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On the enumerative geometry of local curves

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Abstract

This thesis studies the Gromov-Witten and stable pair invariants of local curves. In particular, we give a closed formula for the full descendent stable pair theory of all (non-relative) local curves in terms of the Bethe roots of the quantum intermediate long wave system. In the process, we derive a new explicit description of these Bethe roots, which may be of independent interest. We further deduce rationality, functional equation and a pole restriction for the descendent stable pair theory of local curves as conjectured by Pandharipande and Pixton. Furthermore, we show how the Bethe roots can be used to diagonalize the descedent invariants of the tube and give explicit formulas for the first few descendents. On the Gromov-Witten side, we conjecture that the Gromov-Witten theory of the local elliptic curve is governed by quasi-Jacobi forms. Finally, we compute an infinite series of special cases, which provides evidence for our conjecture.

Zusammenfassung

Diese Doktorarbeit untersucht die Gromov-Witten- und Stabile-Paare-Invarianten lokaler Kurven. Insbesondere geben wir eine geschlossene Formel für die volle Deszendenten-Theorie stabiler Paare aller (nicht-relativen) lokalen Kurven in Termen der Bethe-Wurzeln des quantenmechanischen Intermediate-Long-Wave-Systems an. Im Zuge dessen leiten wir eine neue Charakterisierung der Bethe-Wurzeln her, die von eigenstädigem Interesse sein könnte. Weiterhin beweisen wir die Rationalität, Funktionalgleichung und Polbeschränkung der Stabile-Paare-Invarianten lokaler Kurven gemäß einer Vermutung von Pandharipande und Pixton. Außerdem zeigen wir wie die Bethe-Wurzeln zur Diagonalisierung der Deszendenten-Invarianten der „Tube“ genutzt werden können und geben explizite Formeln für einige der Deszendenten an. Zudem stellen wir die Vermutung auf, dass die Gromov-Witten-Invarianten der lokalen elliptischen Kurve Quasi-Jacobi-Formen sind. Schließlich berechnen wir eine unendliche Reihe von Spezialfällen, wodurch wir diese Vermutung untermauern.

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Published contents

The work presented in Section 2 and Section 3.1 has appeared in the article [96]. The corresponding introductions in Section 1.2.1 and 1.2.2 are also adaptations of the introduction of [96]. Furthermore, our account of quasi-modular and quasi-Jacobi forms in Appendix A is closely modelled after the presentation in [95, Appendix A].

Contents

1 Introduction

1.1 Background and context

1.1.1 History of enumerative geometry

Enumerative geometry goes back to antiquity and is the subfield of algebraic geometry concerned with counting the number of geometric objects satisfying certain constraints. For example: One of the very hard problems in this area was to find the number N_d of rational degree d curves in $\mathbb{P}^2_{\mathbb{C}}$ that pass through $3d-1$ given points in general position. The cases $N_1 = N_2 = 1$ go back to the ancient greeks, while as late as the 1980s only the cases $d \leq 5$ had been determined. In the 1990s, the subject was transformed due to the realization that such curve counts also arise from certain $\mathcal{N} = (2,2)$ *superconformal field theories* in string theory as well as in symplectic geometry. On the physics side, a large but non-rigorous arsenal of computational tools and ideas had by then already been developed - see [37]. Inspired by this, Kontsevich proved:

$$N_d = \sum_{\substack{d_A+d_B=d \\ d_1,d_2>0}} N_{d_1} N_{d_2} d_1^2 d_2 \left(d_2 \binom{3d-4}{3d_1-2} - d_1 \binom{3d-4}{3d_1-1} \right), \tag{1}$$

which recursively determines *all* N_d from the trivial case $N_1 = 1$ (c.f. [44]). One of the main innovations of his proof was the introduction of the moduli space of stable maps:

Theorem 1.1. [8, 44] Let X be a complex projective variety. For all choices of $\beta \in H_2(X,\mathbb{Z})$[1] and integers $g, n \geq 0$, there is a Deligne-Mumford stack $\overline{M}_{g,n}(X,\beta)$ called the *moduli space of stable maps*, whose $\mathbb{C}$-valued points correspond to isomorphism classes of pairs

$$(f : C \to X, p_1, \ldots, p_n \in C)$$

so that

- C is a reduced, connected and projective complex curve whose singularities are nodal.
- the $p_i \in C$ are pairwise distinct closed points called *marked points* that lie in the smooth locus of C.
- the map f is *stable* i.e. there are at most finitely many automorphisms of C fixing f and the p_i.

In particular, morphisms $T \to \overline{M}_{g,n}(X,\beta)$ correspond to flat families of such pairs.

[1] All (co)homology groups will be with $\mathbb{Q}$-coefficients unless otherwise stated.

Remark 1.2. (1) One can relax the definition of stable maps by also allowing disconnected domain curves C so that f collapses no connected component to a point. The space parametrizing such maps is denoted $\overline{M}^{\bullet}_{g,n}(X,\beta)$.

(2) As a moduli stack, the space of stable maps has a universal family of maps

$$\begin{array}{ccc} U_{g,n}(X,\beta) & \xrightarrow{f} & X \\ \pi\downarrow \uparrow s_i & & \\ \overline{M}_{g,n}(X,\beta) & & \end{array} \tag{2}$$

where s_i is the sections induced by the i-th marked point. One usually writes

$$\mathrm{ev}_i = f \circ s_i \colon \overline{M}_{g,n}(X,\beta) \to X$$

for the evaluation of f at the i-th marked point.

For generic points $x_1, \ldots, x_{3d-1} \in \mathbb{P}^2$, one can show that any curve counted in N_d arises uniquely as a stable map and one has

$$\begin{aligned} N_d &= \#\left(\mathrm{ev}_1^{-1}(\{x_1\}) \cap \cdots \cap \mathrm{ev}_{3d-1}^{-1}(\{x_n\})\right) \\ &= \int_{\overline{M}_{0,3d-1}(\mathbb{P}^2,d)} \mathrm{ev}_1^*(\mathrm{pt}) \cdots \mathrm{ev}_{3d-1}^*(\mathrm{pt}) \end{aligned} \tag{3}$$

for $\mathrm{pt} \in H^4(\mathbb{P}^2)$ the point class. This heavily uses the nice structure of $\overline{M}_{0,3d-1}(\mathbb{P}^2, d)$, which turns out to be smooth of dimension $6d-2$.

Having established (3), the identity (1) is now a corollary of a relation in $H^*(\overline{M}_{0,3d-1}(\mathbb{P}^2, d))$ called the *Witten-Dijkgraaf-Verlinde-Verlinde relation*, which can also be viewed as the associativity of the quantum product. See [28, 42] for further details and more on this story.

1.1.2 Gromov-Witten theory

However for general X, the space $\overline{M}_{g,n}(X,\beta)$ may no longer be smooth or even equidimensional and can in fact be arbitrarily ill-behaved (c.f. [104]). As a result, integrals as in (3) tend to be essentially uncomputable and are not expected to have any good properties in general. Surprisingly, most of these problems can be fixed using:

Theorem 1.3. [6, 7] For any smooth and projective complex variety X and discrete data $\beta \in H_2(X,\mathbb{Z})$, $g, n \geq 0$, the space $\overline{M}_{g,n}(X,\beta)$ admits a *perfect obstruction theory*

$$\phi \colon \mathbb{E} \to \mathbb{L}_{\overline{M}_{g,n}(X,\beta)}$$

in the sense of [7], which induces a *virtual fundamental class*

$$\left[\overline{M}_{g,n}(X,\beta)\right]^{\mathrm{vir}} \in H_{2\cdot \mathrm{vdim}}(\overline{M}_{g,n}(X,\beta))$$

in complex homological degree

$$\mathrm{vdim} = (3-\dim X)(g-1) + d_\beta + n \tag{4}$$

where we set $d_\beta = \int_\beta c_1(X)$.

If all components of $\overline{M}_{g,n}(X,\beta)$ have dimension equal to vdim (e.g. if $g=0$ and $X=\mathbb{P}^r$ c.f. [8]), then it follows from Lemma 2.8 that the virtual class is just the usual fundamental class. Otherwise, the virtual class is a more well-behaved alternative. We can therefore generalize the integral (3):

Definition 1.4. Using the notation of Theorem 1.3, let $\gamma_1,\dots,\gamma_n \in H^*(X)$ and $k_1,\dots,k_n \geq 0$ non-negative integers. We denote the corresponding *Gromov-Witten invariant* by

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle_{g,\beta}^{X,\mathrm{GW}} := \int_{\left[\overline{M}_{g,n}(X,\beta)\right]^{\mathrm{vir}}} \psi_1^{k_1}\,\mathrm{ev}_1^*(\gamma_1)\cdots\psi_n^{k_n}\,\mathrm{ev}_n^*(\gamma_n) \tag{5}$$

where $\psi_i = c_1(s_i^*\Omega_\pi) \in H^2(\overline{M}_{g,n}(X,\beta))$ with π and s_i as in (2). We call the formal expressions $\tau_k(\gamma)$ *descendent insertions*. All insertions of the shape $\tau_0(\gamma)$ are furthermore called *primary insertions*.

Remark 1.5. (1) The moduli space of disconnected stable maps $\overline{M}^\bullet_{g,n}(X,\beta)$ also admits a virtual class

$$\left[\overline{M}^\bullet_{g,n}(X,\beta)\right]^{\mathrm{vir}} \in H_{2\cdot \mathrm{vdim}}(\overline{M}^\bullet_{g,n}(X,\beta))$$

and the resulting invariants

(6)

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle_{g,\beta}^{X,\mathrm{GW},\bullet} := \int_{\left[\overline{M}^\bullet_{g,n}(X,\beta)\right]^{\mathrm{vir}}} \psi_1^{k_1}\,\mathrm{ev}_1^*(\gamma_1)\cdots\psi_n^{k_n}\,\mathrm{ev}_n^*(\gamma_n)$$

are called *disconnected Gromov-Witten invariants*. For emphasis, we will sometimes denote the connected invariants of Definition 1.4 by $\langle \tau_{k_1}(\gamma_1)\dots\tau_{k_n}(\gamma_n)\rangle_{g,\beta}^{X,\mathrm{GW},\circ}$. It turns out that connected and disconnected invariants determine each other, see Proposition 4.1.

(2) The case $\dim X = 3$ is especially interesting as the first summand in (4) vanishes. This lets us define generating series

(7)

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle_{\beta}^{X,\mathrm{GW}} := \sum_{g\in\mathbb{Z}} (-1)^{g-1} z^{2g-2} \langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle_{g,\beta}^{X,\mathrm{GW}}$$

of which all coefficients are usually non-zero. Indeed, string theorists are most interested in Gromov-Witten invariants of *Calabi-Yau*[2] threefolds as they are conjectured to encapsulate the six extra dimensions of spacetime - see [16].

(3) Assuming that the γ_i are represented by some generic submanifolds $M_i \subset X$, one can think of (5) as (virtually) counting the number of stable maps
$$f\colon (C, p_1, \dots p_n) \to X$$
so that $f(p_i) \in M_i$ with the ψ_i imposing certain ramification conditions on f (c.f. [76, §1.1]). However, Gromov-Witten invariants are rarely literal curve counts - for one because most naive curve counts do not have finite answers. The advantage of Definition 1.4 is that it always yields rational numbers which are invariant under complex deformation of X. Furthermore, these agree with the corresponding curve counts arising in symplectic geometry (c.f. [47, 98]).

In practice, Gromov-Witten invariants are rather difficult to compute. For instance, Gromov-Witten theory in case $\dim X \leq 1$ has been solved completely (c.f. [43, 75, 76, 77]), but there are few varieties of higher dimension whose standard Gromov-Witten theory is non-trivial and has been fully computed. This is presents a challenge as this thesis is mostly concerned with the case $\dim X = 3$.

One of the most powerful computational tools (and the one that is used most in this thesis) is virtual localization:

1.1.3 Virtual localization

Let X be a smooth, complex, projective variety with an action by a torus $\mathbb{T} = (\mathbb{C}^*)^N$. This induces a natural $\mathbb{T}$-action on $\overline{M}_{g,n}(X,\beta)$ and there is an equivariant virtual class
$$\left[\overline{M}_{g,n}(X,\beta)\right]^{\mathrm{vir},\mathbb{T}} \in H^{\mathbb{T}}_{2\cdot \mathrm{vdim}}(\overline{M}_{g,n}(X,\beta))$$
in equivariant homology[3], which maps to the usual virtual class under the canonical map $H^{\mathbb{T}}_* \to H_*$. One can therefore define *equivariant Gromov-Witten invariants*

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^{X,\mathbb{T}}_{g,\beta} := \int_{\left[\overline{M}_{g,n}(X,\beta)\right]^{\mathrm{vir},\mathbb{T}}} \psi_1^{k_1}\,\mathrm{ev}_1^*(\gamma_1)\cdots\psi_n^{k_n}\,\mathrm{ev}_n^*(\gamma_n), \tag{8}$$

where $\gamma_1, \dots, \gamma_n \in H^*_{\mathbb{T}}(X)$. These invariants take values in $H^{\mathbb{T}}_*(\mathrm{pt}) = \mathbb{Q}[t_1, \dots, t_N]$. The connection to usual Gromov-Witten invariants comes

[2] i.e. smooth and projective X so that $c_1(X) = 0$.
[3] See [2] for an introduction to equivariant (co)homology.

from the fact that

$$\langle \tau_{k_1}(\overline{\gamma}_1) \cdots \tau_{k_n}(\overline{\gamma}_n) \rangle^{X}_{g,\beta} = \langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle^{X,\mathbb{T}}_{g,\beta} \,|_{t_1=\ldots=t_N=0}$$

where $\overline{\gamma}_i$ denotes the image of $\gamma_i \in H^*_{\mathbb{T}}(X)$ under the canonical map $H^*_{\mathbb{T}}(X) \to H^*(X)$, which is often surjective. The virtual localization theorem [32] gives a means of calculating equivariant Gromov-Witten invariants:

(9)

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle^{X,\mathbb{T}}_{g,\beta} = \int_{[\overline{M}_{g,n}(X,\beta)^{\mathbb{T}}]^{\mathrm{vir}}} \frac{\psi_1^{k_1} \operatorname{ev}_1^*(\gamma_1) \cdots \psi_n^{k_n} \operatorname{ev}_n^*(\gamma_n)\Big|_{M^{\mathbb{T}}}}{e(N^{\mathrm{vir}}_{M^{\mathbb{T}}/M})}$$

where $[\overline{M}_{g,n}(X,\beta)^{\mathbb{T}}]^{\mathrm{vir}} \in H_*(\overline{M}_{g,n}(X,\beta)^{\mathbb{T}})$ is the induced virtual class on the fixed locus and $N^{\mathrm{vir}}_{M^{\mathbb{T}}/M} \in K^0_{\mathbb{T}}(M^{\mathbb{T}})$ is the *virtual normal bundle* of the embedding $M^{\mathbb{T}} \hookrightarrow M$. Note that the right hand side remains well-defined in greater generality:

Situation 1. Let X be a smooth, *quasi-projective* complex variety equipped with an action by a torus $\mathbb{T} = (\mathbb{C}^*)^N$ so that the union of all projective $\mathbb{T}$-invariant subcurves

$$X^{(\mathbb{T})} := \bigcup_{\substack{Z \subseteq X \text{ proj} \\ \dim Z \leq 1 \\ \mathbb{T} \cdot Z \subset Z}} Z \subseteq X$$

is a closed subscheme and projective.

Situation 1 guarantees that the fixed locus $\overline{M}_{g,n}(X,\beta)^{\mathbb{T}} = \overline{M}_{g,n}(X^{(\mathbb{T})},\beta)^{\mathbb{T}}$ is a proper moduli space, which enables:

Definition 1.6. In Situation 1 we set

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle^{X,\mathbb{T}}_{g,\beta} := \int_{[\overline{M}_{g,n}(X,\beta)^{\mathbb{T}}]^{\mathrm{vir}}} \frac{\psi_1^{k_1} \operatorname{ev}_1^*(\gamma_1) \cdots \psi_n^{k_n} \operatorname{ev}_n^*(\gamma_n)\Big|_{M^{\mathbb{T}}}}{e(N^{\mathrm{vir}})}$$

which takes values in $\mathsf{Frac}(H^*_{\mathbb{T}}(\mathrm{pt})) = \mathbb{Q}(t_1, \ldots, t_N)$.

Indeed, this is the definition of Gromov-Witten invariants that we will use from now on.

1.1.4 Pandharipande-Thomas invariants

Pandharipande-Thomas (or stable pair) invariants are an alternative to Gromov-Witten theory in case $\dim X = 3$ and have somewhat different properties. More precisely:

Theorem 1.7. [87] Let X be a smooth (quasi-)projective threefold, $n \in \mathbb{Z}$ and $\beta \in H_2(X)$. There is a (quasi-)projective scheme $P_n(X, \beta)$, whose $\mathbb{C}$-valued points correspond to isomorphism classes of *stable pairs* i.e. morphisms $\mathcal{O}_X \xrightarrow{s} F$ of coherent sheaves on X so that

- F is pure of dimension 1 and has proper support.
- the cokernel of s is zero-dimensional.
- we have $\chi(F) = \text{n}$ and $[\text{Supp}(F)] = \beta \in H_2(X)$.

Furthermore, morphisms $T \to P_n(X, \beta)$ correspond to isomorphism classes of flat families of stable pairs.

Furthermore, this space admits a natural perfect obstruction theory

$$\phi \colon \mathbb{E} \to \mathbb{L}_{P_n(X,\beta)}$$

which gives rise to a virtual fundamental class

$$[P_n(X, \beta)]^{\text{vir}} \in H_{2 \cdot \text{vdim}}(P_n(X, \beta))$$

with $\text{vdim} = d_\beta$.

Since this scheme is a fine moduli space, there is a universal stable pair $\mathcal{O}_{X \times P} \to \mathbb{F}$ on $X \times P_n(X, \beta)$. We introduce descendent insertions by[4]

$$\text{ch}_k(\gamma) := (\pi_P)_* \left(\text{ch}_k(\mathbb{F}) \cdot \pi_X^* \gamma\right) \in H^*(P_n(X, \beta))$$

for $k \geq 0$ and $\gamma \in H^*(X)$, where we used the maps

$$\begin{array}{ccc} & X \times P_n(X,\beta) & \\ \swarrow^{\pi_P} & & \searrow^{\pi_X} \\ P_n(X,\beta) & & X \end{array}$$

Since $\mathbb{F}$ has support of codimension 2, we have $\text{ch}_0(\gamma) = \text{ch}_1(\gamma) = 0$.

If X is as in Situation 1, then the fixed locus $P_n(X, \beta)^{\mathbb{T}}$ is projective and carries a natural virtual class. Moreover, the construction of $\text{ch}_k(\gamma)$ can be carried out equivariantly. Hence one defines:

Definition 1.8. In Situation 1, let $\gamma_1, \ldots, \gamma_n \in H^*_{\mathbb{T}}(X)$, $k_1, \ldots, k_n \geq 0$. The corresponding *stable pair invariants* or *Pandharipande-Thomas invariants* are defined by:

$$\langle \text{ch}_{k_1}(\gamma_1) \cdots \text{ch}_{k_n}(\gamma_n) \rangle_{n,\beta}^{X,\text{PT},\mathbb{T}} := \int_{[P_n(X,\beta)^{\mathbb{T}}]^{\text{vir}}} \frac{\text{ch}_{k_1}(\gamma_1) \cdots \text{ch}_{k_n}(\gamma_n) \mid_{P^{\mathbb{T}}}}{e\left(N^{\text{vir}}_{P^{\mathbb{T}}/P}\right)}$$

[4] The pushforward in cohomology is defined as the dual of the pullback in homology which exists as π_P is flat [19, Theorem VIII.5.1].

which one puts as coefficients of the power series

$$(10)\quad \langle \mathrm{ch}_{k_1}(\gamma_1)\cdots \mathrm{ch}_{k_n}(\gamma_n)\rangle_{\beta}^{X,\mathrm{PT},\mathbb{T}} := \sum_{n\in\mathbb{Z}} p^n \langle \mathrm{ch}_{k_1}(\gamma_1)\cdots \mathrm{ch}_{k_n}(\gamma_n)\rangle_{n,\beta}^{X,\mathrm{PT},\mathbb{T}}$$

which takes values in $\mathbb{Q}(t_1,\ldots,t_N)((p))$.

Remark 1.9. (1) Gromov-Witten invariants can be quite far from counts of subcurves due to the presence of contracted components and multiple covers. On the other hand, a stable pair differs from a subcurve only by the finite cokernel of s. As a result, Pandharipande-Thomas invariants are closer to counting subcurves.

One can also define invariants using the Hilbert scheme of subcurves of X, which leads to *Donaldson-Thomas invariants* - see [18, 102]. For reasons that will become clear shortly, we will not study them in this thesis.

Stable pair invariants are in a sense nicer than Gromov-Witten invariants as they are expected to have the following structure:

Conjecture A. [80, 96] In Situation 1:

(1) The stable pair invariants (10) are Laurent expansions in p of rational functions i.e. elements in $\mathbb{Q}(t_1,\ldots,t_N,p)$.

(2) Under the variable change $p\mapsto p^{-1}$ these rational functions transform as follows:

$$\begin{aligned}&\langle \mathrm{ch}_{k_1}(\gamma_1)\cdots \mathrm{ch}_{k_n}(\gamma_n)\rangle_{\beta}^{X,\mathrm{PT},\mathbb{T}}\Big|_{p\mapsto p^{-1}}\\ &= (-1)^{\sum_i k_i} p^{-d_\beta} \langle \mathrm{ch}_{k_1}(\gamma_1)\cdots \mathrm{ch}_{k_n}(\gamma_n)\rangle_{\beta}^{X,\mathrm{PT},\mathbb{T}}\end{aligned}$$

(3) Stable pair invariants (10) may have p-poles only at $p=0$ and where $-p$ is an n-th root of unity for $1\le n\le d(\beta)$. Here, we set

$$d(\beta) := \max\{\, m>0 \mid \beta = m\beta_1+\beta_2 \text{ for } \beta_1,\beta_2 \text{ curve classes}, \beta_1>0\,\}$$

Remark 1.10. (1) For a strictly stronger version of these conjectures involving *connected* stable pairs, see [96].

(2) There is no version of Conjecture A for Gromov-Witten theory - the closest approximation being [72, Conjecture 24]. In terms of formulas, Gromov-Witten invariants of threefolds are rather unpleasant and only nice if they happen to match stable pair invariants particularly closely.

(3) Conjecture A was first stated in its full generality in [80] and was historically a big driving factor for the development of stable pair theory. Indeed, it was noted in [56] that the corresponding Donaldson-Thomas descendent theory is irrational. Stable pairs are much better behaved - in particular rationality of the generating series was first conjectured in [87] with first examples being computed in [88]. For toric threefolds and certain complete intersections the rationality was proved in [84, 86]. The $p \mapsto p^{-1}$ symmetry was first formulated in the Calabi-Yau case in [55] and related to Serre duality in [88]. This led to a proof of rationality and symmetry in the Calabi-Yau case in [13, 103] based on the Behrend function approach to enumerative geometry. However, this approach does not generalize to the non-Calabi-Yau case.

1.1.5 GW/PT correspondence

One of the most profound conjectures in enumerative geometry is the *Gromov-Witten/Pandharipande-Thomas correspondence*, which asserts that the Gromov-Witten and stable pair invariants of a threefold should determine each other. We will now sketch this and refer to [81] for further details.

Recall the universal correspondence matrix[5]

$$\widetilde{K}_{\alpha,\widehat{\alpha}} \in \mathbb{Q}[c_1, c_2, c_3]((z)),$$

which was constructed in [85]. Here we take $\alpha, \widehat{\alpha}$ to be two *weak partitions*, by which we mean non-empty sequences

$$\alpha = \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{\ell(\alpha)} \geq 0 \text{ and } \widehat{\alpha} = \widehat{\alpha}_1 \geq \widehat{\alpha}_2 \geq \ldots \geq \widehat{\alpha}_{\ell(\widehat{\alpha})} \geq 0$$

of non-negative natural numbers. This matrix has the following properties:

Proposition 1.11. [85, Thm 2 and §7.3]

(1) We have $\widetilde{K}_{\alpha,\widehat{\alpha}} = 0$ if $|\alpha| + \ell(\alpha) < |\widehat{\alpha}| + \ell(\widehat{\alpha})$.

(2) The z-coefficients
$$[z^k]\widetilde{K}_{\alpha,\widehat{\alpha}} \in \mathbb{Q}[c_1, c_2, c_3]$$
are homogeneous of degree
$$|\alpha| - |\widehat{\alpha}| - \ell(\alpha) - 2\ell(\widehat{\alpha}) + 3$$
where c_i is defined to have degree i.

(3) We have
$$\widetilde{K}_{(d),(d)} = z^{-d}$$

[5]Our matrix differs from the one introduced in [85] by the variable change $z = iu$ and the use of weak partitions instead of partitions.

Remark 1.12. The universal correspondence matrix is defined in [85] in terms of the capped triple vertex. While it has been partially computed in [63, 72], no full formulas for $\widetilde{K}$ are known.

For any X as in Situation 1, one can consider $\widetilde{K}_{\alpha,\widehat{\alpha}}$ as an element in $H^*_{\mathbb{T}}(X)((z))$ by substituting the equivariant chern classes $c_i^{\mathbb{T}}(X)$ for c_i. Given $\gamma_1, \dots, \gamma_n \in H^*_{\mathbb{T}}(X)$ homogeneous cohomology classes and $\alpha = (\alpha_1, \dots, \alpha_n)$ a weak partition, we can use this to define a formal descendent insertion

$$\overline{\tau_{\alpha_1}(\gamma_1) \cdots \tau_{\alpha_n}(\gamma_n)} := \sum_{\substack{P \text{ set part of} \\ \{1,\dots,n\}}} \operatorname{sgn}(P) \prod_{S \in P} \sum_{\widehat{\alpha}} \tau_{\widehat{\alpha}}\left(\widetilde{K}_{\alpha_S,\widehat{\alpha}} \cdot \gamma_S\right), \tag{11}$$

where $\operatorname{sgn}(P) = \pm 1$ is the sign that arises from permuting the γ_i of odd degree. Furthermore, we set $\alpha_S = (\alpha_i)_{i \in S}$, $\gamma_S = \prod_{i \in S} \gamma_i$ and

$$\tau_{\widehat{\alpha}}(\gamma) := \sum_s \tau_{\widehat{\alpha}_1}(\delta_{1,s}) \cdots \tau_{\widehat{\alpha}_l}(\delta_{l,s}),$$

where

$$\Delta_*^{\text{small}}(\gamma) = \sum_s \delta_{1,s} \otimes \cdots \otimes \delta_{l,s} \in H^*_{\mathbb{T}}(X^l)$$

is the equivariant Künneth decomposition of the pushforward along the small diagonal $\Delta^{\text{small}} \colon X \hookrightarrow X^l$. We are now ready to relate Gromov-Witten and stable pair invariants:

Conjecture B. [55, 56, 85] In Situation 1, for any $\beta \in H_2(X)$, $\gamma_1, \dots, \gamma_l \in H^*_{\mathbb{T}}(X)$ and weak partition $\alpha = (\alpha_1, \dots, \alpha_n)$ we have

$$\begin{aligned} &(-p)^{-d_\beta/2} \left\langle \operatorname{ch}_{\alpha_1+2}(\gamma_1) \cdots \operatorname{ch}_{\alpha_l+2}(\gamma_l) \right\rangle_\beta^{X,\text{PT},\mathbb{T}} \\ &= (-z)^{d_\beta} \left\langle \overline{\tau_{\alpha_1}(\gamma_1) \cdots \tau_{\alpha_l}(\gamma_l)} \right\rangle_\beta^{X,\text{GW},\mathbb{T},\bullet} \end{aligned} \tag{12}$$

with the variable change $p = -e^z$.

Remark 1.13. (1) Note that the variable change $p = -e^z$ is only well-defined if we assume Conjecture A.(1).

(2) By Proposition 1.11.(1),(2), the $\widehat{\alpha}$ occuring in (11) must satisfy

$$|\alpha_S| \geq |\widehat{\alpha}| \text{ and } |\alpha_S| + \ell(\alpha_S) \geq |\widehat{\alpha}| + \ell(\widehat{\alpha}).$$

In particular, the sum over $\widehat{\alpha}$ must therefore be finite. If one of these inequalities becomes an equality, we can use Proposition 1.11.(2) to see that

$$|\alpha_S| - |\widehat{\alpha}| \geq \ell(\alpha_S) + 2\ell(\widehat{\alpha}) - 3 \geq 0$$

$$|\alpha_S| + \ell(\alpha_S) - (|\widehat{\alpha}| + \ell(\widehat{\alpha})) \geq 2\ell(\alpha_S) + \ell(\widehat{\alpha}) - 3 \geq 0$$

and hence we must have $|S| = 1$ and $\widehat{\alpha} = \alpha_S = (d)$ for some $d \geq 0$. This case is described by Proposition 1.11.(3), so we have

$$\overline{\tau_{\alpha_1}(\gamma_1)\cdots\tau_{\alpha_l}(\gamma_l)} = z^{-|\alpha|}\tau_{\alpha_1}(\gamma_1)\cdots\tau_{\alpha_l}(\gamma_l) + \ldots$$

where "…" stands for summands of the shape $c \cdot \tau_{\beta_1}(\delta_1)\cdots\tau_{\beta_l}(\delta_m)$ so that

$$|\alpha| > |\beta| \text{ and } |\alpha| + \ell(\alpha) > |\beta| + \ell(\beta).$$

In particular, this gives

$$\overline{\tau_0(\gamma_1)\cdots\tau_0(\gamma_l)} = \tau_0(\gamma_1)\cdots\tau_0(\gamma_l).$$

There has been a lot of progress towards Conjecture B. In particular, if X is a toric variety, the conjecture is known by [57, 85]. Furthermore, many more examples have been confirmed mostly using degeneration techniques - see for example [61, 69, 79, 86, 89]. Recently, there was a remarkable breakthrough in [90], where Pardon showed:

Theorem 1.14. [90] Let X be a smooth projective semi-Fano[6] threefold, $\gamma_1, \ldots, \gamma_l \in H^*(X)$, $\beta \in H_2(X)$, then:

$$(-p)^{-d_\beta/2}\langle \mathrm{ch}_2(\gamma_1)\cdots\mathrm{ch}_2(\gamma_l)\rangle_\beta^{X,\mathrm{PT}} = (-z)^{d_\beta}\langle \tau_0(\gamma_1)\cdots\tau_0(\gamma_l)\rangle_\beta^{X,\mathrm{GW},\bullet}$$

with the variable change $p = -e^z$.

Remark 1.15. We expect that the results of [90] can be extended to the semi-Fano version of Situation 1 using an approximation argument similar to [90, §4.3].

1.1.6 Pardon's proof of Theorem 1.14

We will briefly outline the gist of the proof as it serves as one of the main motivations for this thesis. The first step is the construction of a ring called the *Grothendieck ring of semi-Fano 1-cycles* $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ in which any projective semi-Fano X with $\beta \in H_2(X)$ and $\gamma_1, \ldots, \gamma_l \in H^*(X)$ naturally induces an element

$$(X, \beta; \gamma_1, \ldots, \gamma_l) \in H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3). \tag{13}$$

Furthermore, this ring admits a bigrading so that (13) has bidegree (d_β, vdim) where the second degree is

$$\mathrm{vdim} := 2d_\beta - \sum_i (\deg(\gamma_i) - 2)$$

[6] i.e. $c_1(X)$ is nef, meaning that $\int_C c_1(X) \geq 0$ for any subcurve $C \subset X$.

$$= \deg \left(\left[\overline{M}^{\bullet}_{g,l}(X,\beta) \right]^{\mathrm{vir}} \cap \prod_i \tau_0(\gamma_i) \right)$$
$$= \deg \left([P_n(X,\beta)]^{\mathrm{vir}} \cap \prod_i \mathrm{ch}_2(\gamma_i) \right),$$

which is hence dubbed the *virtual dimension.* It now turns out that Gromov-Witten and stable pair theory induce ring homomorphisms

$$H^*_c(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3) \xrightarrow{\mathrm{GW}} \mathbb{Q}((z)) \text{ and } H^*_c(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3) \xrightarrow{\mathrm{PT}} \mathbb{Q}((p)), \tag{14}$$

which yield the corresponding primary Gromov-Witten and stable pair invariants when evaluated at (13) - indeed, this holds for any curve counting theory that is deformation invariant and multiplicative under disjoint unions. As a result, the class (13) can be regarded as a kind of universal curve counting invariant. Pardon then shows the following via intricate analytic transversality arguments:

Theorem 1.16. [90, Theorem 1.1] The virtual dimension 0 part of $H^*_c(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ is generated by the so-called *equivariant local curve elements*

$$x_{g,m,k} \in H^*_c(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)_{mk,0}$$

for $g, m, k \geq 0$.

Remark 1.17. (1) Since only classes of virtual dimension 0 have a non-zero image under (14), this expresses all primary semi-Fano curve-counting invariants as polynomials of the corresponding invariants of the $x_{g,m,k}$.

(2) The intuition behind this statement is that morally, the virtual dimension should be the actual dimension of the moduli space of curves in X that we want to count. Thus, in virtual dimension 0, the moduli space should consist of only finitely many curves $C_1, \ldots, C_l \subset X$. By deformation to the normal cone and deformation invariance, the contribution of a fixed curve $C_i \subset X$ must be the same as that of the zero-section $C_i \subset N_{C_i/X}$ to the curve count of $N_{C_i/X}$. The equivariant local curve elements precisely encapsulate the contributions of such zero-sections. We can therefore replace X by $\coprod_{i=1}^{l} N_{C_i/X}$. Since the product in $H^*_c(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ is induced by disjoint union, this implies that the class in $H^*_c(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ corresponding to the curve count is a product of l local curve elements. Pardon's proof of Theorem 1.16 may be viewed as making this very imprecise argument rigorous.

To further explain the $x_{g,m,k}$, recall that a *local curve* is a threefold which is a total space $X = \mathsf{Tot}_C(L_1 \oplus L_2)$ for two line bundles L_1, L_2 of degree l_1, l_2

on a smooth projective curve C. This space is equipped with the $\mathbb{T} = (\mathbb{C}^*)^2$-action scaling the two summands. Since $X^{(\mathbb{T})} = C$ is projective, we are in Situation 1, which allows us to do curve counting on X. The $x_{g,m,k}$ are constructed in [90, §4.3] so as to satisfy

$$\mathrm{GW}(x_{g,m,k}) = \langle 1 \rangle_{m[C]}^{X,\mathrm{GW},\mathbb{T},\bullet}\Big|_{t_1=t_2=1} \text{ and } \mathrm{PT}(x_{g,m,k}) = \langle 1 \rangle_{m[C]}^{X,\mathrm{PT},\mathbb{T}}\Big|_{t_1=t_2=1}$$

for any local curve $X = \mathsf{Tot}_C(L_1 \oplus L_2)$ with

$$g = g(C) \text{ and } k = c_1(TX) = 2 - 2g + l_1 + l_2.$$

The following theorem follows from lengthy degeneration and localization arguments:

Theorem 1.18. [15, 79, 82, 83] For any local curve X, the empty insertions

$$\langle 1 \rangle_m^{X,\mathrm{GW},\mathbb{T},\bullet} \text{ and } \langle 1 \rangle_m^{X,\mathrm{PT},\mathbb{T}}$$

obey Conjectures A and B.

This altogether proves Theorem 1.14 and under the same assumptions also Conjecture A since all[7] statements are compatible with the ring structure and grading of $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ and hold for the local curve elements.

It is yet unclear how to extend this strategy to arbitrary descendent insertions. Indeed, the fact that Conjecture B is not a straight equality, but involves complicated (and unknown) correction terms, gives a hint that the $x_{g,m,k}$ and the empty invariants on local curves may perhaps not be able to capture the general case.

Guided by this line of thought, the main aim of this thesis is to study the full descendent Gromov-Witten and stable pair theories of local curves, which we see as very representative of the behaviour of general threefolds. We further advocate for the notion that the local curve invariants are very complicated, but should nonetheless be fully and explicitly computable.

Our results are as follows:

1.2 Stable pair theory of local curves

Let us first fix some notation. Since this section studies stable pair invariants, we will drop the superscript "PT" from our invariants $\langle \ldots \rangle^{\mathrm{PT}}$. It will also be useful to turn our insertions into generating series

$$\mathrm{ch}_z(\gamma) := \sum_{k \geq 0} z^k \mathrm{ch}_k(\gamma).$$

Furthermore, any boldface letter will refer to a vector whose entries are denoted by the corresponding non-boldface letter e.g. $\mathbf{v} = (v_1, \ldots, v_n)$.

[7] This shows only a slightly weaker version of Conjecture A.(3) - we expect that one can prove the full statement through a more careful study of [90].

1.2.1 Absolute theory

The first main result of this thesis is

Theorem 1.19. Conjecture A holds for all descendent invariants on local curves.

Remark 1.20. In part this was already proved in [82, 83], where rationality is shown for all descendents and symmetry and pole restriction for stationary and even descendents respectively. Their proof relies on a subtle pole cancellation property of the stable pair vertex as well as degeneration, monodromy invariance and localization. As a result, their methods also work in the relative case, which we do not address. However, it does not seem that these methods can yield a full proof of Theorem 1.19.[8]

We will deduce Theorem 1.19 from Theorem 1.24, which even gives a closed formula for these invariants. But before we state that formula, we must first introduce its constituents: The Bethe roots.

1.2.1.1 Bethe roots. A commonly used tool in the theory of quantum integrable systems is the *algebraic Bethe Ansatz*, which goes back to [10, 26, 99]. The key observation is that many integrable systems have an associated system of polynomial equations called *Bethe equations* whose solutions allow one to diagonalize the integrals of motion of that system - see [100] for an introduction. The connection to enumerative geometry was first observed in [67, 68]. The Bethe equations most relevant to us come from the *quantum intermediate long wave* system (ILW_1 in the notation of [51, 52]), which are as follows:

Let K be the field of Puiseux series

$$K = \overline{\mathbb{Q}(t_1,t_2)}\{\{p\}\} \coloneqq \bigcup_{n\geq 1} \overline{\mathbb{Q}(t_1,t_2)}((p^{1/n}))$$

with $\overline{\mathbb{Q}(t_1,t_2)}$ the algebraic closure of $\mathbb{Q}(t_1,t_2)$. In particular, recall that K is algebraically closed [21, Cor 13.15]. For fixed $d \geq 1$ we call a tuple $\mathbf{Y} \in K^d$ *admissible* if $Y_i \neq 0, t_1+t_2$ for any i and $Y_i - Y_{i'} \neq t_1, t_2, t_1+t_2$ for all $i \neq i'$. We are then interested in certain admissible tuples satisfying

$$p = F_i(\mathbf{Y}) \tag{15}$$

for all $i = 1, \ldots, d$, where

$$F_i(\mathbf{Y}) \coloneqq \tfrac{Y_i}{t_1+t_2-Y_i} \prod_{\substack{i'\neq i \\ 0\leq a,b,c\leq 1 \\ (a,b)\neq(0,0)}} \Big((-1)^c(at_1+bt_2) + Y_{i'} - Y_i\Big)^{(-1)^{a+b+c}}. \tag{16}$$

[8]For the pole restriction, the problem is that the algorithm given in [75, 82, 83] relies on inverting the cap matrix (c.f. [82, §9.1]). However, the fact that the entries of a matrix only have certain poles does not imply that the entries of the inverse also only have said poles.

From now on we will simply refer to (15) as the *Bethe equations*. The particular solutions of these equations that are of interest to us are characterized as follows:

Theorem 1.21. For any partition λ of size d there is a tuple $\mathbf{Y}^\lambda := \left(Y_\square^\lambda(p)\right)_{\square\in\lambda}$ of power series $Y_\square^\lambda(p) \in \mathbb{Q}(t_1,t_2)[[p]]$ indexed by the boxes in the Young diagram[9] of λ which is uniquely determined by any of the following equivalent descriptions:

(1) It is the unique admissible solution of (15) in K^d so that

$$Y_{(i,j)}^\lambda(p) = -it_1 - jt_2 + \mathcal{O}\left(p^{>0}\right)$$

for any box $(i,j) \in \lambda$.

(2) Let the sequence $\mathbf{v}^n = (v_\square^n(p))_{\square\in\lambda}$ of tuples of power series $v_\square^n(p) \in \mathbb{Q}(t_1,t_2)[[p]]$ be defined by

$$v_\square^0(p) := 0$$

for $n = 0$ and for $n > 0$ we recursively set

$$v_\square^n(p) := \tfrac{p}{\widetilde{F}_\square^\lambda(\mathbf{Y}^{(\lambda)}(\mathbf{v}^{n-1}))},$$

where

$$\widetilde{F}_\square^\lambda(\mathbf{Y})$$
$$= (-1)^{d-1}\tfrac{Y_\square^{1-\delta_{\square,(0,0)}}}{t_1+t_2-Y_\square} \prod_{\substack{\square\neq\square'\in\lambda \\ 0\leq a,b,c\leq 1 \\ (a,b)\neq(0,0) \\ \square'\neq\square+(-1)^c(a,b)}} (at_1+bt_2+(-1)^c(Y_{\square'}-Y_\square))^{(-1)^{a+b+c}}$$

and $\mathbf{Y}^{(\lambda)}(\mathbf{v}) = \left(Y_\square^{(\lambda)}(\mathbf{v})\right)_{\square\in\lambda}$ has entries given by

$$Y_{(i,j)}^{(\lambda)}(\mathbf{v}) := -it_1 - jt_2 + \sum_{\substack{\lambda/\mu \text{ conn. skew} \\ (i,j)\in\lambda/\mu}} \prod_{\square\in\lambda/\mu} v_\square \tag{17}$$

with the sum running over all connected skew partitions contained in λ. We now have [10]

$$Y_\square^\lambda(p) = \lim_{n\to\infty} Y_\square^{(\lambda)}(\mathbf{v}^n).$$

[9] c.f. Section 2.1.1 for the relevant notation.

[10] All factors occuring in $\widetilde{F}_{(i,j)}^\lambda(\mathbf{Y}^{(\lambda)}(\mathbf{v}^{n-1}))$ have a non-zero p^0-coefficient. It follows from this that $v_\square^n(p) - v_\square^{n-1}(p) = \mathcal{O}(p^n)$ and hence $Y_\square^{(\lambda)}(\mathbf{v}^n) - Y_\square^{(\lambda)}(\mathbf{v}^{n-1}) = \mathcal{O}(p^n)$ for all n. Therefore the limit exists.

(3) One has the following closed formula:

$$Y_\square^\lambda(p) = [\mathbf{v}^0]\left(\left\| \frac{\partial F_\square(\mathbf{Y}^{(\lambda)}(\mathbf{v}))/\partial v_{\square'}}{F_\square(\mathbf{Y}^{(\lambda)}(\mathbf{v}))} \right\| \cdot \prod_{\square\in\lambda} \frac{1}{1-p\cdot F_\square(\mathbf{Y}^{(\lambda)}(\mathbf{v}))}\right)$$

where $\|\ldots\|$ denotes the determinant of a matrix and $[\mathbf{v}^0]$ means taking the coefficient of $\prod_{\square\in\lambda} v_\square^0$ in the expression to the right, all of whose p-coefficients turn out to be Laurent series in $\mathbf{v}$. Furthermore, $\mathbf{Y}^{(\lambda)}(\mathbf{v})$ is as in (17).

Remark 1.22. (1) The uniqueness in (1) is not immediate and part of the statement. Also note that the Bethe equations are symmetric in the Y_i which allows us to use the boxes of λ as indices.

(2) The Bethe roots are usually characterized using (1). However, explicit descriptions like (2) and (3) have to our knowledge not appeared in the literature before.

From now on we will call the $\mathbf{Y}^\lambda$ simply *Bethe roots*. One might hope that they are the only solutions of (15). However, there are more - for example $\mathbf{Y} = (Y_i)_{i=1}^d$ with $Y_i = (t_1+t_2)\frac{(-1)^{d-1}p}{(-1)^{d-1}p+1}$ for all i is one such. In order to further narrow things down, we call a solution $\mathbf{Y} = (Y_i)_{i=1}^d$ *fully admissible* if in addition to being admissible we have $Y_i \neq Y_{i'}$ for all $i \neq i'$. Indeed, it is believed that this get rid of all unwanted solutions.

Conjecture C. [51, §3] Up to permutations of tuple-entries, the $\mathbf{Y}^\lambda$ described in Theorem 1.21 are the only fully admissible solutions of the Bethe equations (15) over K.

This would give us an entirely algebraic characterization of the Bethe roots whereas the descriptions in Theorem 1.21 were all somewhat analytic. Though we can not prove Conjecture C, we will provide the following partial result:

Proposition 1.23. The Bethe equations have only finitely many admissible solutions $\mathbf{Y}$ over K all of which are of the shape

$$Y_i = a_i t_1 + b_i t_2 + \mathcal{O}(p^{>0})$$

for a_i, b_i integers with $|a_i|, |b_i| \leq d$.

1.2.1.2 The main formula.

From now on we will fix a curve C of genus g and

$$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \in H^1(C, \mathbb{Z})$$

a symplectic basis i.e. so that

$$\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0 \text{ and } \alpha_i \cdot \beta_j = \delta_{i,j} \cdot \mathrm{pt}$$

for any $1 \leq i, j \leq g$ and $\mathrm{pt} \in H^2(C, \mathbb{Z})$ the point class. As a result we get a basis $\mathcal{B} = \{ 1, \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \mathrm{pt} \}$ of $H^*(C)$. Let further L_1, L_2 be line bundles of degrees l_1, l_2 respectively and $X = \mathsf{Tot}_C(L_1 \oplus L_2)$ the associated local curve with $\mathbb{T}$-action.

