $v \otimes w - w \otimes v - 2B(v,w)$ for $v, w \in \mathfrak{g}_{\bar{1}}$.

The notation is such that the Weyl algebra $\mathscr{W}(\mathfrak{g}_{\bar{1}})$ over $\mathfrak{g}_{\bar{1}}$ is generated by ∂_k and x_l, *i.e.*, it can be identified with the algebra of differential operators with polynomial coefficients in the variables $x_1, \ldots, x_{mn}$, by identifying ∂_k with the partial derivative $\partial/\partial x_k$ for all $k = 1, \ldots, mn$. In particular, $\mathscr{W}(\mathfrak{g}_{\bar{1}})$ is a Lie algebra with the following commutator relations:

$$[x_k, x_l]_W = 0, \qquad [\partial_k, \partial_l]_W = 0, \qquad [\partial_k, x_l]_W = \delta_{kl},$$

for all $1 \leq k, l \leq mn$.

The Lie algebra $\mathfrak{g}_{\bar{0}}$ can be embedded as a Lie subalgebra in the Weyl algebra $\mathscr{W}(\mathfrak{g}_{\bar{1}})$. To this end, we fix some notation and state some well-known properties. Let $\mathfrak{sp}(\mathfrak{g}_{\bar{1}})$ denote the complex symplectic algebra over the vector space $\mathfrak{g}_{\bar{1}}$. The adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ defines a Lie algebra homomorphism:

$$\nu : \mathfrak{g}_{\bar{0}} \longrightarrow \mathfrak{sp}(\mathfrak{g}_{\bar{1}}),$$

where the symplectic form is given by the restriction of $B(\cdot,\cdot)$ to $\mathfrak{g}_{\bar{1}}$. Next, $\mathfrak{sp}(\mathfrak{g}_{\bar{1}})$ can be embedded in the Weyl algebra $\mathscr{W}(\mathfrak{g}_{\bar{1}})$. Let $S(\mathfrak{g}_{\bar{1}})$ denote the *symmetric algebra* over $\mathfrak{g}_{\bar{1}}$, which we define as the quotient $S(\mathfrak{g}_{\bar{1}}) := T(\mathfrak{g}_{\bar{1}})/J$, where J is the two-sided ideal generated by all elements of the form $u \otimes v - v \otimes u$ for $u, v \in \mathfrak{g}_{\bar{1}}$ (see Example 2.1.2). The symmetric algebra has a natural $\mathbb{Z}$-grading: $S(\mathfrak{g}_{\bar{1}}) = \bigoplus_{n=0}^{\infty} S^n(\mathfrak{g}_{\bar{1}})$, where the vector subspaces $S^n(\mathfrak{g}_{\bar{1}}) = T^n(\mathfrak{g}_{\bar{1}})/(J \cap T^n(\mathfrak{g}_{\bar{1}}))$ are called the *n-th symmetric power of* $\mathfrak{g}_{\bar{1}}$. Let $\mathrm{Sym}(\mathfrak{g}_{\bar{1}})$ denote the *space of symmetric tensors* in $T(\mathfrak{g}_{\bar{1}})$ with natural $\mathbb{Z}$-grading. Then the *symmetrization map*

$$S(\mathfrak{g}_{\bar{1}}) \longrightarrow \mathrm{Sym}(\mathfrak{g}_{\bar{1}}), \qquad x_1 \ldots x_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(k)}$$

is an isomorphism of graded vector spaces, and the composition with the natural projection $\pi : \mathrm{Sym}(\mathfrak{g}_{\bar{1}}) \to \mathscr{W}(\mathfrak{g}_{\bar{1}})$ defines an isomorphism

$$\sigma : S(\mathfrak{g}_{\bar{1}}) \to \mathscr{W}(\mathfrak{g}_{\bar{1}}).$$

The *symmetric square* is the Lie algebra $S^2(\mathfrak{g}_{\bar{1}})$ spanned by $\{x_i x_j, \partial_i \partial_j : i \leq j\} \cup \{\partial_i x_j\}$. Under the isomorphism above, the image of $S^2(\mathfrak{g}_{\bar{1}})$ is the Lie algebra $\sigma(S^2(\mathfrak{g}_{\bar{1}}))$ that is spanned by

$$\sigma(x_i x_j) = x_i x_j, \qquad \sigma(\partial_i \partial_j) = \partial_i \partial_j, \qquad \sigma(\partial_i x_j) = \partial_i x_j - \frac{1}{2}\delta_{ij}$$

in the basis $\{x_i, \partial_j\}$. A direct calculation yields an isomorphism of Lie algebras

$$(\sigma(S^2(\mathfrak{g}_{\bar{1}})), [\cdot,\cdot]_W) \cong (\mathfrak{sp}(\mathfrak{g}_{\bar{1}}), [\cdot,\cdot]).$$

Altogether, combining the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$, together with the isomorphism of Lie algebras, we obtain the Lie algebra homomorphism

$$\alpha : \mathfrak{g}_{\bar{0}} \longrightarrow \mathscr{W}(\mathfrak{g}_{\bar{1}}).$$

For later calculations, the Lie algebra morphism is explicitly given by [67, Equation 10]:

$$\alpha(X) = \sum_{k,j=1}^{mn} (B(X,[\partial_k,\partial_j])x_k x_j + B(X,[x_k,x_j])\partial_k\partial_j) \\ - \sum_{k,j=1}^{mn} 2B(X,[x_k,\partial_j])x_j\partial_k - \sum_{l=1}^{mn} B(X,[\partial_l,x_l])$$

for any $X \in \mathfrak{g}_{\bar{0}}$.

The Lie algebra morphism $\alpha : \mathfrak{g}_{\bar{0}} \to \mathscr{W}(\mathfrak{g}_{\bar{1}})$ allows us to define a diagonal embedding:

$$\mathfrak{g}_{\bar{0}} \to \mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}}), \qquad X \mapsto X \otimes 1 + 1 \otimes \alpha(X).$$

We denote the image of $\mathfrak{g}_{\bar{0}}, \mathfrak{U}(\mathfrak{g}_{\bar{0}})$, and $\mathfrak{Z}(\mathfrak{g}_{\bar{0}})$ by $\mathfrak{g}_{\bar{0},\Delta}, \mathfrak{U}(\mathfrak{g}_{\bar{0},\Delta})$, and $\mathfrak{Z}(\mathfrak{g}_{\bar{0},\Delta})$, respectively. Furthermore, the image of the quadratic Casimir $\Omega_{\mathfrak{g}_{\bar{0}}}$ of $\mathfrak{g}_{\bar{0}}$ is denoted by $\Omega_{\mathfrak{g}_{\bar{0},\Delta}}$. The quadratic Casimir of $\mathfrak{g}$ is denoted by $\Omega_{\mathfrak{g}}$. Then the following lemma follows from a direct calculation.

Lemma 7.1.2 ([65, 90])**.** *Let $\{W_k\}$ be an orthonormal basis for $\mathfrak{g}_{\bar{0}}$ with respect to $B(\cdot,\cdot)$. Then the following assertion holds:*

a) $\Omega_{\mathfrak{g}_{\bar{0}}} = \sum_k W_k^2$.

b) $\Omega_{\mathfrak{g}} = \Omega_{\mathfrak{g}_{\bar{0}}} + 2\sum_i (x_i\partial_i - \partial_i x_i)$.

c) $\Omega_{\mathfrak{g}_{\bar{0},\Delta}} = \sum_k \left(W_k^2 \otimes 1 + 2W_k \otimes \alpha(W_k) + 1 \otimes \alpha(W_k)^2\right)$.

d) $C := \sum_k \alpha(W_k)^2$ *is a constant.*

The Dirac operator

Having fixed the basic notion of the Weyl algebra, we can now introduce the *Dirac operator*.

Definition 7.1.3 ([66])**.** The *(algebraic) Dirac operator* D is the element in $\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$ given by:

$$\mathrm{D} = 2\sum_{k=1}^{mn} (\partial_k \otimes x_k - x_k \otimes \partial_k) \in \mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}}).$$

The operator D is independent of the chosen basis for $\mathfrak{g}_{\bar{1}}$, and it is $\mathfrak{g}_{\bar{0}}$-invariant under the $\mathfrak{g}_{\bar{0}}$-action on $\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$, induced by the adjoint action on both factors [67, Lemma 10.2.1], *i.e.*, $[\mathfrak{g}_{\bar{0}}, \mathrm{D}] = 0$. Additionally, similar to the (algebraic) Dirac operator for reductive Lie algebras, D has a nice square.

Proposition 7.1.4 ([66, Proposition 2])**.** *The Dirac operator* $\mathrm{D} \in \mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$ *has square*

$$\mathrm{D}^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{g}_{\bar{0},\Delta}} - C,$$

where C is the constant of Lemma 7.1.2.

Remark 7.1.5. For explicit calculations, it is particularly useful to express the square of the Dirac operator in the following alternative form:

$$\mathrm{D}^2 = 2\sum_{k,l=1}^{mn} ([\partial_k,\partial_l] \otimes x_k x_l + [x_k,x_l] \otimes \partial_k\partial_l - 2[\partial_k,x_l] \otimes x_k\partial_l) - 4\sum_{k=1}^{mn} x_k\partial_k \otimes 1.$$

For the remainder of this article, we will interpret the Dirac operator as a refined version with respect to the fixed positive system $\mathfrak{n}_{\bar{1}}^{+} = \mathfrak{p}_1 \oplus \mathfrak{q}_2$ introduced in Section 2.1.4. In this regard, we decompose the Dirac operator as $\mathrm{D} = \mathrm{D}^{\mathfrak{p}_1} + \mathrm{D}^{\mathfrak{q}_2}$, where

$$\mathrm{D}^{\mathfrak{p}_1} = 2(\mathrm{d}^{\mathfrak{p}_1} - \delta^{\mathfrak{p}_1}), \qquad \mathrm{d}^{\mathfrak{p}_1} = \sum_{k=1}^{pn} \partial_k \otimes x_k, \qquad \delta^{\mathfrak{p}_1} = \sum_{k=1}^{pn} x_k \otimes \partial_k,$$

$$\mathrm{D}^{\mathfrak{q}_2} = 2(\mathrm{d}^{\mathfrak{q}_2} - \delta^{\mathfrak{q}_2}), \qquad \mathrm{d}^{\mathfrak{q}_2} = \sum_{k=pn+1}^{mn} \partial_k \otimes x_k, \qquad \delta^{\mathfrak{q}_2} = \sum_{k=pn+1}^{mn} x_k \otimes \partial_k.$$

Lemma 7.1.6. *The operators* $\mathrm{d}^{\mathfrak{p}_1}, \mathrm{d}^{\mathfrak{q}_2}$ *and* $\delta^{\mathfrak{p}_1}, \delta^{\mathfrak{q}_2}$ *are* $\mathfrak{k}^{\mathbb{C}}$*-invariant, meaning* $[\mathrm{d}^{\mathfrak{p}_1,\mathfrak{q}_2}, \mathfrak{k}^{\mathbb{C}}] = 0$ *and* $[\delta^{\mathfrak{p}_1,\mathfrak{q}_2}, \mathfrak{k}^{\mathbb{C}}] = 0$.

Proof. We prove the statement only for $\mathrm{d}^{\mathfrak{p}_1} = \sum_{k=1}^{pn} \partial_k \otimes x_k$; the rest follows analogously.

Let $X \in \mathfrak{k}^{\mathbb{C}}$. Since $\mathfrak{k}^{\mathbb{C}}$ leaves $\mathfrak{p}_1$ invariant by Lemma 7.0.1, the commutators in the standard basis are given by

$$[X, x_i] = -2\sum_{k=1}^{pn} B(X, [x_i, \partial_k])x_k, \qquad [X, \partial_i] = 2\sum_{k=1}^{pn} B(X, [\partial_i, x_k])\partial_k$$

for all $1 \leq i \leq pn$. This leads to the following computation:

$$\begin{aligned}
[X, \mathrm{d}^{\mathfrak{p}_1}] &= \sum_{i=1}^{pn} ([X, \partial_i] \otimes x_i + \partial_i \otimes [X, x_i]) \\
&= \sum_{i=1}^{pn} \left(\sum_{k=1}^{pn} (-2B(X, [x_i, \partial_k]) + 2B(X, [\partial_i, x_k])) \right) \partial_i \otimes x_i \\
&= 0,
\end{aligned}$$

where we used in the last equality $B(X, [\partial_i, x_k]) = B(X, [x_k, \partial_i])$ for all $1 \leq i, k \leq pn$. □

7.1.2. Dirac cohomology

The Dirac cohomology assigns to any $\mathfrak{g}$-supermodule M an $\mathfrak{g}_{\bar{0}}$-supermodule $\mathrm{H}_{\mathrm{D}}(M)$, utilizing the $\mathfrak{g}_{\bar{0}}$-invariance of the Dirac operator D. To define it, we consider the natural componentwise action of $\mathrm{D} \in \mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$ on $M \otimes M(\mathfrak{g}_{\bar{1}})$, where $M(\mathfrak{g}_{\bar{1}})$ denotes the oscillator module over $\mathscr{W}(\mathfrak{g}_{\bar{1}})$, which we briefly introduce.

Oscillator module

The Weyl algebra $\mathscr{W}(\mathfrak{g}_{\bar{1}})$, identified with the algebra of differential operators with polynomial coefficients in the variables x_i, where ∂_i corresponds to the partial derivative $\frac{\partial}{\partial x_i}$, has a natural simple module, the *oscillator module* $M(\mathfrak{g}_{\bar{1}}) := \mathbb{C}[x_1, \ldots, x_{mn}]$. The following proposition is standard.

Proposition 7.1.7. *The oscillator module* $M(\mathfrak{g}_{\bar{1}})$ *is a simple* $\mathscr{W}(\mathfrak{g}_{\bar{1}})$*-module.*

We equip $M(\mathfrak{g}_{\bar{1}})$ with a $\mathbb{Z}_2$-grading by declaring $M(\mathfrak{g}_{\bar{1}})_{\bar{0}}$ to be the subspace generated by homogeneous polynomials of even degree, and $M(\mathfrak{g}_{\bar{1}})_{\bar{1}}$ to be the subspace generated by homogeneous polynomials of odd degree.

Additionally, $M(\mathfrak{g}_{\bar{1}})$ carries a unique Hermitian form $\langle\cdot,\cdot\rangle_{M(\mathfrak{g}_{\bar{1}})}$, known as the *Bargmann–Fock Hermitian form* or *Fischer–Fock Hermitian form* [5, 41, 42], such that ∂_k and x_k are adjoint to each other and the following orthogonality relations hold:

$$\langle\prod_{k=1}^{mn} x_k^{p_k}, \prod_{k=1}^{mn} x_k^{q_k}\rangle_{M(\mathfrak{g}_{\bar{1}})} = \begin{cases} \prod_{k=1}^{mn} p_k! & \text{if } p_k = q_k \text{ for all } k, \\ 0 & \text{otherwise.} \end{cases}$$

For the remainder of this article, we regard the oscillator module as a tuple $(M(\mathfrak{g}_{\bar{1}}), \langle\cdot,\cdot\rangle_{M(\mathfrak{g}_{\bar{1}})})$. We also record the adjointness of ∂_k and x_k with respect to $\langle\cdot,\cdot\rangle_{M(\mathfrak{g}_{\bar{1}})}$ in the following lemma, for future reference.

Lemma 7.1.8. *For any $v, w \in M(\mathfrak{g}_{\bar{1}})$, the generators of $\mathscr{W}(\mathfrak{g}_{\bar{1}})$ satisfy the following relations for all $1 \leq k \leq mn$:*

$$\langle\partial_k v, w\rangle_{M(\mathfrak{g}_{\bar{1}})} = \langle v, x_k w\rangle_{M(\mathfrak{g}_{\bar{1}})}, \qquad \langle x_k v, w\rangle_{M(\mathfrak{g}_{\bar{1}})} = \langle v, \partial_k w\rangle_{M(\mathfrak{g}_{\bar{1}})}.$$

Moreover, under the embedding $\alpha : \mathfrak{g}_{\bar{0}} \to \mathscr{W}(\mathfrak{g}_{\bar{1}})$, we may view $M(\mathfrak{g}_{\bar{1}})$ as a $\mathfrak{g}_{\bar{0}}$-module.

Lemma 7.1.9. *The oscillator module $M(\mathfrak{g}_{\bar{1}})$ is a unitarizable $\mathfrak{g}_{\bar{0}}$-module under the action induced by $\alpha : \mathfrak{g}_{\bar{0}} \to \mathscr{W}(\mathfrak{g}_{\bar{1}})$. In particular, $M(\mathfrak{g}_{\bar{1}})$ is $\mathfrak{g}_{\bar{0}}$-semisimple.*

The proof of this lemma follows directly from the explicit form of the embedding $\alpha : \mathfrak{g}_{\bar{0}} \to \mathscr{W}(\mathfrak{g}_{\bar{1}})$ and Lemma 7.1.8. In particular, $M(\mathfrak{g}_{\bar{1}})$ has a natural interpretation as the oscillator module for $\mathfrak{g}_{\bar{0}}$ [80], and its $\mathfrak{g}_{\bar{0}}$-constituents are often referred to as *ladder modules*.

Furthermore, we can view $M(\mathfrak{g}_{\bar{1}})$ as a $(\mathfrak{g}_{\bar{0}}, \mathfrak{k}^{\mathbb{C}})$-module, which allows us to compare the action of $\mathfrak{k}^{\mathbb{C}}$ on $M(\mathfrak{g}_{\bar{1}})$ induced by α with the natural action of $\mathfrak{k}^{\mathbb{C}}$ on $\mathbb{C}[x_1, \dots, x_{mn}]$ arising from the adjoint action. This is the best comparison available, since $\mathfrak{n}_{\bar{1}}^-$ is only $\mathfrak{k}^{\mathbb{C}}$-invariant. These two $\mathfrak{g}_{\bar{0}}$-modules are related by the following proposition.

Proposition 7.1.10. *The action of $\alpha(\mathfrak{k}^{\mathbb{C}})$ on $M(\mathfrak{g}_{\bar{1}})$ and the adjoint action of $\mathfrak{k}^{\mathbb{C}}$ on the polynomial ring $\mathbb{C}[x_1, \dots, x_{mn}]$ differ by a twist of $\mathbb{C}_{-\rho_{\bar{1}}}$. In particular, we have an isomorphism of $\mathfrak{k}^{\mathbb{C}}$-modules:*

$$M(\mathfrak{g}_{\bar{1}}) \cong \mathbb{C}[x_1, \dots, x_{mn}] \otimes \mathbb{C}_{-\rho_{\bar{1}}}.$$

Proof. First, we recall the commutation relations from Lemma 7.0.1:

$$[\partial_k, \partial_l] \in \mathfrak{n}_{\bar{0},n}^+, \quad [x_k, x_l] \in \mathfrak{n}_{\bar{0},n}^-, \quad [x_k, \partial_l] \in \mathfrak{h}, \quad [\mathfrak{k}^{\mathbb{C}}, \mathfrak{n}_{\bar{0},n}^{\pm}] \subset \mathfrak{n}_{\bar{0},n}^{\pm}$$

for all $1 \leq k, l \leq mn$. These relations imply that $B(X, [\partial_k, \partial_l]) = 0$ and $B(X, [x_k, x_l]) = 0$ for any $X \in \mathfrak{k}^{\mathbb{C}}$. Consequently, $\alpha : \mathfrak{g}|_{\mathfrak{k}^{\mathbb{C}}} \to \mathscr{W}(\mathfrak{g}_{\bar{1}})$ is given by

$$\alpha(X) = -\sum_{k,j=1}^{mn} 2B(X, [x_k, \partial_j])x_j\partial_k - \sum_{l=1}^{mn} B(X, [\partial_l, x_l]).$$

We claim that $\alpha(X)P = [X, P]$ for any $X \in \mathfrak{k}^{\mathbb{C}}$ and any polynomial $P \in \mathbb{C}[x_1, \dots, x_{mn}] \otimes \mathbb{C}_{-\rho_{\bar{1}}}$, where $\mathbb{C}_{-\rho_{\bar{1}}}$ is identified with the one-dimensional $\mathfrak{k}^{\mathbb{C}}$-module generated by the constant polynomial 1. By linearity, it is sufficient to consider monomials of the form $P = \prod_{k=1}^{mn} x_k^{r_k} \otimes 1$, where $r_k \in \mathbb{Z}_+$.

Since $\mathfrak{n}_{\bar{1}}^-$ is $\mathfrak{k}^{\mathbb{C}}$-invariant, we have the following expression in the standard basis:

$$[X, x_i] = -2 \sum_{k=1}^{mn} B([X, x_i], \partial_k) x_k = -2 \sum_{k=1}^{mn} B(X, [x_i, \partial_k]) x_k.$$

This implies

$$\alpha(X) = \sum_{k=1}^{mn} [X, x_k] \partial_k - \sum_{l=1}^{mn} B(X, [\partial_l, x_l]).$$

The first term acts non-trivially only on $\prod_{l=1}^{mn} x_l^{r_l}$, while the second term acts as a scalar multiple of the identity. Hence, it suffices to consider the action on 1.

To compute this, we evaluate the action of the two terms separately. First, we have

$$\sum_{k=1}^{mn} [X, x_k] \partial_k P = \sum_{k=1}^{mn} [X, x_k] \frac{r_k}{x_k} \prod_{l=1}^{mn} x_l^{r_l} = \sum_{k=1}^{mn} x_1^{r_1} \cdots [X, x_k^{r_k}] \cdots x_{mn}^{r_{mn}} = [X, \prod_{l=1}^{mn} x_l^{r_l}] = [X, P].$$

Next, we compute the action of $-\sum_{l=1}^{mn} B(X, [x_l, \partial_l])$ on 1. Since $[x_l, \partial_l] \in \mathfrak{h}$, the action is trivial unless $X = H \in \mathfrak{h}$:

$$-\sum_{l=1}^{mn} B(H, [\partial_l, x_l]) \cdot 1 = -\sum_{l=1}^{mn} B(H, E_{ll} + E_{n+l,n+l}) \cdot 1 = -\frac{1}{2} \sum_{\alpha \in \Delta_{\bar{1}}^+} \alpha(H) \cdot 1 = -\rho_{\bar{1}}(H) \cdot 1.$$

This concludes the proof. □

Dirac cohomology

We defined the Dirac operator D as an element of $\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$. As such, it acts naturally on $M \otimes M(\mathfrak{g}_{\bar{1}})$, where M is a $\mathfrak{U}(\mathfrak{g})$-supermodule and $M(\mathfrak{g}_{\bar{1}})$ is the oscillator module. The action is defined componentwise as follows:

$$(X \otimes X')(v \otimes P) := Xv \otimes X'P$$

for all $X \in \mathfrak{U}(\mathfrak{g})$, $X' \in \mathscr{W}(\mathfrak{g}_{\bar{1}})$, and $v \otimes P \in M \otimes M(\mathfrak{g}_{\bar{1}})$. It has been shown that D is $\mathfrak{g}_{\bar{0}}$-invariant under the $\mathfrak{g}_{\bar{0}}$-action on $\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$, which is induced by the adjoint action on both factors. This motivates the following definition.

Definition 7.1.11 ([66])**.** Let M be a $\mathfrak{g}$-supermodule. Consider the action of the Dirac operator $\mathrm{D} \in \mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$ on $M \otimes M(\mathfrak{g}_{\bar{1}})$. The *Dirac cohomology* of M is the $\mathfrak{g}_{\bar{0}}$-supermodule

$$\mathrm{H_D}(M) := \ker \mathrm{D} \,/\, (\ker \mathrm{D} \cap \operatorname{im} \mathrm{D}).$$

Example 7.1.12. Let $\mathcal{H}$ be the trivial $\mathfrak{g}$-supermodule. Then we have the following isomorphism of $\mathfrak{g}_{\bar{0}}$-supermodules

$$\mathrm{H_D}(\mathcal{H}) \cong M(\mathfrak{g}_{\bar{1}}),$$

i.e., $\mathrm{H_D}(\mathcal{H})$ decomposes in a direct sum of unitarizable simple $\mathfrak{g}_{\bar{0}}$-supermodules that occur with multiplicity one, the ladder modules.

In general, Dirac cohomology defines a functor

$$\mathrm{H_D}(\cdot):\ \mathfrak{g}\textbf{-smod} \longrightarrow \mathfrak{g}_{\bar{0}}\textbf{-smod}, \qquad M \mapsto \mathrm{H_D}(M),$$

which admits a natural decomposition, $\mathrm{H_D}(M) = \mathrm{H_D^+}(M) \oplus \mathrm{H_D^-}(M)$, induced by the $\mathbb{Z}_2$-grading of $M(\mathfrak{g}_{\bar{1}})$. We decompose the Dirac operator as $\mathrm{D} = \mathrm{D}^+ + \mathrm{D}^-$, where

$$\mathrm{D}^+ := \mathrm{D}\Big|_{M\otimes M(\mathfrak{g}_{\bar{1}})_{\bar{0}}} : M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{0}} \to M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{1}},$$
$$\mathrm{D}^- := \mathrm{D}\Big|_{M\otimes M(\mathfrak{g}_{\bar{1}})_{\bar{1}}} : M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{1}} \to M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{0}},$$

and define $\mathrm{H_D^+}(M) := \mathrm{H_{D^+}}(M)$ and $\mathrm{H_D^-}(M) := \mathrm{H_{D^-}}(M)$. In what follows, we study different aspects of the Dirac cohomology.

Dirac cohomology and infinitesimal characters

Dirac cohomology demonstrates its full strength when we restrict to $\mathfrak{g}$-supermodules with infinitesimal character, introduced in Section 3.1.4. Here, we say that M, viewed as an $\mathfrak{g}_{\bar{0}}$-module, has an *even infinitesimal character* $\chi^{\bar{0}} : \mathfrak{Z}(\mathfrak{g}_{\bar{0}}) \to \mathbb{C}$ if every $z \in \mathfrak{Z}(\mathfrak{g}_{\bar{0}})$ acts on M as the scalar $\chi^{\bar{0}}(z)$ times the identity. Recall that an even infinitesimal character $\chi^{\bar{0}}$ is determined, up to the dot action of the Weyl group, by some $\lambda \in \mathfrak{h}^*$ [70, Chapter 1.7]. We will denote such a character by $\chi^{\bar{0}}_\lambda$ from here on.

The following theorem is central.

Theorem 7.1.13 ([66, Theorem 6])**.** *For any $z \in \mathfrak{Z}(\mathfrak{g})$, there exists an algebra homomorphism $\zeta : \mathfrak{Z}(\mathfrak{g}) \to \mathfrak{Z}(\mathfrak{g}_{\bar{0}}) \cong \mathfrak{Z}(\mathfrak{g}_{\bar{0},\Delta})$ and a $\mathfrak{g}_{\bar{0}}$-invariant element $a \in \mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$ such that in $\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$ the following holds:*

$$z \otimes 1 = \zeta(z) + Da + aD.$$

Moreover, ζ fits into the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{Z}(\mathfrak{g}) & \xrightarrow{\ \zeta\ } & \mathfrak{Z}(\mathfrak{g}_{\bar{0}}) \\ {\scriptstyle \mathrm{HC}_{\mathfrak{g}}}\downarrow & & \downarrow{\scriptstyle \mathrm{HC}_{\mathfrak{g}_{\bar{0}}}} \\ S(\mathfrak{h})^W & \xrightarrow{\ \mathrm{res}\ } & S(\mathfrak{h})^W, \end{array}$$

where $\mathrm{HC}_{\mathfrak{g}}$ and $\mathrm{HC}_{\mathfrak{g}_{\bar{0}}}$ denote the Harish-Chandra monomorphism for $\mathfrak{g}$ and the Harish-Chandra isomorphism for $\mathfrak{g}_{\bar{0}}$, respectively.

As a direct consequence of Theorem 7.1.13, we obtain the following statement for infinitesimal characters.

Theorem 7.1.14. *Let M be a $\mathfrak{g}$-supermodule with infinitesimal character χ_Λ. If the Dirac cohomology $\mathrm{H_D}(M)$ contains a non-zero $\mathfrak{g}_{\bar{0}}$-supermodule with even infinitesimal character $\chi^{\bar{0}}_\lambda$ for some $\lambda \in \mathfrak{h}^*$, then $\chi_\Lambda(z) = \chi^{\bar{0}}_\lambda(\zeta(z))$ for all $z \in \mathfrak{Z}(\mathfrak{g})$.*

If M is a unitarizable highest weight $\mathfrak{g}$-supermodule, we obtain the following immediate corollary using Proposition 7.2.14.

Corollary 7.1.15. *Let M be a unitarizable highest weight $\mathfrak{g}$-supermodule with highest weight Λ. If the Dirac cohomology $\mathrm{H}_{\mathrm{D}}(M)$ contains a non-zero $\mathfrak{g}_{\bar{0}}$-supermodule with even infinitesimal character $\chi_{\lambda}^{\bar{0}}$ for some $\lambda \in \mathfrak{h}^*$, then*

$$\Lambda - \rho_{\bar{1}} = w(\lambda + \rho_{\bar{0}}) - \rho_{\bar{0}}$$

for some $w \in W$.

Both the square of the Dirac operator (Proposition 7.1.4) and Dirac cohomology (Theorem 7.1.14) are most useful when M decomposes completely under $\mathfrak{g}_{\bar{0}}$. This is the case for unitarizable $\mathfrak{g}$-supermodules.

In the subsequent, we are interested in categorical properties of the Dirac cohomology.

7.1.3. Dirac induction

For general $\mathfrak{g}$-supermodules, Dirac cohomology does not exhibit good functorial behavior. Specifically, as a functor, Dirac cohomology generally lacks an adjoint, and $\mathrm{H}_{\mathrm{D}}(\cdot)$ satisfies a six-term exact sequence only when the supermodules involved admit an infinitesimal character

Lemma 7.1.16. *Let $0 \to M' \xrightarrow{a} M \xrightarrow{b} M'' \to 0$ be a short exact sequence of $\mathfrak{g}$-supermodules that admit an infinitesimal character. Then there exists a six-term exact sequence:*

$$\begin{array}{ccccc} \mathrm{H}_{\mathrm{D}}^{+}(M') & \longrightarrow & \mathrm{H}_{\mathrm{D}}^{+}(M) & \longrightarrow & \mathrm{H}_{\mathrm{D}}^{+}(M'') \\ \uparrow & & & & \downarrow \\ \mathrm{H}_{\mathrm{D}}^{-}(M'') & \longleftarrow & \mathrm{H}_{\mathrm{D}}^{-}(M) & \longleftarrow & \mathrm{H}_{\mathrm{D}}^{-}(M') \end{array}$$

Proof. The proof is similar to the proof of [62, Theorem 8.1]. We tensor the short exact sequence with $M(\mathfrak{g}_{\bar{1}})$ and denote the arrows again by a and b, which get tensored with the identity on $M(\mathfrak{g}_{\bar{1}})$. As a result, we obtain a right exact sequence

$$M' \otimes M(\mathfrak{g}_{\bar{1}}) \to M \otimes M(\mathfrak{g}_{\bar{1}}) \to M'' \otimes M(\mathfrak{g}_{\bar{1}}) \to 0.$$

The horizontal arrows in the diagram are then induced by a and b. We define the vertical arrows. Let $m'' \in M'' \otimes M(\mathfrak{g}_{\bar{1}})$ represent a non-trivial Dirac-cohomology class, so that $\mathrm{D}m'' = 0$. Choose $m \in M \otimes M(\mathfrak{g}_{\bar{1}})$ such that $b(m) = m''$. We can assume $\mathrm{D}^2 m = 0$ (Proposition 7.1.4), as M admits an infinitesimal character. Indeed, assume $\mathrm{D}^2 m = cm$ for some $c \in \mathbb{C}$. Then, by Proposition 7.1.4, we have

$$0 = \mathrm{D}^2 m'' = \mathrm{D}^2 b(m) = b(\mathrm{D}^2 m) = cb(m) = cm''.$$

However, $m'' \neq 0$, yielding $c = 0$. In addition, we have $b(\mathrm{D}m) = \mathrm{D}b(m) = \mathrm{D}m'' = 0$, *i.e.*, $\mathrm{D}m$ lies in $\ker(b) = \mathrm{im}(a)$, and we find some $m' \in M$ with $\mathrm{D}m = a(m')$. The element m' defines a cohomology class, as $a(\mathrm{D}m') = \mathrm{D}a(m') = \mathrm{D}^2 m = 0$, and a is injective. This class is by definition the image of the class of m'' under the connecting homomorphism. This defines both vertical arrows. The exactness of the sequence is a direct calculation and is being omitted. □

This issue motivates an alternative formulation of Dirac cohomology, which coincides with the classical definition when restricted to unitarizable supermodules. The underlying concept and construction trace back to [111]. Following the approach of [111], we construct a functor—also referred to as *Dirac cohomology*—which admits a right adjoint and agrees with $\mathrm{H_D}(\cdot)$ on the subcategory of unitarizable $\mathfrak{g}$-supermodules. This right adjoint is called *Dirac induction*.

Any $\mathfrak{g}$-supermodule M can be extended to a $(\mathfrak{U}(\mathfrak{g})\otimes\mathscr{W}(\mathfrak{g}_{\bar{1}}))$-supermodule via the mapping $M \mapsto M \otimes M(\mathfrak{g}_{\bar{1}})$. Conversely, any $(\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}}))$-supermodule X can be restricted to a $\mathfrak{U}(\mathfrak{g})$-supermodule via the assignment $X \mapsto \mathrm{Hom}_{\mathscr{W}(\mathfrak{g}_{\bar{1}})}(M(\mathfrak{g}_{\bar{1}}), X)$, endowed with the induced $\mathfrak{U}(\mathfrak{g})$-action.

Proposition 7.1.17. *There is an equivalence of categories*

$$\mathfrak{g}\text{-}\boldsymbol{smod} \cong (\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}}))\text{-}\boldsymbol{smod}, \qquad M \mapsto M \otimes M(\mathfrak{g}_{\bar{1}})$$

with inverse $X \mapsto \mathrm{Hom}_{\mathscr{W}(\mathfrak{g}_{\bar{1}})}(M(\mathfrak{g}_{\bar{1}}), X)$.

Proof. It is well known that $\mathfrak{U}(\mathfrak{g})$-**smod** is equivalent to $\mathfrak{g}$-**smod**. For any $\mathfrak{U}(\mathfrak{g})$-supermodule M, we have the following isomorphisms:

$$\mathrm{Hom}_{\mathscr{W}(\mathfrak{g}_{\bar{1}})}(M(\mathfrak{g}_{\bar{1}}), M \otimes M(\mathfrak{g}_{\bar{1}})) \cong M \otimes \mathrm{Hom}_{\mathscr{W}(\mathfrak{g}_{\bar{1}})}(M(\mathfrak{g}_{\bar{1}}), M(\mathfrak{g}_{\bar{1}})) \cong M,$$

where the second isomorphism uses the fact that $M(\mathfrak{g}_{\bar{1}})$ is a simple $\mathscr{W}(\mathfrak{g}_{\bar{1}})$-supermodule (see Proposition 7.1.7). Conversely, for any $(\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}}))$-supermodule X, we have:

$$\mathrm{Hom}_{\mathscr{W}(\mathfrak{g}_{\bar{1}})}(M(\mathfrak{g}_{\bar{1}}), X) \otimes M(\mathfrak{g}_{\bar{1}}) \cong X,$$

where the isomorphism is given by the evaluation map, utilizing the simplicity of $M(\mathfrak{g}_{\bar{1}})$. It is straightforward to see that this map respects the $\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$-action. Altogether, we have shown that $\mathfrak{g}$-**smod** $\cong \mathfrak{U}(\mathfrak{g})$-**smod** $\cong \mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$-**smod**. □

We consider $\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$ as a $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$-module under the diagonal embedding, and denote the subspace of $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$-invariants in $\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$ by $(\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}}))^{\mathfrak{U}(\mathfrak{g}_{\bar{0}})}$. Let $\mathscr{I}$ be the two-sided ideal in $(\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}}))^{\mathfrak{U}(\mathfrak{g}_{\bar{0}})}$ generated by the Dirac operator D. Since the Dirac operator D commutes with $\mathfrak{g}_{\bar{0}}$, the ideal $\mathscr{I}$ is $\mathfrak{g}_{\bar{0}}$-invariant by construction. We define $\mathrm{H}'_{\mathrm{D}}(M)$ as the subspace of $\mathscr{I}$-invariants in $M \otimes M(\mathfrak{g}_{\bar{1}})$ for a given $\mathfrak{g}$-supermodule M, that is,

$$\mathrm{H}'_{\mathrm{D}}(M) := \{v \in M \otimes M(\mathfrak{g}_{\bar{1}}) \ : \ xv = 0 \text{ for all } x \in \mathscr{I}\}.$$

In particular, $\mathrm{H}'_{\mathrm{D}}(M)$ is an $\mathfrak{g}_{\bar{0}}$-supermodule for any $\mathfrak{g}$-supermodule M. For unitarizable $\mathfrak{g}$-supermodules M, we will show that $\mathrm{H_D}(M) \cong \mathrm{H}'_{\mathrm{D}}(M)$ in Corollary 7.2.13. For this reason, we also refer to $\mathrm{H}'_{\mathrm{D}}(\cdot)$ as Dirac cohomology.

Proposition 7.1.18. *The Dirac cohomology* $\mathrm{H}'_{\mathrm{D}}(\cdot)$ *admits a left-adjoint functor* $\mathrm{Ind_D}$: $\mathfrak{g}_{\bar{0},\Delta}$-$\boldsymbol{smod} \to (\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}}))$-$\boldsymbol{smod}$*, called* Dirac induction*, given by*

$$\mathrm{Ind_D}(V) := (\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0},\Delta})(\mathbb{C}1 \oplus \mathscr{I})} V$$

for any $\mathfrak{g}_{\bar{0}}$*-supermodule* V *with trivial* $\mathscr{I}$*-action. In particular,* $\mathrm{H}'_{\mathrm{D}}(\cdot)$ *is left-exact.*

Proof. We can understand $\mathrm{H}'_{\mathrm{D}}(\cdot)$ through the forgetful functor from $\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$ to $\mathfrak{U}(\mathfrak{g}_{\bar{0},\Delta})(\mathbb{C}1 \oplus \mathscr{I})$ and then considering $\mathscr{I}$-invariants. Indeed, $\mathfrak{U}(\mathfrak{g}_{\bar{0},\Delta})(\mathbb{C}1 \oplus \mathscr{I})$ is the $\mathfrak{g}_{\bar{0}}$-invariant subalgebra of $\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$ generated by $\mathfrak{g}_{\bar{0},\Delta}$ and $\mathscr{I}$. This has exactly as left-adjoint $(\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0},\Delta})(\mathbb{C}1 \oplus \mathscr{I})} V$. □

Corollary 7.1.19. *Let M be a $\mathfrak{g}$-supermodule. Then a submodule $V \subset \mathrm{H}_{\mathrm{D}}(M)$ appears as a $\mathfrak{g}_{\bar{0},\Delta}$-constituent if and only if M is a quotient of $\mathrm{Ind}_{\mathrm{D}}(V)$.*

7.2. Dirac operators, Dirac cohomology and unitarizable supermodules

7.2.1. Dirac operators and unitarity

Let M be a $\mathfrak{g}$-supermodule that is $\mathfrak{g}_{\bar{0}}$-semisimple, and let D denote the Dirac operator acting on $M \otimes M(\mathfrak{g}_{\bar{1}})$. If M is unitarizable, we will show that D is self-adjoint, leading to a Parthasarathy–Dirac inequality (see Proposition 7.2.4). Moreover, we demonstrate how the Dirac operator reflects the unitarity of M: specifically, the self-adjointness of D is equivalent to the contravariance of a positive definite Hermitian form. In particular, for any fixed unitarizable supermodule $\mathcal{H}$, we derive a decomposition of $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$ with respect to D^2.

We fix a unitarizable $\mathfrak{g}$-supermodule $\mathcal{H}$, equipped with a positive definite Hermitian form $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. For $M(\mathfrak{g}_{\bar{1}})$, we use the Bargmann–Fock Hermitian form $\langle \cdot, \cdot \rangle_{M(\mathfrak{g}_{\bar{1}})}$, as defined in Section 7.1.2. We then endow the $\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$-supermodule $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$ with the Hermitian form defined by

$$\langle v \otimes P, w \otimes Q \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})} := \langle v, w \rangle_{\mathcal{H}} \cdot \langle P, Q \rangle_{M(\mathfrak{g}_{\bar{1}})},$$

for all $v \otimes P, w \otimes Q \in \mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$. The Hermitian form $\langle \cdot, \cdot \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})}$ is positive definite and unique up to a real scalar, since both $\mathcal{H}$ and $M(\mathfrak{g}_{\bar{1}})$ are highest weight modules with respect to $\mathfrak{U}(\mathfrak{g})$ and $\mathscr{W}(\mathfrak{g}_{\bar{1}})$, respectively.

We extend the conjugate-linear anti-involution ω naturally to $\mathfrak{U}(\mathfrak{g})$. Moreover, ω defines a conjugate-linear anti-involution on the Weyl algebra $\mathscr{W}(\mathfrak{g}_{\bar{1}})$, and we extend it in the obvious way to a conjugate-linear anti-involution on the tensor product $\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$, denoted by the same symbol.

The following lemma is a direct consequence of Lemma 7.1.8.

Lemma 7.2.1. *For all $X \in \mathfrak{g}$ and for all generators ∂_k, x_k of $\mathscr{W}(\mathfrak{g}_{\bar{1}})$, the following identities hold:*

$$\langle (X \otimes x_k)v, w \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})} = \langle v, (\omega(X) \otimes \partial_k)w \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})},$$
$$\langle (X \otimes \partial_k)v, w \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})} = \langle v, (\omega(X) \otimes x_k)w \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})},$$

for all $v, w \in \mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$.

We decompose the Dirac operator D as $\mathrm{D} = \mathrm{D}^{\mathfrak{p}_1} + \mathrm{D}^{\mathfrak{q}_2}$, where $\mathrm{D}^{\mathfrak{p}_1}, \mathrm{D}^{\mathfrak{q}_2} \in \mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$ are $\mathfrak{k}^{\mathbb{C}}$-invariant elements. Recall that

$$\mathrm{D}^{\mathfrak{p}_1} = 2(\mathrm{d}^{\mathfrak{p}_1} - \delta^{\mathfrak{p}_1}), \qquad \mathrm{D}^{\mathfrak{q}_2} = 2(\mathrm{d}^{\mathfrak{q}_2} - \delta^{\mathfrak{q}_2}).$$

By Lemma 7.2.1 and the identity $\omega(x_k) = -\partial_k$ in $\mathfrak{g}$ for all $k = 1, \dots, mn$, the operators $\mathrm{d}^{\mathfrak{p}_1}$ and $\delta^{\mathfrak{p}_1}$, as well as $\mathrm{d}^{\mathfrak{q}_2}$ and $\delta^{\mathfrak{q}_2}$, are adjoint to each other up to sign with respect to the Hermitian form $\langle \cdot, \cdot \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})}$.

Lemma 7.2.2. *The operators* $\mathrm{d}^{\mathfrak{p}_1}$ *and* $\delta^{\mathfrak{p}_1}$*, and similarly* $\mathrm{d}^{\mathfrak{q}_2}$ *and* $\delta^{\mathfrak{q}_2}$*, are adjoint to each other with respect to the Hermitian form* $\langle \cdot, \cdot \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})}$, i.e.,

$$\langle \delta^{\mathfrak{p}_1} v, w \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})} = -\langle v, \mathrm{d}^{\mathfrak{p}_1} w \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})}, \qquad \langle \delta^{\mathfrak{q}_2} v, w \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})} = -\langle v, \mathrm{d}^{\mathfrak{q}_2} w \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})}.$$

In particular, the operators $\mathrm{D}^{\mathfrak{p}_1}$*,* $\mathrm{D}^{\mathfrak{q}_2}$*, and* D *are self-adjoint on* $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$*.*

Corollary 7.2.3. *For all* $k \in \mathbb{Z}_+$*, the operators* D*,* $\mathrm{D}^{\mathfrak{p}_1}$*, and* $\mathrm{D}^{\mathfrak{q}_2}$ *satisfy:*

$$\ker \mathrm{D} = \ker \mathrm{D}^k, \qquad \ker \mathrm{D}^{\mathfrak{p}_1} = \ker \left(\mathrm{D}^{\mathfrak{p}_1} \right)^k, \qquad \ker \mathrm{D}^{\mathfrak{q}_2} = \ker \left(\mathrm{D}^{\mathfrak{q}_2} \right)^k.$$

Another direct consequence of Lemma 7.2.2 is an inequality for D, known as *Parthasarathy's Dirac inequality*, or simply the *Dirac inequality*.

Proposition 7.2.4 (Parthasarathy's Dirac inequality)**.** *The square of the Dirac operator satisfies* $\mathrm{D}^2 \geq 0$ *on* $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$*; that is,*

$$\langle \mathrm{D}^2 v, v \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})} \geq 0 \quad \textit{for all } v \in \mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}).$$

Proof. Let $v \in \mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$ be arbitrary. By Lemma 7.2.2, the Dirac operator D is self-adjoint with respect to the Hermitian form $\langle \cdot, \cdot \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})}$. Moreover, this form is positive definite by construction. Consequently,

$$\langle \mathrm{D}v, \mathrm{D}v \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})} = \langle \mathrm{D}^2 v, v \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})} \geq 0,$$

which proves the claim. □

Let M be a simple $\mathfrak{g}$-supermodule equipped with a positive definite Hermitian form, and assume that M is $\mathfrak{g}_{\bar{0}}$-semisimple. A natural question is: when does the Dirac operator D act as a self-adjoint operator on $M \otimes M(\mathfrak{g}_{\bar{1}})$?

Theorem 7.2.5. *Let* M *be a simple* $\mathfrak{g}$*-supermodule equipped with a positive definite Hermitian form* $\langle \cdot, \cdot \rangle_M$*, such that* $M_{\bar{0}}$ *and* $M_{\bar{1}}$ *are mutually orthogonal. Assume that* M *is* $\mathfrak{g}_{\bar{0}}$*-semisimple. Then the following are equivalent:*

a) $(M, \langle \cdot, \cdot \rangle_M)$ *is a unitarizable* $\mathfrak{g}$*-supermodule.*

b) The Dirac operator D *is self-adjoint with respect to* $\langle \cdot, \cdot \rangle_{M \otimes M(\mathfrak{g}_{\bar{1}})}$*.*

Proof. If M is unitarizable with respect to $\langle \cdot, \cdot \rangle_M$, then D is self-adjoint on $M \otimes M(\mathfrak{g}_{\bar{1}})$ by Lemma 7.2.2.

Conversely, assume D is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{M \otimes M(\mathfrak{g}_{\bar{1}})}$. It suffices to show that the operators $\partial_k \otimes 1$ and $x_k \otimes 1$ act supersymmetrically, since they generate $\mathfrak{g}_{\bar{1}}$ and $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{0}}$. A direct calculation shows:

$$\mathrm{D}(1 \otimes \partial_k) - (1 \otimes \partial_k)\,\mathrm{D} = -2(\partial_k \otimes 1), \qquad (1 \otimes x_k)\,\mathrm{D} - \mathrm{D}(1 \otimes x_k) = 2(x_k \otimes 1).$$

Taking adjoints yields:

$$(\mathrm{D}(1\otimes\partial_k)-(1\otimes\partial_k)\,\mathrm{D})^\dagger=(1\otimes x_k)\,\mathrm{D}-\mathrm{D}(1\otimes x_k),$$

using Lemmas 7.1.8 and 7.2.2.

Consequently, for all $v,w\in M$, we compute:

$$\begin{aligned}\langle\partial_k v,w\rangle_M&=\langle(\partial_k\otimes 1)(v\otimes 1),w\otimes 1\rangle_{M\otimes M(\mathfrak{g}_{\bar 1})}\\&=-\tfrac{1}{2}\langle(\mathrm{D}(1\otimes\partial_k)-(1\otimes\partial_k)\,\mathrm{D})(v\otimes 1),w\otimes 1\rangle_{M\otimes M(\mathfrak{g}_{\bar 1})}\\&=-\tfrac{1}{2}\langle v\otimes 1,((1\otimes x_k)\,\mathrm{D}-\mathrm{D}(1\otimes x_k))(w\otimes 1)\rangle_{M\otimes M(\mathfrak{g}_{\bar 1})}\\&=-\langle v\otimes 1,(x_k\otimes 1)(w\otimes 1)\rangle_{M\otimes M(\mathfrak{g}_{\bar 1})}\\&=-\langle v,x_k w\rangle_M.\end{aligned}$$

This completes the proof. □

In general, the Kac induction of a unitarizable $\mathfrak{g}_{\bar 0}$-supermodule is not itself unitarizable. Moreover, it may happen that a simple $\mathfrak{g}$-supermodule M restricts to a unitarizable $\mathfrak{g}_{\bar 0}$-module, even though M is not unitarizable as a $\mathfrak{g}$-supermodule. The Dirac operator provides a criterion that helps clarify this situation.

Corollary 7.2.6. *Let M be a simple $\mathfrak{g}$-supermodule equipped with a positive definite Hermitian form $\langle\cdot,\cdot\rangle_M$, such that $M_{\bar 0}$ and $M_{\bar 1}$ are mutually orthogonal. Assume that M admits an infinitesimal character and is $\mathfrak{g}_{\bar 0}$-semisimple. Then M is unitarizable as a $\mathfrak{g}$-supermodule if and only if the following conditions hold:*

a) All eigenvalues of D^2 on $M\otimes M(\mathfrak{g}_{\bar 1})$ are positive.

b) For all $v,w\in M\otimes M(\mathfrak{g}_{\bar 1})$, and for each eigenvalue λ^2 of D^2, we have

$$\langle \mathrm{D}v,\mathrm{D}w\rangle_{M\otimes M(\mathfrak{g}_{\bar 1})}=\lambda^2\langle v,w\rangle_{M\otimes M(\mathfrak{g}_{\bar 1})}.$$

Proof. We adapt the argument from [108, Corollary 2]. By assumption and Lemma 7.1.9, the $\mathfrak{g}_{\bar 0}$-module $M\otimes M(\mathfrak{g}_{\bar 1})$ decomposes completely into irreducibles.

Moreover, by Propositions 7.1.4 and 7.2.4, the operator D^2 acts on each $\mathfrak{g}_{\bar 0}$-isotypic component as a scalar $\lambda^2\in\mathbb{R}$. If this is not the case, then M cannot be unitarizable.

On each such component, D has eigenvalues $\pm\lambda$, and these components are mutually orthogonal with respect to the Hermitian form. The operator D is self-adjoint on $\ker(\mathrm{D}^2-\lambda^2)$ if and only if the $+\lambda$ and $-\lambda$ eigenspaces are orthogonal.

Indeed, for any $v\in\ker(\mathrm{D}^2-\lambda^2)$, the span of v and $\mathrm{D}v$ is D-invariant and contains the eigenvectors $\lambda v\pm\mathrm{D}v$ with eigenvalues $\pm\lambda$. These eigenspaces are orthogonal if and only if

$$\langle\lambda v+\mathrm{D}v,\lambda w-\mathrm{D}w\rangle_{M\otimes M(\mathfrak{g}_{\bar 1})}=0$$

for all $v,w\in\ker(\mathrm{D}^2-\lambda^2)$. A direct computation shows that this is equivalent to

$$\langle \mathrm{D}v,\mathrm{D}w\rangle_{M\otimes M(\mathfrak{g}_{\bar 1})}=\lambda^2\langle v,w\rangle_{M\otimes M(\mathfrak{g}_{\bar 1})}.$$

□

We now decompose the $\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$-supermodule $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$, where $\mathcal{H}$ is a fixed unitarizable $\mathfrak{g}$-supermodule, with respect to the Dirac operator D.

Since $\mathcal{H}$ is unitarizable and simple, it is completely reducible as both a $\mathfrak{g}$- and an $\mathfrak{g}_{\bar{0}}$-supermodule (see Proposition 6.1.8). Assume $\mathcal{H}$ decomposes as a direct sum of unitarizable simple $\mathfrak{g}$-supermodules. The oscillator module $M(\mathfrak{g}_{\bar{1}})$ is also unitarizable and completely reducible as an $\mathfrak{g}_{\bar{0}}$-module, decomposing into unitarizable highest weight constituents. Consequently, $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$ is a completely reducible unitarizable $\mathfrak{g}_{\bar{0}}$-supermodule. This decomposition is known to be a direct sum.

On any simple $\mathfrak{g}$-constituent of $\mathcal{H}$, the quadratic Casimir acts as a scalar multiple of the identity. Similarly, on any simple $\mathfrak{g}_{\bar{0}}$-constituent of $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$, the even quadratic Casimir acts as a scalar by Dixmier's Theorem [32, Proposition 2.6.8]. By Theorem 7.1.4, it follows that D^2 is a semisimple operator on $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$, and we may decompose $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$ into eigenspaces $(\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}))(c)$ corresponding to eigenvalue c, that is,

$$\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}) = \bigoplus_{c} (\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}))(c),$$

where the direct sum is orthogonal with respect to the Hermitian form $\langle \cdot, \cdot \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})}$.

Each eigenspace is a $\mathfrak{g}_{\bar{0}}$-supermodule, since $[\mathrm{D}, \mathfrak{g}_{\bar{0}}] = 0$. In particular, the eigenspace with eigenvalue $c = 0$ is precisely the kernel of D, as $\ker \mathrm{D} = \ker \mathrm{D}^k$ for all $k \in \mathbb{Z}_+$.

More generally, such a decomposition with respect to D^2 exists for any $\mathfrak{g}$-supermodule $\mathcal{H}$ of finite length that is $\mathfrak{g}_{\bar{0}}$-semisimple.

Lemma 7.2.7. *Let M be a $\mathfrak{g}$-supermodule that admits an infinitesimal character and is $\mathfrak{g}_{\bar{0}}$-semisimple. Then, as an $\mathfrak{g}_{\bar{0}}$-supermodule, it decomposes into a direct sum of generalized D^2-eigenspaces:*

$$M \otimes M(\mathfrak{g}_{\bar{1}}) = \bigoplus_{c \in \mathbb{C}} (M \otimes M(\mathfrak{g}_{\bar{1}}))(c),$$

where

$$(M \otimes M(\mathfrak{g}_{\bar{1}}))(c) := \left\{ v \in M \otimes M(\mathfrak{g}_{\bar{1}}) \mid \exists n \in \mathbb{Z}_+ \text{ such that } (c \cdot \mathrm{id} - \mathrm{D}^2)^n v = 0 \right\}.$$

The proof proceeds *mutatis mutandis* as in [1, Corollary 3.3].

Lemma 7.2.8. *Let $\mathcal{H}$ be a unitarizable $\mathfrak{g}$-supermodule. Then there is an orthogonal decomposition:*

$$\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}) = \ker \mathrm{D}^2 \oplus \operatorname{im} \mathrm{D}^2,$$

where the direct sum is orthogonal with respect to the Hermitian form $\langle \cdot, \cdot \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})}$.

Proof. First, we recall the eigenspace decomposition with respect to D^2:

$$\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}) = (\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}))(0) \oplus \bigoplus_{c \neq 0} (\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}))(c),$$

where the zero eigenspace satisfies $(\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}))(0) = \ker \mathrm{D}$.

Since D is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})}$, it follows that $\operatorname{im} \mathrm{D} = (\ker \mathrm{D})^{\perp}$, *i.e.*., the image and kernel are orthogonal complements. Hence,

$$\operatorname{im} \mathrm{D} = \ker \mathrm{D}^{\perp} = (\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}))(0)^{\perp} = \bigoplus_{c \neq 0} (\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}))(c).$$

Since $\ker \mathrm{D}^2 = \ker \mathrm{D}$, and $\operatorname{im} \mathrm{D}^2 = \operatorname{im} \mathrm{D}$, this yields the desired orthogonal decomposition:

$$\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}) = \ker \mathrm{D}^2 \oplus \operatorname{im} \mathrm{D}^2 . \qquad \square$$

7.2.2. Dirac cohomology and unitarity

The Dirac cohomology functor

$$\mathrm{H_D}(\cdot) : \mathfrak{g}\textbf{-smod} \to \mathfrak{g}_{\bar{0},\Delta}\textbf{-smod} \cong \mathfrak{g}_{\bar{0}}\textbf{-smod}$$

assigns to a $\mathfrak{g}$-supermodule M its Dirac cohomology,

$$\mathrm{H_D}(M) := \ker \mathrm{D} \,/\, (\ker \mathrm{D} \cap \operatorname{im} \mathrm{D}) \,.$$

In this section, we investigate the Dirac cohomology of unitarizable $\mathfrak{g}$-supermodules and show that, in this setting, Dirac cohomology provides a complete characterization.

Basic properties

For unitarizable supermodules, Dirac cohomology admits a particularly simple description: it coincides with the kernel of the Dirac operator.

Proposition 7.2.9. *Let $\mathcal{H}$ be a unitarizable $\mathfrak{g}$-supermodule. Then*

$$\mathrm{H_D}(\mathcal{H}) = \ker \mathrm{D} \,.$$

Proof. We prove that $\ker \mathrm{D} \cap \operatorname{im} \mathrm{D} = \{0\}$. Let $v \in \ker \mathrm{D} \cap \operatorname{im} \mathrm{D}$, so that $\mathrm{D}\, v = 0$ and there exists $w \in \mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$ with $v = \mathrm{D}\, w$. Since $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$ is equipped with a positive definite Hermitian form, we compute:

$$\langle v, v \rangle = \langle \mathrm{D}\, w, v \rangle = \langle w, \mathrm{D}\, v \rangle = 0.$$

Thus $v = 0$, and the claim follows. □

As an immediate consequence, Dirac cohomology is additive on direct sums of unitarizable supermodules.

Lemma 7.2.10. *Let $\mathcal{H}_1, \mathcal{H}_2$ be unitarizable $\mathfrak{g}$-supermodules. Then*

$$\mathrm{H_D}(\mathcal{H}_1 \oplus \mathcal{H}_2) = \mathrm{H_D}(\mathcal{H}_1) \oplus \mathrm{H_D}(\mathcal{H}_2).$$

We now express the Dirac operator as $\mathrm{D} = \mathrm{D}^{\mathfrak{p}_1} + \mathrm{D}^{\mathfrak{q}_2}$, where $\mathrm{D}^{\mathfrak{p}_1} = 2(\mathrm{d}^{\mathfrak{p}_1} - \delta^{\mathfrak{p}_1})$, $\mathrm{D}^{\mathfrak{q}_2} = 2(\mathrm{d}^{\mathfrak{q}_2} - \delta^{\mathfrak{q}_2})$, and $\mathrm{d}^{\mathfrak{p}_1}, \mathrm{d}^{\mathfrak{q}_2}, \delta^{\mathfrak{p}_1}, \delta^{\mathfrak{q}_2}$ are $\mathfrak{k}^{\mathbb{C}}$-invariant operators. As shown in Lemma 7.2.2, these operators are anti-adjoint to one another with respect to the Hermitian form $\langle \cdot, \cdot \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})}$. This decomposition provides a more explicit understanding of the kernel of D. We now record some structural relations among the constituent operators.

Lemma 7.2.11. *Let $\mathcal{H}$ be a unitarizable $\mathfrak{g}$-supermodule. Then the following assertions hold:*

a) The operators $\mathrm{d}^{\mathfrak{p}_1}, \delta^{\mathfrak{p}_1}, \mathrm{d}^{\mathfrak{q}_2}, \delta^{\mathfrak{q}_2}$ square to zero.

b) The operators $\mathrm{d}^{\mathfrak{p}_1}$ and $\delta^{\mathfrak{q}_2}$, and the operators $\mathrm{d}^{\mathfrak{q}_2}$ and $\delta^{\mathfrak{p}_1}$ commute, i.e., $[\mathrm{d}^{\mathfrak{p}_1}, \delta^{\mathfrak{q}_2}] = 0$ *and* $[\mathrm{d}^{\mathfrak{q}_2}, \delta^{\mathfrak{p}_1}] = 0$.

c) With respect to the form $\langle \cdot, \cdot \rangle_{M \otimes M(\mathfrak{g}_{\bar{1}})}$, the following holds:

(i) $\operatorname{im} \mathrm{d}^{\mathfrak{p}_1}$ *is orthogonal to* $\ker \delta^{\mathfrak{p}_1}$ *and* $\operatorname{im} \delta^{\mathfrak{p}_1}$*;* $\operatorname{im} \delta^{\mathfrak{p}_1}$ *is orthogonal to* $\ker \mathrm{d}^{\mathfrak{p}_1}$*.*

(ii) $\operatorname{im} \mathrm{d}^{\mathfrak{q}_2}$ *is orthogonal to* $\ker \delta^{\mathfrak{q}_2}$ *and* $\operatorname{im} \delta^{\mathfrak{q}_2}$*;* $\operatorname{im} \delta^{\mathfrak{q}_2}$ *is orthogonal to* $\ker \mathrm{d}^{\mathfrak{q}_2}$*.*

d) $\ker(\mathrm{D}^{\mathfrak{p}_1})^2 = \ker \mathrm{D}^{\mathfrak{p}_1} = \ker \mathrm{d}^{\mathfrak{q}_2} \cap \ker \delta^{\mathfrak{p}_1}$*, and* $\ker(\mathrm{D}^{\mathfrak{q}_2})^2 = \ker \mathrm{D}^{\mathfrak{q}_2} = \ker \mathrm{d}^{\mathfrak{q}_2} \cap \ker \delta^{\mathfrak{q}_2}$*.*

Proof. a) This is a direct consequence of the fact that $\mathfrak{p}_{1,2}$ and $\mathfrak{q}_{1,2}$ are abelian Lie super-subalgebras of $\mathfrak{g}$.

b) This is a direct consequence of $[\partial_k, x_l] = 0$ unless $k = l$.

c) We only prove that $\operatorname{im} \mathrm{d}^{\mathfrak{p}_1}$ and $\ker \delta^{\mathfrak{p}_1}$ are orthogonal; the rest can be proven similarly using a) and b). First, let $v \in \operatorname{im} \mathrm{d}^{\mathfrak{p}_1}$ and $w \in \ker \delta^{\mathfrak{p}_1}$. Then there exists a non-trivial $v' \in \mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$ such that $\mathrm{d}^{\mathfrak{p}_1} v' = v$, and consequently by Lemma 7.2.2

$$\langle v, w \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})} = \langle \mathrm{d}^{\mathfrak{p}_1} v', w \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})} = -\langle v', \delta^{\mathfrak{p}_1} w \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})} = 0,$$

i.e., $\langle \ker \mathrm{d}^{\mathfrak{q}_2}, \ker \delta^{\mathfrak{p}_1} \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})} = 0$.

d) The operators $\mathrm{D}^{\mathfrak{p}_1}$ and $\mathrm{D}^{\mathfrak{q}_2}$ are self-adjoint by Lemma 7.2.2, and therefore $\ker \mathrm{D}^{\mathfrak{p}_1} = \ker(\mathrm{D}^{\mathfrak{p}_1})^2$ and $\ker \mathrm{D}^{\mathfrak{q}_2} = \ker(\mathrm{D}^{\mathfrak{q}_2})^2$. We prove that $\ker \mathrm{D}^{\mathfrak{p}_1} = \ker \mathrm{d}^{\mathfrak{p}_1} \cap \ker \delta^{\mathfrak{p}_1}$. Let $v \in \ker \mathrm{D}^{\mathfrak{p}_1}$, then $\mathrm{D}^{\mathfrak{p}_1} v = 2(\mathrm{d}^{\mathfrak{p}_1} - \delta^{\mathfrak{p}_1})v = 0$, *i.e.*, $\mathrm{d}^{\mathfrak{p}_1} v = \delta^{\mathfrak{p}_1} v$. By b), $\operatorname{im} \mathrm{d}^{\mathfrak{p}_1}$ and $\operatorname{im} \delta^{\mathfrak{p}_1}$ are orthogonal to each other, hence $v \in \ker \delta^{\mathfrak{p}_1} \cap \ker \mathrm{d}^{\mathfrak{q}_2}$. The other inclusion is trivial. Analogously, the equality $\ker \mathrm{D}^{\mathfrak{q}_2} = \ker \mathrm{d}^{\mathfrak{q}_2} \cap \ker \delta^{\mathfrak{q}_2}$ follows. □

In summary, we can describe the Dirac cohomology of unitarizable supermodules in terms of $\ker \mathrm{D}^{\mathfrak{p}_1}$ and $\ker \mathrm{D}^{\mathfrak{q}_2}$.

Lemma 7.2.12. *The Dirac cohomology of a unitarizable* $\mathfrak{g}$*-supermodule* $\mathcal{H}$ *is*

$$\mathrm{H}_{\mathrm{D}}(\mathcal{H}) = \ker \mathrm{D}^{\mathfrak{p}_1} \cap \ker \mathrm{D}^{\mathfrak{q}_2}.$$

Proof. We decompose the Dirac operator as $\mathrm{D} = \mathrm{D}^{\mathfrak{p}_1} + \mathrm{D}^{\mathfrak{q}_2}$, such that

$$\ker \mathrm{D} = (\ker \mathrm{D}^{\mathfrak{p}_1} \cap \ker \mathrm{D}^{\mathfrak{q}_2}) \cup \{v \in \mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}) : \mathrm{D}^{\mathfrak{p}_1} v = -\mathrm{D}^{\mathfrak{q}_2} v\}.$$

Assume $v := m \otimes P \in \mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$ satisfies $w := \mathrm{D}^{\mathfrak{p}_1} v = -\mathrm{D}^{\mathfrak{q}_2} v$. Then

$$\begin{aligned} 0 &\leq \langle w, w \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})} \\ &= -\langle \mathrm{D}^{\mathfrak{p}_1} v, \mathrm{D}^{\mathfrak{q}_2} v \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})} \\ &= -\sum_{k=1}^{pn} \sum_{l=pn+1}^{mn} \langle (\partial_k - x_k)m, (\partial_l - x_l)m \rangle_{\mathcal{H}} \langle (x_k - \partial_k)P, (x_l - \partial_l)P \rangle_{M(\mathfrak{g}_{\bar{1}})}. \end{aligned}$$

However, for all $1 \leq k \leq pn$ and $p + 1 \leq l \leq mn$, we have

$$\langle (x_k - \partial_k)P \rangle_{\mathcal{H}}, (x_l - \partial_l)P \rangle_{M(\mathfrak{g}_{\bar{1}})} = 0$$

by the construction of the Bargmann–Fock form and $k \neq l$. We conclude $w = 0$, as the Hermitian form is positive definite, *i.e.*, $\mathrm{D}^{\mathfrak{p}_1} v = 0$ and $\mathrm{D}^{\mathfrak{q}_2} v = 0$ or $v \in (\ker \mathrm{D}^{\mathfrak{p}_1} \cap \ker \mathrm{D}^{\mathfrak{q}_2})$. The assertion now follows from Proposition 7.2.9. □

In Section 7.1.3, we introduced a left exact functor $\mathrm{H}'_{\mathrm{D}}(\cdot)$, which we also referred to as Dirac cohomology. It coincides with the Dirac cohomology $\mathrm{H}_{\mathrm{D}}(\cdot)$ on unitarizable $\mathfrak{g}$-supermodules.

Corollary 7.2.13. *Let $\mathcal{H}$ be a unitarizable $\mathfrak{g}$-supermodule. Then the Dirac cohomologies* $\mathrm{H}_{\mathrm{D}}(\mathcal{H})$ *and* $\mathrm{H}'_{\mathrm{D}}(\mathcal{H})$ *coincide. In particular,* $\mathrm{H}_{\mathrm{D}}(\cdot)$ *is left exact in the category of unitarizable $\mathfrak{g}$-supermodules.*

Proof. The Dirac cohomology of a unitarizable $\mathfrak{g}$-supermodule $\mathcal{H}$ is $\mathrm{H}_{\mathrm{D}}(\mathcal{H}) = \ker \mathrm{D} = \ker \mathrm{D}^2$ by Corollary 7.2.3 and Proposition 7.2.9. Moreover, D^2 commutes with any element of $(\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}}))^{\mathfrak{U}(\mathfrak{g}_{\bar{0}})}$ by Proposition 7.1.4, and it is an element of $\mathscr{I}$, where $\mathscr{I}$ is the two-sided ideal in $(\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}}))^{\mathfrak{U}(\mathfrak{g}_{\bar{0}})}$ generated by the Dirac operator D. The Dirac cohomology $\mathrm{H}'_{\mathrm{D}}(\mathcal{H})$ is defined as the $\mathscr{I}$-invariants in $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$. The relation $\mathrm{H}'_{\mathrm{D}}(\mathcal{H}) \subset \ker \mathrm{D}^2$ is immediate. Conversely, for any $v \in \mathrm{H}_{\mathrm{D}}(\mathcal{H})$, we have $DXv = 0$ for any $X \in (\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}}))^{\mathfrak{U}(\mathfrak{g}_{\bar{0}})}$, as $\mathrm{D}^2 Xv = X\mathrm{D}^2 v = 0$. This shows $\mathrm{H}_{\mathrm{D}}(\mathcal{H}) = \mathrm{H}'_{\mathrm{D}}(\mathcal{H})$. The second assertion follows by Proposition 7.1.18. □

The Dirac cohomology $\mathrm{H}_{\mathrm{D}}(\mathcal{H})$ inherits the structure of a unitarizable $\mathfrak{g}_{\bar{0}}$-supermodule. Since the oscillator module $M(\mathfrak{g}_{\bar{1}})$ is itself a unitarizable $\mathfrak{g}_{\bar{0}}$-supermodule (see Section 7.1.2), it follows that $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$ – under the diagonal embedding – is also a unitarizable $\mathfrak{g}_{\bar{0}}$-supermodule. Given that the Dirac operator D is invariant, it follows that $\ker \mathrm{D}$ likewise carries the structure of a unitarizable $\mathfrak{g}_{\bar{0}}$-supermodule.

Proposition 7.2.14. *The Dirac cohomology* $\mathrm{H}_{\mathrm{D}}(\mathcal{H})$ *of a unitarizable $\mathfrak{g}$-supermodule $\mathcal{H}$ is a unitarizable $\mathfrak{g}_{\bar{0}}$-supermodule. In particular, if $\mathcal{H}$ is simple, it decomposes completely in unitarizable simple $\mathfrak{g}_{\bar{0}}$-supermodules.*

Computation of Dirac cohomology

Having established the basic properties of the Dirac cohomology for unitarizable supermodules, we now proceed to compute it explicitly. We begin by noting that unitarizable $\mathfrak{g}$-supermodules possess non-trivial Dirac cohomology.

Proposition 7.2.15. *Let $\mathcal{H}$ be a unitarizable highest weight $\mathfrak{g}$-supermodule with highest weight Λ. Then* $\mathrm{H}_{\mathrm{D}}(\mathcal{H})$ *contains a highest weight $\mathfrak{g}_{\bar{0}}$-supermodule with highest weight $\Lambda - \rho_{\bar{1}}$ that occurs with multiplicity one. In particular,* $\mathrm{H}_{\mathrm{D}}(\mathcal{H}) \neq \{0\}$.

Proof. Let v_Λ be the highest weight vector of $\mathcal{H}$, meaning that $\mathfrak{n}^+ v_\Lambda = 0$, and specifically $\mathfrak{n}^+_{\bar{1}} v_\Lambda = 0$. Since 1 is constant and annihilated by ∂_k, the vector $v_\Lambda \otimes 1$ lies in the kernel of $\mathrm{D} = 2\sum_{k=1}^{mn}(\partial_k \otimes x_k - x_k \otimes \partial_k)$. Further, we assert that $v_\Lambda \otimes 1$ generates a highest weight $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$-supermodule.

Any element $X \in \mathfrak{g}_{\bar{0}}$ acts on $v_\Lambda \otimes 1$ via the diagonal embedding:

$$Xv_\Lambda \otimes 1 + v_\Lambda \otimes \alpha(X)1,$$

and we recall:

$$\alpha(X) = \sum_{k,j=1}^{mn} (B(X, [\partial_k, \partial_j])x_k x_j + B(X, [x_k, x_j])\partial_k \partial_j) - \sum_{k,j=1}^{mn} 2B(X, [x_k, \partial_j])x_j \partial_k - \sum_{l=1}^{mn} B(X, [\partial_l, x_l]).$$

By Lemma 7.0.1, we have the commutation relations $[\partial_k, \partial_j] \in \mathfrak{n}_{\bar{0}}^+$, $[x_k, x_j] \in \mathfrak{n}_{\bar{0}}^-$ and $[x_k, \partial_j] \in \mathfrak{h}$. Let $X \in \mathfrak{n}_{\bar{0}}^+$. Then the definition of $B(\cdot, \cdot)$ forces

$$Xv_\Lambda \otimes 1 + v_\Lambda \otimes \alpha(X)1 = v_\Lambda \otimes \sum_{k,j=1}^{mn} B(X, [x_k, x_j])\partial_k \partial_j 1 = 0,$$

where we use that $\mathfrak{n}_{\bar{0}}^+ v_\Lambda = 0$.

Any $H \in \mathfrak{h}$ acts on $v_\Lambda \otimes 1$ by

$$\begin{aligned} Hv_\Lambda \otimes 1 + v_\Lambda \otimes \alpha(H)v_\Lambda &= \Lambda(H)v_\Lambda \otimes 1 - v_\Lambda \otimes \sum_{l=1}^{mn} B(H, [\partial_l, x_l])1 \\ &= \Lambda(H)v_\Lambda \otimes 1 - v_\Lambda \otimes \sum_{\alpha \in \Delta_{\bar{1}}^+} \frac{1}{2}\alpha(H)1 \\ &= (\Lambda - \rho_{\bar{1}})(H)(v_\Lambda \otimes 1). \end{aligned}$$

Hence, $v_\Lambda \otimes 1 \in \mathrm{H_D}(\mathcal{H})$ generates a (simple) highest weight $\mathfrak{g}_{\bar{0}}$-supermodule, that is in particular unitarizable by Proposition 7.2.14. The module appears with multiplicity one, as the weight spaces of v_Λ and 1 are one-dimensional. □

The following corollary is an immediate consequence of Proposition 13.2.19.

Corollary 7.2.16. *Let $\mathcal{H}$ be a unitarizable $\mathfrak{g}$-supermodule. Then $\mathrm{H_D}(\mathcal{H})$ is non-trivial.*

To compute $\mathrm{H_D}(\mathcal{H})$ explicitly, we use Proposition 7.1.4. For that, we relate the constant C with the Weyl vector $\rho = \rho_{\bar{0}} - \rho_{\bar{1}}$.

Lemma 7.2.17. *The constant is $C = -(\rho_{\bar{1}} - 2\rho_{\bar{0}}, \rho_{\bar{1}})$.*

Proof. Let $\mathcal{H}$ be a unitary simple highest weight $\mathfrak{g}$-supermodule with highest weight Λ. By Proposition 7.2.15, there exists a $\mathfrak{g}_{\bar{0}}$-supermodule in $\mathrm{H_D}(\mathcal{H})$ with highest weight $\Lambda - \rho_{\bar{1}}$. As $\mathrm{H_D}(\mathcal{H}) = \ker \mathrm{D} = \ker \mathrm{D}^2$, we have by Theorem 7.1.14:

$$0 = -(\Lambda + 2\rho, \Lambda) + (\Lambda - \rho_{\bar{1}} + 2\rho_{\bar{0}}, \Lambda - \rho_{\bar{1}}) + C,$$

and a direct calculation yields

$$(\Lambda - \rho_{\bar{1}} + 2\rho_{\bar{0}}, \Lambda - \rho_{\bar{1}}) = (\Lambda + 2\rho, \Lambda) + (\rho_{\bar{1}} - 2\rho_{\bar{0}}, \rho_{\bar{1}}).$$

This finishes the proof. □

Using the identity $\mathrm{H_D}(\mathcal{H}) = \ker \mathrm{D} = \ker \mathrm{D}^2$ together with the expression $\mathrm{D}^2 = \mathrm{D}^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{g}_{\bar{0}},\Delta} - C$, and Lemma 7.2.17, we obtain the following proposition.