For fixed degree $d \geq 0$, free variables $\mathbf{Y} = (Y_i)_{i=1}^d$ and classes $\gamma_i \in \mathcal{B}$ we define the formal bracket

$$\langle \mathrm{ch}_{z_1}(\gamma_1) \dots \mathrm{ch}_{z_n}(\gamma_n) \rangle_d^{X,\mathrm{form}} \in \mathbb{Q}(t_1, t_2, \mathbf{Y})[[\mathbf{z}]]$$

to be the unique super-commutative expression that vanishes in case

$$|\{ i \mid \gamma_i = \alpha_l \}| \neq |\{ i \mid \gamma_i = \beta_l \}|$$

for some $l = 1, \dots, g$ and is otherwise given by

(18)

$$\left\langle \prod_{i=1}^{a} \mathrm{ch}_{x_i}(1) \cdot \prod_{i=1}^{b} \mathrm{ch}_{y_i}(\mathrm{pt}) \cdot \prod_{l=1}^{g} \Big(\mathrm{ch}_{z_1^l}(\alpha_l) \mathrm{ch}_{w_1^l}(\beta_l) \dots \mathrm{ch}_{z_{c_l}^l}(\alpha_l) \mathrm{ch}_{w_{c_l}^l}(\beta_l) \Big) \right\rangle_d^{X,\mathrm{form}}$$

$$= \prod_{i=1}^{a} x_i \cdot \sum_{\substack{\coprod_{i=-1}^{g} S_i = \{ 1, \dots, a \} \\ \Box_i \in \lambda \text{ for } i \in S_{-1}}} \prod_{i \in S_{-}1} \nabla_i^{\mathbf{Y}} \prod_{i \in S_{-1}} E(x_i, Y_{\Box_i})$$

$$\cdot \prod_{i \in S_0} \Big(l_1 \mathfrak{B}(x_i t_1) + l_2 \mathfrak{B}(x_i t_2) \Big) E(x_i, \mathbf{Y}) \cdot \prod_{i=1}^{b} E(y_i, \mathbf{Y})$$

$$\cdot \prod_{i=1}^{g} \left(\mathbf{z}^i, \mathbf{w}^i; \mathbf{x}_{S_i} \mid \mathbf{Y} \right)_{M(\mathbf{Y})^{-1}} \cdot A(\mathbf{Y})^{g-1} \cdot B_1(\mathbf{Y})^{l_1} \cdot B_2(\mathbf{Y})^{l_2}.$$

Here, we denoted

$$M(\mathbf{Y}) \coloneqq \left(\frac{\partial F_j(\mathbf{Y}) / \partial Y_i}{F_j(\mathbf{Y})} \right)_{i,j},$$

where the $F_i(\mathbf{Y})$ are as in (16) and

$$\nabla_i^{\mathbf{Y}} \coloneqq \sum_{i'} \left(M(\mathbf{Y})^{-1} \right)_{i,i'} \frac{\partial}{\partial Y_{i'}}$$

as well as

$$
\begin{aligned}
A(\mathbf{Y}) &:= \prod_{i=1}^{d} (Y_i \cdot (t_1 + t_2 - Y_i)) \cdot \left\| M(\mathbf{Y}) \right\| \\
&\cdot \prod_{\substack{1\le i,j\le d\\ 0\le a,b\le 1\\ (a,b,i-j)\neq(0,0,0)}} (at_1 + bt_2 + Y_i - Y_j)^{(-1)^{a+b+1}}, \\
B_1(\mathbf{Y}) &:= \prod_{i=1}^{d} (t_1 + t_2 - Y_i)^{-1} \cdot \prod_{i,j=1}^{d} \prod_{b=0}^{1} (t_1 + bt_2 + Y_i - Y_j)^{(-1)^{b+1}}, \\
B_2(\mathbf{Y}) &:= \prod_{i=1}^{d} (t_1 + t_2 - Y_i)^{-1} \cdot \prod_{i,j=1}^{d} \prod_{a=0}^{1} (at_1 + t_2 + Y_i - Y_j)^{(-1)^{a+1}}.
\end{aligned} \tag{19}
$$

We also wrote

$$\mathfrak{B}(t) := \tfrac{1}{e^t - 1} - \tfrac{1}{t} = -\tfrac{1}{2} + \sum_{i\ge 1} B_{2i} \tfrac{t^{2i-1}}{(2i)!}$$

with B_n the n-th Bernoulli number and

$$
\begin{aligned}
(\mathbf{z}, \mathbf{w}; \mathbf{x} \mid \mathbf{Y})_N :=& (-1)^n \sum_{\substack{\mathbf{a}=(a_i)_{i=1}^m,\\ \mathbf{b}=(b_i)_{i=1}^m,\\ \mathbf{c}=(c_i)_{i=1}^n}} \left\| N_{\mathbf{a}\sqcup\mathbf{c};\mathbf{b}\sqcup\mathbf{c}} \right\| \cdot \prod_{i=1}^{n} x_i E(x_i, \mathbf{Y}_{c_i}) \\
&\cdot \prod_{i=1}^{m} z_i w_i E(z_i, \mathbf{Y}_{a_i}) E(w_i, \mathbf{Y}_{b_i})
\end{aligned}
$$

for any $d \times d$-matrix N, vectors $\mathbf{z}, \mathbf{w}$ of length m and $\mathbf{x}$ of length n. The $E(z, \mathbf{Y})$ are defined by

$$
\begin{aligned}
E(z, Y) &:= \tfrac{(1-e^{-t_1 z})(1-e^{-t_2 z})}{t_1 t_2} e^{zY} \\
E(z, \mathbf{Y}) &:= \sum_{i=1}^{d} E(z, Y_i)
\end{aligned} \tag{20}
$$

Note in particular that $\langle \dots \rangle_d^{X,\mathrm{form}}$ is symmetric in the Y_i.

The main result of this section now specifies the relationship of the formal bracket with the actual stable pair invariant:

Theorem 1.24. One can evaluate

$$\langle \mathrm{ch}_{z_1}(\gamma_1) \dots \mathrm{ch}_{z_n}(\gamma_n) \rangle_d^{X,\mathrm{form}}$$

at $\mathbf{Y} = \mathbf{Y}^\lambda$ for any Bethe root $\mathbf{Y}^\lambda$ and we have

(21)

$$\langle \mathrm{ch}_{z_1}(\gamma_1) \dots \mathrm{ch}_{z_n}(\gamma_n) \rangle_d^{X,\mathbb{T}} = p^{d(1-g)} \sum_{\lambda \vdash d} \langle \mathrm{ch}_{z_1}(\gamma_1) \dots \mathrm{ch}_{z_n}(\gamma_n) \rangle_d^{X,\mathrm{form}} |_{\mathbf{Y}=\mathbf{Y}^\lambda} .$$

Remark 1.25. (1) Theorem 1.24 is new even without insertions. We will prove it using a strategy similar to [62], where the fixed locus of the stable pair moduli space is identified as the *double nested Hilbert scheme*. Our formula is made possible by an explicit description of the geometry of the double nested Hilbert scheme.

(2) In [62], the case of no insertions and $t_1 = -t_2$ was considered. In particular, one should be able to obtain [62, Theorem 1.3] as a special case of Theorem 1.24, however one can show that

$$Y^\lambda_{(i,j)}(p)\Big|_{t_1=-t_2} = t_1(i-j)$$

and hence setting $t_1 = -t_2$ makes some numerators and denominators in (19) vanish. This makes it difficult to compute the limit $t_1 \to -t_2$.

The upshot of Theorem 1.24 is the following slogan:

The structure of stable pair invariants
is induced by the structure of the Bethe roots!

Indeed, this is how we will prove Theorem 1.19. Assuming Conjecture C, rationality is immediate, symmetry comes from the invariance of the Bethe equations under the involution

$$t_i \mapsto t_i, p \mapsto p^{-1}, Y_i \mapsto t_1 + t_2 - Y_i$$

and pole restriction is connected to the fact that the $Y^\lambda_\square(p)$ are convergent power series that can be locally analytically continued to any p in

$$\mathbb{C} \setminus \{\, \zeta \mid (-\zeta)^n = 1 \text{ for some } 1 \leq n \leq d \,\}\,.$$

In the absence of Conjecture C, we can give a very similar proof by instead deducing it from certain special cases proved in [82, 83].

1.2.2 Relative theory

Although the above results only apply to absolute local curves, one can still use them to gain information about the relative case:

Consider the threefold $X = \mathbb{C}^2 \times \mathbb{P}^1$ and the smooth divisor $D = \mathbb{C}^2 \times \{0, \infty\}$ together with the diagonal $\mathbb{T} = (\mathbb{C}^*)^2$-action on the $\mathbb{C}^2$-factor. This geometry is often referred to as the *tube*. Recall that one can define moduli spaces $P_n(X/D, d)$ of *relative stable pairs* [48] together with evaluation maps

$$\mathsf{Hilb}^d(\mathbb{C}^2) \xleftarrow{ev_0} P_n(X/D,\beta) \xrightarrow{ev_\infty} \mathsf{Hilb}^d(\mathbb{C}^2)$$

to the Hilbert scheme of points on $\mathbb{C}^2$ and an equivariant virtual class

$$[P_n(X/D,d)]^{vir} \in H^{\mathbb{T}}_*(P_n(X/D,d)) \otimes \mathbb{Q}(t_1,t_2).$$

Using this one can define invariants via[11]

$$\begin{aligned}&\langle\epsilon|\mathrm{ch}_{z_1}(\gamma_1)\dots\mathrm{ch}_{z_n}(\gamma_n)|\delta\rangle^{X/D,T}\\&:=\sum_{n\geq d}p^{n-d}\int_{[P_n(X/D,d)]^{vir}}\mathrm{ch}_{z_1}(\gamma_1)\dots\mathrm{ch}_{z_n}(\gamma_n)ev_0^*(\epsilon)ev_\infty^*(\delta)\in\mathbb{Q}(t_1,t_2)[[p,\mathbf{z}]],\end{aligned}$$

where $\epsilon,\delta\in H^*_{\mathbb{T}}(\mathsf{Hilb}^d(\mathbb{C}^2))$. As shown in [82, 86], these power series are Taylor expansions of rational functions. Moreover, by [78, 79] these invariants also encode quantum multiplication on $\mathsf{Hilb}^n(\mathbb{C}^2)$. Now consider the $\mathbb{Q}(t_1,t_2)((p))$-vector space

$$\mathsf{H}:=H^*_{\mathbb{T}}(\mathsf{Hilb}^d(\mathbb{C}^2))\otimes_{\mathbb{Q}[t_1,t_2]}\mathbb{Q}(t_1,t_2)((p)).$$

We encode the stationary invariants of X/D in terms of endomorphism-valued power series

$$M(\mathbf{z})\in\mathsf{End}\,(\mathsf{H})\,[[\mathbf{z}]]$$

for $\mathbf{z}=(z_1,\dots,z_n)$. Indeed, these are defined by taking the functionals of the shape

$$\begin{aligned}\mathsf{H}\otimes\mathsf{H}&\longrightarrow\mathbb{Q}(t_1,t_2)((p))[[\mathbf{z}]]\\\gamma\otimes\delta&\longmapsto\langle\gamma|\mathrm{ch}_{z_1}(\mathrm{pt})\dots\mathrm{ch}_{z_n}(\mathrm{pt})|\delta\rangle^{X/D,\mathbb{T}}\end{aligned}$$

and using the identification

$$\mathsf{End}(\mathsf{H})=\mathsf{H}^\vee\otimes\mathsf{H}=\mathsf{H}^\vee\otimes\mathsf{H}^\vee=(\mathsf{H}\otimes\mathsf{H})^\vee,$$

which comes from equivariant Poincaré duality for $\mathsf{Hilb}^d(\mathbb{C}^2)$. Recall furthermore that the set of fixed points $\mathsf{Hilb}^d(\mathbb{C}^2)^{\mathbb{T}}$ is in natural bijection with the set of partitions $\lambda\vdash d$ i.e. of size d and the associated classes $[\lambda]\in\mathsf{H}$ of the fixed points form a basis of H.

Theorem 1.26. There is a basis $(v_\lambda)_\lambda$ of H so that

$$v_\lambda=[\lambda]+\mathcal{O}(p)$$

and

$$M(\mathbf{z})v_\lambda=\prod_{i=1}^n E(z_i,\mathbf{Y}^\lambda)v_\lambda$$

for all n and $\lambda\vdash d$, where $\mathbf{Y}^\lambda=(Y^\lambda_\square(p))_{\square\in\lambda}$ is the Bethe root associated to λ and the $E(z,\mathbf{Y})$ are as in (20).

[11] One can show that $P_n(X/D,d)=\emptyset$ for $n<d$.

Various versions of this had already been shown in [1, 27, 52] (see also [93]). Indeed, $M(\mathbf{z})$ is very closely related to the integrals of motion of ILW_1, which is why it is natural to expect that its eigenvalues can be expressed in terms of Bethe roots. We will give a new proof by deducing it from Theorem 1.24. See [97] for a generalization of this approach to the enumerative geometry of Nakajima quiver varieties.

For the rest of this section we aim to study $M(z)$ more concretely, which will also shed new light on the Bethe roots via Theorem 1.26. In order to state our results, we first recall the Fock space description of H, see [35, 66] for more details.

The Fock space $\mathcal{F}$ is freely generated over $\mathbb{Q}$ by commuting creation operators α_{-k} for $k > 0$ acting on the vacuum vector $|\emptyset\rangle \in \mathcal{F}$. There are also annihilation operators α_k for $k > 0$ so that $\alpha_k \cdot |\emptyset\rangle = 0$ with commutation relation

$$[\alpha_k, \alpha_l] = k\delta_{k,-l}. \tag{22}$$

From these we get a new family of operators

$$\alpha_{\mathbf{v}} = \prod_{\substack{i \\ v_i<0}} \alpha_{v_i} \cdot \prod_{\substack{i \\ v_i>0}} \alpha_{v_i}$$

for any integer vector $\mathbf{v} = (v_i)_{i=1}^l \in \mathbb{Z}_{\neq 0}^l$. There is a natural basis of $\mathcal{F}$ given by the vectors of shape

$$|\mu\rangle = \frac{1}{\mathfrak{z}(\mu)} \alpha_{-\mu} |\emptyset\rangle$$

for μ any partition and normalization factor

$$\mathfrak{z}(\mu) = |\mathrm{Aut}(\mu)| \prod_i \mu_i.$$

There is furthermore a natural inner product on $\mathcal{F}$ defined by

$$\langle \mu | \nu \rangle = \frac{\delta_{\mu,\nu}}{\mathfrak{z}(\mu)},$$

for which α_k and α_{-k} are adjoint operators. The significance of $\mathcal{F}$ for us lies in the existence of an isomorphism

$$\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}[t_1, t_2] \cong \bigoplus_{d \geq 0} H^*_{\mathbb{T}}(\mathsf{Hilb}^d(\mathbb{C}^2)) \tag{23}$$

where the graded component $H^*_{\mathbb{T}}(\mathsf{Hilb}^d(\mathbb{C}^2))$ on the right is generated by $|\mu\rangle$ for $|\mu| = d$ on the left. Note that $\alpha_{\mathbf{v}}$ preserves this grading only for $\mathbf{v} \in \mathbb{V}^l$ where $\mathbb{V}^l$ is the set of all vectors $\mathbf{v} \in \mathbb{Z}_{\neq 0}^l$ with $\sum_i v_i = 0$. Moreover, the equivariant Poincaré pairing on the right of (23) corresponds to the pairing

$$\langle \mu | \nu \rangle' = \frac{(-1)^{|\mu| - \ell(\mu)}}{(t_1 t_2)^{\ell(\mu)}} \frac{\delta_{\mu,\nu}}{\mathfrak{z}(\mu)} \tag{24}$$

on $\mathcal{F}$, for which $(\alpha_k)^* = (-1)^{k-1}(t_1t_2)^{\mathrm{sgn}(k)}\alpha_{-k}$. Using the isomorphism (23), we can now use the basis $|\mu\rangle$ to represent $M(z)$ as a matrix with entries

$$M(z)_{\mu,\nu} = (-1)^{|\mu|-\ell(\mu)}(t_1t_2)^{\ell(\mu)}\mathfrak{z}(\mu)\langle\mu|\mathrm{ch}_z(\mathrm{pt})|\nu\rangle^{X/D,\mathbb{T}}.$$

We will now give a partial description of its z-coefficients $M_k = [z^k]M(z)$.

Theorem 1.27. For any $k \geq 0$, there is a unique collection of $f^k(\mathbf{v}) \in \mathbb{Q}(p)[t_1,t_2]$ depending on $\mathbf{v} \in \mathbb{V}^l$ with $2 \leq l \leq k$ which is invariant under permutation of the vector entries and so that

$$M_k = \sum_{\substack{2\leq l\leq k\\ \mathbf{v}\in\mathbb{V}^l}} (-1)^{p(v)-1}(t_1t_2)^{n(v)-1}f^k(\mathbf{v})\frac{\alpha_{\mathbf{v}}}{l!}, \tag{25}$$

where $n(v)$ and $p(v)$ are the number of negative and positive entries of $\mathbf{v}$ respectively. For any $\mathbf{v} \in \mathbb{V}^l$ we can further expand

$$f^k(\mathbf{v}) = \sum_{\substack{a,b\geq 0\\ 2a+b+l=k}} (t_1t_2)^a(t_1+t_2)^b f^k_{a,b}(\mathbf{v})$$

for some $f^k_{a,b}(\mathbf{v}) \in \mathbb{Q}(p)$. The $f^k_{a,b}$ have the following properties:

(1) for any $a, b, \mathbf{v}$ we have:

$$f^k_{a,b}(\mathbf{v})\sum_i v_i^2\frac{(-p)^{v_i}+1}{(-p)^{v_i}-1} = \sum_{\substack{i\\ s+t=v_i\\ \mathrm{sgn}(s)=\mathrm{sgn}(t)}} \mathrm{sgn}(v_i)st f^k_{a-1,b+1}(\mathbf{v}\setminus(v_i)\cup(s,t)) - \sum_{i\neq j}(v_i+v_j)f^k_{a,b+1}(\mathbf{v}\setminus(v_i,v_j)\cup(v_i+v_j)),$$

where $\mathbf{v}\setminus(v_i,v_j)\cup(v_i+v_j) = (v_1,\ldots,\widehat{v_i},\ldots,\widehat{v_j},\ldots,v_n,v_i+v_j)$ and $\mathbf{v}\setminus(v_i)\cup(s,t) = (v_1,\ldots,\widehat{v_i},\ldots,v_n,s,t)$.

(2) One has

$$f^k_{0,k-2}(v,-v) = \frac{v^{k-2}}{(k-1)!}\frac{(-p)^{(k-1)v}-1}{((-p)^v-1)^{k-1}}$$

for any $v > 0$.

(3) for all $0 \leq a \leq \frac{k}{2}$ and $\mathbf{v} \in \mathbb{V}^{k-2a}$:

$$f_{a,0}(\mathbf{v}) = \left(-\frac{1}{4}\right)^a \sum_{\substack{n_l\geq 0\\ \sum_l n_l=a}} \prod_l \frac{v_l^{2n_l}}{(2n_l+1)!}$$

Remark 1.28. (1) Any linear operator $M \in \mathsf{End}(\mathcal{F})$ which preserves the grading (23) must be of the shape

$$M = \sum_{\mathbf{v} \in \mathbb{V}^l} f(\mathbf{v}) \alpha_{\mathbf{v}}$$

for a uniquely determined symmetric f. Imposing a bound $\ell(\mathbf{v}) \leq k$ is however a very strong constraint on M and means that $M|\mu\rangle$ is a linear combination of $|\lambda\rangle$ where λ is obtained from μ by at most $k-2$ cuts and joins.

(2) The functions f^k are all even in the sense that $f^k(-\mathbf{v}) = f^k(\mathbf{v})$ for any $\mathbf{v}$. Indeed, by construction, M_k must be self-adjoint with respect to the Poincaré pairing (24). By applying $(\cdots)^*$ to (25) and using uniqueness of the f^k, this gives us $f^k(-\mathbf{v}) = f^k(\mathbf{v})$ as desired.

(3) As we will see later, equation (1) is equivalent to the fact that M_3 and M_k commute, which follows from degeneration.

While Theorem 1.27 does not fully determine the M_k, only a small number of additional calculations are sufficient to show:

Theorem 1.29.

$$M_0 = M_1 = 0,$$

$$M_2 = \sum_{v>0} \alpha_{-v}\alpha_v,$$

$$M_3 = (t_1+t_2) \sum_{v>0} \tfrac{v}{2} \tfrac{(-p)^v+1}{(-p)^v-1} \alpha_{-v}\alpha_v + \frac{1}{2} \sum_{v_1,v_2>0} (t_1 t_2 \alpha_{-v_1}\alpha_{-v_2}\alpha_{v_1+v_2} - \alpha_{-v_1-v_2}\alpha_{v_1}\alpha_{v_2}),$$

$$\begin{aligned} M_4 = {} & (t_1+t_2)^2 \sum_{v>0} \tfrac{v^2}{6} \tfrac{(-p)^{3v}-1}{((-p)^v-1)^3} \alpha_{-v}\alpha_v - t_1 t_2 \sum_{v>0} \tfrac{v^2}{12} \alpha_{-v}\alpha_v \\ & + \tfrac{t_1+t_2}{8} \sum_{v_1,v_2>0} \left((v_1+v_2) \tfrac{(-p)^{v_1+v_2}+1}{(-p)^{v_1+v_2}-1} + v_1 \tfrac{(-p)^{v_1}+1}{(-p)^{v_1}-1} + v_2 \tfrac{(-p)^{v_2}+1}{(-p)^{v_2}-1} \right) \\ & \cdot (t_1 t_2 \alpha_{-v_1}\alpha_{-v_2}\alpha_{v_1+v_2} - \alpha_{-v_1-v_2}\alpha_{v_1}\alpha_{v_2}) \\ & + \tfrac{1}{6} \sum_{v_1,v_2,v_3>0} \left(t_1^2 t_2^2 \alpha_{-v_1}\alpha_{-v_2}\alpha_{-v_3}\alpha_{v_1+v_2+v_3} + \alpha_{-v_1-v_2-v_3}\alpha_{v_1}\alpha_{v_2}\alpha_{v_3} \right) \\ & - \tfrac{t_1 t_2}{4} \sum_{\substack{v_1,v_2,v_3,v_4>0 \\ v_1+v_2=v_3+v_4}} \alpha_{-v_1}\alpha_{-v_2}\alpha_{v_3}\alpha_{v_4}, \end{aligned}$$

$$M_5 = (t_1+t_2)^3 \sum_{v>0} \tfrac{v^3}{24} \tfrac{(-p)^{4v}-1}{((-p)^v-1)^4} \alpha_{-v}\alpha_v$$

$$
\begin{aligned}
&- \tfrac{t_1 t_2(t_1+t_2)}{24} \sum_{v>0} \left(\sum_{0<i<v} 2i(v-i) \tfrac{(-p)^i+1}{(-p)^i-1} + \tfrac{(5v^2+1)v}{6} \tfrac{(-p)^v+1}{(-p)^v-1} \right) \alpha_{-v}\alpha_v \\
&+ \tfrac{(t_1+t_2)^2}{48} \sum_{v_1,v_2>0} \Big(v_1 v_2 \tfrac{(-p)^{v_1}+1}{(-p)^{v_1}-1} \tfrac{(-p)^{v_2}+1}{(-p)^{v_2}-1} + v_1(v_1+v_2) \tfrac{(-p)^{v_1}+1}{(-p)^{v_1}-1} \tfrac{(-p)^{v_1+v_2}+1}{(-p)^{v_1+v_2}-1} \\
&+ v_2(v_1+v_2) \tfrac{(-p)^{v_2}+1}{(-p)^{v_2}-1} \tfrac{(-p)^{v_1+v_2}+1}{(-p)^{v_1+v_2}-1} + v_1^2 \left(\tfrac{(-p)^{v_1}+1}{(-p)^{v_1}-1} \right)^2 + v_2^2 \left(\tfrac{(-p)^{v_2}+1}{(-p)^{v_2}-1} \right)^2 \\
&+ (v_1+v_2)^2 \left(\tfrac{(-p)^{v_1+v_2}+1}{(-p)^{v_1+v_2}-1} \right)^2 + \tfrac{v_1^2+v_2^2}{2} \Big) \\
&\cdot (t_1 t_2 \alpha_{-v_1}\alpha_{-v_2}\alpha_{v_1+v_2} - \alpha_{-v_1-v_2}\alpha_{v_1}\alpha_{v_2}) \\
&+ \tfrac{t_1+t_2}{96} \sum_{\substack{v_1,v_2,v_3,v_4>0 \\ v_1+v_2=v_3+v_4}} \Bigg(2\sum_{i=1}^{4} v_i \tfrac{(-p)^{v_i}+1}{(-p)^{v_i}-1} + 2(v_1+v_2) \tfrac{(-p)^{v_1+v_2}+1}{(-p)^{v_1+v_2}-1} \\
&+ \sum_{\substack{i=1,2 \\ j=3,4}} (v_i - v_j) \tfrac{(-p)^{v_i-v_j}+1}{(-p)^{v_i-v_j}-1} \Bigg) \alpha_{-v_1}\alpha_{-v_2}\alpha_{v_3}\alpha_{v_4} \\
&+ \tfrac{t_1+t_2}{72} \sum_{\substack{v_1,v_2,v_3,v_4>0 \\ v_1+v_2=v_3+v_4}} \Bigg(2\sum_{i=1}^{3} v_i \tfrac{(-p)^{v_i}+1}{(-p)^{v_i}-1} + 2(v_1+v_2+v_3) \tfrac{(-p)^{v_1+v_2+v_3}+1}{(-p)^{v_1+v_2+v_3}-1} \\
&+ \sum_{1\le i<j\le 3} (v_i+v_j) \tfrac{(-p)^{v_i+v_j}+1}{(-p)^{v_i+v_j}-1} \Bigg) \cdot (\alpha_{-v_1-v_2-v_3}\alpha_{v_1}\alpha_{v_2}\alpha_{v_3} \\
&+ t_1^2 t_2^2 \alpha_{-v_1}\alpha_{-v_2}\alpha_{-v_3}\alpha_{v_1+v_2+v_3}) \\
&+ \tfrac{1}{24} \sum_{v_1,v_2,v_3,v_4>0} (t_1^3 t_2^3 \alpha_{v_1+v_2+v_3+v_4} \prod_{i=1}^{4} \alpha_{-v_i} - \alpha_{-v_1-v_2-v_3-v_4} \prod_{i=1}^{4} \alpha_{v_i}) \\
&+ \frac{t_1 t_2}{12} \sum_{\substack{v_1,v_2,v_3,v_4,v_5>0 \\ v_1+v_2+v_3=v_4+v_5}} (\alpha_{-v_4}\alpha_{-v_5}\alpha_{v_1}\alpha_{v_2}\alpha_{v_3} - t_1 t_2 \alpha_{-v_1}\alpha_{-v_2}\alpha_{-v_3}\alpha_{v_4}\alpha_{v_5}),
\end{aligned}
$$

where $0 \cdot \frac{(-p)^0+1}{(-p)^0-1} := 0$.

Remark 1.30. (1) The formulas for $M_{\le 2}$ are trivial and the one for M_3 was first shown in [78], though our proof of it is different. However, the formulas for M_4 and M_5 are new.

(2) Theorem 1.26 and Theorem 1.29 together determine the first three power sums of the Bethe roots. For example, taking the trace of M_3 yields

$$\sum_{d\ge 1} q^d \sum_{\substack{\lambda\vdash d \\ \square\in\lambda}} Y_\square^\lambda(p) = (t_1+t_2) \prod_{d\ge 1} \frac{1}{1-q^d} \sum_{v>0} \frac{v}{2} \frac{q^v}{1-q^v} \left(v \frac{(-p)^v+1}{(-p)^v-1} + 1 \right),$$

which seems nearly impossible to tell from the Bethe equations alone.

Theorem 1.29 is also a consequence of the following conjecture.

Conjecture D. For any $k \geq 0$ we have

$$f^k_{0,1}(\mathbf{v}) = \tfrac{1}{4} \sum_{S_1 \sqcup S_2 = \{1,\dots,k-1\}} \tfrac{(|S_1|-1)!(|S_2|-1)!}{(k-2)!} \mathbf{v}_{S_1} \tfrac{(-p)^{\mathbf{v}_{S_1}}+1}{(-p)^{\mathbf{v}_{S_1}}-1}$$

$$f^k_{0,2}(\mathbf{v}) = \tfrac{1}{8} \sum_{S_1 \sqcup S_2 \sqcup S_3 = \{1,\dots,k-2\}} \tfrac{(|S_1|-1)!(|S_2|-1)!|S_3|!}{(k-2)!} \cdot \mathbf{v}_{S_1}\mathbf{v}_{S_2} \tfrac{(-p)^{\mathbf{v}_{S_1}}+1}{(-p)^{\mathbf{v}_{S_1}}-1} \tfrac{(-p)^{\mathbf{v}_{S_2}}+1}{(-p)^{\mathbf{v}_{S_2}}-1} + \tfrac{1}{48} \sum_i v_i^2$$

$$\begin{aligned}
f^k_{0,3}(\mathbf{v}) = {} & \tfrac{1}{48} \sum_{S_1 \sqcup S_2 \sqcup S_3 \sqcup S_4 = \{1,\dots,k-3\}} \tfrac{(|S_1|-1)!(|S_2|-1)!(|S_3|-1)!(|S_4|+1)!}{(k-2)!} \\
& \cdot \mathbf{v}_{S_1}\mathbf{v}_{S_2}\mathbf{v}_{S_3} \tfrac{(-p)^{\mathbf{v}_{S_1}}+1}{(-p)^{\mathbf{v}_{S_1}}-1} \tfrac{(-p)^{\mathbf{v}_{S_2}}+1}{(-p)^{\mathbf{v}_{S_2}}-1} \tfrac{(-p)^{\mathbf{v}_{S_3}}+1}{(-p)^{\mathbf{v}_{S_3}}-1} \\
& + \tfrac{1}{16} \sum_{S_1 \sqcup S_2 \sqcup S_3 \sqcup S_4 = \{1,\dots,k-3\}} \tfrac{|S_1|!(|S_2|-1)!|S_3|!(|S_4|-1)!}{(k-2)!} \\
& \cdot (\mathbf{v}_{S_1} + \mathbf{v}_{S_4})\mathbf{v}_{S_2}\mathbf{v}_{S_4} \tfrac{(-p)^{\mathbf{v}_{S_1}+\mathbf{v}_{S_4}}+1}{(-p)^{\mathbf{v}_{S_1}+\mathbf{v}_{S_4}}-1} \tfrac{(-p)^{\mathbf{v}_{S_2}}+1}{(-p)^{\mathbf{v}_{S_2}}-1} \tfrac{(-p)^{\mathbf{v}_{S_4}}+1}{(-p)^{\mathbf{v}_{S_4}}-1} \\
& - \frac{1}{24} \sum_{S_1 \sqcup S_2 \sqcup S_3 = \{1,\dots,k-3\}} \tfrac{|S_1|!|S_2|!(|S_3|-1)!}{(k-2)!} \mathbf{v}_{S_1}(\mathbf{v}_{S_2} + \mathbf{v}_{S_3})\mathbf{v}_{S_3} \tfrac{(-p)^{\mathbf{v}_{S_3}}+1}{(-p)^{\mathbf{v}_{S_3}}-1} \\
& - \tfrac{1}{96} \sum_{S_1 \sqcup S_2 \sqcup S_3 = \{1,\dots,k-3\}} \tfrac{(|S_1|-1)!(|S_2|+1)!(|S_3|-1)!}{(k-2)!} \mathbf{v}_{S_1}\mathbf{v}_{S_2}\mathbf{v}_{S_3} \tfrac{(-p)^{\mathbf{v}_{S_3}}+1}{(-p)^{\mathbf{v}_{S_3}}-1} \\
& - \tfrac{1}{48} \sum_{S_1 \sqcup S_2 \sqcup S_3 = \{1,\dots,k-3\}} \tfrac{(|S_1|-1)!|S_2|!|S_3|!}{(k-2)!} \mathbf{v}_{S_1}\mathbf{v}_{S_2}^2 \tfrac{(-p)^{\mathbf{v}_{S_3}}+1}{(-p)^{\mathbf{v}_{S_3}}-1}
\end{aligned}$$

$$\begin{aligned}
f^k_{1,1}(\mathbf{v}) = {} & -\tfrac{1}{48} \sum_{S_1 \sqcup S_2 = \{1,\dots,k-3\}} \tfrac{(|S_1|-1)!(|S_2|-1)!}{(k-4)!} \mathbf{v}_{S_1} \tfrac{(-p)^{\mathbf{v}_{S_1}}+1}{(-p)^{\mathbf{v}_{S_1}}-1} \\
& - \tfrac{1}{12} \sum_{S_1 \sqcup S_2 \sqcup S_3 = \{1,\dots,k-3\}} \tfrac{|S_1|!|S_2|!(|S_3|-1)!}{(k-2)!} \mathbf{v}_{S_1}(\mathbf{v}_{S_1}^2 - 1) \tfrac{(-p)^{\mathbf{v}_{S_1}+\mathbf{v}_{S_2}}+1}{(-p)^{\mathbf{v}_{S_1}+\mathbf{v}_{S_2}}-1} \\
& - \tfrac{1}{48} \sum_{S_1 \sqcup S_2 = \{1,\dots,k-3\}} \tfrac{|S_1|!|S_2|!}{(k-2)!} \mathbf{v}_{S_1}(\mathbf{v}_{S_1}^2 - 1) \tfrac{(-p)^{\mathbf{v}_{S_1}}+1}{(-p)^{\mathbf{v}_{S_1}}-1} \\
& - \tfrac{1}{4} \sum_{\substack{S_1 \sqcup S_2 = \{1,\dots,k-3\} \\ s+t=\mathbf{v}_{S_1} \\ \operatorname{sgn}(s)=\operatorname{sgn}(t)}} \tfrac{|S_1|!|S_2|!}{(k-2)!} \operatorname{sgn}(\mathbf{v}_{S_1}) \cdot st \cdot \tfrac{(-p)^s+1}{(-p)^s-1},
\end{aligned}$$

where we set $\mathbf{v}_S := \sum_{i \in S} v_i$ and $0 \cdot \frac{(-p)^0+1}{(-p)^0-1} := 0$ as before. Furthermore, these formulas for $f^k_{a,b}(\mathbf{v})$ only hold if $2a+b+l = k$ where $\mathbf{v} \in \mathbb{V}^l$ - otherwise we have $f^k_{a,b}(\mathbf{v}) = 0$.

Remark 1.31. The formulas above were found through extensive computer search and fit a large amount of data. Moreover, it is straightforward to check that Conjecture D is consistent with Theorem 1.27 in the sense that the right hand sides satisfy Theorem 1.27(1) and specialize to Theorem 1.27(2) in case $a = 0$ and $b = k - 2$.

1.3 Gromov-Witten theory of local curves

We will now consider the $\mathbb{T}$-equivariant Gromov-Witten theory of a local curve $X = \mathsf{Tot}_C(L_1 \oplus L_2)$ of genus $g = g(C)$ and $l_i = c_1(L_i)$. As a result, we will drop the superscript "GW" from our invariants. The specific flavor of Gromov-Witten theory, which in our case is most convenient, comes from the moduli space

$$\overline{M}'_{g,n}(X, d).$$

This is the space of stable maps from possibly disconnected curves where every connected component maps nontrivially and receives a marking. The difference between the resulting invariants $\langle \ldots \rangle_d^{X,\prime}$ and the disconnected invariants we encountered earlier is encapsulated by the following consequence of Theorem 1.18 and Theorem 1.24.

Proposition 1.32.

$$\begin{aligned}
&\langle 1 \rangle_d^{X,\mathbb{T},\bullet} \\
&= (-z^2)^{d(g-1)} \left(z e^{z/2}\right)^{-d(l_1+l_2)} \sum_{\lambda \vdash d} A(\mathbf{Y}^\lambda)^{g-1} B_1(\mathbf{Y}^\lambda)^{l_1} B_2(\mathbf{Y}^\lambda)^{l_2} \Big|_{p=-e^z}
\end{aligned}$$

where $\mathbf{Y}^\lambda(p)$ is the Bethe root corresponding to λ.

At first, one might hope that a sufficiently careful spelling out of the localization formula might yield similarly comprehensive results as in Section 1.2.1. Unfortunately, this does not work since the fixed locus $\overline{M}'_{g,n}(X, d)^{\mathbb{T}} = \overline{M}'_{g,n}(C, d)$ is not as nice as the double nested Hilbert scheme and admits no explicit description. As a result, the Gromov-Witten theory of local curves is much more difficult to compute and we will restrict ourselves mostly to $g = 1$ and $l_1 = l_2 = 0$ i.e. the *local elliptic curve* $X = \mathbb{C}^2 \times E$. This is in some sense the easiest local curve as for example Proposition 1.32 simplifies greatly in this case. By expanding Definition 1.6, one can show that the Gromov-Witten theory of X is equivalent to *double Hodge integrals over the elliptic curve*:

Definition 1.33. For $\gamma_1, \ldots, \gamma_n \in H^*(E)$ we set

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x) \prod_{i=1}^n \frac{\gamma_i}{1/z_i - \psi_i} \right\rangle^{E,\prime}$$

$$:= \sum_{\substack{g\in\mathbb{Z}\\ d\geq 0\\ l_1,\dots,l_n\in\mathbb{Z}}} (-1)^{g-1} z^{2g-2} q^d \prod_{i=1}^n z_i^{l_i+1} \left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x) \prod_{i=1}^n \tau_{l_i}(\gamma_i) \right\rangle_{g,d}^{E,\prime}$$
$$\in \mathbb{Q}[[q]][x^\pm, z_1^\pm, \dots, z_n^\pm]((z)),$$

where

$$\mathbb{E}^\vee(x) = \sum_{i=0}^g (-1)^i \lambda_i x^{g-i} \in H^*(\overline{M}'_{g,n}(E,d))[x]$$

with $\lambda_i = c_i(\mathbb{E})$ the i-th chern class of the Hodge bundle $\mathbb{E} = \pi_*\Omega^1_\pi$ of the forgetful map $\pi\colon \overline{M}'_{g,n+1}(E,d) \to \overline{M}'_{g,n}(E,d)$.

See Section 4.1 for more details. What makes the local elliptic curve particularly attractive is the fact that it is based on the Gromov-Witten theory of the elliptic curve, which has been fully computed and is quasi-modular - see [75, 76, 91]. Moreover, in case $x = -1$ one can use Mumford's relation (c.f. [65])

$$\mathbb{E}^\vee(1)\mathbb{E}^\vee(-1) = (-1)^g \tag{26}$$

to remove both Hodge classes. As a result, the explicit formulas for the Gromov-Witten theory of the elliptic curve given in [76, §5] and [91, Proposition 3.3.2] imply:

Theorem 1.34. Let $\mathcal{B} = \{1, \alpha, \beta, \mathrm{pt}\}$ be the basis of $H^*(E)$ introduced in Section 1.2.1.2. For all $\gamma_1, \dots, \gamma_n \in \mathcal{B}$ we have

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1) \prod_{i=1}^n \frac{\gamma_i}{1/z_i - \psi_i} \right\rangle^{E,\prime}$$
$$= -z^{-n} \sum_{\substack{\{1,\dots,n\}=\coprod_{I\in S} I\\ \cup_{i\in I}\gamma_i = \pm\mathrm{pt}}} \mathrm{sgn}(S) \prod_{I\in S} \left(\prod_{i\in I} z_i \left(\sum_{i\in I} z_i\right)^{|I|-2} \right) F_{|S|}\left((\sum_{i\in I} zz_i)_{I\in S} \right),$$

where $\mathrm{sgn}(S)$ is the sign that arises out of super-commuting the γ_i into the shape S. Here, F_n is the *Bloch-Okounkov correlation function*

$$F_n(z_1,\dots,z_n) = \sum_{\sigma\in S_n} \frac{\left\| \left(\frac{\Theta^{j-i+1}(z_{\sigma(1)}+\dots+z_{\sigma(n-j)})}{(j-i+1)!} \right)_{i,j} \right\|}{\Theta(z_{\sigma(1)})\Theta(z_{\sigma(1)}+z_{\sigma(2)})\cdots\Theta(z_{\sigma(1)}+\dots+z_{\sigma(n)})}$$

with $\Theta(z)$ the Jacobi theta function as in Appendix A.

Since Gromov-Witten invariants of the elliptic curve are quasi-modular forms (c.f. Appendix A), one has for all x:

Theorem 1.35. We have

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n}\frac{\gamma_i}{1/z_i-\psi_i}\right\rangle^{E,\prime} \in \mathrm{QMod}[x^{\pm}, z_1^{\pm},\dots,z_n^{\pm}]((z))$$

so that the coefficient of z^k is quasi-modular of weight $k+\sum_i \deg_{\mathbb{R}}(\gamma_i)$. Moreover, there is the following *holomorphic anomaly equation*:

$$\begin{aligned}
&\frac{d}{dG_2}\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n}\frac{\gamma_i}{1/z_i-\psi_i}\right\rangle^{E,\prime}\\
&= -xz^2\left(\sum_{i=1}^{n} z_i\right)^2\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n}\frac{\gamma_i}{1/z_i-\psi_i}\right\rangle^{E,\prime}\\
&\quad -2\sum_{i=1}^{n}\left(\int_E \gamma_i\right) z_i^{-1}\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{l=1}^{n}\frac{\gamma_i^{1-\delta_{i,l}}}{1/z_l-\psi_l}\right\rangle^{E,\prime},
\end{aligned} \tag{27}$$

where the formal derivative with respect to G_2 is taken coefficient-wise.

Remark 1.36. The holomorphic anomaly equation was first proven in [91] and later extended in [70].

As far as we know, this modularity has no clear analogue for stable pair invariants of the local elliptic curve.

It is well-known from the literature on Hodge integrals on $\overline{M}_{g,n}$ that generating series as in Definition 1.33 tend to be most approachable if $z_1,\dots,z_n \in \mathbb{Z}_{>0}$ - see for example [22, 23, 24, 53, 74, 77]. Our main conjecture of this section attempts to make this precise:

Conjecture E. Let $x\in\mathbb{Z}_{\neq 0}$, $z_1,\dots,z_n\in\mathbb{Z}_{>0}$, $\gamma_1,\dots,\gamma_n\in H^*(E)$ and μ a partition of degree $|\mu|=d$ and length $\ell(\mu)=n$.

(1) If $x>0$, then

$$z^{d(x+1)+n}\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n}\frac{\gamma_i}{1/\mu_i-\psi_i}\right\rangle^{E,\prime} \in \mathrm{QJac}[z]$$

is a polynomial in z with coefficients in the ring of *quasi-Jacobi forms* QJac - see Appendix A. Furthermore, the z-degree of the polynomial is at most $d-n$.

(2) If $x<0$, then

$$z^{d(x+1)+n}\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n}\frac{\gamma_i}{1/\mu_i-\psi_i}\right\rangle^{E,\prime} = \sum_{\substack{\mathbf{a}=(a_i)_{i\in S}\\ \sum_{i\in S}a_i=d^2}} \phi_{\mathbf{a}}\cdot\prod_{i\in S}\Theta(iz)^{a_i x}$$

where

$$S = \left\{ 0 \neq \sum_{i \in M} \mu_i \;\middle|\; M \subset \{1, \ldots, n\} \right\}$$

and $\phi_{\mathbf{a}} \in \mathrm{QJac}[z]$ of z-degree at most $n - \sum_i \deg_{\mathbb{C}} \gamma_i$.

In either case, the Hodge integral is homogeneous of weight $\sum_i \deg_{\mathbb{R}}(\gamma_i)$ and index $\frac{xd^2}{2}$. See Remark A.2(4) for the definition of weight and index.

Remark 1.37. (1) The case $x = -1$ follows from Theorem 1.34. We did not mention the case $x = 0$ as it only makes sense for connected Hodge integrals. By [91, Lemma 4.4.1], these are moreover equal to their constant coefficients (67).

(2) Recall that the space $\mathrm{QJac}_{m,k}$ of quasi-Jacobi forms of specified weight and index is finite-dimensional. Conjecture E therefore implies that each such Hodge integral is determined finitely many of its coefficients.

(3) The claim on weight and index follows from Theorem 1.35, (92) and the rest of Conjecture E.

(4) Conjecture E is partially motivated by [71, Conjecture C] which asserts that stable pair invariants (and hence also Gromov-Witten invariants) of compact elliptic fibrations are quasi-Jacobi. Despite the fact that $\mathbb{C}^2 \times E$ is not compact, Hodge integrals as in E still appear in the localization formula for $S \times E$ with S a projective toric surface.

Our main theorem in this section is:

Theorem 1.38. Conjecture E holds if $\mu = (1^n)$ or $\mu = (2)$.

Remark 1.39. Besides Theorem 1.34, Theorem 1.38 is the only evidence of Conjecture E that we currently have. As a result, it is not unlikely that one would have to modify Conjecture E. However, all Hodge integrals of the same degree $|\mu| = d$ must be related to each other via tautological relations similar to those appearing in the proof of Proposition 4.11. It would therefore seem somewhat unexpected if some Hodge integrals behave differently than others.

We will deduce this from the following explicit formulas:

Theorem 1.40. If $x > 0$, we have

$$\begin{aligned} &\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x) \prod_{i=1}^{n} \frac{\mathrm{pt}}{1-\psi_i} \right\rangle^{E,\prime} \\ &= \frac{(-1)^n (n-1)! \Theta(z)^{nx}}{x^{n-1} z^{nx+2n}} \mathrm{Res}_{u_{n-1}=u_n} \cdots \mathrm{Res}_{u_1=u_2} \\ &\sum_{1=l_1<\cdots<l_N=n} \left(\prod_{i \neq j} \frac{\Theta(z+u_i-u_j)}{\Theta(u_i-u_j)} \right)^x \cdot \prod_{m=1}^{N-1} \frac{\mathsf{A}(u_{l_m}-u_{l_{m+1}})^{l_{m+1}-l_m}}{(n-l_m)\cdot(l_{m+1}-l_m)!} \end{aligned} \tag{28}$$

and if $x < 0$,

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n}\frac{\mathrm{pt}}{1-\psi_i}\right\rangle^{E,\prime}$$

$$= 2\frac{(-1)^n(n-1)!\Theta(z)^{nx}}{x^{n-1}z^{nx+2n}}\mathrm{Res}_{u_{n-1}=u_n}\cdots\mathrm{Res}_{u_1=u_2} \tag{29}$$

$$\sum_{\substack{1=l_1<\cdots<l_N=n\\ s_1,\ldots,s_n}}\left(\prod_{i\neq j}\frac{\Theta(z+u_i'-u_j')}{\Theta(u_i'-u_j')}\right)^x\cdot\prod_{m=1}^{N-1}\frac{\mathsf{A}(u'_{l_m}-u'_{l_{m+1}})^{l_{m+1}-l_m}}{(n-l_m)\cdot(l_{m+1}-l_m)!},$$

where all residues are taken for $z \neq 0$. Moreover, we set $u_i' = u_i + s_i z$ and $s_1,\ldots,s_n$ is a sequence of integers so that $s_1 = 0$, $s_2 = 1$ and for any j, we have $\{s_1,\ldots,s_j\} = \mathbb{Z}\cap[a,b]$ for some $a,b\in\mathbb{Z}$. Finally, $\mathsf{A}(u)$ is as in Appendix A.

2 Stable pair theory of absolute local curves

We now turn towards the proof of the results in Section 1.2.1. The content of this section is taken from the paper [96].

2.1 Double nested Hilbert schemes and their irreducible components

In this section we recall the double nested Hilbert scheme of a smooth projective curve and give a description of its irreducible components, which all turn out to have the same dimension (c.f. Proposition 2.4). This fact is what enabled all calculations in this section. In Section 2.1.1 we recall some combinatorial notation necessary for stating this result.

2.1.1 Notation

By a *partition* λ of *size* $d \geq 0$ (or $\lambda \vdash d$ for short) we mean a finite sequence of positive integers $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} > 0$ so that

$$|\lambda| := \sum_{i=0}^{n-1}\lambda_i = d.$$

We call the integer $l(\lambda) := n$ the *length* of λ. Furthermore, we will always identify λ with its *Young diagram* which is the set of all $(i,j)\in\mathbb{N}_0^2$ so that $0 \leq j < \lambda_i$[12]. For any box $(i,j)\in\lambda$ it is often convenient to write $\Box = (i,j)$ when the coordinates i and j are not used.

[12]Note that the top left box is denoted $(0,0)$ and not $(1,1)$ as in most of the combinatorics literature.

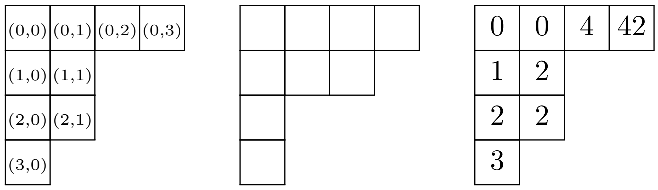

Figure 1: On the left: The Young diagram of $\lambda = (4, 2, 2, 1)$ with coordinates. In the middle: The Young diagram of $\overline{\lambda} = (4, 3, 1, 1)$. On the right: A reverse plane partition on λ

We write $\overline{\lambda}$ for the unique partition whose Young diagram is

$$\{ (i, j) \mid (j, i) \in \lambda \}$$

i.e. the Young diagram of λ flipped along the diagonal. We further denote

$$n(\lambda) = \sum_{(i,j)\in\lambda} i.$$

Recall that a *skew partition* λ/μ is a pair of partitions λ and μ so that $\mu_i \leq \lambda_i$ for any i. This is equivalent to the Young diagram of μ being contained in the Young diagram of λ and we will often identify λ/μ with the complement of Young diagrams $\lambda \setminus \mu$ - in particular we write $|\lambda/\mu| := |\lambda \setminus \mu|$. Note here that a subset $S \subset \lambda$ is a skew partition $S = \lambda/\mu$ if and only if $\square \in S$ and $\square \leq \square' \in \lambda$ imply $\square' \in S$. Unless stated otherwise, we will

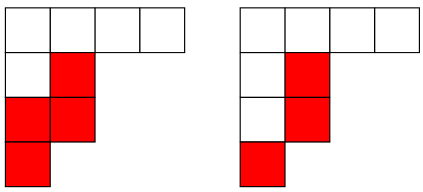

Figure 2: Skew partitions λ/μ_l and λ/μ_r respectively with $\lambda = (4, 2, 2, 1)$, $\mu_l = (4, 1)$, $\mu_r = (4, 1, 1)$ and complements in red. The left one is connected and the right one is disconnected.

from now on require all skew partitions to be *connected* i.e. any two boxes $\square, \square' \in \lambda/\mu$ must be connected via a sequence of boxes in λ/μ in which any two consecutive boxes share an edge.

One can equip $\mathbb{N}_0^2$ with the partial order given by

$$(i, j) \leq (i', j') \text{ iff } i \leq i' \text{ and } j \leq j'.$$

A tuple of natural numbers $\mathbf{n} = (n_\square)_{\square\in\lambda}$ on a partition λ is called a *reverse plane partition* if for any $\square, \square' \in \lambda$

$$\square \leq \square' \text{ implies } n_\square \leq n'_\square.$$

In this case we denote

$$|\mathbf{n}| \coloneqq \sum_{\square\in\lambda} n_\square.$$

Observe furthermore that for any tuple of numbers $\mathbf{m} = \big(m_{\lambda/\mu}\big)_{\lambda/\mu}$ indexed over the connected skew partitions in λ one gets an associated reverse plane partition $\mathbf{n} = (n_\square)_{\square\in\lambda}$ defined by

$$n_\square = \sum_{\square\in\lambda/\mu} m_{\lambda/\mu}.$$

We will abbreviate this relation as $|\mathbf{m}| = \mathbf{n}$ or $\mathbf{m} \vdash \mathbf{n}$. Further, we will write

$$\|\mathbf{m}\| \coloneqq |\mathbf{n}| = \sum_{\lambda/\mu} |\lambda/\mu| \cdot m_{\lambda/\mu}.$$

Lemma 2.1. Any reverse plane partition $\mathbf{n}$ is $|\mathbf{m}|$ for some $\mathbf{m}$ as above. Furthermore, if $n_\square \leq 1$ for all $\square \in \lambda$, then $\mathbf{m}$ is unique.

Proof. First note that any possibly non-connected skew partition λ/μ is uniquely a disjoint union of connected skew partitions. Indeed, for existence note that the connected components λ/μ are connected skew partitions. For uniqueness note that if $\lambda/\mu = \lambda/\mu_1 \sqcup \cdots \sqcup \lambda/\mu_n$, then any $\lambda/\mu_i \subset \lambda/\mu$ must be closed under $\leq$ and $\geq$ hence making it a maximal connected subset. Applying this fact to

$$\lambda/\mu \coloneqq \{\, \square \in \lambda \mid n_\square = 1 \,\}$$

we obtain the second claim.

We show the first claim by induction on $|\mathbf{n}|$. For this let λ/μ_0 be a connected component of $\{\, \square \in \lambda \mid n_\square > 0 \,\}$. One easily sees that $\mathbf{n}'$ defined by

$$n'_\square \coloneqq \begin{cases} n_\square - 1, & \text{if } \square \in \lambda/\mu_0, \\ n_\square, & \text{else} \end{cases}$$

is again a reverse plane partition of smaller size. Hence for any $\mathbf{m}' \vdash \mathbf{n}'$ we get an $\mathbf{m} \vdash \mathbf{n}$ defined by

$$m_{\lambda/\mu} \coloneqq \begin{cases} m'_{\lambda/\mu} + 1, & \text{if } \mu = \mu_0 \\ m'_{\lambda/\mu}, & \text{else.} \end{cases} \quad .$$

□

2.1.2 Double nested Hilbert schemes

For the rest of this section we fix a partition λ, a reverse plane partition $\mathbf{n} = (n_\Box)_{\Box\in\lambda}$ and a smooth projective curve C over the complex numbers. We further denote by

$$C^{(n)} := \mathsf{Hilb}^n(C) = C^n/\mathrm{Sym}_n$$

the Hilbert scheme of n points on C.

Definition 2.2. For any tuple of natural numbers $\mathbf{m} = (m_i)_{i=1}^l$ we write

$$C^{(\mathbf{m})} := \prod_{i=1}^{l} C^{(m_i)}$$

for the product of Hilbert schemes of C.

Furthermore, the *double nested Hilbert scheme* associated to $\mathbf{n}$ is defined as

$$C^{[\mathbf{n}]} := \left\{ (D_\Box)_{\Box\in\lambda} \;\middle|\; \begin{array}{l} D_\Box \subset C \text{ divisor of length } n_\Box \text{ such that} \\ \text{for } \Box \leq \Box' \text{ we have } D_\Box \subset D_{\Box'} \end{array} \right\} \subset C^{(\mathbf{n})}.$$

Remark 2.3. More precisely, $C^{[\mathbf{n}]}$ is the scheme representing the obvious moduli functor. For more details see [62, §2.2].

Given a tuple of nonnegative numbers $\mathbf{m} = (m_{\lambda/\mu})_{\lambda/\mu}$ so that $\mathbf{m} \vdash \mathbf{n}$ we get an induced map

$$\begin{aligned} \phi_{\mathbf{m},\mathbf{n}} \colon C^{(\mathbf{m})} &\longrightarrow C^{[\mathbf{n}]} \\ (D_{\lambda/\mu})_{\lambda/\mu} &\longmapsto \left(\sum_{\Box\in\lambda/\mu} D_{\lambda/\mu} \right)_{\Box\in\lambda} \end{aligned}.$$

Taking the disjoint union over all such tuples we obtain

$$\phi\colon \coprod_{\mathbf{m}\vdash\mathbf{n}} C^{(\mathbf{m})} \longrightarrow C^{[\mathbf{n}]}.$$

Proposition 2.4. The morphism ϕ is birational and both sides are pure of dimension

$$n_{0,0} - \sum_{\substack{(i,j),(k,l)\in\lambda \\ 0\leq a,b\leq 1 \\ (k,l)=(i+a,j+b)}} (-1)^{a+b}(n_{k,l} - n_{i,j}).$$

In particular, the fundamental class of $C^{[\mathbf{n}]}$ can be written as

$$\left[C^{[\mathbf{n}]}\right] = \sum_{\mathbf{m}\vdash\mathbf{n}} \phi_* \left[C^{(\mathbf{m})}\right]$$

Proof. To show that ϕ is birational it suffices to show that ϕ restricts to a bijection $\phi^{-1}(U) \to U$ of dense open subsets and that $C^{[\mathbf{n}]}$ is generically smooth and pure of the desired dimension. Indeed, this would imply that the domain and codomain of ϕ are both pure of the same dimension and by restricting to an appropriate dense open subset of $C^{[\mathbf{n}]}$, [64, Proposition 3.17] then implies that ϕ restricts to a degree 1 map to a normal scheme which must be birational.

We first show that ϕ is surjective, hence take an arbitrary $\mathbf{D} = (D_\square)_{\square\in\lambda} \in C^{[\mathbf{n}]}$. It suffices to treat the case when $\mathbf{D}$ is supported on a single point $x \in C$ since any $\mathbf{D}$ is a sum of such tuples. In this case, $\mathbf{D}$ is in the image of $\phi_{\mathbf{m},\mathbf{n}}$ for any $\mathbf{m} \vdash \mathbf{n}$ and by Lemma 2.1 such an $\mathbf{m}$ exists.

Furthermore, this process gives a unique preimage if $\mathbf{D} \in S$ for $S \subset C^{[\mathbf{n}]}$ the open set of tuples $\mathbf{D}$ for which $D_\square$ is reduced for any $\square \in \lambda$. Denoting by $S' \subset \coprod_{\mathbf{m}\vdash\mathbf{n}} C^{(\mathbf{m})}$ the dense open consisting of tuples of mutually disjoint reduced divisors it follows that $\phi(S') \subset S$ and by surjectivity of ϕ and denseness of S' it follows that S and $\phi^{-1}(S)$ must also be dense. This establishes the generic injectivity.

It remains to check generic smoothness of the double nested Hilbert scheme. For this we consider the closed embedding

$$C^{[\mathbf{n}]} \longrightarrow C^{(n_{0,0})} \times \prod_{\substack{(i,j)\in\lambda\\ j\geq 1}} C^{(n_{i,j}-n_{i,j-1})} \times \prod_{\substack{(i,j)\in\lambda\\ i\geq 1}} C^{(n_{i,j}-n_{i-1,j})} =: X$$

$$(D_{i,j})_{(i,j)\in\lambda} \longmapsto (D_{0,0}, (D_{i,j} - D_{i,j-1})_{(i,j)\in\lambda}, (D_{i,j} - D_{i-1,j})_{(i,j)\in\lambda}).$$

This embedding was already considered in [62, §2.4] in which it was noted that $C^{[\mathbf{n}]}$ is cut out by the set of equations

$$D^1_{i-1,j} + D^2_{i,j} = D^2_{i,j-1} + D^1_{i,j},$$

where $(i,j) \in \lambda$ is any box with $i, j \geq 1$ and we denote a point in X by $(D^0, (D^1_\square)_{\square\in\lambda}, (D^2_\square)_{\square\in\lambda})$. Letting $\mathcal{U} \subset X$ be the open set consisting of tuples of reduced divisors we note that $\mathcal{U} \cap C^{[\mathbf{n}]} \subset C^{[\mathbf{n}]}$ is dense since it contains S defined above. It therefore suffices to show that $\mathcal{U} \cap C^{[\mathbf{n}]}$ is smooth. Under the product of the quotient maps $C^n \twoheadrightarrow C^n/S_n = C^{(n)}$ one can pull $\mathcal{U}$ back to an open set:

$$\widetilde{\mathcal{U}} \subset C^{n_{0,0}} \times \prod_{\substack{(i,j)\in\lambda\\ j\geq 1}} C^{n_{i,j}-n_{i,j-1}} \times \prod_{\substack{(i,j)\in\lambda\\ i\geq 1}} C^{n_{i,j}-n_{i-1,j}}$$

giving an étale cover $\widetilde{\mathcal{U}} \to \mathcal{U}$. This reduces us to showing that the preimage of $\mathcal{U} \cap C^{[\mathbf{n}]}$ in $\widetilde{\mathcal{U}}$, which we denote by $\widetilde{\mathcal{U}} \cap C^{[\mathbf{n}]}$, is smooth. Since smoothness is étale local, we may further assume $C = \mathbb{A}^1$, in which case we denote the coordinates of $\widetilde{\mathcal{U}}$ by $s_l, t_{\square,l}, u_{\square,l}$. The equations cutting out $\widetilde{\mathcal{U}} \cap C^{[\mathbf{n}]}$ therefore become:

$$\begin{aligned} f_{i,j,m} \coloneqq & \sum_{l=0}^{n_{i-1,j}-n_{i-1,j-1}} t^m_{i-1,j,l} + \sum_{l=0}^{n_{i,j}-n_{i-1,j}} u^m_{i,j,l} \\ & - \sum_{l=0}^{n_{i,j-1}-n_{i-1,j-1}} u^m_{i,j-1,l} - \sum_{l=0}^{n_{i,j}-n_{i,j-1}} t^m_{i,j,l} \end{aligned}$$

for $1 \leq m \leq n_{i,j} - n_{i-1,j-1}$. We will now show that the Jacobian of these equations has maximal rank by induction on $|\lambda|$. For this we pick point in $\widetilde{\mathcal{U}} \cap C^{[\mathbf{n}]}$, a box $(i_0, j_0) \in \lambda$ with $(i_0+1, j_0), (i_0, j_0+1) \notin \lambda$ and set $\widetilde{\lambda} \coloneqq \lambda \setminus \{(i_0, j_0)\}$. We now need to show that if for a given tuple of complex numbers $(a_{i,j,m})_{i,j,m}$ one has

$$\sum_{i',j',m'} a_{i',j',m'} \cdot \partial_{t_{i,j,l}} f_{i',j',m'} = 0 \text{ and } \sum_{i',j',m'} a_{i',j',m'} \cdot \partial_{u_{i,j,l}} f_{i',j',m'} = 0$$

at that point for all i, j, l, then we must have $a_{i_0,j_0,m} = 0$ for all m as the claim for $\widetilde{\lambda}$ gives the rest. Indeed, looking at partials with respect to $u_{i_0,j_0,l}$ the above equations yield in particular

$$\sum_m a_{i_0,j_0,m} \cdot m u^{m-1}_{i_0,j_0,l} = 0$$

for all l, which gives $a_{i_0,j_0,m} = 0$ by the distinctness of the $u_{i_0,j_0,l}$ and the invertibility of the Vandermonde matrix. Furthermore, this implies that $C^{[\mathbf{n}]}$ is pure of dimension

$$\begin{aligned} & \dim(\mathcal{U}) - \sum_{\substack{(i,j)\in\lambda \\ i,j\geq 1}} (n_{i,j} - n_{i-1,j-1}) \\ & = n_{0,0} + \sum_{\substack{(i,j)\in\lambda \\ i\geq 1}} (n_{i,j} - n_{i-1,j}) + \sum_{\substack{(i,j)\in\lambda \\ j\geq 1}} (n_{i,j} - n_{i,j-1}) - \sum_{\substack{(i,j)\in\lambda \\ i,j\geq 1}} (n_{i,j} - n_{i-1,j-1}) \end{aligned}$$

as desired. □

Remark 2.5. Note that the proof in particular shows that $C^{[\mathbf{n}]}$ is a local complete intersection. A more thorough study of the geometry of $C^{[\mathbf{n}]}$ has been undertaken in [33], where Proposition 2.4 was independently proved using slightly different methods. In particular, it is shown that $C^{[\mathbf{n}]}$ is connected and reduced and that ϕ is its normalization.

2.2 Description of the fixed loci and virtual normal bundle

In this section we will recall the discussion of [62, §3.3] and explore the consequences that Proposition 2.4 has for our stable pair calculation.

Resuming the notation in the introduction we let $X = \mathsf{Tot}_C(L_1 \oplus L_2)$ be the local curve over C with line bundles L_i of degree l_i and $\mathbb{T} = (\mathbb{C}^*)^2$ the torus acting on X.

In order to compute the equivariant stable pair theory of X we must first compute the fixed locus $P_n(X,d)^{\mathbb{T}}$ of the induced $\mathbb{T}$-action.

Indeed, an element $[\mathcal{O}_X \to \mathcal{F}] \in P_n(X,d)^{\mathbb{T}}$ is the same as an equivariant stable pair. By pushing it down to C and decomposing it into its weight spaces, this must be of the shape

$$s = (s_{i,j})\colon \mathcal{O}_X = \bigoplus_{i,j\geq 0} L_1^{-i} \otimes L_2^{-j} \cdot \mathbf{t}_1^{-i}\mathbf{t}_2^{-j} \longrightarrow \bigoplus_{i,j\geq 0} \mathcal{F}_{i,j} \otimes L_1^{-i} \otimes L_2^{-j} \cdot \mathbf{t}_1^{-i}\mathbf{t}_2^{-j}$$

for some coherent $\mathcal{F}_{i,j}$ on C and morphisms $s_{i,j}\colon \mathcal{O}_C \to \mathcal{F}_{i,j}$. Since $\mathcal{F}$ is of compact support, we must have $\mathcal{F}_{i,j} = 0$ for all but finitely many i,j and the stability is equivalent to each $\mathcal{F}_{i,j}$ being pure of dimension 1 (hence locally free) and each $s_{i,j}$ having finite cokernel. For any given i,j this forces either $\mathcal{F}_{i,j} = \mathcal{O}_C(D_{i,j})$ for some effective divisors $D_{i,j} \subset C$ and $s_{i,j}$ the canonical inclusion or $\mathcal{F}_{i,j} = 0$ and $s_{i,j} = 0$. We write $S \subset \mathbb{N}_0^2$ for the set on which the former happens. The compatibility of s with the multiplication on $\mathcal{O}_X$ is then equivalent to $S = \lambda$ for some partition λ and $D_\square \subset D_{\square'}$ for any $\square \leq \square' \in \lambda$. Therefore $[\mathcal{O}_X \to \mathcal{F}]$ corresponds to an element $(D_\square)_{\square\in\lambda} \in C^{[\mathbf{n}]}$ for $n_\square = \deg D_\square$. Since this argument can also be performed in flat families, one gets:

Proposition 2.6 ([62, Proposition 3.1])**.** The $\mathbb{T}$-fixed locus of $P_n(X,d)$ is a disjoint union

$$P_n(X,d)^{\mathbb{T}} = \coprod_{\lambda\vdash d}\coprod_{\mathbf{n}} C^{[\mathbf{n}]},$$

where the second disjoint union is over those reverse plane partitions $\mathbf{n} = (n_\square)_{\square\in\lambda}$ satisfying

$$d(1-g) - n(\lambda)\cdot l_1 - n(\bar{\lambda})\cdot l_2 + |\mathbf{n}| = n.$$

Furthermore, the K-theory class of the universal stable pair on a component of the fixed locus $C^{[\mathbf{n}]}$ is given by

$$\mathbb{F} = \sum_{(i,j)\in\lambda} \iota_*\mathcal{O}_{C\times C^{[\mathbf{n}]}}(\mathcal{D}_{i,j}) \otimes L_1^{-i} \otimes L_2^{-j} \cdot \mathbf{t}_1^{-i}\mathbf{t}_2^{-j} \in K^0_{\mathbb{T}}(X \times C^{[\mathbf{n}]}),$$

where $\mathcal{D}_{i,j} \subset C \times C^{[\mathbf{n}]}$ is the universal divisor at the box $(i,j) \in \lambda$ and $\iota\colon C \times C^{[\mathbf{n}]} \hookrightarrow X \times C^{[\mathbf{n}]}$ is the inclusion of the zero-section. We wrote $\mathbf{t}_i \in K^0_{\mathbb{T}}(\mathrm{pt})$ for the K-theory classes associated to the standard coordinate representations of $\mathbb{T}$. □

Using this description and Proposition 2.4 one can now express stable pair invariants on X in terms of integrals on symmetric products of C:

Proposition 2.7. Given an insertion of the shape $\gamma = \mathrm{ch}_{z_1}(\gamma_1)\dots\mathrm{ch}_{z_n}(\gamma_n)$ on X, the associated stable pair invariant in degree d can be written as

$$\langle\gamma\rangle_d^{X,T} = p^{d(1-g)}\sum_{\lambda\vdash d}\sum_{\substack{\mathbf{m}=(m_{\lambda/\mu})_{\lambda/\mu}\\ m_{\lambda/\mu}\geq 0}} p^{\|\mathbf{m}\|-n(\lambda)\cdot l_1-n(\bar{\lambda})\cdot l_2}\int_{C^{(\mathbf{m})}}\frac{\widetilde{\gamma}}{e(N_{\mathbf{m}})}. \tag{30}$$

The K-theory class $N_{\mathbf{m}} \in K^0_{\mathbb{T}}\left(C^{(\mathbf{m})}\right)$ is given by

$$\begin{aligned}
&N_{\mathbf{m}}\\
&= \sum_{\substack{(i,j)\in\lambda\\(i,j)\neq 0}} R\pi_*\left(\mathcal{O}_{C\times C^{(\mathbf{m})}}(\mathcal{D}_{i,j})\otimes L_1^{-i}\otimes L_2^{-j}\right)\mathbf{t}_1^{-i}\mathbf{t}_2^{-j}\\
&+ \sum_{(i,j)\in\lambda} R\pi_*\left(\mathcal{O}_{C\times C^{(\mathbf{m})}}(-\mathcal{D}_{i,j})\otimes L_1^{i+1}\otimes L_2^{j+1}\right)\mathbf{t}_1^{i+1}\mathbf{t}_2^{j+1}\\
&- \sum_{\substack{(i,j),(k,l)\in\lambda\\0\leq a,b\leq 1\\(i+a,j+b)\neq(k,l)}}(-1)^{a+b}R\pi_*\Big(\mathcal{O}_{C\times C^{(\mathbf{m})}}(\mathcal{D}_{k,l}-\mathcal{D}_{i,j})\otimes L_1^{i-k+a}\otimes L_2^{j-l+b}\Big)\mathbf{t}_1^{i-k+a}\mathbf{t}_2^{j-l+b},
\end{aligned}$$

where

$$\mathcal{D}_{\Box} = \sum_{\Box\in\lambda/\mu}\mathcal{D}_{\lambda/\mu}$$

is a sum over universal divisors $\mathcal{D}_{\lambda/\mu}\subset C\times C^{(\mathbf{m})}$ and $\pi\colon C\times C^{(\mathbf{m})}\to C^{(\mathbf{m})}$ is the projection onto the second factor. Furthermore, we have

$$\widetilde{\gamma} = \widetilde{\mathrm{ch}_{z_1}(\gamma_1)}\dots\widetilde{\mathrm{ch}_{z_n}(\gamma_n)}\in H^*(C^{(\mathbf{m})}),$$

where

$$\begin{aligned}
&\widetilde{\mathrm{ch}_z(\gamma)}\\
&:= \tfrac{(1-e^{-t_1z})(1-e^{-t_2z})}{t_1t_2}\pi_*\left[\mathrm{ch}_z(\widetilde{\mathbb{F}})\cdot\left(1+z\mathrm{pt}_C\Big(l_1\mathcal{B}(zt_1)+l_2\mathcal{B}(zt_2)\Big)\right)\cdot\pi'^*\gamma\right]
\end{aligned}\tag{31}$$

with $\mathrm{pt}_C\in H^2(C\times C^{(\mathbf{m})})$ the pullback of the point class along the projection $\pi'\colon C\times C^{(\mathbf{m})}\to C$ to the first factor and

$$\widetilde{\mathbb{F}} := \sum_{(i,j)\in\lambda}\mathcal{O}_{C\times C^{[\mathbf{n}]}}(\mathcal{D}_{i,j})\otimes L_1^{-i}\otimes L_2^{-j}\cdot\mathbf{t}_1^{-i}\mathbf{t}_2^{-j}\in K^0_{\mathbb{T}}(C\times C^{[\mathbf{n}]}). \tag{32}$$

Proof. Recall that

$$\langle\gamma\rangle_{n,d}^{X,\mathbb{T}} = \int_{[P_n(X,d)^{\mathbb{T}}]^{vir}}\frac{\gamma|_{P_n(X,d)^{\mathbb{T}}}}{e(N^{vir})}.$$

To define everything on the right hand side recall from [38] that $P_n(X,d)$ has a perfect obstruction theory which is the morphism in $\mathcal{D}^b(P_n(X,d))$ given by the Atiyah class

$$\mathbb{E} = R\mathcal{H}om_\pi(\mathbb{I},\mathbb{I})_0^\vee[-1] \to \mathbb{L}_{P_n(X,d)},$$

where the target is the cotangent complex $\mathbb{L}_{P_n(X,d)}$ of $P_n(X,d)$ with $\mathbb{I} = [\mathcal{O}_{X\times P_n(X,d)} \to \mathcal{F}]$ the universal stable pair on $X \times P_n(X,d)$ and $\pi\colon X\times P_n(X,d) \to P_n(X,d)$ the projection to the second factor. This can be seen to be $\mathbb{T}$-equivariant [94, Example 4.6] so that the invariant part

$$\mathbb{E}^{\mathbb{T}}\Big|_{P_n(X,d)^{\mathbb{T}}} \to \mathbb{L}^{\mathbb{T}}_{P_n(X,d)}\Big|_{P_n(X,d)^{\mathbb{T}}} = \mathbb{L}_{P_n(X,d)^{\mathbb{T}}}$$

of the restriction to the fixed locus is again a perfect obstruction theory and by [7] this induces a virtual class $[P_n(X,d)^{\mathbb{T}}]^{vir} \in H_*(P_n(X,d)^{\mathbb{T}})$ whose restriction to any connected component $C^{[\mathbf{n}]}$ sits in degree $2 \cdot \mathrm{rk}\left(\mathbb{E}^{\mathbb{T}}\big|_{C^{[\mathbf{n}]}}\right)$. Furthermore, the virtual normal bundle is defined as the K-theory class of the non-fixed part:

$$N^{vir} := (\mathbb{E}^\vee\big|_{P_n(X,d)^{\mathbb{T}}})^{mov} \in K^0_{\mathbb{T}}(P_n(X,d)^{\mathbb{T}}).$$

Here, K^0 denotes the K-theory of locally free sheaves as opposed to K_0 the K-theory of coherent sheaves. In [62, Section 4] the following identity in $K_0^{\mathbb{T}}(P_n(X,d)^{\mathbb{T}})$ was shown:

(33)

$$\begin{aligned}
&\mathbb{E}^\vee\big|_{C^{[\mathbf{n}]}}\\
&= \sum_{(i,j)\in\lambda} R\pi_*\left(\mathcal{O}_{C\times C^{[\mathbf{n}]}}(\mathcal{D}_{i,j}) \otimes L_1^{-i} \otimes L_2^{-j}\right)\cdot \mathbf{t}_1^{-i}\mathbf{t}_2^{-j}\\
&+ \sum_{(i,j)\in\lambda} R\pi_*\left(\mathcal{O}_{C\times C^{[\mathbf{n}]]}}(-\mathcal{D}_{i,j}) \otimes L_1^{i+1} \otimes L_2^{j+1}\right)\cdot \mathbf{t}_1^{i+1}\mathbf{t}_2^{j+1}\\
&- \sum_{\substack{(i,j),(k,l)\in\lambda\\ 0\le a,b\le 1}} (-1)^{a+b} R\pi_*\left(\mathcal{O}_{C\times C^{[\mathbf{n}]}}(\mathcal{D}_{k,l}-\mathcal{D}_{i,j}) \otimes L_1^{i-k+a} \otimes L_2^{j-l+b}\right)\cdot \mathbf{t}_1^{i-k+a}\mathbf{t}_2^{j-l+b},
\end{aligned}$$

where the $D_\square \subset C\times C^{[\mathbf{n}]}$ is the universal divisor corresponding to $\square \in \lambda$ and $\pi\colon C\times C^{[\mathbf{n}]} \to C^{[\mathbf{n}]}$ is the projection to the second factor. However, the argument given there also works in $K^0_{\mathbb{T}}(P_n(X,d)^{\mathbb{T}})$ with pushforwards along equivariant perfect morphisms being defined analogously to [9, §IV.2.12]. We now claim that the virtual class $[C^{[\mathbf{n}]}]^{vir} = [C^{[\mathbf{n}]}]$ is just the usual fundamental class. Indeed, using Lemma 2.8 it suffices to show that $C^{[\mathbf{n}]}$ is of dimension $\mathrm{rk}\left(\mathbb{E}^{\mathbb{T}}\big|_{C^{[\mathbf{n}]}}\right)$. Furthermore, one can use Riemann-Roch to calculate this rank:

$$\mathrm{rk}\left(\mathbb{E}^{\mathbb{T}}\Big|_{C^{[\mathbf{n}]}}\right) = \mathrm{rk}\left(R\pi_*\left(\mathcal{O}_{C\times C^{[\mathbf{n}]}}(\mathcal{D}_{0,0})\right)\right)$$

$$- \sum_{\substack{(i,j),(k,l)\in\lambda \\ 0\le a,b\le 1 \\ (k,l)=(i+a,j+b)}} (-1)^{a+b}\mathrm{rk}\left(R\pi_*\left(\mathcal{O}_{C\times C^{[\mathbf{n}]}}\left(\mathcal{D}_{k,l}-\mathcal{D}_{i,j}\right)\right)\right)$$

$$= n_{0,0}+1-g-\sum_{\substack{(i,j),(k,l)\in\lambda \\ 0\le a,b\le 1 \\ (k,l)=(i+a,j+b)}} (-1)^{a+b}\left(n_{k,l}-n_{i,j}+1-g\right)$$

Using Proposition 2.4 it therefore suffices to show that

$$1-\sum_{\substack{(i,j),(k,l)\in\lambda \\ 0\le a,b\le 1 \\ (k,l)=(i+a,j+b)}} (-1)^{a+b}=0.$$

Indeed, one can deduce this from the fact that the number of pairs $(i,j),(k,l)$ for which $(a,b)=(0,0),(1,0),(0,1)$ and $(1,1)$ is $d, d-l(\overline{\lambda})=d-\lambda_0, d-l(\lambda)$ and $d-\lambda_0-l(\lambda)+1$ respectively. This establishes the claim.

Because of the projection formula and Proposition 2.4 we now only need to show that

$$\phi^*_{\mathbf{m},\mathbf{n}}\mathrm{ch}_z(\gamma)=\widetilde{\mathrm{ch}_z(\gamma)} \tag{34}$$

and $L\phi^*_{\mathbf{m},\mathbf{n}}N^{vir}=N_{\mathbf{m}}$. For the second claim one looks at (33) and sees that this would easily follow from $L\phi^*_{\mathbf{m},\mathbf{n}}R\pi_*=R\pi_*L\left(\mathrm{Id}\times\phi_{\mathbf{m},\mathbf{n}}\right)^*$ with the maps coming from the cartesian diagram:

$$\begin{array}{ccc} C\times C^{(\mathbf{m})} & \xrightarrow{\mathrm{Id}\times\phi_{\mathbf{m},\mathbf{n}}} & C\times C^{[\mathbf{n}]} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ C^{(\mathbf{m})} & \xrightarrow{\phi_{\mathbf{m},\mathbf{n}}} & C^{[\mathbf{n}]} \end{array} \tag{35}$$

As π is flat, $\phi_{\mathbf{m},\mathbf{n}}$ and π are Tor-independent and therefore the desired commutativity of pushforward and pullback follows from [9, Proposition IV.3.1.1]. To show (34) we recall that

$$\mathrm{ch}_z(\gamma)=\widetilde{\pi}_*\left(\mathrm{ch}_z(\mathbb{F})\cdot(\widetilde{\pi}')^*\gamma\right),$$

where $\widetilde{\pi}\colon X\times C^{[\mathbf{n}]}\to C^{[\mathbf{n}]}$ and $\widetilde{\pi}'\colon X\times C^{[\mathbf{n}]}\to X$ are the projection to the second and first factor respectively. Using equivariant Grothendieck-Riemann-Roch [3, Corollary after Theorem 1.1] and $\mathbb{F}=\iota_*\widetilde{\mathbb{F}}$ for the zero section $\iota\colon C\times C^{[\mathbf{n}]}\hookrightarrow X\times C^{[\mathbf{n}]}$ we obtain

$$\mathrm{ch}_z(\mathbb{F})=\iota_*\left(\mathrm{ch}_z(\widetilde{\mathbb{F}})\cdot\mathrm{td}^{-1}_{C/X}\right)=\iota_*\left(\mathrm{ch}_z(\widetilde{\mathbb{F}})\cdot\tfrac{(1-e^{-(t_1+l_1\mathrm{pt}_C)z})(1-e^{-(t_2+l_2\mathrm{pt}_C)z})}{(t_1+l_1\mathrm{pt}_C)(t_2+l_2\mathrm{pt}_C)}\right)$$
$$=\tfrac{(1-e^{-t_1z})(1-e^{-t_2z})}{t_1t_2}\iota_*\left[\mathrm{ch}_z(\widetilde{\mathbb{F}})\cdot\left(1+z\mathrm{pt}_C\Big(l_1\mathcal{B}(zt_1)+l_2\mathcal{B}(zt_2)\Big)\right)\right].$$

From the projection formula and $\pi = \widetilde{\pi} \circ \iota$ and $\pi' = \widetilde{\pi}' \circ \iota$ it then follows that

$$\begin{aligned} &\mathrm{ch}_z(\gamma) \\ &= \tfrac{(1-e^{-t_1 z})(1-e^{-t_2 z})}{t_1 t_2} \pi_* \left[\mathrm{ch}_z(\widetilde{\mathbb{F}}) \cdot \left(1 + z\mathrm{pt}_C \left(l_1 \mathcal{B}(zt_1) + l_2 \mathcal{B}(zt_2) \right)\right) \cdot \pi'^* \gamma \right], \end{aligned}$$

where pullbacks and pushforwards are along $\pi\colon C \times C^{[\mathbf{n}]} \to C^{[\mathbf{n}]}$ and $\pi'\colon C \times C^{[\mathbf{n}]} \to C$. It now suffices to show that (35) satisfies $\phi^*_{\mathbf{m},\mathbf{n}} \pi_* = \pi_* \left(\mathrm{Id} \times \phi_{\mathbf{m},\mathbf{n}}\right)^*$ in cohomology, which is dual to $\pi^* \left(\phi_{\mathbf{m},\mathbf{n}}\right)_* = \left(\mathrm{Id} \times \phi_{\mathbf{m},\mathbf{n}}\right)_* \pi^*$ in homology. This however follows from flat base change [19, Theorem VIII.5.1(2)], which finishes the proof. □

The following Lemma was needed in the above proof and is a well-known piece of folklore for which the author claims no originality. However, due to the apparent lack of a reference and for the sake of completeness we give a full proof.

Lemma 2.8. Let X be of finite type over some field k and of pure dimension d together with a perfect obstruction theory $\phi\colon \mathbb{E} \to \mathbb{L}_X$ of rank $\mathrm{rk}\,(\mathbb{E}) = d$. It follows that ϕ is a quasi-isomorphism, X is lci and the virtual class must agree with the fundamental class i.e.

$$[X]^{vir} = [X] \in CH_d(X).$$

Proof. Since this is a local question we may assume that $\mathbb{E} = [E^{-1} \to E^0]$ for E^i locally free and that the morphism $\mathbb{E} \to \mathbb{L}_X$ results from a commutative diagram

$$\begin{array}{ccc} E^{-1} & \xrightarrow{\phi} & E^0 \\ \downarrow & & \downarrow \\ I/I^2 & \xrightarrow{\delta} & \Omega_M|_X \end{array}$$

where we used $\tau_{\geq -1}\mathbb{L}_X = [I/I^2 \to \Omega_M|_X]$ for I the ideal sheaf cutting out a closed embedding $X \hookrightarrow M$ into M non-singular. Recall that $\mathbb{E} \to \tau_{\geq -1}\mathbb{L}_X$ is an obstruction theory if and only if

$$E^{-1} \to E^0 \oplus I/I^2 \to \Omega_M|_X \to 0$$

is exact. After tensoring with the residue field $k(x)$ for some arbitrary $x \in X$ we therefore obtain

$$\dim\left(I \otimes k(x)\right) = \dim\left(I/I^2 \otimes k(x)\right) \leq \dim(M) - d$$

meaning that I is locally generated by at most (indeed exactly) $\dim(M) - d$ elements which must form a regular sequence. Therefore X is a local complete intersection and I/I^2 must be a vector bundle of rank $\dim(M)-d$. It then follows from rank considerations that

$$0 \to E^{-1} \to E^0 \oplus I/I^2 \to \Omega_M|_X \to 0$$

is exact i.e. $\mathbb{E} \to \tau_{\geq -1}\mathbb{L}_X = \mathbb{L}_X$ is a quasi-isomorphism. Finally, [7, Proposition 5.3] and the example following it give us $[X]^{vir} = [X]$. □

2.3 The main computation

2.3.1 Intersection theory of symmetric products of curves

We now want to compute the integrals appearing in Proposition 2.7. In order to do that we first need to recall the intersection theory of the symmetric product of a connected smooth projective curve C of genus g as outlined for example in [4].

For any $m > 0$ and fixed $c \in C$ we have an embedding

$$\begin{aligned} \iota\colon C^{(m-1)} &\hookrightarrow C^{(m)} \\ D &\longmapsto D + c \end{aligned}$$

of a divisor with cohomology class $u \in H^2(C^{(m)})$. Note that u does not depend on the choice of c as C is connected. More generally, for any $n \geq 0$ the subvariety

$$\begin{aligned} \iota_n\colon C^{(m-n)} &\hookrightarrow C^{(m)} \\ D &\longmapsto D + m \cdot c \end{aligned}$$

represents $u^n \in H^{2n}(C^{(m)})$, which in particular implies $\int_{C^{(m)}} u^n = \delta_{m,n}$. Furthermore, we have the Abel-Jacobi maps

$$\begin{aligned} \mathrm{AJ}_m\colon C^{(m)} &\longrightarrow \mathrm{Pic}^0(C) \\ D &\longmapsto \mathcal{O}_C(D - m \cdot c) \end{aligned}$$

to the Jacobian of C, which are compatible with the ι_n. It can be shown that this induces an isomorphism on H^1 and so $H^1(C^{(m)}) = H^1(\mathrm{Pic}^0(C)) = H^1(C)$. Recall that $H^1(C)$ has a symplectic base $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \in H^1(C)$ i.e. so that $\int_C \alpha_i\beta_j = \delta_{i,j}$ and $\int_C \alpha_i\alpha_j = \int_C \beta_i\beta_j = 0$ for all $1 \leq i, j \leq g$. By abuse of notation we will also denote the pullback of this basis by α_i, β_i in $H^1(\mathrm{Pic}^0(C))$ and in $H^1(C^{(m)})$. Furthermore, we will denote by $\theta \in H^2(\mathrm{Pic}^0(C))$ the theta divisor [4, §I.4] as well as its pullback along AJ_m in $H^2(C^{(m)})$. More explicitly, we have

$$\theta = \sum_{i=1}^{g} \alpha_i\beta_i.$$

We can now state Poincaré's formula [4, §I.5] which says

$$\left(\mathrm{AJ}_m\right)_*(1) = \begin{cases} \frac{\theta^{g-m}}{(g-m)!}, & \text{if } m \leq g \\ 0, & \text{otherwise} \end{cases}$$

In particular, one can rewrite this without case distinction as

$$\left(\mathrm{AJ}_m\right)_*(1) = \sum_{\substack{I \subset \{1,\ldots,g\} \\ |I| = g-m}} \prod_{i \in I} \alpha_i \beta_i.$$

The following Lemma will be very useful for computations later on:

Lemma 2.9. For any $m, n \geq 0$ and $I \subset \{1, \ldots, g\}$ we have

$$\int_{C^{(m)}} u^n \prod_{i \in I} \alpha_i \beta_i = \delta_{|I|+n,m}$$

and for any two distinct subsets $I, J \subset \{1, \ldots, g\}$

$$\int_{C^{(m)}} u^n \prod_{i \in I} \alpha_i \prod_{j \in J} \beta_j = 0.$$

Proof. For the first integral note that the second factor is pulled back under the Abel-Jacobi map, which commutes with $\iota_n \colon C^{(m-n)} \subset C^{(m)}$. Using the projection formula we can rewrite the integral as

$$\begin{aligned} \int_{C^{(m)}} u^n \prod_{i \in I} \alpha_i \beta_i &= \int_{\mathrm{Pic}^0(C)} \left(\mathrm{AJ}_{m-n}\right)_*(1) \prod_{i \in I} \alpha_i \beta_i \\ &= \sum_{\substack{J \subset \{1,\ldots,g\} \\ |J| = g-m+n}} \int_{\mathrm{Pic}^0(C)} \prod_{j \in J} \alpha_j \beta_j \prod_{i \in I} \alpha_i \beta_i, \end{aligned}$$

where we used Poincaré's formula in the last equality. Now the only summand that can contribute is the one corresponding to $J = I^c$, which occurs in the sum only if $m - n = |I|$, proving the first claim. The second claim follows by a similar argument. $\square$

Remark 2.10. This means that whenever we are computing an integral on $C^{(m)}$ we can replace the expression $\prod_{i \in I} \alpha_i \beta_i$ by $u^{|I|}$ without changing the value of the integral.

Now denote by $\mathcal{D} \subset C \times C^{(m)}$ the universal divisor of $C^{(m)}$. As shown in [4, p. 354] one can write it in cohomology as

$$\mathcal{D} = m \cdot \mathrm{pt}_C + u + \gamma \tag{36}$$

with pt_C and u implicitly pulled back from C and $C^{(m)}$ respectively while γ is defined as

$$\gamma \coloneqq \sum_{i=1}^{g} (\beta_i \times \alpha_i - \alpha_i \times \beta_i) \in H^2(C \times C^{(m)}).$$

One can check that $\gamma^2 = -2\mathrm{pt}_C \cdot \theta$ and $\mathrm{pt}_C \cdot \gamma = \gamma^3 = 0$. Denoting for any K-theory class $\mathcal{F} \in K^0(X)$ on a scheme X:

$$c_T(\mathcal{F}) \coloneqq \sum_{i \geq 0} c_i(\mathcal{F}) \cdot T^{\mathrm{rk}(\gamma)-i} \in H^*(X)[T^{\pm}]$$

we now have:

Lemma 2.11. For any tuple $\mathbf{m} = (m_i)_{i=1}^n$ of nonnegative integers, $a_1, \ldots, a_n \in \mathbb{Z}$ and L a line bundle of degree l on C:

$$\begin{aligned} &c_T\left(R\pi_*\left(\mathcal{O}_{C\times C^{(\mathbf{m})}}\left(\sum_{i=1}^n a_i\mathcal{D}_i\right) \otimes L\right)\right) \\ &= \exp\left(-\frac{\sum_{k,l=1}^n a_k a_l \theta_{k,l}}{T+\sum_{i=1}^n a_i u_i}\right) \cdot \left(T + \sum_{i=1}^n a_i u_i\right)^{1-g+l+\sum_{i=1}^n a_i m_i}, \end{aligned} \tag{37}$$

where $\pi\colon C \times C^{(\mathbf{m})} \to C^{(\mathbf{m})}$ is the projection to the second factor, $\mathcal{D}_i \subset C \times C^{(\mathbf{m})}$ is the universal divisor divisor of the i-th factor, $\theta_{k,l}$ is defined as

$$\theta_{k,l} = \sum_{i=1}^{g} \alpha_i^k \times \beta_i^l$$

and γ^k is understood to be pulled back from the projection

$$C^{(\mathbf{m})} = \prod_{i=1}^{n} C^{(m_i)} \to C^{(m_k)}$$

to the k-th factor.