Proposition 7.2.18. *Let $\mathcal{H}$ be a unitarizable highest weight $\mathfrak{g}$-supermodule with highest weight Λ. Then a $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\mu)$ in $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$ belongs to $\mathrm{H_D}(M)$ if and only if*

$$(\Lambda + 2\rho, \Lambda) = (\mu + 2\rho, \mu).$$

Proof. The square $\mathrm{D}^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{g}_{\bar{0}},\Delta} + C$ acts on $L_0(\mu)$ by the scalar

$$-(\Lambda + \rho, \Lambda) + (\mu - \rho_{\bar{1}} + 2\rho_{\bar{0}}, \mu - \rho_{\bar{1}}) - (\rho_{\bar{1}} - 2\rho_{\bar{0}}, \rho_{\bar{1}}) = -(\Lambda + 2\rho, \Lambda) + (\mu + 2\rho, \mu),$$

which concludes the proof by $\mathrm{H_D}(\mathcal{H}) = \ker \mathrm{D} = \ker \mathrm{D}^2$. □

The following corollary is straightforward.

Corollary 7.2.19. *Let $\mathcal{H}$ be a unitarizable highest weight $\mathfrak{g}$-supermodule, and let $L_0(\mu)$ be a $\mathfrak{g}_{\bar{0}}$-constituent with highest weight $\mu = \Lambda - \alpha$ for some odd root α, and highest weight vector v_μ. Then the following assertions are equivalent:*

a) $v_\mu \otimes 1$ *belongs to* $\mathrm{H_D}(\mathcal{H})$.

b) $(\Lambda + \rho, \alpha) = 0$.

We now compute the Dirac cohomology of unitarizable simple $\mathfrak{g}$-supermodules, using Vogan's Theorem for Lie superalgebras (see Theorem 7.1.14). Since any unitarizable $\mathfrak{g}$-supermodule is completely reducible and $\mathrm{H_D}(\cdot)$ is additive, it suffices to consider unitarizable simple $\mathfrak{g}$-supermodules.

Theorem 7.2.20. *The Dirac cohomology of a non-trivial unitarizable simple $\mathfrak{g}$-supermodule $L(\Lambda)$ with highest weight Λ is*

$$\mathrm{H_D}(L(\Lambda)) = L_0(\Lambda - \rho_{\bar{1}}).$$

Proof. We decompose $\mathrm{H_D}(L(\Lambda))$ in its $\mathfrak{g}_{\bar{0}}$-constituents by Proposition 7.2.14. Then the module $L_0(\Lambda - \rho_{\bar{1}})$ is a simple constituent by Proposition 7.2.15, and unitarizable. Moreover, it is a relative holomorphic $\mathfrak{g}_{\bar{0}}$-supermodule by Harish-Chandra's condition:

$$(\Lambda - \rho_{\bar{1}} + \rho_{\bar{0}}, \epsilon_1 - \epsilon_m) = \lambda_1 - \lambda_m + m - 1 - n < 0$$

as $n \geq m$ and $\lambda_1 - \lambda_m \leq 0$, *i.e.*, $\Lambda - \rho_{\bar{1}} \in \mathcal{D}$, where $\mathcal{D}$ denotes the set of all Harish-Chandra parameters of relative holomorphic discrete series $\mathfrak{g}_{\bar{0}}$-modules. In particular, as $\Delta^+_{\bar{0}}$ is fixed, $\Lambda - \rho_{\bar{1}}$ is unique in its W-linkage class being the highest weight of a unitarizable highest weight $\mathfrak{g}_{\bar{0}}$-supermodule.

Any simple $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\mu)$ is a highest weight $\mathfrak{g}_{\bar{0}}$-supermodule by Proposition 7.2.14, and the highest weight μ is of the form

$$w \cdot (\Lambda - \rho_{\bar{1}}) = w \cdot \Lambda - \rho_{\bar{1}} = \mu,$$

by Theorem 7.1.14, where $w \in W$ and " $\cdot$ " is the even *dot action* of the Weyl group given by $w \cdot \lambda = w(\lambda + \rho_{\bar{0}}) - \rho_{\bar{0}}$ for any $w \in W$ and $\lambda \in \mathfrak{h}^*$. Thus, by uniqueness, w must be the identity and $\Lambda - \rho_{\bar{1}} = \mu$. In addition, the multiplicity is one by Proposition 7.2.15. This concludes the proof. □

Interestingly, as observed in the proof of Theorem 7.2.20, any $\mathfrak{g}_{\bar{0}}$-constituent of $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$ lies in the relative holomorphic discrete series. We record this observation in the following corollary.

Corollary 7.2.21. *Let $\mathcal{H}$ be a unitarizable $\mathfrak{g}$-supermodule. Then any $\mathfrak{g}_{\bar{0}}$-constituent in $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$ belongs to the relative holomorphic discrete series.*

The $\mathfrak{g}_{\bar{0}}$-supermodule generated by $1 \in M(\mathfrak{g}_{\bar{1}})$ is entirely concentrated in even parity due to the $\mathbb{Z}_2$-grading of $M(\mathfrak{g}_{\bar{1}})$. This leads to the following corollary.

Corollary 7.2.22. *Let $\mathcal{H}$ be a non-trivial unitarizable simple $\mathfrak{g}$-supermodule. Then*

$$\mathrm{H_D}(\mathcal{H}) \cong \mathrm{H_D^+}(\mathcal{H}).$$

Combining Proposition 7.2.18, Corollary 7.2.19, and Theorem 7.2.20, we obtain the following corollary.

Corollary 7.2.23. *Let $\mathcal{H}$ be a unitarizable simple highest weight $\mathfrak{g}$-supermodule with highest weight Λ. If $(\Lambda+\rho,\alpha)=0$ for some odd positive root α, then $L_0(\Lambda-\alpha)$ does not appear as a $\mathfrak{g}_{\bar{0}}$-constituent in $\mathcal{H}$.*

Furthermore, unitarizable $\mathfrak{g}$-supermodules are uniquely determined by their Dirac cohomology.

Theorem 7.2.24. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two unitarizable $\mathfrak{g}$-supermodules. Then $\mathcal{H}_1 \cong \mathcal{H}_2$ as $\mathfrak{g}$-supermodules if and only if $\mathrm{H_D}(\mathcal{H}_1) \cong \mathrm{H_D}(\mathcal{H}_2)$ as $\mathfrak{g}_{\bar{0}}$-supermodules.*

Proof. Unitarizable $\mathfrak{g}$-supermodules are completely reducible (Proposition 13.2.19), and the Dirac cohomology is additive. It is therefore enough to consider unitarizable highest weight $\mathfrak{g}$-supermodules, say $\mathcal{H}_1 = L(\Lambda_1)$ and $\mathcal{H}_2 = L(\Lambda_2)$ for some $\Lambda_1, \Lambda_2 \in \mathfrak{h}^*$.

Then, by Theorem 7.2.20, the associated Dirac cohomology is $\mathrm{H_D}(\mathcal{H}_1) = L_0(\Lambda_1 - \rho_{\bar{1}})$ and $\mathrm{H_D}(\mathcal{H}_2) = L_0(\Lambda_2-\rho_{\bar{1}})$. The statement follows now with Proposition 3.1.13, as two highest weight modules with respect to $\Delta_{\bar{0}}^+$ are isomorphic if and only if they have the same highest weight. ☐

Finally, as a direct application, we calculate the Dirac cohomology of a class of Verma supermodules, for which each simple constituent in its composition series is a unitarizable $\mathfrak{g}$-supermodule. Specifically, any $M(\Lambda)$ has a composition series (Proposition 3.1.15):

$$\{0\} = M_0 \subset M_1 \subset \ldots \subset M_n = M(\Lambda)$$

such that $M_{i+1}/M_i \cong L(\Lambda_i)$ for some simple $\mathfrak{g}$-supermodules $L(\Lambda_i)$, each Λ_i being a simple highest weight $\mathfrak{g}_{\bar{0}}$-supermodule. We assume that each quotient supermodule $L(\Lambda_i)$ is a unitarizable highest weight $\mathfrak{g}$-supermodule with highest weight Λ_i. Furthermore, by [73, Theorem 2.5, Corollary 2.7], we can assume $\Lambda_{i+1}-\Lambda_i$ is a non-trivial sum of pairwise different odd roots. We fix such a Verma supermodule $M(\Lambda)$.

Proposition 7.2.25. *The Dirac cohomology of $M(\Lambda)$ is given by*

$$\mathrm{H_D}(M(\Lambda)) = \bigoplus_i \mathrm{H_D}(L(\Lambda_i)) \cong \bigoplus_i L_0(\Lambda_i - \rho_{\bar{1}}).$$

Proof. The short exact sequence

$$0 \to L(\Lambda_0) \to M_2 \to L(\Lambda_1) \cong M_2/M_1 \to 0$$

induces the following short exact sequence of $\mathfrak{g}_{\bar{0}}$-supermodules (Lemma 7.1.16, Corollary 7.2.22, and Theorem 7.2.20), where parity is left implicit:

$$0 \to L_0(\Lambda_0-\rho_{\bar{1}}) \to \mathrm{H}_{\mathrm{D}}^{+}(M_2) \to L_0(\Lambda_1-\rho_{\bar{1}}) \to 0.$$

The difference $\Lambda_1-\Lambda_0$ is a non-trivial sum of odd roots, and $\Lambda_{0,1}-\rho_{\bar{1}} \in \mathcal{D}$ with respect to the fixed Borel $\mathfrak{b}$. In particular, the Weyl orbit does not contain any other highest weight of a unitarizable $\mathfrak{g}_{\bar{0}}$-supermodules with respect to $\mathfrak{b}$. We conclude that $L_0(\Lambda_0-\rho_{\bar{1}})$ and

$L_0(\Lambda_1 - \rho_{\bar{1}})$ have different infinitesimal characters for $\mathfrak{g}_{\bar{0}}$. By Wigner's Lemma (*cf.* [96, Theorem 1.1]) the exact sequence splits, and

$$\mathrm{H}^+_{\mathrm{D}}(M_2) = \mathrm{H}_{\mathrm{D}}(L(\Lambda_0)) \oplus \mathrm{H}_{\mathrm{D}}(L(\Lambda_1)).$$

On the other hand, we can read off from the six-term exact sequence in Lemma 7.1.16 that $\mathrm{H}^-_{\mathrm{D}}(M_2) = 0$ holds as $\mathrm{H}^-_{\mathrm{D}}(L(\Lambda_{0,1})) = 0$, and therefore

$$\mathrm{H}_{\mathrm{D}}(M_2) = \mathrm{H}_{\mathrm{D}}(L(\Lambda_0)) \oplus \mathrm{H}_{\mathrm{D}}(L(\Lambda_1)).$$

Using the fact that Ext and direct sums commute, along with the ordering of the weights, and the observation that direct sums and Dirac cohomology commute for unitarizable $\mathfrak{g}$-supermodules, the proposition follows. □

7.3. Complementary perspectives

In this section, we study complementary perspectives on Dirac cohomology. In Section 7.3.1, we present a novel characterization of unitarity via the Dirac inequality (Theorem 7.3.2). This leads to an $\mathfrak{g}_{\bar{0}}$-decomposition of unitarizable simple $\mathfrak{g}$-supermodules with respect to the Lie subalgebra $\mathfrak{g}_{\bar{0}}$ (Theorem 7.3.4). In Section 7.3.2, we introduce an analog of Kostant's cohomology and show that, as $\mathfrak{k}^{\mathbb{C}}$-modules, it is isomorphic to Dirac cohomology (Theorem 7.3.11). We then define the Dirac index in Section 7.3.3 and prove that it coincides with the Euler characteristic of Dirac cohomology. Finally, in Section 7.3.4, we derive $\mathfrak{k}^{\mathbb{C}}$-character formulas for unitarizable supermodules using Kostant's cohomology (Theorem 7.3.15) and the Dirac index (Theorem 7.3.18).

7.3.1. $\mathfrak{g}_{\bar{0}}$-Decomposition and unitarity

For a given $\mathfrak{g}$-supermodule M, a key task is to decompose M under $\mathfrak{g}_{\bar{0}}$, as this allows for the reduction of certain problems to the well-studied case of complex reductive Lie algebras. By Proposition 6.1.8, if M is unitarizable, it is completely reducible under $\mathfrak{g}_{\bar{0}}$, and the Dirac inequality holds strictly. This leads to the natural question: given an $\mathfrak{g}_{\bar{0}}$-semisimple $\mathfrak{g}$-supermodule, under what conditions on its $\mathfrak{g}_{\bar{0}}$-constituents is it unitarizable?

In Section 6.1.3, we established that every simple highest weight $\mathfrak{g}$-supermodule M possesses a unique non-degenerate contravariant Hermitian form, derived from the Shapovalov form of the Verma supermodule. Consequently, M is unitarizable if and only if the Hermitian form is positive definite, which occurs precisely when the Dirac inequality holds strictly for each $\mathfrak{g}_{\bar{0}}$-constituent. To establish this result, we require the following lemma.

Lemma 7.3.1. *Let M be a simple highest weight $\mathfrak{g}$-supermodule with highest weight Λ. Assume M is $\mathfrak{g}_{\bar{0}}$-semisimple. Let $L_0(\mu)$ be a simple $\mathfrak{g}_{\bar{0}}$-constituent with highest weight μ and $v \in L_0(\mu) \setminus \{0\}$ some vector. Then the following assertion holds:*

$$((\mu + 2\rho, \mu) - (\Lambda + 2\rho, \Lambda))\langle v, v\rangle_M = 4\sum_{k=1}^{mn} \langle \partial_k v, \partial_k v\rangle_M.$$

Proof. The statement follows directly from the following calculation:

$$\begin{aligned}((\mu+2\rho,\mu)-(\Lambda+2\rho,\Lambda))\langle v,v\rangle_M &= ((\mu+2\rho,\mu)-(\Lambda+2\rho,\Lambda))\langle v\otimes 1, v\otimes 1\rangle_{M\otimes M(\mathfrak{g}_{\bar{1}})}\\ &= \langle \mathrm{D}^2(v\otimes 1), v\otimes 1\rangle_{M\otimes M(\mathfrak{g}_{\bar{1}})}\\ &= -4\sum_{k=1}^{mn}\langle x_k\partial_k v\otimes 1, v\otimes 1\rangle_{M\otimes M(\mathfrak{g}_{\bar{1}})}\\ &= 4\sum_{k=1}^{mn}\langle \partial_k v\otimes 1, \partial_k v\otimes 1\rangle_{M\otimes M(\mathfrak{g}_{\bar{1}})}\\ &= 4\sum_{k=1}^{mn}\langle \partial_k v, \partial_k v\rangle_M.\end{aligned}$$

The first equation follows from definition of the super Hermitian form on $M\otimes M(\mathfrak{g}_{\bar{1}})$, while the second equation follows from Theorem 7.1.4 and the observation that $v\otimes 1$ lies in a $\mathfrak{g}_{\bar{0}}$-submodule of $M\otimes M(\mathfrak{g}_{\bar{1}})$ with highest weight $\mu-\rho_{\bar{1}}$. The second uses the explicit form of D^2 given in Remark 7.1.5, *i.e.*,

$$\begin{aligned}\mathrm{D}^2(v\otimes 1) = \Bigg(2\sum_{k,l=1}^{mn}\Big([\partial_k,\partial_l]\otimes x_k x_l + [x_k,x_l]\otimes\partial_k\partial_l - 2[\partial_k,x_l]\otimes x_k\partial_l\Big)\\ -4\sum_{k=1}^{mn}x_k\partial_k\otimes 1\Bigg)v\otimes 1 = 2\sum_{k,l=1}^{mn}[\partial_k,\partial_l]v\otimes x_k x_l - 4\sum_{k=1}^{mn}x_k\partial_k v\otimes 1,\end{aligned}$$

and

$$\langle 2\sum_{k,l}[\partial_k,\partial_l]v\otimes x_k x_l, v\otimes 1\rangle_{M\otimes M(\mathfrak{g}_{\bar{1}})} = 2\sum_{k,l=1}^{mn}\langle[\partial_k,\partial_l]v,v\rangle_M\langle x_k x_l, 1\rangle_{M(\mathfrak{g}_{\bar{1}})} = 0,$$

since $\langle x_k x_l, 1\rangle_{M(\mathfrak{g}_{\bar{1}})} = \langle x_l, \partial_k 1\rangle_{M(\mathfrak{g}_{\bar{1}})} = 0$ for all $1\le k,l\le mn$. The third equality uses the contravariance of $\langle\cdot,\cdot\rangle_{M\otimes M(\mathfrak{g}_{\bar{1}})}$ and the fact that $\omega(x_k) = -\partial_k$ for all $1\le k\le mn$. □

Theorem 7.3.2. *Let M be a simple highest weight $\mathfrak{g}$-supermodule with highest weight Λ that is $\mathfrak{g}_{\bar{0}}$-semisimple. Then M is unitarizable if and only if the highest weight $\mathfrak{g}_{\bar{0}}$-supermodule $L_0(\Lambda)$ is unitarizable and the Dirac inequality holds strictly on each simple highest weight $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\mu)$ with highest weight $\mu\neq\Lambda$ of M embedded in $M\otimes M(\mathfrak{g}_{\bar{1}})$,* i.e.,

$$(\mu+2\rho,\mu) > (\Lambda+2\rho,\Lambda).$$

Proof. Assume first that M is unitarizable, meaning that the Hermitian form $\langle\cdot,\cdot\rangle_M$ is positive definite; in particular, $L_0(\Lambda)$ is a unitarizable $\mathfrak{g}_{\bar{0}}$-supermodule. Let $v\in M$ with $v\notin L_0(\Lambda)$. Then, $\mathfrak{n}^+_{\bar{1}}v\neq 0$ since $[\mathfrak{n}^+_{\bar{1}},\mathfrak{n}^+_{\bar{1}}] = \mathfrak{n}^+_{0,n}$, meaning v cannot lie in a $\mathfrak{k}^{\mathbb{C}}$-module that generates a proper submodule, as $L(\Lambda)$ is simple. By Lemma 7.3.1, we have

$$((\mu+2\rho,\mu)-(\Lambda+2\rho,\Lambda))\langle v,v\rangle_M = 4\sum_{k=1}^{mn}\langle\partial_k v,\partial_k v\rangle_M,$$

which leads – by the strict positivity of the Hermitian form – to $(\mu+2\rho,\mu)-(\Lambda+2\rho,\Lambda) > 0$, thus establishing one direction of the result.

Assume now that $L_0(\Lambda)$ is unitarizable and that the Dirac inequality holds strictly for each simple highest weight $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\mu)$ with highest weight $\mu \neq \Lambda$. We decompose M under $\mathfrak{g}_{\bar{0}}$ as in Corollary 6.1.19 and note in general (Proposition 3.1.13):

$$M = \mathfrak{U}(\mathfrak{n}^-)v_\Lambda = \bigwedge^\bullet(\mathfrak{n}_{\bar{1}}^-)\mathfrak{U}(\mathfrak{n}_{\bar{0}}^-)v_\Lambda,$$

where v_Λ denotes the highest weight vector of $L(\Lambda)$ and $L_0(\Lambda)$. Moreover, we note that $\mathfrak{U}(\mathfrak{n}_{\bar{0}}^-)v_\Lambda$ belongs to $L_0(\Lambda)$, hence we have there a positive definite Hermitian form by assumption.

First, consider $\bigwedge^1(\mathfrak{n}_{\bar{1}}^-)\mathfrak{U}(\mathfrak{n}_{\bar{0}}^-)v_\Lambda$. Any element $v \in \bigwedge^1(\mathfrak{n}_{\bar{1}}^-)\mathfrak{U}(\mathfrak{n}_{\bar{0}}^-)v_\Lambda$ belongs to an $\mathfrak{g}_{\bar{0}}$-constituent, say $L_0(\Lambda-\alpha)$ for some $\alpha \in \Delta^+$, ensuring that $\partial_k v \in \mathfrak{U}(\mathfrak{n}_{\bar{0}}^-)v_\Lambda$ for $1 \leq k \leq mn$ (*cf.* [73, Corollary 2.7]). Moreover, there exists at least one $1 \leq k \leq mn$ such that $\partial_k v \neq 0$. Otherwise, we have $\mathfrak{n}_{\bar{1}}^+ v = 0$, and therefore $\mathfrak{n}_{\bar{0},n}^+ v = 0$ as $[\mathfrak{n}_{\bar{1}}^+, \mathfrak{n}_{\bar{1}}^+] \subset \mathfrak{n}_{\bar{0},n}^+$. In addition, in $\mathfrak{U}(\mathfrak{k}^{\mathbb{C}})v$, the application of $\mathfrak{U}(\mathfrak{n}_{\bar{0},c}^+)$ leads to some vector $w \in \bigwedge^1(\mathfrak{n}_{\bar{1}}^-)\mathfrak{U}(\mathfrak{n}_{\bar{0}}^-)v_\Lambda$ such that $\mathfrak{n}_{\bar{0}}^+ w = 0$ and $\mathfrak{n}_{\bar{1}}^+ w = 0$, as $[\mathfrak{n}_{\bar{0},c}^+, \mathfrak{n}_{\bar{0},n}^+] = 0$ and $[\mathfrak{n}_{\bar{0},c}^+, \mathfrak{n}_{\bar{1}}^+] \subset \mathfrak{n}_{\bar{1}}^+$. This forces w to vanish under the action of $\mathfrak{U}(\mathfrak{n}^+)$, *i.e.*, it generates a proper super submodule of M. This is a contradiction, as M is simple. Now, by the unitarity of $L_0(\Lambda)$ and Lemma 7.3.1, we conclude that

$$\langle v, v\rangle_M > 0,$$

as the Dirac inequality holds strictly.

Next, consider $\bigwedge^2(\mathfrak{n}_{\bar{1}}^-)\mathfrak{U}(\mathfrak{n}_{\bar{0}}^-)v_\Lambda$ such that any $\partial_k v$ belongs to $\bigwedge^1(\mathfrak{n}_{\bar{1}}^-)\mathfrak{U}(\mathfrak{n}_{\bar{0}}^-)v_\Lambda$ for $v \in \bigwedge^2(\mathfrak{n}_{\bar{1}}^-)\mathfrak{U}(\mathfrak{n}_{\bar{0}}^-)v_\Lambda$. Since any non-trivial element w in $\bigwedge^1(\mathfrak{n}_{\bar{1}}^-)\mathfrak{U}(\mathfrak{n}_{\bar{0}}^-)v_\Lambda$ satisfies $\langle w, w\rangle_M > 0$, we conclude by Lemma 7.3.1, our assumption and a similar argument as above, that $\langle v, v\rangle_M > 0$.

By induction, the non-degenerate contravariant Hermitian form $\langle \cdot, \cdot \rangle_M$ is positive definite, implying that M is unitarizable. □

As a direct implication, we present the $\mathfrak{g}_{\bar{0}}$-decomposition of a general unitarizable simple highest weight $\mathfrak{g}$-supermodule $\mathcal{H}$ with highest weight Λ. Recall that $\mathcal{H}$ has a $\mathfrak{h}$-weight decomposition with $\mathcal{P}_{\mathcal{H}} \subset \Lambda - \mathbb{Z}_+[\Delta^+]$ by Proposition 3.1.13, where $\mathcal{P}_{\mathcal{H}} := \{\beta \in \mathfrak{h}^* : \mathcal{H}^\beta \neq \{0\}\}$ is the set of $\mathfrak{h}$-weights of $\mathcal{H}$ and

$$\Delta_{\bar{1}}^+ = \{\epsilon_i - \delta_j, -\epsilon_k + \delta_l \ : \ 1 \leq i \leq p,\ p+1 \leq l \leq m,\ 1 \leq j,l \leq n\}.$$

Following the formulation of the filtration of Verma supermodules as $\mathfrak{g}_{\bar{0}}$-supermodules in Proposition 6.1.18, let S be the set of subsets of $\Delta_{\bar{1}}^+$, and fix a naturally ordered sequence of subsets such that $\Sigma_{S_i} < \Sigma_{S_j}$ implies $i<j$, where $\Sigma_{S_k} = \sum_{\gamma \in S_k} \gamma$.

First, we combine Corollary 6.1.19 and Corollary 7.2.23 to deduce the following lemma.

Lemma 7.3.3. *Let $\mathcal{H}$ be a unitarizable simple $\mathfrak{g}$-supermodule with highest weight Λ. Then any $\mathfrak{g}_{\bar{0}}$-constituent of $\mathcal{H}$ takes the form $L_0(\Lambda - \sum_{\gamma \in S_j} \gamma)$ for some j, and S_j contains no odd root $\alpha \in \Delta_{\bar{1}}^+$ such that $(\Lambda + \rho, \alpha) = 0$.*

Now, by combining Theorem 7.3.2 and Lemma 7.3.3, we have established the following theorem.

Theorem 7.3.4. *Let $\mathcal{H}$ be a unitarizable highest-weight $\mathfrak{g}$-supermodule with highest weight $\Lambda \in \mathfrak{h}^*$. As a $\mathfrak{g}_{\bar{0}}$-module, $L(\Lambda)$ decomposes as*

$$L(\Lambda)_{\text{ev}} \cong L_0(\Lambda) \oplus \bigoplus_j L_0(\Lambda - \sum_{\gamma \in S_j} \gamma),$$

where we sum over all j such that S_j contains no odd root α with $(\Lambda + \rho, \alpha) = 0$, and the Dirac inequality holds strictly on each $\mathfrak{g}_{\bar{0}}$-constituent:

$$(\Lambda - \sum_{\gamma \in S_j} \gamma + 2\rho, \Lambda - \sum_{\gamma \in S_j} \gamma) > (\Lambda + 2\rho, \Lambda)$$

with $\sum_{\gamma \in S_j} \gamma \neq 0$.

Corollary 7.3.5. *Let $\mathcal{H}$ be a unitarizable simple $\mathfrak{g}$-supermodule. Then $\mathcal{H}$ decomposes in a finite sum of simple highest weight $\mathfrak{g}_{\bar{0}}$-supermodules. The maximal number of $\mathfrak{g}_{\bar{0}}$-constituents is $2^{\dim(\mathfrak{n}_{\bar{1}}^-)}$.*

Combining Theorem 7.3.4 with Blattner's formula in Section 5.3.5, we obtain an explicit formula for the $\mathfrak{k}^{\mathbb{C}}$-types appearing in holomorphic discrete series $\mathfrak{g}$-supermodules.

Corollary 7.3.6. *Let $\mathcal{H}$ be a discrete series $\mathfrak{g}$-supermodule with highest weight $\Lambda \in \mathfrak{h}^*$. Then the irreducible $\mathfrak{k}^{\mathbb{C}}$-module F^μ of highest weight μ appears in $\mathcal{H}$ with multiplicity*

$$\sum_{w \in W} \epsilon(w) Q(w(\mu + \rho_c) - \Lambda - \rho_c) + \sum_j \sum_{w \in W} \epsilon(w) Q(w(\mu + \rho_c) - \Lambda - \sum_{\gamma \in S_j} \gamma - \rho_c),$$

where we sum over all j such that S_j contains no odd root α with $(\Lambda + \rho, \alpha) = 0$ and the Dirac inequality holds strictly on each proper $\mathfrak{g}_{\bar{0}}$-constituent.

7.3.2. Relation to Kostant's cohomology

We define an analog of Kostant's cohomology similarly to [18], which captures the $\mathfrak{k}^{\mathbb{C}}$-module structure of the Dirac cohomology rather than its $\mathfrak{g}_{\bar{0}}$-supermodule structure. When we consider a unitarizable $\mathfrak{g}$-supermodule as a $\mathfrak{k}^{\mathbb{C}}$-module, we neglect parity. We adapt the notation of Section 7.1.1.

Construction of $\mathrm{H}^*(\mathfrak{g}_{+1}, M)$

The Lie superalgebra $\mathfrak{g}$ admits a $\mathbb{Z}_2$-compatible $\mathbb{Z}$-grading. We consider $\mathfrak{g}_{\bar{1}}$ as a $\mathfrak{g}_{\bar{0}}$-module under the supercommutator. Then $\mathfrak{g}_{\bar{1}}$ decomposes into a direct sum of two simple $\mathfrak{g}_{\bar{0}}$-modules, namely $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{+1}$, where

$$\mathfrak{g}_{+1} := \mathfrak{p}_1 \oplus \mathfrak{p}_2, \qquad \mathfrak{g}_{-1} := \mathfrak{q}_1 \oplus \mathfrak{q}_2.$$

Both are, in particular, abelian Lie subsuperalgebras of $\mathfrak{g}$, *i.e.*, $[\mathfrak{g}_{\pm 1}, \mathfrak{g}_{\pm 1}] = 0$, and the associated $\mathbb{Z}$-grading of $\mathfrak{g}$, given by

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{+1},$$

is then compatible with the $\mathbb{Z}_2$-grading on $\mathfrak{g}$.

Let $T(\mathfrak{g}_{\pm 1})$ be the tensor algebra over the complex vector space $\mathfrak{g}_{\pm 1}$, and let $S(\mathfrak{g}_{\pm 1}) = T(\mathfrak{g}_{\pm 1})/I$ be the symmetric algebra with grading $S(\mathfrak{g}_{\pm 1}) = \bigoplus_{n=0}^{\infty} S^n(\mathfrak{g}_{\pm 1})$ introduced in Section 7.1. We note that $I \cap T^1(\mathfrak{g}_{\pm 1}) = \{0\}$, and $\mathfrak{g}_{\pm 1}$ can be identified with $S^1(\mathfrak{g}_{\pm 1})$. Further, let I' denote the two-sided ideal in $T(\mathfrak{g}_{\pm 1})$ generated by $v \otimes v$ for all $v \in \mathfrak{g}_{\pm 1}$. We define the *exterior algebra* by $\bigwedge(\mathfrak{g}_{\pm 1}) := T(\mathfrak{g}_{\pm 1})/I'$. The ideal I' is homogeneous, *i.e.*, $I' = \bigoplus_{n=0}^{\infty}(I' \cap T^n(\mathfrak{g}_{\pm 1}))$, and a $\mathbb{Z}$-grading of $\bigwedge(\mathfrak{g}_{\pm 1})$ is given by

$$\bigwedge(\mathfrak{g}_{\pm 1}) = \bigoplus_{n=0}^{\infty} \bigwedge\nolimits^n(\mathfrak{g}_{\pm 1}), \qquad \bigwedge\nolimits^n(\mathfrak{g}_{\pm 1}) := T^n(\mathfrak{g}_{\pm 1})/(I' \cap T^n(\mathfrak{g}_{\pm 1})).$$

We note that $I' \cap T^1(\mathfrak{g}_{\pm 1}) = \{0\}$, and we can identify $\mathfrak{g}_{\pm 1}$ with $\bigwedge^1(\mathfrak{g}_{\pm 1})$.

We identify the dual space $\mathfrak{g}_{+1}^*$ with $\mathfrak{g}_{-1}$ using the bilinear form $2B(\cdot,\cdot)$. This identification is $\mathfrak{g}_{\bar{0}}$-invariant, given that $B(\cdot,\cdot)$ is $\mathfrak{g}_{\bar{0}}$-invariant. Additionally, we equate $S(\mathfrak{g}_{+1}^*)$ with $S(\mathfrak{g}_{-1})$, the polynomial algebra in the variables $x_1, \dots, x_{pn}, \partial_{pn+1}, \dots, \partial_{mn}$. The universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_{+1})$ can be identified with $\bigwedge(\mathfrak{g}_{+1})$.

To construct the cohomology, we consider the free resolution of $\bigwedge(\mathfrak{g}_{+1})$-modules

$$\dots \xrightarrow{\delta} S^i(\mathfrak{g}_{+1}) \otimes \bigwedge(\mathfrak{g}_{+1}) \xrightarrow{\delta} S^{i-1}(\mathfrak{g}_{+1}) \otimes \bigwedge(\mathfrak{g}_{+1}) \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathbb{C} \otimes \bigwedge(\mathfrak{g}_{+1}) \xrightarrow{\delta} 0$$

with boundary map $\delta := -\mathrm{d}^{\mathfrak{q}_2} + \delta^{\mathfrak{p}_1}$. Here, recall that

$$\mathrm{d}^{\mathfrak{p}_1} = \sum_{k=1}^{pn} \partial_k \otimes x_k, \qquad \delta^{\mathfrak{p}_1} = \sum_{k=1}^{pn} x_k \otimes \partial_k,$$
$$\mathrm{d}^{\mathfrak{q}_2} = \sum_{k=pn+1}^{mn} \partial_k \otimes x_k, \qquad \delta^{\mathfrak{q}_2} = \sum_{k=pn+1}^{mn} x_k \otimes \partial_k.$$

In particular, $\delta^2 = 0$ by Lemma 7.2.11. Fix a $(\mathfrak{g}, \mathfrak{k}^{\mathbb{C}})$-supermodule M, and apply the contravariant functor $\mathrm{Hom}_{\bigwedge(\mathfrak{g}_{+1})}(-, M)$ to the above resolution. Then we arrive at the following complex of $\bigwedge(\mathfrak{g}_{+1})$-modules

$$\dots \xleftarrow{d} \mathrm{Hom}_{\mathbb{C}}(S^{i+1}(\mathfrak{g}_{+1}), M) \xleftarrow{d} \mathrm{Hom}_{\mathbb{C}}(S^i(\mathfrak{g}_{+1}), M) \dots \xleftarrow{d} \mathrm{Hom}_{\mathbb{C}}(S^1(\mathfrak{g}_{+1}), M) \leftarrow 0,$$

where d is the pullback operator of δ, explicitly given by $\mathrm{d} := \mathrm{d}^{\mathfrak{p}_1} - \delta^{\mathfrak{q}_2}$ if we identify $\mathrm{Hom}_{\mathbb{C}}(S^i(\mathfrak{g}_{+1}), M)$ with $M \otimes S^i(\mathfrak{g}_{-1})$ for any i. However, the operators d and δ are only $\mathfrak{k}^{\mathbb{C}}$-equivariant by Lemma 7.1.6.

Lemma 7.3.7. *The operators* d *and* δ *are* $\mathfrak{k}^{\mathbb{C}}$*-invariant,* i.e., $[\mathfrak{k}^{\mathbb{C}}, \mathrm{d}] = 0$ *and* $[\mathfrak{k}^{\mathbb{C}}, \delta] = 0$.

Both supermodules, M and $S^i(\mathfrak{g}_{+1})$, are $\mathfrak{k}^{\mathbb{C}}$-modules, where the action on $S^i(\mathfrak{g}_{+1})$ is induced by the adjoint action of $\mathfrak{k}^{\mathbb{C}}$ on $\mathfrak{g}_{\bar{1}}$. This identifies the spaces

$$\mathrm{C}^i(M) := \mathrm{Hom}_{\mathbb{C}}(S^i(\mathfrak{g}_{+1}), M) \cong M \otimes S^i(\mathfrak{g}_{-1})$$

as $\mathfrak{k}^{\mathbb{C}}$-modules.

The $\mathfrak{k}^{\mathbb{C}}$-invariance of d makes the complex $\mathrm{C} := (\mathrm{C}^i(M), \mathrm{d})$ into a $\mathfrak{k}^{\mathbb{C}}$-module complex with a $\mathfrak{k}^{\mathbb{C}}$-equivariant boundary operator d. The cohomology groups of the complex C will be denoted by $\mathrm{H}^i(\mathfrak{g}_{+1}, M)$. These are naturally $\mathfrak{k}^{\mathbb{C}}$-modules.

Relation to Dirac cohomology

For a given unitarizable $\mathfrak{g}$-supermodule $\mathcal{H}$, we identify d and δ as operators on $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})$, noting that $\mathrm{D} = 2(\mathrm{d} - \delta)$. The Dirac cohomology of $\mathcal{H}$ is $\mathrm{H_D}(\mathcal{H}) = \ker \mathrm{D}$, while the analog of Kostant's cohomology is $\ker \mathrm{d} / \operatorname{im} \mathrm{d}$. Therefore, we aim to compare $\ker \mathrm{D}$ with $\ker \mathrm{d} / \operatorname{im} \mathrm{d}$. We begin by gathering the properties of d and δ.

Lemma 7.3.8. *Let $\mathcal{H}$ be a unitarizable simple $\mathfrak{g}$-supermodule. Then the following assertions hold with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})}$:*

a) d *and* δ *are anti-adjoint to each other.*

b) $\operatorname{im} \mathrm{d}$ *is orthogonal to* $\ker \delta$ *and* $\operatorname{im} \delta$, *while* $\operatorname{im} \delta$ *is orthogonal to* $\ker \mathrm{d}$.

Proof. a) The operators are defined as $\mathrm{d} = \mathrm{d}^{\mathfrak{p}_1} - \delta^{\mathfrak{q}_2}$ and $\delta = -\mathrm{d}^{\mathfrak{q}_2} + \delta^{\mathfrak{p}_1}$. By Lemma 7.2.11, $\mathrm{d}^{\mathfrak{p}_1}$ is anti-adjoint to $\delta^{\mathfrak{p}_1}$ and $\mathrm{d}^{\mathfrak{q}_2}$ is anti-adjoint to $\delta^{\mathfrak{q}_2}$. This proves a).

b) This proof is analogous to the proofs of parts c) and d) in Lemma 7.2.11 and will be omitted. □

Lemma 7.3.9. *Let $\mathcal{H}$ be a unitarizable simple $\mathfrak{g}$-supermodule. Then the following assertions hold with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})}$:*

a) $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}) = \ker \mathrm{D} \oplus \operatorname{im} \mathrm{d} \oplus \operatorname{im} \delta$.

b) $\ker \mathrm{d} = \ker \mathrm{D} \oplus \operatorname{im} \mathrm{d}$.

Proof. a) By Lemma 7.2.8 and Corollary 7.2.3, we have $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}) = \ker \mathrm{D}^2 \oplus \operatorname{im} \mathrm{D}^2 = \ker \mathrm{D} \oplus \operatorname{im} \mathrm{D}^2$. Moreover, $\operatorname{im} \mathrm{d}$ and $\operatorname{im} \delta$ are orthogonal to each other by Lemma 7.3.8, and $\mathrm{d}^2 = 0$ or $\delta^2 = 0$. We conclude:

$$\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}) = \ker \mathrm{D} \oplus \operatorname{im} \mathrm{D}^2 \subset \ker \mathrm{D} \oplus \operatorname{im} \mathrm{D} \subset \ker \mathrm{D} \oplus \operatorname{im} \mathrm{d} \oplus \operatorname{im} \delta,$$

and consequently $\mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}}) = \ker \mathrm{D} \oplus \operatorname{im} \mathrm{d} \oplus \operatorname{im} \delta$.

b) The assertion follows from a) and Lemma 7.3.8 if we show $\ker \mathrm{D} = \ker \mathrm{d} \cap \ker \delta$. However, this assertion is clear since $\mathrm{D} = 2(\delta - \mathrm{d})$ and $\operatorname{im} \mathrm{d}$ and $\operatorname{im} \delta$ are orthogonal. □

Now, Lemma 7.3.9 together with Proposition 7.1.10 allows us to compare the $\mathfrak{k}^{\mathbb{C}}$-module structure of $\mathrm{H_D}(\mathcal{H})$ and $\mathrm{H}^*(\mathfrak{g}_{+1}, \mathcal{H})$. We first need the following lemma that is a straightforward calculation.

Lemma 7.3.10. *The following assertions hold:*

a) $\mathbb{C}[x_1, \ldots, x_{mn}] \cong \mathbb{C}[x_1, \ldots, x_{pn}] \otimes \mathbb{C}[x_{pn+1}, \ldots, x_{mn}]$ *as $\mathfrak{k}^{\mathbb{C}}$-modules with action induced by the commutator of $\mathfrak{k}^{\mathbb{C}}$ on $\mathfrak{g}_{\bar{1}}$.*

b) The $\mathfrak{k}^{\mathbb{C}}$-modules $\mathbb{C}[x_1, \ldots, x_{pn}]$, $\mathbb{C}[x_{pn+1}, \ldots, x_{mn}]$ *and* $\mathbb{C}[\partial_1, \ldots, \partial_{pn}]$, $\mathbb{C}[\partial_{pn+1}, \ldots, \partial_{mn}]$ *are dual to each other, respectively. In particular, they are isomorphic.*

Combining these results, we conclude that Dirac cohomology $\mathrm{H_D}(\mathcal{H})$ and Kostant's cohomology $\mathrm{H}^*(\mathfrak{g}_{+1}, \mathcal{H})$ are isomorphic as $\mathfrak{k}^{\mathbb{C}}$-modules, up to a twist, for unitarizable simple $\mathfrak{g}$-supermodules.

Theorem 7.3.11. *For any unitarizable simple $\mathfrak{g}$-supermodule $\mathcal{H}$ there exists a $\mathfrak{k}^{\mathbb{C}}$-module isomorphism*

$$\mathrm{H_D}(\mathcal{H}) \cong \mathrm{H}^*(\mathfrak{g}_{+1}, \mathcal{H}) \otimes \mathbb{C}_{-\rho_{\bar{1}}}.$$

Proof. First, by Lemma 7.2.9 and Lemma 7.3.9, we have $\mathrm{H_D}(\mathcal{H}) = \ker \mathrm{D} \cong \ker \mathrm{d} / \operatorname{im} \mathrm{d}$.

Second, by Proposition 7.1.10 and Lemma 7.3.10, have the following isomorphisms of $\mathfrak{k}^{\mathbb{C}}$-modules

$$\begin{aligned} M(\mathfrak{g}_{\bar{1}}) &\cong \mathbb{C}[x_1, \dots, x_{mn}] \otimes \mathbb{C}_{-\rho_{\bar{1}}} \\ &\cong (\mathbb{C}[x_1, \dots, x_{pn}] \otimes \mathbb{C}[x_{pn+1}, \dots, x_{mn}]) \otimes \mathbb{C}_{-\rho_{\bar{1}}} \\ &\cong (\mathbb{C}[x_1, \dots, x_{pn}] \otimes \mathbb{C}[\partial_{pn+1}, \dots, \partial_{mn}]) \otimes \mathbb{C}_{-\rho_{\bar{1}}} \\ &\cong \mathbb{C}[x_1, \dots, x_{pn}, \partial_{pn+1}, \dots, \partial_{mn}] \otimes \mathbb{C}_{-\rho_{\bar{1}}}. \end{aligned}$$

By construction of $\mathrm{H}^*(\mathfrak{g}_{+1}, \mathcal{H})$, the statement follows. □

7.3.3. Dirac index

In this section, we associate to any D^2-semisimple $\mathfrak{g}$-supermodule M (e.g., M simple or unitarizable) a virtual character, which we refer to as the *Dirac index*. This index coincides with the Euler characteristic of Dirac cohomology and satisfies several desirable properties. Moreover, for unitarizable simple $\mathfrak{g}$-supermodules, the Dirac index agrees with the Dirac cohomology.

For any $\mathfrak{g}$-supermodule M, the Dirac operator D acts on $M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{0}}$ and $M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{1}}$, interchanging these subspaces. The *Dirac index* of M is defined as the virtual $\mathfrak{g}_{\bar{0}}$-supermodule

$$\mathrm{I}(M) := M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{0}} - M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{1}}.$$

This is an element of the Grothendieck group of $\mathfrak{g}_{\bar{0}}$-supermodules. In contrast, the operator $\mathrm{D} : M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{0},\bar{1}} \to M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{1},\bar{0}}$ gives rise to a decomposition of the Dirac cohomology $\mathrm{H_D}(M)$ into even and odd parts:

$$\mathrm{H_D}(M) = \mathrm{H_D^+}(M) \oplus \mathrm{H_D^-}(M),$$

whose *Euler characteristic* is given by the virtual $\mathfrak{g}_{\bar{0}}$-supermodule

$$\mathrm{H_D^+}(M) - \mathrm{H_D^-}(M).$$

These two virtual $\mathfrak{g}_{\bar{0}}$-supermodules — $\mathrm{I}(M)$ and the Euler characteristic — coincide whenever M is D^2-semisimple.

Proposition 7.3.12. *Let M be a D^2-semisimple $\mathfrak{g}$-supermodule. Then the Dirac index $\mathrm{I}(M)$ is equal to the Euler characteristic of the Dirac cohomology $\mathrm{H_D}(M)$, i.e.,*

$$\mathrm{I}(M) = \mathrm{H_D^+}(M) - \mathrm{H_D^-}(M).$$

Proof. We decompose $M \otimes M(\mathfrak{g}_{\bar{1}})$ into a direct sum of eigenspaces of D^2:

$$M \otimes M(\mathfrak{g}_{\bar{1}}) = (M \otimes M(\mathfrak{g}_{\bar{1}}))(0) \oplus \bigoplus_{c \neq 0} (M \otimes M(\mathfrak{g}_{\bar{1}}))(c).$$

Since D^2 is even, this decomposition is compatible with the $\mathbb{Z}_2$-grading:

$$(M \otimes M(\mathfrak{g}_{\bar{1}}))(c) = (M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{0}})(c) \oplus (M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{1}})(c).$$

Moreover, D commutes with D^2, so it preserves each eigenspace $(M \otimes M(\mathfrak{g}_{\bar{1}}))(c)$. However, D switches parity, inducing isomorphisms

$$\mathrm{D}(c) : (M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{0},\bar{1}})(c) \to (M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{1},\bar{0}})(c),$$

with inverses given by $\frac{1}{c}\,\mathrm{D}(c)$ for $c \neq 0$. Consequently, the contributions from nonzero eigenspaces cancel in the index, and we have

$$M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{0}} - M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{1}} = (M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{0}})(0) - (M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{1}})(0).$$

The Dirac operator D restricts to a differential on $\ker(\mathrm{D}^2)$, and its cohomology is precisely the Dirac cohomology. The result then follows from the Euler–Poincaré principle. □

As a direct consequence, for a unitarizable simple $\mathfrak{g}$-supermodule, the Dirac index coincides with its Dirac cohomology (see Corollary 7.2.22).

Corollary 7.3.13. *Let M be a unitarizable simple $\mathfrak{g}$-supermodule. Then*

$$\mathrm{I}(M) \cong \mathrm{H}_{\mathrm{D}}(M)$$

as $\mathfrak{g}_{\bar{0}}$-supermodules.

Furthermore, the Dirac index commutes with tensoring by finite-dimensional $\mathfrak{g}$-supermodules. This compatibility can be used to study the Dirac cohomology of unitarizable supermodules via translation functors.

Lemma 7.3.14. *Let M be a $\mathfrak{g}$-supermodule, and let F be a finite-dimensional $\mathfrak{g}$-supermodule. Then there is a canonical isomorphism of $\mathfrak{g}_{\bar{0}}$-supermodules*

$$\mathrm{I}(M \otimes F) \cong \mathrm{I}(M) \otimes F.$$

Proof. We compute:

$$\mathrm{I}(M \otimes F) = M \otimes F \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{0}} - M \otimes F \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{1}},$$

while

$$\mathrm{I}(M) \otimes F = (M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{0}} - M \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{1}}) \otimes F.$$

Since F is finite-dimensional, the tensor product distributes over the direct sum and difference, yielding a canonical isomorphism between both expressions. □

7.3.4. Formal characters

We present two formulas for the formal $\mathfrak{k}^{\mathbb{C}}$-character of unitarizable (simple) $\mathfrak{g}$-supermodules using Kostant's cohomology and the Dirac index, respectively.

Fix a unitarizable simple $\mathfrak{g}$-supermodule $\mathcal{H}$ with highest weight Λ. Then each weight space $\mathcal{H}^\mu$ of weight $\mu \in \mathfrak{h}^*$ is finite-dimensional, and we can assign to $\mathcal{H}$ its *formal character*

$$\mathrm{ch}(\mathcal{H}) := \sum_{\mu} \dim(\mathcal{H}^\mu) e^\mu,$$

where we sum over the $\mathfrak{h}$-weight space of $\mathcal{H}$, and e^{μ} is a function on $\mathfrak{h}^*$ that takes value 1 at μ and value 0 at $\lambda \neq \mu$. Equivalently, recall that $\mathcal{H}$ is a Harish-Chandra supermodule (see Section 6.1.5) with $\mathfrak{k}^{\mathbb{C}}$-type decomposition $\mathcal{H} = \oplus_{\lambda} m_{\lambda} F^{\lambda}$ and multiplicities $m_{\lambda} < \infty$, where we neglect parity. Here, F^{λ} is a simple $\mathfrak{k}^{\mathbb{C}}$-module of highest weight λ. In addition, recall that $\mathfrak{k}^{\mathbb{C}}$ satisfies the equal rank condition, that is, $\mathfrak{h} \subset \mathfrak{k}^{\mathbb{C}} \subset \mathfrak{g}_{\bar{0}}$. The formal $\mathfrak{k}^{\mathbb{C}}$-character of $\mathcal{H}$ is the formal series

$$\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\mathcal{H}) := \sum_{\lambda} m_{\lambda}\, \mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(F^{\lambda}),$$

with $\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(F^{\lambda})$ being the formal character of the finite-dimensional simple $\mathfrak{k}^{\mathbb{C}}$-module F^{λ}. In particular, both characters (formally) coincide.

Formal characters and Kostant's cohomology

We study the relation of $\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\mathcal{H})$ and $\mathrm{H}^*(\mathfrak{g}_{+1}, \mathcal{H})$ using partially ideas and constructions presented in [18]. In Section 7.3.2, we introduced the cohomology groups $\mathrm{H}^k(\mathfrak{g}_{+1}, \mathcal{H})$ as the cohomology groups of the complex $\mathrm{C} := (\mathrm{C}^k(\mathcal{H}), \mathrm{d})$ with $\mathrm{C}^k(\mathcal{H}) := \mathrm{Hom}_{\mathbb{C}}(S^k(\mathfrak{g}_{+1}), \mathcal{H}) \cong \mathcal{H} \otimes S^k(\mathfrak{g}_{+1}^*)$. These are $\mathfrak{k}^{\mathbb{C}}$-modules.

Let $\mathrm{H}^k(\mathfrak{g}_{+1}, \mathcal{H})^{\lambda}$ be the weight λ-subspace of $\mathrm{H}^k(\mathfrak{g}_{+1}, \mathcal{H})$ for some weight $\lambda \in \mathfrak{h}^*$. As $\mathcal{H}$ is a Harish-Chandra supermodule, we have for fixed weight λ that $\dim(\mathrm{H}^k(\mathfrak{g}_{+1}, \mathcal{H})^{\lambda}) \neq 0$ only for finitely many k. The Euler-Poincaré principle then implies that

$$\sum_{k=0}^{\infty} (-1)^k \dim(\mathrm{H}^k(\mathfrak{g}_{+1}, \mathcal{H})^{\lambda}) = \sum_{k=0}^{\infty} (-1)^k \dim(\mathrm{C}^k(\mathcal{H})^{\lambda}),$$

and considering their formal characters gives

$$\sum_{k=0}^{\infty} (-1)^k \,\mathrm{ch}\left(\mathrm{H}^k(\mathfrak{g}_{+1}, \mathcal{H})\right) = \sum_{k=0}^{\infty} (-1)^k \,\mathrm{ch}(\mathrm{C}^k(\mathcal{H})) = \mathrm{ch}(\mathcal{H}) \sum_{k=0}^{\infty} (-1)^k \,\mathrm{ch}(S^k(\mathfrak{g}_{+1}^*)).$$

Now, we have as $\mathfrak{k}^{\mathbb{C}}$-modules $S^k(\mathfrak{g}_{+1}^*) \cong S^k(\mathfrak{g}_{-1}) \cong S^k(\mathfrak{q}_1 \oplus \mathfrak{p}_2) = S^k(\mathfrak{n}_{\bar{1}}^-)$ for any k (*cf.* Lemma 7.3.10), and it is well-known that $\sum_{k=0}^{\infty} (-1)^k \,\mathrm{ch}(S^k(\mathfrak{n}_{\bar{1}}^-)) = \frac{1}{\prod_{\gamma \in \Delta_{\bar{1}}^+} (1+e^{-\gamma}))}$, We conclude

$$\sum_{k=0}^{\infty} (-1)^k \,\mathrm{ch}(\mathrm{C}^k(\mathcal{H})) = \frac{\mathrm{ch}(\mathcal{H})}{\prod_{\gamma \in \Delta_{\bar{1}}^+} (1+e^{-\gamma})},$$

and the formal character of $\mathcal{H}$ is given by

$$\mathrm{ch}(\mathcal{H}) = \prod_{\gamma \in \Delta_{\bar{1}}^+} (1+e^{-\gamma}) \sum_{k=0}^{\infty} (-1)^k \,\mathrm{ch}(\mathrm{H}^k(\mathfrak{g}_{+1}, \mathcal{H})).$$

Next, we move, without loss of generality, to formal $\mathfrak{k}^{\mathbb{C}}$-characters. Let $[\mathrm{H}^k(\mathfrak{g}_{+1}, \mathcal{H}) : F^{\mu}]$ denote the multiplicity of the simple (unitarizable highest weight) $\mathfrak{k}^{\mathbb{C}}$-module F^{μ} with highest weight μ in $\mathrm{H}^k(\mathfrak{g}_{+1}, \mathcal{H})$. As $\mathfrak{k}$ is compact, $\mathrm{H}^k(\mathfrak{g}_{+1}, \mathcal{H})$ is completely reducible as a $\mathfrak{k}^{\mathbb{C}}$-module, and we can express $\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\mathrm{H}^k(\mathfrak{g}_{+1}, \mathcal{H}))$ in terms of the multiplicities:

$$\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\mathrm{H}^k(\mathfrak{g}_{+1}, \mathcal{H})) = \sum_{\mu} [\mathrm{H}^k(\mathfrak{g}_{+1}, \mathcal{H}) : F^{\mu}]\, \mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(F^{\mu}).$$

Altogether, we have proven the following theorem.

Theorem 7.3.15. *Let $\mathcal{H}$ be a unitarizable simple $\mathfrak{g}$-supermodule with finite multiplicities. The formal character of $\mathcal{H}$ is*

$$\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\mathcal{H}) = \sum_{\mu}\sum_{k=0}^{\infty}(-1)^k[\mathrm{H}^k(\mathfrak{g}_{+1}, M) : F^{\mu}]\,\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\bigwedge \mathfrak{n}_{\bar{1}}^{-} \otimes F^{\mu}).$$

Formal characters and Dirac index

We give a formula for $\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\mathcal{H})$ using $\mathrm{I}(\mathcal{H})$ for a fixed unitarizable supermodule $\mathcal{H}$ admitting a formal $\mathfrak{k}^{\mathbb{C}}$-character. Recall from Section 7.3.3 the definition:

$$\mathrm{I}(\mathcal{H}) = \mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{0}} - \mathcal{H} \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{1}} = \mathrm{H}_{\mathrm{D}}^{+}(\mathcal{H}) - \mathrm{H}_{\mathrm{D}}^{-}(\mathcal{H}),$$

which is the Euler characteristic of $\mathrm{H}_{\mathrm{D}}(\mathcal{H})$ by Proposition 7.3.12. In terms of characters, this reads

$$\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\mathcal{H})(\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(M(\mathfrak{g}_{\bar{1}})_{\bar{0}}) - \mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(M(\mathfrak{g}_{\bar{1}})_{\bar{1}})) = \mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\mathrm{H}_{\mathrm{D}}^{+}(\mathcal{H})) - \mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\mathrm{H}_{\mathrm{D}}^{-}(\mathcal{H})).$$

The methods and ideas used below are similar to those in [64], but have been adapted to our context.

We consider the finite-dimensional vector space $\mathfrak{n}_{\bar{1}}^{-}$, with basis $\{x_1, \dots, x_{mn}\}$. We are interested in the free resolution of free $\bigwedge \mathfrak{n}_{\bar{1}}^{-}$-modules:

$$\dots \xrightarrow{\delta} S^i(\mathfrak{n}_{\bar{1}}^{-}) \otimes \bigwedge \mathfrak{n}_{\bar{1}}^{-} \xrightarrow{\delta} S^{i-1}(\mathfrak{n}_{\bar{1}}^{-}) \otimes \bigwedge \mathfrak{n}_{\bar{1}}^{-} \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathbb{C} \otimes \bigwedge \mathfrak{n}_{\bar{1}}^{-} \xrightarrow{\delta} 0,$$

where the boundary operator is

$$\delta := \mathrm{d}^{\mathfrak{p}_1} + \mathrm{d}^{\mathfrak{q}_2} = \sum_{i=1}^{mn} \frac{\partial}{\partial x_i} \otimes x_i : S(\mathfrak{n}_{\bar{1}}^{-}) \otimes_{\mathbb{C}} \bigwedge \mathfrak{n}_{\bar{1}}^{-} \longrightarrow S(\mathfrak{n}_{\bar{1}}^{-}) \otimes_{\mathbb{C}} \bigwedge \mathfrak{n}_{\bar{1}}^{-}.$$

The boundary operator δ is invariant under the action of $\mathfrak{k}^{\mathbb{C}}$, *i.e.*, $[\mathfrak{k}^{\mathbb{C}}, \delta] = 0$, as shown in Lemma 7.1.6. Additionally, the proof of the following lemma is *mutatis mutandis* to [67, Proposition 3.3.5].

Lemma 7.3.16. *The following assertion holds:*

$$\ker \delta = \mathrm{im}\, \delta \oplus \mathbb{C}(1 \otimes 1).$$

In particular, the cohomology is generated by $\mathbb{C}(1 \otimes 1)$.

Lemma 7.3.17. *Let $\mathbb{C}_{-\rho_{\bar{1}}}$ be the one-dimensional $\mathfrak{k}^{\mathbb{C}}$-module with weight $-\rho_{\bar{1}}$. Then*

$$\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\bigwedge \mathfrak{n}_{\bar{1}}^{-})(\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(M(\mathfrak{g}_{\bar{1}})_{\bar{0}}) - \mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(M(\mathfrak{g}_{\bar{1}})_{\bar{1}})) = \mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\mathbb{C}_{-\rho_{\bar{1}}}).$$

Proof. By Proposition 7.1.10, we have the following isomorphism of $\mathfrak{k}^{\mathbb{C}}$-modules:

$$\bigwedge \mathfrak{n}_{\bar{1}}^{-} \otimes M(\mathfrak{g}_{\bar{1}}) \cong \bigwedge \mathfrak{n}_{\bar{1}}^{-} \otimes S(\mathfrak{n}_{\bar{1}}^{-}) \otimes \mathbb{C}_{-\rho_{\bar{1}}}.$$

In particular, $\bigwedge \mathfrak{n}_{\bar{1}}^{-} \otimes S(\mathfrak{n}_{\bar{1}}^{-})$ is the complex introduced above with $\mathfrak{k}^{\mathbb{C}}$-invariant boundary operator δ. By the Euler–Poincaré principle,

$$\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\bigwedge \mathfrak{n}_{\bar{1}}^{-})(\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(M(\mathfrak{g}_{\bar{1}})_{\bar{0}}) - \mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(M(\mathfrak{g}_{\bar{1}})_{\bar{1}}))$$

is the Euler characteristic of this complex. However, the cohomology is generated by the vector $1 \otimes 1$ by Lemma 7.3.16, and the statement follows with Proposition 7.1.10 with trivial $\mathfrak{k}^{\mathbb{C}}$-action. □

Theorem 7.3.18. *Let F^ν denote a simple $\mathfrak{k}^\mathbb{C}$-module of highest weight $\nu \in \mathfrak{h}^*$. Define $N(\mu) := \bigwedge \mathfrak{n}_{\bar{1}}^- \otimes F^\mu$, and assume $\mathrm{H}_\mathrm{D}^+(\mathcal{H}) = \sum_\mu F^\mu$ and $\mathrm{H}_\mathrm{D}^-(\mathcal{H}) = \sum_\nu F^\nu$. Then*

$$\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(\mathcal{H}) = \sum_\mu \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(N(\mu+\rho_{\bar{1}})) - \sum_\nu \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(N(\nu+\rho_{\bar{1}})).$$

Proof. By the definition of $N(\mu)$ and the Dirac index, we have

$$\begin{aligned}
&\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(\mathcal{H})\big(\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(M(\mathfrak{g}_{\bar{1}})_{\bar{0}}) - \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(M(\mathfrak{g}_{\bar{1}})_{\bar{1}})\big)\\
&\quad = \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(\mathrm{H}_\mathrm{D}^+(\mathcal{H})) - \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(\mathrm{H}_\mathrm{D}^-(\mathcal{H}))\\
&\quad = \sum_\mu \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(F^\mu) - \sum_\nu \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(F^\nu)\\
&\quad = \big(\sum_\mu \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(N(\mu+\rho_{\bar{1}})) - \sum_\nu \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(N(\nu+\rho_{\bar{1}}))\big)\big(\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(M(\mathfrak{g}_{\bar{1}})_{\bar{0}}) - \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(M(\mathfrak{g}_{\bar{1}})_{\bar{1}})\big),
\end{aligned}$$

which can be rewritten as

$$\big(\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(\mathcal{H}) - \sum_\mu \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(N(\mu+\rho_{\bar{1}})) + \sum_\nu \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(N(\nu+\rho_{\bar{1}}))\big)\big(\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(M(\mathfrak{g}_{\bar{1}})_{\bar{0}}) - \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(M(\mathfrak{g}_{\bar{1}})_{\bar{1}})\big) = 0.$$

We claim that the first factor must be trivial. Assume that it is non-trivial, *i.e.*,

$$\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(V) = \sum_{i=1}^\infty n_i \,\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(F^{\mu_i}) \neq 0, \qquad V := \mathcal{H} - \sum_\mu N(\mu+\rho_{\bar{1}}) + \sum_\nu N(\nu+\rho_{\bar{1}}).$$

Assume we have a $\mathfrak{k}^\mathbb{C}$-type F^ξ in V. Then, $\xi = \Lambda - \sum_j \beta_j - \sum_i \alpha_i$ for positive non-compact roots β_j and positive odd roots α_i. The β_j are not present in $N(\Lambda)$ by construction.

We consider the Weyl vector ρ_n associated to the set of non-compact positive roots (*cf.* Section 2.1.4). Recall that the non-compact positive roots are $\epsilon_k - \epsilon_l$ for $1 \leq k \leq p$ and $p+1 \leq l \leq m$, while the odd positive roots are $\{\epsilon_k - \delta_r, -\epsilon_l + \delta_s : 1 \leq r, s \leq n,\ 1 \leq k \leq p,\ p+1 \leq l \leq m\}$. Then, a direct calculation yields

$$(\beta_j, \rho_n) > 0, \qquad (\alpha_i, \rho_n) > 0, \qquad \forall i, j.$$

We conclude $(\xi, \rho_n) \leq (\Lambda, \rho_n)$.

Without loss of generality, we can assume that $n_1 \neq 0$ and $(\mu_1, \rho_n) \geq (\mu_i, \rho_n)$ for all i. Since $F^{\mu_1} \otimes M(\mathfrak{g}_{\bar{1}})$ contains $F^{\mu_1} \otimes 1 \cong F^{\mu_1 - \rho_{\bar{1}}}$ with multiplicity one, the character of $F^{\mu_1 - \rho_{\bar{1}}}$ appears in $\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(F^{\mu_1})(\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(M(\mathfrak{g}_{\bar{1}})_{\bar{0}}) - \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(M(\mathfrak{g}_{\bar{1}})_{\bar{1}}))$ with coefficient one.

By assumption, the contribution of $\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(F^{\mu_1})(\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(M(\mathfrak{g}_{\bar{1}})_{\bar{0}}) - \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(M(\mathfrak{g}_{\bar{1}})_{\bar{1}}))$ must be canceled in $\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(\mathfrak{n}_{\bar{1}}^-)(\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(M(\mathfrak{g}_{\bar{1}})_{\bar{0}}) - \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(M(\mathfrak{g}_{\bar{1}})_{\bar{1}}))$, *i.e.*, $F^{\mu_1 + \rho_{\bar{1}}}$ must appear in some $\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(F^{\mu_i})(\mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(M(\mathfrak{g}_{\bar{1}})_{\bar{0}}) - \mathrm{ch}_{\mathfrak{k}^\mathbb{C}}(M(\mathfrak{g}_{\bar{1}})_{\bar{1}}))$, that is, in some $F^{\mu_i} \otimes M(\mathfrak{g}_{\bar{1}})$ for some $i > 1$.

The weights of $M(\mathfrak{g}_{\bar{1}})$ are of the form $-\rho_{\bar{1}} - \sum_j \beta_j - \sum_k \alpha_k$, where β_j are distinct non-compact positive roots and α_k are distinct positive odd roots. We conclude that

$$\mu_1 = \mu_i - \sum_j \beta_j - \sum_k \alpha_k.$$

This leads to a contradiction, as it would imply $(\mu_1, \rho_n) < (\mu_i, \rho_n)$. This finishes the proof. ☐

8. Classification of unitarizable supermodules

The study of unitarizable $\mathfrak{g}$-supermodules naturally leads to two fundamental questions: how to classify them and how to describe their structure explicitly. While several classifications have been achieved through detailed combinatorial and computational techniques [22, 48, 53, 73], these approaches often obscure the underlying geometry and lack conceptual clarity. A unifying, geometrically motivated framework, similar to the Enright–Howe–Wallach classification for real simple Lie algebras [38], is still missing. In this chapter, we introduce a novel method based on Dirac operators and the Dirac inequality, yielding a simple classification.

In this chapter, let $\mathfrak{g}$ be a basic classical Lie superalgebra of type $A(m|n)$ or $\mathfrak{sl}(m|n)$. Consider a non-trivial unitarizable simple $\mathfrak{g}$-supermodule $\mathcal{H}$, which we regard as the unique simple quotient $L(\Lambda)$ of a Kac supermodule $K(\Lambda)$. Since $\mathcal{H}$ is non-trivial, its unitarity is defined with respect to a complex-conjugate anti-involution associated with a real form $\mathfrak{su}(p,q|n)$, as described in Section 2.1.4.

Depending on whether both $p, q \neq 0$ or not, we distinguish between two cases. A unitarizable simple $\mathfrak{g}$-supermodule is finite-dimensional if and only if $p = 0$ or $q = 0$; otherwise, it is infinite-dimensional (Proposition 3.1.18). We treat these cases separately, as the corresponding conjugate-linear anti-involutions differ fundamentally in nature. We begin with the finite-dimensional case.

8.1. Classification of unitarizable finite-dimensional supermodules

8.1.1. Paramterization

The simple finite-dimensional supermodules are parameterized by dominant integral weights $\lambda \in \mathfrak{h}^*$, defined with respect to a Borel subalgebra $\mathfrak{b} = \mathfrak{b}_{\bar{0}} \oplus \mathfrak{b}_{\bar{1}}$, that is, weights for which there exists a finite-dimensional simple $\mathfrak{g}_{\bar{0}}$-module of highest weight λ with respect to $\mathfrak{b}_{\bar{0}}$. More precisely, λ is dominant integral if and only if

$$(\lambda + \rho_0, \alpha) \in \mathbb{Z}_{>0} \quad \text{for all } \alpha \in \Delta_{\bar{0}}^+,$$

where $\Delta^+ = \Delta_{\bar{0}}^+ \sqcup \Delta_{\bar{1}}^+$ denotes the positive system determined by $\mathfrak{b}$. We denote the set of $\mathfrak{b}$-dominant integral weights by $P_{\mathfrak{b}}^{++}$, and refer to these as Δ^+-dominant integral weights. Moreover, when $\mathfrak{b}$ is clear from context, we omit the subscript and simply write P^{++}.

For any $\lambda \in P_{\mathfrak{b}}^{++}$, we define $L_{\mathfrak{b}}(\lambda)$ (or simply $L(\lambda)$ when no confusion arises) to be the simple supermodule with highest weight λ with respect to $\mathfrak{b}$ such that the highest weight vector is even. The fixed notation allows us to parameterize the simple finite-dimensional $\mathfrak{g}$-supermodules as follows:

$$\{L(\lambda), \Pi L(\lambda) : \lambda \in P^{++}\}.$$

Finite-dimensional unitarizable $\mathfrak{g}$-supermodules are precisely those defined with respect to one of the conjugate-linear anti-involutions (*cf.* Section 2.1.4)

$$\omega_{\pm}\begin{pmatrix} A & B \\ \hline C & D \end{pmatrix} = \begin{pmatrix} A^{\dagger} & \pm C^{\dagger} \\ \hline \pm B^{\dagger} & D^{\dagger} \end{pmatrix}, \qquad \begin{pmatrix} A & B \\ \hline C & D \end{pmatrix} \in \mathfrak{g}.$$

These conjugate-linear anti-involutions correspond to the compact real form of $\mathfrak{g}_{\bar{0}}$; that is, we may represent the Lie algebra as

$$\mathfrak{g}_{\bar{0}} = \begin{cases} \mathfrak{su}(m)^{\mathbb{C}} \oplus \mathfrak{su}(n)^{\mathbb{C}} \oplus \mathfrak{u}(1)^{\mathbb{C}} & \text{if } m \neq n, \\ \mathfrak{su}(m)^{\mathbb{C}} \oplus \mathfrak{su}(m)^{\mathbb{C}} & \text{if } m = n, \end{cases}$$

to emphasize the real form. In what follows, we consider only the conjugate-linear anti-involution ω_+; the case of ω_- can be treated in a completely analogous manner.

We now provide an explicit parameterization of the finite-dimensional $\mathfrak{g}$-supermodules corresponding to the chosen real form.

Suppose the simple $\mathfrak{g}$-supermodule $\mathcal{H} = L(\Lambda)$ is unitarizable with respect to ω_+. Then $\mathcal{H}$ decomposes under $\mathfrak{g}_{\bar{0}}$ into a direct sum of simple unitarizable $\mathfrak{g}_{\bar{0}}$-modules, which, in the case $m \neq n$, are outer tensor products of simple $\mathfrak{su}(m)^{\mathbb{C}}$-, $\mathfrak{su}(n)^{\mathbb{C}}$-, and $\mathfrak{u}(1)^{\mathbb{C}}$-modules (see Section 6.1.4). Otherwise, they are outer tensor products of simple $\mathfrak{su}(m)^{\mathbb{C}}$- and $\mathfrak{su}(n)^{\mathbb{C}}$-modules.

By the highest weight theorem, the simple $\mathfrak{su}(m)^{\mathbb{C}}$-modules are parameterized by dominant integral weights of the form

$$(-a_1, -a_2, \ldots, -a_m | 0, \ldots, 0),$$

where $a_1, \ldots, a_m$ are positive integers satisfying $0 = -a_1 \geq -a_2 \geq \cdots \geq -a_m$. This choice will mirror the unitarity relation appropriately. Analogously, the simple $\mathfrak{su}(n)^{\mathbb{C}}$-modules are parameterized by dominant integral weights of the form

$$(0, \ldots, 0 | b_1, \ldots, b_n),$$

where $b_1, \ldots, b_n$ are positive integers satisfying $b_1 \geq \cdots \geq b_n = 0$.

The Lie algebra $\mathfrak{u}(1)$ is abelian, so by Schur's lemma, each simple $\mathfrak{u}(1)$-module is one-dimensional and, by unitarity, uniquely determined by a positive real number.

Moreover, unitarity imposes a relation between the a_i and b_j values, determined by ω_+. We summarize these observations in the following lemma.

Lemma 8.1.1. *Let $L_0(\mu)$ be an $\mathfrak{g}_{\bar{0}}$-constituent of a simple ω_+-unitarizable $\mathfrak{g}$-supermodule. Then μ is of the form*

$$\mu = (0, -a_2, \ldots, -a_m | b_1, \ldots, b_{n-1}, 0) + \frac{x}{2}(1, \ldots, 1 | 1, \ldots, 1), \qquad x \in \mathbb{R},$$

where $0 \geq -a_2 \geq \cdots \geq -a_m$ and $b_1 \geq \cdots \geq b_{n-1} \geq 0$ are integers. Moreover, the components of μ satisfy the following unitarity inequalities:

$$\mu_m \leq \cdots \leq \mu_1 \leq \mu'_n \leq \cdots \leq \mu'_1.$$

However, in general, a highest weight $\mathfrak{g}$-supermodule $\mathcal{H}$ whose $\mathfrak{g}_{\bar{0}}$-constituents have highest weights of the form described in Lemma 8.1.1 need not be ω_+-unitarizable. To classify all ω_+-unitarizable simple $\mathfrak{g}$-supermodules, it is necessary to study lines in $\mathfrak{h}^*$ of the form

$$\Lambda(x) = \Lambda_0 + \frac{x}{2}(1, \dots, 1|1, \dots, 1), \qquad x \in \mathbb{R},$$

where $\Lambda_0 := (0, -a_2, \dots, -a_m|b_1, \dots, b_{n-1}, 0)$ for integers satisfying $0 \geq -a_2 \geq \dots \geq -a_m$ and $b_1 \geq \dots \geq b_{n-1} \geq 0$. Note that $(\alpha, (1, \dots, 1|1, \dots, 1)) = 0$ for all even roots $\alpha \in \Delta_{\bar{0}}$.

Moreover, to track the nontrivial components among the a_i and b_j, we define i_0 as the largest integer such that $a_{i_0} = 0$, and k_0 as the smallest integer such that $b_{k_0} = 0$. This can be interpreted in terms of Young diagrams. A *Young diagram* $Y = (x_1, x_2, \dots, x_k)$ is a weakly decreasing sequence of non-negative integers. Its *length* is defined as

$$\text{length}(Y) := \max\{i : x_i \neq 0\}.$$

To each Λ_0, we associate two Young diagrams:

$$\begin{aligned} Y_1(\Lambda_0) &:= (\lambda_1 - \lambda_2, \lambda_1 - \lambda_3, \dots, \lambda_1 - \lambda_m), \\ Y_2(\Lambda_0) &:= (\lambda'_1 - \lambda'_2, \lambda'_2 - \lambda'_3, \dots, \lambda'_{n-1} - \lambda'_n), \end{aligned}$$

and define $\text{len}_i(\Lambda_0) := \text{length}(Y_i(\Lambda_0))$ for $i = 1, 2$. Then

$$i_0 = m - \text{len}_1(\Lambda_0), \qquad k_0 = \text{len}_2(\Lambda_0).$$

Moreover, we define the gap function

$$g_1(\Lambda) := \lambda_m + \lambda'_n.$$

8.1.2. Classification

We fix the standard positive system Δ^+ as described in Section 2.1.4. Our approach is based on the Dirac inequality (see Proposition 7.2.4) and Theorem 7.3.4 adapted to our situation, that is, we fix

$$\partial_{(l-1)n+(k-m)} = \begin{cases} E_{lk} & \text{for } 1 \leq l \leq p, \ m+1 \leq k \leq m+n, \\ E_{kl} & \text{for } p+1 \leq l \leq m, \ m+1 \leq k \leq m+n, \end{cases}$$

$$x_{(l-1)n+(k-m)} = \begin{cases} E_{kl} & \text{for } 1 \leq l \leq p, \ m+1 \leq k \leq m+n, \\ E_{lk} & \text{for } p+1 \leq l \leq m, \ m+1 \leq k \leq m+n. \end{cases}$$

such that $\omega_+(x_k) = \partial_k$ and $B(\partial_k, x_l) = \frac{1}{2}\delta_{kl}$ for $1 \leq k, l \leq mn$. In particular, the ∂_i's span $\mathfrak{g}_{+1}$ and the x_j's span $\mathfrak{g}_{-1}$ and D is anti-self-adjoint rather than self-adjoint (*cf.* Chapter 7). The Dirac inequality now reads

$$(\mu + 2\rho, \mu) < (\Lambda + 2\rho, \Lambda)$$

for any $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\mu)$ of a unitarizable highest weight $\mathfrak{g}$-supermodule with highest weight Λ. Moreover, analogously to Theorem 7.3.2, a simple highest weight $\mathfrak{g}$-supermodule M with highest weight Λ is unitarizable if and only if the highest weight $\mathfrak{g}_{\bar{0}}$-module $L_0(\Lambda)$

is unitarizable, and the Dirac inequality holds strictly for each simple highest weight $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\mu)$ with $\mu \neq \Lambda$ occurring in $M \otimes M(\mathfrak{g}_{\bar{1}})$:

$$(\mu + 2\rho, \mu) < (\Lambda + 2\rho, \Lambda).$$

In what follows, we fix a simple highest weight $\mathfrak{g}$-supermodule $L(\Lambda)$ that is $\mathfrak{g}_{\bar{0}}$-semisimple. Since $L_0(\Lambda)$ must be $\mathfrak{g}_{\bar{0}}$-unitarizable, we consider the family of highest weight $\mathfrak{g}$-supermodules $L(\Lambda(x))$, where the highest weights are given by

$$\Lambda(x) = \Lambda_0 + \frac{x}{2}(1, \ldots, 1|1, \ldots, 1),$$

and Λ_0 is fixed as above (see Lemma 8.1.1). We analyze the Dirac inequality on the $\mathfrak{g}_{\bar{0}}$-constituents of $L(\Lambda(x))$. To that end, we note that the condition on the $\mathfrak{g}_{\bar{0}}$-modules can be further relaxed.