Proof. We closely follow the Proof of [4, Lemma VIII.2.5]. First we write

$$\mathcal{D} = \sum_{i=1}^n a_i\mathcal{D}_i,\ M = \sum_{i=1}^n a_i m_i,\ u = \sum_{i=1}^n a_i u_i,\ \gamma = \sum_{i=1}^n a_i\gamma_i, \theta = \sum_{i,j=1}^n a_i a_j \theta_{i,j}$$

and $\mathcal{F} = R\pi_*\left(\mathcal{O}_{C\times C^{(\mathbf{m})}}(\mathcal{D}) \otimes L\right)$ for short. Using Grothendieck-Riemann-Roch we can further compute

$$ch(\mathcal{F} \otimes \mathcal{O}(-u)) = \pi_*\left(e^{\mathcal{D}+l\mathrm{pt}_C-u} \cdot \mathrm{td}_C\right) = \pi_*\left(e^{(M+l)\mathrm{pt}_C+\gamma} \cdot (1+(1-g)\mathrm{pt}_C)\right)$$

$$= \pi_* \left((1 + (M + l - \theta)\mathrm{pt}_C + \gamma) \cdot (1 + (1 - g)\mathrm{pt}_C) \right)$$
$$= M + l + 1 - g - \theta$$

where we have used $\gamma \cdot \mathrm{pt}_C = 0$, $\gamma^2 = -2\mathrm{pt}_C \cdot \theta$ and $\gamma^3 = 0$. Recall further that the conversion from chern character to chern class reads

$$\sum_{k \geq 0} c_k(\mathcal{F}) \cdot t^k = \exp \left(\sum_{k \geq 1} (-1)^{k-1} (k-1)! \mathrm{ch}_k(\mathcal{F}) \cdot t^k \right)$$

and therefore

$$\begin{aligned} c_T(\mathcal{F} \otimes \mathcal{O}(-u)) &= T^{\mathrm{rk}(\mathcal{F})} \sum_{k \geq 0} c_k(\mathcal{F} \otimes \mathcal{O}(-u)) \cdot T^{-k} \\ &= T^{M+l+1-g} e^{-\frac{\theta}{T}}, \end{aligned}$$

where we have used $\mathrm{rk}(\mathcal{F}) = M + l + 1 - g$ which follows from Riemann-Roch. This finally implies

$$c_T(\mathcal{F}) = c_{T+u}(\mathcal{F} \otimes \mathcal{O}(-u)) = (T + u)^{M+l+1-g} e^{-\frac{\theta}{T+u}}$$

as desired. □

Remark 2.12. Note that one can rewrite (37) as

$$\left(T + \sum_i a_i u_i \right)^{1-g+l} \exp \left(- \sum_{k,l=1}^{n} \theta_{k,l} \frac{\partial f_k(\mathbf{u}) / \partial u_l}{f_k(\mathbf{u})} \right) \prod_{k=1}^{n} f_k(\mathbf{u})^{m_k},$$

where we treat u_l in

$$f_k(\mathbf{u}) = \left(T + \sum_i a_i u_i \right)^{a_k}.$$

as a formal variable before taking derivatives and as a cohomology class afterwards.

In order to be able to work with the $\theta_{k,l}$ as above, we will need the following Lemma

Lemma 2.13. For any tuple $\mathbf{m} = (m_i)_{i=1}^n$ of nonnegative integers and tuples $\mathbf{a}^1 = \left(a_i^1\right)_{i=1}^{s_1}, \ldots, \mathbf{a}^g = (a_i^g)_{i=1}^{s_g}, \mathbf{b}^1 = \left(b_i^1\right)_{i=1}^{t_1}, \ldots, \mathbf{b}^g = (b_i^g)_{i=1}^{t_g}, \mathbf{c}^1 = \left(c_i^1\right)_{i=1}^{t}, \mathbf{c}^2 = \left(c_i^2\right)_{i=1}^{t}$ of numbers in $\{ 1, \ldots, n \}$ we have

$$\int_{C^{(\mathbf{m})}} \prod_{i=1}^{n} u_i^{n_i} \prod_{j=1}^{g} \left(\prod_{i=1}^{r_j} \alpha_j^{a_i^j} \prod_{i=1}^{s_j} \beta_j^{b_i^j} \right) \prod_{i=1}^{t} \theta_{c_i^1, c_i^2} \prod_{i_1, i_2 = 1}^{n} e^{z_{i_1,i_2} \theta_{i_1,i_2}} = 0$$

unless $s_l = t_l$ for all l in which case the integral equals

$$(-1)^{\sum_{i=1}^{g}\frac{r_i\cdot(r_i-1)}{2}} \sum_{\coprod_{i=1}^{g} S_i=\{1,\dots,t\}} [\mathbf{u}^{\mathbf{m}}]\prod_{i=1}^{n} u_i^{n_i+g}\left\|M\right\|^g \prod_{i=1}^{g}\left\|(M^{-1})_{\mathbf{a}^i\sqcup\mathbf{c}^1_{S_i},\mathbf{b}^i\sqcup\mathbf{c}^2_{S_i}}\right\|,$$

where $\sqcup$ denotes the concatenation of tuples and for any set $S = \{s_1 < \cdots < s_l\}$ we defined $\mathbf{c}^j_S := (c^j_{s_i})^l_{i=1}$. Furthermore we set $M = (\delta_{i,j}/u_i + z_{j,i})_{1\le i,j\le n}$ and for any matrix $N = (n_{i,j})_{i,j}$ and tuples of indices $\mathbf{a} = (a_l)_l$, $\mathbf{b} = (b_l)_l$ we denote $N_{\mathbf{a},\mathbf{b}} := \left(n_{a_i,b_j}\right)_{i,j}$.

Proof. Let us first examine the case $t = 0$. In Remark 2.10 we noted how to compute integrals of the above kind. First, we have to express the product of classes pulled back from $\prod_{i=1}^n \mathrm{Pic}^0(C)$ in terms of monomials in the α^i_l, β^i_l. Then we delete all unbalanced monomials and replace all balanced ones by the appropriate powers of u_i. Furthermore, we can write

$$\begin{aligned}(38)\qquad &\prod_{l=1}^{g}\left(\prod_{i=1}^{r_l}\alpha_l^{a_{l,i}}\prod_{i=1}^{s_l}\beta_l^{b_{l,i}}\right)\prod_{i_1,i_2=1}^{n} e^{z_{i_1,i_2}\theta_{i_1,i_2}}\\ &=\prod_{l=1}^{g}\left(\prod_{i=1}^{r_l}\alpha_l^{a_{l,i}}\prod_{i=1}^{s_l}\beta_l^{b_{l,i}}\prod_{i_1,i_2=1}^{n} e^{z_{i_1,i_2}\alpha_l^{i_1}\beta_l^{i_2}}\right)\end{aligned}$$

and the different factors of the outer product do not influence each other during this process. We can therefore assume $g = 1$. Any z-monomial in the above product now corresponds to a directed graph on the vertices $1, \dots, n$ as the occurrence of a $z_{i_1,i_2}\alpha^{i_1}\beta^{i_2}$ can be viewed as an edge from i_2 to i_1. The balancing condition is equivalent to the graph consisting of two parts:

(1) a set of vertex-disjoint non-repeating directed cycles $\mathbf{C}$ on the vertices not in $\mathbf{a} \cup \mathbf{b}$

(2) a set of vertex-disjoint non-self-intersecting directed paths $\mathbf{P}$ which start at the entries in $\mathbf{a}$ and end at the entries of $\mathbf{b}$. Furthermore, we have $r = s$ and therefore there must be a permutation $\sigma \in S_r$ so that the path starting at a_i ends at $b_{\sigma(i)}$ or $a_i = b_{\sigma(i)}$.

Moreover, both sets are not allowed to have common vertices. The pair $\mathbf{F} = (\mathbf{P}, \mathbf{C})$ is precisely what is referred to in [101, Definition 2.3] as a *self-avoiding flow* on the complete directed graph on $\{1, \dots, n\}$ connecting $\mathbf{a}$ to $\mathbf{b}$. By weighting each edge $e = (i \to j)$ by $\mathsf{wt}(e) = -u_i z_{j,i}$ and taking into account the signs produced by rearranging the α's and β's, we can therefore replace (38) by

$$(-1)^{r(r-1)/2}\cdot \sum_{\substack{(\mathbf{F},\sigma)\\ \text{as above}}} \mathrm{sgn}(\mathbf{F})\cdot\prod_{e\in E(\mathbf{F})}\mathsf{wt}(e),$$

where $\operatorname{sgn}(\mathbf{F}) = \operatorname{sgn}(\mathbf{P}) \cdot \operatorname{sgn}(\mathbf{C})$ with $\operatorname{sgn}(\mathbf{P}) = \operatorname{sgn}(\sigma)$ and $\operatorname{sgn}(\mathbf{C}) = (-1)^{|\mathbf{C}|}$ and $(-1)^{r(r-1)/2}$ the sign that arises out of permuting the factors of $\prod_{i=1}^{r} \left(\alpha^{a_i} \beta^{b_i}\right)$ back into the order in which they appear in (38). By [101, Theorem 2.5] this is equal to

$$(-1)^{r(r-1)/2} \cdot \left(\sum_{\mathbf{C}} \operatorname{sgn}(\mathbf{C}) \prod_{e \in E(\mathbf{C})} \mathsf{wt}(e) \right) \cdot \left\| \left(\widetilde{M}^{-1} \right)_{\mathbf{a},\mathbf{b}} \right\|,$$

where $\widetilde{M} = (\delta_{i,j} + u_i z_{j,i})_{1 \leq i,j \leq n}$. Moreover, the second factor is easily seen to be $\left\| \widetilde{M} \right\| = \prod_{i=1}^{n} u_i \cdot \left\| M \right\|$ and so our expression is

$$(-1)^{r(r-1)/2} \cdot \prod_{i=1}^{n} u_i \cdot \left\| M \right\| \cdot \left\| \left(M^{-1} \right)_{\mathbf{a},\mathbf{b}} \right\|,$$

which establishes the claim in the case $t = 0$. For the general claim one simply takes derivatives in z_{i_1,i_2}. For this one uses Jacobi's identity [17, Lemma A.1(e)] which says that for any tuples $\mathbf{a} = (a_1 < \ldots < a_s), \mathbf{b} = (b_1 < \ldots < b_s)$ of elements in $\{ 1, \ldots, n \}$:

$$\left\| M \right\| \cdot \left\| \left(M^{-1} \right)_{\mathbf{a},\mathbf{b}} \right\| = (-1)^{\sum_{i=1}^{r} (a_i + b_i)} \left\| M_{\mathbf{b}^c, \mathbf{a}^c} \right\|$$

and therefore

$$\begin{aligned} &\frac{\partial}{\partial z_{i_1,i_2}} \left(\left\| M \right\| \cdot \left\| \left(M^{-1} \right)_{\mathbf{a},\mathbf{b}} \right\| \right) \\ &= \begin{cases} \left\| M \right\| \cdot \left\| \left(M^{-1} \right)_{\mathbf{a} \sqcup (i_1), \mathbf{b} \sqcup (i_2)} \right\|, & \text{if } i_1 \notin \mathbf{a} \text{ and } i_2 \notin \mathbf{b} \\ 0, & \text{else} \end{cases} \end{aligned}$$

which holds even when the entries of $\mathbf{a}$ and $\mathbf{b}$ are not in increasing order. □

2.3.2 Proof of Theorems 1.24, 1.19(1) and (2)

We now use the results of Section 2.3.1 to derive Theorem 1.24 from Proposition 2.7. In addition to Theorem 1.24 we will show the following reformulation which will be more useful when proving Theorem 1.19(3) later on.

Theorem 2.14. We have

$$\langle \operatorname{ch}_{z_1}(\gamma_1) \ldots \operatorname{ch}_{z_n}(\gamma_n) \rangle_d^{X,\mathbb{T}} = p^{d(1-g)} \sum_{\lambda \vdash d} \langle \operatorname{ch}_{z_1}(\gamma_1) \ldots \operatorname{ch}_{z_n}(\gamma_n) \rangle_\lambda^X \big|_{p_\square = p}$$

where $\langle \dots \rangle_\lambda^X \in \mathbb{Q}(t_1,t_2)[[\mathbf{p}]]$ is the power series in $\mathbf{p} = (p_\square)_{\square\in\lambda}$ determined by super-commutativity and

(39)

$$\left\langle \prod_{i=1}^{a} \mathrm{ch}_{x_i}(1) \cdot \prod_{i=1}^{b} \mathrm{ch}_{y_i}(\mathrm{pt}) \cdot \prod_{l=1}^{g} \Big(\mathrm{ch}_{z_1^l}(\alpha_l)\mathrm{ch}_{w_1^l}(\beta_l)\dots\mathrm{ch}_{z_{c_l}^l}(\alpha_l)\mathrm{ch}_{w_{c_l}^l}(\beta_l)\Big)\right\rangle_\lambda^X$$

$$:= \prod_{i=1}^{a} x_i \cdot \sum_{\substack{\coprod_{i=-1}^{g} S_i = \{1,\dots,a\} \\ \square_i\in\lambda \text{ for } i\in S_{-1}}} \prod_{i\in S_{-1}} \Big(p_{\square_i}\frac{\partial}{\partial p_{\square_i}}\Big) \prod_{i\in S_{-1}} E(x_i, Y^\lambda_{\square_i})$$

$$\cdot \prod_{i\in S_0}\Big(l_1\mathfrak{B}(x_it_1)+l_2\mathfrak{B}(x_it_2)\Big)E(x_i,\mathbf{Y}^\lambda)\cdot\prod_{i=1}^{b}E(y_i,\mathbf{Y}^\lambda)$$

$$\cdot\prod_{i=1}^{g}\Big(\mathbf{z}^i,\mathbf{w}^i;\mathbf{x}_{S_i}\mid\mathbf{Y}^\lambda\Big)_{\widetilde{M}(\mathbf{p})}\cdot A(\mathbf{Y}^\lambda)^{g-1}\cdot B_1(\mathbf{Y}^\lambda)^{l_1}\cdot B_2(\mathbf{Y}^\lambda)^{l_2},$$

where

$$\widetilde{M}(\mathbf{p}) = \left(p_\square\frac{\partial Y^\lambda_{\square'}(\mathbf{p})}{\partial p_\square}\right)_{\square,\square'\in\lambda}$$

and $\mathbf{Y}^\lambda = \big(Y^\lambda_\square(\mathbf{p})\big)_{\square\in\lambda}$ is as in Theorem 2.16.

Proof of Theorem 1.24 and Theorem 2.14. First, we express (30) in terms of the cohomology classes introduced in Section 2.3.1. For this we fix a summand corresponding to a partition λ and a tuple $\mathbf{m} = (m_{\lambda/\mu})_{\lambda/\mu}$ as in Proposition 2.7 and let $\mathbf{n} := |\mathbf{m}|$. It follows from (32) that

$$\mathrm{ch}_z(\widetilde{\mathbb{F}}) = \sum_{(i,j)\in\lambda} \exp\left(z\left(\mathcal{D}_{i,j} - (il_1+jl_2)\mathrm{pt}_C - it_1 - jt_2\right)\right)$$

with $\mathcal{D}_{i,j} = \sum_{(i,j)\in\lambda/\mu}\mathcal{D}_{\lambda/\mu}$ and $\mathcal{D}_{\lambda/\mu}\subset C\times C^{(\mathbf{m})}$ the universal divisors. We recall from (36) that

$$\mathcal{D}_{\lambda/\mu} = m_{\lambda/\mu}\cdot\mathrm{pt}_C + u_{\lambda/\mu} + \gamma_{\lambda/\mu},$$

where

$$\gamma_{\lambda/\mu} = \sum_{l=1}^{g}\left(\beta_l^1\times\alpha_l^{\lambda/\mu} - \alpha_l^1\times\beta_l^{\lambda/\mu}\right).$$

One can check that

$$\gamma_{\lambda/\mu_1}\gamma_{\lambda/\mu_2} = -\mathrm{pt}_C\cdot\left(\theta_{\lambda/\mu_1,\lambda/\mu_2}+\theta_{\lambda/\mu_2,\lambda/\mu_1}\right),$$

which gives

$\mathrm{ch}_z(\widetilde{\mathbb{F}})$

$$
\begin{aligned}
&= \sum_{(i,j)\in\lambda} \exp\left(z \sum_{(i,j)\in\lambda/\mu} (m_{\lambda/\mu}\mathrm{pt}_C + u_{\lambda/\mu} + \gamma_{\lambda/\mu}) - (il_1 + jl_2)\mathrm{pt}_C - it_1 - jt_2\right) \\
&= \sum_{(i,j)\in\lambda} \exp\left(z \sum_{(i,j)\in\lambda/\mu} (u_{\lambda/\mu} + \gamma_{\lambda/\mu}) - it_1 - jt_2\right) \\
&\qquad \cdot \big(1 + z(n_{(i,j)} - il_1 - jl_2)\mathrm{pt}_C\big) \\
&= \sum_{(i,j)\in\lambda} \exp\left(z \sum_{(i,j)\in\lambda/\mu} u_{\lambda/\mu} - it_1 - jt_2\right) \cdot \big(1 + z(n_{(i,j)} - il_1 - jl_2)\mathrm{pt}_C\big) \\
&\cdot \left(1 + z \sum_{(i,j)\in\lambda/\mu} \gamma_{\lambda/\mu} - z^2\mathrm{pt}_C \sum_{(i,j)\in\lambda/\mu_1,\lambda/\mu_2} \theta_{\lambda/\mu_1,\lambda/\mu_2}\right) \\
&= \sum_{(i,j)\in\lambda} \exp\left(z \sum_{(i,j)\in\lambda/\mu} u_{\lambda/\mu} - it_1 - jt_2\right) \cdot \left(1 - z^2\mathrm{pt}_C \sum_{(i,j)\in\lambda/\mu_1,\lambda/\mu_2} \theta_{\lambda/\mu_1,\lambda/\mu_2}\right. \\
&\left. + z \sum_{(i,j)\in\lambda/\mu} \big(\gamma_{\lambda/\mu} + (n_{(i,j)} - il_1 - jl_2)\mathrm{pt}_C\big)\right).
\end{aligned}
$$

Finally, using

$$
\begin{aligned}
\gamma_{\lambda/\mu} \cdot \alpha_l^1 &= \mathrm{pt}_C \cdot \alpha_l^{\lambda/\mu} \\
\gamma_{\lambda/\mu} \cdot \beta_l^1 &= \mathrm{pt}_C \cdot \beta_l^{\lambda/\mu}
\end{aligned}
$$

and (31) we see

$$
\begin{aligned}
\mathrm{ch}_z(1) &= z \sum_{(i,j)\in\lambda} E(z, Y_{i,j}) \cdot \left(n_{i,j} - il_1 - jl_2 + l_1\mathcal{B}(zt_1) + l_2\mathcal{B}(zt_2)\right. \\
&\qquad \left. - z \sum_{(i,j)\in\lambda/\mu_1,\lambda/\mu_2} \theta_{\lambda/\mu_1,\lambda/\mu_2}\right) \\
\mathrm{ch}_z(\alpha_l) &= z \sum_{(i,j)\in\lambda} E(z, Y_{i,j}) \cdot \sum_{(i,j)\in\lambda/\mu} \alpha_l^{\lambda/\mu} \\
\mathrm{ch}_z(\beta_l) &= z \sum_{(i,j)\in\lambda} E(z, Y_{i,j}) \cdot \sum_{(i,j)\in\lambda/\mu} \beta_l^{\lambda/\mu} \\
\mathrm{ch}_z(\mathrm{pt}) &= E(y, \mathbf{Y}),
\end{aligned}
$$

where we wrote $\mathbf{Y} = (Y_\square)_{\square\in\lambda}$ for

$$
Y_{i,j} := -it_1 - jt_2 + \sum_{(i,j)\in\lambda/\mu} u_{\lambda/\mu},
$$

which using (48) can be written as $\mathbf{Y} = \overline{\mathbf{Y}}^{\lambda}(\mathbf{u})$ with $\mathbf{u} = (u_{\lambda/\mu})_{\lambda/\mu}$. Using Lemma 2.11 we can further expand the denominator:

$$\tfrac{1}{e(N_{\mathbf{m}})} = \overline{A}^{g-1} \cdot \overline{B}_1^{l_1} \cdot \overline{B}_2^{l_2} \cdot \prod_{\lambda/\mu} \overline{F}^{\lambda}_{\lambda/\mu}(\mathbf{Y})^{-m_{\lambda/\mu}} \cdot h$$

for

$$\begin{aligned}
\overline{A}(\mathbf{Y}) &\coloneqq \prod_{0 \neq (i,j) \in \lambda} Y_{i,j} \cdot \prod_{(i,j)\in\lambda} (t_1 + t_2 - Y_{i,j}) \\
&\quad \cdot \prod_{\substack{(i,j),(k,l)\in\lambda \\ 0 \leq a,b \leq 1 \\ (k,l) \neq (i+a,j+b)}} (at_1 + bt_2 + Y_{k,l} - Y_{i,j})^{(-1)^{a+b+1}}, \\
\overline{B}_1(\mathbf{Y}) &\coloneqq \prod_{(i,j)\in\lambda} \tfrac{Y_{i,j}^i}{(t_1+t_2-Y_{i,j})^{i+1}} \prod_{\substack{(i,j),(k,l)\in\lambda \\ 0\leq a,b\leq 1}} (at_1 + bt_2 + Y_{k,l} - Y_{i,j})^{(i-k+a)(-1)^{a+b}} \\
\overline{B}_2(\mathbf{Y}) &\coloneqq \prod_{(i,j)\in\lambda} \tfrac{Y_{i,j}^j}{(t_1+t_2-Y_{i,j})^{j+1}} \prod_{\substack{(i,j),(k,l)\in\lambda \\ 0\leq a,b\leq 1}} (at_1 + bt_2 + Y_{k,l} - Y_{i,j})^{(j-l+b)(-1)^{a+b}} \\
h &\coloneqq \exp\left(\sum_{\lambda/\mu,\lambda/\mu'} \theta_{\lambda/\mu,\lambda/\mu'} \cdot \tfrac{\partial \overline{F}^{\lambda}_{\lambda/\mu}(\mathbf{Y})}{\partial u_{\lambda/\mu'}} / \overline{F}^{\lambda}_{\lambda/\mu}(\mathbf{Y}) \right),
\end{aligned}$$

where $\overline{F}^{\lambda}_{\lambda/\mu}(\mathbf{Y})$ is as in (49). Using Lemma 2.13 we can see that the contribution of $\mathbf{m}$ is the coefficient of $\mathbf{u}^{\mathbf{m}}$ in

$$\overline{P}_{\lambda}(\mathbf{u}) \cdot \prod_{\lambda/\mu} \overline{F}^{\lambda}_{\lambda/\mu}(\mathbf{Y})^{-m_{\lambda/\mu}} \cdot \prod_{\lambda/\mu} u_{\lambda/\mu} \cdot \left\| M_{\text{skew}}(\mathbf{u}) \right\|,$$

where

(40)

$$\begin{aligned}
\overline{P}_{\lambda}(\mathbf{u}) = \prod_{i=1}^{a} x_i \cdot &\sum_{\substack{\coprod_{i=0}^{g} S_i = \{1,\dots,a\} \\ (i_n,j_n)\in\lambda \text{ for } n \in S_0}} \prod_{i=1}^{b} E(y_i, \mathbf{Y}) \cdot \prod_{i=1}^{g} \left(\mathbf{z}^i, \mathbf{w}^i; \mathbf{x}_{S_i}\right)' \\
&\cdot \prod_{n\in S_0} E(x_n, Y_{(i_n,j_n)}) \cdot \left(n_{(i_n,j_n)} - i_n t_1 - j_n t_2 + l_1 \mathcal{B}(x_n t_1) + l_2 \mathcal{B}(x_n t_2)\right) \\
&\cdot \left(\overline{A}(\mathbf{Y}) \cdot \prod_{\lambda/\mu} u_{\lambda/\mu} \cdot \left\| M_{\text{skew}}(\mathbf{u}) \right\| \right)^{g-1} \cdot \overline{B}_1(\mathbf{Y})^{l_1} \cdot \overline{B}_2(\mathbf{Y})^{l_2},
\end{aligned}$$

where

$$(\mathbf{z},\mathbf{w};\mathbf{x})' := (-1)^n \sum_{\substack{\boldsymbol{\mu}^{(j)}=(\lambda/\mu_i^{(j)})_i \\ j=1,2,3,4 \\ \square_i^{(1)}\in\lambda/\mu_i^{(1)},\square_i^{(2)}\in\lambda/\mu_i^{(2)} \\ \square_i^{(3)}\in\lambda/\mu_i^{(3)},\lambda/\mu_i^{(4)}}} \left\| \left(M_{\text{skew}}(\mathbf{u})^{-1}\right)_{\boldsymbol{\mu}^{(1)}\sqcup\boldsymbol{\mu}^{(3)};\boldsymbol{\mu}^{(2)}\sqcup\boldsymbol{\mu}^{(4)}} \right\|$$

$$\cdot \prod_{i=1}^{m} z_i w_i E(z_i, Y_{\square_i^{(1)}}) E(w_i, Y_{\square_i^{(2)}}) \cdot \prod_{i=1}^{n} x_i E(x_i, Y_{\square_i^{(3)}})$$

and

$$M_{\text{skew}}(\mathbf{u}) := \left(\frac{\delta_{\lambda/\mu,\lambda/\mu'}}{u_{\lambda/\mu}} + \frac{\partial \overline{F}^{\lambda}_{\lambda/\mu'}(\mathbf{Y})/\partial u_{\lambda/\mu}}{\overline{F}^{\lambda}_{\lambda/\mu'}(\mathbf{Y})} \right)_{\lambda/\mu,\lambda'/\mu'}.$$

By using multivariate Lagrange inversion [31, Theorem A] we see that this is the same as the coefficient of $\overline{\mathbf{p}}^{\mathbf{m}}$ in $\overline{P}_\lambda(\mathbf{u}(\overline{\mathbf{p}}))$ where $\mathbf{u}(\overline{\mathbf{p}}) = \left(u_{\lambda/\mu}(\overline{\mathbf{p}})\right)_{\lambda/\mu}$ is the unique power series in $\overline{\mathbf{p}} = \left(\overline{p}_{\lambda/\mu}\right)_{\lambda/\mu}$ satisfying $u_{\lambda/\mu}(\overline{\mathbf{p}}) = \mathcal{O}(\overline{\mathbf{p}}^{>0})$ and

$$\overline{p}_{\lambda/\mu} = u_{\lambda/\mu}(\overline{\mathbf{p}}) \cdot \overline{F}^{\lambda}_{\lambda/\mu}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}(\overline{\mathbf{p}}))).$$

for any λ/μ. Note however that neither the expression $\overline{P}_\lambda(\mathbf{u})$ nor the accompanying prefactor

$$p^{\|\mathbf{m}\|-n(\lambda)\cdot l_1-n(\bar{\lambda})\cdot l_2} = p^{|\mathbf{n}|-n(\lambda)\cdot l_1-n(\bar{\lambda})\cdot l_2} \tag{41}$$

in (30) depend on $\mathbf{m}$, but rather on $\mathbf{n}$. Therefore we may consider the sum over all coefficients of $\mathbf{p}^{\mathbf{m}}$ with $\mathbf{m} \vdash \mathbf{n}$ for a fixed $\mathbf{n}$. This sum can be expressed as:

$$\sum_{\mathbf{m}\vdash\mathbf{n}} [\overline{\mathbf{p}}^{\mathbf{m}}]\overline{P}_\lambda(\mathbf{u}(\overline{\mathbf{p}})) = [\mathbf{p}^{\mathbf{n}}]\overline{P}_\lambda(\mathbf{u}^\lambda(\mathbf{p}))$$

with $\mathbf{u}^\lambda(\mathbf{p})$ as in Theorem 2.16(3) and $\mathbf{p} = (p_\square)_{\square\in\lambda}$ a new set of variables. As a result, $\mathbf{Y}$ is the multivariate Bethe root $\mathbf{Y}^\lambda(\mathbf{p})$ constructed in Theorem 2.16. Using Lemmas 2.21 and 2.22 we see that

$$\overline{A}(\mathbf{Y}) \cdot \prod_{\lambda/\mu} u^{\lambda}_{\lambda/\mu}(\mathbf{p}) \cdot \left\|M_{\text{skew}}(\mathbf{u}^\lambda(\mathbf{p}))\right\| = A(\mathbf{Y}) \cdot \left\|M(\mathbf{Y})\right\|$$

and

$$(\mathbf{z},\mathbf{w};\mathbf{x})' = (\mathbf{z},\mathbf{w};\mathbf{x} \mid \mathbf{Y})_{M(\mathbf{Y})^{-1}}.$$

By making these substitutions in $\overline{P}_\lambda$ we obtain an expression that only depends on $\mathbf{Y}^\lambda$ as opposed to $\mathbf{u}^\lambda$. In that expression we can furthermore replace $n_{(i,j)}$ by $n_{(i,j)} + it_1 + jt_2$. Because of the shift in the prefactor (41) and

$$\overline{B}_1(\mathbf{Y}) = B_1(\mathbf{Y}) \cdot \prod_{(i,j)\in\lambda} \overline{F}^{\lambda}_{\lambda/\mu}(\mathbf{Y})^i = \prod_{(i,j)\in\lambda} p^i_{(i,j)} \cdot B_1(\mathbf{Y})$$

$$\overline{B}_2(\mathbf{Y}) = B_2(\mathbf{Y}) \cdot \prod_{(i,j)\in\lambda} \overline{F}^{\lambda}_{\lambda/\mu}(\mathbf{Y})^j = \prod_{(i,j)\in\lambda} p^j_{(i,j)} \cdot B_2(\mathbf{Y})$$

we may write

$$\left\langle \prod_{i=1}^{a} \mathrm{ch}_{x_i}(1) \cdot \prod_{i=1}^{b} \mathrm{ch}_{y_i}(\mathrm{pt}) \cdot \prod_{l=1}^{g} \Big(\mathrm{ch}_{z_1^l}(\alpha_l)\mathrm{ch}_{w_1^l}(\beta_l) \dots \mathrm{ch}_{z_{c_l}^l}(\alpha_l)\mathrm{ch}_{w_{c_l}^l}(\beta_l) \Big) \right\rangle_d^X$$
$$= p^{d(1-g)} \sum_{\lambda\vdash d} P_\lambda|_{p_\square = p}$$

for

$$\begin{aligned} P_\lambda = \prod_{i=1}^{a} x_i \cdot & \sum_{\substack{\coprod_{i=-1}^{g} S_i = \{1,\dots,a\} \\ \square_i \in \lambda \text{ for } i\in S_{-1}}} \prod_{i\in S_-1} \left(p_{\square_i} \frac{\partial}{\partial p_{\square_i}} \right) \prod_{i\in S_{-1}} E(x_i, Y^\lambda_{\square_i}) \\ & \cdot \prod_{i\in S_0} \Big(l_1 \mathfrak{B}(x_i t_1) + l_2 \mathfrak{B}(x_i t_2) \Big) E(x_i, \mathbf{Y}^\lambda) \cdot \prod_{i=1}^{b} E(y_i, \mathbf{Y}^\lambda) \\ & \cdot \Big(\mathbf{z}^i, \mathbf{w}^i; \mathbf{x}_{S_i} \mid \mathbf{Y}^\lambda \Big)_{M(\mathbf{Y}^\lambda)^{-1}} \cdot A(\mathbf{Y}^\lambda)^{g-1} \cdot B_1(\mathbf{Y}^\lambda)^{l_1} \cdot B_2(\mathbf{Y}^\lambda)^{l_2}. \end{aligned} \tag{42}$$

In order for this expression to agree with (18) and (39) we need

$$M(\mathbf{Y}^\lambda)^{-1} = \left(p_\square \frac{\partial Y^\lambda_{\square'}}{\partial p_\square} \right)_{\square,\square'\in\lambda},$$

which can be seen by applying partials $p_\square \frac{\partial}{\partial p_\square}$ to the Bethe equations (44) i.e.

$$p_{\square'} = F_{\square'}(\mathbf{Y}^\lambda(\mathbf{p})).$$

□

Using Theorem 1.24 we can now prove Theorem 1.19 part (1) and (2).

Proof of Theorem 1.19(1),(2). We first need to show that there is a locally closed subscheme $\mathbf{Be} \subset \mathbb{A}^d_{\mathbb{Q}(t_1,t_2,p)}$ with the following properties:

(1) Up to rearranging the coordinates, its K-valued points are exactly the Bethe roots for partitions of size d.

(2) It is preserved under the involution

$$t_i \mapsto t_i, p \mapsto p^{-1}, Y_i \mapsto t_1 + t_2 - Y_i.$$

Conjecture C implies that the Bethe equations (closed condition) together with full admissibility (open condition) cut out such a subscheme. In the absence of Conjecture C, we use [83, Theorem 2], which implies that

$$\langle \mathrm{ch}_{z_1}(\mathrm{pt}) \cdots \mathrm{ch}_{z_n}(\mathrm{pt}) \rangle^{\mathbb{C}^2 \times E, \mathbb{T}} \in \mathbb{Q}(t_1, t_2, p)[[z_1, \ldots, z_n]]$$

for E the elliptic curve and

$$\langle \mathrm{ch}_{z_1}(\mathrm{pt}) \cdots \mathrm{ch}_{z_n}(\mathrm{pt}) \rangle^{\mathbb{C}^2 \times E, \mathbb{T}} \Big|_{p \mapsto p^{-1}} = \langle \mathrm{ch}_{-z_1}(\mathrm{pt}) \cdots \mathrm{ch}_{-z_n}(\mathrm{pt}) \rangle^{\mathbb{C}^2 \times E}.$$

Theorem 1.24 also gives

$$\begin{aligned} &[z_1^{k_1} \cdots z_n^{k_n}] \prod_{i=1}^n \frac{t_1 t_2}{(e^{\frac{z_i t_1}{2}} - e^{-\frac{z_i t_1}{2}})(e^{\frac{z_i t_2}{2}} - e^{-\frac{z_i t_2}{2}})} \langle \mathrm{ch}_{z_1}(\mathrm{pt}) \cdots \mathrm{ch}_{z_n}(\mathrm{pt}) \rangle^{\mathbb{C}^2 \times E, \mathbb{T}} \\ &= \sum_{\lambda \vdash d} \prod_{i=1}^n p_{k_i} \left(\mathbf{Y}^\lambda - \frac{t_1 + t_2}{2} \mathbf{1} \right) \end{aligned}$$

with $\mathbf{1} = (1, \ldots, 1)$ and p_i the i-th power sum polynomial. Note furthermore that the left hand side gets multiplied by $(-1)^{k_1 + \ldots + k_n}$ when substituting $p \mapsto p^{-1}$. Since power sum polynomials generate the ring of symmetric functions, we get that

$$E_n^m = \sum_{\lambda \vdash d} \prod_{\square \in \lambda} \left(n \left(Y_\square^\lambda - \frac{t_1 + t_2}{2} \right) + 1 \right)^m$$

is in $\mathbb{Q}(t_1, t_2, p)$ for any $n \in \mathbb{Z}$ and $m \geq 0$ and satisfies $E_n^m|_{p \mapsto p^{-1}} = E_{-n}^m$. It follows from Newton's identities that the polynomial

$$P_n(T) = \prod_{\lambda \vdash d} \left(T - \prod_{\square \in \lambda} \left(n \left(Y_\square^\lambda - \frac{t_1 + t_2}{2} \right) + 1 \right) \right)$$

is in $\mathbb{Q}(t_1, t_2, p)[T]$ and satisfies $P_n|_{p \mapsto p^{-1}} = P_{-n}$. As a result, the subscheme $\mathbf{Be} \subset \mathbb{A}^d_{\mathbb{Q}(t_1, t_2, p)}$ cut out by

$$P_n \left(\prod_i \left(n \left(Y_i - \frac{t_1 + t_2}{2} \right) + 1 \right) \right) = 0 \text{ for all } n \in \mathbb{Z}$$

satisfies (2). Furthermore, for any K-valued point $\mathbf{Y}$ of $\mathbf{Be}$, there must be a Bethe root $\mathbf{Y}^\lambda$ so that

$$\prod_i \left(n \left(Y_i - \frac{t_1 + t_2}{2} \right) + 1 \right) = \prod_{\square \in \lambda} \left(n \left(Y_\square^\lambda - \frac{t_1 + t_2}{2} \right) + 1 \right)$$

for infinitely many $n \in \mathbb{Z}$, which implies $\mathbf{Y} = \mathbf{Y}^\lambda$ up to permutation of coordinates. This gives us $\mathbf{Be}$ as desired.

Since K is algebraically closed and **Be** has finitely many K-points, it follows that **Be** is 0-dimensional and thus all its points are $\overline{\mathbb{Q}(t_1, t_2, p)}$-valued. As the absolute Galois-group

$$G \coloneqq \mathrm{Aut}(\overline{\mathbb{Q}(t_1, t_2, p)}/\mathbb{Q}(t_1, t_2, p))$$

preserves **Be** it thus also preserves the Bethe roots and acts on them by a combination of permuting the coordinates and permuting the partitions. But as (21) is invariant under such permutations, all descendent invariants on local curves must be in $\overline{\mathbb{Q}(t_1, t_2, p)}^G = \mathbb{Q}(t_1, t_2, p)$ as desired.

To prove symmetry, we deduce from (2) that $\mathbf{Y} \mapsto (t_1 + t_2 - Y_i)_i$ is an isomorphism from **Be** to the base-change of **Be** by $\mathbb{Q}(t_1, t_2, p) \to \mathbb{Q}(t_1, t_2, p), p \mapsto p^{-1}$. Hence replacing p by p^{-1} amounts to replacing $Y^\lambda_\square$ by $t_1 + t_2 - Y^\lambda_\square$ since permutations of the partitions or boxes do not matter. Therefore we need to look at all the factors in (18) and see how $\mathbf{Y} \mapsto (t_1 + t_2 - Y_i)_i$ changes them. Indeed, $A(\mathbf{Y})$ and $M(\mathbf{Y})$ stay invariant under this substitution while $B_i(\mathbf{Y})$ gets replaced by $\prod_j F_j(\mathbf{Y})^{-1} B_i(\mathbf{Y})$. In case no descendents of 1 are present this will give part of the p^{-d_β} prefactor noting that $d_{d\cdot[C]}(X) = d \cdot (l_1 + l_2 + 2 - 2g)$. In general we will get extra summands from

$$\left[\nabla^{\mathbf{Y}}_i, \prod_j F_j(\mathbf{Y})^{-1}\right] = -\prod_j F_j(\mathbf{Y})^{-1},$$

which can be absorbed into the product over S_0 by using

$$-\mathcal{B}(x) - 1 = \mathcal{B}(-x).$$

Finally, all descendent variables get negated because of

$$E(z, t_1 + t_2 - X) = E(-z, X).$$

□

Remark 2.15. One way to prove Conjecture C would be to carry out the original version of the algebraic Bethe Ansatz for ILW_1 - see [52] for partial progress towards this. For that one must show that any fully admissible solution of the Bethe equations yields an eigenvector so that the eigenvalues of the integrals of motion are all the symmetric functions in the coordinates of the solution. As a result, each joint eigenspace contains the eigenvector of at most one such solution (up to permutation), which bounds the number of solutions from above by

$$d! \cdot \dim{(\mathrm{ILW}_1)_d} = d! \cdot \#\{\lambda \vdash d\},$$

which is the number of Bethe roots as in Theorem 1.21. Though there are other versions of algebraic Bethe Ansatz such as the Mellin-Barnes approach of [1], they do not imply Conjecture C as far as we know.

One other avenue towards Conjecture C might be to use a strategy similar to the following proof of Proposition 1.23:

Proof of Proposition 1.23. Let $\mathbf{Y} = (Y_i)_{i=1}^d$ be an admissible solution of the Bethe equations over K. We first show $Y_i = \mathcal{O}(p^0)$ for any i. Indeed, let $S \subset \{1, \dots, d\}$ be the set of indices that have negative valuation. We then get

$$\begin{aligned}
p^{|S|} &= \prod_{i \in S} F_i(\mathbf{Y}) \\
&= \prod_{i \in S} \tfrac{Y_i}{t_1+t_2-Y_i} \prod_{\substack{0 \le a,b,c \le 1 \\ (a,b) \neq (0,0) \\ i \in S \\ j \neq i}} ((-1)^c(at_1 + bt_2) + Y_j - Y_i)^{(-1)^{a+b+c}} \\
&= \prod_{i \in S} \tfrac{Y_i}{t_1+t_2-Y_i} \prod_{\substack{0 \le a,b,c \le 1 \\ (a,b) \neq (0,0) \\ i \in S \\ j \notin S}} ((-1)^c(at_1 + bt_2) + Y_j - Y_i)^{(-1)^{a+b+c}}.
\end{aligned}$$

As the right hand side is easily seen to have valuation 0, this must also be true for the left hand side and thus $S = \emptyset$.

Let $\mathbf{Be} \subset K^d$ be the subset of admissible solutions to the Bethe equations. The image of any of the coordinate projections $\pi_i \colon \mathbf{Be} \subset K^d \to K$ has constructible image by Chevalley's theorem [5, Tag 00FE] which must be a finite set or the complement of a finite set. However, since the image also lies in the set of Puiseux series of nonnegative valuation, it could not have been the complement of a finite set. Therefore any coordinate projection must have finite image and so $\mathbf{Be}$ must be a finite set.

To show the claim about the initial coefficients, we use $\mathbf{Y}$ to define a weighted directed graph on the set $\{1, \dots, d\}$ of vertices. An edge going from i to j exists if and only if $Y_j - Y_i$ has p^0-coefficient equal to $-t_1, -t_2$ or $t_1 + t_2$. We then set the weight to be $w_{i \to j} := \nu(g_{i,j}(\mathbf{Y})) > 0$ where ν is the canonical valuation on K. As a result, we have

$$c_i = \sum_{j \to i} w_{j \to i} - \sum_{i \to j} w_{i \to j},$$

where

$$c_i = \begin{cases} 1 - \nu(Y_i), \text{ if } Y_i = \mathcal{O}(p^{>0}) \\ 1 + \nu(t_1 + t_2 - Y_i), \text{ if } Y_i = t_1 + t_2 + \mathcal{O}(p^{>0}) \\ 1, \text{ else.} \end{cases}$$

It now suffices to prove that any vertex i_0 is connected to a vertex j with $Y_j = \mathcal{O}(p^{>0})$. Indeed, take Γ to be the subgraph obtained by starting with

i_0 and repeatedly adjoining all edges that end in a vertex of Γ. It follows that

$$\begin{aligned} 0 &= \sum_{\substack{i \to j \\ \text{in } \Gamma}} (w_{i\to j} - w_{i\to j}) = \sum_{i\in\Gamma} \left(\sum_{j\to i} w_{j\to i} - \sum_{\substack{i\to j \\ \text{in } \Gamma}} w_{i\to j} \right) \\ &\geq \sum_{i\in\Gamma} \left(\sum_{j\to i} w_{j\to i} - \sum_{i\to j} w_{i\to j} \right) = \sum_{i\in\Gamma} c_i \end{aligned}$$

as a result one must have $c_i \leq 0$ for some $i \in \Gamma$ which by the definition of c_i must satisfy $Y_i = \mathcal{O}(p^{>0})$. This concludes the proof. $\square$

2.4 More on Bethe roots

We now want to prove Theorem 1.21 and Theorem 1.19(3). As mentioned in the introduction, one would like to prove Theorem 1.19(3) by first showing that $Y^\lambda_\square(p)$ is holomorphic at $p = 0$ and can be locally extended[13] to all of

$$\mathbb{C} \setminus \{\, \zeta \mid (-\zeta)^n = 1 \text{ for some } 1 \leq n \leq d \,\} .$$

Indeed, if all factors in (18) were polynomial in $\mathbf{Y}^\lambda$, then this would already be enough, however most are merely rational functions and so we have no a priori control over their poles. Luckily, one can circumvent this problem at the cost of working with a multivariate version of the Bethe roots and using (39) instead of (18).