Proposition 8.1.2. *Let M be a simple highest weight $\mathfrak{g}$-supermodule with integral highest weight Λ that is $\mathfrak{g}_{\bar{0}}$-semisimple. Then M is unitarizable if and only if the highest weight $\mathfrak{g}_{\bar{0}}$-module $L_0(\Lambda)$ is unitarizable, and for any $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\Lambda - \alpha)$ with $\alpha \in \Delta_{\bar{1}}^+$, we have*

$$(\Lambda + \rho, \alpha) > 0.$$

Proof. Assume $L_0(\Lambda - \alpha)$ is a non-trivial $\mathfrak{g}_{\bar{0}}$-constituent of $L(\Lambda)$ for some $\alpha \in \Delta_{\bar{1}}^+$. Then the Dirac inequality reads

$$-(\Lambda + 2\rho, \Lambda) + (\Lambda - \alpha + 2\rho, \Lambda - \alpha) = -2(\alpha, \Lambda + \rho) \leq 0.$$

If M is unitarizable, this proves one direction.

Now, assume $L_0(\Lambda)$ is unitarizable, and for any $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\Lambda - \alpha)$ with $\alpha \in \Delta_{\bar{1}}^+$, we have $(\Lambda + \rho, \alpha) > 0$. Recall that any highest weight μ of a non-trivial $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\mu)$ of $L(\Lambda)$ is of the form $\mu = \Lambda - \sum_{\gamma \in S} \gamma$ for some subset $S \subset \Delta_{\bar{1}}^+$. We show that $(\mu + 2\rho, \mu) < (\Lambda + 2\rho, \Lambda)$ holds for any $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\mu)$. Then the statement follows by Theorem 7.3.4.

We consider two cases, namely $|S| = 2$ and $|S| > 2$. Enumerate the elements of S by $\gamma_1, \ldots, \gamma_k$. By assumption and integrality of Λ, we know

$$(\Lambda + \rho, \gamma_i) \in \mathbb{Z}_{\geq 1}, \qquad 1 \leq i \leq k.$$

In particular, if $k \geq 3$, there must be at least one i such that $(\Lambda + \rho, \gamma_i) \geq 2$.

First, assume $k = 2$, that is, $|S| = 2$, and $\mu = \Lambda - \alpha - \beta$ for some $\alpha, \beta \in \Delta_{\bar{1}}^+$. Then we have

$$-(\Lambda + 2\rho, \Lambda) + (\mu + 2\rho, \mu) = 2(\alpha, \beta) - 2(\alpha, \Lambda + \rho) - 2(\beta, \Lambda + \rho).$$

By assumption and integrality of Λ, we have $(\alpha, \Lambda + \rho), (\beta, \Lambda + \rho) \in \mathbb{Z}_{\geq 1}$, while a direct calculation yields $(\alpha, \beta) \in \{-1, 0, 1\}$. We conclude

$$-(\Lambda + 2\rho, \Lambda) + (\mu + 2\rho, \mu) < 0.$$

Next, assume $k \geq 3$, and note $(\sum_{i=1}^k \gamma_i, \sum_{j=1}^k \gamma_j) \leq 2k$ as a direct calculation shows. Then we have

$$-(\Lambda + 2\rho, \Lambda) + (\Lambda - \gamma + 2\rho, \Lambda - \gamma) = (\gamma, \gamma) - 2(\gamma, \Lambda + \rho) \leq 2k - 2\sum_{i=1}^{k}(\gamma_i, \Lambda + \rho) < 0,$$

as $(\gamma_k, \Lambda + \rho) \in \mathbb{Z}_{\geq 1}$, and there exists at least one $1 \leq i \leq k$ with $(\gamma_i, \Lambda + \rho) \geq 2$. □

This leads to a simple and explicit classification of ω_+-unitarizable simple highest weight $\mathfrak{g}$-supermodules with integral dominant highest weights. Recall that the standard odd positive system is given by

$$\Delta_{\bar{1}}^+ = \{\epsilon_i - \delta_j : 1 \leq i \leq m,\ 1 \leq j \leq n\}.$$

By Proposition 8.1.2, it suffices to study the Dirac inequality for highest weights of the form $\Lambda - \alpha$, that is,

$$(\Lambda + \rho, \epsilon_i - \delta_j) > 0 \quad \Leftrightarrow \quad x \geq -m + i + j + a_i - b_j, \tag{8.1.1}$$

where we set $a_1 = 0$ and $b_n = 0$. By the unitarity relations in Lemma 8.1.1, the highest weight must also satisfy

$$\lambda_m(x) \leq \cdots \leq \lambda_1(x) \leq \lambda'_n(x) \leq \cdots \leq \lambda'_1(x),$$

where we write $\Lambda(x) = (\lambda_1(x), \ldots, \lambda_m(x) | \lambda'_1(x), \ldots, \lambda'_n(x))$ for convenience.

For reference, we recall the following lemma.

Lemma 8.1.3. *A finite-dimensional $\mathfrak{g}$-supermodule $L(\Lambda(x))$ decomposes under $\mathfrak{g}_{\bar{0}}$ into a direct sum of simple finite-dimensional $\mathfrak{g}_{\bar{0}}$-modules. Moreover, if $(\Lambda + \rho, \alpha) = 0$ for some odd positive root α, then $L_0(\Lambda - \alpha)$ does not appear as an $\mathfrak{g}_{\bar{0}}$-constituent of $L(\Lambda(x))$.*

We divide the classification into two steps. In the first step, we identify the connected regions of values for x where $L(\Lambda(x))$ is unitarizable or non-unitarizable. To this end, we consider Equation (8.1.1). It is immediate that the strongest restriction on x arises from the $\mathfrak{g}_{\bar{0}}$-constituent $L(\Lambda(x) - \epsilon_m + \delta_n)$. However, if $k_0 < n$, then we may have $\lambda'_n(x) = \cdots = \lambda'_{k_0}(x)$, leading to additional possibilities. By the unitarity conditions, no further restrictions on $\Lambda(x)$ arise. This leads to the following lemma.

Lemma 8.1.4. *The following assertions hold:*

a) If $x < a_m + k_0$, then $L(\Lambda(x))$ is not unitary.

b) If $x > a_m + n$, then $L(\Lambda(x))$ is unitary.

Proof. a) We consider the basic $\mathfrak{g}_{\bar{0}}$-constituents of $L(\Lambda(x))$. By assumption, $(\Lambda(x) + \rho, \epsilon_m - \delta_{k_0}) \neq 0$, which implies that the basic $\mathfrak{g}_{\bar{0}}$-constituent $L^0(\Lambda(x) - \epsilon_m + \delta_{k_0})$ appears and is nontrivial. The Dirac inequality for this constituent reads

$$x - a_m - k_0 \geq 0,$$

which proves the claim by Proposition 8.1.2.

b) The Dirac inequalities for the possible $\mathfrak{g}_{\bar{0}}$-constituents $L_0(\Lambda - \epsilon_i + \delta_j)$ are given by

$$x \geq -m + i + a_i + j - b_j.$$

The maximal lower bound occurs for $i = m$ and $j = n$, yielding

$$x \geq a_m + n,$$

which corresponds to the basic $\mathfrak{g}_{\bar{0}}$-constituent $L^0(\Lambda(x) - \epsilon_m + \delta_n)$. Thus, if $x > a_m + n$, then $(\Lambda(x) + \rho, \epsilon_i - \delta_j) \neq 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$, meaning that $\Lambda(x)$ is typical and all basic $\mathfrak{g}_{\bar{0}}$-constituents appear. Since the Dirac inequality holds strictly for all such constituents, the result follows from Proposition 8.1.2. ☐

It remains to examine the case where $a_m + n \geq x \geq a_m + k_0$. To address this, we use Lemma 8.1.3.

Lemma 8.1.5. *Let $1 \leq k \leq n - k_0 + 1$. Then:*

a) If $x = a_m + n - k + 1$, then $L(\Lambda(x))$ is unitary.

b) If $a_m + n - k < x < a_m + n - k + 1$ (with $1 \leq k \leq n - k_0$), then $L(\Lambda(x))$ is not unitary.

Proof. a) Assume $x = a_m + n - k + 1$ for some $1 \leq k \leq n - k_0 + 1$. Then the Dirac inequality holds strictly for all appearing $\mathfrak{g}_{\bar{0}}$-constituents, except potentially for those of the form $L(\Lambda(x) - \epsilon_m + \delta_{n-l+1})$ with $k \leq l \leq n - k_0 + 1$.

However, by Lemma 8.1.3, the $\mathfrak{g}_{\bar{0}}$-constituents $L(\Lambda(x) - \epsilon_m + \delta_{n-l+1})$ with $k < l \leq n - k_0 + 1$ do not appear, since the corresponding weights are not Δ_c^+-dominant. It remains to consider $L_0(\Lambda(x) - \epsilon_m + \delta_{n-k+1})$, for which

$$(\Lambda(x) + \rho, \epsilon_m - \delta_{n-k+1}) = 0.$$

Thus, this constituent also does not appear, by Lemma 8.1.3. Therefore, $L(\Lambda(x))$ is unitary by Lemma 7.3.2.

b) Now assume $a_m + n - k < x < a_m + n - k + 1$ for some $1 \leq k \leq n - k_0$. The Dirac inequality fails to hold strictly for the basic $\mathfrak{g}_{\bar{0}}$-constituents

$$L_0(\Lambda(x) - \epsilon_m + \delta_{n-k+1}), \ldots, L_0(\Lambda(x) - \epsilon_m + \delta_n),$$

even though it may hold for others such as $L_0(\Lambda(x) - \epsilon_m + \delta_{n-k})$. Therefore, the inequality does not hold strictly for all relevant constituents, and by Proposition 8.1.2, the supermodule $L(\Lambda(x))$ is not unitary. □

Combining the previous lemmas, we obtain a complete classification of the finite-dimensional unitary $\mathfrak{g}$-supermodules.

Theorem 8.1.6. *Let $L(\Lambda(x))$ be a finite-dimensional simple $\mathfrak{g}$-supermodule. Let k_0 be the smallest index such that $b_{k_0} = 0$. If all b_i are zero, set $k_0 = n$. Then $L(\Lambda(x))$ is ω_+-unitarizable if and only if*

$$x \in \{a_m + k_0, a_m + k_0 + 1, \ldots, a_m + n\} \cup (a_m + n, \infty).$$

We may reformulate Theorem 8.1.6 using Young diagrams.

Theorem 8.1.7. *Let $L(\Lambda(x))$ be a finite-dimensional simple $\mathfrak{g}$-supermodule. Then $L(\Lambda(x))$ is ω_+-unitarizable if and only if*

$$g_1(\Lambda) \geq \mathrm{len}_2(\Lambda).$$

8.2. Classification of infinite-dimensional unitarizable supermodules with integral highest weight

We adopt the notation from Sections 2.1.4 and 6.1, and fix the conjugate-linear anti-involution $\omega_{(-,+)}$ corresponding to the real form $\mathfrak{su}(p, q|0, n)$. We also fix the non-standard positive system $\Delta^+ := \Delta^+_{\mathrm{nst}}$ so that the results of Chapter 7 apply.

In this context, the Dirac operator is self-adjoint, and the Dirac inequality takes the form

$$(\mu + 2\rho, \mu) > (\Lambda + 2\rho, \Lambda)$$

for any $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\mu)$ of a unitarizable highest weight $\mathfrak{g}$-supermodule $L(\Lambda)$.

In Section 6.1.4, we parameterized the highest weights of unitarizable simple $\mathfrak{g}_0$-modules as

$$\Lambda = (0, a_2, \dots, a_{m-1}, 0 | b_1, \dots, b_{n-1}, 0) + \frac{\lambda}{2}(1, \dots, 1, -1, \dots, -1 | 0, \dots, 0) + \frac{\alpha}{2}(1, \dots, 1 | 1, \dots, 1),$$

where the integers a_i satisfy $a_{p+1} \geq \cdots \geq a_{m-1} \geq 0 \geq a_2 \geq \cdots \geq a_p$, the integers b_k satisfy $b_1 \geq \cdots \geq b_{n-1}$, and $\alpha, \lambda \in \mathbb{R}$, with $\lambda \leq 0$.

Any highest weight

$$\Lambda = (\lambda_1, \dots, \lambda_p, \lambda_{p+1}, \dots, \lambda_m | \lambda'_1, \dots, \lambda'_n) \in \mathfrak{h}^*$$

of a unitarizable simple $\mathfrak{g}$-supermodule is of this form, satisfies the unitarity conditions in Lemma 6.1.11,

$$\lambda_p \leq \cdots \leq \lambda_1 \leq -\lambda'_1 \leq \cdots \leq -\lambda'_n \leq \lambda_m \leq \cdots \leq \lambda_{p+1},$$

and is Δ_c^+-dominant. To each such weight, we associate three Young diagrams, following the notation of [48]:

$$\begin{aligned} Y_1(\Lambda) &:= (\lambda_1 - \lambda_p, \dots, \lambda_1 - \lambda_2, 0), \\ Y_2(\Lambda) &:= (\lambda_{p+1} - \lambda_m, \dots, \lambda_{m-1} - \lambda_m, 0), \\ Y_3(\Lambda) &:= (\lambda'_1 - \lambda'_n, \dots, \lambda'_{n-1} - \lambda'_n, 0). \end{aligned}$$

Let $\mathrm{len}_i(\Lambda) := \mathrm{length}(Y_i(\Lambda))$ for $i = 1, 2, 3$.

According to the unitarity conditions, there are three key quantities of interest:

$$g_1(\Lambda) := \lambda_m + \lambda'_n, \quad g_2(\Lambda) := \lambda_1 + \lambda'_1, \quad g_3(\Lambda) := \lambda_1 - \lambda_m,$$

with the constraint that $g_3(\Lambda) \leq -\mathrm{len}_1(\Lambda) - \mathrm{len}_2(\Lambda)$, as described in Section 6.1.4.

As shown in Theorem 7.3.2, a simple highest weight $\mathfrak{g}$-supermodule M with highest weight Λ is unitarizable if and only if the highest weight $\mathfrak{g}_{\bar{0}}$-module $L_0(\Lambda)$ is unitarizable, and the Dirac inequality holds strictly for each simple highest weight $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\mu)$ with $\mu \neq \Lambda$ occurring in $M \otimes M(\mathfrak{g}_{\bar{1}})$:

$$(\mu + 2\rho, \mu) < (\Lambda + 2\rho, \Lambda).$$

Analogously to Proposition 8.1.2, these conditions can be relaxed.

Proposition 8.2.1. *Let M be a simple highest weight $\mathfrak{g}$-supermodule with integral highest weight Λ, and assume that M is $\mathfrak{g}_{\bar{0}}$-semisimple. Then M is $\omega_{(-,+)}$-unitarizable if and only if the highest weight $\mathfrak{g}_{\bar{0}}$-module $L_0(\Lambda)$ is unitarizable, and for every $\alpha \in \Delta_{\bar{1}}^+$ such that $L_0(\Lambda - \alpha)$ appears as an $\mathfrak{g}_{\bar{0}}$-constituent of $M \otimes M(\mathfrak{g}_{\bar{1}})$, we have*

$$(\Lambda + \rho, \alpha) < 0.$$

The proof is analogous to that of Proposition 8.1.2 and is therefore omitted. As a direct consequence, we obtain a complete classification.

Theorem 8.2.2. *Let M be a simple highest weight $\mathfrak{g}$-supermodule with integral highest weight Λ, and assume that M is $\mathfrak{g}_{\bar{0}}$-semisimple. Then M is $\omega_{(-,+)}$-unitarizable if and only if the following two conditions hold:*

a) Λ satisfies the unitarity inequalities:

$$\lambda_p \leq \cdots \leq \lambda_1 \leq -\lambda'_1 \leq \cdots \leq -\lambda'_n \leq \lambda_m \leq \cdots \leq \lambda_{p+1}.$$

b) Λ satisfies one of the following two sets of inequalities:

a) $g_2(\Lambda) \leq -\mathrm{len}_1(\Lambda) - q$ and $\mathrm{len}_3(\Lambda) \leq g_1(\Lambda)$,

b) $g_2(\Lambda) \leq -\mathrm{len}_1(\Lambda) - \mathrm{len}_2(\Lambda)$ and $g_1(\Lambda) = \mathrm{len}_3(\Lambda) = 0$.

Proof. By Proposition 8.1.2, it suffices to examine the Dirac inequality

$$(\Lambda + \rho, \alpha) < 0$$

for any $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\Lambda - \alpha)$ with $\alpha \in \Delta^+_{\bar{1}}$. Here, recall that the odd positive system is

$$\Delta^+_{\bar{1}} = \{\epsilon_i - \delta_j : 1 \leq i \leq p,\ 1 \leq k \leq n\} \sqcup \{-\epsilon_k + \delta_l : p+1 \leq k \leq m,\ 1 \leq l \leq n\},$$

The highest weight μ of any $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\mu)$ is Δ^+_c-dominant, which, by the unitarity relations, implies that we only need to consider

$$(\Lambda + \rho, \epsilon_{p-\mathrm{len}_1(\Lambda)} - \delta_1) = \lambda_1 + \lambda'_1 + \mathrm{len}_1(\Lambda) + q,$$
$$(\Lambda + \rho, -\epsilon_m + \delta_{\mathrm{len}_3(\Lambda)}) = -\lambda_m - \lambda'_n + \mathrm{len}_3(\Lambda).$$

If $(\Lambda + \rho, \epsilon_{p-\mathrm{len}_1(\Lambda)} - \delta_1) \leq 0$ and $(\Lambda + \rho, -\epsilon_m + \delta_{\mathrm{len}_3(\Lambda)}) \leq 0$, then the Dirac inequality holds for all other $\mathfrak{g}_{\bar{0}}$-constituents by the unitarity relations. The $\mathfrak{g}_{\bar{0}}$-constituents $L_0(\Lambda - \epsilon_{p-\mathrm{len}_1(\Lambda)} + \delta_1)$ and $L_0(\Lambda + \epsilon_m - \delta_{\mathrm{len}_3(\Lambda)})$ are present if and only if Λ is neither atypical with respect to $\epsilon_{p-\mathrm{len}_1(\Lambda)} - \delta_1$ nor $\epsilon_m - \delta_{\mathrm{len}_3(\Lambda)}$.

We distinguish between two cases: $\mathrm{len}_3(\Lambda) \neq 0$ and $\mathrm{len}_3(\Lambda) = 0$. First, consider $\mathrm{len}_3(\Lambda) \neq 0$. From Proposition 8.1.2 and the equations above, we obtain that M is unitarizable if and only if

$$\lambda_1 + \lambda'_1 + \mathrm{len}_1(\Lambda) + q \leq 0, \qquad -\lambda_m - \lambda'_n + \mathrm{len}_3(\Lambda) \leq 0.$$

Now, consider $\mathrm{len}_3(\Lambda) = 0$. The argument is similar to the above, except when $\mathrm{len}_3(\Lambda) = g_1(\Lambda) = 0$, as in this case there is an interference with the condition on $g_2(\Lambda)$, where $\lambda_m + \lambda'_n = 0$ and $\lambda'_1 = \ldots = \lambda'_n$, *i.e.*, $\lambda_m = -\lambda'_1$, and $g_2(\Lambda) = \lambda_1 + \lambda'_1 = \lambda_1 - \lambda_m$. However, the unitarity condition (6.1.4) in Section 6.1.4 states that

$$g_2(\Lambda) = \lambda_1 + \lambda'_1 = \lambda_1 - \lambda_m \leq -\mathrm{len}_1(\Lambda) - \mathrm{len}_2(\Lambda),$$

and M is unitarizable if and only if $g_2(\Lambda) \leq -\mathrm{len}_1(\Lambda) - \mathrm{len}_2(\Lambda)$ by Proposition 8.1.2. □

Remark 8.2.3. In [80], Kashiwara and Vergne studied the restriction of the oscillator representation (also known as the Segal–Shale–Weil representation) of the metaplectic group — that is, the universal double cover of the symplectic group — to the Lie groups $\mathrm{SU}(p,q)$. They conjectured that any unitary irreducible highest weight representation of $\mathrm{SU}(p,q)$ appears in a k-fold tensor product of oscillator representations for some k. This conjecture was later proven in [71].

In [105], Nishiyama constructed an analogue of the oscillator representation for the symplectic Lie algebra in the setting of orthosymplectic Lie superalgebras $\mathfrak{spo}(2m|2n;\mathbb{R})$, termed the *oscillator supermodule*, which carries a natural unitary structure. Subsequently, in [48], Furutsu and Nishiyama investigated the restriction of the oscillator supermodule to the Lie subsuperalgebras $\mathfrak{su}(p,q|n)$, and proved that any unitarizable simple $\mathfrak{g}$-supermodule that integrates to $\mathrm{SU}(p,q) \times \mathrm{SU}(n)$ appears in a k-fold tensor product of oscillator supermodules for some k. This result generalizes the conjecture of Kashiwara and Vergne to the superalgebra setting.

Using our classification theorem, we confirm the main result of [48] through a different method.

Part III.

Superdimension

9. Methods in representation theory

In this chapter, we introduce the Duflo–Serganova functor and the translation functors, which serve as powerful tools for studying the formal superdimension in the following chapter.

9.1. Duflo–Serganova functor

The *Duflo–Serganova functor*, or simply the *DS functor*, is a symmetric monoidal tensor functor introduced by Michel Duflo and Vera Serganova in [34]. It has emerged as a powerful tool in the study of the representation theory of Lie superalgebras, with applications extending into mathematical physics. Associated to any odd square-zero element, the DS functor relates the $\mathbb{Z}_2$-graded representation categories of Lie superalgebras of different dimensions in a cohomological fashion, while preserving superdimension.

In mathematical physics, the DS functor appears under the pseudonym of *twisting* and plays a central role in reducing (Lagrangian) supersymmetric field theories to simpler *topological* or *holomorphic* models. This correspondence goes beyond analogy; the superconformal index offers a concrete bridge between the representation-theoretic and physical interpretations [139, 140, 23, 119].

In what follows, we give a brief introduction to the DS functor and outline its extension to the unitarizable subcategory. The first part of our exposition is based largely on [34, 51].

9.1.1. Self-commuting variety and associated varieties

The DS functor is defined relative to a chosen element in the self-commuting variety $\mathcal{Y}$, with this choice being unique up to the adjoint $G_{\bar{0}}$-action on $\mathcal{Y}$. Following [34, 51], we introduce the *self-commuting variety* of $\mathfrak{g}$, while for explicit geometric considerations, we refer to [52].

Definition 9.1.1. The self-commuting variety of $\mathfrak{g}$ is the algebraic variety

$$\mathcal{Y} := \{x \in \mathfrak{g}_{\bar{1}} \ : \ [x, x] = 0\}.$$ [1]

On $\mathcal{Y}$, the (complex) Lie group $G_{\bar{0}}$ associated with $\mathfrak{g}_{\bar{0}}$ acts via the adjoint action, making $\mathcal{Y}$ a $G_{\bar{0}}$-invariant Zariski-closed cone in $\mathfrak{g}_{\bar{1}}$. We are interested in the $G_{\bar{0}}$-orbits on $\mathcal{Y}$, which can be equivalently described as the action of the Weyl group W by permutations on the set S of subsets of mutually orthogonal linearly independent roots in $\Delta_{\bar{1}}$. For any

[1] Within Lagrangian field theories exhibiting super Poincaré symmetry, the self-commuting variety—also known as the *nilpotence variety* [35, 37]—was initially introduced to classify the possible (topological and holomorphic) twists of such theories. These twists are closely related to, and in many ways mirror, the role of the DS functor. However, it has since become clear that the nilpotence variety has significantly broader relevance, extending beyond its original scope.

$A = \{\alpha_1, \dots, \alpha_k\} \in S$, choose a non-zero $x_i \in \mathfrak{g}^{\alpha_i}$ and set $x = x_1 + \cdots + x_k \in \mathcal{Y}$. This defines a map

$$\Phi' : S \to \mathcal{Y}/G_{\bar{0}}, \quad \Phi'(A) := G_{\bar{0}}x.$$

If $B = w(A)$ for some $w \in W$ and $A, B \in S$, then $\Phi'(A) = \Phi'(B)$. Thus, Φ' induces a map

$$\Phi : S/W \longrightarrow \mathcal{Y}/G_{\bar{0}},$$

which is indeed a bijection [34, Section 4], *i.e.*, the $G_{\bar{0}}$-orbits on $\mathcal{Y}$ are in one-to-one correspondence with W-orbits in S. We conclude that the self-commuting variety $\mathcal{Y}$ has finitely many $G_{\bar{0}}$-orbits. Moreover, we can assign to any $G_{\bar{0}}$-orbit a number, referred to as its *rank*. Fix $x \in \mathcal{Y}$. Then there exists a $g \in G_{\bar{0}}$ and isotropic, mutually orthogonal, linearly independent roots $\alpha_1, \dots, \alpha_k$ such that

$$\mathrm{Ad}_g(x) = x_1 + \cdots + x_k,$$

where $x_i \in \mathfrak{g}^{\alpha_i}$ (as described above). The number $\mathrm{rk}(x) := k$ is independent of the choice of g and is called the *rank* of x. Equivalently, $\mathrm{rk}(x)$ is given by the rank of x as a linear operator acting in the standard representation.

Given a highest weight $\mathfrak{g}$-supermodule M, we construct a $G_{\bar{0}}$-invariant subvariety of $\mathcal{Y}$, called the *associated variety* $\mathcal{Y}_M$, which provides useful information about atypicality. To this end, we note that any element $x \in \mathcal{Y}$ defines an endomorphism $x_M \in \mathrm{End}_{\mathbb{C}}(M)$ such that $x_M^2 = x_M \circ x_M = 0$. We then set

$$M_x := \ker(x_M)/\operatorname{im}(x_M),$$

and define the *associated variety* of a $\mathfrak{g}$-supermodule M is defined to be the $G_{\bar{0}}$-invariant subvariety

$$\mathcal{Y}_M := \{x \in \mathcal{Y} \ : \ M_x \neq \{0\}\} \subset \mathcal{Y}.$$

The following proposition is a straightforward generalization of Lemma 5.12 and Proposition 5.14 in [24].

Proposition 9.1.2. *Let M be a highest weight $\mathfrak{g}$-supermodule with highest weight Λ.*

a) The associated variety of M is trivial if and only if the highest weight Λ is typical.

b) If α is an odd positive root satisfying $(\Lambda + \rho, \alpha) = 0$, any associated root vector Q_α lies in $\mathcal{Y}_M$. In particular, $\mathcal{Y}_M \neq \{0\}$.

Notably, the unitarity of a supermodule M and the distinction between typicality and atypicality are entirely independent, preventing us from deducing additional properties of $\mathcal{Y}_M$ from the unitary structure.

Finally, we state an important property of translation functors.

Lemma 9.1.3. *Let M be a unitarizable highest weight $\mathfrak{g}$-supermodule with highest weight Λ, F a finite-dimensional $\mathfrak{g}$-supermodule and $\mu \in \mathfrak{h}^*$. Then*

$$\mathcal{Y}_{{}^F T_\Lambda^\mu(M)} \subset \mathcal{Y}_M.$$

Proof. Let $x \in \mathcal{Y} \setminus \mathcal{Y}_M$. Then M is free over $\mathbb{C}[x]$, and $M \otimes F$ is also free over $\mathbb{C}[x]$. Since ${}^F T_\Lambda^\mu(M)$ is a direct summand of $M \otimes F$, it follows that ${}^F T_\Lambda^\mu(M)$ is free over $\mathbb{C}[x]$. Consequently, we conclude that $x \notin \mathcal{Y}_{{}^F T_\Lambda^\mu(M)}$. □

9.1.2. DS functor

In this subsection, we briefly summarize the notion of the DS functor following [34, 128]. In the next subsection, we extend these concepts to unitarizable highest weight $\mathfrak{g}$-supermodules.

First, we assign to any element $x \in \mathcal{Y}$ of the self-commuting variety a Lie superalgebra $\mathfrak{g}_x$. For that, fix $x \in \mathcal{Y}$, and let $k := \mathrm{rk}(x)$. Then

$$\mathfrak{g}_x := \ker(\mathrm{ad}_x)/\operatorname{im}(\mathrm{ad}_x),$$

is a Lie superalgebra, since $[x, \mathfrak{g}] := \operatorname{im}(\mathrm{ad}_x)$ is an ideal in $\ker(\mathrm{ad}_x)$. Naturally, the defect of $\mathfrak{g}$ and $\mathfrak{g}_x$ are related via the rank of x [51, Section 4]:

$$\mathrm{rk}(x) = \operatorname{def}(\mathfrak{g}) - \operatorname{def}(\mathfrak{g}_x).$$

For example, we have $\mathfrak{gl}(m|n)_x \cong \mathfrak{gl}(m-k|n-k)$ and $\mathfrak{sl}(m|n)_x = \mathfrak{sl}(m-k|n-k)$ for $m \neq n$. The Lie superalgebra $\mathfrak{g}_x$ has a Cartan subalgebra

$$\mathfrak{h}_x := (\ker(\alpha_1) \cap \cdots \cap \ker(\alpha_k))/(\mathfrak{h}^{\alpha_1} \oplus \cdots \oplus \mathfrak{h}^{\alpha_k}),$$

where $\mathfrak{h}^{\alpha} = [\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}]$, with set of roots for $(\mathfrak{g}_x, \mathfrak{h}_x)$ given by

$$\Delta_x = \{\alpha \in \Delta : (\alpha, \alpha_i) = 0,\ \alpha \neq \alpha_i,\ i = 1, \ldots, k\}.$$

Second, we define the DS functor. For that, fix a $\mathfrak{g}$-supermodule M, and define

$$M_x := \ker(x_M)/\operatorname{im}(x_M).$$

By definition, M_x is a $\mathfrak{g}_x$-supermodule, since $\ker(x_M)$ is $\ker(\mathrm{ad}_x)$-invariant and $[x, \mathfrak{g}]\ker(x_M) \subset \operatorname{im}(x_M)$. This assignment defines a functor $M \mapsto M_x$ from the category of $\mathfrak{g}$-supermodules to the category of $\mathfrak{g}_x$-supermodules, also denoted by

$$DS_x(M) := M_x.$$

This functor is known as the *Duflo–Serganova functor*, or simply the *DS functor*. As for finite-dimensional $\mathfrak{g}$-supermodules, one can prove the following lemma which encapsulates several important properties.

Lemma 9.1.4 ([51, Section 2])**.**

*a) For any $x \in \mathcal{Y}$ is the Duflo–Serganova functor $DS_x : \mathfrak{g}$-**smod** $\to \mathfrak{g}_x$-**smod** a symmetric monoidal tensor functor, and additive.*

b) The Duflo–Serganova functor commutes with translation functors.

c) Let $x \in \mathcal{Y}$ and let $0 \to M' \xrightarrow{\psi} M \xrightarrow{\phi} M'' \to 0$ be an exact sequence of $\mathfrak{g}$-supermodules. Then there exists an exact sequence of $\mathfrak{g}_x$-supermodules

$$0 \to E \to DS_x(M') \to DS_x(M) \to DS_x(M'') \to \Pi(E) \to 0$$

for some $\mathfrak{g}_x$-supermodule E. In particular, the DS functor is middle exact and satisfies $DS_x(\Pi(M)) = \Pi(DS_x(M))$ for any $\mathfrak{g}$-supermodule M.

Third, we restrict our considerations to highest weight $\mathfrak{g}$-supermodules M. We provide a sufficient condition for $DS_x(M) = \{0\}$ in terms of the rank of x and the degree of atypicality of the infinitesimal character/highest weight of M. The simplest case occurs when M has typical infinitesimal character/highest weight.

Proposition 9.1.5. *Let M be a highest weight $\mathfrak{g}$-supermodule with typical highest weight. Then*

$$DS_x(M) = \{0\}$$

for all $x \in \mathcal{Y} \setminus \{0\}$.

Proof. As the highest weight is typical, the associated variety is trivial by Proposition 9.1.2, *i.e.*, $\mathcal{Y}_M = \{0\}$. By definition, $DS_x(M) = M_x$, and $\mathcal{Y}_M = \{x \in \mathcal{Y} : M_x \neq \{0\}\}$. The statement is now evident. □

Next, we consider the case where the rank of $x \in \mathcal{Y}$ is greater than the atypicality of the highest weight of M. We thus examine a construction first introduced in [34, Section 6]. We also follow the approach outlined in [128].

Let $\mathfrak{U}(\mathfrak{g})^{\mathrm{ad}_x}$ be the subsuperalgebra of ad_x-invariants in $\mathfrak{U}(\mathfrak{g})$, and consider the left ideal $I_x := [x, \mathfrak{U}(\mathfrak{g})]$ in $\mathfrak{U}(\mathfrak{g})$. We define a map $\phi = \pi \circ \iota$ via the sequence

$$\mathfrak{U}(\mathfrak{g}_x) \xrightarrow{\iota} \mathfrak{U}(\mathfrak{g})^{\mathrm{ad}_x} \xrightarrow{\pi} \mathfrak{U}(\mathfrak{g})^{\mathrm{ad}_x}/(I_x \cap \mathfrak{U}(\mathfrak{g})^{\mathrm{ad}_x}),$$

where the inclusion ι and the projection π are homomorphisms of $\mathfrak{g}_x$-supermodules with respect to the adjoint action. This induces an isomorphism of super vector spaces [34, Lemma 6.6]. We consider the natural surjective projection

$$\eta := \phi^{-1} \circ \pi : \mathfrak{U}(\mathfrak{g})^{\mathrm{ad}_x} \to \mathfrak{U}(\mathfrak{g}_x),$$

such that for any $u \in \mathfrak{U}(\mathfrak{g})^{\mathrm{ad}_x}$ and $m \in M_x$ the following holds:

$$um = \eta(u)m.$$

Here, note that $\ker(x_M)$ is invariant under $\mathfrak{U}(\mathfrak{g})^{\mathrm{ad}_x}$ and $I_x \ker(x_M) \subset \mathrm{im}(x_M)$. We use this map to investigate infinitesimal characters. Let $\mathfrak{Z}(\mathfrak{g}_x)$ be the center of $\mathfrak{U}(\mathfrak{g}_x)$, and note that $\mathfrak{Z}(\mathfrak{g})$ is a subsuperalgebra of $\mathfrak{U}(\mathfrak{g})^{\mathrm{ad}_x}$. Since η is a homomorphism of $\mathfrak{g}_x$-supermodules, we have $\eta(\mathfrak{Z}(\mathfrak{g})) \subset \mathfrak{Z}(\mathfrak{g}_x)$. Moreover, the associated dual map

$$\eta^* : \mathrm{Hom}(\mathfrak{Z}(\mathfrak{g}_x), \mathbb{C}) \longrightarrow \mathrm{Hom}(\mathfrak{Z}(\mathfrak{g}), \mathbb{C}),$$

is injective [34, Theorem 6.11].

Fix a highest weight $\mathfrak{g}$-supermodule M, and let χ be its infinitesimal character. For any $z \in \mathfrak{Z}(\mathfrak{g})$ and $m \in \ker(x_M)$, we have

$$\chi(z)m = zm = \eta(z)m \bmod xM,$$

and hence, if M_x contains a submodule with infinitesimal character ξ, we must have $\eta^*(\xi) = \chi$. Moreover, by the definitions of atypicality and $\mathfrak{h}_x$, the degree of atypicality satisfies $\mathrm{at}(\eta^*(\xi)) = \mathrm{at}(\xi) + \mathrm{rk}(x)$. This establishes the following theorem.

Theorem 9.1.6 ([128, Theorem 2.1])**.** *Let M be a simple highest weight $\mathfrak{g}$-supermodule with infinitesimal character χ. Then M_x is a direct sum of $\mathfrak{g}_x$-supermodules admitting generalized infinitesimal characters from $(\eta^*)^{-1}(\chi)$. The degree of atypicality of the infinitesimal characters in $(\eta^*)^{-1}(\chi)$ is equal to the degree of atypicality of χ minus the rank of x.*

The following corollary is immediate.

Corollary 9.1.7. *Let $x \in \mathcal{Y} \setminus \{0\}$ be an element of the self-commuting variety of rank l. Let M be a highest weight $\mathfrak{g}$-supermodule with atypical highest weight of degree k. Assume $k < l$. Then*

$$DS_x(M) = \{0\}.$$

9.1.3. DS functor and unitarity

For a unitarizable highest weight $\mathfrak{g}$-supermodule $\mathcal{H}$ and an element $x \in \mathcal{Y}$ of the self-commuting variety, it remains to determine under which conditions $DS_x(\mathcal{H})$ is unitarizable. In this context, however, the DS functor does not adequately address unitarity, as the real forms of $\mathfrak{g}_x$ are not determined by the conjugate-linear anti-involution ω on $\mathfrak{g}$ that defines unitarity. Indeed, no $x \in \mathcal{Y}$ satisfies $\omega(x) = -x$, as $[x, \omega(x)] \neq 0$ for all $x \in \mathfrak{g}_{\bar{1}}$, that is, $\mathcal{Y}$ has no real locus. More fundamentally, and without any assumptions on $\mathfrak{g}$, any $x \in \mathfrak{g}^\omega$ acts on $\mathcal{H}$ as a (possibly (anti-)imaginary) self-adjoint operator. This implies that $x^2 = -x\omega(x)$ is (up to a phase positive or negative semi-)definite, and vanishes (if and) only if x itself acts by 0. As a consequence, $\mathcal{H}_x = \mathcal{H}$ for any such x because the unitary representation theory of $\mathfrak{g}$ factors through $\mathcal{Y}^\omega$.

To extend the DS functor to incorporate elements that reflect the real form, we follow [51].

Note that $\mathfrak{g}_{\bar{0}}$ has semisimple elements as being reductive. We then define the variety

$$\mathcal{Y}^{\text{hom}} := \{x \in \mathfrak{g}_{\bar{1}} : [x, x] \text{ is semisimple}\},$$

which is stable under $G_{\bar{0}}$, but not closed in $\mathfrak{g}_{\bar{1}}$. Elements in $\mathcal{Y}^{\text{hom}}$ are called *homological*.

Fix some $x \in \mathcal{Y}^{\text{hom}}$, and set $c := [x, x]$. For a $\mathfrak{g}$-supermodule M, let M^c denote the space of c-invariants on M. Then x defines a square-zero endomorphism on M^c, allowing us to consider its cohomology:

$$M_x := (\ker x|_{M^c})/(\operatorname{im} x|_{M^c}).$$

The subsequent lemma follows by the same arguments as those in [34].

Lemma 9.1.8. *For $x \in \mathcal{Y}^{hom}$, the following assertions hold:*

a) $\mathfrak{g}_x$ has the natural structure of a Lie superalgebra.

b) M_x is a $\mathfrak{g}_x$-supermodule.

The Lie superalgebra $\mathfrak{g}_x$ for general $x \in \mathcal{Y}^{\text{hom}}$ has an easy description given in [134]. To this end, we note that c lies in some Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}_{\bar{0}}$, as $\mathfrak{g}_{\bar{0}}$ is semisimple. The Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{t}$ are in general different. In particular, $\mathfrak{g}_{\bar{0}}^c$ is a reductive Lie algebra, and we have a root space decomposition

$$\mathfrak{g}^c = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathfrak{t}^*,\ \alpha(c) = 0} \mathfrak{g}^\alpha.$$

We denote the root system by $\Delta(\mathfrak{g}^c;\mathfrak{t})$. Furthermore, there exist mutually orthogonal, linearly independent isotropic roots $\alpha_1,\dots,\alpha_r$ such that [134, Lemma 3.9]:

$$x = u_{\alpha_1} + \dots + u_{\alpha_r} + z_1 v_{\alpha_1} + \dots + z_r v_{\alpha_r},$$

where $u_{\alpha_i} \in \mathfrak{g}^{\alpha_i} \setminus \{0\}$, $v_{\alpha_i} \in \mathfrak{g}^{-\alpha_i} \setminus \{0\}$, and $z_i \in \mathbb{C}$. In this case, we say x has rank r. Set $t_{\alpha_i} := [u_{\alpha_i}, v_{\alpha_i}] \in \mathfrak{t}$. Then any $\{t_{\alpha_i}, u_{\alpha_i}, v_{\alpha_i}\}$ generate a $\mathfrak{sl}(1|1)$-subalgebra and we conclude that $u \in \mathfrak{sl}(1|1)^k$. We can assume that u is generic, *i.e.*,

$$\ker \mathrm{ad}_c = \ker \mathrm{ad}_{t_{\alpha_1}} \cap \dots \cap \ker \mathrm{ad}_{t_{\alpha_k}}.$$

Then,

$$\mathfrak{t}_x := (\ker \alpha_1 \cap \dots \cap \ker \alpha_k)/(\mathfrak{t}_{\alpha_1} \oplus \dots \oplus \mathfrak{t}_{\alpha_k})$$

is a Cartan subalgebra of $\mathfrak{g}_x$, and

$$\Delta_x = \{\alpha \in \Delta(\mathfrak{g}^c;\mathfrak{t}) : (\alpha,\alpha_i) = 0,\ \alpha \neq \alpha_i,\ i = 1,\dots,k\}$$

is the set of roots. In particular, $\mathfrak{g}_x$ is a Lie subsuperalgebra of $\mathfrak{g}$, and the following result holds [130, Section 5]:

$$\mathfrak{g}_x = DS_x(\mathfrak{g}) = \begin{cases} \mathfrak{sl}(m-k|n-k) & \text{for } m \neq n, \\ \mathfrak{psl}(n-k|n-k) & \text{for } m = n. \end{cases}$$

To describe the real forms of $\mathfrak{g}_x$, we note that we may assume $z_i = \pm 1$, meaning each x has a signature. In the example of $\mathfrak{g}$, we have

$$\mathfrak{g}_x^{\omega_x} = \mathfrak{su}(p-r, q-s|n-k)$$

for some $r \leq p, s \leq q$ with $r + s = k$.

As a result, for any $x \in \mathcal{Y}^{\mathrm{hom}}$, we define a functor

$$DS_x(M) := M_x := (\ker x|_{M^c})/(\mathrm{im}\, x|_{M^c})$$

from the category of $\mathfrak{g}$-supermodules to the category of $\mathfrak{g}_x$-supermodules, which we also call the *DS functor* with respect to x. This functor is again a tensor functor that allows us to study unitarity.

Lemma 9.1.9. *Let $x \in \mathcal{Y}^{hom}$, and assume $\omega(x) = -x$. Then the conjugate-linear anti-involution ω has a well-defined restriction to $\mathfrak{g}_x$, denoted by ω_x.*

Proof. For $A \in \mathfrak{g}_x$, it is enough to show that $\omega(A) \in \mathfrak{g}_x$. However, for any $A \in \ker \mathrm{ad}_x$, we have $\omega(A) \in \ker \mathrm{ad}_x$, as

$$[\omega(A), x] = [\omega(A), -\omega(x)] = -\omega([x, A]) = 0.$$

Note that $\ker \mathrm{ad}_x = \ker \mathrm{ad}_x|_{\mathfrak{g}^c}$. Moreover, if $A \in \mathrm{im}\, \mathrm{ad}_x|_{\mathfrak{g}^c}$, then $\omega(A) \in \mathrm{im}\, \mathrm{ad}_x|_{\mathfrak{g}^c}$, since there exists some $B \in \mathfrak{g}^c$ such that $A = [x, B]$, which implies

$$\omega(A) = \omega([x, B]) = [\omega(B), \omega(x)] = [x, -(-1)^{p(B)}\omega(B)],$$

and $\omega(B) \in \mathfrak{g}^c$ for any $B \in \mathfrak{g}^c$ as $\omega(x) = -x$. This concludes the proof. □

Proposition 9.1.10. *Let $\mathcal{H}$ be a unitarizable $\mathfrak{g}$-supermodule and let $x \in \mathcal{Y}^{hom}$ such that $\omega(x) = -x$. Then $DS_x(\mathcal{H}) = \ker x|_{\mathcal{H}^c}$ is a ω_x-unitarizable $\mathfrak{g}_x$-supermodule. In particular, $DS_x(\mathcal{H})$ is a semisimple $\mathfrak{g}_x$-supermodule.*

Proof. The kernel $\ker x|_{\mathcal{H}^c}$ and image $\operatorname{im} x|_{\mathcal{H}^c}$ are linear subspaces of the pre-super Hilbert space $\mathcal{H}$. In particular, $\mathcal{H}_x$ forms a pre-super Hilbert space with a positive-definite Hermitian form $\langle \cdot, \cdot \rangle$ given by the restriction of the Hermitian form on $\mathcal{H}$. By construction and Lemma 9.1.9, this form is also ω_x-contravariant for $\mathfrak{g}_x$.

Assume that $v \in M^c$ lies in the image of x, *i.e.*, there exists some $w \in \mathcal{H}^c$ with $xw = v$. Then

$$\langle v, v \rangle = \langle xw, xw \rangle = \langle \omega_x(x)xw, w \rangle = \langle -x^2 w, w \rangle = 0$$

as $w \in \mathcal{H}^c$ and $c = [x, x] = 2x^2$. By the positive-definiteness of the Hermitian form, it follows that $v = 0$, hence $\operatorname{im} x|_{\mathcal{H}^c} = 0$, and thus $\mathcal{H}_x = \ker x|_{\mathcal{H}^c}$.

The semisimplicity of $\mathcal{H}_x$ as a $\mathfrak{g}_x$-supermodule follows by Proposition 6.1.8. □

Using Proposition 9.1.10 and Theorem 6.1.11, we conclude the following:

Corollary 9.1.11. *Let $\mathcal{H}$ be a unitarizable $\mathfrak{g}$-supermodule. Then $DS_x(\mathcal{H})$ decomposes in unitarizable highest weight $\mathfrak{g}_x$-supermodules.*

Lemma 9.1.12. *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a unitarizable $\mathfrak{g}$-supermodule and let $x \in \mathcal{Y}^{hom}$ such that $\omega(x) = -x$. Then $c := [x, x]$ is a negative operator, that is,*

$$(cv, v) \leq 0 \qquad \forall v \in \mathcal{H}.$$

Proof. For any $v \in \mathcal{H}$, we have

$$\langle cv, v \rangle = 2\langle x^2 v, v \rangle = 2\langle xv, \omega(x)v \rangle = -2\langle xv, xv \rangle \leq 0$$

by the positive definiteness of $\langle \cdot, \cdot \rangle$. □

For any $x \in \mathcal{Y}^{\text{hom}}$, the element $c := [x, x]$ is semisimple in $\mathfrak{g}_{\bar{0}}$. Since $\mathfrak{g}_{\bar{1}}$ is a semisimple $\mathfrak{g}_{\bar{0}}$-module, we conclude that ad_c is diagonalizable on $\mathfrak{g}$.

Corollary 9.1.13. *Let $x \in \mathcal{Y}^{\text{hom}}$, and set $c := [x, x]$. Then the eigenvalues of ad_c on $\mathfrak{g}$ are negative. In particular, $\mathfrak{U}(\mathfrak{g})^c = \mathfrak{U}(\mathfrak{g}^c)$.*

Proof. Fix a nontrivial unitarizable $\mathfrak{g}$-supermodule $\mathcal{H}$ such that $DS_x(\mathcal{H}) \neq \{0\}$. Let $X \in \mathfrak{g}$ be an eigenvector of ad_c, so that $\operatorname{ad}_c(X) = zX$ for some $z \in \mathbb{C}$. Then, using Lemma 9.1.12, we have for any $v \in DS_x(\mathcal{H})$:

$$0 \geq \langle cXv, Xv \rangle = \langle [c, X]v, Xv \rangle + \langle Xcv, Xv \rangle = \langle [c, X]v, Xv \rangle = z\langle Xv, Xv \rangle.$$

Since $\langle Xv, Xv \rangle \geq 0$, it follows that $z \leq 0$. The second claim follows immediately. □

Corollary 9.1.14. *Let $\mathcal{H}$ be a unitarizable simple $\mathfrak{g}$-supermodule and let $x \in \mathcal{Y}^{hom}$ such that $\omega(x) = -x$. Then*

$$\mathcal{H}^c = \ker x_{\mathcal{H}}.$$

Proof. The inclusion $\ker x_{\mathcal{H}} \subset \mathcal{H}^c$ is immediate, since $c = [x, x] = x^2$. Conversely, for any $v \in \mathcal{H}^c$, we have:

$$0 = \langle cv, v\rangle = -\langle xv, xv\rangle,$$

and thus $xv = 0$ by the positive definiteness of the Hermitian form. This completes the proof. □

Finally, we give a condition for simplicity of $DS_x(\mathcal{H})$ as a $\mathfrak{g}_x$-supermodule. For this purpose, we need the following lemma.

Lemma 9.1.15. *Let $x \in \mathcal{Y}^{\mathrm{hom}}$ such that $x = u_\alpha + z_\alpha v_\alpha$ has rank one and $c = [x, x] \in \mathfrak{h}$. Let $X \in \mathfrak{g}^\xi$ for some $\xi \in \Delta$, and assume $[c, X] = 0$. Then X belongs to $\mathfrak{g}^c \setminus \mathfrak{g}^x$ if and only if $\xi = \pm\alpha$.*

Proof. As $[c, X] = 0$, we have $(\alpha, \xi) = 0$. Assume

$$[x, X] = [u_\alpha, X] + z_\alpha[v_\alpha, X] \neq 0.$$

Then, $\alpha + \xi \in \Delta$ or $-\alpha + \xi \in \Delta$. We distinguish two cases. First, assume ξ is even, such that $\pm\alpha + \xi \in \Delta_{\bar{1}}$. However,

$$(\pm\alpha + \xi, \pm\alpha + \xi) = (\xi, \xi) \neq 0,$$

which is a contraction, as all odd roots are isotropic. Now, assume ξ is odd; hence isotropic. Then $\pm\alpha + \xi \in \Delta_{\bar{0}}$ is isotropic, and again, as the only isotropic roots are odd, implying $\xi = \alpha$ or $\xi = -\alpha$. □

Theorem 9.1.16. *Let $\mathcal{H}$ be a unitarizable simple $\mathfrak{g}$-supermodule and let $x \in \mathcal{Y}^{hom}$ such that $\omega(x) = -x$. Assume the highest weight Λ of $\mathcal{H}$ satisfies $(\Lambda + \rho, \alpha) = 0$ for some odd root α, and $x = u_\alpha + z_\alpha v_\alpha$ has rank one with $c := [x, x] = h_\alpha \in \mathfrak{h}$. Then the following two assertions hold:*

a) $DS_x(\mathcal{H}) \neq 0$ and decomposes in at most two unitarizable highest weight $\mathfrak{g}_x$-supermodules.

b) If α is simple, $DS_x(\mathcal{H})$ is a simple $\mathfrak{g}_x$-supermodule.

Proof. By Proposition 6.1.8 and Proposition 9.1.10, the $\mathfrak{g}_x$-supermodule $DS_x(\mathcal{H})$ decomposes completely in unitarizable highest weight supermodules. Assume v_μ and v_λ are highest weight vectors of unitarizable highest weight $\mathfrak{g}_x$-supermodules in $DS_x(\mathcal{H})$. Then there exists some $X \in \mathfrak{U}(\mathfrak{g})$ such that $v_\lambda = Xv_\mu$, since $\mathcal{H}$ is a simple $\mathfrak{g}$-supermodules.

First, we show that X belongs to $\mathfrak{U}(\mathfrak{g})^c$. For that, we note that $0 = cv_\Lambda = cXv_\mu$. By the PBW Theorem, the element X is a finite linear combination of elements of the form

$$X_1^{r_1} \cdots X_{\dim(\mathfrak{g}_{\bar{0}})}^{r_{\dim(\mathfrak{g}_{\bar{0}})}} Y_1^{s_1} \cdots Y_{\dim(\mathfrak{g}_{\bar{1}})}^{s_{\dim(\mathfrak{g}_{\bar{1}})}}, \quad r_1, \ldots, r_{\dim(\mathfrak{g}_{\bar{0}})} \in \mathbb{Z}_+,\ s_1, \ldots, s_{\dim(\mathfrak{g}_{\bar{1}})} \in \{0, 1\},$$

where $\{X_1, \ldots, X_{\dim(\mathfrak{g}_{\bar{0}})}\}$ is a basis for $\mathfrak{g}_{\bar{0}}$ and $\{Y_1, \ldots, Y_{\dim(\mathfrak{g}_{\bar{1}})}\}$ is a basis for $\mathfrak{g}_{\bar{1}}$. These elements form a basis of $\mathfrak{U}(\mathfrak{g})$. Let X' be any basis element appearing in the decomposition of X. We may assume, without loss of generality, that $X'v_\mu \neq 0$. Then, by assumption, there exist complex numbers z_i and w_j such that $[c, X_i] = z_iX_i$ and $[c, Y_j] = w_jY_j$ for all i, j. A direct calculation yields

$$cX'v_\mu = (\sum_i z_i + \sum_j w_j)X'v_\mu = 0,$$

which forces $\sum_i z_i + \sum_j w_j = 0$, since $X' v_\mu \neq 0$. By Corollary 9.1.13, we conclude $z_i = 0$ and $w_j = 0$ for all i, j, *i.e.*, X' commutes with c. In particular, $X \in \mathfrak{U}(\mathfrak{g})^c$. Again, by Corollary 9.1.13, we conclude $X \in \mathfrak{U}(\mathfrak{g}^c)$.

Next, without loss of generality, we assume that $X \in \mathfrak{U}((\mathfrak{n}^+)^c)$, as $\mathcal{H}$ is a highest weight $\mathfrak{g}$-supermodule. The vectors v_μ and v_λ are highest weight vectors for $\mathfrak{g}_x$, and consequently, are annihilated by the highest weight vectors in $\mathfrak{g}_x$. Using Lemma 9.1.15 and the PBW Theorem, we conclude that

$$X = z + z' X_\alpha, \qquad X_\alpha \in \mathfrak{g}^\alpha, \quad z, z' \in \mathbb{C}.$$

In particular, as $X_\alpha^2 = 0$, there are at most two highest weight vectors. This proves statement a).

To prove b), we note that it is enough to show that $DS_x(\mathcal{H})$ is a highest weight $\mathfrak{g}_x$-supermodule, *i.e.*, $X \in \mathbb{C}$. As α is simple, $v_\Lambda \in DS_x(\mathcal{H})$, and we have for any other highest weight vector $v_\mu \in DS_x(\mathcal{H})$ with $X v_\mu = v_\Lambda$, $X \in \mathfrak{U}((\mathfrak{n}^+)^c)$:

$$\langle v_\Lambda, v_\Lambda \rangle = \langle X_\alpha v_\mu, X_\alpha v_\mu \rangle = \langle X_{-\alpha} X_\alpha v_\mu, v_\mu \rangle = (\mu + \rho, \alpha) \langle v_\mu, v_\mu \rangle = 0,$$

using $\Lambda - \alpha = \mu$, $(\rho, \alpha) = 0$ and $(\Lambda + \rho, \alpha) = 0$. This is a contradiction, forcing $X = z' \in \mathbb{C}$, and $v_\Lambda = z' \cdot v_\mu$. This completes the proof of b). □

We use Theorem 9.1.16 to show that $DS_x(\mathcal{H})$ decomposes into finitely many $\mathfrak{g}_x$-supermodules whenever $\mathcal{H}$ is simple. To this end, we first observe that $DS_x(\mathcal{H}) = \ker x = \mathcal{H}^c$, since c is a negative operator.

Now, consider an element $x = u_1 + u_2 + z_1 v_1 + z_2 v_2 \in \mathcal{Y}^{\mathrm{hom}}$ of rank 2, such that $[x, x] \in \mathfrak{h}$. Then we can write $c = [x, x] = c_1 + c_2$, where $c_1 = [x_1, x_1]$ and $c_2 = [x_2, x_2]$, with $x_i = u_i + z_i v_i$. In particular, c_1 and c_2 are negative operators, and we have

$$DS_x(\mathcal{H}) = \mathcal{H}^{c_1} \cap \mathcal{H}^{c_2} = DS_{x_2}(DS_{x_1}(\mathcal{H})).$$

Using this structure inductively, we obtain the following lemma from Theorem 9.1.16.

Lemma 9.1.17. *Let $\mathcal{H}$ be a unitarizable simple $\mathfrak{g}$-supermodule, and let $x \in \mathcal{Y}^{hom}$ satisfy $\omega(x) = -x$. Suppose that x is of rank k and decomposes as $x = x_1 + \cdots + x_k$. Then*

$$DS_x(\mathcal{H}) = DS_{x_k}(DS_{x_{k-1}}(\dots DS_{x_1}(\mathcal{H}))).$$

In particular, $DS_x(\mathcal{H})$ decomposes into finitely many $\mathfrak{g}_x$-supermodules.

9.2. Translation functors

Unitarizable simple $\mathfrak{g}$-supermodules are of highest weight type and belong to the category of Harish-Chandra supermodules, $\mathcal{M}(\mathfrak{g}, \mathfrak{k})$ (*cf.* Section 6.1.5). We now consider translation functors in $\mathcal{M}(\mathfrak{g}, \mathfrak{k})$.

We begin with a suitable decomposition of any object M in $\mathcal{M}(\mathfrak{g}, \mathfrak{k})$. To do so, we recall that M is $\mathfrak{h}$-semisimple, as $\mathfrak{k}^{\mathbb{C}}$ satisfies the equal rank condition $\mathfrak{h} \subset \mathfrak{k}^{\mathbb{C}} \subset \mathfrak{g}_{\bar{0}}$.

We decompose M under $\mathfrak{h}$ into weight spaces, that is, $M = \oplus_{\lambda \in \mathfrak{h}^*} M^\lambda$.

For any $\lambda \in \mathfrak{h}^*$, let $\chi_\lambda : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$ denote the associated infinitesimal character. We define

$$M_\lambda := \bigoplus_{\nu \in \sum_{\alpha \in \Delta} \mathbb{Z}\alpha} M_\lambda^{\lambda + \nu},$$

where $M_\lambda^{\lambda+\nu} := \{m \in M^{\lambda+\nu} :$ for all $z \in \mathfrak{Z}(\mathfrak{g}),\ \exists n \in \mathbb{Z}_+$ such that $(z - \chi_\lambda(z))^n m = 0\}$.

The space M_λ is a sub $\mathfrak{g}$-supermodule of M by construction, and we say that M_λ has *generalized infinitesimal character* χ_λ. The following lemma is straightforward.

Lemma 9.2.1. *$M_\lambda = M_\mu$ for any two $\lambda, \mu \in \mathfrak{h}^*$ if and only if $\lambda = w(\mu+\rho+\sum_{i=1}^k t_i\alpha_i) - \rho$, where $w \in W$, $t_i \in \mathbb{C}$, and $\alpha_1, \ldots, \alpha_k$ are linearly independent odd roots that satisfy $(\mu + \rho, \alpha_i) = 0$, and $\lambda - \mu \in \Xi$.* [2]

The lemma provides a canonically defined projection pr_λ from any Harish-Chandra supermodule M to the subspace M_λ with generalized infinitesimal character χ_λ.

Let $\lambda, \mu \in \mathfrak{h}^*$, let M be an object in $\mathcal{M}(\mathfrak{g}, \mathfrak{k})$, and let F be a finite-dimensional $\mathfrak{g}$-supermodule. Recall that $\mathcal{M}(\mathfrak{g}, \mathfrak{k})$ is closed under tensoring with finite-dimensional $\mathfrak{g}$-supermodules. The *translation functor* is the exact functor

$$M \mapsto {}^F T_\lambda^\mu(M) := \mathrm{pr}_\mu(M \otimes \mathrm{pr}_\lambda(F))$$

on the category of Harish-Chandra supermodules $\mathcal{M}(\mathfrak{g}, \mathfrak{k})$.

To understand the projection functor, we investigate the Harish-Chandra supermodule $M \otimes F$ for some highest weight Harish-Chandra supermodule M and some finite-dimensional $\mathfrak{g}$-supermodule F. Fix a basis $\{e_1, \ldots, e_n\}$ of F such that e_i is a vector of weight ν_i and $i \leq j$ if $\nu_i \leq \nu_j$. Let P_F denote the set of all weights of F, counted with multiplicity.

Assume that M has a symmetric highest weight Λ, that is, $\overline{\Lambda(\omega(\cdot))} = \Lambda(\cdot)$, so that we can endow M with the ω-contravariant Shapovalov form $\langle \cdot, \cdot \rangle_M$, as introduced in Section 6.1.3. With respect to this form, the generalized eigenspaces of M are orthogonal to each other.

Lemma 9.2.2. *The following assertions hold:*

a) M_λ, M_μ are orthogonal for $\chi_\lambda \neq \chi_\mu$ with respect to $\langle \cdot, \cdot \rangle_M$.

b) M decomposes as a finite direct sum $M = \bigoplus_\lambda M_\lambda$.

Proof. a) Assume $\langle M_\lambda, M_\mu \rangle_M \neq 0$. Then there exists a weight $\nu \in \mathfrak{h}^*$ such that

$$\langle M_\lambda^{\lambda+\nu}, M_\mu^{\mu+\nu} \rangle_M \neq 0.$$

By assumption, there exists an $n \in \mathbb{N}$ such that $(z - \chi_\lambda(z))^n M_\lambda^{\lambda+\nu} = 0$, and consequently, by the contravariance of the Shapovalov form:

$$0 = \langle (z - \chi_\lambda(z))^n M_\lambda^{\lambda+\nu}, M_\mu^{\mu+\nu} \rangle_M = \langle M_\lambda^{\lambda+\nu}, (\omega(z) - \chi_\lambda(\omega(z)))^n M_\mu^{\mu+\nu} \rangle_M.$$

Here, we used the symmetry condition on Λ to deduce that $\overline{\chi_\lambda(z)} = \chi_\lambda(\omega(z))$, and $\omega(z) \in \mathfrak{Z}(\mathfrak{g})$ if and only if $z \in \mathfrak{Z}(\mathfrak{g})$. Moreover, if $\chi_\lambda(z) \neq \chi_\mu(z)$, then $\chi_\lambda(\omega(z)) \neq \chi_\mu(\omega(z))$, and $\omega(z) - \chi_\lambda(\omega(z))$ must act bijectively on $M_\mu^{\mu+\nu}$. We conclude that $(\omega(z) - \chi_\lambda(\omega(z)))^n M_\mu^{\mu+\nu} = M_\mu^{\mu+\nu}$ for all $z \in \mathfrak{Z}(\mathfrak{g})$, *i.e.*, $\langle M_\lambda^{\lambda+\nu}, M_\mu^{\mu+\nu} \rangle_M = 0$, which is a contradiction. This proves the first statement.

b) The second statement is immediate, as $M \in \mathcal{M}(\mathfrak{g}, \mathfrak{k})$ is generated by finitely many weight vectors and has a finite Jordan–Hölder series. □

[2] Recall that Ξ is the root lattice.

Let Λ be the highest weight of M, and let v_Λ be the highest weight vector. The $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$-module $V := \mathfrak{U}(\mathfrak{g}_{\bar{0}})v_\Lambda \subset M$ is a highest weight Harish-Chandra module for $\mathfrak{g}_{\bar{0}}$, and M can be realized as a quotient of the Kac supermodule $K(V)$. Accordingly, any highest weight Harish-Chandra supermodule is Kac-induced from a Harish-Chandra module of $\mathfrak{g}_{\bar{0}}$. The following tensor identity describes the relationship between tensor products with finite-dimensional Harish-Chandra supermodules and Kac induction.

Lemma 9.2.3. *Let F be a finite-dimensional $\mathfrak{g}$-supermodule, and set $F^{\mathfrak{g}_{+1}} := \{v \in F : Xv = 0$ for all $X \in \mathfrak{g}_{+1}\}$. Then there exists a natural equivalence of functors*

$$K(- \otimes F^{\mathfrak{g}_{+1}}) \cong K(-) \otimes F$$

such that for all homogeneous elements $X \in \mathfrak{g}$ and $v \in F$,

$$X \otimes (- \otimes v) \mapsto (X \otimes -) \otimes v + (-1)^{p(X)p(-)}(1 \otimes -) \otimes Xv.$$

Proof. The following are natural isomorphisms of functors:

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{U}(\mathfrak{g})}(K(-) \otimes F, -) &\cong \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g})}(K(-), \mathrm{Hom}_{\mathbb{C}}(F, -)) \\
&\cong \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})}(-, \mathrm{Hom}_{\mathbb{C}}(F, -)^{\mathfrak{g}_{+1}}) \\
&\cong \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})}(-, \mathrm{Hom}_{\mathbb{C}}(F^{\mathfrak{g}_{+1}}, -)) \\
&\cong \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})}(- \otimes F^{\mathfrak{g}_{+1}}, -) \\
&\cong \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g})}(K(- \otimes F^{\mathfrak{g}_{+1}}), -),
\end{aligned}$$

where we use that $\mathfrak{g}_{\bar{0}} \subset \mathfrak{g}$ is a subalgebra, F is finite-dimensional, and the functor $F \mapsto F^{\mathfrak{g}_{+1}}$ is right adjoint to Kac induction. A direct calculation confirms that the induced isomorphism is given by the stated formula. □

It is well-known that every highest weight $\mathfrak{g}_{\bar{0}}$-module has a Jordan–Hölder filtration. Consequently, if V is a highest weight Harish-Chandra module and F a finite-dimensional $\mathfrak{g}$-supermodule, the lemma implies the existence of a Kac filtration for $K(V) \otimes F$.

Lemma 9.2.4. *Let V be a highest weight Harish-Chandra module with highest weight Λ, and let F be a finite-dimensional $\mathfrak{g}$-supermodule. Then $K(V) \otimes F$ has a filtration, natural in V,*

$$\{0\} = N_n \subset N_{n-1} \subset \ldots \subset N_1 = K(V) \otimes F$$

with subquotients N_i/N_{i+1}, which are either trivial or isomorphic to Kac supermodules induced from highest weight $\mathfrak{g}_{\bar{0}}$-modules with highest weight $\Lambda + \nu_i$ for some weight ν_i of F.

Proof. The tensor identity shows

$$K(V) \otimes F \cong K(V \otimes_{\mathbb{C}} F^{\mathfrak{g}_{+1}}).$$

Since V is an $\mathfrak{g}_{\bar{0}}$-module and a Harish-Chandra module, and $F^{\mathfrak{g}_{+1}}$ decomposes as a finite direct sum of finite-dimensional Harish-Chandra modules, in particular, $V \otimes_{\mathbb{C}} F^{\mathfrak{g}_{+1}}$ has a filtration

$$\{0\} = V_n \subset V_{n-1} \subset \ldots \subset V_1 = V \otimes_{\mathbb{C}} F^{\mathfrak{g}_{+1}},$$

with quotients V_i/V_{i+1}, which are either trivial or highest weight $\mathfrak{g}_{\bar{0}}$-modules with highest weight $\Lambda + \nu_i$ [74, Chapter 1]. Set $N_i := K(V_i)$. Then, combining the exactness of Kac induction with the tensor identity, we conclude that the N_i yield the desired filtration. The naturality follows from the explicit description of the V_i given in [74]. □

Corollary 9.2.5. *Let M be a highest weight $\mathfrak{g}$-supermodule with highest weight Λ, and let F be a finite-dimensional $\mathfrak{g}$-supermodule. Then $M \otimes F$ has a filtration with subquotients that are highest weight $\mathfrak{g}$-supermodules with highest weight $\Lambda + \nu$ for some $\nu \in P_F$.*

We now present the main result of this subsection in the following proposition.

Proposition 9.2.6. *Let M be a highest weight Harish-Chandra $\mathfrak{g}$-supermodule with highest weight Λ, and let F be a finite-dimensional $\mathfrak{g}$-supermodule.*

a) $M \otimes F$ is a direct sum of $(M \otimes F)_\mu$ with $\mu \in \Lambda + P_F$ dominant with respect to the set Δ_c^+.

b) Each $(M \otimes F)_\mu$ is generated by $\mathfrak{U}(\mathfrak{n}^-)$ under the projections from the decomposition in a) of $M^\Lambda \otimes F^\nu$, where $\chi_{\Lambda+\nu} = \chi_\mu$ and $\Lambda + \nu - \mu \in \Xi$.

c) If $M \otimes F$ is equipped with the tensor product of the contravariant forms of M and F, then the spaces $(M \otimes F)_\mu$ are orthogonal with respect to this form.

d) Each $(M \otimes F)_\mu$ has a finite Jordan–Hölder series with simple quotients isomorphic to $L(\nu)$ such that $\chi_\nu = \chi_\mu$ and $\nu - \mu \in \Xi$.

e) If $M = L(\Lambda)$ for some $\Lambda \in \mathfrak{h}^*$, then each $(M \otimes F)_\mu$ carries a non-degenerate contravariant form.

Proof. First, observe that all weights of $M \otimes F$ lie in $\Lambda + P_F$, and the tensor product of contravariant forms is contravariant. Additionally, $M \otimes F$ is a member of $\mathcal{M}(\mathfrak{g}, \mathfrak{k})$. This proves the first and third statements by Lemma 9.2.2. Furthermore, for simple highest weight $\mathfrak{g}$-supermodules, we obtain a non-degenerate contravariant form, and the tensor product remains non-degenerate. This concludes the proof of the fifth statement. The second and fourth statements follow from Lemma 9.2.4. □

Proposition 9.2.7. *Let $L(\Lambda)$ be a simple highest weight Harish-Chandra supermodule with highest weight Λ and let F be a finite-dimensional $\mathfrak{g}$-supermodule. Let ν be a weight of F such that $sdim(F^\nu) = (1|0)$ or $(0|1)$. Assume also that $\chi_{\Lambda+\nu'} \neq \chi_{\Lambda+\nu}$ for all weights $\nu' \neq \nu$ of F. Then the highest weight $\mathfrak{g}$-supermodule $(L(\Lambda) \otimes F)_{\Lambda+\nu}$ is isomorphic to $L(\Lambda + \nu)$ or trivial.*

Proof. The space $(L(\Lambda) \otimes F)_{\Lambda+\nu}$ is generated by $\mathfrak{U}(\mathfrak{n}^-)$ from the projections of $L(\Lambda)^\Lambda \otimes F^\nu$, as given in Proposition 9.2.6. Furthermore, this space has dimension $(1|0)$ or $(0|1)$ by Lemma 3.1.13, implying that $(L(\Lambda) \otimes F)_{\Lambda+\nu}$ is either a highest-weight $\mathfrak{g}$-supermodule with weight $\Lambda + \nu$ or trivial. If $(L(\Lambda) \otimes F)_{\Lambda+\nu}$ is non-trivial, it has a non-degenerate contravariant form, and since its radical is the maximal subsupermodule, the radical must be trivial. Consequently, $(L(\Lambda) \otimes F)_{\Lambda+\nu}$ must be simple. □

10. A formal superdimension for unitarizable supermodules

In Section 10.1, we introduce a notion of formal superdimension for a specific class of $\mathfrak{g}$-supermodules, referred to as relative holomorphic discrete series $\mathfrak{g}$-supermodules. Furthermore, we establish key properties of this superdimension and, in Section 10.2, we present a generalization of the Kac–Wakimoto conjecture in this context.

10.1. A formal superdimension

The superdimension of a finite-dimensional supermodule typically compares the sizes of its even and odd components, providing an invariant. For infinite-dimensional $\mathfrak{g}$-supermodules, however, such notions of size are ill-defined, and at first glance, the very idea of a superdimension seems inapplicable. Nevertheless, if the supermodule is unitarizable, the formal dimension associated with certain unitarizable $\mathfrak{g}_{\bar{0}}$-constituents can serve as a suitable substitute.

The goal of this section is to introduce the notion of the *formal superdimension*. This invariant is designed to generalize two key concepts: on the one hand, the classical superdimension of finite-dimensional $\mathfrak{g}$-supermodules; and on the other, the *formal dimension* (also known as the formal degree or *Harish-Chandra degree*) of unitarizable modules over the even subalgebra $\mathfrak{g}_{\bar{0}}$ (see Section 5.4.2). The formal superdimension of a $\mathfrak{g}$-supermodule is defined as the alternating sum of the formal dimensions of its $\mathfrak{g}_{\bar{0}}$-constituents, where we recall that the formal dimension is defined only for those unitarizable $\mathfrak{g}_{\bar{0}}$-modules which integrate to *relative discrete series representations* of a real Lie group G with Lie algebra $\mathfrak{g}_{\bar{0}}^{\mathbb{R}}$. Here, $\mathfrak{g}_{\bar{0}}^{\mathbb{R}}$ denotes the reductive real form $\mathfrak{su}(p,q) \oplus \mathfrak{su}(n) \oplus \mathfrak{u}(1)$ of $\mathfrak{g}_{\bar{0}}$ (see Section 2.1.4), and G is either the matrix Lie group $G_{\bar{0}}^{\mathbb{R}}$, or its simply connected covering group $\widetilde{G}_{\bar{0}}^{\mathbb{R}}$.