2.4.1 Multivariate Bethe roots

We begin by showing a multivariable generalization of Theorem 1.21. For this let $k = \mathbb{Q}(t_1, t_2)$ and λ be a fixed partition. For the rest of this subsection we will fix a collection $\mathbf{p} = (p_\square)_{\square\in\lambda}$ of possibly repeating variables which are otherwise free. Let ν be the non-archimedean valuation on $k[[\mathbf{p}]]$ given for any

$$x = \sum_{\substack{\mathbf{n}=(n_\square)_{\square\in\lambda} \\ n_\square \geq 0}} a_\mathbf{n} \cdot \mathbf{p}^\mathbf{n} \in k[[\mathbf{p}]]$$

by

$$\nu(x) := \inf \left\{ m \;\middle|\; \text{there is } \mathbf{n} \text{ so that } a_\mathbf{n} \neq 0 \text{ and } m = \sum_{\square\in\lambda} n_\square \right\} \in \mathbb{N} \cup \{\infty\}.$$

[13] By this we mean that it can be analytically continued to any simply connected open subset thereof. However, this continuation is usually not unique. Indeed, see [92] for an numerical investigation of the monodromy.

Furthermore let K be any field containing $k[[\mathbf{p}]]$ equipped with an extension of ν which we will also denote by ν. In particular if $n = 1$ one may choose $K = \overline{\mathbb{Q}(t_1, t_2)}\{\{p\}\}$ to be the field of Puiseux series with its canonical valuation.

Theorem 2.16. There is a tuple $\mathbf{Y}^\lambda(\mathbf{p}) = (Y^\lambda_\Box(\mathbf{p}))_{\Box\in\lambda}$ of power series $Y^\lambda_\Box(\mathbf{p}) \in k[[\mathbf{p}]]$ characterized uniquely via any of the following equivalent descriptions:

(1) $\mathbf{Y}^\lambda \in K^d$ is the unique tuple such that

- it is *admissible* in the sense that for any $\Box \in \lambda$ we have $Y^\lambda_\Box \notin \{0, t_1+t_2\}$ and for $\Box \neq \Box' \in \lambda$ we have $Y^\lambda_\Box - Y^\lambda_{\Box'} \notin \{t_1, t_2, t_1+t_2\}$.
- one has

$$\nu\left(Y^\lambda_{(i,j)} + it_1 + jt_2\right) > 0 \tag{43}$$

- it satisfies the *multivariate Bethe equations* i.e. for every $\Box \in \lambda$ we have

$$p_\Box = F_\Box(\mathbf{Y}^\lambda), \tag{44}$$

where

$$F_\Box(\mathbf{Y}) = f_\Box(\mathbf{Y}) \prod_{\Box\neq\Box'\in\lambda} g_{\Box',\Box}(\mathbf{Y})$$

with

$$f_\Box(\mathbf{Y}) = \tfrac{Y_\Box}{t_1+t_2-Y_\Box}$$

and

$$g_{\Box,\Box'}(\mathbf{Y}) = \prod_{\substack{0\leq a,b,c\leq 1\\(a,b)\neq(0,0)}} ((-1)^c(at_1+bt_2) + Y_\Box - Y_{\Box'})^{(-1)^{a+b+c}}$$

(2) It can be written as

$$\mathbf{Y}^\lambda(\mathbf{p}) = \widetilde{\mathbf{Y}}^\lambda(\mathbf{v}^\lambda), \tag{45}$$

where for any tuple $\mathbf{v} = (v_\Box)_{\Box\in\lambda}$ we define $\widetilde{\mathbf{Y}}^\lambda(\mathbf{v}) = (\widetilde{Y}^\lambda_\Box(\mathbf{v}))_{\Box\in\lambda}$ by

$$\widetilde{Y}^\lambda_{(i,j)}(\mathbf{v}) := -it_1 - jt_2 + \sum_{\substack{\lambda/\mu \text{ conn. skew}\\(i,j)\in\lambda/\mu}} \prod_{\Box\in\lambda/\mu} v_\Box \tag{46}$$

and $\mathbf{v}^\lambda = (v^\lambda_\Box)_{\Box\in\lambda}$ is the unique tuple of power series in $k[[\mathbf{p}]]$ so that $v^\lambda_\Box = \mathcal{O}(\mathbf{p}^{>0})$ for all $\Box \in \lambda$ and

$$p_\Box = v^\lambda_\Box \cdot \widetilde{F}^\lambda_\Box(\widetilde{\mathbf{Y}}^\lambda(\mathbf{v}^\lambda)),$$

where
$$\widetilde{F}^\lambda_\Box(\mathbf{Y}) = \tilde{f}_\Box(\mathbf{Y}) \prod_{\Box\neq\Box'\in\lambda} \widetilde{g}_{\Box',\Box}(\mathbf{Y})$$

with
$$\tilde{f}_\Box(\mathbf{Y}) \coloneqq \frac{Y_\Box^{1-\delta_{\Box,(0,0)}}}{t_1+t_2-Y_\Box}$$

and
$$\widetilde{g}_{\Box,\Box'}(\mathbf{Y}) = -\prod_{\substack{0\leq a,b,c\leq 1\\ (a,b)\neq(0,0)\\ \Box\neq\Box'+(-1)^c(a,b)}} (at_1 + bt_2 + (-1)^c(Y_\Box - Y_{\Box'}))^{(-1)^{a+b+c}}.$$

(3) It can be written as

$$\mathbf{Y}^\lambda(\mathbf{p}) = \overline{\mathbf{Y}}^\lambda(\mathbf{u}^\lambda), \tag{47}$$

where for any tuple $\mathbf{u} = (\mathbf{u}_{\lambda/\mu})_{\lambda/\mu}$ we define $\overline{\mathbf{Y}}^\lambda(\mathbf{u}) = (\overline{Y}^\lambda_\Box(\mathbf{u}))_{\Box\in\lambda}$ by

$$\overline{Y}^\lambda_{(i,j)}(\mathbf{u}) \coloneqq -it_1 - jt_2 + \sum_{\substack{\lambda/\mu \text{ conn. skew}\\ (i,j)\in\lambda/\mu}} u_{\lambda/\mu} \tag{48}$$

and where $\mathbf{u}^\lambda = (u^\lambda_{\lambda/\mu})_{\lambda/\mu}$ is the unique tuple of power series in $k[[\mathbf{p}]]$ so that $u^\lambda_{\lambda/\mu} = \mathcal{O}(\mathbf{p}^{>0})$ for all $\Box \in \lambda$ and

$$\prod_{\Box\in\lambda/\mu} p_\Box = u^\lambda_{\lambda/\mu} \cdot \overline{F}^\lambda_{\lambda/\mu}(\overline{\mathbf{Y}}^\lambda(\mathbf{u}^\lambda)),$$

where

$$\overline{F}^\lambda_{\lambda/\mu}(\mathbf{Y}) = \prod_{\Box\in\lambda/\mu} \tilde{f}_\Box(\mathbf{Y}) \prod_{\substack{\Box\in\lambda/\mu\\ \Box'\notin\lambda/\mu}} \widetilde{g}_{\Box',\Box}(\mathbf{Y}). \tag{49}$$

(4) $Y^\lambda_\Box(\mathbf{p})$ is the coefficient of $\prod_{\Box\in\lambda} v_\Box^{-1}$ in

$$\widetilde{Y}^\lambda_\Box(\mathbf{v}) \cdot \left\| \frac{\partial F_{\Box'}(\widetilde{\mathbf{Y}}^\lambda(\mathbf{v}))/\partial v_{\Box''}}{F_{\Box'}(\widetilde{\mathbf{Y}}^\lambda(\mathbf{v}))} \right\| \cdot \prod_{\Box'\in\lambda} \frac{1}{1-p_{\Box'}\cdot F_{\Box'}(\widetilde{\mathbf{Y}}^\lambda(\mathbf{v}))^{-1}}$$

all of whose $\mathbf{p}$-coefficients turn out to be Laurent series in $\mathbf{v}$. Furthermore, $\widetilde{\mathbf{Y}}^\lambda(\mathbf{v})$ is as in (46).

Remark 2.17. Theorem 1.21 follows from Theorem 2.16 by taking $\mathbf{p} = (p, \cdots, p)$. In particular, the recursion of Theorem1.21(2) is just Banach fixed point iteration.

Proof. Indeed, the tuples in (2) and (3) are not just unique over $k[[\mathbf{p}]]$, but also over K. We will only show this for (2) as (3) is similar. For this note that on the set

$$\mathcal{U} := \left\{ \mathbf{v} = (v_\square)_{\square\in\lambda} \in K^d \;\middle|\; \nu(v_\square) > 0 \text{ for all } \square \in \lambda \right\}$$

the map $\mathbf{G}\colon \mathcal{U} \to \mathcal{U}, \mathbf{v} \mapsto (G_\square(\mathbf{v}))_{\square\in\lambda}$ given by

$$G_\square(\mathbf{v}) := p_\square \cdot \widetilde{F}^\lambda_\square(\widetilde{\mathbf{Y}}^\lambda(\mathbf{v}))^{-1}$$

is a contraction with respect to the metric induced by ν. Therefore the Banach fixed point theorem implies that there can only be at most one fixed point and since the complete subset $\mathcal{U} \cap k[[\mathbf{p}]]^d$ is invariant with respect to $\mathbf{G}$ it follows that this fixed point exists and its coordinates must be power series.

To show the uniqueness in (1) and the equivalence of (1), (2) and (3) it will suffice to establish bijections (indeed we will give isomorphisms of varieties) between

$$\mathbf{Be} := \left\{ \mathbf{Y} = (Y_\square)_{\square\in\lambda} \in K^d \;\middle|\; \begin{array}{l} \mathbf{Y} \text{ is admissible and} \\ \text{for all } \square \in \lambda\colon p_\square = F_\square(\mathbf{Y}) \end{array} \right\}$$

and

$$\widetilde{\mathbf{Be}} := \left\{ \mathbf{v} = (v_\square)_{\square\in\lambda} \in K^d \;\middle|\; \begin{array}{l} \widetilde{\mathbf{Y}}^\lambda(\mathbf{v}) \text{ is admissible and for all } \square \in \lambda\colon \\ p_\square = v_\square \cdot \widetilde{F}^\lambda_\square(\widetilde{\mathbf{Y}}^\lambda(\mathbf{v})) \end{array} \right\}$$

and

$$\overline{\mathbf{Be}} := \left\{ \mathbf{u} = (u_{\lambda/\mu})_{\lambda/\mu} \in \prod_{\lambda/\mu} K \;\middle|\; \begin{array}{l} \overline{\mathbf{Y}}^\lambda(\mathbf{u}) \text{ is admissible and for all } \lambda/\mu\colon \\ \prod_{\square\in\lambda/\mu} p_\square = u_\square \cdot \overline{F}^\lambda_\square(\overline{\mathbf{Y}}^\lambda(\mathbf{u})) \end{array} \right\}$$

so that the conditions $\nu(Y_{i,j} + it_1 + jt_2) > 0$, $\nu(v_\square) > 0$ and $\nu(u_{\lambda/\mu}) > 0$ become equivalent and $\mathbf{Y} = \widetilde{\mathbf{Y}}^\lambda(\mathbf{v}) = \overline{\mathbf{Y}}^\lambda(\mathbf{u})$. We claim that

$$\begin{aligned} \widetilde{\mathbf{Be}} &\longleftrightarrow \overline{\mathbf{Be}} \\ \mathbf{v} &\longmapsto \mathbf{u}(\mathbf{v}) = (u_{\lambda/\mu}(\mathbf{v}))_{\lambda/\mu} \\ \mathbf{v}(\mathbf{u}) = (v_\square(\mathbf{u}))_{\square\in\lambda} &\longleftarrow\!\shortmid \mathbf{u} \end{aligned}$$

for

$$v_\square(\mathbf{u}) := p_\square \cdot \widetilde{F}^\lambda_\square(\overline{\mathbf{Y}}^\lambda(\mathbf{u}))^{-1}$$

and

$$u_{\lambda/\mu}(\mathbf{v}) := \prod_{\square\in\lambda/\mu} v_\square$$

is one such bijection. Indeed, we have

$$\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}(\mathbf{v})) = \widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v}),$$

which shows the well-definedness of $\mathbf{u}(\mathbf{v})$ and $\mathbf{v}(\mathbf{u}(\mathbf{v})) = \mathbf{v}$. For the rest can use $\tilde{g}_{\square,\square'}(\mathbf{Y}) = \tilde{g}_{\square',\square}(\mathbf{Y})^{-1}$ to see that

$$\overline{F}^{\lambda}_{\lambda/\mu}(\mathbf{Y}) = \prod_{\square\in\lambda/\mu} \widetilde{F}^{\lambda}_{\square}(\mathbf{Y})$$

and therefore

$$u_{\lambda/\mu}(\mathbf{v}(\mathbf{u})) = \prod_{\square\in\lambda/\mu} \left(p_{\square} \cdot \widetilde{F}^{\lambda}_{\square}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))^{-1}\right) = \prod_{\square\in\lambda/\mu} p_{\square} \cdot \overline{F}^{\lambda}_{\lambda/\mu}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))^{-1} = u_{\lambda/\mu},$$

which gives the rest. Finally, it follows from Lemma 2.20 that

$$\begin{aligned} \widetilde{\mathbf{Be}} &\longrightarrow \mathbf{Be} \\ \mathbf{v} &\longmapsto \widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v}) \end{aligned}$$

is a bijection with inverse $\mathbf{Y} \mapsto \tilde{\mathbf{v}}^{\lambda}(\mathbf{Y})$ and for the unique $\mathbf{v}$ with $\nu(v_{\square}) > 0$ one has $\nu\left(\widetilde{Y}^{\lambda}_{(i,j)}(\mathbf{v}) + it_1 + jt_2\right) > 0$. For the converse observe that for any $\mathbf{Y} = (Y_{\square})_{\square\in\lambda}$ with $\nu\left(Y_{(i,j)} + it_1 + jt_2\right) > 0$ one has $\nu(v_{\square}(\mathbf{Y})) = 1 - \nu(\widetilde{F}^{\lambda}_{\square}(\mathbf{Y})) = 1 > 0$ and therefore $\mathbf{Y}$ is also unique.

It remains to prove the formula in ((4)) for $\mathbf{Y}^{\lambda}$ as in ((1)), ((2)) and ((3)). For this we note that the characterization ((2)) can be seen as an inversion of power series. As $\widetilde{F}^{\lambda}_{\square}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{0})) \neq 0$ we can use multivariate Lagrange inversion in the shape of [31, Theorem A] to conclude that for any tuple of natural numbers $\mathbf{n} = (n_{\square})_{\square\in\lambda}$ we have

$$\begin{aligned} &[\mathbf{p}^{\mathbf{n}}]Y^{\lambda}_{\square}(\mathbf{p}) \\ &= [\mathbf{v}^{\mathbf{n}}]\left(\widetilde{\mathbf{Y}}^{\lambda}_{\square}(\mathbf{v}) \prod_{\square'\in\lambda} \widetilde{F}^{\lambda}_{\square'}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v}))^{-n_{\square'}} \left\| \delta_{\square',\square''} + v_{\square'} \frac{\partial \widetilde{F}^{\lambda}_{\square''}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v}))/\partial v_{\square'}}{\widetilde{F}^{\lambda}_{\square''}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v}))} \right\| \right) \\ &= [\mathbf{v}^{\mathbf{n}-\mathbf{1}}]\left(\widetilde{\mathbf{Y}}^{\lambda}_{\square}(\mathbf{v}) \prod_{\square'\in\lambda} \widetilde{F}^{\lambda}_{\square'}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v}))^{-n_{\square'}} \left\| \frac{\partial F_{\square''}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v}))/\partial v_{\square'}}{F_{\square''}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v}))} \right\| \right) \\ &= [\mathbf{v}^{-\mathbf{1}}]\left(\widetilde{\mathbf{Y}}^{\lambda}_{\square}(\mathbf{v}) \prod_{\square'\in\lambda} F^{\lambda}_{\square'}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v}))^{-n_{\square'}} \left\| \frac{\partial F_{\square''}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v}))/\partial v_{\square'}}{F_{\square''}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v}))} \right\| \right), \end{aligned}$$

where we used $v_{\square} \cdot \widetilde{F}^{\lambda}_{\square}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v})) = F^{\lambda}_{\square}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v}))$ in the second and third equality which follows from Lemma 2.20. After summing over $\mathbf{n}$ we get the claimed formula. □

2.4.2 Proof of Theorem 1.19(3)

For the rest of this section we will fix an embedding $\mathbb{Q}(t_1, t_2) \hookrightarrow \mathbb{C}$. Let $\mathbf{p} = (p_1, \ldots, p_d)$ to be a tuple of free variables (without repetitions). We then take $\mathbf{Y}^\lambda(\mathbf{p})$ to be the multivariate Bethe roots described in Theorem 2.16 where $\mathbf{p}$ is re-indexed by the boxes of λ in an arbitrary way. Our first step is to show:

Lemma 2.18. The coordinates $\mathbf{Y}^\lambda_\square(\mathbf{p}) \in \mathbb{C}[[\mathbf{p}]]$ of the multivariate Bethe roots $\mathbf{Y}^\lambda(\mathbf{p})$ are holomorphic near the origin and can be locally analytically continued to any point in the complement of

$$X = \left\{ (p_i)_i \in \mathbb{C}^d \;\middle|\; \text{there exists } \emptyset \neq S \subset \{1, \ldots, d\} \text{ so that } \prod_{i \in S}(-p_i) = 1 \right\}.$$

Proof. We consider the subset $Z \subset \mathbb{C}^d \times \mathbb{C}^d$ consisting of $(\mathbf{Y}, \mathbf{p})$ so that $\mathbf{Y}$ is admissible, the multivariate Bethe equations

$$p_i = F_i(\mathbf{Y})$$

are satisfied and so that the Jacobian $\left(\frac{\partial F_i(\mathbf{Y})}{\partial Y_j}\right)_{i,j}$ is invertible. As a result, Z is smooth of dimension d. Since the multivariate Bethe roots come as inversions of convergent power series they themselves converge in a small enough neighborhood. Furthermore, they are admissible and have a nonvanishing Jacobian determinant on the level of power series which implies that there is an open subset of $\mathbb{C}^d$ on which they all give sections of $\pi_2 \colon Z \to \mathbb{C}^d$. Therefore π_2 is dominant and on the complement of some big enough proper algebraic subset $Y \subset \mathbb{C}^d$ it is also finite [36, Exercise II.3.7], flat [5, Tag 052A], unramified (characteristic 0) and hence a holomorphic covering. All Bethe roots therefore admit local analytic continuations to any point in this complement.

By Riemann's extension theorem [34, §1] one now only needs to show that these stay bounded when approaching any point in the complement of X. For this let $\mathbf{p}^{(n)}$ be a sequence of points in $\mathbb{C}^d \setminus Y$ converging to a point $\mathbf{p} \in \mathbb{C}^d \setminus X$ and assume that each $Y^\lambda_\square(\mathbf{p}^{(n)})$ either stays bounded or diverges to ∞. Let $S \subset \{1, \ldots, d\}$ be the set of indices where the latter happens. We then have

$$\begin{aligned}\prod_{i \in S} p_i^{(n)} &= \prod_{i \in S} f_i(\mathbf{Y}^\lambda(\mathbf{p}^{(n)})) \prod_{\substack{i \in S \\ j \neq i}} g_{j,i}(\mathbf{Y}^\lambda(\mathbf{p}^{(n)})) \\ &= \prod_{i \in S} f_i(\mathbf{Y}^\lambda(\mathbf{p}^{(n)})) \prod_{\substack{i \in S \\ j \notin S}} g_{j,i}(\mathbf{Y}^\lambda(\mathbf{p}^{(n)})),\end{aligned}$$

where we used $g_{j,i}(\mathbf{Y}) = g_{i,j}(\mathbf{Y})^{-1}$ in the second equality. Since the right hand side converges to $(-1)^{|S|}$ we must have $S = \emptyset$. □

This lets us control the poles of most of the factors appearing in (39). The only ones that need to be dealt with in more detail are A, B_1 and B_2.

Lemma 2.19. For any λ the germs $A(\mathbf{Y}^\lambda(\mathbf{p}))$, $B_1(\mathbf{Y}^\lambda(\mathbf{p}))$ and $B_2(\mathbf{Y}^\lambda(\mathbf{p}))$ can be locally analytically continued to all of $(\mathbb{C}^*)^d \setminus X$ so that the latter two have no zeros. Here X is as in Lemma 2.18.

Proof. Viewing $A(\mathbf{Y})$ purely as an element of $\mathbb{Q}(t_1, t_2, Y_1, \ldots, Y_d)$ it was observed in the proof of Theorem 2.14 that for any partition λ and any re-indexing of $\mathbf{Y}$ by the boxes of λ one can write

$$A(\mathbf{Y}) = \prod_{\substack{(i,j),(k,l)\in\lambda \\ 0\le a,b\le 1 \\ (k,l)\neq(i+a,j+b)}} (at_1 + bt_2 + Y_{k,l} - Y_{i,j})^{(-1)^{a+b+1}} \cdot \left\| M_{\text{skew}}(\mathbf{u}(\widetilde{\mathbf{v}}^\lambda(\mathbf{Y}))) \right\|$$
$$\cdot \prod_{0\neq(i,j)\in\lambda} Y_{i,j} \cdot \prod_{(i,j)\in\lambda} (t_1 + t_2 - Y_{i,j}) \cdot \prod_{\lambda/\mu} u_{\lambda/\mu}(\widetilde{\mathbf{v}}^\lambda(\mathbf{Y}))$$

with notation as in Lemma 2.22. It follows from this that the denominator of $A(\mathbf{Y})$ can therefore consist only of products of expressions of the shape $at_1 + bt_2 + (-1)^c(Y_{(i,j)} - Y_{(k,l)})$ where $(i,j) \neq (k + (-1)^c a, l + (-1)^c b)$. Note that these factors heavily depend on the indexing of the $Y_{(i,j)}$, however $A(\mathbf{Y})$ is symmetric in $\mathbf{Y}$ and so the set of possible factors in the denominator also has to be invariant under index change. Indeed, this excludes all factors and as a result $A(\mathbf{Y})$ must actually be in $\mathbb{Q}(t_1, t_2)[Y_1, \ldots, Y_d]$. Lemma 2.18 now implies the first part of the claim.

For the rest we will only examine B_1 as B_2 is similar. The function $B_1(\mathbf{p}) := B_1(\mathbf{Y}^\lambda(\mathbf{p}))$ is certainly meromorphic. We aim to show that it extends holomorphically to any point $\mathbf{p}_0 \in (\mathbb{C}^*)^d \setminus X$ and is not zero there. For this we define an equivalence relation on $\{1, \ldots, d\}$ by

$$i \sim j \text{ if and only if } Y_i^\lambda(\mathbf{p}_0) - Y_j^\lambda(\mathbf{p}_0) \in \mathbb{Z} \cdot t_1 + \mathbb{Z} \cdot t_2.$$

Let $S_0, S_1, \ldots, S_n$ be the equivalence classes so that S_0 is the set of i with $Y_i^\lambda(\mathbf{p}_0) \in \mathbb{Z} \cdot t_1 + \mathbb{Z} \cdot t_2$ if there are such i and otherwise we artificially set $S_0 = \emptyset$. We further choose elements $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ and take $\iota\colon \{1, \ldots, d\} \to \mathbb{Z}$ to be the map so that for $i \in S_0$ one has

$$Y_i^\lambda(\mathbf{p}_0) + \iota(i)t_1 \in \mathbb{Z} \cdot t_2$$

and for any $i \in S_j$ with $j > 0$ we need

$$Y_i^\lambda(\mathbf{p}_0) - Y_{s_j}^\lambda(\mathbf{p}_0) + \iota(i)t_1 \in \mathbb{Z} \cdot t_2.$$

For $\mathbf{p}$ in a dense open subset of $\mathbb{C}^d$ one now has:

$$\prod_{i=1}^d p_i^{\iota(i)} B_1(\mathbf{p}) = \prod_{i=1}^d F_i(\mathbf{Y}^\lambda(\mathbf{p}))^{\iota(i)} B_1(\mathbf{p})$$

$$= \pm \prod_{i=1}^{d} \frac{Y_i^\lambda(\mathbf{p})^{\iota(i)}}{(t_1+t_2-Y_i^\lambda(\mathbf{p}))^{\iota(i)+1}}$$

$$\cdot \prod_{\substack{i\neq j\\ 0\leq a,b\leq 1}} \Big(at_1 + bt_2 + Y_j^\lambda(\mathbf{p}) - Y_i^\lambda(\mathbf{p})\Big)^{(\iota(i)-\iota(j)+a)(-1)^{a+b}}.$$

As a result, none of the factors in the second product are zero at $\mathbf{p}_0$. The factors in the first product may only vanish for $i \in S_0$ so that $Y_i^\lambda(\mathbf{p}_0) = 0$ or $Y_i^\lambda(\mathbf{p}_0) = t_1 + t_2$. However, in the first case we have $\iota(i) = 0$ and $\iota(i) = -1$ in the second - in each case the vanishing factor is removed. Therefore the whole product is holomorphic and non-vanishing near $\mathbf{p}_0$. □

In case our local curve has genus at least 1 it now follows from (39) and the previous two Lemmas that P_λ is holomorphic on $(\mathbb{C}^*)^d \setminus X$ and the restriction along $p_\square = p$ can only have poles at $p = 0$ or where $-p$ is an n-th root of unity for $1 \leq n \leq d$. This finishes the proof of Theorem 1.19(3) in this case. In order to similarly deduce the $g = 0$ case we would need to know that $A(\mathbf{Y}^\lambda(\mathbf{p}))$ also never vanishes on $(\mathbb{C}^*)^d \setminus X$, but it is not clear to us how to show this. However, since $\mathbb{P}^1$ has only even cohomology classes we can use [82, Theorem 5], which establishes the pole statement in that case.

2.5 Auxiliary lemmas

Let $\mathbb{Q}(t_1, t_2) \subset K$ be any field extension.

Lemma 2.20. The morphism

$$\mathbb{A}_K^d \supset \{\, \mathbf{Y} = (Y_\square)_{\square\in\lambda} \mid \mathbf{Y} \text{ is admissible} \,\} \longrightarrow \mathbb{A}_K^d$$
$$\mathbf{Y} \longmapsto \widetilde{\mathbf{v}}^\lambda(\mathbf{Y}),$$

where $\widetilde{\mathbf{v}}^\lambda(\mathbf{Y}) = (\widetilde{v}_\square^\lambda(\mathbf{Y}))_{\square\in\lambda}$ with

(50)
$$\widetilde{v}_\square^\lambda(\mathbf{Y}) = Y_\square^{\delta_{\square,0}} \prod_{\substack{0\leq a,b,c\leq 1\\ (a,b)\neq(0,0)\\ \square' := \square + (-1)^c(a,b)\in\lambda}} (at_1 + bt_2 + (-1)^c(Y_{\square'} - Y_\square))^{(-1)^{a+b+c}}$$

is an open immersion with image

$$\mathcal{U} = \left\{ \mathbf{v} = (v_\square)_{\square\in\lambda} \;\middle|\; \widetilde{\mathbf{Y}}^\lambda(\mathbf{v}) \text{ is admissible} \right\}$$

on which $\widetilde{\mathbf{Y}}^\lambda$ defined as in (46) is the inverse.

Proof. It suffices to show $\widetilde{\mathbf{Y}}^\lambda(\widetilde{\mathbf{v}}^\lambda(\mathbf{Y})) = \mathbf{Y}$. Indeed, this would imply that the Jacobian of $\mathbf{Y} \mapsto \widetilde{\mathbf{v}}^\lambda(\mathbf{Y})$ is invertible everywhere which makes the map étale and therefore flat. As it is also a monomorphism, [5, Tag 06NC] implies that it must be an open immersion and hence for any $\mathbf{v}$ in its image $\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v})$ must be admissible and $\widetilde{\mathbf{v}}^{\lambda}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v})) = \mathbf{v}$.

Assume the claim has been shown for any partition of smaller size and let $\mathbf{Y} = (Y_\square)_{\square\in\lambda}$ be an admissible tuple and $(i_0, j_0) \in \lambda$ be an arbitrary box. First, we define for any λ/μ

$$\begin{aligned} v_{[\lambda/\mu]} &\coloneqq \prod_{\square\in\lambda/\mu} \widetilde{v}^{\lambda}_{\square}(\mathbf{Y}) \\ &= Y_{(0,0)}^{[(0,0)\in\lambda/\mu]} \prod_{\substack{\square\in\lambda/\mu,\square'\notin\lambda/\mu \\ 0\le a,b,c\le 1 \\ \square'=\square+(-1)^c(a,b)}} (at_1 + bt_2 + (-1)^c(Y_{\square'} - Y_\square))^{(-1)^{a+b+c}}, \end{aligned}$$

where $[P]$ is defined as

$$[P] \coloneqq \begin{cases} 1, & \text{if } P \text{ is true} \\ 0, & \text{if } P \text{ is false} \end{cases} \tag{51}$$

and we used that any factor of (50) occuring in the first product cancels if it involves boxes $\square, \square' \in \lambda/\mu$. We now want to prove

$$Y_{(i_0,j_0)} \overset{!}{=} -i_0t_1 - j_0t_2 + \sum_{(i_0,j_0)\in\lambda/\mu} v_{[\lambda/\mu]}. \tag{52}$$

The claim is trivial if $(i_0, j_0) = (0, 0)$ hence we may assume without losing generality that $i_0 > 0$. Let λ' be the partition $\lambda' = \lambda_1 \geq \ldots \geq \lambda_{l(\lambda)-1}$ which has degree $|\lambda'| = |\lambda| - \lambda_0 < |\lambda|$. We will identify the Young diagram of λ' with the set of boxes $(i, j) \in \lambda$ with $i > 0$. Using this identification we denote $\mathbf{Y}' \coloneqq (Y_\square)_{\square\in\lambda'}$ and

$$v'_{[\lambda'/\mu']} \coloneqq \prod_{\square\in\lambda'/\mu'} \widetilde{v}^{\lambda'}_{\square}(\mathbf{Y}').$$

In this case one can express the right hand side of (52) in the following way:

$$\begin{aligned} &-i_0t_1 - j_0t_2 + \sum_{(i_0,j_0)\in\lambda'/\mu'} \sum_{\substack{\lambda/\mu \text{ s.t.} \\ (\lambda/\mu)\cap\lambda'=\lambda'/\mu'}} v_{[\lambda/\mu]} \\ &= -i_0t_1 - j_0t_2 + Y_{(0,0)} \\ &+ \sum_{(i_0,j_0)\in\lambda'/\mu'} v'_{[\lambda'/\mu']} \left(\frac{t_1+Y_{(1,0)}-Y_{(0,0)}}{Y_{(1,0)}}\right)^{\delta_{\lambda'/\mu',\lambda'}} \prod_{\substack{l=h_{\lambda'/\mu'} \\ l>0}}^{\lambda_1-1} \frac{t_1+Y_{(1,l)}-Y_{(0,l)}}{t_1+t_2+Y_{(1,l)}-Y_{(0,l-1)}} \\ &+ \sum_{(i_0,j_0)\in\lambda'/\mu'} v'_{[\lambda'/\mu']} \left(\frac{t_1+Y_{(1,0)}-Y_{(0,0)}}{Y_{(1,0)}}\right)^{\delta_{\lambda'/\mu',\lambda'}} \sum_{\substack{j=h_{\lambda'/\mu'} \\ j>0}}^{\lambda_1-1} \frac{t_2+Y_{(0,j)}-Y_{(0,j-1)}}{t_1+t_2+Y_{(1,j)}-Y_{(0,j-1)}} \end{aligned}$$

$$\cdot \prod_{\substack{l=h_{\lambda'/\mu'}\\ l>0}}^{j-1} \tfrac{t_1+Y_{(1,l)}-Y_{(0,l)}}{t_1+t_2+Y_{(1,l)}-Y_{(0,l-1)}}$$

$$= -i_0t_1 - j_0t_2 + Y_{(0,0)}$$

$$+ \sum_{(i_0,j_0)\in\lambda'/\mu'} v'_{[\lambda'/\mu']} \left(\tfrac{t_1+Y_{(1,0)}-Y_{(0,0)}}{Y_{(1,0)}}\right)^{\delta_{\lambda'/\mu',\lambda'}} \prod_{\substack{l=h_{\lambda'/\mu'}\\ l>0}}^{\lambda_1-1} \tfrac{t_1+Y_{(1,l)}-Y_{(0,l)}}{t_1+t_2+Y_{(1,l)}-Y_{(0,l-1)}}$$

$$+ \sum_{(i_0,j_0)\in\lambda'/\mu'} v'_{[\lambda'/\mu']} \left(\tfrac{t_1+Y_{(1,0)}-Y_{(0,0)}}{Y_{(1,0)}}\right)^{\delta_{\lambda'/\mu',\lambda'}} \sum_{\substack{j=h_{\lambda'/\mu'}\\ j>0}}^{\lambda_1-1} \left(1 - \tfrac{t_1+Y_{(1,j)}-Y_{(0,j)}}{t_1+t_2+Y_{(1,j)}-Y_{(0,j-1)}}\right)$$

$$\cdot \prod_{\substack{l=h_{\lambda'/\mu'}\\ l>0}}^{j-1} \tfrac{t_1+Y_{(1,l)}-Y_{(0,l)}}{t_1+t_2+Y_{(1,l)}-Y_{(0,l-1)}}$$

$$= -i_0t_1 - j_0t_2 + Y_{(0,0)} + \sum_{(i_0,j_0)\in\lambda'/\mu'\neq\lambda'} v'_{[\lambda'/\mu']} + t_1 + Y_{(1,0)} - Y_{(0,0)}$$

$$= -(i_0-1)t_1 - j_0t_2 + \sum_{(i_0,j_0)\in\lambda'/\mu'} v'_{[\lambda'/\mu']},$$

where we set

$$h_{\lambda'/\mu'} \coloneqq \min\left\{\, j \;\middle|\; (1,j)\in\lambda'/\mu' \,\right\} \in \mathbb{N}_0 \cup \{\,\infty\,\}\,.$$

The claim now follows by induction on $|\lambda|$. □

We now determine the Jacobian determinant of the above bijection. This comes up in the proof of Theorem 2.14.

Lemma 2.21. For any admissible $\mathbf{Y}$ as above, the Jacobian matrix of the above map i.e.

$$M = \left(\tfrac{\partial\widetilde{v}^\lambda_\square(\mathbf{Y})}{\partial Y_{\square'}}\right)_{\square,\square'\in\lambda}$$

has determinant

$$\prod_{\substack{(i,j)\neq(k,l)\in\lambda\\ 0\le a,b\le 1\\ (k,l)=(i+a,j+b)}} (at_1+bt_2+Y_{k,l}-Y_{i,j})^{(-1)^{a+b}}.$$

Proof. We have

$$\|M\| = Y_{0,0}\left\|\left(\tfrac{\partial\widetilde{v}^\lambda_\square(\mathbf{Y})/\partial Y_{\square'}}{\widetilde{v}^\lambda_\square(Y)}\right)_{\square,\square'\in\lambda}\right\|$$

and furthermore

$$\frac{\partial\widetilde{v}^{\lambda}_{\Box}(\mathbf{Y})/\partial Y_{\Box'}}{\widetilde{v}^{\lambda}_{\Box}(\mathbf{Y})} = \frac{\delta_{\Box,0}\delta_{\Box',0}}{Y_{0,0}} + \sum_{\substack{\Box''\neq\Box \text{ s.t.}\\ \exists 0\leq a,b,c\leq 1:\\ \Box'':=\Box+(-1)^c(a,b)\in\lambda}} \frac{(-1)^{a+b}(\delta_{\Box',\Box''}-\delta_{\Box',\Box})}{at_1+bt_2+(-1)^c(Y_{\Box''}-Y_{\Box})}.$$

If one removes the first summand in the above, then the matrix would have determinant 0. Indeed, it is easily checked that $(1,\dots,1)$ is in the kernel. By looking at the Leibniz formula for the determinant it therefore follows that

$$\|M\| = \left\| \left(\frac{\partial\widetilde{v}^{\lambda}_{\Box}(\mathbf{Y})/\partial Y_{\Box'}}{\widetilde{v}^{\lambda}_{\Box}(\mathbf{Y})}\right)_{(0,0)\neq\Box,\Box'\in\lambda} \right\|.$$

We now note that the above expression is a minor of the Laplacian of the undirected weighted graph Γ_λ defined as follows:

Its vertices are given by the boxes in λ and two $\Box,\Box'\in\lambda$ are connected by an edge if there are $0\leq a,b,c\leq 1$ with $(a,b)\neq(0,0)$ so that $\Box' = \Box + (-1)^c(a,b)$. In this case the weight of the corresponding edge is given by

$$w_{\Box,\Box'} = \frac{(-1)^{a+b+1}}{at_1+bt_2+(-1)^c(Y_{\Box'}-Y_{\Box})}.$$

In particular, the whole graph is a union of cycles of length 3 and for each such cycle consisting of the vertices $\Box,\Box',\Box''$ we have

$$w_{\Box,\Box'}w_{\Box',\Box''} + w_{\Box',\Box''}w_{\Box'',\Box} + w_{\Box'',\Box}w_{\Box,\Box'} = 0 \tag{53}$$

or equivalently

$$w^{-1}_{\Box,\Box''} + w^{-1}_{\Box,\Box'} + w^{-1}_{\Box',\Box''} = 0. \tag{54}$$

We will furthermore call cycles consisting of vertices of the shape $(i,j),(i,j+1),(i+1,j+1)\in\lambda$ *upper cycles*. It follows from the weighted matrix tree theorem [12, Theorem II.3.12] that

$$\|M\| = \sum_{\substack{T\subset\Gamma_\lambda\\ \text{spanning tree}}} \prod_{\substack{e=(\Box,\Box')\\ \text{edge in } T}} w_e.$$

We will now define an permutation σ (c.f. Figure 3) on the set of all such spanning trees which will help us remove some of the summands. For this we first fix an ordering on the set of all upper cycles. For a given tree T we then let $C(T)$ be the first upper cycle so that two of its edges are in T. If no such cycle exists we set $\sigma(T) := T$. Otherwise let e_1, e_2, e_3 be the edges of $C(T)$ named in counter-clockwise direction so that e_1 is not in T whereas e_2, e_3 are in T. Removing e_2 from T will turn the tree into a forest consisting of two connected components - one containing e_3 and the other containing

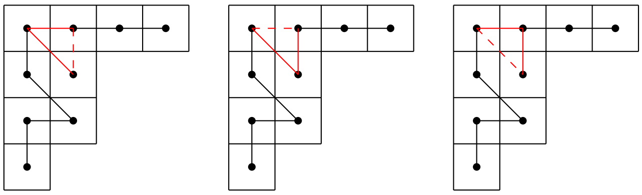

Figure 3: On the left a spanning tree T in Γ_λ for $\lambda = (4,2,2,1)$, in the middle $\sigma(T)$ and to the right $\sigma^2(T)$. The red edges belong to the unique upper cycle intersecting T along two edges.

the vertex incident to both e_1 and e_2. Hence by adding e_1 into the subgraph we obtain another spanning tree different from T which we denote by $\sigma(T)$. Since the upper cycles are pairwise edge-disjoint we get $C(\sigma(T)) = C(T)$ if $\sigma(T) \neq T$ and hence one easily sees that σ has order 3. Moreover it follows from (53) that those spanning trees for which $\sigma(T) \neq T$ cancel in the sum, which yields

$$\|M\| = \sum_{\substack{T\subset\Gamma_\lambda \\ \text{spanning tree} \\ \sigma(T)=T}} \prod_{\substack{e=(\Box,\Box') \\ \text{edge in } T}} w_e.$$

We now claim that those trees T with $\sigma(T) = T$ are the same as sets of non-diagonal edges which have exactly one edge in common with every upper cycle and contain each horizontal or vertical edge that is not part of an upper cycle. If we assume this to be true, then it follows from (54) that

$$\begin{aligned}
&\sum_{\substack{T\subset\Gamma_\lambda \\ \text{spanning tree} \\ \sigma(T)=T}} \prod_{\substack{e=(\Box,\Box') \\ \text{edge in } T}} w_e \\
&= \prod_{(i,j),(i,j+1)\in\lambda} w_{(i,j),(i,j+1)} \prod_{(i,j),(i+1,j)\in\lambda} w_{(i,j),(i+1,j)} \\
&\quad \cdot \prod_{\substack{C=(e_{hor},e_{vert},e_{diag}) \\ \text{upper cycle}}} (w_{e_{hor}}^{-1} + w_{e_{vert}}^{-1}) \\
&= \prod_{(i,j),(i,j+1)\in\lambda} w_{(i,j),(i,j+1)} \prod_{(i,j),(i+1,j)\in\lambda} w_{(i,j),(i+1,j)} \prod_{\substack{C=(e_{hor},e_{vert},e_{diag}) \\ \text{upper cycle}}} (-w_{e_{diag}}^{-1}),
\end{aligned}$$

which is what we wanted to show.

To show the characterization of trees with $\sigma(T) = T$ we first note that each subset as described above is a spanning tree of Γ_λ. Indeed, there are $\sum_{i=1}^{l(\lambda)-1}(\lambda_i - 1)$ many upper cycles and $l(\lambda) - 1 + \sum_{i=0}^{l(\lambda)-1}(\lambda_i - \lambda_{i+1})$ many edges not part of an upper cyle, hence any set of edges as described above has $|\lambda| - 1$ edges and is incident to all boxes, which makes it a spanning tree if it is connected. And indeed one easily sees that each box $(i,j) \in \lambda$ in the subgraph is connected to either $(i+1,j)$ or $(i,j+1)$ and hence each vertex is connected to $(l(\lambda)-1,0)$.

Conversely, the above calculation shows that each spanning tree T with $\sigma(T) = T$ must have exactly one edge in common with any upper cycle and must contain all edges not part of an upper edge. If it contained a diagonal edge e, then we could choose it so that there is no other diagonal edge below the diagonal line going through the vertices of e. Removing e from T would then make its two adjacent vertices (i,j) and $(i+1,j+1)$ lie in two distinct connected components. However, by the same argument as above, both boxes must be connected to $(l(\lambda)-1,0)$ which is a contradiction. This concludes the proof. $\square$

Next, we relate the Jacobian matrices of two kinds of Bethe equations arising during the proof of Theorem 2.14.

Lemma 2.22. For a fixed partition λ, $\mathbf{Y} = (Y_\square)_{\square\in\lambda}$ and $\mathbf{u} = (u_{\lambda/\mu})_{\lambda/\mu}$ free variables we set

$$M_{\text{Bethe}}(\mathbf{Y}) := \left(\frac{\partial F_{\square'}(\mathbf{Y})/\partial Y_\square}{F_{\square'}(\mathbf{Y})}\right)_{\square,\square'\in\lambda}$$

and

$$M_{\text{skew}}(\mathbf{u}) := \left(\frac{\delta_{\lambda/\mu,\lambda/\mu'}}{u_{\lambda/\mu}} + \frac{\partial \overline{F}^{\lambda}_{\lambda/\mu'}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))/\partial u_{\lambda/\mu}}{\overline{F}^{\lambda}_{\lambda/\mu'}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))}\right)_{\lambda/\mu,\lambda'/\mu'}.$$

We now claim that given $\mathbf{v} = (\mathbf{v}_\square)_{\square\in\lambda}$ and $\mathbf{u}(\mathbf{v}) = (u_{\lambda/\mu}(\mathbf{v}))_{\lambda/\mu}$ with $u_{\lambda/\mu}(\mathbf{v}) := \prod_{\square\in\lambda/\mu} v_\square$ we have

(55)
$$\prod_{\lambda/\mu} u_{\lambda/\mu}(\mathbf{v}) \cdot \left\|M_{\text{skew}}(\mathbf{u}(\mathbf{v}))\right\| = \prod_{\square\in\lambda} v_\square \cdot \left\|\left(\frac{\partial \widetilde{Y}^{\lambda}_{\square}(\mathbf{v})}{\partial v_{\square'}}\right)_{\square,\square'\in\lambda}\right\| \cdot \left\|M_{\text{Bethe}}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v}))\right\|$$

and for Q the matrix

$$Q = ([\square \in \lambda/\mu])_{\lambda/\mu,\square}$$

we have

$$Q^T \cdot M_{\text{skew}}(\mathbf{u}(\mathbf{v}))^{-1} \cdot Q = M_{\text{Bethe}}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v}))^{-1}.$$

Proof. From now on we abbreviate $M = M_{\text{skew}}(\mathbf{u}(\mathbf{v}))$ and $N = M_{\text{Bethe}}(\widetilde{\mathbf{Y}}^{\lambda}(\mathbf{v}))$. First we show the claim about determinants. Consider the matrix

$$([\square \geq \square'])_{\square,\square'\in\lambda}$$

with $[P]$ as in (51). Taking a total refinement of the partial ordering on boxes one can realize it as a lower triangular matrix with 1's on the diagonal and hence it is invertible of determinant 1. We will write its inverse as $A = (a_{\square,\square'})_{\square,\square'}$ and extend it to a matrix

$$
A' = \begin{array}{c} \\ \text{box} \\ \text{not box} \end{array} \begin{array}{c} \begin{array}{cc} \text{box} & \text{not box} \end{array} \\ \begin{pmatrix} A & 0 \\ 0 & \mathbb{I} \end{pmatrix} \end{array} \tag{56}
$$

where $\mathbb{I}$ is the identity matrix and "box" denotes the connected skew partitions of λ that are of the shape

$$
\overline{\square} \coloneqq \left\{ \square' \mid \square \leq \square' \right\}
$$

for some box $\square \in \lambda$ and "not box" the other ones. Since we can also write

$$
M = \left(\frac{\delta_{\lambda/\mu,\lambda/\mu'}}{u_{\lambda/\mu}} + \sum_{\square\in\lambda/\mu'} \frac{\partial \widetilde{F}^{\lambda}_{\square}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))/\partial u_{\lambda/\mu}}{\widetilde{F}^{\lambda}_{\square}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))} \right)_{\lambda/\mu,\lambda/\mu'}
$$

this gives

$$
M \cdot A' =
$$

$$
\begin{pmatrix} \left(\frac{a_{\square,\square'}}{u_{\overline{\square}}} + \frac{\partial \widetilde{F}^{\lambda}_{\square'}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))/\partial u_{\overline{\square}}}{\widetilde{F}^{\lambda}_{\square'}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))} \right)_{\square,\square'} & \left(\sum_{\square'\in\lambda/\mu} \frac{\partial \widetilde{F}^{\lambda}_{\square'}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))/\partial u_{\overline{\square}}}{\widetilde{F}^{\lambda}_{\square'}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))} \right)_{\square,\lambda/\mu} \\ \left(\frac{\partial \widetilde{F}^{\lambda}_{\square}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))/\partial u_{\lambda/\mu}}{\widetilde{F}^{\lambda}_{\square}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))} \right)_{\lambda/\mu,\square} & \left(\frac{\delta_{\lambda/\mu,\lambda/\mu'}}{u_{\lambda/\mu}} + \sum_{\square\in\lambda/\mu'} \frac{\partial \widetilde{F}^{\lambda}_{\square}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))/\partial u_{\lambda/\mu}}{\widetilde{F}^{\lambda}_{\square}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))} \right)_{\lambda/\mu,\lambda/\mu'} \end{pmatrix}
$$

where the blocks of the matrix are organized as in (56). Using further row operations we can get rid of most terms in the second i.e. "not box" row. More precisely, writing

$$
B = \begin{array}{c} \\ \text{box} \\ \text{not box} \end{array} \begin{array}{c} \begin{array}{cc} \text{box} & \text{not box} \end{array} \\ \begin{pmatrix} \mathbb{I} & (-[\square \in \lambda/\mu])_{\square,\lambda/\mu} \\ 0 & \mathbb{I} \end{pmatrix} \end{array}
$$

we get

$$
M \cdot A' \cdot B =
$$

$$
\begin{array}{c} \\ \text{box} \\ \text{not box} \end{array} \begin{array}{c} \begin{array}{cc} \text{box} & \text{not box} \end{array} \\ \begin{pmatrix} \left(\frac{a_{\square,\square'}}{u_{\overline{\square}}} + \frac{\partial \widetilde{F}^{\lambda}_{\square'}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))/\partial u_{\overline{\square}}}{\widetilde{F}^{\lambda}_{\square'}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))} \right)_{\square,\square'} & \left(\frac{1}{u_{\overline{\square}}} \sum_{\square'\in\lambda/\mu} a_{\square',\square} \right)_{\square,\lambda/\mu} \\ \left(\frac{\partial \widetilde{F}^{\lambda}_{\square}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))/\partial u_{\lambda/\mu}}{\widetilde{F}^{\lambda}_{\square}(\overline{\mathbf{Y}}^{\lambda}(\mathbf{u}))} \right)_{\lambda/\mu,\square} & \left(\frac{\delta_{\lambda/\mu,\lambda/\mu'}}{u_{\lambda/\mu}} \right)_{\lambda/\mu,\lambda/\mu'} \end{pmatrix} \end{array}
$$

Now we subsitute $\mathbf{u} = \mathbf{u}(\mathbf{v})$ into the matrix and using

$$\frac{\partial}{\partial v_\square} = \sum_{\square \in \lambda/\mu} \frac{u_{\lambda/\mu}}{v_\square} \frac{\partial}{\partial u_{\lambda/\mu}}$$

we see that

$$C \cdot M \cdot A' \cdot B = \begin{array}{c} \\ \text{box} \\ \text{not box} \end{array} \begin{array}{c} \begin{array}{cc} \text{box} & \text{not box} \end{array} \\ \begin{pmatrix} N' & \left(\frac{\partial \widetilde{F}^\lambda_\square(\overline{\mathbf{Y}}^\lambda(\mathbf{u}))/\partial u_{\lambda/\mu}}{\widetilde{F}^\lambda_\square(\overline{\mathbf{Y}}^\lambda(\mathbf{u}))} \right)_{\square,\lambda/\mu} \\ 0 & \left(\frac{\delta_{\lambda/\mu,\lambda/\mu'}}{u_{\lambda/\mu'}} \right)_{\lambda/\mu,\lambda/\mu'} \end{pmatrix} \end{array}$$

for

$$C = \begin{array}{c} \\ \text{box} \\ \text{not box} \end{array} \begin{array}{c} \begin{array}{cc} \text{box} & \text{not box} \end{array} \\ \begin{pmatrix} \left([\square \geq \square'] \frac{u_{\overline{\square'}}}{v_\square} \right)_{\square,\square'} & \left([\square \in \lambda/\mu] \frac{u_{\lambda/\mu}}{v_\square} \right)_{\square,\lambda/\mu} \\ 0 & \mathbb{I} \end{pmatrix} \end{array}$$

and

$$N' = \left(\frac{\delta_{\square,\square'}}{v_\square} + \frac{\partial \widetilde{F}^\lambda_{\square'}(\widetilde{\mathbf{Y}}^\lambda(\mathbf{v}))/\partial v_\square}{\widetilde{F}^\lambda_{\square'}(\widetilde{\mathbf{Y}}^\lambda(\mathbf{v}))} \right)_{\square,\square'}$$

Finally note that because of Lemma 2.20 we have $F_\square(\widetilde{\mathbf{Y}}^\lambda(\mathbf{v})) = v_\square \cdot \widetilde{F}^\lambda_\square(\widetilde{\mathbf{Y}}^\lambda(\mathbf{v}))$ and therefore

$$N' = D \cdot N$$

with

$$D = \left(\frac{\partial \widetilde{Y}^\lambda_{\square'}(\mathbf{v})}{\partial v_\square} \right)_{\square,\square' \in \lambda}.$$

Hence we get

$$\|M\| = \|C\|^{-1} \|CMA'B\| = \|D\| \, \|N\| \prod_{\lambda/\mu} u^{-1}_{\lambda/\mu} \prod_{\square \in \lambda} v_\square$$

which proves the first claim. For the second claim we need to show

$$Q^T M^{-1} Q = N^{-1}$$

for

$$Q = \begin{array}{c} \\ \text{box} \\ \text{not box} \end{array} \begin{array}{c} \text{box} \\ \begin{pmatrix} ([\square \leq \square'])_{\square,\square'} \\ ([\square \in \lambda/\mu])_{\lambda/\mu,\square} \end{pmatrix} \end{array}$$

Indeed, one can verify that

$$Q^T A' B = \begin{array}{c} \\ \text{box} \end{array} \begin{array}{c} \begin{array}{cc} \text{box} & \text{not box} \end{array} \\ \begin{pmatrix} \mathbb{I} & 0 \end{pmatrix} \end{array}$$

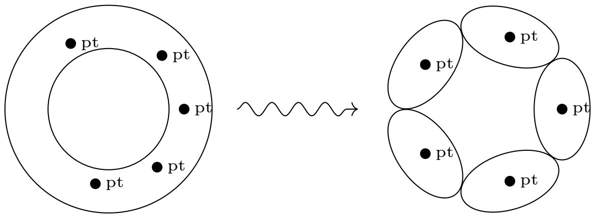

Figure 4: Degeneration of an elliptic curve to a circle of $n = 5$ copies of $\mathbb{P}^1$ joined end to end each receiving one marked point.

and

$$CQ = \begin{array}{cc} & \text{box} \\ \begin{array}{c}\text{box}\\ \text{not box}\end{array} & \begin{pmatrix} D \\ ([\square \in \lambda/\mu])_{\lambda/\mu,\square} \end{pmatrix} \end{array}$$

which implies

$$Q^T M^{-1} Q = Q^T A' B (CMA'B)^{-1} CQ = N^{-1}$$

as desired. $\square$

3 Stable pair theory of relative local curves

This section provides proofs and further details of the results in Section 1.2.2.

3.1 Proof of Theorem 1.26

We will now deduce Theorem 1.26 from Theorem 1.24.

Proof of Theorem 1.26. It suffices to show the claim in case $n = 1$ since

$$M(z_1, \dots, z_n) = M(z_1) \dots M(z_n).$$

Indeed, this follows from [48, §6] applied to the degeneration of $\mathbb{P}^1$ to a chain of n $\mathbb{P}^1$s glued end to end and each point class is lifted to the degneration space so that each Using the degeneration formula we have

and hence the claim only has to be shown for $n = 1$. Furthermore, we can degenerate $\mathbb{C}^2 \times E$ to a circle formed by copies of $\mathbb{C}^2 \times \mathbb{P}^1$ where $\mathbb{C}^2 \times \{\infty\}$ in each copy is identified with $\mathbb{C}^2 \times \{0\}$ inside the next copy - see Figure 4.. This gives

(57)

$$\mathrm{Tr}\big[M(z_1, \dots, z_n)\big] = \langle \mathrm{ch}_{z_1}(\mathrm{pt}) \dots \mathrm{ch}_{z_n}(\mathrm{pt}) \rangle_d^{\mathbb{C}^2 \times E, \mathbb{T}} = \sum_{\lambda \vdash d} \prod_{i=1}^{n} \sum_{\square \in \lambda} E(z_i, Y_\square^\lambda),$$

where we used Theorem 1.24 in the last equality. In particular, if we define M_k so that

$$\tfrac{t_1 t_2}{(1-e^{-t_1 z})(1-e^{-t_2 z})} M(z) = \sum_{k\geq 0} M_k \tfrac{z^k}{k!}$$

then we get

$$\mathrm{Tr}\left[M_k^n\right] = \sum_{\lambda \vdash d} \left(\sum_{\square\in\lambda} \left(Y_\square^\lambda\right)^k \right)^n .$$

It follows that the eigenvalues of M_k must be the sums

$$\sum_{\square\in\lambda} \left(Y_\square^\lambda\right)^k$$

for $\lambda \vdash d$. We now claim that one can choose k so that all of these power sums are distinct and the power sum corresponding to λ has an eigenvector of the shape $v_\lambda = [\lambda] + \mathcal{O}(p)$. For this we recall that

$$P_d(\mathbb{C}^2 \times \mathbb{P}^1/\{0,\infty\}, d) = \mathsf{Hilb}^d(\mathbb{C}^2)$$

and $M_k|_{p=0}$ corresponds to multiplication by $k! \cdot \mathrm{ch}_k(\pi_* \mathcal{O}_Z)$ where $Z \subset \mathbb{C}^2 \times \mathsf{Hilb}^d(\mathbb{C}^2)$ is the universal 0-dimensional subscheme. Furthermore, the fixed points $\lambda \in \mathsf{Hilb}^d(\mathbb{C}^2)$ form an eigenbasis for multiplication by any class. Since

$$H^0\left(\mathcal{O}_\lambda\right) = \bigoplus_{(i,j)\in\lambda} \mathbb{C} \cdot \mathbf{t}_1^{-i} \mathbf{t}_2^{-j}$$

it follows that $M_k|_{p=0}$ has eigenvalue $\sum_{(i,j)\in\lambda} (-it_1 - jt_2)^k$ at $[\lambda]$. To prove the distinctness on power sums we first specialize t_1, t_2 so that the $-it_1 - jt_2$ for $0 \leq i, j \leq d-1$ are distinct nonnegative numbers. One can show that any two sets of nonnegative numbers whose power sums agree for infinitely many powers must be equal[14] - hence any $k_0 \gg 0$ will work. As a result, M_{k_0} has simple spectrum since $M_{k_0}|_{p=0}$ has and is therefore diagonalizable. Furthermore, for any λ we have

$$\left. \sum_{\square\in\lambda} \left(Y_\square^\lambda\right)^{k_0} \right|_{p=0} = \sum_{(i,j)\in\lambda} (-it_1 - jt_2)^{k_0}$$

by Theorem 1.21(1) and so we must have $v_\lambda|_{p=0} = [\lambda]$ for an appropriate eigenvector v_λ with eigenvalue $\sum_{\square\in\lambda} \left(Y_\square^\lambda\right)^{k_0}$. It now remains to show that the v_λ form an eigenbasis of $M(z)$ with eigenvalues as claimed. For this we note that

$$M(z_1)M(z_2) = M(z_1, z_2) = M(z_2, z_1) = M(z_2)M(z_1)$$

[14] This follows from $\max(S) = \lim_{k\to\infty} \left(\sum_{i\in S} i^k\right)^{1/k}$ for any non-empty finite $S \subset \mathbb{R}_{\geq 0}$.

by degeneration and hence $M(z)$ commutes with M_{k_0}. Therefore the v_λ indeed diagonalize $M(z)$ and we can write $a_\lambda(z)$ for the eigenvalue at v_λ. It follows that

$$\mathrm{Tr}\big[M(z)M_{k_0}^n\big] = \sum_{\lambda\vdash d} a_\lambda(z)\left(\sum_{\square\in\lambda}\left(Y_\square^\lambda\right)^{k_0}\right)^n$$

for any $n \geq 0$. But by (57) this must also equal

$$\sum_{\lambda\vdash d}\left(\sum_{\square\in\lambda} E(z, Y_\square^\lambda)\right)\left(\sum_{\square\in\lambda}\left(Y_\square^\lambda\right)^{k_0}\right)^n$$

and as the eigenvalues of M_{k_0} are pairwise distinct, this implies

$$a_\lambda(z) = \sum_{\square\in\lambda} E(z, Y_\square^\lambda)$$

by the invertibility of the Vandermonde matrix. □

Remark 3.1. One could have shortened the proof somewhat by using that M_3 has simple spectrum [78, Proof of Corollary 1]. Recalling that stable pairs on $\mathbb{C}^2 \times C$ are the same thing as quasi-maps from C to $\mathsf{Hilb}^d(\mathbb{C}^2)$ (c.f. [73, Exercise 4.3.22]), this corresponds to the fact that the quantum cohomology of $\mathsf{Hilb}^d(\mathbb{C}^2)$ is generated by divisors. However, we chose to circumvent this fact since the analogous claim for general Nakajima quiver varieties is still a conjecture [58, Question 1]. This makes it possible to repeat the above proof for quasi-maps to quiver varieties, which gives a new proof of the fact that the spectrum of quantum multiplication is described by solutions of Bethe equations. See [97] for more details and consequences.

3.2 Proof of Theorem 1.27 and Theorem 1.29

We will start by proving a version of Theorem 1.27 for the Gromov-Witten theory of the tube i.e.

$$\langle\mu|\tau_k(\mathrm{pt})|\nu\rangle^* := \langle\mu|\tau_k(\mathrm{pt})|\nu\rangle^{\mathbb{C}^2\times\mathbb{P}^1/\{0,\infty\},\mathrm{GW},\mathbb{T},*},$$

where μ and ν are unordered partitions, $* \in \{\circ, \bullet\}$ and the right hand side is defined as in Definition 4.2 and Remark 4.3(3). By Proposition 4.4, one has

(58)
$$\langle\mu|\tau_k(\mathrm{pt})|\nu\rangle_g^{\mathbb{C}^2\times\mathbb{P}^1/\{0,\infty\},\mathrm{GW},\mathbb{T},\circ} = \frac{1}{t_1t_2}\left\langle\mu|\mathbb{E}^\vee(t_1)\mathbb{E}^\vee(t_2)\tau_k(\mathrm{pt})|\nu\right\rangle_g^{\mathbb{P}^1/\{0,\infty\},\mathrm{GW},\circ}.$$

We then define M_k^{GW} to be the matrix with entries

$$\left(M_k^{\mathrm{GW}}\right)_{\mu,\nu} = (-t_1t_2z^2)^{\ell(\mu)}\mathfrak{z}(\mu)\,\langle\mu|\tau_k(\mathrm{pt})|\nu\rangle^\bullet,$$

where the prefactor is chosen so that degeneration becomes matrix multiplication like in the proof of Theorem 1.26.

Lemma 3.2. We have

$$M_k^{\mathrm{GW}} = \sum_{\substack{|\mu|=|\nu| \\ 2\le \ell(\mu)+\ell(\nu)\le k+2}} (-t_1t_2z^2)^{\ell(\mu)} \langle \mu|\tau_k(\mathrm{pt})|\nu\rangle^\circ \alpha_{-\mu}\alpha_\nu, \tag{59}$$

where the connected invariants satisfy:

(1) $\langle \mu|\tau_k(\mathrm{pt})|\nu\rangle^\circ \in \frac{1}{t_1t_2}\cdot \mathbb{Q}((z))[t_1,t_2]$ is homogeneous of degree $k-\ell(\mu)-\ell(\nu)$ in t_1,t_2.

(2) We have

$$\begin{aligned} &-t_1t_2z^2\langle (d)|\tau_k(\mathrm{pt})|(d)\rangle^\circ \\ &= (t_1+t_2)^k[t^k]\sum_{r\ge 1}(dz)^{r-1}\frac{e^{drz}-1}{(e^{dz}-1)^r}\frac{t^{r-1}}{(t+1)\cdots(t+r)} + \mathcal{O}(t_1t_2) \end{aligned}$$

up to terms in $t_1t_2\cdot\mathbb{Q}((z))[t_1,t_2]$.

Proof. It follows from standard considerations that connected and disconnected one-point invariants are related in the following way

$$\langle \mu|\tau_k(\mathrm{pt})|\nu\rangle^\bullet = \sum_{\lambda\le\mu,\nu}\frac{(-t_1t_2z^2)^{-\ell(\lambda)}}{\mathfrak{z}(\lambda)}\langle \mu-\lambda|\tau_k(\mathrm{pt})|\nu-\lambda\rangle^\circ,$$

where we write $\lambda\le\mu$ if $m_l(\lambda)\le m_l(\mu)$ for all l and $\mu-\lambda$ is the unique partition with $m_l(\mu-\lambda)=m_l(\mu)-m_l(\lambda)$ for all l. Here, we used the notation

$$m_l(\lambda)=\#\{i\mid \lambda_i=l\}.$$

On the other hand, one can show that

$$\sum_{\rho,\sigma}c_{\rho,\sigma}\langle\mu|\alpha_{-\rho}\alpha_\sigma|\nu\rangle = \sum_{\lambda\le\mu,\nu}\frac{c_{\mu-\lambda,\nu-\lambda}}{\mathfrak{z}(\lambda)}$$

for any set of constants $(c_{\rho,\sigma})_{\rho,\sigma}$. Since $\left(M_k^{\mathrm{GW}}\right)_{\mu,\nu}=\mathfrak{z}(\mu)\langle\mu|M_k^{\mathrm{GW}}|\nu\rangle$, this implies that

$$M_k^{\mathrm{GW}} = \sum_{\mu,\nu}(-t_1t_2z^2)^{\ell(\mu)}\langle\mu|\tau_k(\mathrm{pt})|\nu\rangle^\circ\alpha_{-\mu}\alpha_\nu$$

From (1) it follows that the connected invariants are 0 unless $\ell(\mu)+\ell(\nu)\le k+2$, so to get (59) it suffices to show (1). This however follows directly from (58). To show (2), we use the degeneration formula

$$\langle\tau_k(\mathrm{pt})\rangle^{\mathbb{C}^2\times E,\mathrm{GW},\mathbb{T},\bullet} = \sum_\mu q^{|\mu|}(-t_1t_2z^2)^{\ell(\mu)}\mathfrak{z}(\mu)\langle\mu|\tau_k(\mathrm{pt})|\mu\rangle^\bullet$$

$$= \sum_{\mu,\lambda} q^{|\mu|+|\lambda|}(-t_1t_2z^2)^{\ell(\lambda)}\frac{\mathfrak{z}(\lambda+\mu)}{\mathfrak{z}(\mu)}\langle\lambda|\tau_k(\mathrm{pt})|\lambda\rangle^\circ$$

arising from the degeneration of the elliptic curve E to the rational nodal curve. We can isolate the summands coming from

$$c_d := [t_1^k](-t_1t_2z^2)\langle(d)|\tau_k(\mathrm{pt})|(d)\rangle^\circ$$

by looking at the t_1^k coefficient, which is

$$\begin{aligned}
[t_1^k]\langle\tau_k(\mathrm{pt})\rangle^{\mathbb{C}^2\times E,\mathrm{GW},\mathbb{T},\bullet} &= \sum_{\mu,d} q^{d+|\lambda|}\frac{\mathfrak{z}((d)+\mu)}{\mathfrak{z}(\mu)}c_d \\
&= \sum_{(m_l)_{l\geq 1},d} q^{d+\sum_l l\cdot m_l}(m_l+1)dc_d \\
&= \prod_{d\geq 1}\frac{1}{1-q^d}\sum_{d\geq 1} dc_d\frac{q^d}{1-q^d}
\end{aligned}$$

and hence

$$[t_1^k]\langle\tau_k(\mathrm{pt})\rangle_d^{\mathbb{C}^2\times E,\mathrm{GW},\mathbb{T},\circ} = \sum_{m|d} mc_m. \tag{60}$$

Furthermore, one has

$$\begin{aligned}
&\langle\tau_k(\mathrm{pt})\rangle_d^{\mathbb{C}^2\times E,\mathrm{GW},\mathbb{T},\circ} \\
&= \sum_{\substack{g\geq 0\\ a,b\geq 1\\ a+b=k+2}} (-1)^{g-1+a+b}t_1^{a-1}t_2^{b-1}z^{2g-2}\left\langle\lambda_{g-a}\lambda_{g-b}\tau_k(\mathrm{pt})\right\rangle_{g,d}^{E,\mathrm{GW},\circ},
\end{aligned}$$

where the summands with $a=0$ or $b=0$ were excluded as they vanish due to [91, Lemma 4.4.1]. We can therefore replace the left hand side of (60) by a Hodge integral, which leads to

$$\sum_{g\geq 0} z^{2g-2}\left\langle\lambda_{g-1}\mathbb{E}^\vee(1)\tau_k(\mathrm{pt})\right\rangle_{g,d}^E = \sum_{m|d} mc_m.$$

The left hand side can be expressed as

$$\left\langle\lambda_{g-1}\mathbb{E}^\vee(1)\tau_k(\mathrm{pt})\right\rangle_{g,d}^E = \int_{\overline{M}_{g,1}} \lambda_{g-1}\mathbb{E}^\vee(1)\psi_1^k\mathcal{C}_{g,d}(\mathrm{pt}),$$

where we denoted the Gromov-Witten classes of the elliptic curve by

$$\mathcal{C}_{g,d}(\gamma_1,\ldots,\gamma_n) = \pi_*\left([\overline{M}_{g,n}(E,d)]^{vir}\cap \mathrm{ev}_1^*(\gamma_1)\cdots\mathrm{ev}_n^*(\gamma_n)\right) \tag{61}$$

for $\pi\colon \overline{M}_{g,n}(E,d) \to \overline{M}_{g,n}$ the forgetful map. Here we used that $\pi^*(\psi_1) = \psi_1$ without boundary terms as all maps from rational curves to E are constant. Using [70, Example 6.9], we get

$$\int_{\overline{M}_{g,1}} \lambda_{g-1}\mathbb{E}^\vee(1)\psi_1^k \mathcal{C}_{g,d}(\mathrm{pt}) = (-1)^{g-1}\frac{4g}{B_{2g}}\sum_{m|d} m^{2g-1}\int_{\overline{M}_{g,1}} \lambda_g\lambda_{g-1}\mathbb{E}^\vee(1)\psi_1^k.$$

We hence have

$$\begin{aligned}
c_d &= \sum_{g\geq 0}(-1)^{g-1}\frac{4g}{B_{2g}}(dz)^{2g-2}\int_{\overline{M}_{g,1}} \lambda_g\lambda_{g-1}\mathbb{E}^\vee(1)\psi_1^k \\
&= \sum_{g\geq 0}(-1)^{g}\frac{4g}{B_{2g}}(dz)^{2g-2}\frac{|B_{2g}|}{2g}[t^k x^{2g-1}]\sum_{r\geq 1}\left(\frac{x}{e^x-1}\right)^r \frac{t^{r-1}}{(t+1)\cdots(t+r)} \\
&= [t^k]\sum_{r\geq 1}(dz)^{r-1}\frac{e^{drz}-1}{(e^{dz}-1)^r}\frac{t^{r-1}}{(t+1)\cdots(t+r)},
\end{aligned}$$

where we used [25, Theorem 3] and [25, Theorem 3 in Appendix] in the second equality and replaced $\left(\frac{x}{e^x-1}\right)^r$ by

$$\frac{1}{2}\left(\left(\frac{x}{e^x-1}\right)^r - \left(\frac{-x}{e^{-x}-1}\right)^r\right) = \frac{x^r}{2}\frac{1-e^{rx}}{(e^x-1)^r}$$

in the third equality to kill all even powers of x. This concludes the proof. □

Using this, we will now prove Theorem 1.27.

Proof of Theorem 1.27. By [86, §3], the relative GW/PT correspondence holds for $\mathbb{C}^2\times\mathbb{P}^1$ relative to $\{0,\infty\}$. Together with degeneration, this implies that under the variable change $p = -e^z$:

$$M_k = \sum_{k_1,\dots,k_n\geq 0} \widetilde{K}_{(k-2),(k_1,\dots,k_n)}(t_1t_2)^{n-1}z^{-L}M^{\mathrm{GW}}_{k_1}\cdots M^{\mathrm{GW}}_{k_n}z^L, \tag{62}$$

where $\widetilde{K}$ is the correspondence matrix of Section 1.1.5 evaluated at $c_i = c_i(T_{\mathbb{C}^2\times\mathbb{P}^1}[-0-\infty]) = c_i(T_{\mathbb{C}^2}\oplus\mathcal{O})$ i.e.

$$c_1 = t_1+t_2,\ c_2 = t_1t_2 \text{ and } c_3 = 0$$

and L is the linear operator determined by

$$L|\mu\rangle = \ell(\mu)|\mu\rangle.$$

Note also that the power $(t_1t_2)^{n-1}$ comes from the equivariant diagonal class of $\mathbb{C}^2$. This implies that M_k is indeed of the shape (25), where the bound

$\ell(\mathbf{v}) \leq k$ holds because $\widetilde{K}_{(k-2),(k_1,\ldots,k_n)} = 0$ unless $k \geq \sum_i (k_i + 2)$, which follows from Proposition 1.11(2). Furthermore, Lemma 3.2(1) implies that $f^k(\mathbf{v})$ must be a symmetric polynomial in t_1, t_2 of degree $k - \ell(\mathbf{v})$.

To prove Theorem 1.27(2), we observe that any term of the shape $c \cdot \alpha_{-v}\alpha_v$ in (62) arises either if $n = 1$ or if $n > 1$ and several α's annihilated via the commutation relation (22). However, the latter case only contributes terms to $f^k(v, -v)$ that are divisible by $t_1 t_2$ and hence can be ignored. By setting $t_1 = 1$ and $t_2 = 0$ to weed out terms divisible by $t_1 t_2$, it follows that

$$\begin{aligned} f^k_{0,k-2}(v, -v) &= \sum_{k' \geq 0} \widetilde{K}_{(k-2),k'}\Big|_{c_1=1, c_2=c_3=0} \\ &\quad \cdot [t^{k'}] \sum_{r \geq 1} (vz)^{r-1} \frac{(-p)^{vr} - 1}{((-p)^v - 1)^r} \frac{t^{r-1}}{(t+1)\cdots(t+r)} \\ &= \frac{v^{k-2}}{(k-1)!} \frac{(-p)^{(k-1)v} - 1}{((-p)^v - 1)^{k-1}}, \end{aligned}$$

where the last equality uses Lemma 3.4. To show Theorem 1.27(3), we first observe that M_k is independent of p in case we specialize $t_1 + t_2 = 0$. Indeed, via a cosection argument as in [84, §4.3] one can see that the virtual class on $P_n(\mathbb{C}^2 \times \mathbb{P}^1/\{0, \infty\}, d)$ vanishes if $n > d$. Moreover, as observed in the proof of Theorem 1.26 above, we have

$$P_d(\mathbb{C}^2 \times \mathbb{P}^1/\{0, \infty\}, d) = \mathsf{Hilb}^d(\mathbb{C}^2)$$

and the fixed points give a basis $[\lambda] \in H^*_{\mathbb{T}}(\mathsf{Hilb}^d(\mathbb{C}^2))$ indexed by partitions $\lambda \vdash d$ so that $M(z)|_{p=0}$ has eigenvalue

$$\frac{(1 - e^{-t_1 z})(1 - e^{-t_2 z})}{t_1 t_2} \sum_{(i,j) \in \lambda} e^{(-it_1 - jt_2)z}$$

at $[\lambda]$. The restriction $M(z)|_{t_1 = -t_2 = 1}$ therefore has eigenvalue

$$\varsigma(z) \cdot \sum_{i \geq 0} e^{(\lambda_i - i - \frac{1}{2})z} - 1$$

at $[\lambda]|_{t_1 = -t_2 = 1}$, where

$$\varsigma(z) = e^{z/2} - e^{-z/2}. \tag{63}$$

Furthermore, [49, 54] tell us that the restriction $[\lambda]|_{t_1 = -t_2 = 1}$ is proportional to the fermionic Fock space basis (which corresponds to the Schur functions). By comparing eigenvalues and [76, (2.9)] it hence follows that

$$M(z)|_{t_1 = -t_2 = 1} = \varsigma(z)\mathcal{E}_0(z) - 1,$$

where $\mathcal{E}_0(z)$ is the operator defined in [76, §2.2.1]. More generally, there are operators $\mathcal{E}_r(z)$ for any $r \in \mathbb{Z}$ which are determined by their commutation relation

$$[\alpha_k, \mathcal{E}_r(z)] = \varsigma(kz)\mathcal{E}_{k+r}(z)$$

and the vacuum expectation

$$\langle\emptyset|\mathcal{E}_r(z)|\emptyset\rangle = \frac{\delta_{r,0}}{\varsigma(z)}.$$

As a result of this, it is straightforward to show

$$\langle\mu|\, M(z)|_{t_1=-t_2=1}\, |\nu\rangle = \sum_{\lambda\leq\mu,\nu} \frac{\prod_i zS((\mu-\lambda)_i z)\cdot\prod_i zS((\mu-\lambda)_i z)}{\mathfrak{z}(\lambda)\cdot|\mathrm{Aut}(\mu-\lambda)|\cdot|\mathrm{Aut}(\nu-\lambda)|},$$

where

$$S(z) = \frac{\varsigma(z)}{z} \tag{64}$$

and similar to the proof of Lemma 3.2, we get

$$\begin{aligned} M(z)|_{t_1=-t_2=1} &= \sum_{\mu,\nu}\prod_i zS(\mu_i z)\prod_i zS(\nu_i z)\frac{\alpha_{-\mu}\alpha_\nu}{|\mathrm{Aut}(\mu)|\cdot|\mathrm{Aut}(\nu)|} \\ &= \sum_{\mathbf{v}\in\mathbb{V}^l} z^l \prod_i S(v_i z)\frac{\alpha_{\mathbf{v}}}{l!} \end{aligned}$$

Taking z^k-coefficients therefore gives

$$f_{a,0}(\mathbf{v}) = (-1)^a[z^{2a}]\prod_i S(v_i z) = \left(-\frac{1}{4}\right)^a \sum_{\substack{n_l\geq 0\\ \sum_l n_l = a}} \prod_l \frac{v_l^{2n_l}}{(2n_l+1)!}$$

as desired. □

Remark 3.3. Denote by $\mathcal{V} = \pi_*\mathcal{O}_{\mathcal{Z}}$ the universal bundle of rank n on $\mathsf{Hilb}^n(\mathbb{C}^2)$, where $\mathcal{Z} \subset \mathbb{C}^2 \times \mathsf{Hilb}^n(\mathbb{C}^2)$ is the universal subscheme and

$$\pi\colon \mathbb{C}^2 \times \mathsf{Hilb}^n(\mathbb{C}^2) \to \mathsf{Hilb}^n(\mathbb{C}^2)$$

is the projection to the second factor. The last part of the above proof then expresses the cup products

$$\mathrm{ch}_k(\mathcal{V})\cup\colon H^*_{\mathbb{T}}(\mathsf{Hilb}^n(\mathbb{C}^2)) \to H^*_{\mathbb{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))$$

for $t_1 = -t_2$ in terms of Nakajima operators. Indeed, this is already known and the above argument appears for example in [72, Proof of Proposition 12]. We nonetheless included it since we were unable to find a canonical reference.

In the above proof, we relied on the following special case of the GW/PT correspondence.

Lemma 3.4. We have

$$\widetilde{K}_{(k),(l)}\Big|_{c_2=c_3=0} = c_1^{k-l} z^{-k} \sum_{1\leq i_1<\ldots<i_{k-l}\leq k} \frac{1}{i_1\cdots i_{k-l}}$$

if $k \geq l$ and $\widetilde{K}_{(k),(l)}\Big|_{c_2=c_3=0} = 0$ otherwise. Furthermore

$$\sum_{l=0}^{k} \widetilde{K}_{(k),(l)}\Big|_{c_2=c_3=0} [t^l] \sum_{r=1}^{k+1} \frac{(zt)^{r-1}}{(1+tc_1)\cdots(r+tc_1)} a_r = \frac{a_{k+1}}{(k+1)!}$$

for all $a_1, \ldots, a_{k+1}$.

Proof. It suffices to show the second identity as it implies the first one by [29, Appendix]. For that we use the description of the correspondence matrix $\widetilde{K}$ in case $c_3 = c_1c_2$ given in [63, 72]. This involves a set of formal descendents $\mathfrak{a}_k(\gamma)$ on the Gromov-Witten side, which in case $c_2 = 0$ is determined by

$$\sum_{r\geq 0} t^r \tau_r(\gamma) = \sum_{r>0} \frac{(tz)^{r-1}}{(1+tc_1)\cdots(r+tc_1)} \mathfrak{a}_r(\gamma)$$

and by [63, (1.14)], we must[15] have

$$\sum_{l=0}^{k} \widetilde{K}_{(k),(l)}\Big|_{c_2=c_3=0} \tau_l(\gamma) = \frac{\mathfrak{a}_{k+1}(\gamma)}{(k+1)!},$$

which concludes the proof. □

We are now ready to give a proof of Theorem 1.29.

Proof of Theorem 1.29. It is straightforward to verify that our formulas satisfy all conclusions of Theorem 1.27. By Lemma 3.6, this implies that they must also have the correct $f^k_{0,k-3}$, which proves our formulas for $M_{\leq 4}$. For M_5, we can again use Lemma 3.6 to see that it suffices to check that we have the correct $f^5_{0,1}(1,1,-1,-1)$ and $f^5_{1,1}(1,-1)$. Since these values are determined by $M_5|_{\mathsf{Hilb}^{\leq 2}}$, we can use degeneration and [83, Theorem 1] to express these in terms of $M_{<5}|_{\mathsf{Hilb}^{\leq 2}}$, which we already know. This concludes the proof. □

Remark 3.5. The above strategy would not allow one to prove any potential formula for M_6. Indeed, one can show that Theorem 1.27 in this case still leaves infinitely many degrees of freedom undetermined.

[15] Indeed, the computation in [63, §9.2.1] shows that modulo c_2 all contributions but one are of the shape $\mathfrak{a}_{k_1}\cdots\mathfrak{a}_{k_n}$ for $n \geq 2$ - even in the equivariant case.

However, Theorem 1.27 does determine the $f^k_{a,b}$ to the following extent:

Lemma 3.6. Let $k \geq 0$ and $\widetilde{f}^k_{a,b}(\mathbf{v}) \in \mathbb{Q}(p)$ be another set of rational functions indexed by a, b and symmetric in $\mathbf{v} \in \mathbb{V}^{k-2a-b}$. If it satisfies Theorem 1.27(1),(2) i.e.

(1) for any $a, b, \mathbf{v}$

$$\widetilde{f}^k_{a,b}(\mathbf{v}) \sum_i v_i^2 \frac{(-p)^{v_i}+1}{(-p)^{v_i}-1} = \sum_{\substack{i \\ s+t=v_i \\ \mathrm{sgn}(s)=\mathrm{sgn}(t)}} \mathrm{sgn}(v_i) st \widetilde{f}^k_{a-1,b+1}(\mathbf{v} \setminus (v_i) \cup (s,t))$$
$$- \sum_{i \neq j} (v_i + v_j) \widetilde{f}^k_{a,b+1}(\mathbf{v} \setminus (v_i, v_j) \cup (v_i + v_j)).$$

(2) for any $v > 0$

$$\widetilde{f}^k_{0,k-2}(v, -v) = \frac{v^{k-2}}{(k-1)!} \frac{(-p)^{(k-1)v} - 1}{((-p)^v - 1)^{k-1}}$$

then it follows that $\widetilde{f}^k_{0,b}(\mathbf{v}) = f^k_{0,b}(\mathbf{v})$ for all b and generic $\mathbf{v}$. Here, we call $\mathbf{v}$ *generic* if there is no proper subset $\emptyset \neq S \subsetneq \{1, \ldots, k-b\}$ with $\mathbf{v}_S = \sum_{i \in S} v_i = 0$.

Moreover, if for any fixed a and b we know that

- $\widetilde{f}^k_{a-1,b} = f^k_{a-1,b}$ and $\widetilde{f}^k_{a,b-1} = f^k_{a,b-1}$.
- $\widetilde{f}^k_{a,b}(\mathbf{v}) = f^k_{a,b}(\mathbf{v})$ for all $\mathbf{v}$ off-diagonal i.e. not of the shape $\mathbf{v} = (v_1, \ldots, v_{\frac{k-2a-b}{2}}, -v_1, \ldots, -v_{\frac{k-2a-b}{2}})$ up to permutation.
- $\widetilde{f}^k_{a,b}(\mathbf{1}) = f^k_{a,b}(\mathbf{1})$ where $\mathbf{1} = (1^{\frac{k-2a-b}{2}}, (-1)^{\frac{k-2a-b}{2}})$ i.e. 1 and -1 occur $\frac{k-2a-b}{2}$ many times each.

then we must have $\widetilde{f}^k_{a,b} = f^k_{a,b}$.

Proof. We prove the first claim by downward induction on b, where the case $b = k-2$ holds by (2). Note that any join of a generic $\mathbf{v}$ is still generic. For $b < k-2$ we can therefore use induction for the right hand side of (1) to get

$$\widetilde{f}^k_{0,b}(\mathbf{v}) \sum_i v_i^2 \frac{(-p)^{v_i}+1}{(-p)^{v_i}-1} = f^k_{0,b}(\mathbf{v}) \sum_i v_i^2 \frac{(-p)^{v_i}+1}{(-p)^{v_i}-1}.$$

By looking at poles one can see that $\sum_i v_i^2 \frac{(-p)^{v_i}+1}{(-p)^{v_i}-1} = 0$ if and only if $\mathbf{v} = (v_1, \ldots, v_{\frac{k-b}{2}}, -v_1, \ldots, -v_{\frac{k-b}{2}})$ up to permutation - in particular, since $\mathbf{v}$ is generic and $b < k-2$ we get $\widetilde{f}^k_{0,b}(\mathbf{v}) = f^k_{0,b}(\mathbf{v})$ as desired.

For the second claim, we note that (1) gives us

$$\sum_{i\neq j}(v_i{+}v_j)\widetilde{f}^k_{a,b+1}(\mathbf{v}\backslash(v_i,v_j)\cup(v_i{+}v_j)) = \sum_{i\neq j}(v_i{+}v_j)f^k_{a,b+1}(\mathbf{v}\backslash(v_i,v_j)\cup(v_i{+}v_j)) \tag{65}$$

for any $\mathbf{v}$. From this it suffices to deduce $\widetilde{f}^k_{a,b}(\lambda,-\lambda) = f^k_{a,b}(\lambda,-\lambda)$ for λ a partition of length $k-2a-b$. We will do this by induction on $|\lambda|$, the case $|\lambda| = k-2a-b$ holding by assumption. Now, assume that the claim has been shown for λ' with $|\lambda'| < |\lambda|$ and let i_0 be maximal so that $\lambda_{i_0} > 1$. We then let

$$\mathbf{v} = (\lambda_0, \ldots, \lambda_{i_0+1}, \lambda_{i_0} - 1, 1^{m_1(\lambda)+1}, -\lambda).$$

For this choice of $\mathbf{v}$, (65) boils down to $\widetilde{f}^k_{a,b}(\lambda,-\lambda) = f^k_{a,b}(\lambda,-\lambda)$ modulo off-diagonal and lower degree summands, which concludes the proof. □

4 Gromov-Witten theory of local curves

This section contains the proofs of the results stated in Section 1.3. In the process, we also derive formulas for certain Gromov-Witten invariants of the tube geometry as well as certain triple Hodge integrals on the moduli space of curves. See Theorem 4.14 and Corollary 4.16 respectively.

4.1 Generalities

We start by giving further details on the discussion at the beginning of Section 1.3. For that, we will first recall the precise connection between the three different kinds of Gromov-Witten theory considered in this thesis. Let $X, \mathbb{T}$ be as in Situation 1 and $\gamma_1, \ldots, \gamma_n \in H^*_{\mathbb{T}}(X)$ be homogeneous generators of $H^*_{\mathbb{T}}(X)$ as a $H^*_{\mathbb{T}}(\mathrm{pt})$-module. We then have the following formal descendent insertion

$$\gamma = \sum_{\substack{i\geq 0\\ 1\leq j\leq n}} t_{i,j}\tau_i(\gamma_j)$$

with $t_{i,j}$ free variables satisfying

$$t_{i_1,j_1}t_{i_2,j_2} = (-1)^{\deg(\gamma_{j_1})\cdot\deg(\gamma_{j_2})}t_{i_2,j_2}t_{i_1,j_1}.$$

There are three Gromov-Witten partition functions defined by

$$\mathcal{Z}^* = \sum_{\substack{g\in\mathbb{Z}\\ \beta\in\mathrm{Eff}(X)}} (-1)^{g-1}z^{2g-2}q^\beta \left\langle \exp(\gamma)\right\rangle^{X,\mathbb{T},*}_{g,\beta}$$

for $* \in \{\circ, \bullet, \prime\}$ and $\mathrm{Eff}(X) \subset H_2(X,\mathbb{Z})$ the submonoid generated by effective curve classes.

Proposition 4.1. We have

$$\mathcal{Z}^\bullet = \exp(\mathcal{Z}^\circ)$$
$$\mathcal{Z}^\bullet = \mathcal{Z}_\emptyset \cdot \mathcal{Z}',$$

where $\mathcal{Z}_\emptyset = \mathcal{Z}^\bullet|_{t_{i,j}=0}$.

Proof. One can show that

$$\overline{M}^\bullet_{g,n}(X,\beta) = \coprod_{\substack{\coprod_{s=1}^l I_s=\{1,\ldots,n\} \\ \sum_{s=1}^l g_s=g+l-1 \\ \sum_{s=1}^l \beta_s=\beta}} \prod_{s=1}^{l} \overline{M}^\circ_{g_s,I_s}(X,\beta_s)/\mathrm{Aut}((g_s,n_s,\beta_s)_{s,I_s=\emptyset}) \tag{66}$$

and same for $\overline{M}'_{g,n}(X,\beta)$, where one additionally requires $I_s \neq \emptyset$ for all s. The identities for the $\mathcal{Z}^*$ follow from the fact that the virtual classes of both sides of (66) agree. □

Up to the datum of $\mathcal{Z}_\emptyset$ all three flavours of Gromov-Witten theory are thus equivalent to each other.

For computational purposes we will also need to use *relative Gromov-Witten theory*. For this, let X be smooth and projective and $D \subset X$ a smooth divisor with components $D_1,\ldots,D_m$. Furthermore, let $\beta \in H_2(X)$ and $\mathbf{a}_1,\ldots,\mathbf{a}_m$ be ordered partitions of same size $(\beta.D)$. For $* \in \{\circ,\bullet,\prime\}$ we denote by $\overline{M}^*_{g,n}(X/D,\beta,\mathbf{a}_1,\ldots,\mathbf{a}_m)$ the moduli space (c.f. [45, 46]) of n-pointed genus g *relative stable maps* with appropriate connectedness condition to a target expansion of X with contact profile $\mathbf{a}_i$ along D_i. This space comes equipped with a virtual class

$$\left[\overline{M}^*_{g,n}(X/D,\beta,\mathbf{a}_1,\ldots,\mathbf{a}_m)\right]^{\mathrm{vir}} \in H_{2\cdot \mathrm{vdim}}(\overline{M}_{g,n}(X/D,\beta,\mathbf{a}_1,\ldots,\mathbf{a}_m))$$

in complex degree

$$\mathrm{vdim} = (3-\dim X)(g-1) + \int_\beta (c_1(X)-D) + n + \ell(\mathbf{a}_1) + \ldots + \ell(\mathbf{a}_m).$$

As in the absolute case, there are evaluation maps at the markings

$$\mathrm{ev}_i \colon \overline{M}^*_{g,n}(X/D,\beta,\mathbf{a}_1,\ldots,\mathbf{a}_m) \to X$$

for $i=1,\ldots,n$. Therefore we can define

Definition 4.2. Let $\gamma_1,\ldots\gamma_n \in H^*(X)$, $k_1,\ldots,k_n \geq 0$ and $\gamma \in H^*(\overline{M}^*_{g,n}(X/D,\beta,\mathbf{a}_1,\ldots,\mathbf{a}_m))$. We set

$$\langle \gamma \cdot \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)|\,\mathbf{a}_1,\ldots,\mathbf{a}_m\rangle^{X/D,*}_{g,\beta}$$

$$:= \int_{\left[\overline{M}^*_{g,n}(X/D,\beta,\mathbf{a}_1,\ldots,\mathbf{a}_m)\right]^{\mathrm{vir}}} \gamma \cdot \prod_{i=1}^{n} \psi_i^{k_i} \operatorname{ev}_i^*(\gamma_i)$$

for $* \in \{\circ, \bullet, \prime\}$. If $\mu_1, \ldots, \mu_m$ are unordered partitions of same size $(\beta.D)$, then we define the following unordered version of relative invariants

$$\begin{aligned} &\langle \gamma \cdot \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) | \, \mu_1, \ldots, \mu_m \rangle^{X/D,*}_{g,\beta} \\ &:= \prod_{i=1}^{m} \frac{1}{|\mathrm{Aut}(\mu_i)|} \langle \gamma \cdot \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) | \, \boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m \rangle^{X/D,*}_{g,\beta}. \end{aligned}$$

Remark 4.3. (1) Similar to Definition 1.6, if X is in Situation 1, then this definition can also be carried out equivariantly. For $D = \emptyset$, this recovers the usual non-relative Gromov-Witten invariants.