10.1.1. A superdimension formula

The proposal for the formal superdimension arises from the natural combination of the Harish-Chandra degree for $\mathfrak{g}_{\bar{0}}$-modules from Section 5.4.2 with the insights from Sections 7 and 6.1.4. To fix notation, we recall that if $\mathcal{H}$ is a non-trivial unitarizable simple $\mathfrak{g}$-supermodule, then

a) $\mathcal{H}$ is a simple highest weight supermodule with some highest weight $\Lambda \in \mathfrak{h}^*$;

b) $\mathcal{H}$ is $\mathfrak{g}_{\bar{0}}$-semisimple, with each $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\Lambda_j)$ being a unitarizable highest weight $\mathfrak{g}_{\bar{0}}$-supermodule concentrated in degree $p(\Lambda_j) = \sum_{k=1}^{n} (\Lambda_j - \Lambda, \delta_k) \mod 2$ relative to the highest weight vector. Moreover, each $\mathfrak{g}_{\bar{0}}$-constituent is a tensor product of a unitarizable simple $\mathfrak{su}(p,q)$-module (which is finite-dimensional if and only if $p = 0$ or $q = 0$), a (finite-dimensional) simple $\mathfrak{su}(n)$-module and a (one-dimensional) simple $\mathfrak{u}(1)$-module.

c) If $p, q \neq 0$, the $\mathfrak{g}_{\bar{0}}$-constituents $L_0(\Lambda_j)$ belong to the relative holomorphic discrete series precisely is they satisfy the Harish-Chandra condition of Theorem 5.4.3,

$$(\Lambda_j + \rho_{\bar{0}}, \beta) < 0, \qquad \text{for all } \beta \in \Delta_n^+.$$

In this case they possess a formal dimension $d(\Lambda_j) \in \mathbb{R}_+$ given by Theorem 5.4.5,

$$d(\Lambda_j) = \prod_{\alpha \in \Delta_c^+} \frac{(\Lambda_j + \rho_c, \alpha)}{(\rho_c, \alpha)} \prod_{\beta \in \Delta_n^+} \frac{|(\Lambda_j + \rho_{\bar{0}}, \beta)|}{(\rho_{\bar{0}}, \beta)}.$$

Recall that this formal dimension serves as a substitute for the dimension of infinite-dimensional unitarizable $\mathfrak{g}_{\bar{0}}$-modules and coincides with the dimension or degree of the representation for finite-dimensional unitarizable $\mathfrak{g}_{\bar{0}}$-modules.

If a unitarizable simple highest weight $\mathfrak{g}_{\bar{0}}$-module does not belong to the relative holomorphic discrete series, there is no concept of a formal dimension. This naturally leads to the definition of *(relative) holomorphic discrete series supermodule*.

Definition 10.1.1. Let $\mathcal{H}$ be a unitarizable highest weight $\mathfrak{g}$-supermodule. Then, $\mathcal{H}$ is called:

a) a *relative holomorphic discrete series supermodule* if each $\mathfrak{g}_{\bar{0}}$-constituent is a Harish-Chandra supermodule corresponding to a relative holomorphic discrete series representation of $\widetilde{G}^{\mathbb{R}}_{\bar{0}}$.

b) a *holomorphic discrete series supermodule* if each $\mathfrak{g}_{\bar{0}}$-constituent is a Harish-Chandra supermodule corresponding to a holomorphic discrete series representation of $G^{\mathbb{R}}_{\bar{0}}$.

(Relative) holomorphic discrete series supermodules carry a natural formal superdimension. All we have to do is weight by the parity $(-1)^{p(\Lambda_j)}$.

Definition 10.1.2. Let $\mathcal{H}$ be a relative holomorphic $\mathfrak{g}$-supermodule. Then the real value

$$\operatorname{sdim}(\mathcal{H}) := \sum_j \operatorname{sdim}(L_0(\Lambda_j)), \qquad \operatorname{sdim}(L_0(\Lambda_j)) := (-1)^{p(\mathcal{H})+p(\Lambda_j)} d(\Lambda_j),$$

where the sum runs over the highest weights of all $\mathfrak{g}_{\bar{0}}$-constituents $L_0(\Lambda_j)$ of $\mathcal{H}$, is called the *formal superdimension* of $\mathcal{H}$.

For simplicity, we assume for the rest of this chapter that $L_0(\Lambda)$ has even parity, such that

$$\operatorname{sdim}(L_0(\Lambda_j)) = (-1)^{p(\Lambda_j)} d(\Lambda_j)$$

for any $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\Lambda_j)$.

Furthermore, an explicit formula for the superdimension of relative holomorphic discrete series $\mathfrak{g}$-supermodules is provided by Theorem 7.3.2 together with Theorem 5.4.5.

Remark 10.1.3. Any unitarizable simple highest weight $\mathfrak{g}_{\bar{0}}$-module is the outer tensor product of simple unitary highest weight representations of $\mathfrak{su}(p,q)^{\mathbb{C}}$, $\mathfrak{su}(n)^{\mathbb{C}}$, and $\mathfrak{u}(1)^{\mathbb{C}}$ (if $m \neq n$; see Section 6.1.4). Since $\mathfrak{u}(1)$ is abelian, any simple $\mathfrak{u}(1)$-module is one-dimensional and does not contribute to the dimension.

Let us denote $\mathfrak{L} := \mathfrak{su}(p,q)^{\mathbb{C}}$ and $\mathfrak{R} := \mathfrak{su}(n)^{\mathbb{C}}$, and write the highest weight as $\mu = (\mu^{\mathfrak{L}} \mid \mu^{\mathfrak{R}})$, with $\mu^{\mathfrak{L}} = (\mu_1, \ldots, \mu_m)$ and $\mu^{\mathfrak{R}} = (\mu'_1, \ldots, \mu'_n)$, as in Section 6.1.4. Then $\mu^{\mathfrak{L}}$ is the highest weight of a Harish-Chandra module over $\mathfrak{L}$, while $\mu^{\mathfrak{R}}$ is the highest weight of a finite-dimensional simple $\mathfrak{R}$-module. With this notation, the simple $\mathfrak{g}_{\bar{0}}$-module decomposes as

$$L_0(\mu) \cong L_0(\mu^{\mathfrak{L}}; \mathfrak{L}) \boxtimes L_0(\mu^{\mathfrak{R}}; \mathfrak{R}) \boxtimes \mathbb{C}_{\mu},$$

and the associated formal dimension factorizes as

$$d(\pi_{\mu}) = d(\pi_{\mu^{\mathfrak{L}}}) \cdot d(\pi_{\mu^{\mathfrak{R}}}),$$

where $d(\pi_{\mu^{\mathfrak{R}}})$ is given by Weyl's dimension formula, *i.e.*,

$$d(\pi_{\mu^{\mathfrak{R}}}) = \dim(L_0(\mu^{\mathfrak{R}})),$$

and $d(\pi_{\mu^{\mathfrak{L}}})$ is the formal dimension associated with the Harish-Chandra module $L_0(\mu^{\mathfrak{L}})$ in Theorem 5.4.5.

In particular, to compute the formal superdimension $\mathrm{sdim}(M)$ of a unitarizable $\mathfrak{g}$-supermodule M, it suffices to decompose M under $\mathfrak{L}$ as

$$M|_{\mathfrak{L}} = \bigoplus_{\mu} L_0(\mu^{\mathfrak{L}}; \mathfrak{L}) \boxtimes X_{\mu},$$

where the X_{μ} are finite-dimensional as super vector subspaces of the space

$$\bigoplus_{\nu} L_0(\nu^{\mathfrak{R}}; \mathfrak{R}) \boxtimes \mathbb{C}_{\nu}.$$

Each space X_{μ} carries a well-defined superdimension, and to each $L_0(\mu^{\mathfrak{L}}; \mathfrak{L})$ we assign its formal dimension. The formal superdimension $\mathrm{sdim}(M)$ is then given by the alternating sum over these components.

Next, we consider examples. The following example is immediate.

Example 10.1.4. We consider the case of compact real forms $\mathfrak{su}(m|n)$, which occurs when either $p = 0$ or $q = 0$. In this case, the formal superdimension of any unitarizable simple supermodule coincides with its superdimension by construction.

Now, we consider the cases $p, q \neq 0$, *i.e.*, infinite-dimensional supermodules. Section 6.1.4 allows for a straightforward computation of the superdimension of relative holomorphic discrete $\mathfrak{g}$-supermodules with respect to the fixed positive system $\mathfrak{n}^+_{\bar{1},\mathrm{nst}}$.

Example 10.1.5. Consider $\mathfrak{su}(1,1|2)$, *i.e.*, $m = n = 2$ and $p = q = 1$. The Weyl vectors are given by

$$\rho_{\bar{0}} = \frac{1}{2}(1,-1|1,-1), \qquad \rho_c = \frac{1}{2}(0,0|1,-1), \qquad \rho = \frac{1}{2}(-1,1|1,-1).$$

We are interested in cases where the highest weight is typical, 1-atypical, or maximally atypical.

a) For a typical highest weight $\Lambda = (-4, 5|3, 2)$, the unitarizable simple $\mathfrak{g}$-supermodule $L(\Lambda)$ decomposes as

$$\begin{aligned} L(\Lambda)_{\text{ev}} =& L_0(-4,5|3,2) \oplus L_0(-4,6|3,1) \oplus L_0(-5,5|3,3) \\ & \oplus L_0(-5,5|4,2) \oplus L_0(-4,6|2,2) \oplus L_0(-5,6|3,2) \\ & \oplus L_0(-5,6|4,1) \oplus L_0(-6,5|4,3) \oplus L_0(-4,7|2,1) \\ & \oplus L_0(-5,6|3,2) \oplus L_0(-6,6|4,2) \oplus L_0(-5,7|2,2) \\ & \oplus L_0(-5,7|3,1) \oplus L_0(-6,6|3,3) \oplus L_0(-6,7|3,2). \end{aligned}$$

The superdimension is computed as

$$\begin{aligned} \operatorname{sdim} L(\Lambda) &= 16 - 27 - 9 - 27 - 9 + 20 + 40 + 20 + 20 + 20 - 33 - 11 \\ &\quad - 33 - 11 + 24 \\ &= 0, \end{aligned}$$

where the formal dimensions of the $\mathfrak{g}_{\bar{0}}$-constituents are listed as above.

b) For a 1-atypical highest weight $\Lambda = (-4, 5|4, 2)$, the unitarizable simple $\mathfrak{g}$-supermodule $L(\Lambda)$ decomposes as

$$\begin{aligned} L(\Lambda)_{\text{ev}} =& L_0(-4,5|4,2) \oplus L_0(-4,6|4,1) \oplus L_0(-5,5|4,3) \\ & \oplus L_0(-4,6|3,2) \oplus L_0(-5,6|4,2) \oplus L_0(-4,7|3,1) \\ & \oplus L_0(-5,6|3,3) \oplus L_0(-5,7|3,2). \end{aligned}$$

The superdimension is

$$\operatorname{sdim}(L(\Lambda)) = 24 - 36 - 18 - 18 + 30 + 30 + 10 - 22 = 0,$$

where the formal dimensions of the $\mathfrak{g}_{\bar{0}}$-constituents are listed as above.

c) For a maximally atypical highest weight $\Lambda = (-4, 5|4, -5)$, the unitarizable simple $\mathfrak{g}$-supermodule $L(\Lambda)$ decomposes as

$$L(\Lambda)_{\text{ev}} = L_0(\Lambda) \oplus L_0(-5,5|4,-4) \oplus L_0(-4,6|3,-5) \oplus L_0(-5,6|3,-4).$$

The superdimension is

$$\operatorname{sdim} L(\Lambda) = 80 - 81 - 81 + 80 = -2,$$

where the formal dimensions of the $\mathfrak{g}_{\bar{0}}$-constituents are listed as above.

Example 10.1.6. Now, consider $\mathfrak{su}(1,1|3)$ with complexification $\mathfrak{sl}(2|3)$. The Weyl vectors are given by

$$\rho_{\bar{0}} = \frac{1}{2}(1,-1|1,0,-1), \qquad \rho_c = (0,0,|1,0,-1), \qquad \rho = (-1,1|1,0,-1).$$

For the maximally atypical highest weight $\Lambda = (-1, 1|1, 0, -1)$, the unitarizable highest weight $\mathfrak{g}$-supermodule $L(\Lambda)$ decomposes as

$$\begin{aligned} L(\Lambda)_{\text{ev}} =& L_0(-1,1|1,0,-1) \oplus L_0(-2,1|1,1,-1) \oplus L_0(-2,1|1,0,0) \\ & \oplus L_0(-1,2|0,0,-1) \oplus L_0(-1,2|1,-1,-1) \oplus L_0(-3,1|1,1,0) \\ & \oplus L_0(-2,2|0,0,0) \oplus L_0(-2,2|1,0,-1) \oplus L_0(-1,3|0,-1,-1) \\ & \oplus L_0(-3,2|1,0,0) \oplus L_0(-2,3|0,0,-1) \oplus L_0(-3,3|0,0,0). \end{aligned}$$

The superdimension is

$$\operatorname{sdim} L(\Lambda) = 8 - 12 - 6 - 6 - 12 + 9 + 3 + 24 + 9 - 12 - 12 + 5 = -2,$$

where the formal dimensions of the $\mathfrak{g}_{\bar{0}}$-constituents are listed as above.

In the examples discussed above, we observed that the formal superdimension is trivial unless the highest weight is maximally atypical. This naturally leads to the conjecture that the superdimension of any (relative) holomorphic discrete series $\mathfrak{g}$-supermodule is trivial unless the highest weight is maximally atypical.

We examine this phenomenon — an analog of the Kac–Wakimoto conjecture — in the following sections.

Formal Q-superdimension

In Section 9.1, we introduced the Duflo–Serganova functor, which assigns to each $\mathfrak{g}$-supermodule M a corresponding $\mathfrak{g}_x$-supermodule $DS_x(M)$. If M is finite-dimensional, the DS functor preserves the superdimension, *i.e.*, $\operatorname{sdim} M = \operatorname{sdim} DS_x(M)$ for any $x \in \mathcal{Y}$. However, the DS functor preserves unitarity if and only if it is defined with respect to $\mathcal{Y}^{\text{hom}}$.

Fix an element $Q \in \mathcal{Y}$ such that $x := Q - \omega(Q) \in \mathcal{Y}^{\text{hom}}$ and $\omega(x) = -x$. Assume that Q has rank k. Define

$$H := \frac{1}{2}[x, x] = [Q, \omega(Q)],$$

which is a semisimple element in $\mathfrak{g}_{\bar{0}}$. The image of the DS functor on $\mathfrak{g}$ is given by

$$\mathfrak{g}_x := DS_x(\mathfrak{g}) = \begin{cases} \mathfrak{sl}(m-k \mid n-k), & \text{if } m \neq n, \\ \mathfrak{psl}(n-k \mid n-k), & \text{if } m = n. \end{cases}$$

The corresponding real form is

$$\mathfrak{g}_x^{\omega_x} = \mathfrak{su}(p-r, q-s \mid n-k),$$

for some $r \leq p$, $s \leq q$ such that $r + s = k$. Using Lemma 9.1.17 and Proposition 9.1.10, we may define a *Q-superdimension* by assigning to the $\mathfrak{g}_x$-supermodule $DS_x(\mathcal{H})$ a well-defined superdimension, as described earlier.

Definition 10.1.7. Let $\mathcal{H}$ be a simple unitarizable $\mathfrak{g}$-supermodule. The superdimension of the $\mathfrak{g}_x$-supermodule $DS_x(\mathcal{H})$ is called the *formal Q-superdimension* of $\mathcal{H}$.

We will study the formal Q-superdimension and its close relationship with the Q-Witten index in Part IV.

10.1.2. Characteristics of the superdimension

We collect here several important properties of the formal superdimension. First, since unitarizable $\mathfrak{g}$-supermodules are completely reducible, the superdimension is additive.

Lemma 10.1.8. *Let M, M', and M'' be unitarizable $\mathfrak{g}$-supermodules fitting into a short exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

Then

$$\mathrm{sdim}(M) = \mathrm{sdim}(M') + \mathrm{sdim}(M'').$$

Next, the superdimension behaves well under tensoring with suitable finite-dimensional supermodules. In particular, if $\mathcal{H} \otimes F$ admits a filtration by holomorphic discrete series supermodules, then $\mathrm{sdim}(\mathcal{H} \otimes F)$ is well-defined, and we can compare it to the product $\mathrm{sdim}(\mathcal{H}) \cdot \mathrm{sdim}(F)$.

Lemma 10.1.9. *Let $\mathcal{H}$ be a holomorphic discrete series $\mathfrak{g}$-supermodule with highest weight Λ, and let F be a finite-dimensional simple $\mathfrak{g}$-supermodule with weight set P_F. Assume that $\mathcal{H} \otimes F$ admits a filtration whose subquotients are holomorphic discrete series supermodules of the form $L(\Lambda + \nu)$ for $\nu \in P_F$. Then*

$$\mathrm{sdim}(\mathcal{H} \otimes F) = \mathrm{sdim}(\mathcal{H}) \cdot \mathrm{sdim}(F).$$

Proof. First, since $\mathcal{H}$ is a unitarizable simple $\mathfrak{g}$-supermodule and F is simple and finite-dimensional, both are $\mathfrak{g}_{\bar{0}}$-semisimple. Hence, we may write their $\mathfrak{g}_{\bar{0}}$-decompositions as:

$$\mathcal{H} = \bigoplus_{\mu} (-1)^{p(\mu)} m(\mu; \mathcal{H})\, L_0(\mu), \qquad F = \bigoplus_{\nu} (-1)^{p(\nu)} m(\nu; F)\, F_0(\nu),$$

where $m(\mu; \mathcal{H})$ and $m(\nu; F)$ denote multiplicities. Tensoring over $\mathfrak{g}_{\bar{0}}$, we obtain:

$$\mathcal{H} \otimes F = \bigoplus_{\mu,\nu} (-1)^{p(\mu)+p(\nu)} m(\mu; \mathcal{H}) m(\nu; F)\, L_0(\mu) \otimes F_0(\nu).$$

In particular, the superdimensions are given by:

$$\mathrm{sdim}(\mathcal{H}) = \sum_{\mu} m(\mu; \mathcal{H}) \cdot \mathrm{sdim}(L_0(\mu)), \qquad \mathrm{sdim}(F) = \sum_{\nu} m(\nu; F) \cdot \mathrm{sdim}(F_0(\nu)).$$

To proceed, we reformulate the problem in terms of characters. For this, we adopt the notation of Section 5.4, and consider, without loss of generality, the group $G'_{\bar{0}} = \mathrm{SU}(p,q) \times \mathrm{SU}(n)$ and its universal covering group $\tilde{G}'_{\bar{0}}$. Let π_μ denote the holomorphic discrete series representation associated with $L_0(\mu)$—equivalently, a discrete series representation of $\tilde{G}'_{\bar{0}}$ by Remark 5.4.8. Let $\Theta_{L_0(\mu)}$ (or Θ_μ) denote the associated Harish-Chandra character, and let $\tilde{\Theta}_{L_0(\mu)}$ (or $\tilde{\Theta}_\mu$) denote its associated L-packet. Then, by Proposition 5.4.9, the formal dimension of π_μ is given by

$$d(\mu) = \lim_{g \to e_{G'_{\bar{0}}},\, g \in T' \cap G'_{\bar{0},reg}} \tilde{\Theta}_\mu(g).$$

Likewise, we view $F_0(\nu)$ as a finite-dimensional irreducible representation of $\tilde{G}'_{\bar{0}}$, with character $\chi_{F_0(\nu)}$. Then,

$$\dim(F_0(\nu)) = \lim_{g \to e_{G'_{\bar{0}}}} \chi_{F_0(\nu)}(g).$$

This allows us to recast the problem in terms of limits of character values near the identity.

By assumption, together with the Jantzen–Zuckerman translation principle [141, Lemma 3.4], the additivity of characters, and the Weyl group invariance of the weight set $P_{F_0(\nu)}$, we obtain the following:

$$\begin{aligned}
\operatorname{sdim}\mathcal{H}\cdot\operatorname{sdim}F &= \sum_{\mu,\nu} m(\mu;\mathcal{H})m(\nu;F)(-1)^{p(\mu)+p(\nu)}d(\mu)\cdot d(\nu)\\
&= \sum_{\mu,\nu} m(\mu;\mathcal{H})m(\nu;F)(-1)^{p(\mu)+p(\nu)} \lim_{g\to e_{\tilde{G}'_{\bar 0}},\ g\in\tilde{T}'\cap\tilde{G}'_{\bar 0,reg}} \tilde{\Theta}_{L_0(\mu)}(g)d(\nu)\\
&= \lim_{g\to e_{\tilde{G}'_{\bar 0}},\ g\in\tilde{T}'\cap\tilde{G}'_{\bar 0,reg}} \sum_{\mu,\nu} m(\mu;\mathcal{H})m(\nu;F)(-1)^{p(\mu)+p(\nu)} \sum_{w\in W/W_c} \Theta_{w.\mu}(g)\chi_{F_0(\nu)}(g)\\
&= \lim_{g\to e_{\tilde{G}'_{\bar 0}},\ g\in\tilde{T}'\cap\tilde{G}'_{\bar 0,reg}} \sum_{\mu,\nu} m(\mu;\mathcal{H})m(\nu;F)(-1)^{p(\mu)+p(\nu)} \sum_{w\in W/W_c} \Theta_{L_{w.\mu}\otimes F_0(\nu)}(g)\\
&= \lim_{g\to e_{\tilde{G}'_{\bar 0}},\ g\in\tilde{T}'\cap\tilde{G}'_{\bar 0,reg}} \sum_{\mu,\nu} m(\mu;\mathcal{H})m(\nu;F)(-1)^{p(\mu)+p(\nu)} \sum_{w\in W/W_c}\ \sum_{\xi(\nu)\in P_{F_0(\nu)}} \Theta_{L_{w.\mu+\xi(\nu)}}(g)\\
&= \lim_{g\to e_{\tilde{G}'_{\bar 0}},\ g\in\tilde{T}'\cap\tilde{G}'_{\bar 0,reg}} \sum_{\mu,\nu} (-1)^{p(\mu)+p(\nu)} m(\mu;\mathcal{H})m(\nu;F) \sum_{\xi(\nu)\in P_{F_0(\nu)}} \tilde{\Theta}_{\mu+\xi(\nu)}\\
&= \sum_{\mu,\nu} (-1)^{p(\mu)+p(\nu)} m(\mu;\mathcal{H})m(\nu;F) \sum_{\xi(\nu)\in P_{F_0(\nu)}} d(\mu+\xi(\nu))\\
&= \sum_{\mu,\nu} (-1)^{p(\mu)+p(\nu)} m(\mu;\mathcal{H})m(\nu;F) \dim(L_0(\mu)\otimes F_0(\nu))\\
&= \operatorname{sdim}\mathcal{H}\otimes F.
\end{aligned}$$

□

Next, we use the structure provided by the DS functor to investigate the formal superdimension. While the DS functor does not preserve superdimension in general, it serves as a valuable tool for reducing considerations to unitarizable simple supermodules with maximally atypical highest weight.

As discussed in Section 5.4.3, any relative holomorphic discrete series $\mathfrak{g}$-supermodule $\mathcal{H}$ decomposes under the action of $\mathfrak{K}$ into finitely many finite-dimensional simple $\mathfrak{K}$-modules. These modules carry a natural (ordinary) dimension, which makes it necessary to examine the role of the semisimple Lie algebra $\mathfrak{L}$ more closely.

Fix an element $x\in\mathcal{Y}$ such that x commutes with $\mathfrak{n}^+_{\mathfrak{L}} := \mathfrak{n}^+_{\bar 0}\cap\mathfrak{L}$. Then x induces a short exact sequence of $\mathfrak{g}$-supermodules:

$$0 \longrightarrow \ker x_{\mathcal{H}} \longrightarrow \mathcal{H} \longrightarrow \Pi\ \operatorname{im} x_{\mathcal{H}} \longrightarrow 0,$$

which splits as a short exact sequence of super vector spaces:

$$\mathcal{H} \cong \ker x_{\mathcal{H}} \oplus \Pi\ \operatorname{im} x_{\mathcal{H}}.$$

Moreover, since x commutes with $\mathfrak{n}^+_{\mathfrak{L}}$, we obtain a direct sum decomposition of the $\mathfrak{n}^+_{\mathfrak{L}}$-invariants:

$$\mathcal{H}^{\mathfrak{n}^+_{\mathfrak{L}}} = (\ker x_{\mathcal{H}})^{\mathfrak{n}^+_{\mathfrak{L}}} \oplus \Pi\,(\operatorname{im} x_{\mathcal{H}})^{\mathfrak{n}^+_{\mathfrak{L}}}.$$

Lemma 10.1.10. *Let $\mathcal{H}$ be a relative holomorphic discrete series $\mathfrak{g}$-supermodule. Then there is an isomorphism of $\mathfrak{U}(\mathfrak{L})$-modules*

$$\mathcal{H} \cong \langle \ker(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}} \rangle_{\mathfrak{L}} \oplus \langle \Pi \ \mathrm{im}(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}} \rangle_{\mathfrak{L}},$$

where $\langle V \rangle_{\mathfrak{L}} := \mathfrak{U}(\mathfrak{L}) \cdot V$ denotes the $\mathfrak{U}(\mathfrak{L})$-submodule of $\mathcal{H}$ generated by the subspace $V \subset \mathcal{H}$.

Proof. The $\mathfrak{n}_{\mathfrak{L}}^{+}$-invariants of $\mathcal{H}$ are precisely the highest weight vectors of its $\mathfrak{L}$-constituents. As explained in Section 6.1.4, there are only finitely many such highest weight vectors (up to scalar), and each generates an irreducible $\mathfrak{L}$-constituent under the action of $\mathfrak{U}(\mathfrak{L})$. Moreover, the $\mathfrak{L}$-constituents are pairwise orthogonal with respect to the Hermitian form on $\mathcal{H}$.

To prove the lemma, it suffices to show that the modules $\langle \ker(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}} \rangle_{\mathfrak{L}}$ and $\langle \Pi \ \mathrm{im}(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}} \rangle_{\mathfrak{L}}$ have no $\mathfrak{L}$-constituents in common. Suppose, for contradiction, that there exists a constituent $L_0(\mu; \mathfrak{L})$ appearing in both submodules. Then its highest weight vector v_μ would lie in both $\ker(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}}$ and $\Pi \ \mathrm{im}(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}}$, and hence in their intersection. However,

$$v_\mu \in \ker(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}} \cap \Pi \ \mathrm{im}(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}} = \{0\},$$

which is a contradiction. □

Each of the submodules $\langle \ker(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}} \rangle_{\mathfrak{L}}$ and $\langle \mathrm{im}(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}} \rangle_{\mathfrak{L}}$ carries a natural superdimension (see Remark 10.1.3). Indeed, we have the decompositions:

$$\langle \ker(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}} \rangle_{\mathfrak{L}} = \bigoplus_{\mu} L_0(\mu^{\mathfrak{L}}; \mathfrak{L}) \boxtimes X_\mu, \qquad \langle \mathrm{im}(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}} \rangle_{\mathfrak{L}} = \bigoplus_{\nu} L_0(\nu^{\mathfrak{L}}; \mathfrak{L}) \boxtimes X_\nu,$$

where each X_μ, X_ν is a super vector subspace of $\bigoplus_\xi L_0(\xi^{\mathfrak{R}}; \mathfrak{R}) \boxtimes \mathbb{C}_\xi$. In particular, they are finite-dimensional. The associated superdimensions are given by:

$$\mathrm{sdim}\langle \ker(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}} \rangle_{\mathfrak{L}} = \sum_{\mu} (-1)^{p(\mu)} \, d(\mu^{\mathfrak{L}}) \cdot \dim(X_\mu),$$
$$\mathrm{sdim}\langle \mathrm{im}(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}} \rangle_{\mathfrak{L}} = \sum_{\nu} (-1)^{p(\nu)} \, d(\nu^{\mathfrak{L}}) \cdot \dim(X_\nu).$$

Note that we do not need to explicitly consider the $\mathfrak{R}$-modules, since $\mathcal{H}$ decomposes under $\mathfrak{R}$ into finitely many finite-dimensional simple modules, each of which carries an ordinary (well-defined) dimension. We conclude the following lemma.

Lemma 10.1.11. *Let $\mathcal{H}$ be a relative holomorphic discrete series $\mathfrak{g}$-supermodule. Then*

$$\mathrm{sdim}(\mathcal{H}) = \mathrm{sdim}\langle \ker(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}} \rangle_{\mathfrak{L}} - \mathrm{sdim}\langle \mathrm{im}(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}} \rangle_{\mathfrak{L}}.$$

We now obtain the following vanishing result for the superdimension.

Theorem 10.1.12. *Let $\mathcal{H}$ be a relative holomorphic discrete series $\mathfrak{g}$-supermodule with highest weight Λ. If Λ is not maximally atypical, then*

$$\mathrm{sdim}(\mathcal{H}) = 0.$$

Proof. Fix an element $x \in \mathcal{Y}$ of maximal rank such that x commutes with $\mathfrak{n}_{\mathfrak{L}}^{+}$. A direct calculation yields that such an element exists. By assumption and Corollary 9.1.7, we have $DS_x(\mathcal{H}) = \{0\}$. It follows that $\ker x_{\mathcal{H}} = \operatorname{im} x_{\mathcal{H}}$, and therefore also

$$\ker(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}} = \operatorname{im}(x_{\mathcal{H}})^{\mathfrak{n}_{\mathfrak{L}}^{+}}.$$

The claim then follows immediately from Lemma 10.1.11. □

Corollary 10.1.13. *Let $\mathcal{H}$ be a relative holomorphic discrete series $\mathfrak{g}$-supermodule with typical highest weight Λ, and let F be a finite-dimensional $\mathfrak{g}$-supermodule. Then for all $\mu \in \mathfrak{h}^*$, we have*

$$\operatorname{sdim}\left({}^{F}T_{\Lambda}^{\mu}(\mathcal{H})\right) = 0.$$

As a consequence, we conclude that the formal superdimension of a relative holomorphic discrete series $\mathfrak{g}$-supermodule with highest weight Λ is trivial whenever $\operatorname{at}(\Lambda) < \operatorname{def}(\mathfrak{g}) = \min(m, n)$.

For the remainder of this article, and without loss of generality, we restrict our attention to relative holomorphic discrete series $\mathfrak{g}$-supermodules with maximally atypical highest weight. In the next subsection, we describe and classify these supermodules.

10.1.3. Intermezzo: Maximal atypical relative holomorphic discrete series $\mathfrak{g}$-supermodules

The notions of unitarity and atypicality for a highest weight $\mathfrak{g}$-supermodule are entirely independent of each other. However, atypicality imposes an integral structure on the highest weight, meaning that $(\lambda, \alpha) \in \mathbb{Z}$ for all $\alpha \in \Delta_{\bar{0}}$, for some highest weight $\lambda \in \mathfrak{h}^*$.

Lemma 10.1.14. *Let $\mathcal{H}$ be a unitarizable simple $\mathfrak{g}$-supermodule with highest weight Λ. Assume $\mathfrak{g} \neq \mathfrak{sl}(m|1), \mathfrak{sl}(1|n)$. If Λ is maximally atypical, then Λ is integral.*

Proof. Unitarity is defined with respect to a real form $\mathfrak{su}(p,q|n)$ for $p+q=m$, with Lie subalgebra $\mathfrak{su}(p,q|n)_{\bar{0}} = \mathfrak{su}(p,q) \oplus \mathfrak{su}(n) \oplus \mathfrak{u}(1)$, where the $\mathfrak{u}(1)$ is only present if $p+q \neq n$. By the parameterization of unitarizable simple $\mathfrak{g}$-supermodules in Section 6.1.4, the highest weight is of the form

$$\Lambda = \left(\frac{\lambda}{2} - a_1, \frac{\lambda}{2} - a_2, \ldots, \frac{\lambda}{2} - a_p, -\frac{\lambda}{2} + a_{p+1}, \ldots, -\frac{\lambda}{2} + a_{p+q}|0, \ldots, 0\right),$$

where $a_1, \ldots, a_p$ and $a_{p+1}, \ldots, a_{p+q}$ are positive integers such that $0 \leq a_2 \leq \ldots \leq a_p$ and $a_{p+1} \geq a_{p+2} \geq \ldots \geq a_{p+q-1} \geq a_{p+q} = 0$, and λ belongs to the set

$$(-\infty, -m + x - (r-1)) \cup \{-m + x - (r-1), -m + x - (r-1) + 1, \ldots, -m + x\}.$$

Consequently, it suffices to show that λ is integral, *i.e.*, $\lambda = (\Lambda, \epsilon_1 - \epsilon_m) \in \mathbb{Z}$. To simplify the notation, we also write $\Lambda = (\lambda_1, \ldots, \lambda_m|\lambda'_1, \ldots, \lambda'_n)$ such that $(\Lambda + \rho, \epsilon_1 - \epsilon_m) = \lambda_1 - \lambda_m$ (see Section 6.1.4).

If Λ is maximally atypical, there exist some $1 \leq k, l \leq n$, such that $(\Lambda + \rho, \epsilon_1 - \delta_k) = 0$ and $(\Lambda + \rho, \epsilon_m - \delta_l) = 0$, *i.e.*,

$$\lambda'_k = -\lambda_1 - p + k, \qquad \lambda'_l = -\lambda_m - p - n - m - l.$$

As $(0,\dots,0|\lambda'_1,\dots,\lambda'_n)$ is the highest weight of a simple $\mathfrak{su}(n)^{\mathbb{C}}$-module, the weight is integral. We conclude

$$\lambda'_l - \lambda'_k = -\lambda_m + \lambda_1 - n - m - l + k \in \mathbb{Z},$$

and $\lambda = (\Lambda, \epsilon_1 - \epsilon_m) = \lambda_1 - \lambda_m \in \mathbb{Z}$, as $m, n, l, k \in \mathbb{Z}_+$. □

For integral weights $\Lambda \in \mathfrak{h}^*$, we obtain a further refinement. Define

$$\mathbf{1}_{m|n} := \sum_{i=1}^{m} \epsilon_i - \sum_{j=1}^{n} \delta_j = (1,\dots,1 \mid -1,\dots,-1).$$

Then, there exists a scalar $a(\Lambda) \in \mathbb{C}$ such that

$$\Lambda = \tilde{\Lambda} + a(\Lambda)\mathbf{1}_{m|n},$$

where $\tilde{\Lambda} = (\tilde{\lambda}_1,\dots,\tilde{\lambda}_m \mid \tilde{\lambda}'_1,\dots,\tilde{\lambda}'_n) \in \mathfrak{h}^*$ satisfies $\tilde{\lambda}_i, \tilde{\lambda}'_j \in \mathbb{Z}$ for all i, j. If Λ is the highest weight of a unitarizable simple $\mathfrak{g}$-supermodule $\mathcal{H}$, then up to tensoring with the one-dimensional $\mathfrak{g}$-supermodule of weight $a(\Lambda)\mathbf{1}_{m|n}$, we may assume that $\mathcal{H}$ has highest weight $\tilde{\Lambda}$ with integral entries.

In the following, we assume for convenience that the highest weight Λ of a unitarizable $\mathfrak{g}$-supermodule satisfies $\lambda_i, \lambda'_j \in \mathbb{Z}$. This assumption is not essential and may be omitted without affecting the results; it is introduced purely to simplify notation.

With this setup, Theorem 8.2.2 provides a complete parameterization of all relative holomorphic discrete series $\mathfrak{g}$-supermodules, including those with maximally atypical highest weight.

Finally, the unitarity relations in Lemma 6.1.13 impose a specific structure on any maximal atypical highest weight Λ.

Lemma 10.1.15. *Let $\mathcal{H}$ be a unitarizable simple $\mathfrak{g}$-supermodule with highest weight Λ. Then the following assertions hold:*

a) If $(\Lambda + \rho, \epsilon_i - \delta_j) = 0$ for $1 \leq i \leq p$, then $1 \leq j \leq p - i + 1$.

b) If $(\Lambda + \rho, \epsilon_i - \delta_j) = 0$ for $p + 1 \leq i \leq m$, then $n + p - i + 1 \leq j \leq n$.

Proof. We consider the case $(\Lambda + \rho, \epsilon_i - \delta_j) = 0$ for $1 \leq i \leq p$, that is,

$$0 = (\Lambda + \rho, \epsilon_i - \delta_j) = \lambda_i + \lambda'_j + p - j - i + 1 \Leftrightarrow -\lambda'_j = \lambda_i + p - j - i + 1.$$

The unitarity relations (see a) in Theorem 8.2.2) force $p - j - i + 1 \geq 0$, *i.e.*, $1 \leq j \leq p - i + 1$. If $p + 1 \leq i \leq m$, we have

$$0 = (\Lambda + \rho, \epsilon_i - \delta_j) = \lambda_i + \lambda'_j + p + n - i - j \Leftrightarrow -\lambda'_j = \lambda_i + p + n - i - j.$$

Again, the unitarity relations require $-\lambda_j \leq \lambda_i$, *i.e.*, $p + n - i - j + 1 \leq 0$. We conclude $p + n - i + 1 \leq j \leq n$. □

Now, the subsequent corollary is immediate using Lemma 10.1.15 and the unitarity relations in Theorem 8.2.2.

Corollary 10.1.16. *Let $\mathcal{H}$ be a unitarizable simple $\mathfrak{g}$-supermodule with maximal atypical highest weight Λ. Then Λ satisfies:*

$$\lambda_1 = \ldots = \lambda_p = -\lambda'_1 = \ldots = -\lambda'_p,$$
$$\lambda_{p+1} = \ldots = \lambda_m = -\lambda'_n = \ldots = -\lambda'_{n-q+1}.$$

In particular, if $m = n$, the highest weight Λ is of the form

$$\Lambda = (\lambda_1, \ldots, \lambda_1, \lambda_m, \ldots, \lambda_m | -\lambda_1, \ldots, -\lambda_1, -\lambda_m, \ldots, -\lambda_m).$$

Furthermore, by definition of Δ^+_{nst}, the following corollary is immediate.

Corollary 10.1.17. *Let $\mathcal{H}$ be a unitarizable simple $\mathfrak{g}$-supermodule with maximal atypical highest weight Λ. If $L_0(\Lambda)$ is a relative holomorphic discrete series $\mathfrak{g}_{\bar{0}}$-module, then $\mathcal{H}$ is a relative holomorphic discrete series $\mathfrak{g}$-supermodule.*

10.2. Kac–Wakimoto conjecture

In this section, we prove an analog of the Kac–Wakimoto conjecture for the formal superdimension of relative holomorphic discrete series $\mathfrak{g}$-supermodules over $\mathfrak{g} := \mathfrak{sl}(n|n)$. That is, we show that the formal superdimension of a relative holomorphic discrete series $\mathfrak{g}$-supermodule $\mathcal{H} = L(\Lambda)$ is trivial unless Λ is maximally atypical. Our method generalizes directly to the general case $\mathfrak{sl}(m|n)$, but for simplicity, we focus on $m = n$. Additionally, we may assume $p, q \neq 0$, otherwise the statement follows directly from Example 10.1.4. In particular, this excludes the case $\mathfrak{sl}(1|1)$. We begin with an illustrative example.

10.2.1. An illustrative example

Let $\mathfrak{g} = \mathfrak{sl}(2|2)$. Recall the standard and non-standard positive systems in Table 10.2.1, where we also list the maximally atypical highest weights Λ.

Data	Non-standard system	Standard system
$\Delta^+_{\bar{0}}$	$\{\epsilon_1 - \epsilon_2,\ \delta_1 - \delta_2\}$	$\{\epsilon_1 - \epsilon_2,\ \delta_1 - \delta_2\}$
$\Delta^+_{\bar{1}}$	$\{\epsilon_1 - \delta_1,\ \epsilon_1 - \delta_2,\ -\epsilon_2 + \delta_1,\ -\epsilon_2 + \delta_2\}$	$\{\epsilon_1 - \delta_1,\ \epsilon_1 - \delta_2,\ \epsilon_2 - \delta_1,\ \epsilon_2 - \delta_2\}$
$\rho_{\bar{0}}$	$(\frac{1}{2}, -\frac{1}{2} \mid \frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2} \mid \frac{1}{2}, -\frac{1}{2})$
$\rho_{\bar{1}}$	$(1, -1 \mid 0, 0)$	$(1, 1 \mid -1, -1)$
$\rho = \rho_{\bar{0}} - \rho_{\bar{1}}$	$(-\frac{1}{2}, \frac{1}{2} \mid \frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{2}, -\frac{3}{2} \mid \frac{3}{2}, \frac{1}{2})$
Maximally atypical Λ	$(\lambda_1, \lambda_2 \mid -\lambda_1, -\lambda_2)$, $\lambda_i \in \mathbb{C}$	$(\lambda_1, \lambda_2 \mid -\lambda_1 - 1, -\lambda_2 + 1)$, $\lambda_i \in \mathbb{C}$

Table 10.1.: Non-standard and standard positive systems in $\mathfrak{sl}(2|2)$

In Section 10.1, we worked with the non-standard positive system such that any maximal atypical highest weight of some unitarizable simple $\mathfrak{g}$-supermodule is of the form

$$\Lambda = (\lambda_1, \lambda_2 | -\lambda_1, -\lambda_2).$$

We consider all such Λ such that $L_0(\Lambda)$ is isomorphic to the Harish-Chandra module of a holomorphic discrete series representation of the underlying Lie group such that $L(\Lambda)$ is a relative holomorphic discrete series $\mathfrak{g}$-supermodule (see Corollary 10.1.17).

In this section, we find it helpful to translate $L(\Lambda; \Delta^+_{\mathrm{nst}})$ to $L(\Lambda'; \Delta^+_{\mathrm{st}})$ using odd reflection functors (see Lemma 6.1.14). The positive systems Δ^+ and Δ^+_{st} are related via the composition $r_{\epsilon_2-\delta_2} \circ r_{\epsilon_2-\delta_1}$ of odd reflections, so that

$$\Delta^+_{\mathrm{nst}} = r_{\epsilon_2-\delta_2} \circ r_{\epsilon_2-\delta_1}(\Delta^+_{\mathrm{st}}).$$

In particular, if Λ is maximally atypical for Δ^+_{nst}, we have

$$L(\Lambda; \Delta^+_{\mathrm{nst}}) \cong \Pi L(\Lambda'; \Delta^+_{\mathrm{st}}), \quad \Lambda' = \Lambda + \epsilon_2 - \delta_1 = (\lambda_1, \lambda_2 + 1 | -\lambda_1 - 1, -\lambda_2).$$

Additionally, the $\mathfrak{g}_{\bar{0}}$-constituents (neglecting parity) of $L(\Lambda; \Delta^+_{\mathrm{nst}})$ and $\Pi L(\Lambda'; \Delta^+_{\mathrm{st}})$ are isomorphic, and we conclude

$$\operatorname{sdim} L(\Lambda; \Delta^+_{\mathrm{nst}}) = -\operatorname{sdim} L(\Lambda'; \Delta^+_{\mathrm{st}}).$$

This can be seen concretely in Figure 10.2.1, where we write down the $\mathfrak{g}_{\bar{0}}$-constituents of both supermodules.

Next, we assign to $L(\Lambda'; \Delta^+_{\mathrm{nst}})$ a natural finite-dimensional supermodule. For this purpose, we note that (λ_1, λ_2) is the highest weight of a discrete series $\mathfrak{su}(1,1)$-module, and there exists a unique element in the Weyl orbit that belongs to a finite-dimensional $\mathfrak{su}(1,1)$-module with respect to $\Delta^+_{\bar{0}}$:

$$w \cdot \Lambda'|_{\mathfrak{su}(1,1)} = w(\Lambda'|_{\mathfrak{su}(1,1)} + \rho_{\bar{0}}|_{\mathfrak{su}(1,1)}) - \rho_{\bar{0}}|_{\mathfrak{su}(1,1)} = (\lambda_2, \lambda_1 + 1)$$

with $w \in W$ satisfying $w\Delta^+_n = \Delta^-_n$ and $w\Delta^+_c = \Delta^+_c$. The Weyl element w is unique. In particular,

$$L_0(\mu; \Delta^+_{\mathrm{nst}}) \cong \Pi L_0(w(\mu); \Delta^+_{\mathrm{st}})$$

for all $\mathfrak{g}_{\bar{0}}$-constituents. We set

$$\Lambda'_{\mathrm{fd}} := (\lambda_2, \lambda_1 + 1 | -\lambda_1 - 1, -\lambda_2),$$

which is a maximally atypical dominant integral weight; in particular, $L(\Lambda'_{\mathrm{fd}})$ is a simple finite-dimensional $\mathfrak{g}$-supermodule. The decomposition under $\mathfrak{g}_{\bar{0}}$ is:

$\mathbf{L}(\boldsymbol{\Lambda}'_{\mathrm{fd}})$

$(\lambda_2, \lambda_1 + 1 \mid -\lambda_1 - 1, -\lambda_2)$

$(\lambda_2 - 1, \lambda_1 + 1 \mid -\lambda_1, -\lambda_2)$ $\quad$ $(\lambda_2, \lambda_1 \mid -\lambda_1 - 1, -\lambda_2 + 1)$

$(\lambda_2 - 1, \lambda_1 \mid -\lambda_1, -\lambda_2 + 1)$

$\mathbf{L}(\mathbf{\Lambda}'; \mathbf{\Delta}^+_{\mathrm{st}})$

$$(\lambda_1, \lambda_2 + 1 \mid -\lambda_1 - 1, -\lambda_2)$$

$$(\lambda_1 - 1, \lambda_2 + 1 \mid -\lambda_1 - 1, -\lambda_2 + 1) \qquad (\lambda_1, \lambda_2 \mid -\lambda_1, -\lambda_2)$$

$$(\lambda_1 - 1, \lambda_2 \mid -\lambda_1, -\lambda_2 + 1)$$

$\mathbf{L}(\mathbf{\Lambda}; \mathbf{\Delta}^+_{\mathrm{nst}})$

$$(\lambda_1, \lambda_2 \mid -\lambda_1, -\lambda_2)$$

$$(\lambda_1 - 1, \lambda_2 \mid -\lambda_1, -\lambda_2 + 1) \qquad (\lambda_1, \lambda_2 + 1 \mid -\lambda_1 - 1, -\lambda_2)$$

$$(\lambda_1 - 1, \lambda_2 + 1 \mid -\lambda_1 - 1, -\lambda_2 + 1)$$

Figure 10.1.: Comparison of $\mathfrak{g}_{\bar{0}}$-constituents for $L(\Lambda; \Delta^+_{\mathrm{nst}})$ and $L(\Lambda'; \Delta^+_{\mathrm{st}})$.

We now compare both $\mathfrak{g}$-supermodules, *i.e.*, $L(\Lambda')$ and $L(\Lambda'_{\text{fd}})$. For this purpose, recall that $\mathfrak{k} = \mathfrak{su}(2) \otimes \mathfrak{u}(1)$. Define

$$L(\mu)_{\mathfrak{k}} := \sum_j L_0(\mu_j; \mathfrak{k}^{\mathbb{C}}),$$

which consists precisely of the K-types with highest weights Λ_j in $L(\Lambda)$ — that is, the K-types occurring as highest weights of the $\mathfrak{g}_{\bar{0}}$-constituents. Then, as $\mathfrak{k}^{\mathbb{C}}$-supermodules, we have the isomorphism:

$$L(\Lambda')_{\mathfrak{k}} \cong L(\Lambda'_{\text{fd}})_{\mathfrak{k}},$$

based on the following identifications:

$$\begin{aligned} L_0(\Lambda'; \mathfrak{k}^{\mathbb{C}}) &\cong L_0(\Lambda'_{\text{fd}}; \mathfrak{k}^{\mathbb{C}}), \\ L_0(\Lambda' - \gamma_1; \mathfrak{k}^{\mathbb{C}}) &\cong L_0(\Lambda'_{\text{fd}} - w(\gamma_1); \mathfrak{k}^{\mathbb{C}}), \\ L_0(\Lambda' - \gamma_2; \mathfrak{k}^{\mathbb{C}}) &\cong L_0(\Lambda'_{\text{fd}} - w(\gamma_2); \mathfrak{k}^{\mathbb{C}}), \\ L_0(\Lambda' - \gamma_1 - \gamma_2); \mathfrak{k}^{\mathbb{C}}) &\cong L_0(\Lambda'_{\text{fd}} - w(\gamma_1) - w(\gamma_2); \mathfrak{k}^{\mathbb{C}}), \end{aligned} \tag{10.2.1}$$

where $\gamma_1 := \epsilon_1 - \delta_2$ and $\gamma_2 := \epsilon_2 - \delta_1$.

Concerning the formal superdimension, we conclude that $L(\Lambda')$ and $L(\Lambda'_{\text{fd}})$ only differ in the "non-compact" part. However, we have by construction for any $\mathfrak{g}_{\bar{0}}$-constituent $L_0(\Lambda'_{\text{fd}} - w(\gamma))$:

$$\begin{aligned} \prod_{\beta \in \Delta_n^+} \frac{(\Lambda'_{\text{fd}} - \omega(\gamma) + \rho_{\bar{0}}, \beta)}{(\rho_{\bar{0}}, \beta)} &= \prod_{\beta \in \Delta_n^+} \frac{(w(\Lambda' - \gamma + \rho_{\bar{0}}), \beta)}{(\rho_{\bar{0}}, \beta)} \\ &= \prod_{\beta \in \Delta_n^+} \frac{(\Lambda' - \gamma + \rho_{\bar{0}}, -\beta)}{(\rho_{\bar{0}}, \beta)} \\ &= \prod_{\beta \in \Delta_n^+} \frac{|(\Lambda' - \gamma + \rho_{\bar{0}}, \beta)|}{(\rho_{\bar{0}}, \beta)}, \end{aligned} \tag{10.2.2}$$

i.e., the "dimension" of the non-compact parts coincide. We conclude:

$$\begin{aligned} \operatorname{sdim} L(\Lambda') = {} & (\lambda_2 - \lambda_1) \cdot \dim L_0(\Lambda'; \mathfrak{k}^{\mathbb{C}}) \\ & - (\lambda_2 - \lambda_1 + 1) \cdot \dim L_0(\Lambda' - \gamma_1; \mathfrak{k}^{\mathbb{C}}) \\ & - 2(\lambda_2 - \lambda_1 - 1) \cdot \dim L_0(\Lambda' - \gamma_2; \mathfrak{k}^{\mathbb{C}}) \\ & + (\lambda_2 - \lambda_1) \cdot \dim L_0(\Lambda' - \gamma_1 - \gamma_2; \mathfrak{k}^{\mathbb{C}}), \\ \operatorname{sdim} L(\Lambda'_{\text{fd}}) = {} & (\lambda_2 - \lambda_1) \cdot \dim L_0(\Lambda'_{\text{fd}}; \mathfrak{k}^{\mathbb{C}}) \\ & - (\lambda_2 - \lambda_1 + 1) \cdot \dim L_0(\Lambda'_{\text{fd}} - w(\gamma_1); \mathfrak{k}^{\mathbb{C}}) \\ & - (\lambda_2 - \lambda_1 - 1) \cdot \dim L_0(\Lambda'_{\text{fd}} - w(\gamma_2); \mathfrak{k}^{\mathbb{C}}) \\ & + (\lambda_2 - \lambda_1) \cdot \dim L_0(\Lambda'_{\text{fd}} - w(\gamma_1) - w(\gamma_2); \mathfrak{k}^{\mathbb{C}}), \end{aligned}$$

which yields with Equation 10.2.1:

$$\operatorname{sdim} L(\Lambda') = \operatorname{sdim} L(\Lambda'_{\text{fd}}).$$

This proves the Kac–Wakimoto conjecture for $\mathfrak{sl}(2|2)$ by [128]. We now generalize this idea to arbitrary $\mathfrak{sl}(n|n)$.

10.2.2. The case $m = n$

Let $\mathfrak{g} := \mathfrak{sl}(n|n)$. Without loss of generality, we consider the standard positive system $\Delta^+ := \Delta'_{\mathrm{st}}$, as $\operatorname{sdim} L(\Lambda;\Delta^+_{\mathrm{nst}}) = \pm \operatorname{sdim} L(\Lambda';\Delta^+_{\mathrm{st}})$ for any relative holomorphic discrete series $\mathfrak{g}$-supermodule $L(\Lambda;\Delta^+_{\mathrm{nst}})$. In particular, any maximal atypical highest weight Λ of a unitarizable simple $\mathfrak{g}$-supermodule (with respect to Δ^+_{st}) is then of the form (analogously to Corollary 10.1.16):

$$\Lambda = (\underbrace{\lambda_1,\ldots,\lambda_1}_{p\text{ times}},\underbrace{\lambda_2,\ldots,\lambda_2}_{q\text{ times}}\,|\,\underbrace{-\lambda_1-q,\ldots,-\lambda_1-q}_{p\text{ times}},\underbrace{-\lambda_2+p,\ldots,-\lambda_2+p}_{q\text{ times}}).$$

Recall that Λ is integral by Lemma 10.1.14. By Corollary 10.1.17, we may assume that Λ is the Harish-Chandra parameter of a relative holomorphic discrete series representation of the underlying Lie group. To any such Λ, we naturally assign a dominant integral and maximal atypical weight Λ_{fd}. For this purpose, we consider the Harish-Chandra parameter

$$\Lambda|_{\mathfrak{su}(p,q)} = (\underbrace{\lambda_1,\ldots,\lambda_1}_{p\text{ times}},\underbrace{\lambda_2,\ldots,\lambda_2}_{q\text{ times}})$$

of a holomorphic discrete series $\mathfrak{su}(p,q)$-module. There exists a unique finite-dimensional highest weight representation for $\Delta^+_{\bar{0}}$ in the Weyl orbit of Λ, that is, there exists a unique $w \in W$ such that

$$w.\Lambda|_{\mathfrak{su}(p,q)} = w(\Lambda + \rho_{\bar{0}}|_{\mathfrak{su}(p,q)}) - \rho_{\bar{0}}|_{\mathfrak{su}(p,q)} = (\underbrace{\lambda_2-p,\ldots,\lambda_2-p}_{q\text{ times}},\underbrace{\lambda_1+q,\ldots,\lambda_1+q}_{p\text{ times}}).$$

Precisely, w is the composition of permutations such that

$$w(\lambda_1,\ldots,\lambda_p,\lambda_{p+1},\ldots,\lambda_{p+q}) = (\lambda_{p+1},\ldots,\lambda_{p+q},\lambda_1,\ldots,\lambda_p),$$

and it is immediate that $w\Delta^+_n = \Delta^-_n$ and $w\Delta^+_c = \Delta^+_c$. We define

$$\Lambda_{\mathrm{fd}} := (\underbrace{\lambda_2-p,\ldots,\lambda_2-p}_{q\text{ times}},\underbrace{\lambda_1+q,\ldots,\lambda_1+q}_{p\text{ times}}\,|\,\underbrace{-\lambda_1-q,\ldots,-\lambda_1-q}_{p\text{ times}},\underbrace{-\lambda_2+p,\ldots,-\lambda_2+p}_{q\text{ times}}),$$

which is Δ^+-dominant integral; hence, it is the highest weight of a finite-dimensional $\mathfrak{g}$-supermodule $L(\Lambda_{\mathrm{fd}})$.

We now compare $L(\Lambda)$ and $L(\Lambda_{\mathrm{fd}})$.

Lemma 10.2.1. *Let γ be the sum of pairwise distinct odd positive roots. Then $L_0(\Lambda-\gamma)$ is a $\mathfrak{g}_{\bar{0}}$-constituent of $L(\Lambda)$ if and only if $L_0(\Lambda_{\mathrm{fd}} - w(\gamma))$ is a $\mathfrak{g}_{\bar{0}}$-constituent of $L(\Lambda_{\mathrm{fd}})$. In particular,*

$$L(\Lambda)_{\mathfrak{k}} \cong L(\Lambda_{\mathrm{fd}})_{\mathfrak{k}}.$$

Proof. First, we show that $\Lambda-\gamma$ is Δ^+_c-dominant integral and Δ^-_n-anti-dominant integral if and only if $\Lambda_{\mathrm{fd}} - w(\gamma)$ is $\Delta^+_{\bar{0}}$-dominant integral. First, let $\alpha \in \Delta^+_c$, and note $w(\alpha) = \alpha$ and $(\alpha,\rho_n) = 0$. We have

$$\begin{aligned}(\Lambda_{\mathrm{fd}} - \omega(\gamma) + \rho_c, \alpha) &= (w(\Lambda-\gamma+\rho_{\bar{0}}) - \rho_n, \alpha)\\ &= (w(\Lambda-\gamma+\rho_{\bar{0}}),\alpha)\\ &= (\Lambda-\gamma+\rho_{\bar{0}},\alpha)\\ &= (\Lambda-\gamma+\rho_c,\alpha),\end{aligned}$$

i.e., $\Lambda - \gamma$ and $\Lambda_{\text{fd}} - w(\gamma)$ have the same integral $\mathfrak{k}^{\mathbb{C}}$-Dynkin labels and hence, they are highest weights of isomorphic $\mathfrak{k}^{\mathbb{C}}$-modules.

Next, let $\beta \in \Delta_n^+$. Note $w(\beta) = -\beta'$ for some $\beta' \in \Delta_n^+$, and $w\Delta_n^+ = \Delta_n^-$. Then, a similar argumentation as above shows that $\Lambda - \gamma$ is Δ_n^+-anti-dominant integral if and only if $\Lambda_{\text{fd}} - w(\gamma)$ is Δ_n^+-dominant integral. In particular, $\Lambda_{\text{fd}} - w(\gamma)$ is $\Delta_{\bar{0}}^+$-dominant integral.

By Theorem 7.3.4, all $\mathfrak{g}_{\bar{0}}$-constituents of $L(\Lambda)$ are of the form $L_0(\Lambda - \gamma)$, where γ is a sum of pairwise distinct odd roots α such that $(\Lambda + \rho, \alpha) \neq 0$. Moreover, $\Lambda - \gamma$ is Δ_c^+-dominant integral and Δ_n^+-anti-dominant integral.

On the other hand, by Theorem 2.5 and Corollary 2.7 in [73] together with Lemma 9.2.1 in [99], $L(\Lambda_{\text{fd}})$ decomposes under $\mathfrak{g}_{\bar{0}}$ in simple highest weight $\mathfrak{g}_{\bar{0}}$-modules $L_0(\mu')$ such that $\mu' = \Lambda_{\text{fd}} - \gamma'$, γ' being a sum of pairwise distinct odd roots α such that $(\Lambda_{\text{fd}} + \rho, \alpha) \neq 0$ and μ' is $\Delta_{\bar{0}}^+$-dominant integral.

It is now immediate that $L_0(\Lambda - \gamma)$ is a $\mathfrak{g}_{\bar{0}}$-constituent of $L(\Lambda)$ if and only if $L_0(\Lambda_{\text{fd}} - w(\gamma))$ is a $\mathfrak{g}_{\bar{0}}$-constituent of $L(\Lambda_{\text{fd}})$. This finishes the proof. □

Furthermore, the following lemma is a direct consequence of Equation 10.2.2 and the proof of Lemma 10.2.1.

Lemma 10.2.2. *Let $L_0(\Lambda - \gamma)$ be a $\mathfrak{g}_{\bar{0}}$-constituent of $L(\Lambda)$ for some sum of pairwise distinct odd roots γ. Then*

$$\prod_{\beta \in \Delta_n^+} \frac{|(\Lambda - \gamma + \rho_{\bar{0}}, \beta)|}{(\rho_{\bar{0}}, \beta)} = \prod_{\beta \in \Delta_n^+} \frac{(\Lambda_{\text{fd}} - \omega(\gamma) + \rho_{\bar{0}}, \beta)}{(\rho_{\bar{0}}, \beta)},$$

Combining Lemma 10.2.1 and Lemma 10.2.2, we conclude the following theorem.

Theorem 10.2.3. *Let $L(\Lambda)$ be a holomorphic discrete series $\mathfrak{g}$-supermodule with maximally atypical highest weight Λ. Let $L(\Lambda_{\text{fd}})$ be the associated finite-dimensional $\mathfrak{g}$-supermodule with maximal atypical highest weight Λ_{fd}. Then*

$$\operatorname{sdim} L(\Lambda) = \operatorname{sdim} L(\Lambda_{\text{fd}}).$$

Altogether, combining Lemma 10.1.14 with Theorem 10.2.3 and the Kac–Wakimoto conjecture for finite-dimensional supermodules [128], we obtain the following theorem.

Theorem 10.2.4. *Let $\mathcal{H}$ be a relative holomorphic discrete series $\mathfrak{g}$-supermodule with highest weight Λ. Then $\operatorname{sdim} \mathcal{H} = 0$ unless Λ is maximally atypical.*

Part IV.

Physics and Indices

11. A math–physics dictionary

The main objective of this part of the thesis is to explain the representation-theoretic foundations of the superconformal index in mathematically precise and intelligible terms. This aims to facilitate the exchange of insights and to draw lessons for possible applications in other areas.

This requires, above all, the translation of physical terminology into mathematical language — the essential goal of this chapter.

In Section 11.1, we begin by outlining the technical assumptions that underpin our treatment of indices for unitarizable supermodules. While our approach is admittedly ad hoc and not entirely satisfactory, we aim to emphasize the broader relevance of defining indices for Lie superalgebras more generally.

Section 11.2 introduces a dictionary that connects various physical concepts to the mathematical notions of atypicality and the DS functor (see Section 9.1). This framework is motivated by a key physical insight that, until now, lacked a precise mathematical expression: the continuity of the fragmentation and recombination process at the boundary of the unitarity region, as discussed in Section 11.3.

To clarify this, we offer a geometric description of the unitarity region in slices of weight space, parametrized continuously by dimension and R-charge. We then provide a detailed and rigorous account of the fragmentation/recombination phenomenon at the boundary of this region, formulated through the decomposition of Kac modules with atypical highest weights. Finally, we relate these ideas to the established physics concepts of protected or short supermodules. Our central result is that maximally atypical supermodules are, in fact, absolutely protected.

This chapter is an adaptation of an earlier version available on arXiv (see [126]), and represents joint research with Johannes Walcher.

11.1. Working assumptions

While this paper is rooted in representation theory, our primary interest lies in *analytic* aspects if we consider physical applications. We won't cover every technical detail, but we want to clarify the overall picture.

The physical states of a supersymmetric quantum field theory form a super Hilbert space $\mathcal{H}$, and the quantum fields are operator-valued distributions in $\mathrm{End}(\mathcal{H})$, defined over the underlying superspacetime. The (super) Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ of a theory is a (super)module over its (super)symmetry algebra $\mathfrak{g}$, and it's typically unitarizable (*i.e.*, compatible with a real structure). But it's not irreducible — otherwise the dynamics would be too simple. Instead, $\mathcal{H}$ is built from simple $\mathfrak{g}$-supermodules, subject to physical constraints like locality, and changes continuously as external parameters vary.

To make this idea precise, we introduce a topological space $\mathbb{H}$ of "coupling constants", and think of a "family of physical theories" as providing (among other things) a continuous map

$$T : \mathbb{H} \to \mathrm{Hom}(\mathfrak{g}^{\mathbb{R}}, \mathfrak{u}(\mathcal{H}))$$

into the space of unitarizable $\mathfrak{g}$-supermodule structures on a fixed super Hilbert space $\mathcal{H}$, which is identified with the representations of the underlying real Lie algebra $\mathfrak{g}^{\mathbb{R}}$ in the space $\mathfrak{u}(\mathcal{H})$ of skew-Hermitian operators on $\mathcal{H}$. Equivalently, we can describe this situation geometrically as a bundle of unitarizable $\mathfrak{g}$-supermodules over $\mathbb{H}$, where the underlying Hilbert space bundle is trivial.

As an example, take $\mathfrak{g} = \mathfrak{sl}(4|4)$ and $\mathfrak{g}^{\mathbb{R}} = \mathfrak{su}(2,2|4)$. Then $\tau \in \mathbb{H}$ (the complex upper half-plane) can be interpreted as the complexified gauge coupling of $\mathcal{N} = 4$ super Yang–Mills theory with a fixed compact gauge group (e.g., $\mathrm{SU}(N)$ for $N \in \mathbb{N}$). The map $T(\tau)$ arises from the supersymmetric path integral over fields on a Lorentzian four-manifold with appropriate boundary conditions. Unitarity and continuity are justified through standard, though non-rigorous, physical arguments.

A key challenge is that the topology on $\mathrm{Hom}(\mathfrak{g}^{\mathbb{R}}, \mathfrak{u}(\mathcal{H}))$ depends on the choice of topology for $\mathfrak{u}(\mathcal{H})$, for which there is no canonical choice in physics. To avoid this ambiguity, we assume that each $T(\tau)$ decomposes into a direct sum of simple $\mathfrak{g}$-supermodules (for a definition of a Hilbert space direct sum, see Section 5.2) — an assumption supported, for instance, by a discrete spectrum condition. This condition is expected to hold in radial quantization or when spacetime has compact spatial slices, both of which are natural from a physical perspective.

As described in Section 6.1, each simple unitarizable $\mathfrak{g}$-supermodule decomposes further into weight spaces — eigenspaces of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. This gives rise to a map

$$\Theta : \mathrm{Hom}(\mathfrak{g}^{\mathbb{R}}, \mathfrak{u}(\mathcal{H})) \to (\mathfrak{h}^*)^{\mathbb{Z}_+}/S_\infty,$$

where $(\mathfrak{h}^*)^{\mathbb{Z}_+}$ denotes sequences of weights, *i.e.*, it consists of all sequences $(\lambda_1, \lambda_2, \dots)$ with $\lambda_i \in \mathfrak{h}^*$, and S_∞ is the infinite symmetric group acting by permutation. We endow the quotient with the natural topology and assume that Θ is continuous with respect to the topology on $\mathrm{Hom}(\mathfrak{g}^{\mathbb{R}}, \mathfrak{u}(\mathcal{H}))$, that is, we assume that the topology on $\mathrm{Hom}(\mathfrak{g}^{\mathbb{R}}, \mathfrak{u}(\mathcal{H}))$ is fine enough to allow Θ to be continuous.

This continuity ensures that the indices constructed in Section 12 depend continuously on the supermodule structure, and can thus serve as invariants — assuming the physical theory makes T continuous. In effect, we use $(\mathfrak{h}^*)^{\mathbb{Z}_+}/S_\infty$ as a (topological) replacement for $\mathrm{Hom}(\mathfrak{g}^{\mathbb{R}}, \mathfrak{u}(\mathcal{H}))$.

11.2. A dictionary for supermodules

This section serves as a conceptual and notational bridge between the mathematics and physics literature on supermodules.

Long and short supermodules. Our discussion begins with Corollary 7.3.5, that is, any unitarizable highest weight $\mathfrak{g}$-supermodule $L(\Lambda)$ decomposes in a finite sum of unitarizable simple highest weight $\mathfrak{g}_{\bar{0}}$-modules, and the number of $\mathfrak{g}_{\bar{0}}$-constituents lies between $2^{\#\Delta_{\bar{1}}^+ - \#A_\Lambda}$ and $2^{\dim(\mathfrak{g}_{-1})}$, with maximum attained precisely if Λ is typical. Following physics, we call a unitarizable simple $\mathfrak{g}$-supermodule with the maximal number of $\mathfrak{g}_{\bar{0}}$-constituents a *long supermodule*, all others *short*. If the number of $\mathfrak{g}_{\bar{0}}$-constituents is smaller than $2^{\dim(\mathfrak{g}_{-1})-1}$, the supermodule might also be called *ultra-short*. A unitarizable simple highest weight supermodule is long precisely if it is isomorphic to a Kac supermodule with typical highest weight, and short precisely if it is isomorphic to the quotient of a Kac supermodule with atypical highest weight by the radical of the Kac–Shapovalov form. In this context, non-zero elements of the Kac supermodule belonging to the radical of the Kac–Shapovalov form are also referred to as *null vectors*.

Region of unitarity. In Section 6.1.4, we parametrized the unitarizable $\mathfrak{g}_{\bar{0}}$-modules by highest weights of the form

$$\left(\frac{\lambda}{2}-a_1, \frac{\lambda}{2}-a_2, \dots, \frac{\lambda}{2}-a_p, -\frac{\lambda}{2}+a_{p+1}, \dots, -\frac{\lambda}{2}+a_{p+q} | b_1, \dots, b_n\right) + \alpha(1, \dots, 1|1, \dots, 1),$$

where $a_1, \dots, a_p$ and $a_{p+1}, \dots, a_{p+q}$ are non-negative integers such that $0 \leq a_2 \leq \dots \leq a_p$ and $a_{p+1} \geq a_{p+2} \geq \dots \geq a_{p+q-1} \geq a_{p+q} = 0$, $b_1, \dots, b_n$ are non-negative integers such that $b_1 \geq \dots \geq b_n = 0$, $\alpha \in \mathbb{R}_{\geq 0}$, and λ belongs to the set

$$(-\infty, -m+x-(r-1)) \cup \{-m+x-(r-1), -m+x-(r-1)+1, \dots, -m+x\}.$$

The numbers a_i and b_j are referred to as *spin quantum numbers* and *R-symmetry quantum numbers*, respectively. The real numbers λ and α are referred to as *scaling dimension* and *R-charge*, respectively.

We will refer to the set of all such Λ as the *set of $\mathfrak{g}_{\bar{0}}$-unitarity for fixed spin and R-symmetry quantum numbers* and denote it as $\Gamma_0^{(a,b)}$. The full set of $\mathfrak{g}_{\bar{0}}$-unitarity Γ_0 is then defined as the union of all $\Gamma_0^{(a,b)}$, which is geometrically a collection of half-spaces for fixed spin and R-symmetry quantum numbers, together with some lines.

The set of $\mathfrak{g}$-unitarity for fixed spin and R-symmetry quantum numbers, which we denote by $\Gamma^{(a,b)}$, consists of a congruent subcone of $\Gamma_0^{(a,b)}$ (see Chapter 8), together with some additional half lines and possibly isolated points. The set of $\mathfrak{g}$-unitarity $\Gamma \subset \mathfrak{h}^*$ is the union of all $\Gamma^{(a,b)}$. Note that in general, there are two different isomorphism classes of unitarizable simple supermodules for each $\Lambda \in \Gamma$, related by parity reversal. Under appropriate circumstances, physical considerations may fix a section of this $\mathbb{Z}/2$ bundle over Γ. The (relative, topological) interior of $\Gamma^{(a,b)}$, where λ and α can be varied independently, is a (relatively) open cone. We will denote it by $\mathcal{C}^{(a,b)}$ and, borrowing physics terminology, call it the *region of unitarity for fixed spin and R-symmetry quantum numbers*. It consists entirely of typical weights in the sense of, as can be shown with the help of the Dirac inequality (see Proposition 7.2.4).

Lemma 11.2.1. *Let $\mathcal{H}$ be a simple highest weight $\mathfrak{g}$-supermodule with highest weight Λ. If $(\Lambda+\rho, \epsilon_p-\delta_1) < 0$ and $(\Lambda+\rho, \epsilon_{p+1}-\delta_n) > 0$, then $\mathcal{H}$ is unitarizable. Moreover, $\mathcal{H}$ is unitarizable and has typical highest weight if and only if Λ satisfies both inequalities. Namely,*

$$\mathcal{C} := \bigsqcup_{(a,b)} \mathcal{C}^{(a,b)} = \{\Lambda \in \Gamma : (\Lambda+\rho, \epsilon_p-\delta_1) < 0,\ (\Lambda+\rho, \epsilon_{p+1}-\delta_n) > 0\}.$$

A weight at the (relative, topological) boundary of $\mathcal{C}$, denoted by $\partial\mathcal{C}$, is said to live *at the unitarity bound*. This is a subset of

$$\{\Lambda \in \Gamma : (\Lambda+\rho, \epsilon_p-\delta_1) = 0 \text{ or } (\Lambda+\rho, \epsilon_{p+1}-\delta_n) = 0\},$$

but in general not identical to it (see [126]). Finally, we set $\overline{\mathcal{C}} := \mathcal{C} \cup \partial\mathcal{C}$, which is a closed subspace of Γ, and refer to $\Lambda \in \overline{\mathcal{C}}$ as *within the unitarity bound*. This, together with Lemma 11.2.1, implies the following statements.

Lemma 11.2.2. *Let $\Lambda \in \Gamma$ be the highest weight of a unitarizable highest weight $\mathfrak{g}$-supermodule.*

a) $\Lambda \in \mathcal{C}$ if and only if Λ is typical. In particular, $K(\Lambda) \cong L(\Lambda)$.

b) If $\Lambda \in \partial\mathcal{C}$, then Λ is either 1-atypical or 2-atypical. If $n = 1$, Λ is 1-atypical. In particular, $K(\Lambda)$ is not simple.

Bogomol'nyi–Prasad–Sommerfield or 'BPS'. The interplay between self-commutativity and unitarity plays a central role in the physical constructions of invariants and measures of atypicality. A widely encountered concept is that of a BPS state. The basic intuition is that multiplet shortening (as discussed above) arises due to null vectors in the Kac supermodule. This phenomenon is manifested in the unitarizable simple quotient, which contains non-trivial subspaces that are annihilated by more supercharges than required. Here, a supercharge is just an element of $\mathfrak{g}_{\bar{1}}$. To make this more precise, let M be a highest weight $\mathfrak{g}$-supermodule. Then, as a consequence of the weight space decomposition $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$ (see Lemma 3.1.13), for any vector $v_\lambda \in M^\lambda$, and any odd root α, we have that (at least) either $Q_\alpha v_\lambda = 0$ or $Q_{-\alpha} v_\lambda = 0$, where Q_α denotes a generator of the root space $\mathfrak{g}^\alpha$. If M is the standard Verma supermodule, exactly one of these is satisfied for non-zero v_λ, and any additional vanishing is a consequence of atypicality. This leads to the following definition.

Definition 11.2.3. Let $\mathcal{H}$ be a unitarizable $\mathfrak{g}$-supermodule. For any $v \in \mathcal{H}$, we denote by $\operatorname{ann}(v) := \{x \in \mathfrak{g}_{+1} | xv = 0\}$ the *odd annihilator* of v. Then a non-zero vector $v \in \mathcal{H}$ is called a *BPS state* if $\dim(\operatorname{ann}(v)) > \frac{\dim \mathfrak{g}_{+1}}{2} = \#\Delta_{\bar{1}}^+$.

Lemma 11.2.4 ([24])**.** *Let $\mathcal{H}$ be a unitarizable simple $\mathfrak{g}$-supermodule.*

a) $v \in \mathcal{H}$ is a BPS state if and only if there exists an odd positive root $\alpha \in \Delta_{\bar{1}}^+$ with non-zero root vector Q_α such that $Q_\alpha v = 0$, but $v \neq 0 \in \ker Q_\alpha / \operatorname{im} Q_\alpha$.

b) If $\mathcal{H}$ has atypical highest weight Λ, and $(\Lambda + \rho, \alpha) = 0$ for some odd root α, then $\ker Q_\alpha / \operatorname{im} Q_\alpha \neq 0$ is non-trivial.

c) $\mathcal{H}$ is a long supermodule if and only if $\ker Q_{\mathcal{H}} = \operatorname{im} Q_{\mathcal{H}}$ for all square-zero $Q \in \mathfrak{g}_{+1}$.

We conclude with the following comparison between various math and physics notions that we have touched upon so far.

Proposition 11.2.5. *Let M be a unitarizable simple $\mathfrak{g}$-supermodule. Then the following assertions are equivalent:*

a) $\mathcal{H}$ is a short supermodule.

b) The highest weight of $\mathcal{H}$ is atypical.

c) $\mathcal{H}$ contains a BPS state.

d) $\mathcal{H}$ is protected as a direct summand in families of unitarizable $\mathfrak{g}$-supermodules.

In particular, long supermodules do not contain BPS states.

One might expect that if $\mathcal{H}$ contains BPS states, then in particular the highest weight state should be one. In other words, that $\mathcal{H}$ is a short supermodule if and only if the highest weight state is BPS. However, this is not necessarily true and in any case depends on the choice of positive system [126].

In physics, BPS states are further classified by comparing the size of the annihilator with the total number of supercharges. In other words, if $v \in \mathcal{H}$ is a BPS state one defines the *(degree of) BPS-ness* by

$$\deg_{\mathrm{BPS}}(v) = \frac{\dim(\mathrm{ann}(v)) - \dim \mathfrak{g}_{+1}}{\dim \mathfrak{g}_{+1}}.$$

The intuition is that the more supercharges annihilate a state, the more it is protected against deformations. Thus, if $\mathcal{H}$ is a unitarizable simple supermodule, one expects a close relationship between the maximal *BPS-ness* of any state in $\mathcal{H}$ and the degree of atypicality of the highest weight, *i.e.*, the maximal number of *mutually orthogonal* roots with $(\Lambda + \rho, \alpha) = 0$. For finite-dimensional supermodules, it is indeed known that the two notions are equivalent concepts. We conjecture that this also extends to the infinite-dimensional case.

11.3. Fragmentation and recombination

If $L(\Lambda)$ is unitarizable, but Λ is atypical (*i.e.*, $\Lambda \in \Gamma \setminus \mathcal{C}$), the Kac supermodule $K(\Lambda)$ is not simple, but also does not split into a direct sum of simple supermodules. Instead, as $K(\Lambda)$ is of highest weight type, it has a Jordan–Hölder series $K_0 = K(\Lambda) \supsetneq K_1 \supsetneq \cdots \supsetneq K_M = 0$ such that each K_{i-1}/K_i is isomorphic to a simple highest weight $\mathfrak{g}$-supermodule $L(\Lambda_i)$, with highest weight Λ_i of the form $\Lambda - \Sigma_S$ where $\Sigma_S := \sum_{\alpha \in S} \alpha$ is a sum of mutually distinct odd positive roots, for some $S \subseteq \Delta_1^+$ (*cf.* discussion around Theorem 7.3.2). The Λ_i will all be atypical, but it is difficult to describe in general how many and which ones occur. Also the composition factors $L(\Lambda_i) = K_{i-1}/K_i$ for $i = 2, \ldots, M$ need not be unitarizable. It is an instructive exercise to show that this does not happen within the unitarity bound.

Lemma 11.3.1. *For $n > 1$, the simple highest weight supermodules in the Jordan–Hölder series of $K(\Lambda)$ for $\Lambda \in \partial\mathcal{C}$ are*

a) $L(\Lambda)$ and $L(\Lambda + \epsilon_p - \delta_1)$ if $\Lambda \in \partial\mathcal{C}$ satisfies only $(\Lambda + \rho, \epsilon_p - \delta_1) = 0$.

b) $L(\Lambda)$ and $L(\Lambda - \epsilon_{p+1} + \delta_n)$ if $\Lambda \in \partial\mathcal{C}$ satisfies only $(\Lambda + \rho, \epsilon_{p+1} - \delta_n) = 0$.

c) $L(\Lambda), L(\Lambda + \epsilon_p - \delta_1)$, $L(\Lambda - \epsilon_{p+1} + \delta_n)$, and $L(\Lambda + (\epsilon_p - \delta_1) + (-\epsilon_{p+1} + \delta_n))$ if $\Lambda \in \partial\mathcal{C}$ satisfies both $(\Lambda + \rho, \epsilon_p - \delta_1) = 0$ and $(\Lambda + \rho, \epsilon_{p+1} - \delta_n) = 0$.

As a consequence, letting

$$\mathrm{gr}\, K(\Lambda) := \bigoplus_{i=1}^{M} K_{i-1}/K_i \cong \bigoplus_{i=1}^{M} L(\Lambda_i), \tag{11.3.1}$$

denote the graded $\mathfrak{g}$-supermodule associated to the Jordan–Hölder series, and note that the parity of the $L(\Lambda_i)$ is determined by the parity of $K(\Lambda)$ and the number of odd roots in $\Sigma_S = \Lambda - \Lambda_i$. This is not necessarily equal to the grading induced from the Jordan–Hölder series, which in general is neither $\mathbb{Z}/2$-graded (nor unique for that matter). We obtain,

Lemma 11.3.2. *Let $\Lambda \in \overline{\mathcal{C}}$. Then $\operatorname{gr} K(\Lambda)$ is a unitarizable $\mathfrak{g}$-supermodule.*

If $L(\Lambda) = K_0/K_1$ is unitarizable, then K_1 is the radical of the Kac–Shapovalov form on $K(\Lambda)$ by Proposition 6.1.17. As a consequence, assuming the factors $L(\Lambda_i)$ for $i = 2, \dots, M$ remain unitarizable, their Hermitian form is *not* induced from the Kac–Shapovalov form on $K(\Lambda)$. Remarkably [81], this can be remedied in a natural way in the consideration of *continuous families* of unitarizable supermodules in the sense of Section 11.1. Concretely, let $(\Lambda^{(k)})_{k=1,2,\dots} \subset \mathcal{C}$ be a sequence of typical weights with unitarizable highest weight modules $L(\Lambda^{(k)})$, and $\lim_{k\to\infty} \Lambda^{(k)} = \Lambda_0 \in \partial\mathcal{C}$ at the unitarity bound, where the limit is taken in the usual topology on $\mathfrak{h}^*$. Then, viewing the $L(\Lambda^{(k)}) = K(\Lambda^{(k)})$ as a sequence of $\mathfrak{g}$-supermodule structures on a fixed underlying (pre-)Hilbert space $\mathcal{H}$, it can be seen that the associated weight space decomposition (see Proposition 3.1.13) agrees in the limit $k \to \infty$ with the weight space decomposition on the supermodule $K(\Lambda_0)$. In particular, this induces a non-degenerate Hermitian form on $K(\Lambda_0) \cong \mathcal{H}$, which by uniqueness must agree with the Kac–Shapovalov form on the composition factors. In the sense of section 11.1, the assignment $\Lambda \mapsto \operatorname{gr} K(\Lambda)$ is continuous on $\overline{\mathcal{C}}$, and

$$\lim_{\Lambda\to\Lambda_0} \operatorname{gr} K(\Lambda) = \operatorname{gr} K(\Lambda_0). \tag{11.3.2}$$

Following physics terminology, we think of the "filling in" of $\operatorname{gr} K(\Lambda_0)$ in a continuous family as the *fragmentation of the long supermodule $K(\Lambda)$ as $\Lambda \to \Lambda_0$ hits the unitarity bound*, or conversely as *recombination of the short constituents when Λ moves away from it*. It is important to emphasize that this *does not* imply that the constituents of $\operatorname{gr} K(\Lambda_0)$ can only appear together in a continuous family. Indeed, the "fragments" can move around independently of each other, as long as their highest weight remains in $\Gamma \setminus \mathcal{C}$. If $n > 1$ and $\Lambda_0 \in \overline{\mathcal{C}}$, Lemma 11.3.1, implies

$$\lim_{\Lambda\to\Lambda_0} \operatorname{gr} K(\Lambda) = \begin{cases} L(\Lambda_0) \oplus L(\Lambda_0 - \gamma_1) & \text{if } (\Lambda_0+\rho,\gamma_1) = 0,\ (\Lambda_0+\rho,\gamma_2) \neq 0, \\ L(\Lambda_0) \oplus L(\Lambda_0 - \gamma_2) & \text{if } (\Lambda_0+\rho,\gamma_1) \neq 0,\ (\Lambda_0+\rho,\gamma_2) = 0, \\ L(\Lambda_0) \oplus L(\Lambda_0 - \gamma_1 - \gamma_2) & \\ \quad \oplus L(\Lambda_0 - \gamma_1) \oplus L(\Lambda_0 - \gamma_2) & \text{if } (\Lambda_0+\rho,\gamma_1) = 0,\ (\Lambda_0+\rho,\gamma_2) = 0, \end{cases}$$

where $\gamma_1 := -\epsilon_p + \delta_1$ and $\gamma_2 := \epsilon_{p+1} - \delta_n$, and the $\mathbb{Z}/2$-grading is left implicit, as usual.

The statement that unitarizable simple $\mathfrak{g}$-supermodules with atypical highest weight (that is, $\Lambda \in \Gamma \setminus \mathcal{C}$) cannot, on their own, "return" to $\mathcal{C}$, and that their conformal dimension is entirely fixed by their superconformal R-charge, is known in physics as the principle that short supermodules are protected. Those supermodules that never participate in any fragmentation or recombination process are referred to as absolutely protected. According to the above, a unitarizable simple supermodule is absolutely protected if and only if it does not appear as a composition factor in the Jordan–Hölder series of any $K(\Lambda_0)$ with $\Lambda_0 \in \partial\mathcal{C}$. This condition can be made more explicit as follows.

Lemma 11.3.3. *Let $L(\Lambda)$ be a unitarizable simple highest weight $\mathfrak{g}$-supermodule. Then $L(\Lambda)$ is absolutely protected if and only if $(\Lambda+\rho, \epsilon_p - \delta_1) > 0$ and $(\Lambda+\rho, \epsilon_{p+1} - \delta_n) < 0$ holds.*

Corollary 11.3.4. *If $m, n \geq 3$, then maximally atypical unitarizable simple highest weight $\mathfrak{g}$-supermodules $L(\Lambda)$ are absolutely protected.*

Proof. Since $m, n \geq 3$, it follows from the assumption that $\mathrm{at}(\Lambda) > 2$. Consider the roots $\gamma_1 := -\epsilon_p + \delta_1$ and $\gamma_2 := \epsilon_{p+1} - \delta_n$. Then $(\gamma_1, \gamma_2) = 0$, showing that γ_1 and γ_2 are orthogonal. This implies that the $\mathfrak{g}$-supermodules appearing in the Jordan–Hölder series of $K(\Lambda_0)$, for $\Lambda_0 \in \partial\mathcal{C}$, all share the same degree of atypicality — specifically, either 1 or 2 (see Lemma 11.3.1). In particular, this shows that $L(\Lambda)$ cannot occur as a simple quotient in the Jordan–Hölder series of $K(\Lambda_0)$ for any $\Lambda_0 \in \overline{\mathcal{C}}$. □

12. Indices

In earlier chapters, we have seen that the representation theory of $\mathfrak{g} = \mathfrak{sl}(m|n)$ with real form $\mathfrak{g}^\omega = \mathfrak{su}(p, q|n)$ exhibits key parallels to that of superconformal algebras such as $\mathfrak{su}(2, 2|\mathcal{N})$ for $1 \leq \mathcal{N} \leq 4$, which are central to many developments in mathematical physics. In this context, an *index* is a quantitative measure of those components of a unitarizable $\mathfrak{g}$-supermodule that are invariant under continuous deformations. Following the physics literature [98, 118], we refer to any invariant count of short supermodules as a *KMMR index*. It has been shown that such indices can be extracted from the *superconformal index*, defined as a regularized Witten index [138] associated to any choice of $Q \in \mathcal{Y}$. In this chapter, we formalize both notions and prove their equivalence for general unitarizable $\mathfrak{sl}(m|n)$-supermodules. Additionally, we demonstrate that the formal superdimension, introduced in Section 10.1, also qualifies as an index under this definition, and clarify its connection to both the KMMR and Witten indices.

This chapter is an adaptation of an earlier version available on arXiv (see [126]), and represents joint research with Johannes Walcher.

12.1. KMMR indices

For simplicity, we restrict our attention to unitarizable $\mathfrak{g}$-supermodules that decompose entirely as a countable direct sum of simple $\mathfrak{g}$-supermodules; see Section 11.1. To define a "counting index", we additionally assume that only finitely many atypical simple $\mathfrak{g}$-supermodules appear in the decomposition. We denote this category by $(\mathfrak{g}, \omega)$-$\mathbf{usmod}'$, and, as before, omit the inner product $\langle \cdot, \cdot \rangle$ and the structure map $\mathfrak{g} \to \mathrm{End}(M)$ in the notation for its objects $M \in (\mathfrak{g}, \omega)$-$\mathbf{usmod}'$.

We will also correspondingly restrict the notion of a "continuous family of unitarizable $\mathfrak{g}$-supermodules" to mean a continuous map $T : \mathbb{H} \to \mathrm{Hom}'(\mathfrak{g}^{\mathbb{R}}, \mathfrak{u}(\mathcal{H}))$ into the space of ω-unitarizable $\mathfrak{g}$-supermodule structures on a fixed Hilbert space, which decompose into a direct sum of simple supermodules, with only finitely many atypical ones. The topology should be sufficiently fine to ensure continuity of the fragmentation process described in Section 11.3. It is worth noting that, in such a topology, $\mathrm{Hom}'(\mathfrak{g}^{\mathbb{R}}, \mathfrak{u}(\mathcal{H}))$ is not closed in $\mathrm{Hom}(\mathfrak{g}^{\mathbb{R}}, \mathfrak{u}(\mathcal{H}))$. There is a natural tautological map $\mathrm{Hom}'(\mathfrak{g}^{\mathbb{R}}, \mathfrak{u}(\mathcal{H})) \to (\mathfrak{g}, \omega)$-$\mathbf{usmod}'$, which we will denote, with a slight abuse of notation, by $(\rho, \mathcal{H}) \mapsto \mathcal{H}$.

Definition 12.1.1. A *Kinney–Maldacena–Minwalla–Raju index* [81], or *KMMR index* for short, is a map $I : (\mathfrak{g}, \omega)$-$\mathbf{usmod}' \to \mathbb{Z}$, $M \mapsto I(M)$ that is

a) additive, *i.e.*, $I(M_1 \oplus M_2) = I(M_1) + I(M_2)$, and

b) such that the induced map $\mathrm{Hom}'(\mathfrak{g}^{\mathbb{R}}, \mathfrak{u}(\mathcal{H})) \to \mathbb{Z}$, $(\rho, \mathcal{H}) \mapsto I(\mathcal{H})$ is continuous.

Lemma 12.1.2. *An additive map $I : (\mathfrak{g}, \omega)$-**usmod**$' \to \mathbb{Z}$ is specified uniquely by its values on simple unitarizable supermodules $L(\Lambda)$ for all $\Lambda \in \Gamma$. It is a KMMR index if and only if*

a) $I(K(\Lambda)) = 0$ *for all typical weights* $\Lambda \in \mathcal{C}$.

b) For all weights $\Lambda_0 \in \partial\mathcal{C}$ at the unitarity bound, $\sum_{i=1}^{M} I(L(\Lambda_i)) = 0$, where $\operatorname{gr} K(\Lambda_0) = \bigoplus_{i=1}^{M} L(\Lambda_i)$ is the fragmentation of the Kac supermodule.

Proof. Existence and uniqueness are obvious. For a), it suffices to notice that if $\Lambda_* \in \mathcal{C}$ is any typical weight, we have

$$I\Big(\bigoplus_{i=1}^{\infty} K(\Lambda_*)\Big) = I\Big(\bigoplus_{i=2}^{\infty} K(\Lambda_*)\Big) + I(K(\Lambda_*)) = I\Big(\bigoplus_{i=1}^{\infty} K(\Lambda_*)\Big) + I(K(\Lambda_*)).$$

Statement b) is equivalent to the continuity of the fragmentation process described in Section 11.3, see Equations (11.3.1) and (11.3.2). □

Lemma 12.1.3. *The set of all KMMR indices forms naturally a finitely generated free $\mathbb{Z}$-module. We denote it by $\mathcal{I}$, and allow extending scalars as*

$$\mathcal{I}_{\mathbb{Q}} := \mathcal{I} \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad \mathcal{I}_{\mathbb{R}} := \mathcal{I} \otimes_{\mathbb{Z}} \mathbb{R}.$$

Remark 12.1.4. In situations such as those considered in [81], only supermodules that respect spin-statistics are physically relevant. This restriction can accordingly reduce the set $\mathcal{I}$.

This identification will be convenient for the comparison with supertraces and superdimensions, to which we now turn.

12.2. The character-valued Witten index

For any element Q of the self-commuting variety $\mathcal{Y}$ acting on a fixed (non-trivial) unitarizable $\mathfrak{g}$-supermodule $\mathcal{H} \in (\mathfrak{g}, \omega)$-**usmod**$'$, its adjoint with respect to the Hermitian inner product $\langle \cdot, \cdot \rangle$ is related to its conjugate under the anti-involution ω defining the real form $\mathfrak{su}(p, q|n)$ of $\mathfrak{g}$ via $Q^\dagger = \omega(Q)$ (see Section 6.1). As a consequence, $x := i(Q + Q^\dagger)$ satisfies $\omega(x) = -x$ and $c := x^2 := \frac{1}{2}[x, x]$ is semisimple on account of $\omega(c) = c^\dagger = -c$. In the notation of Section 9.1.3, $x \in (\mathcal{Y}^{\text{hom}})^\omega$, with $\operatorname{rk}(x) = \operatorname{rk}(Q)$.

Lemma 12.2.1. *Let $\mathcal{H}$ be a unitarizable $\mathfrak{g}$-supermodule, $Q \in \mathcal{Y}$. Set $\Xi = -c = [Q, Q^\dagger]$.*

a) Ξ is a positive operator, i.e., *$\langle v, \Xi v \rangle \geq 0$ for all $v \in \mathcal{H}$.*

b) Ξ is self-adjoint. In particular, $\mathcal{H}$ decomposes completely in eigenspaces for Ξ,

$$\mathcal{H} = \bigoplus_{\xi} \mathcal{H}(\xi), \qquad \mathcal{H}(\xi) := \{v \in \mathcal{H} : \Xi v = \xi v\},$$

and each eigenvalue ξ is a non-negative real number.

c) There exists a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ with $\Xi \in \mathfrak{t}$, and for any such Cartan subalgebra, the $\mathfrak{t}$-weight spaces of $\mathcal{H}$ are eigenspaces of Ξ.

Proof. The contravariance of the Hermitian form implies the following for all $v \in \mathcal{H}$:

$$\langle v, \Xi v\rangle = \langle v, [Q, Q^\dagger]v\rangle = \langle v, QQ^\dagger v\rangle + \langle v, Q^\dagger Q v\rangle = \langle Qv, Qv\rangle + \langle Q^\dagger v, Q^\dagger v\rangle \geq 0,$$

where we used $(Q^\dagger)^\dagger = Q$, and the last inequality follows from the positive definiteness of the Hermitian form. This implies a). b) follows from $\omega(c) = -c$, together with the assumption that $\mathcal{H}$ decomposes discretely under $\mathfrak{g}$ and the spectral theorem. c) is then obvious. □

In physics, the positivity of Ξ is referred to as a *BPS bound associated to the supercharge* Q, and one expects to study the resulting *BPS states* (elements of $\mathcal{H}(0)$; see Section 11.2) by considering the supertrace of the operator $e^{-\beta\Xi}$ for real $\beta > 0$ [138]. The exponential operator is formally defined by the power series

$$e^{-\beta\Xi}v := \sum_{k=0}^{\infty} \frac{(-\beta)^k}{k!}\Xi^k v, \qquad v \in \mathcal{H}.$$

This expression is well-defined by (5.2.1), since Ξ is an even operator and $\mathcal{H}$ decomposes completely into unitarizable simple highest weight $\mathfrak{g}_{\bar{0}}$-supermodules.

The supertrace $\mathrm{str}_{\mathcal{H}}(e^{-\beta\Xi})$, however, is not necessarily well-defined. A mathematically clever — but physically naive — assumption would be that $e^{-\beta\Xi}$ is of trace class (it is clearly bounded), which would imply convergence of the supertrace. However, trace-classness requires that the eigenspaces of Ξ be finite-dimensional, an assumption that is generally too strong, although it may hold in special situations. Instead, to resolve the issue of the infinite degeneracy of $\mathcal{H}(0)$, one introduces a refined invariant: the *character-valued Witten index*. To clarify this notion, let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\Xi \in \mathfrak{t}$. From the discussion of the ω-compatible Duflo–Serganova functor in Section 9.1.3, recall that if $\mathrm{rk}(Q) = k$, then $\mathfrak{g}_x := DS_x(\mathfrak{g}) \cong \mathfrak{sl}(m-k \mid n-k)$ with real form $\mathfrak{g}_x^{\omega_x} \cong \mathfrak{su}(p-r, q-s \mid n-k)$, for some integers $r \leq p$, $s \leq q$ with $r + s = k$.

Viewing $\mathfrak{g}_x$ as a Lie subsuperalgebra of $\mathfrak{g}$, the Cartan subalgebra $\mathfrak{t}$ induces a Cartan subalgebra $\mathfrak{t}_x \subset \mathfrak{g}_x$. Let $T_x \subset G_{\bar{0}}$ denote the analytic complex Lie subgroup corresponding to $\mathfrak{t}_x$. We denote by $\mathfrak{t}_x^{reg}$ the set of regular elements in $\mathfrak{t}_x$, and by T_x^{reg} the set of regular elements in T_x. Note that $T_x^{reg} \subset T_x$ is open and dense. In the physics literature, linear coordinates on $\mathfrak{t}_x$ are often referred to as *fugacities*. We also fix a positive root system Δ_x^+ for $\mathfrak{g}_x$.

Proposition 12.2.2. *Let V be a unitarizable highest weight $\mathfrak{g}_x$-supermodule. Assume $X \in \mathfrak{t}_x^{reg}$ satisfies $\alpha(X) > 0$ for all $\alpha \in \Delta_x^+$. Then e^X is trace class and*

$$\mathrm{str}_V\, e^X \in \mathbb{R}.$$

Proof. Let $\Lambda \in \mathfrak{t}_x^*$ denote the highest weight of V. We consider the triangular decomposition $\mathfrak{g}_x = \mathfrak{n}_x^- \oplus \mathfrak{t}_x \oplus \mathfrak{n}_x^+$, where $\mathfrak{n}_x^\pm := \sum_{\alpha \in \Delta_x^\pm} \mathfrak{g}_x^\alpha$. Then, as a $\mathfrak{t}_x$-module, the supermodule V is a quotient of the $\mathfrak{t}_x$-module

$$\mathfrak{U}(\mathfrak{n}_x^-) \otimes \mathbb{C}_\Lambda \cong \bigwedge(\mathfrak{n}_{x,\bar{0}}^-) \otimes S(\mathfrak{n}_{x,\bar{1}}^-) \otimes \mathbb{C}_\Lambda \cong S(\mathfrak{n}_x^-) \otimes \mathbb{C}_\Lambda.$$

Here, note that $S(\mathfrak{n}_x^-)$ is the symmetric superalgebra over the super vector space $\mathfrak{n}_x^-$. Now, it is enough to show that e^X is trace class on $\mathfrak{U}(\mathfrak{n}_x^-) \otimes \mathbb{C}_\Lambda$, as quotients have smaller multiplicities. For that, we consider the following decomposition of the $\mathfrak{t}_x$-module $S(\mathfrak{n}_x^-)$:

$$S(\mathfrak{n}_x^-) \cong \bigotimes_{\alpha \in \Delta_x^+} S(\mathfrak{g}^{-\alpha}) \cong \bigotimes_{\alpha \in \Delta_x^+} S(\mathbb{C}_{-\alpha}).$$

The trace of e^X on any $S(\mathbb{C}_{-\alpha})$ is

$$\operatorname{tr}_{S(\mathbb{C}_{-\alpha})} e^X = \begin{cases} \sum_{k=0}^{\infty} e^{-k\alpha(X)} = \frac{1}{1-e^{-\alpha(X)}}, & \alpha \in \Delta_{x,\bar{0}}, \\ 1 + e^{-\alpha(X)}, & \alpha \in \Delta_{x,\bar{1}}. \end{cases}$$

We conclude

$$\operatorname{tr}_{S(\mathfrak{n}_x^-)\otimes \mathbb{C}_\Lambda} e^X = e^{\Lambda(X)} \prod_{\alpha \in \Delta_{x,\bar{1}}^+} (1 + e^{-\alpha(X)}) \prod_{\alpha \in \Delta_{x,\bar{0}}} \frac{1}{1 - e^{-\alpha(X)}} < \infty,$$

i.e., e^X is trace class on V. With respect to the weight space decomposition $V = \bigoplus_{\lambda \in P_V} V^\lambda$ for some countable set P_V, the supertrace

$$\operatorname{str}_V e^X = (-1)^V e^{\Lambda(X)} \sum_{\lambda \in P_V} (-1)^{\lambda - \Lambda} m(\lambda) e^{-\Lambda(X) + \lambda(X)},$$

where $m(\lambda) = \dim V^\lambda$, is dominated by $\operatorname{tr}_V e^X$, and real. □

Proposition 12.2.2 defines, for any unitarizable highest weight $\mathfrak{g}_x$-supermodule V, a conjugation-invariant continuous function on the subset

$$T_x^{reg,+} := \{e^X \in T_x^{reg} \mid \alpha(X) > 0 \text{ for all } \alpha \in \Delta_x^+\},$$

given by

$$\chi_V^x : T_x^{reg,+} \to \mathbb{R}, \qquad \chi_V^x(e^X) := \operatorname{str}_V(e^X),$$

which we refer to as the *supercharacter* of V. The set of all such supercharacters forms a ring, called the *supercharacter ring* and denoted by $X^*(T_x^{reg,+})$. This motivates the following definition.

Definition 12.2.3. Let M be a unitarizable highest weight $\mathfrak{g}$-supermodule, and let Q be an element of the self-commuting variety. Using the decomposition $DS_x(M) = \bigoplus_i V_i$ into finitely many $\mathfrak{g}_x$-supermodules (see Lemma 9.1.17), we define the Q-*Witten index* of M as the $\mathfrak{g}_x$-supercharacter

$$I_M^W(Q, \cdot) := \sum_i \chi_{V_i}^x(\cdot) = \operatorname{str}_{DS_x(M)}(\cdot) \in X^*(T_x^{reg,+}).$$

The fact that the Witten index detects short (or protected) unitarizable supermodules is reflected in the following statement.

Lemma 12.2.4. *Let M be a unitarizable highest weight $\mathfrak{g}$-supermodule with highest weight Λ, and $Q \in \mathcal{Y}$. Assume $\operatorname{at}(\Lambda) < \operatorname{rk}(Q)$. Then $I_M^W(Q) = 0$. In particular, if $\Lambda \in \mathcal{C}$, then $I_M^W(Q) = 0$ for any $Q \in \mathcal{Y}$.*

Proof. This follows at once from Corollary 9.1.7. □

Lemma 12.2.4 allows us to extend the Q-Witten index additively to any unitarizable supermodule $\mathcal{H} \in (\mathfrak{g}, \omega)$-**usmod**′ with only finitely many atypical simple components, in a well-defined manner. This extension permits a direct comparison with the formulation of the index as a supertrace over the entire supermodule, a perspective commonly adopted in the physics literature — for instance, in the case of $\mathfrak{psu}(2, 2|4)$ as discussed in [81].

Proposition 12.2.5. *Let $\mathcal{H} \in (\mathfrak{g}, \omega)$-**usmod**′, and $Q \in \mathcal{Y}$. Then the Q-*Witten index *of $\mathcal{H}$ satisfies*

$$I^W_{\mathcal{H}}(Q, X) = \mathrm{str}_{\mathcal{H}}\, e^{-\beta\Xi+X}$$

for any $X \in \mathfrak{t}^{reg,+}_x = \{X \in \mathfrak{t}^{reg}_x : \alpha(X) > 0\ \forall\alpha \in \Delta^+_x\}$, and any positive β.

Proof. With respect to the decomposition in Lemma 12.2.1 (b), the supertrace vanishes outside of the zero eigenspace $\mathcal{H}(0) = \ker Q \cap \ker Q^\dagger$ of Ξ, which is identified with the DS cohomology $DS_x(\mathcal{H})$ according to Proposition 9.1.10. This follows from a standard argument that is akin to the Hodge decomposition, and mostly formal under our assumptions. Note that $[X, \Xi] = 0$. On $\mathcal{H}(0)$, the supertrace is well-defined by Lemma 12.2.2, and equal to the Witten index by definition. Note that in particular, the supertrace is independent of β as advertised. □

It is interesting to observe that the supertrace formula for the Witten index could be extended to a larger class of unitarizable $\mathfrak{g}$-supermodules, provided the sum of supercharacters over atypical components converges in the appropriate topology. We finish with the equivalence between the Q-Witten index and the KMMR index.

Proposition 12.2.6. *For any $Q \in \mathcal{Y}$, $X \in \mathfrak{t}^{reg,+}_x$, the assignment $(\mathfrak{g}, \omega)$-**usmod**′ $\to \mathbb{R}$, $M \mapsto I^W_M(Q, X)$ is a real KMMR index.*

Proof. The vanishing on typical components following from Lemma 12.2.4, we explain "continuity in the $\mathfrak{g}$-supermodule structure on $\mathcal{H}$", *i.e.,* of the Witten index evaluated at $X \in \mathfrak{t}^{reg,+}_x$, viewed as a function

$$I^W_{\mathcal{H}}(Q, X) : \mathrm{Hom}'(\mathfrak{g}^{\mathbb{R}}, \mathfrak{u}(\mathcal{H})) \to \mathbb{R}$$

Topologically, $\mathrm{Hom}'(\mathfrak{g}^{\mathbb{R}}, \mathfrak{u}(\mathcal{H}))$ is $(\mathfrak{h}^*)^{\mathbb{Z}_+}/S_\infty$, while $\mathrm{Hom}'(\mathfrak{g}^{\mathbb{R}}_x, \mathfrak{u}(DS_x(\mathcal{H}))) \cong (\mathfrak{t}^*_x)^{\mathbb{Z}_+}/S_\infty$ if $\Xi \in \mathfrak{t}$ (see Section 11.1). Focusing on a single fragmentation process starting from $\Lambda \in \mathcal{C}$, we have a convergent sum

$$\mathrm{str}_{K(\Lambda)}\, e^{-\beta\Xi+X} = \sum_{\lambda\in P_{K(\Lambda)}\subset\mathfrak{h}^*} (-1)^{\lambda-\Lambda} m(\lambda) e^{\lambda(-\beta\Xi+X)}.$$

This is clearly continuous on $\mathrm{Hom}'(\mathfrak{g}^{\mathbb{R}}, \mathfrak{u}(K(\Lambda))$, where $K(\Lambda)$ is viewed as a fixed Hilbert space (*cf.* Section 11.3). This finishes the proof. □

Theorem 12.2.7. *Any KMMR index is a real linear combination of Witten indices.*

Proof. Let $I : (\mathfrak{g}, \omega)$-**usmod**′ $\to \mathbb{Z}$ denote a KMMR index. By our standing assumptions and the additivity of KMMR and Witten indices, it suffices to prove that for any unitarizable simple $\mathfrak{g}$-supermodule $L(\Lambda)$ with $I(L(\Lambda)) \neq 0$, there exists a Q-Witten index such that

$$I(L(\Lambda)) = c_{X,\Lambda} \cdot I^W_{L(\Lambda)}(Q, X).$$

for some $X \in \mathfrak{t}_x^{reg,+}$, $c_{X,\Lambda} \in \mathbb{R}$. By Lemma 12.1.2, we know that Λ is atypical; that is, there exists an odd root $\alpha \in \Delta_1^+$ such that $(\Lambda + \rho, \alpha) = 0$. Define Q to be an associated root vector, and set $x := i(Q + Q^\dagger)$ so that $\Xi = -\frac{1}{2}[x, x] \in \mathfrak{h}$. Then $DS_x(L(\Lambda))$ decomposes into either one or two unitarizable $\mathfrak{g}_x$-supermodules (see Theorem 9.1.16).

The claim follows if we can show that $I^W_{L(\Lambda)}(Q, X) \neq 0$ for at least one $X \in \mathfrak{t}_x^{reg,+}$. However, $I^W_{L(\Lambda)}(Q, \cdot)$ is either the supercharacter of a nontrivial unitarizable simple $\mathfrak{g}_x$-supermodule, or the sum of two nontrivial and non-equivalent unitarizable simple $\mathfrak{g}_x$-supermodules. In either case, the supercharacter is nonzero. This completes the proof. □

12.3. Formal superdimension and Witten index

In this section, we establish the connection between the Witten index and the formal superdimension for (relative) holomorphic discrete series $\mathfrak{g}$-supermodules in the case where $m, n \geq 3$ (see Lemma 11.3.4 and Chapter 10). To begin, we record that the formal superdimension $\mathrm{sdim}(\cdot)$ also serves as a KMMR index on the subspace of all relative holomorphic discrete series $\mathfrak{g}$-supermodules. This gives rise to a new KMMR index that, to our knowledge, has not previously appeared in the literature.

Proposition 12.3.1. *If $m, n \geq 3$, the formal superdimension is an element of $\mathcal{I}_{\mathbb{R}}$ on the subspace of relative holomorphic discrete series $\mathfrak{g}$-supermodules.*

Proof. By Section 10.2, the superdimension is trivial on all unitarizable simple $\mathfrak{g}$-supermodules except the maximally atypical ones. These do not belong to $\overline{\mathcal{C}}$ by Corollary 11.3.4 and are isolated. Consequently, $\mathrm{sdim}(\cdot)$ is continuous on the subspace of relative holomorphic discrete series $\mathfrak{g}$-supermodules. □

Next, one may motivate the search for a relation between the Q-Witten index and the formal superdimension from the formula (Proposition 12.2.5)

$$I^W_{\mathcal{H}}(Q, X) = \mathrm{str}_{\mathcal{H}}\, e^{-\beta\Xi + X}, \qquad (\text{where } Q \in \mathcal{Y},\ \Xi = [Q, Q^\dagger],\ X \in \mathfrak{t}_x^{reg,+}),$$

which bears a strong resemblance to the Weyl character formula for finite-dimensional representations of Lie groups, where taking the limit $X \to 0$ recovers the dimension of the representation. For infinite-dimensional representations, this will work best in the framework of Harish-Chandra characters and L-packets, as reviewed in Section 5.4.

If M is a relative holomorphic discrete series $\mathfrak{g}$-supermodule with highest weight Λ, and Q, x as before, by Lemma 9.1.17, we can decompose its DS-twist

$$DS_x(M) = \bigoplus_i L_x(\Lambda_i) \qquad \Lambda_i \in \mathfrak{t}_x^*$$

into finitely many relative holomorphic discrete series $\mathfrak{g}_x$-supermodules ($L_x(\Lambda_i) = V_i$ in the notation of Section 12.2). It turn, we write

$$L_x(\Lambda_i)_{\mathrm{ev}} = \bigoplus_j L_{x,\bar{0}}(\Lambda_{i;j})$$

for the decomposition of the $L_x(\Lambda_i)$ under $\mathfrak{g}_{x,\bar{0}}$. If $X \in \mathfrak{t}_x^{reg,+}$, by Proposition 12.2.2 the operator e^X is trace class on any $L_x(\Lambda_i)$, and hence on any $L_{x,\bar{0}}(\Lambda_{i;j})$, such that we can express the trace of e^X on any $L_{x,\bar{0}}(\Lambda_{i;j})$ by [101, Lemma 1.5]:

$$\operatorname{tr}_{L_{x,\bar{0}}(\Lambda_{i;j})} e^X = d(\pi_{\Lambda_{i;j}}) \int_{Z_x \backslash \tilde{G}^{\mathbb{R}}_{x,\bar{0}}} \langle \pi_{\Lambda_{i;j}}(g^{-1})v, e^X \pi_{\Lambda_{i;j}}(g^{-1})v \rangle \; \mathrm{d}\mu(Z_x g)$$

for some fixed element $v \in L_{x,\bar{0}}(\Lambda_{i;j})$ with $\|v\| = 1$. Then, setting

$$c(X; \Lambda_{i;j}) := \int_{Z_x \backslash \tilde{G}^{\mathbb{R}}_{x,\bar{0}}} \langle \pi_{\Lambda_{i;j}}(g^{-1})v, e^X \pi_{\Lambda_{i;j}}(g^{-1})v \rangle \; \mathrm{d}\mu(Z_x g),$$

we conclude that the Witten index of M is the formal dimension of the $\mathfrak{g}_{x,\bar{0}}$-constituents of $DS_x(M)$ with summands weighted by the $c(X; \Lambda_{i;j})$.

Lemma 12.3.2. *The Q-Witten index of $M = L(\Lambda)$ is*

$$I^W_M(Q, X) = \sum_{i,j} (-1)^{\Lambda - \Lambda_{i;j}} d(\pi_{\Lambda_{i;j}}) c(X; \Lambda_{i;j}).$$

This allows in principle an analytic study of the Q-Witten index in the limit $X \to 0$ by evaluating $c(X; \Lambda_{i;j})$. Alternatively, returning to

$$I^W_M(Q, X) = \sum_{i,j} (-1)^{\Lambda - \Lambda_{i;j}} \operatorname{tr}_{L_{0,x}(\Lambda_{i;j})} e^X \tag{12.3.1}$$

we may interpret $\operatorname{tr}_{L_{0,x}(\Lambda_{i;j})} e^X$ (in a distributional sense) as the Harish-Chandra character of $\pi_{\Lambda_{i;j}}$, based on the fact that $e^X \in T'^{reg}_x$. To take the limit $X \to 0$, we construct for each $\mathfrak{g}_{x,\bar{0}}$-constituent the associated L-packet by summing over the Weyl group orbit. Writing

$$\widetilde{\Theta}_{DS_x(M)} := \sum_{i,j} (-1)^{\Lambda - \Lambda_{i;j}} \sum_{w \in W_x / W_{x,c}} \Theta_{\pi_{w\Lambda_{i;j}}}$$

and

$$\tilde{I}^W_M(Q, X) := \widetilde{\Theta}_{DS_x(M)}(e^X)$$

in combination with Proposition 5.4.9, the definition of the superdimension and Equation (12.3.1), we obtain our final result.

Theorem 12.3.3. *Let M be a holomorphic discrete series $\mathfrak{g}$-supermodule, Q, x as above. Then,*

$$\operatorname{sdim}(DS_x(M)) = \lim_{X \to 0} \tilde{I}^W_M(Q, X).$$

Part V.

Addendum

13. Cubic Dirac operators and Dirac cohomology for basic classical Lie superalgebras

This chapter is an adaptation of an earlier version available on arXiv (see [106]), and represents joint research with Simone Noja and Raphael Senghaas, with equal contributions from all three authors.

In this chapter, let $(\mathfrak{g} := \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, [\cdot,\cdot])$ denote a basic classical Lie superalgebra with non-degenerate, invariant, and consistent bilinear form $(\cdot,\cdot)$ (see Section 2.1.3). Kac established that the complete list includes all simple Lie algebras along with the following types of Lie superalgebras [76]:

$$A(m|n), \quad B(m|n), \quad C(n), \quad D(m|n), \quad F(4), \quad G(3), \quad D(2,1;\alpha).$$

The general linear Lie superalgebra $\mathfrak{gl}(m|n)$ for $m, n \geq 1$ is also considered basic classical. Throughout, we assume $\mathfrak{g}_{\bar{1}} \neq 0$ and $\alpha \in \mathbb{R}$ for $D(2,1;\alpha)$, ensuring that $\mathfrak{g}$ is also a contragredient Lie superalgebra. For the basic classical Lie superalgebras $A(m|n)$ with $m \neq n$, $B(m|n)$, $C(n+1)$, $D(m|n)$ with $m \neq n+1$, $F(4)$, and $G(3)$, we use the *Killing form* as the non-degenerate, invariant, and consistent bilinear form $(\cdot,\cdot)$, given by

$$(x,y) := \mathrm{str}(\mathrm{ad}_x \circ \mathrm{ad}_y), \qquad x, y \in \mathfrak{g},$$

where $\mathrm{ad}_z(\cdot) := [z,\cdot]$ denotes the adjoint representation, and $\mathrm{str}(\cdot)$ denotes the supertrace. For the remaining basic classical Lie superalgebras, the Killing form vanishes identically, and an alternative form can be constructed in an ad hoc manner [76]. We choose the form constructed in [99, Section 5.4], and for simplicity, we refer to this form as the Killing form.

Furthermore, let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and denote the set of roots by $\Delta := \Delta(\mathfrak{g},\mathfrak{h})$, so that $\mathfrak{g}$ has the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^{\alpha}$$

with corresponding root spaces $\mathfrak{g}^{\alpha} := \{X \in \mathfrak{g} : [H,X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$. Note that $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}$, since $\mathfrak{g}$ is basic classical. Moreover, we fix some positive system Δ^+, and associated *Weyl vector* ρ. Furthermore, $\mathfrak{g}$ has a *triangular decomposition*

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \qquad \mathfrak{n}^{\pm} := \sum_{\alpha \in \Delta^+} \mathfrak{g}^{\pm\alpha}$$

such that the associated *Borel subalgebra* is $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$.

Finally, recall that the *Weyl group* $W^{\mathfrak{g}}$ of $\mathfrak{g}$ is defined to be the Weyl group of the underlying Lie algebra $\mathfrak{g}_{\bar{0}}$.

13.1. Preliminiaries

13.1.1. Parabolic subalgebras and parabolic induction

There are essentially two ways to define parabolic subalgebras of $\mathfrak{g}$, which we refer to as the classical approach via a parabolic set of roots [7, 39], and the hyperplane approach [30]. However, the two approaches are not equivalent, and the latter appears more natural [29]. We follow the approach in [30].

Let X be the finite-dimensional real vector space $X := \mathbb{R} \otimes_{\mathbb{Z}} Q$, where Q is the abelian group generated by Δ.

Definition 13.1.1. A partition $\Delta = \Delta_T^- \sqcup \Delta_T^0 \sqcup \Delta_T^+$ of the set of roots Δ is called a *triangular decomposition* T if there exists a functional $l : X \to \mathbb{Z}$ such that

$$\Delta_T^0 = \ker l \cap \Delta, \qquad \Delta_T^{\pm} = \{\alpha \in \Delta : l(\alpha) \gtrless 0\}.$$

For any triangular decomposition T of Δ, we obtain a triangular decomposition of $\mathfrak{g}$, that is, a decomposition $\mathfrak{g} = \mathfrak{g}_T^+ \oplus \mathfrak{g}_T^0 \oplus \mathfrak{g}_T^-$, where

$$\mathfrak{g}_T^+ := \bigoplus_{\alpha \in \Delta_T^+} \mathfrak{g}^\alpha, \qquad \mathfrak{g}_T^0 := \bigoplus_{\alpha \in \Delta_T^0} \mathfrak{g}^\alpha, \qquad \mathfrak{g}_T^- := \bigoplus_{\alpha \in \Delta_T^-} \mathfrak{g}^\alpha.$$

The subset $P_T = \Delta_T^0 \sqcup \Delta_T^+$ is called a *principal parabolic subset*. The following lemma is straightforward.

Lemma 13.1.2 ([29]). *Every principal parabolic subset $P \subset \Delta$ is a* parabolic subset, *i.e., the following conditions hold:*

a) $\Delta = P \sqcup -P$, *and*

b) $\alpha, \beta \in P$ *with* $\alpha + \beta \in \Delta$ *implies* $\alpha + \beta \in P$.

This leads us to the definition of a parabolic subalgebra.

Definition 13.1.3. Let T be a triangular decomposition of Δ. The Lie subsuperalgebra $\mathfrak{p}_T := \mathfrak{g}_T^0 \oplus \mathfrak{g}_T^+$ is called a *parabolic subalgebra* of $\mathfrak{g}$.

Remark 13.1.4. For any triangular decomposition $\mathfrak{p}$, the space $\mathfrak{p}_T \cap \mathfrak{g}_{\bar{0}}$ is a parabolic subalgebra of $\mathfrak{g}_{\bar{0}}$ [30, Section 5].

A *root subalgebra* is a Lie subsuperalgebra $\mathfrak{q} \subset \mathfrak{g}$ such that

$$\mathfrak{q} = (\mathfrak{q} \cap \mathfrak{h}) \oplus (\bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha)$$

for some subset $\Sigma \subset \Delta$. In particular, for a given triangular decomposition T with principal parabolic subset P_T, the parabolic subalgebra is

$$\mathfrak{p}_T = \mathfrak{h} \oplus (\bigoplus_{\alpha \in P_T} \mathfrak{g}^\alpha).$$

Any parabolic subalgebra $\mathfrak{p}_T$ admits a Levi decomposition. For the parabolic set of roots $P_T \subset \Delta$, we define $L_T := P_T \cap (-P_T)$ as the *Levi component*, $U_T := P_T \setminus (-P_T)$ as the *nilradical*, and $P_T = L_T \sqcup U_T$ as the *Levi decomposition*. The associated root subalgebras

$$\mathfrak{l}_T := \mathfrak{h} \oplus (\bigoplus_{\alpha \in L_T} \mathfrak{g}^\alpha), \qquad \mathfrak{u}_T := \bigoplus_{\alpha \in U_T} \mathfrak{g}^\alpha,$$

are called the *Levi subalgebra* and *nilradical* of $\mathfrak{p}_T$. A direct calculation yields that $\mathfrak{u}_T$ is an ideal in $\mathfrak{p}_T$. The *Levi decomposition* of $\mathfrak{p}_T$ takes the form of the semidirect product $\mathfrak{p}_T := \mathfrak{l}_T \ltimes \mathfrak{u}_T$. If a parabolic subalgebra $\mathfrak{p}_T$ is fixed, we omit the subscript T and denote it simply by $\mathfrak{p}$.

Any parabolic subalgebra $\mathfrak{p}_T$ has an *opposite parabolic subalgebra* $\overline{\mathfrak{p}}_T$, such that $\mathfrak{g} = \mathfrak{p}_T + \overline{\mathfrak{p}}_T$. The opposite parabolic subalgebra is $\overline{\mathfrak{p}}_T = \mathfrak{l}_T \ltimes \overline{\mathfrak{u}}_T$, where $\overline{\mathfrak{u}}_T$ is the root subalgebra corresponding to $U_T^- := (-P_T) \setminus P_T$.

Furthermore, the Levi subalgebra has a proper root system $\Delta(\mathfrak{l}_T; \mathfrak{h})$, since $\mathfrak{h} \subset \mathfrak{l}$, which is a subset of Δ. We denote the associated Weyl group by $W^{\mathfrak{l}_T}$. The positive system Δ_T^+ induces a positive system for P_T, L_T and U_T by $P_T^+ := P_T \cap \Delta^+$, $L_T^+ := L_T \cap \Delta^+$ and $U_T^+ := U_T \cap \Delta^+$. We set $(U_T^+)_{\bar{0},\bar{1}}$ and $(L_T^+)_{\bar{0},\bar{1}}$ for the associated even and odd parts. Further, we define $\rho^{\mathfrak{l}_T} := \rho_{\bar{0}}^{\mathfrak{l}_T} - \rho_{\bar{1}}^{\mathfrak{l}_T}$ and $\rho^{\mathfrak{u}_T} := \rho_{\bar{0}}^{\mathfrak{u}_T} - \rho_{\bar{1}}^{\mathfrak{u}_T}$ for

$$\rho_{\bar{0},\bar{1}}^{\mathfrak{l}_T} := \frac{1}{2} \sum_{\alpha \in (L_T^+)_{\bar{0},\bar{1}}} \alpha, \qquad \rho_{\bar{0},\bar{1}}^{\mathfrak{u}_T} := \frac{1}{2} \sum_{\alpha \in (U_T^+)_{\bar{0},\bar{1}}} \alpha.$$

Finally, fix a parabolic subalgebra $\mathfrak{p} := \mathfrak{p}_T$ for some parabolic set P with Levi decomposition $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$ and opposite parabolic subalgebra $\overline{\mathfrak{p}} = \mathfrak{l} \ltimes \overline{\mathfrak{u}}$. Set $\mathfrak{s} := \mathfrak{u} \oplus \overline{\mathfrak{u}}$. By construction of $\mathfrak{l}$ and $\mathfrak{s}$, the space $\mathfrak{s}$ is the orthogonal complement of $\mathfrak{l}$ with respect to $(\cdot, \cdot)$, and we have a direct sum decomposition

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{s},$$

where we use $(\mathfrak{g}^\alpha, \mathfrak{g}^\beta) = 0$ unless $\alpha = -\beta \in \Delta$ (cf. Proposition 2.1.15). In particular, the restriction of $(\cdot, \cdot)$ to $\mathfrak{l}$ and $\mathfrak{s}$ remains non-degenerate. In the next section, we will show that $\mathfrak{g}$ is an example of a quadratic Lie superalgebra and introduce a cubic Dirac operator associated to this decomposition.

Parabolic induction

We fix a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$. We are interested in *weight supermodules*, that is, supermodules M where $\mathfrak{h}$ acts semisimply:

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M^\mu, \qquad M^\mu = \{m \in M : hm = \mu(h)m \text{ for every } h \in \mathfrak{h}\}.$$

The elements $\mu \in \mathfrak{h}^*$ with $M^\mu \neq \{0\}$ are called *weights* of M, while M^μ is called *weight space* associated to μ.

Let V be a weight $\mathfrak{l}$-supermodule. Via the projection $\mathfrak{p} \to \mathfrak{l}$, we naturally extend V to a $\mathfrak{p}$-supermodule, where the nilradical $\mathfrak{u}$ acts trivially on V. Conversely, given a non-trivial $\mathfrak{p}$-supermodule M, the space of $\mathfrak{u}$-invariants is non-zero and carries the structure of both a $\mathfrak{p}$- and $\mathfrak{l}$-supermodule.

Lemma 13.1.5. *Any simple weight $\mathfrak{l}$-supermodule V is a simple $\mathfrak{p}$-supermodule with the trivial action of $\mathfrak{u}$. Conversely, if V is a simple weight $\mathfrak{p}$-supermodule, then $\mathfrak{u}$ acts trivially on V, and V is a simple weight $\mathfrak{l}$-supermodule.*

Proof. Given a simple weight $\mathfrak{l}$-supermodule V, we can extend V to a simple weight $\mathfrak{p}$-supermodule by let $\mathfrak{u}$ acting trivially, since $\mathfrak{u} \subset \mathfrak{p}$ is an ideal.

Let V be a simple weight $\mathfrak{p}$-supermodule, and denote by T the triangular decomposition defining $\mathfrak{p}$ with linear map $l : X \to \mathbb{Z}$. Fix a weight μ of V and define two subsupermodules

$$U := \bigoplus_{\nu \in \mathfrak{h}^*,\ l(\nu) \geq l(\mu)} V^\nu, \qquad W := \bigoplus_{\nu \in \mathfrak{h}^*,\ l(\nu) > l(\mu)} V^\nu.$$

As μ is a weight of V, the subsupermodule U is non-trivial, while $W^{\mathfrak{g}}$ is proper. Since V is simple, we must have $U = V$ and $W = \{0\}$. Consequently, by induction, $\mathfrak{u}$ acts trivially on V, and V is a simple $\mathfrak{l}$-supermodule. □

For a simple weight $\mathfrak{l}$-supermodule V, considered equivalently as a simple weight $\mathfrak{p}$-supermodule, the *parabolically induced supermodule (or generalized Verma supermodule)* is defined to be

$$M_{\mathfrak{p}}(V) := \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} V.$$

The following proposition is standard.

Proposition 13.1.6. *The $\mathfrak{g}$-supermodule $M_{\mathfrak{p}}(V)$ has a unique maximal proper subsupermodule. In particular, $M_{\mathfrak{p}}(V)$ has a unique simple quotient $L_{\mathfrak{p}}(V)$.*

The simple weight $\mathfrak{g}$-supermodules $L_{\mathfrak{p}}(V)$ exhaust all simple weight $\mathfrak{g}$-supermodules. To establish this, we introduce the following notations. For a fixed triangular decomposition T of Δ, we write $\Delta^T = \Delta_0^T \sqcup \Delta_1^T$ for the set of roots of $\mathfrak{g}_T^0$. A triangular decomposition T of Δ is called *good*, if the following holds [30]:

a) The monoid generated by Δ_0^T is a group, denoted by Q_0^T.

b) For any $\beta \in \Delta_1^T$, there exists some $m > 0$ such that $m\beta \in Q_0^T$.

In this case, the Levi subalgebra $\mathfrak{l}_T = \mathfrak{g}_T^0$ is called *good*. The good triangular decompositions of $\mathfrak{g}$ were classified in [30, Section 7].

Let $\mathfrak{l}_T$ be a good Levi subalgebra. A weight $\mathfrak{l}_T$-supermodule V is called *cuspidal* if for any $\alpha \in \Delta_0^T$ the associated root vector e_α acts injectively on V.

Theorem 13.1.7 ([30, Theorem 6.1])**.** *Let M be a simple $\mathfrak{g}$-supermodule. Then there exists a parabolic subalgebra $\mathfrak{p}$ with good Levi subalgebra $\mathfrak{l}$, and a cuspidal $\mathfrak{l}$-supermodule V, such that*

$$M \cong L_{\mathfrak{p}}(V).$$

Parabolic category $\mathcal{O}^{\mathfrak{p}}$

Categorically, the parabolic induction can be naturally studied in the *parabolic BGG category* $\mathcal{O}^{\mathfrak{p}}$, which is a variant of the super BGG category $\mathcal{O}$, determined by a (fixed) parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$. Following [93], the category $\mathcal{O}^{\mathfrak{p}}$ is the full subcategory of **$\mathfrak{g}$-smod** whose objects are the $\mathfrak{g}$-supermodules satisfying the following three properties:

1. M is a finitely generated $\mathfrak{U}(\mathfrak{g})$-supermodule.

2. Viewed as a $\mathfrak{U}(\mathfrak{l}_{\bar{0}})$-module, M decomposes in a direct sum of finite-dimensional simple modules.

3. M is locally $\mathfrak{u}$-finite in the sense that $\dim(\mathfrak{U}(\mathfrak{u})v) < \infty$ for all $v \in M$.

This is an abelian subcategory of **$\mathfrak{g}$-smod** closed under the parity switching functor Π. For further categorical properties of $\mathcal{O}^{\mathfrak{p}}$, we refer to [93]. However, it is important for us to note that the simple objects in $\mathcal{O}^{\mathfrak{p}}$ are precisely given by $L_{\mathfrak{p}}(V)$, where V is a simple $\mathfrak{l}$-supermodule.

13.1.2. A note on real forms and Cartan automorphisms

To investigate aspects of unitarity of supermodules over $\mathfrak{g}$, it is necessary to consider real forms $\mathfrak{g}^{\mathbb{R}}$ of $\mathfrak{g}$ (see Section 2.1.4). For this purpose, we recall

$$\begin{aligned} \mathrm{aut}_{2,4}(\mathfrak{g}) &:= \{\theta \in \mathrm{aut}^{\mathbb{R}}_{2,4}(\mathfrak{g}) : \theta \text{ is } \mathbb{C}\text{-linear}\}, \\ \overline{\mathrm{aut}}_{2,s}(\mathfrak{g}) &:= \{\theta \in \mathrm{aut}^{\mathbb{R}}_{2,s}(\mathfrak{g}) : \theta \text{ is conjugate-linear}\}, \end{aligned}$$

such that real structures ϕ on $\mathfrak{g}$ are elements in $\overline{\mathrm{aut}}_{2,2}(\mathfrak{g})$, and the spaces of fixed points $\mathfrak{g}^{\phi}$ are *real forms*. Consequently, up to equivalence, there is a bijective correspondence [131, 116, 40, 20]

$$\{\text{real forms } \mathfrak{g}^{\mathbb{R}} \text{ of } \mathfrak{g}\} \leftrightarrow \{\theta \in \mathrm{aut}_{2,4}(\mathfrak{g})\}.$$

In what follows, we do not distinguish between isomorphic real forms, or equivalently, we do not distinguish between involutions on $\mathfrak{g}$ that are conjugate by $\mathfrak{g}$-automorphisms.

Later, we will use the fact that the real forms $\overline{\mathrm{aut}}_{2,2}(\mathfrak{g})$ of $\mathfrak{g}$ and the set $\mathrm{aut}_{2,4}(\mathfrak{g})$ are related.

Proposition 13.1.8 ([40])**.** *There exists a unique $\omega \in \overline{\mathrm{aut}}_{2,4}(\mathfrak{g})$ (up to inner automorphisms of $\mathfrak{g}$), a positive system Δ^+ and suitable root vectors $e_{\pm\alpha}$ for $\alpha \in \Delta^+$ such that*

$$\omega(e_{\pm\alpha}) = -e_{\mp\alpha} \ \forall\alpha \text{ even simple}, \quad \omega(e_{\pm\alpha}) = \pm e_{\mp\alpha} \ \forall\alpha \text{ odd simple}.$$

Moreover, the following two assertions hold:

a) ω induces a bijection

$$\overline{\mathrm{aut}}_{2,2}(\mathfrak{g}) \setminus \{\theta : \theta|_{\mathfrak{g}_{\bar{0}}} = \omega|_{\mathfrak{g}_{\bar{0}}}\} \to \mathrm{aut}_{2,4}(\mathfrak{g}), \qquad \theta \mapsto \omega^{-1} \circ \theta.$$

b) For the Killing form $(\cdot,\cdot)$ on $\mathfrak{g}$, we have $\overline{(X,Y)} = (\omega(X), \omega(Y))$ for all $X, Y \in \mathfrak{g}$, and $(\cdot, \omega(\cdot))$ is positive definite.

Remark 13.1.9. The positive system Δ^+ is the *distinguished positive system*, meaning that there is exactly one odd positive root that cannot be expressed as the sum of two other positive roots.

We now fix a real form $\mathfrak{g}^{\mathbb{R}}$ of a basic classical Lie superalgebras $\mathfrak{g}$, *i.e.*, $\mathfrak{g}^{\mathbb{R}}$ is the subspace of fixed points of some $\theta \in \mathrm{aut}_{2,4}(\mathfrak{g})$. We denote by $\sigma := \omega \circ \theta \in \overline{\mathrm{aut}}_{2,2}(\mathfrak{g})$ the associated conjugate-linear involution on $\mathfrak{g}$ (see Proposition 13.1.8 above). We say θ is a *Cartan automorphism* on $\mathfrak{g}$ or $\mathfrak{g}^{\mathbb{R}}$, if

$$B_{\theta}(\cdot,\cdot) := -(\cdot, \theta(\cdot)) \tag{13.1.1}$$

is an inner product on $\mathfrak{g}^{\mathbb{R}}$. Given $\theta \in \mathrm{aut}_{2,4}(\mathfrak{g})$, there exists a unique real form $\mathfrak{g}^{\mathbb{R}}$ such that θ restricts to a Cartan automorphism on $\mathfrak{g}^{\mathbb{R}}$. Conversely, any real form $\mathfrak{g}$ has a unique Cartan automorphisms θ [20, Theorem 1.1]. In the following, we may assume that θ associated to $\mathfrak{g}^{\mathbb{R}}$ is a Cartan automorphism.

13.1.3. Clifford superalgebras

Clifford and exterior superalgebras.

Fix a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$, and let $\mathfrak{g} := \mathfrak{l} \oplus \mathfrak{s}$ be the induced decomposition of $\mathfrak{g}$. Recall that $\mathfrak{s} = \mathfrak{u} \oplus \bar{\mathfrak{u}}$, where $\bar{\mathfrak{u}}$ is the nilradical of the opposite parabolic subalgebra $\bar{\mathfrak{p}} = \mathfrak{l} \ltimes \bar{\mathfrak{u}}$. Let $T(\mathfrak{s})$ denote the tensor algebra over the super vector space $\mathfrak{s}$, and let $I(\mathfrak{s})$ be the two-sided ideal generated by elements of the form

$$v \otimes w + (-1)^{p(v)p(w)} w \otimes v - 2(v,w)1_{T(\mathfrak{s})}$$

for any $v, w \in \mathfrak{s}$ and where $(\cdot,\cdot)$ denotes the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{s}$. The *Clifford superalgebra* is defined as the quotient $C(\mathfrak{s}) := T(\mathfrak{s})/I(\mathfrak{s})$. In other words, if we naturally identify $\mathfrak{s}$ as a subspace of $C(\mathfrak{s})$, the Clifford superalgebra is generated by $\mathfrak{s}$ with the relations

$$vw + (-1)^{p(v)p(w)} wv = 2(v,w)1_{T(\mathfrak{s})},$$

where vw represents the Clifford multiplication. The Clifford superalgebra $C(\mathfrak{s})$ naturally inherits a $\mathbb{Z}_2$-grading from the tensor superalgebra $T(\mathfrak{s})$. This arises from the natural $\mathbb{Z}\times\mathbb{Z}_2$-grading on $T(\mathfrak{s})$, where the degree of $v_1\otimes\cdots\otimes v_n$ is defined as $(n, p(v_1)+\cdots+p(v_n))$.

Additionally, the Clifford superalgebra is characterized by a universal property.

Lemma 13.1.10. *Let $\mathcal{A}$ be an associative superalgebra with unit $1_{\mathcal{A}}$. Assume there exists a morphism of super vector spaces $\phi : \mathfrak{s} \to \mathcal{A}$ with $\phi(v)\phi(w) + (-1)^{p(v)p(w)}\phi(w)\phi(v) = 2(v,w)1_{\mathcal{A}}$ for any $v, w \in \mathfrak{s}$. Then ϕ extends uniquely to a superalgebra morphism $\phi : C(\mathfrak{s}) \to \mathcal{A}$, denoted by the same symbol.*

The proof of the lemma is straightforward and will be omitted. As super vector spaces, the Clifford superalgebra and the exterior superalgebra are isomorphic under the Chevalley identification. The *exterior superalgebra* is $\bigwedge \mathfrak{s} := T(\mathfrak{s})/J(\mathfrak{s})$, where $J(\mathfrak{s})$ is the two-sided ideal generated by

$$v \otimes w + (-1)^{p(v)p(w)} w \otimes v$$

for any $v, w \in \mathfrak{s}$. In particular, under the natural embedding $\mathfrak{s} \hookrightarrow \bigwedge \mathfrak{s}$, the Clifford superalgebra is generated by $\mathfrak{s}$ with relations

$$v \wedge w + (-1)^{p(v)p(w)} w \wedge v = 0,$$

with $\wedge$ being the exterior multiplication.

On $\bigwedge \mathfrak{s}$, we have two natural operators. For any $v \in \mathfrak{s}$, let $\epsilon(v)$ denote the left exterior multiplication on $\bigwedge \mathfrak{s}$. Moreover, the derivation

$$\iota(v)(v_1 \otimes \ldots \otimes v_l) := \sum_{k=1}^{l} (-1)^{k-1}(-1)^{p(v)(p(v_1)+\ldots+p(v_{k-1}))}(v, v_k) v_1 \otimes \ldots \otimes \hat{v}_k \otimes \ldots \otimes v_l$$

on $T(\mathfrak{s})$ leaves $J(\mathfrak{s})$ invariant and descends to a derivation on $\bigwedge \mathfrak{s}$. We define $\gamma : \mathfrak{s} \to \mathrm{End}(\bigwedge \mathfrak{s})$ by $\gamma(v) := \epsilon(v) + \iota(v)$, which satisfies $\gamma(v)\gamma(w) + (-1)^{p(v)p(w)}\gamma(w)\gamma(v) = 2(v,w)$ for any $v, w \in \mathfrak{s}$. Thus, by the universal property of the Clifford superalgebra, this realizes $\bigwedge \mathfrak{s}$ as a $C(\mathfrak{s})$-supermodule, *i.e.*, $\gamma : C(\mathfrak{s}) \to \mathrm{End}(\bigwedge \mathfrak{s})$.

Theorem 13.1.11 ([79]). *The map $\eta : C(\mathfrak{s}) \to \bigwedge \mathfrak{s}$ with $\eta(v) := \gamma(v)1_{\bigwedge V}$ is a super vector space isomorphism. Moreover, the inverse map is given by the quantization map $\sum_n Q_n : \bigwedge \mathfrak{s} \to C(\mathfrak{s})$ with*

$$Q_n(v_1 \wedge \ldots \wedge v_n) := \frac{1}{n!} \sum_{\sigma \in S_n} p(\sigma; v_1, \ldots, v_n) v_{\sigma(1)} \ldots v_{\sigma(n)},$$

where

$$p(\sigma; v_1, \ldots, v_n) = \operatorname{sgn}(\sigma) \prod_{1 \leq i < j \leq n,\ \sigma^{-1}(i) > \sigma^{-1}(j)} (-1)^{p(v_i)p(v_j)}$$

Remark 13.1.12. If $v_1, \ldots, v_n$ span an isotropic subspace of $\mathfrak{s}$, we have that $v_{\sigma(1)} \ldots v_{\sigma(n)} = p(\sigma; v_1, \ldots, v_n) v_1 \ldots v_n$ and hence in this case

$$Q_n(v_1 \wedge \cdots \wedge v_n) = v_1 \ldots v_n.$$

We will now provide an explicit realization of the Clifford algebra $C(\mathfrak{s})$ which will be used in a later stage. To this end, we first fix a basis $\eta_1, \ldots, \eta_m$ of $\bar{\mathfrak{u}}_{\bar{0}}$ and a basis $x_1, \ldots, x_n$ of $\bar{\mathfrak{u}}_{\bar{1}}$. We shall denote by $\bigwedge \bar{\mathfrak{u}}$ the exterior superalgebra over $\bar{\mathfrak{u}}$ according to the previous definition, where we recall that

$$\eta_i \wedge \eta_j = -\eta_j \wedge \eta_i, \quad x_i \wedge x_j = x_j \wedge x_i, \quad x_i \wedge \eta_j = -\eta_j \wedge x_i.$$

We now define the $\mathbb{C}$-linear operators $\{\frac{\partial}{\partial x_i}\}$ and $\{\frac{\partial}{\partial \eta_j}\}$ acting on $\bar{\mathfrak{u}}$ as

$$\frac{\partial}{\partial x_i}(x_k) := -\delta_{ik}, \quad \frac{\partial}{\partial x_i}(\eta_l) := 0, \quad \frac{\partial}{\partial \eta_j}(x_k) := 0, \quad \frac{\partial}{\partial \eta_j}(\eta_l) := \delta_{jl}.$$

Note that the unexpected minus sign appearing in the above equation is justified by the choice $(\bar{u}_i, u_j) = \delta_{ij}$ together with the identification made. Further, upon identifying the above operators with the basis elements in $\mathfrak{u}$, *i.e.*, $\frac{\partial}{\partial \eta_j} \leftrightsquigarrow \frac{1}{2}\bar{\eta}_i$ and $\frac{\partial}{\partial x_i} \leftrightsquigarrow \frac{1}{2}\bar{x}_i$, there is a natural action of $\mathfrak{s} = \mathfrak{u} \oplus \bar{\mathfrak{u}}$ on $\bigwedge \bar{\mathfrak{u}}$, where $\bar{\mathfrak{u}}$ acts by multiplication operators $\eta_j \wedge \cdot$, $x_i \wedge \cdot$, and $\bar{\mathfrak{u}}$ acts by $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \eta_j}$ in the obvious manner.

Under the above identifications, using that $(\partial_{\eta_i}, \eta_j) = (\eta_j, \partial_{\eta_i}) = \delta_{ij}$ and $(\partial_{x_i}, x_j) = -(x_j, \partial_{x_i})$, a direct calculation yields the following lemma.

Lemma 13.1.13. *The Clifford superalgebra $C(\mathfrak{s})$ is isomorphic to the superalgebra generated by $\{x_i, \frac{\partial}{\partial x_j} : 1 \leq i, j \leq m\}$ and $\{\eta_i, \frac{\partial}{\partial \eta_j} : 1 \leq i, j \leq n\}$ with quadratic relations*

$$x_i x_j - x_j x_i = 0, \quad \eta_i \eta_j + \eta_j \eta_i = 0, \quad \frac{\partial}{\partial x_i} x_j - x_j \frac{\partial}{\partial x_i} = -\delta_{ij}, \quad \frac{\partial}{\partial \eta_i} \eta_j + \eta_j \frac{\partial}{\partial \eta_i} = \delta_{ij},$$
$$x_i \eta_j + \eta_j x_i = 0, \quad x_i \frac{\partial}{\partial \eta_j} + \frac{\partial}{\partial \eta_j} x_i = 0, \quad \frac{\partial}{\partial x_i} \eta_j + \eta_j \frac{\partial}{\partial x_i} = 0, \quad \frac{\partial}{\partial x_i} \frac{\partial}{\partial \eta_j} + \frac{\partial}{\partial \eta_j} \frac{\partial}{\partial x_i} = 0.$$

Embedding of $\mathfrak{l}$ into $C(\mathfrak{s})$.

Next, we aim at constructing an explicit embedding of $\mathfrak{l}$ into the Clifford algebra $C(\mathfrak{s})$. To this end, we recall that we fixed a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$ such that $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{s}$ with respect to $(\cdot, \cdot)$. The restriction of $(\cdot, \cdot)$ to $\mathfrak{l}$ and $\mathfrak{s}$, denoted by $(\cdot, \cdot)_{\mathfrak{l}}$ and

$(\cdot,\cdot)_{\mathfrak{s}}$ respectively, is still non-degenerate. Recall that the *orthosymplectic superalgebra* of $\mathfrak{s}$ is

$$\mathfrak{osp}(\mathfrak{s}) := \{T \in \mathrm{End}(\mathfrak{s}) : (T(v), w)_{\mathfrak{s}} + (-1)^{p(T)p(v)}(v, T(w))_{\mathfrak{s}} = 0 \text{ for all } v, w \in \mathfrak{s}\},$$

equipped with the natural supercommutator.

Lemma 13.1.14. *The adjoint action of $\mathfrak{l}$ on $\mathfrak{s}$ induces a morphism of superalgebras $\nu : \mathfrak{l} \to \mathfrak{osp}(\mathfrak{s})$.*

Proof. The natural adjoint action of $\mathfrak{l}$ on $\mathfrak{s}$ defines a representation $\nu : \mathfrak{l} \to \mathfrak{osp}(\mathfrak{s})$, since for any $X \in \mathfrak{l}$ and any $u, u' \in \mathfrak{s}$ we have

$$\begin{aligned}(\mathrm{ad}_X u, u') &= ([X,u], u') = (X, [u,u']) = -(-1)^{p(u)p(u')}(X, [u', u]) \\ &= -(-1)^{p(u)p(u')}([X,u'], u) = -(-1)^{p(u)p(u')+p(u)p([X,u'])}(u, \mathrm{ad}_X u') \\ &= -(-1)^{p(X)p(u)}(u, \mathrm{ad}_X u'),\end{aligned}$$

which concludes the verification. □

For the orthosymplectic $\mathbb{Z}_2$-graded representation $\nu : \mathfrak{l} \to \mathfrak{osp}(\mathfrak{s})$, we define the *moment map* $\mu : \mathfrak{s} \times \mathfrak{s} \to \mathfrak{l}$ to be the bilinear map given by

$$(x, \mu(v,w))_{\mathfrak{l}} = (\nu(x)v, w)_{\mathfrak{s}}$$

for all $v, w \in \mathfrak{s}$ and $x \in \mathfrak{l}$. The moment map is even and skew-supersymmetric, hence it descends to a map defined on $\bigwedge^2(\mathfrak{s})$. Post-composing with $\nu : \mathfrak{l} \to \mathfrak{osp}(\mathfrak{s})$, yields a map $\mu : \bigwedge^2(\mathfrak{s}) \to \mathfrak{osp}(\mathfrak{s})$, which we call the moment map associated to $\mathfrak{l}$, and denote by the same symbol as above with a mild notational abuse.

Proposition 13.1.15 ([95, Proposition 2.13])**.** *The moment map $\mu : \bigwedge^2(\mathfrak{s}) \to \mathfrak{osp}(\mathfrak{s})$ associated to $\mathfrak{l}$ satisfies*

$$\mu(x,y)(z) = (y,z)_{\mathfrak{s}} x - (-1)^{p(y)p(z)}(x,z)_{\mathfrak{s}} y$$

for all $x, y, z \in \mathfrak{s}$. Moreover, μ is an isomorphism of super vector spaces, and it satisfies for any $T \in \mathfrak{osp}(\mathfrak{s}) \subset \mathfrak{s} \otimes \mathfrak{s}^*$

$$\mu^{-1}(T) = \frac{1}{2}\sum_{i=1}^{2s} T(e_i^*) \wedge e_i, \quad Q_2(\mu^{-1}(T)) = \frac{1}{4}\sum_{i=1}^{2s}(T(e_i^*)e_i - (-1)^{p(e_i)p(T(e_i^*))} e_i T(e_i^*)),$$

where $\{e_i\}$ is a basis of $\mathfrak{s}$ with dual basis $\{e_i^*\}$ and $s = m + n$.

Note that in the previous proposition one looks at $\mathfrak{osp}(\mathfrak{s})$ as a certain subset of $\mathfrak{s} \otimes \mathfrak{s}^*$, and it makes sense to consider an action of $T \in \mathfrak{osp}(\mathfrak{s})$ on e_i^*, upon using $(\cdot,\cdot)_{\mathfrak{s}}$.
We now define $\nu_* : \mathfrak{l} \to \bigwedge^2(\mathfrak{s})$ as the composition of ν and the inverse of the moment map μ^{-1}: in the remainder of this section we will provide an explicit characterization of this map, that will be used later on in the paper.

Recall that $\mathfrak{s} = \mathfrak{s}_{\bar{0}} \oplus \mathfrak{s}_{\bar{1}}$ with $\mathfrak{s}_{\bar{0}} = \mathfrak{u}_{\bar{0}} \oplus \bar{\mathfrak{u}}_{\bar{0}}$ and $\mathfrak{s}_{\bar{1}} = \mathfrak{u}_{\bar{1}} \oplus \bar{\mathfrak{u}}_{\bar{1}}$. We let $b_1, \ldots, b_{s_0}$ be a basis of $\mathfrak{u}_{\bar{0}}$, and $\psi_1, \ldots, \psi_{s_1}$ be a basis of $\mathfrak{u}_{\bar{1}}$, and accordingly we denote by $\bar{b}_1, \ldots, \bar{b}_{s_0}$ and

$\overline{\psi}_1, \ldots, \overline{\psi}_{s_1}$ the bases of $\overline{\mathfrak{u}}_0$ and $\overline{\mathfrak{u}}_1$ respectively, so that one has $(\bar{b}_i, b_j) = \delta_{ij}$, $(\overline{\psi}_i, \psi_j) = \delta_{ij}$ and $(\mathfrak{s}_{\bar{0}}, \mathfrak{s}_{\bar{1}}) = 0$. In particular, note $(\bar{b}_i, b_j) = (b_j, \bar{b}_i)$ and $(\overline{\psi}_i, \psi_j) = -(\psi_j, \overline{\psi}_i)$.

We set $\mathfrak{s}^* = \underline{\mathrm{Hom}}(\mathfrak{s}, \mathbb{C})$, where $\underline{\mathrm{Hom}}(\cdot,\cdot)$ denotes the inner Hom, that is the set of all linear maps from $\mathfrak{s}$ to $\mathbb{C}$. We equip $\mathfrak{s}^*$ with the natural $\mathbb{Z}_2$-grading, and we define $x^*(y) := (x, y)_{\mathfrak{s}}$ for any $x^* \in \mathfrak{s}^*$ and $y \in \mathfrak{s}$. We may identify $\mathfrak{s}$ with $\mathfrak{s}^*$ under $(\cdot,\cdot)_{\mathfrak{s}}$. A direct calculation yields the following relations:

$$b_i^* = \bar{b}_i, \quad (\bar{b}_i)^* = b_i, \quad \psi_j^* = \overline{\psi}_j, \quad (\overline{\psi}_j)^* = -\psi_j, \qquad 1 \le i \le s_0,\ 1 \le j \le s_1.$$

Upon decomposing $\mathfrak{s} = \mathfrak{u} \oplus \overline{\mathfrak{u}}$, the relations above can be rewritten as

$$u^* = \overline{u}, \qquad (\overline{u})^* = (-1)^{p(\overline{u})} u = (-1)^{p(u)} u$$

for all $u \in \mathfrak{u}$ and $\overline{u} \in \overline{\mathfrak{u}}$.