(2) The above discussion can be carried out more generally. In particular, Proposition 4.1 also works in the relative case and one could put insertions on the relative markings in Definition 4.2. However, we will use neither generalization.

(3) In accordance with the stable pair case, we will also denote the invariants of local $\mathbb{C}^2 \times \mathbb{P}^1$ relative to 0 and ∞ by

$$\langle \mathbf{a} | \cdots | \mathbf{b} \rangle^{\mathbb{C}^2 \times \mathbb{P}^1/\{0,\infty\},\mathbb{T},*} = \langle \cdots | \, \mathbf{a}, \mathbf{b} \rangle^{\mathbb{C}^2 \times \mathbb{P}^1/\{0,\infty\},\mathbb{T},*}.$$

In this chapter, we will consider $X = \mathbb{C}^2 \times C$ relative to $D' = \mathbb{C}^2 \times D$ for C a smooth projective curve and $D = \{p_1, \ldots, p_m\} \subset C$ a reduced divisor. As in the introduction, we will equip X with its canonical $\mathbb{T} = (\mathbb{C}^*)^2$-action, which we need in order for its relative Gromov-Witten invariants to be well-defined.

Proposition 4.4. For cohomology classes $\gamma_1, \ldots, \gamma_n \in H^*(C)$, $k_1, \ldots, k_n \geq 0$ and ordered partitions $\mathbf{a}_1, \ldots, \mathbf{a}_m$ of same size d, we have:

$$\begin{aligned} &\langle \tau_{k_1}(\gamma_1) \ldots \tau_{k_n}(\gamma_n) | \, \mathbf{a}_1 \ldots \mathbf{a}_m \rangle^{X/D',\mathbb{T},*}_{g,d} \\ &= \frac{1}{t_1 t_2} \left\langle \mathbb{E}^\vee(t_1) \mathbb{E}^\vee(t_2) \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \right| \mathbf{a}_1, \ldots, \mathbf{a}_m \rangle^{C/D,*}_{g,d} \\ &= t_1^c t_2^{-1} \left\langle \mathbb{E}^\vee(1) \mathbb{E}^\vee(\tfrac{t_2}{t_1}) \tau_{k_1}(\gamma_1) \ldots \tau_{k_n}(\gamma_n) \right| \mathbf{a}_1, \ldots, \mathbf{a}_m \right\rangle^{C/D,*}_{g,d}, \end{aligned}$$

where $c = \sum_i (k_i + \deg_{\mathbb{C}}(\gamma_i) - 1) - d\chi(C \setminus D) - \sum_i \ell(\mathbf{a}_i) + 1$ and

$$\mathbb{E}^\vee(x) = \sum_{i=0}^{g} (-1)^i \lambda_i x^{g-i} \in H^*(\overline{M}_{g,n}(C/D, d, \mathbf{a}_1, \ldots, \mathbf{a}_m))[x],$$

where $\lambda_i = c_i(\mathbb{E})$ is the i-th chern class of the Hodge bundle $\mathbb{E} = \pi_* \Omega^1_\pi$ of the forgetful map

$$\pi \colon \overline{M}^*_{g,n+1}(C/D, d, \mathbf{a}_1, \ldots, \mathbf{a}_m) \to \overline{M}^*_{g,n}(C/D, d, \mathbf{a}_1, \ldots, \mathbf{a}_m).$$

Proof. The first equality follows from the definition. The second equality follows by noting that multiplying each cohomology class γ in the bracket by $t_1^{\deg_{\mathbb{C}}(\gamma)}$ is the same as multiplying the entire expression by t_1^{vdim}. □

As claimed in the introduction, this reduces us to studying double Hodge integrals over curves:

Definition 4.5. For $\gamma_1, \dots, \gamma_n \in H^*(C)$ and $\mathbf{a}_1, \dots, \mathbf{a}_m$ ordered partitions. Then we set:

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x) \prod_{i=1}^n \frac{\gamma_i}{1/z_i - \psi_i} \middle| \mathbf{a}_1, \dots, \mathbf{a}_m \right\rangle_d^{C/D,*}$$
$$:= \sum_{\substack{g \in \mathbb{Z} \\ l_1,\dots,l_n \in \mathbb{Z}}} (-1)^{g-1} z^{2g-2} \prod_i z_i^{l_i+1} \left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x) \prod_i \tau_{l_i}(\gamma_j) \middle| \mathbf{a}_1, \dots, \mathbf{a}_m \right\rangle_{g,d}^{C/D,*}$$
$$\in \mathbb{Q}[x^\pm, z_1^\pm, \dots, z_n^\pm, (z_i + z_j)^{-1}][((z)),$$

where $* \in \{ \circ, \bullet, \prime \}$ and negative descendents are defined as in [72, 91]. If $m > 0$, we will omit d as it is determined by the $\mathbf{a}_i$.

Remark 4.6. If $* = \circ$, then the negative descendents only give non-zero contributions in case $g = d = 0$ and $n \leq 2$, where we have

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x) \frac{\gamma}{1/z_1 - \psi_1} \middle| \emptyset \right\rangle_{0,0}^{C/D,\circ} = \frac{1}{z_1} \int_C \gamma$$

and

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x) \frac{\gamma_1}{1/z_1 - \psi_1} \frac{\gamma_2}{1/z_2 - \psi_2} \middle| \emptyset \right\rangle_{0,0}^{C/D,\circ} = \frac{z_1 z_2}{z_1 + z_2} \int_C \gamma_1 \cup \gamma_2.$$

In particular, this extends the formula
(67)
$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x) \prod_{i=1}^n \frac{\gamma_i}{1/z_i - \psi_i} \middle| \emptyset \right\rangle_{0,0}^{C/D,\circ} = z_1 \cdots z_n \left(\sum_i z_i \right)^{n-3} \int_C \gamma_1 \cup \cdots \cup \gamma_n$$

for $n \geq 3$ to the case $n = 1, 2$.

If $D = \emptyset$, Proposition 1.32 determines the empty contribution $\mathcal{Z}_\emptyset$. Hence it follows from Proposition 4.1 and Proposition 4.4 (or by a more direct argument) that all three versions of Hodge integrals are equivalent. As a result, we will not consider the $\bullet$-version any more. The other two versions are related as follows:

Corollary 4.7. We have:

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^n \frac{\gamma_i}{1/z_i-\psi_i}\right\rangle_d^{C,\prime}$$

$$= \sum_{\substack{\{1,\dots,n\}=\coprod_{I\in S} I\\ \sum_{I\in S} d_I=d}} x^{1-|S|}\mathrm{sgn}(S)\prod_{I\in S}\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i\in I}\frac{\gamma_i}{1/z_i-\psi_i}\right\rangle_{d_I}^{C,\circ},$$

where we sum over those set partitions so that $\emptyset\notin S$ and $\mathrm{sgn}(S)$ is the sign that arises from super-commuting the γ_i.

4.2 Invariants of local $\mathbb{P}^1$ relative to 0 and ∞

Using degeneration, one can see that the Gromov-Witten theory of local $\mathbb{P}^1$ relative to 0 and ∞ is determined by the invariants

$$\left\langle \mathbf{a}\left|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1/\mu_0-\psi_1}\frac{1}{1/\mu_1-\psi_2}\cdots\frac{1}{1/\mu_l-\psi_{l+1}}\right|\mathbf{b}\right\rangle^{\mathbb{P}^1/\{0,\infty\},\circ} \tag{68}$$

for μ a partition. This section studies such expressions - partly with the aim of using them to gain information about the local elliptic curve in Section 4.3. As a result, all invariants in this section will be connected and we will drop the superscript "$\circ$". The following Proposition and its proof are taken from the authors master's thesis.

Proposition 4.8. [95, Lemma 5.11] For $\mathbf{a}=(a_i)_{i=1}^n$ and $\mathbf{b}=(b_i)_{i=1}^m$ ordered partitions of the same size:

$$\left\langle \mathbf{a}\left|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\right|\mathbf{b}\right\rangle^{\mathbb{P}^1/\{0,\infty\}} = -z^{-2}S(z)^x\prod_{i=1}^n S(a_iz)\prod_{i=1}^m S(b_iz),$$

where $S(z)$ is as in (64).

Proof. We denote the size by $d=|\mathbf{a}|=|\mathbf{b}|$. Using Mumford's relation (26) and [76, (3.11)], we see that the claim holds for $x=-1$. Hence, we only need to show that $F(x)$ is proportional to $S(z)^x$ in x. To this end, we consider the invariant for fixed d and g

$$\left\langle (a_1,H),\dots,(a_n,H)\right|\mathbb{E}^\vee(x)\left|(b_1,H),\dots,(b_m,H)\right\rangle_{g,(d,1)}^{\mathbb{P}^1\times\mathbb{P}^1/\{0,\infty\}\times\mathbb{P}^1,\sim},$$

which is rubber in the first factor and $H\in H^2(\mathbb{P}^1)$ is the hyperplane class in the second factor. In fact, this invariant is independent of x as $(-1)^g\lambda_g$ is the only summand of $\mathbb{E}^\vee(x)$ that gives a nonzero contribution. We now equip the second variable with an action by $\mathbb{T}=\mathbb{C}^*$ so that $T_{\mathbb{P}^1,0}$ becomes the standard $\mathbb{T}$-representation, which corresponds to the generator $t \in H^2_{\mathbb{T}}(\mathrm{pt})$. Moreover, we lift H to $H^2_{\mathbb{T}}(\mathbb{P}^1)$ so that $H|_0 = t$ and $H|_\infty = 0$. The only fixed loci which contribute in localization consist of a tube with degree one in the second factor and two curves D_1 and D_2, where D_1 has genus g_1 and maps of degree d onto $\mathbb{P}^1 \times \{0\}$ and D_2 is a curve of genus g_2 with a constant map to $\mathbb{P}^1 \times \{\infty\}$ and empty ramification profile. See Figure 5. By specializing

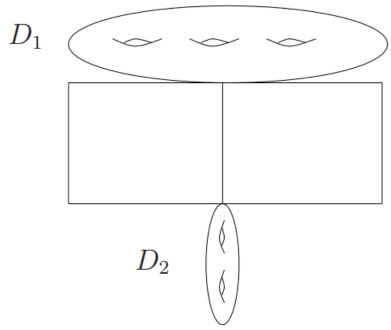

Figure 5: Fixed locus in rubber localization[16]

$t = 1$, we get:

$$
\begin{aligned}
&\sum_{g_1+g_2=g} \left\langle \mathbf{a} \,\middle|\, \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{1}{1-\psi_1} \,\middle|\, \mathbf{b} \right\rangle^{\mathbb{P}^1/\{0,\infty\},\sim}_{g_1,d} \\
&\cdot \left\langle \emptyset \,\middle|\, \mathbb{E}^\vee(1)\mathbb{E}^\vee(-x)\frac{1}{1-\psi_1} \,\middle|\, \emptyset \right\rangle^{\mathbb{P}^1/\{0,\infty\},\sim}_{g_2,0} \\
&= \sum_{g_1+g_2=g} \left\langle \mathbf{a} \,\middle|\, \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1} \,\middle|\, \mathbf{b} \right\rangle^{\mathbb{P}^1/\{0,\infty\}}_{g_1,d} \\
&\cdot \left\langle \emptyset \,\middle|\, \mathbb{E}^\vee(1)\mathbb{E}^\vee(-x)\frac{\mathrm{pt}}{1-\psi_1} \,\middle|\, \emptyset \right\rangle^{\mathbb{P}^1/\{0,\infty\}}_{g_2,0} \\
&= \sum_{g_1+g_2=g} \left\langle \mathbf{a} \,\middle|\, \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1} \,\middle|\, \mathbf{b} \right\rangle^{\mathbb{P}^1/\{0,\infty\},}_{g_1,d} \\
&\cdot \int_{\overline{M}_{g_2,1}} \mathbb{E}^\vee(0)\mathbb{E}^\vee(1)\mathbb{E}^\vee(-x)\frac{1}{1-\psi_1},
\end{aligned}
$$

where we used rigidification [59, Lemma 2] in the second equality. The last Hodge integral was implicitly computed as $g_x(z)$ in the proof of [24, Proposition 3]:

$$
\sum_{g\geq 0}(-1)^g z^{2g} \int_{\overline{M}_{g,1}} \mathbb{E}^\vee(0)\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{1}{1-\psi_1} = S(z)^x
$$

[16]This is Figure 2 from [60], where essentially the same kind of localization is considered.

and so $F(x)S(z)^{-x}$ is constant in x, which concludes the proof. $\square$

This settles the case $|\mu| = 1$. For $|\mu| > 1$, the above localization argument fails and it is somewhat unclear how to compute more cases of (68). Using large amounts of computer data, we were however able to find the following conjectural formulas:

Conjecture F. For any $x \in \mathbb{Z}_{<0}$, we have:

$$\begin{aligned} &\left\langle \mathbf{a} \left| \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1/2-\psi} \right| \mathbf{b} \right\rangle^{\mathbb{P}^1/\{0,\infty\}} \\ &= \frac{S(2z)^{2x}\prod_i S(a_i z)\prod_i S(b_i z)}{xz^3\binom{-2x}{-x}} \sum_{i=0}^{-x} \binom{-2x}{-x+i} \\ &\cdot [t^0]\left(\prod_i \left(t^{a_i/2}+t^{-a_i/2}\right)\prod_i \left(t^{b_i/2}+t^{-b_i/2}\right) \frac{\left(\frac{e^z-t}{1-te^z}\right)^i - \left(\frac{1-te^z}{e^z-t}\right)^i}{2} \right) \end{aligned} \tag{69}$$

for $\mathbf{a} = (a_i)_i$ and $\mathbf{b} = (b_i)_i$ ordered partitions of the same size.

Remark 4.9. (1) For $x = -1$, Conjecture F follows from Mumford's relation (26) and [76, (3.11)].

(2) It follows from the definitions that the z-coefficients of the left hand side are polynomials in x. This is not as obvious for the right hand side, where it follows from Lemma 4.15.

Though we do not know how to prove Conjecture F, we will provide some evidence for it later on - see Theorem 4.14.

4.3 Invariants of the local elliptic curve

In this section we use Proposition 4.8 to derive some formulas for double Hodge integrals on the elliptic curve. The following proposition is an intermediate step and was obtained in collaboration with Jan-Willem van Ittersum:

Proposition 4.10. For any $x \in \mathbb{Z} \setminus \{0\}$ we have

$$\begin{aligned} &\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^n \frac{\mathrm{pt}}{1-\psi_i} \right\rangle^{E,\prime} \\ &= \frac{(-1)^n\Theta(z)^{nx}}{x^{n-1}z^{nx+2n}} \left[\prod_{1\le i\ne j\le n} \left(\frac{\Theta(z+u_i-u_j)}{\Theta(u_i-u_j)}\right)^x \right]_{p^0}, \end{aligned} \tag{70}$$

where we set $p_i = e^{u_i}$ and take the coefficient of $\prod_i p_i^0$ in the sense of (80).

Proof. We prove this by degenerating the elliptic curve to a circle of n copies of $\mathbb{P}^1$ joined end to end with each receiving a single point insertion as depicted in Figure 4. We then apply the degeneration formula of [45, 46] to this degeneration, which expresses the above Gromov-Witten invariant of E in terms of Gromov-Witten invariants of $\mathbb{P}^1$ relative to 0 and ∞. For each of the occuring stable maps to a single $\mathbb{P}^1$, there is a unique component which receives the marking. It further follows from [70, Lemma 1] that

$$\left\langle \mathbf{a} \,\middle|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\middle|\, \mathbf{b}\right\rangle_g^{\mathbb{P}^1/\{0,\infty\}} = \begin{cases} \frac{1}{a}, & \text{if } g=0, \mathbf{a}=\mathbf{b}=(a), \\ 0, & \text{else,} \end{cases} \tag{71}$$

which forces all other components to be tubes. As a result, the contributions of the degeneration formula consist of n many factors of

$$\left\langle \mathbf{a} \,\middle|\, \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi} \,\middle|\, \mathbf{b}\right\rangle^{\mathbb{P}^1/\{0,\infty\}},$$

which are connected to each other via contributions of chains of tubes going around the circle. Here a chain of degree a tubes of length m contributes a factor of $a^{m+1}\frac{1}{a^m}=a$, which results from (71) and the $m+1$ glueing points each contributing a factor of a. In total, this gives

$$\begin{aligned}&\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^n \frac{\mathrm{pt}}{1-\psi_i}\right\rangle^{E,\prime} \\ &= x^{1-n} \sum_{\substack{(a_l^{i,j})_{l=1}^{n_{i,j}},(b_l^{i,j})_{l,i,j}^{n_{i,j}} \\ \forall i:\ \sum_{l,j} a_l^{i,j}=\sum_{l,j} a_l^{j,i} \\ a_l^{i,j},b_l^{i,j}>0 \text{ and } b_l^{i,j}\equiv j-i(n)}} q_n^{\sum_{l,i,j} a_l^{i,j}b_l^{i,j}} (-xz^2)^{\sum_{i,j} n_{i,j}} \prod_{l,i,j} a_l^{i,j} \\ &\cdot \prod_{i,j=1}^n \frac{1}{n_{i,j}!} \prod_{i=1}^n \left\langle ((a_l^{i,j})_l)_j \,\middle|\, \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi} \,\middle|\, ((a_l^{j,i})_l)_j \right\rangle^{\mathbb{P}^1/\{0,\infty\}},\end{aligned}$$

where q_n is a new variable satisfying $q_n^n = q$. The sum arises from numbering the components of the circle of $\mathbb{P}^1$s by $i=1,\dots,n$ in counter-clockwise direction. We then define $n_{i,j}$ to be the number of chains of tubes wrapping around the circle in counter-clockwise direction from i to j, with $a_l^{i,j}$ the degree of the l-th chain and $b_l^{i,j}$ one more than the length of that chain. It follows from this that $b_l^{i,j} \equiv j-i$ modulo n and one can show that any collection of $(a_l^{i,j})_{l=1}^{n_{i,j}}$ and $(b_l^{i,j})_{l=1}^{n_{i,j}}$ satisfying the above conditions corresponds to such a geometric contribution with total degree $\frac{\sum_{l,i,j} a_l^{i,j}b_l^{i,j}}{n}$. Using Proposition 4.8, we get

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^n \frac{\mathrm{pt}}{1-\psi_i}\right\rangle^{E,\prime}$$

$$= \frac{(-1)^n}{x^{n-1}z^{2n}} S(z)^{nx} \sum_{\substack{(a_l^{i,j})_{l=1}^{n_{i,j}},(b_l^{i,j})_{l,i,j}^{n_{i,j}} \\ \forall i:\ \sum_{l,j} a_l^{i,j} = \sum_{l,j} a_l^{j,i} \\ a_l^{i,j}, b_l^{i,j}>0 \text{ and } b_l^{i,j} \equiv j-i(n)}} \prod_{i,j=1}^{n} \frac{1}{n_{i,j}!} \prod_{l=1}^{n_{i,j}} (-xz^2) a_l^{i,j} S(a_l^{i,j} z)^2 q_n^{a_l^{i,j} b_l^{i,j}}$$

$$= \frac{(-1)^n}{x^{n-1}z^{2n}} \left[\exp\left(x \sum_{i,j=1}^{n} C_{j-i(n)}\left(z, q_n, \frac{p_i}{p_j}\right)\right)\right]'_{p^0},$$

where the bracket $[\cdots]'_{p^0}$ denotes the act of expanding the power series in q_n and taking the coefficient of $\prod_{i=1}^n p_i^0$ in each q_n^k-coefficient. Note here that those q_n^k-coefficients are Laurent polynomials in $p_1, \dots, p_n$. We also wrote

$$C_{\gamma(n)}(z,q,p) = \delta_{\gamma,0} \log S(z) - \sum_{\substack{a,b>0 \\ b \equiv \gamma(n)}} \frac{\varsigma(az)^2}{a} \left(pq^b\right)^a$$

for any $\gamma \in \mathbb{Z}$. From this we get

$$\exp\left(C_{\gamma(n)}(z,q,p)\right) = S(z)^{\delta_{\gamma,0}} \prod_{\substack{b>0 \\ b\equiv\gamma(n)}} \frac{(1-e^z pq^b)(1-e^{-z}pq^b)}{(1-pq^b)^2}.$$

Using this, one can phrase $[\cdots]'_{p^0}$ in a more analytic way by noting that the q-expansion inside the bracket converges if

$$\left|\frac{p_i}{p_j} q_n^b\right| < 1 \text{ for all } i,j \text{ and } b>0 \text{ with } b \equiv j-i(n)$$

for $x \geq 0$ and

$$\left|e^{\pm z} \frac{p_i}{p_j} q_n^b\right| < 1 \text{ for all } i,j \text{ and } b>0 \text{ with } b \equiv j-i(n)$$

for $x < 0$. Hence $[\cdots]'_{p^0}$ can also be interpreted as taking the coefficient of $\prod_{i=1}^n p_i^0$ in the Laurent expansion in the respective domains. Moreover, setting $\tilde{p}_i = p_i q_n^i$ gives

$$\exp\left(\sum_{i,j=1}^{n} C_{j-i(n)}(z, q_n, \tfrac{\tilde{p}_i}{\tilde{p}_j})\right)$$
$$= \frac{\Theta(z)^n}{z^n} \prod_{\substack{1\leq i<j\leq n \\ l>0}} \frac{(1-e^z \frac{p_i}{p_j} q^{l-1})(1-e^{-z}\frac{p_i}{p_j}q^{l-1})(1-e^z\frac{p_j}{p_i}q^l)(1-e^{-z}\frac{p_j}{p_i}q^l)}{(1-\frac{p_i}{p_j}q^{l-1})^2(1-\frac{p_j}{p_i}q^l)^2}$$
$$= \frac{\Theta(z)^n}{z^n} \prod_{1\leq i<j\leq n} \frac{\Theta(z+u_i-u_j)\Theta(-z+u_i-u_j)}{\Theta(u_i-u_j)^2}$$

$$= \frac{\Theta(z)^n}{z^n} \prod_{1\le i\ne j\le n} \frac{\Theta(z+u_i-u_j)}{\Theta(u_i-u_j)},$$

where we wrote $p_i = e^{u_i}$ and used

$$\Theta(z) = -e^{-z/2} \prod_{l>0} \frac{(1-e^z q^{l-1})(1-e^{-z}q^l)}{(1-q^l)^2}$$

in the second equality as well as $\Theta(-z) = -\Theta(z)$ in the third. By keeping track of the shift in the p_i, we therefore get

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x) \prod_{i=1}^n \frac{\mathrm{pt}}{1-\psi_i} \right\rangle^{E,\prime}$$
$$= \frac{(-1)^n \Theta(z)^{nx}}{x^{n-1} z^{nx+2n}} \left[\prod_{1\le i\ne j\le n} \left(\frac{\Theta(z+u_i-u_j)}{\Theta(u_i-u_j)} \right)^x \right]''_{p^0},$$

where $[\cdots]''_{p^0}$ denotes taking the coefficient of $\prod_{i=1}^n p_i^0$ in the Fourier expansion in u in the domain defined by

$$|q| < \left| \frac{p_i}{p_j} q \right| < 1 \text{ for all } 1 \le j < i \le n$$

if $x \ge 0$ and

$$|q| < \left| e^{\pm z} \frac{p_i}{p_j} q \right| < 1 \text{ for all } 1 \le j < i \le n$$

if $x < 0$. Using the notation of Section 4.5, we have $[\cdots]''_{p^0} = [\cdots]_{p^0,\mathrm{Id}}$, where $\mathrm{Id} \in S_n$ is the identity permutation. Since the function inside the bracket is symmetric in the u_i, we get $[\cdots]_{p^0,\mathrm{Id}} = [\cdots]_{p^0}$, which concludes the proof. □

We are now ready to give a proof of Theorem 1.40. We recall its statement for the sake of convenience:

Theorem 1.40. If $x > 0$, we have

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x) \prod_{i=1}^n \frac{\mathrm{pt}}{1-\psi_i} \right\rangle^{E,\prime}$$
$$(28) \quad = \frac{(-1)^n (n-1)! \Theta(z)^{nx}}{x^{n-1} z^{nx+2n}} \mathrm{Res}_{u_{n-1}=u_n} \cdots \mathrm{Res}_{u_1=u_2}$$
$$\sum_{1=l_1<\cdots<l_N=n} \left(\prod_{i\ne j} \frac{\Theta(z+u_i-u_j)}{\Theta(u_i-u_j)} \right)^x \cdot \prod_{m=1}^{N-1} \frac{\mathsf{A}(u_{l_m}-u_{l_{m+1}})^{l_{m+1}-l_m}}{(n-l_m)\cdot(l_{m+1}-l_m)!}$$

and if $x < 0$,

$$
\begin{aligned}
&\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^n \frac{\mathrm{pt}}{1-\psi_i}\right\rangle^{E,\prime} \\
(29)\qquad &= 2\frac{(-1)^n(n-1)!\Theta(z)^{nx}}{x^{n-1}z^{nx+2n}}\mathrm{Res}_{u_{n-1}=u_n}\cdots\mathrm{Res}_{u_1=u_2} \\
&\sum_{\substack{1=l_1<\cdots<l_N=n\\ s_1,\ldots,s_n}}\left(\prod_{i\neq j}\frac{\Theta(z+u_i'-u_j')}{\Theta(u_i'-u_j')}\right)^x\cdot\prod_{m=1}^{N-1}\frac{\mathsf{A}(u'_{l_m}-u'_{l_{m+1}})^{l_{m+1}-l_m}}{(n-l_m)\cdot(l_{m+1}-l_m)!},
\end{aligned}
$$

where all residues are taken for $z\neq 0$. Moreover, we set $u_i' = u_i + s_i z$ and $s_1,\ldots,s_n$ is a sequence of integers so that $s_1 = 0$, $s_2 = 1$ and for any j, we have $\{s_1,\ldots,s_j\} = \mathbb{Z}\cap[a,b]$ for some $a,b\in\mathbb{Z}$. Finally, $\mathsf{A}(u)$ is as in Appendix A.

Proof of Theorem 1.40. Using the terminology of Section 4.5, it follows from (89) that

$$
F(\mathbf{u};z,\tau) = \left(\prod_{i\neq j}\frac{\Theta(z+u_i-u_j)}{\Theta(u_i-u_j)}\right)^x
$$

is Λ_τ-invariant and has a pole datum given by $S_{i,j} = \{0\}$ for $x>0$ and $S_{i,j} = \{\pm 1\}$ for $x<0$. Hence (28) follows directly from Corollary 4.21 and Proposition 4.10. For $x<0$, this however gives us:

$$
\begin{aligned}
&\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^n \frac{\mathrm{pt}}{1-\psi_i}\right\rangle^{E,\prime} \\
(72)\qquad &= \frac{(-1)^n(n-1)!\Theta(z)^{nx}}{x^{n-1}z^{nx+2n}}\mathrm{Res}_{u_{n-1}=u_n}\cdots\mathrm{Res}_{u_1=u_2} \\
&\sum_{\substack{1=l_1<\cdots<l_N=n\\ s_1,\ldots,s_n}}\left(\prod_{i\neq j}\frac{\Theta(z+u_i'-u_j')}{\Theta(u_i'-u_j')}\right)^x\cdot\prod_{m=1}^{N-1}\frac{\mathsf{A}(u'_{l_m}-u'_{l_{m+1}})^{l_{m+1}-l_m}}{(n-l_m)\cdot(l_{m+1}-l_m)!},
\end{aligned}
$$

where $u_i' = u_i + s_i z$ and $s_1,\ldots,s_n\in\mathbb{Z}$ are so that $s_1 = 0$ and for any $j>1$ we either have $|s_j - s_r| = 1$ for some $r<j$ or $j = l_{t+1}$ for some $t\geq 1$ and $s_j = s_{l_t}$. Note now that the summands with $s_2 = 0$ all vanish. Indeed, in this case $F(\mathbf{u}';z,\tau)$ has positive order at $u_1 = u_2$ and the factor containing the A's only has negative order if $l_{m+1} = 2$, in which case it has order -1. As a result, the product is holomorphic at $u_1 = u_2$ and so $\mathrm{Res}_{u_1=u_2} = 0$. Furthermore, the summand corresponding to $s_1,\ldots,s_n$ is the same as the one corresponding to $-s_1,\ldots,-s_n$. This is because

$$
\mathrm{Res}_{z=a}f(z) = -\mathrm{Res}_{z=-a}f(-z).
$$

Denoting by $G(\mathbf{u}')$ the function that we are taking the residue of, we therefore see that

$$\begin{aligned}&\mathrm{Res}_{u_{n-1}=u_n}\cdots\mathrm{Res}_{u_1=u_2}G(\mathbf{u}') = (-1)^{n-1}\mathrm{Res}_{u_{n-1}=u_n}\cdots\mathrm{Res}_{u_1=u_2}G(-\mathbf{u}'')\\ &= \mathrm{Res}_{u_{n-1}=u_n}\cdots\mathrm{Res}_{u_1=u_2}G(\mathbf{u}'')\end{aligned}$$

where $u_i'' = u_i - s_i z$ and the last equality used $G(-\mathbf{u}) = (-1)^{n-1}G(\mathbf{u})$, which follows from $\mathsf{A}(-u) = -\mathsf{A}(u)$. This redundancy can be removed by requiring $s_2 = 1$ and giving the right hand side in (72) a factor of 2. Moreover, if $s_2 = 1$, then the above conditions on s_i are equivalent to those stated in Theorem 1.40. Note also that the requirement $z \neq 0$ coming from Section 4.5 is only necessary for (29) and not for (28) since the latter has no poles that involve z. □

Using slightly different methods, we also derived recursive formulas for certain Hodge integrals in [95, Appendix B]. We will now recall this result as well as its proof. Meanwhile we freely use the notation of Appendix A.

Proposition 4.11. [95, Proposition B.1] For $x \in \mathbb{Z}_{\neq 0}$, we have the following formulas:

(1) For $x > 0$ we have:

$$\begin{aligned}\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{1}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\right\rangle^{E,\prime} &= \frac{\Theta^{4x}4^x}{x\binom{2x}{x}z^{2x+4}}a(x)\\ \left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\right\rangle^{E,\prime} &= -\frac{\Theta^{4x}4^{x+1}}{x\binom{2x}{x}z^{2x+4}}\Big[G_2 a(x) + b(x)\Big]\\ \left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1/2-\psi_1}\right\rangle^{E,\prime} &= \frac{\Theta^{4x}}{z^{2x+3}}\Big[(\mathsf{A}+2zG_2)a(x) + z2b(x) + 2c(x)\Big]\end{aligned}$$

for quasi-Jacobi forms $a(x), b(x), c(x) \in \mathbb{Q}[\wp, \wp', G_4]$ of weights $2x-2$, $2x$ and $2x-1$ respectively. They are characterized by

$$a(1) = -\frac{1}{4},\ b(1) = -\frac{1}{8}\wp,\ c(1) = 0$$

and the recursive formula:

$$\begin{aligned}a(x) &= \frac{1}{2x}Da(x-1) + \frac{4x-3}{2x}\wp a(x-1) + \frac{1}{x}b(x-1)\\ b(x) &= \frac{1}{2}\wp a(x) - D_z c(x) + (x-\frac{1}{2})c(x-1)\wp'\\ c(x) &= \frac{1}{2x}Dc(x-1) + \frac{4x-3}{2x}\wp c(x-1) + \frac{1}{8x(x-1/2)}D_z a(x)\end{aligned}$$

for all $x \geq 2$, where D is the operator

$$D = -D_\tau + \mathsf{A}D_z - 2G_2 wt$$

of degree 2. It is easily checked that D indeed preserves $\mathbb{Q}[\wp, \wp', G_4]$.

(2) Likewise, formulas for $x < 0$ are given by:

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{1}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\right\rangle^{E,\prime} = \frac{\Theta(2z)^{2x+1}}{4z^{2x+4}\Theta^{4(x+1)}}\Big[2zd(x) - e(x)\Big]$$

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\right\rangle^{E,\prime} = \frac{\Theta(2z)^{2x+1}}{\Theta^{4(x+1)}z^{2x+4}}\Big[\mathsf{A}d(x) + G_2e(x) + f(x)\Big]$$

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1/2-\psi_1}\right\rangle^{E,\prime} = \frac{\Theta(2z)^{2x+1}}{\Theta^{4(x+1)}4^{x+1}\binom{-2x}{-x}z^{2x+3}}d(x),$$

where again $d(x), e(x), f(x) \in \mathbb{Q}[\wp, \wp', G_4]$ are of weights $-4(x+1)$, $-4x-5$ and $-4x-3$ and determined by

$$d(-1) = -2,\ e(-1) = 0,\ f(-1) = 0$$

and the recursive formula:

$$\begin{aligned} d(x) &= D(x)d(x+1) \\ e(x) &= -\frac{2}{x}\wp' d(x+1) + D(x)e(x+1) \\ f(x) &= D(x)f(x+1) - \frac{1}{x}\wp\wp' d(x+1), \end{aligned}$$

which holds for $x < -1$. Here, $D(x)$ is the operator

$$D(x) = \frac{1}{x}\wp' D_z + 2\wp''\frac{x+3/2}{x}$$

Sketch of proof. It will be somewhat more convenient to instead work with the connected invariants:

$$F(x) := \left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{1}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\right\rangle^{E,\circ} = \left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{1}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\right\rangle^{E,\prime}$$

$$\begin{aligned} G(x) &:= \left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\right\rangle^{E,\circ} \\ &= \left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\right\rangle^{E,\prime} - \frac{\Theta^{2x}}{xz^{2x+4}} \end{aligned}$$

$$H(x) := \left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1/2-\psi_1}\right\rangle^{E,\circ} = \left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1/2-\psi_1}\right\rangle^{E,\prime}.$$

These are also well-defined for $x = 0$, in which case [91, Lemma 4.4.1] implies

$$F(0) = -\frac{1}{2z^2}, G(0) = 0, H(0) = -\frac{1}{2z^2}.$$

We will now use certain tautological relations on $\overline{M}_{g,2}$ to derive the above recursions. In our case, these relations come from $\overline{M}_{g,2}(\mathbb{P}^1, 2)$. More specifically, we look at the bundle

$$E(n) := R^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(n)$$

for $n = 0, -1$, which comes from $\mathbb{P}^1$ via the maps:

$$\begin{array}{ccc} \overline{M}_{g,3}(\mathbb{P}^1,2) & \xrightarrow{f} & \mathbb{P}^1 \\ \downarrow{\scriptstyle\pi} & & \\ \overline{M}_{g,2}(\mathbb{P}^1,2) & & \end{array}$$

where π is the map forgetting the third marking and f is the evaluation at that marking. We now equip $\mathbb{P}^1$ with a $\mathbb{T} = \mathbb{C}^*$-action so that $T_{\mathbb{P}^1,0}$ is the standard $\mathbb{T}$-representation with first chern class $t \in H^2_{\mathbb{T}}(\mathrm{pt})$. For any integer $x \in \mathbb{Z}$, we can turn $\mathcal{O}_{\mathbb{P}^1}(n)$ into an equivariant bundle so that $\mathcal{O}_{\mathbb{P}^1}(n)|_0$ has weight $(n+x)t$ and $\mathcal{O}_{\mathbb{P}^1}(n)|_\infty$ has weight xt. We further denote by $H_0, H_\infty \in H^2_{\mathbb{T}}(\mathbb{P}^1)$ the two lifts of the points class so that $H_0|_\infty = H_\infty|_0 = 0$ and $H_0|_0 = -H_\infty|_\infty = t$. Using Riemann-Roch, we see that $E(n)$ also becomes an equivariant bundle of rank g if $n = 0$ and rank $g+1$ if $n = -1$. As a result, the following integrals are independent of t and hence also of x:

$$\int_{[\overline{M}_{g,2}(\mathbb{P}^1,2)]^{vir,T}} ev_1^*(H_\infty^{3+n})ev_2^*(H_0)e^T(E(n))p^*C_g(\mathrm{pt},1)$$
$$\int_{[\overline{M}_{g,2}(\mathbb{P}^1,2)]^{vir,T}} ev_1^*(H_\infty^{2+n})\psi_2 ev_2^*(H_0)e^T(E(n))p^*C_g(\mathrm{pt},1)$$
$$\int_{[\overline{M}_{g,2}(\mathbb{P}^1,2)]^{vir,T}} ev_1^*(H_\infty^{2+n})ev_2^*(H_\infty)e^T(E(n))p^*C_g(\mathrm{pt},\mathrm{pt}).$$

Here we wrote
$$C_g(\cdots) = \sum_{d\geq 0} q^d C_{g,d}(\cdots)$$
for $C_{g,d}$ the Gromov-Witten classes of the elliptic curve as in (61). Moreover, we pull these classes back along the forgetful map $p\colon \overline{M}_{g,2}(\mathbb{P}^1,2) \to \overline{M}_{g,2}$. We now set $t = 1$ and compute the above expressions using localization. Note for this that the Euler class $e^T(E(n))$ can be computed using the normalization sequence on every fixed locus (c.f. [24, §2]). Using Theorem 1.34 and Mumford's relation (26) we can deduce

$$F(-1) = -\frac{1}{z\Theta(2z)}, G(-1) = \frac{1}{z^2\Theta^2} - \frac{2\mathsf{A}}{z^2\Theta(2z)}, H(-1) = -\frac{1}{z\Theta(2z)}$$

and hence compute the three integrals for $x = -1$ and all g. After summing over g, the independence of x then gives the following two systems of linear equations:

$$\begin{aligned} &F(-x)\Big[\frac{x\Theta^{2x-2}}{z^{2x-2}} + x(1-x)z^2G(x-1)\Big] \\ &+ G(-x)\Big[x(x-1)z^2F(x-1)\Big] \\ &+ H(-x)\Big[(2-4x)H(x-1)\Big] = \frac{(x-1)z^{2x-2}}{\Theta^{2x}}F(x-1), \end{aligned}$$

$$
\begin{aligned}
&F(-x)\Big[x(x-1)z^2(zD_z+2)G(x-1)+2x\frac{\Theta^{2x-2}}{z^{2x}}((x-1)z\mathsf{A}-x)\Big]\\
&+G(-x)\Big[x(1-x)z^2(zD_z+2)F(x-1)\Big]\\
&+H(-x)\Big[(4x-2)(zD_z+1)H(x-1)\Big]\\
&=\frac{(1-x)z^{2x-2}}{\Theta^{2x}}(zD_z+2)F(x-1)+\frac{2}{xz^3\Theta(2z)},\\
&F(-x)\Big[4x(1-x)\frac{D_\tau\Theta\Theta^{2x-3}}{z^{2x}}-2x(x-1)z^2D_\tau G(x-1)\Big]\\
&+G(-x)\Big[x\frac{\Theta^{2x-2}}{z^{2x}}+x(x-1)z^2(G(x-1)+2D_\tau F(x-1))\Big]\\
&+H(-x)\Big[(4-8x)D_\tau H(x-1)\Big]\\
&=(x-1)\frac{z^{2x-2}}{\Theta^{2x}}(G(x-1)+2D_\tau F(x-1))+\frac{1}{z^4\Theta^2}-2\frac{\mathsf{A}}{z^4\Theta(2z)}
\end{aligned}
$$

and

$$
\begin{aligned}
&F(x)\Big[-\frac{z^{2x-2}}{\Theta^{2x}}+xz^2G(-x)\Big]\\
&+G(x)\Big[-xz^2F(-x)\Big]\\
&+H(x)\Big[-4H(-x)\Big]=\frac{\Theta^{2x}}{z^{2x+2}}F(-x),\\
&F(x)\Big[-xz^2(zD_z+2)G(-x)+2\frac{z^{2x-2}}{\Theta^{2x}}(-xz\mathsf{A}+x-1)\Big]\\
&+G(x)\Big[xz^2(zD_z+2)F(-x)\Big]\\
&+H(x)\Big[4(zD_z+1)H(-x)\Big]=-\frac{\Theta^{2x}}{z^{2x+2}}(zD_z+2)F(-x),\\
&F(x)\Big[2xz^2D_\tau G(-x)+4x\frac{z^{2x-2}D_\tau\Theta}{\Theta^{2x+1}}\Big]\\
&+G(x)\Big[\frac{z^{2x-2}}{\Theta^{2x}}-xz^2(G(-x)+2D_\tau F(-x))\Big]\\
&+H(x)\Big[-8D_\tau H(-x)\Big]=\frac{\Theta^{2x}}{z^{2x+2}}(G(-x)+2D_\tau F(-x)).
\end{aligned}
$$

The derivatives D_z and D_τ come from the dilaton and divisor equations respectively (note that both hold on the level of cycles). The z^{-2}-coefficient of the two systems have determinants $2x^2(1-2x)$ and -2 respectively. Hence $F(x)$, $G(x)$ and $H(x)$ are uniquely determined by these equations and the values for $x=0,-1$. Now one simply inserts the claimed recursions for $x<0$ and $x>0$ into these equations and shows inductively that they are indeed satisfied. □

Remark 4.12. The tautological relations here are the ones used by Okounkov and Pandharipande to prove the Mariño-Vafa formula in [74].

We are now ready to show Theorem 1.38.

Proof of Theorem 1.38. By Proposition 4.11, we see that Theorem 1.38 holds for

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1/2-\psi}\right\rangle^{E,\prime}.$$

Moreover, we have

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\gamma}{1/2-\psi}\right\rangle^{E,\prime} = 0$$

for all $\gamma \in H^{\leq 1}(E)$. In case $\gamma = \alpha, \beta$, this follows from the algebraicity of the virtual class, whereas $\gamma = 1$ vanishes because of [40, Proposition 2]. It therefore remains to consider $\mu = (1^n)$. For this, we first claim that any Hodge integral

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n}\frac{\gamma_i}{1-\psi_i}\right\rangle^{E,\prime}$$

for $\gamma_i \in \{1, \alpha, \beta, \mathrm{pt}\}$ can be written as a linear combination of Hodge integrals with $\gamma_i \in \{1, \mathrm{pt}\}$. Indeed, there can not be more than one α as one could otherwise swap two α-insertions at the cost of introducing a sign

$$\begin{aligned}&\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\cdots\frac{\alpha}{1-\psi_i}\frac{\alpha}{1-\psi_{i+1}}\cdots\right\rangle^{E,\prime}\\ &= -\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\cdots\frac{\alpha}{1-\psi_i}\frac{\alpha}{1-\psi_{i+1}}\cdots\right\rangle^{E,\prime},\end{aligned}$$

which shows that the invariant must vanish. Furthermore, all invariants with an unequal number of α's and β's must be zero due to the algebraicity of the virtual class. Hence the only nonzero Hodge integrals with odd insertions are of the shape

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n}\frac{\mathrm{pt}}{1-\psi_i}\prod_{j=1}^{m}\frac{1}{1-\psi_{j+n}}\frac{\alpha}{1-\psi_{n+m+1}}\frac{\beta}{1-\psi_{n+m+2}}\right\rangle^{E,\prime}.$$

By [40, Proposition 2], this is equal to

$$\frac{1}{n+1}\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n+1}\frac{\mathrm{pt}}{1-\psi_i}\prod_{j=1}^{m+1}\frac{1}{1-\psi_{j+n}}\right\rangle^{E,\prime},$$

which reduces to the case of even cohomology insertions. In this case, it follows from (27) that

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n}\frac{\mathrm{pt}}{1-\psi_i}\prod_{j=1}^{m}\frac{1}{1-\psi_{j+n}}\right\rangle^{E,\prime}$$
$$=\frac{n!}{(-2)^m(n+m)!}\left(\left(\frac{d}{dG_2}\right)_z+xz^2(n+m)^2\right)^m\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n+m}\frac{\mathrm{pt}}{1-\psi_i}\right\rangle^{E,\prime}.$$

If we assume that Theorem 1.38 holds for point descendents, then by (92) we get

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n}\frac{\mathrm{pt}}{1-\psi_i}\prod_{j=1}^{m}\frac{1}{1-\psi_{j+n}}\right\rangle^{E,\prime}$$
$$=\frac{n!}{(-2)^m(n+m)!}\left(\frac{d}{dG_2}-2z\frac{d}{d\mathsf{A}}\right)^m\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n+m}\frac{\mathrm{pt}}{1-\psi_i}\right\rangle^{E,\prime},$$

which implies Theorem 1.38 if $x<0$ and also for $x>0$ if one has

$$\frac{d}{d\mathsf{A}}\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n}\frac{\mathrm{pt}}{1-\psi_i}\right\rangle^{E,\prime}=0.$$

Finally, one can deduce the point descendent case by explicitly expanding the residues in Theorem 1.40. The vanishing of $\frac{d}{d\mathsf{A}}$ for $x>0$ follows from the identity

(73) $$\frac{d}{d\mathsf{A}}[u^k]F(u+z)=[u^k]\left(2muF(u+z)+\left(\frac{d}{d\mathsf{A}}F\right)(u+z)\right),$$

where F is a meromorphic quasi-Jacobi form of index m. This is a consequence of (93). ☐

From Theorem 1.40, we can derive the following formulas, which will be useful for providing evidence for Conjecture F in the next section.