We first identify a suitable basis of $\mathfrak{osp}(\mathfrak{s}) \subset \mathfrak{s} \otimes \mathfrak{s}^* = \mathrm{End}(\mathfrak{s})$. For that, we define the usual dual basis by $y^\vee(x) := \delta_{xy}$ for any basis elements x, y, and then extend by linearity. Given such basis elements $x, y \in \mathfrak{s}$, elements of the form $x \otimes y^\vee$ provide a basis of $\mathrm{End}(\mathfrak{s})$.

For any two basis elements $z_1, z_2 \in \mathfrak{s}$, a direct calculation yields

$$(x^* \otimes y^\vee(z_1), z_2) = \begin{cases} 1 & \text{if } x = z_2 \text{ and } y = z_1, \\ 0 & \text{otherwise,} \end{cases}$$
$$(z_1, y^* \otimes x^\vee(z_2)) = \begin{cases} (-1)^{p(y)} & \text{if } x = z_2 \text{ and } y = z_1, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, we find that $x^* \otimes y^\vee - (-1)^{p(x)p(y)} y^* \otimes x^\vee \in \mathfrak{osp}(\mathfrak{s})$. By Proposition 13.1.15 we have an isomorphism of super vector spaces $\mathfrak{osp}(\mathfrak{s}) \to \bigwedge^2(\mathfrak{s})$, that in turn induces a Lie superalgebra morphism $\mathfrak{osp}(\mathfrak{s}) \to C(\mathfrak{s})$. Realizing $C(\mathfrak{s})$ as in Lemma 13.1.13 and using Proposition 13.1.15, the basis vectors of $\mathfrak{osp}(\mathfrak{s})$ are mapped as follows (up to an overall $\frac{1}{2}$ normalization):

$$b_i \otimes (b_j)^\vee - \bar{b}_j \otimes (\bar{b}_i)^\vee \mapsto \frac{1}{2}\left(\frac{\partial}{\partial \eta_i}\eta_j - \eta_j \frac{\partial}{\partial \eta_i}\right), \quad b_i \otimes (\overline{b_j})^\vee - b_j \otimes (\overline{b_i})^\vee \mapsto \frac{1}{2}\left(\frac{\partial}{\partial \eta_i}\frac{\partial}{\partial \eta_j} - \frac{\partial}{\partial \eta_j}\frac{\partial}{\partial \eta_i}\right),$$
$$\overline{b_i} \otimes (b_j)^\vee - \overline{b_j} \otimes (b_i)^\vee \mapsto \frac{1}{2}(\eta_i\eta_j - \eta_j\eta_i), \quad b_i \otimes (\psi_j)^\vee - \overline{\psi_j} \otimes (\bar{b}_i)^\vee \mapsto \frac{1}{2}\left(\frac{\partial}{\partial \eta_i}x_j - x_j\frac{\partial}{\partial \eta_i}\right),$$
$$\overline{b_i} \otimes (\psi_j)^\vee - \overline{\psi_j} \otimes (b_i)^\vee \mapsto \frac{1}{2}(\eta_i x_j - x_j \eta_i), \quad b_i \otimes (\overline{\psi}_j)^\vee + \psi_j \otimes (\bar{b}_i)^\vee \mapsto -\frac{1}{2}\left(\frac{\partial}{\partial \eta_i}\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}\frac{\partial}{\partial \eta_i}\right),$$
$$\bar{b}_i \otimes (\overline{\psi}_j)^\vee + \psi_j \otimes (b_i)^\vee \mapsto -\frac{1}{2}\left(\eta_i \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}\eta_i\right), \quad \psi_i \otimes (\psi_j)^\vee - \overline{\psi}_j \otimes (\overline{\psi}_i)^\vee \mapsto \frac{1}{2}\left(\frac{\partial}{\partial x_i}x_j + x_j\frac{\partial}{\partial x_i}\right),$$
$$\psi_i \otimes (\overline{\psi}_j)^\vee + \psi_j \otimes (\overline{\psi}_i)^\vee \mapsto -\frac{1}{2}\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\right), \quad \overline{\psi}_i \otimes (\psi_j)^\vee + \overline{\psi}_j \otimes (\psi_i)^\vee \mapsto \frac{1}{2}(x_i x_j + x_j x_i),$$

where the generators η_j and x_i together with their derivations $\frac{\partial}{\partial \eta_j}$ and $\frac{\partial}{\partial x_i}$, are given as in Lemma 13.1.13. As described in Lemma 13.1.14, we have a Lie superalgebra morphism $\nu : \mathfrak{l} \to \mathfrak{osp}(\mathfrak{s})$, which in turn defines an embedding $\mathfrak{l} \hookrightarrow C(\mathfrak{s})$. Since $\mathfrak{u}$ and $\overline{\mathfrak{u}}$ are preserved by the action of $\mathfrak{l}$ on $\mathfrak{s}$, the image of $\mathfrak{l}$ is contained in the span

$$\left\langle \eta_i \frac{\partial}{\partial \eta_j} - \frac{\partial}{\partial \eta_j}\eta_i,\ \eta_i \frac{\partial}{\partial x_k} - \frac{\partial}{\partial x_k}\eta_i, \right.$$
$$\left. x_k \frac{\partial}{\partial \eta_i} - \frac{\partial}{\partial \eta_i} x_k,\ x_l \frac{\partial}{\partial x_k} + \frac{\partial}{\partial x_k} x_l : 1 \leq k, l \leq s_1,\ 1 \leq i, j \leq s_0 \right\rangle.$$

Concretely, for any $X \in \mathfrak{l}$, we have $[X, \mathfrak{u}] \subset \mathfrak{u}$ and $[X, \overline{\mathfrak{u}}] \subset \overline{\mathfrak{u}}$, such that

$$\begin{aligned} [X, \bar{b}_i] &= \sum_{k=1}^{s_0} \alpha_{ik}(X)\bar{b}_k + \sum_{l=1}^{s_1} \beta_{il}(X)\overline{\psi}_l, \\ [X, \overline{\psi}_j] &= \sum_{k=1}^{s_0} \gamma_{jk}(X)\bar{b}_k + \sum_{l=1}^{s_1} \delta_{jl}(X)\overline{\psi}_l. \end{aligned}$$

for some complex coefficients $\alpha_{ik}(X), \beta_{il}(X), \gamma_{jk}(X)$ and $\delta_{jl}(X)$, that can be determined by applying $(\cdot, b_k)$ and $(\cdot, \psi_l)$ on both sides of the previous expressions as to get

$$\begin{aligned} \alpha_{ik}(X) &= (X, [\bar{b}_i, b_k]), \quad \beta_{il}(X) = (X, [\bar{b}_i, \psi_l]), \\ \gamma_{jk}(X) &= (X, [\overline{\psi}_j, b_k]), \quad \delta_{jl}(X) = (X, [\overline{\psi}_j, \psi_l]). \end{aligned} \tag{13.1.2}$$

Finally, letting X_i be a basis for $\mathfrak{l}$, we let the structure constant f_{ij}^k and c_{ij}^k be defined by

$$[X_i, \bar{b}_j] = \sum_k f_{ij}^k \bar{b}_k + \sum_{k'} c_{ij}^{k'} \overline{\psi}_k,$$

and similarly, we let g_{ij}^k and d_{ij}^k be defined by

$$[X_i, \overline{\psi}_j] = \sum_k g_{ij}^k \bar{b}_k + \sum_{k'} d_{ij}^{k'} \overline{\psi}_{k'}.$$

Upon using these, if follows that

$$\begin{aligned} \mathrm{ad}_{X_i} =& \sum_{k=1}^{s_0} \left(\sum_{j=1}^{s_0} f_{ij}^k (\bar{b}_k \otimes (\bar{b}_j)^\vee - b_j \otimes (b_k)^\vee) + \sum_{j'=1}^{s_1} g_{ij'}^k (\bar{b}_k \otimes (\overline{\psi}_{j'})^\vee + \psi_{j'} \otimes (b_k)^\vee) \right) \\ &+ \sum_{k'=1}^{s_1} \left(\sum_{j=1}^{s_0} c_{ij}^{k'} (\overline{\psi}_{k'} \otimes (\bar{b}_j)^\vee - b_j \otimes (\psi_{k'})^\vee) + \sum_{j'=1}^{s_1} d_{ij'}^{k'} (\overline{\psi}_{k'} \otimes (\overline{\psi}_{j'})^\vee - \psi_{j'} \otimes (\psi_k)^\vee) \right) \end{aligned}$$

In turn, upon the identifications above, one has

$$\begin{aligned} \nu_*(X_i) =& \frac{1}{2} \sum_{k=1}^{s_0} \left(\sum_{j=1}^{s_0} f_{ij}^k (\eta_k \frac{\partial}{\partial \eta_j} - \frac{\partial}{\partial \eta_j}\eta_k) - \sum_{j'=1}^{s_1} g_{ij'}^k (\eta_k \frac{\partial}{\partial x_{j'}} - \frac{\partial}{\partial x_{j'}}\eta_k) \right) \\ &+ \frac{1}{2} \sum_{k'} \left(\sum_{j=1}^{s_0} c_{ij}^{k'} (x_{k'} \frac{\partial}{\partial \eta_j} - \frac{\partial}{\partial \eta_j} x_{k'}) - \sum_{j'=1}^{s_1} d_{ij'}^{k'} (x_{k'} \frac{\partial}{\partial x_{j'}} + \frac{\partial}{\partial x_{j'}} x_{k'}) \right). \end{aligned}$$

In turn, for general $X \in \mathfrak{l}$, upon using Equation (13.1.2) one gets the following expression:

$$\begin{aligned}\nu_*(X) = &\sum_{k=1}^{s_0}\left(\sum_{j=1}^{s_0}(X,[\overline{b}_j,b_k])\eta_k\frac{\partial}{\partial\eta_j} - \sum_{j'=1}^{s_1}(X,[\overline{\psi}_{j'},b_k])\eta_k\frac{\partial}{\partial x_{j'}}\right)\\ &+\sum_{k'=1}^{s_1}\left(\sum_{j=1}^{s_0}(X,[\overline{b}_j,\psi_{k'}])x_{k'}\frac{\partial}{\partial\eta_j} - \sum_{j'=1}^{s_1}(X,[\overline{\psi}_{j'},\psi_{k'}])x_{k'}\frac{\partial}{\partial x_{j'}}\right)\\ &+\frac{1}{2}\left(-\sum_{k=1}^{s_0}(X,[\overline{b}_k,b_k]) + \sum_{k'=1}^{s_1}(X,[\overline{\psi}_{k'},\psi_{k'}])\right).\end{aligned}$$

Denoting the bases of $\mathfrak{u}$ and $\overline{\mathfrak{u}}$ by $u_1,\dots,u_s$ and $\overline{u}_1,\dots,\overline{u}_s$ respectively, we may rewrite Equation (13.1.3) in the following compact fashion.

Lemma 13.1.16. *The map $\nu_* : \mathfrak{l} \to \bigwedge^2(\mathfrak{s}) \subset C(\mathfrak{s})$ is given by*

$$\begin{aligned}\nu_*(X) &= \frac{1}{2}\sum_{j,k=1}^{s}(X,[\overline{u}_j,u_k])(-1)^{p(u_j)}\overline{u}_k u_j + \begin{cases}\rho^{\mathfrak{u}}(X) & \text{if } X\in\mathfrak{h},\\ 0 & \text{else.}\end{cases}\\ &= \frac{1}{2}\sum_{j,k=1}^{s}(X,[u_k,\overline{u}_j])(-1)^{p(u_j)}u_j\overline{u}_k - \begin{cases}\rho^{\mathfrak{u}}(X) & \text{if } X\in\mathfrak{h},\\ 0 & \text{else.}\end{cases}\end{aligned}$$

Proof. We consider the first equality. For that, it is immediate that

$$\frac{1}{2}\left(-\sum_{k=1}^{s_0}(X,[\overline{b}_k,b_k]) + \sum_{k'=1}^{s_1}(X,[\overline{\psi}_{k'},\psi_{k'}])\right) = -\frac{1}{2}\sum_{k=1}^{s}(-1)^{p(u_k)}(X,[\overline{u}_k,u_k]),$$

and it remains to prove the equality

$$-\frac{1}{2}\sum_{k=1}^{s}(-1)^{p(u_k)}(X,[\overline{u}_k,u_k]) = \begin{cases}\rho^{\mathfrak{u}}(X) & \text{if } X\in\mathfrak{h},\\ 0 & \text{else.}\end{cases}$$

To this end, let $\alpha_1,\dots,\alpha_s$ denote the set of positive roots such that $\mathfrak{u} = \bigoplus_{k=1}^{s}\mathfrak{g}^{\alpha_k}$. In particular, we have $\overline{\mathfrak{u}} = \bigoplus_{k=1}^{s}\mathfrak{g}^{-\alpha_k}$. By Proposition 2.1.15, the weight spaces of $\mathfrak{g}$ are one-dimensional.

We may assume, without loss of generality, that $u_i \in \mathfrak{g}^{\alpha_i}$ and $\overline{u}_j \in \mathfrak{g}^{-\alpha_j}$. Then, by Proposition 2.1.15, we have $[\overline{u}_i, u_i] \in \mathfrak{h}$, and $(X,[\overline{u}_i,u_i])$ is trivial unless $X \in \mathfrak{h}$ by consistency. If $X \in \mathfrak{h}$, the invariance of $(\cdot,\cdot)$ and the root space decomposition yield:

$$(X,[\overline{u}_i,u_i]) = ([X,\overline{u}_i],u_i) = -(\alpha_i(X)\overline{u}_i,u_i) = -\alpha_i(X)(\overline{u}_i,u_i) = -\alpha_i(X),$$

such that

$$-\frac{1}{2}\sum_{k=1}^{s}(-1)^{p(u_i)}(X,[\overline{u}_k,u_k]) = \frac{1}{2}\sum_{k=1}^{s_0}\alpha_k(X) - \frac{1}{2}\sum_{l=s_0+1}^{s}\alpha_l(X) = \rho^{\mathfrak{u}_{\bar{0}}}(X) - \rho^{\mathfrak{u}_{\bar{1}}}(X) = \rho^{\mathfrak{u}}.$$

Finally, the second equality is a straightforward computation using the definition of the Clifford superalgebra $C(\mathfrak{s})$ and will be omitted. □

13.2. Cubic Dirac operators and Dirac cohomology

In this section, we briefly introduce cubic Dirac operators $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ for Levi subalgebras $\mathfrak{l}$ of basic classical Lie superalgebras $\mathfrak{g}$ and discuss their main properties. Later on, in Theorem 13.2.10 and Lemma 13.2.11, we prove that the cubic Dirac operator $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ possesses a convenient decomposition into $\mathfrak{l}$-invariant summands, which will be investigated in a later section. Next, in Subsection 13.2.2, we introduce the oscillator supermodules $M(\mathfrak{s}), \overline{M}(\mathfrak{s})$ and characterize it as an $\mathfrak{l}$-supermodule in view of Lemma 13.1.16. Given any $\mathfrak{g}$-supermodule M, there is a natural action of the cubic Dirac operator $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ on $M\otimes\overline{M}(\mathfrak{s})$, which allows us to introduce a cohomology theory $M\mapsto H_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$, called the Dirac cohomology of the supermodule M, see Definition 13.2.22. In the last part of the section, we focus on the Dirac cohomology of supermodules admitting an infinitesimal character. In particular, we prove a super-analog of the Casselmann–Osborne Lemma in Theorem 13.2.27. Finally, in the last subsection, we briefly discuss some homological properties of Dirac cohomology.

13.2.1. Cubic Dirac operators

Definition and first properties.

Fix a parabolic subalgebra $\mathfrak{p}=\mathfrak{l}\ltimes\mathfrak{u}$, and let $\mathfrak{g}:=\mathfrak{l}\oplus\mathfrak{s}$ be the induced decomposition of $\mathfrak{g}$. Recall $\mathfrak{s}=\mathfrak{u}\oplus\overline{\mathfrak{u}}$ is even dimensional. For convenience, let $2s=2s_0+2s_1:=\dim(\mathfrak{s})$ with $2s_0=\dim(\mathfrak{s}_{\bar 0})$ and $2s_1=\dim(\mathfrak{s}_{\bar 1})$ with $s,s_0,s_1\in\mathbb{Z}_+$. Moreover, we recall that the restriction of $(\cdot,\cdot)$ to $\mathfrak{s}$ gives a non-degenerate supersymmetric invariant bilinear form, denoted by $(\cdot,\cdot)_{\mathfrak{s}}$, that allows to identify $\mathfrak{s}$ and its dual space $\mathfrak{s}^*$. In the following, we will fix an orthogonal basis $\{X_1,\dots,X_{2s}\}$ of $\mathfrak{s}$ with dual basis $\{X_1^*,\dots,X_{2s}^*\}$.

Basic classical Lie superalgebras $\mathfrak{g}$ are examples of *quadratic Lie superalgebras* [79], which have the property that there exists a unique element $\phi\in(\bigwedge^3\mathfrak{g})_{\bar 0}$, called *fundamental* 3-*form*, such that the following holds for all $X,Y,Z\in\mathfrak{g}$:

a) $(\phi, X\wedge Y\wedge Z)=-\frac{1}{2}([X,Y],Z)$.

b) $[X,Y]=2\iota(X)\iota(Y)\phi$.

c) $\phi^2=\frac{1}{24}\operatorname{str}\operatorname{ad}(\Omega_{\mathfrak{g}})$.

In a), we extended the supersymmetric non-degenerate invariant bilinear form $(\cdot,\cdot)$ for $\mathfrak{g}$ to $\bigwedge^3\mathfrak{g}$. The fundamental 3-form ϕ is uniquely determined by its projection $\phi_{\mathfrak{s}}$ to $\mathfrak{s}$, along the decomposition $\mathfrak{g}=\mathfrak{l}\oplus\mathfrak{s}$ [79, Remark 4.1]. With respect to the above fixed basis, $\phi_{\mathfrak{s}}$ reads

$$\phi_{\mathfrak{s}}=-\frac{1}{12}\sum_{1\le i,j,k\le 2s}(-1)^{p(X_i)p(X_j)+p(X_k)p(X_k)}([X_i,X_j],X_k)X_i^*\wedge X_j^*\wedge X_k^*,$$

and it lies in $(\bigwedge^3\mathfrak{s})_{\bar 0}^{\mathfrak{l}}$, meaning that $\phi_{\mathfrak{s}}$ is $\mathfrak{l}$-invariant under the natural action given by the commutator [79, Section 4]. Since the X_i's are orthogonal, we also have

$$\phi_{\mathfrak{s}}=-\frac{1}{12}\sum_{1\le i,j,k\le 2s}(-1)^{p(X_i)p(X_j)+p(X_k)p(X_k)}([X_i,X_j],X_k)X_i^*X_j^*X_k^*.$$

in $C(\mathfrak{s})$. In a fashion analogous to [89], we use $\phi_{\mathfrak{s}}$ to give the following definition, as suggest in [79, Section 5].

Definition 13.2.1. The *cubic Dirac operator* $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ is the element in $\mathfrak{U}(\mathfrak{g})\otimes C(\mathfrak{s})$ given by

$$\mathrm{D}(\mathfrak{g},\mathfrak{l}) := \sum_{i=1}^{2s} X_i \otimes X_i^* + 1\otimes \phi_{\mathfrak{s}}.$$

A direct calculation shows that $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ is independent of the choice of basis, hence the definition is well-posed. We now define a diagonal embedding $\alpha : \mathfrak{l}\to\mathfrak{U}(\mathfrak{g})\otimes C(\mathfrak{s})$ by

$$X \mapsto X\otimes 1 + 1\otimes \nu_*(X), \qquad X\in\mathfrak{l}.$$

The explicit form of ν_* is given in Lemma 13.1.16. We will denote the image of $\mathfrak{l}$, $\mathfrak{U}(\mathfrak{l})$ and $\mathfrak{Z}(\mathfrak{l})$ under α by $\mathfrak{l}_\Delta, \mathfrak{U}(\mathfrak{l}_\Delta)$ and $\mathfrak{Z}(\mathfrak{l}_\Delta)$, respectively. Furthermore, the image of the quadratic Casimir $\Omega_{\mathfrak{l}}$ of $\mathfrak{l}$ will be denoted by $\Omega_{\mathfrak{l},\Delta}$. Finally, the quadratic Casimir of $\mathfrak{g}$ will be denoted by $\Omega_{\mathfrak{g}}$.
The following results recollect some important properties of the cubic Dirac operator, namely that $\mathrm{D}(\mathfrak{g},\mathfrak{l})^2$ is $\mathfrak{l}$-invariant and has a nice square.

Lemma 13.2.2 ([79, Lemma 6.1])**.** *The cubic Dirac operator* $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ *is* $\mathfrak{l}$*-invariant under the* $\mathfrak{l}$*-action on* $\mathfrak{U}(\mathfrak{g})\otimes C(\mathfrak{s})$*, which is induced by the adjoint action on both factors,* i.e., $[X,\mathrm{D}(\mathfrak{g},\mathfrak{l})] = X\,\mathrm{D}(\mathfrak{g},\mathfrak{l}) + (-1)^{p(X)}\,\mathrm{D}(\mathfrak{g},\mathfrak{l})X = 0$ *for all* $X\in\mathfrak{l}$.

Theorem 13.2.3 ([79, Theorem 1.3])**.** *The cubic Dirac operator* $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ *has square*

$$\mathrm{D}(\mathfrak{g},\mathfrak{l})^2 = \Omega_{\mathfrak{g}}\otimes 1 - \Omega_{\mathfrak{l},\Delta} + c(1\otimes 1),$$

where c *is a constant given by* $c = \frac{1}{24}(\operatorname{tr}\operatorname{ad}_{\mathfrak{g}}(\Omega_{\mathfrak{g}}) - \operatorname{tr}\operatorname{ad}_{\mathfrak{l}}(\Omega_{\mathfrak{l}}))$.

For the remainder of this article, we refer to the square of the Dirac operator $\mathrm{D}(\mathfrak{g},\mathfrak{l})^2$ as the *Laplace operator*, denoting it by

$$\boldsymbol{\Delta} := \mathrm{D}(\mathfrak{g},\mathfrak{l})^2.$$

In Theorem 13.2.3, the constant c has an explicit formulation in terms of the Weyl vectors ρ and $\rho^{\mathfrak{l}}$ by applying an argument similar to the one in [89, Proposition 1.84], and a modified formula of Freudenthal and de Vries for Lie superalgebras [95, Theorem 1]. In particular, the following holds.

Lemma 13.2.4. *In terms of Weyl vectors, the constant* $c = \frac{1}{24}(\operatorname{tr}\operatorname{ad}_{\mathfrak{g}}(\Omega_{\mathfrak{g}}) - \operatorname{tr}\operatorname{ad}_{\mathfrak{l}}(\Omega_{\mathfrak{l}}))$ *is given by*

$$c = (\rho,\rho) - (\rho^{\mathfrak{l}},\rho^{\mathfrak{l}}).$$

Decomposition of $\mathrm{D}(\mathfrak{g},\mathfrak{l})$.

We now aim at decomposing the Dirac operator into smaller $\mathfrak{l}$-invariant pieces. To this end, let us fix a basis $u_1,\dots,u_s$ of $\mathfrak{u}$, with dual basis $u_1^* = \overline{u}_1,\dots,u_s^* = \overline{u}_s$ of $\mathfrak{u}^*\cong\overline{\mathfrak{u}}$. Then $\mathfrak{s}$ has basis and dual basis given by

$$\begin{aligned}
&X_1 = u_1,\dots,X_s = u_s, \quad X_{s+1} = \overline{u}_1,\dots,X_{2s} = \overline{u}_s,\\
&X_1^* = \overline{u}_1,\dots,X_s^* = \overline{u}_s, \quad X_{s+1}^* = (-1)^{p(u_1)}u_1,\dots,X_{2s}^* = (-1)^{p(u_s)}u_s,
\end{aligned}$$

where we use Equation (13.1.3). In particular,

$$\sum_{i=1}^{2s} X_i \otimes X_i^* = \sum_{i=1}^{s} u_i \otimes \overline{u}_i + \sum_{i=1}^{s} (-1)^{p(u_i)} \overline{u}_i \otimes u_i =: A + \bar{A},$$

where A and $\bar{A}$ denote the two above summands, respectively. The following Lemma shows that also A and $\bar{A}$ are $\mathfrak{l}$-invariant.

Lemma 13.2.5. *The elements $A, \bar{A} \in \mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s})$ are $\mathfrak{l}$-invariant.*

Proof. It is enough to prove the statement for $A = \sum_{i=1}^{s} u_i \otimes \overline{u}_i$, as the proof for $\bar{A}$ is analogous. First, note that any $X \in \mathfrak{s}$ can be written as

$$X = \sum_{i=1}^{2s} (X_i^*, X) X_i = \sum_{i=1}^{2s} (X, X_i) X_i^*,$$

and $[\mathfrak{l}, \mathfrak{u}] \subseteq \mathfrak{u}$, $[\mathfrak{l}, \overline{\mathfrak{u}}] \subseteq \overline{\mathfrak{u}}$. Consequently, for any $X \in \mathfrak{l}$, we have

$$[X, u_i] = \sum_{j=1}^{s} (\overline{u}_j, [X, u_i]) u_j, \qquad [X, \overline{u}_i] = \sum_{j=1}^{s} ([X, \overline{u}_i], u_j) \overline{u}_j.$$

Fix some $X \in \mathfrak{l}$, and consider

$$[\alpha(X), A] = \sum_{i=1}^{s} [X, u_i] \otimes \overline{u}_i + \sum_{i=1}^{s} (-1)^{p(\nu_*(X)) p(u_i)} u_i \otimes [X, \overline{u}_i].$$

Using invariance of $(\cdot, \cdot)$ and supersymmetry, the first summand can be rewritten as

$$\begin{aligned}
\sum_{i=1}^{s} [X, u_i] \otimes \overline{u}_i &= \sum_{i,j=1}^{s} (\overline{u}_j, [X, u_i]) u_j \otimes \overline{u}_i = \sum_{i,j=1}^{s} u_j \otimes (\overline{u}_j, [X, u_i]) \overline{u}_i \\
&= - \sum_{i,j=1}^{s} (-1)^{p(\nu_*(X)) p(u_j)} u_j \otimes ([X, \overline{u}_j], u_i) \overline{u}_i \\
&= - \sum_{j=1}^{s} (-1)^{p(\nu_*(X)) p(u_j)} u_j \otimes [X, \overline{u}_j],
\end{aligned}$$

which forces $[\alpha(X), A] = 0$. We conclude that A is $\mathfrak{l}$-invariant. $\square$

Next, we decompose the fundamental 3-form $\phi_{\mathfrak{s}}$. The spaces $\mathfrak{u}, \overline{\mathfrak{u}}$ are isotropic subspaces with respect to $(\cdot, \cdot)$, *i.e.*, we have for any i, j, k $([u_i, u_j], u_k) = 0$ and $([\overline{u}_i, \overline{u}_j], \overline{u}_k) = 0$. As a result, we may decompose $\phi_{\mathfrak{s}}$ as $\phi_{\mathfrak{s}} = a + \overline{a}$, where

$$\begin{aligned}
a &= -\frac{1}{4} \sum_{i,j,k=1}^{s} (-1)^{p(u_i)p(u_j) + p(u_k) + p(u_i) + p(u_j)} ([\overline{u}_i, \overline{u}_j], u_k) u_i \wedge u_j \wedge \overline{u}_k, \\
\overline{a} &= -\frac{1}{4} \sum_{i,j,k=1}^{s} (-1)^{p(u_i)p(u_j)} ([u_i, u_j], \overline{u_k}) \overline{u}_i \wedge \overline{u}_j \wedge u_k.
\end{aligned}$$

Here, the combinatorial factor 6 arises from summing over all permutations of X_i, X_j, X_k, each contributing with the same sign, while the factor $\frac{1}{2}$ accounts for summation over all pairs i, j. Moreover, we used that $(\overline{u}_i)^* = (-1)^{p(u_i)} u_i$, as in Equation (13.1.3).

We now want to express a and $\overline{a}$ as elements in the Clifford superalgebra $C(\mathfrak{s})$. For that, we first need the following technical lemma.

Lemma 13.2.6. *Let $u_1, \dots, u_s$ be a basis of $\mathfrak{u}$ with dual basis $\overline{u}_1, \dots, \overline{u}_s$. Then the element*

$$\Sigma := \sum_{i,j=1}^{s} (-1)^{p(u_i)+p(u_j)}([\overline{u}_i, \overline{u}_j], u_i)u_j$$

vanishes identically, i.e., $\Sigma \equiv 0$.

Proof. The vanishing of Σ will follow from $\mathfrak{l}$-invariance. Indeed, let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ and assume Σ is $\mathfrak{l}$-invariant. Since the $\mathfrak{h}$-invariant elements of $\mathfrak{g}$ are precisely $\mathfrak{h}$, and $\mathfrak{h} \subset \mathfrak{l}$, any $\mathfrak{l}$-invariant element must therefore be in $\mathfrak{h}$. Given that $\mathfrak{u} \cap \mathfrak{h} = 0$, we conclude that $\Sigma = 0$.

We are then left to prove that Σ is $\mathfrak{l}$-invariant. For this, first, we show that the map $\psi : \mathfrak{s}^{\otimes 4} \to \mathfrak{u}$, defined by $x \otimes y \otimes z \otimes w \mapsto x(y, [z, w])$, is $\mathfrak{l}$-equivariant. Here, $\otimes$ denotes the $\mathbb{Z}_2$-graded tensor product, and $\mathfrak{s} \otimes \mathfrak{s}$ is the $\mathfrak{l}$-supermodule with $\mathfrak{l}$-action given by (see Section 3.1.1)

$$X(v \otimes w) := Xv \otimes w + (-1)^{p(X)p(v)} v \otimes Xw, \quad X \in \mathfrak{l},\ v, w \in \mathfrak{s}.$$

Let $X \in \mathfrak{l}$, then one computes

$$\begin{aligned} \psi(X(x \otimes y \otimes z \otimes w)) = [X, x]([z, w], y) + x(-1)^{p(X)p(x)} \big(([X, y], [z, w]) \\ +(-1)^{p(X)p(y)}(y, [[X, z], w]) + (-1)^{p(X)(p(y)+p(z))}(y, [z, [X, w]])\big) . \end{aligned}$$

As a consequence, proving equivariance reduces to check that

$$\begin{aligned} &([X, y], [z, w]) + (-1)^{p(X)p(y)}(y, [[X, z], w]) + (-1)^{p(X)(p(y)+p(z))}(y, [z, [X, w]]) \\ =&([X, y], [z, w]) + (-1)^{p(X)p(y)}(y, [X, [z, w]]) = 0, \end{aligned}$$

where we used

$$[X, [z, w]] = [[X, z], w] + (-1)^{p(X)p(z)}[z, [X, w]]$$

by the super Jacobi identity, and

$$([X, y], [z, w]) + (-1)^{p(X)p(y)}(y, [X, [z, w]]) = 0$$

by the $\mathfrak{l}$-invariance of $(\cdot, \cdot)$.

Next, we construct an $\mathfrak{l}$-invariant element in $\mathfrak{s}^{\otimes 4}$, which is mapped under ψ to an $\mathfrak{l}$-invariant element in $\mathfrak{u}$ by $\mathfrak{l}$-equivariance.

We first claim that the map $\tau : \overline{\mathfrak{u}} \to \mathfrak{u}^*$, defined by $\overline{u} \mapsto (\overline{u}, \cdot)$, is an $\mathfrak{l}$-equivariant isomorphism. Whilst it is clear that the map is an isomorphism of super vector spaces, it remains to show $\mathfrak{l}$-equivariance. For this, let $f \in \mathfrak{u}^*$, $v \in \mathfrak{u}$. Then we have:

$$\begin{aligned} \tau([X, f])(v) = ([X, f], v) = -(-1)^{p(X)p(f)}(f, [X, v]) = -(-1)^{p(X)p(f)}\tau(f)([X, v]) \\ = (X\tau(f))(v). \end{aligned}$$

Now, consider the invariant element $\mathrm{id} \otimes \mathrm{id} \in \mathrm{End}(\mathfrak{u}) \otimes \mathrm{End}(\mathfrak{u})$. Under the identification $\mathrm{End}(\mathfrak{u}) \cong \mathfrak{u} \otimes \mathfrak{u}^*$ with a basis $(u_i)_{i=1,\dots,s}$ and dual basis $(u_i^*)_{i=1,\dots,s}$, this can be expressed as:

$$\sum_{i,j=1,\dots,s} u_i \otimes u_i^* \otimes u_j \otimes u_j^*.$$

Applying the natural $\mathfrak{l}$-supermodule isomorphism of $\mathfrak{u}\otimes\mathfrak{u}^*\otimes\mathfrak{u}\otimes\mathfrak{u}^*$, given by $x\otimes\nu\otimes y\otimes\mu\mapsto(-1)^{p(\nu)p(y)}x\otimes y\otimes\mu\otimes\nu$, the above is mapped to

$$\sum_{i,j=1,\dots,s}(-1)^{p(u_i)p(u_j)}u_i\otimes u_j\otimes u_i^*\otimes u_j^*.$$

In turn, under the map $\mathrm{id}\otimes\mathrm{id}\otimes\tau^{-1}\otimes\tau^{-1}:\mathfrak{u}\otimes\mathfrak{u}\otimes\mathfrak{u}^*\otimes\mathfrak{u}^*\to\mathfrak{u}\otimes\mathfrak{u}\otimes\overline{\mathfrak{u}}\otimes\overline{\mathfrak{u}}$, the previous becomes

$$\sum_{i,j=1}^{s}(-1)^{p(u_i)p(u_j)}u_i\otimes u_j\otimes\overline{u}_i\otimes\overline{u}_j,$$

which gives the desired element in $\mathfrak{s}^{\otimes 4}$ under inclusion. Applying now ψ to (13.2.1), we obtain the $\mathfrak{l}$-invariant element:

$$\begin{aligned}\sum_{i,j=1}^{s}(-1)^{p(u_i)p(u_j)}u_i(u_j,[\overline{u}_i,\overline{u}_j]) &= \sum_{i,j=1}^{s}(-1)^{p(u_i)p(u_j)}(-1)^{p(u_j)(p(u_i)+p(u_j))}([\overline{u}_i,\overline{u}_j],u_j)u_i\\ &= \sum_{i,j=1}^{s}(-1)^{p(u_i)p(u_j)+p(u_j)}([\overline{u}_j,\overline{u}_i],u_j)u_i.\end{aligned}$$

Notably, $([\overline{u}_j,\overline{u}_i],u_j)$ is non-zero only if $p(u_j)=p(u_j)+p(u_i)$, leading to $p(u_i)=0$. It follows that

$$\Sigma=\sum_{i,j=1}^{s}(-1)^{p(u_j)+p(u_i)}([\overline{u}_j,\overline{u}_i],u_j)u_i$$

is $\mathfrak{l}$-invariant. This finishes the proof. □

Having this technical result available, we are ready to give the following characterization of the elements a and $\overline{a}$ inside $C(\mathfrak{s})$.

Lemma 13.2.7. *In the Clifford superalgebra $C(\mathfrak{s})$, the elements a and $\overline{a}$ are given by*

$$\begin{aligned}a&=-\frac{1}{4}\sum_{1\le i,j\le s}(-1)^{p(u_i)p(u_j)+p(u_i)+p(u_j)}[\overline{u}_i,\overline{u}_j]\,u_iu_j,\\ \overline{a}&=-\frac{1}{4}\sum_{1\le i,j\le s}(-1)^{p(u_i)p(u_j)}[u_i,u_j]\overline{u}_i\overline{u}_j.\end{aligned}$$

Proof. We only prove the expression for a, as the proof for $\overline{a}$ is analogous. First, we note that under the quantization map of Theorem 13.1.11 the following holds:

$$u_i\wedge u_j\wedge\overline{u}_k\mapsto u_iu_j\overline{u}_k+(-1)^{p(u_k)}(\delta_{ik}(-1)^{p(u_i)p(u_j)}u_j-\delta_{jk}u_i).$$

Thus, in the Clifford superalgebra $C(\mathfrak{s})$ we can write

$$\begin{aligned}a=&-\frac{1}{4}\sum_{i,jk=1}^{s}(-1)^{p(u_i)p(u_j)+p(u_i)+p(u_j)+p(u_k)}([\overline{u}_i,\overline{u}_j],u_k)u_iu_j\overline{u}_k\\ &+\frac{1}{2}\sum_{i,j=1}^{s}(-1)^{p(u_i)+p(u_j)}([\overline{u}_i,\overline{u}_j],u_i)u_j\\ =&-\frac{1}{4}\sum_{i,j=1}^{s}(-1)^{p(u_i)p(u_j)+p(u_i)+p(u_j)}[\overline{u}_i,\overline{u}_j]u_iu_j,\end{aligned}$$

where the second summand vanishes by Lemma 13.2.6. □

Remark 13.2.8. We can rewrite a and $\bar{a}$ as

$$a = -\frac{1}{4}\sum_{1\le i,j\le s}(-1)^{p(u_i)p(u_j)}u_iu_j[\overline{u}_i,\overline{u}_j],$$
$$\bar{a} = -\frac{1}{4}\sum_{1\le i,j\le s}(-1)^{p(u_i)p(u_j)+p(u_i)+p(u_j)}\overline{u}_i\overline{u}_j[u_i,u_j].$$

This follows from a straightforward computation in $C(\mathfrak{s})$ using the properties of $(\cdot,\cdot)$ and will be omitted.

Further, a direct but lengthy calculation yields the following lemma.

Lemma 13.2.9. *The elements $a, \overline{a} \in C(\mathfrak{s})$ are invariant under the adjoint action of $\mathfrak{l}$.*

Altogether, relying on the above results, we can define the following $\mathfrak{l}$-invariant elements $C, \bar{C} \in \mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s})$:

$$C := A + 1 \otimes a, \qquad \bar{C} := \bar{A} + 1 \otimes \overline{a},$$

which can be used to decompose the cubic Dirac operator. We summarize this discussion in the following theorem, whose proof is an immediate consequence of the above lemmas.

Theorem 13.2.10. *The cubic Dirac operator has the following decomposition in $\mathfrak{l}$-invariant elements*

$$\mathrm{D}(\mathfrak{g},\mathfrak{l}) = C + \bar{C}.$$

We conclude this subsection showing that each of the $\mathfrak{l}$-invariant summand of $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ is nilpotent.

Lemma 13.2.11. *The square of $C, \bar{C}$ is*

$$C^2 = 0, \qquad \bar{C}^2 = 0.$$

Proof. The idea of the proof is similar to the one of Proposition 2.6 in [62]. We define

$$E := \frac{1}{2}\sum_{i=1}^{s} 1 \otimes u_i\overline{u}_i$$

and compute the commutator $[E, C] = [E, A] + [E, 1 \otimes a]$:

$$\begin{aligned}
[E,A] &= EA - AE \\
&= \frac{1}{2}\sum_{i,j=1}^{s} u_i \otimes (u_j\overline{u}_ju_i - u_iu_j\overline{u}_j) \\
&= \frac{1}{2}\sum_{i,j=1}^{s} u_i \otimes (u_j\overline{u}_ju_i + (-1)^{p(u_i)p(u_j)}u_ju_i\overline{u}_j) \\
&= \frac{1}{2}\sum_{i,j=1}^{s} u_i \otimes (u_j\overline{u}_ju_i + (-1)^{p(u_i)p(u_j)}u_j(-(-1)^{p(u_i)p(u_j)}\overline{u}_ju_i + 2\delta_{ij})) \\
&= \sum_{j=1}^{s} u_j \otimes u_j,
\end{aligned}$$

which equals A under the natural identification u_j with $\overline{u}_j = u_j^*$. Analogously, $[E, 1 \otimes a] = 1 \otimes a$, *i.e.*, $[E, C] = C$. Moreover, a direct calculation yields $[E, \bar{C}] = -\bar{C}$ and therefore $[E, C^2] = 2C^2$, $[E, \bar{C}^2] = -2\bar{C}^2$.

However, set $D := \mathrm{D}(\mathfrak{g}, \mathfrak{l})$ and $\bar{D} := C - \bar{C}$ such that $C = \frac{1}{2}(D + \bar{D})$ and $\bar{C} = \frac{1}{2}(D - \bar{D})$. Then

$$4C^2 = (D + \bar{D})^2 = D^2 + \bar{D}^2 = (D - \bar{D})^2 = 4\bar{C}^2,$$

since $D\bar{D} + \bar{D}D = [E, D^2] = 0$ by Theorem 13.2.3. This forces $C^2 = \bar{C}^2 = 0$. □

13.2.2. Dirac cohomology

We define Dirac cohomology with respect to a cubic Dirac operator $\mathrm{D}(\mathfrak{g}, \mathfrak{l})$ for some parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$ and study its homological properties. Furthermore, we state Vogan's Theorem to examine the Dirac cohomology of supermodules with infinitesimal character by formulating a Casselman–Osborne Lemma.

Oscillator supermodule

There exists a natural simple supermodule for the Clifford superalgebra $C(\mathfrak{s})$, the *oscillator supermodule*, which we will construct.

The super vector space $\mathfrak{s}$ decomposes in its even and odd parts as $\mathfrak{s} = \mathfrak{s}_{\bar{0}} \oplus \mathfrak{s}_{\bar{1}}$, where in turn $\mathfrak{s}_{\bar{0}} = \mathfrak{u}_{\bar{0}} \oplus \overline{\mathfrak{u}}_{\bar{0}}$ and $\mathfrak{s}_{\bar{1}} = \mathfrak{u}_{\bar{1}} \oplus \overline{\mathfrak{u}}_{\bar{1}}$. Note that these decompositions are not direct sum decompositions with respect to $(\cdot, \cdot)$.

We consider the Clifford superalgebra $C(\mathfrak{s}) = C(\mathfrak{s}_{\bar{0}}) \otimes C(\mathfrak{s}_{\bar{1}})$, and treat the Clifford algebra $C(\mathfrak{s}_{\bar{0}})$ and the Weyl algebra $C(\mathfrak{s}_{\bar{1}})$ separately.

First, we consider the Clifford algebra $C(\mathfrak{s}_{\bar{0}})$. The subspaces $\mathfrak{u}_{\bar{0}}$ and $\overline{\mathfrak{u}}_{\bar{0}}$ of $\mathfrak{s}_{\bar{0}}$ define isotropic and complementary subspaces of $\mathfrak{s}_{\bar{0}}$ with respect to $(\cdot, \cdot)_{\mathfrak{s}}$. We fix a basis $u_1, \ldots, u_{s_0}$ of $\mathfrak{u}_{\bar{0}}$ and $\overline{u}_1, \ldots, \overline{u}_{s_0}$ of $\overline{\mathfrak{u}}_{\bar{0}}$ such that $(\overline{u}_j, u_i) = (u_i, \overline{u}_j) = \delta_{ij}$ for all $1 \leq i, j \leq s_0$. We may also identify $\overline{\mathfrak{u}}_{\bar{0}}$ with the dual space $\mathfrak{u}_{\bar{0}}^*$ under $(\cdot, \cdot)_{\mathfrak{s}}$, such that the dual basis u_i^* is $\overline{u}_i$ (*cf.* Equation (13.1.3)).

We are interested in

$$S^{\mathfrak{g},\mathfrak{l}} := \bigwedge \mathfrak{u}_{\bar{0}}, \qquad \overline{S}^{\mathfrak{g},\mathfrak{l}} := \bigwedge \overline{\mathfrak{u}}_{\bar{0}}.$$

Without loss of generality, we focus on $\overline{S}^{\mathfrak{g},\mathfrak{l}}$, which is relevant for later applications. However, the following discussion applies to both $S^{\mathfrak{g},\mathfrak{l}}$ and $\overline{S}^{\mathfrak{g},\mathfrak{l}}$.

On $\bigwedge \overline{\mathfrak{u}}_{\bar{0}}$, we have a natural action of $\mathfrak{s}_{\bar{0}}$, where $\overline{u} \in \overline{\mathfrak{u}}_{\bar{0}}$ acts as the left exterior multiplication $\epsilon(\overline{u})$, and $u \in \mathfrak{u}_{\bar{0}}$ acts as the contraction $\iota(u)$ defined in Equation (13.1.3). By the universal property of the Clifford superalgebra $C(\mathfrak{s}_{\bar{0}})$, this extends to an action of $C(\mathfrak{s}_{\bar{0}})$ on $\overline{S}^{\mathfrak{g},\mathfrak{l}}$, which realizes $\overline{S}^{\mathfrak{g},\mathfrak{l}}$ as a $C(\mathfrak{s}_{\bar{0}})$-module, called *spin module*. The following lemma is standard.

Lemma 13.2.12. *The spin module $\overline{S}^{\mathfrak{g},\mathfrak{l}}$ is the unique simple $C(\mathfrak{s}_{\bar{0}})$-module, up to isomorphism. Additionally, $\overline{S}^{\mathfrak{g},\mathfrak{l}}$ contains a highest weight vector with respect to $\Delta(\mathfrak{l}; \mathfrak{h})^+$, whose weight is $\rho^{\mathfrak{u}_{\bar{0}}}$.*

Remark 13.2.13. Equivalently, we may consider $\overline{S}^{\mathfrak{g},\mathfrak{l}}$ as the left-ideal in $C(\mathfrak{s}_{\bar{0}})$ generated by the element $u := u_1 \ldots u_{s_0}$ such that $\overline{S}^{\mathfrak{g},\mathfrak{l}} = (\bigwedge \overline{\mathfrak{u}}_{\bar{0}})u$ and the action is given by (left) Clifford multiplication. Note that the Clifford product and the exterior product coincide on $\mathfrak{u}_{\bar{0}}$ and $\overline{\mathfrak{u}}_{\bar{0}}$, since they are isotropic.

The spin module $\overline{S}^{\mathfrak{g},\mathfrak{l}}$ is equipped with a non-degenerate Hermitian form $\langle \cdot, \cdot \rangle_{\overline{S}^{\mathfrak{g},\mathfrak{l}}}$, such that u_i and $\overline{u}_i$ are adjoint to each other. We fix a real form $\mathfrak{g}^{\mathbb{R}}$ of $\mathfrak{g}$ defined with respect to a Cartan automorphism $\theta \in \mathrm{aut}_{2,4}(\mathfrak{g})$, such that

$$B_\theta(\cdot,\cdot) := -(\cdot, \theta(\cdot))$$

defines an inner product on $\mathfrak{g}^{\mathbb{R}}$ (see Section 13.1.2). We uniquely extend the inner product $B_\theta(\cdot,\cdot)$ to a Hermitian form on $\mathfrak{g}$. Restricting this form to $\mathfrak{s}$, we denote it by the same symbol, $B_\theta(\cdot,\cdot)$, by abuse of notation. We may assume that $u_1, \ldots, u_{s_0}, \overline{u}_1, \ldots, \overline{u}_{s_0}$ is an orthonormal basis of $(\mathfrak{s}_{\bar{0}}, B_\theta(\cdot,\cdot))$. Then

$$1 = B_\theta(u_i, u_i) = -(u_i, \theta(u_i)) = (u_i, u_i^*), \qquad 1 = B_\theta(\overline{u}_i, \overline{u}_i) = -(\overline{u}_i, \theta(\overline{u}_i)) = (\overline{u}_i, \overline{u}_i^*),$$

and $-\theta(u_i) = u_i^* = \overline{u}_i$ for all $1 \leq i \leq s_0$.

On $T^n(\overline{\mathfrak{u}}_{\bar{0}})$, we consider the bilinear form

$$\langle v_1 \otimes \ldots \otimes v_n, w_1 \otimes \ldots \otimes w_n \rangle_{\bigwedge} := \sum_{\sigma \in S_n} \tilde{B}(p(\sigma; v_1, \ldots, v_n) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}, w_1 \otimes \ldots \otimes w_n),$$

where

$$\tilde{B}(v_1 \otimes \ldots \otimes v_n, w_1 \otimes \ldots \otimes w_n) := \prod_{i=0}^{n-1} B_\theta(v_{n-i}, w_{1+i}).$$

A direct calculation shows that $\langle \cdot, \cdot \rangle_{\bigwedge}$ descends to a Hermitian form on $\overline{S}^{\mathfrak{g},\mathfrak{l}}$, denoted by $\langle \cdot, \cdot \rangle_{\overline{S}^{\mathfrak{g},\mathfrak{l}}}$ in what follows. By construction, the following holds true.

Lemma 13.2.14. *a) If we consider $\overline{S}^{\mathfrak{g},\mathfrak{l}}$ as a super vector space with obvious $\mathbb{Z}_2$-grading, $\langle \cdot, \cdot \rangle_{\overline{S}^{\mathfrak{g},\mathfrak{l}}}$ is a super positive definite super Hermitian form, that is,*

$$\langle v, w \rangle_{\overline{S}^{\mathfrak{g},\mathfrak{l}}} = (-1)^{p(v)p(w)} \overline{\langle w, v \rangle}_{\overline{S}^{\mathfrak{g},\mathfrak{l}}}, \quad v, w \in \overline{S}^{\mathfrak{g},\mathfrak{l}},$$

$\langle v, w \rangle_{\overline{S}^{\mathfrak{g},\mathfrak{l}}} = 0$ whenever $p(v) \neq p(w)$, and $\langle \cdot, \cdot \rangle_{\overline{S}^{\mathfrak{g},\mathfrak{l}}}$ is positive definite on $\overline{S}^{\mathfrak{g},\mathfrak{l}}_{\bar{0}}$ and $-i$-times positive definite on $\overline{S}^{\mathfrak{g},\mathfrak{l}}_{\bar{1}}$.

b) The adjoint of u_i with respect to $\langle \cdot, \cdot \rangle_{\overline{S}^{\mathfrak{g},\mathfrak{l}}}$ is $\theta(u_i) = -\overline{u}_i$, and the adjoint of $\overline{u}_i$ is $\theta(\overline{u}_i) = -u_i$.

Second, we consider the *Weyl algebra* $C(\mathfrak{s}_{\bar{1}})$. To this end, we note that $(\cdot,\cdot)|_{\mathfrak{s}_{\bar{1}}}$ is a symplectic form on $\mathfrak{s}_{\bar{1}}$, and $\mathfrak{s}_{\bar{1}} = \mathfrak{u}_{\bar{1}} \oplus \overline{\mathfrak{u}}_{\bar{1}}$ is a complete polarization—that is, $\mathfrak{u}_{\bar{1}}$ and $\overline{\mathfrak{u}}_{\bar{1}}$ are maximal isotropic subspaces. We refer to these spaces collectively as X and Y, assigning one to each. Further, we fix a basis $e_1, \ldots, e_{s_1}$ of X with dual basis $f_1, \ldots, f_{s_1}$ of Y, such that the Weyl algebra $\mathscr{W}(\mathfrak{g}_{\bar{1}}) := C(\mathfrak{s}_{\bar{1}})$ over $\mathfrak{s}_{\bar{1}}$ is generated by e_k and f_l.

The Weyl algebra acts naturally on $\mathbb{C}[X] \cong \mathrm{Sym}(X)$, where elements of X acts by multiplication and the action of Y is given as follows:

$$f_i \cdot e_j := (f_i, e_j)_{\mathfrak{s}}, \qquad 1 \leq i, j \leq s_1.$$

We call $\mathbb{C}[X]$ *oscillator module*. Any element in $\mathbb{C}[X]$ that is annihilated by all f_i is necessarily constant. We conclude that the maximal proper submodule of $\mathbb{C}[X]$ is zero, and $\mathbb{C}[X]$ is a simple module over $\mathscr{W}(\mathfrak{g}_{\bar{1}})$.

Lemma 13.2.15. *The oscillator module $\mathbb{C}[X]$ is a simple $\mathscr{W}(\mathfrak{g}_{\bar{1}})$-module.*

To treat $X = \mathfrak{u}_{\bar{1}}$ and $X = \bar{\mathfrak{u}}_{\bar{1}}$ separately, we introduce the following notation:

$$M(\mathfrak{s}_{\bar{1}}) := \mathbb{C}[\mathfrak{u}_{\bar{1}}] = \mathrm{Sym}(\mathfrak{u}_{\bar{1}}), \qquad \overline{M}(\mathfrak{s}_{\bar{1}}) := \mathbb{C}[\bar{\mathfrak{u}}_{\bar{1}}] = \mathrm{Sym}(\bar{\mathfrak{u}}_{\bar{1}}).$$

For simplicity and clarity, we consider only $\overline{M}(\mathfrak{s}_{\bar{1}})$ in the following. The oscillator module $M(\mathfrak{s}_{\bar{1}})$ can be treated analogously, and in particular, all results hold for $M(\mathfrak{s}_{\bar{1}})$ as well.

We introduce an appropriate notation following Section 13.1.3. Fix a basis $\partial_1, \ldots, \partial_{s_1}$ of $\mathfrak{u}_{\bar{1}}$, and a basis $x_1, \ldots, x_{s_1}$ of $\bar{\mathfrak{u}}_{\bar{1}}$ such that

$$(x_k, \partial_l) = \frac{1}{2}\delta_{kl}.$$

Then the Weyl algebra $\mathscr{W}(\mathfrak{g}_{\bar{1}})$ can be identified with the algebra of differential operators with polynomial coefficients in the variables $x_1, \ldots, x_{s_1}$, by identifying ∂_k with the partial derivative $\partial/\partial x_k$ for all $k = 1, \ldots, s_1$. In particular, $\mathscr{W}(\mathfrak{g}_{\bar{1}})$ forms a Lie algebra with the following commutator relations:

$$[x_k, x_l]_W = 0, \qquad [\partial_k, \partial_l]_W = 0, \qquad [x_k, \partial_l]_W = \delta_{kl},$$

for all $1 \leq k, l \leq s_1$. As a Lie algebra of differential operators, the action of the Weyl algebra on $\overline{M}(\mathfrak{s}_{\bar{1}}) := \mathbb{C}[x_1, \ldots, x_{s_1}]$ is natural.

We give $\overline{M}(\mathfrak{s}_{\bar{1}})$ a $\mathbb{Z}_2$-grading by declaring $\overline{M}(\mathfrak{s}_{\bar{1}})_{\bar{0}}$ to be the subspace generated by homogeneous polynomials of even degree, and $\overline{M}(\mathfrak{s}_{\bar{1}})_{\bar{1}}$ to be the subspace generated by homogeneous polynomials of odd degree.

Furthermore, $\overline{M}(\mathfrak{s}_{\bar{1}})$ carries a Hermitian form $\langle \cdot, \cdot \rangle_{\overline{M}(\mathfrak{s}_{\bar{1}})}$, namely the *Bargmann–Fock Hermitian form* or *Fischer–Fock Hermitian form*, that is uniquely determined by

$$\langle \prod_{k=1}^{s_1} x_k^{p_k}, \prod_{k=1}^{s_1} x_k^{q_k} \rangle_{\overline{M}(\mathfrak{s}_{\bar{1}})} = \begin{cases} \prod_{k=1}^{s_1} p_k! & \text{if } p_k = q_k \text{ for all } k, \\ 0 & \text{otherwise.} \end{cases}$$

The form is positive definite and consistent, *i.e.*, one has $\langle \overline{M}(\mathfrak{s}_{\bar{1}})_{\bar{0}}, \overline{M}(\mathfrak{s}_{\bar{1}})_{\bar{1}} \rangle_{\overline{M}(\mathfrak{s}_{\bar{1}})} = 0$. In particular, for any $v, w \in \overline{M}(\mathfrak{s}_{\bar{1}})$, the generators of $\mathscr{W}(\mathfrak{g}_{\bar{1}})$ satisfy the following relations for all $1 \leq k \leq s_1$:

$$\langle \partial_k v, w \rangle_{\overline{M}(\mathfrak{s}_{\bar{1}})} = \langle v, x_k w \rangle_{\overline{M}(\mathfrak{s}_{\bar{1}})}, \qquad \langle x_k v, w \rangle_{\overline{M}(\mathfrak{s}_{\bar{1}})} = \langle v, \partial_k w \rangle_{\overline{M}(\mathfrak{s}_{\bar{1}})}.$$

We deduce the following lemma.

Lemma 13.2.16. *For $\langle \cdot, \cdot \rangle_{\overline{M}(\mathfrak{s}_{\bar{1}})}$, the adjoint of x_k is ∂_k, and the adjoint of ∂_k is x_k for all $1 \leq k \leq s_1$.*

Finally, combining the previous constructions, we define the *oscillator supermodules* over $C(\mathfrak{s})$ as

$$M(\mathfrak{s}) := S^{\mathfrak{g},\mathfrak{l}} \otimes M(\mathfrak{s}_{\bar{1}}), \qquad \overline{M}(\mathfrak{s}) := \overline{S}^{\mathfrak{g},\mathfrak{l}} \otimes \overline{M}(\mathfrak{s}_{\bar{1}}).$$

Here, we equip $S^{\mathfrak{g},\mathfrak{l}}$ and $\overline{S}^{\mathfrak{g},\mathfrak{l}}$ with the $\mathbb{Z}_2$-grading induced by the natural $\mathbb{Z}_2$-grading of $T(\mathfrak{u}_{\bar{0}})$ and $T(\bar{\mathfrak{u}}_{\bar{0}})$, such that $S^{\mathfrak{g},\mathfrak{l}} = S^{\mathfrak{g},\mathfrak{l}}_{\bar{0}} \oplus S^{\mathfrak{g},\mathfrak{l}}_{\bar{1}}$ and $\overline{S}^{\mathfrak{g},\mathfrak{l}} = \overline{S}^{\mathfrak{g},\mathfrak{l}}_{\bar{0}} \oplus \overline{S}^{\mathfrak{g},\mathfrak{l}}_{\bar{1}}$. This makes $M(\mathfrak{s})$ and $\overline{M}(\mathfrak{s})$ into $C(\mathfrak{s})$-supermodules by posing $M(\mathfrak{s})_{\bar{0},\bar{1}} := S^{\mathfrak{g},\mathfrak{l}}_{\bar{0},\bar{1}} \otimes M(\mathfrak{s}_{\bar{1}})$ and $\overline{M}(\mathfrak{s})_{\bar{0},\bar{1}} := \overline{S}^{\mathfrak{g},\mathfrak{l}}_{\bar{0},\bar{1}} \otimes \overline{M}(\mathfrak{s}_{\bar{1}})$.

We conclude this section by describing the properties of $\overline{M}(\mathfrak{s})$. The supermodule $M(\mathfrak{s})$ can be treated analogously. First, we note $\overline{M}(\mathfrak{s})$ is $\mathfrak{h}$-semisimple. We define the set of $\mathfrak{h}$-weights of $\overline{M}(\mathfrak{s})$ by $\mathcal{P}_{\overline{M}(\mathfrak{s})} := \{\mu \in \mathfrak{h}^* : \overline{M}(\mathfrak{s})^\mu \neq \{0\}\}$, where $\overline{M}(\mathfrak{s})^\mu$ is the weight space of weight μ. Then the sets of $\mathfrak{h}$-weights of $\overline{M}(\mathfrak{s})$ is more precisely

$$\mathcal{P}_{\overline{M}(\mathfrak{s})} = \{\rho^{\mathfrak{u}} - \mathbb{Z}_+[A] : A \subset \Delta^+ \setminus \Delta(\mathfrak{l};\mathfrak{h})^+\},$$

where $\mathbb{Z}_+[A] := \sum_{\xi \in A} \mathbb{Z}_+ \xi$.

By Lemma 13.2.12 and Proposition 13.2.15, one immediately has the following.

Lemma 13.2.17. *The $C(\mathfrak{s})$-supermodule $\overline{M}(\mathfrak{s})$ is simple.*

Second, $\overline{M}(\mathfrak{s})$ carries a natural non-degenerate Hermitian form

$$\langle v \otimes P, w \otimes Q\rangle_{\overline{M}(\mathfrak{s})} := \langle v, w\rangle_{\overline{S}^{\mathfrak{g},\mathfrak{l}}} \langle P, Q\rangle_{\overline{M}(\mathfrak{s}_{\bar{1}})}.$$

for $v \otimes P, w \otimes Q \in \overline{M}(\mathfrak{s})$. The properties of the form are given in the following straightforward lemma.

Lemma 13.2.18. *The Hermitian form $\langle\cdot,\cdot\rangle_{\overline{M}(\mathfrak{s})}$ on $\overline{M}(\mathfrak{s})$ is non-degenerate, supersymmetric and consistent. Moreover, the adjoint of any basis element $u \in \mathfrak{u}$ is*

$$u^\dagger = -(-1)^{p(u)}\overline{u}.$$

Finally, we consider $M(\mathfrak{s})$ and $\overline{M}(\mathfrak{s})$ as $\mathfrak{l}$-supermodules under the Lie algebra morphism $\nu_* : \mathfrak{l} \to C(\mathfrak{s})$ introduced in Section 13.1.3, and described explicitly in Lemma 13.1.16. As a first result, we show that $M(\mathfrak{s})$ and $\overline{M}(\mathfrak{s})$ are completely reducible as $\mathfrak{l}$-supermodules. This follows directly from Proposition 6.1.8, provided we show that $M(\mathfrak{s})$ and $\overline{M}(\mathfrak{s})$ are unitarizable $\mathfrak{l}$-supermodules.

We consider unitarity with respect to a fixed Cartan automorphism $\theta \in \mathrm{aut}_{2,4}(\mathfrak{g})$ defining the real form $\mathfrak{g}^{\mathbb{R}}$, such that $B_\theta(\cdot,\cdot)$ is an inner product on $\mathfrak{g}^{\mathbb{R}}$. In particular, $\mathfrak{g}^{\mathbb{R}} = \mathfrak{l}^{\mathbb{R}} \oplus \mathfrak{s}^{\mathbb{R}}$. Then, by Lemma 13.2.14 and the definition of $\langle\cdot,\cdot\rangle_{\overline{M}(\mathfrak{s})}$, it is enough to show

$$\langle \nu_*(X)(v \otimes P), (w \otimes Q)\rangle_{\overline{M}(\mathfrak{s})} = -(-1)^{p(X)p(v)}\langle v \otimes P, \nu_*(X)(w \otimes Q)\rangle_{\overline{M}(\mathfrak{s})}$$

for any homogeneous $X \in \mathfrak{l}^{\mathbb{R}}$ and $v \otimes P, w \otimes Q \in \overline{M}(\mathfrak{s})$. Similarly, for $M(\mathfrak{s})$.

However, this is immediate by the explicit form of $\nu_*(X)$ given in Lemma 13.1.16 and the application of Lemma 13.2.14 and Lemma 13.2.16. We conclude the following proposition.

Proposition 13.2.19. *The supermodules $M(\mathfrak{s})$ and $\overline{M}(\mathfrak{s})$ are unitarizable $\mathfrak{l}$-supermodules. In particular, $M(\mathfrak{s})$ and $\overline{M}(\mathfrak{s})$ are completely reducible as $\mathfrak{l}$-supermodules.*

As an $\mathfrak{l}$-supermodule, $M(\mathfrak{s})$ and $\overline{M}(\mathfrak{s})$ have another elegant description as the exterior superalgebra over $\mathfrak{u}$ and $\overline{\mathfrak{u}}$, respectively.

Proposition 13.2.20. *There are $\mathfrak{l}$-supermodule isomorphisms*

$$M(\mathfrak{s}) \cong \bigwedge \mathfrak{u} \otimes \mathbb{C}_{-\rho^{\mathfrak{u}}} \cong \bigwedge \mathfrak{u}_{\bar{0}} \otimes \mathrm{Sym}(\mathfrak{u}_{\bar{1}}) \otimes \mathbb{C}_{-\rho^{\mathfrak{u}}},$$
$$\overline{M}(\mathfrak{s}) \cong \bigwedge \overline{\mathfrak{u}} \otimes \mathbb{C}_{\rho^{\mathfrak{u}}} \cong \bigwedge \overline{\mathfrak{u}}_{\bar{0}} \otimes \mathrm{Sym}(\overline{\mathfrak{u}}_{\bar{1}}) \otimes \mathbb{C}_{\rho^{\mathfrak{u}}},$$

where the action of $\mathfrak{l}$ on $M(\mathfrak{s})$ and $\overline{M}(\mathfrak{s})$ is induced by ν_* as in Lemma 13.1.16, and the action of $\mathfrak{l}$ on $\bigwedge \mathfrak{u}_{\bar{0}} \otimes \mathrm{Sym}(\mathfrak{u}_{\bar{1}}) \otimes \mathbb{C}_{-\rho^{\mathfrak{u}}}$ and $\bigwedge \overline{\mathfrak{u}}_{\bar{0}} \otimes \mathrm{Sym}(\overline{\mathfrak{u}}_{\bar{1}}) \otimes \mathbb{C}_{\rho^{\mathfrak{u}}}$ is induced by the adjoint action.

Proof. We prove the statement for $\overline{M}(\mathfrak{s})$. The proof for $M(\mathfrak{s})$ is analogous and will be omitted.

We consider $\bar{\mathfrak{u}}$ as an $\mathfrak{l}$-supermodule under the adjoint action, which naturally makes the exterior superalgebra $\bigwedge \bar{\mathfrak{u}} = \bigwedge \bar{\mathfrak{u}}_{\bar{0}} \otimes \mathrm{Sym}(\bar{\mathfrak{u}}_{\bar{1}})$ into an $\mathfrak{l}$-supermodule. Let $X \in \mathfrak{l}$ and let $\eta_{i_0} \wedge \cdots \wedge \eta_{i_n} \otimes x_{j_0} \ldots x_{j_m} \in \bigwedge \bar{\mathfrak{u}}_{\bar{0}} \otimes \mathrm{Sym}(\bar{\mathfrak{u}}_{\bar{1}})$. Then X acts as

$$X(\eta_{i_0} \wedge \cdots \wedge \eta_{i_n} \otimes x_{j_0} \ldots x_{j_m}) = \sum_{t=0}^{n} (-1)^t [X, \eta_{i_k}] \eta_{i_0} \wedge \ldots \hat{\eta}_{i_t} \cdots \wedge \eta_{i_n} \otimes x_{j_0} \ldots x_{j_m}$$
$$+ (-1)^{n+1} \sum_{t'=0}^{m} [X, x_{j_{t'}}] \eta_{i_0} \wedge \cdots \wedge \eta_{i_n} \otimes x_{j_0} \ldots \hat{x}_{j_{t'}} \ldots x_{j_m},$$

where $\hat{x}$ and $\hat{\eta}$ indicate that the corresponding term is omitted.

Now, consider the action of $\mathfrak{l}$ on $\overline{M}(\mathfrak{s})$ induced by ν_*, as given explicitly in Lemma 13.1.16. By Theorem 13.1.11, there exists an isomorphism $Q : \bigwedge \bar{\mathfrak{u}} \to C(\mathfrak{s})$ of super vector spaces, providing a basis of $\overline{M}(\mathfrak{s})$ consisting of elements of the form

$$Q(\eta_{i_0} \wedge \cdots \wedge \eta_{i_n} \otimes x_{j_0} \ldots x_{j_m}) = \eta_{i_0} \ldots \eta_{i_n} x_{j_0} \ldots x_{j_m},$$

using isotropy of $\bar{\mathfrak{u}}$ (*cf.* Remark 13.1.12). Here, $\overline{M}(\mathfrak{s})$ is regarded as a natural quotient of $C(\mathfrak{s})$. We have:

$$\begin{aligned}
&\nu_*(X) Q(\eta_{i_0} \wedge \cdots \wedge \eta_{i_n} \otimes x_{j_0} \ldots x_{j_m}) \\
&\quad = \sum_{k=1}^{s_0} \left(\sum_{t=0}^{n} (X, [\bar{b}_{i_t}, b_k]) \eta_k (-1)^t \eta_{i_0} \ldots \hat{\eta}_{i_t} \ldots x_{j_m} \right. \\
&\qquad \left. + \sum_{t'=0}^{m} (X, [\bar{\psi}_{t'}, b_k]) \eta_k (-1)^{n+1} \eta_{i_1} \ldots \hat{x}_{j_{t'}} \ldots x_{j_m} \right) \\
&\qquad + \sum_{k'=1}^{s_1} \left(\sum_{t=0}^{n} (X, [\bar{b}_{i_t}, \psi_{k'}]) x_{k'} (-1)^t \eta_{i_1} \ldots \hat{\eta}_{i_t} \ldots x_{j_m} \right. \\
&\qquad \left. + \sum_{t'=0}^{m} (X, [\bar{\psi}_{t'}, \psi_{k'}]) x_{k'} (-1)^{n+1} \eta_{i_1} \ldots \hat{x}_{j_{t'}} \ldots x_{j_m} \right) \\
&\qquad + \rho^{\mathfrak{u}}(X) \eta_{i_1} \ldots x_{j_n} \\
&\qquad = Q \left(\sum_{t=0}^{n} (-1)^t [X, \eta_{i_t}] \eta_{i_1} \wedge \ldots \hat{\eta}_{i_t} \ldots x_{j_m} \right. \\
&\qquad \left. + \sum_{t'=0}^{m} (-1)^{n+1} [X, x_{i_{t'}}] \eta_{i_1} \wedge \ldots \hat{x}_{j_{t'}} \ldots x_{j_m} + \rho^{\mathfrak{u}}(X) \eta_{i_0} \wedge \ldots x_{j_m} \right),
\end{aligned}$$

where $\rho^{\mathfrak{u}}(X)$ is only present if $X \in \mathfrak{h}$. Concerning the signs, recall, that by (13.1.3) we have the action $\frac{\partial}{\partial x_i} x_k = -\delta_{ik}$. Here, we use the notation of Section 13.1.3, and the proof of Lemma 13.1.16 to obtain $\rho^{\mathfrak{u}}$ together with (13.1.3) in the final step. We conclude that both actions coincide up to a twist by $\mathbb{C}_{\rho^{\mathfrak{u}}}$. This finishes the proof. □

The superspaces $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ are finite-dimensional $\mathfrak{l}$-supermodules, and the adjoint action of $\mathfrak{l}$ preserves the natural $\mathbb{Z}$-grading of $\bigwedge \mathfrak{u}$ and $\bigwedge \bar{\mathfrak{u}}$. Combining Proposition 13.2.19 and Proposition 13.2.20, we obtain the following corollary.

Corollary 13.2.21. *The* $\mathfrak{l}$*-supermodules* $M(\mathfrak{s})$ *and* $\overline{M}(\mathfrak{s})$ *decompose completely into finite-dimensional simple* $\mathfrak{l}$*-supermodules.*

For the remainder of this article, we focus on $\overline{M}(\mathfrak{s})$, as this supermodule is best suited for studying highest weight $\mathfrak{g}$-supermodules.

Dirac cohomology: definition and first results

For any $\mathfrak{g}$-supermodule M, the cubic Dirac operator $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ acts naturally on the $\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s})$-supermodule $M \otimes \overline{M}(\mathfrak{s})$ by componentwise action. In particular, as $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ and $\mathfrak{l}$ commute, the kernel $\ker \mathrm{D}(\mathfrak{g},\mathfrak{l})$ naturally carries the structure of an $\mathfrak{l}$-supermodule. This gives rise to a definition of Dirac cohomology analogous to [66], as suggested in [79].

Definition 13.2.22. The *Dirac cohomology* $H_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ of a $\mathfrak{g}$-supermodule M is the $\mathfrak{l}$-supermodule

$$H_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) := \ker \mathrm{D}(\mathfrak{g},\mathfrak{l}) / \ker \mathrm{D}(\mathfrak{g},\mathfrak{l}) \cap \operatorname{im} \mathrm{D}(\mathfrak{g},\mathfrak{l}).$$

The Dirac cohomology $H_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ has a natural decomposition induced by the $\mathbb{Z}_2$-grading of $\overline{M}(\mathfrak{s})$. Accordingly, we decompose the Dirac operator as $\mathrm{D}(\mathfrak{g},\mathfrak{l}) = \mathrm{D}(\mathfrak{g},\mathfrak{l})^+ + \mathrm{D}(\mathfrak{g},\mathfrak{l})^-$, where

$$\begin{aligned} \mathrm{D}(\mathfrak{g},\mathfrak{l})^+ &:= \mathrm{D}(\mathfrak{g},\mathfrak{l})|_{M\otimes\overline{M}(\mathfrak{s})_{\bar{0}}} : M \otimes \overline{M}(\mathfrak{s})_{\bar{0}} \to M \otimes \overline{M}(\mathfrak{s})_{\bar{1}}, \\ \mathrm{D}(\mathfrak{g},\mathfrak{l})^- &:= \mathrm{D}(\mathfrak{g},\mathfrak{l})|_{M\otimes\overline{M}(\mathfrak{s})_{\bar{1}}} : M \otimes \overline{M}(\mathfrak{s})_{\bar{1}} \to M \otimes \overline{M}(\mathfrak{s})_{\bar{0}}, \end{aligned} \tag{13.2.1}$$

and define $\mathrm{H}^+_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) := \mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})^+}(M)$ and $\mathrm{H}^-_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) := \mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})^-}(M)$, so that

$$\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) := \mathrm{H}^+_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) + \mathrm{H}^-_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M).$$

We are particularly interested in admissible $(\mathfrak{g},\mathfrak{l})$-supermodules, *i.e.*, $\mathfrak{g}$-supermodules that are $\mathfrak{l}$-semisimple. For these supermodules, the Dirac cohomology is $\mathfrak{l}$-semisimple by Proposition 13.2.19, as $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ commutes with the action of $\mathfrak{l}$.

Lemma 13.2.23. *Let* M *be an admissible* $(\mathfrak{g},\mathfrak{l})$*-supermodule. Then* $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ *is a semisimple* $\mathfrak{l}$*-supermodule, that is,* $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ *is completely reducible as an* $\mathfrak{l}$*-supermodule.*

Given the nice square of the Dirac operator in Theorem 13.2.3, the Dirac cohomology reveals its full potential when the supermodule under consideration possesses an infinitesimal character. This will be explored in the subsequent section. Before addressing this topic, we establish that $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ naturally induces a cohomology on the space $(\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s}))^{\mathfrak{l}}$ of $\mathfrak{l}$-invariants in $\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s})$, as shown in [79].

We equip $\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s})$ with the $\mathbb{Z}_2$-grading induced by $C(\mathfrak{s})$, that is, $(\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s}))_{\bar{0},\bar{1}} := \mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s})_{\bar{0},\bar{1}}$. The diagonal embedding $\alpha : \mathfrak{l} \to \mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s})$, as defined in equation (13.2.1), endows $\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s})$ with the structure of an $\mathfrak{l}$-supermodule via the adjoint action. Hence, it makes sense to consider the space of $\mathfrak{l}$-invariants in $\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s})$:

$$(\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s}))^{\mathfrak{l}} := \{A \in \mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s}) : [\alpha(X), A] = 0 \text{ for all } X \in \mathfrak{l}\},$$

which is still endowed with the obvious $\mathbb{Z}_2$-grading. This enters the following construction, due to [79]. Let $\hat{d}$ be the operator acting on $\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s})$ as

$$\hat{d}(A) := [\mathrm{D}(\mathfrak{g},\mathfrak{l}), A] = \mathrm{D}(\mathfrak{g},\mathfrak{l})A - (-1)^{p(A)} A\, \mathrm{D}(\mathfrak{g},\mathfrak{l})$$

for any homogeneous $A \in \mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s})$. Since $\mathrm{D}(\mathfrak{g}, \mathfrak{l})$ is $\mathfrak{l}$-invariant, the operator $\hat{d}$ can be restricted to an operator

$$d : (\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s}))^{\mathfrak{l}} \to (\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s}))^{\mathfrak{l}},$$

which is proved to be nilpotent, *i.e.*, $d^2 = 0$, and hence it makes sense to define its cohomology to be the quotient $\ker d / \operatorname{im} d$, see [79]. Recalling that we denoted by $\mathfrak{Z}(\mathfrak{l}_\Delta)$ the image under the diagonal embedding α of $\mathfrak{Z}(\mathfrak{l})$, the center of the universal enveloping superalgebra $\mathfrak{U}(\mathfrak{l})$, one has the following characterization for the cohomology of d.