Corollary 4.13. For $x\in\mathbb{Z}_{<0}$ we have

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1/2-\psi}\right\rangle^{E,\prime}=\frac{\Theta(z)^{2x}}{2x\binom{-2x}{-x}z^{2x+3}}\mathrm{Res}_{u=0}\left(\frac{\Theta(u+2z)\Theta(u)}{4\Theta(u+z)^2}\right)^x$$
$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{1}{1-\psi_2}\right\rangle^{E,\prime}=\frac{\Theta(z)^{2x}}{xz^{2x+4}}\mathrm{Res}_{u=0}\left(\frac{\Theta(u+2z)\Theta(u)}{\Theta(u+z)^2}\right)^x(u+z)$$
$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\right\rangle^{E,\prime}=\frac{2\Theta(z)^{2x}}{xz^{2x+4}}\mathrm{Res}_{u=0}\left(\frac{\Theta(u)\Theta(u+2z)}{\Theta(u+z)^2}\right)^x\mathsf{A}(u+z),$$

where all residues are taken for $z \neq 0$ and and for $x \in \mathbb{Z}_{>0}$:

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{1}{1-\psi_2}\right\rangle^{E,\prime} = \frac{\Theta(z)^{2x}}{2xz^{2x+4}}\mathrm{Res}_{u=0}\left(\frac{\Theta(u+z)\Theta(u-z)}{\Theta(u)^2}\right)^x u$$
$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\right\rangle^{E,\prime} = \frac{\Theta(z)^{2x}}{xz^{2x+4}}\mathrm{Res}_{u=0}\left(\frac{\Theta(u-z)\Theta(u+z)}{\Theta(u)^2}\right)^x \mathsf{A}(u)$$

Proof. The formulas for $\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\right\rangle^{E,\prime}$ are special cases of Theorem 1.40. Using the holomorphic anomaly equation (27), we get

$$\begin{aligned}&\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{1}{1-\psi_2}\right\rangle^{E,\prime}\\ &= -\left(xz^2+\frac{1}{4}\left(\frac{d}{dG_2}\right)_z\right)\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\right\rangle^{E,\prime},\end{aligned}$$

where $\left(\frac{d}{dG_2}\right)_z$ is the z-coefficientwise holomorphic anomaly operator for quasi-modular forms (not to be confused with $\frac{d}{dG_2}$ acting on QJac). The last formula follows from Proposition 4.11(2), which implies that

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1/2-\psi}\right\rangle' = \frac{z}{4^{x+1}\binom{-2x}{-x}}\frac{d}{d\mathsf{A}}\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\right\rangle'.$$

The right hand side is then simplified using (73). □

4.4 Applications and consequences

Using the Corollary 4.13, we can deduce the following formulas for invariants of the tube:

Theorem 4.14. Conjecture F holds for $\mathbf{a} = \mathbf{b} = \emptyset$ and $\mathbf{a} = \mathbf{b} = (1)$. We also have the following identities: For $x < 0$:

$$\begin{aligned}&\langle\emptyset|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{1}{1-\psi_2}|\emptyset\rangle^{\mathbb{P}^1/\{0,\infty\}}\\ &= \frac{(2S(2z))^{2x}}{xz^3}f_{-x}(z) - \frac{(2S(2z))^{2x}}{2z^4}\sum_{i+j=-x}\frac{f_i(z)f_j(z)}{ij},\end{aligned}$$

where

$$f_n(z) = \sum_{i=0}^{n}\binom{2n}{n+i}\varsigma(2iz) \tag{74}$$

with $\varsigma(z) = e^{z/2} - e^{-z/2}$ as in (63) and for $x \geq 0$:

$$\langle\emptyset|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{1}{1-\psi_2}|\emptyset\rangle^{\mathbb{P}^1/\{0,\infty\}} = -\frac{S(z)^{2x}}{x\binom{2x}{x}z^2}\sum_{i=0}^{x} iS(iz)^2\binom{2x}{x+i}$$

$$\langle (1)|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{1}{1-\psi_2}|(1)\rangle^{\mathbb{P}^1/\{0,\infty\}}$$
$$= \frac{S(z)^{2x+2}(x+1)}{2xz^2\binom{2x}{x}}\left[4\sum_{i=2}^{x}\binom{2x}{x+i}\frac{iS(iz)^2}{(i-1)(i+1)}\right.$$
$$\left.-S(z)^2\binom{2x}{x+1}\left(1+2\sum_{k=1}^{x+1}\frac{1}{k}\right)\right].$$

Proof of Theorem 4.14. In principle, one should be able to show that the above formulas follow from Proposition 4.11. We will however find it easier to work with Corollary 4.13 instead. For the most part, this proof consists of checking that the above formulas agree with the $q^{\leq 1}$-coefficients of those in Corollary 4.13, which must hold by degeneration. For instance, in case $x < 0$, Corollary 4.13 and taking q^0-coefficients yield

(75)
$$\left\langle \emptyset \left| \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1/2-\psi}\right| \emptyset \right\rangle^{\mathbb{P}^1/\{0,\infty\}} = \frac{\varsigma(z)^{2x}}{2x\binom{-2x}{-x}z^{2x+3}}\mathrm{Res}_{u=0}\left(\frac{\varsigma(u+2z)\varsigma(u)}{4\varsigma(u+z)^2}\right)^x.$$

To show the empty set case of Conjecture F, we therefore have to show

$$\mathrm{Res}_{u=0}\left(\frac{\varsigma(z)^2\varsigma(u+2z)\varsigma(u)}{\varsigma(2z)^2\varsigma(u+z)^2}\right)^x = f_{-x}(z), \tag{76}$$

where $f_{-x}(z)$ is as in (74). From Lemma 4.15 we can deduce the following differential equation:

$$D_z^2 f_n(z) = n^2 f_n(z) - 2n(2n-1)f_{n-1}(z). \tag{77}$$

To check that this differential equation also holds for the left hand side of (76), one uses Lagrange inversion to see that

$$\mathrm{Res}_{u=0}\left(\frac{\varsigma(z)^2\varsigma(u+2z)\varsigma(u)}{\varsigma(2z)^2\varsigma(u+z)^2}\right)^x = -x[u^{-x}]F(u,z),$$

where $F(u,z)$ is the compositional inverse in u of $G(u,z) = \frac{\varsigma(z)^2\varsigma(u+2z)\varsigma(u)}{\varsigma(2z)^2\varsigma(u+z)^2}$. Equation (77) is therefore equivalent to

$$D_z^2 F(u,z) = \left(uD_u + u^2D_u^2 - 4u^3D_u^2 - 6u^2D_u\right)F(u,z). \tag{78}$$

By using

$$D_uF(u,z) = \frac{1}{(D_uG)(F(u,z),z)}, D_zF(u,z) = -\frac{(D_zG)(F(u,z),z)}{(D_uG)(F(u,z),z)}$$

to express the derivatives of F in terms of the derivatives of G, we see that this is equivalent to

$$2(D_uG)(D_zG)D_uD_zG - (D_uG)^2D_z^2F - (D_zG)^2D_u^2G$$

$$= G(D_uG)^2 - G^2D_u^2G + 4G^3D_u^2G - 6G^2(D_uG)^2,$$

which holds by explicit computation. Since (77) is second order and both sides of (76) are odd functions in z, it remains only to check that the z^1-coefficients on both sides match as desired. The simplest[17] way to see this is to use the fact that the left hand side of (75) has z^{-2}-coefficient $\frac{1}{2}$.

For the degree 1 case, degeneration and Corollary 4.13 give

$$\begin{aligned}
&\langle (1)|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1/2-\psi}|(1)\rangle^{\mathbb{P}^1/\{0,\infty\}} \\
&= -\frac{1}{xz^2}[q^1]\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1/2-\psi}\right\rangle^{E,\prime} \\
&= \frac{\varsigma(z)^{2x+2}}{x\binom{-2x}{-x}z^{2x+5}}\mathrm{Res}_{u=0}\left(\frac{\varsigma(u+2z)\varsigma(u)}{4\varsigma(u+z)^2}\right)^x\left(1+\frac{e^{u+z}+e^{-u-z}}{2}\right) \\
&= \frac{S(2z)^{2x}S(z)^2}{x\binom{-2x}{-x}z^3}\left(f_{-x}(z) - \frac{x}{2}[u^{-x}]\varsigma(2F(u,z)+2z)\right),
\end{aligned}$$

where the last equality uses Lagrange-Bürmann inversion. Using $G(u,z) = \frac{\varsigma(z)^2}{\varsigma(2z)^2}\left(1-\frac{\varsigma(z)^2}{\varsigma(u+z)^2}\right)$ we can further deduce that

$$\begin{aligned}
\varsigma(2F(u,z)+2z) &= \varsigma(F(u,z)+z)\sqrt{4+\varsigma(F(u,z)+z)^2} \\
&= \frac{\varsigma(2z)\sqrt{1-4u}}{1-\frac{\varsigma(2z)^2}{\varsigma(z)^2}u}.
\end{aligned}$$

On the other hand, Conjecture F predicts that

$$\begin{aligned}
&\langle (1)|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1/2-\psi}|(1)\rangle^{\mathbb{P}^1/\{0,\infty\}} \\
&= \frac{S(2z)^{2x}S(z)^2}{x\binom{-2x}{-x}z^3}\left(f_{-x}(z)+\frac{\varsigma(2z)}{2}D_zf_{-x}(z)\right).
\end{aligned}$$

To prove this prediction it thus suffices to show that

$$\frac{\sqrt{1-4u}}{1-\frac{\varsigma(2z)^2}{\varsigma(z)^2}u} - 1 = D_zF(u,z).$$

Indeed, one can check that both sides satisfy (78) and are even in z. Hence it suffices to check that the z^0-coefficients agree, which follows from

$$\frac{1}{\sqrt{1-4u}} = \sum_{n\geq 0}\binom{2n}{n}u^n.$$

[17]Note that $\mathrm{Res}_{u=0}$ and $[z^1]$ do not commute as the residue has to be taken in the domain $z \neq 0$.

Taking the q^0-coefficient of the second equation in Theorem 4.14 yields

$$\left\langle \emptyset|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{1}{1-\psi_2}|\emptyset\right\rangle^{\mathbb{P}^1/\{0,\infty\}}$$
$$= \frac{\varsigma(z)^{2x}}{xz^{2x+4}}\mathrm{Res}_{u=0}\left(\frac{\varsigma(u+2z)\varsigma(u)}{\varsigma(u+z)^2}\right)^x (u+z)$$
$$= \frac{\varsigma(2z)^{2x}}{xz^{2x+4}}\mathrm{Res}_{u=0}G(u,z)^x(u+z) = -\frac{\varsigma(2z)^{2x}}{z^{2x+4}}[u^{-x}]\left(\frac{F(u,z)^2}{2}+zF(u,z)\right).$$

The desired formula for the Hodge integral now follows from (76).

For $x \geq 0$, we first show that

$$\langle\emptyset|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{1}{1-\psi_2}|\emptyset\rangle^{\mathbb{P}^1/\{0,\infty\}} = -\frac{S(z)^{2x}}{xz^2\binom{2x}{x}}\sum_{i=0}^{x} iS(iz)^2\binom{2x}{x+i},$$

which by Corollary 4.13 is equivalent to

$$-\frac{1}{\binom{2x}{x}}\sum_{i=0}^{x}\frac{\varsigma(iz)^2}{i}\binom{2x}{x+i} = \mathrm{Res}_{u=0}\left(\frac{\varsigma(u+z)\varsigma(u-z)}{\varsigma(u)^2}\right)^x\frac{u}{2}$$

(79)
$$= \mathrm{Res}_{u=0}\left(\frac{\sqrt{\varsigma(u+z)\varsigma(u-z)}}{\varsigma(u)}\right)^{2x}\frac{u}{2}$$
$$= \frac{x}{2}[u^{2x}]H(u,z),$$

where $H(u,z)$ is the compositional inverse in u of $\frac{\varsigma(u)}{\sqrt{\varsigma(u+z)\varsigma(u-z)}}$. As above, one can check this by matching up the z^2-coefficients and checking that the differential equation

$$D_z^2 g_x(z) = x^2(g_x(z) - g_{x-1}(z)),$$

which holds for the left hand side of (79) by Lemma 4.15, also holds for the right hand side. The formula for $\langle(1)|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{1}{1-\psi_2}|(1)\rangle^{\mathbb{P}^1/\{0,\infty\}}$ can be proved in a similar manner. □

The above proof used the following purely combinatorial lemma, for which we could not find an adequate reference:

Lemma 4.15. Let $\mathbf{c} = c_0, c_1, \dots$ be an arbitrary infinite sequence of numbers and $n \in \mathbb{Z}$. For any $x \geq 0$ we let

$$p_n^{\mathbf{c}}(x) = \frac{1}{\binom{2x}{x}}\sum_{i=0}^{x}\binom{2x}{x+i}c_i i^{2n+1}.$$

This satisfies the recursion

$$p_{n+1}^{\mathbf{c}}(x) = x^2\left(p_n^{\mathbf{c}}(x) - p_n^{\mathbf{c}}(x-1)\right)$$

for all n and $x > 0$. Let further $\mathbf{1} = 1, 1, 1, \ldots$ the sequence consisting of 1s. If $n \geq 0$, then $p^{\mathbf{1}}_n$ is polynomial of degree $n+1$ in x and we have the special values

$$p^{\mathbf{1}}_0(x) = \frac{x}{2} \text{ and } p^{\mathbf{1}}_{-1}(x) = \frac{1}{2}\sum_{k=1}^{x}\frac{1}{k}$$

Proof. The claim $p^{\mathbf{1}}_0(x) = \frac{x}{2}$ follows from the identity

$$\binom{2x}{x}\frac{x}{2} - \binom{2x}{x+m}\frac{x-m}{2} = \sum_{i=0}^{m}\binom{2x}{x+i}i,$$

which one can show by induction on m. The recursion relation is easily checked and together with $p^{\mathbf{1}}_0(x) = \frac{x}{2}$ implies the other claims. □

We close this section by noting that Theorem 1.40 implies the following formula for certain *triple Hodge integrals* on the moduli space of curves $\overline{M}_{g,n} = \overline{M}^{\circ}_{g,n}(\mathrm{pt}, 0)$:

Corollary 4.16. For $x > 0$, we have

$$\begin{aligned}
&\sum_{g\geq 0}(-1)^{g-1}z^{2g-2}\int_{\overline{M}_{g,n}}\mathbb{E}^\vee(0)\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n}\frac{1}{1-\psi_i} = \\
&= \frac{(-1)^n(n-1)!S(z)^{nx}}{x^{n-1}z^{2n}}\mathrm{Res}_{u_{n-1}=u_n}\cdots\mathrm{Res}_{u_1=u_2} \\
&\sum_{1=l_1<\cdots<l_N=n}\left(\prod_{i\neq j}\frac{\varsigma(z+u_i-u_j)}{\varsigma(u_i-u_j)}\right)^x\cdot\prod_{m=1}^{N-1}\frac{(u_{l_m}-u_{l_{m+1}})^{l_{m+1}-l_m}}{(n-l_m)\cdot(l_{m+1}-l_m)!},
\end{aligned}$$

where $\mathbb{E}^\vee(x) \in H^*(\overline{M}_{g,n})[x]$ is defined in the same way as above. For $x < 0$, we have:

$$\begin{aligned}
&\sum_{g\geq 0}(-1)^{g-1}z^{2g-2}\int_{\overline{M}_{g,n}}\mathbb{E}^\vee(0)\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\prod_{i=1}^{n}\frac{1}{1-\psi_i} = \\
&= 2\frac{(-1)^n(n-1)!S(z)^{nx}}{nx^{n-1}z^{2n}}\mathrm{Res}_{u_{n-1}=u_n}\cdots\mathrm{Res}_{u_1=u_2} \\
&\sum_{\substack{1=l_1<\cdots<l_N=n\\ s_1,\ldots,s_n}}\left(\prod_{i\neq j}\frac{\varsigma(z+u'_i-u'_j)}{\varsigma(u'_i-u'_j)}\right)^x\cdot\prod_{m=1}^{N-1}\frac{(u'_{l_m}-u'_{l_{m+1}})^{l_{m+1}-l_m}}{(n-l_m)\cdot(l_{m+1}-l_m)!},
\end{aligned}$$

where all residues are taken for $z \neq 0$ and the s_i are as in Theorem 1.40.

The proof, which to some extent already appeared above, consists of first using 27 to obtain formulas for

$$\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\prod_{i=2}^{n}\frac{1}{1-\psi_i}\right\rangle^{E,\prime} = \left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\prod_{i=2}^{n}\frac{1}{1-\psi_i}\right\rangle^{E,\circ},$$

where the equality follows from 4.7 and [40, Proposition 2]. By taking the q^0 coefficient of this and using the description for degree 0 Gromov-Witten theory given in [30, §2], we get the desired claim.

4.5 Details on p^0-coefficients

In this section, we make the p^0-coefficients that appear in Theorem 4.10 precise and more explicit - the main result being Corollary 4.21. This section is also heavily inspired by and closely modelled on [70, Appendix A] with most arguments being almost identical. We nonetheless present them in full detail for the convenience of the reader. Throughout, we use the notations

$$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

for the upper half plane and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Our main objects of study are holomorphic functions of the following kind:

Situation 2. Let $n \geq 2$ and $F(u_1, \ldots, u_n; z, \tau)$ be a holomorphic function on a domain of the shape

$$U_{\mathbf{S}} = \left\{ (\mathbf{u}; z, \tau) \in \mathbb{C}^n \times \mathbb{C}^* \times \mathbb{H} \;\middle|\; \begin{array}{l} \text{for all } i \neq j, \lambda \in \Lambda_\tau \text{ and } m \in S_{i,j} : \\ u_i \neq u_j + mz + \lambda \end{array} \right\}$$

for finite sets $S_{i,j} \subset \mathbb{Z}$ for $1 \leq i \neq j \leq n$. We call a choice of $\mathbf{S} = (S_{i,j})_{i,j}$ so that F is holomorphic on $U_{\mathbf{S}}$ a *pole datum* for F if $S_{j,i} = -S_{i,j}$. We furthermore require that

$$F(\mathbf{u} + \lambda; z, \tau) = F(\mathbf{u}; z, \tau)$$

for any $\lambda \in \Lambda_\tau^n$ where $\Lambda_\tau = 2\pi i\mathbb{Z} + 2\pi i\tau\mathbb{Z}$.

Because of translation invariance, any such function can be written as

$$F(\mathbf{u}; z, \tau) = \overline{F}(\mathbf{p}; z, \tau)$$

for $\mathbf{p} = (p_1, \ldots, p_n)$ with $p_i = e^{u_i}$ and $\overline{F}$ holomorphic on a certain subdomain of $(\mathbb{C}^*)^{n+1} \times \mathbb{H}$. We would like to take the constant coefficient in the Laurent series expansion of $\overline{F}$ in $\mathbf{p}$ around the origin, but because of the existence of poles, such an expansion may not exist on all of $(\mathbb{C}^*)^{n+1} \times \mathbb{H}$. It does however exist on domains which are unions of products of annuli in $(\mathbb{C}^*)^n$ and do not touch any poles. In particular, for any permutation $\sigma \in S_n$, we have a Fourier expansion

$$F(\mathbf{u}; z, \tau) = \sum_{\mathbf{m} \in \mathbb{Z}^n} a_{\mathbf{m}}^{(\sigma)}(z, \tau) p_1^{m_1} \cdots p_n^{m_n}$$

on the domain

$$U_\sigma^{(\mathbf{S})} := \left\{ (\mathbf{u}; z, \tau) \in U_{\mathbf{S}} \;\middle|\; \begin{array}{l} \text{for } 1 \leq j < i \leq n \text{ } \textit{and } m \in S_{\sigma^{-1}(i),\sigma^{-1}(j)} : \\ 0 < \mathsf{Re}(u_{\sigma^{-1}(i)} - u_{\sigma^{-1}(j)} - mz) < 2\pi \mathsf{Im}(\tau) \end{array} \right\}.$$

We view the coefficients $a_{\mathbf{m}}^{(\sigma)}$ as germs of holomorphic functions on a punctured neighborhood of $\{0\} \times \mathbb{H} \subset \mathbb{C} \times \mathbb{H}$. Note furthermore that they are independent of the choice of pole datum $\mathbf{S}$. We can therefore define

$$[F]_{p^0,\sigma} := a_{\mathbf{0}}^{(\sigma)}$$

and

$$[F]_{p^0} := \frac{1}{n!} \sum_{\sigma \in S_n} [F]_{p^0,\sigma}. \tag{80}$$

Our main goal in this section will be to study such constant coefficients.

Lemma 4.17. For any $\sigma \in S_n$ we have

$$[F]_{p^0,\sigma} = [F]_{p^0,\tilde{\sigma}}$$

for $\tilde{\sigma}$ a cyclic permutation of σ.

Proof. Let $\tilde{\sigma} \in S_n$ be the unique permutation so that

$$\tilde{\sigma}(i) \equiv \sigma(i) + 1 \pmod{n}.$$

For any $(\mathbf{a}; z, \tau) \in U_\sigma^{(\mathbf{S})}$ we have

$$[F]_{p^0,\sigma} = \frac{1}{(2\pi i)^n} \int_{\mathsf{C}_{a_1}} \cdots \int_{\mathsf{C}_{a_n}} F(\mathbf{u}; z, \tau) du_n \cdots du_1,$$

where C_a is the line segment going from a to $a + 2\pi i$. By Λ_τ-invariance of F, this integral does not change if we replace $\mathbf{a}$ by $\widetilde{\mathbf{a}}$ where

$$\widetilde{a}_l = \begin{cases} a_l, & \text{if } \sigma(l) \neq n \\ a_l + 2\pi i \tau, & \text{if } \sigma(l) = n. \end{cases}$$

However, if z was small enough, then we have $(\widetilde{\mathbf{a}}; z, \tau) \in U_{\tilde{\sigma}}^{(\mathbf{S})}$, which shows that $[F]_{p^0,\sigma} = [F]_{p^0,\tilde{\sigma}}$ as desired. □

The following proposition will make these Fourier coefficients more explicit. For that, we denote by

$$\mathrm{Res}_{u_a = u_b} F(u_1, \ldots, u_n; z, \tau)$$

the (one-variable) residue in u_a at $u_a = u_b$ on $U_{\mathbf{S}}$. This residue again satisfies Situation 2 and has pole datum $\widetilde{\mathbf{S}}$, where $\widetilde{\mathbf{S}}$ is determined by $\widetilde{S}_{i,j} = S_{i,j}$ for $i, j \neq b$ and $\widetilde{S}_{b,j} = S_{a,j} \cup S_{b,j}$. Note also that we needed to impose $z \neq 0$ in the definition of $U_{\mathbf{S}}$ as the residue might otherwise not even be a continuous function as the example

$$\mathrm{Res}_{u_1 = u_2} \frac{1}{z + u_1 - u_2} = \delta_{z,0}$$

shows.

Proposition 4.18. Let F be as in Situation 2 and S a pole datum for F with $0 \in S_{1,n}$. We then have

(81)

$$[F]_{p^0,\sigma} = \sum_{l\geq 1}\sum_{\substack{\mathbf{i}=(i_1,\dots,i_l)\\ s_1,\dots,s_l}} \left[\mathrm{Res}_{u_{i_{l-1}}=u_{i_l}}\cdots \mathrm{Res}_{u_{i_1}=u_{i_2}} F(\mathbf{u}')\cdot \binom{\mathsf{A}_{1,n}(\mathbf{u}')+l-2-n_{\mathbf{i},\sigma}}{l-1}\right]_{p^0,\sigma},$$

where the inner sum runs over all non-recurring[18] sequences $i_1,\dots,i_l \in \{1,\dots,n\}$ with $i_1 = 1$ and $i_l = n$ and integers s_j so that $s_1 = 0$ and for any $j > 1$ there is some $r < j$ so that $s_j - s_r \in S_{i_j,i_r}$. We further wrote $n_{\mathbf{i},\sigma}$ for the number of $1 \leq m \leq l-1$ so that $\sigma(i_{m+1}) > \sigma(i_m)$ and $\mathbf{u}' = (u'_i)_i$ for the shift

$$u'_j = \begin{cases} u_j + s_r z, & \text{if } j = i_r \\ u_j + s_l z, & \text{if } j \notin \{i_1,\dots,i_l\}. \end{cases}$$

Remark 4.19. Note that the functions on the right hand side of (81) satisfy Situation 2 as they are again Λ_τ-invariant and have a pole datum $\widetilde{S}$ determined by $\widetilde{S}_{i,j} = S_{i,j}$ for $i,j \notin \{i_1,\dots,i_l\}$ and $\widetilde{S}_{n,j} = \bigcup_{r=1}^{l}(S_{i_r,j} + s_r - s_l)$. Their p^0 coefficients are therefore well-defined.

Proof. We will prove (81) by showing the following by induction on L:

(82)

$$[F]_{p^0,\sigma} = \sum_{l=0}^{L}\sum_{\substack{\mathbf{i}=(i_1,\dots,i_l)\\ s_1,\dots,s_l}} \left[\mathrm{Res}_{u_{i_{l-1}}=u_{i_l}}\cdots \mathrm{Res}_{u_{i_1}=u_{i_2}} F(\mathbf{u}')\cdot \binom{\mathsf{A}_{1,n}(\mathbf{u}')+l-2-n_{\mathbf{i},\sigma}}{l-1}\right]_{p^0,\sigma}$$

where notations are as in (81) except for the non-recurring sequences $i_1,\dots,i_l \in \{1,\dots,n\}$, where $i_l = n$ is imposed only if $l < L$, but we still require $i_1 = 1$ and $i_m \neq n$ for all l and $m < l$.

The case $L = 0$ is trivial. Assume therefore that (82) holds for L and we want to deduce it for $L+1$. Consider any summand for which $l = L$ and $i_l \neq n$. The function inside of the bracket

$$\mathrm{Res}_{u_{i_{l-1}}=u_{i_l}}\cdots \mathrm{Res}_{u_{i_1}=u_{i_2}} F(\mathbf{u}')\cdot \binom{\mathsf{A}_{1,n}(\mathbf{u}')+l-2-n_{\mathbf{i},\sigma}}{l-1} \tag{83}$$

almost satisfies Situation 2 except for the invariance under translation by $2\pi i\tau$ in u_{i_l} and u_n. It does however admit a pole datum $\widetilde{\mathbf{S}}$ determined

[18] i.e. $i_s \neq i_t$ if $s \neq t$.

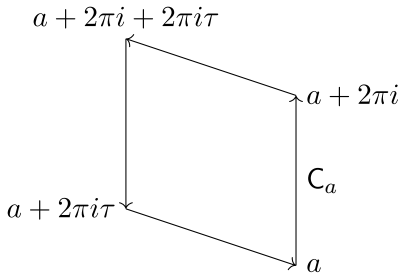

Figure 6: The closed path B_a

by $\widetilde{S}_{i,j} = S_{i,j}$ for $i,j \notin \{i_1,\dots,i_l\}$ and $\widetilde{S}_{i_l,j} = \bigcup_{r=1}^{l}(S_{i_r,j}+s_l-s_r)$. Its $[\cdots]_{p^0,\sigma}$ is therefore well-defined and is also the p^0-coefficient of

(84)
$$\frac{1}{2\pi i}\int_{\mathsf{C}_a} \mathrm{Res}_{u_{i_{l-1}}=u_{i_l}}\cdots\mathrm{Res}_{u_{i_1}=u_{i_2}} F(\mathbf{u}')\cdot\binom{\mathsf{A}_{1,n}(\mathbf{u}')+l-2-n_{\mathbf{i},\sigma}}{l-1}du_{i_l}$$

for any $a\in\mathbb{C}$ and other variables so that

$$(u_1,\dots,u_{i_l-1},a,u_{i_l+1},\dots,u_n;z,\tau)\in U_\sigma^{(\widetilde{\mathbf{S}})}.$$

As before, the path C_a denotes the line segment from a to $a+2\pi i$. Using the Λ_τ-invariance of F and (90) it follows that (84) is equal to

$$\frac{1}{2\pi i}\int_{\mathsf{B}_a}\mathsf{H}(\mathbf{u})du_{i_l},$$

where

$$\mathsf{H}(\mathbf{u}) = \mathrm{Res}_{u_{i_{l-1}}=u_{i_l}}\cdots\mathrm{Res}_{u_{i_1}=u_{i_2}} F(\mathbf{u}')\cdot\binom{\mathsf{A}_{1,n}(\mathbf{u}')+l-1-n_{\mathbf{i},\sigma}}{l}$$

and B_a is the closed contour depicted in Figure 6. By the residue theorem, we have

(85)
$$\frac{1}{2\pi i}\int_{\mathsf{B}_a}\mathsf{H}(\mathbf{u})du_{i_l} = \sum_{j,m,\lambda}\mathrm{Res}_{u_{i_l}=u_j+mz+\lambda}\mathsf{H}(\mathbf{u}),$$

where the sum goes over all u_{i_l}-poles in the interior of B_a, which are exactly points of the shape

$$u_{i_l} = u_j+mz+\lambda$$

for $j\notin\{i_1,\dots,i_l\}$, $m\in\widetilde{S}_{i_l,j}$ and appropriate $\lambda\in\Lambda_\tau$. Since $\mathsf{H}(\mathbf{u})$ is invariant under translation by $2\pi i$, we may take $\lambda = 2\pi i\tau m'$ for some $m'\in\mathbb{Z}$. By requiring $|\mathrm{Re}(z)|$ to be small enough, we can furthermore force $m'=1$ if $\sigma(j)>\sigma(i_l)$ and $m'=0$ if $\sigma(j)<\sigma(i_l)$ meaning that $n_{\mathbf{i}',\sigma} = n_{\mathbf{i},\sigma}+m'$ for

$\mathbf{i}' = (i_1, \ldots, i_l, j)$. Using $\operatorname{Res}_{z=r+s} f(z) = \operatorname{Res}_{z=r} f(z+s)$ repeatedly, (85) simplifies to

$$\sum_{\substack{j \notin \{i_1, \ldots, i_l\} \\ m \in \widetilde{S}_{i_l, j}}} \operatorname{Res}_{u_{i_l} = u_j} \cdots \operatorname{Res}_{u_{i_1} = u_{i_2}} F(\mathbf{u}'') \cdot \binom{\mathsf{A}_{1,n}(\mathbf{u}'') + l - 1 - n_{\mathbf{i}', \sigma}}{l} \tag{86}$$

where $\mathbf{u}'' = (u''_i)_i$ is defined by

$$u''_i = \begin{cases} u'_i + mz, & \text{if } i \in \{i_1, \ldots, i_l\} \\ u'_i, & \text{else.} \end{cases}$$

Since $[\cdots]_{p^0, \sigma}$ and the residues are invariant under shift of all variables, we can instead also take

$$u''_i = \begin{cases} u'_i, & \text{if } i \in \{i_1, \ldots, i_l\} \\ u'_i - mz, & \text{else.} \end{cases}$$

and the p^0-coefficient of (86) would have been the same. Defining $s_{l+1} = s_l - m$ yields the $l = L + 1$ summands in (82), which finishes the induction. □

Summing over all σ yields:

Proposition 4.20. Let F be as in Situation 2 and pole datum S with $0 \in S_{1,n}$. Then

$$[F]_{p^0} = \sum_{l \geq 1} \sum_{\substack{\mathbf{i} = (i_1, \ldots, i_l) \\ s_1, \ldots, s_l}} \left[\operatorname{Res}_{u_{i_{l-1}} = u_{i_l}} \cdots \operatorname{Res}_{u_{i_1} = u_{i_2}} F(\mathbf{u}') \cdot \frac{\mathsf{A}_{1,n}(\mathbf{u}')^{l-1}}{(l-1)!} \right]_{p^0} \tag{87}$$

with notations and summation variables are as in Proposition 4.18.

Proof. By Lemma 4.17, we have

$$[F]_{p^0} = \frac{1}{n!} \sum_{\sigma \in S_n} [F]_{p^0, \sigma} = \frac{1}{(n-1)!} \sum_{\substack{\sigma \in S_n \\ \sigma(n) = n}} [F]_{p^0, \sigma}.$$

Using Proposition 4.18, we get

$$[F]_{p^0} = \frac{1}{(n-1)!} \sum_{l \geq 1} \sum_{\substack{\mathbf{i} = (i_1, \ldots, i_l) \\ s_1, \ldots, s_l}} \sum_{\substack{\sigma \in S_n \\ \sigma(n) = n}} \left[\operatorname{Res}_{u_{i_{l-1}} = u_{i_l}} \cdots \operatorname{Res}_{u_{i_1} = u_{i_2}} F(\mathbf{u}') \cdot \binom{\mathsf{A}_{1,n}(\mathbf{u}') + l - 2 - n_{\mathbf{i}, \sigma}}{l - 1} \right]_{p^0, \sigma}.$$

Note that the functions inside of the brackets on the right hand side only depend on u_i for $i \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_{l-1}\}$. From this, one can see that given a choice of $\mathbf{i}$ and $s_1, \ldots, s_l$ as above, any two permutations $\sigma, \sigma' \in S_n$ that induce the same ordering on $\{1, \ldots, n-1\} \setminus \{i_1, \ldots, i_{l-1}\}$ will yield equal summands on the right hand side. We can therefore split the inner most sum into a sum over all permutations τ of $\{i_1, \ldots, i_{l-1}\}$, all permutations ρ of $\{1, \ldots, n-1\} \setminus \{i_1, \ldots, i_{l-1}\}$ and all the $\binom{n-1}{l-1}$ ways of shuffling these together. This gives us

$$[F]_{p^0} = \sum_{l \geq 1} \sum_{\substack{\mathbf{i}=(i_1,\ldots,i_l) \\ s_1,\ldots,s_l}} \sum_{\rho \in S_{n-l}} \frac{1}{(n-l)! \cdot (l-1)!}$$

$$\left[\sum_{\tau \in S_{l-1}} \operatorname{Res}_{u_{i_{l-1}}=u_{i_l}} \cdots \operatorname{Res}_{u_{i_1}=u_{i_2}} F(\mathbf{u}') \cdot \binom{\mathsf{A}_{1,n}(\mathbf{u}') + l - 2 - n_{\mathbf{i},\tau}}{l-1} \right]_{p^0,\rho}.$$

The proposition now follows from Worpitzky's identity

$$\sum_{\tau \in S_{l-1}} \binom{x + l - 2 - a_\tau}{l-1} = x^{l-1},$$

where a_τ is the number of all $1 \leq i \leq l-2$ so that $\tau(i+1) > \tau(i)$. □

The main case that we want to apply Proposition 4.20 to is (70), which has a lot of symmetry. This yields the following simplification:

Corollary 4.21. Let F be as in Situation 2 and invariant under permutation of the u_i. Let furthermore $S \subset \mathbb{Z}$ be so that $S_{i,j} = S$ for all $i \neq j$ is a pole datum for F. Then we have

(88)
$$[F]_{p^0} = (n-1)! \sum_{\substack{1=l_1<l_2<\ldots<l_N=n \\ s_1,\ldots,s_n}} \operatorname{Res}_{u_{n-1}=u_n} \cdots \operatorname{Res}_{u_1=u_2}$$
$$F(\mathbf{u}') \cdot \prod_{m=1}^{N-1} \frac{\mathsf{A}_{l_m,l_{m+1}}(\mathbf{u}')^{l_{m+1}-l_m}}{(n-l_m) \cdot (l_{m+1}-l_m)!},$$

where the $s_1, \ldots, s_n$ are integers so that $s_1 = 0$ and for any $j > 1$ one of the following must hold:

- $j = l_{t+1}$ for some $t \geq 0$ and $s_j = s_{l_t}$.
- There is some $r < j$ so that $s_j - s_r \in S$.

We further wrote $\mathbf{u}' = (u'_j)_j$, where $u'_j = u_j + s_j z$.

Remark 4.22. Note that the right hand side of (88) only depends on z and τ. Indeed, the only other variable that it a priori depends on is u_n in which it is holomorphic on all of $\mathbb{C}$ for fixed z and τ. By Λ_τ invariance it must however be constant in u_n.

Proof. Note that (88) follows from repeatedly applying the following claim:

Claim. Let F be as in Situation 2 and invariant under permuting the u_i for $i \geq 2$. Let $S_1, S_2 \subset \mathbb{Z}$ be so that $S_{1,j} = S_{i,1} = S_1$ and $S_{i,j} = S_2$ for $i, j \geq 2$ determines a pole datum for F. Then

$$[F]_{p^0} = \sum_{l \geq 1} \sum_{s_1, \ldots, s_l} \left[\mathrm{Res}_{u_{l-1}=u_l} \cdots \mathrm{Res}_{u_1=u_2} F(\mathbf{u}') \cdot \binom{n-1}{l-1} \frac{\mathsf{A}_{1,l}(\mathbf{u}')^{l-1}}{n-1} \right]_{p^0},$$

where the s_j are integers so that $s_1 = 0$ and for any $j > 1$ we must have one of:

- There is some $1 < r < j$ so that $s_j - s_r \in S_2$
- $s_j \in S_1$
- $j = l$ and $s_j = 0$.

We furthermore wrote and $\mathbf{u}' = (u'_j)_j$ with

$$u'_j = \begin{cases} u_j + s_j z, & \text{if } j \leq l \\ u_j, & \text{else.} \end{cases}$$

This claim directly follows from Proposition 4.20 by using that any of the $\frac{(n-2)!}{(n-l)!}$ many non-recurring sequences $1 = i_1, \ldots, i_l$ of length l which occur there give the same contribution as the case $i_j = j$ due to symmetry. The exceptional case of $s_l = 0$ is due to the fact that we have to include 0 in $S_{1,l}$. □

A Quasi-modular and quasi-Jacobi forms

This appendix recalls basic facts about the theories of quasi-modular and quasi-Jacobi forms, which we use throughout Section 4. We only collect the bare essentials that are important for this thesis. For more on quasi-modular forms, see in particular [11, 14, 41] and [20] for Jacobi forms. Quasi-Jacobi forms were first introduced in [50]. See also [39, §2].

A.1 Quasi-modular forms

For us, the ring of *quasi-modular forms* will be the subring

$$\mathrm{QMod} = \mathbb{Q}[G_k \mid k \geq 2 \text{ even}] \subset \mathbb{Q}[[q]]$$

generated by the *Eisenstein series*

$$G_k = -\frac{B_k}{2k} + \sum_{n \geq 1} \Big(\sum_{d \mid n} d^{k-1} \Big) q^n$$

for even $k \geq 2$ with B_k the kth Bernoulli number. The Bernoulli numbers are determined by the identity

$$\sum_{k\geq 0} \frac{B_k}{k!} z^k = \frac{z}{e^z - 1},$$

which is the convention that is used throughout this thesis.

One can also characterize $\mathrm{QMod} \otimes_{\mathbb{Q}} \mathbb{C}$ as the ring of holomorphic functions on the upper half plane that satisfy certain transformation properties - see [11, 14, 41] for details. From this, one can derive that QMod admits a natural grading called *weight*:

$$\mathrm{QMod} = \bigoplus_{k\geq 0} \mathrm{QMod}_k$$

so that $G_k \in \mathrm{QMod}_k$. Furthermore, QMod is freely generated by G_2, G_4, G_6, which allows one to define the formal derivative $\frac{d}{dG_2}$, which is called *holomorphic anomaly operator*. Moreover, the derivative $D_\tau := q\frac{d}{dq}$ preserves QMod and satisfies the relation

$$\left[\frac{d}{dG_2}, D_\tau\right] = -2 \cdot \mathsf{wt},$$

which turns QMod into an sl_2-representation.

A.2 Quasi-Jacobi forms

The most important example of a quasi-Jacobi form is the Jacobi Θ-function

$$\Theta(z,\tau) = \varsigma(z) \prod_{k\geq 1} \frac{(1-q^k e^z)(1-q^k e^{-z})}{(1-q^k)^2} = z e^{-2\sum_{k\geq 1} G_{2k} \frac{z^{2k}}{(2k)!}},$$

where we wrote $\varsigma(z) = e^{z/2} - e^{-z/2}$ and $q = e^{2\pi i \tau}$. All other quasi-Jacobi forms can be expressed in terms of Θ. For example, we have

$$\mathsf{A} = \frac{D_z \Theta}{\Theta} = \frac{1}{z} - 2\sum_{k\geq 1} G_{2k} \frac{z^{2k-1}}{(2k-1)!},$$

where $D_z = \frac{d}{dz}$ and $\wp$ the Weierstrass $\wp$-function, which we can write as

$$\wp = -2G_2 - D_z \mathsf{A}.$$

We also denote $\wp' = D_z \wp$.

Definition A.1. The ring of *quasi-Jacobi forms* is the subring

$$\mathrm{QJac} \subset \mathbb{Q}[\Theta, \mathsf{A}, G_2, \wp, \wp', G_4] \subset \mathbb{Q}[[q]]((z))$$

of power series in z and $q = e^{2\pi i\tau}$ which are holomorphic as functions $(z,\tau) \in \mathbb{C}\times\mathbb{H} \to \mathbb{C}$. The ring is doubly graded

$$\mathrm{QJac} = \bigoplus_{k,m} \mathrm{QJac}_{k,m}$$

by *weight* k and *index* m, which is specified on generators as follows:

form	weight	index
Θ	-1	$1/2$
A	1	0
G_2	2	0
$\wp$	2	0
$\wp'$	3	0
G_4	4	0

Remark A.2. (1) Similar to quasi-modular forms, one can also define quasi-Jacobi forms as functions that satisfy certain transformation laws. For example:

(89)
$$\Theta(z + 2\pi i\tau\lambda + 2\pi i\mu, \tau) = (-1)^{\lambda+\mu} e^{-\lambda z} q^{-\lambda^2/2}\Theta(z,\tau) \text{ for } \lambda,\mu\in\mathbb{Z},$$
$$\Theta\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \frac{e^{\frac{cz^2}{2\pi i(c\tau+d)}}}{c\tau+d}\Theta(z,\tau) \text{ for } \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}).$$

This determines the transformation laws of all other quasi-Jacobi forms. In particular:

$$\mathsf{A}(z + 2\pi i\lambda + 2\pi i\tau\mu, \tau) = \mathsf{A}(z,\tau) - \lambda \text{ for } \lambda,\mu\in\mathbb{Z}$$
$$\mathsf{A}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\left(\mathsf{A}(z,\tau) + \frac{cz}{2\pi i(c\tau+d)}\right) \tag{90}$$
$$\text{for } \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}).$$

(2) One can show

$$\mathrm{QJac}_{*,0} = \mathrm{QMod}_*.$$

Indeed, the Weierstrass equation

$$G_6 = \frac{12}{7}\wp^3 - \frac{3}{7}(\wp')^2 - \frac{60}{7}\wp G_4$$

gives $G_6 \in \mathrm{QJac}_{6,0}$ and hence "$\supset$".

(3) The weight of a given quasi-Jacobi form $\phi \in \mathrm{QJac}_{k,m}$ can be seen from the z-expansion. Indeed, we always have:

$$\phi(z) = \sum_{g \gg -\infty} a_g z^g \tag{91}$$

with $a_g \in \mathrm{QMod}_{g+k}$ (c.f. [20, Theorem 3.1]).

(4) Since all quasi-Jacobi forms are invariant under $z \mapsto z + 2\pi i$, it follows that z is algebraically independent over QJac. As a result, one can extend the double-grading to QJac$[z^{\pm}]$:

form	weight	index
Θ	-1	$1/2$
A	1	0
G_2	2	0
$\wp$	2	0
$\wp'$	3	0
G_4	4	0
z	-1	0

(5) QJac is closed under D_z and $D_\tau = q\frac{d}{dq}$, which have degrees $(1,0)$ and $(2,0)$ respectively. Furthermore, $\Theta, \mathsf{A}, G_2, \wp, \wp', G_4$ are algebraically independent, which allows one to define *holomorphic anomaly operators* $\frac{d}{d\mathsf{A}}$ and $\frac{d}{dG_2}$. These have degrees $(-1,0)$ and $(-2,0)$ respectively. Note here that $\frac{d}{dG_2}$ is not the same as applying the holomorphic anomaly operator of Section A.1 on each z-coefficient of (91). Indeed, denoting the latter operator by $\left(\frac{d}{dG_2}\right)_z$:

$$\left(\frac{d}{dG_2}\right)_z = -2z^2 \cdot \mathsf{ind} - 2z\frac{d}{d\mathsf{A}} + \frac{d}{dG_2}. \tag{92}$$

We also have the following commutation relations:

$$\begin{aligned} \left[\frac{d}{dG_2}, D_\tau\right] &= -2 \cdot \mathsf{wt}, \left[\frac{d}{d\mathsf{A}}, D_z\right] = 2 \cdot \mathsf{ind} \\ \left[\frac{d}{dG_2}, D_z\right] &= -2\frac{d}{dA}, \left[\frac{d}{d\mathsf{A}}, D_\tau\right] = D_z. \end{aligned} \tag{93}$$

(6) There is a notion of Hecke-operators for Jacobi forms - one of which is

$$\phi(z) \mapsto \phi(n \cdot z)$$

for $n \geq 0$ (see [20]), which also extends to quasi-Jacobi forms and maps $\mathrm{QJac}_{k,m}$ to QJac_{k,n^2m}. For instance:

$$\Theta(2z) = -\Theta^4\wp'.$$

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