Theorem 13.2.24 ([79, Theorem 6.2])**.** *The cohomology of d is isomorphic to $\mathfrak{Z}(\mathfrak{l}_\Delta)$,* i.e.,

$$\ker d = \mathfrak{Z}(\mathfrak{l}_\Delta) \oplus \operatorname{im} d.$$

Infinitesimal characters and Dirac cohomology

We now aim to extend the above discussion to supermodules admitting an infinitesimal character, which we introduced in Section 3.1.4. We start with the following straightforward corollary of Theorem 13.2.24.

Corollary 13.2.25. *Any element $z \otimes 1 \in \mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s})$ with $z \in \mathfrak{Z}(\mathfrak{g})$ can be written as*

$$z \otimes 1 = \eta_{\mathfrak{l}}(z) + \mathrm{D}(\mathfrak{g}, \mathfrak{l})A + A\,\mathrm{D}(\mathfrak{g}, \mathfrak{l}),$$

for some $A \in (\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s}))^{\mathfrak{l}}$, and some unique $\eta_{\mathfrak{l}}(z) \in \mathfrak{Z}(\mathfrak{l}_\Delta) \cong \mathfrak{Z}(\mathfrak{l})$.

The previous Corollary 13.2.25 introduces a map $\eta_{\mathfrak{l}} : \mathfrak{Z}(\mathfrak{g}) \to \mathfrak{Z}(\mathfrak{l})$, which we now describe. Letting $\mathfrak{g}$ be decomposed as $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{s}$, where we note that $\mathfrak{h} \subset \mathfrak{l}$ by definition. Moreover, let $HC_{\mathfrak{g},\mathfrak{l}}$ denote the Harish-Chandra monomorphism and $\mathrm{res} : S(\mathfrak{h})^{W^{\mathfrak{g}}} \to S(\mathfrak{h})^{W^{\mathfrak{l}}}$ be the restriction map.

Proposition 13.2.26. *The map $\eta_{\mathfrak{l}} : \mathfrak{Z}(\mathfrak{g}) \to \mathfrak{Z}(\mathfrak{l})$ is an algebra homomorphism. Further, the following diagram is commutative.*

$$\begin{array}{ccc} \mathfrak{Z}(\mathfrak{g}) & \xrightarrow{\eta_{\mathfrak{l}}} & \mathfrak{Z}(\mathfrak{l}) \\ {\scriptstyle HC_{\mathfrak{g}}}\downarrow & & \downarrow{\scriptstyle HC_{\mathfrak{l}}} \\ S(\mathfrak{h})^{W^{\mathfrak{g}}} & \xrightarrow{\mathrm{res}} & S(\mathfrak{h})^{W^{\mathfrak{l}}} \end{array}$$

Proof. For any $z_1, z_2 \in \mathfrak{Z}(\mathfrak{g})$, we find $A_1, A_2 \in (\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s}))^{\mathfrak{l}}$ such that

$$z_1 \otimes 1 = \eta_{\mathfrak{l}}(z_1) + d(A_1), \qquad z_2 \otimes 1 = \eta_{\mathfrak{l}}(z_2) + d(A_2).$$

Multiplication yields

$$z_1 z_2 \otimes 1 = \eta_{\mathfrak{l}}(z_1)\eta_{\mathfrak{l}}(z_2) + \eta_{\mathfrak{l}}(z_1)d(A_2) + d(A_1)\eta_{\mathfrak{l}}(z_2) + d(A_1)d(A_2).$$

However, by Corollary 13.2.25, we have $d(\eta_{\mathfrak{l}}(z_1)) = d(\eta_{\mathfrak{l}}(z_2)) = 0$, and consequently

$$z_1 z_2 \otimes 1 = \eta_{\mathfrak{l}}(z_1)\eta_{\mathfrak{l}}(z_2) + d(\eta_{\mathfrak{l}}(z_1)A_2 + A_1\eta_{\mathfrak{l}}(z_2) + Ad(A'))$$

with $\eta_{\mathfrak{l}}(z_1)A_2 + A_1\eta_{\mathfrak{l}}(z_2) + Ad(A') \in (\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s}))^{\mathfrak{l}}$. Again, by Corollary 13.2.25, we have

$$z_1 z_2 \otimes 1 = \eta_{\mathfrak{l}}(z_1 z_2) + d(A)$$

for some $A \in (\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s}))^{\mathfrak{l}}$, *i.e.*, $\eta_{\mathfrak{l}}(z_1)\eta_{\mathfrak{l}}(z_2) = \eta_{\mathfrak{l}}(z_1 z_2)$.

For the second part of the proof, we note $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}$, and we identify $S(\mathfrak{h})^{W^{\mathfrak{g}}}$ and $S(\mathfrak{h})^{W^{\mathfrak{l}}}$ with the algebraic varieties $\mathfrak{h}^*/W^{\mathfrak{g}}$ and $\mathfrak{h}^*/W^{\mathfrak{l}}$. Let $\zeta : S(\mathfrak{h})^{W^{\mathfrak{g}}} \to S(\mathfrak{h})^{W^{\mathfrak{l}}}$ denote the homomorphism induced by $\eta_{\mathfrak{l}}$ under the Harish-Chandra monomorphisms, and let $\zeta' : \mathfrak{h}^*/W^{\mathfrak{l}} \to \mathfrak{h}^*/W^{\mathfrak{g}}$ be the associated morphism of the algebraic varieties. To show that the diagram commutes, it suffices to prove that ζ' is the inclusion map, *i.e.*, $\zeta'(\lambda) = \lambda$ for all $\lambda \in \mathfrak{h}^*/W^{\mathfrak{l}}$.

For any $\lambda \in \mathfrak{h}^*$, we define a highest weight $\mathfrak{g}$-supermodule M as in Equation (3.1.3). As stated explicitly in Proposition 13.3.7, $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ contains a non-trivial simple weight $\mathfrak{l}$-supermodule generated by $v_\lambda \otimes v_{\overline{M(\mathfrak{s})}}$ of weight $\lambda + \rho^{\mathfrak{u}}$. However, on $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$, we conclude from $z = \eta_{\mathfrak{l}}(z)$ the following:

$$(\lambda + \rho)(z) = ((\lambda + \rho^{\mathfrak{u}}) + \rho^{\mathfrak{l}})(\eta_{\mathfrak{l}}(z)) = (\lambda + \rho)(\eta_{\mathfrak{l}}(z)),$$

i.e., $\zeta'(\mu) = \mu$ for all $\mu = \lambda + \rho$. This completes the proof. □

As a result, we obtain a super-analog of the Casselman–Osborne Lemma (*cf.* [82, 97]), that will play a crucial role in what follows. More precisely, we have the following theorem, which gives a Lie superalgebra version of [33, Theorem 4.3].

Theorem 13.2.27. *Let M be a $\mathfrak{g}$-supermodule with infinitesimal character χ_λ. Then $z \otimes 1 \in \mathfrak{Z}(\mathfrak{g}) \otimes 1$ acts as $\eta_{\mathfrak{l}}(z) \otimes 1$ on $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$. In particular, if V is an $\mathfrak{l}$-subsupermodule of $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ with infinitesimal character $\chi^{\mathfrak{l}}_\mu$ for $\mu \in \mathfrak{h}^*$, then $\chi_\lambda = \chi^{\mathfrak{l}}_\mu \circ \eta_{\mathfrak{l}}$.*

Proof. Let V be as above and fix some non-trivial $v \in V$. By Corollary 13.2.25, there exists some $A \in (\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{s}))^{\mathfrak{l}}$ such that for any $z \in \mathfrak{Z}(\mathfrak{g})$:

$$z \otimes 1 - \chi_\lambda(z) = (\eta_{\mathfrak{l}}(z) - \chi^{\mathfrak{l}}_\mu(\eta_{\mathfrak{l}}(z))) + d(A) + (\chi^{\mathfrak{l}}_\mu(\eta_{\mathfrak{l}}(z)) - \chi_\lambda(z)),$$

where we identify $\mathfrak{Z}(\mathfrak{l}) \cong \mathfrak{Z}(\mathfrak{l}_\Delta)$. Applying both sides of this identity to v, we obtain

$$(\chi^{\mathfrak{l}}_\mu(\eta_{\mathfrak{l}}(z)) - \chi_\lambda(z))v \mod \ker \mathrm{D}(\mathfrak{g},\mathfrak{l}) \cap \mathrm{im}\, \mathrm{D}(\mathfrak{g},\mathfrak{l}) = 0.$$

This concludes the proof by Proposition 13.2.26. □

The previous result, combined with Proposition 13.2.19, yields the following corollary.

Corollary 13.2.28. *Let M be a $\mathfrak{g}$-supermodule in $\mathcal{O}^{\mathfrak{p}}$ with infinitesimal character. Then $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ is a completely reducible $\mathfrak{l}_{\bar{0}}$-module.*

Homological properties

We now briefly discuss the homological properties of the Dirac cohomology, seen as a functor from the category of $\mathfrak{g}$-supermodules to the category of $\mathfrak{l}$-supermodules. In general, it has neither a right nor a left adjoint, but it satisfies a six-term exact sequence.

Lemma 13.2.29. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $\mathfrak{g}$-supermodules having an infinitesimal character. Then there exists a natural six-term exact sequence*

$$\begin{array}{ccccc} & \mathrm{H}^+_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(A) & \longrightarrow & \mathrm{H}^+_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(B) & \\ \nearrow & & & & \searrow \\ \mathrm{H}^-_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(C) & & & & \mathrm{H}^+_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(C) \\ \nwarrow & & & & \swarrow \\ & \mathrm{H}^-_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(B) & \longleftarrow & \mathrm{H}^-_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(A) & \end{array}$$

The proof of the above statement is similar to our proof of follows from a similar argument to the one used in the proof of Theorem 8.1 in [62],, relying on the fact that the supermodules involved have an infinitesimal character, together with Theorem 13.2.3, and that supermodule morphisms are even.

Second, we consider the *Euler characteristic* of the Dirac cohomology when restricted to admissible $(\mathfrak{g}, \mathfrak{l})$-supermodules M. For any such supermodule, the Dirac operator $\mathrm{D}(\mathfrak{g}, \mathfrak{l})$ acts on $M \otimes \overline{M}(\mathfrak{s})_{\bar{0}}$ and $M \otimes \overline{M}(\mathfrak{s})_{\bar{1}}$, interchanging these two spaces. The *Dirac index* $\mathrm{I}(M)$ of M is defined as the difference of $\mathfrak{l}$-supermodules

$$\mathrm{I}(M) := M \otimes \overline{M}(\mathfrak{s})_{\bar{0}} - M \otimes \overline{M}(\mathfrak{s})_{\bar{1}}.$$

This Dirac index $\mathrm{I}(M)$ is a *virtual* $\mathfrak{l}$-supermodule, meaning that it is an integer combination of finitely many $\mathfrak{l}$-supermodules – notice that $\mathrm{I}(M)$ is an element in the Grothendieck group of the Abelian category of admissible $(\mathfrak{g}, \mathfrak{l})$-supermodules, in fact it is additive with respect to short exact sequences. In contrast, as seen above, we can use $\mathrm{D}(\mathfrak{g}, \mathfrak{l}) : M \otimes \overline{M}(\mathfrak{s})_{\bar{0},\bar{1}} \to M \otimes \overline{M}(\mathfrak{s})_{\bar{1},\bar{0}}$ to decompose the Dirac cohomology $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ into two parts, $\mathrm{H}^+_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ and $\mathrm{H}^-_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$. The *Euler characteristic* of $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ is given by the virtual $\mathfrak{l}$-supermodule $\mathrm{H}^+_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) - \mathrm{H}^-_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$. These two virtual $\mathfrak{l}$-supermodules, $\mathrm{I}(M)$ and the Euler characteristic, are equal if M has an infinitesimal character.

Proposition 13.2.30. *Let M be an admissible $(\mathfrak{g}, \mathfrak{l})$-supermodule that admits an infinitesimal character. Then the Dirac index $\mathrm{I}(M)$ is equal to the Euler characteristic of the Dirac cohomology $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ of M,* i.e.,

$$\mathrm{I}(M) = \mathrm{H}^+_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) - \mathrm{H}^-_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M).$$

Proof. Since M admits an infinitesimal character, we can decompose $M \otimes \overline{M}(\mathfrak{s})$ into a direct sum of eigenspaces of $\mathrm{D}(\mathfrak{g}, \mathfrak{l})^2$ by Theorem 13.2.3 (see below in Section 13.3.1 for a proof), namely,

$$M \otimes \overline{M}(\mathfrak{s}) = (M \otimes \overline{M}(\mathfrak{s}))(0) \oplus \sum_{r \neq 0} (M \otimes \overline{M}(\mathfrak{s}))(r),$$

where $(M \otimes \overline{M}(\mathfrak{s}))(r)$ denotes an eigenspace with eigenvalue $r \in \mathbb{C}$. The operator $\mathrm{D}(\mathfrak{g}, \mathfrak{l})^2$ is even, so the decomposition is compatible with the decomposition into even and odd parts of $\overline{M}(\mathfrak{s})$:

$$(M \otimes \overline{M}(\mathfrak{s}))(r) = (M \otimes \overline{M}(\mathfrak{s})_{\bar{0}})(r) \oplus (M \otimes \overline{M}(\mathfrak{s})_{\bar{1}})(r).$$

On the other hand, the Dirac operator $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ preserves each eigenspace because $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ and $\mathrm{D}(\mathfrak{g},\mathfrak{l})^2$ commute. However, $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ switches parity, defining maps

$$\mathrm{D}(\mathfrak{g},\mathfrak{l})(r) : (M \otimes \overline{M}(\mathfrak{s})_{\bar{0},\bar{1}})(r) \to (M \otimes \overline{M}(\mathfrak{s})_{\bar{1},\bar{0}})(r),$$

which are isomorphisms with inverses $1/r\,\mathrm{D}(\mathfrak{g},\mathfrak{l})(r)$ for $r \neq 0$. Consequently, we have

$$M \otimes \overline{M}(\mathfrak{s})_{\bar{0}} - M \otimes \overline{M}(\mathfrak{s})_{\bar{1}} = (M \otimes \overline{M}(\mathfrak{s})_{\bar{0}})(0) - (M \otimes \overline{M}(\mathfrak{s})_{\bar{1}})(0).$$

The Dirac operator $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ acts as a differential on $\ker(\mathrm{D}(\mathfrak{g},\mathfrak{l})^2)$, and the associated cohomology is the Dirac cohomology. The result follows from the Euler–Poincaré principle. □

13.3. Dirac cohomology and highest weight supermodules

In this subsection, we build on the super-analog of the Casselman–Osborne Lemma (Theorem 13.2.27) as a key tool for investigating the Dirac cohomology of highest-weight $\mathfrak{g}$-supermodules, with a particular emphasis on finite-dimensional cases.

Specifically, in Section 13.3.2, we prove that the Dirac cohomology of highest-weight $\mathfrak{g}$-supermodules is always non-trivial (Proposition 13.3.7). In Section 13.3.3, we refine this analysis to the finite-dimensional setting, where we perform explicit computations of Dirac cohomology.

More precisely, we determine the Dirac cohomology of finite-dimensional supermodules over classical Lie superalgebras of type 1 (*i.e.*, $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{g} = A(m|n)$ or $\mathfrak{g} = C(n)$) with a typical highest weight (Theorem 13.3.10), and of finite-dimensional simple objects in $\mathcal{O}^{\mathfrak{p}}$ (Theorem 13.3.11).

13.3.1. Decomposition of $M \otimes \overline{M}(\mathfrak{s})$

Let M be a $(\mathfrak{g},\mathfrak{l})$-supermodule that admits Jordan–Hölder series with simple quotients isomorphic to highest weight $\mathfrak{g}$-supermodules. Then any simple quotient admits an infinitesimal character χ_Λ, and the quadratic Casimir $\Omega_{\mathfrak{g}}$ acts as the scalar multiple of $(\Lambda + 2\rho, \Lambda)$. In addition, $\overline{M}(\mathfrak{s})$ is a completely reducible $\mathfrak{l}$-supermodule by Proposition 13.2.19.

As a consequence, we will decompose $M \otimes \overline{M}(\mathfrak{s})$ into generalized eigenspaces of the Laplace operator $\boldsymbol{\Delta} = \mathrm{D}(\mathfrak{g},\mathfrak{l})^2$, given by

$$M \otimes \overline{M}(\mathfrak{s}) = \bigoplus_{r \in \mathbb{C}} (M \otimes \overline{M}(\mathfrak{s}))(r),$$

where the *generalized eigenspaces* are defined as

$$(M \otimes \overline{M}(\mathfrak{s}))(r) := \{v \in M \otimes \overline{M}(\mathfrak{s}) : \exists n := n(v) \in \mathbb{Z}_+ \text{ s.t. } (r \cdot \mathrm{id}_{M \otimes \overline{M}(\mathfrak{s})} - \boldsymbol{\Delta})^n v = 0\}.$$

An element $v \in (M \otimes \overline{M}(\mathfrak{s}))(r)$ is called a *generalized eigenvector* with *generalized eigenvalue* r. To achieve this decomposition, we first decompose $M \otimes \overline{M}(\mathfrak{s})$ with respect to infinitesimal characters.

Lemma 13.3.1. *The $\mathfrak{l}$-supermodule $M \otimes \overline{M}(\mathfrak{s})$ has a direct sum decomposition*

$$M \otimes \overline{M}(\mathfrak{s}) = \bigoplus_{\nu \in \mathfrak{h}^*} (M \otimes \overline{M}(\mathfrak{s}))^{\eta^{\mathfrak{l}}_\nu},$$

where $(M \otimes \overline{M}(\mathfrak{s}))^{\eta^{\mathfrak{l}}_{\nu}}$ *is the generalized infinitesimal character subspace*

$(M \otimes \overline{M}(\mathfrak{s}))^{\eta^{\mathfrak{l}}_{\nu}} := \{v \in M \otimes \overline{M}(\mathfrak{s}) :$ *for all* $z \in \mathfrak{Z}(\mathfrak{l})$ *exists* $n := n(z,v)$ *s.t.* $(z - \chi^{\mathfrak{l}}_{\nu}(z))^n v = 0\}$.

Proof. The $\mathfrak{l}$-supermodule $M \otimes \overline{M}(\mathfrak{s})$ is $\mathfrak{h}$-semisimple, hence, we can consider its weight space decomposition

$$M \otimes \overline{M}(\mathfrak{s}) = \bigoplus_{\mu \in \mathfrak{h}^*} (M \otimes \overline{M}(\mathfrak{s}))^{\mu},$$

and any weight space $(M \otimes \overline{M}(\mathfrak{s}))^{\mu}$ is finite-dimensional. Consequently, each weight space decomposes in generalized eigenspaces for some $z \in \mathfrak{Z}(\mathfrak{l})$, which concludes the proof. □

By Theorem 13.2.3 and a direct calculation, on any $(M \otimes \overline{M}(\mathfrak{s}))^{\eta^{\mathfrak{l}}_{\nu}}$, the cubic Dirac operator $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ acts as a scalar multiple of

$$c_{\nu} := \chi_{\Lambda}(\Omega_{\mathfrak{g}}) - \chi^{\mathfrak{l}}_{\nu}(\Omega_{\mathfrak{l}}) - c,$$

where the value c is given explicitly in Lemma 13.2.4.

Proposition 13.3.2. *The* $\mathfrak{l}$*-supermodule decomposes into a direct sum of generalized* $\mathbf{\Delta}$*-eigenspaces*

$$M \otimes \overline{M}(\mathfrak{s}) = \bigoplus_{r \in \mathbb{C}} (M \otimes \overline{M}(\mathfrak{s}))(r).$$

In particular, the following decomposition holds:

$$M \otimes \overline{M}(\mathfrak{s}) = \ker \mathbf{\Delta} \oplus \operatorname{im} \mathbf{\Delta}.$$

Proof. The lemma is a direct consequence of Lemma 13.3.1 and Equation (13.3.1) with

$$(M \otimes \overline{M}(\mathfrak{s}))(r) = \bigoplus_{\nu \in \mathfrak{h}^* \,:\, c_{\nu} = r} (M \otimes \overline{M}(\mathfrak{s}))^{\chi^{\mathfrak{l}}_{\nu}}.$$

This concludes the proof. □

As a direct consequence, we provide an equivalent description of the Dirac cohomology. Let $\mathrm{D}(\mathfrak{g},\mathfrak{l})'$ denote the Dirac operator restricted to $\ker \mathbf{\Delta}$. Since $(\mathrm{D}(\mathfrak{g},\mathfrak{l})')^2 = 0$, we have $\operatorname{im} \mathrm{D}(\mathfrak{g},\mathfrak{l})' \subset \ker \mathrm{D}(\mathfrak{g},\mathfrak{l})'$, allowing us to define the cohomology

$$\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})'}(M) := \ker \mathrm{D}(\mathfrak{g},\mathfrak{l})' / \operatorname{im} \mathrm{D}(\mathfrak{g},\mathfrak{l})'.$$

This cohomology is isomorphic to the Dirac cohomology $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ as $\ker \mathrm{D}(\mathfrak{g},\mathfrak{l}) \subset \ker \mathbf{\Delta}$.

Corollary 13.3.3. *As* $\mathfrak{l}$*-supermodules, the following isomorphism holds:*

$$\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})'}(M) \cong \mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M).$$

Moreover, we obtain a relation between $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ and the decomposition of $M \otimes \overline{M}(\mathfrak{s})$.

Lemma 13.3.4. *The following three assertions are equivalent:*

a) $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) = \ker \mathrm{D}(\mathfrak{g},\mathfrak{l})$.

b) $\ker \mathrm{D}(\mathfrak{g},\mathfrak{l}) \cap \operatorname{im} \mathrm{D}(\mathfrak{g},\mathfrak{l}) = \{0\}$.

c) $M \otimes \overline{M}(\mathfrak{s}) = \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \oplus \operatorname{im} \mathrm{D}(\mathfrak{g}, \mathfrak{l})$.

Proof. The assertion b) follows from a) by definition of Dirac cohomology. To deduce c) from b), we note that

$$M \otimes \overline{M}(\mathfrak{s}) = \ker \boldsymbol{\Delta} \oplus \operatorname{im} \boldsymbol{\Delta}.$$

by Proposition 13.3.2, and $\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \subset \ker \boldsymbol{\Delta}$, $\operatorname{im} \boldsymbol{\Delta} \subset \operatorname{im} \mathrm{D}(\mathfrak{g}, \mathfrak{l})$. We show $\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) = \ker \boldsymbol{\Delta}$ and $\operatorname{im} \mathrm{D}(\mathfrak{g}, \mathfrak{l}) = \operatorname{im} \boldsymbol{\Delta}$.

Assume $v \in \ker \boldsymbol{\Delta} \setminus \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l})$. Then $w := \mathrm{D}(\mathfrak{g}, \mathfrak{l})v \neq 0$, by assumption, and $\mathrm{D}(\mathfrak{g}, \mathfrak{l})w = 0$. However, $w \in \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \cap \operatorname{im} \mathrm{D}(\mathfrak{g}, \mathfrak{l})$, which is trivial. We conclude $\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) = \ker \boldsymbol{\Delta}$. Similarly, the equality $\operatorname{im} \mathrm{D}(\mathfrak{g}, \mathfrak{l}) = \operatorname{im} \boldsymbol{\Delta}$ holds.

Finally, the assertion a) follows from c) by definition again. □

13.3.2. Generalities on Dirac cohomology of highest weight supermodules

We now characterize the kernel of the Laplace operator for admissible highest weight $(\mathfrak{g}, \mathfrak{l})$-supermodules and establish that the Dirac cohomology of highest weight $\mathfrak{g}$-supermodules is non-trivial.

Let M be an admissible highest weight $(\mathfrak{g}, \mathfrak{l})$-supermodule with highest weight $\Lambda \in \mathfrak{h}^*$. The explicit form of the Laplace operator $\boldsymbol{\Delta}$, as provided in Theorem 13.2.3, is given by

$$\boldsymbol{\Delta} = \Omega_{\mathfrak{g}} \otimes 1 - \Omega_{\mathfrak{l},\Delta} + c(1 \otimes 1),$$

where $c = (\rho, \rho) - (\rho^{\mathfrak{l}}, \rho^{\mathfrak{l}})$. According to Lemma 3.1.19, the action of $\Omega_{\mathfrak{g}}$ on M is the scalar $(\Lambda + 2\rho, \Lambda)$ times the identity. Furthermore, as a $\mathfrak{l}$-supermodule, $M \otimes \overline{M}(\mathfrak{s})$ fully decomposes into simple highest weight $\mathfrak{l}$-supermodules. By applying Lemma 3.1.19, we get the following result.

Lemma 13.3.5. *Let M be an admissible highest weight $(\mathfrak{g}, \mathfrak{l})$-supermodule with highest weight $\Lambda \in \mathfrak{h}^*$. Let N be a simple $\mathfrak{l}$-constituent of $M \otimes \overline{M}(\mathfrak{s})$ with highest weight $\nu \in \mathfrak{h}^*$. Then $\boldsymbol{\Delta}$ acts on N as the scalar multiple*

$$(\Lambda + 2\rho, \Lambda) - (\nu + 2\rho^{\mathfrak{l}}, \nu) + (\rho, \rho) - (\rho^{\mathfrak{l}}, \rho^{\mathfrak{l}})$$

of the identity. In particular, a simple $\mathfrak{l}$-constituent N of highest weight ν belongs to $\ker \boldsymbol{\Delta}$ *if and only if*

$$(\Lambda + 2\rho, \Lambda) + (\rho, \rho) = (\nu + 2\rho^{\mathfrak{l}}, \nu) + (\rho^{\mathfrak{l}}, \rho^{\mathfrak{l}}).$$

Corollary 13.3.6. *Let M be an admissible highest weight $(\mathfrak{g}, \mathfrak{l})$-supermodule with highest weight $\Lambda \in \mathfrak{h}^*$, and let N be a simple $\mathfrak{l}$-supermodule constituent of $M \otimes \overline{M}(\mathfrak{s})$ with highest weight $\nu \in \mathfrak{h}^*$. Assume there exist $w \in W^{\mathfrak{g}}$, $t_1, \dots, t_k \in \mathbb{C}$, and linearly independent odd isotropic roots $\alpha_1, \dots, \alpha_k \in \Delta^+$ satisfying $(\Lambda + \rho, \alpha_i) = 0$ for all i, such that*

$$\nu = w\Big(\Lambda + \rho + \sum_{i=1}^{k} t_i \alpha_i\Big) - \rho^{\mathfrak{l}}.$$

Then N belongs to $\ker \boldsymbol{\Delta}$.

Finally, we show that the Dirac cohomology of generic highest weight supermodules is non-trivial.

Proposition 13.3.7. *Let M be a highest weight $\mathfrak{g}$-supermodule with highest weight Λ. Then $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ is non-trivial.*

Proof. Let v_M be the highest weight vector of M, and let $v_{\overline{M}(\mathfrak{s})} = 1 \otimes 1 \in \overline{M}(\mathfrak{s})$. In particular, $\mathfrak{u}$ annihilates v_M and $v_{\overline{M}(\mathfrak{s})}$, respectively, where for the latter we may identify elements of $\mathfrak{u}$ with $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial \eta_j}$, as in Section 13.1.3.

First, we show that $v_M \otimes v_{\overline{M}(\mathfrak{s})} \in M \otimes \overline{M}(\mathfrak{s})$ lies in $\ker \mathrm{D}(\mathfrak{g},\mathfrak{l})$. As in Section 13.2.1, we decompose $\mathrm{D}(\mathfrak{g},\mathfrak{l}) = C + \bar{C}$, where $C = A + 1 \otimes a$ and $\bar{C} = \bar{A} + 1 \otimes \bar{a}$, with

$$A = \sum_{i=1}^{s} u_i \otimes \overline{u}_i, \qquad \bar{A} = \sum_{i=1}^{s} (-1)^{p(u_i)} \overline{u}_i \otimes u_i,$$

$$a = -\frac{1}{4} \sum_{1 \le i,j \le s} (-1)^{p(u_i)p(u_j)+p(u_i)+p(u_j)} [\overline{u}_i, \overline{u}_j]\, u_i u_j,$$

$$\bar{a} = -\frac{1}{4} \sum_{1 \le i,j \le s} (-1)^{p(u_i)p(u_j)} [u_i, u_j] \overline{u}_i \overline{u}_j.$$

Since $u_i v_M = 0$ and $u_i v_{\overline{M}(\mathfrak{s})} = 0$ for all $i = 1, \ldots, s$, it is immediate that A, $\bar{A}$, and a annihilate $v_M \otimes v_{\overline{M}(\mathfrak{s})}$. To see that $\bar{a}$ annihilates $v_M \otimes v_{\overline{M}(\mathfrak{s})}$, we note that $[u_i, u_j] \in \mathfrak{u}$ for all $1 \le i, j \le s$, and we can rewrite $\bar{a}$ as (Remark 13.2.8)

$$\bar{a} = -\frac{1}{4} \sum_{1 \le i,j \le s} (-1)^{p(u_i)p(u_j)+p(u_i)+p(u_j)} \overline{u}_i \overline{u}_j [u_i, u_j].$$

We conclude that $v_M \otimes v_{\overline{M}(\mathfrak{s})} \in \ker \mathrm{D}(\mathfrak{g},\mathfrak{l})$.

Furthermore, under the identification of $\mathfrak{u}$ with elements of the form $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial \eta_j}$ acting on $\bigwedge \overline{\mathfrak{u}}$, the discussion in Section 13.1.3 makes it immediate that $v_{\overline{M}(\mathfrak{s})}$, and therefore $v_M \otimes v_{\overline{M}(\mathfrak{s})}$, cannot lie in the image of $\mathrm{D}(\mathfrak{g},\mathfrak{l})$. This concludes the proof. □

13.3.3. Dirac cohomology and finite-dimensional supermodules

In this subsection, we specify to the Dirac cohomology of finite-dimensional simple admissible $(\mathfrak{g},\mathfrak{l})$-supermodules, where $\mathfrak{g}$ is a basic classical Lie superalgebra of type 1. In particular, we will compute the Dirac cohomology of those with typical highest weight and finite-dimensional simple objects in $\mathcal{O}^{\mathfrak{p}}$.

In the following, throughout this subsection, we let $\mathfrak{g}$ be a basic classical Lie superalgebra of type 1, *i.e.*, $\mathfrak{g}$ is $\mathfrak{gl}(m|n)$, $A(m|n)$, or $C(n)$. The simple finite-dimensional supermodules are parameterized by dominant integral weights $\lambda \in \mathfrak{h}^*$ with respect to some Borel subalgebra $\mathfrak{b} = \mathfrak{b}_{\bar{0}} \oplus \mathfrak{b}_{\bar{1}}$, *i.e.*, those weights for which there exists a finite-dimensional simple $\mathfrak{g}_{\bar{0}}$-module with highest weight λ for $\mathfrak{b}_{\bar{0}}$. More precisely, λ is dominant integral if and only if

$$(\lambda + \rho_{\bar{0}}, \alpha) > 0 \text{ for all } \alpha \in \Delta^+_{\bar{0}},$$

where $\Delta^+ = \Delta^+_{\bar{0}} \sqcup \Delta^+_{\bar{1}}$ is the positive system with respect to $\mathfrak{b}$. We denote the set of $\mathfrak{b}$-dominant integral weights by $P^{++}_{\mathfrak{b}}$, and call them also Δ^+-dominant integral. Moreover, if $\mathfrak{b}$ is clear from the context, we omit the subscript and simply write P^{++}.

For any $\lambda \in P_{\mathfrak{b}}^{++}$, we define $L_{\mathfrak{b}}(\lambda)$ (or simply $L(\lambda)$ when no confusion arises) to be the simple supermodule with highest weight λ with respect to $\mathfrak{b}$ such that the highest weight vector is even. The fixed notation allows us to parameterize the simple finite-dimensional $\mathfrak{g}$-supermodules as follows:

$$\{L(\lambda), \Pi L(\lambda) : \lambda \in P^{++}\}.$$

In the following, we assume that $M := L(\lambda)$, $\lambda \in P_{\mathfrak{b}}^{++}$, is an admissible finite-dimensional simple $(\mathfrak{g}, \mathfrak{l})$-supermodule, where $\mathfrak{b}$ is the distinguished Borel subalgebra. The *distinguished Borel subalgebra* is defined with respect to the *distinguished positive system* Δ^+, that is, the positive system with the smallest number of odd roots. For this system, as a direct calculation shows, we have

$$(\rho_{\bar{1}}, \alpha) \begin{cases} = 0 & \text{if } \alpha \in \Delta_{\bar{0}}, \\ > 0 & \text{if } \alpha \in \Delta_{\bar{1}}^+. \end{cases}$$

The $\mathfrak{l}$-supermodule $M \otimes \overline{M}(\mathfrak{s})$ decomposes completely in finite-dimensional weight $\mathfrak{l}$-supermodules by Proposition 13.2.19. Each finite-dimensional simple weight $\mathfrak{l}$-supermodule is of highest weight type with respect to some positive system $\Delta(\mathfrak{l};\mathfrak{h})^+$. The associated highest weight $\mu \in \mathfrak{h}^*$ is called $\Delta(\mathfrak{l};\mathfrak{h})^+$*-dominant integral*. If ν is the $\Delta(\mathfrak{l};\mathfrak{h})^+$-dominant integral, we denote the associated simple $\mathfrak{l}$-supermodule by $L_{\mathfrak{l}}(\nu)$.

For the remainder, we fix a positive system $\Delta(\mathfrak{l};\mathfrak{h})^+$, which is contained in the distinguished positive system Δ^+. In particular, if M is a highest weight $\mathfrak{g}$-supermodule with highest weight λ with respect to Δ^+, then λ is the highest weight with respect to $\Delta(\mathfrak{l};\mathfrak{h})^+$ of an $\mathfrak{l}$-constituent of M. For any $\mu \in \mathfrak{h}^*$, we define

$$W_\lambda^{\mathfrak{l},1} := \{w \in W^{\mathfrak{l}} : w(\lambda + \rho^{\mathfrak{l}}) \text{ is } \Delta(\mathfrak{l};\mathfrak{h})^+\text{-dominant integral}\}.$$

Note that $w\Delta^+$ contains $\Delta(\mathfrak{l};\mathfrak{h})^+$ whenever $w \in W_\lambda^{\mathfrak{l},1}$.

We are now ready to compute the Dirac cohomology. We use that $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ decomposes completely in simple finite-dimensional highest weight $\mathfrak{l}$-supermodules, and, crucially, Theorem 13.2.27. We start with two lemmas that will enter the proofs of our main result.

Lemma 13.3.8. *Let M be an admissible finite-dimensional simple $(\mathfrak{g}, \mathfrak{l})$-supermodule with highest weight Λ. Then $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ decomposes in a direct sum of simple finite-dimensional highest weight $\mathfrak{l}$-supermodules each of with highest weight ν of the form*

$$\nu = w\left(\Lambda + \rho + \sum_{i=1}^{k} t_i \alpha_i\right) - \rho^{\mathfrak{l}},$$

for some $w \in W^{\mathfrak{l}}$, $t_i \in \mathbb{C}$ and isotropic odd roots α_i in $\Delta(\mathfrak{l};\mathfrak{h})$ satisfying $(\Lambda + \rho, \alpha_i) = 0$.

Proof. As M is a highest weight $\mathfrak{g}$-supermodule with highest weight Λ, the Dirac cohomology contains a simple highest weight $\mathfrak{l}$-supermodule with highest weight $\Lambda + \rho^{\mathfrak{u}}$ by Proposition 13.3.7. If V is another simple highest weight $\mathfrak{g}$-supermodule with highest weight ν, Theorem 13.2.27 dictates for any $z \in \mathfrak{Z}(\mathfrak{l})$ on $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ the equality,

$$\chi^{\mathfrak{l}}_{\Lambda+\rho^{\mathfrak{u}}}(z) = \chi^{\mathfrak{l}}_{\Lambda+\rho^{\mathfrak{u}}}(\eta_{\mathfrak{l}}(z)) = \chi^{\mathfrak{l}}_{\nu}(\eta_{\mathfrak{l}}(z)) = \chi^{\mathfrak{l}}_{\nu}(z),$$

which concludes the proof with Corollary 3.1.23 and $\rho = \rho^{\mathfrak{l}} + \rho^{\mathfrak{u}}$. $\square$

Lemma 13.3.9. *Let M be an admissible finite-dimensional simple $\mathfrak{g}$-supermodule with highest weight Λ. Then any simple $\mathfrak{l}$-constituent in $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ appears with multiplicity one.*

Proof. Let V be a (non-trivial) simple $\mathfrak{l}$-constituent of $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ with highest weight ν. By Lemma 13.3.8, the highest weight ν is of the form

$$\nu = w\Big(\Lambda + \rho + \sum_{i=1}^{k} t_i\alpha_i\Big) - \rho^{\mathfrak{l}},$$

for some $w \in W^{\mathfrak{l}}$, $t_i \in \mathbb{C}$ and isotropic odd roots α_i in $\Delta(\mathfrak{l};\mathfrak{h})$ satisfying $(\Lambda + \rho, \alpha_i) = 0$. We may assume $w \in W_{\mu}^{\mathfrak{l},1}$ for $\mu = \Lambda + \rho^{\mathfrak{u}} + \sum_{i=1}^{k} t_i\alpha_i$. To prove the lemma, we have to show that

$$w(\lambda + \rho + \sum_{i=1}^{k} t_i\alpha_i) - \rho^{\mathfrak{l}} = w(\lambda + \sum_{i=1}^{k} t_i\alpha_i) + \rho^{w\mathfrak{u}}$$

appears with multiplicity one. Here, we denote by $\rho^{w\mathfrak{u}}$ the Weyl element with respect to $w(\Delta^+ \setminus \Delta(\mathfrak{l};\mathfrak{h})^+)$, and note that $w\rho^{\mathfrak{l}} = \rho^{\mathfrak{l}}$.

Assume this is not the case. Then, as we are dealing with highest weight supermodules, there exists $A \subset w(\Delta^+ \setminus \Delta(\mathfrak{l};\mathfrak{h})^+)$ and $B \subset w\Delta^+$ with

$$w(\Lambda + \sum_{i=1}^{k} t_i\alpha_i) + \rho^{w\mathfrak{u}} = (w(\Lambda + \sum_{i=1}^{k} t_i\alpha_i) - \mathbb{Z}_+[B]) + (\rho^{w\mathfrak{u}} - \mathbb{Z}_+[A]),$$

where $\mathbb{Z}_+[A] := \sum_{\xi\in A} \mathbb{Z}_+\xi$ and $\mathbb{Z}_+[B] := \sum_{\zeta\in A} \mathbb{Z}_+\zeta$. This forces $\mathbb{Z}_+[A] + \mathbb{Z}_+[B] = 0$. Now, if we are taking the inner product with $\rho_{\bar{0}}$ and use the invariance of $(\cdot,\cdot)$ under the action of the Weyl group together with Equation (13.3.3), we deduce that neither A nor B can contain odd roots. On the other hand, taking then the inner product with $w(\lambda + \rho_{\bar{0}})$, we obtain

$$\sum_{\alpha\in A} (w(\lambda + \rho_{\bar{0}}), \alpha) + \sum_{\beta\in B} (w(\lambda + \rho_{\bar{0}}), \beta) = 0,$$

and each summand is strictly positive as no odd roots appear and $w(\Lambda + \rho_0)$ is dominant integral. Consequently, $A = \emptyset$ and $B = \emptyset$. This concludes the proof. $\square$

The above lemmas enter the proof of the following Theorem, which is one of the main result of this section.

Theorem 13.3.10. *Let M be an admissible finite-dimensional simple $(\mathfrak{g},\mathfrak{l})$-supermodule with typical highest weight Λ. Then*

$$\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) = \bigoplus_{w\in W^{\mathfrak{l},1}_{\Lambda+\rho^{\mathfrak{u}}}} L_{\mathfrak{l}}(w(\Lambda+\rho) - \rho^{\mathfrak{l}}).$$

Proof. Let V be a non-trivial $\mathfrak{l}$-constituent in $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ with highest weight ν. By atypicality of Λ, Lemma 13.3.8 and Lemma 13.3.9, the simple finite-dimensional $\mathfrak{l}$-supermodule V appears with multiplicity one and has highest weight

$$\nu = w(\Lambda + \rho) - \rho^{\mathfrak{l}}$$

for some $w \in W_\Lambda^{\mathfrak{l},1}$. Consequently, it is enough to show that each $w(\Lambda+\rho)-\rho^{\mathfrak{l}}$ appears as a weight in $M \otimes \overline{M}(\mathfrak{s})$. However, this is immediate as

$$w(\Lambda+\rho)-\rho^{\mathfrak{l}} = w(\Lambda) + (w(\rho)-\rho^{\mathfrak{l}}) = w(\Lambda) + w(\rho^{\mathfrak{u}})$$

is a sum of extreme weights in M and $\overline{M}(\mathfrak{s})$, respectively, which are $\Delta(\mathfrak{l};\mathfrak{h})^+$-dominant integral. ☐

These results for general finite-dimensional admissible $(\mathfrak{g},\mathfrak{l})$-supermodules allow us to compute the Dirac cohomology of finite-dimensional simple objects in $\mathcal{O}^{\mathfrak{p}}$ for some parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$ with reductive $\mathfrak{l}_{\bar{0}}$.

Dirac cohomology and finite-dimensional simple objects in $\mathcal{O}^{\mathfrak{p}}$

Let $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$ be a parabolic subalgebra with reductive $\mathfrak{l}_{\bar{0}}$. This is the case if the Levi subalgebra is good [30, Section 7]. Recall that by Theorem 13.1.7, we may consider any finite-dimensional simple M as the unique simple quotient of a parabolically induced supermodule, where $\mathfrak{p}$ has good Levi subalgebra $\mathfrak{l}$.

By definition of $\mathcal{O}^{\mathfrak{p}}$ in Section 13.1.1, any object decomposes completely in finite-dimensional simple $\mathfrak{l}_{\bar{0}}$-modules. In particular, if M belongs to $\mathcal{O}^{\mathfrak{p}}$, the supermodules $M \otimes \overline{M}(\mathfrak{s})$ and $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ decompose completely in finite-dimensional $\mathfrak{l}_{\bar{0}}$-modules by a similar argumentation as above.

Let $M := L(\Lambda) \in \mathcal{O}^{\mathfrak{p}}$ be a finite-dimensional simple object with highest weight $\Lambda \in \mathfrak{h}^*$ with respect to some positive system. Fix a positive system $\Delta(\mathfrak{l};\mathfrak{h})^+$. Then, noting $\mathfrak{h} \subset \mathfrak{l}_{\bar{0}}$, we may identify $\Delta(\mathfrak{l}_{\bar{0}};\mathfrak{h})$ with $\Delta(\mathfrak{l};\mathfrak{h})_{\bar{0}}$ and $\Delta(\mathfrak{l}_{\bar{0}};\mathfrak{h})^+$ with $\Delta(\mathfrak{l};\mathfrak{h})_{\bar{0}}^+$.

Adapting Theorem 13.2.27 and Lemma 13.3.8 to the new setting, we conclude that any $\mathfrak{l}_{\bar{0}}$-constituent in $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ has highest weight

$$\nu = w(\Lambda + \rho^{\mathfrak{u}} + \rho^{\mathfrak{l}_{\bar{0}}}) - \rho^{\mathfrak{l}_{\bar{0}}}$$

for some $w \in W_\lambda^{\mathfrak{l}_{\bar{0}},1} = \{w \in W^{\mathfrak{g}} : w(\lambda+\rho^{\mathfrak{l}_{\bar{0}}}) \text{ is } \Delta(\mathfrak{l}_{\bar{0}};\mathfrak{h})^+\text{-dominant integral}\}$. No isotropic roots appear as $\mathfrak{l}_{\bar{0}}$ is purely even. Now, a straightforward modification of the proofs of Lemma 13.3.9 and Theorem 13.3.10 leads to the following theorem.

Theorem 13.3.11. *Let $M \in \mathcal{O}^{\mathfrak{p}}$ be simple and finite-dimensional with highest weight Λ. Then*

$$\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) = \bigoplus_{w \in W_{\Lambda+\rho^{\mathfrak{u}}}^{\mathfrak{l}_{\bar{0}},1}} L_{\mathfrak{l}_{\bar{0}}}(w(\Lambda+\rho^{\mathfrak{u}}+\rho^{\mathfrak{l}_{\bar{0}}})-\rho^{\mathfrak{l}_{\bar{0}}}).$$

13.4. Dirac cohomology and Kostant's cohomology

In this section, we study the relation between Dirac cohomology and Kostant's (co)homology. For that, we briefly introduce Kostant's (co)homology in Subsection 13.4.1. Then Proposition 13.2.20 allows us to decompose the cubic Dirac operator in terms of the boundary and coboundary operators of Kostant's (co)homology. This decomposition is used to deduce an embedding of the Dirac cohomology into Kostant's (co)homology in Theorem 13.4.10.

Afterward, we study Dirac cohomology for unitarizable supermodules with respect to Hermitian real forms, introduced in Subsection 13.4.3. In this case, we show that Dirac cohomology and Kostant's (co)homology are isomorphic as supermodules over the Levi subalgebra.

Finally, as an application, we study the Dirac cohomology of simple weight supermodules in Subsection 13.4.4. These supermodules are particularly interesting because they are $\mathfrak{h}$-semisimple with finite-dimensional weight spaces. We establish in Theorem 13.4.20 that weight supermodules have trivial Dirac cohomology unless they are highest weight supermodules.

13.4.1. Kostant's $\mathfrak{u}$-cohomology and $\overline{\mathfrak{u}}$-homology

We start fixing some notations. Let M be a $\mathfrak{g}$-supermodule that is $(\mathfrak{g}, \mathfrak{l})$-admissible, *i.e.*, M is a $\mathfrak{g}$-supermodule that is $\mathfrak{l}$-semisimple. Fix a parabolic subalgebra $\mathfrak{p} := \mathfrak{l} \ltimes \mathfrak{u}$ with opposite parabolic subalgebra $\overline{\mathfrak{p}} = \mathfrak{l} \ltimes \overline{\mathfrak{u}}$. Recall that $\mathfrak{l}$ is the Levi subalgebra and $\mathfrak{u}, \overline{\mathfrak{u}}$ are the nilradicals of $\mathfrak{p}$ and $\overline{\mathfrak{p}}$, respectively.

Recall that the exterior superalgebras $\bigwedge \mathfrak{u}$ and $\bigwedge \overline{\mathfrak{u}}$ over the super vector spaces $\mathfrak{u}$ and $\overline{\mathfrak{u}}$ inherit a natural $\mathbb{Z}$-grading induced by the $\mathbb{Z}$-grading of the tensor superalgebras $T(\mathfrak{u})$ and $T(\overline{\mathfrak{u}})$, respectively. More precisely, let V be either $\mathfrak{u}$ or $\overline{\mathfrak{u}}$, then the *exterior n-power* is the super vector space (see Example 2.1.2)

$$\bigwedge^n V = V^{\otimes n}/\mathfrak{J}_n,$$

where $\mathfrak{J}_n$ is the subspace of $V^{\otimes n}$ generated by the elements of the form

$$v_1 \otimes \cdots \otimes v_n - (-1)^{p(\sigma)} \sigma \cdot v_1 \otimes \cdots \otimes v_n, \qquad \sigma \in S_n,\ v_i \in V.$$

Here, $p(\sigma)$ denotes the parity of the permutation σ.

After establishing the notation, we introduce Kostant's (co)homology, see [17, Section 6.4], which is a natural generalization of the classical Lie algebra (co)homology theory, see for example [82] or [25] for the super case. We define the space of *p-chains*, $C_p(\overline{\mathfrak{u}}, M)$, and the space of *$p$-cochains*, $C^p(\mathfrak{u}, M)$, as

$$C_p(\overline{\mathfrak{u}}, M) := \bigwedge^p \overline{\mathfrak{u}} \otimes M, \qquad C^p(\mathfrak{u}, M) := \mathrm{Hom}_{\mathbb{C}}(\bigwedge^p \mathfrak{u}, M)$$

for any $p \in \mathbb{Z}_+$. Note that $C_p(\overline{\mathfrak{u}}, M)$ and $C^p(\mathfrak{u}, M)$ are naturally $\mathfrak{p}$-supermodules, and if M is finite-dimensional, there is a natural identification $C_p(\overline{\mathfrak{u}}, M) \cong C^p(\mathfrak{u}, M)$. Moreover, we set $C_* := \sum_{p \geq 0} C_p(\overline{\mathfrak{u}}, M)$, and $C^* := \sum_{p \geq 0} C^p(\mathfrak{u}, M)$, which are $\mathfrak{p}$-supermodules by construction.

Define the *boundary operator* $d := \sum_p d_p : C_*(\overline{\mathfrak{u}}, M) \to C_*(\overline{\mathfrak{u}}, M)$ by setting

$$\begin{aligned} d_p(x_1 \ldots x_p \otimes v) := \\ &\sum_{s=1}^{p} (-1)^{s + p(x_s) \sum_{i=s+1}^{p} p(x_i)} x_1 \ldots \hat{x}_s \ldots x_p \otimes x_s v \\ &+ \sum_{1 \leq s < t \leq p} (-1)^{s+t+p(x_s) \sum_{i=1}^{s-1} p(x_i) + p(x_t) \sum_{j=1}^{t-1} p(x_j) + p(x_s)p(x_t)} [x_s, x_t] x_1 \ldots \hat{x}_s \ldots \hat{x}_t \ldots x_p \otimes v \end{aligned}$$

for $x_i \in \mathfrak{u}$ homogeneous and $v \in M$. Here, $\hat{x}$ indicates that the corresponding term x is omitted. A straightforward calculation yields $d_{p-1} \circ d_p = 0$ for any $p \in \mathbb{Z}_+$. The p-th

$\overline{\mathfrak{u}}$-*homology group* with coefficients in M, denoted by $H_p(\overline{\mathfrak{u}}, M)$, is defined to be the p-th homology group of the following chain complex:

$$\ldots \to C_p(\overline{\mathfrak{u}}, M) \xrightarrow{d_p} C_{p-1}(\overline{\mathfrak{u}}, M) \xrightarrow{d_{p-1}} \ldots \xrightarrow{d_2} \overline{\mathfrak{u}} \otimes V \xrightarrow{d_1} M \xrightarrow{d_0} 0,$$

that is, $H_p(\overline{\mathfrak{u}}, M) := \ker d_p / \operatorname{im} d_{p+1}$ for $p \in \mathbb{Z}_+$. We refer to $H_p(\overline{\mathfrak{u}}, M)$ as *Kostant's homology* for M.

Define the *coboundary operator* $\partial := \sum_p \partial_p : C^*(\mathfrak{u}, M) \to C^*(\mathfrak{u}, M)$ by

$$(\partial_p f)(x_1, \ldots, x_{p+1}) :=$$
$$\sum_{s=1}^{p+1} (-1)^{s+1+p(x_s)(p(f)+\sum_{i=1}^{s-1} p(x_i))} x_s f(x_1, \ldots, x_{s-1}, \hat{x}_s, x_{s+1}, \ldots, x_{p+1})$$
$$+ \sum_{s<t} (-1)^{s+t+p(x_s)\sum_{i=1}^{s-1} p(x_i)+p(x_t)\sum_{j=1}^{t} p(x_j)+p(x_s)p(x_t)} f([x_s, x_t], x_1, \ldots, \hat{x}_s, \ldots, \hat{x}_t, \ldots, x_{p+1}),$$

for homogeneous $x_i \in \mathfrak{u}$. The coboundary operator ∂ satisfies $\partial_p \circ \partial_{p-1} = 0$ for all $p \in \mathbb{Z}_+$. The p-th $\mathfrak{u}$-*cohomology group* with coefficients in M, denoted by $H^p(\mathfrak{u}, M)$, is defined to be the p-th cohomology group of the following chain complex:

$$0 \xrightarrow{\partial_{-1}} M \xrightarrow{\partial_0} C^1(\mathfrak{u}, M) \xrightarrow{\partial_1} \ldots \xrightarrow{\partial_{p-1}} C^p(\mathfrak{u}, M) \xrightarrow{\partial_p} C^{p+1}(\mathfrak{u}, M) \xrightarrow{\partial_{p+1}} \ldots,$$

i.e., $H^p(\mathfrak{u}, M) := \ker \partial_p / \operatorname{im} \partial_{p-1}$.We refer to $H^p(\mathfrak{u}, M)$ as *Kostant's cohomology* for M. As in [66], we will consider the operator $\delta = -2d$. It is clear, that δ defines the same cohomology as d.

A direct calculation yields the subsequent lemma.

Lemma 13.4.1. *The boundary operator $d : C_*(\overline{\mathfrak{u}}, M) \to C_*(\overline{\mathfrak{u}}, M)$ and the coboundary operator $\partial : C^*(\mathfrak{u}, M) \to C^*(\mathfrak{u}, M)$ are $\mathfrak{l}$-supermodule morphisms. In particular, $H_p(\overline{\mathfrak{u}}, M)$ and $H^p(\mathfrak{u}, M)$ are $\mathfrak{l}$-supermodules.*

Remark 13.4.2. The boundary operators ∂ is even a $\mathfrak{p}$-supermodule morphism.

Having introduced Kostant's homology and cohomology, we now describe their relation in the case M is an admissible $(\mathfrak{g}, \mathfrak{l})$-supermodule, following [17]. In this case, M is also an admissible $(\mathfrak{g}, \mathfrak{h})$-supermodule, that is $M = \oplus_{\mu \in \mathfrak{h}^*} M^\mu$ with $\dim(M^\mu) < \infty$. To any such M, we can assign a *dual supermodule*. Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be the Chevalley automorphism of $\mathfrak{g}$, and for any $\mu \in \mathfrak{h}^*$, let $(M^\mu)^*$ denote the dual space of the finite-dimensional weight space M^μ. The *dual* of M is the $\mathfrak{g}$-supermodule

$$M^\vee := \bigoplus_{\mu \in \mathfrak{h}^*} (M^\mu)^*,$$

with $\mathfrak{g}$-action given by $(X \cdot f)(v) := (-1)^{p(X)p(f)-1} f(\tau(X) \cdot v)$.

Lemma 13.4.3 ([17, Theorem 6.22])**.** *Let M be an admissible $(\mathfrak{g}, \mathfrak{l})$-supermodule and $p \in \mathbb{Z}_+$. Then, as semisimple $\mathfrak{l}$-supermodules, we have the following isomorphisms:*

$$H_p(\overline{\mathfrak{u}}, M^\vee) \cong H^p(\mathfrak{u}, M).$$

In particular, if M is a simple highest weight $\mathfrak{g}$-supermodule, we have

$$H_p(\overline{\mathfrak{u}}, M) \cong H^p(\mathfrak{u}, M).$$

13.4.2. Relation to Dirac cohomology

We construct an explicit embedding of Dirac cohomology into Kostant's cohomology for admissible $(\mathfrak{g},\mathfrak{l})$-supermodules. This involves identifying $M\otimes\overline{M}(\mathfrak{s})$ with $M\otimes\bigwedge\overline{\mathfrak{u}}\otimes\mathbb{C}_{\rho^{\mathfrak{u}}}$ as $\mathfrak{l}$-supermodules (*cf.* Proposition 13.2.20) and interpreting the operators C and $\bar{C}$ (*cf.* Theorem 13.2.10) as the boundary and coboundary operators, respectively.

Proposition 13.4.4. *Let M be an admissible $(\mathfrak{g},\mathfrak{l})$-supermodule. Then under the action of $\mathfrak{U}(\mathfrak{g})\otimes C(\mathfrak{s})$ on $M\otimes\overline{M}(\mathfrak{s})$, and the identification in Proposition 13.2.20, the operators C and $\bar{C}$ act as $\delta=-2d$ and ∂, respectively. In particular, $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ acts as $\partial-2d=\partial+\delta$.*

Proof. Under the identification $\overline{M}(\mathfrak{s})\otimes M\cong\bigwedge\overline{\mathfrak{u}}\otimes\mathbb{C}_{\rho^{\mathfrak{u}}}\otimes M$ of $\mathfrak{l}$-supermodules, we compare the actions of C, $\bar{C}$, d, and ∂. We explicitly perform the calculation for C and the boundary operator d, while the other cases follow by a similar line of argument.

By swapping the order of $\mathfrak{U}(\mathfrak{g})$ and $C(\mathfrak{s})$, we may rewrite $C=A'+a\otimes 1$ with

$$A'=\sum_i(-1)^{p(u_i)}u_i\otimes\overline{u}_i=\sum_i(\overline{u}_i)^*\otimes\overline{u}_i,$$

where we note that exchanging the factors of the tensor product introduces an extra factor $(-1)^{p(u_i)}$ in the sum, and $(\overline{u}_i)^*=(-1)^{p(u_i)}u_i$. To deduce the action of C, we recall the action of $u\in\mathfrak{u}$ and $\overline{u}\in\overline{\mathfrak{u}}$ on $Y=Y_1\dots Y_n\in\overline{M}(\mathfrak{s})$:

$$\overline{u}\cdot Y:=\overline{u}\wedge Y,$$
$$u\cdot Y:=2\sum_{t=1}^{n}(-1)^{t+1+p(u)\sum\limits_{k=1}^{t-1}p(Y_t)}(u,Y_t)Y_1\dots\hat{Y}_t\dots Y_n.$$

For general $\xi=Y_1\dots Y_N\otimes v\in\overline{M}(\mathfrak{s})\otimes M$, we find

$$A'\xi=\sum_{i=1}^{s}2\sum_{t=1}^{n}(-1)^{t+1+p(u_i)\sum\limits_{k=1}^{t-1}p(Y_t)}((\overline{u}_i)^*,Y_t)Y_1\dots\hat{Y}_t\dots Y_n\otimes\overline{u}_iv.$$

Here, we use that $((\overline{u}_i)^*,Y_t)$ is only non-zero if $p(u_i)=p(Y_t)$, and hence we can replace $p(u_i)$ by $p(Y_t)$ in the exponent. Then, since $\sum_i((\overline{u}_i)^*,Y_t)\overline{u}_i=(-1)^{p(Y_t)p(Y_t)}Y_t$, we see that this is equal to minus twice the first sum in the expression for d in Equation (13.4.1).

It remains to identify the action of the cubic term a. In order to compute a, we fix some notation. For $Y_1\dots Y_n$, let

$$\hat{Y}_{t,r}=\gamma(t,r)Y_1\dots\hat{Y}_t\dots\hat{Y}_r\dots Y_n,$$

where

$$\gamma(t,r)=\begin{cases}1 & \text{if } t<r,\\ -(-1)^{p(Y_t)p(Y_r)} & \text{if } t>r,\\ 0 & \text{if } t=r.\end{cases}$$

This allows us to write

$$u_iu_jY_1\dots Y_n=4\sum_{t,r}(-1)^{t+r+p(u_j)\sum\limits_{k=1}^{t-1}p(Y_t)+p(u_i)\sum\limits_{l=1}^{r-1}p(Y_l)}((\overline{u}_i)^*,Y_t)((\overline{u}_j)^*,Y_r)\hat{Y}_{t,r},$$

and $a \otimes 1$ acts on $Y_1 \dots Y_n \otimes v$ as

$$-\frac{1}{4}4 \sum_{i,j,t,r} (-1)^{t+r+p(u_j)\sum_{k=1}^{t-1} p(Y_t)+p(u_i)\sum_{l=1}^{r-1} p(Y_l)} ((\overline{u}_i)^*, Y_t)((\overline{u}_j)^*, Y_r)[\overline{u}_j, \overline{u}_i]\hat{Y}_{t,r} \otimes v.$$

Again, we can replace $p(u_i)$ by $p(Y_t)$ and $p(u_j)$ by $p(Y_r)$. Upon summing $\sum_i ((\overline{u}_i)^*, Y_\kappa)\overline{u}_i = (-1)^{p(Y_\kappa)p(Y_\kappa)} Y_\kappa$ for $\kappa \in \{r, t\}$, we obtain

$$-\sum_{t,r} (-1)^{t+r+p(Y_t)\sum_{k=1}^{t} p(Y_t)+p(Y_r)\sum_{l=1}^{r} p(Y_l)+p(Y_r)p(Y_t)} [Y_t, Y_r]\hat{Y}_{t,r} \otimes v.$$

This expression remains invariant under exchanging the roles of k and l and vanishes for $k = l$. We conclude that it equals twice the sum restricted to $k < l$, i.e., minus twice the second sum in the expression for d. This finishes the proof. □

The subsequent corollary is immediate.

Corollary 13.4.5. *Let M be an admissible $(\mathfrak{g}, \mathfrak{l})$-supermodule. Then, as $\mathfrak{l}$-supermodules, we have the following isomorphisms*

$$H(C, M \otimes \overline{M}(\mathfrak{s})) \cong H_*(\overline{\mathfrak{u}}, M) \otimes \mathbb{C}_{\rho^{\mathfrak{u}}}, \qquad H(\bar{C}, M \otimes \overline{M}(\mathfrak{s})) \cong H^*(\mathfrak{u}, M) \otimes \mathbb{C}_{\rho^{\mathfrak{u}}}$$

We now rely on Proposition 13.4.4 and Corollary 13.4.5 to construct an embedding of the Dirac cohomology in Kostant's cohomology. As a first step, we study the relation between $\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}), \ker \Delta, \ker \delta$ and $\ker \partial$. More precisely, we establish that the map

$$\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \to \ker \Delta \cap \ker \partial, \quad x \mapsto x$$

is a well-defined $\mathfrak{l}$-equivariant injective map. The proof is similar to the proof of Lemma 4.6 and Lemma 4.7 in [68].

To this end, we use the fact that, as $\mathfrak{l}$-supermodules (see Proposition 13.2.20),

$$M \otimes \overline{M}(\mathfrak{s}) \cong M \otimes \bigwedge \overline{\mathfrak{u}} \otimes \mathbb{C}_{\rho^{\mathfrak{u}}}.$$

The exterior superalgebra $\bigwedge \overline{\mathfrak{u}}$ over the super vector space $\mathfrak{u}$ inherits a natural $\mathbb{Z}$-grading induced by the $\mathbb{Z}$-grading of the tensor superalgebra $T(\mathfrak{u})$, which induces an $\mathfrak{l}$-invariant increasing filtration

$$\{0\} \subset \bigwedge^0 \overline{\mathfrak{u}} \subset \bigoplus_{i=0}^{1} \bigwedge^i \overline{\mathfrak{u}} \subset \dots \subset \bigoplus_{i=0}^{s} \bigwedge^i \overline{\mathfrak{u}} = \bigwedge \overline{\mathfrak{u}},$$

and an $\mathfrak{l}$-invariant decreasing filtration

$$\bigwedge \overline{\mathfrak{u}} = \bigoplus_{i=0}^{s} \bigwedge^i \overline{\mathfrak{u}} \supset \bigoplus_{i=1}^{s} \bigwedge^i \overline{\mathfrak{u}} \supset \dots \supset \bigwedge^s \overline{\mathfrak{u}} \supset \{0\}.$$

These $\mathfrak{l}$-invariant increasing/decreasing filtrations induce $\mathfrak{l}$-invariant increasing/decreasing filtration of $M \otimes \bigwedge \overline{\mathfrak{u}} \otimes \mathbb{C}_{\rho^{\mathfrak{u}}}$:

$$\{0\} = X_{-1} \subset X_0 \subset X_1 \subset \dots \subset X_s = M \otimes \bigwedge \overline{\mathfrak{u}} \otimes \mathbb{C}_{\rho^{\mathfrak{u}}},$$
$$M \otimes \bigwedge \overline{\mathfrak{u}} \otimes \mathbb{C}_{\rho^{\mathfrak{u}}} = X^0 \supset X^1 \supset \dots \supset X^s \supset X^{s+1} = \{0\},$$

where $X_k := \bigoplus_{i=0}^{k} M \otimes \bigwedge^i \bar{\mathfrak{u}}$ and $X^k := \bigoplus_{i=k}^{s} M \otimes \bigwedge^i \bar{\mathfrak{u}}$.

The following lemma is an immediate consequence of the fact that the Dirac operator $\mathrm{D}(\mathfrak{g}, \mathfrak{l})$ is $\mathfrak{l}$-invariant.

Lemma 13.4.6. *Let M be an admissible $(\mathfrak{g}, \mathfrak{l})$-supermodule. Then the following two assertions hold:*

a) $\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l})$ *has an increasing $\mathfrak{l}$-invariant filtration*

$$\{0\} = (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))_{-1} \subset (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))_0 \subset (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))_1 \subset \ldots \subset (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))_s = \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l})$$

with $(\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))_k := \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \cap X_k$ *for* $0 \leq k \leq s$.

b) $\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l})$ *has a decreasing $\mathfrak{l}$-invariant filtration*

$$\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) = (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))^0 \supset (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))^1 \supset \ldots \supset (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))^s \supset \{0\}$$

with $(\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))^k := \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \cap X^k$ *for* $0 \leq k \leq s$.

The $\mathfrak{l}$-invariant increasing/decreasing filtrations of $\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l})$ induce gradings of $\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l})$, namely

$$\operatorname{gr} \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) := \bigoplus_{k=0}^{s} (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))^k / (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))^{k+1},$$
$$\operatorname{Gr} \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) := \bigoplus_{k=0}^{s} (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))_k / (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))_{k-1}.$$

By construction, $\mathfrak{l}$ leaves $(\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))^k / (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))^{k+1}$ and $(\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))_k / (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))_{k-1}$ invariant, leading to the following lemma.

Lemma 13.4.7. *The $\mathfrak{l}$-equivariant maps*

$$\operatorname{gr} : \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \to \operatorname{gr} \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}), \quad \operatorname{Gr} : \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \to \operatorname{Gr} \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l})$$

are isomorphisms.

In turns, the previous $\mathfrak{l}$-equivariant maps gr and Gr enter the proof of following lemma.

Lemma 13.4.8. *Let M be an admissible $(\mathfrak{g}, \mathfrak{l})$-supermodule. Then there are injective $\mathfrak{l}$-supermodule homomorphisms*

$$f : \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \hookrightarrow \ker \Delta \cap \ker \delta, \qquad g : \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \hookrightarrow \ker \Delta \cap \ker \partial$$

given by

$$f \circ \operatorname{gr} = \bigoplus_{k=0}^{s} f_k, \quad f_k(x_k + x_{k+1} + \ldots + x_s) := x_k$$

with $x_k + \ldots + x_s \in (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))^k / (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))^{k+1}$ *and*

$$g \circ \operatorname{Gr} = \bigoplus_{k=0}^{s} g_k, \quad g_k(x_1 + x_2 + \ldots + x_k) := x_k$$

for $x_1 + \ldots + x_k \in (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))_k / (\ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}))_{k-1}$.

Proof. We only prove that $g = \bigoplus_{k=0}^{s} g_k : \ker \mathrm{D}(\mathfrak{g},\mathfrak{l}) \hookrightarrow \ker \Delta \cap \ker \partial$ is an injective $\mathfrak{l}$-supermodule homomorphism. The proof for $f : \ker \mathrm{D}(\mathfrak{g},\mathfrak{l}) \hookrightarrow \ker \Delta \cap \ker \delta$ is analogously and will be omitted.

First, f is well-defined as for any $x := x_1 + \ldots + x_k \in (\ker \mathrm{D}(\mathfrak{g},\mathfrak{l}))_k$ with $x_i \in M \otimes \bigwedge^i \bar{\mathfrak{u}}$ for $0 \leq i \leq k$, we have by Proposition 13.4.4 $\mathrm{D}(\mathfrak{g},\mathfrak{l}) = \partial - 2d$ and thus

$$\mathrm{D}(\mathfrak{g},\mathfrak{l})(x) = -2d^1(x_1) + \ldots - 2d^k(x_k) + \partial^1(x_1) + \ldots + \partial^k(x_k) = 0.$$

By degree reasons, we conclude $\partial^k(x_k) = 0$, as it is of degree $k+1$. This shows that $g_k : (\ker \mathrm{D}(\mathfrak{g},\mathfrak{l}))_k / (\ker \mathrm{D}(\mathfrak{g},\mathfrak{l}))_{k-1} \to \ker \partial^k$ is well-defined, and hence, $g = \bigoplus_{k=0}^{s} g_k$ is well-defined. Moreover, by Lemma 13.4.6 and Lemma 13.4.7, the map g is $\mathfrak{l}$-equivariant.

Second, the map g is injective as $g_k(x) = 0$ implies $x_k = 0$, that is, $x \in (\ker \mathrm{D}(\mathfrak{g},\mathfrak{l}))_{k-1}$.

Finally, the image of g is $\ker \Delta \cap \ker \partial$, as for any $x = x_1 + \ldots + x_k \in (\ker \mathrm{D}(\mathfrak{g},\mathfrak{l}))_k$, we have $\mathbf{\Delta}(x_i) = 0$, $1 \leq i \leq k$, since $\mathbf{\Delta} = -2(d\partial + \partial d)$ preserves the degree. □

By combining Lemma 13.4.8 with $\mathrm{D}(\mathfrak{g},\mathfrak{l}) = \partial - 2d = \partial + \delta$ and $\mathrm{D}(\mathfrak{g},\mathfrak{l})^2 = 2(\partial\delta + \delta\partial)$, we conclude the following Lemma.

Lemma 13.4.9. *Let M be an admissible $(\mathfrak{g},\mathfrak{l})$-supermodule. Then*

$$\ker \mathrm{D}(\mathfrak{g},\mathfrak{l}) = \ker \partial \cap \ker \delta.$$

Theorem 13.4.10. *Let $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$ be a parabolic subalgebra, and let M be an admissible simple $(\mathfrak{g},\mathfrak{l})$-supermodule. Then there exist injective $\mathfrak{l}$-supermodule morphisms*

$$\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) \hookrightarrow H^*(\mathfrak{u}, M), \qquad \mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) \hookrightarrow H_*(\bar{\mathfrak{u}}, M).$$

Proof. We note that the Casimir operators $\Omega_{\mathfrak{g}}$ and $\Omega_{\mathfrak{l}}$ act semisimply on M by assumption. Then, using Proposition 13.3.2, we have the decomposition

$$M \otimes \overline{M}(\mathfrak{s}) \cong \ker \mathbf{\Delta} \oplus \operatorname{im} \mathbf{\Delta},$$

Furthermore, we may consider $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})'}(M)$ instead of $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M)$ (*cf.* Corollary 13.3.3), where $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})'}(M) = \ker \mathrm{D}(\mathfrak{g},\mathfrak{l})' / \operatorname{im} \mathrm{D}(\mathfrak{g},\mathfrak{l})'$ and $\mathrm{D}(\mathfrak{g},\mathfrak{l})'$ denotes the restriction of the Dirac operator to $\ker \mathbf{\Delta}$.

Since ∂ commutes with $\mathbf{\Delta}$, as a direct calculation shows, we can restrict ∂ to $\ker \mathbf{\Delta}$, denoted by ∂', and define the associated cohomology $\ker \partial' / \operatorname{im} \partial'$. This cohomology is naturally an $\mathfrak{l}$-subsupermodule of $H^*(\mathfrak{u}, M)$, and it is enough to show the existence of an injective $\mathfrak{l}$-supermodule morphism

$$\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) \cong \mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})'}(M) \hookrightarrow \ker \partial' / \operatorname{im} \partial'.$$

For simplicity, we set $V := \ker \mathbf{\Delta}$. The idea of the proof is to decompose $\ker \mathrm{D}(\mathfrak{g},\mathfrak{l})'$, $\ker \partial'$, $\operatorname{im} \mathrm{D}(\mathfrak{g},\mathfrak{l})'$, and $\operatorname{im} \partial'$ into suitable $\mathfrak{l}$-supermodules, leveraging the $\mathfrak{l}$-semisimplicity of $M \otimes \overline{M}(\mathfrak{s})$ to compare the corresponding components.

We start by considering the following two short exact sequences of $\mathfrak{l}$-supermodules, recalling that $\mathrm{D}(\mathfrak{g},\mathfrak{l}), \partial$ and d commute with $\mathfrak{l}$ and switch parity:

$$0 \to \ker \mathrm{D}(\mathfrak{g},\mathfrak{l})' \to V \to \Pi \operatorname{im} \mathrm{D}(\mathfrak{g},\mathfrak{l})' \to 0, \qquad 0 \to \ker \partial' \to V \to \Pi \operatorname{im} \partial' \to 0.$$

Here, Π denotes as usual the parity switching functor. By semi-simplicity, the short exact sequences split as $\mathfrak{l}$-supermodules:

$$V \cong \ker \mathrm{D}(\mathfrak{g},\mathfrak{l})' \oplus \Pi \operatorname{im} \mathrm{D}(\mathfrak{g},\mathfrak{l})', \qquad V \cong \ker \partial' \oplus \Pi \operatorname{im} \partial'.$$

Next, we decompose $\ker \partial'$. In the following, all isomorphisms are $\mathfrak{l}$-supermodule isomorphisms unless otherwise stated. For the decomposition, we use Lemma 13.4.9 to see $\ker \mathrm{D}(\mathfrak{g},\mathfrak{l})' \subset \ker \partial'$. Then there exists an $\mathfrak{l}$-invariant subspace X such that

$$\ker \partial' \cong \ker \mathrm{D}(\mathfrak{g},\mathfrak{l})' \oplus X,$$

which forces in particular

$$\Pi \operatorname{im} \mathrm{D}(\mathfrak{g},\mathfrak{l})' \cong X \oplus \Pi \operatorname{im} \partial',$$

by the decomposition of V above. On the other hand, $\operatorname{im} \mathrm{D}(\mathfrak{g},\mathfrak{l})' \subset \ker \mathrm{D}(\mathfrak{g},\mathfrak{l})'$ such that we find an $\mathfrak{l}$-supermodule Y with

$$\ker \mathrm{D}(\mathfrak{g},\mathfrak{l})' \cong \operatorname{im} \mathrm{D}(\mathfrak{g},\mathfrak{l})' \oplus Y,$$

which yields $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})'}(M) \cong Y$, and

$$\ker \partial' \cong \Pi X \oplus Y \oplus \operatorname{im} \partial'.$$

This induces directly an embedding of $\mathfrak{l}$-supermodules

$$\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})'}(M) \cong \ker \mathrm{D}(\mathfrak{g},\mathfrak{l})' / \operatorname{im} \mathrm{D}(\mathfrak{g},\mathfrak{l})' \cong Y \hookrightarrow \Pi X \oplus Y \cong \ker \partial' / \operatorname{im} \partial'.$$

This concludes the proof. □

13.4.3. Hermitian real forms and unitarizable supermodules

We now aim at comparing the Dirac cohomology and the Kostant's $\mathfrak{u}$-cohomology for a special kind of supermodules, namely *unitarizable* supermodules over basic classical Lie superalgebras $\mathfrak{g}$. As shown [13], these supermodules are particularly relevant, as they admit a geometric realization as superspaces of sections of certain holomorphic super vector bundles on Hermitian superspaces. Furthermore, as an application, we will consider weight supermodules, and show that their Dirac cohomology is trivial unless they are of highest weight type. On our way to the above results, we start reviewing Hermitian real forms in the next subsection, following in particular the results of Fioresi and collaborators, in [13] and [21, 40].

Hermitian real forms

We fix a real form $\mathfrak{g}^{\mathbb{R}}$ of a basic classical Lie superalgebra $\mathfrak{g}$, i.e., $\mathfrak{g}^{\mathbb{R}}$ is the subspace of fixed points of some $\theta \in \mathrm{aut}_{2,4}(\mathfrak{g})$, and we denote by $\sigma := \omega \circ \theta \in \overline{\mathrm{aut}}_{2,2}(\mathfrak{g})$ the associated conjugate-linear involution on $\mathfrak{g}$ (see Proposition 13.1.8 above). In the following, we may assume that θ associated to $\mathfrak{g}^{\mathbb{R}}$ is a Cartan automorphism (*cf.* Section 13.1.2).

Following [13, 21], we now extend the concept of Hermitian semisimple Lie algebras over $\mathbb{C}$ to basic classical Lie superalgebras.

First, the Lie subalgebra $\mathfrak{g}_{\bar{0}}^{\mathbb{R}} \subset \mathfrak{g}^{\mathbb{R}}$ is either semisimple or reductive with one-dimensional center, since $\mathfrak{g}$ is basic classical. In general, we have the decomposition

$$\mathfrak{g}_{\bar{0}}^{\mathbb{R}} = \mathfrak{g}_{\bar{0}}^{\mathbb{R},\mathrm{ss}} \oplus \mathfrak{z}(\mathfrak{g}_{\bar{0}}^{\mathbb{R}}),$$

where $\mathfrak{g}_{\bar{0}}^{\mathbb{R},\mathrm{ss}} := [\mathfrak{g}_{\bar{0}}^{\mathbb{R}}, \mathfrak{g}_{\bar{0}}^{\mathbb{R}}] \subset \mathfrak{g}_{\bar{0}}^{\mathbb{R}}$ is the commutator subalgebra, *i.e.*, the semisimple part, and $\mathfrak{z}(\mathfrak{g}_{\bar{0}}^{\mathbb{R}})$ the center. As we shall see, there exists a notion of Hermiticity for $\mathfrak{g}_{\bar{0}}^{\mathbb{R},\mathrm{ss}}$.

On $\mathfrak{g}_{\bar{0}}^{\mathbb{R}}$, the Cartan automorphism $\theta \in \mathrm{aut}_{2,4}(\mathfrak{g})$ is an involution, such that we have the following decomposition:

$$\mathfrak{g}_{\bar{0}}^{\mathbb{R}} = \mathfrak{k}^{\mathbb{R}} \oplus \mathfrak{p}_{\bar{0}}^{\mathbb{R}},$$

where $\mathfrak{k}^{\mathbb{R}}$ is the eigenspace of θ with eigenvalue $+1$, and $\mathfrak{p}_{\bar{0}}^{\mathbb{R}}$ is the eigenspace with eigenvalue -1. Complexification yields

$$\mathfrak{g}_{\bar{0}} = \mathfrak{k} \oplus \mathfrak{p}_{\bar{0}}, \tag{13.4.1}$$

with $\mathfrak{k}$ and $\mathfrak{p}_{\bar{0}}$ being the complexifications of $\mathfrak{k}^{\mathbb{R}}$ and $\mathfrak{p}_{\bar{0}}^{\mathbb{R}}$, respectively. This is a *Cartan decomposition* for $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{0}}^{\mathbb{R}}$, respectively.

Definition 13.4.11 ([21])**.** A real form $\mathfrak{g}^{\mathbb{R}}$ of $\mathfrak{g}$ is called *Hermitian* if the following two conditions hold:

a) $\mathfrak{g}_{\bar{0}}^{\mathbb{R},\mathrm{ss}}$ is a Hermitian Lie algebra, *i.e.*, $\theta_{\bar{0}} := \theta|_{\mathfrak{g}_{\bar{0}}^{\mathbb{R}}}$ induces a Cartan decomposition $\mathfrak{g}_{\bar{0}}^{\mathbb{R},\mathrm{ss}} = \mathfrak{k}' \oplus \mathfrak{p}'$, where $\mathfrak{k}', \mathfrak{p}'$ are the $\theta_{\bar{0}}|_{\mathfrak{g}_{\bar{0}}^{\mathbb{R},\mathrm{ss}}}$-eigenspaces with eigenvalue 1 and -1, respectively, such that the adjoint representation of $\mathfrak{k}'$ on $\mathfrak{p}'$ has two simple components.

b) $\mathrm{rank}\, \mathfrak{g}_{\bar{0}} = \mathrm{rank}\, \mathfrak{k}$.

The Hermitian real forms $\mathfrak{g}^{\mathbb{R}}$ of basic classical Lie superalgebras $\mathfrak{g}$ are summarized in the following table [13, 40], where we emphasize that $\mathfrak{g}^{\mathbb{R}}$ is uniquely determined by the indicated real Lie subalgebra $\mathfrak{g}_{\bar{0}}^{\mathbb{R}}$.

Hermitian real forms $\mathfrak{g}^{\mathbb{R}}$ have a Cartan decomposition

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{k}^{\mathbb{R}} \oplus \mathfrak{p}^{\mathbb{R}},$$

where $\mathfrak{p}^{\mathbb{R}} := \mathfrak{p}_{\bar{0}}^{\mathbb{R}} \oplus \mathfrak{g}_{\bar{1}}^{\mathbb{R}}$, and such that $\mathfrak{g}_{\bar{0}}^{\mathbb{R}} = \mathfrak{k}^{\mathbb{R}} \oplus \mathfrak{p}_{\bar{0}}^{\mathbb{R}}$ is a Cartan decomposition for $\mathfrak{g}_{\bar{0}}^{\mathbb{R}}$. We denote the complexification of $\mathfrak{p}^{\mathbb{R}}$ by $\mathfrak{p}$, and note that the Cartan decomposition extends to $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}.$$

Both decompositions are compatible with σ, as θ and ω commute, and $B_\sigma(\cdot,\cdot) = (\cdot, \sigma(\cdot))$ is positive definite on $\mathfrak{k}$ and negative definite on $\mathfrak{p}$, which justifies the name.

For convenience, we set $\mathfrak{p}_{\bar{1}} := \mathfrak{g}_{\bar{1}}$. We may decompose $\mathfrak{p}$ into two $\mathfrak{k}$-stable subspaces, $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, which we now describe.

The even rank condition implies $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}_{\bar{0}} \subset \mathfrak{g}$. The root system Δ_c for $(\mathfrak{h}, \mathfrak{k})$ is a subset of $\Delta_{\bar{0}}$. We call a root $\alpha \in \Delta_{\bar{0}}$ *compact* if $\alpha \in \Delta_c$, or equivalently, if the associated root vector lies in $\mathfrak{k}$; otherwise, the root is referred to as *non-compact*. The set of non-compact

$\mathfrak{g}$	$\mathfrak{g}_{\bar{0}}^{\mathbb{R}}$
$\mathfrak{sl}(m\|n)$	$\mathfrak{su}(p, m-p) \oplus \mathfrak{su}(n) \oplus i\mathbb{R}$ $\mathfrak{su}(p, m-p) \oplus \mathfrak{su}(r, n-r) \oplus i\mathbb{R}$
$B(n\|m)$, $D(n\|m)$	$\mathfrak{sp}(m, \mathbb{R})$ $\mathfrak{so}(p) \oplus \mathfrak{sp}(m, \mathbb{R})$ $\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(m)$ $\mathfrak{so}(q, 2) \oplus \mathfrak{sp}(m, \mathbb{R})$
$C(m)$	$\mathfrak{sp}(m, \mathbb{R}) \oplus \mathfrak{so}(2)$
$D(2, 1; \alpha)$	$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{sl}(2, \mathbb{R})$
$F(4)$	$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(7)$ $\mathfrak{su}(2) \oplus \mathfrak{so}(5, 2)$
$G(3)$	$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{g}_c$

Table 13.1.: Hermitian real Lie superalgebras. For $B(n|m)$, the values of p and q are $p = 2n + 1$ and $q = 2n - 1$, while for $D(n|m)$, $p = 2n$ and $q = 2n - 2$. Moreover, the Lie algebra $\mathfrak{g}_C$ denotes the compact real form of G_2.

roots is $\Delta_n := \Delta \backslash \Delta_c$, such that $\Delta = \Delta_c \sqcup \Delta_n$. In particular, all odd roots are non-compact. For a fixed positive system Δ^+, we set

$$\mathfrak{p}^+ := \sum_{\alpha \in \Delta_n^+} \mathfrak{g}^{\alpha}, \qquad \mathfrak{p}^- := \sum_{\alpha \in \Delta_n^+} \mathfrak{g}^{-\alpha}, \tag{13.4.2}$$

where $\Delta_n^+ := \Delta_n \cap \Delta^+$. Then $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$.

Theorem 13.4.12 ([21, 13])**.** *There exists a positive system $\Delta^+ \subset \Delta$, called* admissible*, such that the following two assertions hold:*

a) $\mathfrak{p}^+$ is $\mathfrak{k}$-stable, that is, $[\mathfrak{k}, \mathfrak{p}^+] \subset \mathfrak{p}^+$.

b) $\mathfrak{p}^+$ is a Lie subsuperalgebra, that is, $[\mathfrak{p}^+, \mathfrak{p}^+] \subset \mathfrak{p}^+$.

A complete list of admissible systems can be found in [13]. In the following, we fix an admissible positive system for $\mathfrak{g}$, denoted by Δ^+. Then, note that $\mathfrak{k} \ltimes \mathfrak{p}^+$ is a Lie subsuperalgebra, and $[\mathfrak{k}, \mathfrak{p}^-] \subset \mathfrak{p}^-$, $[\mathfrak{p}^-, \mathfrak{p}^-] \subset \mathfrak{p}^-$.

$\mathfrak{p}^-$-cohomology, Dirac cohomology and Hodge decomposition

Having prepared our setting in the previous subsection, we are now ready to study the cohomology of unitarizable supermodules. In particular, in the case $\mathcal{H}$ is a simple unitarizable $\mathfrak{g}$-supermodule, we will show that $\mathcal{H} \otimes \overline{M}(\mathfrak{s})$ decomposes as $\mathcal{H} \otimes \overline{M}(\mathfrak{s}) = \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \oplus \mathrm{im}\, \mathrm{D}(\mathfrak{g}, \mathfrak{l})$, and hence in this case one has that $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(\mathcal{H}) = \ker \mathrm{D}(\mathfrak{g},\mathfrak{l})$, see Proposition 13.4.15. Furthermore, Theorem 13.4.17 can be seen as a Hodge decomposition-like result for the cubic Dirac operator. In particular, it shows that the Dirac cohomology of simple unitarizable supermodules is isomorphic (as $\mathfrak{l}$-supermodules) to the Kostant $\overline{\mathfrak{u}}$-cohomology (or $\overline{\mathfrak{u}}$-homology), up to a twist by $\mathbb{C}_{\rho^{\mathfrak{u}}}$.

We fix our notation as follows. Let $\mathfrak{g}^{\mathbb{R}}$ be a Hermitian real form of $\mathfrak{g}$ with associated Cartan automorphism $\theta \in \mathrm{aut}_{2,4}(\mathfrak{g})$. Let $B_\theta(\cdot,\cdot)$ denote the inner product for $\mathfrak{g}^{\mathbb{R}}$ defined in Equation (13.1.1). The associated Cartan decomposition reads $\mathfrak{g}^{\mathbb{R}} = \mathfrak{k}^{\mathbb{R}} \oplus \mathfrak{p}^{\mathbb{R}}$, and we consider the complexification yielding a Cartan decomposition for $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (*cf.* Equation (13.4.2)).

We fix the parabolic subalgebra $\mathfrak{q} := \mathfrak{k} \ltimes \mathfrak{p}^+$ with $\mathfrak{l} = \mathfrak{k}$ and $\mathfrak{u} = \mathfrak{p}^+$, which is well-defined by Theorem 13.4.12. The choice of the parabolic subalgebra $\mathfrak{q} = \mathfrak{k} \ltimes \mathfrak{p}^+$ leads to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ with $\mathfrak{s} := \mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. The parabolic subalgebra $\mathfrak{q}$ is an example of a θ-stable subsuperalgebra, that is, θ preserves $\mathfrak{k}, \mathfrak{p}^+$ and $\mathfrak{p}^-$, which is immediate as $\mathfrak{p}$ is the θ-eigenspace with eigenvalue -1 and $\mathfrak{k}$ is the θ-eigenspace with eigenvalue $+1$. Moreover, by the definition of σ and the action of ω on weight spaces given in Proposition 13.1.8, we conclude $\sigma(\mathfrak{p}^{\pm}) = \mathfrak{p}^{\mp}$.

We restrict B_θ to $\mathfrak{p}^{\mathbb{R}} (= \mathfrak{s}^{\mathbb{R}})$, and fix some orthonormal basis $Z_1, \ldots, Z_{2s}$. Then $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are spanned by

$$u_j := \frac{Z_{2j-1} + iZ_{2j}}{\sqrt{2}}, \quad \overline{u}_j := \frac{Z_{2j-1} - iZ_{2j}}{\sqrt{2}} \tag{13.4.3}$$

respectively, for $j = 1 \ldots, s$, as a direct calculation yields. In particular, $\sigma(u_j) = \overline{u}_j$ for all $j = 1, \ldots, s$.

We study the action of u_j and $\overline{u}_j$ on $\mathcal{H} \otimes \overline{M}(\mathfrak{s})$ for some unitarizable $\mathfrak{g}$-supermodule $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$. For that, we associate to $\mathcal{H} \otimes \overline{M}(\mathfrak{s})$ the non-degenerate super Hermitian product

$$\langle v \otimes P, w \otimes Q\rangle_{\mathcal{H}\otimes\overline{M}(\mathfrak{s})} := \langle v, w\rangle_{\mathcal{H}} \langle P, Q\rangle_{\overline{M}(\mathfrak{s})}$$

for any $v \otimes P, w \otimes Q \in \mathcal{H} \otimes \overline{M}(\mathfrak{s})$. We refer to Section 13.2.2 for an explicit realization of $\langle\cdot,\cdot\rangle_{\overline{M}(\mathfrak{s})}$. By construction, we study the action componentwise.

Lemma 13.4.13. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a unitarizable $\mathfrak{g}^{\mathbb{R}}$-supermodule. Then the following holds for all $j = 1, \ldots, s$:*

$$u_j^\dagger = -\overline{u}_j$$

Proof. As $\mathcal{H}$ is a unitarizable $\mathfrak{g}^{\mathbb{R}}$-supermodule, the orthonormal basis $Z_1, \ldots, Z_{2s}$ of $\mathfrak{p}^{\mathbb{R}}$ satisfies

$$Z_j^\dagger = -Z_j, \quad j = 1, \ldots, 2s.$$

The statement follows with Equation (13.4.3). □

For $(\overline{M}(\mathfrak{s}), \langle\cdot,\cdot\rangle_{\overline{M}(\mathfrak{s})})$, we described the adjoint of any u_j already in Lemma 13.2.18, namely, we have

$$u_j^\dagger = -(-1)^{p(u_j)}\overline{u}_j$$

for all $j = 1, \ldots, s$. By combining Lemma 13.2.18 and Lemma 13.4.13, we have proven the following lemma.

Lemma 13.4.14. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be a unitarizable $\mathfrak{g}$-supermodule. Then the cubic Dirac operator $\mathrm{D}(\mathfrak{g},\mathfrak{l})$ is anti-selfadjoint with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}\otimes\overline{M}(\mathfrak{s})}$. In particular,*

$$\ker \mathrm{D}(\mathfrak{g},\mathfrak{l}) = \ker \mathrm{D}(\mathfrak{g},\mathfrak{l})^k$$

for any $k \in \mathbb{Z}_+$.

In turns, this leads to the following decomposition.

Proposition 13.4.15. *Let $\mathcal{H}$ be a simple unitarizable $\mathfrak{g}$-supermodule. Then*

$$\mathcal{H} \otimes \overline{M}(\mathfrak{s}) = \ker \mathrm{D}(\mathfrak{g},\mathfrak{l}) \oplus \operatorname{im} \mathrm{D}(\mathfrak{g},\mathfrak{l}).$$

In particular, the Dirac cohomology of a simple unitarizable $\mathfrak{g}$-supermodule $\mathcal{H}$ is

$$\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(\mathcal{H}) = \ker \mathrm{D}(\mathfrak{g},\mathfrak{l}).$$

Proof. First, the statement of Proposition 13.3.2 holds more generally for any admissible $(\mathfrak{g},\mathfrak{l})$-supermodule which has infinitesimal character - the argument follows from similar lines. Consequently, it is enough to prove $\operatorname{im} \mathrm{D}(\mathfrak{g},\mathfrak{l}) \cap \ker \mathrm{D}(\mathfrak{g},\mathfrak{l}) = \{0\}$.

Let $v \in \operatorname{im} \mathrm{D}(\mathfrak{g},\mathfrak{l}) \cap \ker \mathrm{D}(\mathfrak{g},\mathfrak{l})$. Then there exists some $w \in \mathcal{H}\otimes\overline{M}(\mathfrak{s})$ such that $\mathrm{D}(\mathfrak{g},\mathfrak{l})w = v$, and by positive definiteness of $(\langle\cdot,\cdot\rangle_{\mathcal{H}\otimes\overline{M}(\mathfrak{s})})_{0,1}$ and Lemma 13.4.14, we have

$$0 \leq (-i)^{p(v)}\langle v, v\rangle = (-i)^{p(v)}\langle \mathrm{D}(\mathfrak{g},\mathfrak{l})w, v\rangle = -(-i)^{p(v)}\langle w, \mathrm{D}(\mathfrak{g},\mathfrak{l})v\rangle = 0.$$

This forces $v = 0$. ☐

We now consider the decomposition $\mathrm{D}(\mathfrak{g},\mathfrak{l}) = C + \bar{C}$ as in Equation (13.2.1). A direct calculation yields that the adjoint of C is $-\bar{C}$. More precisely, the following lemma holds.

Lemma 13.4.16. *The following assertions hold:*

a) $\ker \mathrm{D}(\mathfrak{g},\mathfrak{l}) = \ker C \cap \ker \bar{C}$.

b) $\operatorname{im} C$ *is orthogonal to* $\ker \bar{C}$ *and* $\operatorname{im} \bar{C}$, *while* $\operatorname{im} \bar{C}$ *is orthogonal to* $\ker C$.

Proof. a) As $\mathrm{D}(\mathfrak{g},\mathfrak{l}) = C + \bar{C}$, the inclusion $\ker C \cap \ker \bar{C} \subset \ker \mathrm{D}(\mathfrak{g},\mathfrak{l})$ is clear. Assume $\mathrm{D}(\mathfrak{g},\mathfrak{l})v = 0$ for some $v \in \mathcal{H}\otimes\overline{M}(\mathfrak{s})$, *i.e.*, $Cv = -\bar{C}v$. Consequently,

$$(-i)^{p(Cv)}\langle Cv, Cv\rangle_{\mathcal{H}\otimes\overline{M}(\mathfrak{s})} = (-i)^{p(Cv)}\langle Cv, -\bar{C}v\rangle_{\mathcal{H}\otimes\overline{M}(\mathfrak{s})} = (-i)^{p(Cv)}\langle C^2 v, v\rangle_{\mathcal{H}\otimes\overline{M}(\mathfrak{s})} = 0,$$

where we use $C^2 = 0$ by Lemma 13.2.11. In particular, $v \in \ker C$ by super positive definiteness. Analogously, $v \in \ker \bar{C}$.

b) Let $v \in \operatorname{im} C$ and $w \in \ker \bar{C}$. We show $\langle v, w\rangle_{\mathcal{H}\otimes\overline{M}(\mathfrak{s})} = 0$. As $v \in \operatorname{im} C$, there exists some $v' \in \mathcal{H}\otimes\overline{M}(\mathfrak{s})$ such that $Cv' = v$. We conclude

$$\langle v, w\rangle_{\mathcal{H}\otimes\overline{M}(\mathfrak{s})} = \langle Cv', w\rangle_{\mathcal{H}\otimes\overline{M}(\mathfrak{s})} = -\langle v', \bar{C}w\rangle_{\mathcal{H}\otimes\overline{M}(\mathfrak{s})} = 0,$$

since $w \in \ker \bar{C}$. Analogously, one can prove that $\operatorname{im} \bar{C}$ is orthogonal to $\ker C$.

We show that $\operatorname{im} C$ and $\operatorname{im} \bar{C}$ are orthogonal. Let $v \in \operatorname{im} C \cap \operatorname{im} \bar{C}$, then there exists $v_C, v_{\bar{C}} \in \mathcal{H}\otimes\overline{M}(\mathfrak{s})$ with $Cv_C = v$ and $\bar{C}v_{\bar{C}} = v$. We conclude

$$(-i)^{p(v)}\langle v, v\rangle_{\mathcal{H}\otimes\overline{M}(\mathfrak{s})} = (-i)^{p(v)}\langle Cv_C, \bar{C}v_{\bar{C}}\rangle_{\mathcal{H}\otimes\overline{M}(\mathfrak{s})} = -(-i)^{p(v)}\langle C^2 v_C, v_{\bar{C}}\rangle_{\mathcal{H}\otimes\overline{M}(\mathfrak{s})} = 0,$$

by Lemma 13.2.11. Hence, by super positive definiteness of $\langle\cdot,\cdot\rangle_{\mathcal{H}\otimes\overline{M}(\mathfrak{s})}$, we have $v = 0$. ☐

By combining all the previous results, we are now in the position to prove the following theorem, which is the main result of the present section.

Theorem 13.4.17. *Let $\mathcal{H}$ be a simple unitarizable $\mathfrak{g}$-supermodule. Then the following assertions hold:*

a) $\mathcal{H} \otimes \overline{M}(\mathfrak{s}) = \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \oplus \operatorname{im} C \oplus \operatorname{im} \bar{C}$.

b) $\ker C = \ker D \oplus \operatorname{im} C$.

c) $\ker \bar{C} = \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \oplus \operatorname{im} \bar{C}$.

In particular, we have an isomorphism of $\mathfrak{l}$-supermodules

$$\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(\mathcal{H}) \cong H^*(\mathfrak{u}, \mathcal{H}) \otimes \mathbb{C}_{\rho^{\mathfrak{u}}}.$$

Proof. By Proposition 13.4.15, the decomposition $\mathrm{D}(\mathfrak{g}, \mathfrak{l}) = C + \bar{C}$ and Lemma 13.4.16, we have

$$\mathcal{H} \otimes \overline{M}(\mathfrak{s}) = \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \oplus \operatorname{im} \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \subset \ker \mathrm{D}(\mathfrak{g}, \mathfrak{l}) \oplus \operatorname{im} C \oplus \operatorname{im} \bar{C},$$

i.e., $\operatorname{im} \mathrm{D}(\mathfrak{g}, \mathfrak{l}) = \operatorname{im} C \oplus \operatorname{im} \bar{C}$. This proves a). The assertions b) and c) follows with a) and Lemma 13.4.16.

The isomorphisms are now a direct consequence of b), c) and Proposition 13.2.20. □

13.4.4. Application: Dirac cohomology and weight supermodules

In this section, we prove that the Dirac cohomology of simple weight supermodules is trivial unless they are of highest weight type. This generalizes the result for reductive Lie algebras over $\mathbb{C}$ in [69] to basic classical Lie superalgebras.

First, any simple weight $\mathfrak{g}$-supermodule M admits an infinitesimal character such that $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) \subset H^*(\mathfrak{u}, M)$ by Theorem 13.4.10. By Proposition 13.3.7, we already know that $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) \neq \{0\}$ if M is of highest weight type. We show that $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) = \{0\}$ unless M is of highest weight type. To this end, we identify $H^i(\mathfrak{u}, M)$ with $\mathrm{Ext}^i_{\mathfrak{u}}(\mathbb{C}, M)$ for any $i > 0$. To compute $\mathrm{Ext}^i_{\mathfrak{u}}(\mathbb{C}, M)$, we use the subsequent lemma.

Lemma 13.4.18. *Let M be a weight $\mathfrak{g}$-supermodule. Assume that there exists a positive root α such that $e_{-\alpha}$ acts injectively on M. Then there exists an injective resolution of M*

$$0 \to M \to I_0 \to I_1 \to \dots$$

such that $e_{-\alpha}$ acts injectively on every I_i for $i \in \mathbb{Z}_+$.

A proof of the above follows is given in [106]. Let M be a simple weight $\mathfrak{g}$-supermodule. Assume M is not of highest weight type. By Theorem 13.1.7, M is (isomorphic to) the unique simple quotient $L_{\mathfrak{p}}(V)$ of a parabolically induced $\mathfrak{g}$-supermodule $M_{\mathfrak{p}}(V)$, where $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$ is a parabolic subalgebra with a good Levi subalgebra, and V is a cuspidal $\mathfrak{l}$-supermodule. Recall that an $\mathfrak{l}$-supermodule is called cuspidal if for any $\alpha \in \Delta(\mathfrak{l}; \mathfrak{h})_{\bar{0}}$ the associated root vector e_α acts injectively on V.

As $\mathfrak{p}$ is parabolic, it contains a Borel subalgebra $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$ (*cf.* Lemma 13.1.2), where $\mathfrak{n}^+$ is the maximal radical of $\mathfrak{b}$. Moreover, $\mathfrak{l} \neq \mathfrak{h}$ by Section 3.1.3. We conclude $\mathfrak{l} \cap \mathfrak{n}^+ \neq \{0\}$. In particular, we have proven the following lemma.

Lemma 13.4.19. *There exists a root $\alpha \in \Delta_{\bar{0}}^{+}(\mathfrak{l}, \mathfrak{h})$ such that $e_{-\alpha}$ acts injectively on M.*

Combining Lemma 13.4.18 and Lemma 13.4.19, we conclude the subsequent theorem.

Theorem 13.4.20. *Let M be a simple weight $\mathfrak{g}$-supermodule. Then $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) = \{0\}$ unless M is a highest weight $\mathfrak{g}$-supermodule.*

Proof. If M is a highest weight module, we have $\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(M) \neq 0$ by Proposition 13.3.7. Assume M is not a highest weight supermodule. Then there exists some $\alpha \in \Delta(\mathfrak{l}, \mathfrak{h})$ such that $e_{-\alpha}$ acts injectively. On the other hand, by Lemma 13.4.18, there exists an injective resolution of M

$$0 \to M \to I_0 \to I_1 \to \dots$$

such that $e_{-\alpha}$ acts injectively on any I_i. Hence, the space of $\mathfrak{g}$-invariants $I_i^{\mathfrak{g}}$ is trivial for all $i = 0, 1, 2, \dots$. In particular, $\{0\} = \mathrm{Ext}^i_{\mathfrak{u}}(\mathbb{C}, M) = H^i(\mathfrak{u}, M)$. The statement follows from Theorem 13.4.10. □

List of Symbols

The following list presents a selection of the most frequently used symbols and notations that appear throughout this document.

$\bar{0}, \bar{1}$	residue classes of even and odd integers in $\mathbb{Z}/2\mathbb{Z}$
$[1 \otimes 1]$	highest weight vector in Verma supermodule
$\lvert \cdot \rvert$	absolute value
$(\cdot)^*$	super-adjoint
$[\cdot, \cdot]$	Lie superbracket or supercommutator
$\overline{(\cdot)}$	complex conjugate
$\langle \cdot \rangle_{\mathfrak{g}}$	cyclic (super)module over $\mathfrak{g}$
$(\cdot)^\dagger$	adjoint with respect to an inner product or conjugate adjoint
$(\cdot, \cdot)$	non-degenerate supersymmetric invariant bilinear form on $\mathfrak{g}$
$\langle \cdot, \cdot \rangle_{\mathcal{H}}$	positive definite Hermitian form over the super Hilbert space $\mathcal{H}$
$\int^{\oplus}$	direct integral
$\ltimes$	semi-direct product
$\lvert\lvert \cdot \rvert\rvert$	norm
$\oplus$	direct sum
$\bigoplus_{i \in I}$	Hilbert space direct sum
$\hat{\otimes}$	$\mathbb{Z}_2$-graded tensor product
$\otimes$	tensor product over the field $\mathbb{K}$
$a, \overline{a}$	decomposition summands: $1 \otimes \phi_{\mathfrak{s}} = a + \overline{a}$
$A, \overline{A}$	decomposition summands: $C = A + 1 \otimes a$, $\overline{C} = \overline{A} + 1 \otimes \overline{a}$
Ad	adjoint representation of a Lie (super)group
ad	adjoint representation of a Lie (super)algebra
$A(m\lvert m)$	Lie superalgebra $\mathfrak{sl}(m\lvert m)/\mathbb{C}E_{m\lvert m}$
$A(m\lvert n)$	Lie superalgebras $\mathfrak{sl}(m\lvert n)$ for $m \neq n$

$\mathrm{at}(\cdot)$	degree of atypicality
$\mathfrak{b}$	Borel subalgebra
Ber	Berezinian
$B(\cdot,\cdot)$	$\mathbb{K}$-bilinear form
$B(m\|n)$	Lie superalgebras $\mathfrak{osp}(2m+1\|2n)$ for $m \geq 0$, $n \geq 1$
c	semisimple element $c = [x,x]$ for $x \in \mathcal{Y}^{\mathrm{hom}}$
$\mathcal{C}$	region of unitarity
$\mathcal{C}^{(a,b)}$	region of unitarity with fixed spin and R-symmetry quantum numbers
$\mathcal{O}$	BGG category
$C, \overline{C}$	decomposition summands $\mathrm{D}(\mathfrak{g},\mathfrak{l}) = C + \overline{C}$
$\mathbb{C}$	complex numbers
$\mathbb{C}_\lambda$	one-dimensional supermodule of weight λ
$\mathrm{ch}_{\mathfrak{k}^{\mathbb{C}}}(\cdot)$	formal character
$\overline{C}$	topological closure of $\mathcal{C}$
$C(n)$	Lie superalgebras $\mathfrak{osp}(2\|2n-2)$ for $n \geq 2$
$\mathrm{Coind}_{\mathfrak{a}}^{\mathfrak{g}}(\cdot)$	coinduction from a (super)module of a sub(super)algebra $\mathfrak{a}$ to $\mathfrak{g}$
$\mathrm{Cone}(\mathcal{G})$	cone of a Lie supergroup
$C_p(\overline{\mathfrak{u}}, M)$	Kostant's p-chains
$C(\pi_1,\pi_2)$	space of intertwining operators between $(\pi_1,\mathcal{H}_1)$ and $(\pi_2,\mathcal{H}_2)$
$C^p(\mathfrak{u}, M)$	Kostant's p-cochains
$C(\mathfrak{s})$	Clifford superalgebra
C	Weyl chamber or Kostant's constant
$\mathfrak{d}$	Cartan subalgebra of diagonal elements in $\mathfrak{gl}(m\|n)$
$\mathcal{D}$	set of relative holomorphic discrete series representations
d	Kostant's boundary operator
d	grading operator or Kostant's boundary operator
$\mathrm{H_D}(\cdot)$	Dirac cohomology
$\mathrm{H}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(\cdot)$	Dirac cohomology with respect to the cubic Dirac operator $\mathrm{D}(\mathfrak{g},\mathfrak{l})$
$\mathrm{def}(\mathfrak{g})$	defect of a Lie superalgebra $\mathfrak{g}$

$\deg_{\mathrm{BPS}}(\cdot)$	degree of BPS-ness
δ	Kostant's coboundary operator
det	determinant
$\mathrm{d}g$	Haar measure of the Lie group G
$d(\pi_\Lambda), d(\Lambda)$	formal dimension
$\dim(V)$	dimension of a (super) vector space V
D	Dirac operator
$D(\mathfrak{g}, \mathfrak{l})$	cubic Dirac operator
$\mathrm{D}^{\pm}$	$\mathrm{D}^{\pm} : (\cdot) \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{0},\bar{1}} \to (\cdot) \otimes M(\mathfrak{g}_{\bar{1}})_{\bar{1},\bar{0}}$
$D(m\|n)$	Lie superalgebras $\mathfrak{osp}(2m\|2n)$ for $m \geq 2$, $n \geq 1$
$\mathrm{d}\pi$	derived representation of a group representation π
$DS_x(\cdot)$	Duflo-Serganova functor with respect to $x \in \mathcal{Y}$
e_G	identity element of a Lie group G
$E_{k,l}$	matrix with 1 at row k and column l
$\mathrm{End}_{\mathbb{K}}(\cdot)$	endomorphisms of a super vector space
ev_g	evaluation map at $g \in G$
$\exp,\ e^{(\cdot)}$	exponential map
F	fermion number
$G_{\bar{0}}$	analytic Lie group associated with $\mathfrak{g}_{\bar{0}}$
$\mathfrak{g}_{\bar{0}}$	even part of the Lie superalgebra
G^0	connected component of the identity of G
$\mathfrak{g}_{\bar{0},\Delta}$	diagonal embedding of $\mathfrak{g}_{\bar{0}}$ in $\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$
$\mathfrak{g}_{\bar{0}}$**-mod**	category of $\mathfrak{g}_{\bar{0}}$-modules
$\mathfrak{g}_{\bar{0}}$**-smod**	category of $\mathfrak{g}_{\bar{0}}$-supermodules
$\mathfrak{g}_{\bar{1}}$	odd part of the Lie superalgebra
$\mathfrak{g}$	Lie algebra or Lie superalgebra
G	Lie group
$G^{\mathbb{C}}$	complexification of a Lie group G
$\mathfrak{g}^{\mathbb{C}}$	complexification of the real Lie (super)algebra $\mathfrak{g}$

$\widehat{G}$	unitary dual of G
$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$	$\mathbb{Z}_2$-compatible $\mathbb{Z}$-grading of a Lie superalgebra $\mathfrak{g}$
$(\mathfrak{g}, \omega)$-**usmod**	category of unitarizable $(\mathfrak{g}, \omega)$-supermodules
$(\mathfrak{g}, \omega)$-**usmod**$'$	$(\mathfrak{g}, \omega)$-supermodules with only finitely many atypical constituents
$\mathrm{GL}(m\|n)$	general linear Lie supergroup
$\mathfrak{gl}(m\|n)$	general linear Lie superalgebra
$g_{p,q}$	Riemannian metric with signature (p, q)
$\mathrm{gr}(\cdot)$	associated graded (super)module
$\mathfrak{g}^{\mathbb{R}}$	real form of the Lie superalgebra
G_{reg}	set of regular elements of a Lie group G
$\mathbf{G}$	Lie group or its universal cover
$(G, \mathfrak{g})$	super Harish-Chandra pair of a Lie supergroup $\mathcal{G}$
$\mathfrak{g}$-**smod**	category of $\mathfrak{g}$-supermodules
$\mathfrak{g}^{\mathrm{ss}}$	semisimple part of a reductive Lie algebra $\mathfrak{g}$
$\mathcal{G}$	Lie supergroup
$\widetilde{G}$	universal cover of the Lie group G
$\mathfrak{g}_x$	Lie superalgebra $DS_x(\mathfrak{g})$
G_x	analytic Lie group of $\mathfrak{g}_x$
$H(2)$	space of Hermitian 2×2 matrices
$\mathrm{HC}_{\mathfrak{g}}$	Harish-Chandra morphism for the Lie (super)algebra $\mathfrak{g}$
$H_*(\overline{\mathfrak{u}}, \cdot)$	Kostant's $\overline{\mathfrak{u}}$-homology
$\mathcal{H}_d$	discrete part of the unitary representation $\mathcal{H}$
$\mathrm{H}^{\pm}_{\mathrm{D}(\mathfrak{g},\mathfrak{l})}(\cdot)$	Dirac cohomologies of $\mathrm{D}(\mathfrak{g}, \mathfrak{l})^{\pm}$
$\mathrm{H}_{\mathrm{D}^{\pm}}(\cdot)$	Dirac cohomology with respect to $\mathrm{D}^{\pm}$
$\mathbb{H}$	upper half-plane or space of coupling constants
$\mathfrak{h}$	Cartan subalgebra
$\mathcal{H}^K$	space of K-finite vectors in the unitary representation $\mathcal{H}$
$\mathcal{H}$	super Hilbert space or unitarizable supermodule over $\mathfrak{g}$
$\mathrm{H}^*(\mathfrak{g}_{+1}, \cdot)$	Kostant's cohomology

$\mathrm{Hom}(\cdot,\cdot)$	vector space of homomorphisms
$\underline{\mathrm{Hom}}(\cdot,\cdot)$	inner Hom
$\mathcal{H}_{[\pi]}$	isotypic component of the unitary representation $\mathcal{H}$
$H^*(\mathfrak{u},\cdot)$	Kostant's $\mathfrak{u}$-cohomology
$\mathrm{I}(\cdot)$	Dirac index or KMMR index
id	identity operator
I	identity matrix
im	image
$\mathrm{Ind}_{\mathrm{D}}(\cdot)$	Dirac induction
$I^W_{\mathcal{H}}(Q)$	Q-Witten index on $\mathcal{H}$
$\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{a}}(\cdot)$	induction functor from $\mathfrak{a}$ to $\mathfrak{g}$
$I_{p,q}$	diagonal matrix $\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$
$\mathrm{ISO}(1,d-1)$	Poincaré Lie group of d-dimensional spacetime
$\mathfrak{iso}(1,d-1)$	Poincaré Lie algebra of d-dimensional spacetime
K	maximal compact subgroup
$K(\cdot)$	Kac supermodule
ker	kernel
$\mathfrak{k}$	(real) maximal compact subalgebra
$\mathbb{K}$	field of real or complex numbers
$K(\Lambda)$	Kac supermodule of highest weight $\Lambda\in\mathfrak{h}^*$
$\mathbb{K}[x_1,\ldots,x_n]$	polynomial ring in indeterminates $x_1,\ldots,x_n$ over the field $\mathbb{K}$
$\mathbb{K}^{m\|n}$	super vector space over $\mathbb{K}$ of superdimension $(m\|n)$
$L_0(\Lambda)$	simple $\mathfrak{g}_{\bar{0}}$-(super)module of highest weight Λ
$L^2(G)$	Hilbert space of square-integrable functions on G
$\mathfrak{L}$	Lie algebra $\mathfrak{su}(p,q)^{\mathbb{C}}$
$\mathrm{len}(\Lambda)$	length of $Y(\Lambda)$
$\mathfrak{l}$	Levi subalgebra or supertranslation algebra
$L(\Lambda), L^{\mathfrak{b}}(\Lambda)$	unique simple quotient of $M(\Lambda)$ or $K(\Lambda)$
$L_{\mathfrak{p}}(V)$	simple quotient of a parabolic Verma supermodule

$\mathfrak{L}$	superconformal algebra
$M(\Lambda), M^{\mathfrak{b}}(\Lambda)$	Verma supermodule for $\Lambda \in \mathfrak{h}^*$ with respect to $\mathfrak{b}$
$M^{\mathfrak{a}}$	fixed point set of a sub(super)algebra of a supermodule
$\mathrm{Mat}(m\|n)$	Lie supergroup of supermatrices
$\mathrm{Mat}(m \times n; \mathbb{K})$	space of $m \times n$ matrices over $\mathbb{K}$
M^c	set of fixed points in M under the action of c
$M(\mathfrak{g}, \mathbb{K})$	category of Harish-Chandra supermodules
M_{ev}	$\mathfrak{g}$-supermodule M considered under $\mathfrak{g}_{\bar{0}}$ after neglecting parity
$\mathbb{M}$	Minkowski space
M^{λ}	weight space of weight λ of the supermodule M
$\mathrm{Mod}_{\mathfrak{g},K}$	category of $(\mathfrak{g}, K)$-modules
$M(\mathfrak{g}_{\bar{1}})$	oscillator module over $\mathscr{W}(\mathfrak{g}_{\bar{1}})$
$M, (\rho, M)$	supermodule
$M_{\mathfrak{p}}(V)$	parabolic Verma supermodule
$M(\mathfrak{s}), \overline{M}(\mathfrak{s})$	oscillator modules
$m(\cdot)$	multiplicity function
$\mathfrak{n}^{\pm}$	subalgebra of positive and negative root spaces
$\mathcal{O}_{\chi}$	block of the BGG category $\mathcal{O}$ with infinitesimal character χ
$\mathcal{O}^{\mathfrak{p}}$	parabolic category $\mathcal{O}$
$\mathrm{O}(p, q)$	indefinite orthogonal group
$\mathfrak{osp}(V), \mathfrak{spo}(V)$	orthosymplectic Lie superalgebra over the super vector space V
$\mathrm{Osp}(m\|2n)$	orthosymplectic Lie supergroup
$\mathfrak{O}$	oscillator supermodule
$\mathcal{O}_{\mathcal{M}}$	structure sheaf of a supermanifold $\mathcal{M}$
$p(\cdot)$	parity function
$\overline{\mathfrak{p}}$	opposite parabolic subalgebra
P_M	set of weights of a weight supermodule M
$\mathbb{P}_n$	smooth projective space of dimension n
$\mathfrak{p}$	parabolic subalgebra

$\mathfrak{p}^{\pm}$	sum of root spaces of positive/negative non-compact roots
pr	projection operator
$P_{\mathfrak{b}}^{++}$	set of Δ^+-dominant integral weights
$\mathfrak{psl}(n\|n)$	projective special linear Lie superalgebra
$\mathfrak{psu}(p,q\|r,s)$	projective unitary special linear Lie superalgebra of signature $(p,q\|r,s)$
P, P_T	parabolic set
Q	supercharge or element of $\mathcal{Y}$
$\mathfrak{R}$	Lie algebra $\mathfrak{su}(n)^{\mathbb{C}}$
r_α	even or odd reflection
$\mathrm{rk}(\cdot)$	rank of a Lie algebra, Lie group, or matrix
$\mathrm{Res}_{\mathfrak{a}}^{\mathfrak{g}}(\cdot)$	restriction functor from $\mathfrak{g}$ to $\mathfrak{a}$
$\mathbb{R}$	field of real numbers
$\mathbb{R}_{>0}$	positive real numbers
$\mathfrak{s}$	sum $\mathfrak{u} \oplus \overline{\mathfrak{u}}$
$\mathbf{sAlg}_{\mathbb{K}}$	category of superalgebras over $\mathbb{K}$
$s\mathcal{O}$	super BGG category
$\mathrm{sdim}(V)$	superdimension of a super vector space V
$\mathfrak{siso}(V)$	Poincaré Lie superalgebra
$\mathfrak{siso}(d,\mathbb{C})$	complex super Poincaré Lie algebra
S_k	symmetric group on $\{1,\ldots,k\}$
$\mathbf{sLie}_{\mathbb{K}}$	category of Lie supergroups over $\mathbb{K}$
$\mathrm{SL}(n,\mathbb{K})$	special linear Lie group of degree n over $\mathbb{K}$
$\mathrm{SL}(m\|n)$	special linear Lie supergroup
$\mathfrak{sl}(n,\mathbb{K})$	special linear Lie algebra of degree n over $\mathbb{K}$
$\mathbf{sMan}_{\mathbb{K}}$	category of supermanifolds over $\mathbb{K}$
$\mathrm{soc}(\cdot)$	socle of a (super)module
$\mathfrak{so}(n,\mathbb{K})$	special orthogonal Lie algebra over $\mathbb{K}$
$\mathrm{SO}(n,\mathbb{K})$	special orthogonal group of degree n over $\mathbb{K}$
$\mathrm{SO}(p,q)$	special orthogonal indefinite group

$\mathrm{Spin}(V)$	spin group
$S^{\mathfrak{g},\mathfrak{l}}, \overline{S}^{\mathfrak{g},\mathfrak{l}}$	spin modules with respect to $\mathfrak{u}$ and $\overline{\mathfrak{u}}$
$\mathfrak{sp}(V)$	symplectic Lie algebra over the vector space V
S	set of subsets of $\Delta_{\bar{1}}^{+}$ or supercharge
str	supertrace
$\mathrm{SU}(p,q)$	indefinite special unitary group
$S(V)$	(super)symmetric algebra over the (super) vector space V
$\mathfrak{su}(p,q)$	special indefinite unitary Lie algebra of signature (p,q)
$\mathfrak{su}(p,q\|r,s)$	special unitary indefinite Lie superalgebras
$\mathbf{sVect}_{\mathbb{K}}$	category of super vector spaces over the field $\mathbb{K}$
$\mathrm{Sym}(V)$	space of symmetric tensors in $T(V)$
T	triangular decomposition of a root system or Cartan subgroup
$T(V)$	tensor superalgebra
${}^{F}T_{\lambda}^{\mu}$	translation functor w.r.t. a finite-dimensional supermodule F
tr	trace
$\mathfrak{t}$	Cartan subalgebra
$\mathfrak{t}_x^{\mathrm{reg},+}$	regular and positive elements of the Cartan subalgebra $\mathfrak{t}_x$ of $\mathfrak{g}_x$
$T_x\mathcal{M}$	tangent space at x
$\mathfrak{U}(\mathfrak{g}_{\bar{0},\Delta})$	diagonal embedding of $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$ in $\mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$
$\mathfrak{U}(\mathfrak{g})$	universal enveloping Lie superalgebra
$U(\mathcal{H})$	space of unitary operators on the Hilbert space $\mathcal{H}$
$\mathfrak{u}(n)$	unitary Lie algebra of degree n
$\mathfrak{u}(p,q\|r,s)$	unitary indefinite Lie superalgebras
$\mathfrak{u}(\mathcal{H})$	space of skew-Hermitian operators on $\mathcal{H}$
$\mathfrak{u}, \overline{\mathfrak{u}}$	nilpotent radical of $\mathfrak{p}$ and $\overline{\mathfrak{p}}$
$V_{\bar{0}}$	even part of the super vector space V
$V_{\bar{1}}$	odd part of the super vector space V
V	real or complex super vector space
V^{*}	dual of the super vector space V

V^∞	space of smooth vectors in a (super) vector space V
v_Λ	highest weight vector with highest weight Λ
W_c	Weyl group of Δ_c
$w \cdot \lambda$	dot action of the Weyl group
$\mathscr{W}(\mathfrak{g}_{\bar{1}})$	Weyl algebra over the vector space $\mathfrak{g}_{\bar{1}}$
$W, W^{\mathfrak{g}}$	Weyl group of a Lie (super)algebra $\mathfrak{g}$
$X^*(T^{\mathbb{C}})$	character group
x_M	endomorphism on M associated with $x \in \mathfrak{g}$
$\mathcal{Y}$	self-commuting variety
$\mathcal{Y}^{\mathrm{hom}}$	rank variety
$Y(\Lambda)$	Young diagram of a certain weight Λ
$\mathcal{Y}_M$	associated variety of a supermodule M
Z	center of the Lie group G
$\mathfrak{z}(\mathfrak{g})$	center of $\mathfrak{U}(\mathfrak{g})$
$\pi(\lambda, \Delta^+)$	limit of discrete series representations
$\mathbb{Z}_2$	ring of integers modulo 2
$\mathbb{Z}$	ring of integers
$\mathbb{Z}_{\leq 0}$	set of non-positive integers
$\mathbb{Z}_+$	set of non-negative integers
$\mathbb{Z}_{\geq 0}$	set of non-negative integers
$\bigwedge V$	exterior superalgebra
$\psi(\cdot, \cdot)$	super Hermitian form
α	root or diagonal embedding $\mathfrak{g}_{\bar{0}} \to \mathfrak{U}(\mathfrak{g}) \otimes \mathscr{W}(\mathfrak{g}_{\bar{1}})$
$\boldsymbol{\alpha}$	central $\mathfrak{u}(1)$-charge
$\chi^{\bar{0}}$	even infinitesimal character
χ_Λ	infinitesimal character with respect to Λ
χ_V^x	supercharacter of the supermodule V at x
Δ	root system
$\boldsymbol{\Delta}$	Laplace operator

Δ_c	set of compact roots
$\Delta(\mathfrak{g};\mathfrak{h})$	root system for the Cartan subalgebra $\mathfrak{h}$
δ_j	dual basis element of $H = \mathrm{diag}(0,\dots,0\vert 0,\dots,0,1,0,\dots,0)$
Δ_n	set of non-compact roots
Δ^+_{nst}	non-standard positive system
$\Delta^+_{\bar{0}}$	even positive system
$\Delta^+_{\bar{1}}$	odd positive system
Δ^+	positive root system
Δ^+_{st}	standard positive system
$\Delta^{0,\pm}_T$	root system with respect to a triangular decomposition T
Δ_x	root system of $DS_x(\mathfrak{g})$
$\Delta_{\bar{0}}$	set of even roots
$\Delta_{\bar{1}}$	set of odd roots
ϵ_i	dual basis element of $H = \mathrm{diag}(0,\dots,0,1,0,\dots,0\vert 0,\dots,0)$
Γ	set of $\mathfrak{g}$-unitarity or spin pairings
Γ_0	set of $\mathfrak{g}_{\bar{0}}$-unitarity
$\Gamma_0^{(a,b)}$	set of $\mathfrak{g}_{\bar{0}}$-unitarity with fixed spin and R-symmetry quantum numbers
$\Gamma^{(a,b)}$	set of $\mathfrak{g}$-unitarity with fixed spin and R-symmetry quantum numbers
$\mathfrak{g}^\omega$	real form associated with a conjugate-linear anti-involution ω
Λ	highest weight
λ	central charge in $\mathfrak{su}(p,q)$
Λ_j	highest weight of a $\mathfrak{g}_{\bar{0}}$-constituent
ω	conjugate-linear anti-involution
$\Omega_\mathfrak{g}$	quadratic Casimir element of $\mathfrak{U}(\mathfrak{g})$
$\omega_{(-,+)}$	conjugate-linear anti-involution associated with $\mathfrak{su}(p,q\vert 0,n)$
$\omega_{(+,-)}$	conjugate-linear anti-involution associated with $\mathfrak{su}(p,q\vert n,0)$
$\omega_\pm$	conjugate-linear anti-involutions associated with $\mathfrak{su}(m\vert n,0)$ and $\mathfrak{su}(m\vert 0,n)$
Φ	morphism of supermanifolds

ϕ	fundamental 3-form in $(\bigwedge^3 \mathfrak{g})_{\bar{0}}$
$\phi_{\mathfrak{s}}$	restriction of ϕ to $\mathfrak{s}$
Π	parity reversal functor
$\Pi = (\pi, \rho^{\pi})$	representation of a Lie supergroup
$\uppi$	simple system
$(\pi_\Lambda, \mathcal{H}_\Lambda)$	(rel.) holomorphic discrete series with HC parameter Λ
$\rho = \rho_{\bar{0}} - \rho_{\bar{1}}$	Weyl vector
ρ_c	Weyl vector with respect to Δ_c^+
$\rho_{\bar{0}}$	even Weyl vector
$\rho^{\mathfrak{l}}$	Weyl vector with respect to the Levi subalgebra
ρ_n	Weyl vector with respect to Δ_n^+
$\rho^{\mathfrak{u}}$	Weyl vector with respect to the nilpotent radical
σ	conjugate-linear involution
Θ, Θ_Λ	Harish-Chandra character
θ	infinitesimal Cartan involution or Cartan automorphism
$\widetilde{\Theta}$	L-packet
Ξ	root lattice or Hamiltonian

Bibliography

[1] S. Afentoulidis-Almpanis. Dirac cohomology for the BGG category $\mathcal{O}$. *Indag. Math. (N.S.)*, 35(2):205–229, 2024.

[2] A. Alldridge. Fréchet globalisations of Harish-Chandra supermodules. *Int. Math. Res. Not.*, 2017(17):5182–5232, 2017.

[3] M. Atiyah and W. Schmid. A geometric construction of the discrete series for semisimple Lie groups. Harmonic analysis and representations of semisimple Lie groups, Lect. NATO Adv. Study Inst., Liège 1977, 317-378 (1980)., 1980.

[4] M. W. Baldoni. General representation theory of real reductive Lie groups. In *Representation theory and automorphic forms. Proceedings of an instructional conference, Edinburgh, UK, March 17–29, 1996*, pages 61–72. Providence, RI: American Mathematical Society; Edingurgh: International Centre for Mathematical Sciences, 1997.

[5] V. Bargmann. On a Hilbert space of analytic functions and an associated integral transform. *Comm. Pure Appl. Math.*, 14:187–214, 1961.

[6] F. A. Berezin. *Introduction to superanalysis. Ed. by A. A. Kirillov. Transl. from the Russian by J. Niederle and R. Kotecký, ed. and rev. by Dimitri Leites*, volume 9 of *Math. Phys. Appl. Math.* Kluwer Academic Publishers, Dordrecht, 1987.

[7] N. Bourbaki. *Lie groups and Lie algebras. Chapters 1–3.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.

[8] J.-P. Bourguignon, T. Branson, A. H. Chamseddine, O. Hijazi, and R. J. Stanton, editors. *Dirac operators: yesterday and today. Proceedings of the summer school and workshop, Beirut, Lebanon, August 27–September 7, 2001.* Somerville, MA: International Press, 2005.

[9] J. Brundan. Representations of the general linear Lie superalgebra in the BGG category $\mathcal{O}$. In *Developments and retrospectives in Lie theory. Algebraic methods. Retrospective selected papers based on the presentations at the seminar "Lie groups, Lie algebras and their representations", 1991–2014*, pages 71–98. Cham: Springer, 2014.

[10] C. Carmeli. *Super Lie groups: structure and representations*, volume 532. PhD Thesis, University of Genoa, 2006.

[11] C. Carmeli, G. Cassinelli, A. Toigo, and V. S. Varadarajan. Unitary representations of super Lie groups and applications to the classification and multiplet structure of super particles. *Commun. Math. Phys.*, 263(1):217–258, 2006.

[12] C. Carmeli, L. Caston, and R. Fioresi. *Mathematical foundations of supersymmetry*. EMS Ser. Lect. Math. Zürich: European Mathematical Society (EMS), 2011.

[13] C. Carmeli, R. Fioresi, and V. S. Varadarajan. Highest weight Harish-Chandra supermodules and their geometric realizations. *Transform. Groups*, 25(1):33–80, 2020.

[14] C. Carmeli, R. Fioresi, and V. S. Varadarajan. Unitary Harish-Chandra representations of real supergroups. *J. Noncommut. Geom.*, 17(1):287–303, 2023.

[15] E. Cartan. *The theory of spinors*. Dover Books on Advanced Mathematics. Dover Publications, Inc., New York, 1981. With a foreword by Raymond Streater, A reprint of the 1966 English translation.

[16] C.-W. Chen and V. Mazorchuk. Simple supermodules over Lie superalgebras. *Trans. Am. Math. Soc.*, 374(2):899–921, 2021.

[17] S.-J. Cheng and W. Wang. *Dualities and representations of Lie superalgebras*, volume 144 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[18] S.-J. Cheng and R. B. Zhang. Analogue of Kostant's $\mathfrak{u}$-cohomology formula for the general linear superalgebra. *Int. Math. Res. Not.*, 2004(1):31–53, 2004.

[19] C. C. Chevalley. *The algebraic theory of spinors*. Columbia University Press, New York, 1954.

[20] M.-K. Chuah. Cartan automorphisms and Vogan superdiagrams. *Math. Z.*, 273(3-4):793–800, 2013.

[21] M.-K. Chuah and R. Fioresi. Hermitian real forms of contragredient Lie superalgebras. *J. Algebra*, 437:161–176, 2015.

[22] C. Córdova, T. T. Dumitrescu, and K. Intriligator. Multiplets of superconformal symmetry in diverse dimensions. *J. High Energy Phys.*, 2019(3):143, 2019. Id/No 163.

[23] K. J. Costello. Notes on supersymmetric and holomorphic field theories in dimensions 2 and 4. *Pure Appl. Math. Quart.*, 09(1):73–165, 2013.

[24] K. Coulembier and V. Serganova. Homological invariants in category $\mathcal{O}$ for the general linear superalgebra. *Trans. Am. Math. Soc.*, 369(11):7961–7997, 2017.

[25] C. A. Cremonini, P. A. Grassi, and S. Noja. Cohomology of Lie superalgebras: forms, integral forms and coset superspaces. *J. Lie Theory*, 33(2):567–608, 2023.

[26] V. D.A. Dirac operator and unitary representations. *3 talks at MIT Lie groups seminar*, Fall 1997.

[27] P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, and E. Witten, editors. *Quantum fields and strings: a course for mathematicians. Vol. 1, 2.* American Mathematical Society, Providence, RI; Institute

for Advanced Study (IAS), Princeton, NJ, 1999. Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997.

[28] B. DeWitt. *Supermanifolds.* Camb. Monogr. Math. Phys. Cambridge: Cambridge University Press, 2nd ed. edition, 1992.

[29] I. Dimitrov, V. Futorny, and D. Grantcharov. Parabolic sets of roots. In *Groups, rings and group rings*, volume 499 of *Contemp. Math.*, pages 61–73. Amer. Math. Soc., Providence, RI, 2009.

[30] I. Dimitrov, O. Mathieu, and I. Penkov. On the structure of weight modules. *Trans. Amer. Math. Soc.*, 352(6):2857–2869, 2000.

[31] J. Dixmier. *C^*-algebras. Translated by Francis Jellett*, volume 15 of *North-Holland Math. Libr.* Elsevier (North-Holland), Amsterdam, 1977.

[32] J. Dixmier. *Enveloping algebras*, volume 11 of *Grad. Stud. Math.* Providence, RI: AMS, American Mathematical Society, 1996.

[33] C.-P. Dong and J.-S. Huang. Jacquet modules and Dirac cohomology. *Adv. Math.*, 226(4):2911–2934, 2011.

[34] M. Duflo and V. Serganova. On associated variety for Lie superalgebras. Preprint, arXiv:math/0507198 [math.RT] (2005), 2005.

[35] R. Eager, I. Saberi, and J. Walcher. Nilpotence varieties. *Annales Henri Poincare*, 22(4):1319–1376, 2021.

[36] R. Eager and J. Schmude. Superconformal indices and M2-branes. *J. High Energy Phys.*, 2015(12):38, 2015. Id/No 62.

[37] C. Elliott, P. Safronov, and B. R. Williams. A taxonomy of twists of supersymmetric Yang–Mills theory. *Selecta Math.*, 28(4):73, 2022.

[38] T. Enright, R. Howe, and N. Wallach. A classification of unitary highest weight modules. In *Representation theory of reductive groups (Park City, Utah, 1982)*, volume 40 of *Progr. Math.*, pages 97–143. Birkhäuser Boston, Boston, MA, 1983.

[39] S. Eswara Rao and V. Futorny. Integrable modules for affine Lie superalgebras. *Trans. Amer. Math. Soc.*, 361(10):5435–5455, 2009.

[40] R. Fioresi and F. Gavarini. Real forms of complex Lie superalgebras and supergroups. *Comm. Math. Phys.*, 397(2):937–965, 2023.

[41] E. Fischer. über algebraische Modulsysteme und lineare homogene partielle Differentialgleichungen mit konstanten Koeffizienten. *J. Reine Angew. Math.*, 140:48–82, 1911.

[42] V. Fock. Verallgemeinerung und lösung der diracschen statistischen gleichung. *Zeitschrift für Physik*, 49:339–357, 1928.

[43] V. Fock. Konfigurationsraum und zweite Quantelung. *Z. Phys.*, 75:622–647, 1932.

[44] G. B. Folland. *A course in abstract harmonic analysis.* Textb. Math. Boca Raton, FL: CRC Press, 2nd updated edition edition, 2016.

[45] D. S. Freed. *Five lectures on supersymmetry.* American Mathematical Society, Providence, RI, 1999.

[46] T. Friedrich. *Dirac operators in Riemannian geometry. Transl. from the German by Andreas Nestke*, volume 25 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2000.

[47] S. Fubini and E. Rabinovici. Superconformal quantum mechanics. *Nucl. Phys. B*, 245:17, 1984.

[48] H. Furutsu and K. Nishiyama. Classification of irreducible super-unitary representations of $\mathfrak{su}(p, q/n)$. *Commun. Math. Phys.*, 141(3):475–502, 1991.

[49] H. Furutsu and K. Nishiyama. Realization of irreducible unitary representations of $\mathfrak{osp}(M/N; \mathbb{R})$ on Fock spaces. In *Representation theory of Lie groups and Lie algebras. Proceedings of the conference, Fuji-Kawaguchiko, Japan, August 31–September 3, 1990*, pages 1–21. Hackensack, NJ: World Scientific, 1992.

[50] M. Gorelik. On the ghost centre of Lie superalgebras. *Ann. Inst. Fourier (Grenoble)*, 50(6):1745–1764, 2000.

[51] M. Gorelik, C. Hoyt, V. Serganova, and A. Sherman. The duflo–serganova functor, vingt ans aprés. *Journal of the Indian Institute of Science*, 102(3):961–1000, 2022.

[52] C. Gruson. Self-commuting cone for basic classical and strange Lie superalgebras. *Ann. Inst. Fourier*, 50(3):807–831, 2000.

[53] M. Günaydin and D. Volin. The complete unitary dual of non-compact Lie superalgebra $\mathfrak{su}(p, q|n)$ via the generalised oscillator formalism, and non-compact Young diagrams. *Comm. Math. Phys.*, 367(3):873–939, 2019.

[54] Harish-Chandra. Representations of semisimple Lie groups. VI. *Proc. Nat. Acad. Sci. U.S.A.*, 40:1078–1080, 1954.

[55] Harish-Chandra. Discrete series for semisimple Lie groups. I: Construction of invariant eigendistributions. *Acta Math.*, 113:241–318, 1965.

[56] Harish-Chandra. Discrete series for semisimple Lie groups. II: Explicit determination of the characters. *Acta Math.*, 116:1–111, 1966.

[57] Harish-Chandra. Two theorems on semi-simple Lie groups. *Ann. Math. (2)*, 83:74–128, 1966.

[58] T. Heidersdorf and R. Weissauer. *Cohomological tensor functors on representations of the general linear supergroup*, volume 1320 of *Mem. Am. Math. Soc.* Providence, RI: American Mathematical Society (AMS), 2021.

[59] R. A. Herb and J. A. Wolf. The Plancherel theorem for general semisimple groups. *Compositio Math.*, 57(3):271–355, 1986.

[60] P. Hochs and H. Wang. Orbital integrals and K-theory classes. *Ann. K-Theory*, 4(2):185–209, 2019.

[61] G. Hochschild. Semisimplicity of 2-graded Lie algebras. *Ill. J. Math.*, 20:107–123, 1976.

[62] J.-S. Huang, P. Pandˇzić, and D. Renard. Dirac operators and Lie algebra cohomology. *Represent. Theory*, 10:299–313, 2006.

[63] J.-S. Huang, P. Pandˇzić, and D. Vogan. On classifying unitary modules by their Dirac cohomology. *Sci. China Math.*, 60(11):1937–1962, 2017.

[64] J.-S. Huang, P. Pandˇzić, and F. Zhu. Dirac cohomology, K-characters and branching laws. *Amer. J. Math.*, 135(5):1253–1269, 2013.

[65] J.-S. Huang and P. Pandžić. Dirac cohomology, unitary representations and a proof of a conjecture of Vogan. *J. Amer. Math. Soc.*, 15(1):185–202, 2002.

[66] J.-S. Huang and P. Pandžić. Dirac cohomology for Lie superalgebras. *Transform. Groups*, 10(2):201–209, 2005.

[67] J.-S. Huang and P. Pandžić. *Dirac operators in representation theory.* Basel: Birkhäuser, 2006.

[68] J.-S. Huang and W. Xiao. Dirac cohomology of highest weight modules. *Selecta Math. (N.S.)*, 18(4):803–824, 2012.

[69] J.-S. Huang and W. Xiao. Dirac cohomology and orthogonality relations for weight modules. *Pac. J. Math.*, 319(1):129–152, 2022.

[70] J. E. Humphreys. *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, volume 94 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2008.

[71] H. P. Jakobsen. On singular holomorphic representations. *Invent. Math.*, 62(1):67–78, 1980.

[72] H. P. Jakobsen. The last possible place of unitarity for certain highest weight modules. *Math. Ann.*, 256(4):439–447, 1981.

[73] H. P. Jakobsen. *The full set of unitarizable highest weight modules of basic classical Lie superalgebras*, volume 532 of *Mem. Am. Math. Soc.* Providence, RI: American Mathematical Society (AMS), 1994.

[74] J. C. Jantzen. *Moduln mit einem höchsten Gewicht*, volume 750 of *Lect. Notes Math.* Springer, Cham, 1979.

[75] V. Kac. Representations of classical Lie superalgebras. In *Differential geometrical methods in mathematical physics, II (Proc. Conf., Univ. Bonn, Bonn, 1977)*, volume 676 of *Lecture Notes in Math.*, pages 597–626. Springer, Berlin, 1978.

[76] V. G. Kac. Lie superalgebras. *Advances in Math.*, 26(1):8–96, 1977.

[77] V. G. Kac. Laplace operators of infinite-dimensional lie algebras and theta functions. *Proceedings of the National Academy of Sciences*, 81(2):645–647, 1984.

[78] V. G. Kac and M. Wakimoto. Integrable highest weight modules over affine superalgebras and number theory. In *Lie theory and geometry*, volume 123 of *Progr. Math.*, pages 415–456. Birkhäuser Boston, Boston, MA, 1994.

[79] Y. F. Kang and Z. Q. Chen. Dirac operators on quadratic lie superalgebras. *Acta Math. Sinica, English Series*, 37(8):1229–1253, 2021.

[80] M. Kashiwara and M. Vergne. On the Segal-Shale-Weil representations and harmonic polynomials. *Invent. Math.*, 44:1–47, 1978.

[81] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju. An Index for 4 dimensional super conformal theories. *Commun. Math. Phys.*, 275:209–254, 2007.

[82] A. W. Knapp. *Lie groups, Lie algebras, and cohomology*, volume 34 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1988.

[83] A. W. Knapp. *Representation theory of semisimple groups. An overview based on examples. With a new preface by the author.* Princeton, NJ: Princeton University Press, paperback ed. edition, 2001.

[84] A. W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Prog. Math.* Boston, MA: Birkhäuser, 2nd ed. edition, 2002.

[85] A. W. Knapp and P. E. Trapa. Representations of semisimple Lie groups. In *Representation theory of Lie groups. Lecture notes from the Graduate summer school program, Park City, UT, USA, July 13–31, 1998*, pages 5–87. Providence, RI: American Mathematical Society (AMS), 2000.

[86] A. W. Knapp and D. A. j. Vogan. *Cohomological induction and unitary representations*, volume 45 of *Princeton Math. Ser.* Princeton, NJ: Univ. Press, 1995.

[87] B. Kostant. Graded manifolds, graded Lie theory, and prequantization. Differ. geom. Meth. math. Phys., Proc. Symp. Bonn 1975, Lect. Notes Math. 570, 177-306 (1977)., 1977.

[88] B. Kostant. Graded manifolds, graded Lie theory, and prequantization. In *Differential geometrical methods in mathematical physics (Proc. Sympos., Univ. Bonn, Bonn, 1975)*, volume Vol. 570 of *Lecture Notes in Math.*, pages 177–306. Springer, Berlin-New York, 1977.

[89] B. Kostant. A cubic dirac operator and the emergence of euler number multiplets of representations for equal rank subgroups. *Duke Math. J.*, 100(3):447–501, 1999.

[90] B. Kostant. The Weyl algebra and the structure of all Lie superalgebras of Riemannian type. *Transform. Groups*, 6(3):215–226, 2001.

[91] J. L. Koszul. Graded manifolds and graded Lie algebras. Geometry and physics, Proc. int. Meet., Florence/Italy 1982, 71-84 (1983)., 1983.

[92] H. B. j. Lawson. The Atiyah-Singer index theorem and applications. In *Dirac operators: yesterday and today. Proceedings of the summer school and workshop, Beirut, Lebanon, August 27–September 7, 2001*, pages 117–144. Somerville, MA: International Press, 2005.

[93] V. Mazorchuk. Parabolic category $\mathscr{O}$ for classical Lie superalgebras. In *Advances in Lie superalgebras*, volume 7 of *Springer INdAM Ser.*, pages 149–166. Springer, Cham, 2014.

[94] S. Mehdi. Parthasarathy Dirac operators and discrete series representations. In *Connected at infinity II. A selection of mathematics by Indians*, pages 130–151. New Delhi: Hindustan Book Agency, 2013.

[95] P. Meyer. Cubic Dirac operators and the strange Freudenthal–de Vries formula for colour Lie algebras. *Transform. Groups*, 27(4):1307–1336, 2022.

[96] D. Miličić. Variations on a Casselman-Osborne theme. In *Developments and retrospectives in Lie theory*, volume 38 of *Dev. Math.*, pages 275–289. Springer, Cham, 2014.

[97] D. Miličić. Variations on a Casselman-Osborne theme. In *Developments and retrospectives in Lie theory*, volume 38 of *Dev. Math.*, pages 275–289. Springer, Cham, 2014.

[98] S. Minwalla. Restrictions imposed by superconformal invariance on quantum field theories. *Adv. Theor. Math. Phys.*, 2(4):783–851, 1998.

[99] I. M. Musson. *Lie superalgebras and enveloping algebras*, volume 131 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2012.

[100] W. Nahm. Supersymmetries and their representations. In *The world in eleven dimensions: supergravity, supermembranes and M-theory.*, pages 7–24. Bristol: IoP, Institute of Physics Publishing, 1999.

[101] K.-H. Neeb. Square integrable highest weight representations. *Glasgow Math. J.*, 39(3):295–321, 1997.

[102] K.-H. Neeb. *Holomorphy and convexity in Lie theory*, volume 28 of *De Gruyter Expositions in Mathematics.* Walter de Gruyter & Co., Berlin, 2000.

[103] K.-H. Neeb. Unitary representation theory, January 2024. Lecture notes. Available online.

[104] K.-H. Neeb and H. Salmasian. Lie supergroups, unitary representations, and invariant cones. In *Supersymmetry in mathematics and physics. UCLA Los Angeles, USA 2010. Papers based on the presentations at the workshop, Februar 2010.*, pages 195–239. Berlin: Springer, 2011.

[105] K. Nishiyama. Oscillator representations for orthosymplectic algebras. *Journal of Algebra*, 129(1):231–262, 1990.

[106] S. Noja, S. Schmidt, and R. Senghaas. Cubic dirac operators and dirac cohomology for basic classical lie superalgebras, 2025. arXiv:2503.04187.

[107] D. Ž. Đoković and G. Hochschild. Semisimplicity of 2-graded Lie algebras. II. *Ill. J. Math.*, 20:134–143, 1976.

[108] P. Pandˇzić. Dirac operators and unitarizability of Harish-Chandra modules. *Math. Commun.*, 15(1):273–279, 2010.

[109] P. Pandˇzić. Classifying $A_{\mathfrak{q}}(\lambda)$ modules by their Dirac cohomology. In *Lie theory and its applications in physics*, volume 191 of *Springer Proc. Math. Stat.*, pages 387–398. Springer, Singapore, 2016.

[110] P. Pandžić, A. Prlić, V. Souček, and V. Tuček. On the classification of unitary highest weight modules. *arXiv e-prints*, May 2023.

[111] P. Pandžić and D. Renard. Dirac induction for Harish-Chandra modules. *J. Lie Theory*, 20(4):617–641, 2010.

[112] M. Parker. Classification of real simple Lie superalgebras of classical type. *J. Math. Phys.*, 21:689–697, 1980.

[113] R. Parthasarathy. Dirac operator and the discrete series. *Ann. of Math. (2)*, 96:1–30, 1972.

[114] R. Parthasarathy. Dirac operator and the discrete series. *Ann. of Math. (2)*, 96:1–30, 1972.

[115] R. Parthasarathy. Classification of discrete series by minimal K-type. *Represent. Theory*, 19:167–185, 2015.

[116] F. Pellegrini. Real forms of complex Lie superalgebras and complex algebraic supergroups. *Pacific J. Math.*, 229(2):485–498, 2007.

[117] A. Rogers. *Supermanifolds. Theory and applications.* Singapore: World Scientific, 2007.

[118] C. Romelsberger. Counting chiral primaries in N = 1, d=4 superconformal field theories. *Nucl. Phys. B*, 747:329–353, 2006.

[119] I. Saberi and B. R. Williams. Twisting pure spinor superfields, with applications to supergravity. *Pure Appl. Math. Quart.*, 20(2):645–701, 2024.

[120] C. Sachse. *Global analytic approach to super Teichmüller spaces.* Leipzig: Univ. Leipzig, Fakultät für Mathematik und Informatik (Dissertation), 2007.

[121] M. Scheunert. *The theory of Lie superalgebras. An introduction*, volume 716 of *Lect. Notes Math.* Springer, Cham, 1979.

[122] W. Schmid. On the characters of the discrete series. The Hermitian symmetric case. *Invent. Math.*, 30:47–144, 1975.

[123] S. Schmidt. Dirac Cohomology and Unitarizable Supermodules over Lie Superalgebras of Type $A(m|n)$, 2024. arXiv:2411.09404.

[124] S. Schmidt. Classification of unitarizable $\mathfrak{sl}(m|n)$-supermodules. In preparation, 2025.

[125] S. Schmidt. A superdimension for unitarizable $\mathfrak{sl}(m|n)$-supermodules. In preparation, 2025.

[126] S. Schmidt and J. Walcher. An index for unitarizable $sl(m|n)$-supermodules, 2025. arXiv:2504.09161.

[127] M. Schottenloher. *A mathematical introduction to conformal field theory. Based on a series of lectures given at the Mathematisches Institut der Universität Hamburg*, volume 759 of *Lect. Notes Phys.* Berlin: Springer, 2nd revised ed. edition, 2008.

[128] V. Serganova. On the superdimension of an irreducible representation of a basic classical Lie superalgebra. In *Supersymmetry in mathematics and physics. UCLA Los Angeles, USA 2010. Papers based on the presentations at the workshop, Februar 2010.*, pages 253–273. Berlin: Springer, 2011.

[129] V. Serganova. Representations of Lie superalgebras. In *Perspectives in Lie theory*, volume 19 of *Springer INdAM Ser.*, pages 125–177. Springer, Cham, 2017.

[130] V. Serganova and A. Sherman. Localization theorem for homological vector fields. *Comm. Math. Phys.*, 405(2):Paper No. 39, 26, 2024.

[131] V. V. Serganova. Classification of simple real Lie superalgebras and symmetric superspaces. *Funktsional. Anal. i Prilozhen.*, 17(3):46–54, 1983.

[132] A. Sergeev. *Enveloping algebra centre for Lie superalgebras GL and Q.* PhD thesis, Ph. D. thesis, Moscow State University, Moscow, 1987.

[133] A. Sergeev. Invariant functions and laplace-casimir operators on lie superalgebra. In *Seminar on supermanifold*, volume 32, pages 44–95, 1988.

[134] A. Sherman. On symmetries of the duflo-serganova functor, 2022.

[135] S. Shnider. The superconformal algebra in higher dimensions. *Lett. Math. Phys.*, 16(4):377–383, 1988.

[136] V. S. Varadarajan. *Supersymmetry for mathematicians: an introduction*, volume 11 of *Courant Lecture Notes in Mathematics.* New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2004.

[137] E. G. Vishnyakova. On complex Lie supergroups and split homogeneous supermanifolds. *Transform. Groups*, 16(1):265–285, 2011.

[138] E. Witten. Constraints on supersymmetry breaking. *Nuclear Phys. B*, 202(2):253–316, 1982.

[139] E. Witten. Supersymmetry and Morse theory. *J. Diff. Geom.*, 17(4):661–692, 1982.

[140] E. Witten. Topological Quantum Field Theory. *Commun. Math. Phys.*, 117:353, 1988.

[141] G. Zuckerman. Tensor products of finite and infinite dimensional representations of semisimple Lie groups. *Ann. of Math. (2)*, 106(2):295–308, 1977.